Global solutions of the nonlinear Schrödinger equation with multiplicative noise

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M.Sc. Fabian Hornung

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Referent:	Prof. Dr. Lutz Weis
Korreferent:	PD Dr. Peer Christian Kunstmann
Korreferent:	Prof. Dr. Zdzislaw Brzezniak

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Abstract (English Version)

In this thesis, we investigate existence and uniqueness of solutions to the stochastic nonlinear Schrödinger equation (NLS), i.e. the NLS perturbed by a multiplicative noise.

First, we present a fixed point argument based on deterministic and stochastic Strichartz estimates. In this way, we prove local existence and uniqueness of stochastically strong solutions of the stochastic NLS with nonlinear Gaussian noise for initial values in $L^2(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d)$, respectively. Using a stochastic generalization of mass conservation, we show that the L^2-solution exists globally under an additional restriction of the noise.

In the second part, we develop a general existence theory for global martingale solutions of the stochastic NLS with a saturated Gaussian multiplicative noise. The proof is based on a modified Galerkin approximation and a limit process due to the tightness of the approximated solutions. As an application, we get existence results for the stochastic defocusing and focusing NLS and fractional NLS on various geometries like bounded domains with Dirichlet or Neumann boundary conditions as well as compact Riemannian manifolds.

The martingale solution constructed by the Galerkin method is not necessarily unique. For this reason, we independently show pathwise uniqueness of solutions to the stochastic NLS with linear conservative Gaussian noise. The proof works in special cases as 2D and 3D Riemannian manifolds and is based on spectrally localized Strichartz estimates.

In the last chapter, we replace the Gaussian noise by a Poisson noise in the Marcus form and transfer the proof of existence of martingale solutions to this case.

Abstract (German Version)

Wir untersuchen die Existenz und Eindeutigkeit von Lösungen der stochastischen nichtlinearen Schrödinger-Gleichung (NLS), d.h. einer durch multiplikatives Rauschen gestörten NLS.

Zunächst präsentieren wir ein Fixpunktargument basierend auf deterministischen und stochastischen Strichartz-Abschätzungen. Auf diese Art und Weise zeigen wir die lokale Existenz und Eindeutigkeit starker Lösungen der stochastischen NLS mit nichtlinearem Gaußschen Rauschen für Anfangswerte in $L^2(\mathbb{R}^d)$ beziehungsweise $H^1(\mathbb{R}^d)$. Basierend auf einer stochastischen Verallgemeinerung der Massenerhaltung beweisen wir anschließend, dass die L^2-Lösung für alle Zeiten existiert, falls das Rauschen einer zusätzlichen Einschränkung genügt.

Im zweiten Teil der Arbeit entwickeln wir eine allgemeine Existenztheorie für Martingallösungen der stochastischen NLS mit global Lipschitz-stetigem Gaußschem Rauschen. Der Beweis beruht auf einer modifizierten Galerkin-Approximation, der Straffheit der Näherungslösungen und einem Grenzwertprozess. Als Anwendung des allgemeinen Resultats erhalten wir Existenzresultate für die stochastische defokussierende und fokussierende NLS sowie die gebrochene NLS für verschiedene Geometrien wie beschränkte Gebiete mit Dirichlet- oder Neumann-Randbedingungen und kompakte Riemannsche Mannigfaltigkeiten.

Die Martingallösung, die wir durch das Galerkin-Verfahren erhalten, ist nicht notwendigerweise eindeutig bestimmt. Dies motivert uns, die pfadweise Eindeutigkeit von Lösungen der stochastischen NLS mit linearem, konservativem Gaußschen Rauschen zu gesondert zu zeigen. Der Beweis funktioniert in Spezialfällen wie zwei- oder dreidimensionalen Riemannschen Mannigfaltigkeiten und beruht auf spektral lokalisierten Strichartz-Abschätzungen.
Im letzten Kapitel ersetzen wir das Gaußsche durch ein Poissonsches Rauschen und übertragen den Existenzbeweis für Martingallösungen.

Contents

1. Introduction

This thesis is devoted to existence and uniqueness theorems for the *stochastic nonlinear Schrödinger equation*

$$\begin{cases} \mathrm{i}\partial_t u(t,x) = -\Delta_x u(t,x) \pm |u(t,x)|^{\alpha-1}u(t,x) + \text{“noise”}, \\ \quad u(0,x) = u_0(x). \end{cases} \tag{1.1}$$

The noise is Gaussian and Poissonian and acts as a random potential used to incorporate spatial and temporal fluctuations of certain parameters in a physical model. Typically, the noise depends on the solution u itself and is therefore called *multiplicative.* As its deterministic counterpart, (1.1) occurs in applications such as nonlinear optics in a Kerr-medium as well as the description of Bose-Einstein condensates and deep-water waves. For example, the cubic equation

$$\mathrm{i}\partial_t u(t,x) = -\Delta_x u(t,x) - |u(t,x)|^2 u(t,x) + \partial_t \sigma(x,t) u(t,x), \qquad t \geq 0, \quad x \in \mathbb{R}^2, \tag{1.2}$$

with linear noise was proposed in [10] as a model for Scheibe aggregates under random temperature effects. Here, $\partial_t \sigma$ is a real-valued Gaussian process with correlation

$$\mathbb{E}\big[\partial_t \sigma(x_1,t_1)\partial_t \sigma(x_2,t_2)\big] = c(x_1,x_2)\delta(t_1-t_2), \qquad t_1,t_2 \geq 0, \quad x_1,x_2 \in \mathbb{R}^2.$$

In this thesis, we consider spatially correlated noise

$$\sigma(t,x) = \sum_{m=1}^{\infty} e_m(x)\beta_m(t), \qquad t \geq 0, \quad x \in \mathbb{R}^2,$$

with essentially bounded coefficients $(e_m)_{m\in\mathbb{N}}$ and a sequence $(\beta_m)_{m\in\mathbb{N}}$ of independent Brownian motions. In this case, the spatial correlation reads

$$c(x_1,x_2) = \sum_{m=1}^{\infty} e_m(x_1)e_m(x_2), \qquad t \geq 0, \quad x_1,x_2 \in \mathbb{R}^2.$$

Since the Brownian motion is not differentiable in time, the expression $\partial_t\sigma(x,t)$ only represents a distribution and the question arises how to interpret the product on the right hand site of (1.2). We use the formulation as a Stratonovich stochastic evolution equation

$$\begin{cases} \mathrm{d}u(t) = \Big(\mathrm{i}\Delta u(t) \pm \mathrm{i}|u(t)|^{\alpha-1}u(t)\Big)\mathrm{d}t - \mathrm{i}\displaystyle\sum_{m=1}^{\infty} e_m u(t) \circ \mathrm{d}\beta_m(t), \\ u(0) = u_0, \end{cases} \tag{1.3}$$

in a Hilbert space containing the spatial dependence. The Stratonovich differential $\circ$ is defined through its connection

$$-\mathrm{i}\sum_{m=1}^{\infty} e_m u \circ \mathrm{d}\beta_m = -\frac{1}{2}\sum_{m=1}^{\infty} e_m^2 u - \mathrm{i}\sum_{m=1}^{\infty} e_m u \,\mathrm{d}\beta_m$$

to the Itô differential. The NLS with Stratonovich noise is the natural generalization of the deterministic equation since the L^2-norm of the solution is conserved. For this reason, the real-valued noise is often called *conservative* and the quantity $|u|^2$ is a probability density if the initial value is also normalized in L^2.

Apart from the NLS with Gaussian noise, we also treat the NLS perturbed by discrete random jumps. In the physical literature, this has been proposed in [125] and [126] as a model to incorporate amplification of a signal in a fiber at random isolated locations caused by material inhomogeneities. In this case, the term σ in the problem (1.2) has the from

$$\sigma(t) = \sum_{m=1}^{N} e_m L_m(t), \qquad t \geq 0,$$

with spatial coefficients e_m and an $\mathbb{R}^N$-valued Lévy process $(L(t))_{t\geq 0}$ of pure jump type. The problem is formulated as stochastic evolution equation

$$\begin{cases} \mathrm{d}u(t) = \Big(\mathrm{i}\Delta u(t) \pm \mathrm{i}|u(t)|^{\alpha-1}u(t)\Big)\mathrm{d}t - \mathrm{i}\sum_{m=1}^{N} e_m u(t) \diamond \mathrm{d}L_m(t), \\ u(0) = u_0, \end{cases} \tag{1.4}$$

with the Marcus product $\diamond$ which guarantees the conservation of the L^2-norm of solutions and is therefore the best substitute for the Stratonovich product in the case of discontinuous noise.

Historic sketch and overview of the literature

To give an idea of the techniques which are relevant for solving stochastic equations like (1.3) and (1.4), we would like to give a historical and methodical overview of the previous mathematical research. In detail, we describe the analysis of the deterministic NLS and the study of stochastic partial differential equations (SPDE) since these fields particularly influenced this thesis.

The deterministic NLS

$$\begin{cases} \partial_t v(t,x) = \mathrm{i}\Delta_x v(t,x) - \mathrm{i}\lambda|v(t,x)|^{\alpha-1}v(t,x), \\ v(0,x) = v_0(x), \end{cases} \tag{1.5}$$

has been a very rich subject of study for many mathematicians and physicists in the previous decades. On the one hand, this has been motivated by its appearance in applications, where the dimensions $d = 1, 2, 3$ and the power $\alpha = 3$ are important. On the other hand, the mathematical interest in the NLS comes from the difficulties posed by the combination of the linear part without regularization effect and the power-nonlinearity. Moreover, the NLS serves as a model dispersive Hamiltonian partial differential equation since it has a particularly strong dispersive behavior and is technically simpler than comparable equations like the nonlinear wave equation, the Korteweg-de Vries equation and the wave maps equation. The most important focus of mathematical research on the NLS has been the appearance of different phenomena like global wellposedness and blow-up depending on the choice of the parameters $\lambda \in \{-1, 1\}$ and $\alpha > 1$.

Let us describe the main properties of the NLS. As a consequence of the Hamiltonian structure, sufficiently smooth solutions v obey the conservation laws

$$\mathcal{M}(v) := \|v\|_{L^2}^2 \equiv \mathrm{const},$$

$$\mathcal{E}(v) := \frac{1}{2} \| (-\Delta)^{\frac{1}{2}} v\|_{L^2}^2 - \frac{\lambda}{\alpha+1} \|v\|_{L^{\alpha+1}}^{\alpha+1} \equiv \text{const.} \tag{1.6}$$

Typically, $\mathcal{M}(v)$ and $\mathcal{E}(v)$ are called *mass* and *energy*. The conservation of mass indicates that L^2 is a natural space to look for global solutions of (1.5). We observe that the parameter λ enters into the energy. Nonlinearities with $\lambda = -1$ are called *defocusing* and the notion *focusing* refers to the case $\lambda = 1$. The sum of mass and energy dominates the H^1-norm if $\lambda = -1$ and in view of the Sobolev embedding $H^1 \hookrightarrow L^{\alpha+1}$, the energy is well-defined in H^1 for

$$\alpha \in \left(1, 1 + \frac{4}{(d-2)_+}\right]. \tag{1.7}$$

Moreover, the NLS (1.5) is invariant under the scaling

$$v(t,x) \mapsto \mu^{\frac{2}{\alpha-1}} v(\mu^2 t, \mu x) \tag{1.8}$$

and the energy tolerates the scaling if and only if $\alpha = 1 + \frac{4}{d-2}$ for $d \geq 3$. These observations lead to the conjecture that (1.7) is the right range of exponents to study global wellposedness of the NLS in H^1. The mass is invariant under the scaling (1.8) for $\alpha = 1 + \frac{4}{d}$ and thus, wellposedness in L^2 can be studied for $\alpha \in (1, 1 + \frac{4}{d}]$. For further details on the scaling heuristic, we refer to Tao [114].

Starting in the 1970s, there have been many attempts to use the conservation laws (1.6) for an existence theory of (1.5) based on the following strategy:

1) Choose a suitable approximation of (1.5).
2) Deduce variants of the conservation laws (1.6) for the approximation and use them for uniform bounds.
3) Pass to the limit via a compactness argument and obtain a solution of (1.5).

The most popular choice in point 1) has been the Galerkin method in Gajewski [55], Ginibre/Velo [58] and Vladimirov [127]. We also would like to mention different approximation techniques like mollifying in Ginibre/Velo [57] or Yosida type approximations in Cazenave's monograph [36] and in Okazawa/Suzuki/Yokota [102]. The advantage of the procedure described above lies in the fact that it only employs basic tools from the theory of partial differential equations and functional analysis and hence, it can be formulated for various geometries and boundary conditions. However, this method leads to a solution which is not necessarily unique and only weakly continuous in H^1. In a second step, uniqueness can be approached by the formula

$$\|v_1(t) - v_2(t)\|_{L^2}^2 = 2 \operatorname{Re} \int_0^t \big\langle v_1(s) - v_2(s), -\mathrm{i}|v_1(s)|^{\alpha-1} v_1(s) + \mathrm{i}|v_2(s)|^{\alpha-1} v_2(s) \big\rangle \mathrm{d}s \tag{1.9}$$

for the difference of two solutions v_1 and v_2 of (1.5) and improvements of the classical Gronwall argument. As a key ingredient in this argument, one has to show that solutions have spatial L^p-estimates of the form

$$\|u\|_{L^2(J, L^p)} \lesssim 1 + (|J|p)^{\frac{1}{2}} \tag{1.10}$$

for all $p \in (1, \infty)$. In one and two dimensions, the Sobolev inequality and its limiting case, the Moser-Trudinger inequality, provide these bounds and as a consequence, the uniqueness of weak solutions in $L_t^\infty H_x^1$. We refer to Vladimirov, [127], and Ogawa/Ozawa, [100],[101], for articles in this direction. We further remark that, originally, the strategy to use estimates of the type (1.10) to prove uniqueness was developed by Yudovitch, [131], for the Euler equation.

To overcome the deficits of the previous strategy and get unique strong solutions, an alternative approach the NLS has been developed in the mid of the 1980s. The idea was to construct local solutions up to a maximal existence time T^* via the fixed point equation

$$v(t) = e^{\mathrm{i}t\Delta}v_0 - \mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-s)\Delta}|v(s)|^{\alpha-1}v(s)\mathrm{d}s, \qquad t \in [0, T^*), \tag{1.11}$$

and a contraction argument. Due to the algebra property of L^2-based Sobolev spaces H^s for $s > \frac{d}{2}$ and the fact that $\mathrm{i}\Delta$ generates a unitary C_0-semigroup, the abstract theory for evolution equations guarantees a unique local solution for initial values in H^s. In [24], Brézis and Gallouet combined an argument of this type with the conservation laws to prove global wellposedness in H^2. At lower regularity, however, it is not clear how to find local solutions in the first place since the nonlinearity maps neither L^2 nor H^1 into itself.

At this point, another characteristic property of the NLS comes into play: the dispersive character of the linear part. Physically, this means that waves of different frequency propagate with different velocity and thus, wave packages spread out to infinity while the complete mass is constant. In the model case $M = \mathbb{R}^d$, this can be mathematically expressed by the estimates

$$\left\|e^{\mathrm{i}t\Delta}v_0\right\|_{L^2} = \|v_0\|_{L^2}, \qquad \left\|e^{\mathrm{i}t\Delta}v_0\right\|_{L^\infty} \le (4\pi|t|)^{-\frac{d}{2}} \|v_0\|_{L^1}, \qquad t \neq 0. \tag{1.12}$$

The second estimate reflects a gain of spatial integrability by the solution of the linear equation which can be improved to gain integrability in space-time using tools from harmonic analysis and interpolation theory. As a result, we obtain the Strichartz estimates

$$\|t \mapsto e^{\mathrm{i}t\Delta}v_0\|_{L_t^{q_1}L_x^{p_1}} \lesssim \|v_0\|_{L_x^2}, \qquad \left\|t \mapsto \int_0^t e^{\mathrm{i}(t-s)\Delta}f(s)\mathrm{d}s\right\|_{L_t^{q_1}L_x^{p_1}} \lesssim \|f\|_{L_t^{q_2'}L_x^{p_2'}} \tag{1.13}$$

for exponent pairs $(p_i, q_i) \in [2, \infty]^2$ with

$$\frac{2}{q_i} + \frac{d}{p_i} = \frac{d}{2}, \qquad (q_i, p_i, d) \neq (2, \infty, 2), \qquad i = 1, 2. \tag{1.14}$$

The first estimate in (1.13) occurred in the special case $p_1 = q_1 = 2 + \frac{4}{d}$ in the article [111] by Strichartz. Later on, Yajima [129] and Ginibre/Velo [58], [59] obtained the general Strichartz estimates for the free evolution and the convolution term in (1.13) for non-endpoints, i.e. $q_1, q_2 \neq 2$. Finally, Keel and Tao proved the endpoint case in [76]. Based on Strichartz estimates and the conservation laws (1.6), a unique global solution $v \in C_b(\mathbb{R}, H^1(\mathbb{R}^d)) \cap L^q(\mathbb{R}, W^{1,\alpha+1}(\mathbb{R}^d))$ of the NLS can be constructed for an initial value $v_0 \in H^1(M)$ if

$$\alpha \in \begin{cases} (1, 1 + \frac{4}{(d-2)_+}), & \lambda = -1, \\ (1, 1 + \frac{4}{d}), & \lambda = 1. \end{cases}$$

As another benefit of this argument besides the uniqueness and the strong continuity of the solutions, we would like to mention that it can also be used to obtain an L^2-theory for the NLS. For $\alpha \in (1, 1 + \frac{4}{d})$, one can prove global wellposedness in $C_b(\mathbb{R}, L^2(\mathbb{R}^d)) \cap L^q(\mathbb{R}, L^{\alpha+1}(\mathbb{R}^d))$. For critical exponents, the technique presented above only leads to local wellposedness since the blow-up criterion is not strong enough to be accessed with the conservation laws. The fixed point argument can be traced back to the articles [119] by Tsutsumi and [75] by Kato. We also refer to Cazenave [36], Tao [114] and Linares/Ponce [88] for a detailed treatment of the deterministic NLS as well as other dispersive equations. These monographs also include an overview of the progress on the critical NLS on the full space $\mathbb{R}^d$ in the last twenty years.

We would like to mention another development in the research on the NLS which has strongly influenced this thesis. Unlike the case of the full space $\mathbb{R}^d$ described above, the dispersion of the Schrödinger group is far less well understood on other geometries like Riemannian manifolds or domains in $\mathbb{R}^d$. Obviously, a global dispersive estimate as in (1.12) cannot be true for operators like the Neumann Laplacian on a bounded domain or the Laplace-Beltrami operator on a compact manifold since constant functions are integrable in this case and solve the linear Schrödinger equation. For compact manifolds, however, Burq, Gérard and Tzvetkov, [35], were able to prove a spectrally localized short-time dispersive estimate

$$\|e^{\mathrm{i}t\Delta_g}\varphi(h^2\Delta_g)v_0\|_{L^\infty} \lesssim |t|^{-\frac{d}{2}}\|v_0\|_{L^1}, \qquad t\in[-\alpha h,\alpha h]\setminus\{0\}, \tag{1.15}$$

for some $\alpha>0$ and all $h\in(0,1]$. Combining this result with the abstract Strichartz estimates by Keel and Tao [76] and Littlewood-Paley theory, they could prove Strichartz estimates of the type

$$\|t\mapsto e^{\mathrm{i}t\Delta}v_0\|_{L^q(I,L^p)}\lesssim\|v_0\|_{H^{\frac{1}{q}}}, \qquad \left\|t\mapsto\int_0^t e^{\mathrm{i}(t-s)\Delta}f(s)\mathrm{d}s\right\|_{L^q(I,L^p)}\lesssim\|f\|_{L^1(I,H^{\frac{1}{q}})} \tag{1.16}$$

for exponents (p,q) as in (1.14) with $p<\infty$. In particular, the weaker dispersive behavior is reflected in the regularity loss $\frac{1}{q}$ and the fact that the convolution estimate does not involve general exponents on the right hand side. These deficits restrict the application area of Strichartz estimates for the construction of local strong solutions to higher regularity compared to the $\mathbb{R}^d$-case. Burq, Gérard and Tzvetkov obtained a unique solution

$$v\in C([-T_0,T_0],H^s(M))\cap L^p(-T_0,T_0;L^\infty(M))$$

of (1.5) for $v_0\in H^s(M)$, $p>\alpha-1$ and

$$s>\frac{d}{2}-\frac{1}{\max\{\alpha-1,2\}}.$$

In particular, the conservation laws (1.6) only yield global wellposedness in dimension one and two. Remarkably, (1.16) could be used in three dimensions to provide the L^p-estimates for the uniqueness argument based on (1.9). Compared to the Sobolev-type arguments used before this reflects a gain of $\frac{1}{2}$-regularity by Strichartz estimates. In their article [16], Bernicot and Samoyeau generalized (1.16) to manifolds with bounded geometry under a slightly higher loss $\frac{1}{q}+\varepsilon$ of regularity. For similar estimates on domains and manifolds with boundary, we refer to Anton [4] and Blair/Smith/Sogge [18], [19].

Besides the theory of the deterministic NLS, the second branch of mathematical analysis underlying this thesis is the theory of stochastic partial differential equations (SPDE). Typically, these equations are formulated as Hilbert or Banach space valued stochastic differential equations For example, the integral form

$$u(t)=u_0+\int_0^t\Big(\mathrm{i}\Delta u(s)-\mathrm{i}\lambda|u(s)|^{\alpha-1}u(s)-\frac{1}{2}\sum_{m=1}^\infty e_m^2u(s)\Big)\mathrm{d}s-\mathrm{i}\sum_{m=1}^\infty\int_0^t e_mu(s)\,\mathrm{d}\beta_m(s) \tag{1.17}$$

and the mild form

$$u(t)=e^{\mathrm{i}t\Delta}u_0+\int_0^t e^{\mathrm{i}(t-\cdot)\Delta}\Big(-\mathrm{i}\lambda|u|^{\alpha-1}u-\frac{1}{2}\sum_{m=1}^\infty e_m^2u\Big)\mathrm{d}s-\mathrm{i}\sum_{m=1}^\infty\int_0^t e^{\mathrm{i}(t-\cdot)\Delta}e_mu\,\mathrm{d}\beta_m \tag{1.18}$$

are two equivalent ways of viewing our problem (1.3). The last equation somehow corresponds to the deterministic fixed point problem (1.11). Both (1.17) and (1.18) employ the Itô integral which goes back to the pioneering works [68] and [69] by Kiyosi Itô in the finite dimensional setting. Since his construction merely depends on the existence of an inner product, the construction of the stochastic integral was soon generalized to general Hilbert spaces. For a comprehensive treatment of SPDE in Hilbert spaces, we refer to the monographs [40] by da Prato and Zabczyk, [90] by Liu and Röckner and [37] by Chow. We also highlight some results on stochastic integrals in Banach spaces which are useful to estimate the stochastic convolution term in (1.18). In the class of martingale type 2 Banach spaces which include L^p for $p \geq 2$, the stochastic integral was investigated among others by Dettweiler [46] and Brzezniak [25], [26]. In [124], van Neerven, Veraar and Weis were able to generalize the stochastic integral to UMD spaces. In the particular case of L^p-spaces for $p \in (1, \infty)$, Antoni improved this theory in his diploma thesis [5] and his PhD thesis [6] by proving a stronger maximal inequality for the stochastic integral.

There is another branch of stochastic analysis which should be mentioned here. The Brownian motion in Itô's stochastic integration theory could be replaced by more general continuous semi-martingales and, more importantly in view of problem (1.4), by Lévy processes. We refer to the monographs [74] by Karatzas and Shreve and [7] by Applebaum for the finite-dimensional case. Stochastic PDE with Lévy noise are treated comprehensively by Peszat and Zabczyk [107].

Let us continue with an overview of the literature dealing with the stochastic NLS (1.3) and (1.4) with a focus on the techniques which have been developed so far. First, we would like to remark that real-valued constant coefficients e_m are not interesting since in this case, the equation can be reduced to the deterministic NLS by a simple gauge transform

$$u(t) = e^{-\mathrm{i} \sum_{m=1}^{\infty} e_m \beta_m(t)} y(t). \tag{1.19}$$

For more general noise, however, the mathematical study of the stochastic NLS is also strongly inspired by the deterministic NLS and employs the methods we described above. In particular, generalizations of the conservation laws (1.6) and stochastic analogues of the Strichartz estimates in (1.13) and (1.16) are crucial to transfer deterministic techniques to the stochastic case. For example, the mass of a solution to the stochastic NLS satisfies the evolution formula

$$\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 - 2 \sum_{m=1}^{\infty} \int_0^t \operatorname{Re} \left(u(s), \mathrm{i} e_m u(s) \right)_{L^2} \mathrm{d}\beta_m(s) \tag{1.20}$$

almost surely for all $t \geq 0$. Thus, we have to give up the property of mass conservation in the non-conservative case with $\operatorname{Re} e_m \neq 0$ for some $m \in \mathbb{N}$. At least, the mass is still a martingale and thus constant in the expected value. According to [11] and [14], this kind of noise represents a stochastic continuous measurement along the observables e_m, $m \in \mathbb{N}$. Similarly, the deterministic energy conservation has a generalization in form of an evolution formula containing stochastic and deterministic integrals. Under suitable assumptions the terms induced by the noise can be treated as perturbations of the main part of the equation. In the case of linear noise, for example, the additional terms behave reasonably well and can be treated in a Gronwall argument. This results in estimates of the type

$$\mathbb{E}\big[\sup_{t \in [0,T]} \|u(t)\|_{L^2}^q \big] \leq C_1 e^{C_2 T}, \qquad \mathbb{E}\big[\sup_{t \in [0,T]} \|u(t)\|_{H^1}^q \big] \leq C_3 e^{C_4 T} \tag{1.21}$$

for all $q \in [1, \infty)$ and $T > 0$. These estimates are still sufficient to globalize local solutions and to prove existence by the compactness method. The appearance of the expected value in (1.21) is quite characteristic for the theory. This is due to the fact that pathwise estimates for the Itô integral are generally not available.

As in the deterministic case, most of the interest has been caught by the problem on $\mathbb{R}^d$. The research started with the article [41] by de Bouard and Debussche who studied the stochastic NLS with conservative linear noise in $L^2(\mathbb{R}^d)$. They transfered the fixed point argument we explained above to (1.18) using the inequality

$$\left\| t \mapsto \sum_{m=1}^{\infty} \int_0^t e^{\mathrm{i}(t-s)\Delta} e_m u(s)\,\mathrm{d}\beta_m(s) \right\|_{L^r(\Omega,L^q(0,T;L^p))} \lesssim \|u\|_{L^r(\Omega,L^\infty(0,T;L^2))} \tag{1.22}$$

to complement the deterministic Strichartz estimates from (1.13). In this way, they were able to estimate the third term on the right hand side of (1.18). The proof of (1.22) is based on the Burkholder inequality in $L^p(\mathbb{R}^d)$ and the dispersive estimate from (1.12). We observe that similarly to (1.21), the expected value shows up in the estimate (1.22) and Strichartz estimates do not gain integrability in Ω. Nevertheless, de Bouard and Debussche managed to close a fixed point argument for an approximated equation arising by a cut-off of the nonlinearity. The solution of this equation solves the original one up to a certain stopping time and therefore, it is a local solution. Although this leads to a more complicated blow-up alternative including the $L_t^q L_x^{\alpha+1}$-norm, the authors were able to show that the solution exists globally. Since the estimate (1.22) does not work for arbitrary Strichartz pairs (p,q), the authors had to impose an unsatisfactory additional restriction of the admissible exponents. Similar results were obtained by de Bouard, Debussche in [43] for the conservative stochastic NLS in $H^1(\mathbb{R}^d)$. Subsequently, the same authors studied blow-up behavior in [42],[44].

In [30], Brzeźniak and Millet derived the estimate

$$\left\| t \mapsto \sum_{m=1}^{\infty} \int_0^t e^{\mathrm{i}(t-s)\Delta} b_m(s)\,\mathrm{d}\beta_m(s) \right\|_{L^r(\Omega,L^q(0,T;L^p))} \lesssim \|(b_m)_m\|_{L^r(\Omega,L^2([0,T]\times\mathbb{N};L^2))} \tag{1.23}$$

for the stochastic convolution associated to the Schrödinger group. Compared to (1.22), (1.23) has two important advantages. On the one hand, it is true for arbitrary Strichartz pairs (p,q) and an L_t^2-norm appears on the right-hand side instead of an L_t^∞-norm. This reflects a gain of integrability in both time and space and makes it possible to deal with nonlinear noise. On the other hand, the proof is based on the Strichartz estimate for the free evolution in (1.13) and also works if one has only access to Strichartz estimates with loss of regularity as (1.16). The estimate (1.22) does not enjoy this flexibility since the dispersive estimate is a particular feature of the Schrödinger group on $\mathbb{R}^d$. Brzezniak and Millet used their stochastic Strichartz estimate to generalize the argument by Burq, Gérard and Tzvetkov to the stochastic setting and proved global existence and uniqueness of the stochastic NLS with nonlinear noise on $2D$ compact manifolds. In one of the main results of this thesis, see Theorem 1, we use (1.23) in $\mathbb{R}^d$ to improve the results from [41] significantly.

Motivated by the goals to get rid of the restriction of the exponents from [41] and to incorporate non-conservative noise, Barbu, Röckner and Zhang approached the problem (1.3) on $\mathbb{R}^d$ in their article [11]. For a finite dimensional noise $W = \sum_{m=1}^{M} e_m \beta_m$, they reduced (1.3) to a non-autonomous NLS with random coefficients via a generalization of the transform (1.19). The authors call this procedure *rescaling approach.* Generally speaking, the main advantage of this strategy is the fact that the equation can be solved pathwisely. This allows to use the fixed point argument for the deterministic NLS without the cut-off procedure by de Bouard and Debussche as soon as Strichartz estimates for non-autonomous operators of the form

$$A(s) := \mathrm{i}\,(\Delta + b(s)\cdot\nabla + c(s)) \tag{1.24}$$

are available. In particular, the full range of subcritical exponents $\alpha \in (1, 1+\frac{4}{d})$ is accessible and a transfer of the argument to higher regularity is easier compared to [41]. This has been investigated by the same authors in [12]. The Strichartz estimates for (1.24) can be obtained from Marzuola, Metcalfe and Tataru [95]. In this context, we would like to mention the recent preprint [134] by Zhang who adapts the argument from [95] to obtain pathwise Strichartz estimates with an emphasis on the rescaling approach for stochastic dispersive equations on $\mathbb{R}^d$. Most notably, the rescaling approach can be used to show that large conservative noise has a stabilizing effect on the NLS in the sense that it prevents blow-up with a high probability. In [13], Barbu, Röckner and Zhang discovered this effect for the focusing NLS in $H^1(\mathbb{R}^d)$ with super-critical $\alpha \in (1+\frac{4}{d}, 1+\frac{4}{(d-2)_+})$. However, the assumptions on b and c in the articles [95] and [134] lead to severe additional requirements on the noise coefficients e_m, $m \in \mathbb{N}$, which can be viewed as the main disadvantage of this approach besides the fact that the transformation only works for linear noise.

