

Quantum Optical Realization of Arbitrary Linear Transformations Allowing for Loss and Gain

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Unitary transformations are routinely modeled and implemented in the field of quantum optics. In contrast, nonunitary transformations, which can involve loss and gain, require a different approach. In this work, we present a universal method to deal with nonunitary networks. An input to the method is an arbitrary linear transformation matrix of optical modes that does not need to adhere to bosonic commutation relations. The method constructs a transformation that includes the network of interest and accounts for full quantum optical effects related to loss and gain. Furthermore, through a decomposition in terms of simple building blocks, it provides a step-by-step implementation recipe, in a manner similar to the decomposition by Reck *et al.* [*Experimental Realization of Any Discrete Unitary Operator*, *Phys. Rev. Lett.* **73**, 58 (1994)] but applicable to nonunitary transformations. Applications of the method include the implementation of positive-operator-valued measures and the design of probabilistic optical quantum information protocols.

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I. INTRODUCTION

Transformations between sets of orthogonal input and output modes are ubiquitous in optics and quantum information technology. In particular, linear transformations between the amplitudes of the input and output modes are used to perform a variety of tasks, e.g., to operate single-qubit gates or to model the action of physical elements such as beam splitters [1]. Mathematically, a linear transformation can be expressed as a transformation matrix T relating the mean fields of the m optical input modes $1 \text{ in} \dots m \text{ in}$ with those of the n optical output modes $1 \text{ out} \dots n \text{ out}$:

$$\begin{pmatrix} \langle \hat{a}_{1 \text{ out}} \rangle \\ \vdots \\ \langle \hat{a}_{n \text{ out}} \rangle \end{pmatrix} = T \begin{pmatrix} \langle \hat{a}_{1 \text{ in}} \rangle \\ \vdots \\ \langle \hat{a}_{m \text{ in}} \rangle \end{pmatrix}. \quad (1)$$

Among such transformations, unitary optical networks, for which T is a unitary matrix that also relates the annihilation operators themselves and not only their expectation values, are routinely used in optical quantum information processing. Unitary networks conserve the number of photons, and their implementation in terms of basic building blocks, namely, phase shifters acting on individual modes and beam splitters mixing two modes at a time, is well understood [2,3]. However, as unitarity imposes restrictions on the transformation matrix, unitary networks can be considered as a special case of linear networks.

Relaxing the restrictions unlocks fascinating opportunities for new transformations, including the options of loss and gain [4–14]. One noteworthy class of such networks consists of asymmetric nonunitary beam splitters, which can allow highly tunable quantum interference [14].

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Among the symmetric beam splitters, an example of a nonunitary beam splitter that has attracted particular interest is the 2×2 transformation given by the matrix $T = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ [4,6,8,12,13]. A device with this action can be thought of as a lossy beam splitter. It exhibits a striking, apparently nonlinear, behavior when one photon is incident on each input: Either both photons are lost, or neither of them is lost.

Even though the initial interest in devices such as this one was primarily theoretical, the technical capabilities in the design and fabrication of novel and nanostructured materials are now making elements with such properties possible [12,13,15–18]. Nonunitary transformation matrices also prove useful in modeling the inevitable imperfections of real optical elements that show a wavelength-dependent behavior [4]. A further reason for stepping outside the framework of unitary networks is that transformations may have an unequal number of input and output modes of interest, a clear indicator of nonunitarity. Two particularly simple examples are Y junctions in integrated optics and absorptive polarizers, which feature two orthogonal input modes but only one output mode.

For a quantum optical description of such transformations, the relationship of Eq. (1) does not suffice. Additionally, a relationship between annihilation and creation operators is required. It would be tempting to simply drop the expectation values in Eq. (1), but the modes associated with nonunitary networks generally would not fulfill the required bosonic commutation relations. Hence, from now on, we drop the expectation values and take T to be a transformation between the annihilation operators of interest, with the understanding that it is an incomplete transformation: Ancilla modes need to be introduced in the mathematical description to faithfully reproduce or predict the full quantum optical transformation. Although this is straightforward for the simple examples of Y junctions and polarizers, a systematic method to deal with larger-scale problems would be desirable.

In this paper, we investigate whether such a strategy is possible for all linear transformation matrices, how many ancilla modes are needed for any given case, and how a full enlarged quantum optical network can be mathematically represented and physically realized.

Related problems have previously been studied in a number of works. In Refs. [9,10], Miller demonstrates how to construct universal linear transformation machines in a classical optics picture, where the mean fields are of interest, so a modulation of field amplitudes is possible without the need to take into account quantum optical effects. The Bloch-Messiah reduction also shows how a decomposition into basic building blocks can be found, and it includes a rigorous quantum optical description. However, it begins with the complete transformation matrix respecting bosonic commutation relations (a linear unitary Bogoliubov transformation) rather than a partial network [19]. Allowing

nonunitary partial networks as an input, He *et al.* and Knöll and co-workers presented techniques to find corresponding enlarged transformations, but they do not allow for transformations that include both loss and gain [5,7,20].

In this article, we put forward a systematic method for dealing with linear transformation matrices of any size, allowing for the option of loss and gain. The method combines a singular value decomposition of the partial network and the single-mode treatment presented in Ref. [21] to provide full information about the transformation; therefore, the quantum optical output state can be calculated for any input state. In addition, as a generalization of the seminal decomposition in Ref. [2] or the more recent variant of Ref. [3] to nonunitary networks, our method shows how to realize transformations in terms of the basic building blocks of phase shifters, beam splitters, and parametric amplifiers.

We discuss possible applications of nonunitary networks, which include the implementation of positive-operator-valued measures (POVMs) and probabilistic optical quantum information protocols. The physical realization of small circuits could be achieved with bulk optics, whereas integrated optics would be naturally suited as a platform for larger-scale networks. In the appendixes, we demonstrate the method on several examples, including the lossy beam splitter with the apparent nonlinear action described earlier. The lossy beam splitter example illustrates how devices made of exotic materials can be replaced by standard optical circuits.

