Orbit Closures in a Hurwitz Space of Translation Surfaces of Genus Three

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Dipl.-Math. Benjamin Peters

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1. Referent:	Prof. Dr. Frank Herrlich
2. Referentin:	Prof. Dr. Gabriela Weitze-Schmithüsen, Universität des Saarlandes

Introduction

The key to understand the behavior of geodesic flows on a single translation surface is to understand its $\mathrm{GL}_2^+(\mathbb{R})$-orbit closure. In 1989, a first glimpse of this connection was discovered by Veech [Vee89]: The famous Veech dichotomy states that on a Veech surface, either all geodesics in a given direction are closed or all geodesics in a given direction are dense and uniformly distributed.

In a similar vein, Masur [Mas06] showed that if the vertical geodesic flow on a translation surface is not dense and not uniformly distributed, then the $\left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right)$-orbit of this translation surface leaves every compact set in the corresponding stratum. Hence to construct a translation surface with a non-dense flow, we could instead construct a translation surface such that its orbit leaves every compact set in the stratum.

The classification of $\mathrm{GL}_2^+(\mathbb{R})$-orbit closures is far from complete. Every translation surface of genus 1 has a closed orbit and is a Veech surface. Translation surfaces of genus 2 provide a much richer theory and were classified by Calta and McMullen [Cal04; McM07]. Using Prym varieties McMullen [McM06] constructed families of closed orbits in genus 2, 3 and 4. Using similar methods, Lanneau and Nguyen [LN17] describe orbit closures coming from Prym eigenforms in the strata $\mathcal{H}(2,2)^{\mathrm{odd}}$ and $\mathcal{H}(1,1,2)$.

The groundbreaking work of Eskin, Mirzakhani and Mohammadi [EMM15] offers ways to construct orbit closures different from Prym eigenforms and Teichmüller curves. Their Fields medal winning work proved what Zorich [Zor14] calls a "magic wand": $\mathrm{GL}_2^+(\mathbb{R})$-orbit closures are affine invariant submanifolds.

Affine invariant submanifolds open new doors to the study of orbit closures. Lanneau, Nguyen, Möller and Wright showed that in special orbit closures there are at most finitely many (primitive) Teichmüller curves [LM17; LNW15]. Another helpful theorem is the cylinder deformation theorem of Wright [Wri15a], which says that distorting only some cylinders of a translation surface by a matrix in $\mathrm{GL}_2^+(\mathbb{R})$, yields a translation surface in the orbit closure of the first translation surface. Using this theorem, Mirzakhani and Wright classified orbit closures of full rank [MW18], Aulicino and Nguyen and classified orbit closures in genus 3 of rank 2 [AN16a; AN16b; ANW16] and Nguyen and Wright showed that in the hyperelliptic component of $\mathcal{H}(4)$ orbits are either closed or dense [NW13]. Furthermore, Apisa showed that in the hyperelliptic components of $\mathcal{H}(2g-2)$ and $\mathcal{H}(g-1,g-1)$ orbit closures of higher rank are covering constructions [Api15].

Despite this far from complete list of results, the classification of orbit closures is still a wide open question.

We study orbit closures in the principal stratum $\mathcal{H}(1,1,1,1)$. Herrlich and Schmithüsen studied the Wollmilchsau and a special Hurwitz space consisting of translation surfaces of genus three with four simple singularities [HS07a; HS08]. The special feature of this

Hurwitz space is that there exists a family of Teichmüller curves intersecting the Teichmüller curve of the Wollmilchsau. Moreover, the union of these Teichmüller curves is dense in the Hurwitz space.

We will study which other orbit closures occur in a slightly larger Hurwitz space H. Additionally, we enrich this Hurwitz space with the translation structures coming from the covered tori and denote it by ΩH. Forgetting the covering information, we obtain the subspace $\Omega\mathcal{L}$ of the moduli space of translation surfaces. A *translation* is an affine automorphism of a translation surface in $\Omega\mathcal{L}$ with derivative I and a *rotation* is one with derivative $-I$. In this thesis we show the following results:

Theorem 1. *There exists a descending chain of affine invariant submanifolds $\Omega\mathcal{L}_i$ of $\Omega\mathcal{L}$ of dimension $5-i$ for $i=1,2,3$, each described by rotations and translations.*

Using Ratner's theorem we can deduce the existence of affine invariant submanifold in $\Omega\mathcal{L}$, and more general in every Hurwitz space of coverings of tori, of arbitrary dimension. However, we study affine invariant submanifolds given by rotations and translations. In this case Ratner does not help. For more than four branch points such a chain does not even exist any more, see Section 7.1.

Every connected component of an affine invariant submanifold is an orbit closure of a single translation surface. We classify the connected components of these spaces.

Theorem 2. *The space ΩH is connected. The spaces ΩH_i have $i+1$ connected components for $i=1,2,3$, distinguished by the monodromy of the covering.*

All these subspaces are constructed using translations and rotations of the torus that can be lifted to automorphisms of the covering surface. We show that there are no other affine invariant submanifolds which are constructed using translations and rotations.

Theorem 3. *All affine invariant submanifolds of $\Omega\mathcal{L}$, which are described by rotations and translations of the covering surface, are $\Omega\mathcal{L}$, $\Omega\mathcal{L}_1$, $\Omega\mathcal{L}_2$ and $\Omega\mathcal{L}_3$.*

There is one more affine invariant submanifold of $\Omega\mathcal{L}_1$, which is described by translations of the torus that do not lift to the covering surface.

In $\Omega\mathcal{L}_3$ every translation surface is a Veech surface. We compute their Veech groups. Finally, we explain why different generalizations of our approach do not yield results as nice as in this thesis.

The structure of this thesis is as follows:

Chapter 1 to 5 is about fundamentals. In the first chapter we briefly discuss Riemann surfaces and algebraic curves and why they are the same. In the second chapter we introduce translation surfaces and, most importantly, affine invariant submanifolds. In the third chapter we discuss covering theory and Hurwitz spaces. In chapter 4 we introduce the notation for all finite groups we will need and in chapter 5 we summarize facts about the Wollmilchsau.

Chapter 6 is the main part of this thesis. We introduce the Hurwitz space ΩH and discuss its properties. The crucial tool is the polygon decomposition of translation surfaces in $\Omega\mathcal{L}$. This gives us the ideas to construct subspaces and enables us to prove our theorems. In Section 6.4 we prove Theorems 1 and 2. After that, we compute the Veech groups of translation surfaces in $\Omega\mathcal{L}_3$. Finally, we prove the first half of Theorem 3 in Section 6.6.

Chapter 7 illustrates the difficulties in generalizing our attempt. Firstly, we increase the number of branch points. We show that for more than four branch points the equivalent of Theorem 1 does not hold any more. Secondly, in Section 7.2 we study affine invariant submanifolds in $\Omega\mathcal{L}$ that are given by an automorphism of the torus, which has no lift on the covering surface. From this section the second half of Theorem 3 follows. Finally, we study coverings of translation surfaces of genus two.

At this point I want to express my gratitude to all those who have supported me and made this thesis possible: First and foremost, *Frank Herrlich*, who introduced me to this topic and, with his famous open door policy, gave me at the same time all the freedom to work on my own and helped me to answer all the questions I had.

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Contents

1 Riemann surfaces and algebraic curves

In this section we recall the basic definitions of Riemann surfaces and algebraic curves. Furthermore, we justify that we use the terms "compact Riemann surface" and "non-singular projective curve over $\mathbb{C}$" interchangeably.

1.1 Riemann surfaces

Let us briefly recall the basic notions of a Riemann surface and a holomorphic differential on it.

Definition 1.1. A *Riemann surface* is a 2-dimensional manifold together with a complex atlas whose transition maps are biholomorphic.

A definition of a holomorphic differential on a Riemann surface is given by Miranda [Mir95]. This may not be the shortest definition, but it is given with a nice motivation.

Definition 1.2 (Miranda [Mir95]). Let $V \subseteq \mathbb{C}$ be an open subset of the complex plane. A *holomorphic 1-form* is an expression $\omega = f(z)dz$, where f is a holomorphic function on V.

Let $\omega_1 = f(z)dz$ and $\omega_2 = g(w)dw$ be holomorphic 1-forms on V_1 and V_2 with coordinates z and w and let $T\colon V_2 \to V_1$ be holomorphic. We say T *transforms* ω_1 to ω_2 if $g(w) = f(T(w))T'(w)$.

Now let X be a Riemann surface. A *(holomorphic) 1-form* or *(holomorphic) differential form* on X is a collection of holomorphic 1-forms $\{\, \omega_\Phi \,\}_\Phi$, one for each chart $\Phi\colon U \to V$, on the target V. For two overlapping charts the holomorphic 1-forms are transformed into each other by the transition map.

Hence locally a differential form is of the form $f dz$ for some holomorphic map f. The *order* of a holomorphic differential at some point $p \in X$ is defined to be the order of the corresponding function f at p. This notion is independent of the choice of local coordinates. If the order is n we call p a *zero* of order n. Furthermore, let us denote the set of non-zero holomorphic differential forms on a subset $U \subseteq X$ of a Riemann surface X by

$$\Omega(U) = \{\, \omega \text{ holomorphic differential form on } U \,\} \setminus \{\, 0 \,\}.$$

This defines a sheaf $\Omega = \Omega_X$, which is called the *sheaf of nonzero holomorphic differential forms* on X.

Proposition 1.3 (Lamotke [Lam09]). *Let X be a compact Riemann surface of genus g. Then the set of holomorphic differentials $\Omega(X) \cup \{\, 0 \,\}$ is a g-dimensional vector space and the sum of orders of the zeros of a holomorphic differential $\omega \in \Omega(X)$ is $2g - 2$.*

According to Miranda [Mir95], we can pull back differential forms. For a given holomorphic map $F\colon X \to Y$ the pullback is defined as $F^*\colon \Omega(Y) \to \Omega(X)$ by applying F on every chart. Furthermore, we can integrate differential forms along paths. Integration along paths is done locally in every chart. It makes no difference whether we integrate the pullback of a differential form or integrate along the pushforward of the path, i. e. for a path γ in X and a differential $\omega \in \Omega(Y)$, we have

$$\int_\gamma F^*\omega = \int_{F_*\gamma} \omega.$$

Integration along homotopic and homologous paths yields the same result. So the notion of the integral extends to elements in the fundamental group or in the first homology group.

The moduli space of compact Riemann surfaces

$$M_g = \{ \, X \text{ compact Riemann surface of genus } g \, \} \, / \sim$$

consists of all Riemann surfaces of a given genus g. Two surfaces are equivalent if they are biholomorphic. For $g \geq 2$, the dimension of M_g is $3g-3$, and the moduli space of elliptic curves M_1 can be identified via the j-invariant with the complex plane $\mathbb{C}$.

Here and subsequently, all Riemann surfaces are compact and hence we use "Riemann surface" instead of "compact Riemann surface".

1.2 Algebraic curves

The basic definitions of algebraic geometry are found in the book of Hartshorne [Har77]. For simplicity of notation we write "curve" instead of "non-singular, projective variety of dimension 1". Since we work over the complex numbers we talk about Riemann surfaces instead of curves most of the time. This is justified by the following explanation.

In 1851 Riemann [Rie51] showed the Riemann existence theorem. It states that on each compact Riemann surface there is a non-constant meromorphic function. Hence it is a covering of the Riemann sphere $\mathbb{P}^1\mathbb{C}$. This shows that the field of all meromorphic functions of the surface is a finite algebraic extension of the function field $\mathbb{C}(x)$ and hence an algebraic curve over the complex numbers. On the other hand, every algebraic curve over $\mathbb{C}$ can be embedded in some projective space and thus can be given a complex structure. Using Serre's GAGA [Ser56], sheaves in the analytic category and sheaves in the algebraic category can be identified.

In what follows, a curve will always be a projective, non-singular curve over the complex numbers and hence we will use the terms "curve" and "Riemann surface" interchangeably.

Recall that a *hyperelliptic involution* of a curve X of genus $g \geq 2$ is an involution $\tau\colon X \to X$ with $2g+2$ fixed points. The quotient $X/\langle\tau\rangle$ is the projective space $\mathbb{P}^1\mathbb{C}$. A curve is called *hyperelliptic* if it has a hyperelliptic involution in its automorphism group, or if it is a covering of the Riemann sphere of degree 2. A hyperelliptic curve is given by an equation

$$y^2 = f(x),$$

where f is a polynomial of degree $2g+1$ or $2g+2$.

2 Translation surfaces

In this section we define translation surfaces – the subject of this work – and state some of their fundamental properties. Furthermore, we discuss affine invariant submanifolds, which are what Zorich calls a "magic wand" [Zor14].

2.1 Translation surfaces, strata and Veech surfaces

We give three definitions of translation surfaces and define their moduli space and its stratification. Moreover, we discuss how the group $\mathrm{GL}_2^+(\mathbb{R})$ of real matrices with positive determinant acts on the space of translation surfaces.

We start with the most geometric definition of a translation surface. We take a collection of polygons in $\mathbb{R}^2$ such that for every edge there is one parallel edge of the same length. Then we glue each edge to exactly one edge by a translation. The resulting object is a surface and is called a translation surface. The following definition is more precise.

Definition 2.1 (Masur [Mas06]). Let $P_1, \ldots, P_n$ be a collection of finitely many polygons in the plane. Let P_i^* be the polygon P_i without its vertices and $D = \bigcup_{i=1}^n \partial P_i^*$ the union of the edges of the polygons. Fix an orientation of the plane and of each edge as well as an involution $T \colon D \to D$, such that the restriction of T to the interior of an edge is the translation to an edge with opposite orientation. We denote the gluing instructions given by the polygons and the gluing map T by ω. If the surface

$$X = \left(\coprod_{i=1}^n P_i^* \right) / T$$

is connected, the surface together with the gluing instruction (X, ω) is called *translation surface.*

We abbreviate $X = (X, \omega)$, if it is not ambiguous.

With this definition we can glue the opposite sides of a square and get a torus. This is the best-studied example of a translation surface. More complicated and more interesting is the double n-gon constructed by Veech [Vee89] in the following way. For odd n, take two regular n-gons, rotate one copy by π and glue parallel edges of equal length. This yields a translation surface of genus $\frac{n-1}{2}$. Via the gluing all vertices get identified and thus the angle at a vertex is $(n-2)\pi$. This is a so-called singularity, which we define rigorously later. The case $n = 5$ is drawn in Figure 2.1.

It is good, but not always true, to think of singularities as the vertices of the polygons defining the translation surface. Since the gluing of the polygons is given by translations,

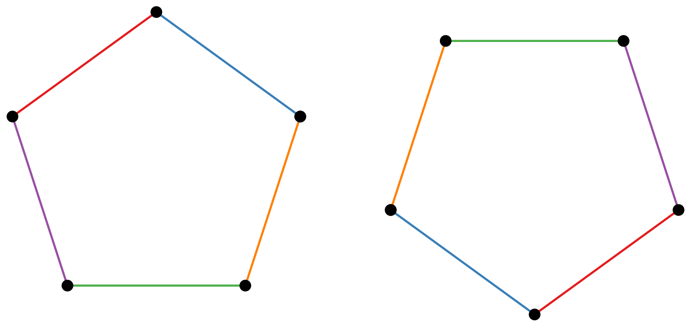


Figure 2.1: Each pair of edges with the same color is glued together. This gives rise to a translation surface called the Veech pentagon. In the gluing process, all vertices get identified and form a singularity with angle 6π.

we can give an atlas of a translation surface without its singularities whose transition maps are translations. On the other hand, given a surface together with an atlas whose transition maps are translations, we can triangulate this surface to get a polygonial decomposition.

Definition 2.2. A *translation surface* is a compact surface such that there exists an atlas on all but finitely many points whose transition maps are translations.

In particular, a translation surface is a Riemann surface. Locally it looks like the complex plane and hence we can pull back the differential dz on every chart. Since the transition maps are translations, this gives rise to a holomorphic differential on the whole translation surface. On the other hand, let X be a Riemann surface and $\omega \in \Omega(X)$ a holomorphic differential. For each chart (U, z) we fix a point $p_0 \in U$ and define a new chart

$$U \ni p \mapsto \int_{p_0}^{p} \omega,$$

i.e. we integrate along some path going from p_0 to p. This provides a translation atlas of $X \setminus \{\text{ zeros of } \omega\,\}$. For more details, see the survey of Hubert and Schmidt [HS06].

Definition 2.3. Let X be a Riemann surface and $\omega \in \Omega(X)$ a holomorphic differential. The pair (X, ω) is called a *translation surface* and the zeros of ω are called *singularities*.

With the above considerations we have sketched that all three definitions of translation surfaces coincide. With more effort one can show that the notion of singularity is the same in all three definitions. A zero of order n of ω gives rise to an angle $2(n+1)\pi$ in the polygon description. For our purpose, the most important ones are the description by polygons and the description by holomorphic differentials.

Naturally, we are interested in maps between translation surfaces.

Definition 2.4. A map $(X, \omega) \to (Y, \upsilon)$ is called *affine map* if it is locally given by $z \mapsto Az + b$ for $A \in \mathrm{GL}_2(\mathbb{R})$ and $b \in \mathbb{R}^2$. If $A = I$, the map is called *translation* and *rotation* if $A = -I$. Let us denote by $\mathrm{Aff}(X, \omega)$ the set of all affine maps $(X, \omega) \to (X, \omega)$ and by $\mathrm{Aff}^+(X, \omega)$ the set of all orientation preserving affine maps $(X, \omega) \to (X, \omega)$.

The pullback of a translation structure via a translation gives the same translation structure, i.e. $t^*\omega = \omega$ for a translation t and $\omega \in \Omega(X)$. Since the transition maps are translations, the matrix A of an affine map does not depend on the chosen chart. Hence we have a well-defined map

$$D\colon \operatorname{Aff}(X,\omega) \to \mathrm{GL}_2(\mathbb{R})$$

sending an affine map to its linear part. This map is called the *derivative*. Note that the surface X is compact, thus the matrix has to preserve the area and hence is of determinant ± 1. Furthermore, the derivative of an orientation preserving affine map is a matrix in $\mathrm{SL}_2(\mathbb{R})$. Hence we have the map

$$D\colon \operatorname{Aff}^+(X,\omega) \to \mathrm{SL}_2(\mathbb{R}).$$

Now consider the space

$$\Omega M_g = \{\, (X,\omega) \mid X \in M_g,\ \omega \in \Omega(X) \,\} / \sim$$

of translation surfaces of genus g. Two translation surfaces are considered equivalent if and only if there is a translation between them. This is the *moduli space of translation surfaces* of genus g. It is a holomorphic fiber bundle over the moduli space M_g.

Corollary 2.5 (Gauß-Bonnet formula). *Let $(X,\omega) \in \Omega M_g$ be a translation surface of genus g and let $\alpha_1, \dots, \alpha_n$ be the orders of the zeros of ω. Then*

$$\sum_{i=1}^{n} \alpha_i = 2g - 2.$$

Proof. By Proposition 1.3 the orders of zeros of ω add up to $2g-2$. □

Note that the zeros of the holomorphic differential are the singularities of the corresponding translation surface. Hence we have a stratification of the moduli space of translation surfaces ΩM_g into *strata* $\mathcal{H}(\alpha)$ of translation surfaces with fixed numbers of singularities, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a partition of $2g-2$. For example, the double 5-gon is in the stratum $\mathcal{H}(2)$. The torus is in the stratum $\mathcal{H}(\emptyset) = \Omega M_1$, which is the whole moduli space of translation surfaces of genus 1. A holomorphic differential in ΩM_1 cannot have a zero, hence it is of the form λdz for some $\lambda \in \mathbb{C}^\times$. Every such differential is valid, hence $\Omega M_1 = \mathbb{C}^\times \times \mathbb{C}$.

Proposition 2.6 (Veech [Vee86]). *Each stratum $\mathcal{H}(\alpha)$ is a complex orbifold of dimension $2g + n - 1$.*

Idea of proof. We will just sketch the idea of the proof according to Wright [Wri15b]. Let S_g be a surface of genus g and $\Sigma \subseteq S_g$ a finite set of marked points. We study the infinite degree covering $\widetilde{\mathcal{H}}(\alpha)$ of the stratum $\mathcal{H}(\alpha)$ consisting (up to some equivalence) of triples (X,ω,f). Here, $f\colon S_g \to X$ is a homeomorphism sending marked points to

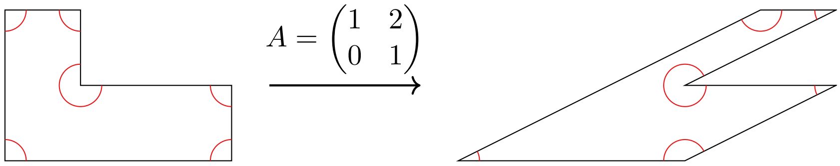


Figure 2.2: On this translation surface, opposite edges are glued. Its single singularity of angle 6π is marked in red. The effect of applying the matrix A is shown.

singularities. Now we fix a basis $\{ \gamma_1, \dots, \gamma_{2g+n-1} \}$ of relative homology $H_1(S, \Sigma, \mathbb{Z})$ to get charts

$$\tilde{\mathcal{H}}(\alpha) \to \mathbb{C}^{2g+n-1}, \qquad (X, \omega, f) \mapsto \left(\int_{f_* \gamma_i} \omega \right)_{i=1}^{2g+n-1}.$$

These coordinates are called *period coordinates* and make $\tilde{\mathcal{H}}(\alpha)$ into an orbifold. To see this one can use Veech's zippered rectangle construction, see e.g. Yoccoz [Yoc10]. Factoring out the mapping class group, this atlas descends to an atlas of $\mathcal{H}(\alpha)$ away from the fixed points of the action. $\square$

One interesting concept closely related to the dynamics of translation surfaces is the action of the matrix group $\mathrm{GL}_2^+(\mathbb{R})$ on the moduli space of translation surfaces. Given a translation surface in its polygon decomposition, we can apply a matrix to these polygons. Polygons are mapped to polygons and parallel sides of equal length are mapped to parallel sides of equal length. Hence we can apply the same gluing as before and get a new translation surface. It has the same number of singularities with the same order as before. For an example of this action see Figure 2.2. In conclusion, this gives an action of $\mathrm{GL}_2^+(\mathbb{R})$ on each stratum $\mathcal{H}(\alpha)$.

This action can be expressed via holomorphic differentials and translation atlases as well.

Due to Masur [Mas82], most translation surfaces have large orbits, i.e. their orbits are dense in the whole stratum they live in. Those special translation surfaces whose orbits are so small that they are closed deserve their own name.

Definition 2.7. Let (X, ω) be a translation surface. The stabilizer of the $\mathrm{GL}_2^+(\mathbb{R})$-action is denoted by $\Gamma(X, \omega) \subseteq \mathrm{SL}_2(\mathbb{R})$ and called *Veech group*. If the Veech group is a lattice in $\mathrm{SL}_2(\mathbb{R})$, the translation surface is called a *Veech surface* or *lattice surface*.

The name of this group is coined by the ground breaking work of Veech [Vee89]. The Veech group of a translation surface is a discrete, non-cocompact subgroup of $\mathrm{SL}_2(\mathbb{R})$. The Veech group may be seen as the image of the derivative $D(\mathrm{Aff}^+(X, \omega))$ of all affine maps of X. There are many equivalent descriptions of a surface being a Veech surface and we will just state some of them. Let us define a *saddle connection* to be a geodesic segment connecting a singularity to some singularity with no singularity in its interior. Since the transition maps are translations, we have a well-defined image of this geodesic segment in $\mathbb{R}^2$ starting at the origin. This image is called the *holonomy vector* of the saddle connection. Denote by $\mathrm{hol}(X, \omega)$ the set of all holonomy vectors of saddle connections on the translation surface (X, ω).

Proposition 2.8 (Smillie-Weiss [SW10]). *Let (X, ω) be a translation surface. Then the following are equivalent:*

a) (X, ω) is a Veech surface.

b) The $\mathrm{GL}_2^+(\mathbb{R})$-orbit of (X, ω) is closed.

c) The set $\{ u \wedge v \mid u, v \in \mathrm{hol}(X, \omega) \}$ is a discrete set of numbers.

The wedge product $u \wedge v$ can be thought of, up to the factor $e_1 \wedge e_2$ with the standard basis vectors e_1 and e_2, as the determinant $\det(u, v)$. Since a fixed factor does not change the discreteness, we can omit this factor.

Furthermore, the famous Veech dichotomy gives a first glimpse at the connections between dynamics on one single surface and dynamics on the whole moduli space.

Proposition 2.9 (Veech dichotomy [Vee89]). *Let (X, ω) be a Veech surface. Then for each direction θ the geodesic flow in direction θ is periodic or it is uniquely ergodic.*

This means that starting from any point in (X, ω) in direction θ and going in the same direction all the time, the resulting path is either closed or it is dense and uniformly distributed. The reader interested in the connection between the $\mathrm{GL}_2^+(\mathbb{R})$-orbit of a translation surface and its dynamical behavior will find more information, and more literature, in the survey by Zorich [Zor06].

2.2 Origamis

The easiest example of a translation surface is the torus. Its Veech group is $\mathrm{SL}_2(\mathbb{Z})$. A finite covering of the torus ramified over exactly one point has a Veech group which is commensurable to $\mathrm{SL}_2(\mathbb{Z})$, i.e. its intersection with $\mathrm{SL}_2(\mathbb{Z})$ has finite index in the Veech group and in $\mathrm{SL}_2(\mathbb{Z})$. This special property motivates the following definition.

Definition 2.10. An *origami* is a finite unramified covering of the once-punctured torus.

There is exactly one translation structure on the origami such that the covering map is a translation. This is the translation structure we choose. By work of Gutkin and Judge [GJ00] a translation surface is an origami if and only if its Veech group is commensurable to $\mathrm{SL}_2(\mathbb{Z})$. An equivalent condition is that there is a parallelogram that tiles the surface. This last property is the reason for the playful name origami coined by Lochak [Loc05]. Observe that every origami is a Veech surface.

One remarkable property of origamis is that they give rise to curves in moduli space. More generally, given a translation surface (X, ω) we study its $\mathrm{GL}_2^+(\mathbb{R})$-orbit. As a Riemann surface, X is stabilized by the orthogonal group $\mathrm{O}^+(2)$ with positive determinant. Hence we have a map

$$\mathrm{GL}_2^+(\mathbb{R}) / \mathrm{O}^+(2) = \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}(2) = \mathbb{H} \to T_{g,n}$$

from the upper half plane into the Teichmüller space of X. A holomorphic, isometric embedding $\mathbb{H} \to T_{g,n}$ is called a *Teichmüller embedding*. Its image is called *Teichmüller disk*. The above map is a Teichmüller embedding, see e.g. Herrlich [Her12]. The projection of the Teichmüller disk into moduli space is almost always something strange. In a few happy cases the projection is a curve. Then this curve is called *Teichmüller curve*.

Proposition 2.11. *Let (X, ω) be a translation surface. The above construction yields a Teichmüller curve if and only if (X, ω) is a Veech surface. In this case, the Teichmüller curve is birationally equivalent to $\mathbb{H}/\,\Gamma(X, \omega)$.*

Proof. See e.g. Herrlich and Schmithüsen [HS07b]. □

The next advantage of origamis is that it is remarkably easy to compute their Veech groups. Let (X, ω) be an origami and let $q\colon X^* \to E^*$ be the unramified covering belonging to this origami. Then the map q yields a natural inclusion into the free group

$$q_* \pi_1(X^*) = U \subseteq F_2 = \pi_1(E^*)$$

of finite index. On the other hand, each such inclusion $U \subseteq F_2$ into the free group gives a covering of the once-punctured torus E^*. So we can describe an origami as a subgroup of the free group F_2. Let $\beta\colon \operatorname{Aut}(F_2) \to \operatorname{GL}_2(\mathbb{Z}) = \operatorname{Out}(F_2)$ be the natural projection and define $\operatorname{Aut}^+(F_2) = \beta^{-1}(\operatorname{SL}_2(\mathbb{Z}))$ to be the group of orientation preserving automorphisms. Now define

$$\operatorname{Stab}(U) = \left\{ \gamma \in \operatorname{Aut}^+(F_2) \;\middle|\; \gamma(U) = U \right\},$$

the stabilizer of U.

Proposition 2.12 (Schmithüsen [Sch04]). *Let $q\colon X \to E$ be an origami and $U = q_* \pi_1(X^*) \subseteq F_2$. Then the Veech group of the origami is given by*

$$\Gamma(X, \omega) = \beta(\operatorname{Stab}(U)),$$

the image of the stabilizer of U in $\operatorname{SL}_2(\mathbb{Z})$.

This theorem gives an algorithm to compute the Veech group of any origami elaborated in Schmithüsen's paper [Sch04]. There the description of an origami as a finite index subgroup of the free group F_2 can also be found. Observe that such an algorithm does not exist for general translation surfaces and depends heavily on the extra data carried by an origami.

For a more thorough treatment of origamis and Teichmüller curves see the introductory texts by Herrlich and Schmithüsen [Her12; HS07b].

2.3 Affine invariant submanifolds

So far all we know about $\mathrm{GL}_2^+(\mathbb{R})$-orbits is that they are nice for Veech surfaces, because their projection in moduli space is a curve. Most other orbits are dense in the stratum they belong to. Hence it seems like a good idea to study orbit closures and not just orbits. This is justified by the Fields-medal winning result of Eskin, Mirzakhani and Mohammadi [EMM15], which shows that orbit closures are so-called affine invariant submanifolds. In this section we want to define what Zorich [Zor14] calls a "magic wand" and state some of its properties.

Firstly, let us recall absolute and relative homology following Hatcher [Hat15]. Let X be a topological space and define a *singular n-simplex* to be a continuous map $\sigma\colon \Delta^n \to X$ from an n-simplex to X. Let $C_n(X)$ be the free group generated by the singular n-simplices. We have a boundary map

$$\partial_n\colon C_n(X) \to C_{n-1}(X), \qquad \sigma \mapsto \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_n]},$$

where $[v_0,\ldots,\hat{v_i},\ldots,v_n]$ is the $(n-1)$-simplex omitting the vertex v_i. This equips $C_\bullet(X)$ with the structure of a chain complex. We define the absolute *n-th homology group* by $H_n(X,\mathbb{Z}) = \ker(\partial_n)/\operatorname{im}(\partial_{n+1})$.

Now let $\Sigma \subseteq X$ be a subset of X. The complex $C_\bullet(\Sigma)$ is a sub chain complex of $C_\bullet(X)$ and note that the boundary map descends to a boundary map of the quotient $\partial_n^\Sigma\colon C_n(X)/C_n(\Sigma) \to C_{n-1}(X)/C_{n-1}(\Sigma)$. We define the *$n$-th relative homology group* of X relative to Σ by

$$H_n(X,\Sigma,\mathbb{Z}) = \ker(\partial_n^\Sigma)/\operatorname{im}(\partial_{n+1}^\Sigma).$$

The relative homology group of a surface of genus g with n punctures is a free abelian group in $2g+n-1$ generators. Those generators can be thought of as a symplectic basis of the fundamental group and, by ordering the punctures, $n-1$ paths from one puncture to the next. We will make this more clear in the following easy, yet important example.

Example 2.13. Let E be a torus and $\bar{\Sigma} = \{\, \bar{P}, \bar{Q}, -\bar{Q}, -\bar{P} \,\} \subseteq E$. The questionable names of the points are chosen, since later we need points with these names and looking up the relations will be easier this way.

We are interested in the relative homology group $H_1(X,\bar{\Sigma},\mathbb{Z})$ of the torus relative to the set $\bar{\Sigma}$. Observe that $C_2(\bar{\Sigma}) = C_1(\bar{\Sigma})$ is the free abelian group over $\bar{\Sigma}$. Let Δ be a 2-simplex with vertices x, y and z. Then

$$\partial_2(\Delta) = xy - yz + zx,$$

where xy denotes the (oriented) 1-simplex from x to y. Furthermore,

$$\partial_1(xy) = x - y.$$

What is the kernel of ∂_1? Loops having the same beginning and end point, $x = y$, are in the kernel. Moreover, paths with beginning and end point in $\bar{\Sigma}$ have image in $C_0(\bar{\Sigma})$ and thus are in the kernel. So the kernel

$$\ker(\partial_1) = \{\, \text{loops and paths with beginning and end point in } \bar{\Sigma} \,\}$$

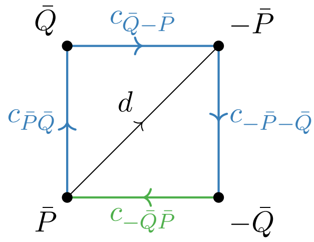
consists of all loops as well as of all paths with endpoints in the set $\bar{\Sigma}$. Then we need to calculate the image of ∂_2. Since $C_2(\bar{\Sigma}) = C_1(\bar{\Sigma})$ the image consists of all alternating boundaries of triangles, just as in absolute homology, and of all constant paths in $\bar{\Sigma}$. Summing up we have

$$H_1(E, \bar{\Sigma}, \mathbb{Z}) = \left\{ \text{ loops and nonconstant paths with beginning and end point in } \bar{\Sigma} \right\} / \sim,$$

where two paths are equivalent if there exist triangles with alternating orientation connecting those paths.

The current situation is sketched in Figure 2.3. A basis of absolute homology, labeled by $\bar{a}$ and $\bar{b}$, is given by the red paths and a basis of relative homology by the red and blue paths, where the blue paths are labeled by $c_{\bar{P}\bar{Q}}$, $c_{\bar{Q}-\bar{P}}$ and $c_{-\bar{P}-\bar{Q}}$, each connecting two points in $\bar{\Sigma}$. We now compute the green paths, whose labels are explained in Figure 2.3.

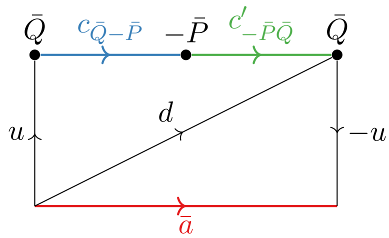
Let us write d for the diagonal path from $\bar{P}$ to $-\bar{P}$, so we have the rectangle



and this gives us $c_{\bar{P}\bar{Q}} + c_{\bar{Q}-\bar{P}} = d$ and $c_{-\bar{P}-\bar{Q}} + c_{-\bar{Q}\bar{P}} = -d$ with respect to the orientation of d. Thus we have

$$c_{-\bar{Q}\bar{P}} = -c_{\bar{P}\bar{Q}} - c_{\bar{Q}-\bar{P}} - c_{-\bar{P}-\bar{Q}}.$$

Similarly, orienting d from the lower left to the upper right corner, the rectangle



gives us the two equations

$$u + c'_{-\bar{P}\bar{Q}} + c_{\bar{Q}-\bar{P}} = d \quad \text{and} \quad \bar{a} - d = -u.$$

Thus we have $\bar{a} - c_{\bar{Q}-\bar{P}} = c'_{-\bar{P}\bar{Q}}$.

In the same spirit, we can calculate the elements $c'_{\bar{Q}\bar{P}}$ and $c'_{-\bar{Q}-\bar{P}}$. Furthermore, we can show that every horizontal edge is homologous to $\bar{a}$ and every vertical edge is homologous

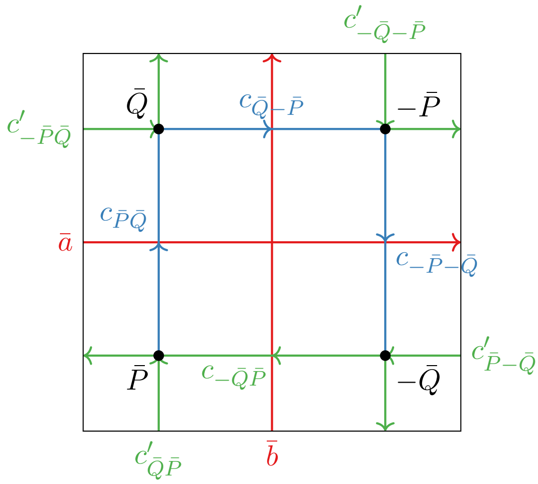


Figure 2.3: A basis of the relative homology of the four times punctured torus is depicted in red and blue. Other elements are green.

to $\bar{b}$. All in all, we have

$$\begin{aligned} c_{-\bar{Q}\bar{P}} &= -c_{\bar{P}\bar{Q}} - c_{\bar{Q}-\bar{P}} - c_{-\bar{P}-\bar{Q}}, \\ c'_{-\bar{P}\bar{Q}} &= \bar{a} - c_{\bar{Q}-\bar{P}}, \\ c'_{\bar{Q}\bar{P}} &= \bar{b} - c_{\bar{P}\bar{Q}} \quad \text{and} \\ c'_{-\bar{Q}-\bar{P}} &= -\bar{b} - c_{-\bar{P}-\bar{Q}}. \end{aligned}$$

Having seen these explicit calculations, let us return to our actual topic. We need relative homology to define period coordinates. Let (X, ω) be a translation surface and fix a basis $\{\, \alpha_1, \dots, \alpha_m \,\}$ of relative homology. By Proposition 2.6 the map

$$(X, \omega) \mapsto \left(\int_{\alpha_i} \omega \right)_{i=1}^{m}$$

can be extended to a neighborhood of (X, ω) in its stratum. These maps make a stratum into an orbifold and are called *period coordinates*. Implicitly this says that a basis of relative homology can be canonically transformed onto nearby surfaces.

Definition 2.14 (Wright [Wri15b]). An *affine invariant submanifold* $\mathcal{L}$ is a suborbifold of a stratum $\mathcal{H}$ that satisfies the following conditions:

a) There exists a manifold $\mathcal{N}$ and a proper immersion $f\colon \mathcal{N} \to \mathcal{H}$ into a stratum with image $f(\mathcal{N}) = \mathcal{L}$.

b) For every point in $\mathcal{N}$ there exists a neighborhood U such that $f(U)$ can be described by $\mathbb{R}$-linear homogeneous equations in period coordinates.

The first half of this definition is rather technical and we may think of an affine invariant submanifold as a subset of a stratum.

Proposition 2.15 (Eskin-Mirzakhani-Mohammadi [EMM15]). *Any* $\mathrm{GL}_2^+(\mathbb{R})$*-invariant, closed set is a finite union of affine invariant submanifolds. In particular, every orbit closure is an affine invariant submanifold.*

To indicate why this seminal result is so important we state some of its implications. More elaborated surveys are from Wright or Zorich [Wri15b; Zor14].

Firstly, the only previously known result about orbit closures was the study of genus 2 translation surfaces by McMullen and Calta [Cal04; McM07]. Due to them, in genus 2 we can write down a list of all possible orbit closures of translation surfaces. So far, such a classification is not even achieved in genus 3.

Secondly, it is fairly easy to see that affine invariant submanifolds are in fact $\mathrm{GL}_2^+(\mathbb{R})$-invariant [Wri15b, Proposition 3.5]. To prove the converse one needs roughly 250 pages. Hence the terms "closed orbit" and "2-dimensional affine invariant submanifold" describe the same objects.

One remarkable theorem using affine invariant submanifolds is the Cylinder Deformation Theorem by Wright [Wri15a]. Roughly, this theorem says that if we deform just some cylinders of a translation surface by a matrix in $\mathrm{GL}_2^+(\mathbb{R})$, the resulting translation surface is in the orbit closure of the first one. Subsequent papers by Aulicino, Nguyen and Wright classify all orbit closures in two of the three connected components of $\mathcal{H}(4)$ [ANW16; NW13].

Another theorem of importance by Wright [Wri14], generalizing work by Masur and Veech, shows that almost every translation surface in an affine invariant submanifold is generic, i.e. its orbit closure is the whole affine invariant submanifold. One of its corollaries specifies explicit conditions for a surface to be generic.

Corollary 2.16 (Wright [Wri14]). *A translation surface whose period coordinates are linearly independent over* $\overline{\mathbb{Q}}$ *is generic, i.e. its orbit closure is as large as possible.*

More classicaly, orbit closures are studied using Ratner's theorem. Many thanks to Martin Möller for pointing this out.

Proposition 2.17 (Ratner). *Let* G *be a finite dimensional, connected Lie group and* $\Gamma \subseteq G$ *a lattice. Let* $U \subseteq G$ *be a subgroup of* G *generated by unipotent elements. For every* $x \in G$ *there exists a group* $U \subseteq H \subseteq G$ *such that*

$$\overline{U \cdot [x]} = H \cdot [x] \subseteq G/\Gamma.$$

This theorem has applications for translation surfaces. Let $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. For $t \in \mathbb{R}$ define $u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and let U be the group generated by u_t. Using Ratner's theorem one can show that every U-orbit is either closed or dense in $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$. This is an answer to the question wether the geodesic flow in a given direction is ergodic or periodic.

For our purposes, another application is of interest. Let $G = \mathrm{SL}_2(\mathbb{R}) \rtimes (\mathbb{R}^2)^n$ be the semi-direct product of $\mathrm{SL}_2(\mathbb{R})$ with n copies of $\mathbb{R}^2$. We define the semi-direct product by the multiplication

$$(A, x_1, \dots, x_n) \cdot (B, y_1, \dots, y_n) = (AB, x_1 + Ay_1, \dots, x_n + Ay_n).$$

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z}) \rtimes (\mathbb{Z}^2)^n$ be a lattice in G. Then G/Γ describes the space of tori with n marked points. We regard the action of $U = \mathrm{SL}_2(\mathbb{R})$ on G/Γ given by

$$A \cdot (B, y_1, \dots, y_n) = (AB, Ay_1, \dots, Ay_n).$$

By Ratner's theorem there is for every (B, y) a group H such that

$$\overline{\mathrm{SL}_2(\mathbb{R})(B, y)} = H(B, y).$$

We want to understand the dimension of this group. We already know that for a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ we have $(AB, Ay) = (B, Ay)$ for every $B \in \mathrm{SL}_2(\mathbb{R})$. Let us write $y_1 = (a_1, b_1)$. Then the first coordinate of Ay_1 is in the set $a_1\mathbb{Z} + b_1\mathbb{Z}$. This set is dense in $\mathbb{R}$ if and only if $\mathbb{Z} + \frac{b_1}{a_1}\mathbb{Z}$ is dense in $\mathbb{R}$ and this happens if and only if $\frac{b_1}{a_1} \notin \mathbb{Q}$. Since the second coordinate is of the same form, either $\mathrm{SL}_2\, y_1$ is dense or discrete in $\mathbb{R}^2$. It is dense if and only if a_1 and b_1 are $\mathbb{Q}$-linearly independent, i. e. y_1 is a generic point. Hence for every generic point the group H has to have a corresponding component of dimension 2 and for every non-generic point a component of dimension 0. Furthermore, H contains $\mathrm{SL}_2(\mathbb{R})$. Hence it is of dimension 2 plus the number of generic points in y.