To the best of our knowledge, there is only one article so far which employs an approximation argument to construct an analytically weak solution of a stochastic NLS. In [77], Keller and Lisei transfered the classical Galerkin argument by Gajewski [54] to the stochastic setting and obtained existence and uniqueness for the NLS on a closed interval with Neumann boundary conditions.

Stochastic nonlinear Schrödinger equations with jump noise as in (1.4) are less well studied in the literature compared to their Gaussian counterpart (1.3). In [45], de Bouard and Hausenblas consider a similar problem as (1.4) with more general assumptions on the noise. They also allow more general jumps in a function space $Z \hookrightarrow L^2(\mathbb{R}^d)$ and particularly, infinite dimensional Marcus noise is admissible in their framework. For $\gamma < 1$ for initial values $u_0 \in H^1(\mathbb{R}^d)$ which additionally satisfy $|\cdot| u_0 \in L^2(\mathbb{R}^d)$, the authors obtain a martingale solution with càdlàg-paths in $H^\gamma(\mathbb{R}^d)$. Other articles like [125] and [126] by Villarroel and Montero studied the NLS with jump noise with a focus on modeling aspects and the qualitative behavior of solutions rather than developing a wellposedness theory.

Formulation of the problems

To prepare the presentation of the contents and the main results, we introduce some notations and give a precise unified formulation of the equations we consider in this thesis and solve in various special cases.

Suppose that u_0 is an initial value, $\lambda \in \{-1, 1\}$, $\alpha > 1$ and A is a non-negative selfadjoint operator on a Hilbert space $L^2(M)$. Moreover, we take a sequence of independent Brownian motions $(\beta_m)_{m\in\mathbb{N}}$ and coefficient functions $e_m : M \to \mathbb{C}$ as well as $g : [0,\infty) \to \mathbb{R}$ specifying the multiplicative Gaussian noise. In this setting, we consider the Itô stochastic evolution equation

$$\begin{cases} \mathrm{d}u = \Big(-\mathrm{i}Au - \mathrm{i}\lambda |u|^{\alpha-1}u - \frac{1}{2}\sum_{m=1}^\infty |e_m|^2 g(|u|^2)^2 u \Big)\mathrm{d}t - \mathrm{i}\sum_{m=1}^\infty e_m g(|u|^2) u \,\mathrm{d}\beta_m, \\ u(0) = u_0. \end{cases} \tag{1.25}$$

To model random jumps, we employ a compensated time-homogeneous Poisson random measure on $\mathbb{R}^N$ for some $N \in \mathbb{N}$ which induces a Lévy process L via

$$L(t) = (L_1(t), \dots, L_N(t))^T = \int_0^t \int_{\{|l|\le 1\}} l\, \tilde{\eta}(\mathrm{d}s, \mathrm{d}l), \qquad t \ge 0.$$

Using the Marcus product $\diamond$ which can be viewed as an analogue of the Stratonovich noise in the discontinuous case, we treat the equation

$$\begin{cases} \mathrm{d}u(t) = \big(-\mathrm{i}Au(t) - \mathrm{i}\lambda|u|^{\alpha-1}u\big)\mathrm{d}t - \mathrm{i}\sum_{m=1}^{N} e_m u(t-) \diamond \mathrm{d}L_m(t), \\ u(0) = u_0. \end{cases} \tag{1.26}$$

Content and main results of this thesis

In order to improve the presentation of the main results, we have outsourced various contents to the second chapter. There, we prepare the study of the stochastic NLS by providing the most important solution concepts, formulae for the mass of solutions as well as deterministic and stochastic Strichartz estimates in a unified framework. Moreover, we present compactness results in particular function spaces suitable for the Gaussian and the Poissonian noise together to highlight the similarities, but also the differences between the continuous and the càdlàg case.

The third chapter is based on the article [62] by the author of this thesis. We study the problem (1.25) in the most classical situation with $A = -\Delta$ and initial values $u_0 \in L^2(\mathbb{R}^d)$ and $u_0 \in H^1(\mathbb{R}^d)$. We allow the particularly difficult case of power-type nonlinear noise, i.e.

$$g(r) = r^{\frac{\gamma-1}{2}}, \qquad \gamma \geq 1.$$

Our approach to the problem is inspired by de Bouard and Debussche [41],[43]. However, we replace their estimate (1.22) of the stochastic convolution by the improved one (1.23) due to Brzezniak and Millet. This leads to different conditions of the coefficients e_m, $m \in \mathbb{N}$, and more notably, the complete range of exponents and nonlinear noise with $\gamma > 1$. Our results for $u_0 \in L^2(\mathbb{R}^d)$ can be combined in the following theorem.

Theorem 1. Let $u_0 \in L^2(\mathbb{R}^d)$, $A = -\Delta$ and $(e_m)_{m\in\mathbb{N}} \subset L^\infty(\mathbb{R}^d)$ with $\sum_{m=1}^\infty \|e_m\|_{L^\infty}^2 < \infty$. Then, the following assertions hold:

a) Let $\alpha \in (1, 1+\frac{4}{d}]$ and $\gamma \in [1, 1+\frac{2}{d}]$. Then, there exists a unique local solution of (1.25) in $L^2(\mathbb{R}^d)$. Both stochastically and analytically, the solution is understood in the strong sense from Definition 2.1.

b) Let $\alpha \in (1, 1+\frac{4}{d})$ and $\gamma = 1$. Then, the solution from a) is global.

c) Let e_m be real valued for each $m \in \mathbb{N}$, $\alpha \in (1, 1+\frac{4}{d})$ and

$$1 < \gamma < \frac{\alpha-1}{\alpha+1}\frac{4+d(1-\alpha)}{4\alpha+d(1-\alpha)} + 1.$$

Then, the solution from a) is global.

Above, we presented the approaches of the articles [41] by de Bouard and Debussche and [11] by Barbu, Röckner and Zhang who also considered the stochastic NLS in $\mathbb{R}^d$. Now, we would like to classify Theorem 1 in view of [41] and [11]. In terms of the exponents α in the deterministic part of the equation, the Theorem 1 is identical to the result in [11]. However, it improves [41] since de Bouard and Debussche additionally assume $\alpha \in (1, 1+\frac{2}{d-1})$ for $d \geq 3$. Moreover, Theorem 1 is the first result which incorporates nonlinear noise excluded inherently by the rescaling approach and the fixed point argument based on (1.22). The assumption Barbu, Röckner and Zhang impose on the coefficients reads

$$e_m \in C_b^\infty(\mathbb{R}^d), \qquad \lim_{|\xi|\to\infty} \eta(\xi)\left(|e_m(\xi)| + |\nabla e_m(\xi)| + |\Delta e_m(\xi)|\right) = 0$$

with

$$\eta(\xi) := \begin{cases} 1+|\xi|^2, & d \neq 2, \\ (1+|\xi|^2)(\log(2+|\xi|^2))^2, & d = 2. \end{cases}$$

This reflects a significant restriction to regular and decaying coefficients. For technical reasons, the authors replace the series in (1.25) by a finite sum. In [133], they remark that the infinite case can also be handled under the strong summability condition

$$\sum_{m=1}^\infty \|e_m\|_{L^\infty}^2 < \infty, \qquad \sum_{m=1}^\infty \|\partial^\beta e_m\|_{L^\infty}^2 < \infty$$

for all multi-indices β. The assumptions by de Bouard and Debussche correspond to the square function estimate

$$\left\| \Big(\sum_{m=1}^\infty |e_m|^2 \Big)^{\frac{1}{2}} \right\|_{L^2(M)\cap L^{2+\delta}(\mathbb{R}^d)} < \infty \tag{1.27}$$

for some $\delta > 2(d-1)$. There is no strict inclusion between (1.27) and $\sum_{m=1}^\infty \|e_m\|_{L^\infty}^2 < \infty$ from Theorem 1. However, one might say that the latter condition is more natural since it guarantees that each e_m is a multiplier on $L^2(\mathbb{R}^d)$. Furthermore, the L^∞-assumption leads to the fact that a Hilbert-Schmidt operator $B(u) : \ell^2(\mathbb{N}) \to L^2(\mathbb{R}^d)$ can be defined by

$$B(u) f_m := e_m u, \qquad m \in \mathbb{N},$$

for the solution u and the canonical ONB $(f_m)_{m\in\mathbb{N}}$ of $\ell^2(\mathbb{N})$. This is useful since it allows to construct the stochastic integrals in (1.17) and (1.18) in $L^2(\mathbb{R}^d)$.

Let us state and review our results for initial values $u_0 \in H^1(\mathbb{R}^d)$.

Theorem 2. Let $u_0 \in H^1(\mathbb{R}^d)$, $A = -\Delta$ and assume $\sum_{m=1}^\infty \left(\|e_m\|_{L^\infty} + \|\nabla e_m\|_F\right)^2 < \infty$, where

$$F := \begin{cases} L^d(\mathbb{R}^d), & d \geq 3, \\ L^{2+\varepsilon}(\mathbb{R}^d), & d = 2, \\ L^2(\mathbb{R}^d), & d = 1, \end{cases}$$

for some $\varepsilon > 0$. Let $\alpha \in (1, 1+\frac{4}{d}] \cup (2, 1+\frac{4}{(d-2)_+}]$ and $\gamma \in [1, 1+\frac{2}{d}] \cup (2, 1+\frac{2}{(d-2)_+}]$. Then, there is a unique local solution of (1.25) in $H^1(\mathbb{R}^d)$. Both stochastically and analytically, the solution is understood in the strong sense from Definition 2.1.

This result is quite similar to Theorem 1 a) apart from the gap in the ranges for the exponents α and γ. This gap occurs due to technical difficulties arising if one combines the truncation method to deal with the stochastic terms with the fixed point argument in a ball which is usually used in the deterministic case. A similar, but even stronger restriction was observed by de Bouard and Debussche in their H^1-article [43]. With the rescaling approach of Barbu, Röckner and Zhang, see [12], the gap can be avoided and they obtain local wellposedness including a result on pathwise continuous dependence for all $\alpha \in (1, 1+\frac{4}{(d-2)_+})$. Concerning the other aspects, the comparison of Theorem 2 with [12] and [43] is similar to the L^2-case. The main advantages of Theorem 2 are the more convenient assumptions on the coefficients e_m, $m \in \mathbb{N}$, and the fact that it is the first result for nonlinear noise. Furthermore, we remark that Theorem 2 can also be used for global existence if one proves a rigorous evolution formula for the energy as in [12].

The contents of the fourth and fifth chapter are motivated by the question whether the global existence and uniqueness result by Brzezniak and Millet [30] on 2D compact manifolds could be generalized to higher dimensions. We recall from the overview of the deterministic NLS that this is significantly harder compared to the $\mathbb{R}^d$-case where the dimension only enters in the conditions for the admissible exponents. The difficulties arise from the loss in the Strichartz estimates and the weaker convolution estimate in (1.16) that make a fixed point argument less attractive. In fact, one cannot avoid to use Sobolev embeddings $H^{s,q} \hookrightarrow L^\infty$ which get more and more restrictive in higher dimensions. Thus, we follow a different strategy and separate the proofs of existence and uniqueness which are contained in the fourth and fifth chapter, respectively. In Chapter 4, we construct a solution via an approximation argument purely based on stochastic variants of the conservation laws (1.6) and observe that the manifold structure is not needed at all. As a main result, we get the existence of a martingale solution in the energy space in a quite general framework containing the stochastic NLS on compact Riemannian manifolds and bounded domains as the leading examples. Similar to Brzezniak and Millet, we treat a nonlinear noise under Lipschitz assumptions allowing e.g.

$$g(r) = \frac{r}{1+\sigma r}, \qquad g(r) = \frac{r(2+\sigma r)}{(1+\sigma r)^2}, \qquad g(r) = \frac{\log(1+\sigma r)}{1+\log(1+\sigma r)}, \qquad r \in [0,\infty),$$

for a constant $\sigma > 0$. Let us formulate this in the following theorem which is a generalization of the results in the preprint [29] by Brzezniak, Weis and the author of this thesis, where we only considered linear noise.

Theorem 3. Suppose that a) or b) or c) is true.

a) Let M be a compact Riemannian manifold, $A = -\Delta_g$ and $E_A = H^1(M)$.

b) Let $M \subset \mathbb{R}^d$ be a bounded domain, $A = -\Delta_D$ be the Dirichlet-Laplacian and $E_A = H^1_0(M)$.

c) Let $M \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $A = -\Delta_N$ be the Neumann-Laplacian and $E_A = H^1(M)$.

Choose the nonlinearity from $i)$ or $ii)$.

i) $F(u) = |u|^{\alpha-1}u$ with $\alpha \in \left(1, 1 + \frac{4}{(d-2)_+}\right)$,

ii) $F(u) = -|u|^{\alpha-1}u$ with $\alpha \in \left(1, 1 + \frac{4}{d}\right)$.

Assume $u_0 \in E_A$ and that the coefficients satisfy $\sum_{m=1}^\infty \|e_m\|_F^2 < \infty$ for

$$F := \begin{cases} H^{1,d}(M) \cap L^\infty(M), & d \geq 3, \\ H^{1,q}(M), & d = 2, \\ H^1(M), & d = 1, \end{cases} \tag{1.28}$$

for some $q > 2$ in the case $d = 2$. Suppose that $g : [0,\infty) \to \mathbb{R}$ is continuously differentiable and satisfies

$$\sup_{r>0} |g(r)| < \infty, \qquad \sup_{r>0} r|g'(r)| < \infty.$$

Then, (1.25) has a global martingale solution $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbb{F}}, u\right)$ in E_A which satisfies $u \in C_w([0,T], E_A)$ almost surely and $u \in L^q(\tilde{\Omega}, L^\infty(0,T;E_A))$ for all $q \in [1,\infty)$.

Notably, the different Laplacians in the settings a), b), c) of Theorem 3 can be replaced by more general elliptic operators. At least by changing the scale of exponents, the same result also applies to the fractional stochastic NLS, see Corollary 4.32 below.

We shortly sketch the proof of Theorem 3. In a first step, we approximate the original problem (1.25) by a modified Faedo-Galerkin equation

$$\begin{cases} \mathrm{d}u_n = \Big(-\mathrm{i}Au_n - \mathrm{i}\lambda P_n\left[|u_n|^{\alpha-1}u_n\right] - \frac{1}{2}\sum_{m=1}^{\infty} S_n\left[|e_m|^2 g(|u_n|^2)^2 u_n\right]\Big)\mathrm{d}t \\ \qquad\qquad - \mathrm{i}\sum_{m=1}^{\infty} S_n\left[e_m g(|u_n|^2)u_n\right]\mathrm{d}\beta_m, \\ u_n(0) = S_n u_0, \end{cases} \tag{1.29}$$

in a finite dimensional subspace H_n of $L^2(M)$ spanned by some eigenvectors of A. Here, $P_n : L^2(M) \to H_n$ is the standard orthogonal projection and $S_n : L^2(M) \to H_n$ is a selfadjoint operator derived from the Littlewood-Paley-decomposition associated to A. The reason for using the operators $(S_n)_{n\in\mathbb{N}}$ lies in the uniform estimate

$$\sup_{n\in\mathbb{N}} \|S_n\|_{\mathcal{L}(L^{\alpha+1})} < \infty$$

that turns out to be necessary in the estimates of the noise and which is false if one replaces S_n by P_n. Roughly speaking, S_n arises by smoothing the characteristic function which is associated to P_n via the functional calculus. This allows us to apply spectral multiplier theorems and get the uniform $L^{\alpha+1}$-boundedness. On the other hand, the orthogonal projections P_n are used in the deterministic part, because they do not destroy the cancellation effects which lead to the mass and energy conservation (1.6) in the deterministic setting. Combining the Itô formulae for mass and energy with Gronwall arguments, we obtain uniform a priori estimate

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\sup_{t\in[0,T]} \|u_n(t)\|_{E_A}^2\Big] < \infty \tag{1.30}$$

for every $T > 0$. Together with the Aldous condition $[A]$, a stochastic version of equicontinuity, the estimate (1.30) leads to the tightness of the sequence $(u_n)_{n\in\mathbb{N}}$ in the locally convex space

$$Z_T := C([0,T], E_A^*) \cap L^{\alpha+1}(0,T;L^{\alpha+1}(M)) \cap C_w([0,T], E_A),$$

where $C_w([0,T], E_A)$ denotes the space of continuous functions with respect to the weak topology in E_A. The construction of a martingale solution is similar to [31] and employs a limit argument based on Jakubowski's extension of the Skorohod Theorem to non-metric spaces and the Martingale Representation Theorem from [40], Chapter 8.

Theorem 3 holds in a very general setting, but it suffers from two the characteristic defects of solutions to stochastic PDE constructed by an approximation argument: On the one hand, u is only a martingale solution, i.e. stochastically weak solution. On the other hand, it is a priori unclear if u is unique. In view of the Yamada-Watanabe Theorem, see [85], [108] and [115] for results of this type which hold in infinite dimensions, one can overcome both of these defects by proving pathwise uniqueness. In the special case of two- and three-dimensional Riemannian manifolds, this is the content of the main result in Chapter 5.

Theorem 4. Let (M, g) be a Riemannian manifold without boundary of dimension $d \in \{2, 3\}$ and $A := -\Delta_g$ be the Laplace-Beltrami operator on M. Suppose that $g(r) = r$ for all $r \geq 0$ and $(e_m)_{m\in\mathbb{N}} \subset L^\infty(\mathbb{R}^d)$ is a sequence of real-valued functions.

a) Suppose that $d = 2$, M has bounded geometry and satisfies the doubling property. Let $\alpha \in (1, \infty)$ and

$$s \in \begin{cases} (1 - \frac{1}{2\alpha}, 1] & \text{for } \alpha \in (1, 3], \\ (1 - \frac{1}{\alpha(\alpha-1)}, 1] & \text{for } \alpha > 3. \end{cases}$$

We choose $q := \frac{2}{s}$ for $s < 1$ and $q > 2$ arbitrary if $s = 1$ and assume

$$\sum_{m=1}^{\infty} \|e_m\|^2_{L^\infty \cap H^{s,q}} < \infty.$$

Then, the solutions of (1.25) are pathwise unique in $L^r(\Omega, L^\beta(0, T; H^s(M)))$ for $r > \alpha$ and $\beta \geq \max\{\alpha, 2\}$.

b) Let $d = 3$ and $\alpha \in (1, 3]$. Suppose that M is compact and

$$\sum_{m=1}^{\infty} \left(\|e_m\|_{L^\infty} + \|\nabla e_m\|_{L^3}\right)^2 < \infty.$$

Then, solutions of (1.25) are pathwise unique in $L^2(\Omega, L^\infty(0, T; H^1(M)))$.

Note that in contrast to the existence, the uniqueness result does not distinguish between focusing and defocusing nonlinearities. However, it is restricted to low-dimensional Riemannian manifolds since Strichartz estimates are used in the proof. We would like remark that this is not necessary in 2D for $\alpha \in (1, 3]$ since the argument based on the Moser-Trudinger inequality described above in the deterministic overview can be transfered to the stochastic setting. For this result, we refer to Theorem 5.4 which contains uniqueness for bounded domains, for example. Besides the geometrical restrictions, we would like to point out that all our uniqueness results in Chapter 5 are only true for linear conservative noise. This is due to the fact that the proofs crucially rely on the formula (1.9) for the difference of two solutions without any stochastic integral. In this way, we can avoid using expected values and get pathwise integrability estimates of the type (1.10) for large p. As in the deterministic case described above, this estimate used in (1.9) finally leads to uniqueness by an improved Gronwall argument. In three dimensions, the proof of (1.10) is inspired by Burq, Gérard and Tzvetkov's uniqueness result in the deterministic case, see [35]. The techniques are spectrally localized Strichartz estimates emerging from (1.15), Bernstein inequalities and the Littlewood-Paley decomposition. The noise term is controlled by a localized analogue of (1.23).

In the two dimensional case, we use the global Strichartz estimates (1.16) with loss rather than the spectrally localized ones to prove $u \in L^q(0, T; L^\infty(M))$. This has the advantage that the uniqueness follows directly from the Gronwall Lemma and holds for all $\alpha \in (1, \infty)$. Moreover, the argument is not as sharp as in 3D such that we are able to deal with the additional loss from Bernicot and Samoyeau [16] and allow possibly non-compact manifolds with bounded geometry.

The last chapter differs from the previous ones since it is devoted to the equation (1.26). We consider linear conservative noise of jump type in the Marcus canonical form and aim for a general existence result similar to Theorem 3 in the Gaussian case. Let us state the main result.

Theorem 5. In the setting of Theorem 3, we suppose that a) or b) or c) is true and choose the nonlinearity from $i)$ or $ii)$. Let $(L(t))_{t\geq 0}$ be an $\mathbb{R}^N$-valued Lévy process of pure jump type given by

$$L(t)=\int_0^t\int_{\{|l|\leq 1\}} l\,\tilde{\eta}(\mathrm{d}s,\mathrm{d}l),$$

where $\tilde{\eta}$ is a compensated time homogeneous Poisson random measure on $\mathbb{R}^N$. Take real-valued functions $(e_m)_{m=1}^N\subset F$ as in (1.28). Then, the problem (1.26) has a global martingale solution $\left(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}},\tilde{L},\tilde{\mathbb{F}},u\right)$ in E_A. The process u is almost surely weakly càdlàg in E_A and satisfies $u\in L^q(\tilde{\Omega},L^\infty(0,T;E_A))$ for all $q\in[1,\infty)$.

Since the Marcus form of equations with jump noise is not as common as the Gaussian Stratonovich noise, we would like to explain how to understand the equation (1.26). A solution of this problem is defined via the integral equation

$$\begin{aligned}u(t)=u_0-\mathrm{i}\int_0^t\left(Au(s)+\lambda|u(s)|^{\alpha-1}u(s)\right)\,\mathrm{d}s+\int_0^t\int_{\{|l|\leq 1\}}\left[e^{-\mathrm{i}\mathcal{B}(l)}u(s-)-u(s-)\right]\,\tilde{\eta}(\mathrm{d}s,\mathrm{d}l)\\ +\int_0^t\int_{\{|l|\leq 1\}}\left\{e^{-\mathrm{i}\mathcal{B}(l)}u(s)-u(s)+\mathrm{i}\mathcal{B}(l)u(s)\right\}\,\nu(\mathrm{d}l)\mathrm{d}s \qquad (1.31)\end{aligned}$$

with

$$\mathcal{B}(l)=\sum_{m=1}^N l_m e_m,\qquad l\in\mathbb{R}^N.$$

The proof of the existence is based on uniform estimates for the finite dimensional approximation

$$\begin{aligned}u_n(t)=P_nu_0-\mathrm{i}\int_0^t\left(Au_n(s)+\lambda P_n\left[|u_n(s)|^{\alpha-1}u_n(s)\right]\right)\,\mathrm{d}s\\ +\int_0^t\int_{\{|l|\leq 1\}}\left[e^{-\mathrm{i}\mathcal{B}_n(l)}u_n(s-)-u_n(s-)\right]\,\tilde{\eta}(\mathrm{d}s,\mathrm{d}l)\\ +\int_0^t\int_{\{|l|\leq 1\}}\left\{e^{-\mathrm{i}\mathcal{B}_n(l)}u_n(s)-u_n(s)+\mathrm{i}\mathcal{B}_n(l)u_n(s)\right\}\,\nu(\mathrm{d}l)\mathrm{d}s \qquad (1.32)\end{aligned}$$

of problem (1.26), where we choose the same spaces H_n and operators P_n and S_n as in the Gaussian setting and denote $\mathcal{B}_n(l)=\sum_{m=1}^N l_mS_ne_mS_n$ for $n\in\mathbb{N}$ and $l\in\mathbb{R}^N$. Most of the differences to the latter case have their origin in the fact that now, we deal with càdlàg-functions instead of continuous ones. Nevertheless, it is possible to obtain tightness criteria in

$$Z_T^{\mathbb{D}}:=\mathbb{D}([0,T],E_A^*)\cap L^{\alpha+1}(0,T;L^{\alpha+1}(M))\cap\mathbb{D}_w\left([0,T],E_A\right)$$

instead of Z_T and use a variant of the Skorohod-Jakubowski Theorem for the limiting procedure.

In the appendix, we finally provide background information on topics like stochastic integration, fractional domains of selfadjoint operators, basic notions of Riemannian geometry and function spaces on manifolds. Essentially, these contents are known from the literature and presented with a particular emphasis on the results needed in this thesis.

Notational remarks

Let us briefly introduce some notations used throughout this thesis which are valid unless we state otherwise in the particular chapter.

- We consider a finite time horizon $T > 0$ and denote the set of extended natural numbers $\mathbb{N} \cup \{0\} \cup \{\infty\}$ by $\bar{\mathbb{N}}$.
- The minimum and maximum of $x, y \in \mathbb{R}$ are denoted by $x \wedge y := \min\{x, y\}$ and $x \vee y := \max\{x, y\}$. We write $a_+ := a \vee 0$ for the positive part of $a \in \mathbb{R}$.
- We assume that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a *filtered probability space with the usual conditions*, i.e. $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ such that
  - i) for each $t \in [0, T]$, $\mathcal{F}_t$ contains all $\mathbb{P}$-nullsets,
  - ii) the filtration is right-continuous, i.e.

$$\mathcal{F}_t = \bigcap_{s\in(t,T]} \mathcal{F}_s, \qquad t \in [0, T].$$

- If $(A, \mathcal{A})$ is a measurable space and $X : \Omega \to A$ is a random variable, then the law of X on A is denoted by $\mathbb{P}^X$.
- If functions $a, b \geq 0$ satisfy the inequality $a \leq C(A)b$ with a constant $C(A) > 0$ depending on the expression A, we write $a \lesssim b$ and sometimes $a \lesssim_A b$ if the dependence on A shall be highlighted. Given $a \lesssim b$ and $b \lesssim a$, we write $a \eqsim b$.
- For two Banach spaces E, F over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, we denote by $\mathcal{L}(E, F)$ the space of linear bounded operators $B : E \to F$ and abbreviate $\mathcal{L}(E) := \mathcal{L}(E, E)$ as well as $E^* := \mathcal{L}(E, \mathbb{K})$. We write

$$\langle x, x^* \rangle := x^*(x), \qquad x \in E, \quad x^* \in E^*,$$

  and $E \hookrightarrow F$, if E is continuously embedded in F; i.e. $E \subset F$ with natural embedding $j \in \mathcal{L}(E, F)$.
- In a Hilbert space H, the inner product is typically denoted by $\left(\cdot, \cdot\right)_H$. If the scalar field is $\mathbb{C}$, the inner product is linear in the first and anti-linear in the second component. We use the notation $\mathrm{HS}(H_1, H_2)$ for the space of Hilbert-Schmidt-operators between Hilbert spaces H_1 and H_2.
- In this thesis, (M, Σ, μ) often denotes a σ-finite measure space. Typically, M is equipped with a topology and Σ is the Borel σ-algebra. By A, we denote a non-negative selfadjoint operator on $L^2(M)$ and E_A stands for the domain of $(\mathrm{Id} + A)^{\frac{1}{2}}$.

2. Preliminaries

The purpose of this chapter is to provide a solid foundation for the mathematical theory which will be developed in the Chapters 3 to 6. We introduce several solution concepts which will reappear on various occasions. Moreover, we present tools like stochastic Strichartz estimates, mass formulae and tightness criteria. We decided to outsource these contents to an independent chapter to highlight their significance for the study of the stochastic NLS and to improve the presentation of the proofs of the mains results.

2.1. Solution concepts

This section is devoted to different concepts of solutions to a general stochastic nonlinear Schrödinger equation with Gaussian noise

$$\begin{cases} \mathrm{d}u(t) = (-\mathrm{i}Au(t) + F(u(t)))\,\mathrm{d}t + B(u(t))\mathrm{d}W(t), \qquad t \in [0,T], \\ u(0) = u_0. \end{cases} \tag{2.1}$$

To this end we fix a complex separable Hilbert space X, a real separable Hilbert space Y, a non-negative selfadjoint operator $A : X \supset D(A) \to X$ and an Y-valued cylindrical Wiener process W. Let $(X_\theta)_{\theta\in\mathbb{R}}$ be the scale of fractional domains of A introduced in Appendix A.3. To make use of the stochastic integration theory in real Hilbert spaces treated in Appendix A.1, we equip H with the real inner product $\operatorname{Re}\left(\cdot,\cdot\right)_H$. Moreover,

$$F : X_1 \to X, \qquad B : X_1 \to \mathrm{HS}(Y,X)$$

are supposed to be possibly nonlinear maps and $u_0 \in X_1$. As usual in the field of stochastic differential equations, (2.1) is understood in the integral sense

$$u(t) = u_0 + \int_0^t \left[-\mathrm{i}Au(s) + F(u(s))\right]\mathrm{d}s + \int_0^t B(u(s))\mathrm{d}W(s) \tag{2.2}$$

using the Bochner-integral, see for example [48], and the stochastic integral in Hilbert space. In the second part of this section, we introduce two notions of uniqueness for (2.1) and finally, we show how to reformulate (2.2) in the mild form based on the Schrödinger group.

The first two solution concepts, namely the (analytically) strong and weak solution, are build on (2.2), but they differ in the regularity of the paths. Since it is convenient to allow solutions without integrability in Ω, we will used the L^0-notation from Definition A.15.