II. RESULTS

We begin by outlining the basic structure of the method, illustrated in Fig. 1. Starting with the partial network T , a singular value decomposition is performed, which yields three main components: U , D , and W . The singular value decomposition is particularly useful as each main

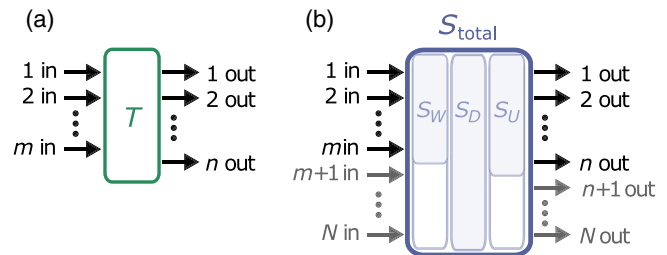


FIG. 1. The concept of mode transformations. (a) The linear network T specifies a mapping from m input modes to n output modes and may be characterized by a nonunitary matrix. (b) The full network S_{total} includes the nominal modes of T , as well as ancilla modes, which account for any losses and gains in T . The transformation S_{total} consists of three main components, of which only the second involves a coupling between the nominal modes and ancilla modes.

component is well suited to be further decomposed into a sequence of operations in the form of simple building blocks. Each of these building blocks corresponds to a physical operation and has a known complete quantum optical description.

Importantly, since U and W are unitary, they can physically be implemented with phase shifters and beam splitters using the techniques of Ref. [2] or [3]. These two main components only involve the nominal modes and can be understood as an initial conversion from the input modes to another basis, the modulation basis, and a final conversion from the modulation basis to the output modes. The modulation takes place in D , the second main component, and includes interactions with ancilla modes. Specifically, each operation here corresponds to a singular value, and each singular value that is different from 1 results in the interaction of a nominal mode with a vacuum ancilla, either through a beam splitter or a parametric amplifier.

Combining all of the individual operations provides the quantum optical description of the overall transformation, which we denote by S_{total} .

A. Preliminaries

As a basis for the detailed description of the method in Sec. II B, it is useful to first establish some terminology and a single-mode framework following Ref. [21], i.e., the case with a single nominal input mode and a single nominal output mode. In the general multimode treatment put forward in the present article, we make extensive use of these basic single-mode tools.

1. Quasiunitarity

A $2N \times 2N$ -dimensional matrix S is quasiunitary if

$$SGS^\dagger = G, \quad (2)$$

where G is defined as the $2N \times 2N$ diagonal matrix with the first N diagonal elements equal to 1 and the last N diagonal elements equal to -1 [21,22].

2. Properties of partial and full transformations

The input of the method is the partial network T , a complex matrix of any size without any conditions on its elements. We call T a partial network because, in general, T on its own is not enough to predict the quantum optical output for an arbitrary input. For instance, the noise due to vacuum fluctuations in ancilla modes is neglected, and this noise impacts quantum properties of light such as the degree of squeezing. One of the aims of the method is to construct another network, S_{total} , which contains T as its upper-left block and includes the ancilla modes, such that it can be used as a quantum optical model of the transformation T (Fig. 2). The matrix S_{total} relates the

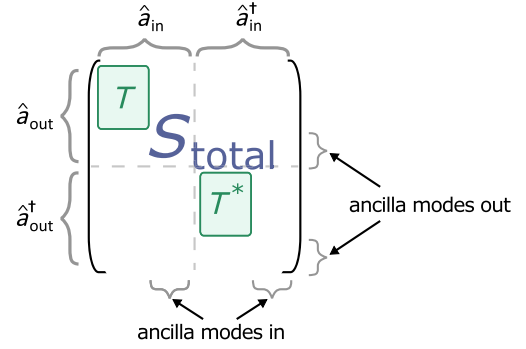


FIG. 2. The structure of the matrix S_{total} . The matrix elements represent the coupling between the respective input modes (columns) and output modes (rows). Viewed as a 2×2 block matrix, nonzero elements in the off-diagonal blocks are responsible for active elements in the implementation. The transformation of interest, T , is contained in the upper-left block.

input and output creation and annihilation operators in the following way:

$$\begin{pmatrix} \hat{a}_{1 \text{ out}} \\ \vdots \\ \hat{a}_{N \text{ out}} \\ \hat{a}_{1 \text{ out}}^\dagger \\ \vdots \\ \hat{a}_{N \text{ out}}^\dagger \end{pmatrix} = S_{\text{total}} \begin{pmatrix} \hat{a}_{1 \text{ in}} \\ \vdots \\ \hat{a}_{N \text{ in}} \\ \hat{a}_{1 \text{ in}}^\dagger \\ \vdots \\ \hat{a}_{N \text{ in}}^\dagger \end{pmatrix}. \quad (3)$$

It is $2N \times 2N$ dimensional, where, in general, $N \geq \max(m, n)$ because of the possible inclusion of ancilla modes. A requirement on S_{total} is that it must fulfill the quasiunitarity equation (2) so that its modes are bosonic, i.e., the creation and annihilation operators fulfill the standard bosonic commutation relations $[\hat{a}_i, \hat{a}_j] = 0$, $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$. The reason that creation operators are included in the description is that active elements associated with gain lead to a coupling of creation and annihilation operators. In fact, whether the transformation contains only passive elements or includes active elements can be recognized based on the off-diagonal blocks of S_{total} when viewed as a 2×2 block matrix: A passive transformation has zeros for these blocks.