To construct an $\mathrm{GL}_2^+(\mathbb{R})$-orbit closure of (complex) dimension k in G we take some arbitrary torus and choose $k-2$ generic branch points. By the above considerations this gives us a $(k-2)+2 = k$-dimensional group H and thus a k-dimensional orbit closure. Thus we have shown the following.

Proposition 2.18. *In the space of translation tori with n marked points,* $\mathrm{GL}_2^+(\mathbb{R})$*-orbit closures of every possible dimension exist.*

3 Hurwitz spaces

In this chapter we briefly recall some covering theory and introduce the monodromy group. Branched coverings behave like coverings outside of a special set of points, the branch points. These branch points have fewer preimages than the degree of the covering. This is where ramification occurs. The set of all branched coverings with restricted ramification behavior is called a Hurwitz space. We talk briefly about classical Hurwitz spaces, which contain covers of the Riemann sphere. Then we consider Hurwitz spaces of coverings of elliptic curves.

3.1 Covering theory

Let us recall the basic definitions and statements of covering theory. For more details see Hatcher and Miranda [Hat15; Mir95]. In this section all spaces are topological spaces, mostly denoted by X, Y or Z.

The *fundamental group* of a topological space X with base point $x \in X$ is the group $\pi_1(X, x)$ consisting of all homotopy classes of loops starting in x. This is in fact a group. Given a map $f\colon X \to Y$ we get a push forward map $f_*\colon \pi_1(X, x) \to \pi_1(Y, f(x))$ by mapping a loop γ to the loop $f \circ \gamma$. If X is connected, for all $x_0, x_1 \in X$ the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic. Hence we often omit the base point and write $\pi_1(X) = \pi_1(X, x)$.

Definition 3.1. A *covering (map)* is a continuous map $p\colon X \to Y$ of topological spaces such that every point $y \in Y$ has a neighborhood U whose preimage $p^{-1}(U)$ is a union of disjoint open subsets in X, each of which is mapped homeomorphically onto U by p.

The cardinality of a fiber $p^{-1}(y)$ of $y \in Y$ does not depend on y, is called the *degree* of the covering $p\colon X \to Y$ and is denoted by $\deg(p)$. By convention, all our covering spaces will be connected. A simply connected covering space is called *universal covering.* It is universal in the sense that it covers every covering of the base space. In our setting, the universal covering exists.

One important feature of a covering is that a path on the covered space can be lifted to a path on the covering space. This follows from the following, more general result.

Proposition 3.2 (Lifting property). *Fix a base point $z \in Z$. A map $f\colon Z \to Y$ can be* lifted *along a covering $p\colon X \to Y$, i.e. there exists a map $\tilde{f}\colon Z \to X$ such that the diagram*

$$\begin{tikzcd} & X \arrow[d, "p"] \\ Z \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes, if and only if $f_*\pi_1(Z, z) \subseteq p_*\pi_1(X, x)$ and $f(z) = p(x)$.

The map $\tilde{f}$ is called a *lift* of f. A lift is not unique, but two lifts coincide if they coincide in one point. If f is a path, $Z = [0, 1]$ is simply connected and hence every path can be lifted to the covering space. From the proposition it easily follows that a map $f\colon Y \to Y$ can be lifted along a covering $p\colon X \to Y$ to a map $X \to X$ if and only if $f_*p_*\pi_1(X, x_1) \subseteq p_*\pi_1(X, x_2)$ with $fp(x_1) = p(x_2)$ for base points $x_1, x_2 \in X$.

For a covering map $p\colon X \to Y$ the induced map $p_*\colon \pi_1(X) \to \pi_1(Y)$ is injective. Its image consists of all those loops in Y whose lifts are loops in X. A lift of the identity is called *deck transformation* and the group of those maps will be denoted by $\mathrm{Deck}(X/Y)$ or $\mathrm{Deck}(p)$. A covering is called *normal* if the deck transformation group acts transitively on one (or each) fiber. This is equivalent to the condition that $p_*\pi_1(X) \subseteq \pi_1(Y)$ is a normal subgroup.

Let $H \subseteq \pi_1(Y, y)$ be a subgroup of the fundamental group of Y with base point y and let $\tilde{Y}$ be the universal cover of Y. Then the space $\tilde{Y}/H$ is a covering space of Y since $\tilde{Y}/\pi_1(Y, y) = Y$. Hence every subgroup of $\pi_1(Y, y)$ gives a covering of Y. On the other hand, every covering $p\colon X \to Y$ gives us a subgroup $p_*\pi_1(X, x) \subseteq \pi_1(Y, y)$. All in all, this gives a bijection between the set of all isomorphism classes of coverings of Y with the set of all conjugacy classes of subgroups of $\pi_1(Y, y)$. Here, two coverings $p\colon X \to Y$ and $p'\colon X' \to Y$ are *isomorphic* if there is a homeomorphism $X \to X'$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "\cong"] \arrow[dr, "p"'] & & X' \arrow[dl, "p'"] \\ & Y & \end{tikzcd}$$

commutes.

Let $p\colon X \to Y$ be a finite covering of degree d. Consider the fiber $p^{-1}(y) = \{ y_1, \ldots, y_d \}$ over some point $y \in Y$ and a loop γ in Y based at y. This loop can be lifted to a path $\tilde{\gamma}$. Fixing $\tilde{\gamma}(0) = y_i$ this lift is unique and its endpoint is some y_j for $j \in \{ 1, \ldots, d \}$. Doing this for all indices yields a permutation and gives rise to a homomorphism $\pi_1(Y, y) \to S_d$.

Definition 3.3. The group homomorphism

$$\mu\colon \pi_1(Y, y) \to S_d$$

is called the *monodromy map*. Its image is the *monodromy group*.

A subgroup of S_d is called *transitive* if for each pair of indices $\{ i, j \}$ there is a permutation σ in this group such that $\sigma(i) = j$. If Y is connected, the image of the monodromy map is transitive.

3.2 Branched coverings

We now apply the above theory to holomorphic maps between Riemann surfaces, see Miranda [Mir95]. From now on, all spaces will be Riemann surfaces, which we denote mostly by capital letters X, Y and E. Unfortunately, most holomorphic maps are no covering maps. To see this, let $p\colon X \to Y$ be a non-constant holomorphic map and $x \in X$. In a neighborhood of x the map has the form $z \mapsto z^n$. The integer n is called the *multiplicity* or *ramification index* of p at x, is denoted by $\mathrm{mult}_x(p)$ and depends on x. But the set of points with multiplicity strictly greater than 1 is a discrete set and outside of this set the map p is a covering map. Furthermore, the degree of p can be calculated as the sum $\sum_{x \in p^{-1}(y)} \mathrm{mult}_x(p)$ of all multiplicities of points in the fiber over a point y and is independent of y.

Definition 3.4. A *branched covering* of a Riemann surface is a holomorphic non-constant map $p\colon X \to Y$ between Riemann surfaces. A point $x \in X$ is called *ramification point* if it has multiplicity greater than one and a point $y \in Y$ is called *branch point* if it is the image of a ramification point.

Let $p\colon X \to Y$ be a branched covering and $B \subseteq Y$ be the set of branch points. Then the map $p\colon X \setminus p^{-1}(B) \to Y \setminus B$ is a (topological) covering as defined in the previous section. We will call a covering *simply branched* if each fiber of a branch point has $\deg(p) - 1$ elements.

Proposition 3.5 (Riemann-Hurwitz formula). *Let $p\colon X \to Y$ be a branched covering. Then*

$$2g(X) - 2 = \deg(p)\,(2g(Y) - 2) + \sum_{x \in X} (\mathrm{mult}_x(p) - 1),$$

where $g(X)$ and $g(Y)$ denote the genus of X and Y, respectively.

In the case of a simply branched covering we have $\mathrm{mult}_x(p) = 2$ for each ramification point $x \in X$. Therefore it suffices to know three of the following four informations to compute the fourth: The degree of the covering, the genus of the covered surface, the genus of the covering surface and the number of ramification points.

The deck transformation group of a branched covering is defined to be the deck transformation group of the corresponding (unbranched) covering. Furthermore, a branched covering is normal if and only if its unbranched variant is normal. The monodromy of a branched covering $p\colon X \to Y$ is the group homomorphism

$$\mu\colon \pi_1(Y \setminus B, y_0) \to S_d,$$

which is defined as before.

Lemma 3.6 (Żołądek [Żoł02]). *The branched covering $X \to Y$ is normal if and only if the deck transformation group and the image of the monodromy map are isomorphic.*

For the next proposition, let us call two holomorphic maps $X \to Y$ and $X' \to Y$ *equivalent* if their corresponding covering maps are isomorphic. Two group homomorphisms from the same source to S_d are called *equivalent* if they are conjugated in S_d.

Proposition 3.7 (Miranda [Mir95]). *Let Y be a compact Riemann surface and $B \subseteq Y$ a finite subset. We have a bijection of the sets*

$$\{ \text{ holomorphic maps } p\colon X \to Y \text{ of degree } d \text{ whose branch points lie in } B \,\} / \sim$$

and

$$\{ \text{ group homomorphism } \mu\colon \pi_1(Y \setminus B, y_0) \to S_d \text{ with transitive image} \,\} / \sim .$$

3.3 Hurwitz spaces

In 1891 Hurwitz [Hur91] answered the question of how many coverings with given ramification data of the Riemann sphere exist. He showed the above proposition for $Y = \mathbb{P}^1\mathbb{C}$. This led to the so-called (classical) *Hurwitz space*

$$H_{d,g}(\mathbb{P}^1\mathbb{C}) = \left\{ p\colon X \to \mathbb{P}^1\mathbb{C} \text{ simply branched covering} \;\middle|\; g(X) = g, \deg(p) = d \right\}$$

of all simply branched coverings of the Riemann sphere of degree d and of genus g. By the Riemann-Hurwitz formula from Proposition 3.5, we can calculate the number of ramification points of the covering. Hurwitz showed that for all d and g the spaces $H_{d,g}(\mathbb{P}^1\mathbb{C})$ are connected complex manifolds.

In the following we generalize the classical Hurwitz space. The first step is to look at coverings of surfaces other than the Riemann sphere. Following Fulton [Ful69], we sketch why these are complex manifolds. The next step is to vary the covered surface and look at all coverings of a family of surfaces. Berstein and Edmons [BE84] showed that for a genus 1 base curve the Hurwitz space of primitive, simply branched coverings is always connected. A more algebraic approach by Bujokas [Buj15] shows the same and gives a classification of the connected components, when the coverings do not need to be primitive. Finally, Magaard, Shaska, Shpectorov and Völklein [Mag+02] describe all automorphism groups of genus 3 surfaces as well as the loci described by those groups.

Let Y be a Riemann surface, r a positive integer and let $\Sigma^r Y = Y^r/S_r$ be the r-fold symmetric product of Y. Define the *discriminant locus*

$$\Delta = \{ \, (y_1, \ldots, y_r) \in \Sigma^r Y \mid \text{there is } i \neq j \text{ such that } y_i = y_j \, \}$$

and let $B \in \Sigma^r Y \setminus \Delta$. Then Fulton [Ful69] defines the *Hurwitz space*

$$H(d, B, Y) = \{ \, p\colon X \to Y \text{ branched covering with branch locus } B \mid \deg(p) = d \, \} / \sim$$

of branched coverings of degree d with branch locus B and the *Hurwitz space*

$$H(d, r, Y) = \{ \, p\colon X \to Y \text{ branched covering with } r \text{ branch points} \mid \deg(p) = d \, \} / \sim$$

of branched coverings of degree d with r branch points. In both cases, two coverings are equivalent if they are isomorphic as defined in Section 3.1. Define the map

$$\Psi_r \colon H(d,r,Y) \to \Sigma^r Y \setminus \Delta,$$

which assigns to a covering its set of branch points. For every branch locus $B \in \Sigma^r Y \setminus \Delta$ we have $\Psi_r^{-1}(B) = H(d,B,Y)$. Wanting the map Ψ_r to be a covering, we choose the following topology for $\Sigma^r Y \setminus \Delta$: Let $U_1, \dots, U_r$ be disjoint, simply connected, open subsets in Y and define the set

$$N(U_1, \dots, U_r) = \{ \, (p_1, \dots, p_r) \in \Sigma^r Y \setminus \Delta \mid p_i \in U_i \, \} \, .$$

These sets form a basis of the topology of $\Sigma^r Y \setminus \Delta$. Via pulling back along Ψ_r, this gives a topology on $H(d,r,Y)$. Now fix $U_1, \dots, U_r$ and let $U = \bigcup_{i=1}^r U_i$ be their union. Let B and $B' \in N(U_1, \dots, U_r)$. Then the maps $Y \setminus B \to Y \setminus U$ and $Y \setminus B' \to Y \setminus U$ are deformation retractions and hence the fundamental groups $\pi_1(Y \setminus B)$ and $\pi_1(Y \setminus B')$ are both isomorphic to $\pi_1(Y \setminus U)$. Each subgroup $H \subseteq \pi_1(Y \setminus B)$ of the fundamental group corresponds to a covering $p_H \colon X_H \to Y \setminus B$. On the one hand, by the above isomorphism, $H \subseteq \pi_1(Y \setminus U)$ and H corresponds to a covering $p'_H \colon X'_H \to Y \setminus U$. On the other hand, by restricting p_H to the preimage $p_H^{-1}(Y \setminus U)$, we get a covering of $Y \setminus U$ described by the group H. Hence it is the same as the covering p'_H, i.e. the covering of $Y \setminus U$ is a restriction of the covering of $Y \setminus B$. By symmetry arguments, the covering of $Y \setminus U$ is a restriction of the covering of $Y \setminus B'$ as well. Thus we have a bijection

$$H(d,B,Y) \to H(d,B',Y) \quad \text{for} \quad B, B' \in N(U_1, \dots, U_r).$$

For a covering $p \in H(d,B,Y)$ let us denote by $p_{B'} \in H(d,B',Y)$ the covering assigned to p by the above bijection. Hence the preimage

$$\Psi_r^{-1}\left(N(U_1, \dots, U_r)\right) = \coprod_{p \in H(d,B,Y)} \left\{ \, p_{B'} \;\middle|\; B' \in N(U_1, \dots, U_r) \, \right\}$$

of an open set in $\Sigma^r Y \setminus \Delta$ is a disjoint union of sets homeomorphic to $N(U_1, \dots, U_r)$. Thus Ψ_r is a covering map. The natural complex structure of $\Sigma^r Y$ makes $H(d,r,Y)$ into a complex manifold.

Next we vary the covered surface. We fix the genus of the covered surface, the genus of the covering surface, the degree of the covering and assume that our covering is simply branched.

Definition 3.8. The space

$$H_{d,g,h} = \{ \, p \colon X \to Y \text{ simply branched covering} \mid \deg(p) = d, g(X) = g, g(Y) = h \, \} \, / \sim,$$

which consists of all simply branched coverings of degree d from a surface of genus g to one of genus h, is called *Hurwitz space*. Two coverings $p \colon X \to Y$ and $p' \colon X' \to Y'$ are *equivalent* if and only if there are isomorphisms $X \to X'$ and $Y \to Y'$ such that the diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ Y & \longrightarrow & Y' \end{array}$$

commutes.

Observe that by the Riemann-Hurwitz formula in Proposition 3.5 we can equivalently describe this space by the degree d of the covering, the genus g of the covering surface and the number of ramification points. A covering $p\colon X \to Y$ is called *primitive* if it does not factor into an unramified covering $Y' \to Y$ and a ramified one $X \to Y'$. Note that if for every $y \in B$ the fiber $p^{-1}(y)$ has exactly one element, the covering p is primitive.

Proposition 3.9 (Berstein-Edmonds [BE84]). *Let X and Y be connected surfaces and $p\colon X \to Y$ be a simply branched covering of degree d. Then p is primitive if and only if the monodromy map is surjective.*

Let us now restrict to coverings of the torus, thus $h = 1$.

Proposition 3.10. *The Hurwitz space $H_{d,g,1}$ is a complex manifold.*

Proof. We already know that $H_{d,g}(E)$ is a complex manifold for each $E \in M_1$. The space $H_{d,g,1}$ is a fiber bundle over the complex manifold M_1 and the fibers $H_{d,g}(E)$ are complex manifolds as well. Hence it is a complex manifold. $\square$

Berstein and Edmonds [BE84] showed that the space

$$H^0_{d,g,1} = \{\, p \in H_{d,g,1} \mid p \text{ primitive} \,\}$$

of all primitive coverings of tori is connected. They used topological methods similar to those of Hurwitz.

Algebraically, the same result was shown by Bujokas in his thesis [Buj15]. Moreover, he classified the connected components of the Hurwitz space $H_{d,g,1}$ in two ways. On the one hand, by the type of the cokernel of the push forward p_* of the first homology group. This is the direct product of two cyclic groups $C_{d_1} \times C_{d_2}$ with $d_1 \mid d_2$. On the other hand, let e be the maximal degree of an isogeny over which the covering factors and let m denote the maximal natural number for which the covering factors over the multiplication by $[m]$. Then the above cokernel can be written as $C_m \times C_{\frac{e}{m}}$.

Proposition 3.11 (Berstein-Edmonds [BE84], Bujokas [Buj15]). *The connected components of $H_{d,g,1}$ are separated by the isomorphism type of the cokernel of the push forward p_*. Moreover, we have a bijection between the connected components of $H_{d,g,1}$ and pairs (e, m), where $m^2 \mid e \mid d$. In particular, the space of primitive coverings $H^0_{d,g,1}$ is connected.*

Finally, we want to describe Hurwitz spaces the way Magaard et al. [Mag+02] did. Let G be a finite group. Then a curve X is called a *G-curve* if there exists an injective map $G \hookrightarrow \mathrm{Aut}(X)$. Two G-curves X and X' are equivalent if there is a G-equivariant isomorphism $X \to X'$. Let $C_1, \ldots, C_r$ be non-trivial conjugacy classes in G and $C = (C_1, \ldots, C_r)$. For $r = 0$ we have $C = \emptyset$. Given a branched covering $p\colon X \to X/G$, a *distinguished inertia group generator* at $x \in X$ is an element in G which acts on the tangent space at x as multiplication by $\exp\left(\frac{2\pi\mathrm{i}}{\mathrm{mult}_x(p)}\right)$.

Definition 3.12. Let G be a finite group and X a curve.

a) A G-curve X is *of ramification type* (g, G, C) if the genus of X is g and the covering $X \to X/G$ is ramified at r points such that C_i is the conjugacy class of the distinguished inertia group generator at the i-th branch point.

b) The space $H(g, G, C)$ of all equivalence classes of G-curves of ramification type (g, G, C) is called *Hurwitz space.*

Denote by c_i the order of an element in C_i and call $c = (c_1, \ldots, c_r)$ the *signature* of the G-curve. The genus of X/G and the dimension of the space of all G-curves with given ramification type only depends on the signature c, not on C.

We denote the map forgetting the G-action by

$$\mathcal{F}\colon H(g, G, C) \to M_g$$

and the map, which maps a G-curve X to the quotient X/G with its set of branch points marked, by

$$\mathcal{F}_0\colon H(g, G, C) \to M_{g_0, r}$$

This enables us to calculate the dimension of these Hurwitz spaces.

Proposition 3.13 (Bertin-Romagny [BR96]). *The set $H(g, G, C)$ is a quasi-projective variety. The morphism $\mathcal{F}_0$ is surjective and both $\mathcal{F}$ and $\mathcal{F}_0$ are finite. If nonempty, the dimension of each connected component of $\mathcal{F}(H(g, G, C))$ is $3g_0 - 3 + r$.*

Furthermore, the morphism $\mathcal{F}\colon H(g, G, C) \to M_g$ is unramified.

The notion of unramified is defined by Grothendieck in EGA [Gro67]. Most importantly, interpreting the Hurwitz space $H(g, G, C)$ as a complex manifold, the algebraic notion unramified translates into the differential geometric notion of immersion, i.e. the induced map on the tangent spaces $T_1 H(g, G, C) \to T_1 M_g$ is injective. See for example [FL80].

Furthermore, Magaard et al. [Mag+02] study which groups appear as full automorphism groups of curves. They give a complete list of all possible automorphism groups of curves of genus 3. Furthermore, for each automorphism group they calculate the dimension of the locus of all curves with these automorphisms and write down equations defining the curves in this locus. In genus 3 most of this was known before by Komiya and two Kuribayashis [KK79; KK90].

Proposition 3.14 (Magaard et al. [Mag+02]). *For non-hyperelliptic surfaces of genus 3, we state in Table 3.1 their possible equations, their automorphism groups and the dimension of the locus of all surfaces with corresponding automorphism group. Each*

G	Equation $0 =$	Dimension
C_2	$x^4 + x^2(y^2 + a) + by^4 + cy^3 + dy^2 + ey + g$, either $e = 1$ or $g = 1$	4
V_4	$x^4 + y^4 + ax^2y^2 + bx^2 + cy^2 + 1$	3
D_8	$x^4 + y^4 + ax^2y^2 + b(x^2 + y^2) + 1$	2
$D_8 \times_Z C_4$	$y^4 - x(x-1)(x-\lambda)$	1
S_4	$x^4 + y^4 + a(x^2y^2 + x^2 + y^2) + 1$	1
C_3	$y^3 - x(x-1)(x-s)(x-t)$	2
C_6	$y^3 - x(x-1)(x-s)(x+1-s)$	1
S_3	$a(x^4 + y^4 + 1) + b(x^2y^2 + x^2 + y^2) + c(x^2y + y^2x + xy)$	3

Table 3.1: For a non-hyperelliptic surface of genus 3, we denote in the first column its automorphism group, in the second its equation and in the third the dimension of the locus of such curves. All variables are complex numbers.

G	Equation $y^2 =$	Dimension of locus
C_2	$x(x-1)(x^5 + ax^4 + bx^3 + cx^2 + dx + e)$	5
V_4	$(x^2-1)(x^6 + ax^4 + bx^2 + c)$	3
C_2^3	$(x^4 + ax^2 + 1)(x^4 + bx^2 + 1)$	2
$D_8 \times C_2$	$x^8 + ax^4 + 1$	1
C_4	$x(x^2-1)(x^4 + ax^2 + b)$	2
$C_2 \times C_4$	$x(x^2-1)(x^4 + ax^2 + 1)$	1
D_{12}	$x(x^6 + ax^3 + 1)$	1

Table 3.2: For a hyperelliptic surface of genus 3, we denote in the first column its automorphism group, in the second its equation and in the third the dimension of the locus of such curves. All variables are complex numbers.

locus is connected. We do leave out those exceptional automorphism groups, whose corresponding loci consist of one point only.

For hyperelliptic surfaces of genus 3, we state in Table 3.2 their possible equations, their automorphism groups and the dimension of the locus of all surfaces with corresponding automorphism group. Each locus is connected. We do leave out those exceptional automorphism groups, whose corresponding loci consist of one point only.

The groups appearing in the tables will be described more precisely in Chapter 4. For a first idea it is enough to say that C_n is the cyclic group with n elements, D_n the dihedral group with n elements, S_n the symmetric group on n elements and $V_4 = C_2 \times C_2$ the Klein four-group.

In fact, we are only interested in one Hurwitz space: The Hurwitz space $H_{2,3,1}$ of all coverings of degree 2 from a surface of genus 3 to a torus. We use both, the connectedness argument of Bujokas and the dimension formulas as well as the equations and automorphism groups from Magaard et al. Hence we have to show that $H_{2,3,1}$ is a Hurwitz space of G-curves of some ramification type.

Lemma 3.15. *The Hurwitz spaces $H_{2,3,1}$, $H^0_{2,3,1}$ and $H(3, C_2, (-1,-1,-1,-1))$ coincide.*

Proof. Let $p\colon X \to E$ be in $H_{2,3,1}$. Since $\deg(p) = 2$, the covering is normal. By Lemma 3.6 we have $\mathrm{Deck}(p) = \mu(\pi_1(E \setminus B)) \subseteq S_2$ and $|\mathrm{Deck}(p)| = 2$. Hence the monodromy map is surjective and by Proposition 3.9 the covering is primitive. This implies

$$H_{2,3,1} = H^0_{2,3,1}.$$

We now show that $H_{2,3,1} \subseteq H(g, G, C)$ and calculate g, G and C: The quotient $X/\mathrm{Deck}(p)$ is E and we set $G = \mathrm{Deck}(p) = C_2$. Furthermore, $g(X) = g = 3$. By the Riemann-Hurwitz formula we have four ramification points, hence $r = 4$. The generator of C_2 acts on the tangent space of a ramification point by $\mathrm{e}^{\frac{2\pi \mathrm{i}}{e}}$ for some natural number e. Since the generator is of order 2, $e \in \{1, 2\}$. Because the multiplicity is greater than 1, $e = 2$ for each ramification point. Hence $C = (-1,-1,-1,-1)$ and $H_{2,3,1} \subseteq H(3, C_2, (-1,-1,-1,-1))$.

On the other hand, $H(3, C_2, (-1,-1,-1,-1)) \subseteq H_{2,3,1}$: Define $E = X/C_2$. The covering $X \to E$ is of degree 2 and has four ramification points, each with ramification index 2. Hence the Riemann-Hurwitz formula gives

$$4 = 2 \cdot (2g(E) - 2) + \sum_{i=1}^{4} 1,$$

showing $g(E) = 1$.

Up to now, we showed that each covering in $H(3, C_2, (-1,-1,-1,-1))$ is a covering in $H_{2,3,1}$ and vice versa. We still have to check that the equivalence relations on both spaces coincide. On the one hand, let $p\colon X \to E$ and $p'\colon X' \to E'$ be equivalent, i.e. there exist biholomorphic maps $\Phi\colon X \to X'$ and $\Psi\colon E \to E'$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "p"] & X' \arrow[d, "p'"] \\ E \arrow[r, "\Psi"] & E' \end{tikzcd}$$

commutes. We have to show that Φ is G-equivariant for $G = C_2$. Let $i\colon G \to \mathrm{Aut}(X)$ and $i'\colon G \to \mathrm{Aut}(X')$ be embeddings. For $g \in i(G)$ we have

$$p' \circ \Phi(gx) = \Psi \circ p(gx) = \Psi \circ p(x)$$

since g is a deck transformation. Hence $\Phi(gx)$ and $\Phi(x)$ are in the fiber $p'^{-1}(\Psi \circ p(x))$. So there exists a deck transformation $g' \in i'(G)$ such that $\Phi(gx) = g'\Phi(x)$. Thus Φ is G-equivariant.

On the other hand, let X and X' in $H(g,G,C)$ be two equivalent curves of type (g,G,C) and let $\Phi\colon X \to X'$ be a G-equivariant isomorphism. We define the surfaces $E = X/G$ and $E' = X'/G$ and the map

$$\Psi\colon E \to E', \qquad x \cdot G \mapsto \Phi(x) \cdot G.$$

This map is well defined, since if $x = x' \cdot g$ are in the same G-orbit, we have

$$\Psi(x \cdot G) = \Phi(x) \cdot G \ni \Phi(x) = \Phi(x' \cdot g) = \Phi(x') \cdot g \in \Phi(x') \cdot G = \Psi(x' \cdot G).$$

for a $g' \in G$. Furthermore, Ψ is biholomorphic because Φ is. □

3.4 Hurwitz spaces of translation coverings

In this section we want to equip the Hurwitz spaces discussed before with additional structure. These Hurwitz spaces give rise to nontrivial affine invariant submanifolds.

Definition 3.16. Let $H_{d,g,h}$ be the Hurwitz space consisting of all simply branched coverings of degree d from a surface of genus g to one of genus h. We define the *Hurwitz space of translation coverings* by

$$\Omega H_{d,g,h} = \{\, (p, X, \omega, Y, \upsilon) \mid (p, X, Y) \in H_{d,g,h}, \upsilon \in \Omega(Y), \omega = p^*\upsilon \,\} \,/\, \sim,$$

consisting of all translation coverings, which, after forgetting the translation structure, belong to the Hurwitz space $H_{d,g,h}$. We call two translation coverings $(p, X, \omega, Y, \upsilon)$ and $(p', X', \omega', Y', \upsilon')$ equivalent if and only if there exist translations $X \to X'$ and $Y \to Y'$ such that the diagram

$$\begin{tikzcd} (X,\omega) \arrow[r] \arrow[d, "p"] & (X',\omega') \arrow[d, "p'"] \\ (Y,\upsilon) \arrow[r] & (Y',\upsilon') \end{tikzcd}$$

commutes.

The map $\pi_H\colon \Omega H_{d,g,h} \to H_{d,g,h}$ makes $\Omega H_{d,g,h}$ naturally into a topological space. There are more important maps: We have the forgetful map $\Omega\mathcal{F}\colon \Omega H_{d,g,h} \to \Omega M_g$ given by $\Omega\mathcal{F}(p, X, \omega, Y, \upsilon) = (X, \omega)$. We denote the image of $\Omega\mathcal{F}$ by $\Omega\mathcal{L}_{d,g,h}$. Furthermore, recall the forgetful map $\mathcal{F}\colon H_{d,g,h} \to M_g$ mapping (p, X, Y) to X. We denote the image of $\mathcal{F}$ by $\mathcal{L}_{d,g,h}$. Finally, we have the map $\pi_{\mathcal{L}}\colon \Omega\mathcal{L}_{d,g,h} \to \mathcal{L}_{d,g,h}$ given by $\pi_{\mathcal{L}}(X, \omega) = X$. These maps make the diagram

$$\begin{tikzcd}
\Omega H_{d,g,h} \arrow[rrr, "\Omega\mathcal{F}"] \arrow[ddd, "\pi_H"'] & & & \Omega\mathcal{L}_{d,g,h} \arrow[ddd, "\pi_{\mathcal{L}}"] \\
& (p, X, \omega, Y, \upsilon) \arrow[r, mapsto] \arrow[d, mapsto] & (X, \omega) \arrow[d, mapsto] & \\
& (p, X, Y) \arrow[r, mapsto] & X & \\
H_{d,g,h} \arrow[rrr, "\mathcal{F}"] & & & \mathcal{L}_{d,g,h}
\end{tikzcd}$$

commutative and have nice properties. To simplify notation we omit the indices and write $H = H_{d,g,h}$.

Proposition 3.17. *Let $H = H_{d,g,h}$ be a Hurwitz space.*

a) The map $\pi_H\colon \Omega H \to H$ defines a fiber bundle with fiber $\pi_H^{-1}(p, X, Y) = p^*\Omega(Y)$.

b) The space ΩH is the pullback $\Omega\mathcal{L} \times_{\mathcal{L}} H$ along the maps $\mathcal{F}\colon H \to \mathcal{L}$ and $\pi_{\mathcal{L}}\colon \Omega\mathcal{L} \to \mathcal{L}$.

c) The map $\Omega\mathcal{F}\colon \Omega H \to \Omega\mathcal{L}$ is a proper immersion.

d) The map $\pi_{\mathcal{L}}\colon \Omega\mathcal{L} \to \mathcal{L}$ defines a fiber bundle.

Proof. a) Let $U(p, X, Y)$ be a neighborhood of a covering $(p, X, Y) \in H$. We are interested in the fiber

$$\pi_H^{-1}(U(p, X, Y)) = \{\ (p', X', \omega', Y', \upsilon') \mid (p', X', Y') \in U(p, X, Y),\ \upsilon' \in \Omega(Y'),\ \omega' = p'^*\upsilon'\ \}.$$

The map $\pi_0\colon \Omega M_h \to M_h$ is a fiber bundle with fiber $\pi_0^{-1}(Y) = \Omega(Y)$. This gives, for a neighborhood $U(Y)$ of Y, a homeomorphism $\phi\colon \pi_0^{-1}(U(Y)) \to U(Y) \times \Omega(Y)$. In other words, a translation structure on a surface gives us translation structures in a neighborhood of this surface. Hence for $Y' \in U(Y)$ and $\upsilon \in \Omega(Y)$ we find via $\varphi^{-1}(Y', \upsilon)$ a translation structure υ' on Y'. By pulling back we get a translation structure $\omega' = p'^*\upsilon'$ for a covering $p\colon X \to Y$. This leads to a homeomorphism

$$\pi_H^{-1}(U(p, X, Y)) \to U(p, X, Y) \times p^*\Omega(Y),$$

which makes $\pi_H\colon \Omega H \to H$ into a fiber bundle with fiber $p^*\Omega(Y)$.

b) We have to check that ΩH has the universal property of a fiber product. Given a manifold J and maps $f\colon J \to \Omega\mathcal{L}$ and $g\colon J \to H$ we have to show that there exists a unique map $h\colon J \to \Omega H$ making the diagram

$$\begin{tikzcd}
J \arrow[drr, bend left, "f"] \arrow[dr, dashed, "h"] \arrow[ddr, bend right, "g"'] & & \\
& \Omega H \arrow[r, "\Omega\mathcal{F}"] \arrow[d, "\pi_H"] & \Omega\mathcal{L} \arrow[d, "\pi_{\mathcal{L}}"] \\
& H \arrow[r, "\mathcal{F}"] & \mathcal{L}
\end{tikzcd}$$

commutative. Since $\mathcal{F}(g(j)) = \pi_{\mathcal{L}}(f(j)) = X$ we can assume $f(j) = (X, \omega)$ and $g(j) = (p, X, E)$. Then the only way to define h is by

$$h(j) = (p, X, \omega, Y, \upsilon)$$

with $p^*\upsilon = \omega$. The differential η is unique since from $p^*\upsilon_1 = p^*\upsilon_2$ follows $\upsilon_1 \circ p = \upsilon_2 \circ p$. Since p is surjective, $\upsilon_1 = \upsilon_2$.

c) By Proposition 3.13 the map $\mathcal{F}\colon H \to \mathcal{L}$ is a unramified morphism and hence a (differential geometric) immersion. Being unramified is invariant under base change, so $\Omega\mathcal{F}$ is unramified and hence a (differential geometric) immersion.

d) By a theorem of Ehresmann [Ehr48], to see that $\pi_{\mathcal{L}}$ is a fiber bundle, it is sufficient to show that $\pi_{\mathcal{L}}$ is a surjective proper submersion. Because the tangent spaces TH and $T\mathcal{L}$ have the same dimension and because $\mathcal{F}$ and $\Omega\mathcal{F}$ are immersions according to c), they are submersions. Furthermore, π_H is a fiber bundle and thus a submersion. The following commutative diagram of tangent spaces

$$\begin{tikzcd}
T\Omega H \arrow[r, two heads] \arrow[d, two heads] & T\Omega\mathcal{L} \arrow[d] \\
TH \arrow[r, two heads] & T\mathcal{L}
\end{tikzcd}$$

shows that the map $\pi_{\mathcal{L}}$ is a submersion. Moreover, it is surjective by definition. Finally, since π_H is a fiber bundle, it is proper. Since $\mathcal{F}$ is finite, it is proper in the algebro-geometric sense and by SGA1 [Gro71, XII, Prop 3.2] also in the topological sense. Then $\pi_{\mathcal{L}}^{-1}(K) = \Omega\mathcal{F} \circ \pi_H^{-1} \circ \mathcal{F}^{-1}(K)$ is compact for each compact set K and hence $\pi_{\mathcal{L}}$ is proper. $\square$

This enables us to show that $\Omega\mathcal{L}_{d,g,1}$ is an affine invariant submanifold whose dimension depends only on the number of branch points.

Corollary 3.18. *Let n be the number of branch points of a covering in $\Omega H_{d,g,1}$.*

a) The space $\Omega H_{d,g,1}$ is a connected complex manifold of dimension $n+1$.

b) The space $\Omega\mathcal{L}_{d,g,1}$ is an affine invariant submanifold of dimension $n+1$.

Proof. a) By Proposition 3.10 $H_{d,g,1}$ is a manifold, by Proposition 3.9 it is connected and by Proposition 3.13 it is of dimension $\dim H_{d,g,1} = \dim M_{1,n} = n$.

The space $\Omega H_{d,g,1}$ is a fiber bundle over the complex manifold $H_{d,g,1}$ with fibers $\pi_H^{-1}(E) = \Omega(E) = \mathbb{C}^\times$ which are one-dimensional complex connected manifolds. Thus the fiber bundle itself is a complex connected manifold of dimension $n+1$.

b) A sketch of the proof is carried out by Wright [Wri15b].

By a) $\Omega H_{d,g,1}$ is a manifold. Furthermore, the map $\Omega\mathcal{F}\colon \Omega H_{d,g,1} \to \Omega\mathcal{L}_{d,g,1}$ is a proper immersion. Hence we only have to check that $\Omega\mathcal{L}_{d,g,1}$ is locally described by homogeneous $\mathbb{R}$-linear equations in period coordinates. The immersion $\Omega\mathcal{F}$ is locally injective. Thus, when working with a translation surface (X,ω), we can always assume it is equipped with a unique covering denoted by $p\colon (X,\omega) \to (E,\eta)$.

Let $(X,\omega) \in \Omega\mathcal{L}_{d,g,1}$ and let Σ be the set of singularities of ω and $\bar{\Sigma} = p(\Sigma)$ the set of branch points. Since $\omega = p^*\eta$ we have

$$\int_c \omega = \int_c p^*\eta = \int_{p_* c} \eta.$$

The kernel of the map $p_*\colon H_1(X,\Sigma,\mathbb{Z}) \to H_1(E,\bar{\Sigma},\mathbb{Z})$ is of dimension

$$\dim H_1(X,\Sigma,\mathbb{C}) - \dim H_1(E,\bar{\Sigma},\mathbb{C}) = (2g+n-1) - (n+1) = 2g-2.$$

Extending the basis of the kernel $\{\, c_1,\dots,c_{2g-2} \,\}$ to a basis of the whole relative homology $H_1(X,\Sigma,\mathbb{Z})$, we get systems of linear equations

$$\int_{c_i} \omega = 0, \qquad i = 1,\dots,2g-2.$$

This system describes a linear subspace of dimension $(2g+n-1)-(2g-2) = n+1$. This shows that $\Omega\mathcal{L}_{d,g,1}$ is an affine invariant submanifold of dimension at most $n+1$.

On the other hand, $\Omega\mathcal{L}_{d,g,1}$ is at least $(n+1)$-dimensional: The space ΩM_1 is 2-dimensional and we may choose freely, after fixing one branch point, $n-1$ branch points. This gives us at least $n+1$ degrees of freedom. $\square$

Although $\Omega H_{d,g,h}$ is not a manifold anymore for $h > 1$ the same procedure works to produce affine invariant submanifolds. By picking a finite covering $\widetilde{\Omega H}_{d,g,h}$ of $\Omega H_{d,g,h}$ we get an immersion $\widetilde{\Omega H}_{d,g,h} \to \Omega H_{d,g,h} \to \Omega\mathcal{L}_{d,g,h}$. Note that $\Omega H_{d,g,h}$ does not need to be connected for $d > 2$ or $h > 1$.

4 Finite groups

In this chapter we define the groups we need in this thesis.

We denote by C_n the cyclic group of order n, by S_n the symmetric group on n elements and by D_n the dihedral group with n elements given by

$$D_n = \langle x, a \mid a^{\frac{n}{2}} = x^2 = 1,\ xax^{-1} = a^{-1} \rangle$$

for even n.

All the following groups can be found in the large book by Hall and Senior [HS64]. Denote by $V_4 = C_2 \times C_2$ the Klein four-group. The group $D_8 \times_Z C_4$ is the central product of the dihedral group D_8 and the cyclic group C_4. It can also be seen as the direct product of those with amalgamation by C_2. We have a presentation

$$D_8 \times_Z C_4 = \langle a, x, y \mid a^4 = x^2 = 1,\ a^2 = y^2,\ xax^{-1} = a^{-1},\ ay = ya,\ xy = yx \rangle.$$

In the book by Hall and Senior this group corresponds to the group of order 16 with number 8. Moreover, we are interested in a presentation of the direct product

$$D_8 \times C_2 = \langle a, x, y \mid a^4 = x^2 = y^2 = 1,\ xax = a^{-1},\ ya = ay,\ xy = yx \rangle$$

of the dihedral group D_8 and the cyclic group C_2. In the book by Hall and Senior this group corresponds to the group of order 16 with number 6. Furthermore, by

$$Q = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk = -1 \rangle$$

we denote the quaternion group. In the book by Hall and Senior this group corresponds to the group of order 8 with number 5.

In the following diagram we sketch how some of these groups are contained in each other. Subgroups are indicated by arrows. The order of the groups increases from left to right and is 2, 4, 8 and 16, respectively.

$$\begin{tikzcd}
 & & Q \arrow[dr] & \\
 & & D_8 \arrow[r] \arrow[ddr] & D_8 \times_Z C_4 \\
C_2 \arrow[r] \arrow[uurr] & V_4 \arrow[ur] \arrow[dr] & & \\
 & & C_2^3 \arrow[r] & D_8 \times C_2
\end{tikzcd}$$

5 The Wollmilchsau

In this chapter we sketch some results of Herrlich and Schmithüsen [HS07a; HS08]. We define the Wollmilchsau, consider a Hurwitz space in which it is contained and talk about other origamis, whose Teichmüller curves form a dense subset of this Hurwitz space.

The *Wollmilchsau* is the translation surface described by Figure 5.1. Edges with same labels are glued and points with the same name coincide. By the Euler formula the genus of the Wollmilchsau is 3.

The quaternion group Q acts by left multiplication on the squares of the Wollmilchsau. The quotient of the Wollmilchsau by the quaternion group Q is a torus and the corresponding covering has one ramification point. Hence the Wollmilchsau is an origami. There is one more automorphism that rotates each square by π around the four vertices. Observe that its square is the map -1. Hence the whole automorphism group is $D_8 \times_Z C_4$, the central product of the dihedral group D_8 and the cyclic group C_4. Another point of view is to regard the Wollmilchsau as a covering of degree 2 of the torus with four ramification points. To see this note that the map -1 interchanges the two polygons and has four fixed points, namely the vertices.

The Veech group of the Wollmilchsau is $\mathrm{SL}_2(\mathbb{Z})$ and its Teichmüller curve consists of those curves W_λ given by the equation $y^4 = x(x-1)(x-\lambda)$. For every W_λ there is an involution with four fixed-points such that the quotient of W_λ by this involution does not depend on λ. Such an involution is not in Q. If the branch points of the corresponding covering are N-torsion points, multiplying by N gives rise to a new origami. Since there are infinitely many torsion points, we get infinitely many origamis. For every origami obtained in this way, its Teichmüller curve intersects the one of the Wollmilchsau. These infinitely many Teichmüller curves form a dense subset of a subspace of the Hurwitz space $H_{2,3,1}$, which is the closure of all coverings with symmetric branch points and a non-hyperelliptic covering surface. The first condition implies that the Klein four-group V_4 is contained in the automorphism group of the surface.

Using explicit formulas, Herrlich and Schmithüsen compute the equations for each curve in this special subspace and the dimension of this space. These coincide with the equation and dimension in Table 3.1 for $G = V_4$. Furthermore, they compute the Veech groups of origamis whose Teichmüller curves intersect the one of the Wollmilchsau.