Definition 2.1. Let $u_0 \in X_1$, $F : X_1 \to X$ and $B : X_1 \to \mathrm{HS}(Y,X)$ continuous.

a) A triple $\left(u, (\tau_n)_{n\in\mathbb{N}}, \tau_\infty\right)$ consisting of

   i) a process $u : \Omega \times [0,T] \to X_1$ which is adapted and continuous in X_θ for all $\theta \in [0,1)$ and satisfies $u \in L^1(0,T;X_1)$ almost surely,

ii) stopping times $\tau_n, \tau_\infty \in [0,T]$ with $\tau_n \nearrow \tau_\infty$ almost surely for $n \to \infty$,

is called *(analytically) weak solution* of (2.1) in X_1 if we have

$$F(u) \in L^0_{\mathbb{F}}(\Omega, L^1(0,\tau_n;X)), \qquad B(u) \in L^0_{\mathbb{F}}(\Omega, L^2(0,\tau_n;\mathrm{HS}(Y,X)))$$

and (2.2) holds almost surely in X on $\{t \le \tau_n\}$ for all $n \in \mathbb{N}$.

b) An (analytically) weak solution $\left(u, (\tau_n)_{n\in\mathbb{N}}, \tau_\infty\right)$ (2.1) in X_1 is called *(analytically) strong* if u is continuous and adapted in X_1.

c) The solutions from part a) and b) are also called analytically weak (strong) and *stochastically strong.*

d) A strong (weak) solution $\left(u, (\tau_n)_{n\in\mathbb{N}}, \tau_\infty\right)$ is called *global* if we have $\tau_\infty = T$ almost surely.

e) Let U be a subset of $L^0(\Omega, L^1(0,T;X_1))$. Assume that given two strong (weak) solutions $\left(u, (\tau_n)_{n\in\mathbb{N}}, \tau_\infty\right)$ and $\left(v, (\sigma_n)_{n\in\mathbb{N}}, \sigma_\infty\right)$ of (2.1) with $u, v \in U$, we have $u(t) = v(t)$ almost surely on $\{t < \tau_\infty \wedge \sigma_\infty\}$. Then, the strong (weak) solutions of (2.1) are called *unique* in U.

If there is no risk of ambiguity, we skip the sequence $(\tau_n)_{n\in\mathbb{N}}$ and simply write (u, τ_∞) . The characteristic property of the solution concept we introduce next, the *martingale solution*, has a stochastic nature. It is weaker compared to Definition 2.1, where the stochastic setting, i.e. the probability space and the cylindrical Wiener process, was given and we looked for a stochastic process u. Now, the stochastic setting is part of the martingale solution. The freedom to enlarge the probability space and choose the Wiener process W will be very useful in the approximation argument in the fourth chapter based on tightness and Skohorod's theorem.

Definition 2.2. Let $u_0 \in X_1$, $F : X_1 \to X$ and $B : X_1 \to \mathrm{HS}(Y,X)$ continuous. A system $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbb{F}}, u, \tau\right)$ with

- a probability space $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\right)$;
- a filtration $\tilde{\mathbb{F}} = \left(\tilde{\mathcal{F}}_t\right)_{t\in[0,T]}$ with the usual conditions;
- a Y-valued cylindrical Wiener $\tilde{W}$ process on $\tilde{\Omega}$ adapted to $\tilde{\mathbb{F}}$;
- an analytically strong (weak) solution (u, τ_∞) of (2.1) in X_1 with respect to $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbb{F}})$;

is called analytically strong (weak) *martingale solution* of (2.1) in X_1. Another notion which is frequently used is analytically strong (weak) and *stochastically weak* solution. If u is a global solution, we write $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbb{F}}, u)$.

In the framework of martingale solutions, there are several possibilities how to define uniqueness. This is due to the fact that solutions may be defined on different probability spaces which complicates the natural understanding of uniqueness via the indistinguishability of stochastic processes from Definition 2.1.

Definition 2.3. Let U be a subset of $L^1(0,T;X_1)$ and $r \in [0,\infty)$.

a) The solutions of problem (2.1) are called *pathwise unique in $L^r_\omega U$* if given two analytically weak global martingale solutions $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbb{F}}, u_j\right)$ of (2.1) in X_1 with $u_j \in L^r(\tilde{\Omega}, U)$ for $j = 1,2$, we have $u_1(t) = u_2(t)$ almost surely for all $t \in [0,T]$.

b) The solutions of (2.1) are called *unique in law in* $L^r_\omega U$ if for two analytically weak global martingale solutions $(\Omega_j, \mathcal{F}_j, \mathbb{P}_j, W_j, \mathbb{F}_j, u_j)$ of (2.1) in X_1 with $u_j \in L^r(\Omega_j, U)$ for $j = 1, 2$, we have $\mathbb{P}_1^{u_1} = \mathbb{P}_2^{u_2}$ in $C([0,T], X)$.

A very popular conclusion in the field of stochastic PDE states, roughly speaking:

$$\text{existence of a stochastically weak solution and pathwise uniqueness} \\ \Rightarrow \text{existence of stochastically strong solution.} \tag{2.3}$$

This statement gives additional importance to uniqueness results in the stochastic context. Results like (2.3) were first established by Yamada and Watanabe in [130] in the finite dimensional case and have been carried over to stochastic PDE in the weak or mild formulation by [108], [103], [85] and [115] to list some articles without pretense of completeness. In the following Theorem, we state (2.3) rigorously in our framework.

Theorem 2.4 (Yamada-Watanabe). *Let* $\rho \in (0,1)$ *and* U *be a subset of* $L^1(0,T;X_1)$. *We assume:*

i) $F : X_\rho \to X$ *is strongly measurable and bounded on bounded subsets of* X_ρ;

ii) $B : X_\rho \to \mathcal{L}(Y, X)$ *is* Y*-strongly measurable and bounded on bounded subsets of* X_ρ.

Then, the following assertions hold:

a) Pathwise uniqueness in $L^r_\omega U$ *implies uniqueness in law in* $L^r_\omega U$.

b) Suppose that there exists an analytically weak global martingale solution $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbb{F}}, u, T\right)$ *of* (2.1) *with* $u \in L^r(\tilde{\Omega}, U)$. *Furthermore, we assume that the solutions of* (2.1) *are pathwise unique in* $L^r_\omega U$. *Then, there exists a stochastically strong and analytically weak global solution* (u, T) *of* (2.1) *with* $u \in L^r(\Omega, U)$.

Proof. The assumptions allow a direct application of Theorem 5.3 and Corollary 5.4 in [85]. □

In the next Lemma, we prepare the fixed point argument of the third chapter by deriving the equivalence of the *mild formulation* of the linear stochastic Schrödinger equation with the standard formulation via the Itô process. For the notation $L^0_{\mathbb{F}}(\Omega, L^p(0,T;E))$ for a Banach space E and $p \in [1, \infty)$ which will be used below, we refer to Definition A.15.

Lemma 2.5. *Let* $u_0 \in X$, $F \in L^0_{\mathbb{F}}(\Omega, L^1(0,T;X))$ *and* $B \in L^0_{\mathbb{F}}(\Omega, L^2(0,T;\mathrm{HS}(Y,X)))$. *Then, the following are equivalent:*

a) $u \in L^0_{\mathbb{F}}(\Omega, L^1(0,T;X_1))$ *is an Itô process in* X *with*

$$u(t) = u_0 + \int_0^t \left[-\mathrm{i}Au(s) + F(s)\right] \mathrm{d}s + \int_0^t B(s) \mathrm{d}W(s) \tag{2.4}$$

almost surely for all $t \in [0,T]$.

b) $u \in L^0_{\mathbb{F}}(\Omega, L^1(0,T;X_1))$ *satisfies*

$$u(t) = e^{-\mathrm{i}tA} u_0 + \int_0^t e^{-\mathrm{i}(t-s)A} F(s) \mathrm{d}s + \int_0^t e^{-\mathrm{i}(t-s)A} B(s) \mathrm{d}W(s) \tag{2.5}$$

almost surely for all $t \in [0,T]$.

Proof. By Proposition A.41 and Stone's Theorem, $\left(e^{-\mathrm{i}tA}\right)_{t\in\mathbb{R}}$ can be extended to a unitary C_0-group $(T(t))_{t\in\mathbb{R}}$ on X_{-1} with generator $-\mathrm{i}A_{-1}$ and $D(A_{-1}) = X$.
$a) \Rightarrow b)$: We apply the Itô formula from [34], Theorem 2.4, to $f \in C^{1,2}([0,t] \times X, X_{-1})$ defined by

$$f(s,x) := T(t-s)x, \qquad s \in [0,t], \quad x \in X,$$

and obtain

$$\begin{aligned} u(t) =& T(t)u_0 + \int_0^t \mathrm{i}A_{-1}T(t-s)u(s)\mathrm{d}s + \int_0^t T(t-s)\left[-\mathrm{i}Au(s) + F(s)\right]\mathrm{d}s \\ &+ \int_0^t T(t-s)B(s)\mathrm{d}W(s) \\ =& T(t)u_0 + \int_0^t T(t-s)F(s)\mathrm{d}s + \int_0^t T(t-s)B(s)\mathrm{d}W(s) \end{aligned}$$

in X_{-1} for all $t \in [0,T]$ almost surely. By Lemma A.6 and the continuity of the processes on LHS and RHS which is a consequence of Proposition A.13, the null set can be chosen independently of $t \in [0,T]$. By the regularity of u_0, F and B, we obtain (2.5) as equation in X.

$b) \Rightarrow a)$: Inserting (2.5) yields

$$\begin{aligned} -\int_0^t \mathrm{i}A_{-1}u(s)\mathrm{d}s =& -\int_0^t \mathrm{i}A_{-1}e^{-\mathrm{i}sA}u_0\mathrm{d}s - \int_0^t \mathrm{i}A_{-1}\int_0^s e^{-\mathrm{i}(s-r)A}F(r)\mathrm{d}r\mathrm{d}s \\ &- \int_0^t \mathrm{i}A_{-1}\int_0^s e^{-\mathrm{i}(s-r)A}B(r)\mathrm{d}W(r)\mathrm{d}s \end{aligned}$$

almost surely in X_{-1} for all $t \in [0,T]$. Due to Hille's Theorem and Proposition A.14, we can interchange A_{-1} with the integrals and by

$$\begin{aligned} &\|(s,r) \mapsto \mathrm{i}A_{-1}e^{-\mathrm{i}(s-r)A}B(r)\|_{L^1_s(0,t;L^2_r(0,t,\mathrm{HS}(Y,X_{-1}))} \\ &\quad\lesssim \|(s,r) \mapsto e^{-\mathrm{i}(s-r)A}B(r)\|_{L^1_s(0,t;L^2_r(0,t,\mathrm{HS}(Y,X))} = \|(s,r) \mapsto B(r)\|_{L^1_s(0,t;L^2_r(0,t,\mathrm{HS}(Y,X))} \\ &\quad\le T\|B\|_{L^2(0,T,\mathrm{HS}(Y,X))} < \infty \qquad \text{a.s.} \end{aligned}$$

and similarly,

$$\|(s,r) \mapsto \mathrm{i}A_{-1}e^{-\mathrm{i}(s-r)A}F(r)\|_{L^1_s(0,t;L^1_r(0,t,\mathrm{HS}(Y,X_{-1}))} \lesssim T\|F\|_{L^1(0,T,\mathrm{HS}(Y,X))} < \infty \qquad \text{a.s.},$$

we can employ the deterministic and stochastic Fubini Theorems, see [122], to get

$$\begin{aligned} -\int_0^t \mathrm{i}A_{-1}u(s)\mathrm{d}s =& -\int_0^t \mathrm{i}A_{-1}e^{-\mathrm{i}sA}u_0\mathrm{d}s - \int_0^t\int_r^t \mathrm{i}A_{-1}e^{-\mathrm{i}(s-r)A}F(r)\mathrm{d}s\mathrm{d}r \\ &- \int_0^t\int_r^t \mathrm{i}A_{-1}e^{-\mathrm{i}(s-r)A}B(r)\mathrm{d}s\mathrm{d}W(r) \end{aligned}$$

in X_{-1} for all $t \in [0,T]$ almost surely. Note that now, the null set may depend on t. Next, we simplify the inner integrals by

$$-\int_0^t \mathrm{i}A_{-1}e^{-\mathrm{i}sA}u_0\mathrm{d}s = T(t)u_0 - u_0,$$

$$-\int_r^t \mathrm{i}A_{-1}e^{-\mathrm{i}(s-r)A}F(r)\mathrm{d}s = T(t-r)F(r) - F(r),$$
$$-\int_r^t \mathrm{i}A_{-1}e^{-\mathrm{i}(s-r)A}B(r)\mathrm{d}s = T(t-r)B(r) - B(r)$$

and conclude

$$-\int_0^t \mathrm{i}A_{-1}u(s)\mathrm{d}s = T(t)u_0 - u_0 + \int_0^t \left[T(t-r)F(r) - F(r)\right]\mathrm{d}r + \int_0^t \left[T(t-r)B(r) - B(r)\right]\mathrm{d}W(r)$$

in X_{-1} for all $t \in [0,T]$ almost surely and using (2.5) again,

$$u(t) = u_0 - \int_0^t \mathrm{i}A_{-1}u(s)\mathrm{d}s + \int_0^t F(r)\mathrm{d}r + \int_0^t B(r)\mathrm{d}W(r) \tag{2.6}$$

in X_{-1} for all $t \in [0,T]$ almost surely. By the almost sure continuity of the processes on the LHS and RHS of (2.6) and Lemma A.6, the identity (2.6) holds almost surely for all $t \in [0,T]$, i.e. with a null set independent of t. By the regularity of u_0, u, F and B, we get (2.4) as an equation in X. □

2.2. The mass of solutions to the stochastic NLS

In the study of the NLS, the L^2-norm, often called mass, plays a particularly important role since it is conserved by the solutions of the deterministic NLS. We motivate this by the following formal calculation. Assume that a sufficiently smooth function u, for example $u \in C^1([0,T], L^2(\mathbb{R}^d)) \cap C([0,T], H^2(\mathbb{R}^d))$, solves

$$\partial_t u(t,x) = \mathrm{i}\Delta u(t,x) - \mathrm{i}\lambda |u(t,x)|^{\alpha-1}u(t,x), \qquad t \in \mathbb{R}, \quad x \in \mathbb{R}^d.$$

Then, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^2}^2 &= 2\,\mathrm{Re}\left(u(t), \partial_t u(t)\right)_{L^2} = 2\,\mathrm{Re}\int_{\mathbb{R}^d} u(t,x)\left(-\mathrm{i}\Delta\overline{u(t,x)} + \mathrm{i}\lambda|u(t,x)|^{\alpha-1}\overline{u(t,x)}\right)\mathrm{d}x \\ &= 2\,\mathrm{Re}\int_{\mathbb{R}^d} \mathrm{i}|\nabla u(t,x)|^2\mathrm{d}x + 2\,\mathrm{Re}\int_{\mathbb{R}^d} \mathrm{i}\lambda|u(t,x)|^{\alpha+1}\mathrm{d}x = 0, \qquad t \in [0,T]. \end{aligned} \tag{2.7}$$

In this section, we rigorously prove similar formulae for the stochastic NLS which are useful

- to globalize local solutions in the third chapter;
- as foundation of the uniqueness proofs in the fifth chapter.

Naturally, we will use the Itô formula to substitute the differentiation in (2.7) and an regularization procedure based on Yosida approximations to get similar identities for solutions which are only in L^2. This strategy is classical and for the stochastic NLS, it was used in [11].

In the following, M will be a σ-finite measure space, $A : L^2(M) \supset D(A) \to L^2(M)$ will be a non-negative selfadjoint operator. In the main theorem of this section, we prove a general formula for the L^2-norm of the difference of solutions to the stochastic NLS.

Theorem 2.6. *Let $\alpha > 1$ and $\gamma \geq 1$ such that*

$$X_1 \hookrightarrow L^{\alpha+1}(M) \qquad X_1 \hookrightarrow L^{2\gamma}(M).$$

For each $p \in \left\{\alpha+1, \frac{\alpha+1}{\alpha}, 2\gamma, \frac{2\gamma}{2\gamma-1}\right\}$, we assume that there is a C_0-semigroup $(T_p(t))_{t\geq 0}$ on $L^p(M)$ which is consistent with $\left(e^{-tA}\right)_{t\geq 0}$, i.e.

$$T_p(t)f = e^{-tA}f, \qquad f \in L^2(M) \cap L^p(M).$$

We denote the generator of T_p by $-A_p$. Let $F : [0,T] \times L^{\alpha+1}(M) \to L^{\frac{\alpha+1}{\alpha}}(M)$ satisfy

$$\|F(s,u)\|_{L^{\frac{\alpha+1}{\alpha}}(M)} \lesssim \|u\|^{\alpha}_{L^{\alpha+1}(M)}, \qquad u \in L^{\alpha+1}(M),$$

and take

$$\mu_j \in L^0_{\mathbb{F}}(\Omega, L^{\frac{2\gamma}{2\gamma-1}}(0,T; L^{\frac{2\gamma}{2\gamma-1}}(M))), \qquad B_j \in L^0_{\mathbb{F}}(\Omega, L^2(0,T; \mathrm{HS}(Y, L^2(M))))$$

Assume that u_j for $j = 1, 2$, satisfy

$$u_j \in L^0_{\mathbb{F}}\big(\Omega, C([0,T], L^2(M)) \cap L^{\alpha+1}(0,T; L^{\alpha+1}(M)) \cap L^{2\gamma}(0,T; L^{2\gamma}(M))\big)$$

and

$$u_j(t) = u_j(0) + \int_0^t [-\mathrm{i}A_{-1}u_j(s) - \mathrm{i}F(s,u_j(s)) + \mu_j(s)]\,\mathrm{d}s - \mathrm{i}\int_0^t B_j(s)\mathrm{d}W(s)$$

in X_{-1} almost surely for all $t \in [0,T]$. Then, $w := u_1 - u_2$ has the representation

$$\begin{aligned}
\|w(t)\|^2_{L^2} =& \|w(0)\|^2_{L^2} + 2\int_0^t \mathrm{Re}\langle w(s), -\mathrm{i}F(s,u_1(s)) + \mathrm{i}F(s,u_2(s))\rangle_{L^{\alpha+1}, L^{\frac{\alpha+1}{\alpha}}}\,\mathrm{d}s \\
&+ 2\int_0^t \mathrm{Re}\langle w(s), \mu_1(s) - \mu_2(s)\rangle_{L^{2\gamma}, L^{\frac{2\gamma}{2\gamma-1}}}\,\mathrm{d}s \\
&- 2\int_0^t \mathrm{Re}\left(w(s), \mathrm{i}\,(B_1(s) - B_2(s))\,\mathrm{d}W(s)\right)_{L^2} \\
&+ \sum_{m=1}^{\infty}\int_0^t \|B_1(s)f_m - B_2(s)f_m\|^2_{L^2}\mathrm{d}s
\end{aligned} \tag{2.8}$$

almost surely for all $t \in [0,T]$.

Before we continue with the proof of the Theorem, we illustrate the assumption on the existence of consistent semigroups by the following remark.

Remark 2.7. Let M be an open subset of a metric measure space $(\tilde{M}, \rho)$ with the doubling property, i.e. $\mu(B(x,r)) < \infty$ for all $x \in \tilde{M}$ and $r > 0$ and $\mu(B(x,2r)) \lesssim \mu(B(x,r))$. Suppose that the heat semigroup $\left(e^{-tA}\right)_{t\geq 0}$ on $L^2(M)$ has *upper Gaussian bounds*, i.e. for all $t > 0$ there is a measurable function $p(t,\cdot,\cdot) : M \times M \to \mathbb{R}$ with

$$e^{-tA}f(x) = \int_M p(t,x,y)f(y)\mu(dy), \quad t > 0, \quad \text{a.e. } x \in M,$$

for all $f \in H$ and

$$|p(t,x,y)| \le \frac{C}{\mu(B(x,t^{\frac{1}{m}}))} \exp\left\{-c\left(\frac{\rho(x,y)^m}{t}\right)^{\frac{1}{m-1}}\right\},$$

for all $t>0$ and almost all $(x,y) \in M \times M$ with constants $c, C > 0$ and $m \ge 2$.

Then, $\left(e^{-tA}\right)_{t\ge 0}$ can be extended from $L^2(M) \cap L^p(M)$ to a bounded analytic semigroup on $L^p(M)$ for all $p \in [1,\infty)$. We refer to [104], Corollary 7.5, for a proof of this assertion. In this thesis, the doubling property is typically satisfied and the choices for A have Gaussian bounds; for example the Laplacian $-\Delta$ on $\mathbb{R}^d$, the Laplace-Beltrami operator $-\Delta_g$ on a compact Riemannian manifold (M,g) or the Laplacian on a domain $M \subset \mathbb{R}^d$ under various boundary conditions.

Proof of Theorem 2.6. Step 1. For each $p \in \left\{\alpha+1, \frac{\alpha+1}{\alpha}, 2\gamma, \frac{2\gamma}{2\gamma-1}\right\}$, there are $M_p \ge 1$ and $\omega_p \ge 0$ with $\|T_p(t)\|_{\mathcal{L}(L^p)} \le M_p e^{\omega_p t}$ for all $t \ge 0$. Consequently, we have $(\omega_p, \infty) \subset \rho(-A_p)$ with the uniform estimate

$$\|\lambda\left(\lambda+A_p\right)^{-1}\|_{\mathcal{L}(L^p)} \le \frac{\lambda M_p}{\lambda-\omega_p} \le 2M_p, \qquad \lambda \ge 2\omega_p. \tag{2.9}$$

Moreover, the convergence

$$\|\lambda\left(\lambda+A_p\right)^{-1} f - f\|_{L^p} = \|\left(\lambda+A_p\right)^{-1} A_p f\|_{L^p} \le \frac{M_p}{\lambda-\omega_p}\|A_p f\|_{L^p} \xrightarrow{\lambda\to\infty} 0$$

holds for $f \in \mathcal{D}(A_p)$ and (2.9) yields

$$\lambda\left(\lambda+A_p\right)^{-1} f \xrightarrow{\lambda\to\infty} f \quad \text{in} \quad L^p(M), \quad f \in L^p(M). \tag{2.10}$$

Let us recall from Proposition A.41 that A_{-1} is a non-negative selfadjoint operator on X_{-1}. We define $R_\lambda : X_{-1} \to L^2(M)$ by

$$R_\lambda f := \lambda\left(\lambda+A_{-1}\right)^{-1} f, \qquad f \in X_{-1},\ \lambda > 0.$$

By $e^{-tA_{-1}}|_{L^2(M)} = e^{-tA}$ and the Laplace transform, we get

$$R_\lambda f = \lambda\left(\lambda+A\right)^{-1} f, \qquad R_\lambda A_{-1} f = A R_\lambda f, \qquad f \in L^2(M),$$

as well as

$$\begin{aligned} R_\lambda f \to f \quad \text{in} \quad L^2(M), \quad \lambda\to\infty, \quad f \in L^2(M), \\ \sup\left\{\|R_\lambda\|_{\mathcal{L}(L^2)} :\ \lambda>0\right\} \le 1. \end{aligned} \tag{2.11}$$

Moreover, R_λ is defined on $L^{\frac{\alpha+1}{\alpha}}(M)$ since we have $L^{\frac{\alpha+1}{\alpha}}(M) \hookrightarrow X_{-1}$. We take $f \in L^2(M) \cap L^{\frac{\alpha+1}{\alpha}}(M)$ and by the consistency of the semigroups, we obtain the identity

$$\lambda\left(\lambda+A_{\frac{\alpha+1}{\alpha}}\right)^{-1} f = \lambda\int_0^\infty e^{-\lambda t} T_{\frac{\alpha+1}{\alpha}}(t) f \mathrm{d}t = \lambda\int_0^\infty e^{-\lambda t} e^{-tA} f \mathrm{d}t = \lambda\left(\lambda+A\right)^{-1} f = R_\lambda f$$

for $\lambda > \omega_{\frac{\alpha+1}{\alpha}}$. Since $L^2(M) \cap L^{\frac{\alpha+1}{\alpha}}(M)$ is dense in $L^{\frac{\alpha+1}{\alpha}}(M)$ and the operators

$$R_\lambda|_{L^{\frac{\alpha+1}{\alpha}}(M)} : L^{\frac{\alpha+1}{\alpha}}(M) \to X_{-1}, \qquad \lambda\left(\lambda+A_{\frac{\alpha+1}{\alpha}}\right)^{-1} : L^{\frac{\alpha+1}{\alpha}}(M) \to X_{-1}$$

are bounded, we conclude that

$$R_\lambda f = \lambda \left(\lambda + A_{\frac{\alpha+1}{\alpha}}\right)^{-1} f, \qquad f \in L^{\frac{\alpha+1}{\alpha}}(M),\ \lambda > \omega_{\frac{\alpha+1}{\alpha}}.$$

Hence, (2.9) and (2.10) yield

$$\begin{aligned} R_\lambda f \xrightarrow{\lambda\to\infty} f \quad &\text{in} \quad L^{\frac{\alpha+1}{\alpha}}(M), \quad f \in L^{\frac{\alpha+1}{\alpha}}(M), \\ \sup\left\{\|R_\lambda\|_{\mathcal{L}(L^{\frac{\alpha+1}{\alpha}})} : \lambda \geq 2\omega_{\frac{\alpha+1}{\alpha}}\right\} &\leq 2M_{\frac{\alpha+1}{\alpha}}. \end{aligned} \tag{2.12}$$

To estimate the $L^{\alpha+1}$-norm of R_λ, we will use the part of A in $L^{\alpha+1}(M)$ defined by

$$A_{\alpha+1,0} f = Af, \qquad f \in D(A_{\alpha+1,0}) := \left\{f \in D(A) \cap L^{\alpha+1}(M) : Af \in L^{\alpha+1}(M)\right\}.$$

The operator $A_{\alpha+1,0}$ is the generator of a C_0-semigroup on $L^{\alpha+1}(M) \cap L^2(M)$ and in particular, we have

$$\begin{aligned} R_\lambda f \xrightarrow{\lambda\to\infty} f \quad &\text{in} \quad L^{\alpha+1}(M) \cap L^2(M), \quad f \in L^{\alpha+1}(M) \cap L^2(M), \\ \sup\left\{\|R_\lambda\|_{\mathcal{L}(L^{\alpha+1}\cap L^2)} : \lambda \geq 2\omega_{\alpha+1}\right\} &\leq 2M_{\alpha+1}. \end{aligned} \tag{2.13}$$

Analogously, one can show (2.12) for the exponent $\frac{2\gamma}{2\gamma-1}$ instead of $\frac{\alpha+1}{\alpha}$ and (2.13) with $\alpha+1$ replaced by 2γ.

Set $w := u_1 - u_2$ and fix

$$\lambda > 2\omega_{\max} := 2\max\left\{\omega_{\alpha+1}, \omega_{\frac{\alpha+1}{\alpha}}, \omega_{2\gamma}, \omega_{\frac{2\gamma}{2\gamma-1}}\right\}.$$

Then, the process $R_\lambda w$ has the representation

$$\begin{aligned} R_\lambda w(t) =& R_\lambda w(0) + \int_0^t \left[-\mathrm{i} R_\lambda A_{-1} w(s) - \mathrm{i} R_\lambda F(s, u_1(s)) + \mathrm{i} R_\lambda F(s, u_2(s))\right] \mathrm{d}s \\ &+ \int_0^t \left[R_\lambda \mu_1(s) - R_\lambda \mu_2(s)\right] \mathrm{d}s - \mathrm{i} \int_0^t \left[R_\lambda B_1(s) - R_\lambda B_2(s)\right] \mathrm{d}W(s) \end{aligned} \tag{2.14}$$

almost surely in $L^2(M)$ for all $t \in [0,T]$, where we used the dual versions of the embeddings $X_1 \hookrightarrow L^{\alpha+1}(M)$ and $X_1 \hookrightarrow L^{2\gamma}(M)$ to ensure that the terms $F(\cdot, u_j)$ and μ_j are in $L^2(M)$ for $j = 1, 2$. The function $\mathcal{M} : L^2(M) \to \mathbb{R}$ defined by $\mathcal{M}(v) := \|v\|_{L^2}^2$ is twice continuously Fréchet-differentiable with

$$\mathcal{M}'[v] h_1 = 2\operatorname{Re}\left(v, h_1\right)_{L^2}, \qquad \mathcal{M}''[v]\left[h_1, h_2\right] = 2\operatorname{Re}\left(h_1, h_2\right)_{L^2}$$

for $v, h_1, h_2 \in L^2(M)$. Therefore, we get

$$\begin{aligned} \|R_\lambda w(t)\|_{L^2}^2 =& \|R_\lambda w(0)\|_{L^2}^2 \\ &+ 2\int_0^t \operatorname{Re}\left(R_\lambda w(s), -\mathrm{i} R_\lambda A_{-1} w(s) - \mathrm{i} R_\lambda F(s, u_1(s)) + \mathrm{i} R_\lambda F(s, u_2(s))\right)_{L^2} \mathrm{d}s \\ &+ 2\int_0^t \operatorname{Re}\left(R_\lambda w(s), R_\lambda\left[\mu_1(s) - \mu_2(s)\right]\right)_{L^2} \mathrm{d}s \\ &- 2\int_0^t \operatorname{Re}\left(R_\lambda w(s), \mathrm{i} R_\lambda\left[B_1(s) - B_2(s)\right] \mathrm{d}W(s)\right)_{L^2} \\ &+ \sum_{m=1}^\infty \int_0^t \|R_\lambda\left[B_1(s) f_m - B_2(s) f_m\right]\|_{L^2}^2 \mathrm{d}s \end{aligned} \tag{2.15}$$

almost surely for all $t \in [0,T]$.