3. Single-mode loss

A single lossy channel characterized by $T = \sigma$, where $\sigma \in \mathbb{R}$, $0 \leq \sigma < 1$, can be implemented using a lossless beam splitter with an ancilla mode \hat{a}_2 initialized in its vacuum state. The transformation of the modes is then generated by a beam-splitter Hamiltonian $\hat{H} = i\phi(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1)$, with $\cos \phi = \sigma$ and $\sin \phi = \sqrt{1 - \sigma^2}$ representing the transmission and reflection amplitudes of the beam splitter,

respectively. The connection between this Hamiltonian and the corresponding transformation matrix

$$S = \begin{pmatrix} \sigma & \sqrt{1-\sigma^2} & 0 & 0 \\ -\sqrt{1-\sigma^2} & \sigma & 0 & 0 \\ 0 & 0 & \sigma & \sqrt{1-\sigma^2} \\ 0 & 0 & -\sqrt{1-\sigma^2} & \sigma \end{pmatrix}, \quad (4)$$

such that

$$\begin{pmatrix} \hat{a}_{1\text{out}} \\ \hat{a}_{2\text{out}} \\ \hat{a}_{1\text{out}}^\dagger \\ \hat{a}_{2\text{out}}^\dagger \end{pmatrix} = S \begin{pmatrix} \hat{a}_{1\text{in}} \\ \hat{a}_{2\text{in}} \\ \hat{a}_{1\text{in}}^\dagger \\ \hat{a}_{2\text{in}}^\dagger \end{pmatrix}, \quad (5)$$

is described in Ref. [21], pp. 1215–1216 (see also [23]).

4. Single-mode gain

Similarly, for a single channel with gain given by $T = \sigma$, where $\sigma \in \mathbb{R}$, $\sigma > 1$, we introduce an ancilla mode \hat{a}_2 , initially in vacuum. Gain can be realized with a parametric amplifier with a Hamiltonian $\hat{H} = i\xi(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2)$, where $\cosh \xi = \sigma$. The corresponding enlarged transformation

$$S = \begin{pmatrix} \sigma & 0 & 0 & \sqrt{\sigma^2-1} \\ 0 & \sigma & \sqrt{\sigma^2-1} & 0 \\ 0 & \sqrt{\sigma^2-1} & \sigma & 0 \\ \sqrt{\sigma^2-1} & 0 & 0 & \sigma \end{pmatrix} \quad (6)$$

can be constructed as described in Ref. [21], p. 1217.

5. Single-mode phase shift

Complex transformations involve phase shifts $T = e^{i\varphi}$ ($\varphi \in \mathbb{R}$) for which no ancilla mode is required. The Hamiltonian is $\hat{H} = -\varphi \hat{a}_1^\dagger \hat{a}_1$, and the transformation takes the simple form of $\begin{pmatrix} \hat{a}_{1\text{out}} \\ \hat{a}_{1\text{out}}^\dagger \end{pmatrix} = S \begin{pmatrix} \hat{a}_{1\text{in}} \\ \hat{a}_{1\text{in}}^\dagger \end{pmatrix}$, with

$$S = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}. \quad (7)$$

B. Method

The method consists of the steps illustrated in Fig. 3 and described below:

- (a) *Step 1, singular value decomposition of T* : A singular value decomposition provides the main components

$$T = UDW, \quad (8)$$

where U and W are unitary matrices and D is a diagonal matrix with non-negative real diagonal elements.

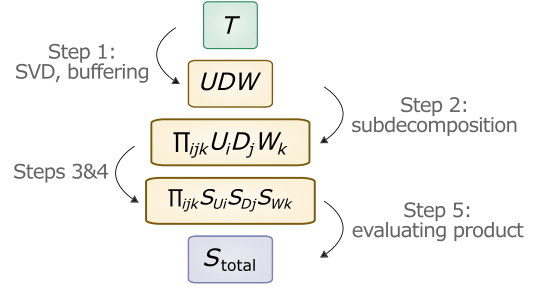


FIG. 3. Overview of the method. Starting with T , a singular value decomposition, followed by a padding with identity matrix elements to make all components the same size, provides the main components U , D , and W . In step 2, a further decomposition of U and W into simple blocks is found through the technique in Ref. [2] or [3], and D is written as a product of single-mode modulations. For each of these, a corresponding enlarged transformation is determined in steps 3 and 4. Finally, the overall transformation S_{total} is obtained as the product of all the individual transformations.

- (b) *Step 1b, if T is not square*: The method can be applied to transformations of arbitrary dimensionality, including those given by nonsquare matrices. Such transformations apparently correspond to unequal numbers of input and output modes, which is an incomplete description in quantum mechanics as it can neglect necessary sources of quantum noise. For this reason, nonsquare transformations definitely require either ancilla input modes or ancilla output modes so that the number of inputs matches the outputs. In addition, both square and nonsquare transformations may require what we refer to as full ancilla modes, which will be discussed later.

A singular value decomposition of a nonsquare $n \times m$ matrix provides a square $n \times n$ matrix U , a diagonal $n \times m$ matrix D , and another square $m \times m$ matrix W . The impact of the missing input or output modes can be naturally taken into account through augmentation of the matrices U , D , and W to the $\max(m, n) \times \max(m, n)$ size, by padding them with the corresponding elements of the identity matrix as the last rows and columns, where required. The following steps 2–5 should be applied to the augmented matrices, which we still call U , D , W for simplicity.

An example of an application of the method to a nonsquare matrix is shown in Appendix C 1.

- (c) *Step 2, subdecomposition of all three matrices*: We further decompose the two unitary matrices U and W by the established methods of Ref. [2] or [3], and thereby write the main components as the products $U = \prod_i U_i$ and $W = \prod_k W_k$, respectively. All of the matrices U_i and W_k correspond to simple physical operations of phase shifters and beam splitters. The diagonal matrix D can be decomposed into a product of matrices $D = \prod_j D_j$, where each D_j is the identity matrix with element (j, j) replaced by $D_{j,j}$. Overall, we obtain $T = \prod_{ijk} U_i D_j W_k$.