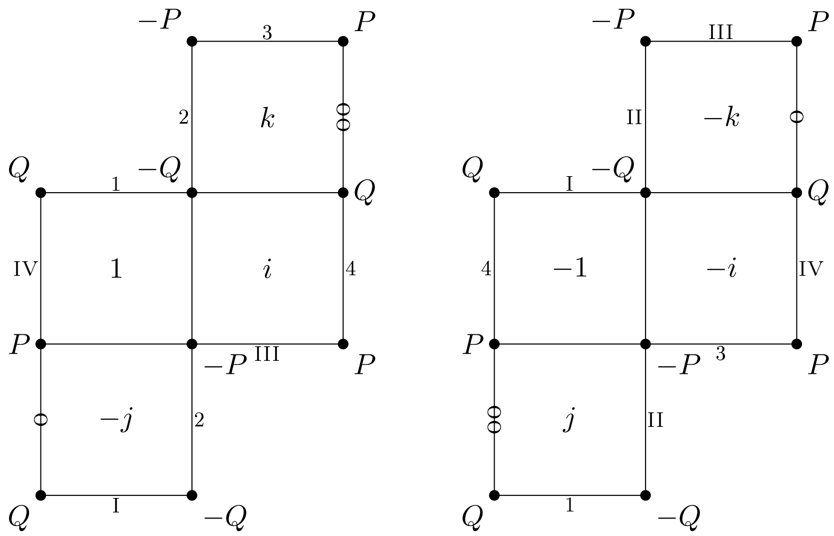


Figure 5.1: The Wollmilchsau. Edges with the same label are glued. The quaternion group acts by left multiplication on the set of squares.

6 A Hurwitz space of translation surfaces

In the previous chapter, we discussed a special Hurwitz space discovered by Herrlich and Schmithüsen [HS07a]. Its remarkable property is that it is the closure of an infinite union of Teichmüller curves, each one intersecting the Teichmüller curve of the Wollmilchsau. This Hurwitz space can be described as the space of all coverings of degree 2 of a torus with four ramification points such that the covering surfaces are not hyperelliptic and such that the branch points are symmetric with respect to some origin. The last condition is equivalent to the condition that the automorphism group contains the Klein four-group V_4.

In this chapter we reinvent this Hurwitz space, starting from a more general perspective with the Hurwitz space $H_{2,3,1}$ of all coverings of the torus of degree 2 with four ramification points. This space contains the above Hurwitz space as a subspace of codimension 1. Our viewpoint differs greatly from the original one. We equip each covering in $H_{2,3,1}$ with a natural translation structure arising from the one of the covered torus. Hence we can use more elementary methods to obtain results about the connected components and the dimension of the Hurwitz space. Then we investigate subspaces of this Hurwitz space where the relations of the branch points are restricted. The first subspace is the Hurwitz space defined by Herrlich and Schmithüsen, except that we permit hyperelliptic surfaces. Thereafter, each restriction on the set of branch points gives an extra automorphism. We compute the connected components and dimensions of the subspaces given by those automorphisms. Furthermore, their images in ΩM_3 are not only loci given by a fixed automorphism group, but also affine invariant submanifolds.

Finally, we show that there are no affine invariant submanifolds in the Hurwitz space of translation coverings that are described by their automorphism groups other than the ones we constructed. The Wollmilchsau is a translation surface in one of those subspaces. We find three siblings of the Wollmilchsau and compute their Veech groups.

6.1 A special Hurwitz space

In this section we compute properties of the Hurwitz space $H_{2,3,1}$. By adding an additional translation structure to each covering in this space, the results follow elementarily. Most of them follow as well from more general statements by Bujokas, Berstein-Edmonds and Wright [BE84; Buj15; Wri15b].

Recall the definition of the Hurwitz space $H_{d,g,h}$ in Definition 3.8 and the forgetful map $\mathcal{F}\colon H_{d,g,h} \to M_g$ defined in Section 3.3.

Definition 6.1. The Hurwitz space

$$H = H_{2,3,1} = \Big\{ \ (p, X, E) \mid X \in M_3, \ E \in M_{1,1} \\ p\colon X \to E \text{ simply ramified covering of degree } 2 \Big\}$$

consists of all coverings of degree 2 of a Riemann surface of genus 3 over an elliptic curve.

Denote by $\mathcal{F}\colon H \to M_3$ the forgetful morphism and its image by

$$\mathcal{L} = \mathcal{L}_{2,3,1} = \mathcal{F}(H).$$

It consists of those Riemann surfaces of genus 3 which are coverings of degree 2 of an elliptic curve.

By the Riemann-Hurwitz-formula in Proposition 3.5 we can describe the space H equivalently by coverings of degree 2 with four ramification points. Since a covering in H is of degree 2 and hence normal, the deck transformation group is C_2, the cyclic group of order 2. Here and subsequently, let σ denote the generator of the deck transformation group. Thus the generic automorphism group of a Riemann surface X in $\mathcal{L}$ is

$$\mathrm{Aut}(X) = C_2.$$

By Proposition 3.11 we know that H is a connected complex manifold. Due to Proposition 3.13 its dimension is $3 \cdot 1 - 3 + 4 = 4$.

Given the automorphism σ, we can compute explicit formulas for curves in $\mathcal{L}$. Those can be found in Tables 3.1 and 3.2 for the non-hyperelliptic and the hyperelliptic case, respectively. Nevertheless, we make our computation explicit.

Proposition 6.2. *Let $X \in \mathcal{L}$ be a non-hyperelliptic Riemann surface of genus 3. Then it is the zero set of the homogeneous polynomial*

$$f = x^4 + x^2(y^2 + az^2) + by^4 + cy^3z + dy^2z^2 + eyz^3 + gz^4,$$

where not $e = g = 0$. The dimension of the locus $\mathcal{L}$ is 4, hence the complex numbers a, b, c, d, e and g are not independent.

Proof. Since the genus of X is $g(X) = 3$, the canonical embedding embeds X into $\mathbb{P}^2\mathbb{C}$. So there is a homogeneous polynomial $f \in \mathbb{C}[x, y, z]$ of degree $\deg(f) = 4$ such that $V(f) = X$, see e.g. [Har77]. We may assume that $\sigma\colon x \mapsto -x$. Hence in the polynomial f only even powers of x are necessary. Let f be given by

$$f = a_0x^4 + x^2(a_1y^2 + a_2yz + a_3z^2) + a_4y^4 + a_5y^3z + a_6y^2z^2 + a_7yz^3 + a_8z^4$$

with coefficients $a_i \in \mathbb{C}$, $i = 0, 1, \ldots, 8$.

Firstly, observe that $a_1 = a_2 = a_3 = 0$ would imply the existence of an automorphism $x \mapsto \zeta_4 x$ of order 4. In the generic case, this does not exist. For $a_0 = 0$ we get that $x^2(a_1y^2 + a_2yz + a_3z^2) = p(x : y : z)$ for a homogeneous polynomial p. Substituting $a_1y^2 + a_2yz + a_3z^2 = z^2$, we have $p(x : y : z) = x^2z^2$ and hence X is hyperelliptic, a

contradiction. Thus $a_0 \neq 0$ and we normalize $a_0 = 1$. Now assume $a_7 = a_8 = 0$. Then we have
$$f = a_0x^4 + x^2(a_1y^2 + a_2yz + a_3z^2) + a_4y^4 + a_5y^3z + a_6y^2z^2.$$
Every fixed point of σ has to fulfill $x = 0$. For $z = 1$ this gives
$$f(0 : y : 1) = y^2(a_4y^2 + a_5yz + a_6z^2),$$
which has three solutions and for $z = 0$ we have
$$f(0 : 1 : 0) = a_4,$$
which is a solution only if $a_4 = 0$. Thus σ has only three fixed points in the generic case, a contradiction.

We may assume that $a_1 \neq 0$. If $a_1 = 0$, interchange y and z. Then a_1 is substituted by a_3. If furthermore $a_3 = 0$, we substitute y with $y + z$ and thus a_2yz with $a_2yz + a_2z^2$ and $a_2 \neq 0$, since $a_1 = a_2 = a_3 = 0$ is false. By substituting y with $y + \lambda z$ we get
$$x^2(a_1y^2 + a_2yz + a_3z^2) \mapsto x^2 \left(a_1y^2 + (2\lambda a_1 + a_2)yz + (a_1\lambda^2 + a_2\lambda + a_3)\right) z^2.$$
Choose λ such that $2\lambda a_1 + a_2 = 0$, then the polynomial is transformed to
$$f = x^4 + x^2(a_1y^2 + a_3z^2) + a_4y^4 + a_5y^3z + a_6y^2z^2 + a_7yz^3 + a_8z^4.$$
Since we assumed $a_1 \neq 0$, we may subsitute y by $\frac{1}{\sqrt{a_1}}y$ and hence get $a_1 = 1$.

Note that the dimension of the locus is 4. This can be seen as follows. The automorphism σ comes from an automorphism $\sigma\colon \mathbb{P}^2\mathbb{C} \to \mathbb{P}^2\mathbb{C}$ given by $\sigma(x : y : z) = (-x : y : z)$. This automorphism has the fixed point set $V(x) \cup \{ (1 : 0 : 0) \}$. To change the equation f into a nicer form we are using automorphisms in $\mathrm{Aut}(\mathbb{P}^2\mathbb{C}) = \mathrm{PGL}_3(\mathbb{C})$ which leave the images of f invariant under σ. Equivalently, we use automorphisms which leave the fixed point set of σ invariant. Hence they are of the form
$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$$
with λ, a, b, c, $d \in \mathbb{C}$. Using these automorphisms we can eliminate five parameters and four remain. $\square$

Proposition 6.3. *Let $X \in \mathcal{L}$ be a hyperelliptic Riemann surface of genus 3. Then it is the zero set of the polynomial*
$$y^2z^6 = (x^2 - z^2)(x^6 + ax^4z^2 + bx^2z^4 + cz^6),$$
where $a, b, c \in \mathbb{C}$.

Proof. A hyperelliptic curve X of genus g is defined by an equation $y^2z^6 = f(x,z)$ with $\deg(f) = 2g+2$. The hyperelliptic involution is given by $y \mapsto -y$. Since $X \in \mathcal{L}$, we may assume that $\sigma\colon x \mapsto -x$. Thus only even powers of x occur in the polynomial f.

By substituting x by some multiple of itself we may assume that

$$y^2z^6 = \prod_{i=1}^{4}(x^2 - a_iz^2).$$

By nonsingularity we may assume $a_1 \neq 0$ and hence substitute z by $\sqrt{a_1}z$ and y by $y\sqrt{a_1}^{-3}$. We get

$$y^2z^6 = (x^2 - z^2)\prod_{i=2}^{4}(x^2 - a_iz^2) = (x^2 - z^2)(x^6 + ax^4z^2 + bx^2z^4 + cz^6).$$ □

Observe that there is, as pointed out in the proof, a hidden dependence in the first equation. The space H is of dimension 4, but we can choose 5 coefficients. Furthermore, the dimension of the hyperelliptic locus is less than the dimension of the non-hyperelliptic locus. This is reasonable, since the automorphism group of hyperelliptic curves in $\mathcal{L}$ has to contain the Klein four-group $V_4 = C_2 \times C_2$. The generic, non-hyperelliptic curve has just one automorphism of order 2, hence this space is less restrictive.

Instead of holomorphic structures, we use the extra information given by translation structures. Hence we start with a torus equipped with a translation structure and regard all coverings of degree 2 with four ramification points. We pull back the translation structure along the covering, getting a translation structure on the covering surface as well as a translation covering.

Definition 6.4. Define the *Hurwitz space of translations coverings* to be the set

$$\Omega H = \{\, (p, X, \omega, E, \eta) \mid (p, X, E) \in H,\ \eta \in \Omega(E),\ p^*\eta = \omega \,\}$$

of all translation coverings coming from coverings in H. Observe that by definition p is a translation, i.e. $Dp = I$. Let us denote by

$$\Omega\mathcal{L} = \{\, (X, \omega) \mid \text{there exist } p, E, \eta \text{ such that } (p, X, \omega, E, \eta) \in \Omega H \,\}$$

the space of all translation surfaces in ΩM_3 that admit a translation covering of degree 2 of a torus ramified over four points.

From now on, we denote the set of ramification points in X by $\Sigma = \{\, P, Q, R, S \,\}$ and the set of branch points in E by $\bar{\Sigma} = \{\, \bar{P}, \bar{Q}, \bar{R}, \bar{S} \,\}$.

In the more general case in Section 3.4 we already discussed Hurwitz spaces of translation coverings. Using this language, we have $\Omega H = \Omega H_{2,3,1}$ and $\Omega\mathcal{L} = \Omega\mathcal{L}_{2,3,1}$. Let us recall our main result.

Proposition 6.5. *The space of translation surfaces $\Omega\mathcal{L}$ is a five-dimensional affine invariant submanifold of the stratum $\mathcal{H}(1,1,1,1)$.*

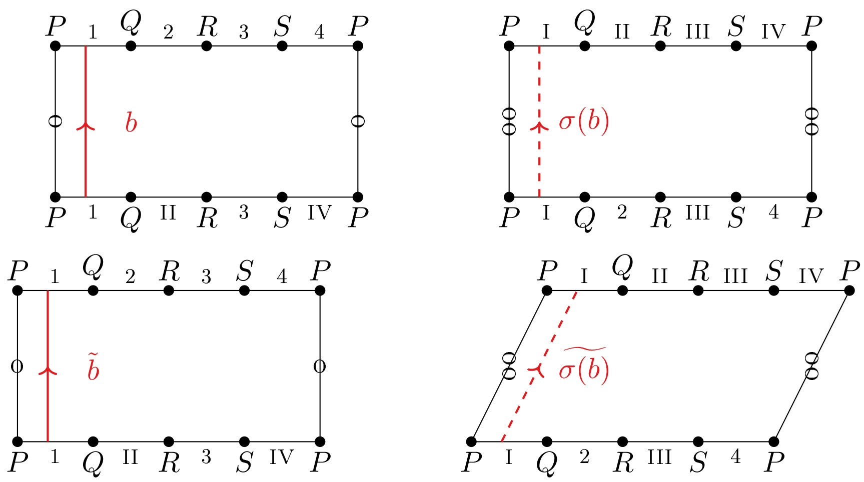


Figure 6.1: In the first row, a translation surface in the stratum $\mathcal{H}(1,1,1,1)$ with an automorphism σ interchanging both polygons is sketched. In the second row, a translation surface in the same stratum without this automorphism is sketched. An element b of the relative homology group, its image under σ and their natural extensions onto the nearby surface are sketched.

Proof. See Corollary 3.18. □

In the proof of Corollary 3.18 we have used that the forgetful maps $\mathcal{F}\colon H \to \mathcal{L}$ and $\Omega\mathcal{F}\colon \Omega H \to \Omega\mathcal{L}$ are immersions. Hence their images are not necessarily manifolds, but may contain orbifold points. In the language of translation surfaces, these orbifold points can be described quite easily.

In the following, we illustrate why $\Omega\mathcal{L}$ is part of the singular locus of the orbifold $\mathcal{H}(1,1,1,1)$. Let $(X,\omega) \in \Omega\mathcal{L}$ be a translation surface and fix a basis of relative homology. Applying the extra automorphism σ to this basis, we get another basis. Integrating along both bases yields the same period coordinates. If, on the other hand, we transform (X,ω) to a surface nearby, which does not have an extra automorphism, integrating along both bases gives us different period coordinates.

An example is depicted in Figure 6.1. We see two translation surfaces in the principal stratum $\mathcal{H}(1,1,1,1)$. The first one has an automorphism σ interchanging both polygons. The second translation surface is in a small neighborhood of the first one, but does not have this automorphism. We choose a path b and its image $\sigma(b)$. Due to Proposition 2.6 we can extend those paths naturally to paths $\tilde{b}$ and $\widetilde{\sigma(b)}$ on the second, nearby surface. Whether we integrate b or $\sigma(b)$ does make no difference. But on the second surface, integrating along $\tilde{b}$ or $\widetilde{\sigma(b)}$ makes a difference. Hence locally the space $\Omega\mathcal{L}$ looks like $\mathbb{C}^5/C_2$ and thus, as a subset of the whole stratum, consists of orbifold points. But regarding $\Omega\mathcal{L}$ on its own, most points are manifold points. Later, we show that the singular locus is of codimension 1 in $\Omega\mathcal{L}$.

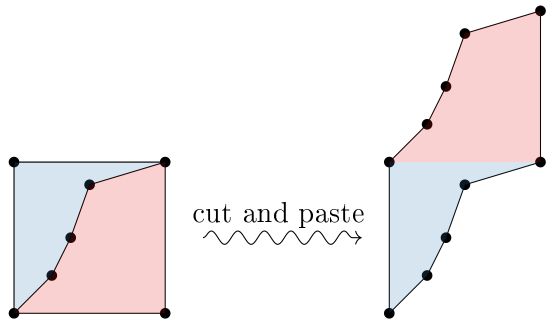


Figure 6.2: A square torus is cut into a polygon.

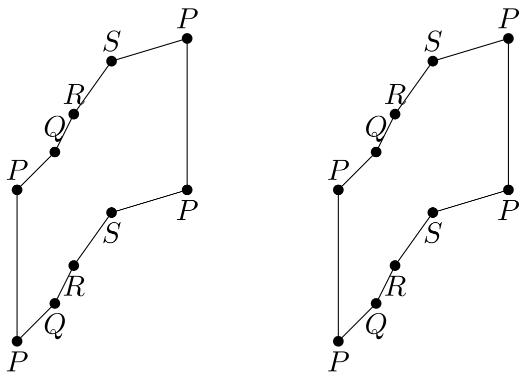


Figure 6.3: A polygon consisting of two copies of the square torus whose vertices are labeled by P, Q, R and S.

6.1.1 Taking advantage of the translation structure

In the remainder of this work, a description by polygons of translation coverings in ΩH is crucial. Firstly, we sketch the rough idea of this polygon construction and then we present a rigorous proof thereof.

We fix a torus and cut and reglue this torus as sketched in Figure 6.2. Then we take two copies of this torus as in Figure 6.3 and label the vertices by P, Q, R and S. We have to glue those two polygons such that the angle at every labeled point is 4π. We can choose to either glue the leftmost vertical edge to the right vertical edge of the same or the other polygon. Furthermore, we can glue the top edge connecting P and Q to the bottom edge connecting P and Q in the left or the right polygon. If we would glue the edge from Q to R to the edge on the same polygon as we glued the edge connecting P and Q, the angle around Q would be 2π. Hence every possible gluing is determined by the first two choices. All in all, this gives us four possible gluings, which are sketched in Figure 6.4.

We give a more thorough treatment of the above discussion here. We describe the space of polygons needed to construct every translation covering in ΩH. Let P be a polygon. We can describe P by its vertices $v_1, \ldots, v_n$ in clockwise orientation. On the

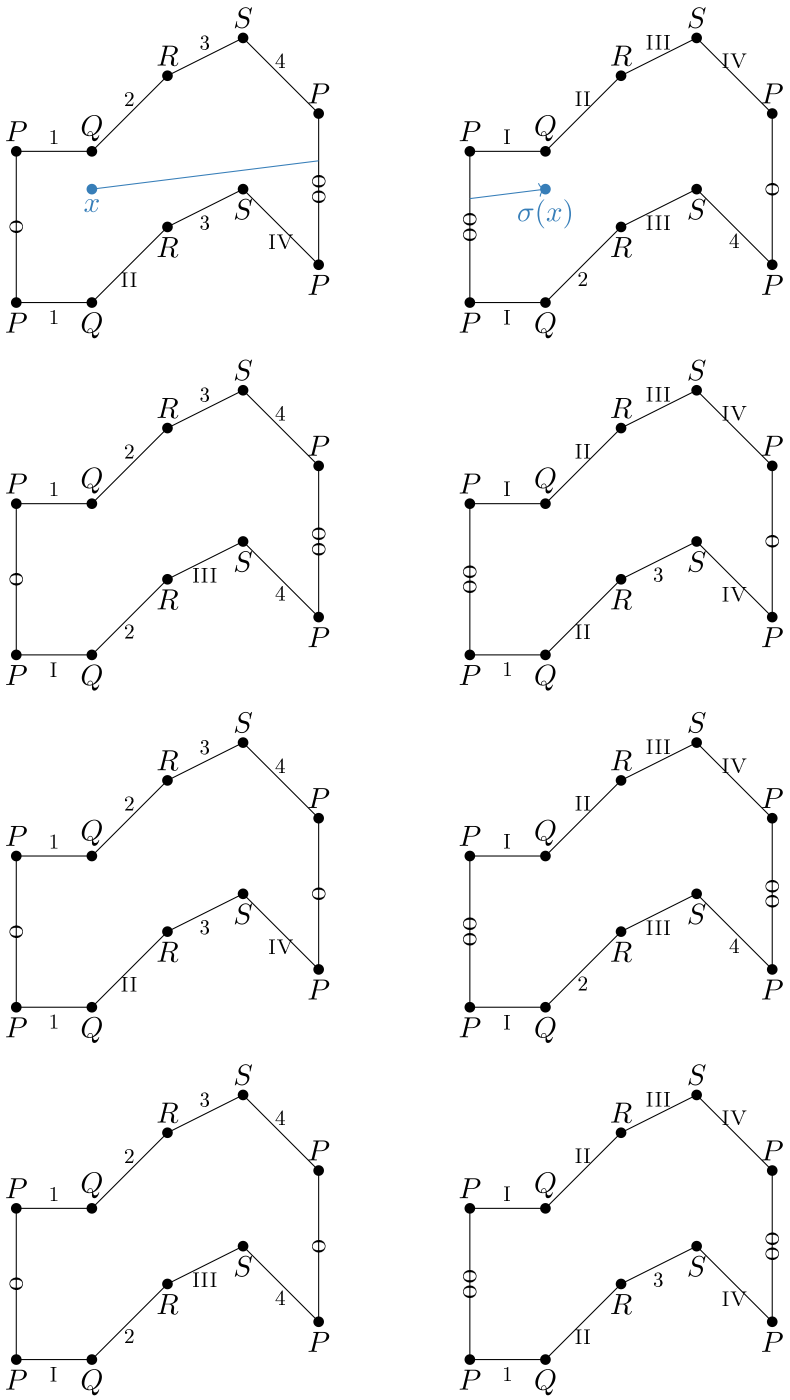


Figure 6.4: In each row, a translation surface is depicted. All of them can be described by one polygon. Edges with same markings are glued. The deck transformation σ interchanges the two copies of the polygon and is sketched on the first translation surface.

one hand, two consecutive vertices give us an edge $e_i = v_{i+1} - v_i$. On the other hand, with the convention $v_1 = 0$, we regain every vertex v_i as the sum of preceding edges $v_i = \sum_{j=1}^{i-1} e_j$. Therefore we can describe P equivalently either by its edges or by its vertices. If a polygon is given by its edges $e_1, \dots, e_n$, we will denote those as a tuple in $\mathbb{C}^n$. For a tuple $(e_1, \dots, e_n) \in \mathbb{C}^n$ we get a path by drawing the vector e_1 starting in 0, then drawing the vector e_2 starting in e_1 and so forth. However, most tuples $(e_1, \dots, e_n) \in \mathbb{C}^n$ do not produce a polygon, because the path they form is not closed. The space of (not necessarily convex) polygons with n vertices is given by

$$\mathcal{P}_n = \left\{ (e_1, \dots, e_n) \in \mathbb{C}^n \;\middle|\; \sum_{i=1}^{n} e_i = 0 \right\},$$

consisting of all tuples in $\mathbb{C}^n$ such that the sum of their entries is zero. As this set is the zero set of a polynomial, it is a closed submanifold of $\mathbb{C}^n$. Furthermore, we want to glue our polygons to translation surfaces. So we need for every edge one parallel edge of the same length to which it can be glued. Inspired by our pictures, for even n we define

$$\mathcal{P}_n^s = \left\{ (e_1, \dots, e_n) \in \mathcal{P}_n \;\middle|\; e_1 = -e_{\frac{n}{2}+1},\; e_i = -e_{n-i+2} \text{ for } 2 \leq i \leq \frac{n}{2} \right\}.$$

This space consists of all polygons whose upper half is a translate of the lower half. Those polygons will be called *symmetric*. The defining conditions are closed conditions, so this is a closed subspace of the space of polygons $\mathcal{P}_n$. The dimension of $\mathcal{P}_n^s$ is $\frac{n}{2}$.

The polygons in $\mathcal{P}_n$ may have self-intersections. We do not want to forbid these per se, but we just allow very simple self-intersections. To see why this is necessary, imagine that the branch point $\bar{Q}$ is on top of the point $\bar{P}$. In the construction described above, the path from $\bar{P}$ to $\bar{Q}$ will be part of the vertical path from $\bar{P}$ to $\bar{P}$. This may be thought of as a degeneration of the polygon, see e.g. Figure 6.5. It is also possible, that the points $\bar{P}$, $\bar{Q}$ and $\bar{R}$ are on top of each other. These types of self-intersections are precisely the ones we want to allow. Let us denote by $v_i + e_i$ the image of the path defined by $t \mapsto v_i + te_i$ for $t \in (0,1)$. We define the set

$$\begin{aligned} \mathcal{P}_n^o = \{ \; (e_1, \dots, e_n) \in \mathcal{P}_n \mid \exists j_0 \leq \frac{n}{2} : & \; v_i + e_i \subseteq v_1 + e_1 \; \forall i \leq j_0 \text{ and} \\ & (v_i + e_i) \cap (v_j + e_j) = \emptyset \; \forall 2 \leq i < j \leq n \text{ or } i = 1, j > j_0, \\ & \overline{v_i + e_i} \cap \overline{v_j + e_j} = \emptyset \; \forall |i - j| \neq 1 \text{ and } i < j > j_0 \}, \end{aligned}$$

consisting of all polygons in $\mathcal{P}_n$ for which the first $\frac{n}{2}$ edges may be contained in the first edge but with no other self-intersections. Those polygons will be called *semi-simple*.

Definition 6.6. We denote by $\mathcal{P}_n^{s,o} = \mathcal{P}_n^s \cap \mathcal{P}_n^o$ the space of all symmetric, semi-simple polygons.

For simplicity of notation, we write $\mathcal{P} = \mathcal{P}_{10}^{s,o}$.

In the following we want to show that every translation covering in ΩH can be represented by a polygon in $\mathcal{P}$ and a choice of four possible gluings.

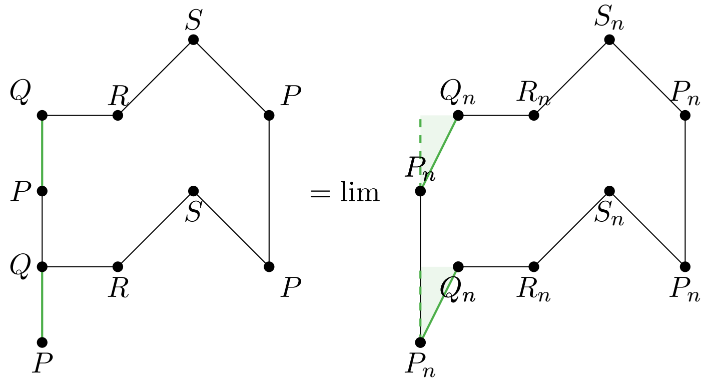


Figure 6.5: The degenerate polygon on the left can be thought of as the limit of the polygon on the right where the green edge converges along the light green face to the dashed edge.

Lemma 6.7. *Every four-punctured torus with translation structure can be represented by a polygon in $\mathcal{P}$ whose vertices are the punctures.*

Proof. Without loss of generality we may assume the torus to be the standard torus $\mathbb{C}/\mathbb{Z} + \mathrm{i}\mathbb{Z}$. Let $\mathbb{C}$ be the universal covering of the torus chosen such that the origin is mapped to one marked point $\bar{P}$. Assume that no two marked points have the same real part. In the universal covering $\mathbb{C}$ we draw a straight path going up and/or right from the origin to one preimage of a marked point that has the smallest positive real part. Again, from this point we draw a straight path going up and/or right to the point with the next largest real part. We go on in this manner until we reach a point which is mapped to $\bar{P}$ again and then draw a straight path upwards until we reach the next point mapped to $\bar{P}$. Then, we draw from the origin upwards until we reach another point mapped to $\bar{P}$ and repeat the procedure explained above. This gives us a closed polygon. The obvious gluing instruction gives us our torus.

If two preimages $\tilde{P}$ and $\tilde{Q}$ of marked points have the same real part, we have to draw a straight line going up. Without loss of generality we may assume $\tilde{P} = 0$, which gives us a degenerate polygon as seen in Figure 6.5. The edge going from $\tilde{P}$ to another preimage $\tilde{P}_1$ of the same point has to be glued to the right side of the polygon. At the same time the edge going up from $\tilde{P}$ to $\tilde{Q}$ lies on this edge, so it has to be glued to the right side as well. Hence we cut off the polygon's little arm and glue it to the right side.

If all four preimages of the marked points have the same real part, our construction fails. Instead we add 1 to one of these preimages. It still gets mapped to the same point and our construction works as before. $\square$

Proposition 6.8. *We have a surjective map $\mathcal{P} \times \{1, 2, 3, 4\} \to \Omega H$.*

Proof. Let $N \in \mathcal{P}$ be a polygon and take two copies N_1 and N_2 of N. We label the edges of N_i clockwise by a_i, b_i, c_i, d_i, e_i, $-a_i$, $-e_i$, $-d_i$, $-c_i$ and $-b_i$ for $i = 1, 2$. Let us think of the edges a_i as the "vertical" edges connecting P to P. Assume that the polygon N is non-degenerate. We may glue the edge a_1 to the edge $-a_1$ or $-a_2$. Then we are free to glue b_1 to either $-b_1$ or $-b_2$. After that, there is no choice anymore. If we glued b_1 and c_1 both to $-b_1$ and $-c_1$, the angle around the vertex in between would be 2π instead of 4π. Hence the choice of how to glue a_1 and b_1 determines how to glue the other edges. We define the gluing map T_i by

$$T_i(a_1) = \begin{cases} -a_1, & i = 3,4 \\ -a_2, & i = 1,2 \end{cases} \qquad \text{and}$$
$$T_i(b_1) = \begin{cases} -b_1, & i = 1,3 \\ -b_2, & i = 2,4 \end{cases}.$$

For a non-degenerate polygon N and $i \in \{1,2,3,4\}$ this gives us the translation surface $(N_1 \times N_2)/T_i$.

We want the map $\mathcal{P} \times \{1,2,3,4\} \to \Omega H$ to be continuous and this determines the map on degenerate polygons. To illustrate this, assume that the point Q lies above P. See Figures 6.5 and 6.6. Take a series of non-degenerate polygons (N_n) with points (Q_n) converging to the polygon N and the point Q. For each N_n the gluing defined above works. As (Q_n) converges to Q, the edges $(-b_1)_n$ converge to a subset of the edge a_1. Since $-b_1 \subseteq a_1$, the edge $-b_1$ has to be glued to some subedge of $T_i(a_1)$. Because of coninuity we have

$$T_i(a_1) \supseteq T_i(-b_1) = \lim_{n\to\infty} T_i\left((-b_1)_n\right).$$

In other words, the edge $-b_1$ has to be glued to a subedge of the edge to which a_1 gets glued. At the same time it has to be glued to the same edge as every $(-b_1)_n$. So we add a new point Q on the edge $T_i(a_1)$ and glue the edge $-b_1$ to the subedge of $T_i(a_1)$ between P and Q. More graphically, we cut off the little arm forming the edge from P to Q and translate it onto the edge $T_i(a_1)$ such that the point P fits in. We have to add the point Q on this edge. This new edge is glued to the edge $\lim_{n\to\infty} T_i\left((-b_1)_n\right)$.

The above construction gives us a translation surface. In Figure 6.4 every row corresponds to one translation surface covering a fixed torus. The four possible gluings correspond, from top to bottom, to the gluing maps T_1 to T_4. We see that the angle at each vertex is 4π. So those translation surfaces are in the stratum $\mathcal{H}(1,1,1,1)$. Furthermore, we define a translation σ interchanging the two polygons. The quotient by the group generated by σ is one polygon with Roman and Arabic numerals as well as o and oo identified. Clearly, the quotient is a torus. Since the translation σ has four fixed points and is of order 2, this shows that the map $\mathcal{P} \to \Omega H$ is well defined.

It is surjective since for each torus we can give four different coverings with the given ramification data: By Proposition 3.7 there are as many coverings of a given surface with fixed degree and branch points as there are monodromy maps with transitive image. In our case the degree is $d = 2$ and hence being transitive is the same as being surjective. The fundamental group of the four-times punctures torus is generated by four loops around the branch points, which are mapped to $(1\,2)$, and two loops generating the fundamental group of the torus, whose images we can choose freely. So there exist four different monodromy maps and hence four different coverings. □

The fibers of the map $\mathcal{P} \times \{ 1, 2, 3, 4 \} \to \Omega H$ are at least countable: For a covering $p\colon (X, \omega) \to (E, \eta)$ we construct a polygon by looking at the universal cover of the torus E. We mark the preimages of the branch points $\bar{P}$, $\bar{Q}$, $\bar{R}$ and $\bar{S}$. Then, starting in one preimage of $\bar{P}$ we draw an edge to one preimage of the next point, say $\bar{Q}$. As depicted in Figure 6.7, there are countably many ways to draw this edge. None of these edges interferes with an edge from a preimage of $\bar{Q}$ to one of $\bar{R}$. Hence we can draw countably many polygons all describing the same torus.

We now show explicitly that the space ΩH is connected. Every polygon in $\mathcal{P}$ can be continuously transformed into a rectangle by just flattening out all the angles. To show that the Hurwitz space of translation coverings ΩH is connected, we only need to show that we can transform translation coverings given by the same polygon, but with different gluings, into each other.

Proposition 6.9. *The Hurwitz space of translation coverings ΩH is connected.*

Proof. By the above argument, we can transform every translation covering in one of the forms seen in Figure 6.8. The first one is glued by T_2 and the second one by T_1. Cutting and gluing in the indicated way gives us an isomorphic translation covering. So we have a path from any translation covering glued by T_2 to any translation covering glued by T_1.

The second row gives an isomorphism from a translation covering glued by T_3 to one glued by T_4 and the third row gives an isomorphism from a translation covering glued by T_1 to one glued by T_3.

Combining those, we find a path from any translation covering in ΩH to any other translation covering in ΩH, thus ΩH is path connected and hence connected.

6.2 A subspace of codimension one

In this section we discuss in more detail the Hurwitz space first mentioned by Herrlich and Schmithüsen [HS07a; HS08]. They described it as the non-hyperelliptic component of the Hurwitz space of coverings of degree 2 of the torus whose branch points are symmetric or, equivalently, by those coverings which allow a lift of the multiplication by -1. Among other things, they showed that this space is a 3-dimensional manifold and computed equations for each surface therein. The most remarkable property of this space is that its projection into the moduli space contains a dense subset consisting of Teichmüller curves, which all intersect the Teichmüller curve of the Wollmilchsau.

We start in the same vein as Herrlich and Schmithüsen and define a subspace of H by restricting the branch points to symmetric branch points. This is equivalent to the fact that the multiplication by -1 can be lifted. Thereafter, we use translation coverings instead of holomorphic coverings and thus obtain a 4-dimensional affine invariant

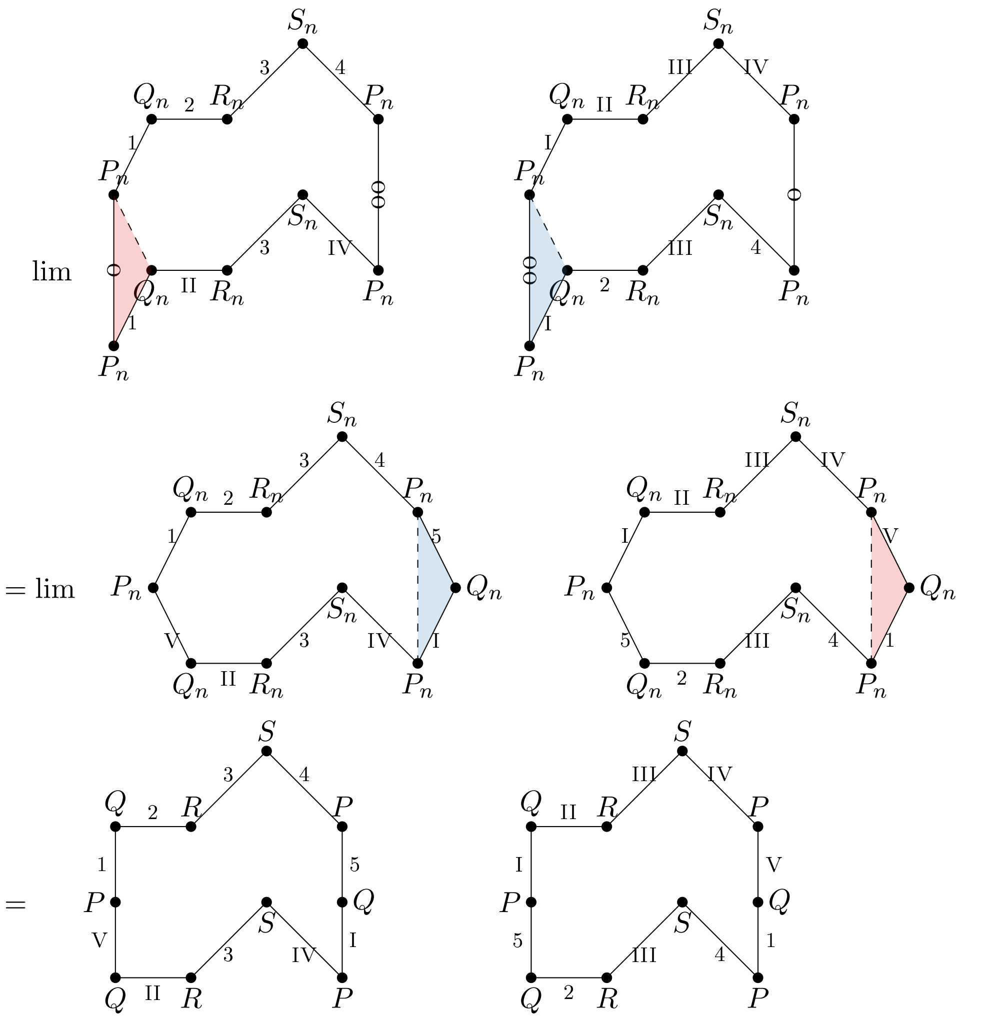


Figure 6.6: The bottom translation surface can be thought of as the limit of the upper one, where the limit is as in Figure 6.5. The indicated cutting and gluing procedure, via a detour along the middle picture, shows how the gluing on the degenerate translation surface is achieved.

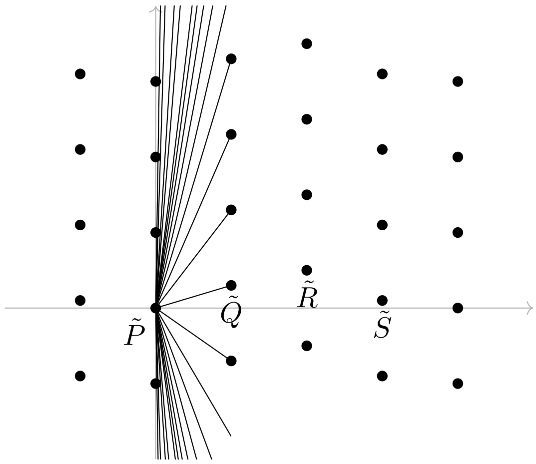


Figure 6.7: This is the universal covering of a torus with preimages of the branch points marked. Starting from one preimage of $\bar{P}$, say $\tilde{P}$, there are countably many ways to draw an edge to a preimage of $\bar{Q}$.

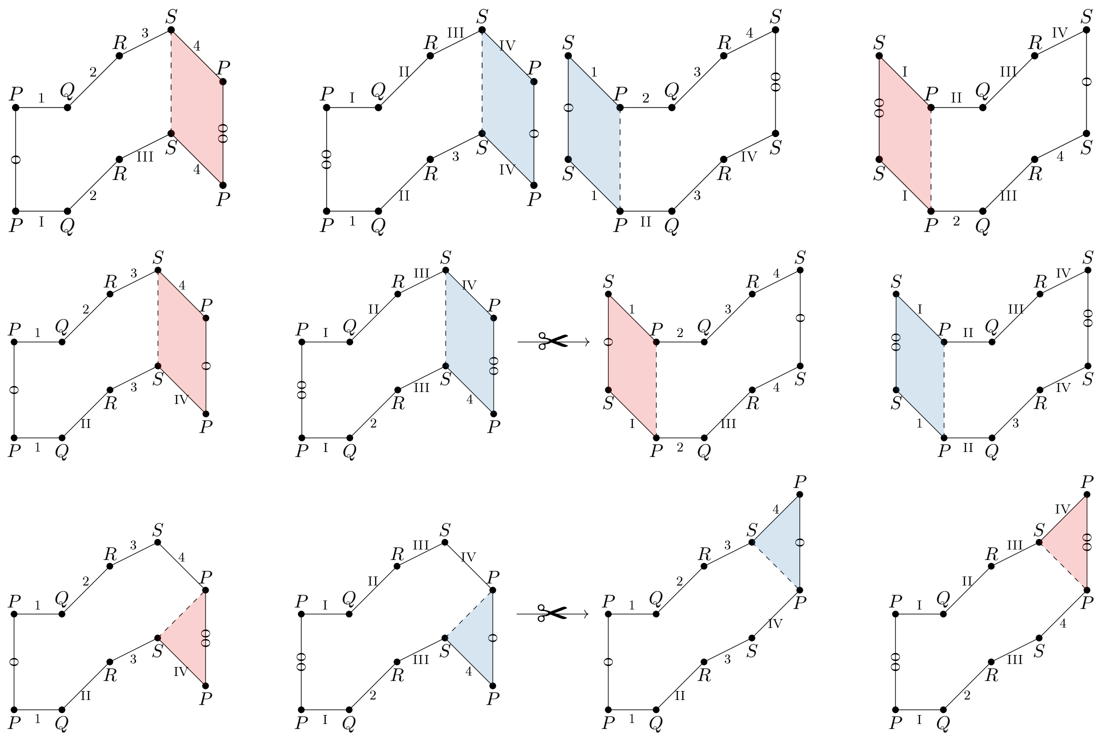


Figure 6.8: Three isomorphisms of translation surfaces, which interchange every possible gluing.

submanifold of $\Omega\mathcal{L}$. This point of view enables us to reprove the existence of a lift of the multiplication by -1 and to describe the connected components of this space.

Definition 6.10. For a covering $p\colon X \to E$ in H let $\bar{\Sigma} = \{ \bar{P}, \bar{Q}, \bar{R}, \bar{S} \}$ denote the set of branch points in E and let $\bar{O} \in E$ be its origin. We define the subspace

$$H_1 = \{ (p, X, E) \in H \mid \bar{P} + \bar{S} = \bar{Q} + \bar{R} = \bar{O} \}$$

of H consisting of all coverings with two pairs of symmetric branch points. We define its image in the moduli space M_3 to be

$$\mathcal{L}_1 = \mathcal{F}(H_1),$$

consisting of those Riemann surfaces that admit a covering in H_1.