Step 2. In the following, we deal with the behavior of the terms in (2.15) for $\lambda \to \infty$. Since R_λ and A commute, we get

$$\big(R_\lambda w(s), -\mathrm{i} R_\lambda A w(s)\big)_{L^2} = \big(R_\lambda w(s), -\mathrm{i} A R_\lambda w(s)\big)_{L^2} = 0, \quad s \in [0,T], \quad \lambda > 0. \tag{2.16}$$

For $s \in [0,T]$, we have

$$\begin{aligned} &\operatorname{Re}\big(R_\lambda w(s), -\mathrm{i} R_\lambda F(s,u_1(s)) + \mathrm{i} R_\lambda F(s,u_2(s))\big)_{L^2} \\ &\qquad \xrightarrow{\lambda\to\infty} \operatorname{Re}\langle w(s), -\mathrm{i} F(s,u_1(s)) + \mathrm{i} F(s,u_2(s))\rangle_{L^{\alpha+1}, L^{\frac{\alpha+1}{\alpha}}} \end{aligned} \tag{2.17}$$

by (2.12). We estimate

$$\begin{aligned} &|\operatorname{Re}\langle R_\lambda w(s), -\mathrm{i} R_\lambda F(s,u_1(s)) + \mathrm{i} R_\lambda F(s,u_2(s))\rangle_{L^{\alpha+1}, L^{\frac{\alpha+1}{\alpha}}}| \\ &\qquad \le 4 M_{\alpha+1} M_{\frac{\alpha+1}{\alpha}} \|w(s)\|_{L^{\alpha+1}\cap L^2} \|-\mathrm{i} F(s,u_1(s)) + \mathrm{i} F(s,u_2(s))\|_{L^{\frac{\alpha+1}{\alpha}}(M)} \\ &\qquad \lesssim \|w(s)\|_{L^{\alpha+1}\cap L^2} \left(\|u_1(s)\|_{L^{\alpha+1}(M)}^\alpha + \|u_2(s)\|_{L^{\alpha+1}(M)}^\alpha\right) \in L^1(0,T) \quad \text{a.s.} \end{aligned}$$

for $\lambda > 2\omega_{\max}$ and thus, Lebesgue's Theorem yields

$$\begin{aligned} &\int_0^t \operatorname{Re}\big(R_\lambda w(s), -\mathrm{i} R_\lambda F(s,u_1(s)) + \mathrm{i} R_\lambda F(s,u_2(s))\big)_{L^2} \mathrm{d}s \\ &\qquad \xrightarrow{\lambda\to\infty} \int_0^t \operatorname{Re}\langle w(s), -\mathrm{i} F(s,u_1(s)) + \mathrm{i} F(s,u_2(s))\rangle_{L^{\alpha+1}, L^{\frac{\alpha+1}{\alpha}}} \mathrm{d}s \end{aligned} \tag{2.18}$$

almost surely for all $t \in [0,T]$. In the same way, one can also deduce

$$\begin{aligned} &\int_0^t \operatorname{Re}\big(R_\lambda w(s), R_\lambda\left[\mu_1(s) - \mu_2(s)\right]\big)_{L^2} \mathrm{d}s \\ &\qquad \xrightarrow{\lambda\to\infty} \int_0^t \operatorname{Re}\langle w(s), \mu_1(s) - \mu_2(s)\rangle_{L^{2\gamma}, L^{\frac{2\gamma}{2\gamma-1}}} \mathrm{d}s. \end{aligned} \tag{2.19}$$

From (2.11), we infer the pointwise convergence

$$\|R_\lambda\left[B_1(s) f_m - B_2(s) f_m\right]\|_{L^2} \xrightarrow{\lambda\to\infty} \|B_1(s) f_m - B_2(s) f_m\|_{L^2}, \qquad m \in \mathbb{N}, \quad s \in [0,T],$$

and the estimate

$$\begin{aligned} \sum_{m=1}^\infty \|R_\lambda\left[B_1(s) f_m - B_2(s) f_m\right]\|_{L^2}^2 &= \|R_\lambda\left[B_1(s) - B_2(s)\right]\|_{\mathrm{HS}(Y,L^2)}^2 \\ &\le \|B_1(s) - B_2(s)\|_{\mathrm{HS}(Y,L^2)}^2 \in L^1(0,T) \quad \text{a.s.} \end{aligned}$$

Together, this leads to

$$\sum_{m=1}^\infty \int_0^t \|R_\lambda\left[B_1(s) f_m - B_2(s) f_m\right]\|_{L^2}^2 \mathrm{d}s \xrightarrow{\lambda\to\infty} \sum_{m=1}^\infty \int_0^t \|B_1(s) f_m - B_2(s) f_m\|_{L^2}^2 \mathrm{d}s \tag{2.20}$$

almost surely for all $t \in [0,T]$ by Lebesgue's Theorem. For the stochastic term, we fix $K \in \mathbb{N}$ and define the stopping time

$$\tau_K := \inf\left\{t \in [0,T] : \|w(t)\|_{L^2} + \|B_1 - B_2\|_{L^2(0,t;\mathrm{HS}(Y,L^2))} > K\right\} \wedge T.$$

Then, we use

$$\operatorname{Re}\left(R_\lambda w(s), \mathrm{i} R_\lambda\left[B_1(s) f_m-B_2(s) f_m\right]\right)_{L^2} \xrightarrow{\lambda \rightarrow \infty} \operatorname{Re}\left(w(s), \mathrm{i}\left[B_1(s) f_m-B_2(s) f_m\right]\right)_{L^2} \quad \text{a.s.,}$$

for $m \in \mathbb{N}$ and $s \in[0, T]$ and

$$\begin{aligned} & \left|\mathbf{1}_{\{s \leq \tau_K\}} \operatorname{Re}\left(R_\lambda w(s), \mathrm{i} R_\lambda\left[B_1(s) f_m-B_2(s) f_m\right]\right)_{L^2}\right|^2 \\ & \qquad \leq \mathbf{1}_{\{s \leq \tau_K\}}\|w(s)\|_{L^2}^2\left\|B_1(s) f_m-B_2(s) f_m\right\|_{L^2}^2 \end{aligned}$$

together with

$$\begin{aligned} & \left\|\mathbf{1}_{\{\cdot \leq \tau_K\}}\|w\|_{L^2}^2\left\|B_1(s) f_m-B_2(s) f_m\right\|_{L^2}^2\right\|_{L^1(\Omega \times[0, T] \times \mathbb{N})} \\ & \leq \tilde{\mathbb{E}} \int_0^{\tau_K}\|w(s)\|_{L^2}^2 \sum_{m=1}^{\infty}\left\|B_1(s) f_m-B_2(s) f_m\right\|_{L^2}^2 \mathrm{d} s \leq K^4<\infty \end{aligned}$$

to get

$$\operatorname{Re}\left(R_\lambda w, \mathrm{i} R_\lambda\left[B_1(s) f_m-B_2(s) f_m\right]\right)_{L^2} \xrightarrow{\lambda \rightarrow \infty} \operatorname{Re}\left(w, \mathrm{i} B_1(s) f_m-\mathrm{i} B_2(s) f_m\right)_{L^2}$$

in $L^2(\Omega, L^2([0, \tau_K] \times \mathbb{N}))$. The Itô isometry and the Doob inequality yield

$$\int_0^{\cdot} \operatorname{Re}\left(R_\lambda w(s), \mathrm{i} R_\lambda\left[B_1(s)-B_2(s)\right] \mathrm{d} W(s)\right)_{L^2} \rightarrow \int_0^{\cdot} \operatorname{Re}\left(w(s), \mathrm{i}\left[B_1(s)-B_2(s)\right] \mathrm{d} W(s)\right)_{L^2}$$

in $L^2(\Omega, C([0, \tau_K]))$ for $\lambda \rightarrow \infty$. After passing to a subsequence, we get

$$\int_0^t \operatorname{Re}\left(R_\lambda w(s), \mathrm{i} R_\lambda B w(s) \mathrm{d} W(s)\right)_{L^2} \xrightarrow{\lambda \rightarrow \infty} \int_0^t \operatorname{Re}\left(w(s), \mathrm{i} B w(s) \mathrm{d} W(s)\right)_{L^2} \tag{2.21}$$

almost surely in $\{t \leq \tau_K\}$. By

$$\bigcup_{K \in \mathbb{N}}\{t \leq \tau_K\}=[0, T] \qquad \text{a.s.,}$$

we conclude that (2.21) holds almost surely on $[0, T]$.

Step 3. Using 2.12 for the convergence of the initial value and (2.16), (2.18), (2.19), (2.20) and (2.21) in (2.15), we obtain the assertion. □

We continue with three Corollaries of the previous Theorem. In the first one, we state a representation formula for the mass of a solution to the stochastic NLS with nonlinear Stratonovich noise.

Corollary 2.8. *Let $\alpha>1$ and $\gamma \geq 1$ such that*

$$X_1 \hookrightarrow L^{\alpha+1}(M) \cap L^{2 \gamma}(M).$$

We assume that for each $p \in\left\{\alpha+1, \frac{\alpha+1}{\alpha}, 2 \gamma, \frac{2 \gamma}{2 \gamma-1}\right\}$, there is a C_0-semigroup $\left(T_p(t)\right)_{t \geq 0}$ on $L^p(M)$ which is consistent with $\left(e^{-t A}\right)_{t \geq 0}$. Let $F:[0, T] \times L^{\alpha+1}(M) \rightarrow L^{\frac{\alpha+1}{\alpha}}(M)$ satisfy

$$\|F(s, u)\|_{L^{\frac{\alpha+1}{\alpha}}} \lesssim\|u\|_{L^{\alpha+1}}^\alpha, \qquad \operatorname{Re}\langle\mathrm{i} u, F(s, u)\rangle_{L^{\alpha+1}, L^{\frac{\alpha+1}{\alpha}}}=0, \qquad u \in L^{\alpha+1}(M),\ s \in[0, T],$$

and choose $g:[0,T]\times[0,\infty)\to\mathbb{R}$ *such that* $|g(s,x)|\lesssim x^{\frac{\gamma-1}{2}}$. *We take*

$$u\in L^0_{\mathbb{F}}(\Omega,C([0,T],L^2(M))\cap L^{\alpha+1}(0,T;L^{\alpha+1}(M))\cap L^{2\gamma}(0,T;L^{2\gamma}(M)))$$

and define $B\in L^0_{\mathbb{F}}(\Omega,L^2(0,T;\mathrm{HS}(Y,L^2(M)))$ *by*

$$B(s)f_m:=e_m g(s,|u(s)|^2)u(s),\qquad m\in\mathbb{N},\quad s\in[0,T],$$

for a sequence $(e_m)_{m\in\mathbb{N}}\subset L^\infty(M)$ *with* $\sum_{m=1}^\infty\|e_m\|_{L^\infty}^2<\infty$. *We assume that the identity*

$$\begin{aligned}u(t)=&u_0+\int_0^t\left[-\mathrm{i}A_{-1}u(s)-\mathrm{i}F(s,u(s))-\frac{1}{2}\sum_{m=1}^\infty|e_m|^2\left\{g(s,|u(s)|^2)\right\}^2u(s)\right]\mathrm{d}s\\&-\mathrm{i}\int_0^t B(s)\mathrm{d}W(s)\end{aligned}\tag{2.22}$$

is satisfied in X_{-1} *almost surely for all* $t\in[0,T]$. *Then, we have*

$$\|u(t)\|_{L^2}^2=\|u_0\|_{L^2}^2-2\int_0^t\mathrm{Re}\left(u(s),\mathrm{i}B(s)\mathrm{d}W(s)\right)_{L^2}\tag{2.23}$$

almost surely for all $t\in[0,T]$.

Proof. We denote

$$\mu_1(s):=-\frac{1}{2}\sum_{m=1}^\infty|e_m|^2\left\{g(s,|u(s)|^2)\right\}^2u(s).$$

We observe $\mu_1\in L^0_{\mathbb{F}}(\Omega,L^{\frac{2\gamma}{2\gamma-1}}(0,T;L^{\frac{2\gamma}{2\gamma-1}}(M)))$ and $B\in L^0_{\mathbb{F}}(\Omega,L^2(0,T;\mathrm{HS}(Y,L^2(M))))$ as a consequence of $u\in L^0_{\mathbb{F}}(\Omega,L^{2\gamma}(0,T;L^{2\gamma}(M)))$ and the growth bound on g. Hence, we can apply Theorem 2.6 for $u_1:=u$, $u_2:=0$ and obtain

$$\begin{aligned}\|u(t)\|_{L^2}^2=&\|u_0\|_{L^2}^2-2\int_0^t\mathrm{Re}\langle u(s),\mathrm{i}F(s,u(s))\rangle_{L^{\alpha+1},L^{\frac{\alpha+1}{\alpha}}}\mathrm{d}s\\&-\sum_{m=1}^\infty\int_0^t\mathrm{Re}\langle u(s),|e_m|^2\left\{g(s,|u(s)|^2)\right\}^2u(s)\rangle_{L^{2\gamma},L^{\frac{2\gamma}{2\gamma-1}}}\mathrm{d}s\\&+2\int_0^t\mathrm{Re}\left(u(s),\mathrm{i}B(s)\mathrm{d}W(s)\right)_{L^2}\\&+\sum_{m=1}^\infty\int_0^t\|e_m g(s,|u(s)|^2)u(s)\|_{L^2}^2\mathrm{d}s\end{aligned}$$

almost surely for all $t\in[0,T]$. This formula simplifies to due the cancellations

$$\mathrm{Re}\langle u(s),\mathrm{i}F(s,u(s))\rangle_{L^{\alpha+1},L^{\frac{\alpha+1}{\alpha}}}=0,$$

$$\begin{aligned}\|e_m g(s,|u(s)|^2)u(s)\|_{L^2}^2&=\mathrm{Re}\int_M|e_m|^2\left\{g(s,|u(s)|^2)\right\}^2|u|^2\mathrm{d}x\\&=\mathrm{Re}\langle u(s),|e_m|^2\left\{g(s,|u(s)|^2)\right\}^2u(s)\rangle_{L^{2\gamma},L^{\frac{2\gamma}{2\gamma-1}}}\end{aligned}$$

and we get

$$\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 + 2\int_0^t \operatorname{Re}\big(u(s), \mathrm{i}B(s)\mathrm{d}W(s)\big)_{L^2}$$

almost surely for all $t \in [0,T]$. □

As a special case of the previous Corollary (set $\gamma = 1$ and $g = 1$), we obtain the evolution formula the mass of a solution to the stochastic NLS with linear noise.

Corollary 2.9. *Let $\alpha > 1$ such that $X_1 \hookrightarrow L^{\alpha+1}(M)$. We assume that for each $p \in \left\{\alpha+1, \frac{\alpha+1}{\alpha}\right\}$, there is a C_0-semigroup $(T_p(t))_{t\geq 0}$ on $L^p(M)$ which is consistent with $\left(e^{-tA}\right)_{t\geq 0}$. Let $F : [0,T] \times L^{\alpha+1}(M) \to L^{\frac{\alpha+1}{\alpha}}(M)$ satisfy*

$$\|F(s,u)\|_{L^{\frac{\alpha+1}{\alpha}}} \lesssim \|u\|_{L^{\alpha+1}}^{\alpha}, \qquad \operatorname{Re}\langle \mathrm{i}u, F(s,u)\rangle_{L^{\alpha+1}, L^{\frac{\alpha+1}{\alpha}}} = 0, \qquad u \in L^{\alpha+1}(M).$$

We take $u \in L^0_{\mathbb{F}}(\Omega, C([0,T], L^2) \cap L^{\alpha+1}(0,T; L^{\alpha+1}))$ and define $B \in L^0_{\mathbb{F}}(\Omega, L^2(0,T; \mathrm{HS}(Y, L^2))$ by

$$B(s)f_m := B_m u(s), \qquad m \in \mathbb{N}, \quad s \in [0,T],$$

for a sequence $(B_m)_{m\in\mathbb{N}} \subset \mathcal{L}(L^2(M))$ with $\sum_{m=1}^\infty \|B_m\|_{\mathcal{L}(L^2)}^2 < \infty$. Moreover, we assume that

$$u(t) = u_0 + \int_0^t \left[-\mathrm{i}A_{-1}u(s) - \mathrm{i}F(s,u(s)) - \frac{1}{2}\sum_{m=1}^\infty B_m^* B_m u(s)\right] \mathrm{d}s - \mathrm{i}\int_0^t B(s)\mathrm{d}W(s)$$

is satisfied in X_{-1} almost surely for all $t \in [0,T]$. Then, we have

$$\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 - 2\int_0^t \operatorname{Re}\big(u(s), \mathrm{i}B(s)\mathrm{d}W(s)\big)_{L^2} \tag{2.24}$$

almost surely for all $t \in [0,T]$.

In a typical uniqueness proof, see Chapter 5, one considers a suitable norm of the difference of two solutions and wants to prove that it equals zero. The following Corollary is the basis of this type of argument. Since it will be crucial that there are no stochastic integrals in the formula, we have to restrict ourselves to the case of linear conservative noise.

Corollary 2.10. *Let $\alpha > 1$ with $X_1 \hookrightarrow L^{\alpha+1}(M)$. We assume that for $p \in \left\{\alpha+1, \frac{\alpha+1}{\alpha}\right\}$, there is a C_0-semigroup $(T_p(t))_{t\geq 0}$ on $L^p(M)$ which is consistent with $\left(e^{-tA}\right)_{t\geq 0}$ and denote the generator of T_p by $-A_p$. Let $F : [0,T] \times L^{\alpha+1}(M) \to L^{\frac{\alpha+1}{\alpha}}(M)$ satisfy*

$$\|F(s,u)\|_{L^{\frac{\alpha+1}{\alpha}}(M)} \lesssim \|u\|_{L^{\alpha+1}(M)}^{\alpha}, \qquad u \in L^{\alpha+1}(M), \quad s \in [0,T].$$

Assume that $u_j \in C([0,T], L^2(M)) \cap L^{\alpha+1}(0,T; L^{\alpha+1}(M))$ for $j = 1,2$ almost surely satisfy

$$u_j(t) = u_0 \int_0^t \left[-\mathrm{i}Au_j(s) - \mathrm{i}F(s,u_j(s)) + \mu(u_j(s))\right] \mathrm{d}s - \mathrm{i}\int_0^t Bu_j(s)\mathrm{d}W(s)$$

in X_{-1} almost surely for all $t \in [0,T]$, where we used the operators $B \in \mathcal{L}(L^2(M), \mathrm{HS}(Y, L^2(M)))$ and $\mu \in \mathcal{L}(L^2(M))$ given by

$$B(u)f_m := B_m u, \qquad \mu(u) := -\frac{1}{2}\sum_{m=1}^\infty B_m^2 u, \qquad u \in L^2(M).$$

Then, if the operators B_m, $m \in \mathbb{N}$ are symmetric, we have

$$\|u_1(t) - u_2(t)\|_{L^2}^2 = 2\int_0^t \mathrm{Re}\langle u_1(s) - u_2(s), -\mathrm{i}F(s, u_1(s)) + \mathrm{i}F(s, u_2(s))\rangle_{L^{\alpha+1}, L^{\frac{\alpha+1}{\alpha}}} \mathrm{d}s \tag{2.25}$$

almost surely for all $t \in [0, T]$.

Proof. The Corollary follows from Theorem 2.6 by $w(0) = 0$ for $w := u_1 - u_2$ and

$$\int_0^t \mathrm{Re}\left(w(s), \mathrm{i}Bw(s)\mathrm{d}W(s)\right)_{L^2} = 0, \qquad t \in [0, T],$$

where we used the symmetry of B_m, $m \in \mathbb{N}$. The cancellation of $\mu(u)$ and the correction term in the Itô formula can be seen as in the proof of Corollary 2.8. □

We close this section with a small Lemma which will help us throughout the whole thesis when formulae like (2.23) are used for Gronwall arguments.

Lemma 2.11. *Let $r \in [1, \infty)$, $q \in (1, \infty)$, $\varepsilon > 0$, $T > 0$ and $X \in L^r(\Omega, L^\infty(0, T))$. Then, we have*

$$\|X\|_{L^r(\Omega, L^q(0,t))} \le \varepsilon\|X\|_{L^r(\Omega, L^\infty(0,t))} + \varepsilon^{1-q}\frac{1}{q}\left(1 - \frac{1}{q}\right)^{q-1} \int_0^t \|X\|_{L^r(\Omega, L^\infty(0,s))}\mathrm{d}s, \quad t \in [0, T].$$

Proof. As a consequence of Young's inequality, we obtain

$$a^{1-\frac{1}{q}} b^{\frac{1}{q}} \le \varepsilon a + \varepsilon^{1-q}\frac{1}{q}\left(1 - \frac{1}{q}\right)^{q-1} b, \qquad a, b \ge 0, \quad \varepsilon > 0. \tag{2.26}$$

Then, interpolation of $L^q(0, t)$ between $L^\infty(0, t)$ and $L^1(0, t)$ and (2.26) yield

$$\|X\|_{L^q(0,t)} \le \|X\|_{L^\infty(0,t)}^{1-\frac{1}{q}} \|X\|_{L^1(0,t)}^{\frac{1}{q}} \le \varepsilon\|X\|_{L^\infty(0,t)} + \varepsilon^{1-q}\frac{1}{q}\left(1 - \frac{1}{q}\right)^{q-1} \|X\|_{L^1(0,t)}.$$

Now, we take the $L^r(\Omega)$-norm and apply Minkowski's inequality to get

$$\begin{aligned}\|X\|_{L^r(\Omega, L^q(0,t))} &\le \varepsilon\|X\|_{L^r(\Omega, L^\infty(0,t))} + \varepsilon^{1-q}\frac{1}{q}\left(1 - \frac{1}{q}\right)^{q-1} \int_0^t \|X(s)\|_{L^r(\Omega)}\mathrm{d}s \\ &\le \varepsilon\|X\|_{L^r(\Omega, L^\infty(0,t))} + \varepsilon^{1-q}\frac{1}{q}\left(1 - \frac{1}{q}\right)^{q-1} \int_0^t \|X\|_{L^r(\Omega, L^\infty(0,s))}\mathrm{d}s.\end{aligned}$$

□

2.3. Deterministic and stochastic Strichartz estimates

In this section, we collect Strichartz estimates for the Schrödinger group. They express a gain of integrability as a consequence of the dispersive nature of the linear Schrödinger equation. In a fixed point argument, this property will help us to deal with power type nonlinearities. Moreover, Strichartz estimates will be important when we prove uniqueness of the solutions obtained from the Galerkin approximation technique. Unfortunately, Strichartz estimates typically depend on the underlying geometry. Thus, we have to leave the rather general setting of the previous sections and consider special selfadjoint operators A. In view of the applications in this thesis, we will restrict ourselves to Laplacians on the full space and on Riemannian manifolds with bounded geometry.

Let us start with the scaling condition for the pairs of exponent appearing in Strichartz estimates.

Definition 2.12. A pair $(p,q) \in [2,\infty]^2$ is called *admissible* if

$$\frac{2}{q} + \frac{d}{p} = \frac{d}{2}, \qquad (q,p,d) \neq (2,\infty,2).$$

In Figure 2.1, we visualize the admissible pairs in different dimensions. They correspond to line segments which are closed for $d \neq 2$. In contrast, the end point associated to the pair $(\infty, 2)$ is excluded for $d = 2$.

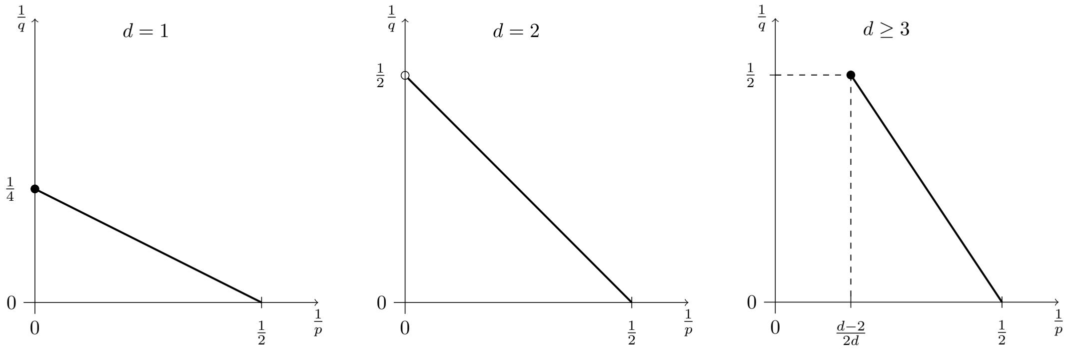

Figure 2.1.: Admissible pairs $(p,q) \in [2,\infty]^2$.

Figure 2.1 indicates that the set $\left\{(\frac{1}{p},\frac{1}{q}) : (p,q) \text{ admissible}\right\}$ is convex. This leads to the following interpolation Lemma.

Lemma 2.13. *Let M be a σ-finite measure space and $J \subset \mathbb{R}$ be an interval. For admissible pairs $(p,q), (p_1,q_1), (p_2,q_2) \in [2,\infty]^2$ with $p_2 < p < p_1$ and*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$$

for some $\theta \in (0,1)$, we have

$$\|u\|_{L^q(J,L^p)} \leq \|u\|^{\theta}_{L^{q_1}(J,L^{p_1})} \|u\|^{1-\theta}_{L^{q_2}(J,L^{p_2})}, \qquad u \in L^{q_1}(J, L^{p_1}(M)) \cap L^{q_2}(J, L^{p_2}(M)).$$

Proof. Obviously, the Strichartz scaling condition yields $q_1 < q < q_2$ and by a straightforward computation, we get

$$\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

From Lyapunov's inequality and Hölder with exponents $\frac{q_1}{q\theta}$ and $\frac{q_2}{q(1-\theta)}$, we infer

$$\|u\|^q_{L^q(J,L^p)} \leq \int_J \|u(s)\|^{q\theta}_{L^{p_1}} \|u(s)\|^{q(1-\theta)}_{L^{p_2}} \mathrm{d}s \leq \|u\|^{q\theta}_{L^{q_1}(J,L^{p_1})} \|u\|^{q(1-\theta)}_{L^{q_2}(J,L^{p_2})}.$$

□

We continue with the prototypical Strichartz estimates for Schrödinger group on $\mathbb{R}^d$.

Proposition 2.14. *Let $(p_j, q_j) \in [2,\infty]^2$, $j = 1, 2$, be admissible pairs and $J \subset \mathbb{R}$ be an interval with $0 \in J$. Then, the following estimates hold for $k \in \{0, 1\}$:*

a) $\|t \mapsto e^{\mathrm{i}t\Delta} x\|_{L^{q_1}(J,W^{k,p_1})} \lesssim \|x\|_{H^k}, \qquad x \in H^k(\mathbb{R}^d);$

b) $\left\|t \mapsto \int_0^t e^{\mathrm{i}(t-s)\Delta} f(s) \mathrm{d}s\right\|_{L^{q_1}(J,W^{k,p_1})} \lesssim \|f\|_{L^{q_2'}(J,W^{k,p_2'})}, \qquad f \in L^{q_2'}(J, W^{k,p_2'}(\mathbb{R}^d)).$

Furthermore, $t \mapsto e^{\mathrm{i}t\Delta}x$ and $t \mapsto \int_0^t e^{\mathrm{i}(t-s)\Delta} f(s)\mathrm{d}s$ are elements of $C_b(J, H^k(\mathbb{R}^d))$ and we have

c) $\|t \mapsto e^{\mathrm{i}t\Delta} x\|_{C_b(J,H^k)} \lesssim \|x\|_{H^k}, \qquad x \in H^k(\mathbb{R}^d);$

d) $\left\|t \mapsto \int_0^t e^{\mathrm{i}(t-s)\Delta} f(s) \mathrm{d}s\right\|_{C_b(J,H^k)} \lesssim \|f\|_{L^{q_2'}(J,W^{k,p_2'})}, \qquad f \in L^{q_2'}(J, W^{k,p_2'}(\mathbb{R}^d)).$

The implicit constants in a)-d) are independent of J and k.

Proof. These estimates are well-known, see for example [36], Theorem 2.3.3. □

The following Propositions are devoted to Strichartz estimates for the Laplace-Beltrami operator on manifolds. From now on, let M be a d-dimensional Riemannian manifold such that

$$M \text{ is complete and connect, has a positive injectivity radius and a bounded geometry.} \tag{2.27}$$

For a definition of these notions, we refer to Appendix A.4. Moreover, we equip M with the canonical volume μ and suppose that M satisfies the doubling property: For all $x \in \tilde{M}$ and $r > 0$, we have $\mu(B(x,r)) < \infty$ and

$$\mu(B(x,2r)) \lesssim \mu(B(x,r)). \tag{2.28}$$

We start with the deterministic homogeneous Strichartz estimate due to Bernicot and Samoyeau from [16], Corollary 6.2. In contrast to the flat case from Proposition 2.14, we have to accept a regularity loss of $\frac{1+\varepsilon}{q}$ derivatives in the Strichartz estimates since the dispersive behavior of the Schrödinger group is not as strong as before.

Proposition 2.15. *Let M be a d-dimensional Riemannian manifold with* (2.27) *and* (2.28). *Let $\varepsilon > 0$ and $(p, q) \in [2,\infty) \times [2,\infty]$ be admissible. Then,*

$$\|t \mapsto e^{\mathrm{i}t\Delta_g} x\|_{L^q(0,T;L^p(M))} \lesssim \|x\|_{H^{\frac{1+\varepsilon}{q}}(M)}, \qquad x \in H^{\frac{1+\varepsilon}{q}}(M). \tag{2.29}$$

The implicit constant C depends on T and ε with $C \to \infty$ as $\varepsilon \to 0$.

From Lemma 2.15, one can deduce the following Strichartz estimates in fractional Sobolev spaces.

Lemma 2.16. *Let M be a d-dimensional Riemannian manifold with* (2.27) *and* (2.28). *Let $(p, q) \in [2,\infty) \times [2,\infty]$ be admissible, $\varepsilon \in (0, q-1)$, and $\theta \in (\frac{1+\varepsilon}{q}, 1]$.*

a) We have the homogeneous Strichartz estimate

$$\|e^{\mathrm{i}t\Delta_g} x\|_{L^q(0,T;H^{\theta-\frac{1+\varepsilon}{q},p})} \lesssim_{T,\varepsilon} \|x\|_{H^\theta}, \qquad x \in H^\theta(M). \tag{2.30}$$

b) *We have the inhomogeneous Strichartz estimate*

$$\left\| \int_0^{\cdot} e^{\mathrm{i}(\cdot-\tau)\Delta_g} f(\tau)\mathrm{d}\tau \right\|_{L^q(0,T;H^{\theta-\frac{1+\varepsilon}{q},p})} \lesssim_{T,\varepsilon} \|f\|_{L^1(0,T;H^\theta)} \tag{2.31}$$

for $f \in L^1(0,T;H^\theta(M))$.