- (d) *Step 3, determining the dimensionality of the enlarged system and assigning modes:* The dimensionality of the enlarged matrices is $2N \times 2N$, with N given by $N \equiv n_N + n_A$, where $n_N \equiv \max(m, n)$ is the number of nominal modes, i.e., the number of modes explicitly included in T , and n_A is the number of singular values of T not equal to 1. Modes 1 to n_N are associated with the nominal modes, while modes $n_N + 1$ to N are associated with full ancilla modes, by which we denote those modes that are added throughout the whole transformation, not just as inputs or outputs to match the number of input and output modes, as described in step 1b. Each nominal mode j has its own corresponding ancilla mode m_{Aj} if the j th singular value of T differs from 1.
- (e) *Step 4, finding associated quasiunitary matrices:* We construct a matrix S_{U_i} for each U_i and, similarly, a matrix S_{W_k} for each W_k . Here, S_{U_i} and S_{W_k} are defined as

$$S_{U_i} \equiv \begin{pmatrix} U_i & 0 & 0 & 0 \\ 0 & I_{n_A} & 0 & 0 \\ 0 & 0 & U_i^* & 0 \\ 0 & 0 & 0 & I_{n_A} \end{pmatrix} \quad (9)$$

and

$$S_{W_k} \equiv \begin{pmatrix} W_k & 0 & 0 & 0 \\ 0 & I_{n_A} & 0 & 0 \\ 0 & 0 & W_k^* & 0 \\ 0 & 0 & 0 & I_{n_A} \end{pmatrix}, \quad (10)$$

respectively, with U_i and W_k being the $n_N \times n_N$ matrices from step 2, I_{n_A} being the $n_A \times n_A$ identity matrix, and the 0s being matrices of the appropriate size filled with zeros. In addition, we construct a matrix S_{D_j} for each D_j . For the special case in which the j th singular value $\sigma_j = 1$, S_{D_j} is the $N \times N$ identity matrix and therefore not needed. Otherwise, if $\sigma_j \neq 1$, the matrix S_{D_j} is the $N \times N$ identity matrix with the elements corresponding to the intersection of rows and columns j , m_{Aj} , $j + N$, $m_{Aj} + N$ replaced by

$$\begin{pmatrix} \sigma_j & \sqrt{1 - \sigma_j^2} & 0 & 0 \\ -\sqrt{1 - \sigma_j^2} & \sigma_j & 0 & 0 \\ 0 & 0 & \sigma_j & \sqrt{1 - \sigma_j^2} \\ 0 & 0 & -\sqrt{1 - \sigma_j^2} & \sigma_j \end{pmatrix} \quad (11)$$

if $\sigma_j < 1$, and

$$\begin{pmatrix} \sigma_j & 0 & 0 & \sqrt{\sigma_j^2 - 1} \\ 0 & \sigma_j & \sqrt{\sigma_j^2 - 1} & 0 \\ 0 & \sqrt{\sigma_j^2 - 1} & \sigma_j & 0 \\ \sqrt{\sigma_j^2 - 1} & 0 & 0 & \sigma_j \end{pmatrix} \quad (12)$$

if $\sigma_j > 1$. This also allows us to deal with transformations that combine loss in some modes with gain in others, which previously proposed methods did not accommodate. An example can be found in Appendix B 2.

- (f) *Step 5, multiplication of quasiunitary matrices to obtain the overall transformation:* We obtain the overall enlarged transformation as

$$S_{\text{total}} = \prod_{ijk} S_{U_i} S_{D_j} S_{W_k}. \quad (13)$$

A proof that S_{total} fulfills the quasiunitarity equation (2) and contains T as its upper-left block can be found in Appendix A, and an example decomposition is shown in Appendix B 1.

- (g) *Implementation of the decomposition in terms of simple building blocks:* The full decomposition $S_{\text{total}} = \prod_{ijk} S_{U_i} S_{D_j} S_{W_k}$ provides a recipe for an implementation in terms of the simple building blocks of phase shifters, beam splitters, and parametric amplifiers, as each of the matrices in the decomposition directly corresponds to such a building block. The factors S_{U_i} and S_{W_k} correspond to beam splitters and phase shifters involving the nominal modes, i.e., the first n_N modes. The factors S_{D_j} that differ from the identity correspond to beam splitters and parametric amplifiers, each involving one of the nominal modes and one of the full ancilla modes.

III. DISCUSSION

Section II has shown how a full enlarged quantum optical network can be mathematically represented and physically realized. Now, we are also in a position to answer the remaining questions from the Introduction. Contrary to conclusions of earlier works devoted to setups with either loss or gain alone, any transformation is available. The decomposition works for all linear networks as an input since a singular value decomposition can be performed for any complex matrix. This means that, in principle, any transformation can be realized, even if the practical implementation of arbitrary two-mode squeezing is technically challenging [24].

The number of required ancilla modes is tied to the dimensionality of T if it is not square, as well as to its

singular values. A nonsquare $n \times m$ transformation T leads to $(m - n)$ output ancilla modes if $m > n$, or to $(n - m)$ input ancilla modes if $n > m$. In addition to these input or output ancilla modes, full ancilla modes are introduced, and their number is equal to the number of singular values of T that are not equal to 1. Each singular value below (above) 1 entails a beam-splitter operation (parametric amplification) with such an ancilla mode. For the special case in which T is square and all of its singular values are equal to 1, no ancilla modes are needed because T is unitary, and then the method can be reduced to the known unitary decompositions (Ref. [2] or [3]). Upper bounds on the number of elemental building blocks required when using the scheme depend on the dimensionality of T in the following way: The maximum number of variable beam splitters needed to implement the unitary blocks U and W is $n(n - 1)/2 + m(m - 1)/2$, while the maximum number of phase shifters is $n(n + 1)/2 + m(m + 1)/2$. Additionally, up to $\min(m, n)$ elements are required to implement D ; these elements are either beam splitters or parametric amplifiers. Hence, the number of parametric amplifiers only scales linearly with the size of the transformation matrix.

A unitary network followed by photon detection in the different modes can be used to implement a projective measurement in a Hilbert space with a dimensionality matching the unitary network. In the context of generalized measurements, it is possible for a POVM to have a number of measurement outcomes that is larger than the dimensionality of the system. The Naimark dilation theorem guarantees that such a POVM can be implemented as a projective measurement in an enlarged Hilbert space [25]. Our method can be used to find a Naimark extension, which provides a suitable enlarged unitary transformation for this projective measurement (see Appendix C 1).

Another possible application of the method lies in the construction of probabilistic optical quantum information protocols. Starting with a general transformation matrix, by formulating the action of the protocol as a mapping from a given set of input states to a set of desired output states, a system of possibly nonlinear equations for the elements of T can be constructed. A solution of the system of equations defines a network that performs the protocol, and the method can then be used to find an implementation of that network (for an example, see Appendix C 2).