From now on we write $\bar{S} = -\bar{P}$ and $\bar{R} = -\bar{Q}$. We assume that the branch points are not 2-torsion points. If one point, say $\bar{P}$, was a 2-torsion point, we would have $\bar{P} = -\bar{P} = \bar{S}$. Thus the covering would have at most three ramification points in contradiction to our definition of H.

Following Herrlich and Schmithüsen [HS07a], we describe this space by the possibility to lift the multiplication by -1 to the covering.

Proposition 6.11. *Let $[-1]\colon E \to E$ denote the multiplication by -1 on an elliptic curve E. Then the Hurwitz space*

$$H_1 = \{ (p, X, E) \in H \mid \text{ there exists a lift } \tau\colon X \to X \text{ of } [-1]\colon E \to E \}$$

can be identified with the space of all coverings, to which the multiplication by -1 can be lifted.

Here and subsequently, we denote by τ the lift of $[-1]$.

Proof. Let $p\colon X \to E$ be a covering in the right hand set and let τ be a lift of $[-1]$. Then $[-1](\bar{\Sigma}) \subseteq \bar{\Sigma}$. Note that $\bar{\Sigma} = \{ \bar{P}, \bar{Q}, -\bar{P}, -\bar{Q} \}$ has four elements. If $[-1]$ had a fixed point in $\bar{\Sigma}$, say $\bar{P}$, then $\bar{P} = [-1](\bar{P}) = -\bar{P}$. Hence $\bar{\Sigma}$ has only three elements, a contradiction. Thus $[-1]$ has no fixed points in $\bar{\Sigma}$, which shows that $\bar{P}$, $[-1](\bar{P}) = -\bar{P}$, $\bar{Q}$ and $[-1](\bar{Q}) = -\bar{Q}$ are four different branch points. The branch points fulfill the relation $\bar{P} + (-\bar{P}) = \bar{Q} + (-\bar{Q}) = \bar{O}$. Thus $p \in H_1$.

Now let $p\colon X \to E$ be a covering in H_1 having branch points with the relation $\bar{P} + \bar{S} = \bar{Q} + \bar{R} = \bar{O}$. We follow the proof by Herrlich and Schmithüsen [HS07a]. Let $X^* = X \backslash \Sigma$ and $E^* = E \backslash \bar{\Sigma}$ be the corresponding punctured surfaces. Observe that $[-1]$ also is an automorphism of E^*. We want to show that the automorphism $[-1]$ lifts to an automorphism τ of X (or X^*). Thus we want to show that there exists an automorphism $\tau\colon X^* \to X^*$ such that the diagram

$$\begin{array}{ccc} X^* & \xrightarrow{\tau} & X^* \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ E^* & \xrightarrow{[-1]} & E^* \end{array}$$

is commutative. By the lifting property stated after Proposition 3.2, we have to show

$$[-1]_* p_* \pi_1(X^*) \subseteq p_* \pi_1(X^*).$$

To simplify notation, we use loops instead of their equivalence classes. The group $p_*\pi_1(X^*)$ consists of those loops in $\pi_1(E^*)$ that are lifted to loops in $\pi_1(X^*)$, compare Section 3.1. The kernel of the monodromy map $\mu\colon \pi_1(E^*) \to S_2$ consists of those loops that are lifted to loops in $\pi_1(X^*)$. So $p_*\pi_1(X^*) = \ker(\mu)$. Thus we have to check whether multiplying by -1 changes the monodromy type of a given loop in E^*. We choose generators of the fundamental group of E^* with base point $\bar{O}$. Let α and β be two loops generating the fundamental group of the (non-punctured) torus such that they are invariant under multiplication by -1. Then pick counterclockwise loops $\ell_{\bar{P}}$, $\ell_{\bar{Q}}$, $\ell_{-\bar{Q}}$ and $\ell_{-\bar{P}}$ around the branch points $\bar{P}$, $\bar{Q}$, $-\bar{Q}$ and $-\bar{P}$. These six loops generate the fundamental group $\pi_1(E^*)$. Since $[-1]$ maps $\bar{P}$ to $-\bar{P}$, $\bar{Q}$ to $-\bar{Q}$ and vice versa, we have

$$[-1]_*(\ell_{\bar{P}}) = \ell_{-\bar{P}} \quad \text{and} \quad [-1]_*(\ell_{\bar{Q}}) = \ell_{-\bar{Q}}.$$

Thus the monodromy of the loops around the branch points is not changed by $[-1]$. Furthermore, since α and β are invariant under $[-1]$, we have

$$[-1]_*(\alpha) = -\alpha \quad \text{and} \quad [-1]_*(\beta) = -\beta. \qquad \square$$

Since μ maps into S_2 and therein every element is self-inverse, $[-1]_* \ker(\mu) \subseteq \ker(\mu)$ and there exists a lift $\tau^*\colon X^* \to X^*$ that can be extended to an automorphism $\tau \in \mathrm{Aut}(X)$.

Corollary 6.12. *The automorphism group of a covering in H_1 or a surface in $\mathcal{L}_1$ contains the Klein four-group V_4.*

Proof. Let $p\colon X \to E$ be a covering in H_1. We have to show that the generator σ of the deck transformation group and the rotation τ commute and that τ is of order 2.

Let $X^* = X \setminus \Sigma$. We have $p \circ \sigma = p$ and by definition $p \circ \tau = [-1] \circ p$, so

$$p \circ \tau(x) = [-1] \circ p(x) = [-1] \circ p \circ \sigma(x) = p \circ \tau \circ \sigma(x) \quad \text{for} \quad x \in X^*.$$

Thus for $x \in X$, $\tau \circ \sigma(x)$ and $\tau(x)$ are in the same fiber, i.e. they differ by the deck transformation σ. Hence $\tau \circ \sigma = \sigma \circ \tau$.

The fixed points of τ are in the fibers of the fixed points of $[-1]$. Those are 2-torsion points and thus different from the points in $\bar{\Sigma}$. Therefore the fixed points of τ are not in the set of ramification points Σ. For a fixed point $F \in X^*$ of τ we have $\tau^2(F) = F$. Hence τ^2 has a fixed point and τ^2 is a lift of id_E, which implies $\tau^2 = \mathrm{id}_X$. $\square$

According to the Riemann-Hurwitz formula in Proposition 3.5, the involution τ has either no, four or eight fixed points. For a fixed point $\bar{F}$ of $[-1]$ let $\{ F, \sigma(F) \} = p^{-1}(\bar{F})$ be its fiber. We have

$$p \circ \tau(F) = [-1] \circ p(F) = [-1](\bar{F}) = \bar{F}$$

and hence τ acts on the fiber of the fixed point $\bar{F}$. If τ had no fixed point, then it would act non-trivially on the fiber of $\bar{F}$. Hence

$$\tau(F) = \sigma(F) \quad \text{and} \quad \tau(\sigma(F)) = F$$

and $\sigma\tau$ has a fixed point. By substituting τ with $\sigma\tau$ we can always assume that τ has a fixed point.

Corollary 6.13. *The equation of a generic covering in H_1 or a generic surface in $\mathcal{L}_1$ depends on the number of fixed points of τ. If τ has four fixed points, the generic surface is not hyperelliptic and is given by*

$$x^4 + y^4 + ax^2y^2 + bx^2 + cy^2 + 1 = 0, \quad a, b, c \in \mathbb{C}.$$

If τ has eight fixed points, the surface is hyperelliptic and is given by

$$y^2 = (x^2 - 1)(x^6 + ax^4 + bx^2 + c), \quad a, b, c \in \mathbb{C}.$$

In both cases, the dimension of the locus of those curves is 3.

Proof. The generic automorphism group of a surface in $\mathcal{L}_1$ is the Klein four-group V_4. The only possible hyperelliptic involutions are τ and $\sigma\tau$. By our discussion above, $\sigma\tau$ cannot have eight fixed points. Hence τ has eight fixed points if and only if the surface is hyperelliptic. Then both equations and the dimensions are given in Tables 3.1 and 3.2.

The first equation and its dimension also were computed by Herrlich and Schmithüsen [HS07a]. We determined the second equation in Proposition 6.3, too. □

As before, we now equip coverings in H_1 with an extra translation structure and study the Hurwitz space of translation coverings. We show that it gives us an affine invariant submanifold of dimension 4. Furthermore, we explicitly describe its connected components using a description of this space by polygons.

Definition 6.14. The Hurwitz space of translation coverings

$$\Omega H_1 = \left\{ (p, X, \omega, E, \eta) \in \Omega H \;\middle|\; \bar{P} + \bar{S} = \bar{Q} + \bar{R} = \bar{O} \right\}$$

contains all translation coverings in ΩH with two pairs of symmetric branch points. Equivalently, it contains all translation coverings in ΩH to which there exists a lift τ of the rotation $[-1]$. Its image in the moduli space of translation surfaces

$$\Omega\mathcal{L}_1 = \Omega\mathcal{F}(\Omega H_1)$$

consists of all translation surfaces admitting a translation covering in ΩH_1.

Observe that the derivative $D[-1]$ is $-I$ and thus $D\tau = -I$, i.e. τ is a rotation.

We want to show that ΩH_1 is an affine invariant submanifold of dimension 4. If we show that $H_1 \subseteq H$ is a submanifold, we can restrict the immersion $\Omega\mathcal{F}\colon \Omega H \to \Omega\mathcal{L}$ to ΩH_1 and it only remains to compute linear equations in period coordinates.

Lemma 6.15. *The subspace $H_1 \subseteq H$ is a submanifold.*

Proof. Our aim is to construct for every covering a neighborhood U in H and a chart $\varphi\colon U \to \mathbb{C}^5$ such that $\varphi(U \cap H_1)$ is the intersection of $\mathbb{C}^5$ with some linear subspace.

For this purpose, let us recall the definition of the topology on $H(2,4,E)$, the Hurwitz space of coverings of E of degree 2 with four ramification points, given in Section 3.3. Let e_1, e_2, e_3 and $e_4 \in E$ and let U_i be a simply connected neighborhood of $e_i \in E$ for $i = 1,2,3,4$, such that $U_i \cap U_j = \emptyset$ for $i \neq j$. The sets

$$N(U_1,\dots,U_4) = \{\, (e_1,e_2,e_3,e_4) \in \Sigma^4 E \setminus \Delta \mid e_i \in U_i \,\}$$

define a basis of the topology of $\Sigma^4 E \setminus \Delta$, the fourfold symmetric product of E with its diagonal removed. Furthermore, the map

$$\Psi_4\colon H(2,4,E) \to \Sigma^4 E \setminus \Delta$$

is a covering, which maps every covering of E to the set of its branch points.

We fix a covering $p\colon X \to E$ in $H(2,4,E)$. Since Ψ_4 is a covering map, we may choose a small enough neighborhood U of p such that $\Psi_4(U)$ is biholomorphic to U. On this open set we define a chart via

$$\varphi\colon U \to \Psi_4(U) \to \mathbb{C}^4,$$

where the last map is defined as follows: Denote by $\bar{e}_i$ a representative in $\mathbb{C}$ of the branch point e_i, $i = 1,2,3,4$, then define

$$(e_1,e_2,e_3,e_4) \mapsto (\bar{e}_1,\bar{e}_2,\bar{e}_1+\bar{e}_4,\bar{e}_2+\bar{e}_3).$$

Let us denote by $H_1(2,4,E)$ the space of coverings E of degree 2 with four ramification points having symmetric branch points as in H_1. For a neighborhood U, chosen as above, we have

$$\Psi_4\colon U \cap H_1(2,4,E) \to \{\, (e_1,e_2,e_3,e_4) \in N(U_1,\dots,U_4) \mid e_1+e_4,\ e_2+e_3 \in \Gamma \,\}$$

for $E = \mathbb{C}/\Gamma$. Hence for a covering $p \in H_1(2,4,E)$ we can choose the map φ such that

$$\varphi(p) = (\bar{e}_1,\bar{e}_2,0,0).$$

This implies that $\varphi(U \cap H_1(2,4,E))$ is the intersection of $\mathbb{C}^4$ with a two-dimensional linear subspace and thus $H_1(2,4,E)$ is a submanifold of $H(2,4,E)$.

So far, so good. But actually we are interested in the space H_1, not $H_1(2,4,E)$. As before, choose a neighborhood $U \subseteq H(2,4,E)$ of a covering $p\colon X \to E$. For every nearby torus E' we find a matrix A such that $A \cdot E = E'$. Hence for every covering $p_0 \in U$ with branch point set $\bar{\Sigma}$ we get a new covering of $A \cdot E$ with branch point set $A \cdot \bar{\Sigma}$. These coverings form a neighborhood $\tilde{U}$ of $p\colon X \to E$ in H. The chart $\varphi\colon \tilde{U} \to \mathbb{C}^5$ is given by mapping a covering $p\colon X \to E = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ with branch points e_1, e_2, e_3 and e_4 to

$$\varphi(p) = (\tau,\bar{e}_1,\bar{e}_2,\bar{e}_1+\bar{e}_4,\bar{e}_2+\bar{e}_3).$$

Because the relations of the branch points are linear equations, they are invariant under multiplication by a matrix. Hence we can choose φ such that

$$\varphi(p) = (\tau, \bar{e}_1, \bar{e}_2, 0, 0)$$

for a covering $(p, X, E) \in H_1$. Hence $\varphi(\tilde{U} \cap H_1)$ is the intersection of $\mathbb{C}^5$ with a three-dimensional linear subspace and thus H_1 is a submanifold of H. $\square$

Recall that we have computed generators and explicit relations of the relative homology group $H_1(E, \bar{\Sigma}, \mathbb{Z})$ of the torus E relative to the set of branch points $\bar{\Sigma}$ in Example 2.13. In particular, recall the following notations: The paths $\bar{a}$ and $\bar{b}$ form a basis of the absolute homology group $H_1(E, \mathbb{Z})$ of the torus and for $x, y \in E$ the paths c_{xy} are geodesic paths going from x to y.

Proposition 6.16. *The space $\Omega\mathcal{L}_1$ is an affine invariant submanifold of dimension* 4.

Proof. By Lemma 6.15, H_1 is a submanifold of H. Hence ΩH_1 is a submanifold of ΩH. Moreover, the restriction of the map $\Omega\mathcal{F}\colon \Omega H_1 \to \Omega\mathcal{L}_1$ is an immersion into the principal stratum $\mathcal{H}(1,1,1,1)$. Thus we only have to show that we can describe $\Omega\mathcal{L}_1$ locally by linear equations in period coordinates. In a small enough neighborhood of a translation covering, the immersion $\Omega\mathcal{F}$ is injective. Thus, when working with a translation surface (X, ω) we can always assume it is equipped with a unique covering denoted by $p\colon (X, \omega) \to (E, \eta)$.

We choose linearly independent relative homology classes in $H_1(X, \Sigma, \mathbb{Z})$, which project via p to

$$\bar{a},\ \bar{b},\ c_{\bar{P}\bar{Q}},\ c_{\bar{Q}-\bar{P}},\ c_{-\bar{P}-\bar{Q}},$$

described above and in Example 2.13. We name them

$$a,\ b,\ c_{PQ},\ c_{Q-P},\ c_{-P-Q},$$

respectively. Observe that none of these elements is in the kernel of p_* since they project to non-trivial paths on E. Thus we can extend these five elements to a basis of nine-dimensional relative homology by adding elements in the four-dimensional kernel $\ker(p_*)$. We denote the important integrals by

$$\int\limits_a \omega = A, \quad \int\limits_b \omega = B, \quad \int\limits_{c_{PQ}} \omega = C_1, \quad \int\limits_{c_{Q-P}} \omega = C_2 \quad \text{and} \quad \int\limits_{c_{-P-Q}} \omega = C_3.$$

Because the rotation τ is a lift of $[-1]$ and p a translation covering, we have the identity $\tau^*\omega = \tau^* p^* \eta = p^*[-1]^*\eta = -p^*\eta = -\omega$. For each $c \in H_1(X, \Sigma, \mathbb{Z})$,

$$-\int\limits_c \omega = \int\limits_c \tau^*\omega = \int\limits_c p^*[-1]^*\eta = \int\limits_{[-1]_* p_* c} \eta$$

follows. Plugging in the basis of the relative homology group and using the relations from 2.13, we obtain the following equations:

$$\begin{aligned}
C_1 &= \int\limits_{c_{PQ}} \omega = -\int\limits_{c_{PQ}} \tau^*\omega = -\int\limits_{[-1]_* c_{\bar{P}\bar{Q}}} \eta = -\int\limits_{c_{-\bar{P}-\bar{Q}}} \eta = -\int\limits_{c_{-P-Q}} \omega = -C_3,\\
C_2 &= \int\limits_{c_{Q-P}} \omega = -\int\limits_{c_{-QP}} \omega = \int\limits_{c_{PQ}} \omega + \int\limits_{c_{Q-P}} \omega + \int\limits_{c_{-P-Q}} \omega = C_1 + C_2 + C_3,\\
C_3 &= \int\limits_{c_{-P-Q}} \omega = -\int\limits_{c_{PQ}} \omega = -C_1,\\
A &= \int\limits_{a} \omega = -\int\limits_{a} \tau^*\omega = -\int\limits_{-a} \omega = A \quad \text{and}\\
B &= \int\limits_{b} \omega = -\int\limits_{b} \tau^*\omega = -\int\limits_{-b} \omega = B. \qquad \square
\end{aligned}$$

Combining all these equations, the only restriction we get is $C_1 + C_3 = 0$. This is one more restriction than in $\Omega\mathcal{L}$, thus in period coordinates $\Omega\mathcal{L}_1$ is a hyperplane in $\mathbb{C}^5$ and hence its dimension is less or equal to $5 - 1 = 4$.

On the other hand, $\Omega\mathcal{L}_1$ is at least four-dimensional: The moduli space ΩM_1 of translation structures on tori is 2-dimensional. Moreover, we can fix an origin and choose two branch points $\bar{P}$ and $\bar{Q}$. The other two points are given by definition as $-\bar{P}$ and $-\bar{Q}$. Choosing the torus and two branch points gives us at least four degrees of freedom.

In the next step, we show that the space ΩH_1 is not connected any more. In fact, one can already see from the equations given in Corollary 6.13 that there must be at least two connected components in $\mathcal{L}_1$, distinguished by being hyperelliptic or not. We show that there are exactly those two connected components in the Hurwitz space of translation surfaces ΩH_1. To proceed we describe the space ΩH_1 by polygons.

Recall from Proposition 6.8 that the polygons in $\mathcal{P}$ defining ΩH are given by five complex parameters. We denoted those vectors by a, b, c, d and e with $\partial(a) = \bar{P} - \bar{P}$, $\partial(b) = \bar{P} - \bar{Q}$, $\partial(c) = \bar{Q} - (-\bar{Q})$, $\partial(d) = (-\bar{Q}) - (-\bar{P})$ and $\partial(e) = (-\bar{P}) - \bar{P}$. The rotation $[-1]\colon E \to E$ interchanges the points $\bar{P}$ and $-\bar{P}$ as well as $\bar{Q}$ and $-\bar{Q}$. Thus the edges c and e are mapped to themselves, whereas the edges b and d are interchanged. This is sketched in Figure 6.9.

Thus the polygons describing translation coverings in ΩH_1 have to be in the space of polygons

$$\mathcal{P}_1 = \{\, (a, b, c, d, e) \in \mathcal{P} \mid b = d \,\},$$

where the second and fourth edge are represented by the same vector in $\mathbb{C}$.

Proposition 6.17. *We have a surjective map $\mathcal{P}_1 \times \{\, 1, 2, 3, 4 \,\} \to \Omega H_1$.*

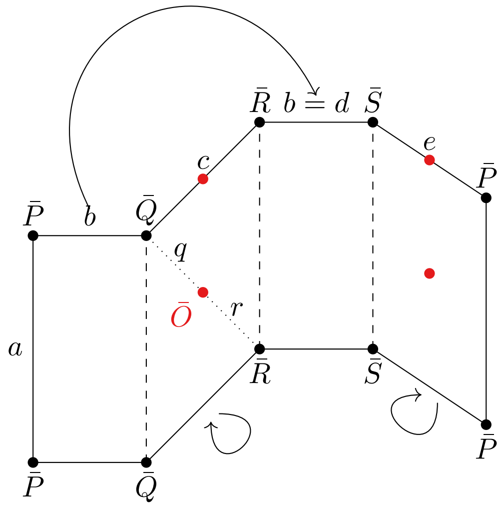


Figure 6.9: The rotation $[-1]$ interchanges the first and third parallelogram and leaves the second and fourth one invariant. Its fixed points are marked in red. The edges of the polygon are labeled fitting the description.

Proof. By the above discussion, every polygon representing a translation covering in ΩH_1 has to live in $\mathcal{P}_1$.

It suffices to show that every polygon in $\mathcal{P}_1$ gives rise to four different translation coverings in ΩH_1, since by Proposition 3.7 used as in the proof of Proposition 6.8, there are exactly four such translation coverings. Obviously, there are four coverings in ΩH. Hence we have to prove the existence of a lift τ of $[-1]$ on each of them. We can do this in two ways: Firstly, choose one of the fixed points of $[-1]$, see Figure 6.9, to be the origin. Say we choose the one between $\bar Q$ and $\bar R$. By definition $\bar P + \bar R = \bar O$. Let us denote by q the edge from the origin to the left upper $\bar Q$ and by r the edge from the origin to the right lower $\bar R$. We have

$$\bar S = q + c + d = q + c + b \quad \text{and} \quad \bar P = r - c - b$$

giving us $\bar P + \bar S = \bar O$. Hence $[-1]$ leaves the set of branch points invariant and its lift τ exists.

Secondly, we can give for every translation structure explicit lifts of $[-1]$. See Figure 6.10: The map τ is determined by deciding whether the parallelogram labeled by B is mapped to itself or to B'. We choose $\tau(B) = B$. Then, in all cases we have $\tau(A) = C$, $\tau(B') = B'$ and $\tau(A') = C'$. The image of D depends on the gluing. In the first two cases, $\tau(D) = D'$ and in the last two cases, $\tau(D) = D$ and $\tau(D') = D'$. □

In Figure 6.10 we see that the map τ has either four or eight fixed points. The latter implies that τ is a hyperelliptic involution. This is an elementary confirmation of the

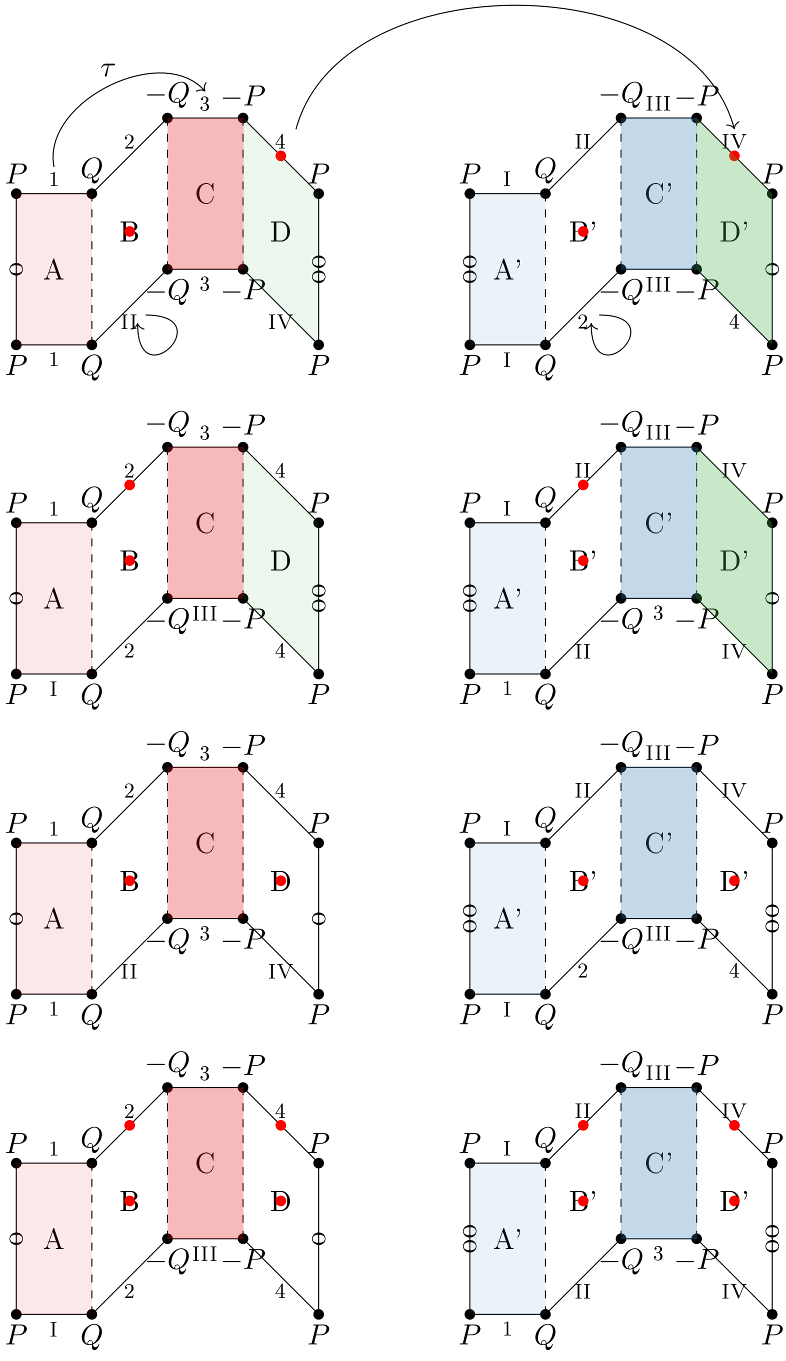


Figure 6.10: In the same order as in Figure 6.4, in every row is a translation surface in ΩH_1. The map τ is sketched and maps a parallelogram to a more saturated parallelogram of the same color. The fixed points of τ are marked in red.

results we presented before. From this it follows that the spaces ΩH_1 and $\Omega\mathcal{L}_1$ cannot be connected. Any continuous path from one surface to another transforms the involution τ to another involution with the same number of fixed points. We can distinguish the connected components by counting the fixed points of τ. To see this, we need to find better paths than those from Proposition 6.9, because these disturb the symmetries of translation coverings in ΩH_1. The idea for the constructions of those paths comes from the following group-theoretical statement.

Lemma 6.18. *Let $\mu\colon F_5 = \langle x, y, z, a, b\rangle \to S_2$ be a surjective homomorphism such that $\mu(x) = \mu(y) = \mu(z) = (1\,2)$. There are four such maps, which are determined by the images of a and b. The kernels of those four maps are*

$$\begin{aligned}
U_1 &= \langle xx, xy, yx, xz, zx, ax, xa, bx, xb\rangle, && \textit{if}\ \ \mu(a) = \mu(b) = (1\,2),\\
U_2 &= \langle xx, xy, yx, xz, zx, ax, xa, b, xbx\rangle, && \textit{if}\ \ \mu(a) = (1\,2),\ \mu(b) = \mathrm{id},\\
U_3 &= \langle xx, xy, yx, xz, zx, a, xax, bx, xb\rangle, && \textit{if}\ \ \mu(a) = \mathrm{id},\ \mu(b) = (1\,2) \quad \textit{and}\\
U_4 &= \langle xx, xy, yx, xz, zx, a, xax, b, xbx\rangle, && \textit{if}\ \ \mu(a) = \mu(b) = \mathrm{id}\,.
\end{aligned}$$

Those kernels are free groups in nine generators. Furthermore, they are isomorphic as subgroups of the free group F_5. The isomorphisms $\varphi_i \in \mathrm{Aut}(F_5)$ are given by $\varphi_i(x) = x$, $\varphi_i(y) = y$, $\varphi_i(z) = z$ for $i = 1, 2, 3, 4$ and

$$\begin{aligned}
&\varphi_1\colon U_1 \to U_2, && \varphi_1(a) = a, && \varphi_1(b) = ab,\\
&\varphi_2\colon U_1 \to U_3, && \varphi_2(a) = xa, && \varphi_2(b) = b,\\
&\varphi_3\colon U_3 \to U_4, && \varphi_3(a) = a, && \varphi_3(b) = xb \ \textit{and}\\
&\varphi_4\colon U_4 \to U_2, && \varphi_4(a) = xa, && \varphi_4(b) = b.
\end{aligned}$$

This technical lemma is filled with life by the observation that such maps μ are exactly the monodromy maps of coverings $p\colon X \to E$ from a Riemann surface of genus 3 to an elliptic curve of degree 2 with four ramification points. By Proposition 3.7 there exist only four such coverings for fixed ramification data. In the above setting, we interpret the elements x, y and z as loops around branch points, for instance $x = \ell_{\bar{P}}$ is the loop in E around the branch point $\bar{P}$. The elements a and b may be seen as the horizontal and vertical loops $\bar{\alpha}$ and $\bar{\beta}$ of the torus going through a fixed point of the involution $[-1]$. With this in mind, compare the groups U_i with the pictures of the translation surfaces in Figure 6.4 or Figure 6.10. In the first picture, the lift of the vertical loop $\bar{\beta}$ and the lift of the horizontal loop $\bar{\alpha}$ are not closed, hence the kernel of the monodromy map belonging to the first picture is the first group U_1. Likewise, the second, third and fourth group belongs to the second, third and fourth picture, respectively.

For this reason let us fix the following notation: A translation covering in ΩH is *of (monodromy) type U_i* (or just i) if the kernel of the monodromy map $p_*\pi_1(X) = \ker(\mu)$ is the group U_i. Moreover, let us denote by $\ell_{\bar{P}}$, $\ell_{\bar{Q}}$ and $\ell_{-\bar{Q}}$ the counterclockwise loops around the branch points $\bar{P}$, $\bar{Q}$ and $-\bar{Q}$ corresponding to x, y and z.

Proof of Lemma. By the theorem of Nielsen-Schreier, see Schreier's paper [Sch27], a subgroup of index d in a free group of rank k is free of rank $1 + d(k-1)$. Since μ is surjective, the kernel is of index 2 in F_5. So it is a free group of rank $1+2(5-1)=9$. It is easily seen that every element in U_i is in the kernel of the corresponding map μ and that there are no relations between the generators, so the groups U_i are free in the given generators.

We verify that all those maps are surjective, by showing that every generator is in the image. Hence they have to be bijective. The difference between two sets of generators are always two elements. Hence we only have to show that those two missing generators are in the image of the corresponding map. For the map φ_1 we have

$$\varphi_1((xa)^{-1}xb) = a^{-1}x^{-1}xab = b \quad \text{and}$$
$$\varphi_1(x^2(ax)^{-1}bx) = xxx^{-1}a^{-1}abx = xbx,$$

showing that the map φ_1 is surjective. Similarly, we have

$$\begin{aligned} &\varphi_2(ax) = xax, && \varphi_2(x^{-2}xa) = a, \\ &\varphi_3(x^{-2}xb) = b, && \varphi_3(bx) = xbx, \\ &\varphi_4(x^{-2}xax) = ax \quad \text{and} && \varphi_4(a) = xa. \end{aligned}$$

All the given maps are surjective and the claim follows. □

Proposition 6.19. *The Hurwitz space of translation surfaces ΩH_1 has two connected components. They are distinguished by the number of fixed points of τ.*

More precisely, one connected component consists of the coverings of type 1, 2 and 3. The second connected component consists of the coverings of type 4. Every covering surface in the second component is hyperelliptic.

Proof. As discussed before, there must be at least two connected components. We construct paths that connect any translation covering in ΩH_1 of type U_1 with any translation covering of type U_2 and U_3. The path has to linger in ΩH_1 the whole time, i.e. it has to respect the extra symmetry. Looking at Figure 6.8, the first two paths do not respect the symmetry: If before the cutting the two edges $b = d$ coincide, then after cutting and gluing the edges b and d no longer coincide. The idea for paths that work comes from the rather technical Lemma 6.18. We have to translate the maps $\varphi_1 \colon U_1 \to U_2$ and $\varphi_2 \colon U_1 \to U_3$ into a cutting and gluing instruction.

Let $\bar{\alpha}$ and $\bar{\beta}$ be the horizontal and vertical loop of the torus. Together with the loops $\ell_{\bar{P}}$, $\ell_{\bar{Q}}$ and $\ell_{-\bar{Q}}$ around the branch points they form a basis of the fundamental group of the punctured torus.

The map $\varphi_1 \colon U_1 \to U_2$ maps the vertical path $\bar{\beta}$ to $\bar{\alpha}\bar{\beta}$, see Figure 6.11. Denote by $(a, b, c, d, e) \in \mathcal{P}_1$ a polygon defining a translation covering in ΩH_1. The translation surface consists of two copies of this polygon, where we label the edges of the first polygon by a_1, b_1, c_1, d_1 and e_1 and the edges of the second polygon by a_2, b_2, c_2, d_2 and e_2. The resulting covering is of type U_2 if the edges $-c_1$ and c_1 are glued. It is of type U_1 if the edge $-c_1$ is glued to the edge c_2. Hence starting in U_2, we cut along the lift $\alpha\beta$ of $\bar{\alpha}\bar{\beta}$ which connects the edges $-c_1$ and c_2. Then we reglue to get a surface in U_1 as follows:

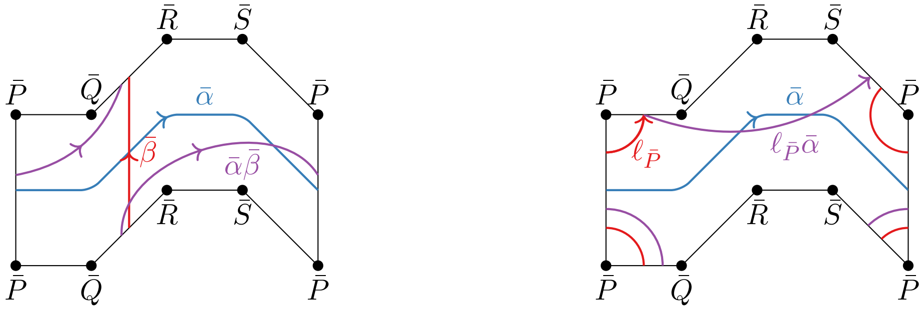


Figure 6.11: On the left, in the fundamental group the combination of the paths $\bar{\alpha}$ and $\bar{\beta}$ gives the path $\bar{\alpha}\bar{\beta}$. On the right, combining $\bar{\alpha}$ and the loop $\ell_{\bar{P}}$ gives $\ell_{\bar{P}}\bar{\alpha}$. For simplicity, the base point is not fixed.

See Figure 6.12, where the purple path is the path $\alpha\beta$. In the first step, we transform the surface such that the dashed line is vertical and all edges are horizontal. We then reglue the polygon such that the path $\alpha\beta$ is in one polygon. Transforming the edges again we return to the polygon we began with, but now of type U_1. In every transformation we do not change the relation of the edges $b_1 = d_1$.

In the same spirit, the map $\varphi_2\colon U_1 \to U_3$ maps the horizontal path $\bar{\alpha}$ to the path $\ell_{\bar{P}}\bar{\alpha}$. Again, we want its lift $\ell_P\alpha$ to be in only one polygon. But comparing the right picture in 6.11 and the last operation in Figure 6.8, we see that this is exactly what happened when showing the connectedness of ΩH. By cutting off the right lower part in Figure 6.8, which contains the purple path, we see that the two edges b_1 and d_1 are not affected. Thus we obtain a path in ΩH_1 that connects a translation covering of type U_1 with one of type U_3. $\square$

6.3 A subspace of codimension two

In the last section we have seen that demanding an extra symmetry yields a 4-dimensional affine invariant submanifold in the principal stratum $\mathcal{H}(1,1,1,1)$. In this section we introduce another additional symmetry. Firstly, by a relation of the branch points, secondly by a restriction on the defining polygons and thirdly by an extra automorphism. With the second and third description we compute the automorphism group of each generic surface depending on the monodromy type. This shows that this new Hurwitz space of translation surfaces has three connected components and that it is an affine invariant submanifold of dimension 3 in $\mathcal{H}(1,1,1,1)$. Finally, we calculate under which conditions translation surfaces in this space are Veech surfaces and under which they are origamis.

Definition 6.20. For a covering $p\colon X \to E$ in H_1, let $\bar{P}$, $\bar{Q}$, $-\bar{Q}$ and $-\bar{P}$ denote the branch points in E. We define the subspace

$$H_2 = \left\{ (p, X, E) \in H_1 \;\middle|\; \bar{P} + \bar{Q} = -\bar{P} - \bar{Q} \right\}$$

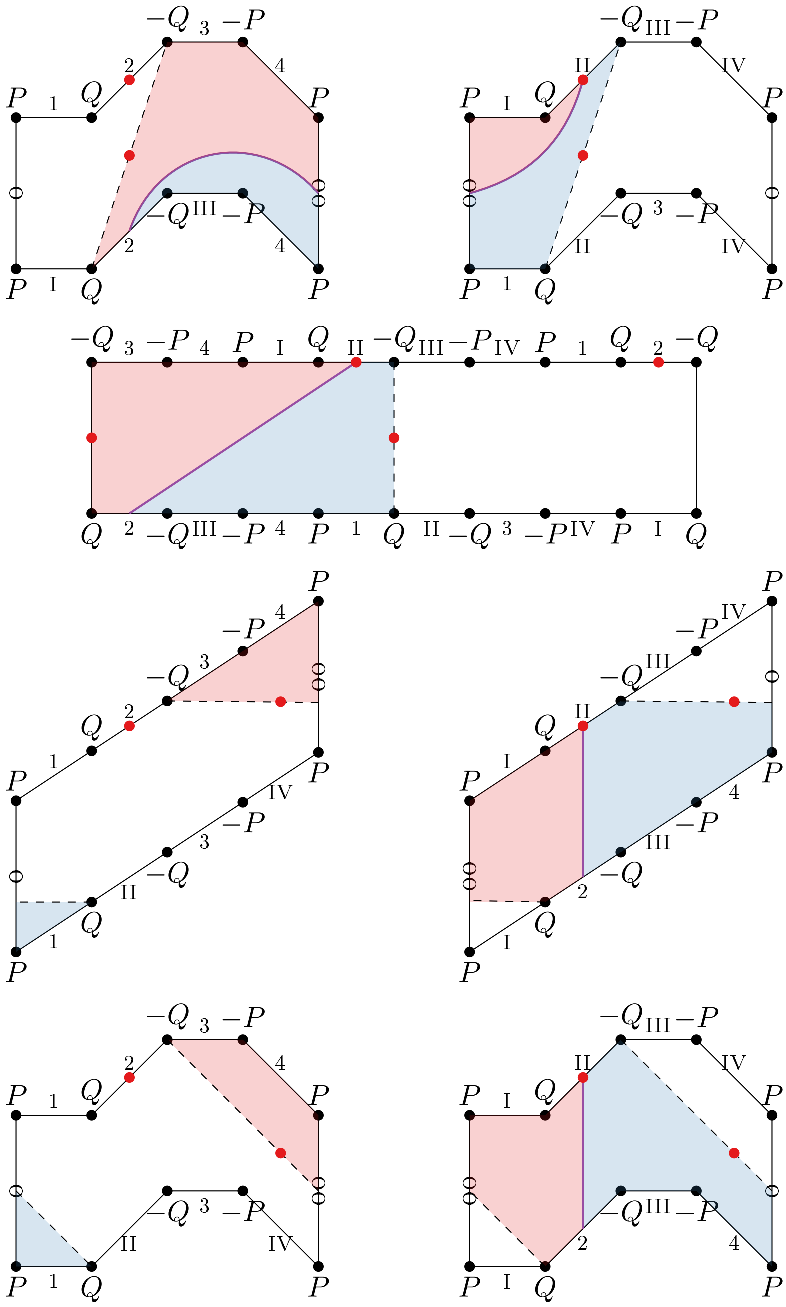


Figure 6.12: This picture shows the transformation from a surface of type U_2 to one of type U_1 as described in the proof of Proposition 6.28: We bash the surface into a rectangle, cut it in half and shear it a bit. Finally, we bring it back into the form we began with. The fixed points of τ are red. The purple curve is the path $\alpha\beta$.

of all coverings such that the point $\bar{P}+\bar{Q}$ is a 2-torsion point. Its image

$$\mathcal{L}_2 = \mathcal{F}(H_2)$$

in the moduli space of Riemann surfaces of genus 3 consists of those Riemann surfaces admitting a covering in H_2.

As before, we are mainly interested in the Hurwitz space of translation coverings belonging to H_2. Again, we can use polygons to describe the translation coverings arising from H_2. These polygons have one more restriction, which stems from the condition $\bar{P}+\bar{Q} = -\bar{P}-\bar{Q}$. This point of view has the further advantage that our proofs are quite elementary.

Definition 6.21. The Hurwitz space of translation surfaces

$$\Omega H_2 = \left\{ (p, X, \omega, E, \eta) \in \Omega H_1 \;\middle|\; \bar{P}+\bar{Q} = -\bar{P}-\bar{Q} \right\}$$

contains all translation coverings in ΩH_1 such that the point $\bar{P}+\bar{Q}$ is a 2-torsion point. Its image

$$\Omega\mathcal{L}_2 = \Omega\mathcal{F}(\Omega H_2)$$

in the moduli space of translation surfaces of genus 3 consists of those translation surfaces that admit a translation covering in ΩH_2.

Lemma 6.22. *Every torus that is covered by a translation covering in ΩH_2 can be described using a polygon in*

$$\mathcal{P}_2 = \{ (a, b, c, d, e) \in \mathcal{P} \mid b = d, c = e \}.$$

Figure 6.13 shows an example of a torus described by a polygon in $\mathcal{P}_2$. Its edges are marked accordingly.

Proof. Let E be a torus covered by a translation covering in ΩH_2 with branch points $\bar{P}$, $\bar{Q}$, $-\bar{Q}$ and $-\bar{P}$. Writing the edges as the vectors between two vertices, we get

$$c = -\bar{Q} - (\bar{Q}) = -2\bar{Q} \quad \text{and} \quad e = \bar{P} - (-\bar{P}) = 2\bar{P}.$$

As $\bar{P}+\bar{Q}$ is a 2-torsion point, $e - c = 2(\bar{P}+\bar{Q}) = 0$. Furthermore, since $\mathcal{P}_2 \subseteq \mathcal{P}_1$, we have $b = d$. $\square$

Corollary 6.23. *We have a surjective map $\mathcal{P}_2 \times \{1,2,3,4\} \to \Omega H_2$.*

Proof. The map is the restriction of the map defined in Proposition 6.8. We show that it is surjective: By Lemma 6.22 every covered torus can be described as a polygon in $\mathcal{P}_2$. We take two copies of this polygon and glue them via the gluings given in Proposition 6.8. These gluing are depicted in 6.15. Hence this yields four different translation coverings of one torus with fixed branch points. As in the proof of Proposition 6.8, there exist exactly four different coverings of a fixed torus with fixed branch points. $\square$

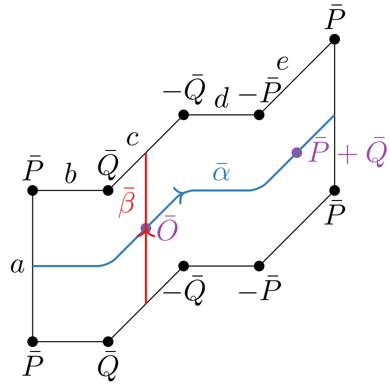


Figure 6.13: An elliptic curve with its origin $\bar O$ and the point $\bar P + \bar Q$ marked in purple. This surface is covered by a surface in ΩH_2. Two loops of the fundamental group are plotted in blue and red. Note that the edges fulfill $b = d$ and $c = e$.