Proof. ad a). The Propositions $A.53$ a) and 2.15 yield

$$\begin{aligned}\|e^{\mathrm{i}t\Delta_g}x\|_{L^q(0,T;H^{\theta-\frac{1+\varepsilon}{q},p})} &\eqsim \|(1-\Delta_g)^{\frac{\theta}{2}-\frac{1+\varepsilon}{2q}} e^{\mathrm{i}t\Delta_g}x\|_{L^q(0,T;L^p)} \\ &= \|e^{\mathrm{i}t\Delta_g}(1-\Delta_g)^{\frac{\theta}{2}-\frac{1+\varepsilon}{2q}}x\|_{L^q(0,T;L^p)} \\ &\lesssim_{T,\varepsilon} \|(1-\Delta_g)^{\frac{\theta}{2}-\frac{1+\varepsilon}{2q}}x\|_{H^{\frac{1+\varepsilon}{2q}}} \eqsim \|x\|_{H^\theta}.\end{aligned}$$

ad b). From (2.30) and Minkowski's inequality, we get

$$\left\| \int_0^{\cdot} e^{\mathrm{i}(\cdot-\tau)\Delta_g} f(\tau)\mathrm{d}\tau \right\|_{L^q(0,T;H^{\theta-\frac{1+\varepsilon}{q},p})} \lesssim_{T,\varepsilon} \left\| \int_0^{\cdot} e^{-\mathrm{i}\tau\Delta_g} f(\tau)\mathrm{d}\tau \right\|_{H^\theta} \lesssim \|f\|_{L^1(0,T;H^\theta)}.$$

□

In the special case of a compact manifold M, we will use the following spectrally localized Strichartz estimates due to Burq, Gérard and Tzvetkov.

Proposition 2.17. *Let M be a compact d-dimensional Riemannian manifold and take admissible pairs $(p_1,q_1),(p_2,q_2) \in [2,\infty]^2$. Then, for any $\varphi \in C_c^\infty(\mathbb{R})$, there are $\beta > 0$, $C_1 > 0$ and $C_2 > 0$ such that the following assertions hold for all $h \in (0,1]$:*

a) *For any interval J of length $|J| \leq \beta h$ and $x \in L^2(M)$*

$$\|t \mapsto e^{\mathrm{i}t\Delta_g}\varphi(h^2\Delta_g)x\|_{L^{q_1}(J,L^{p_1})} \leq C\|x\|_{L^2}.$$

b) *For any interval J of length $|J| \leq \frac{\beta h}{2}$ and $f \in L^{q_2'}(J,L^{p_2'}(M))$*

$$\left\| t \mapsto \int_{-\infty}^t e^{\mathrm{i}(t-s)\Delta_g}\varphi(h^2\Delta_g)f(s)\mathrm{d}s \right\|_{L^{q_1}(J,L^{p_1})} \leq C\|\varphi(h^2\Delta_g)f\|_{L^{q_2'}(J,L^{p_2'})}.$$

Proof. ad a). See [35], Proposition 2.9. The result follows from the dispersive estimate for the Schrödinger group from [35], Lemma 2.5, and an application of Keel-Tao's Theorem from [76] with $U(t) = e^{\mathrm{i}t\Delta_g}\tilde{\varphi}(h^2\Delta_g)\mathbf{1}_J(t)$ for some $\tilde{\varphi} \in C_c^\infty(\mathbb{R})$ with $\tilde{\varphi} = 1$ on $\operatorname{supp}(\varphi)$.

ad b). See [35], Lemma 3.4. □

In the following Remark, we would like to provide some background information about the previous Propositions.

Remark 2.18. In the special case of a compact manifold M, Burq, Gérard and Tzvetkov used Littlewood-Paley Theory and the spectrally localized estimate from Proposition 2.17 a) to prove a sharp version of Proposition 2.15 with $\varepsilon = 0$, see [35], Theorem 1. Similarly, the proof of Proposition 2.15 is based on an analogue of the spectrally localized estimate for $\varepsilon > 0$. The restriction to $p < \infty$ is due to the Littlewood-Paley characterization of L^p-spaces which fails for $p = \infty$.

In the second part of this section, we aim for a Strichartz estimate for the stochastic convolution. Originally this estimate is due to Brzezniak and Millet, [30], Theorem 3.10, but we present two alternative proofs. The first one is based on the strong BDG-inequality from Theorem A.20. The second one employs a duality argument and the surjectivity of the Itô isomorphism. To give a unified proof for all kinds of deterministic Strichartz estimates, we work in the following setting.

Assumption 2.19. i) Let $p \in [2,\infty)$, (M,Σ,μ) be a σ-finite measure space and A be a closed operator on $L^2(M)\cap L^p(M)$ with $-1\in\rho(A)$. For $\theta\geq 0$, we define a Banach space by

$$H^{\theta,p,A} := \left\{ x\in L^p(M):\ (\operatorname{Id}+A)^{\frac{\theta}{2}}\, x \in L^p(M)\right\}$$

equipped with the norm $\|x\|_{H^{\theta,p,A}} := \|\,(\operatorname{Id}+A)^{\frac{\theta}{2}}\, x\|_{L^p}$.

ii) Let $J=[a,b]\subset[0,T]$ be a closed interval and $(U(t))_{t\in J}$ a strongly continuous family of bounded operators on $H^{\mu,2}(M)$ for some $\mu\geq 0$. Furthermore, we assume that

$$(\mathcal{U}x)(s) := U(s)x, \qquad x\in H^{\mu,2}(M), \quad s\in J,$$

defines a bounded operator $\mathcal{U}\in\mathcal{L}(H^{\mu,2}(M), L^q(J;L^p(M)))$ for some $q\in[2,\infty)$.

iii) We assume that A commutes with U, i.e. $U(t)x\in\mathcal{D}(A)$ and $U(t)Ax = AU(t)x$ for $x\in\mathcal{D}(A)$ and $t\in J$.

Obviously, the notation $H^{\theta,p,A}$ is motivated by the connection to fractional Sobolev spaces. Indeed, we have $H^{\theta,p,-\Delta} = H^{\theta,p}(\mathbb{R}^d)$ and $H^{\theta,p,-\Delta_g} = H^{\theta,p}(M)$ for Riemannian manifolds M satisfying (2.27). The next definition provides the space in which the stochastic convolution will be defined.

Definition 2.20. Let $r\in[1,\infty)$, $\theta_1\geq\mu$ and $\theta_2\geq 0$. By $L^r_{\mathbb{F}}(\Omega, C(J,H^{\theta_1,2,A})\cap L^q(J,H^{\theta_2,p,A}))$, we denote the set of all $u\in L^r(\Omega, C(J,H^{\theta_1,2,A})\cap L^q(J,H^{\theta_2,p,A}))$ such that the continuous representant in $L^2(M)$ is an $\mathbb{F}$-adapted process and there is also an $\mathbb{F}$-predictable process $\tilde{u}:[0,T]\times\Omega\to H^{\theta_2,p,A}$ which represents u.

We remark that $L^r_{\mathbb{F}}(\Omega, C(J,H^{\theta_1,2,A})\cap L^q(J,H^{\theta_2,p,A}))$ is a Banach space since it is a closed subspace of $L^r(\Omega, C(J,H^{\theta_1,2,A})\cap L^q(J,H^{\theta_2,p,A}))$. Moreover, we recall that the space of stochastically integrable processes in $H^{\theta_2,p,A}$ is denoted by $\mathcal{M}^r_{\mathbb{F},Y}(J,H^{\theta,2,A})$. Next, we present a result that lays the foundations for the subsequent estimates of stochastic convolutions.

Theorem 2.21. *In the setting of Assumption 2.19, the map*

$$K\Phi(t) = U(t)\int_a^t \Phi(s)\mathrm{d}W(s), \qquad t\in J, \quad \Phi\in\mathcal{M}^r_{\mathbb{F},Y}(J,H^{\theta,2,A}),$$

defines a bounded operator

$$K:\quad \mathcal{M}^r_{\mathbb{F},Y}(J,H^{\theta,2,A}) \to L^r_{\mathbb{F}}(\Omega, C(J,H^{\theta,2,A})\cap L^q(J,H^{\theta-\mu,p,A}))$$

for all $\theta\geq\mu$.

Proof. Step 1. We start with the case $\theta=\mu$ and define

$$K_1\Phi(\cdot,\tilde{t}) := \int_a^{\tilde{t}} \mathcal{U}\Phi(s)\mathrm{d}W(s) = \sum_{n=0}^{N}\sum_{m=1}^{M}\mathbf{1}_{A_{m,n}}\sum_{k=1}^{K}\left[W(\tilde{t}\wedge t_n)y_k - W(\tilde{t}\wedge t_{n-1})y_k\right]\mathcal{U}x_{k,m,n}$$

for $\tilde{t} \in J$ and for an elementary process

$$\Phi(s,\omega) := \sum_{n=0}^{N} \mathbf{1}_{(t_{n-1},t_n]}(s) \sum_{m=1}^{M} \mathbf{1}_{A_{m,n}}(\omega) \sum_{k=1}^{K} y_k \otimes x_{k,m,n}$$

in $\mathcal{M}^r_{\mathbb{F},Y}(J, H^{\mu,2,A})$. Moreover, we have

$$K\Phi(t) = \sum_{n=0}^{N} \sum_{m=1}^{M} \mathbf{1}_{A_{m,n}} \sum_{k=1}^{K} \left[W(t \wedge t_n)y_k - W(t \wedge t_{n-1})y_k\right] U(t)x_{k,m,n}. \tag{2.32}$$

In particular, we observe that the maps $t \mapsto K_1\Phi(t,t)$ and $t \mapsto K\Phi(t)$ coincide in $L^q(J, L^p(M))$ and in $C(J, H^{\mu,2,A})$. By $\sup_{t\in J} \|U(t)\|_{\mathcal{L}(H^{\mu,2})} < \infty$ and the BDG-inequality for the $H^{\mu,2,A}$-valued stochastic integral, we infer

$$\mathbb{E}\|K\Phi\|^r_{C(J,H^{\mu,2})} \lesssim \mathbb{E}\left\|\int_a^{\cdot} \Phi(s)\mathrm{d}W(s)\right\|^r_{C(J,H^{\mu,2})} \lesssim \mathbb{E}\|\Phi\|^r_{L^2(J,\mathrm{HS}(Y,H^{\mu,2}))}.$$

To get a similar estimate in $L^q(J, L^p(M))$, we employ Theorem A.20, i.e. the strong BDG-inequality in mixed L^p-spaces, and obtain

$$\begin{aligned}\mathbb{E}\|K\Phi\|^r_{L^q(J,L^p)} &\leq \mathbb{E}\left\|\sup_{\tilde{t}\in J} |K_1\Phi(\cdot,\tilde{t})|\right\|^r_{L^q(J,L^p)} = \mathbb{E}\left\|\sup_{\tilde{t}\in J}\left|\int_a^{\tilde{t}} \mathcal{U}\Phi(s)\mathrm{d}W(s)\right|\right\|^r_{L^q(J,L^p)}\\ &\lesssim \mathbb{E}\left\|\left(\int_J \|\left(\mathcal{U}\Phi(s)f_m\right)_{m\in\mathbb{N}}\|^2_{\ell^2(\mathbb{N})}\mathrm{d}s\right)^{\frac{1}{2}}\right\|^r_{L^q(J,L^p)}\end{aligned}$$

Since we have $p, q \geq 2$, we can apply the Minkowski inequality and afterwards, the deterministic Strichartz estimate from Assumption 2.19 ii) yields

$$\begin{aligned}\mathbb{E}\|K\Phi\|^r_{L^q(J,L^p)} &\lesssim \mathbb{E}\left(\sum_{m=1}^{\infty}\int_J \|\mathcal{U}\Phi(s)f_m\|^2_{L^q(J,L^p)}\mathrm{d}s\right)^{\frac{r}{2}} \lesssim \mathbb{E}\left(\sum_{m=1}^{\infty}\int_J \|\Phi(s)f_m\|^2_{H^{\mu,2}}\mathrm{d}s\right)^{\frac{r}{2}}\\ &= \mathbb{E}\|\Phi\|^r_{L^2(J,\mathrm{HS}(Y,H^{\mu,2}))}.\end{aligned}$$

In particular, the estimate

$$\mathbb{E}\|K\Phi\|^r_{L^q(J,L^p)\cap C(J,H^{\mu,2})} \lesssim \mathbb{E}\|\Phi\|^r_{L^2(J,\mathrm{HS}(Y,L^2))}$$

holds and obviously, $K\Phi$ is $\mathbb{F}$-predictable in $H^{\mu,2,A}$ by pathwise continuity and adaptedness. By continuous extension, we obtain these properties for all $\Phi \in \mathcal{M}^r_{\mathbb{F},Y}(J, H^{\mu,2,A})$.

Step 2. It remains to prove that there is an $\mathbb{F}$-predictable representant of $K\Phi$ in $L^p(M)$. Since $K\Phi$ is predictable in $H^{\mu,2,A}$ and $H^{\mu,2,A} \hookrightarrow L^2(M)$, we obtain that the map

$$[0,t] \times \Omega \ni (s,\omega) \mapsto \left(K\Phi(s,\omega), \Psi\right)_{L^2}$$

is $\mathcal{B}([0,t])\otimes\mathcal{F}_t$-measurable for all $t \in J$ and $\Psi \in L^2(M)\cap L^{p'}(M)$. Moreover, $L^p(M)$ is separable, $L^2(M) \cap L^{p'}(M)$ dense in $L^{p'}(M)$ and we have

$$\left(K\Phi(s,\omega), \Psi\right)_{L^2} = \langle K\Phi(s,\omega), \Psi\rangle_{L^p,L^{p'}}$$

for almost all $(s,\omega) \in J \times \Omega$. By the Pettis measurability Theorem, see [48], Chapter II, Theorem 1.2, we infer that there is a representant of $K\Phi$ such that

$$[0,t] \times \Omega \ni (s,\omega) \mapsto K\Phi(s,\omega) \in L^p(M)$$

is strongly $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable. Hence, $K\Phi$ is $\mathbb{F}$-predictable in $L^p(M)$.

Step 3. The case $\theta > \mu$ is a direct consequence of Step 1 and 2 since A is a closed operator commuting with $\mathcal{U}$. Indeed, we can interchange $(\mathrm{Id} + A)^{\frac{\theta-\mu}{2}}$ with the stochastic integral by Proposition A.14 and obtain

$$(\mathrm{Id} + A)^{\frac{\theta-\mu}{2}} K\Phi = \mathcal{U} \int_a^{\cdot} (\mathrm{Id} + A)^{\frac{\theta-\mu}{2}} \Phi(s) \mathrm{d}W(s) = K((\mathrm{Id} + A)^{\frac{\theta-\mu}{2}} \Phi).$$

By the previous results,

$$\begin{aligned}\|K\Phi\|_{L^r(\Omega, C(J,H^{\theta,2}) \cap L^q(J,H^{\theta-\mu,p}))} &\lesssim \|(\mathrm{Id} + A)^{\frac{\theta-\mu}{2}} \Phi\|_{L^r(\Omega, L^2(J, \mathrm{HS}(Y, H^{\mu,2})))} \\ &= \|\Phi\|_{L^r(\Omega, L^2(J, \mathrm{HS}(Y, H^{\theta,2})))}.\end{aligned}$$

□

As an immediate consequence of the previous Theorem, we obtain the Strichartz estimate for the stochastic convolution.

Corollary 2.22. *Let $r \in [1,\infty)$ and $\theta \geq \mu$. Suppose that Assumption 2.19 holds with a unitary C_0-group $(U(t))_{t\in\mathbb{R}}$ and generator $-\mathrm{i}A$. Then, the stochastic convolution*

$$K_{Stoch}\Phi(t) = \int_a^t U(t-s)\Phi(s) \mathrm{d}W(s), \qquad t \in J, \quad \Phi \in \mathcal{M}^r_{\mathbb{F},Y}(J, H^{\theta,2,A}),$$

has a continuous version in $H^{\theta,2,A}$ which is in $L^r_{\mathbb{F}}(\Omega, C(J, H^{\theta,2,A}) \cap L^q(J, H^{\theta-\mu,p,A}))$ and satisfies the estimate

$$\|K_{Stoch}\Phi\|_{L^r(\Omega, C(J,H^{\theta,2}) \cap L^q(J,H^{\theta-\mu,p}))} \lesssim \|\Phi\|_{L^r(\Omega, L^2(J, \mathrm{HS}(Y, H^{\theta,2})))}. \tag{2.33}$$

Proof. The restriction of $(U(t))_{t\in\mathbb{R}}$ to $H^{\theta,2,A}$ is also a unitary C_0-group. By Theorem A.13, the stochastic convolution has a continuous modification in $H^{\mu,2,A}$. Moreover, we have

$$U(t) \int_a^t U(-s)\Phi(s) \mathrm{d}W(s) = \int_a^t U(t-s)\Phi(s) \mathrm{d}W(s)$$

in $H^{\mu,2,A}$ for all $t \in [0,T]$ almost surely. By the continuity of the processes on the LHS and RHS of the equation and Lemma A.6, the Ω-nullset can be chosen independently of $t \in [0,T]$. Finally, the estimate (2.33) follows from Theorem 2.21 and the continuity of the operators $U(-s)$, $s \in J$. □

In the following Corollary, we apply the results from Theorem 2.21 and Corollary 2.22 to concrete situations where we have deterministic Strichartz estimates.

Corollary 2.23. *Let $J \subset [0,T]$ be a closed interval, $r \in [1,\infty)$ and (p,q) admissible.*

a) For $k \in \{0,1\}$*, we have*

$$\left\| \int_0^{\cdot} e^{\mathrm{i}(\cdot - s)\Delta} B(s) \mathrm{d}W(s) \right\|_{L^r(\Omega, L^q(J, W^{k,p}(\mathbb{R}^d)) \cap C(J, H^k))} \lesssim \|B\|_{L^r(\Omega, L^2(J, \mathrm{HS}(Y, H^k)))}$$

for all $B \in \mathcal{M}^r_{\mathbb{F},Y}(J, H^k(\mathbb{R}^d))$.

b) Let M *be a Riemannian manifold satisfying* (2.27) *and* (2.28)*. Let* $\varepsilon > 0$ *and* $\theta \geq \frac{1+\varepsilon}{q}$*. Then, we have*

$$\left\| \int_0^{\cdot} e^{\mathrm{i}(\cdot - s)\Delta_g} B(s) \mathrm{d}W(s) \right\|_{L^r(\Omega, L^q(J, H^{\theta - \frac{1+\varepsilon}{q}, p}) \cap C(J, H^{\theta,p}))} \lesssim_\varepsilon \|B\|_{L^r(\Omega, L^2(J, \mathrm{HS}(Y, H^\theta)))}$$

for all $B \in \mathcal{M}^r_{\mathbb{F},Y}(J, H^\theta(M))$.

c) Let M *be a compact Riemannian manifold of dimension* d*. Let* $\varphi \in C_c^\infty(\mathbb{R})$ *and* $\beta > 0$ *as in Lemma* 2.17*. Let* $h \in (0,1]$ *and* $J \subset [0,T]$ *be an interval of length* $|J| \leq \beta h$ *and* $\chi_h \in C_c^\infty(\mathbb{R})$ *with* $\operatorname{supp}(\chi_h) \subset J$*. Then, we have*

$$\begin{aligned}\left\| \int_{\inf J}^{\cdot} e^{\mathrm{i}(\cdot - s)\Delta} \chi_h(s) \varphi(h^2 \Delta_g) B(s) \mathrm{d}W(s) \right\|_{L^r(\Omega, L^q(J, L^p) \cap C(J, L^2))} \\ \lesssim \|\varphi(h^2\Delta_g) B\|_{L^r(\Omega, L^2(J, \mathrm{HS}(Y, L^2)))}\end{aligned}$$

for all $B \in \mathcal{M}^r_{\mathbb{F},Y}(J, L^2(M))$.

Proof. ad a). We apply Corollary 2.22 with $\mu = 0$, $\theta = k$ and $U(t) = e^{it\Delta}$ for $t \in \mathbb{R}$. The homogeneous Strichartz estimates holds due to Proposition 2.14.

ad b). This is a consequence of Proposition 2.15 and Corollary 2.22 applied to $\mu = \frac{1+\varepsilon}{q}$ and $U(t) = e^{it\Delta_g}$.

ad c). Set $\mu = \theta = 0$ and

$$U(t) = \mathbf{1}_J(t) e^{it\Delta_g} \tilde{\varphi}(h^2 \Delta_g), \qquad t \in [0,T],$$

for some $\tilde{\varphi} \in C_c^\infty(\mathbb{R})$ with $\tilde{\varphi} = 1$ on $\operatorname{supp}(\varphi)$. The homogeneous Strichartz estimate for $U(\cdot)$ is guaranteed by Proposition 2.17, but it is not possible to apply Corollary 2.22 directly since U does not possess the group property. Nevertheless, we have the identity

$$\begin{aligned}\int_{\inf J}^t e^{\mathrm{i}(t-s)\Delta_g} \chi_h(s) \varphi(h^2\Delta_g) B(s) \mathrm{d}W(s) &= \int_0^t U(t) U(s)^* \chi_h(s) \varphi(h^2 \Delta_g) B(s) \mathrm{d}W(s) \\ &= U(t) \int_0^t U(s)^* \chi_h(s) \varphi(h^2\Delta_g) B(s) \mathrm{d}W(s)\end{aligned}$$

for all $t \in J$ almost surely. By Theorem A.13, the LHS has a continuous modification in $L^2(M)$ and the RHS is continuous as a consequence of Theorem 2.21. Thus, we can choose the Ω-nullset independent of time and Theorem 2.21 yields

$$\begin{aligned}&\left\| \int_{\inf J}^{\cdot} e^{\mathrm{i}(\cdot - s)\Delta} \chi_h(s) \varphi(h^2\Delta_g) B(s) \mathrm{d}W(s) \right\|_{L^r(\Omega, C(J, L^2) \cap L^q(J, L^p))} \\ &\quad \lesssim \|s \mapsto e^{\mathrm{i}s\Delta_g} \chi_h(s) \varphi(h^2\Delta_g) B(s)\|_{L^r(\Omega, L^2(J, \mathrm{HS}(Y, L^2)))} \lesssim \|\varphi(h^2\Delta_g) B\|_{L^r(\Omega, L^2(J, \mathrm{HS}(Y, L^2)))}.\end{aligned}$$

□

If $\mathbb{F}$ is the Brownian filtration, i.e. the augmentation of

$$\mathcal{F}_t^\beta := \sigma\left(W(s) : s \in [0,t]\right), \qquad t \in [0,T],$$

we can give the following alternative proof of the stochastic Strichartz estimate. For simplicity, we restrict ourselves to the case of Corollary 2.23, a), i.e. $U(t) = e^{\mathrm{i}t\Delta}$ on $L^2(\mathbb{R}^d)$.

Second proof of the stochastic Strichartz estimates. Step 1. In view of the estimates

$$\begin{aligned}\mathbb{E}\left\|t \mapsto \int_0^t e^{\mathrm{i}(t-s)\Delta} B(s) \mathrm{d}W(s)\right\|_{L^q(0,T;L^p)}^r &\le \mathbb{E}\left\|t \mapsto \sup_{t_0 \in [0,T]} \left|\int_0^{t_0} e^{\mathrm{i}(t-s)\Delta} B(s) \mathrm{d}W(s)\right|\right\|_{L^q(0,T;L^p)}^r \\ &\lesssim \mathbb{E}\left\|\int_0^T e^{\mathrm{i}(t-s)\Delta} B(s) \mathrm{d}W(s)\right\|_{L^q(0,T;L^p)}^r\end{aligned}$$

as a consequence of the maximal inequality from Theorem A.20, it is sufficient to prove

$$\left\|\int_0^T e^{\mathrm{i}(\cdot-s)\Delta} B(s) \mathrm{d}W(s)\right\|_{L^r(\Omega,L^q(J,L^p))} \lesssim \|B\|_{L^r(\Omega,L^2(J,\mathrm{HS}(Y,L^2)))}.$$

Step 2. From the Itô isomorphism, see Corollary A.19 and in particular, its surjectivity in the case of the Brownian filtration, we infer

$$\begin{aligned}&\left\|\int_0^T e^{\mathrm{i}(\cdot-s)\Delta} B(s) \mathrm{d}W(s)\right\|_{L^r(\Omega,L^q(J,L^p))} \\ &\qquad \eqsim \sup\left\{\left|\mathbb{E}\left\langle \int_0^T e^{\mathrm{i}(\cdot-s)\Delta} B(s) \mathrm{d}W(s), \int_0^T \Phi \mathrm{d}W\right\rangle_{L^qL^p}\right| : \|\Phi\|_{\mathcal{M}^{r'}_{\mathbb{F},Y}(0,T;L^{q'}(0,T;L^{p'}))} \le 1\right\}.\end{aligned} \tag{2.34}$$

Let us define the bilinear form

$$a(B,\Phi) := \mathbb{E}\sum_{m=1}^\infty \int_0^T \int_0^T \langle e^{-\mathrm{i}s\Delta} B(s) f_m, e^{-\mathrm{i}t\Delta}\Phi(s,t) f_m\rangle \mathrm{d}t\mathrm{d}s \tag{2.35}$$

for $B \in \mathcal{M}^r_{\mathbb{F},Y}(0,T;L^2(\mathbb{R}^d))$ and $\Phi \in \mathcal{M}^{r'}_{\mathbb{F},Y}(0,T;L^{q'}(0,T;L^{p'}(\mathbb{R}^d)))$. By the Itô formula for the Banach space duality duality, see [34], Corollary 2.6 and equation (2.6), we can simplify

$$\mathbb{E}\left\langle \int_0^T e^{\mathrm{i}(\cdot-s)\Delta} B(s) \mathrm{d}W(s), \int_0^T \Phi \mathrm{d}W\right\rangle_{L^qL^p} = \mathbb{E}\sum_{m=1}^\infty \int_0^T \langle e^{\mathrm{i}(\cdot-s)\Delta} B(s) f_m, \Phi(s) f_m\rangle_{L^qL^p} \mathrm{d}s.$$

Hence, it is sufficient to prove

$$\sup\left\{|a(B,\Phi)| : \|\Phi\|_{\mathcal{M}^{r'}_{\mathbb{F},Y}(0,T;L^{q'}(0,T;L^{p'}))} \le 1\right\} \le \|B\|_{L^r(\Omega,L^2(J,\mathrm{HS}(Y,L^2)))}. \tag{2.36}$$

Step 3. We fix $B \in \mathcal{M}^r_{\mathbb{F},Y}(0,T;L^2(\mathbb{R}^d))$ and $\Phi \in \mathcal{M}^{r'}_{\mathbb{F},Y}(0,T;L^{q'}(0,T;L^{p'}(\mathbb{R}^d)))$ and recall that the deterministic Strichartz estimate is equivalent to the dual version

$$\left\|\int_0^T e^{-\mathrm{i}t\Delta} g(t) \mathrm{d}t\right\|_{L^2} \lesssim \|g\|_{L^{q'}(0,T;L^{p'})}, \qquad g \in L^{q'}(0,T;L^{p'}). \tag{2.37}$$

By (2.37), we obtain

$$\begin{aligned}|a(B,\Phi)| &= \left|\mathbb{E}\sum_{m=1}^\infty \int_0^T \left\langle e^{-\mathrm{i}s\Delta}B(s)f_m, \int_0^T e^{-\mathrm{i}t\Delta}\Phi(s,t)f_m\,\mathrm{d}t\right\rangle \mathrm{d}s\right| \\ &\lesssim \mathbb{E}\sum_{m=1}^\infty \int_0^T \|B(s)f_m\|_{L^2}\|\Phi(s,\cdot)f_m\|_{L^{q'}(0,T;L^{p'})}\mathrm{d}s \\ &\lesssim \|B\|_{L^r(\Omega,L^2(0,T;\mathrm{HS}(Y,L^2)))}\|(\Phi f_m)_m\|_{L^{r'}(\Omega,L^2([0,T]\times\mathbb{N},L^{q'}(0,T;L^{p'})))}.\end{aligned}$$

Next, we employ the Minkowski inequality with $p',q'\le 2$ as well as the Itô isomorphism (A.11) to deduce

$$\begin{aligned}|a(B,\Phi)| &\le \|B\|_{L^r(\Omega,L^2(0,T;\mathrm{HS}(Y,L^2)))}\left\|\left(\sum_{m=1}^\infty\int_0^T|\Phi f_m|^2\mathrm{d}s\right)^{\frac12}\right\|_{L^{r'}(\Omega,L^{q'}(0,T;L^{p'}))} \\ &= \|B\|_{L^r(\Omega,L^2(0,T;\mathrm{HS}(Y,L^2)))}\|\Phi\|_{\mathcal{M}^{r'}_{\mathbb{F},Y}(0,T;L^{q'}(0,T;L^{p'}))},\end{aligned}$$

which yields (2.36). □

Remark 2.24. Although this proof is obviously more complicated than the first one and only holds for the Brownian filtration, we decided to present it here to point out the remarkable connection of the stochastic Strichartz estimate with the bilinear form a from (2.35). This is similar to the well-known TT^*-argument, see for example [113], Lemma 4.3.4, which links deterministic Strichartz estimates with the inequality

$$|b(f,g)|\lesssim \|f\|_{L^{q'}(\mathbb{R},L^{p'})}\|g\|_{L^{\tilde{q}'}(\mathbb{R},L^{\tilde{p}'})}$$

for the bilinear form defined by

$$b(f,g) := \int_{\mathbb{R}}\int_{\mathbb{R}}\langle e^{-\mathrm{i}t\Delta}f(t), e^{-\mathrm{i}s\Delta}g(s)\rangle\mathrm{d}s\mathrm{d}t.$$

2.4. Skorohod-Jakubowski Theorem and Tightness Criteria

In this section, we lay the foundation for the construction of a martingale solution to the stochastic NLS in the Chapters 4 and 6. We present two variants of the Skorohod-Jakubowski Theorem which enable us to extract an almost surely converging subsequence of a tight sequence of random variables. Afterwards, we deduce a tightness criterium in a space of continuous functions suitable for the Gaussian noise from Chapter 4. To be able to deal with jump noise, we generalize this criterium to càdlàgfunctions.