Although the decomposition always provides a full quantum optical transformation with the dependence of the mean output fields on the mean input fields as specified by the partial network T , the implementation is not unique. This is already evident from the simplest nonunitary “network,” a single channel with loss or gain. As discussed in Ref. [21], the same mean field could be achieved by including excess gain and loss that compensate each other’s effect on the mean field, at the expense of a reduction in the purity of the state. Given that this leaves the first moment of the field invariant but changes higher-order moments, it

presents an opportunity to tailor the higher-order moments. It is an interesting question (beyond the scope of the present article) whether the multimode control over first moments of the field provided by the method could be extended to higher-order moments.

IV. CONCLUSION

In summary, we have presented a way to describe and implement an arbitrary linear optical transformation, which can have any size and does not need to be complete in the sense that its modes fulfill bosonic commutation relations. This is achieved by finding a transformation in an enlarged space that includes the network of interest. The ancilla modes included in the description enable rigorous quantum optical modeling of the gains and losses in the network. In addition, a decomposition into the basic building blocks of beam splitters, phase shifters, and parametric amplifiers is obtained. This shows a way to implement the network that could physically be realized with integrated optics. We have discussed the role that the singular values of the transformation matrix play with respect to the number and type of ancilla modes. The method could prove useful for the implementation of POVMs, the design of probabilistic optical quantum information protocols, and, more generally, in any application that involves nonunitary networks.

We provide a MATLAB code for numerically implementing the full decomposition on GitHub [34].

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APPENDIX A: PROOF THAT THE PRODUCT $S_{\text{total}} = \prod_{ijk} S_{U_i} S_{D_j} S_{W_k}$ RESULTS IN A QUASIUNITARY MATRIX WITH T AS ITS UPPER-LEFT BLOCK

First, it should be noted that the individual S matrices (S_{U_i} , S_{D_j} , and S_{W_k}) fulfill Eq. (2). The product of two matrices that fulfill Eq. (2) is another quasiunitary matrix, which can be seen as follows.

Let A and B fulfill Eq. (2). Then,

$$M = AB,$$

$$MGM^\dagger = (AB)G(AB)^\dagger = A(BGB^\dagger)A^\dagger = AGA^\dagger = G.$$

Therefore, S_{total} is quasiunitary.

The second part of the proof is that the product of the individual S matrices has T as its upper-left block.

We have $T = UDW = \prod_{ijk} U_i D_j W_k$, and $S_{\text{total}} = \prod_{ijk} S_{U_i} S_{D_j} S_{W_k}$. Because of the block structure of the matrices S_{U_i} and S_{W_k} ,

$$\prod_i S_{U_i} = \begin{pmatrix} \prod_i U_i & 0 & 0 & 0 \\ 0 & I_{n_A} & 0 & 0 \\ 0 & 0 & \prod_{i'} U_{i'}^* & 0 \\ 0 & 0 & 0 & I_{n_A} \end{pmatrix},$$

and similarly,

$$\prod_k S_{W_k} = \begin{pmatrix} \prod_k W_k & 0 & 0 & 0 \\ 0 & I_{n_A} & 0 & 0 \\ 0 & 0 & \prod_{k'} W_{k'}^* & 0 \\ 0 & 0 & 0 & I_{n_A} \end{pmatrix}.$$

The components S_{D_j} corresponding to D_j generally do not have the same structure. The matrix S_{D_j} is the identity matrix if the j th singular value of T , $\sigma_j = 1$. Otherwise, if $\sigma_j \neq 1$, j and m_{A_j} are the mode numbers corresponding to the nominal mode and ancilla mode, respectively, of the j th singular value. Then, each matrix S_{D_j} is the identity matrix with the elements corresponding to the intersection of rows and columns j , m_{A_j} , $j + N$, $m_{A_j} + N$ replaced as given by expressions (11) and (12).

The fact that $S_{\text{total}} = \prod_{ijk} S_{U_i} S_{D_j} S_{W_k}$ has $T = \prod_{ijk} U_i D_j W_k$ as its upper-left block can be shown by observing the structure of the matrix as the multiplication is carried out. Let us consider the multiplication by starting from the rightmost matrix, sequentially multiplying from the left by the other matrices as specified, and denoting the product after x steps as S_x . The rows of S_x that deviate from those of the identity matrix are of interest at different stages of the multiplication, i.e., for different x . Let x_1 equal the number of matrices in the decomposition of W . For $S_{x_1} = \prod_k S_{W_k}$, we have already seen that the upper-left block of S_{x_1} is the product of the upper-left blocks of the components and that the only elements that deviate from

the identity matrix are contained in the blocks $(1:n_N, 1:n_N)$ and $(1+N:n_N+N, 1+N:n_N+N)$ [26]. Now, as each S_{D_j} is multiplied from the left, there are at most two new rows of the resulting matrix that can deviate from the identity: rows m_{A_j} and $(m_{A_j} + N)$ when $\sigma_j \neq 1$. Let x_2 lie between x_1 and the number of matrices in the decomposition of DW . After each step, the upper-left block of S_{x_2} is the product of the upper-left blocks of the components because the elements $(m_{A_j}, 1:n_N)$ and $(m_{A_j} + N, 1:n_N)$ of S_{x_2-1} are zero. This is essentially due to the fact that a unique ancilla mode is assigned to each singular value that is different from 1. After having multiplied through the individual S matrices corresponding to DW , for $\prod_i S_{U_i}$ we again have the block structure that guarantees that the upper-left block of S_{total} is T .

APPENDIX B: EXAMPLES

We demonstrate the method using two examples. First, we discuss how the lossy beam splitter with apparent nonlinearity can be constructed with standard optical elements. We then apply the method to an arbitrary 2×2 transformation, which may combine loss and gain in different modes, to obtain an analytic decomposition.