Now we describe the space ΩH_2 by its automorphisms. We do this explicitly by specifying an automorphism of the torus which can be lifted to an automorphism of the covering surface.

Lemma 6.24. *Let $p\colon (X,\omega) \to (E,\eta)$ be a translation covering in ΩH_2 with branch points $\bar P$, $\bar Q$, $-\bar P$ and $-\bar Q$ in E. The translation*

$$t_{\bar P+\bar Q}\colon E \to E, \qquad x \mapsto x + \bar P + \bar Q$$

can be lifted to a translation on the translation surface (X,ω).

Proof. Similarly to Proposition 6.11 we have to show that the monodromy of the fundamental group of the punctured torus is invariant under the map $t_{\bar P+\bar Q}$. We construct a basis for which this holds.

We denote by $\bar O$ the base point of the torus, which is a fixed point of $[-1]$. We have $t_{\bar P+\bar Q}(\bar O) = \bar P + \bar Q$. This is a 2-torsion point and thus another fixed point of $[-1]$.

Choose $\bar\alpha$ to be the "horizontal" loop of the torus that contains the two fixed points $\bar O$ and $\bar P + \bar Q$ of $[-1]$ as sketched in Figure 6.13. Then

$$t_{\bar P+\bar Q}(\bar O) = \bar P + \bar Q \in \alpha \quad \text{and} \quad t_{\bar P+\bar Q}(\bar P + \bar Q) = \bar O \in \alpha.$$

Since $t_{\bar P+\bar Q}$ is a translation, it leaves the loop $\bar\alpha$ and its monodromy invariant.

We define $\bar\beta$ to be the "vertical" loop of the torus. It starts in $\bar O$, so it is mapped by the translation $t_{\bar P+\bar Q}$ to a vertical loop starting in $\bar P + \bar Q$. Thus $\bar\beta$ is mapped two parallelograms to the right by $t_{\bar P+\bar Q}$. We can translate $\bar\beta$ one parallelogram to the right by concatenating it with the loop around $-\bar Q$. For a visualization see Figure 6.14. Thus the image of $\bar\beta$ under a translation by two parallelograms to the right is given by $\bar\beta \circ \ell_{-\bar Q} \circ \ell_{-\bar P}$. Hence the monodromy of $t_{\bar P+\bar Q}(\bar\beta)$ is computed by

$$\mu(t_{\bar P+\bar Q}(\bar\beta)) = \mu(\bar\beta)\mu(\ell_{-\bar P})\mu(\ell_{-\bar Q}) = \mu(\bar\beta)(1\,2)(1\,2) = \mu(\bar\beta)$$

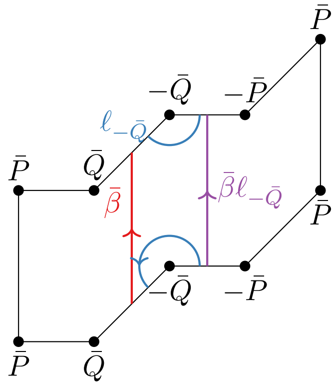


Figure 6.14: A torus with some loops. Combining the loops $\bar{\beta}$ and $\ell_{-\bar{Q}}$ gives the purple loop. For better visualization, the loops all have different base points.

and thus is left invariant.

Because the loops around the branch points are mapped to loops around branch points, their monodromy does not change and the claim follows. □

Now we reformulate the definition of the space ΩH_2. Instead of using the relation of the branch points, we lift the above translation $t_{\bar{P}+\bar{Q}}$ and use this automorphism as the defining property of ΩH_2.

Proposition 6.25. *The Hurwitz space of translation coverings*

$$\Omega H_2 = \Big\{ \ (p, X, \omega, E, \eta) \in \Omega H_1 \ \Big| \ \textit{there exists } \varphi \in \mathrm{Aut}(X),\ \varphi(\pm P) = \mp Q, \\ \varphi(\pm Q) = \mp P,\ D\varphi = I,\ \varphi \circ \sigma = \sigma \circ \varphi \ \Big\}$$

can be identified with the space of all coverings that have an extra translation φ. This translation φ commutes with the deck transformation and interchanges the ramification points in the given way.

Here and subsequently, we denote this particular translation by φ.

Proof. Let $(p, X, \omega, E, \eta) \in \Omega H_2$. Then we have $\bar{P} + \bar{Q} = -\bar{P} - \bar{Q}$. According to Lemma 6.24, there exists a lift φ of $t_{\bar{P}+\bar{Q}}$. Since the covering is a translation covering, $D\varphi = I$. Furthermore, $\varphi\sigma = \sigma\varphi$, since the map φ is a lift. Finally, we have

$$t_{\bar{P}+\bar{Q}}(\bar{P}) = \bar{P} + \bar{P} + \bar{Q} = \bar{P} - \bar{P} - \bar{Q} = -\bar{Q}$$

and thus $\varphi(P) = -Q$. In the same manner, one can show that the remaining points are mapped appropriately.

On the other hand, let p be a translation covering in ΩH_1 and let $\varphi\colon X \to X$ be a map with the prescribed properties. Because φ and σ commute, φ descends to a translation $\bar{\varphi}\colon E \to E$. It fulfills

$$\bar{\varphi}(\bar{P}) = -\bar{Q} \quad \text{and} \quad \bar{\varphi}(-\bar{P}) = \bar{Q}.$$

Hence it is a translation by $-\bar{P} - \bar{Q}$ and, at the same time, by $\bar{P} + \bar{Q}$. Thus $\bar{P} + \bar{Q} = -\bar{P} - \bar{Q}$. $\square$

We now give an alternative, more elementary proof of Proposition 6.25, which yields a description of the automorphism group and of the connected components of ΩH_2. Using Corollary 6.23, we can draw explicit pictures of translation coverings in ΩH_2. The translation $t_{\bar{P}+\bar{Q}}\colon E \to E$ on the torus maps one fixed point of $[-1]$ to another one. Thus it maps the left half of the polygon to the right half. We can choose one of two possible lifts, which differ by the deck transformation σ. In Figure 6.15 we choose the lift $\varphi\colon X \to X$ of $t_{\bar{P}+\bar{Q}}$ such that it maps the left half of the left polygon to the right half of the left polygon. In the first two cases the map φ has order 4, because the right edge of the hexagon A gets mapped to the right edge of $\varphi(A)$, labeled by *oo*. Hence applying φ^2 to A, the left edge of A is mapped to the edge labeled by *oo*, the left edge of the hexagon $\varphi^2(A)$. Similarly, in the last two cases the automorphism φ has order 2.

Proposition 6.26. *The automorphism group of a generic translation covering in ΩH_2 or of a generic translation surface in $\Omega\mathcal{L}_2$ depends on the monodromy of the covering. It is either the dihedral group D_8 with eight elements or the elementary abelian group C_2^3 with eight elements. In the latter case, the corresponding surface is hyperelliptic.*

Proof. We write out the proof in the first case and leave the other cases to the reader. According to the first picture in Figure 6.15 we see that $\varphi^2 = \sigma$. Observe that

$$\begin{aligned}[-1] \circ t_{\bar{P}+\bar{Q}} \circ [-1](\bar{x}) &= [-1] \circ t_{\bar{P}+\bar{Q}}(-\bar{x}) = [-1](-\bar{x} + \bar{P} + \bar{Q}) \\ &= \bar{x} - \bar{P} - \bar{Q} = \bar{x} + \bar{P} + \bar{Q} = t_{\bar{P}+\bar{Q}}(\bar{x}).\end{aligned}$$

Hence $[-1] \circ t_{\bar{P}+\bar{Q}} \circ [-1]^{-1} = t_{\bar{P}+\bar{Q}}$. The map $\sigma\varphi = \varphi^{-1}$ is a lift of the right hand side of the equation. The map $\tau \circ \varphi \circ \tau^{-1}$ is a lift of the left hand side. If they coincide in one point, they are equal. Let O be the fixed point of τ plotted in 6.15. In this picture, we see that $\tau\varphi\tau^{-1}(O) = \varphi^{-1}(O)$ and hence $\tau\varphi\tau^{-1} = \varphi^{-1}$. So the automorphism group is given by the relations

$$\mathrm{Aut}(X) = \langle \tau, \varphi \mid \varphi^4 = \tau^2 = \mathrm{id},\ \tau\varphi\tau^{-1} = \varphi^{-1} \rangle.$$

Comparing with Chapter 4 we see that this is exactly the dihedral group D_8.

Similarly, in the second case the automorphism group is again D_8. In the third and fourth case, φ and τ commute. Thus the automorphism group is the elementary abelian group C_2^3.

Furthermore, the third and fourth case are hyperelliptic: From Figure 6.10 and Corollary 6.13 we know that the fourth one is hyperelliptic. In Figure 6.15 we see that in the third case the involution $\tau\varphi$ has eight fixed points. Hence those surfaces are hyperelliptic as well. $\square$

Corollary 6.27. *If the automorphism group of a surface in $\mathcal{L}_2$ contains the dihedral group D_8, the generic curve is not hyperelliptic and is given by*

$$x^4 + y^4 + ax^2y^2 + b(x^2 + y^2) + 1 = 0, \qquad a, b \in \mathbb{C}.$$

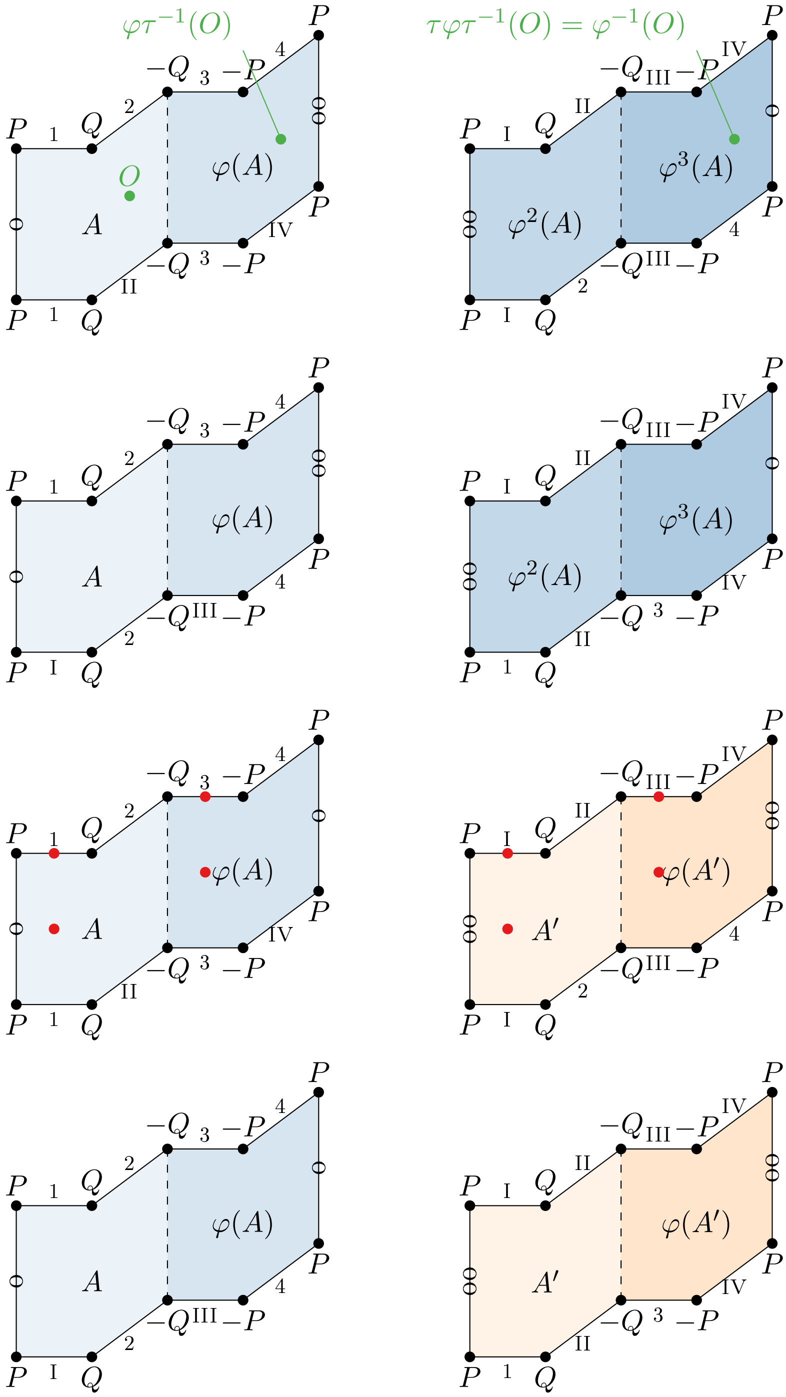


Figure 6.15: In every row is a translation surface in ΩH_2 of type 1, 2, 3 and 4, respectively. The translation φ maps a region to a more saturated region of the same color. For a translation surface of type 1, the points O and $\tau\varphi\tau^{-1}(O)$ are sketched in green. For one of type 3, the fixed points of $\tau\varphi$ are sketched in red.

If, on the other hand, the automorphism group contains C_2^3, the generic curve is hyperelliptic and is given by

$$y^2 = (x^4 + ax^2 + 1)(x^4 + bx^2 + 1), \qquad a, b \in \mathbb{C}.$$

In both cases, the dimension of the locus of those curves is 2.

Proof. The generic automorphism group of a covering and hence of a surface is, by Proposition 6.26, either D_8 or C_2^3. In the last case, the surface is hyperelliptic. The dimensions and equations follow from Tables 3.1 and 3.2. □

This implies that the dimension of the space $\Omega\mathcal{L}_2$ and the Hurwitz space ΩH_2 is 3. We will prove this independently in Proposition 6.31.

Proposition 6.28. *The Hurwitz space of translation surfaces ΩH_2 has three connected components. They are distinguished by the number of fixed points of τ and $\varphi\tau$.*

Note that translation coverings of type 1 and 2 are in one connected component. Translation coverings of type 3 and 4 each form a connected, hyperelliptic component.

Proof. Again, we use the structure of the translation coverings as sketched in Figure 6.15. Looking at Proposition 6.9 and Figure 6.12, we can use the exactly same procedure to construct a path from a surface of type 1 to one of type 2.

There cannot be less connected components, because the map $\varphi\tau$ has eight fixed points if and only if the surface is of type 3. A continuous path connecting a surface of type 3 to one of another type preserves the number of fixed points of $\varphi\tau$. □

Since the map $\pi_{H_2} : \Omega H_2 \to H_2$ has connected fibers, the Hurwitz space of translation surfaces ΩH_2 cannot have more connected components than the Hurwitz space H_2. Hence H_2 must at least have three connected components. Furthermore, each continuous path between two translation coverings yields, via π_{H_2}, a continuous path between the two corresponding (holomorphic) coverings. Hence H_2 and ΩH_2 do have the same number of connected components. Nevertheless, $\mathcal{L}_2$ and $\Omega\mathcal{L}_2$ just have two connected components.

Proposition 6.29. *The space of translation surfaces $\Omega\mathcal{L}_2$ and its image $\mathcal{L}_2$ in the moduli space both have two connected components. They are distinguished by being hyperelliptic or not.*

Proof. We have to construct a path from a translation surface of type 3 to one of type 4, since it follows from Proposition 6.28 that the non-hyperelliptic part of $\Omega\mathcal{L}_2$ is connected. In Figure 6.16 we see how a translation surface of type 4 is transformed into one of type 3: Cut off the last square in each polygon and glue it in front. Then we relabel the vertices and edges. By deforming the polygon we may assume both translation surfaces are of the depicted form. Note that in the first row the hyperelliptic involution is τ and in the second row it is $\varphi\tau$.

Since the map $\pi_{\mathcal{L}_2} : \Omega\mathcal{L}_2 \to \mathcal{L}_2$ has connected fibers, we get that $\mathcal{L}_2$ has two connected components as well. □

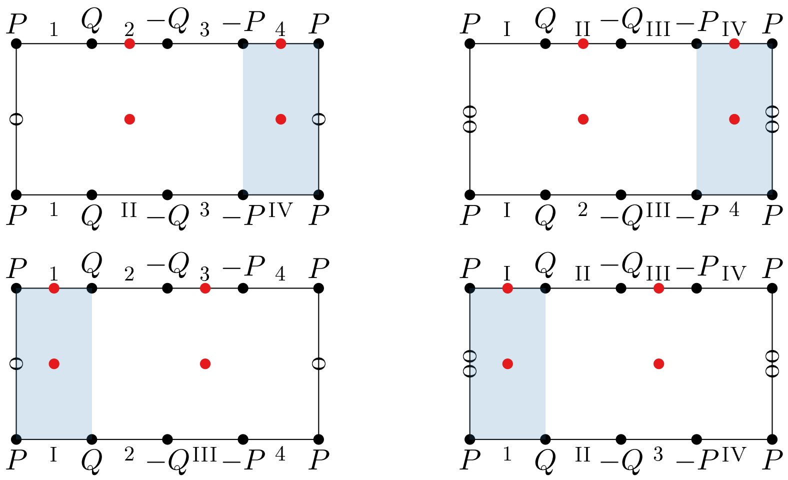


Figure 6.16: A path between two hyperelliptic translation surfaces in $\Omega\mathcal{L}_2$. We cut off and glue the blue rectangle in the upper row to the left. Relabeling gives the second row.

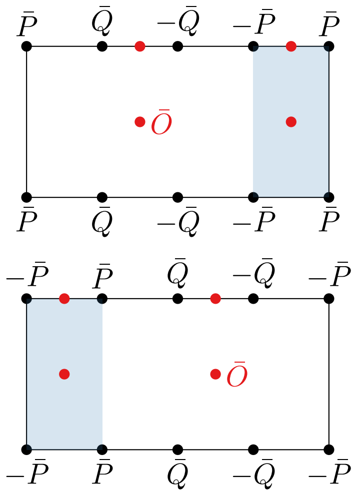


Figure 6.17: Not a path between two coverings. We cut off the blue rectangle and glue it to the left. The origin $\bar{O}$ does not change and hence the red points in both pictures are the fixed points of the rotation $[-1]$.

Why does this procedure not work in ΩH_2? If we would do the same thing, we had to cut the covered torus in the same manner, see Figure 6.17. This cutting and gluing does not change the origin $\bar{O}$. From the origin we can (up to a transposition of $\bar{P}$ and $\bar{Q}$) deduce the branch points. Hence in both rows, the map whose fixed point set consists of the red points is the rotation $[-1]$. The lift of both maps is τ. But to fit together with the covering the lift of the map in the second row should be $\varphi\tau$, see Figure 6.17. A contradiction. In other words, the covering remembers the order of the branch points (at least up to a transposition), whereas forgetting the covering forgets the order of the points as well.

To finish this section we show that the space $\Omega\mathcal{L}_2$ is an affine invariant submanifold of dimension 3 of $\mathcal{H}(1,1,1,1)$. Thus $\Omega\mathcal{L}_2$ is of codimension 2 in $\Omega\mathcal{L}$, justifying its name.

Lemma 6.30. *The subspace $H_2 \subseteq H$ is a submanifold.*

Proof. The proof works as the proof of Lemma 6.15. Hence we only show how we must alter a chart φ to make this work.

Let p be a covering and let $e_1 = \bar{P}$, $e_2 = \bar{Q}$, $e_3 = -\bar{Q}$ and $e_4 = -\bar{P}$ be the branch points in $E = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$. We define the chart φ such that

$$\varphi(p) = (\tau, \bar{e}_1, \bar{e}_1 + \bar{e}_4, \bar{e}_2 + \bar{e}_3, \bar{e}_1 + \bar{e}_2 - \bar{e}_3 - \bar{e}_4).$$

In H_2 we have the relations $e_1 + e_4 = \bar{O}$, $e_2 + e_3 = \bar{O}$ and $e_1 + e_2 - e_3 - e_4 = \bar{O}$. Thus for a covering $p \in H_2$ the chart φ is given by

$$\varphi(p) = (\tau, \bar{e}_1, 0, 0, 0).$$

Hence locally H_2 is the intersection of $\mathbb{C}^5$ with a two-dimensional linear subspace showing that H_2 is a submanifold of H. $\square$

In particular, the proof shows that the Hurwitz space H_2 is not only a submanifold of H, but also of H_1.

Proposition 6.31. *The space $\Omega\mathcal{L}_2 \subset \Omega\mathcal{L}$ is an affine invariant submanifold of the principal stratum $\mathcal{H}(1,1,1,1)$ of dimension* 3.

Proof. The Hurwitz space of translation surface ΩH_2 is a submanifold of ΩH, because due to Lemma 6.30 H_2 is a submanifold of H. Furthermore, we can restrict the immersion $\Omega\mathcal{F}\colon \Omega H \to \Omega\mathcal{L}$ to an immersion $\Omega H_2 \to \Omega\mathcal{L}_2$ into the principal stratum $\mathcal{H}(1,1,1,1)$. Hence we only need to show that the image of some translation covering under the forgetful map $\Omega\mathcal{F}$ is given by linear equations in period coordinates of the right dimension. In a small enough neighborhood of a translation covering the immersion $\Omega\mathcal{F}$ is injective. Thus, when working with a translation surface $(X,\omega) \in \Omega\mathcal{L}_2$ we can always assume that it is equipped with a unique translation covering denoted by $p\colon (X,\omega) \to (E,\eta)$.

Let us recall the notation introduced in Proposition 6.16: The elements in the absolute homology group a and $b \in H_1(X,\mathbb{Z})$ project via $p\colon X \to E$ to $\bar{a}$ and $\bar{b}$, the generators of the homology group $H_1(E,\mathbb{Z})$ of the torus. The arcs from one ramification point to another are called c_{PQ}, c_{Q-P} and c_{-P-Q}. Their projections going from one branch point to another are called $c_{\bar{P}\bar{Q}}$, $c_{\bar{Q}-\bar{P}}$ and $c_{-\bar{P}-\bar{Q}}$. Finally, we denote the integrals by

$$\int\limits_a \omega = A, \quad \int\limits_b \omega = B, \quad \int\limits_{c_{PQ}} \omega = C_1, \quad \int\limits_{c_{Q-P}} \omega = C_2 \quad \text{and} \quad \int\limits_{c_{-P-Q}} \omega = C_3.$$

Because p is a translation covering and φ a translation, we have $\varphi^*\omega = \omega$ and $p^*\eta = \omega$. Thus for each $c \in H_1(X, \Sigma, \mathbb{Z})$ we have

$$\int\limits_c \omega = \int\limits_c \varphi^*\omega = \int\limits_{\varphi_* c} \omega = \int\limits_{p_*\varphi_* c} \eta = \int\limits_{t_{\bar{P}+\bar{Q}*}p_* c} \eta.$$

Moreover, by Example 2.13, we compute the relations

$$\begin{aligned}
&t_{\bar{P}+\bar{Q}*}\bar{a} = \bar{a}, \qquad t_{\bar{P}+\bar{Q}*}\bar{b} = \bar{b},\\
&t_{\bar{P}+\bar{Q}*}c_{\bar{P}\bar{Q}} = -c_{-\bar{P}-\bar{Q}},\\
&t_{\bar{P}+\bar{Q}*}c_{\bar{Q}-\bar{P}} = c'_{-\bar{P}\bar{Q}} = \bar{a} - c_{\bar{Q}-\bar{P}} \quad \text{and}\\
&t_{\bar{P}+\bar{Q}*}c_{-\bar{P}-\bar{Q}} = -c_{\bar{P}\bar{Q}},
\end{aligned}$$

where $c'_{-\bar{P}\bar{Q}}$ is the path from $-\bar{P}$ to $\bar{Q}$. In Example 2.13 we computed how to write it as a linear combination of basis vectors. For example, the third equation follows because $t_{\bar{P}+\bar{Q}}(\bar{P}) = -\bar{Q}$ and $t_{\bar{P}+\bar{Q}}(\bar{Q}) = -\bar{P}$. Hence the geodesic $c_{\bar{P}\bar{Q}}$ gets mapped to a geodesic going from $-\bar{Q}$ to $-\bar{P}$, but in the same direction as $c_{\bar{P}\bar{Q}}$, so the image is $-c_{-\bar{P}-\bar{Q}}$. These equations give us

$$\begin{aligned}
A &= \int\limits_a \omega = \int\limits_{t_{\bar{P}+\bar{Q}*}\bar{a}} \eta = \int\limits_{\bar{a}} \eta = \int\limits_a \omega = A,\\
B &= \int\limits_b \omega = \int\limits_{t_{\bar{P}+\bar{Q}*}\bar{b}} \eta = \int\limits_{\bar{b}} \eta = B
\end{aligned}$$

as well as the more interesting equations

$$\begin{aligned}
C_1 &= \int\limits_{c_{PQ}} \omega = \int\limits_{t_{\bar{P}+\bar{Q}*}c_{\bar{P}\bar{Q}}} \eta = -\int\limits_{c_{-\bar{P}-\bar{Q}}} \eta = -\int\limits_{c_{-P-Q}} \omega = -C_3,\\
C_2 &= \int\limits_{c_{Q-P}} \omega = \int\limits_a \omega - \int\limits_{c_{Q-P}} \omega = A - C_2 \quad \text{and}\\
C_3 &= \int\limits_{c_{-P-Q}} \omega = -\int\limits_{c_{PQ}} \omega = -C_1.
\end{aligned}$$

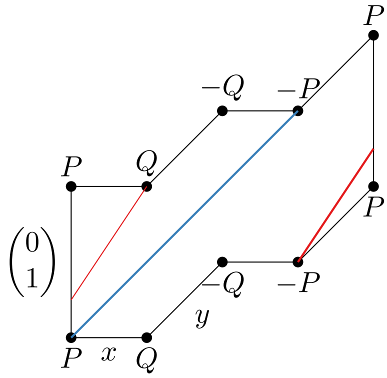


Figure 6.18: A polygon describing a translation surface in standard form. The holonomy of the blue saddle connection is $x + y + (0,1)^\top + x$ and the holonomy of the red one is $(0,1)^\top + y$.

Overall, we have the equations $C_1 + C_3 = 0$ and $2C_2 = A$. Note that $C_1 + C_3 = 0$ is the equation already given by τ in the proof of Proposition 6.16. Hence we have one more restriction than in the case of $\Omega\mathcal{L}_1$. In period coordinates this describes a hyperplane in $\mathbb{C}^4$, making $\Omega\mathcal{L}_2$ an at most 3-dimensional affine invariant submanifold.

On the other hand, the space is at least 3-dimensional. The moduli space of translation tori ΩM_1 is 2-dimensional. We may choose one branch point freely, say $\bar{P}$. This determines the other three branch points because the fourth one is $-\bar{P}$ and the condition $\bar{P} + \bar{Q} = -\bar{P} - \bar{Q}$ implies $2\bar{P} = 2\bar{Q}$, which yields $\bar{Q}$ and thus also $-\bar{Q}$. $\square$

In the previous proof we have seen that the translation $\varphi\colon X \to X$ gives us a subspace of codimension 2 in ΩH, without explicitly requiring the existence of the rotation τ. This manifests itself in the proof by the condition $C_1 + C_3 = 0$. We interpret this geometrically as follows: We draw a polygon that admits a translation φ as in the definition of ΩH_2. The edges b and d coincide, since $\varphi(P) = -Q$ and $\varphi(Q) = -P$. Thus we can construct the involution τ as in Figure 6.15 and the existence of the translation φ implies the existence of the map τ.

Finally, according to Wright [Wri14] the $\mathrm{GL}_2^+(\mathbb{R})$-orbit of almost every translation surface in an affine invariant submanifold is as large as possible. In our case, almost every translation surface in $\Omega\mathcal{L}_2$ has a 3-dimensional orbit closure in $\Omega\mathcal{L}_2$. The $\mathrm{GL}_2^+(\mathbb{R})$-orbit of any other translation surface is closed and hence the corresponding surface is a Veech surface. A translation surface in $\Omega\mathcal{L}_2$ is in *standard form* if the torus is the standard torus and the branch point $\bar{Q}$ is to the right of $\bar{P}$. Hence we can find a polygon as depicted in Figure 6.18 which describes the translation surface. Let us denote by $a = (0,1)^\top$, $x = (x_1, 0)$ and $y = (y_1, y_2) \in \mathbb{R}^2$ the holonomy vectors described in the figure. In every $\mathrm{GL}_2^+(\mathbb{R})$-orbit we find a translation surface in standard form. For these translation surfaces, we describe explicitly when they are Veech surfaces.

Proposition 6.32. *Let $(X, \omega) \in \Omega\mathcal{L}_2$ be a translation surface given in standard form. Denote by $a = (0,1)^\top$, $x = (x_1, 0)$ and $y = (y_1, y_2) \in \mathbb{R}^2$ the holonomy vectors of* *the saddle connections leaving P to the north, P to the east and going from Q to $-Q$, respectively.*

Then (X, ω) is a Veech surface if and only if y_2 and $\frac{y_1}{x_1}$ are rational.

Proof. For a visualization see Figure 6.18.

Using Proposition 2.8 a translation surface is a Veech surface if and only if the set of wedge products of holonomy vectors of saddle connections

$$\{ \operatorname{hol}(\gamma_1) \wedge \operatorname{hol}(\gamma_2) \mid \gamma_1, \gamma_2 \text{ saddle connections of } X \}$$

is discrete in $\mathbb{R}$. Since a saddle connection has to connect two singularities, the most basic holonomy vectors are of the form x, y, $x+y$ or $(0,1)$. The more complicated holonomy vectors are $\mathbb{Z}$-linear combinations of those. Note that neither $2x$ nor $2y$ are holonomy vectors of saddle connections. Hence the holonomy vector of a saddle connection γ has to be of the form

$$\operatorname{hol}(\gamma) = m(x+y) + n(0,1) + ex + fy, \qquad m, n \in \mathbb{Z}, e, f \in \{0, 1\}.$$

Computing the wedge of the holonomy vectors of two saddle connections γ_1 and γ_2 with appropriate indices gives us

$$\begin{aligned}
\operatorname{hol}(\gamma_1) \wedge \operatorname{hol}(\gamma_2) &= \det\big(m_1(x+y) + e_1 x + f_1 y + (0, n_1), m_2(x+y) + e_2 x + f_2 y + (0, n_2)\big) \\
&= \det \begin{pmatrix} (m_1+e_1)x_1 + (m_1+f_1)y_1 & (m_2+e_2)x_1 + (m_2+f_2)y_1 \\ (m_1+f_1)y_2 + n_1 & (m_2+f_2)y_2 + n_2 \end{pmatrix} \\
&= (e_1 f_2 + e_1 m_2 - e_2 f_1 - e_2 m_1 - f_1 m_2 + f_2 m_1) x_1 y_2 \\
&+ (e_1 n_2 - e_2 n_1 + m_1 n_2 - m_2 n_1) x_1 + (f_1 n_2 - f_2 n_1 + m_1 n_2 - m_2 n_1) y_1.
\end{aligned}$$

On the one hand, by dividing by $x_1 \neq 0$ we see that for rational y_2, $\frac{y_1}{x_1} \in \mathbb{Q}$ the set of holonomy vecotrs is discrete, because it is of the form $\mathbb{Z} + q\mathbb{Z}$ for some rational number $q \in \mathbb{Q}$.

On the other hand, for $y_2 \neq 0$ we divide by $x_1 y_2$. The division by a number does not change wether a subset of $\mathbb{R}$ is dense or not. For $f_1 = n_1 = m_1 = m_2 = 0$ we compute

$$\frac{\operatorname{hol}(\gamma_1) \wedge \operatorname{hol}(\gamma_2)}{x_1 y_2} = e_1 f_2 + (e_1 n_2) \frac{1}{y_2} \in \mathbb{Z} + \frac{1}{y_2}\mathbb{Z}.$$

Hence the set

$$\left\{ \frac{\operatorname{hol}(\gamma_1) \wedge \operatorname{hol}(\gamma_2)}{x_1 y_2} \;\middle|\; f_1 = n_1 = m_1 = m_2 = 0 \right\} = \mathbb{Z} + \frac{1}{y_2}\mathbb{Z}$$

is dense in $\mathbb{R}$ if and only if $\frac{1}{y_2} \notin \mathbb{Q}$. Thus if X is a Veech surface we have $\frac{1}{y_2} \in \mathbb{Q}$. Assuming $e_1 = n_1 = m_1 = m_2 = 0$, we have

$$\frac{\operatorname{hol}(\gamma_1) \wedge \operatorname{hol}(\gamma_2)}{x_1 y_2} = -e_2 f_1 + (f_1 n_2) \frac{y_1}{x_1 y_2} \in \mathbb{Z} + \frac{y_1}{x_1 y_2}\mathbb{Z}.$$

The set
$$\left\{ \frac{\mathrm{hol}(\gamma_1) \wedge \mathrm{hol}(\gamma_2)}{x_1 y_2} \;\middle|\; e_1 = n_1 = m_1 = m_2 = 0 \right\} = \mathbb{Z} + \frac{y_1}{x_1 y_2}\mathbb{Z}$$
is dense in $\mathbb{R}$ if and only if $\frac{y_1}{x_1 y_2} \notin \mathbb{Q}$. Thus if X is a Veech surface we have $\frac{y_1}{x_1 y_2} \in \mathbb{Q}$ as well as $\frac{1}{y_2} \in \mathbb{Q}$, which leads to $\frac{x_1}{y_1} \in \mathbb{Q}$.

Finally, assume $y_2 = 0$. Then, after division by x_1, we have

$$\begin{aligned} \mathrm{hol}(\gamma_1) \wedge \mathrm{hol}(\gamma_2) &= (e_1 n_2 - e_2 n_1 + m_1 n_2 - m_2 n_1) + (f_1 n_2 - f_2 n_1 + m_1 n_2 - m_2 n_1)\frac{y_1}{x_1} \\ &\in \mathbb{Z} + \frac{y_1}{x_1}\mathbb{Z}. \end{aligned}$$

This gives us a dense subset of $\mathbb{R}$ if and only if $\frac{y_1}{x_1} \notin \mathbb{Q}$. Hence, if X is a Veech surface we have $\frac{y_1}{x_1} \in \mathbb{Q}$. □

Corollary 6.33. *Every Veech surface in $\mathcal{L}_2$ is an origami.*

Proof. We study the branch points of the covered torus.

Let X be a Veech surface in standard form. Then we have $y_2 = \frac{m}{n}$ and $\frac{y_1}{x_1} = \frac{p}{q}$ for some $m, n, p, q \in \mathbb{Z}$. The torus is given by $\mathbb{R}^2/\Gamma$ with $\Gamma = (0,1)^\top\mathbb{Z} + (2x_1 + 2y_1, 0)^\top\mathbb{Z}$ and representatives of the branch points are

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \text{ and } \begin{pmatrix} y_1 + x_1 \\ y_2 \end{pmatrix}.$$

Note that a vector (v, w) corresponds to the zero vector if and only if $w \in \mathbb{Z}$ and $v \in (x_1 + y_1)2\mathbb{Z}$. Observe that

$$2(x_1 + y_1)\mathbb{Z} = 2\left(x_1 + \frac{x_1 y_1}{x_1}\right)\mathbb{Z} = 2x_1\left(1 + \frac{p}{q}\right)\mathbb{Z} \ni 2x_1(p+q) \text{ and} \tag{6.1}$$
$$2(x_1 + y_1)\mathbb{Z} = 2\left(\frac{x_1 y_1}{y_1} + y_1\right)\mathbb{Z} = 2y_1\left(\frac{q}{p} + 1\right)\mathbb{Z} \ni 2y_1(q+p). \tag{6.2}$$

Hence after multiplying by n, the representatives of the branch points are

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} nx_1 \\ 0 \end{pmatrix}, \begin{pmatrix} ny_1 \\ m \end{pmatrix} \text{ and } \begin{pmatrix} n(y_1 + x_1) \\ m \end{pmatrix}.$$

After multiplying by $2(p+q)$ they are given by

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2x_1(p+q)n \\ 0 \end{pmatrix}, \begin{pmatrix} 2y_1(p+q)n \\ m \end{pmatrix} \text{ and } \begin{pmatrix} (2y_1 + 2x_1)n(p+q) \\ m \end{pmatrix}.$$

Because of Equations (6.1) and (6.2) all representatives are in Γ and hence by multiplying with $2n(p+q)$ we map all branch points to the origin.

The map $[2n(p+q)] \circ p \colon X \to E$ is a once-ramified covering and thus X is an origami. □

6.4 A subspace of codimension three

This section resembles the last section. By an additional relation of the branch points we obtain a subspace ΩH_3 of the Hurwitz space of translation surfaces ΩH_2. We describe this space in terms of polygons, which are degenerate. The degeneration is an implication of the symmetries the polygons have. These symmetries imply the existence of an additional translation giving us three equivalent definitions of the subspace ΩH_3. Combining these, we show that the subspace ΩH_3 has four connected components and that its image under the forgetful map is an affine invariant submanifold of dimension 2.

Definition 6.34. For a covering $p\colon X \to E$ in H_2, denote by $\bar{P}$, $\bar{Q}$, $-\bar{Q}$ and $-\bar{P}$ the branch points in E. We define the Hurwitz space

$$H_3 = \left\{ (p, X, E) \in H_2 \;\middle|\; \bar{P} - \bar{Q} = \bar{Q} - \bar{P} \right\}$$

of all coverings, where the point $\bar{P} - \bar{Q}$ is a 2-torsion point. Its image under the forgetful map

$$\mathcal{L}_3 = \mathcal{F}(H_3) = \{ X \in \mathcal{L}_2 \mid \text{ there exists } p, E, \text{ such that } (p, X, E) \in H_3 \}$$

contains all Riemann surfaces of genus 3 that admit a covering of degree 2 of an elliptic curve ramified over four points such that the sum of any two is a 2-torsion point.

We are interested in the Hurwitz space of translation surfaces belonging to H_3.

Definition 6.35. The Hurwitz space of translation coverings

$$\Omega H_3 = \left\{ (p, X, \omega, E, \eta) \in \Omega H_2 \;\middle|\; \bar{P} - \bar{Q} = \bar{Q} - \bar{P} \right\}$$

contains all translation surfaces in ΩH_2 that have the extra symmetry describing H_3. Its image under the forgetful map

$$\Omega\mathcal{L}_3 = \Omega\mathcal{F}(\Omega H_3)$$

contains all translation surfaces in the principal stratum $\mathcal{H}(1,1,1,1)$ admitting a translation covering of degree 2 of the torus ramified over four points such that the sum of any two branch points is a 2-torsion point.

Next we want to understand this space in terms of polygons. For a given covering $p\colon (X, \omega) \to (E, \eta)$ in ΩH_3, we describe the torus as a polygon. For the branch points $\bar{P}$, $\bar{Q}$, $-\bar{Q}$ and $-\bar{P}$, let us denote

$$u = \bar{P} - \bar{Q} = \bar{Q} - \bar{P}, \quad v = \bar{P} + \bar{Q} = -\bar{P} - \bar{Q} \quad \text{and} \quad w = u + v.$$

From these relations, the formulas

$$\begin{array}{llll}
\bar{P} + u = \bar{Q}, & \bar{Q} + u = \bar{P}, & -\bar{Q} + u = -\bar{P}, & -\bar{P} + u = -\bar{Q}, \\
\bar{P} + v = -\bar{Q}, & \bar{Q} + v = -\bar{P} & -\bar{Q} + v = \bar{P}, & -\bar{P} + v = \bar{Q}, \\
\bar{P} + w = -\bar{P}, & \bar{Q} + w = -\bar{Q} & -\bar{Q} + w = \bar{Q} \text{ and} & -\bar{P} + w = \bar{P}
\end{array}$$

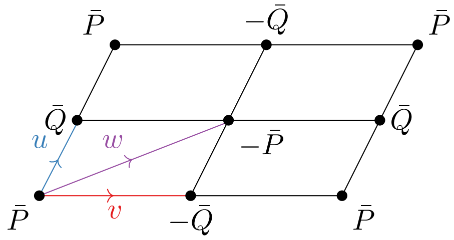


Figure 6.19: This torus belongs to a covering in ΩH_3. The edges u, u' and v are marked. By cutting and gluing we can transform this torus into the left one in Figure 6.20.

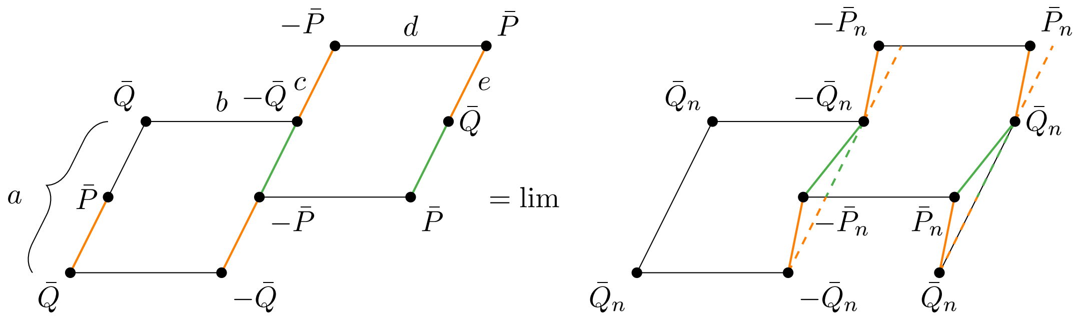


Figure 6.20: The right picture degenerates to the left one by letting the orange and green paths converge to the corresponding dashed paths. Cutting off the arising arm we get the left picture. The edges of the left polygon are labeled.

follow. Hence we draw the torus by starting in a point $\bar{P}$ and going along the vector u to reach $\bar{Q}$. Starting in $\bar{P}$ and going along the vector v, we reach $-\bar{Q}$. Along the vector w we reach $-\bar{P}$. The resulting torus is sketched in Figure 6.19. Via cutting and gluing this torus can be transformed into the left one in Figure 6.20. There we see how this torus arises via a degeneration of the right torus: The green and orange edges converge to the corresponding dashed edges. There remains an orange arm in the lower right corner which we cut off and glue to the left side. The colored edges in the left picture indicate this. Furthermore, in the left picture we see that the (oriented) edges of this polygon in $\mathcal{P}$ fulfill the relation $a = -2e$. This also follows from the formulas above by noting that $a = 2u$ and $e = -u$. So we define the set

$$\mathcal{P}_3 = \{\, (a, b, c, d, e) \in \mathcal{P} \mid b = d, c = e, a = -2e \,\}$$

of polygons which define such a torus.