Definition 2.25. Let Z be a topological space equipped with a σ-algebra $\mathcal{Z}$ which contains the topology.

a) We say that a set $\{f_i : i\in I\}$ of functions $f_i : Z\to\mathbb{R}$ separates points of Z if $x\neq y\in Z$ implies $f_i(x)\neq f_i(y)$ for some $i\in I$.
b) A sequence $(\mathbb{P}_n)_{n\in\mathbb{N}}$ of probability measures on $(Z,\mathcal{Z})$ is called *tight* if for every $\varepsilon>0$, there is a compact set $K_\varepsilon\subset Z$ with

$$\inf_{n\in\mathbb{N}}\mathbb{P}_n(K_\varepsilon)\ge 1-\varepsilon.$$

A sequence $(X_n)_{n\in\mathbb{N}}$ of Z-valued random variables is called tight if $\left(\mathbb{P}^{X_n}\right)_{n\in\mathbb{N}}$ is tight.

In metric spaces, one can apply the Prokhorov Theorem II.6.7 from [106] and the Skorohod Theorem 6.7 from [17] to show that tightness implies almost sure convergence of a subsequence. Since we will be faced with non-metric spaces, we will use the following generalization of this classical procedure.

Theorem 2.26 (Skorohod-Jakubowski). *Let $\mathcal{X}$ be a topological space such that there is a sequence of continuous functions $f_m : \mathcal{X} \to \mathbb{R}$ that separates points of $\mathcal{X}$. Let $\mathcal{A}$ be the σ-algebra generated by $(f_m)_m$ and $(\mu_n)_{n\in\mathbb{N}}$ be a tight sequence of probability measures on $(\mathcal{X}, \mathcal{A})$.*
Then, there are a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$, random variables X_k, X for $k \in \mathbb{N}$ on a common probability space $(\tilde{\Omega}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$ with $\tilde{\mathbb{P}}^{X_k} = \mu_{n_k}$ for $k \in \mathbb{N}$, and $X_k \to X$ $\tilde{\mathbb{P}}$-almost surely for $k \to \infty$.

We stated Theorem 2.26 in the form of [31]. For the original source, however, we refer to [72]. Starting from [32], this theorem has been frequently used as a tool for the compactness method for stochastic partial differential equations. For the application to the NLS with jump noise, we also state the following variant of Motyl, [99], Appendix B, Corollary 2.

Corollary 2.27. *Let $\mathcal{X}_1$ be a complete separable metric space and $\mathcal{X}_2$ a topological space such that there is a sequence of continuous functions $f_m : \mathcal{X}_2 \to \mathbb{R}$ that separates points of $\mathcal{X}_2$. Define $\mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2$ and equip $\mathcal{X}$ with the topology induced by the canonical projections $\pi_j : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}_j$ for $j = 1, 2$. Let $(\chi_n)_{n\in\mathbb{N}}$ be a tight sequence of random variables $\chi_n : \Omega \to (\mathcal{X}, \mathcal{B}(\mathcal{X}_1) \otimes \mathcal{A})$, where $\mathcal{A}$ is the σ-algebra generated by f_m, $m \in \mathbb{N}$. Assume that there is a random variable η in $\mathcal{X}_1$ such that $\mathbb{P}^{\pi_1 \circ \chi_n} = \mathbb{P}^{\eta}$ for all $n \in \mathbb{N}$.*

Then, there are a subsequence $(\chi_{n_k})_{k\in\mathbb{N}}$ and random variables $\tilde{\chi}_k$, $\tilde{\chi}$ in $\mathcal{X}$ for $k \in \mathbb{N}$ on a common probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with

i) $\tilde{\mathbb{P}}^{\tilde{\chi}_k} = \mathbb{P}^{\chi_{n_k}}$ *for $k \in \mathbb{N}$,*

ii) $\tilde{\chi}_k \to \tilde{\chi}$ *in $\mathcal{X}$ almost surely for $k \to \infty$,*

iii) $\pi_1 \circ \tilde{\chi}_k = \pi_1 \circ \tilde{\chi}$ *almost surely.*

We continue with a Lemma that gives us additional information on the topological assumption in the previous results.

Lemma 2.28. *Let X be a separable locally convex topological vector space with Hausdorff-property.*

a) Then, there is a sequence $F := \{f_m : m \in \mathbb{N}\}$ in X^* which generates the Borel σ-algebra $\mathcal{B}(X)$ and separates points of X.

b) For each topological space which has the property from a), compactness and sequential compactness coincide.

Proof. Assertion b) is stated in [72]. So, we concentrate on a). Let us choose a dense sequence $(x_n)_{n\in\mathbb{N}} \subset X$ and denote a family of seminorms generating the topology in X by $(p_k)_{k\in\mathbb{N}}$. As a consequence of the Hahn-Banach Theorem in locally convex spaces, see [128], Theorem VIII. 2.8, there is $F = \{\ell_{n,k} : n, k \in \mathbb{N}\} \subset X^*$ with

$$p_k(x_n) = \ell_{n,k}(x_n), \qquad \sup\{\ell_{n,k}(x) : p_k(x) \leq 1\} = 1.$$

In particular, we get $p_k(x) = \sup_{n\in\mathbb{N}} \ell_{n,k}(x)$ for each $x \in X$ and thus,

$$\overline{B}_k(x, r) := \{y \in X : p_k(y - x) \leq r\} = \bigcap_{n\in\mathbb{N}} \{y \in X : \ell_{n,k}(y - x) \leq r\} \in \sigma(F).$$

Since each open ball is the union of countably many closed ones, also $B_k(x_n, r)$ for $k, n \in \mathbb{N}$ and rational $r > 0$ is contained in $\sigma(F)$. These sets build a basis of the topology in X such that we obtain $\mathcal{B}(X) \subset \sigma(F)$. The other implication " $\supset$ " is obvious, since each $f \in F$ is continuous and therefore Borel-measurable.
To prove that F separates points, we take $x \neq y \in X$. By the Hausdorff-property, there is $k \in \mathbb{N}$ with $0 < p_k(x - y)$, see [128], Lemma VIII.1.4. Hence, there is an $n \in \mathbb{N}$ such that $\ell_{n,k}(x - y) > 0$. □

In the following subsections, we would like to apply Theorem 2.26 and Corollary 2.27 in concrete functional settings associated to the stochastic NLS to get the tightness criteria mentioned above.

2.4.1. Tightness in a space of continuous functions

Throughout this section, M is a finite measure space, $A : L^2(M) \supset D(A) \to L^2(M)$ is a non-negative selfadjoint operator with the scale $(X_\theta)_{\theta \in \mathbb{R}}$ of fractional domains from Appendix A.3. We denote $E_A := X_{\frac{1}{2}}$ and $E_A^* := X_{-\frac{1}{2}}$. In view of the applications in Chapter 3, it is convenient to equip $L^2(M)$, E_A and E_A^* with the real inner products $\operatorname{Re}\left(\cdot,\cdot\right)_{L^2}$, $\operatorname{Re}\left(\cdot,\cdot\right)_{E_A}$ and $\operatorname{Re}\left(\cdot,\cdot\right)_{-\frac{1}{2}}$, respectively. In particular, the notation E_A^* is justified since E_A and $X_{-\frac{1}{2}}$ are dual in the sense that each real-valued linear functional f on E_A has the representation $f = \operatorname{Re}\langle \cdot, y_f \rangle_{\frac{1}{2},-\frac{1}{2}}$ for some $y_f \in X_{-\frac{1}{2}}$.

The goal of this section is to find a criterion for tightness of random variables in

$$Z_T := C([0,T], E_A^*) \cap L^{\alpha+1}(0,T; L^{\alpha+1}(M)) \cap C_w([0,T], E_A)$$

for $\alpha > 1$ under the assumption that E_A is compactly embedded in $L^{\alpha+1}(M)$. This will enable us to apply Theorem 2.26. The first definition tells us how to interpret Z_T as a topological space.

Definition 2.29. Let $(X_1, \mathcal{O}_1)$ and $(X_2, \mathcal{O}_2)$ be topological spaces. In this thesis, we always equip $X_1 \cap X_2$ with the *supremum-topology*, i.e. the smallest topology that contains $\tilde{\mathcal{O}}_1$ and $\tilde{\mathcal{O}}_2$, where

$$\tilde{\mathcal{O}}_1 = \{O \cap X_2 : O \in \mathcal{O}_1\}, \qquad \tilde{\mathcal{O}}_2 = \{O \cap X_1 : O \in \mathcal{O}_2\}.$$

The first two spaces in the definition of Z_T are Banach spaces and thus, we consider the topologies induced by the norms. The third space $C_w([0,T], E_A)$ is understood in the sense of the following definition.

Definition 2.30. Let X be a Banach space with separable dual X^*.

a) We define

$$C_w([0,T], X) := \left\{ u : [0,T] \to X \,\middle|\, [0,T] \ni t \to \langle u(t), x^* \rangle \in \mathbb{C} \text{ is cont. for all } x^* \in X^* \right\}$$

and equip $C_w([0,T], X)$ with the locally convex topology induced by the family P of seminorms given by

$$P := \{p_{x^*} : x^* \in X^*\}, \qquad p_{x^*}(u) := \sup_{t \in [0,T]} |\langle u(t), x^* \rangle|.$$

b) For $r > 0$, we consider the ball $\mathbb{B}_X^r := \{u \in X : \|u\|_X \le r\}$ and define

$$C([0,T], \mathbb{B}_X^r) := \Big\{ u \in C_w([0,T], X) : \sup_{t \in [0,T]} \|u(t)\|_X \le r \Big\}.$$

Remark 2.31. By the separability of X^*, the weak topology on $\mathbb{B}_X^r$ is metrizable and a metric is given by

$$q(x_1, x_2) = \sum_{k=1}^{\infty} 2^{-k} |\langle x_1 - x_2, x_k^* \rangle|, \qquad x_1, x_2 \in X,$$

for a dense sequence $(x_k^*)_{k \in \mathbb{N}} \in \left(B_{X^*}^1\right)^{\mathbb{N}}$, see [23], Theorem 3.29. In particular, the notation in Definition 2.30 is justified in the sense that $C([0,T], \mathbb{B}_X^r)$ coincides with $C([0,T], M)$ for $(M, d) = (\mathbb{B}_X^r, q)$. Since M is also separable by [23], Theorem 3.26, $C([0,T], \mathbb{B}_X^r)$ is a complete, separable metric space with metric

$$\rho(u, v) := \sup_{t \in [0,T]} q(u(t), v(t)), \qquad u, v \in C([0,T], \mathbb{B}_{E_A}^r).$$

We continue with some auxiliary results.

Lemma 2.32. *Let $r > 0$ and $u_n, u \in C_w([0,T], X)$ with $\sup_{t \in [0,T]} \|u_n(t)\|_X \le r$ and $u_n \to u$ in $C_w([0,T], X)$. Then, we have $u_n \to u$ in $C([0,T], \mathbb{B}_X^r)$.*

Proof. By Lebesgue's Convergence Theorem,

$$\rho(u_n, u) \le \sum_{k=1}^{\infty} 2^{-k} \sup_{t \in [0,T]} |\langle u_n(t) - u(t), x_k^* \rangle| \to 0, \qquad n \to \infty,$$

where we used the definition of convergence in $C_w([0,T], X)$ for fixed $k \in \mathbb{N}$ and

$$\sup_{t \in [0,T]} |\langle u_n(t) - u(t), x_k^* \rangle| \le \left(\sup_{t \in [0,T]} \|u_n(t)\|_X + \sup_{t \in [0,T]} \|u(t)\|_X \right) \|x_k^*\|_{X^*} \le 2r.$$

□

Lemma 2.33 (Strauss). *Let X, Y be Banach spaces with $X \hookrightarrow Y$. Then, we have the inclusion*

$$L^\infty(0,T; X) \cap C_w([0,T], Y) \subset C_w([0,T], X).$$

Proof. See [116], Chapter 3, Lemma 1.4. □

The following Lemma can be found in [89], p. 58. Since the reference does not contain a proof, we give it for the convenience of the reader.

Lemma 2.34 (Lions). *Let X, X_0, X_1 be Banach spaces with $X_0 \hookrightarrow X \hookrightarrow X_1$, where the first embedding is compact. Furthermore, we assume that X_0, X_1 are reflexive and $p \in [1, \infty)$. Then, for each $\varepsilon > 0$ there is $C_\varepsilon > 0$ with*

$$\|x\|_X^p \le \varepsilon \|x\|_{X_0}^p + C_\varepsilon \|x\|_{X_1}^p, \qquad x \in X_0.$$

Proof. Step 1: Let $p = 1$ and assume that the assertion does not hold, i.e. there is $\varepsilon_0 > 0$ such that for each $n \in \mathbb{N}$ we can choose $x_n \in X_0 \setminus \{0\}$ with

$$\|x_n\|_X > \varepsilon_0 \|x_n\|_{X_0} + n\|x_n\|_{X_1}. \tag{2.38}$$

We define a normed sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ in X_0 by $\tilde{x}_n := x_n\|x_n\|_{X_0}^{-1}$ for $n \in \mathbb{N}$. By the reflexivity of X_0 there is an $x \in X_0$ and a subsequence again denoted by $(\tilde{x}_n)_{n\in\mathbb{N}}$ with $\tilde{x}_n \rightharpoonup x$ in X_0 for $n \to \infty$. The compactness of $X_0 \hookrightarrow X$ yields $\tilde{x}_n \to x$ in X for $n \to \infty$. Due to the embedding $X \hookrightarrow X_1$, the strong convergence also holds true in X_1. As a consequence of Assumption (2.38) and the fact that $(\tilde{x}_n)_{n\in\mathbb{N}}$ is bounded in X, there is a constant $C \in (\varepsilon_0, \infty)$ such that

$$C \geq \|\tilde{x}_n\|_X > \varepsilon_0 + n\|\tilde{x}_n\|_{X_1} \tag{2.39}$$

for all $n \in \mathbb{N}$. Hence,

$$\|\tilde{x}_n\|_{X_1} < \frac{C - \varepsilon_0}{n} \to 0$$

for $n \to \infty$. Thus, we get $x = 0$ and therefore $\tilde{x}_n \to 0$ in X, which is a contradiction to $\|\tilde{x}_n\|_X > \varepsilon_0 > 0$ for all $n \in \mathbb{N}$, see (2.39).

Step 2: For arbitrary $p \in [1, \infty)$ and $\varepsilon > 0$, we set $\tilde{\varepsilon} := \left(\frac{\varepsilon}{2^{p-1}}\right)^{\frac{1}{p}}$ and apply the first step for $\tilde{\varepsilon}$. With $C_\varepsilon := 2^{p-1}C_{\tilde{\varepsilon}}^p$, we obtain

$$\begin{aligned}\|x\|_X^p \leq (\tilde{\varepsilon}\|x\|_{X_0} + C_{\tilde{\varepsilon}}\|x\|_{X_1})^p &\leq 2^{p-1}\left(\tilde{\varepsilon}^p\|x\|_{X_0}^p + C_{\tilde{\varepsilon}}^p\|x\|_{X_1}^p\right) \\ &= \varepsilon\|x\|_{X_0}^p + C_\varepsilon\|x\|_{X_1}^p.\end{aligned}$$

□

We continue with a criterion for convergence of a sequence in $C([0,T], \mathbb{B}_{E_A}^r)$.

Lemma 2.35. *Let $r > 0$ and $(u_n)_{n\in\mathbb{N}} \subset L^\infty(0,T;E_A) \cap C([0,T], E_A^*)$ be a sequence with*

a) $\sup_{n\in\mathbb{N}} \|u_n\|_{L^\infty(0,T;E_A)} \leq r$,

b) $u_n \to u$ *in* $C([0,T], E_A^*)$ *for* $n \to \infty$.

Then $u_n, u \in C([0,T], \mathbb{B}_{E_A}^r)$ for all $n \in \mathbb{N}$ and $u_n \to u$ in $C([0,T], \mathbb{B}_{E_A}^r)$ for $n \to \infty$.

Proof. The Strauss-Lemma 2.33 and the assumptions guarantee that

$$u_n \in C([0,T], E_A^*) \cap L^\infty(0,T;E_A) \subset C_w([0,T], E_A)$$

for all $n \in \mathbb{N}$ and $\sup_{t\in[0,T]} \|u(t)\|_{E_A} \leq r$. Hence, we infer $u_n \in C([0,T], \mathbb{B}_{E_A}^r)$ for all $n \in \mathbb{N}$. For $h \in E_A$, we have

$$\sup_{s\in[0,T]} |\mathrm{Re}\langle u_n(s) - u(s), h\rangle| \leq \|u_n - u\|_{C([0,T],E_A^*)}\|h\|_{E_A} \xrightarrow{n\to\infty} 0.$$

By Assumption a) and Banach-Alaoglu, we get a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and $v \in L^\infty(0,T;E_A)$ with $u_{n_k} \rightharpoonup^* v$ in $L^\infty(0,T;E_A)$ and by the uniqueness of the weak star limit in $L^\infty(0,T;E_A^*)$, we conclude $u = v \in L^\infty(0,T;E_A)$ with $\|u\|_{L^\infty(0,T;E_A)} \leq r$.
Let $\varepsilon > 0$ and $h \in E_A^*$. By the density of E_A in E_A^*, we choose $h_\varepsilon \in E_A$ with $\|h - h_\varepsilon\|_{E_A^*} \leq \frac{\varepsilon}{4r}$ and obtain for large $n \in \mathbb{N}$

$$|\mathrm{Re}\langle u_n(s) - u(s), h\rangle| \leq |\mathrm{Re}\langle u_n(s) - u(s), h - h_\varepsilon\rangle| + |\mathrm{Re}\langle u_n(s) - u(s), h_\varepsilon\rangle|$$

$$\leq \|u_n(s) - u(s)\|_{E_A} \|h - h_\varepsilon\|_{E_A^*} + |\mathrm{Re}\langle u_n(s) - u(s), h_\varepsilon\rangle|$$
$$\leq 2r\frac{\varepsilon}{4r} + \frac{\varepsilon}{2} = \varepsilon$$

independent of $s \in [0,T]$. This implies $\sup_{s\in[0,T]} |\mathrm{Re}\langle u_n(s) - u(s), h\rangle| \to 0$ for $n \to \infty$ and all $h \in E_A^*$, i.e. $u_n \to u$ in $C_w([0,T], E_A)$. By Lemma 2.32, we obtain the assertion. □

Let us recall that M is a finite measure space and the embedding $E_A \hookrightarrow L^{\alpha+1}(M)$ is compact. This leads to the following criterion for compactness in Z_T which is a variant of the Arzela-Ascoli Theorem.

Proposition 2.36. *Let K be a subset of Z_T and $r > 0$ such that*

a) $\sup_{u\in K} \|u\|_{L^\infty(0,T;E_A)} \leq r$;

b) K is equicontinuous in $C([0,T], E_A^*)$, i.e.

$$\lim_{\delta\to 0} \sup_{u\in K} \sup_{|t-s|\leq\delta} \|u(t) - u(s)\|_{E_A^*} = 0.$$

Then, K is relatively compact in Z_T.

Proof. Step 1. First, we show that there is a sequence of continuous real-valued functions on Z_T which generates the Borel σ-algebra and separates points. We set

$$Z_1 := C([0,T], E_A^*), \qquad Z_2 := L^{\alpha+1}(0,T; L^{\alpha+1}(M)), \qquad Z_3 := C_w([0,T], E_A)$$

and note that by Lemma 2.28 a), there are sequences $(f_{m,j})_{m\in\mathbb{N}} \subset Z_j^*$ that separate points of Z_j and generate $\mathcal{B}(Z_j)$ for $j = 1,2,3$. We define $F_j = \{f_{m,j}|_{Z_T} : m \in \mathbb{N}\}$ and $F = F_1 \cup F_2 \cup F_3$. By the definition of the supremum-topology, we get

$$\mathcal{B}(Z_T) = \sigma\big(\bigcup_{j=1,2,3} \mathcal{B}(Z_j)|_{Z_T}\big) = \sigma\big(\bigcup_{j=1,2,3} \left(\sigma\left(f_{m,j} : m \in \mathbb{N}\right)\right)|_{Z_T}\big)$$
$$= \sigma\big(\bigcup_{j=1,2,3} \sigma\left(f_{m,j}|_{Z_T} : m \in \mathbb{N}\right)\big) = \sigma(F).$$

Step 2: Let K be a subset of Z_T such that the assumptions $a)$ and $b)$ are fulfilled. By Lemma 2.28 b), it suffices to show that K is sequentially relatively compact. We choose a sequence $(z_n)_{n\in\mathbb{N}} \subset K$. We want to construct a subsequence converging in $L^{\alpha+1}(0,T; L^{\alpha+1}(M))$, $C([0,T], E_A^*)$ and $C_w([0,T], E_A)$.

By $a)$, we can choose a null set I_n for each $n \in \mathbb{N}$ with $\|z_n(t)\|_{E_A} \leq r$ for all $t \in [0,T] \setminus I_n$. The set $I := \bigcup_{n\in\mathbb{N}} I_n$ is also a null set and for each $t \in [0,T] \setminus I$, the sequence $(z_n(t))_{n\in\mathbb{N}}$ is bounded in E_A. Let $(t_j)_{j\in\mathbb{N}} \subset [0,T] \setminus I$ be a sequence that is dense in $[0,T]$.

From the compactness of $E_A \hookrightarrow L^{\alpha+1}(M)$ and the continuity of $L^{\alpha+1}(M) \hookrightarrow E_A^*$, we infer that the embedding $E_A \hookrightarrow E_A^*$ is also compact. Therefore, we can choose for each $j \in \mathbb{N}$ a Cauchy subsequence in E_A^* again denoted by $(z_n(t_j))_{n\in\mathbb{N}}$. By a diagonalisation argument, one obtains a common Cauchy subsequence $(z_n(t_j))_{n\in\mathbb{N}}$.

Let $\varepsilon > 0$. Assumption $b)$ yields $\delta > 0$ with

$$\sup_{u\in K} \sup_{|t-s|\leq\delta} \|u(t) - u(s)\|_{E_A^*} \leq \frac{\varepsilon}{3}. \tag{2.40}$$

Let us choose finitely many open balls $U_\delta^1, \dots, U_\delta^L$ of radius δ covering $[0,T]$. By density, each of these balls contains an element of the sequence $(t_j)_{j\in\mathbb{N}}$, say $t_{j_l} \in U_\delta^l$ for $l \in \{1,\dots,L\}$. In particular, the sequence $(z_n(t_{j_l}))_{n\in\mathbb{N}}$ is Cauchy for all $l \in \{1,\dots,L\}$. Hence,

$$\|z_n(t_{j_l}) - z_m(t_{j_l})\|_{E_A^*} \le \frac{\varepsilon}{3}, \qquad l = 1,\dots,L, \tag{2.41}$$

if we choose $m,n \in \mathbb{N}$ sufficiently large. Now, we fix $t \in [0,T]$ and take $l \in \{1,\dots,L\}$ with $|t_{j_l} - t| \le \delta$. We use (2.40) and (2.41) to get

$$\|z_n(t) - z_m(t)\|_{E_A^*} \le \|z_n(t) - z_n(t_{j_l})\|_{E_A^*} + \|z_n(t_{j_l}) - z_m(t_{j_l})\|_{E_A^*} + \|z_m(t_{j_l}) - z_m(t)\|_{E_A^*} \le \varepsilon. \tag{2.42}$$

This means that $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T],E_A^*)$ since the estimate (2.42) is uniform in $t \in [0,T]$.

Step 3: The first step yields $z \in C([0,T],E_A^*)$ with $z_n \to z$ in $C([0,T],E_A^*)$ for $n \to \infty$ and assumption a) implies, that there is $r > 0$ with $\sup_{n\in\mathbb{N}} \|z_n\|_{L^\infty(0,T;E_A)} \le r$.
Therefore, we obtain $z \in C([0,T],\mathbb{B}_{E_A}^r)$ and $z_n \to z$ in $C([0,T],\mathbb{B}_{E_A}^r)$ for $n \to \infty$ by Lemma 2.35. Hence, $z_n \to z$ in $C_w([0,T],E_A)$.

Step 4: We fix again $\varepsilon > 0$. By the Lions Lemma 2.34 with $X_0 = E_A$, $X = L^{\alpha+1}(M)$,
$X_1 = E_A^*$, $p = \alpha+1$ and $\varepsilon_0 = \frac{\varepsilon}{2T(2r)^{\alpha+1}}$ we get

$$\|v\|_{L^{\alpha+1}(M)}^{\alpha+1} \le \varepsilon_0 \|v\|_{E_A}^{\alpha+1} + C_{\varepsilon_0} \|v\|_{E_A^*}^{\alpha+1} \tag{2.43}$$

for all $v \in E_A$. The first step allows us to choose $n,m \in \mathbb{N}$ large enough that

$$\|z_n - z_m\|_{C([0,T],E_A^*)}^{\alpha+1} \le \frac{\varepsilon}{2C_{\varepsilon_0} T}.$$

The special choice $v(t) = z_n(t) - z_m(t)$ for $t \in [0,T]$ in (2.43) and integration with respect to time yields

$$\begin{aligned}
\|z_n - z_m\|_{L^{\alpha+1}(0,T;L^{\alpha+1}(M))}^{\alpha+1} &\le \varepsilon_0 \|z_n - z_m\|_{L^{\alpha+1}(0,T;E_A)}^{\alpha+1} + C_{\varepsilon_0} \|z_n - z_m\|_{L^{\alpha+1}(0,T;E_A^*)}^{\alpha+1} \\
&\le \varepsilon_0 T \|z_n - z_m\|_{L^\infty(0,T;E_A)}^{\alpha+1} + C_{\varepsilon_0} T \|z_n - z_m\|_{C([0,T],E_A^*)}^{\alpha+1} \\
&\le \varepsilon_0 T (2r)^{\alpha+1} + C_{\varepsilon_0} T \|z_n - z_m\|_{C([0,T],E_A^*)}^{\alpha+1} \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence, the sequence $(z_n)_{n\in\mathbb{N}}$ is also Cauchy in $L^{\alpha+1}(0,T;L^{\alpha+1}(M))$. □

To transfer the previous result to the stochastic setting, we introduce the Aldous condition which can be viewed as a stochastic analogue to equi-rightcontinuity.

Definition 2.37. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of stochastic processes in a Banach space E. Assume that for every $\varepsilon > 0$ and $\eta > 0$ there is $\delta > 0$ such that for every sequence $(\tau_n)_{n\in\mathbb{N}}$ of $[0,T]$-valued stopping times, one has

$$\sup_{n\in\mathbb{N}} \sup_{0<\theta\le\delta} \mathbb{P}\left\{\|X_n((\tau_n+\theta)\wedge T) - X_n(\tau_n)\|_E \ge \eta\right\} \le \varepsilon.$$

In this case, we say that $(X_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition $[A]$.

The following Lemma from [99], Appendix A, gives us a useful consequence of the Aldous condition $[A]$.

Lemma 2.38. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of stochastic processes in a Banach space E satisfying the Aldous condition $[A]$. Then, for every $\varepsilon > 0$, there exists a Borel measurable subset $A_\varepsilon \subset C([0,T],E)$ such that*

$$\mathbb{P}^{X_n}(A_\varepsilon) \geq 1-\varepsilon, \qquad \lim_{\delta\to 0} \sup_{u\in A_\varepsilon} \sup_{|t-s|\leq\delta} \|u(t)-u(s)\|_E = 0.$$

The deterministic compactness result in Proposition 2.36 and the last Lemma can be used to get the following criterion for tightness in Z_T.

Proposition 2.39. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of adapted continuous E_A^*-valued processes satisfying the Aldous condition $[A]$ in E_A^* and*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\|X_n\|_{L^\infty(0,T;E_A)}^2\right] < \infty.$$

Then the sequence $\left(\mathbb{P}^{X_n}\right)_{n\in\mathbb{N}}$ is tight in Z_T.

Proof. Let $\varepsilon > 0$ and $R_1 := \left(\frac{2}{\varepsilon}\sup_{n\in\mathbb{N}} \mathbb{E}\left[\|X_n\|_{L^\infty(0,T;E_A)}^2\right]\right)^{\frac{1}{2}}$. Using the Tschebyscheff inequality, we obtain

$$\mathbb{P}\left\{\|X_n\|_{L^\infty(0,T;E_A)} > R_1\right\} \leq \frac{1}{R_1^2}\mathbb{E}\left[\|X_n\|_{L^\infty(0,T;E_A)}^2\right] \leq \frac{\varepsilon}{2}.$$

We set $B := \left\{u \in L^\infty(0,T;E_A) : \|u\|_{L^\infty(0,T;E_A)} \leq R_1\right\}$. By Lemma 2.38, one can use the Aldous condition $[A]$ to get a Borel subset A of $C([0,T],E_A^*)$ with

$$\mathbb{P}^{X_n}(A) \geq 1-\frac{\varepsilon}{2}, \quad n\in\mathbb{N}, \qquad \lim_{\delta\to 0} \sup_{u\in A} \sup_{|t-s|\leq\delta} \|u(t)-u(s)\|_{E_A^*} = 0.$$

From the Strauss Lemma 2.33 and $E_A \hookrightarrow L^{\alpha+1}(M)$, we infer $A\cap B \subset Z_T$. We define $K := \overline{A\cap B}$ where the closure is understood in Z_T. The set K is compact in Z_T by Proposition 2.36 and we can estimate

$$\mathbb{P}^{X_n}(K) \geq \mathbb{P}^{X_n}(A\cap B) \geq \mathbb{P}^{X_n}(A) - \mathbb{P}^{X_n}(B^c) \geq 1-\frac{\varepsilon}{2}-\frac{\varepsilon}{2} = 1-\varepsilon, \qquad n\in\mathbb{N}.$$

□

We close this section with the following Corollary which brings together the ingredients we prepared.

Corollary 2.40. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of adapted E_A^*-valued processes satisfying the Aldous condition $[A]$ in E_A^* and*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\|X_n\|_{L^\infty(0,T;E_A)}^2\right] < \infty.$$

Then, there are a subsequence $(X_{n_k})_{k\in\mathbb{N}}$, a second probability space $(\tilde{\Omega},\tilde{\mathbb{F}},\tilde{\mathbb{P}})$ and Borel-measurable random variables $\tilde{X}_k, \tilde{X} : \tilde{\Omega} \to Z_T$ for $k\in\mathbb{N}$ such that $\tilde{\mathbb{P}}^{\tilde{X}_k} = \mathbb{P}^{X_{n_k}}$ for $k\in\mathbb{N}$, and $\tilde{X}_k \to \tilde{X}$ $\tilde{\mathbb{P}}$-almost surely in Z_T for $k\to\infty$.