1. Lossy beam splitter with apparent nonlinear action

Here, the method is applied to decompose the 2×2 transformation $T = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ into simple building blocks. We begin with a singular value decomposition of $T = UDW$, which gives $U = (1/\sqrt{2}) \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $W = (1/\sqrt{2}) \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$. Since T is square, no augmentation of U , D , or W is required. Further decomposition provides $U = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$, while W is already a beam splitter, one of the basic building blocks. The matrix D does not need to be decomposed further because of its simple form: The diagonal element 0 in D represents a complete attenuation of a mode and constitutes the only singular value different from 1. One can thus proceed to identify the number of ancilla modes $n_A = 1$, so that $N = 3$ and the dimensionality of the corresponding S matrix is 6×6 ,

$$\begin{pmatrix} \hat{a}_{1\text{out}} \\ \hat{a}_{2\text{out}} \\ \hat{a}_{3\text{out}} \\ \hat{a}_{1\text{out}}^\dagger \\ \hat{a}_{2\text{out}}^\dagger \\ \hat{a}_{3\text{out}}^\dagger \end{pmatrix} = S \begin{pmatrix} \hat{a}_{1\text{in}} \\ \hat{a}_{2\text{in}} \\ \hat{a}_{3\text{in}} \\ \hat{a}_{1\text{in}}^\dagger \\ \hat{a}_{2\text{in}}^\dagger \\ \hat{a}_{3\text{in}}^\dagger \end{pmatrix},$$

with the nominal modes \hat{a}_1 and \hat{a}_2 , and the ancilla mode \hat{a}_3 .

We continue to identify the S_U , S_D , and S_W matrices corresponding to individual operations based on Eqs. (9)–(11):

$$\prod_{i=1}^2 S_{U_i} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$S_W = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$S_D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

The total transformation matrix

$$S_{\text{total}} = \prod_i S_{U_i} S_D S_W = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{-1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{-1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

indeed contains T as its upper-left block and is consistent with the scattering matrix given in Ref. [12]. Figure 4(b) shows the setup after simplifications, such as rewriting the

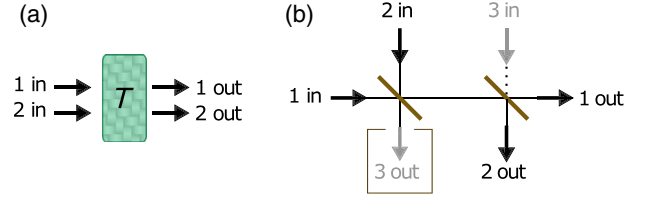


FIG. 4. Implementation of the lossy beam splitter with apparent nonlinear loss. (a) One approach would be to implement the transformation directly by a single device. The special transformation coefficients may be achieved with a novel material, e.g., a metamaterial. (b) The decomposition reveals a much simpler implementation consisting of two 50:50 beam splitters (along with single-mode phase shifts omitted from the diagram) and elucidates the simple role quantum interference plays with respect to the “nonlinear loss.”

beam splitter between modes 2 and 3 from S_D in terms of a swap operation, which means an exchange between the labels of the two modes. The setup reveals that the apparent nonlinear loss is simply the result of photon bunching due to two-photon quantum interference at the first beam splitter; one of the output ports of the beam splitter is discarded, which leads to either both or neither of the two photons emerging in the nominal output modes 1 and 2.

2. General 2×2 linear transformation

We now turn to a more general case of an arbitrary 2×2 linear transformation matrix $T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$, with complex elements $t_{ij} = |t_{ij}|e^{i\varphi_{ij}}$, $\varphi_{ij} \in \mathbb{R}$. Although the method always provides an easy way to obtain a decomposition numerically, in this low-dimensional case, an analytical solution, depicted in Fig. 5, can also be found. We represent the solution in terms of the following matrices: rotations by a beam splitter of real coefficients

$$BS(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

and single-mode phase shifts

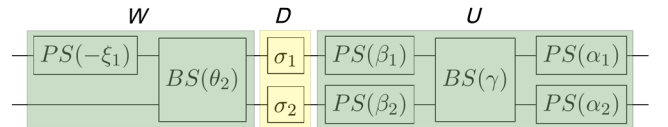


FIG. 5. Implementation of an arbitrary 2×2 transformation. The two green rectangles represent the unitary components W and U of the singular value decomposition. The diagonal part D , marked in yellow, corresponds to single-mode modulations by σ_j , which are realized by coupling to ancilla modes.

$$PS_1(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix}, \quad PS_2(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}.$$

$$T_{\text{re}} = BS(\theta_1).D.BS(\theta_2), \quad (\text{B2})$$

To solve this case analytically, one can transform the T matrix to a real form T_{re} through the following sequence of operations:

- (1) Cancel phases in the left column

$$\begin{aligned} T &\rightarrow T_1 = PS_1(-\varphi_{11}).PS_2(-\varphi_{21}).T \\ &= \begin{pmatrix} |t_{11}| & |t_{12}|e^{i(\varphi_{12}-\varphi_{11})} \\ |t_{21}| & |t_{22}|e^{i(\varphi_{22}-\varphi_{21})} \end{pmatrix}, \end{aligned}$$

where matrix multiplication is indicated by “.” for clarity.

- (2) Rotate the matrix to make the bottom-left component zero

$$T_1 \rightarrow T_2 = BS(\vartheta).T_1 = \begin{pmatrix} \tilde{t}_{11} & \tilde{t}_{12} \\ 0 & \tilde{t}_{22} \end{pmatrix},$$

where $\vartheta = \arctan[(|t_{21}|)/(|t_{11}|)]$ and

$$\begin{aligned} \tilde{t}_{11} &= |t_{11}| \cos \vartheta + |t_{21}| \sin \vartheta, \\ \tilde{t}_{12} &= |t_{12}| \cos \vartheta e^{i(\varphi_{12}-\varphi_{11})} + |t_{22}| \sin \vartheta e^{i(\varphi_{22}-\varphi_{21})}, \\ \tilde{t}_{22} &= -|t_{12}| \sin \vartheta e^{i(\varphi_{12}-\varphi_{11})} + |t_{22}| \cos \vartheta e^{i(\varphi_{22}-\varphi_{21})}. \end{aligned}$$

Note that \tilde{t}_{11} is real and non-negative, which we emphasize below by explicitly writing $\tilde{t}_{11} = |\tilde{t}_{11}|$.