Proposition 6.36. *We have a surjective map $\mathcal{P}_3 \times \{\, 1, 2, 3, 4 \,\} \to \Omega H_3$.*

Proof. Let $p\colon (X, \omega) \to (E, \eta)$ be a translation covering in ΩH_3. We have just seen that the torus E is represented by a polygon of the desired form. We can state explicitly four different gluings, see Figure 6.21, which yields four coverings of E of degree 2 with four ramification points. For now, ignore all the colors and the labels of the parallelograms, because Ψ is not yet defined.

We have seen before, compare Propositions 3.7 and 6.8, that there are exactly four translation coverings for each given set of covering data. Thus we have found all. $\square$

The gluing given in Figure 6.21 is no coincidence. It arises via the limit construction seen in Figure 6.6. In our case, as in Figure 6.20, the edges $-2e_n$ converge to the edge a.

A question we should have addressed earlier must be answered now: How do the maps φ and τ look like on a surface coming from a degenerate polygon? The short answer is given in Figure 6.22: The first row is a translation surface of monodromy type 1, the second one is of type 4. The arrows indicate which parallelogram is mapped to which by τ and the colors do the same for φ. Note that the maps τ and φ look the same on translation surfaces of type 1 and 2, and they do look the same on translation surfaces of type 3 and 4.

Let us start with the rotation τ: See Figure 6.10 for its original definition. The map τ is given as the rotation around the midpoint of an edge between Q and $-Q$ and is a lift of $[-1]$. We fix this lift by giving an explicit point of rotation. In Figure 6.22 we choose the red point in the left polygon. Then the map τ is rotates each parallelogram by π and maps it to the parallelogram indicated by the arrow. The same map is obtained in another way: We degenerate a translation surface in $\Omega\mathcal{L}_1$ to one in $\Omega\mathcal{L}_3$. The parallelograms in Figure 6.10 labeled by A, C, A' and C' degenerate to lines. The degeneration of τ is exactly the map described above.

The translation φ can be studied similarly: Recall its original definition in Figure 6.15 as a lift of the translation $t_{\bar{P}+\bar{Q}}$. It maps the point P to $-Q$ and the point Q to $-P$. We choose the lift of $t_{\bar{P}+\bar{Q}}$ such that in Figure 6.22 every parallelogram is mapped to a more saturated parallelogram of the same color. The same map is obtained in another way: We degenerate a translation surface in $\Omega\mathcal{L}_2$ to one in $\Omega\mathcal{L}_3$. Since the hexagon labeled by A in Figure 6.15 degenerates to a parallelogram, φ behaves as described above.

A closer study reveals that none of the properties of τ and φ change: The order and the relations of τ and φ are the same, for example for a translation surface of type 1 we have $\tau\varphi = \sigma\varphi\tau$, $\tau^2 = \mathrm{id}$ and $\varphi^2 = \sigma$.

In what follows, we use the polygon description of coverings in ΩH_3 to state their generic automorphism groups and to show that ΩH_3 has four connected components. Using the automorphism groups, we describe the space $\Omega\mathcal{L}_3$ as an affine invariant submanifold of dimension 2 in $\mathcal{H}(1,1,1,1)$. We start by giving a description of the space ΩH_3 in terms of an additional automorphism, which is the lift of a translation.

Lemma 6.37. *Let $p\colon (X,\omega) \to (E,\eta)$ be a translation covering in ΩH_3 with branch points $\bar{P}$, $\bar{Q}$, $-\bar{P}$ and $-\bar{Q}$ in E. The translation*

$$t_{\bar{P}-\bar{Q}}\colon E \to E, \qquad x \mapsto x + \bar{P} - \bar{Q}$$

can be lifted to a translation on X.

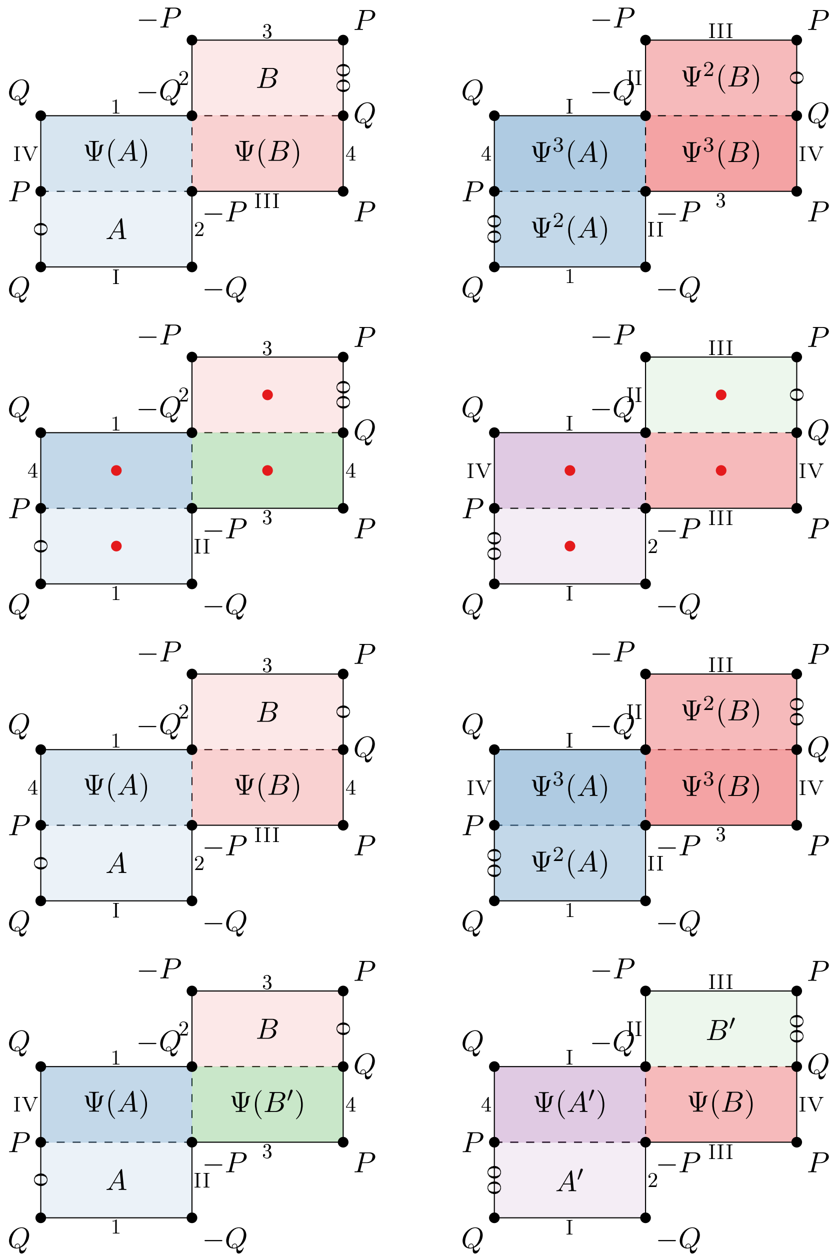


Figure 6.21: In every row a translation covering in ΩH_3 of type 1, 2, 3 and 4, respectively, is sketched. The colored squares indicate the map Ψ: A parallelogram is mapped to a more saturated parallelogram of the same color. In the second picture the fixed points of $\Psi \circ \tau$ are indicated in red.

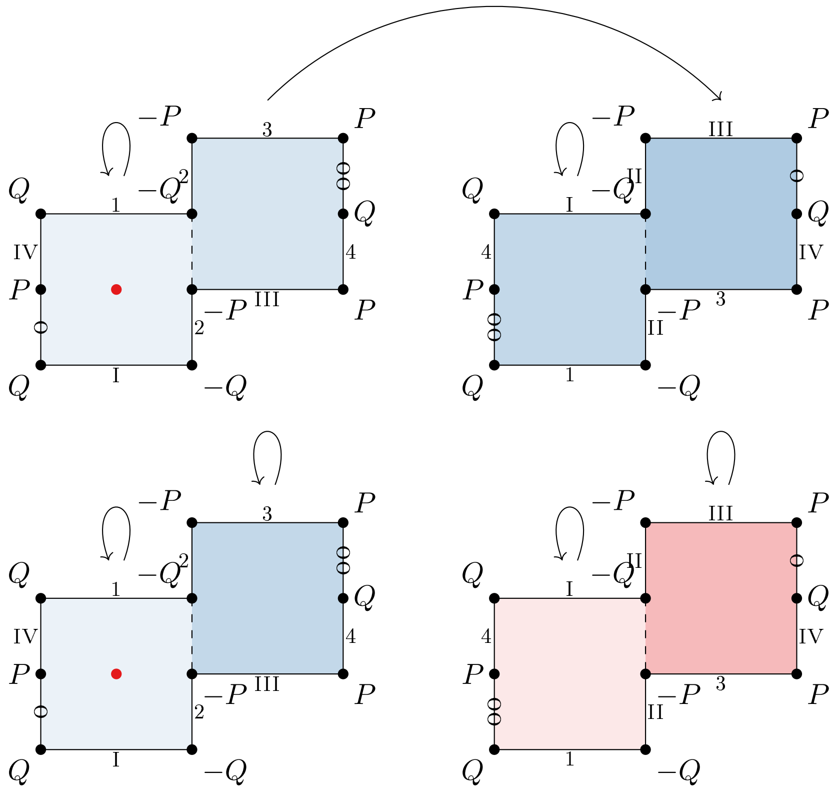


Figure 6.22: In the first row, the maps τ and φ are sketched on a degenerate translation surface of type 1 (they do the same for type 2). In the second row, the maps τ and φ are sketched on a degenerate translation surface of type 4 (they do the same for type 3).

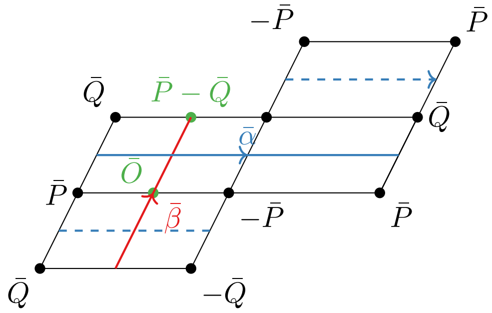


Figure 6.23: The base point of this elliptic curve is $\bar{O}$. Two loops are sketched. The translation $t_{\bar{P}-\bar{Q}}$ leaves the "vertical" loop $\bar{\beta}$ invariant and maps the "horizontal" loop $\bar{\alpha}$ to the dashed blue loop.

Proof. Given the translation $t_{\bar{P}-\bar{Q}}\colon E \to E$, from $\bar{P}-\bar{Q} = \bar{Q}-\bar{P}$ the identities

$$\begin{aligned} t_{\bar{P}-\bar{Q}}(\bar{P}) &= \bar{P}+\bar{P}-\bar{Q} = \bar{P}+\bar{Q}-\bar{P} = \bar{Q}, \quad & t_{\bar{P}-\bar{Q}}(\bar{Q}) &= \bar{P}, \\ t_{\bar{P}-\bar{Q}}(-\bar{Q}) &= -\bar{P} \quad \text{and} & t_{\bar{P}-\bar{Q}}(-\bar{P}) &= -\bar{Q} \end{aligned}$$

follow. Hence the translation maps branch points to branch points. We need to show that $t_{\bar{P}-\bar{Q}*}p_*\pi_1(X^*) \subseteq p_*\pi_1(X^*)$ or, equivalently, that the translation $t_{\bar{P}-\bar{Q}}$ acts on the kernel of the monodromy map $\mu\colon \pi_1(E^*) \to S_2$. Obviously, the loops around the branch points are mapped to loops around branch points and their monodromy is not changed. We explicitly describe a basis of the fundamental group whose monodromy is not changed by the translation $t_{\bar{P}-\bar{Q}}$.

Let $\bar{O}$ be a fixed point of $[-1]$, say between $\bar{P}$ and $-\bar{P}$ as sketched in Figure 6.23. Choose $\bar{\beta}$ to be the "vertical" loop going through $\bar{O}$ and $\bar{P}-\bar{Q} = t_{\bar{P}-\bar{Q}}(\bar{O})$. Hence $t_{\bar{P}-\bar{Q}}(\bar{\beta}) = \bar{\beta}$ and its monodromy is not changed. Choose $\bar{\alpha}$ to be the "horizontal" loop as drawn in Figure 6.23. It is mapped to the dashed blue loop $\bar{\alpha} \circ \ell_{-\bar{Q}} \circ \ell_{\bar{Q}}$. Thus $\mu(t_{\bar{P}-\bar{Q}}(\bar{\alpha})) = \mu(\bar{\alpha})(1\,2)^2 = \mu(\bar{\alpha})$, the monodromy of $\bar{\alpha}$ is not changed by $t_{\bar{P}-\bar{Q}}$. $\square$

Proposition 6.38. *The Hurwitz space of translation coverings*

$$\Omega H_3 = \Big\{\ (p, X, \omega, E, \eta) \in \Omega H_2 \ \Big|\ \textit{there exists } \Psi \in \operatorname{Aut}(X),\ \Psi(\pm P) = \pm Q, \\ \Psi(\pm Q) = \pm P,\ D\Psi = I,\ \Psi \circ \sigma = \sigma \circ \Psi\ \Big\}$$

can be identified with the space of all translation coverings that have an extra translation Ψ. This automorphism Ψ commutes with the deck transformation σ and interchanges the ramification points as described above.

Proof. Let p be a translation covering in the right-hand set and let $\Psi\colon X \to X$ be an automorphism as described above. Since Ψ and σ commute, Ψ descends to a translation $t\colon E \to E$ of the torus given by $x \mapsto b + x$. This translation fulfills

$$t(\bar{P}) = b + \bar{P} = \bar{Q} \quad \text{and} \quad t(\bar{Q}) = b + \bar{Q} = \bar{P}$$

and hence $b = \bar{Q} - \bar{P} = \bar{P} - \bar{Q}$. This relation shows that the translation covering p is in ΩH_3.

Now let $p \in \Omega H_3$ be a translation covering. We have $\bar{P} - \bar{Q} = \bar{Q} - \bar{P}$ and then, by Lemma 6.37, there exists a translation $t_{\bar{P}-\bar{Q}}$ which can be lifted to a map with the desired properties. Hence p is in the right-hand side. $\square$

More elementarily, we describe the map Ψ explicitly. The base point $\bar{O}$ of the torus is the green point depicted in Figure 6.23. Via the translation $t_{\bar{P}-\bar{Q}}$ it is mapped to the other green point $t_{\bar{P}-\bar{Q}}(\bar{O})$. Similarly, $t^2_{\bar{P}-\bar{Q}}(\bar{O}) = \bar{O}$. We lift this map to a covering in ΩH_3. We label the lower left parallelogram of the left polygon by A as in Figure 6.21. By choosing whether the parallelogram A is mapped to the parallelogram on top of it, or to the corresponding parallelogram in the right polygon, we fix the lift Ψ of $t_{\bar{P}-\bar{Q}}$. To be consistent with our previous work, we choose the former. In the first row of Figure 6.21, we observe that the edge labeled by I on the left polygon is mapped to the dashed edge above of it. Hence, the edge labeled by I on the right polygon is mapped onto the same dashed line, thus the parallelogram below of it is $\Psi^{-1}(A)$. By similar arguments, one deduces that the map looks as given in Figure 6.21. The first row describes a translation covering of type 1, the second of type 2 and so forth. The map Ψ maps each parallelogram to a more saturated parallelogram of the same color. We can deduce several interesting facts about Ψ from these pictures, which we use in the next proposition to describe the generic automorphism group of each translation covering or translation surface.

Proposition 6.39. *The automorphism group of a generic translation covering in ΩH_3 or a generic translation surface in $\Omega\mathcal{L}_3$ is of order 16 and depends on its monodromy. It is either the central product of the dihedral 8-group and the 4-cyclic group, $D_8 \times_Z C_4$, or the direct product of the dihedral 8-group and the 2-cyclic group, $D_8 \times C_2$. In the latter case, the surface is hyperelliptic.*

Proof. Let (X, ω) be a translation surface in $\Omega\mathcal{L}_3$ of type 1. In the first picture of Figure 6.21, we note that $\Psi^2 = \sigma$ and Ψ is of order 4. Using both, Figures 6.21 and 6.22, we check the following: The map $\Psi\varphi$ maps the parallelogram labeled by A to the one labeled by $\Psi^2(B)$. But the map $\varphi\Psi$ maps the parallelogram labeled by A to the one labeled by B. Hence they differ by σ and we have $\Psi\varphi = \varphi\Psi\sigma$. Similarly, we confirm $\Psi\tau = \tau\Psi\sigma$. From Proposition 6.26 we recall the relation $\tau\varphi = \varphi\tau\sigma$ and that $\varphi^2 = \sigma$. Note that σ is in the center of the automorphism group.

Comparing with Chapter 4 we want to show that

$$\begin{aligned}\mathrm{Aut}(X, \omega) &= \langle a, x, y | a^4 = x^2 = 1,\ a^2 = y^2,\ xax^{-1} = a^{-1},\ ya = ay,\ xy = xy\rangle \\ &= D_8 \times_Z C_4.\end{aligned}$$

We choose $a = \varphi\Psi$, $x = \varphi\tau$, $y = \tau\varphi\Psi$ and compute, by using the relations described above and $\sigma^2 = \mathrm{id}$, that all relations are fulfilled:

$$\begin{aligned}
a^2 &= \varphi\Psi\varphi\Psi = \varphi\varphi\Psi\Psi\sigma = \sigma^3 = \sigma \quad \text{and hence} \\
a^4 &= \mathrm{id}\,. \quad \text{Furthermore} \\
x^2 &= \varphi\tau\varphi\tau = \varphi\varphi\tau\tau\sigma = \sigma^2 = \mathrm{id}, \\
y^2 &= \tau\varphi\Psi\tau\varphi\Psi = \tau^2\varphi\Psi\varphi\Psi\sigma^2 = \varphi\Psi\varphi\Psi = a^2, \\
xax^{-1} &= \varphi\tau\varphi\Psi\varphi\tau = \varphi\varphi\tau\Psi\varphi\tau\sigma = \tau\Psi\varphi\tau = \Psi\varphi\tau\tau\sigma^2 = \Psi\varphi = \sigma\varphi\Psi = a^{-1}, \\
ay &= \varphi\Psi\tau\varphi\Psi = \tau\varphi\Psi\varphi\Psi\sigma^2 = ya \quad \text{and finally} \\
xy &= \varphi\tau\tau\varphi\Psi = \varphi\tau\Psi\tau\varphi\sigma^2 = \varphi\tau\tau\varphi\Psi\sigma^4 = yx.
\end{aligned}$$

Similarly, for a translation surface of type 2 the claim follows by choosing

$$a = \varphi, \quad x = \tau, \quad y = \tau\Psi$$

and by noting the relations $\Psi\tau = \tau\Psi$ and $\Psi\varphi = \varphi\Psi\sigma$. For a translation covering of type 3 we choose

$$a = \Psi, \quad x = \tau, \quad y = \tau\varphi$$

with the relations $\Psi\tau = \tau\Psi\sigma$ and $\Psi\varphi = \varphi\Psi\sigma$. Finally, for a translation covering of type 4 we choose

$$a = \varphi\Psi, \quad x = \varphi, \quad y = \tau$$

with the relations $\Psi\tau = \tau\Psi$ and $\Psi\varphi = \varphi\Psi\sigma$. For any translation surface $(X,\omega) \in \Omega\mathcal{L}_3$ of type 2, 3 or 4 with generators chosen as above we then get

$$\begin{aligned}
\mathrm{Aut}(X,\omega) &= \langle\, a, x, y \;\mid a^4 = x^2 = y^2 = 1,\ xax = a^{-1},\ ya = ay,\ xy = yx\rangle \\
&= D_8 \times C_2.
\end{aligned}$$

□

Translation surfaces of type 3 and 4 are hyperelliptic by Propositions 6.9 and 6.26. Translation surfaces of type 2 are hyperelliptic, because the map $\Psi\tau$ is of order 2 and has eight fixed points. These fixed points are the red points in the second row in Figure 6.21.

Corollary 6.40. *The equation of a surface in $\mathcal{L}_3$ depends on the automorphism group of it. If it is $D_8 \times_Z C_4$, the central product of the dihedral group of order 8 and the cyclic group of order 4, the generic surface is given by*

$$y^4 = x(x-1)(x-\lambda), \qquad \lambda \in \mathbb{C}.$$

If, on the other hand, the automorphism group is $D_8 \times C_2$, the direct product of the dihedral group of order 8 and the cyclic group of order 2, the generic surface is hyperelliptic and is given by

$$y^2 = x^8 + ax^4 + 1, \qquad a \in \mathbb{C}.$$

In both cases the dimension of the locus of those surfaces is 1.

Proof. Using Proposition 6.39 and comparing the possible automorphism groups with the Tables 3.1 and 3.2, we get these equations as well as the dimensions of the loci. □

From this follows that the dimension of ΩH_3 has to be 2. We will prove this independently in Proposition 6.43. Furthermore, the first equation is also shown by Herrlich and Schmithüsen [HS08], since it is the Teichmüller curve of the Wollmilchsau.

Corollary 6.41. *The Hurwitz space ΩH_3 of translation surfaces has four connected components. They can be distinguished by the number of fixed points of τ, $\varphi\tau$ and $\Psi\tau$.*

Note that each translation surface belonging to a covering in ΩH_3 of type 2, 3 or 4 is hyperelliptic.

Proof. There cannot be a continuous path between surfaces with different automorphism groups, so there is no path from a translation surface of type 1 to any other. By Proposition 6.28 there are no paths between surfaces of type 2, 3 and 4. Hence we have four connected components. □

In Corollary 6.52 we will see that $\Omega\mathcal{L}_3$ has only two connected components. But the previous corollary gives us that H_3 has and ΩH_3 each have four connected components. Hence the forgetful map merges the three hyperelliptic components of H_3 into a single connected component of $\mathcal{L}_3$. Furthermore, combining this corollary with Propositions 6.9, 6.19 and 6.28 we get our desired result:

Theorem 2. *The space ΩH is connected. The spaces ΩH_i have $i+1$ connected components for $i = 1, 2, 3$. These components can be distinguished by the monodromy of the covering or by the number of fixed points of τ, $\varphi\tau$ and $\Psi\tau$, respectively.*

As in the previous sections, we show that the subspace $H_3 \subseteq H$ is a submanifold. This enables us to prove that ΩH_3 is an affine invariant submanifold by using the restriction of the immersion $\Omega\mathcal{F}\colon \Omega H \to \Omega\mathcal{L}$.

Lemma 6.42. *The subspace $H_3 \subseteq H$ is a submanifold.*

Proof. The proof works as the proof of Lemmas 6.15 and 6.30. Hence we only show how we must alter a chart φ to make this work.

Let p be a covering and let $e_1 = \bar{P}$, $e_2 = \bar{Q}$, $e_3 = -\bar{Q}$ and $e_4 = -\bar{P}$ be the branch points in $E = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$. We define the chart φ such that

$$\varphi(p) = (\tau, \bar{e}_1 + \bar{e}_4, \bar{e}_2 + \bar{e}_3, \bar{e}_1 + \bar{e}_2 - \bar{e}_3 - \bar{e}_4, \bar{e}_1 - \bar{e}_2 + \bar{e}_3 - \bar{e}_4).$$

This map is still bijective, because the linear map

$$(\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4) \mapsto (\bar{e}_1 + \bar{e}_4, \bar{e}_2 + \bar{e}_3, \bar{e}_1 + \bar{e}_2 - \bar{e}_3 - \bar{e}_4, \bar{e}_! - \bar{e}_2 + \bar{e}_3 - \bar{e}_4)$$

corresponds to the regular matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

In H_3, the branch points fulfill the relations $e_1+e_3 = \bar{O}$, $e_2+e_4 = \bar{O}$ and $e_1-e_2+e_3-e_4 = \bar{O}$. Thus for a covering $p \in H_3$ the chart φ is given by

$$\varphi(p) = (\tau, 0, 0, 0, 0).$$

Hence locally H_3 is the intersection of $\mathbb{C}^5$ with a 1-dimensional linear subspace showing that H_3 is a submanifold of H. $\square$

Proposition 6.43. *The space $\Omega\mathcal{L}_3 \subseteq \Omega\mathcal{L}$ is an affine invariant submanifold of the principal stratum $\mathcal{H}(1,1,1,1)$ of dimension* 2.

Proof. By Lemma 6.42, $H_3 \subseteq H_2$ is a submanifold and so is $\Omega H_3 \subseteq \Omega H_2$. We restrict the immersion $\Omega\mathcal{F}$ to this subspace and thus the only thing we need to show is that the image of some translation covering under the forgetful map $\Omega\mathcal{F}$ of ΩH_3 is given by linear equations in period coordinates of the right dimension. In a small enough neighborhood of a translation covering the immersion $\Omega\mathcal{F}$ is injective. Thus, when working with a translation surface $(X, \omega) \in \Omega\mathcal{L}_3$, we may assume that it is equipped with a unique translation covering denoted by $p\colon (X, \omega) \to (E, \eta)$.

Recall the notation introduced in Propositions 6.16 and 6.31: The loops a and b project to a basis $\bar{a}$ and $\bar{b}$ of the absolute homology group of the torus. The arcs c_{PQ}, c_{Q-P} and c_{-P-Q} and the elements in the kernel of p_* extend this to a basis of the relative homology group $H_1(X, \Sigma, \mathbb{Z})$. Denote by $c_{\bar{P}\bar{Q}}$, $c_{\bar{Q}-\bar{P}}$ and $c_{-\bar{P}-\bar{Q}}$ the projections of the arcs via p to the torus. Integrating ω over those paths gives the complex numbers A, B, C_1, C_2 and C_3 respectively. In Figure 2.3 the translation $t_{\bar{P}-\bar{Q}}$ acts by a vertical translation mapping $\bar{P}$ to $\bar{Q}$. Hence using the relations computed in Example 2.13 we establish

$$\begin{aligned}
&t_{\bar{P}-\bar{Q}*}\bar{a} = \bar{a}, && t_{\bar{P}-\bar{Q}*}\bar{b} = \bar{b},\\
&t_{\bar{P}-\bar{Q}*}c_{\bar{P}\bar{Q}} = c'_{\bar{Q}\bar{P}} = \bar{b} - c_{\bar{P}\bar{Q}}, && t_{\bar{P}-\bar{Q}*}c_{\bar{Q}-\bar{P}} = -c_{-\bar{Q}\bar{P}} = c_{\bar{P}\bar{Q}} + c_{\bar{Q}-\bar{P}} + c_{-\bar{P}-\bar{Q}},\\
&t_{\bar{P}-\bar{Q}*}c_{-\bar{P}-\bar{Q}} = c'_{-\bar{Q}-\bar{P}} = -\bar{b} - c_{-\bar{P}-\bar{Q}}.
\end{aligned}$$

Integration yields the equations

$$\begin{aligned}
A &= \int_a \omega = \int_a \Psi^*\omega = \int_{t_{\bar{P}-\bar{Q}*}\bar{a}} \eta = \int_a \omega = A,\\
B &= \int_b \omega = \int_{\Psi_* b} \omega = B,\\
C_1 &= \int_{c_{PQ}} \omega = \int_{\Psi_* c_{PQ}} \omega = \int_b \omega - \int_{c_{PQ}} \omega = B - C_1,\\
C_2 &= \int_{c_{Q-P}} \omega = \int_{\Psi_* c_{Q-P}} \omega = \int_{c_{PQ}} \omega + \int_{c_{Q-P}} \omega + \int_{c_{-P-Q}} \omega
\end{aligned}$$

$= C_1 + C_2 + C_3$ and

$$C_3 = \int\limits_{c_{-P-Q}} \omega = \int\limits_{\Psi_* c_{-P-Q}} \omega = -\int\limits_{b} \omega - \int\limits_{c_{-P-Q}} \omega = -B - C_3.$$

This gives us the equations $B - 2C_1 = 0$, $C_1 + C_3 = 0$ and $B + 2C_3 = 0$. Plugging in $C_1 = -C_3$ we see that the first and the last equation are the same. By using the automorphism φ we retrieve the equations given in $\Omega\mathcal{L}_2$, namely $C_1 + C_3 = 0$ and $2C_2 = A$. Thus we have one restriction more as in $\Omega\mathcal{L}_2$. In period coordinates this extra restriction describes a hyperplane in $\mathbb{C}^3$. This shows that $\Omega\mathcal{L}_3$ is an affine invariant submanifold of dimension at most 2.

Its dimension is at least 2 because we are free to choose a torus only. Having chosen a torus, both points $\bar{P} + \bar{Q}$ and $\bar{P} - \bar{Q}$ are 2-torsion points. There are just four of them. Having chosen those the sum, $(\bar{P} + \bar{Q}) + (\bar{P} - \bar{Q}) = 2\bar{P}$ gives us the point $\bar{P}$ and then the others follow as well. $\square$

This especially means that each connected component of $\Omega\mathcal{L}_3$ is a closed $\mathrm{GL}_2^+(\mathbb{R})$-orbit of a single translation surface. Furthermore, every translation surface in $\Omega\mathcal{L}_3$ is a Veech surface, thus its Veech group is a lattice in $\mathrm{SL}_2(\mathbb{R})$.

Combining this proposition with Propositions 6.5, 6.16 and 6.31 we get our desired result:

Theorem 1. *There exists an descending chain of affine invariant submanifolds $\Omega\mathcal{L}_i$ of $\Omega\mathcal{L}$ of every possible dimension $5-i$ for $i = 1, 2, 3$, each given by translations or rotations.*

Let us summarize briefly what happened so far: We constructed the Hurwitz space of translation coverings ΩH. We found a sequence of subspaces $\Omega H \supseteq \Omega H_1 \supseteq \Omega H_2 \supseteq \Omega H_3$ of decreasing dimensions, each given by translations and rotations. We computed the connected components of every of these spaces as stated in Theorem 2. We showed that these space give rise to affine invariant submanifolds and hence proved Theorem 1. In conclusion, we described explicitly orbit closures of every possible dimension in ΩH. In particular, we found closed $\mathrm{GL}_2^+(\mathbb{R})$-orbits. In the next section, we compute the Veech groups of a translation surface in each of these orbits and their Teichmüller curves.

6.5 The Wollmilchsau and its siblings

The Wollmilchsau is discussed by Herrlich and Schmithüsen [HS07a; HS08] and summarized in Chapter 5. They show several of its remarkable properties. It is given by the polygon

$$\left(\begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \in \mathcal{P}_3$$

glued by the first gluing rule and thus of type 1. Hence it is a translation surface in the non-hyperelliptic connected component of ΩH_3. It is depicted in Figure 5.1.

Herrlich and Schmithüsen compute its automorphism group, its equation, its Veech group and show that there are infinitely many other origamis whose Teichmüller curves intersect the one of the Wollmilchsau. All these Teichmüller curves form a dense subset of the non-hyperelliptic component of the Hurwitz space H_1.

We call the translation surfaces given by the same polygon as the Wollmilchsau, but of different type, *siblings* of the Wollmilchsau. We compute their Veech groups with the same methods used by Herrlich and Schmithüsen [HS07a]. Recall that any translation surface in $\Omega\mathcal{L}_3$ is in a $\mathrm{GL}_2^+(\mathbb{R})$-orbit of the Wollmilchsau or one of its siblings. Hence its Veech group is conjugated to one of the Veech groups we will calculate.

To compute the Veech group we need some preparations. Most importantly, recall Proposition 2.12: Given an origami $q\colon X \to E$, denote by $q\colon X^* \to E^*$ its unramified correspondent. This yields the inclusion $q_*\pi_1(X^*) \subseteq \pi_1(E^*) = F_2 = \langle x, y\rangle$. We define $\beta\colon \mathrm{Aut}(F_2) \to \mathrm{Out}(F_2) = \mathrm{GL}_2(\mathbb{Z})$ to be the natural projection, which is explicitly given by

$$\beta(\gamma) = \begin{pmatrix} \#_x(\gamma(x)) & \#_x(\gamma(y)) \\ \#_y(\gamma(x)) & \#_y(\gamma(y)) \end{pmatrix}, \tag{6.3}$$

where $\#_x(w)$ and $\#_y(w)$ denote the number of x respectively y appearing in the word w. Here x^{-1} counts as -1. Let us denote by $\mathrm{Aut}^+(F_2) = \beta^{-1}(\mathrm{SL}_2(\mathbb{Z}))$ the group of orientation preserving automorphisms of the free group F_2 and define the stabilizer of U by $\mathrm{Stab}(U) = \left\{ \gamma \in \mathrm{Aut}^+(F_2) \mid \gamma(U) = U) \right\}$. Then Proposition 2.12 states that

$$\Gamma(X, \omega) = \beta(\mathrm{Stab}(U)).$$

Hence we can compute the Veech group of an origami by computing the stabilizer of the fundamental group.

Let us recall our setting. We are interested in the Veech groups of all translation surfaces in $\Omega\mathcal{L}_3$. Since the Hurwitz space of translation coverings ΩH_3 is an affine invariant submanifold of dimension 2, each of its connected components is a closed $\mathrm{GL}_2^+(\mathbb{R})$-orbit. Hence we have to pick a translation covering in each connected component of ΩH_3 and regard the corresponding translation surface in $\Omega\mathcal{L}_3$. The Veech groups of two different translation surfaces in one orbit are conjugated, so we may as well choose the Wollmilchsau and its siblings.

Let $p\colon X \to E$ in ΩH_3 be a covering of the Wollmilchsau or one of its siblings. The torus is given by $E^2 = \mathbb{C}/(2\mathbb{Z} + 2\mathrm{i}\mathbb{Z})$ and consists of four squares. By translating the torus we can assume that

$$\bar{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; \bar{Q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; -\bar{P} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } -\bar{Q} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus the multiplication $[2]\colon E^2 \to E^2$ by 2 on the torus maps all branch points to the origin. Note that this does not contradict our requirement that the branch points are not 2-torsion points, because we look at the translated torus. Nonetheless, the map

$$q = [2] \circ p\colon X \to E^2 \to E$$

is a covering ramified over one point and thus an origami. Denoting the punctured surfaces by a star $*$, we have the unramified covering $q\colon X^* \to E^{2*} \to E^*$. Furthermore, for $U = q_*\pi_1(X^*)$ and $K_2 = [2]_*\pi_1(E^{2*})$, by covering theory, we have the inclusions

$$U = q_*\pi_1(X^*) \subseteq K_2 = [2]_*\pi_1(E^{2*}) \subseteq \pi_1(E^*) = F_2.$$

Let x and y be the generators of F_2. As a subgroup of the free group F_2, the fundamental group K_2 of the four-times punctured torus is generated by the loops x^2, y^2 and the loops around the four branch points. Define $\ell_{m,n} = x^m y^n [x, y] y^{-n} x^{-m}$ for $m, n \in \mathbb{Z}$, then

$$K_2 = \langle x^2, y^2, \ell_{0,0}, \ell_{1,0}, \ell_{0,1} \rangle$$

is a free group of rank 5. The loop $\ell_{1,1} \in K_2$ satisfies the relation $[x, y] = \ell_{0,0}\ell_{1,0}\ell_{0,1}\ell_{1,1}$.

We need a description of the fundamental groups of the Wollmilchsau and its siblings and the stabilizers of their fundamental groups. We proceed in two steps: Firstly, we compute a group G, containing the stabilizer $\mathrm{Stab}(U)$, and the index $[G : \mathrm{Stab}(U)]$ of G in the stabilizer. Secondly, we give a subgroup of $H \subseteq \mathrm{Stab}(U)$ contained in the stabilizer with index $[G : H] = [G : \mathrm{Stab}(U)]$. Thus $H = \mathrm{Stab}(U)$ and this yields the Veech group.

To calculate U, we use that $p_*\pi_1(X^*)$ is the kernel of the monodromy $\mu\colon \pi_1(E^{2*}) \to S_2$. The monodromy of the loops $\ell_{m,n}$ has to be non-trivial, because they are loops around the branch points. Thus there are only four possible monodromy maps μ given by

$$\begin{aligned}
&\mu_1\colon x^2 \mapsto (1\,2), && y^2 \mapsto (1\,2),\\
&\mu_2\colon x^2 \mapsto \mathrm{id}, && y^2 \mapsto \mathrm{id},\\
&\mu_3\colon x^2 \mapsto \mathrm{id}, && y^2 \mapsto (1\,2) \text{ and}\\
&\mu_4\colon x^2 \mapsto (1\,2), && y^2 \mapsto \mathrm{id}\,.
\end{aligned}$$

We call their kernels U_1, U_2, U_3 and U_4, respectively. Note that the kernel U_i corresponds to the translation surfaces in the i-th row sketched in Figures 6.4, 6.10, 6.15 and 6.21. Furthermore, the U_i coincide with those defined in Lemma 6.18. Hence if the kernel of a monodromy map is U_i, the corresponding translation covering is of type i. We denote the Wollmilchsau and its siblings by $X_i^* = \mathbb{H}/U_i$ for $i = 1, 2, 3, 4$, where we identify U_i with the deck transformation group of the universal covering. When it is not important which one we mean, we just write $X^* = \mathbb{H}/U$.

Let $\mathbb{H} \to X^*$ be a universal covering. Then we can lift the translation structure on X^* to one on $\mathbb{H}$. With respect to this translation structure we define the group $\mathrm{Aff}^+(\mathbb{H})$ of orientation preserving affine maps on the upper half plane. Since $\mathbb{H}$ is simply connected, every affine map of X^*, E^{2*} or E^* can be lifted to $\mathbb{H}$. In Lemma 6.46 we show that every affine map of $\mathbb{H}$ descends to an affine map on E^{2*}. Hence every affine map on X^* descends to an affine map on E^{2*} by first lifting it to $\mathbb{H}$ and then projecting it down onto E^{2*}. The interesting question is: Which affine maps on E^{2*} can be lifted to X^*? We can extend the affine map $E^{2*} \to E^{2*}$ to an affine diffeomorphism $E^2 \to E^2$. A necessary condition for this map to have a lift is that it leaves the branch points invariant. Thus it is reasonable to define the group

$$G^\star = \left\{ \bar f \in \mathrm{Diff}^+(E^2) \;\middle|\; \tilde f(z) = Az + e,\ A \in \mathrm{SL}_2(\mathbb{R}),\ e \in \mathbb{R}^2,\ \bar f(\bar\Sigma) = \bar\Sigma \right\}$$

of those orientation preserving diffeomorphisms of the torus that leave the set of branch points invariant and whose lifts to the $\mathbb{C}$ are affine. Note that the derivative of a map in $G^\star$ is in $\mathrm{SL}_2(\mathbb{R})$, since the maps are orientation and volume preserving automorphisms.

Lemma 6.44. *Fix a universal covering $\pi\colon \mathbb{C} \to E^2$. Let $\tilde\Sigma = \pi^{-1}(\bar\Sigma)$ be the preimage of $\bar\Sigma$. The group*

$$G^\star = \left\{ \bar f \in \mathrm{Diff}^+(E^2) \;\middle|\; \tilde f(z) = Az + e,\ A \in \mathrm{SL}_2(\mathbb{Z}),\ e \in \tilde\Sigma \right\}$$

can be described by all affine maps whose lift has linear part in $\mathrm{SL}_2(\mathbb{Z})$ and translation part in $\tilde\Sigma$.

Proof. Let $\bar f \in G^\star$ be such that $\tilde f(z) = Az + e$. We write shortly $\bar f(\bar P)$ for $\pi(\tilde f(\tilde P))$. For $\tilde P = (0,0)^\top$ we compute $\bar f(\bar P) = \pi(e)$ and hence $e \in \tilde\Sigma$. Furthermore, $\bar f(\bar Q) = \pi(\tilde f(\tilde Q)) = \pi(A\tilde Q + e) \in \bar\Sigma$ if and only if $\pi(A\tilde Q) \in \bar\Sigma$. We compute

$$\pi(A\tilde Q) = \pi\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} 1+2m \\ 2n\end{pmatrix}\right) = \pi\left(\begin{pmatrix} a+2ma+2nb \\ c+2mc+2nd\end{pmatrix}\right) = \begin{pmatrix} a \\ c\end{pmatrix} + 2\mathbb{Z}^2.$$

Thus $\bar f(\bar Q) \in \bar\Sigma$ if and only if $a, c \in \mathbb{Z}$. Similarly, $\bar f(-\bar Q) \in \bar\Sigma$ if and only if $b, d \in \mathbb{Z}$. $\square$

Since we want to compare a group of affine maps with a group of automorphisms of F_2, we need a tool which identifies them. For this purpose, we define the map

$$\star\colon\ \mathrm{Aff}^+(\mathbb{H}) \to \mathrm{Aut}^+(F_2), \quad f \mapsto (f_\star\colon \sigma \mapsto f \circ \sigma \circ f^{-1}).$$

Here we identify $\pi_1(E^*) = F_2 = \mathrm{Deck}(\mathbb{H}/E^*)$: For the following notation, see Figure 6.24. Let $e_0 \in E$ be the base point and denote by $p_0\colon \mathbb{H} \to E^*$ a universal covering. Let $z_0 \in \mathbb{H}$ be such that $p_0(z_0) = e_0$. Then every loop $\ell \in \pi_1(E^*) = F_2$ can be lifted to a loop $\tilde\ell$ such that $\tilde\ell(0) = z_0$. We define the deck transformation ℓ_D in z_0 to be $\ell_D(z_0) = \tilde\ell(1)$. If $z \in \mathbb{H}$ with $p_0(z) = e \neq e_0$, we choose a path α from z to z_0. We lift $-p_0(\alpha)$ to a path β with $\beta(0) = \tilde\gamma(1)$ and define $\ell_D(z) = \beta(1)$. Note that $p_0(\ell_D(z)) = e$. One can check that the map ℓ_D is a deck transformation.

Lemma 6.45 (Schmithüsen [Sch04]). *The map $\star\colon\ \mathrm{Aff}^+(\mathbb{H}) \to \mathrm{Aut}^+(F_2)$ is an isomorphism and the diagram*

$$\begin{tikzcd} \mathrm{Aff}^+(\mathbb{H}) \arrow[r, "D"] \arrow[d, "\star"] & \mathrm{SL}_2(\mathbb{Z}) \\ \mathrm{Aut}^+(F_2) \arrow[ur, "\beta"] & \end{tikzcd}$$

commutes. Furthermore, for a subgroup $H \subseteq F_2$ an affine automorphism $\tilde f \in \mathrm{Aff}^+(\mathbb{H})$ descends to $\mathbb{H}/H$ if and only if $\tilde f_\star(H) = H$.