Proof. The assertion is an immediate consequence of Proposition 2.39 and Theorem 2.26. □

2.4.2. Tightness in a space of càdlàg functions

We continue the study of tightness criteria. Since we will also consider the stochastic NLS with jump noise, we need a generalization of the previous results to spaces of càdlàg functions. As in the previous section, M is a finite measure space and A is a non-negative selfadjoint operator with the scale $(X_\theta)_{\theta\in\mathbb{R}}$ of fractional domains. We denote $E_A := X_{\frac{1}{2}}$ and $E_A^* := X_{-\frac{1}{2}}$. Furthermore, we take $\alpha > 1$ and assume that E_A is compactly embedded in $L^{\alpha+1}(M)$. Throughout the section, $(\mathbb{S}, \mathrm{d})$ denotes a complete, separable metric space.

Definition 2.41. a) The space of all *càdlàg functions* $f : [0,T] \to \mathbb{S}$, i.e. f is right-continuous with left limit in every $t \in [0,T]$, is denoted by $\mathbb{D}([0,T], \mathbb{S})$.

b) For $u \in \mathbb{D}([0,T], \mathbb{S})$ and $\delta > 0$, we define the *modulus*

$$w_{\mathbb{S}}(u,\delta) := \inf_{\Pi_\delta} \max_{t_j \in Q} \sup_{t,s\in[t_{j-1},t_j)} \mathrm{d}(u(t), u(s)),$$

where Π_δ is the set of all partitions $Q = \{0 = t_0 < t_1 < \dots < t_N = T\}$ of $[0,T]$ with

$$t_{j+1} - t_j \geq \delta, \qquad j = 0, \dots, N-1.$$

The following Proposition is about the so-called *Skorohod topology* on $\mathbb{D}([0,T], \mathbb{S})$.

Proposition 2.42. *a) We denote the set of increasing homeomorphisms of* $[0,T]$ *by* Λ. *If we equip* $\mathbb{D}([0,T], \mathbb{S})$ *with the metric defined by*

$$\rho(u,v) := \inf_{\lambda\in\Lambda} \left[\sup_{t\in[0,T]} \mathrm{d}(u(t), v(\lambda(t))) + \sup_{t\in[0,T]} |t - \lambda(t)| + \sup_{s\neq t} \left| \log \frac{\lambda(t)-\lambda(s)}{t-s} \right| \right]$$

for $u, v \in \mathbb{D}([0,T], \mathbb{S})$, *we obtain a complete, separable metric space.*

b) A sequence $(u_n)_{n\in\mathbb{N}} \in \mathbb{D}([0,T], \mathbb{S})^{\mathbb{N}}$ *is convergent to* $u \in \mathbb{D}([0,T], \mathbb{S})$ *in the metric* ρ *if and only if there is* $(\lambda_n) \in \Lambda^{\mathbb{N}}$ *with*

$$\sup_{t\in[0,T]} |\lambda_n(t) - t| \to 0, \qquad \sup_{t\in[0,T]} \mathrm{d}(u_n(\lambda_n(t)), u(t)) \to 0, \qquad n \to \infty.$$

Proof. See [17], page 123 and following. □

As an analogue to $C_w([0,T], X)$ we also define the space of càdlàg functions w.r.t. the weak topology.

Definition 2.43. Let X be a reflexive, separable Banach space.

a) Then, we define $\mathbb{D}_w([0,T], X)$ as the space of all $u : [0,T] \to X$ such that

$$[0,T] \ni t \to \langle u(t), x^* \rangle \in \mathbb{R} \text{ is càdlàg for all } x^* \in X^*.$$

We equip $\mathbb{D}_w([0,T], X)$ with the weakest topology such that the map

$$\mathbb{D}_w([0,T], X) \ni u \mapsto \langle u(\cdot), x^* \rangle \in \mathbb{D}([0,T], \mathbb{R})$$

is continuous for all $x^* \in X^*$.

b) For $r>0$, we consider the ball $\mathbb{B}_X^r := \{u \in X : \|u\|_X \le r\}$ and define

$$\mathbb{D}([0,T],\mathbb{B}_X^r) := \left\{ u \in \mathbb{D}_w\left([0,T],X\right) : \sup_{t\in[0,T]} \|u(t)\|_X \le r \right\}.$$

In the following remark, we show that $\mathbb{D}([0,T],\mathbb{B}_X^r)$ is a complete separable metric space. This will illustrate why we assumed the reflexivity of X in the definition $\mathbb{D}_w\left([0,T],X\right)$ whereas it was not needed for the continuous analogue $C_w([0,T],X)$.

Remark 2.44. By reflexivity, X^* is also separable, see [23], Corollary 3.27. Thus, the weak topology on $\mathbb{B}_X^r$ is metrizable via

$$q(x_1,x_2) = \sum_{k=1}^{\infty} 2^{-k} |\langle x_1 - x_2, x_k^* \rangle|, \qquad x_1, x_2 \in X,$$

for a dense sequence $(x_k^*)_{k\in\mathbb{N}} \in \left(B_{X^*}^1\right)^{\mathbb{N}}$. We would like to show that $\mathbb{D}([0,T],\mathbb{B}_X^r)$ coincides with $\mathbb{D}([0,T],\mathbb{S})$ for $(\mathbb{S},d) = (\mathbb{B}_X^r, q)$, which justifies the notation in Definition 2.43. In particular, $\mathbb{D}([0,T],\mathbb{B}_X^r)$ is a complete, separable metric space by Proposition 2.42.

To show this, we note that the right-continuity of $\langle u(\cdot), x^* \rangle$ for all $x^* \in X^*$ is equivalent to the right-continuity of u in $(\mathbb{B}_X^r, q)$ by the definition of q. It is also easy to see that the existence of left limits transfers from $(\mathbb{B}_X^r, q)$ to $\langle \cdot, x^* \rangle$ for all $x^* \in X^*$.

For the converse direction, let $t_n \nearrow t$. Then, for each $x^* \in X^*$, there is $\gamma_{x^*} \in \mathbb{R}$ with $\langle u(t_n), x^* \rangle \to \gamma_{x^*}$. Since X is reflexive, $x^* \mapsto \gamma_{x^*}$ is linear and $|\gamma_{x^*}| \le r\|x^*\|_{X^*}$, there is a $v \in X$ such that $\gamma_{x^*} = \langle v, x^* \rangle$. Hence, $q(u(t_n), v) \to 0$ by Lebesgue.

The final goal of this section consists in applying the Skorohod-Jakubowski Theorem in the variant of Corollary 2.27 to the space

$$Z_T^{\mathbb{D}} := \mathbb{D}([0,T],E_A^*) \cap L^{\alpha+1}(0,T;L^{\alpha+1}(M)) \cap \mathbb{D}_w\left([0,T],E_A\right) := Z_1^{\mathbb{D}} \cap Z_2^{\mathbb{D}} \cap Z_3^{\mathbb{D}}.$$

As the first ingredient, we investigate, whether $Z_T^{\mathbb{D}}$ possesses the crucial topological property, i.e. whether we can find a countable set F of real-valued continuous functions on $Z_T^{\mathbb{D}}$ separating points. Moreover, we determine an appropriate σ-algebra $\mathcal{A}$ on $Z_T^{\mathbb{D}}$. Of course, it would be natural to equip $Z_T^{\mathbb{D}}$ with the Borel σ-algebra $\mathcal{B}(Z_T^{\mathbb{D}})$, but it turns out that $\mathcal{A}$ will be strictly contained in $\mathcal{B}(Z_T^{\mathbb{D}})$.

Given real-valued functions f_m on a topological space Z, we will frequently use the notation $f = (f_1, f_2, \dots)$ and the fact that $\sigma(f_m : m \in \mathbb{N}) = f^{-1}(\mathcal{B}(\mathbb{R}^\infty))$, where $\mathbb{R}^\infty$ is equipped with the locally convex topology induced by the seminorms $p_k(x) := |x_k|$ for $k \in \mathbb{N}$.

Lemma 2.45. *Let X be a set and $f_m : X \to \mathbb{R}$, $m \in \mathbb{N}$. Let $\mathcal{O}_X$ be the coarsest topology such f_m is continuous for all $m \in \mathbb{N}$. Then, we have*

$$\mathcal{B}(X) := \sigma(\mathcal{O}_X) = \sigma(f_m : m \in \mathbb{N}).$$

Proof. The direction " $\supset$ " is obvious by the continuity of f_m for $m \in \mathbb{N}$. In view of the good set principle, it is sufficient for the other inclusion to show that each $O \in \mathcal{O}_X$ is contained in $f^{-1}(\mathcal{B}(\mathbb{R}^\infty))$. Since each $O \in \mathcal{O}_X$ is of the form

$$O = \bigcup_{i\in I} \bigcap_{k=1}^{K} f^{-1}(O_{i,k}), \qquad O_{i,k} \text{ open in } \mathbb{R}^\infty,$$

see [53], Proposition 4.4, we can write represent O as the inverse image of the open set $\bigcup_{i\in I}\bigcap_{k=1}^{K} O_{i,k}$ under the continuous function f, which verifies the assertion. □

Lemma 2.46. *There is a countable family F of real-valued continuous functions on $Z_T^{\mathbb{D}}$ that separates points of $Z_T^{\mathbb{D}}$ and generates the σ-algebra*

$$\mathcal{A} = \sigma\left(\mathcal{B}(Z_1^{\mathbb{D}} \cap Z_2^{\mathbb{D}})|_{Z_T^{\mathbb{D}}} \cup \sigma(F_3)\right), \tag{2.44}$$

where F_3 consists of real-valued continuous functions on $Z_3^{\mathbb{D}}$ separating points of $Z_3^{\mathbb{D}}$.

Proof. Step 1. For each Z_i, we give a sequence $(f_{m,i})_{m\in\mathbb{N}}$ of continuous functions $f_{m,i} : Z_i \to \mathbb{R}$ separating points and determine the generated σ-algebras.

Let $\{\varphi_k : k \in \mathbb{N}\}$ be a sequence with $\|\varphi_k\|_{E_A} \le 1$ and $\|x\|_{E_A^*} = \sup_{k\in\mathbb{N}} |\operatorname{Re}\langle x, \varphi_k\rangle|$ for all $x \in E_A^*$. Let $\{t_l : l \in \mathbb{N}\}$ be dense in $[0,T]$. We set

$$f_{k,l,1}(u) := \operatorname{Re}\langle u(t_l), \varphi_k\rangle, \qquad u \in Z_1^{\mathbb{D}}, \quad k,l \in \mathbb{N},$$

and the enumeration of $(f_{k,l,1})_{k,l\in\mathbb{N}}$ will be called $(f_{m,1})_{m\in\mathbb{N}}$. For $n \in \mathbb{N}$, we denote

$$\pi_{t_1,\dots,t_n} : Z_1^{\mathbb{D}} \to (E_A^*)^n, \qquad u \mapsto (u(t_1), \dots, u(t_n)).$$

From [71], Corollary 2.4, we know that

$$\mathcal{B}(Z_1^{\mathbb{D}}) = \sigma(\pi_{t_1,\dots,t_n} : n \in \mathbb{N}).$$

Since $\pi_{t_1,\dots,t_n}$ is strongly measurable in $(E_A^*)^n$ if and only if the map

$$Z_1^{\mathbb{D}} \ni u \mapsto \operatorname{Re}\langle \pi_{t_1,\dots,t_n}(u), (\varphi_{k_1}, \dots, \varphi_{k_n})\rangle_{(E_A^*)^n,(E_A)^n} = \sum_{j=1}^{n} f_{k_j,j,1}(u)$$

is measurable for all $k_1, \dots, k_n \in \mathbb{N}$, we deduce $\mathcal{B}(Z_1^{\mathbb{D}}) = \sigma(f_{k,l,1} : k \in \mathbb{N},\ l \in \mathbb{N})$. By the choice of φ_k, $k \in \mathbb{N}$, we obtain that $f_{k,l,1}(u) = 0$ for all $k,l \in \mathbb{N}$ implies $u(t_l) = 0$ for all $l \in \mathbb{N}$. Since $(t_l)_{l\in\mathbb{N}}$ is dense and u right-continuous, this yields $u = 0$ and thus, $(f_{m,1})_{m\in\mathbb{N}}$ separates points in $Z_1^{\mathbb{D}}$. Moreover, functions of this form are continuous since the convergence $u_n \to u$ in $Z_1^{\mathbb{D}}$ implies pointwise convergence $u_n(t) \to u(t)$ in E_A^* for all $t \in [0,T]$. In particular, $f_{k,l,1}(u_n) \to f_{k,l,1}(u)$ for all $k,l \in \mathbb{N}$.

The existence of $(f_{m,2})_{m\in\mathbb{N}}$ is a consequence of the Hahn-Banach-Theorem in $Z_2^{\mathbb{D}}$ as we have proved in Lemma 2.28.

Let $\{h_k : k \in \mathbb{N}\}$ and $\{t_l : l \in \mathbb{N}\}$ be dense subsets of E_A^* and $[0,T]$, respectively. We set

$$f_{k,l,3}(u) := \operatorname{Re}\langle u(t_l), h_k\rangle, \qquad u \in Z_3^{\mathbb{D}}, \quad k,l \in \mathbb{N},$$

and denote the enumeration of $(f_{k,l,3})_{k,l\in\mathbb{N}}$ by $(f_{m,3})_{m\in\mathbb{N}}$. By the definition of the topology in $Z_3^{\mathbb{D}}$ and the fact that convergence in $\mathbb{D}([0,T],\mathbb{R})$ implies pointwise convergence, we obtain that $f_{m,3}$ is continuous. Suppose that $f_{m,3}(u_1) = f_{m,3}(u_2)$ for all $u_1, u_2 \in Z_3^{\mathbb{D}}$. From the right-continuity of the map $[0,T] \ni t \mapsto \operatorname{Re}\langle u_j(t), h_k\rangle$ and the density of $(t_l)_l$ as well as $(h_k)_k$, we infer $u_1(t) = u_2(t)$ for all $t \in \mathbb{N}$, i.e. $(f_{m,3})_{m\in\mathbb{N}}$ separates points in $Z_3^{\mathbb{D}}$.

Step 2. We define $F_j := \left\{f_{m,j}|_{Z_T^{\mathbb{D}}} : m \in \mathbb{N}\right\}$ for $j = 1,2,3$ and set $\mathcal{A} := \sigma(F)$, where $F := F_1 \cup F_2 \cup F_3$. We would like to prove (2.44). Above, we obtained $\sigma(f_{m,j} : m \in \mathbb{N}) = \mathcal{B}(Z_j)$ for $j = 1,2$. Since we have

$$\sigma\big(f_{m,j}|_{Z_1^{\mathbb{D}}\cap Z_2^{\mathbb{D}}} : m \in \mathbb{N}\big) = \sigma(f_{m,j} : m \in \mathbb{N})|_{Z_1^{\mathbb{D}}\cap Z_2^{\mathbb{D}}}$$

and

$$\mathcal{B}(Z_1^{\mathbb{D}} \cap Z_2^{\mathbb{D}}) = \sigma\Big(\bigcup_{j=1,2} \mathcal{B}(Z_j)|_{Z_1^{\mathbb{D}} \cap Z_2^{\mathbb{D}}} \Big),$$

we conclude

$$\mathcal{B}(Z_1^{\mathbb{D}} \cap Z_2^{\mathbb{D}}) = \sigma\Big(\bigcup_{j=1,2} \sigma\left(f_{m,j}|_{Z_1^{\mathbb{D}} \cap Z_2^{\mathbb{D}}} : m \in \mathbb{N} \right) \Big)$$

and thus,

$$\mathcal{B}(Z_1^{\mathbb{D}} \cap Z_2^{\mathbb{D}})|_{Z_T^{\mathbb{D}}} = \sigma\left(f_{m,1}|_{Z_T^{\mathbb{D}}}, f_{m,2}|_{Z_T^{\mathbb{D}}} : m \in \mathbb{N} \right) = \sigma(F_1 \cup F_2).$$

Similarly, we obtain $\mathcal{A} = \sigma\big(\mathcal{B}(Z_1^{\mathbb{D}} \cap Z_2^{\mathbb{D}})|_{Z_T^{\mathbb{D}}} \cup \sigma(F_3)\big)$.

□

Remark 2.47. By Lemma 2.45, we have $\sigma\left(f_{m,3} : m \in \mathbb{N}\right) = \sigma(\tilde{\mathcal{O}}_{Z_3^{\mathbb{D}}})$, where $\tilde{\mathcal{O}}_{Z_3^{\mathbb{D}}}$ is the coarsest topology such that $f_{m,3}$ is continuous for each $m \in \mathbb{N}$. In particular, we have

$$\sigma\left(f_{m,3} : m \in \mathbb{N}\right) \subsetneq \mathcal{B}(Z_3^{\mathbb{D}}),$$

since convergence in $\mathbb{D}([0,T],\mathbb{R})$ implies pointwise convergence, but not vice versa. In particular, we get $\mathcal{A} = \mathcal{B}(\tilde{Z}_T)$, where $\tilde{Z}_T$ is the topological space arising when we replace the topology on $Z_3^{\mathbb{D}}$ by $\tilde{\mathcal{O}}_{Z_3^{\mathbb{D}}}$.

Our study of compactness in $Z_T^{\mathbb{D}}$ will be based on the following classical result which can be viewed as a càdlàg-analogue to the Arzela-Ascoli Theorem. For a proof, we refer to [97], chapter 2.

Proposition 2.48. *Let $\mathbb{S}$ be a metric space. Then, a set $A \subset \mathbb{D}([0,T],\mathbb{S})$ has compact closure if and only if it satisfies the following conditions:*

i) There is a dense subset $J \subset [0,T]$ such that for every $t \in J$, the set $\{u(t) : u \in A\}$ has compact closure in $\mathbb{S}$.

ii) $\lim_{\delta \to 0} \sup_{u \in A} w_{\mathbb{S}}(u,\delta) = 0$.

Next, we repeat a criterion for convergence of a sequence in $\mathbb{D}([0,T],\mathbb{B}_{E_A}^r)$ from [99], Lemma 3.3.

Lemma 2.49. *Let $r > 0$ and take $u_n : [0,T] \to E_A$ for $n \in \mathbb{N}$ such that*

a) $\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \|u_n(t)\|_{E_A} \leq r$,

b) $u_n \to u$ in $\mathbb{D}([0,T],E_A^*)$ for $n \to \infty$.

Then $u_n, u \in \mathbb{D}([0,T],\mathbb{B}_{E_A}^r)$ for all $n \in \mathbb{N}$ and $u_n \to u$ in $\mathbb{D}([0,T],\mathbb{B}_{E_A}^r)$ for $n \to \infty$.

The previous results culminate in the following deterministic compactness criterion. Let us recall that M is a finite measure space and the embedding $E_A \hookrightarrow L^{\alpha+1}(M)$ is supposed to be compact.

Proposition 2.50. *Let K be a subset of $Z_T^{\mathbb{D}}$ and $r > 0$ such that*

a) $\sup_{z\in K} \sup_{t\in[0,T]} \|z(t)\|_{E_A} \leq r$;

b) $\lim_{\delta\to 0}\sup_{z\in K} w_{E_A^*}(z,\delta)=0$.

Then, K is relatively compact in $Z_T^{\mathbb{D}}$.

Proof. Let K be a subset of $Z_T^{\mathbb{D}}$ such that the assumptions $a)$ and $b)$ are fulfilled. From Lemma 2.28 b) and Lemma 2.46, we know that it is sufficient to check that K is sequentially relatively compact. Let $(z_n)_{n\in\mathbb{N}}\subset K$.

Step 1: The relative compactness of K in $\mathbb{D}([0,T],E_A^*)$ is an immediate consequence of Proposition 2.48 and the compactness of E_A in E_A^*. Hence, we can take a subsequence again denoted by $(z_n)_{n\in\mathbb{N}}$ and $z\in\mathbb{D}([0,T],E_A^*)$ with $z_n\to z$ in $\mathbb{D}([0,T],E_A^*)$. By Lemma 2.49, we get $z_n\to z$ in $\mathbb{D}_w\left([0,T],E_A\right)$ and $\sup_{t\in[0,T]}\|z(t)\|_{E_A}\le r$.

Step 2: We fix $\varepsilon>0$. As in the proof of Proposition 2.36, we get

$$\|v\|_{L^{\alpha+1}(M)}^{\alpha+1}\le\varepsilon_0\|v\|_{E_A}^{\alpha+1}+C_{\varepsilon_0}\|v\|_{E_A^*}^{\alpha+1}$$

for $\varepsilon_0=\frac{\varepsilon}{2T(2r)^{\alpha+1}}$ and all $v\in E_A$. Integration with respect to time yields

$$\|z_n-z\|_{L^{\alpha+1}(0,T;L^{\alpha+1}(M))}^{\alpha+1}\le\varepsilon_0\|z_n-z\|_{L^{\alpha+1}(0,T;E_A)}^{\alpha+1}+C_{\varepsilon_0}\|z_n-z\|_{L^{\alpha+1}(0,T;E_A^*)}^{\alpha+1};$$

$$\varepsilon_0\|z_n-z\|_{L^{\alpha+1}(0,T;E_A)}^{\alpha+1}\le\varepsilon_0 T\|z_n-z\|_{L^{\infty}(0,T;E_A)}^{\alpha+1}\le\varepsilon_0 T\left(2r\right)^{\alpha+1}\le\frac{\varepsilon}{2}.$$

By [17], page 124, equation (12.14), convergence in $\mathbb{D}([0,T],E_A^*)$ implies $z_n(t)\to z(t)$ in E_A^* for almost all $t\in[0,T]$. By Assumption $a)$, Lebesgue's Theorem yields $z_n\to z$ in $L^{\alpha+1}(0,T;E_A^*)$. Hence,

$$\limsup_{n\to\infty}\|z_n-z\|_{L^{\alpha+1}(0,T;L^{\alpha+1}(M))}^{\alpha+1}\le\frac{\varepsilon}{2}$$

for all $\varepsilon>0$ and thus, the sequence $(z_n)_{n\in\mathbb{N}}$ is also convergent to u in $L^{\alpha+1}(0,T;L^{\alpha+1}(M))$. □

The following Lemma (see [99], Lemma A.7) gives us a useful consequence of the Aldous condition $[A]$ from Definition 2.37.

Lemma 2.51. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of adapted, càdlàg stochastic processes in a Banach space E which satisfies the Aldous condition $[A]$. Then, for every $\varepsilon>0$ there exists a measurable subset $A_\varepsilon\subset\mathbb{D}([0,T],E)$ such that*

$$\mathbb{P}^{X_n}(A_\varepsilon)\ge 1-\varepsilon,\qquad \lim_{\delta\to 0}\sup_{u\in A_\varepsilon} w_E(u,\delta)=0.$$

The deterministic compactness result in Proposition 2.50 and the last Lemma can be used to get the following criterion for tightness in $Z_T^{\mathbb{D}}$.

Proposition 2.52. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of adapted càdlàg E_A^*-valued processes satisfying the Aldous condition $[A]$ in E_A^* and*

$$\sup_{n\in\mathbb{N}}\mathbb{E}\left[\sup_{t\in[0,T]}\|X_n(t)\|_{E_A}^2\right]<\infty.$$

Then, the sequence $\left(\mathbb{P}^{X_n}\right)_{n\in\mathbb{N}}$ is tight in $Z_T^{\mathbb{D}}$.

Proof. The assertion follows from a similar reasoning as in Proposition 2.39, where the Strauss Lemma 2.33 can be substituted by Lemma 2.49. □

We continue with a short interlude in measure theory. Let S be a Polish space with metric ρ, i.e. a separable, complete metric space equipped with the Borel σ-algebra $\mathcal{B}(S)$. Let us denote the set of all finite measures on S by $\mathcal{M}_+(S)$ and define the Prokhorov-metric

$$\tilde{\pi}(\mu,\nu) := \inf\left\{\varepsilon > 0 : \quad \nu(A) \le \mu(A^\varepsilon) + \varepsilon \text{ and } \mu(A) \le \nu(A^\varepsilon) + \varepsilon \quad \forall A \in \mathcal{B}(S)\right\}$$

for $\mu, \nu \in \mathcal{M}_+(S)$, where $A^\varepsilon := \{x \in S \,|\, \exists a \in A : \rho(x,a) < \varepsilon\}$. From [17], p. 72 and 73, we infer that $\left(\mathcal{M}_+(S), \tilde{\pi}\right)$ is a complete separable metric space. We fix a σ-finite measure ϑ on S and a sequence $(S_n)_{n\in\mathbb{N}} \subset \mathcal{B}(S)$ such that $S_n \nearrow S$ and $\vartheta(S_n) < \infty$ for all $n \in \mathbb{N}$. Then, we denote the set of all $\bar{\mathbb{N}}$-valued Borel measures ξ on S with $\xi(S_n) < \infty$ for all $n \in \mathbb{N}$ by $M_{\bar{\mathbb{N}}}^{\vartheta}(S)$.

Lemma 2.53. *Together with the metric*

$$\pi(\xi_1,\xi_2) := \sum_{n=1}^{\infty} 2^{-n} \min\left\{1, \tilde{\pi}(\xi_1(\cdot \cap S_n), \xi_2(\cdot \cap S_n))\right\}, \qquad \xi_1, \xi_2 \in M_{\bar{\mathbb{N}}}^{\vartheta}(S),$$

$M_{\bar{\mathbb{N}}}^{\vartheta}(S)$ *is a complete separable metric space.*

Proof. We denote the set of all $\mathbb{N}\cup\{0\}$-valued measures on S by $M_{\mathbb{N}\cup\{0\}}(S)$. This set is separable and complete since it is closed in $(\mathcal{M}_+(S), \tilde{\pi})$. Then, we equip $\mathcal{M} := \left(M_{\mathbb{N}\cup\{0\}}(S)\right)^{\mathbb{N}}$ with the metric

$$\mathrm{d}((\mu_n)_{n\in\mathbb{N}}, (\nu_n)_{n\in\mathbb{N}}) := \sum_{n=1}^{\infty} 2^{-n} \min\left\{1, \tilde{\pi}(\mu_n, \nu_n)\right\}.$$

We show that $(\mathcal{M}, d)$ is separable and complete. Let us take a Cauchy sequence $\left(\mu_n^k\right)_{n\in\mathbb{N},k\in\mathbb{N}} \subset \mathcal{M}^{\mathbb{N}}$. For each $n \in \mathbb{N}$, we get $\tilde{\pi}(\mu_n^k, \mu_n^l) \to 0$ for $k, l \to \infty$. The completeness of $M_{\mathbb{N}\cup\{0\}}(S)$ in the Prokhorov metric yields $(\mu_n)_{n\in\mathbb{N}} \in \mathcal{M}$ with $\tilde{\pi}(\mu_n^k, \mu_n) \to 0$ for $k \to \infty$ and each $n \in \mathbb{N}$. Now, we deduce $\mathrm{d}(\left(\mu_n^k\right)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}}) \to 0$ from Lebesgue's Theorem and thus, $\mathcal{M}$ is complete. The separability of $\mathcal{M}$ is a consequence of the fact that $M_{\mathbb{N}\cup\{0\}}(S)$ is separable. The map

$$I : M_{\bar{\mathbb{N}}}^{\vartheta}(S) \to \mathcal{M}, \qquad I(\mu) := (\mu(\cdot \cap S_n))_{n\in\mathbb{N}},$$

defines a homeomorphism onto a closed subset of $\mathcal{M}$ and thus, $M_{\bar{\mathbb{N}}}^{\vartheta}(S)$ is also complete and separable. □

Let η be a time-homogeneous Poisson random measure on $\mathbb{R}^N$ with intensity measure ν. Finally, we would like to apply Corollary 2.27 with

$$\mathcal{X}_1 := M_{\bar{\mathbb{N}}}^{\nu}([0,T] \times \mathbb{R}^N), \qquad \mathcal{X}_2 := Z_T^{\mathbb{D}},$$

and a sequence $(X_n)_{n\in\mathbb{N}}$ of processes as in Proposition 2.52. For convenience, we use the abbreviation $M_{\bar{\mathbb{N}}}^{\nu}([0,T] \times \mathbb{R}^N) := M_{\bar{\mathbb{N}}}^{\mathrm{Leb}\otimes\nu}([0,T] \times \mathbb{R}^N)$. Combining Corollary 2.27, Lemma 2.46 and Proposition 2.52, we get the following Corollary as the main result of this section.

Corollary 2.54. *Let η be a random variable in $M_{\bar{\mathbb{N}}}^{\nu}([0,T]\times\mathbb{R}^N)$ and $(X_n)_{n\in\mathbb{N}}$ be a sequence of adapted càdlàg E_A^*-valued processes satisfying the Aldous condition $[A]$ in E_A^* and*

$$\sup_{n\in\mathbb{N}}\mathbb{E}\left[\sup_{t\in[0,T]}\|X_n(t)\|_{E_A}^2\right]<\infty.$$

We equip $Z_T^{\mathbb{D}}$ with the σ-algebra $\mathcal{A}$ from (2.44). *Then, there are a probability space $(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathbb{P}})$, a subsequence $(X_{n_k})_{k\in\mathbb{N}}$ and random variables $v, v_k:\bar{\Omega}\to Z_T$ and $\bar{\eta}_k,\bar{\eta}:\bar{\Omega}\to M_{\bar{\mathbb{N}}}^{\nu}([0,T]\times\mathbb{R}^N)$ with*

i) $\bar{\mathbb{P}}^{(\bar{\eta}_k,v_k)}=\mathbb{P}^{(\eta,X_{n_k})}$ *for* $k\in\mathbb{N}$,

ii) $(\bar{\eta}_k,v_k)\to(\bar{\eta},v)$ *in* $M_{\bar{\mathbb{N}}}^{\nu}([0,T]\times\mathbb{R}^N)\times Z_T$ *almost surely for* $k\to\infty$,

iii) $\bar{\eta}_k=\bar{\eta}$ *almost surely.*

3. The fixed point method for the stochastic NLS on the full space

In this chapter, we study existence and uniqueness for the following stochastic NLS

$$\begin{cases} \mathrm{d}u = \Big(\mathrm{i}\Delta u - \mathrm{i}\lambda|u|^{\alpha-1}u - \frac{1}{2}\sum_{m=1}^{\infty}|e_m|^2|u|^{2(\gamma-1)}u\Big)\mathrm{d}t - \mathrm{i}\sum_{m=1}^{\infty} e_m|u|^{\gamma-1}u\,\mathrm{d}\beta_m, \\ u(0) = u_0, \end{cases} \tag{3.1}$$

with $\lambda \in \{-1,1\}$, $\alpha > 1$, $\gamma \geq 1$, $(e_m)_{m\in\mathbb{N}} \subset L^\infty(\mathbb{R}^d,\mathbb{C})$ and independent Brownian motions $(\beta_m)_{m\in\mathbb{N}}$. We consider initial values in $L^2(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d)$. As a particular feature of this equation, we observe the power-type noise which is somehow similar to the deterministic nonlinearity.