- (3) Cancel phases in the right column,

$$\begin{aligned} T_2 &\rightarrow T_3 = PS_1(-\xi_1).PS_2(-\xi_2).T_2 \\ &= \begin{pmatrix} |\tilde{t}_{11}|e^{-i\xi_1} & |\tilde{t}_{12}| \\ 0 & |\tilde{t}_{22}| \end{pmatrix}, \end{aligned}$$

with $\xi_j = \arg \tilde{t}_{j2}$.

- (4) Cancel the remaining phase in the left column,

$$T_3 \rightarrow T_{\text{re}} = T_3.PS_1(\xi_1) = \begin{pmatrix} |\tilde{t}_{11}| & |\tilde{t}_{12}| \\ 0 & |\tilde{t}_{22}| \end{pmatrix}.$$

Finally, the transformed real matrix reads

$$\begin{aligned} T_{\text{re}} &= PS_1(-\xi_1).PS_2(-\xi_2).BS(\vartheta) \\ &\times PS_1(-\varphi_{11}).PS_2(-\varphi_{21}).T.PS_1(\xi_1). \quad (\text{B1}) \end{aligned}$$

A singular value decomposition of the resulting real 2×2 matrix is especially simple, with the unitary components given as two beam-splitter rotations. In this particular case, we make use of the fact that one of the components is 0 and obtain

where

$$\begin{aligned} \theta_j &= \frac{(-1)^j}{2} \arg(q_j + 2p_j i), \\ p_j &= |\tilde{t}_{1,3-j}\tilde{t}_{3-j,2}|, \\ q_j &= |\tilde{t}_{11}|^2 - |\tilde{t}_{22}|^2 + (-1)^{j-1}|\tilde{t}_{12}|^2. \end{aligned}$$

The matrix of singular values determines the required degree of attenuation or amplification,

$$\begin{aligned} D &= \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \\ \sigma_j &= \sqrt{\frac{s + (-1)^{j-1}\sqrt{q_j^2 + 4p_j^2}}{2}}, \\ s &= |\tilde{t}_{11}|^2 + |\tilde{t}_{22}|^2 + |\tilde{t}_{12}|^2. \end{aligned}$$

Finally, a combination of Eqs. (B1) and (B2) yields the decomposition of the original matrix T ,

$$\begin{aligned} T &= \underbrace{PS_2(\varphi_{21}).PS_1(\varphi_{11}).BS(-\vartheta).PS_2(\xi_2).PS_1(\xi_1).BS(\theta_1)}_U \\ &\times \underbrace{D.BS(\theta_2).PS_1(-\xi_1)}_W. \end{aligned}$$

Note that the matrix U can be further simplified to

$$\begin{aligned} U &= PS_1 \left(\underbrace{\varphi_{11} + \xi_1 + \frac{\alpha + \beta}{2}}_{\alpha_1} \right) . PS_2 \left(\underbrace{\varphi_{21} + \xi_2 - \frac{\alpha + \beta}{2}}_{\alpha_2} \right) \\ &\times BS(\gamma).PS_1 \left(\underbrace{\frac{\alpha - \beta}{2}}_{\beta_1} \right) . PS_2 \left(\underbrace{\frac{\beta - \alpha}{2}}_{\beta_2} \right), \end{aligned}$$

$$\alpha = \arg(\cos \vartheta \cos \theta_1 + \sin \vartheta \sin \theta_1 e^{i(\xi_2 - \xi_1)}),$$

$$\beta = \arg(\cos \vartheta \sin \theta_1 - \sin \vartheta \cos \theta_1 e^{i(\xi_2 - \xi_1)}),$$

$$\gamma = \arccos(|\cos \vartheta \cos \theta_1 + \sin \vartheta \sin \theta_1 e^{i(\xi_2 - \xi_1)}|).$$

The construction of the S network depends on the singular values $\sigma_{1,2}$ and can be obtained from Eqs. (9)–(12). The dimensionality of S is at most 8×8 since there is one ancilla mode per singular value that is not equal to 1. For a particular example, the case of a transformation combining loss in mode 1 ($\sigma_1 < 1$) with gain in mode 2 ($\sigma_2 > 1$), the submatrices read

1. POVMs

A POVM is determined by a set of positive semidefinite operators $\{E_i\}_{i=1}^m$, which sum to identity $\sum_{i=1}^m E_i = I_n$ and represent generalized measurements in an n -dimensional Hilbert space [27]. Here, I_n denotes the n -dimensional identity matrix. An active field of research has been focused on the physical implementation of POVMs [28–32]. One of the strategies is based on Naimark’s dilation theorem. According to the theorem, any POVM can be realized as a projective measurement in an enlarged Hilbert space \mathcal{H} [25]. However, the theorem does not itself provide a general recipe to find the extension to \mathcal{H} , called Naimark extension.

To see how our method can be exploited to find Naimark extensions, let us focus on the important case of rank-one POVMs. The operators forming rank-one POVMs correspond to projectors $E_i = |\phi^{(i)}\rangle\langle\phi^{(i)}|$ on, in general, non-orthogonal vectors $|\phi^{(i)}\rangle$ in the original Hilbert space. A Naimark extension can be found by augmenting the vectors $|\phi^{(i)}\rangle$ to size m so that they become orthogonal in \mathcal{H} . For this purpose, let us define a rectangular $n \times m$ matrix with columns given by the n -dimensional vectors $|\phi^{(i)}\rangle$:

$$T = \begin{pmatrix} \phi_1^{(1)} & \dots & \phi_1^{(m)} \\ \vdots & & \vdots \\ \phi_n^{(1)} & \dots & \phi_n^{(m)} \end{pmatrix},$$

such that $TT^\dagger = I$. Here, $\phi_j^{(i)}$ stand for elements of $|\phi^{(i)}\rangle$. A singular value decomposition of $T = UDW$ provides a unitary $n \times n$ matrix U , an $n \times m$ matrix D , and a unitary $m \times m$ matrix W . Note that since $TT^\dagger = I$, all the singular values of T are equal to 1. This means that the dimensionality of the Naimark extension found with this method is m , and the number of ancilla output modes is $m - n$. Next, let us pad the matrices of smaller dimensionalities with elements of the identity matrix, in accordance with step 1b of the method. As a result, we obtain the enlarged $m \times m$ matrices:

$$U \rightarrow \begin{pmatrix} U & 0 \\ 0 & I_{m-n} \end{pmatrix}, \quad D \rightarrow I_m,$$

and W does not require any modification. The product $UDW = UW$ is unitary and becomes an m -dimensional Naimark extension of T , which can be directly decomposed into building blocks with the methods of Reck *et al.* [2] or Clements *et al.* [3]. This procedure allows us to design a network for an arbitrary rank-one POVM.