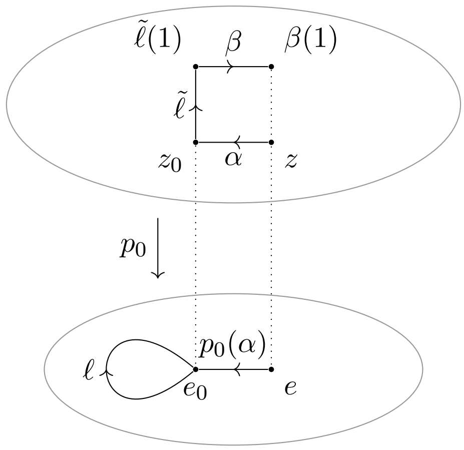


Figure 6.24: The loop ℓ gives rise to a deck transformation of the universal covering p_0, which maps z to $\beta(1)$.

From this lemma we deduce that, as we claimed before, every affine diffeomorphism of $\mathbb{H}$ descends to one of E^2 and compute the image of $G^\star$ under the map $\star$.

Lemma 6.46 (Herrlich, Schmithüsen [HS07a]). *Every affine, orientation preserving diffeomorphism $\tilde{f} \in \mathrm{Aff}^+(\mathbb{H})$ descends to an affine diffeomorphism $\bar{f}\colon E^2 \to E^2$.*

Proof. We show that $K_2 = [2]_*\pi_1(E^*)$ is a characteristic subgroup of F_2, i.e. that it is mapped to itself by every automorphism of F_2. Then by definition $\tilde{f}_\star(K_2) = K_2$ and by Lemma 6.45 the claim follows.

The group K_2 is the kernel of the homomorphism

$$F_2 \to V_4, \qquad x \mapsto (1,0), \quad y \mapsto (0,1).$$

Given any surjective homomorphism $F_2 \to V_4$, one can verify that its kernel is K_2. Hence for any automorphism of F_2, the concatenation with the above map is a surjective homomorphism $F_2 \to V_4$. Thus the kernel is not changed. $\square$

We write $\ell_{m,n} \in \bar{\Sigma}$ if the loop $\ell_{m,n}$ is a loop around a point in $\bar{\Sigma}$. In our case, this is true for every $m, n \in \mathbb{Z}$. But the following results can be proven similarly for tori coming from multiplication by some larger $k \in \mathbb{N}$.

Lemma 6.47 (Herrlich, Schmithüsen [HS07a]). *Define the groups*

$$G = \left\{ \gamma \in \mathrm{Aut}^+(F_2) \mid \gamma(\ell_{m,n}) \in \bar{\Sigma} \iff \ell_{m,n} \in \bar{\Sigma} \right\} \quad \textit{and}$$
$$\mathrm{Stab}(U)^\star = \left\{ \tilde{f} \in \mathrm{Aff}^+(\mathbb{H}) \;\middle|\; \bar{f} \textit{ lifts to } X \right\},$$

where $\bar{f}\colon E^2 \to E^2$ is the projection of $\tilde{f}$. Then the map $\star\colon \mathrm{Aff}^+(\mathbb{H}) \to \mathrm{Aut}^+(F_2)$ induces isomorphisms

$$G^\star \to G \quad \textit{and} \quad \mathrm{Stab}(U)^\star \to \mathrm{Stab}(U).$$

Proof. Every $\bar{f} \in G^\star$ can be restricted to an affine map $E^{2*} \to E^{2*}$. This map can be lifted to $\mathbb{H}$. We show that $\bar{f}(\bar{\Sigma}) = \bar{\Sigma}$ is equivalent to the fact that $\tilde{f}_\star(\ell) \in \bar{\Sigma}$ if and only if $\ell \in \bar{\Sigma}$:

Let ℓ be a loop in E^{2*} and let $\tilde{f} \in \mathrm{Aff}^+(\mathbb{H})$. For $z \in \mathbb{H}$ we have the map

$$\tilde{f}_\star(\ell)(z) = \tilde{f}\ell_D\tilde{f}^{-1}(z) = \tilde{f}\beta(1)$$

with $\tilde{\ell}(0) = z_0$, $\beta(0) = \tilde{\ell}(1)$ and $\alpha(0) = \tilde{f}^{-1}(z)$. Furthermore, note that

$$(\bar{f}\ell)_D(z) = \beta'(1)$$

with $\tilde{f}\tilde{\ell}(0) = \tilde{f}(z_0)$, $\beta'(0) = \tilde{f}\tilde{\ell}(1)$ and $\alpha'(0) = z$. We define $\beta = \tilde{f}^{-1}\beta'$ and $\alpha = \tilde{f}^{-1}\alpha'$ and have $\tilde{\ell}(0) = z_0$, $\beta(0) = \tilde{\ell}(1)$ and $\alpha(0) = \tilde{f}^{-1}(z)$. Hence $\tilde{f}_\star(\ell) = (\bar{f}\ell)_D$. This shows that

$$\tilde{f}_\star(\ell) \in \bar{\Sigma} \quad \text{if and only if} \quad \bar{f}\ell \in \bar{\Sigma}. \tag{6.4}$$

For $\tilde{f}_\star \in G$ we have $\tilde{f}_\star(\ell) \in \bar{\Sigma}$ if and only if $\ell \in \bar{\Sigma}$. By Equation (6.4) this is equivalent to $\ell \in \bar{\Sigma}$ and hence $\bar{f} \in G^\star$. On the other hand, for $\bar{f} \in G^\star$ we have $\bar{f}\ell \in \bar{\Sigma}$ if and only if $\ell \in \bar{\Sigma}$. By Equation (6.4) this is equivalent to $\tilde{f}_\star(\ell) \in \bar{\Sigma}$ and hence $\tilde{f}_\star \in G$. Hence the map $\star\colon G^\star \to G$ is well defined and surjective. It is injective, because it is the restriction of an injective map.

The second claim follows, since $\bar{f}$ lifts to X if and only if $\tilde{f}$ descends to $\mathbb{H}/U$. By Lemma 6.45 this is equivalent to $\tilde{f}_*(U) = U$. Hence $\star$: $\mathrm{Stab}(U)^\star \to \mathrm{Stab}(U)$ is well defined and surjective. The map is injective, because it is the restriction of an injective map. □

Our next goal is to compute the index of $[G : \mathrm{Stab}(U)] = [G^\star : \mathrm{Stab}(U)^\star]$ in terms of automorphisms of the free group. Then we define a subgroup $H \subseteq \mathrm{Stab}(U)$ by giving explicit generators. Using the isomorphism $\star$, we compute the index $[G : H]$ and show that it equals $[G : \mathrm{Stab}(U)]$. Hence $H = \mathrm{Stab}(U)$ and the Veech group is given as the image of H under the map $\beta\colon \mathrm{Aut}^+(F_2) \to \mathrm{SL}_2(\mathbb{Z})$. For this purpose, we need a technical lemma.

Lemma 6.48. *Let x and y be the generators of F_2 and define $\ell_{p,q} = x^p\, y^q\, [x, y]\, y^{-q}\, x^{-p}$. We have the following equations for all $p, q, n \in \mathbb{Z}$ and $m \in \mathbb{N}$:*

$$y^n\, \ell_{0,q} = \ell_{0,q+n}\, y^n \tag{6.5}$$

$$x^n\, \ell_{p,q} = \ell_{p+n,q}\, x^n \tag{6.6}$$

$$x\, y = \ell_{0,0}\, y\, x \tag{6.7}$$

$$x\, y^m = \ell_{0,0} \cdots \ell_{0,m-1} y^m x \tag{6.8}$$

$$y\, x^2 = \ell_{0,0}^{-1}\, \ell_{1,0}^{-1}\, x^2\, y \tag{6.9}$$

$$x\, y^{-1} = y\, x\, \ell_{-1,-1}\, y^{-2} \tag{6.10}$$

Proof. We have

$$y^n\, \ell_{0,q} = y^n\, y^q\, x\, y\, x^{-1}\, y^{-1}\, y^{-q} = y^{n+q}\, x\, y\, x^{-1}\, y^{-1}\, y^{-q-n}\, y^n.$$

Thus the first equation follows. Similarly, the second one is obtained. For the third one we compute

$$x\,y = x\,y\,x^{-1}\,y^{-1}\,y\,x = \ell_{0,0}\,y\,x.$$

The fourth equation follows inductively:

$$\begin{aligned} x\,y^m = x\,y^{m-1}\,y &= \ell_{0,0}\cdots\ell_{0,m-2}\,y^{m-1}\,x\,y = \ell_{0,0}\cdots\ell_{0,m-2}\,y^{m-1}\ell_{0,0}\,y\,x \\ &= \ell_{0,0}\cdots\ell_{0,m-1}\,y^m\,x \end{aligned}$$

The fifth equation can by computed by using Equations (6.6) and (6.7):

$$y\,x^2 = y\,x\,x = \ell_{0,0}^{-1}\,x\,y\,x = \ell_{0,0}^{-1}\,x\,\ell_{0,0}^{-1}\,x\,y = \ell_{0,0}^{-1}\,\ell_{1,0}^{-1}\,x^2\,y$$

Finally, the last equation follows from Equations (6.5) and (6.6):

$$x\,y^{-1} = x\,y\,x^{-1}\,y^{-1}\,y\,x\,y^{-1}\,y^{-1} = \ell_{0,0}\,y\,x\,y^{-2} = y\,x\,\ell_{-1,-1}\,y^{-2}.$$

The step $x\,\ell_{0,0}^{-1} = \ell_{1,0}^{-1}\,x$ follows by using Equation (6.6) with $n = -1$ and taking the inverse. □

Proposition 6.49. *The index* $[G : \mathrm{Stab}(U_i)] = 3$ *for* $i = 2, 3, 4$ *and* $G = \mathrm{Stab}(U_1)$*.*

Proof. We use the exact same technique as Herrlich and Schmithüsen [HS07a]. The group G acts on the set $\{ U_1, U_2, U_3, U_4 \}$. The index $[G : \mathrm{Stab}(U_i)]$ is, by the orbit counting theorem, the size of the orbit $G \cdot U_i$ for $i = 1, 2, 3, 4$. Hence we want to find maps in $G \subseteq \mathrm{Aut}^+(F_2)$ interchanging U_2, U_3 and U_4.

Firstly, let us discuss the case U_1. Assume there would be a map $U_i \to U_1$ for some $i = 2, 3, 4$. Then this map would descend to an affine map $X_i = \mathbb{H}/U_i \to \mathbb{H}/U_1 = X_1$. By conjugation, this gives an isomorphism $\mathrm{Aut}(X_i) \to \mathrm{Aut}(X_1)$. But by Proposition 6.39 these groups are non-isomorphic, a contradiction! Thus the orbit of U_1 is $G \cdot U_1 = \{ U_1 \}$. This implies $\mathrm{Stab}(U_1) = G$.

Now we state the maps interchanging U_2, U_3 and U_4. We define the two automorphisms

$$\gamma_1 \colon \begin{cases} x \mapsto x\,y\,x^{-1}, \\ y \mapsto x^{-1} \end{cases} \quad \text{and} \quad \gamma_2 \colon \begin{cases} x \mapsto x, \\ y \mapsto x\,y \end{cases}.$$

We have to check that both are in the group G. To see this, we show that the corresponding affine maps are in $G^\star$. The linear part is found by applying $\beta\colon \mathrm{Aut}(F_2) \to \mathrm{GL}_2(\mathbb{Z})$ as defined in Equation (6.3) and has to be in $\mathrm{SL}_2(\mathbb{Z})$. The translation part can by seen by computing $\gamma_i(\ell_{0,0})$ for $i = 1, 2$. If $\gamma_i(\ell_{0,0}) = \ell_{p,q}$ for some $p, q \in \mathbb{Z}$, the affine map maps $\bar{P} = (0, 0)$ to the branch point $(p, q) \in \bar{\Sigma}$. Hence the translation part is in $\bar{\Sigma}$ and the affine map in G.

The corresponding matrices are given by

$$\beta(\gamma_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \beta(\gamma_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Furthermore, we compute

$$\gamma_1(\ell_{0,0}) = \ell_{0,0} \quad \text{and} \quad \gamma_2(\ell_{0,0}) = x\,x\,y\,x^{-1}\,y^{-1}\,x^{-1} = \ell_{1,0}.$$

We calculate how these maps act on the set $\{\, U_1, U_2, U_3, U_4 \,\}$ of kernels of monodromy maps. The map γ_1 interchanges U_4 and U_3: By Equation (6.8) we have

$$\begin{aligned}&\gamma_1(x^2) = x\,y\,x^{-1}\,x\,y\,x^{-1} = x\,y^2\,x^{-1} = \ell_{0,0}\,\ell_{0,1}\,y^2\,x\,x^{-1} = \ell_{0,0}\,\ell_{0,1}\,y^2 \quad \text{and}\\ &\gamma_1(y^2) = x^{-2}.\end{aligned}$$

Hence $\mu(\gamma_1(x^2)) = (1\,2)(1\,2)\mu(y^2)$ and $\mu(\gamma_1(y^2)) = \mu(x^2)$. Recall that U_3 and U_4 are given by

$$U_3\colon \begin{cases} \mu(x^2) = \mathrm{id} \\ \mu(y^2) = (1\,2) \end{cases} \quad \text{and} \quad U_4\colon \begin{cases} \mu(x^2) = (1\,2) \\ \mu(y^2) = \mathrm{id} \end{cases},$$

thus we have $\gamma_1(U_4) = U_3$.

The map γ_2 interchanges U_2 and U_3: According to Equations (6.7) and (6.9) we have

$$\begin{aligned}&\gamma_2(x^2) = x^2 \quad \text{and}\\ &\gamma_2(y^2) = x\,y\,x\,y = \ell_{0,0}\,y\,x^2\,y = \ell_{0,0}\,\ell_{0,0}^{-1}\,\ell_{1,0}^{-1}\,x^2\,y^2 = \ell_{1,0}^{-1}\,x^2\,y^2.\end{aligned}$$

Therefore $\mu(\gamma_2(x^2)) = \mu(x^2)$ and $\mu(\gamma_2(y^2)) = (1\,2)\mu(x^2)\mu(y^2)$. Recall that U_2 and U_3 are given by

$$U_2\colon \begin{cases} \mu(x^2) = \mathrm{id} \\ \mu(y^2) = \mathrm{id} \end{cases} \quad \text{and} \quad U_3\colon \begin{cases} \mu(x^2) = \mathrm{id} \\ \mu(y^2) = (1\,2) \end{cases},$$

thus we have $\gamma_1(U_2) = U_3$.

Summing up, $\{\, U_2, U_3, U_4 \,\}$ is one orbit of length three. □

Proposition 6.50. *Let $X_i = \mathbb{H}/U_i$ for $i = 1, 2, 3, 4$. The Veech groups of the Wollmilchsau and its siblings are given by*

a) $\Gamma(X_1) = \mathrm{SL}_2(\mathbb{Z})$*,*

b) $\Gamma(X_2) = \Theta = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; a+b+c+d \equiv 0 \mod 2 \right\}$*,*

c) $\Gamma(X_3) = \Gamma^0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; b \equiv 0 \mod 2 \right\}$ *and*

d) $\Gamma(X_4) = \Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \mod 2 \right\}$*.*

Proof. We use Proposition 2.12. Hence $\Gamma(X_i) = \beta(\mathrm{Stab}(U_i))$ for $i = 1, 2, 3, 4$. We write down generators of the stabilizers $\mathrm{Stab}(U_i)$ by showing that they generate a subgroup of $\mathrm{Stab}(U_i)$ of index 3 in G for $i = 2, 3, 4$ and the whole group G for $i = 1$. By

Proposition 6.49 this suffices. Then we compute the index in $\mathrm{SL}_2(\mathbb{Z})$ by using the map $\beta\colon \mathrm{Aut}^+(F_2) \to \mathrm{SL}_2(\mathbb{Z})$.

a) Since $G = \mathrm{Stab}(U_1)$ and $\beta(G) = \mathrm{SL}_2(\mathbb{Z})$ this is obvious. This was already shown by Herrlich and Schmithüsen [HS08].

b) Recall that U_2 is given by $\mu(x^2) = \mu(y^2) = \mathrm{id}$. We define the maps

$$\gamma_1\colon \begin{cases} x \mapsto y^{-1} \\ y \mapsto x \end{cases} \quad \text{and} \quad \gamma_2\colon \begin{cases} x \mapsto x \\ y \mapsto x^2\, y \end{cases}.$$

Obviously, the map γ_1 stabilizes U_2, because it interchanges the monodromy. To see that γ_2 stabilizes U_2, we check that $\mu(\gamma_2(y^2)) = \mathrm{id}$. By Equation (6.9) we have

$$\gamma_2(y^2) = x^2\, y\, x^2\, y = x^2\, \ell_{0,0}^{-1}\, \ell_{1,0}^{-1}\, x^2\, y\, y$$

and thus $\mu(\gamma_2(y^2)) = \mu(x^2)^2\,(1\,2)^2\,\mu(y^2) = \mathrm{id}$.

For $i = 1, 2$ we define $A_i = \beta(\gamma_i)$ giving us

$$A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Those two matrices generate the theta group Θ, see e.g. Busam and Freitag [BF09]. Thus $\Theta \subseteq \Gamma(X_2)$ is a subgroup of the Veech group of index 3 in $\mathrm{SL}_2(\mathbb{Z})$. Since the index $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(X_2)] = 3$, we have $\Theta = \Gamma(X_2)$.

d) Recall that U_4 is given by $\mu(x^2) = (1\,2)$ and $\mu(y^2) = \mathrm{id}$. We define the maps

$$\gamma_1\colon \begin{cases} x \mapsto x, \\ y \mapsto x\, y \end{cases} \quad \text{and} \quad \gamma_2\colon \begin{cases} x \mapsto xy^2, \\ y \mapsto x^{-1}\, y^{-1} \end{cases}.$$

Clearly, $\mu(\gamma_1(x^2)) = \mu(x^2)$. By showing that $\gamma_1(y^2) = x\, y\, x\, y$ has trivial monodromy, we see that γ_1 is in the stabilizer of U_4. Using Equations (6.5) to (6.8) we have

$$\gamma_1(y^2) = x\, y\, x\, y = x\, y\, \ell_{0,0}\, y\, x = \ell_{1,1}\, x\, y\, y\, x = \ell_{1,1}\, x\, y^2\, x = \ell_{1,1}\, \ell_{0,0}\, \ell_{0,1}\, y^2\, x\, x. \tag{6.11}$$

Hence $\mu(\gamma_1(y^2)) = (1\,2)^3 \mu(y^2)\mu(x^2) = (1\,2)^4 = \mathrm{id}$.

Furthermore, we show that $\mu(\gamma_2(x^2)) = (1\,2)$ and $\mu(\gamma_2(y^2)) = \mathrm{id}$. We use Equations (6.5) and (6.7) to (6.9) to compute

$$\gamma_2(x^2) = x\, y^2\, x\, y^2 = \ell_{0,0}\, \ell_{0,1}\, y^2\, x\, x\, y^2 \text{ and}$$
$$\gamma_2(y^2) = x^{-1}\, y^{-1}\, x^{-1}\, y^{-1} = (y\, x\, y\, x\,)^{-1} = (y\, \ell_{0,0}\, y\, x\, x\,)^{-1} = x^{-2}\, y^{-2}\, \ell_{0,1}^{-1}.$$

As requested, we have shown that $\mu(\gamma_2(x^2)) = (1\,2)^2 \mu(y^2)^2 \mu(x^2) = (1\,2)$ as well as $\mu(\gamma_2(y^2)) = (1\,2)\mu(x^2)\mu(y^2) = \mathrm{id}$.

Define $A_i = \beta(\gamma_i)$ for $i = 1, 2$, then

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}.$$

Busam and Freitag [BF09] give two generators of $\Gamma_0(2)$:

$$B_1 = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix}.$$

One can compute that $B_1 = A_1^{-1}A_2^{-1}A_1$ and $B_2 = A_1^{-1}A_2A_1^2$ as well as $A_1 = B_1B_2$ and $A_2 = B_1B_2B_1^{-1}B_2^{-1}B_1^{-1}$. Hence A_1 and A_2 generate $\Gamma_0(2)$, which is a subgroup of the Veech group. By Shimura [Shi71], $\Gamma_0(2)$ is of index 3 in $\mathrm{SL}_2(\mathbb{Z})$ and thus by Proposition 6.49, $\Gamma_0(2) = \Gamma(X_4)$.

c) Recall that U_3 is given by $\mu(x^2) = \mathrm{id}$ and $\mu(y^2) = (1\,2)$. Define the maps

$$\gamma_1\colon \begin{cases} x \mapsto xy, \\ y \mapsto y \end{cases} \quad \text{and} \quad \gamma_2\colon \begin{cases} x \mapsto xy^{-1}, \\ y \mapsto x^2y^{-1} \end{cases}.$$

Obviously $\mu(\gamma_1(y^2)) = (1\,2)$. To see that γ_1 is in the stabilizer of U_3 we have to show that $\mu(\gamma_1(x^2)) = \mathrm{id}$. By Equation (6.11) we have

$$\gamma_1(x^2) = x\,y\,x\,y = \ell_{0,0}\,\ell_{0,1}\,y^2\,x^2\,\ell_{1,1}.$$

Thus $\mu(\gamma_1(x^2) = (1\,2)^3\mu(x^2)\mu(y^2) = (1\,2)\mu(y^2) = \mathrm{id}$.

For γ_2 to be in the stabilizer of U_3, we have to show that $\mu(\gamma_2(x^2)) = \mathrm{id}$ and $\mu(\gamma_2(y^2)) = (1\,2)$. By Equations (6.7) and (6.10) we have

$$\begin{aligned} \gamma_2(x^2) &= x\,y^{-1}\,x\,y^{-1} = x\,y^{-1}\,y\,x\,\ell_{-1,-1}\,y^{-2} = x^2\,\ell_{-1,-1}\,y^{-2} \\ \gamma_2(y^2) &= x^2\,y^{-1}\,x^2\,y^{-1} = x^2\,y^{-1}\,x\,(x\,y^{-1}) = x^2\,y^{-1}\,(x\,y)\,x\,\ell_{-1,-1}\,y^{-2} \\ &= x^2\,y^{-1}\,\ell_{0,0}\,y\,x\,x\,\ell_{-1,-1}\,y^{-2} = x^2\,\ell_{0,-1}\,x^2\,\ell_{-1,-1}\,y^{-2} \end{aligned}$$

Hence $\mu(\gamma_2(x^2)) = \mu(x^2)(1\,2)\mu(y^2) = \mathrm{id}$ and $\mu(\gamma_2(y^2)) = \mu(x^2)^2(1\,2)^2\mu(y^2) = (1\,2)$ as desired.

Define $A_i = \beta(\gamma_i)$ for $i = 1, 2$. Then

$$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}.$$

The transposed matrices $A_1^\top$ and $A_2^\top$ generate the group $\Gamma_0(2)$. Hence A_1 and A_2 generate the group $\Gamma^0(2)$. The index in $\mathrm{SL}_2(\mathbb{Z})$ is again 3 since the map

$$\mathrm{SL}_2(\mathbb{Z})/\Gamma_0(2) \to \mathrm{SL}_2(\mathbb{Z})/\Gamma^0(2), \quad A\cdot\Gamma_0(2) \mapsto \left(A^{-1}\right)^\top \Gamma^0(2)$$

is bijective.

For another computation of the index see e.g. Busam and Freitag [BF09]. □

Corollary 6.51. *The siblings of the Wollmilchsau X_2, X_3 and X_4 have conjugated Veech groups.*

Proof. Recall the maps

$$\gamma_1\colon \begin{cases} x \mapsto x\,y\,x^{-1}, \\ y \mapsto x^{-1} \end{cases} \quad \text{and} \quad \gamma_2\colon \begin{cases} x \mapsto x, \\ y \mapsto x\,y \end{cases}.$$

defined in the proof of Proposition 6.49. They give rise to the matrices

$$\beta(\gamma_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \beta(\gamma_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It is easy to see that the groups $\Gamma_0(2)$ and $\Gamma^0(2)$ are conjugated using the first matrix. Conjugating a matrix in $\Gamma^0(2)$ by the matrix $\beta(\gamma_2)$, we get a matrix of the form

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+c & -a+b-c+d \\ c & -c+d \end{pmatrix}.$$

The sum of the entries is $b+2d \equiv b \equiv 0 \mod 2$, since $b \equiv 0 \mod 2$. Thus the conjugated matrix is in Θ. $\square$

Corollary 6.52. *The space $\Omega\mathcal{L}_3$ has two connected components.*

Proof. The siblings of the Wollmilchsau are in the same $\mathrm{GL}_2^+(\mathbb{R})$-orbit in $\mathcal{L}_3$. Moreover, they are hyperelliptic and the Wollmilchsau is not, hence the Wollmilchsau has to be in another orbit. $\square$

Corollary 6.53. *Let $X_i = \mathbb{H}/U_i$ for $i = 1, 2, 3, 4$. The Teichmüller curves $C(X_i)$ of the origamis X_i, $i = 1, 2, 3, 4$, all have genus* 0.

Proof. We have $C(X_i) = \mathbb{H}/\,\Gamma(X_i)$. Then $C(X_1) = \mathbb{H}/\,\mathrm{SL}_2(\mathbb{Z})$ is the affine line. We look-up in Shimura's book [Shi71] that the curve $\mathbb{H}/\Gamma_0(2)$ also has genus 0, but more cusps. Since Θ and $\Gamma^0(2)$ are conjugated to $\Gamma_0(2)$, the curves $\mathbb{H}/\Gamma^0(2)$ and $\mathbb{H}/\Theta$ have the same genus. $\square$

As stated in Chapter 5, there are infinitely many Teichmüller curves intersecting the Teichmüller curve of the Wollmilchsau. But the proof in the paper of Herrlich and Schmithüsen [HS08] cannot be transferred to the siblings of the Wollmilchsau: Let us denote by W_λ a surface in the Teichmüller curve of the Wollmilchsau and by E_λ the torus given by $y^2 = x(x-1)(x-\lambda)$. A crucial step in their proof is to show that the Jacobian of W_λ is isogenous to $E_{-1} \times E_{-1} \times E_\lambda$. There are plenty of automorphisms of order 2 with four fixed points on W_λ, which give rise to a covering $W_\lambda \to E$ for an elliptic curve E. In the lucky case that we find an automorphism of W_λ that descends to an automorphism of order 4 on E, we immediately have $E = E_{-1}$. This leads to the isogeny between the Jacobian and $E_{-1} \times E_{-1} \times E_\lambda$.

For the Wollmilchsau, one sees that the map $\sigma\varphi\tau$ is an involution with four fixed points. The map $\tau\varphi\Psi$ is in the normalizer of $\sigma\varphi\tau$ and hence descends to an automorphism of E. One can check that it has order 4, which yields $E = E_{-1}$ as desired.

The situation is quite different for the siblings of the Wollmilchsau. One can compute the normalizers of all elements of order 2 in the corresponding automorphism group. The result is surprisingly simple: In the notation of the proof of Proposition 6.39, the normalizers are generated either by σ, x and y or by σ, xa^{-1} and y. In both cases, the normalizer is isomorphic to C_2^3 and has no element of order 4. Hence there is no element of order 4 on the covered elliptic curve and it seems plausible, that we cannot decompose the Jacobian as nicely as for the Wollmilchsau.

Although this does not prove that there are not infinitely many Teichmüller curves intersecting the Teichmüller curve of the siblings of the Wollmilchsau, it strongly hints into this direction.

6.6 Invariant loci

So far we have shown that the Hurwitz space of translation surfaces $\Omega\mathcal{L}$ contains affine invariant submanifolds $\Omega\mathcal{L}_3 \subseteq \Omega\mathcal{L}_2 \subseteq \Omega\mathcal{L}_1 \subseteq \Omega\mathcal{L}$ of descending dimensions. In particular, $\Omega\mathcal{L}$ contains orbit closures of every possible dimension. It is not clear whether there are orbit closures other than the ones we listed here. The subspaces of ΩH, which we created, can be described by their automorphism groups. The automorphisms are either translations or rotations, hence affine and holomorphic. Moreover, these maps can be extended along the whole $\mathrm{GL}_2^+(\mathbb{R})$-orbit of each translation covering. This motivates the following definition.

Definition 6.54. Let G be a group. A set of translation surfaces is called *Ω-invariant (G-)locus*, if the affine automorphism group of every translation surface in it contains G. By forgetting the translation structure, each automorphism is a holomorphic one, thus we have an inclusion of G into the holomorphic automorphism group of the surface or covering. Finally, these inclusions should be *G-compatible*, i.e. for each translation surface (X, ω), each matrix $A \in \mathrm{GL}_2^+(\mathbb{R})$ and each affine homeomorphism $(X, \omega) \to A \cdot (X, \omega)$ there are embeddings $G \hookrightarrow \mathrm{Aut}(X, \omega)$ and $G \hookrightarrow \mathrm{Aut}(A \cdot (X, \omega))$ such that the following diagram

$$\begin{tikzcd} G \arrow[r, hook] \arrow[dr, hook] & \mathrm{Aut}(X,\omega) \arrow[d] \\ & \mathrm{Aut}(A \cdot (X,\omega)) \end{tikzcd}$$

commutes.

In other words, every automorphism of a translation surface in an Ω-invariant locus is holomorphic and affine. Moreover, it can be extended along the whole $\mathrm{GL}_2^+(\mathbb{R})$-orbit.

Referring to the previous sections, the space $\Omega\mathcal{L}_1$ is the Ω-invariant V_4-locus of $\Omega\mathcal{L}$. The space $\Omega\mathcal{L}_2$ is a union of the Ω-invariant D_8-locus and of the Ω-invariant C_2^3-locus of $\Omega\mathcal{L}$. Finally, the space $\Omega\mathcal{L}_3$ is a union of the Ω-invariant $(D_8 \times C_2)$-locus and of the Ω-invariant $(D_8 \times_Z C_4)$-locus of $\Omega\mathcal{L}$. In this section we show that those are all affine invariant submanifolds of $\Omega\mathcal{L}$ that are Ω-invariant loci.

Lemma 6.55. *Let G be a finite group and let (X, ω) be a translation surface in an Ω-invariant G-locus. For $f \in G$, the corresponding map $f\colon (X, \omega) \to (X, \omega)$ is either a rotation or a translation, i.e. $Df = \pm I$.*

Proof. Since f is holomorphic, affine and orientation preserving, we have $Df \in \mathrm{SO}(2)$. For $A \in \mathrm{GL}_2^+(\mathbb{R})$ the automorphism f extends to an affine map $f_A\colon A{\cdot}(X, \omega) \to A{\cdot}(X, \omega)$. Its derivative is

$$Df_A = ADfA^{-1}.$$

Hence f_A is an automorphism of $A \cdot X$ if and only if its derivative $Df_A \in \mathrm{SO}(2)$. But $Df_A = ADfA^{-1} \in \mathrm{SO}(2)$ if and only if either $A \in \mathrm{SO}(2)$ or Df is in the center of $\mathrm{GL}_2^+(\mathbb{R})$. Since A is not necessarily orthogonal, Df is in the center of $\mathrm{GL}_2^+(\mathbb{R})$. Because $\det(Df) = 1$, $Df = \pm I$. $\square$

Lemma 6.56. *For each Ω-invariant G-locus of $\Omega\mathcal{L}$, G has no element of order* 3.

Proof. Let $f\colon (X, \omega) \to (X, \omega)$ be an automorphism of order 3. By Lemma 6.55 it is a rotation or translation, i.e. $Df = \pm I$. Since $f^3 = \mathrm{id}$, we have $Df^3 = I$ and hence f is a translation.

Let $\Sigma = \{ P, Q, R, S \}$ be the set of singularities and assume $f(P) = P$. Then there is a neighborhood of U of P which gets translated to a neighborhood of P. Hence $f|_U = \sigma|_U$ or $f|_U = \mathrm{id}|_U$, where σ denotes the deck transformation. Since f is holomorphic, $f = \sigma$ or $f = \mathrm{id}$. None of them is of order 3, which is a contradiction.

Now assume $f(P) \neq P$. We have $f^3(P) = P$. But $f^2(P) \neq P$, because otherwise $f^2 = \sigma$ or $f^2 = \mathrm{id}$ by the above reasoning. Furthermore, $f^2(P) \neq f(P)$ since otherwise $P = f(P)$. Let us name $f(P) = Q$, $f^2(P) = R$. Then the set

$$\{ P, f(P), f^2(P) \} = \{ P, Q, R \} \subsetneq \{ P, Q, R, S \} = \Sigma$$

has three elements. It follows that $f(S) = S$. This is again a contradiction. $\square$

The elements of a group describing an Ω-invariant locus are affine and holomorphic maps. Using the previous lemmas, we show that most possible automorphism groups from Tables 3.1 and 3.2 do not define an Ω-invariant locus in $\Omega\mathcal{L}$.

Proposition 6.57. *The only affine invariant submanifolds of $\Omega\mathcal{L}$ forming an Ω-invariant locus are those we constructed.*

Proof. Recall the automorphism groups of surfaces of genus 3 listed in Tables 3.1 and 3.2.

Every automorphism has to extend to the whole $\mathrm{GL}_2^+(\mathbb{R})$-orbit. By Lemma 6.56 there is no element of order 3 in the automorphism group. Hence we can exclude the groups C_3, C_6, S_3, S_4 and D_{12} as possible automorphism groups.

In the hyperelliptic case, there are only three groups left: C_2, C_4 and $C_2 \times C_4$. The first one is generated by the hyperelliptic involution and hence cannot contain the deck transformation σ. The group C_4 has to contain the hyperelliptic involution and the involution σ. But C_4 contains only one element of order 2. The group $C_2 \times C_4$ can be

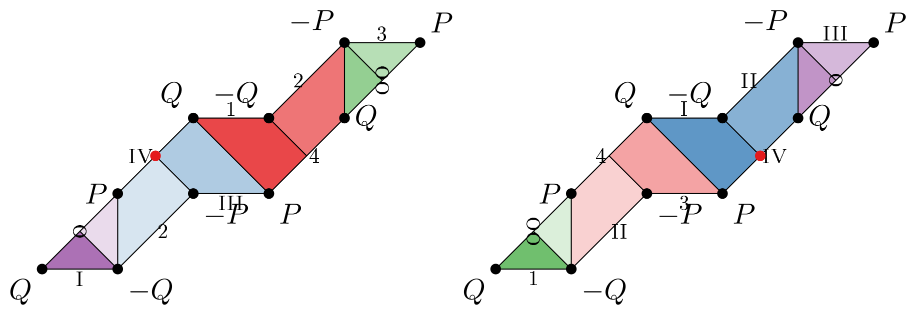


Figure 6.25: On this special translation surface we have a rotation around the marked red point with angle $\frac{\pi}{2}$. Polygons of the same color get mapped onto each other by the map $f \in C_2 \times C_4$ from least to most saturated color.

excluded by comparing with the list in Magaard's paper [Mag+02]: The quotient of the automorphism group by the hyperelliptic involution is the Klein four-group V_4. Hence the hyperelliptic involution in $C_2 \times C_4$ is an element of order 2 in C_4. Thus there has to be a map $f \in C_4$ with $f^2 = \tau$. Hence $Df^2 = -I$, so $Df \notin \{ I, -I \}$. By Lemma 6.55 such a map cannot be extended on the whole $\mathrm{GL}_2^+(\mathbb{R})$-orbit.

In the non-hyperelliptic case we excluded all possible groups, so there is nothing to show. $\square$

In fact, there is a nice picture of the map $f \in C_2 \times C_4$, see Figure 6.25. But one easily sees that the map heavily depends on the symmetries and cannot be extended to a translation surface distorted by the action of a non-orthogonal matrix.

7 Generalizations

In this chapter we briefly discuss the difficulties of generalizing the concepts presented before.

7.1 More points

The most naive attempt to generalize is to add more branch points. The main difference is that the resulting polygons have more edges and vertices. In the following we want to show, that for more than four branch points we cannot use our previous methodes to construct affine invariant submnaifolds of codimension 1.

We are interested in covering of degree 2 of the torus with n branch points. By the Riemann-Hurwitz formula the number of branch points is even. We define the Hurwitz space

$$H^n = \Big\{ (p, X, E) \;\Big|\; p\colon X \to E \text{ simply ramified covering,} \\ g(X) = \frac{n}{2} + 1,\ E \in M_{1,1},\ \deg(p) = 2 \Big\}$$

of all coverings of the torus simply ramified at n points. Let us denote the set of branch points by $\bar{\Sigma}$. Choosing a translation structure of the torus, we define the Hurwitz space of translation coverings

$$\Omega H^n = \{ (p, X, \omega, E, \eta) \mid (p, X, E) \in H^n,\ \eta \in \Omega(E),\ p^*\eta = \omega \}\,.$$

Similar to our construction in Chapter 6 we can find polygons describing translation surfaces in the Hurwitz space ΩH^n, see e. g. Figure 7.1. By Proposition 3.13 the dimension

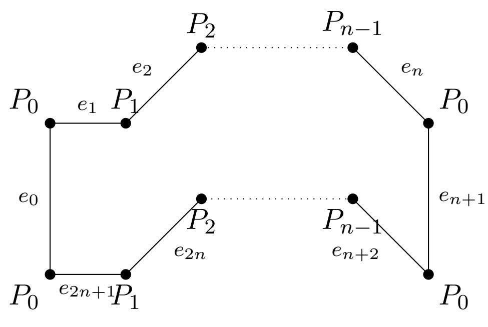


Figure 7.1: A polygon with $2n + 2$ edges defining a translation surface in ΩH^n.

of H^n is given by $\dim H^n = n$ and hence $\dim \Omega H^n = n+1$. As before, let us define the subspace

$$\Omega H^n_{+,k} = \{\ (p,X,\omega,E,\eta) \in \Omega H^n \mid \exists f\colon (X,\omega) \to (X,\omega), Df = I, \text{ which descends to } \bar{f}\colon (E,\eta) \to (E,\eta) \text{ such that } \bar{f}^k = \mathrm{id}\ \}$$

consisting of those translation coverings, which admit a translation descending to a translation of order k on the torus. Equivalently, the translation $\bar{f}$ lifts to a translation $\tilde{f}\colon \mathbb{C} \to \mathbb{C}$ by the preimage of a k-torsion point.

Furthermore, we define the subspace

$$\Omega H^n_{-,k} = \{\ (p,X,\omega,E,\eta \in \Omega H^n) \mid \exists f\colon (X,\omega) \to (X,\omega), Df = -I, \text{ which descends to } \bar{f}\colon (E,\eta) \to (E,\eta) \text{ s. th. } |\operatorname{Fix}(\bar{f}) \cap \bar{\Sigma})| = k\ \}$$

consisting of those translation coverings, which admit a rotation descending to a rotation of the torus. The number of fixed points of $\bar{f}$ which are branch points, is k.

Proposition 7.1. *Assume $\Omega H^n_{J,k}$ is nonempty for $J \in \{+,-\}$ and $n,k \in \mathbb{N}$. Then it is an affine invariant submanifold and its dimension is given by*

a) $\dim \Omega\mathcal{L}^n_{+,k} = \frac{n}{k} + 1$ *for* $1 \le k \le n$,

b) $\dim \Omega\mathcal{L}^n_{-,k} = \frac{n-k}{2} + 2$ *for* $n \ge k \in \{0,2,4\}$.

Note that we do not claim that $\Omega H^n_{J,k}$ exists. For $J = -$ the number k is restricted to 0, 2 or 4. For $J = +$ the space $\Omega H^4_{I,4}$ does not exist, because the translation by a 4-torsion point does not lift to a translation of the covering space. This is partly discussed in the next section.

Proof. a) The translation $\bar{f}\colon (E,\eta) \to (E,\eta)$ is of order k and acts on the set of branch points. The $\bar{f}$-orbit of a branch point has length k, hence we are free to choose $\frac{n}{k}$ branch points and two vectors defining the torus. We fix one branch point by translating the torus and hence remain with $\frac{n}{k} + 2 - 1$ degrees of freedom. Thus the dimension of $\Omega H^n_{I,k}$ is at most $\frac{n}{k} + 1$.

On the other hand, let us choose the following basis of relative homology of the torus relative the the set of branch points $\bar{\Sigma}$: Let us fix one representative $\bar{P}_j$ for each $\bar{f}$-orbit of branch points and denote the number of orbits by r. We define $\bar{a}$ and $\bar{b}$ to be the generators of the absolute homology group of the torus. Then we define the path

$$\bar{c}^i_j \text{ to be a geodesic path from } \bar{f}^{i-1}\bar{P}_j \text{ to } \bar{f}^i\bar{P}_j$$

for $i = 1,\dots,k$ and $j = 1,\dots,r$. Note that $r = \frac{n}{k}$. Since $\sum_{i=1}^k \bar{c}^i_j$ is a closed path, it is of the form $-m\bar{a} - n\bar{b}$ for integers $m,n \in \mathbb{Z}$. Moreover, we define

$$\bar{c}_\ell \text{ to be the geodesic path from } \bar{f}^{k-1}\bar{P}_\ell \text{ to } \bar{P}_{\ell+1}$$

for $\ell = 1, \dots, r-1$. The set

$$\left\{\, \bar{a}, \bar{b}, \bar{c}_j^i, \bar{c}_\ell \;\middle|\; i = 1, \dots, k-1,\ j = 1, \dots, r,\ \ell = 1, \dots, r-1 \,\right\}$$

is a basis of the relative homology group $H_1(E, \bar{\Sigma}, \mathbb{Z})$ of the torus E relative to the set of branch points $\bar{\Sigma}$. We denote fixed lifts of those paths by omitting the bar. Because $\bar{c}_j^i = \bar{f}_*^{i-1} \bar{c}_j^1$ we immediately see that

$$\int_{c_j^1} \omega = \int_{c_j^2} \omega = \cdots = \int_{c_j^k} \omega.$$

These equations are equivalent to

$$\int_{c_j^1} \omega - \int_{c_j^2} \omega = 0, \quad \dots \quad , \int_{c_j^{k-2}} \omega - \int_{c_j^{k-1}} \omega = 0 \tag{7.1}$$

and, since $\bar{f}_* \bar{c}_j^{k-1} = \bar{c}_j^k = -\sum_{i=1}^{k-1} \bar{c}_j^i + m\bar{a} + n\bar{b}$, to

$$\sum_{i=1}^{k} \int_{c_j^i} \omega - m \int_a \omega + n \int_b \omega = k \int_{c_j^1} \omega - m \int_a \omega + n \int_b \omega = 0 \tag{7.2}$$

for all $j = 1, \dots, r$.