2. Design of probabilistic protocols

Here, we demonstrate how the method can be used in the design of probabilistic optical quantum logic gates. We

illustrate the design on the example of a two-qubit controlled- Z gate and show a systematic way to find the setup presented in Ref. [33]. A two-qubit controlled- Z gate can be implemented with two photons and four optical modes. The control qubit is encoded by one photon within the first two modes (called the control modes), while the target qubit is encoded by another photon in the last two modes (the target modes). The goal is to construct a transformation using passive optical elements, such that it implements a controlled phase flip, given that both the input and output states fulfill the condition that there is one photon in the control modes and one photon in the target modes.

Our starting point is the desired effect on two-photon states: For the four different input states below and only considering outputs according to the postselection condition of having one photon in a control mode and the other photon in a target mode, we want the circuit to output the following states:

$$\begin{aligned} \hat{a}_{cH\text{in}}\hat{a}_{tH\text{in}} &\rightarrow -k\hat{a}_{cH\text{out}}\hat{a}_{tH\text{out}}, \\ \hat{a}_{cH\text{in}}\hat{a}_{tV\text{in}} &\rightarrow k\hat{a}_{cH\text{out}}\hat{a}_{tV\text{out}}, \\ \hat{a}_{cV\text{in}}\hat{a}_{tH\text{in}} &\rightarrow k\hat{a}_{cV\text{out}}\hat{a}_{tH\text{out}}, \\ \hat{a}_{cV\text{in}}\hat{a}_{tV\text{in}} &\rightarrow k\hat{a}_{cV\text{out}}\hat{a}_{tV\text{out}}, \end{aligned} \quad (\text{C1})$$

where the four modes are denoted cH , cV , tH , tV , after horizontal and vertical polarization in the control and target modes. The real constant $k \in (0, 1]$ allows for the possibility of the protocol being probabilistic, with a success rate of k^2 . The above transformations involve four input and output modes, so the transformation we seek has the general form

$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{pmatrix}.$$

Since we assume that the setup will be passive, we know that S_{total} will be block diagonal and can be written as

$$S_{\text{total}} = \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix},$$

where A is a unitary matrix that contains T as its upper-left block, relating annihilation operators as follows:

$$\begin{pmatrix} \hat{a}_{cH\text{out}} \\ \hat{a}_{cV\text{out}} \\ \hat{a}_{tH\text{out}} \\ \hat{a}_{tV\text{out}} \\ \vdots \end{pmatrix} = \begin{pmatrix} T & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \hat{a}_{cH\text{in}} \\ \hat{a}_{cV\text{in}} \\ \hat{a}_{tH\text{in}} \\ \hat{a}_{tV\text{in}} \\ \vdots \end{pmatrix}.$$

Based on the constraints of Eq. (C1), the elements of T need to be determined. To do this, it is useful to write the annihilation operators of the input modes in terms of those of the output modes. The unitary matrix A can simply be inverted to write the input modes in terms of the output modes, and we obtain

$$\begin{pmatrix} \hat{a}_{cH\text{in}} \\ \hat{a}_{cV\text{in}} \\ \hat{a}_{tH\text{in}} \\ \hat{a}_{tV\text{in}} \\ \vdots \end{pmatrix} = \begin{pmatrix} T^\dagger & A_{21}^\dagger \\ A_{12}^\dagger & A_{22}^\dagger \end{pmatrix} \begin{pmatrix} \hat{a}_{cH\text{out}} \\ \hat{a}_{cV\text{out}} \\ \hat{a}_{tH\text{out}} \\ \hat{a}_{tV\text{out}} \\ \vdots \end{pmatrix}. \quad (\text{C2})$$

Using Eq. (C1) together with Eq. (C2), we obtain a set of nonlinear equations, of which one solution is

$$T = \begin{pmatrix} t_{11} & 0 & t_{13} & 0 \\ 0 & t_{11} & 0 & 0 \\ t_{31} & 0 & -\frac{t_{13}t_{31}}{2t_{11}} & 0 \\ 0 & 0 & 0 & -\frac{t_{13}t_{31}}{2t_{11}} \end{pmatrix}, \quad k = -\frac{1}{2}t_{13}t_{31}.$$

There are three free parameters, t_{11} , t_{13} , t_{31} , and the success probability of the protocol, k^2 , depends on two of these parameters. Moreover, the singular values of T depend on the parameters. We need all the singular values to be less than or equal to 1 so that the circuit is a passive network, but we would like as many of the values as possible to be 1 so that the number of ancilla modes is minimized. A suitable choice of parameters is $t_{11} = \sqrt{\frac{1}{3}}$, $t_{13} = t_{31} = \sqrt{\frac{2}{3}}$. This choice results in the success probability of the protocol $k^2 = \frac{1}{9}$ and the singular values $(1, 1, \sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}})$, which show that two ancilla modes are required. From here, the decomposition method can be used to find the physical realization of the matrix

$$T = \begin{pmatrix} \sqrt{\frac{1}{3}} & 0 & \sqrt{\frac{2}{3}} & 0 \\ 0 & \sqrt{\frac{1}{3}} & 0 & 0 \\ \sqrt{\frac{2}{3}} & 0 & -\sqrt{\frac{1}{3}} & 0 \\ 0 & 0 & 0 & -\sqrt{\frac{1}{3}} \end{pmatrix},$$

which finally provides the scheme of Ref. [33].

[1] By the terms ‘‘linear transformation’’ and ‘‘linear network’’ we refer to transformations for which the expectation values of the fields are related by a linear transformation between

the input and output modes and the annihilation operators of the output modes have the same linear dependence on the input annihilation operators.

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