For $\ell = 1, \dots, r-1$ the path $\bar{f}_* \bar{c}_\ell$ is a geodesic path from $\bar{P}_\ell$ to $\bar{f}(\bar{P}_{\ell+1})$ and hence

$$\bar{f}_* \bar{c}_\ell - \sum_{i=1}^{k-1} \bar{c}_\ell^i - \bar{c}_\ell - \bar{c}_{\ell+1}^1 = -m\bar{a} - n\bar{b}$$

is a closed path. This gives us the equation

$$\sum_{i=1}^{k-1} \int_{c_j^i} \omega + \int_{c_{j+1}^1} \omega + m \int_a \omega + n \int_b \omega = 0. \tag{7.3}$$

By Equations (7.1) and (7.2) we get a system of linear equations in $\int_{c_j^i} \omega$ for $i = 1, \dots, k-1$ and $j = 1, \dots, r$. The dependence on $\int_a \omega$ and $\int_b \omega$ is postponed. As a matrix it is of the form

$$A_j = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \\ k & & & & \end{pmatrix} \in \mathbb{Z}^{(k-1)\times(k-1)}$$

for $j = 1, \dots, r$. The columns describe the equations in $\int_{c_j^1} \omega, \dots, \int_{c_j^{k-1}} \omega$. Similarly, by Equation (7.3) and postponing the dependence on the absolute homology classes, for every $j = 1, \dots, r-1$ we get a matrix

$$B_j = \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix} \in \mathbb{Z}^{1\times k},$$

where the columns describe the equations in $\int_{c_j^1}\omega,\dots,\int_{c_j^{k-1}}\omega$ and in $\int_{c_{j+1}^1}\omega$. Now we collect the dependence of the above equations on $\int_a\omega$ and $\int_b\omega$. By Equations (7.2) and (7.3) we get a matrix $C\in\mathbb{Z}^{k(r-1)-1\times 2}$, where the j-th row is of the form

$$C_j=\begin{cases}(m,n), & \text{if } j\in k\mathbb{Z}\cup(k\mathbb{Z}-1),\\ (0,0), & \text{otherwise.}\end{cases}$$

The matrix A defined by

$$\begin{pmatrix} \boxed{C} & \boxed{A_1} & & & & \\ & \boxed{B_1} & & & & \\ & & \boxed{A_2} & & & \\ & & \boxed{B_2} & & & \\ & & & \ddots & \ddots & \\ & & & & & \boxed{A_r} \end{pmatrix}$$

is an integer $(rk-1\times r(k-1)+2)$-matrix. It describes the system of linear equations belonging to (7.1), (7.2), (7.3), where the columns belong to the variables $\int_a\omega$, $\int_b\omega$, $\int_{c_1^1}\omega$, $\dots$, $\int_{c_1^{k-1}}\omega$, $\int_{c_1^1}\omega$,$\dots$, $\int_{c_r^{k-1}}\omega$. It is of rank $r(k-1)$. The basis of relative homology has $r(k-1)+r-1+2=r(k-1)+r+1$ elements and hence the linear equations describe a subspace of dimension

$$r(k-1)+r+1-r(k-1)=r+1=\frac{n}{k}+1.$$

Hence, if not empty, $\Omega H^n_{I,k}$ gives rise to an affine invariant submanifold $\Omega\mathcal{L}^n_{I,k}$ of dimension $\frac{n}{k}+1$.

b) The rotation $\bar f\colon(E,\eta)\to(E,\eta)$ is of order 2 and acts on the set of branch points. The number of fixed points of $\bar f$ is, according to the Riemann-Hurwitz formula, even and at most four. We choose freely one branch point for each non-trivial $\bar f$-orbit. The number of non-trivial $\bar f$-orbits is given by

$$\frac{n-\#\{\,\text{fixed points of }\bar f\text{ , which are branch points}\,\}}{2}=\frac{n-k}{2}.$$

A trivial $\bar f$-orbit allows us to choose from a finite set and hence does not increase our degrees of freedom. Furthermore, we can choose two vectors defining the torus. Thus we have at least $\frac{n-k}{2}+2$ degrees of freedom.

On the other hand, let us denote by $\bar P_1,\dots,\bar P_r$ and by $\bar P^1,\dots,\bar P^k$ branch points such that r is minimal with

$$\bar\Sigma=\langle\bar f\rangle\{\,\bar P_1,\dots,\bar P_r\,\}\cup\{\,\bar P^1,\dots,\bar P^k\,\}.$$

The first r points are branch points, which are not fixed points of $\bar{f}$ and the last k points are branch points and fixed points of $\bar{f}$. Furthermore, let us choose a basis of relative homology by defining

$$\bar{c}_j \text{ to be a geodesic path from } \bar{P}_j \text{ to } \bar{f}\bar{P}_j$$

for $1 \leq j \leq r$ and

$$\bar{c}^j \text{ to be a geodesic path from } \bar{P}^j \text{ to } \bar{P}^{j+1}$$

for $1 \leq j \leq k-1$. If $k = 0$ no path of the second type exists. Moreover, we define the path

$$\bar{d}_j \text{ to be a geodesic path from } \bar{f}\bar{P}_j \text{ to } \bar{P}_{j+1}$$

for $1 \leq j \leq r-1$ and, for $k \neq 0$,

$$\bar{d}_r \text{ to be a geodesic path from } \bar{f}\bar{P}_r \text{ to } \bar{P}^1.$$

Finally, we denote by $\bar{a}$ and $\bar{b}$ the two generators of the absolute homology group of E. Hence the set

$$\left\{ \bar{a}, \bar{b}, \bar{c}_i, \bar{c}^j, \bar{d}_\ell \;\middle|\; i = 1, \ldots, r,\ j = 1, \ldots, k,\ \ell = 1, \ldots, r-1 \text{ and up to } r \text{ if } k \neq 0 \right\}$$

describes a basis of the relative homology group $H_1(E, \bar{\Sigma}, \mathbb{Z})$. Their lifts to X are denoted by omitting the bar.

This gives us, again, a bunch of equations. We have that $\bar{f}^*\eta = -\eta$ for every every holomorphic differential η on E. Because we have $\bar{f}_*\bar{c}^j = -\bar{c}^j$ we get the uninteresting equations

$$\int_{c^j} \omega = -\int_{c^j} f^*\omega = -\int_{f_*c^j} \omega = -\int_{-c^j} \omega = \int_{c^j} \omega$$

for every $1 \leq j \leq k$. Since $\bar{f}_*\bar{d}_\ell = \bar{c}_\ell + \bar{d}_\ell + \bar{c}_{\ell+1}$ we get the equations

$$2\int_{d_\ell} \omega + \int_{c_\ell} \omega + \int_{c_{\ell+1}} \omega = 0 \tag{7.4}$$

for every $1 \leq \ell \leq r-1$. For $k \neq 0$ we get more equations: Because the path $\bar{c}^j - \bar{f}_*\bar{c}^j = -m_j\bar{a} - n_j\bar{b}$ is closed we have

$$\int_{c^j} \omega - \int_{f_*c^j} \omega = 2\int_{c^j} \omega = m_j \int_a \omega + n_j \int_b \omega \tag{7.5}$$

for every $1 \leq j \leq k$ and for some integers $m_j, n_j \in \mathbb{Z}$. Furthermore, we have $f_*d_r = c_r + d_r$ and hence

$$2\int_{d_r} \omega + \int_{c_r} \omega = 0. \tag{7.6}$$

As a system of linear equations in period coordinates, for $k = 0$ only Equation (7.4) matters and hence this system is given by the $(r-1) \times (2r-1)$-matrix

$$\begin{pmatrix} 1 & 1 & & & & 1 & & & \\ & 1 & 1 & & & & 1 & & \\ & & \ddots & \ddots & & & & \ddots & \\ & & & 1 & 1 & & & & 1 \end{pmatrix},$$

where the columns describe the linear equations in $\int_{c_1} \omega, \ldots, \int_{c_r} \omega, \int_{d_1} \omega, \ldots, \int_{d_{r-1}} \omega$. The rank of the matrix is $r-1$ and the basis of relative homology has $r+r-1+2 = 2r+1$ elements, hence it defines a subspace of dimension $r+2 = \frac{n}{2}+2$.

For $k \neq 0$, in addition to Equation (7.4), which gives the same matrix as before, also Equations (7.5) and (7.6) matter. Thus the system of linear equations defined by

$$\left(\begin{array}{cccccccccccccccc} 1 & 1 & & & & 1 & & & & & & & & & & \\ & 1 & 1 & & & & 1 & & & & & & & & & \\ & & \ddots & \ddots & & & & \ddots & & & & & & & & \\ & & & 1 & 1 & & & & 1 & & & & & & & \\ & & & 1 & & & & & & 2 & & & & & & \\ & & & & & & & & & & 2 & & & & m_1 & n_1 \\ & & & & & & & & & & & 2 & & & m_2 & n_2 \\ & & & & & & & & & & & & \ddots & & \vdots & \vdots \\ & & & & & & & & & & & & & 2 & m_k & n_k \end{array}\right)$$

is an integer $(r+k) \times (2r+k+2)$-matrix. The columns describe the linear equations in $\int_{c_1} \omega, \ldots, \int_{c_r} \omega, \int_{d_1} \omega, \ldots, \int_{d_{r-1}} \omega, \int_{d_r} \omega, \int_{c^1} \omega, \ldots, \int_{c^k} \omega, \int_a \omega, \int_b \omega$. The rank of this matrix is $r+k$ and the basis of relative homology has $2r+k+2$ elements, hence it describes a subspace of dimension $r+2 = \frac{n-k}{2}+2$.

In summary we have shown that $\Omega\mathcal{L}^n_{-I,k}$ is an affine invariant submanifold of dimension at least $\frac{n-k}{2}+2$. $\square$

As before, the forgetful map $\mathcal{F}\colon \Omega H^n \to \Omega\mathcal{L}^n$ gives us affine invariant submanifolds. This enables us to prove that, using our previous construction, we are not able to construct affine invariant submanifolds of codimension 1 in $\Omega\mathcal{L}^n$.

Corollary 7.2. *For $n \geq 6$ there is no affine invariant submanifold of $\Omega\mathcal{L}^n$ of codimension 1 which is an Ω-invariant locus.*

Proof. The space $\Omega\mathcal{L}^n$ is an affine invariant submanifold of dimension $n+1$.

An affine invariant submanifold and Ω-invariant locus of $\Omega\mathcal{L}^n$ is described by an additional automorphism, which is a translation or a rotation.

If it is a translation, it descends to a translation of order k on the torus. Hence the dimension of the locus is given by $\frac{n}{k}+1$. Let us assume $k>1$, because otherwise the translation is a deck transformation. Hence we have

$$\frac{n}{k}+1=(n+1)-1=n \text{ if and only if } n=\frac{k}{k-1}.$$

Since n is an integer, this is true if and only if $k=2$ and $n=2$.

If the automorphism is a rotation, let k be the number of fixed points of $\bar{f}$ which are branch points. Hence the dimension of the locus is given by $\frac{n-k}{2}+2$. We have

$$\frac{n-k}{2}+2=n \text{ if and only if } 4-k=n,$$

thus if and only if $(n,k)=(4,0)$ or $(2,2)$. □

7.2 Weakening the meaning

In the previous chapter we constructed affine invariant submanifolds described by automorphisms of the covering surface. But to construct linear equations in period coordinates we only need automorphisms of the covered surface. In this section we study which affine invariant submanifolds of $\Omega\mathcal{L}_1$ are described by automorphisms of the torus. The restriction to subspaces of $\Omega\mathcal{L}_1$ is necessary for our construction, but it seems plausible that without this restriction no new subspaces arise.

On the torus E with four marked points $\bar{P}$, $\bar{Q}$, $-\bar{P}$ and $-\bar{Q}$, we look at the translation $h\colon E\to E$ given by $h(\bar{P})=\bar{Q}$, $h(\bar{Q})=-\bar{Q}$, $h(-\bar{Q})=-\bar{P}$ and $h(-\bar{P})=\bar{P}$. It is of the form

$$h(z)=z+b \quad \text{with} \quad b=\bar{Q}-\bar{P}=-2\bar{Q}=2\bar{P}.$$

This gives a relation of the branch points. Note that $8\bar{P}=\bar{O}$ and hence h is the translation by a 4-torsion point.

The other way around, given these relations there exists a translation h with the above properties. For a visualization see Figure 7.2.

As in Chapter 6, we consider translation coverings of degree 2 of the torus with four ramification points. Can the translation h be lifted along such a covering? The map h can be lifted if and only if it leaves the kernel of the monodromy map invariant. Let $\bar{b}$ be the vertical path in the fundamental group $\pi_1(E^*)$ of the punctured torus as depicted in Figure 7.2. This loop gets mapped to a loop translated one square to the right. Hence $h(\bar{\beta})=\ell_{-\bar{Q}}\circ\bar{\beta}$. Since the monodromy of a loop around a branch point is never trivial, we have $\mu(h(\bar{\beta}))=(1\,2)\mu(\bar{\beta})\neq\mu(\bar{\beta})$. Hence the translation h cannot be lifted to a covering of the torus.

Nevertheless, this automorphism of the torus gives rise to an affine invariant submanifold of the principal stratum $\mathcal{H}(1,1,1,1)$.

Proposition 7.3. *The subspace of $\Omega\mathcal{L}$, consisting of all translation surfaces coming from translation coverings with branch points that fulfill the condition $\bar{Q}-\bar{P}=-2\bar{Q}=2\bar{P}$, is an affine invariant submanifold of $\mathcal{H}(1,1,1,1)$ of dimension 2.*

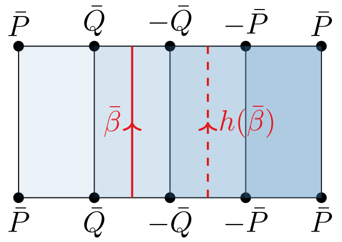


Figure 7.2: The translation h is sketched: It maps each parallelogram to the more saturated parallelogram, i.e. one to the right. Furthermore, $\bar{\beta}$ is an element of the fundamental group and of the homology group of E.

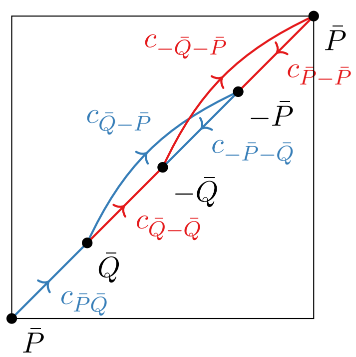


Figure 7.3: A torus, which admits a translation h. In blue, basis elements of the relative homology group are sketched. In red, their pictures are depicted.

Proof. The corresponding Hurwitz space of translation coverings is a submanifold of ΩH: Recall that the charts defined in Lemma 6.15 were given by mapping a covering to its set of branch points and then to some linear combination of these. We have to alter this map slightly to get the right notion of a chart. A chart is given by

$$\varphi\colon (p, X, E) \mapsto \bar{\Sigma} \mapsto (\tau, 2\bar{e}_1 + 2\bar{e}_2, 3\bar{e}_2 + \bar{e}_4, \bar{e}_1 + \bar{e}_4, \bar{e}_2 + \bar{e}_3),$$

where $\bar{\Sigma} = \{\, \bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4 \,\}$ is the set of branch points. Due to the relations of the branch points we have

$$\varphi(p, X, E) = (\tau, 0, 0, 0, 0).$$

Thus locally this space is the intersection of $\mathbb{C}^5$ with a linear subspace. By adding the translation structure we make this space into a submanifold of ΩH.

The restriction of the forgetful map $\Omega H \to \Omega\mathcal{L}$ to this subspace is a proper immersion. Hence we only need to show that the image of some translation covering under the forgetful map $\Omega\mathcal{F}$ is given by linear equations in period coordinates of the right dimension. In a small enough neighborhood of a translation covering the immersion $\Omega\mathcal{F}$ is injective. Thus, when working with a translation surface (X, ω) we can always assume that it is equipped with a unique translation covering denoted by $p\colon (X, \omega) \to (E, \eta)$.

We choose a basis of the relative homology group as in Example 2.13. The loops $\bar{a}$ and $\bar{b}$ denote the "horizontal" and "vertical" loop of the torus and the paths $c_{\bar{P}\bar{Q}}$, $c_{\bar{Q}-\bar{P}}$ and $c_{-\bar{P}-\bar{Q}}$ are paths connecting two branch points. We still have that $h(\bar{b})$ and $h(\bar{a})$ are homologous to $\bar{b}$ and $\bar{a}$, respectively. The lifts of those paths are denoted by removing the bars.

For a better visualization see Figure 7.3. The blue paths are elements of the basis of relative homology and their images are depicted in red. One sees that

$$h(c_{\bar{P}\bar{Q}}) = c_{\bar{Q}-\bar{Q}} = c_{\bar{Q}-\bar{P}} + c_{-\bar{P}-\bar{Q}}.$$

Moreover, as in Example 2.13 one can compute

$$\begin{aligned}
c_{\bar{P}-\bar{P}} &= -a - b + c_{\bar{P}\bar{Q}} + c_{\bar{Q}-\bar{P}}, \text{ which leads to} \\
h(c_{-\bar{Q}\bar{P}}) &= -c_{\bar{P}-\bar{P}} - c_{-\bar{P}-\bar{Q}} = a + b - c_{\bar{P}\bar{Q}} - c_{\bar{Q}-\bar{P}} - c_{-\bar{P}-\bar{Q}} \quad \text{and} \\
h(c_{\bar{Q}-\bar{P}}) &= c_{\bar{P}-\bar{P}} = -a - b + c_{\bar{P}\bar{Q}} + c_{\bar{Q}-\bar{P}}.
\end{aligned}$$

We denote the integral over a, b, c_{PQ}, c_{Q-P} and c_{-P-Q} by A, B, C_1, C_2 and C_3, respectively. We compute

$$\begin{aligned}
C_1 &= \int\limits_{c_{PQ}} \omega = \int\limits_{c_{PQ}} p^*\eta = \int\limits_{c_{\bar{P}\bar{Q}}} \eta = \int\limits_{c_{\bar{P}\bar{Q}}} h^*\eta = \int\limits_{h_* c_{\bar{P}\bar{Q}}} \eta \\
&= \int\limits_{c_{\bar{Q}-\bar{P}}} \eta + \int\limits_{c_{-\bar{P}-\bar{Q}}} \eta = \int\limits_{c_{Q-P}} \omega + \int\limits_{c_{-P-Q}} \omega = C_2 + C_3.
\end{aligned}$$

With similar computations we get

$$\begin{aligned}
A &= A, \\
B &= B, \\
C_2 &= A + B - C_1 - C_2 - C_3 \quad \text{and} \\
C_3 &= -A - B + C_1 + C_2.
\end{aligned}$$

This yields three linearly independent equations. The dimension of $\Omega\mathcal{L}$ is 5, hence these equations describe an affine invariant submanifold of dimension at most $5 - 3 = 2$.

Its dimension is at least 2, since in Figure 7.2 we can choose the vertical edge and the first horizontal edge freely. □

Let us point out that this subspace lives in the space $\Omega\mathcal{L}_1$, because the rotation $[-1]$ exists on every translation covering in this space. This can be seen by explicitly defining the rotation in Figure 7.2. Another way to see this is to manipulate the linear equations, which yields $C_1 + C_3 = 0$, as in the proof of Proposition 6.16.

In the following we discuss in which sense we have a complete list of affine invariant submanifolds in $\Omega\mathcal{L}$. For technical reasons we work in $\Omega\mathcal{L}_1$, but probably a similar result holds in $\Omega\mathcal{L}$. We study which affine invariant submanifolds of $\Omega\mathcal{L}_1$, given by translations and rotations of the torus, exist. The surprising answer is that no new submanifolds appear in this list. They only differ from the ones we constructed by a relabeling of the ramification points.

In Table 7.1 we write down all possible, non-trivial translations of the torus by giving the images of the branch points. In the first four columns we denote the images of the branch points. In the fifth column we write down the relation of the branch points or, equivalently, the translation part of the translation. Normally, we label the points by $\bar{P}$, $\bar{Q}$, $-\bar{Q}$ and $-\bar{P}$. Changing these names does not change the translation surface. In which way we relabel the branch points is described in the sixth column. No change is denoted by -. With this new order of points, the translation corresponds to one of the three translations we discussed so far. This is denoted in the last column. For example, the seventh row in the table reads as follows: The translation t maps each point to its negative. Then it has to be of the form $z \mapsto z + b$ with $b = -2\bar{P} = 2\bar{P} = 2\bar{Q} = -2\bar{Q}$. Furthermore, if we interchange $\bar{P}$ with $-\bar{Q}$, we see that $t(\bar{P})$ is given by what $-\bar{Q}$ gets mapped to, hence $t(\bar{P}) = \bar{Q}$. And $t(\bar{Q})$ gets mapped, after interchanging the points, to $\bar{P}$. Hence this map is, after relabeling the branch points, the automorphism which lifts to Ψ.

In the space $\Omega\mathcal{L}_1$ we can assume without loss of generality, by multiplying with $[-1]$, that every affine holomorphic automorphism t is a translation. Hence we have shown the following proposition.

Proposition 7.4. *The only affine invariant submanifolds of $\Omega\mathcal{L}_1$ that can be described by automorphisms of the torus are those we constructed so far.*

Combining this with Proposition 6.57 we get our desired result:

$t(\bar{P})$	$t(\bar{Q})$	$t(-\bar{Q})$	$t(-\bar{P})$	translation b	order	translation
$\bar{Q}$	$\bar{P}$	$-\bar{P}$	$-\bar{Q}$	$\bar{Q}-\bar{P}=\bar{P}-\bar{Q}$	-	Ψ
$\bar{Q}$	$-\bar{Q}$	$-\bar{P}$	$\bar{P}$	$\bar{Q}-\bar{P}=2\bar{P}$	-	h
$\bar{Q}$	$-\bar{P}$	$\bar{P}$	$-\bar{Q}$	$\pm\bar{P}\pm\bar{Q}=\pm\bar{P}\mp\bar{Q}$	$\bar{P},\bar{Q},-\bar{P},-\bar{Q}$	h
$-\bar{Q}$	$\bar{P}$	$-\bar{P}$	$\bar{Q}$	$\pm\bar{P}\pm\bar{Q}=\pm\bar{P}\mp\bar{Q}$	$\bar{P},-\bar{Q},-\bar{P},\bar{Q}$	h
$-\bar{Q}$	$-\bar{P}$	$\bar{P}$	$\bar{Q}$	$-\bar{Q}-\bar{P}=\bar{Q}+\bar{P}$	-	φ
$-\bar{Q}$	$-\bar{P}$	$\bar{Q}$	$\bar{P}$	$-\bar{Q}-\bar{P}=2\bar{Q}$	$\bar{P},-\bar{P},\bar{Q},-\bar{Q}$	h
$-\bar{P}$	$-\bar{Q}$	$\bar{Q}$	$\bar{P}$	$\pm 2\bar{P}=\pm 2\bar{Q}$	$-\bar{Q},\bar{Q},\bar{P},-\bar{P}$	Ψ
$-\bar{P}$	$\bar{P}$	$\bar{Q}$	$-\bar{Q}$	$-2\bar{P}=2\bar{Q}=\bar{P}-\bar{Q}$	-	h^{-1}
$-\bar{P}$	$-\bar{Q}$	$\bar{P}$	$\bar{Q}$	$-2\bar{P}=-2\bar{Q}=\bar{P}+\bar{Q}$	$\bar{P},-\bar{P},\bar{Q},-\bar{Q}$	h

Table 7.1: A list of all possible translations t of a torus covered by a translation surface in $\Omega\mathcal{L}_1$ that leave the set of branch points invariant. The first four columns indicate how the branch points are mapped, the fifth column describes the translation part of t and the sixth column shows how the order of the branch points has to be changed to retrieve the well-known map written in the last column.

Theorem 3. *The only affine invariant submanifolds of $\Omega\mathcal{L}$ forming an Ω-invariant locus are $\Omega\mathcal{L}_i$ for $i = 1, 2, 3$. The only other affine invariant submanifold of $\Omega\mathcal{L}_1$, which is described by an automorphism of the torus that extends along the whole $\mathrm{GL}_2^+(\mathbb{R})$-orbit, is the one constructed above.*

Finally, one can use the algorithm of Schmithüsen [Sch04] to compute the Veech group of a translation surface given by the automorphism h. It is the congruence subgroup $\Gamma^0(4)$ and is of index 6 in $\mathrm{SL}_2(\mathbb{Z})$.

7.3 Higher base genus

To generalize our construction to higher base genus, we must be able to draw polygon decompositions of the covered translation surfaces. Due to McMullen [McM07], such a decomposition exists for genus $g = 2$. Roughly, this works as follows: Every translation surface of genus 2 is hyperelliptic and has a saddle connection that is not fixed by the hyperelliptic involution. Either the saddle connection is a loop, in which case the resulting translation surface has a single singularity of order 2, or the saddle connection is not closed, in which case the resulting translation surface has two singularities of order 1. Cutting along this saddle connection and its image yields a torus and a cylinder in the first case and two tori in the second case. In Figure 7.4 an example for either case is depicted.

For genus $g > 2$ just some sporadic examples of polygon decomposition of translation surfaces are known. Some examples in genus 3 can be found in the work of Aulicino,

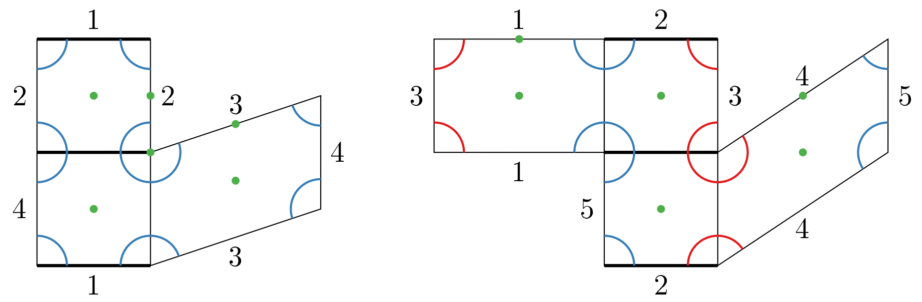


Figure 7.4: The left translation surfaces is in the stratum $\mathcal{H}(2)$ and the right one in $\mathcal{H}(1,1)$. The hyperelliptic involution interchanges the thick edges. Cutting along these edges yields a torus and a cylinder in the left picture and two tori in the right picture. The fixed points of the hyperelliptic involution are marked in green.

Nguyen and Wright [AN16b; ANW16; NW13]. We restrict ourselves to coverings of surfaces of genus 2.

Consequently, let (Y, υ) be a translation surface of genus 2 and let $p\colon X \to Y$ be a covering from a surface of genus 3. By the Riemann-Hurwitz formula from Proposition 3.5 we have

$$4 = 2 \cdot \deg(p) + \sum_{P \in Y} (e_P(p) - 1).$$

Since $\deg(p) > 1$, we have $\deg(p) = 2$ and thus the covering is unramified. We pull back the translation structure υ on Y to one on X. Since p is unramified, every point in Y has two preimages. Hence a covering of a translation surface in the stratum $\mathcal{H}(1,1)$ is in $\mathcal{H}(1,1,1,1)$ and one of a translation surface in the stratum $\mathcal{H}(2)$ is in $\mathcal{H}(2,2)$.

Firstly, we want to count how many unramified coverings of degree 2 of a surface of genus 2 exist. By Proposition 3.7, there is a bijection between the set of all coverings $p\colon X \to Y$ and the set of all surjective group homomorphisms $\mu\colon \pi_1(Y) \to S_2$. Because the fundamental group of Y is given by

$$\pi_1(Y) = \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1] \cdot [a_2, b_2] = 1 \rangle$$

and the codomain is abelian, there are no restrictions for a map $\{ a_1, a_2, b_1, b_2 \} \to S_2$ to induce a homomorphism $\pi_1(Y) \to S_2$. For the map to be surjective, we exclude the case $\mu(a_1) = \mu(a_2) = \mu(b_1) = \mu(b_2) = \mathrm{id}$. Hence there exist $16 - 1 = 15$ possible maps, giving us 15 possible coverings. In the next step, we will construct 15 translation coverings for a given translation surface in $\mathcal{H}(1,1)$ and for one in $\mathcal{H}(2)$.

We start with a translation surface of genus 2 in the stratum $\mathcal{H}(2)$. We take two copies of the polygon describing the covered surface, see for example the left picture in Figure 7.4. We label the edges of the first copy by the Arabic numerals 1, 2, 3 and 4 and the edges of the second copy by the Roman numerals I, II, III and IV. Then we define a gluing map

$$T\colon \{ 1, 2, 3, 4 \} \to \{ 1, 2, 3, 4 \} \cup \{ \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV} \}.$$

no	3	5	both
	3-III	5-V	3-III, 5-V
1-I, 2-II	1-I, 2-II, 3-III	1-I, 2-II, 5-V	1-I, 2-II, 3-III, 5-V
1-I, 4-IV	1-I, 4-IV, 3-III	1-I, 4-IV, 5-V	1-I, 4-IV, 3-III, 5-V
2-II, 4-IV	2-II, 4-IV, 3-III	2-II, 4-IV, 5-V	2-II, 4-IV, 3-III, 5-V
3 possibilities	4 possibilities	4 possibilities	4 possibilities

Table 7.2: All possible gluing for a covering of a translation surface in $\mathcal{H}(1,1)$, which is in $\mathcal{H}(1,1,1,1)$.

The map T assigns to an Arabic numeral a Roman one if the edge labeled by the Arabic numeral is glued via a translation to the edge labeled by the Roman numeral. Otherwise it is the identity. Note that we always have two edges with the same label, but since we glue by translations, it is clear which edge has to be glued to which edge. We abbreviate the map T by denoting just the pairs of Arabic and Roman numerals that are glued by T. For example, the translation surface belonging to 4-IV is drawn in Figure 7.5. One can check that every non-trivial gluing, i.e. $T \neq \mathrm{id}$, gives rise to a translation surface in $\mathcal{H}(2,2)$.

For a translation surface in $\mathcal{H}(1,1)$ a similar approach works: We take two copies of a polygon describing this translations surface and label the edges of the first copy by 1 to 5 and the edges of the second copy by I to V. We define a gluing map

$$T\colon \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\} \cup \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{V}\}.$$

As before, if an edge labeled by an Arabic numeral is glued via a translation to an edge labeled by a Roman numeral, the map T assigns this Arabic numeral to the corresponding Roman numeral. We abbreviate the map T by denoting just the pairs of Arabic and Roman numerals that are glued by T. For example, the translation surface belonging to 5-V can be seen in Figure 7.6.

In this case, not every non-trivial gluing gives a translation surface in the stratum $\mathcal{H}(1,1,1,1)$. For example the assignment 1-I gives a translation surface of genus 4 in the stratum $\mathcal{H}(3,3)$. One can check that the gluing 3-III, the gluing 5-V and the gluing 3-III, 5-V each yield a translation surface in the stratum $\mathcal{H}(1,1,1,1)$. By adding or, if possible, removing two pairs of numerals from a valid gluing, we obtain another gluing giving us a translation surface in $\mathcal{H}(1,1,1,1)$. Table 7.2 contains and counts all possible gluings. For a systematic approach, every column starts with one valid gluing (except for the first one, which starts with the non-valid trivial gluing) and adds or subtracts two pairs of gluing in each step.

On the one hand, for a given translation surface of genus 2 we can draw 15 explicit coverings. On the other hand, by counting maps from the fundamental group to the symmetric group S_2, there exist at most 15 different coverings. Summing up, we found all unramified translation coverings of degree 2 of a translation surface of genus 2.

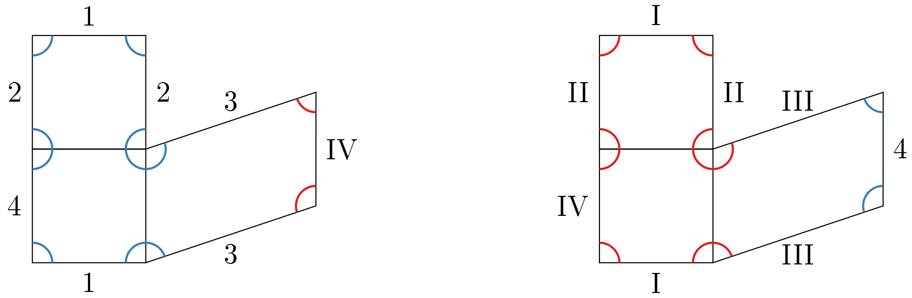


Figure 7.5: A covering of degree 2 of a translation surface in $\mathcal{H}(2)$ is in the stratum $\mathcal{H}(2,2)$.

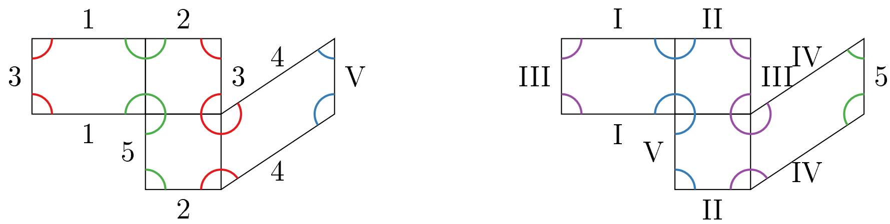


Figure 7.6: A covering of degree 2 of a translation surface in $\mathcal{H}(1,1)$ is in the stratum $\mathcal{H}(1,1,1,1)$.

Let us denote by ΩH^2 the space of all unramified translation coverings of degree 2 covering a translation surface in $\mathcal{H}(2)$ and by $\Omega H^{1,1}$ the space of all unramified translation coverings of degree 2 covering a translation surface in $\mathcal{H}(1,1)$. The images of these spaces under the forgetful map are denoted by $\Omega\mathcal{L}^2$ and $\Omega\mathcal{L}^{1,1}$, respectively.

Proposition 7.5. *The spaces $\Omega\mathcal{L}^2$ and $\Omega\mathcal{L}^{1,1}$ are affine invariant submanifolds of dimension 4 and 5, respectively.*

Proof. In Section 3.3 we have shown that for a fixed surface Y the space $H(d,r,Y)$, consisting of coverings of degree d of Y with r ramification points, is a complex manifold. Since ΩH^2 and $\Omega H^{1,1}$ are fiber bundles over $H(2,0,Y)$, they are complex manifolds as well. Exactly as in the proof of Proposition 6.5 one can show that the forgetful map $\Omega\mathcal{F}\colon \Omega H^\alpha \to \Omega\mathcal{L}^\alpha$ is an immersion for $\alpha \in \{(1,1),2\}$. Hence we only have to check that we can describe the spaces $\Omega\mathcal{L}^2$ and $\Omega\mathcal{L}^{1,1}$ locally by linear equations in period coordinates. Recall that the forgetful map is locally injective, hence given a translation surface (X,ω) we find a unique covering $p\colon (X,\omega) \to (Y,\upsilon)$.

Let $\bar{\Sigma}$ be the set of singularities of (Y,υ), $p\colon X \to Y$ a covering of degree 2 and $\Sigma = p^{-1}(\bar{\Sigma})$. Consider the map

$$p_*\colon H_1(X,\Sigma,\mathbb{Z}) \to H_1(Y,\bar{\Sigma},\mathbb{Z}).$$

If $c \in H_1(X, \Sigma, \mathbb{Z})$ is in the kernel of p_*, we have

$$\int_c \omega = \int_c p^* \upsilon = \int_{p_* c} \upsilon = 0.$$

Hence both spaces are affine invariant submanifolds and their dimension is at most the dimension of their images, i.e. the dimension of $\mathbb{Z}^{2g+|\Sigma|-1}$. For $(Y, \upsilon) \in \mathcal{H}(2)$ we have $|\bar{\Sigma}| = 1$ and $|\Sigma| = 2$ and hence $\mathrm{im}(p_*) = \mathbb{Z}^4$. For $(Y, \upsilon) \in \mathcal{H}(1,1)$ we have $|\bar{\Sigma}| = 2$ and $|\Sigma| = 4$ and hence $\mathrm{im}(p_*) = \mathbb{Z}^5$. Thus dimensions of $\Omega\mathcal{L}^2$ and $\Omega\mathcal{L}^{1,1}$ are 4 and 5, respectively.

On the other hand, the dimension is at least 4 and 5, respectively, since the strata $\mathcal{H}(2)$ and $\mathcal{H}(1,1)$ are of dimension 4 and 5, respectively. □

In the search for affine invariant submanifolds of this space we cannot proceed as in the case of coverings of the torus. The main reason is that the automorphism group of a translation surface of genus 2 is at most the Klein four-group V_4. A proof thereof is given by Herrlich, Kappes and Schmithüsen [HKWS08]. One generator of the automorphism group, which always exists, is the hyperelliptic involution τ. In Figure 7.4 the hyperelliptic involution can be seen in the following way: Rotate each parallelogram by π, then reglue such that the polygon is the same as before. The fixed points of the hyperelliptic involution are marked in green. Hence we have 6 fixed points and the described map is the hyperelliptic involution.

If a translation surface in $\mathcal{H}(1,1)$ is symmetric enough, we have a second map φ, which can be chosen to be a translation. In Figure 7.7 a translation is sketched. It maps a rectangle to the other rectangle of the same color. Such a translation can only exist if the defining polygon is rather symmetric: the edges labeled by 1 and 4 as well as the edges labeled by 3 and 5 have to be parallel and of equal length. By using the coverings we constructed above, this gives us, at least intuitively, a three-dimensional subspace of $\mathcal{H}(1,1,1,1)$.

Proposition 7.6. *The locus $\Omega\mathcal{L}_2^{1,1}$ of all translation surfaces in $\Omega\mathcal{L}^{1,1}$ with automorphism group V_4 is an affine invariant submanifold of $\mathcal{H}(1,1,1,1)$ of dimension* 3*.*

Proof. If both generators of the automorphism group are rotations, multiplying them gives a translation. Hence we may assume that our second generator is a translation. In the right picture of Figure 7.4, we see three vertical cylinders. By gluing arguments, every translation of this translation surface has to map the middle cylinder onto itself. Hence the only possible translation is the one sketched in Figure 7.7.

Let $\varphi\colon Y \to Y$ denote this translation. We choose a basis of relative homology a_1, a_2, b_1, b_2 and c as sketched in Figure 7.7. We observe that $\varphi(a_1) = a_1$, $\varphi(b_1) = b_2$ and $\varphi(b_2) = b_1$. One can check, similar to Example 2.13, that $\varphi(c)$ is homologous to c. Similarly, but more elaborate, $\varphi(a_2) = a_1 - a_2$. Hence

$$(p, X, \omega, Y, \upsilon) \mapsto (a_1 - 2a_2, b_1 - b_2, a_2, b_2, c) \in \mathbb{C}^5$$

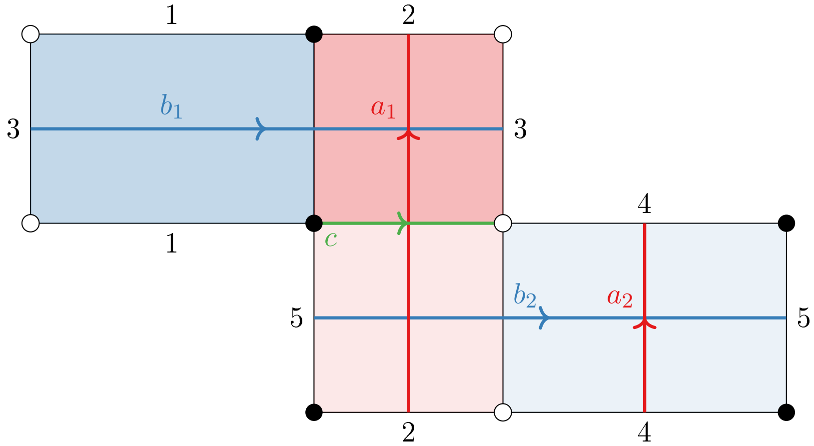


Figure 7.7: A symmetric translation surface in $\Omega\mathcal{L}(1,1)$ with maximal automorphism group. The translation φ is of order 2 and maps each parallelogram to the more saturated parallelogram of the same color. The marked paths form a basis of the relative homology group.

defines a chart of $\Omega H^{1,1}$, where we map an element in the relative homology group via integration onto a vector in $\mathbb{C}$. Due to the above observations, for a translation covering in $\Omega H_2^{1,1}$, the chart is given by

$$(p, X, \omega, Y, \upsilon) \mapsto (0, 0, a_2, b_2, c).$$

This shows that locally $\Omega H_2^{1,1}$ is the intersection of $\mathbb{C}^5$ with a linear subspace and thus it is a submanifold of $\Omega H^{1,1}$. Hence we can restrict the immersion $\Omega H^{1,1} \to \mathcal{H}(1,1,1,1)$ to $\Omega H_2^{1,1}$ and it only remains to show that we can describe $\Omega\mathcal{L}_2^{1,1}$ by linear equations in period coordinates.

By a tilde we denote the lift of an element in the relative homology group of Y to the one of X. Integrating these paths, the only interesting equations we get are

$$\int_{\tilde{a}_2} \omega = \int_{\tilde{a}_2} p^*\upsilon = \int_{a_2} \upsilon = \int_{a_2} \varphi^*\upsilon = \int_{\varphi_* a_2} \upsilon = \int_{a_1} \upsilon - \int_{a_2} \upsilon = \int_{\tilde{a}_1} \omega - \int_{\tilde{a}_2} \omega \quad \text{and}$$

$$\int_{\tilde{b}_1} \omega = \int_{\varphi_* b_1} \upsilon = \int_{b_2} \upsilon = \int_{\tilde{b}_2} \omega.$$

Let us denote the integral over $\tilde{b}_i$ by B_i and the integral over $\tilde{a}_i$ by A_i for $i = 1, 2$. Then we can denote the above equations shortly as

$$2A_2 - A_1 = 0 \quad \text{and} \quad B_1 = B_2.$$

These are two independent equations describing an affine invariant submanifold of at most dimension $5 - 2 = 3$.

Furthermore, the dimension is at least 3, since in Figure 7.7 we have three independent edges. $\square$

Let us observe three things. Firstly, the map φ cannot be lifted to every covering. More precisely, the map φ can be lifted if and only if the gluing is given by 1-I, 4-IV or 3-III, 5-V or 1-I, 3-III, 4-IV, 5-V. This implies that the space $\Omega H_2^{1,1}$ is not connected.

Secondly, there are no more affine invariant submanifolds of $\Omega\mathcal{L}^2$ or $\Omega\mathcal{L}^{1,1}$ that are described by automorphisms of the covered surface. In particular, we cannot construct a descending chain of affine invariant submanifolds given by automorphisms, such that the codimension of two consecutive submanifolds is 1. Moreover, this procedure does not yield Teichmüller curves as in the case of translation coverings of the torus discussed in Section 6.4.

Thirdly, in contrast to the case of translation coverings of tori, we actually see the discrepancy between holomorphic automorphisms and affine holomorphic automorphisms. The automorphism group of a translation surface of genus 2 is C_2 or V_4, but there are considerably more possible automorphism groups for a Riemann surface of genus 2, see for example Cardona et al. [Car+99]. The last observation can be strengthened: Every automorphism of a torus is an affine map, since it can be lifted to an automorphism of the complex plane $\mathbb{C}$. This automorphism is of the form $z \mapsto az + b$ and thus affine.

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