For the standard NLS, i.e. $e_m = 0$ for all $m \in \mathbb{N}$, global wellposedness in $L^2(\mathbb{R}^d)$ for $\alpha \in (1, 1+\frac{4}{d})$ and local wellposedness in the critical case $\alpha = \alpha_c := 1+\frac{4}{d}$ are classical results which can be found e.g. in the monographs [36],[88],[114]. In $H^1(\mathbb{R}^d)$, similar results hold for $\alpha \in (1, 1+\frac{4}{(d-2)_+})$ and $\alpha = 1+\frac{4}{(d-2)_+}$. Comparing the degree of the deterministic terms in (3.1), we observe that the value $2(\gamma-1)$ plays the same role as $\alpha-1$. We start with a remark to make this more precise by a formal calculation which transfers the invariance of the deterministic NLS under the scaling

$$v(t,x) \mapsto v_\theta(t,x) := \theta^{\frac{2}{\alpha-1}} v(\theta^2 t, \theta x), \qquad v_0(x) \mapsto v_{\theta,0}(x) := \theta^{\frac{2}{\alpha-1}} v_0(\theta^{-1}x), \tag{3.2}$$

to the stochastic setting.

Remark 3.1. We assume that $e_m(\theta x) = e_m(x)$ for all $\theta > 0$ and $x \in \mathbb{R}^d$. Let u be a solution to (3.1), i.e.

$$u(t) = u_0 + \int_0^t \Big(\mathrm{i}\Delta u - \mathrm{i}\lambda|u|^{\alpha-1}u - \frac{1}{2}\sum_{m=1}^{\infty}|e_m|^2|u|^{2(\gamma-1)}u\Big)\mathrm{d}s - \mathrm{i}\sum_{m=1}^{\infty}\int_0^t e_m|u|^{\gamma-1}u\,\mathrm{d}\beta_m$$

almost surely for all $t \geq 0$. For all $\theta > 0$, the sequence $(\tilde{\beta}_m^\theta)_{m\in\mathbb{N}}$ defined by $\tilde{\beta}_m^\theta(t) = \theta^{-1}\beta(\theta^2 t)$ for $t \geq 0$ consists of independent Brownian motions. The stochastic integral satisfies the inequality

$$\theta\sum_{m=1}^{\infty}\int_0^t B_m(\theta^2 s)\mathrm{d}\tilde{\beta}_m^\theta(s) = \sum_{m=1}^{\infty}\int_0^{\theta^2 t} B_m(s)\mathrm{d}\beta_m(s) \tag{3.3}$$

almost surely for all $t \geq 0$, which can be easily checked for simple processes and transfered to general integrands by approximation. From (3.3) and a change of variables in the Bochner integral, we infer

$$u(\theta^2 t) = u_0 + \theta^2\int_0^t \Big(\mathrm{i}\Delta u(\theta^2 s) - \mathrm{i}\lambda|u(\theta^2 s)|^{\alpha-1}u(\theta^2 s) - \frac{1}{2}\sum_{m=1}^{\infty}|e_m|^2|u(\theta^2 s)|^{2(\gamma-1)}u(\theta^2 s)\Big)\mathrm{d}s$$

$$- \mathrm{i}\theta \sum_{m=1}^{\infty} \int_0^t e_m |u(\theta^2 s)|^{\gamma-1} u(\theta^2 s) \, \mathrm{d}\tilde{\beta}_m^{\theta}.$$

almost surely for all $t \geq 0$. Now, we set $\tilde{u}_\theta(x,t) := u(\theta x, \theta^2 t)$ for $x \in \mathbb{R}^d$ and $t \geq 0$ and obtain

$$\tilde{u}_\theta(t) = u_0(\theta\cdot) + \int_0^t \Big(\mathrm{i}\Delta \tilde{u}_\theta - \mathrm{i}\lambda\theta^2 |\tilde{u}_\theta|^{\alpha-1}\tilde{u}_\theta - \frac{1}{2}\sum_{m=1}^{\infty} |e_m(\theta\cdot)|^2 \theta^2 |\tilde{u}_\theta|^{2(\gamma-1)}\tilde{u}_\theta \Big) \mathrm{d}s$$
$$- \mathrm{i} \sum_{m=1}^{\infty} \int_0^t e_m(\theta\cdot)\theta |\tilde{u}_\theta|^{\gamma-1}\tilde{u}_\theta \, \mathrm{d}\tilde{\beta}_m^{\theta}$$

almost surely for all $t \geq 0$. We abbreviate $u_\theta := \theta^{\frac{2}{\alpha-1}} \tilde{u}_\theta$ and multiply the previous equation with $\theta^{\frac{2}{\alpha-1}}$ to deduce

$$u_\theta(t) = \theta^{\frac{2}{\alpha-1}} u_0(\theta\cdot) + \int_0^t \Big(\mathrm{i}\Delta u_\theta - \mathrm{i}\lambda |u_\theta|^{\alpha-1} u_\theta - \frac{1}{2}\sum_{m=1}^{\infty} |e_m|^2 \theta^{2-\frac{4(\gamma-1)}{\alpha-1}} |u_\theta|^{2(\gamma-1)} u_\theta \Big) \mathrm{d}s$$
$$- \mathrm{i} \sum_{m=1}^{\infty} \int_0^t e_m \theta^{1-\frac{2(\gamma-1)}{\alpha-1}} |\tilde{u}_\theta|^{\gamma-1}\tilde{u}_\theta \, \mathrm{d}\tilde{\beta}_m^{\theta}$$

almost surely for all $t \geq 0$. Thus, the system $\big(\Omega, \mathcal{F}, \mathbb{P}, (\tilde{\beta}_m^{\theta})_{m\in\mathbb{N}}, (\mathcal{F}_{\theta^2 t})_{t\geq 0}, u_\theta\big)$ is a martingale solution of (3.1) with initial value $\theta^{\frac{2}{\alpha-1}} u_0(\theta\cdot)$ for $1 = \frac{2(\gamma-1)}{\alpha-1}$. The latter condition is equivalent to $\gamma = \frac{\alpha+1}{2}$.

In view of the previous calculation and the fact that the L^2-norm is invariant under the scaling (3.2) if and only if $\alpha = 1 + \frac{4}{d}$ it is natural to call the stochastic NLS (3.1) *mass-critical* if the exponents are given by $\alpha = 1 + \frac{4}{d}$ and $\gamma = 1 + \frac{2}{d}$. The stochastic NLS with $\alpha = 1 + \frac{4}{(d-2)_+}$ and $\gamma = 1 + \frac{2}{(d-2)_+}$ is called *energy-critical* since in this case, the energy is scaling invariant.

In this chapter, we prove local existence and uniqueness in $L^2(\mathbb{R}^d)$ for all subcritical and critical exponents $\alpha \in (1, 1+\frac{4}{d}]$ and $\gamma \in [1, 1+\frac{2}{d}]$ under modest assumptions on the coefficients e_m, $m \in \mathbb{N}$. This reflects a significant improvement of the previous results since (3.1) has only been treated for $\gamma = 1$ so far. We refer to the introduction of this thesis for a more detailed overview of the literature on this problem. Moreover, we prove a global result in $L^2(\mathbb{R}^d)$ for all subcritical α under a substantial restriction of the admissible exponents γ. In $H^1(\mathbb{R}^d)$, we prove local existence and uniqueness, but we are not able to cover all exponents. Let us recall our results which have already been stated in the introduction, see Theorem 1.

Theorem 3.2. *Let $u_0 \in L^2(\mathbb{R}^d)$, $\lambda \in \{-1, 1\}$, $(\beta_m)_{m\in\mathbb{N}}$ be a sequence of independent Brownian motions and $(e_m)_{m\in\mathbb{N}} \subset L^\infty(\mathbb{R}^d)$ with $\sum_{m=1}^{\infty} \|e_m\|_{L^\infty}^2 < \infty$. Then, the following assertions hold:*

a) Let $\alpha \in (1, 1+\frac{4}{d}]$ and $\gamma \in [1, 1+\frac{2}{d}]$. Then, there is a unique local solution of (3.1) *in $L^2(\mathbb{R}^d)$. Both stochastically and analytically, the solution is understood in the strong sense from Definition 2.1.*

b) Let $\alpha \in (1, 1+\frac{4}{d})$ and $\gamma = 1$. Then, the solution from a) is global.

c) Let e_m be real valued for each $m \in \mathbb{N}$, $\alpha \in (1, 1+\frac{4}{d})$ and

$$1 < \gamma < \frac{\alpha-1}{\alpha+1} \frac{4+d(1-\alpha)}{4\alpha+d(1-\alpha)} + 1.$$

Then, the solution from a) is global.

Theorem 3.3. *Let $u_0 \in H^1(\mathbb{R}^d)$, $\lambda \in \{-1,1\}$, $(\beta_m)_{m\in\mathbb{N}}$ be a sequence of independent Brownian motions and suppose that we have $\sum_{m=1}^\infty \left(\|e_m\|_{L^\infty} + \|\nabla e_m\|_F\right)^2 < \infty$, where*

$$F := \begin{cases} L^d(\mathbb{R}^d), & d \geq 3, \\ L^{2+\varepsilon}(\mathbb{R}^d), & d = 2, \\ L^2(\mathbb{R}^d), & d = 1, \end{cases}$$

for some $\varepsilon > 0$. Let $\alpha \in (1, 1+\frac{4}{d}] \cup (2, 1+\frac{4}{(d-2)_+}]$ and $\gamma \in [1, 1+\frac{2}{d}] \cup (2, 1+\frac{2}{(d-2)_+}]$. Then, there is a unique local solution of (3.1) *in $H^1(\mathbb{R}^d)$. Both stochastically and analytically, the solution is understood in the strong sense from Definition 2.1.*

The chapter is structured as follows. In the first section, we prove part a) of Theorem 3.2 and the second one is devoted to b) and c). In the third section, we prove Theorem 3.3.

3.1. Local existence and uniqueness in $L^2(\mathbb{R}^d)$

In this section, we prove Theorem 3.2 a). On a technical level, the case $\gamma = 1$ is significantly simpler. However, we would like to treat $\gamma = 1$ and $\gamma \neq 1$ at once to keep the presentation at a reasonable length. Moreover, it is possible to substitute the sequence $(\beta_m)_{m\in\mathbb{N}}$ of independent Brownian motions by a cylindrical Wiener process W on a real valued Hilbert space Y with ONB $(f_m)_{m\in\mathbb{N}}$. We refer to Example A.3 for this correspondence. To incorporate these two aspects, we solve the slightly more general problem

$$\begin{cases} \mathrm{d}u = \left[\mathrm{i}\Delta u - \mathrm{i}\lambda|u|^{\alpha-1}u + \mu_1\left(|u|^{2(\gamma-1)}u\right) + \mu_2(u)\right]\mathrm{d}t - \mathrm{i}\left[B_1\left(|u|^{\gamma-1}u\right) + B_2 u\right]\mathrm{d}W, \\ u(0) = u_0 \in L^2(\mathbb{R}^d), \end{cases} \tag{3.4}$$

for $\gamma > 1$, where $B_1, B_2 : L^2(\mathbb{R}^d) \to \mathrm{HS}(Y, L^2(\mathbb{R}^d))$ are linear bounded operators defined by

$$B_1(u)f_m := e_m u, \qquad B_2(u)f_m := B_m u, \qquad u \in L^2(\mathbb{R}^d), \quad m \in \mathbb{N}, \tag{3.5}$$

with

$$\sum_{m=1}^\infty \|e_m\|_{L^\infty}^2 < \infty, \qquad \sum_{m=1}^\infty \|B_m\|_{\mathcal{L}(L^2)}^2 < \infty.$$

Moreover, we denote

$$\mu_1 := -\frac{1}{2}\sum_{m=1}^\infty |e_m|^2, \qquad \mu_2 := -\frac{1}{2}\sum_{m=1}^\infty B_m^* B_m. \tag{3.6}$$

Before we proceed with the proof, we briefly sketch our strategy. First, we truncate the nonlinearities and look for a mild solution of

$$\begin{cases} \mathrm{d}u_n = \left(\mathrm{i}\Delta u_n - \mathrm{i}\lambda\varphi_n(u_n,\cdot)|u_n|^{\alpha-1}u_n + [\varphi_n(u_n,\cdot)]^2\mu_1(|u_n|^{2(\gamma-1)}u_n) + \mu_2(u_n)\right)\mathrm{d}t \\ \qquad\quad - \mathrm{i}\left(\varphi_n(u_n,\cdot)B_1\left(|u_n|^{\gamma-1}u_n\right) + B_2 u_n\right)\mathrm{d}W, \\ u(0) = u_0, \end{cases} \tag{3.7}$$

for fixed $n \in \mathbb{N}$. The truncation is given by $\varphi_n(u_n, t) = \theta_n(Z_t(u_n))$ for a process

$$Z_t(u_n) := \|u_n\|_{L^q(0,t;L^{\alpha+1})} + \|u_n\|_{L^{\tilde{q}}(0,t;L^{2\gamma})} \tag{3.8}$$

and

$$\theta_n(x) := \begin{cases} 1, & x \in [0,n], \\ 2 - \dfrac{x}{n}, & x \in [n, 2n], \\ 0, & x \in [2n, \infty). \end{cases} \tag{3.9}$$

The functions θ_n and θ_n^2 obey the Lipschitz conditions

$$|\theta_n(x) - \theta_n(y)| \le \frac{1}{n}|x-y|, \qquad \left|[\theta_n(x)]^2 - [\theta_n(y)]^2\right| \le \frac{2}{n}|x-y| \quad x, y \ge 0. \tag{3.10}$$

In Figure 3.1, we sketch θ_n and θ_n^2. It is beneficial to use the squared cut-off function in the correction term since (3.7) has still Stratonovich structure. This ensures that the L^2-norm of the solution u_n is conserved as long as e_m is real-valued for each $m \in \mathbb{N}$.

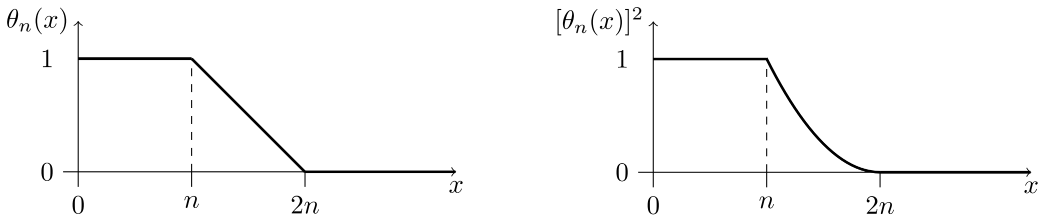

Figure 3.1.: Cut-off functions θ_n and θ_n^2.

In (3.8), $q, \tilde{q} \in (2, \infty)$ are chosen according to

$$\frac{2}{q} + \frac{d}{\alpha+1} = \frac{d}{2}, \qquad \frac{2}{\tilde{q}} + \frac{d}{2\gamma} = \frac{d}{2}. \tag{3.11}$$

Hence, $(\alpha+1, q)$ and $(2\gamma, \tilde{q})$ are Strichartz pairs. In order to construct a solution of (3.7), we use a fixed point argument in the natural space $L^q(\Omega, E_{[0,T]})$, where

$$E_{[a,b]} := Y_{[a,b]} \cap C([a,b], L^2(\mathbb{R}^d)), \qquad Y_{[a,b]} := \begin{cases} L^q(a,b; L^{\alpha+1}(\mathbb{R}^d)), & \alpha+1 \ge 2\gamma, \\ L^{\tilde{q}}(a,b; L^{2\gamma}(\mathbb{R}^d)), & \alpha+1 < 2\gamma, \end{cases} \tag{3.12}$$

for $0 \le a \le b \le T$. The argument is based on the Strichartz estimates from Proposition 2.14 and Corollary 2.22 and the truncation replaces the restriction to balls in E_T used in the deterministic setting. Since the solution of (3.7) also solves the untruncated problem up to the stopping time

$$\tau_n := \inf\{t \ge 0 : Z_t(u_n) \ge n\} \wedge T, \tag{3.13}$$

this yields a local solution u to (3.4) up to time $\tau_\infty := \sup_{n\in\mathbb{N}} \tau_n$. The uniqueness of the solution to (3.4) can be reduced to the uniqueness of (3.7). In the critical setting $\alpha = 1 + \frac{4}{d}$ or $\gamma = 1 + \frac{2}{d}$, a similar argument yields a local solution. Note that in this case, we use the truncation φ_ν for a small $\nu \in (0,1)$ instead of φ_n for a large $n \in \mathbb{N}$.

We remark that in the critical case, the Strichartz exponents for time and space coincide and we get $Y_{[a,b]} = L^{2+\frac{4}{d}}(a,b;L^{2+\frac{4}{d}}(\mathbb{R}^d))$. A further relationship between the spaces from above is clarified by the following interpolation Lemma.

Lemma 3.4. *We have*

$$E_{[a,b]} \hookrightarrow L^q(a,b;L^{\alpha+1}(\mathbb{R}^d)) \cap L^{\tilde{q}}(a,b;L^{2\gamma}(\mathbb{R}^d)).$$

Proof. We treat $\alpha+1 \geq 2\gamma$. The other case can be proved analogously. From Lemma 2.13, we infer

$$\|u\|_{L^{\tilde{q}}(a,b;L^{2\gamma})} \leq \|u\|_{L^\infty(a,b;L^2)}^{1-\theta} \|u\|_{L^q(a,b;L^{\alpha+1})}^{\theta} \leq \|u\|_{E_{[a,b]}}$$

for $u \in E_{[a,b]}$ and thus, we have

$$\|u\|_{L^q(a,b;L^{\alpha+1})} + \|u\|_{L^{\tilde{q}}(a,b;L^{2\gamma})} \leq 2\|u\|_{E_{[a,b]}}, \qquad u \in E_{[a,b]}.$$

□

Furthermore, we abbreviate $Y_r := Y_{[0,r]}$ and $E_r := E_{[0,r]}$ for $r>0$. Let τ be an $\mathbb{F}$-stopping time and $p \in (1,\infty)$. Then, we denote by $\mathbb{M}_{\mathbb{F}}^p(\Omega, E_{[0,\tau]})$ the space of processes $u : [0,T]\times\Omega \to L^2(\mathbb{R}^d)\cap L^{2\gamma}(\mathbb{R}^d)$ with continuous paths in $L^2(\mathbb{R}^d)$ which are $\mathbb{F}$-adapted in $L^2(\mathbb{R}^d)$ and $\mathbb{F}$-predictable in $L^{2\gamma}(\mathbb{R}^d)$ such that

$$\|u\|_{\mathbb{M}_{\mathbb{F}}^p(\Omega,E_{[0,\tau]})}^p := \mathbb{E}\left[\sup_{t\in[0,\tau]} \|u(t)\|_{L^2}^p + \|u\|_{Y_\tau}^p\right] < \infty.$$

Often, we abbreviate $u \in \mathbb{M}_{\mathbb{F}}^p(\Omega, E_\tau) := \mathbb{M}_{\mathbb{F}}^p(\Omega, E_{[0,\tau]})$. Moreover, we say $u \in \mathbb{M}_{\mathbb{F}}^p(\Omega, E_{[0,\tau)})$ if there is a sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times with $\tau_n \nearrow \tau$ almost surely as $n\to\infty$ such that $u \in \mathbb{M}_{\mathbb{F}}^p(\Omega, E_{[0,\tau_n]})$ for all $n \in \mathbb{N}$.

The first following Lemma contains the differentiability properties of the power type nonlinearities in L^p-spaces which will be needed frequently in this chapter. Since these properties are also important throughout the thesis and do not depend on the underlying space $\mathbb{R}^d$, we formulate the results in a more general setting.

Lemma 3.5. *Let $(S,\mathcal{A},\mu)$ be a measure space and $\alpha>1$.*

a) Let $p>1$. The map $G_1 : L^p(S) \to \mathbb{R}$ defined by $G_1(u) := \|u\|_{L^p(S)}^p$ is continuously Fréchet differentiable with

$$G_1'[u]h = \operatorname{Re}\int_S |u|^{\alpha-1} u\overline{h}\,\mathrm{d}\mu$$

for all $u,h \in L^p(S)$.

b) Let $p>\alpha$ and $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be continuously differentiable. Assume that there is $C>0$ with

$$|\Phi(z)| \leq C|z|^\alpha, \qquad |\Phi'(z)| \leq C|z|^{\alpha-1}, \qquad z \in \mathbb{C}.$$

Then, the map

$$G : L^p(S) \to L^{\frac{p}{\alpha}}(S), \qquad G(u) := \Phi_1(u) + \mathrm{i}\Phi_2(u)$$

is continuously Fréchet differentiable with

$$G'[u]h = [\Phi_1'(u) + \mathrm{i}\Phi_2'(u)]\, h, \qquad u, h \in L^p(S).$$

In particular, we have $\|G'[u]\|_{L^p \to L^{\frac{p}{\alpha}}} \le C\|u\|_{L^p}^{\alpha-1}$ *for* $u \in L^p(S)$ *and*

$$\|G(u) - G(v)\|_{L^{\frac{p}{\alpha}}} \lesssim \left(\|u\|_{L^p} + \|v\|_{L^p}\right)^{\alpha-1} \|u - v\|_{L^p}, \qquad u, v \in L^p(S). \tag{3.14}$$

Proof. This lemma is well known, see for example the lecture notes [66], Lemma 9.2. □

The next Lemma is the justification of solving (3.4) via a fixed point argument. We use the following abbreviations for $r > 0$ and $t \in [0, T]$:

$$K^n_{det}u(t) := -\,\mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-s)\Delta} \left[\varphi_n(u,s)|u(s)|^{\alpha-1}u(s)\right] \mathrm{d}s, \tag{3.15}$$

$$K^n_{Strat}u(t) := \int_0^t e^{\mathrm{i}(t-s)\Delta} \left[\mu_1 \left([\varphi_n(u_n,t)]^2\, |u(s)|^{2(\gamma-1)}u(s)\right) + \mu_2\left(u(s)\right)\right] \mathrm{d}s, \tag{3.16}$$

$$K^n_{stoch}u(t) := -\,\mathrm{i} \int_0^t e^{\mathrm{i}(t-s)\Delta} \left[B_1\left(\varphi_n(u,s)|u(s)|^{\gamma-1}u(s)\right) + B_2 u(s)\right] \mathrm{d}W(s). \tag{3.17}$$

Lemma 3.6. *Let* $\alpha \in (1, 1+\frac{4}{d}]$, $\gamma \in (1, 1+\frac{2}{d}]$ *and* $p \in (1,\infty)$. *Then,* $u^n \in \mathbb{M}^p_{\mathbb{F}}(\Omega, E_T)$ *is a global strong solution of* (3.7) *in* $L^2(\mathbb{R}^d)$, *if and only if*

$$u^n = U(\cdot)u_0 + K^n_{det}u^n + K^n_{Strat}u^n + K^n_{stoch}u^n \tag{3.18}$$

holds almost for all $t \in [0, T]$.

Proof. For $s \in [0, T]$, we set $\delta := 1 + \frac{d}{4}(1-\alpha) > 0$ and $\tilde{\delta} = 1 + \frac{d}{2}(1-\gamma)$ as well as

$$F(s) := -\,\mathrm{i}\left[\varphi_n(u^n,s)|u^n(s)|^{\alpha-1}u^n(s)\right] + \left[\mu_1\left([\varphi_n(u_n,t)]^2\,|u^n(s)|^{2(\gamma-1)}u^n(s)\right) + \mu_2\left(u^n(s)\right)\right],$$
$$B(s) := -\,\mathrm{i}\left[B_1\left(\varphi_n(u^n,s)|u^n(s)|^{\gamma-1}u^n(s)\right) + B_2 u^n(s)\right]$$

Based on (3.5), (3.6) and the Hölder inequality, we estimate

$$\|\varphi_n(u)|u|^{\alpha-1}u\|_{L^{q'}(0,T;L^{\frac{\alpha+1}{\alpha}})} \le \|u\|^{\alpha}_{L^q(0,\tau;L^{\alpha+1})} T^{\delta},$$

$$\left\|\mu_1\left([\varphi_n(u)]^2|u|^{2(\gamma-1)}u\right)\right\|_{L^{\tilde{q}'}(0,T;L^{\frac{2\gamma}{2\gamma-1}})} \lesssim \|u\|^{2\gamma-1}_{L^{\tilde{q}}(0,\tau;L^{2\gamma})} T^{\tilde{\delta}},$$

$$\|\mu_2(u)\|_{L^1(0,T;L^2)} \lesssim T\|u\|_{L^\infty(0,T;L^2)},$$

$$\begin{aligned}
&\|B_1\left(\varphi_n(u)|u|^{\gamma-1}u\right) + B_2(u)\|_{L^2(0,T;\mathrm{HS}(Y,L^2))} \\
&\quad\le \left(\sum_{m=1}^{\infty} \|e_m\|^2_{L^\infty(\mathbb{R}^d)}\right)^{\frac{1}{2}} \|\varphi_n(u)|u|^{\gamma-1}u\|_{L^2(0,T;L^2)} + \left(\sum_{m=1}^{\infty} \|B_m\|^2_{\mathcal{L}(L^2)}\right)^{\frac{1}{2}} \|u\|_{L^2(0,T;L^2)} \\
&\quad\lesssim \|u\|_{L^{2\gamma}(0,T;L^{2\gamma}))} + r^{\frac{1}{2}}\|u\|_{L^\infty(0,T;L^2)}.
\end{aligned}$$

Hence, we obtain $F \in L^1(0,T;X)$ and $B \in L^2(0,T;\mathrm{HS}(Y,X))$ almost surely. Hence, we can apply Lemma 2.5 with $X = H^{-2}(\mathbb{R}^d)$ and $Af = -\Delta f$ for $f \in L^2(\mathbb{R}^d) := D(A)$. □

In the following Proposition, we state existence and uniqueness for (3.7) in the subcritical case.

Proposition 3.7. *Let $\alpha \in (1, 1+\frac{4}{d})$, $\gamma \in (1, 1+\frac{2}{d})$ and $p \in (1,\infty)$. Then, there is a unique global strong solution (u^n, T) of* (3.7) *in* $L^2(\mathbb{R}^d)$.

Proof. We fix $n \in \mathbb{N}$ and construct the solution from the assertion inductively.
Step 1: We look for a fixed point of the operator given by

$$K^n u := e^{\mathrm{i}\cdot\Delta} u_0 + K^n_{det} u + K^n_{Strat} u + K^n_{stoch} u, \qquad u \in \mathbb{M}^p_{\mathbb{F}}(\Omega, E_r),$$

where $r > 0$ will be chosen small enough. Let $u \in \mathbb{M}^p_{\mathbb{F}}(\Omega, E_r)$. A pathwise application of Proposition 2.14 and integration over Ω yields

$$\|e^{\mathrm{i}\cdot\Delta} u_0\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,E_r)} \lesssim \|u_0\|_{L^2}.$$

We define a stopping time by

$$\tau := \inf\left\{t \geq 0 : \|u\|_{L^q(0,t;L^{\alpha+1})} + \|u\|_{L^{\tilde{q}}(0,t;L^{2\gamma})} \geq 2n\right\} \wedge r$$

and set

$$\delta := 1 + \frac{d}{4}(1-\alpha) \in (0,1), \qquad \tilde{\delta} = 1 + \frac{d}{2}(1-\gamma) \in (0,1).$$

We estimate

$$\begin{aligned}\|K^n_{det} u\|_{E_r} \lesssim & \|\varphi_n(u)|u|^{\alpha-1}u\|_{L^{q'}(0,r;L^{\frac{\alpha+1}{\alpha}})} \leq \||u|^{\alpha-1}u\|_{L^{q'}(0,\tau;L^{\frac{\alpha+1}{\alpha}})} \\ \leq & \|u\|^{\alpha}_{L^q(0,\tau;L^{\alpha+1})} \tau^{\delta} \leq (2n)^{\alpha} r^{\delta}\end{aligned}$$

using Proposition 2.14 b) and d) and the Hölder inequality with exponents $\frac{q-1}{\alpha}$ and $\frac{1}{\delta}$. In the same spirit, we get

$$\begin{aligned}\|K^n_{Strat} u\|_{E_r} \lesssim & \left\|\mu_1\left([\varphi_n(u)]^2 |u|^{2(\gamma-1)} u\right)\right\|_{L^{\tilde{q}'}(0,r;L^{\frac{2\gamma}{2\gamma-1}})} + \|\mu_2(u)\|_{L^1(0,r;L^2)} \\ \leq & \frac{1}{2}\sum_{m=1}^{\infty} \|e_m\|^2_{L^\infty(\mathbb{R}^d)} \||u|^{2(\gamma-1)}u\|_{L^{\tilde{q}'}(0,\tau;L^{\frac{2\gamma}{2\gamma-1}})} + \frac{1}{2}\sum_{m=1}^{\infty} \|B_m\|^2_{\mathcal{L}(L^2)} r \|u\|_{L^\infty(0,r;L^2)} \\ \lesssim & \|u\|^{2\gamma-1}_{L^{\tilde{q}}(0,\tau;L^{2\gamma})} \tau^{\tilde{\delta}} + r\|u\|_{L^\infty(0,r;L^2)} \leq (2n)^{2\gamma-1} r^{\tilde{\delta}} + r\|u\|_{L^\infty(0,r;L^2)}.\end{aligned}$$

Integrating over Ω yields

$$\|K^n_{det} u\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,E_r)} \lesssim (2n)^{\alpha} r^{\delta}, \qquad \|K^n_{Strat} u\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,E_r)} \lesssim (2n)^{2\gamma-1} r^{\tilde{\delta}} + r\|u\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,E_r)}.$$

By Corollary 2.23, we obtain

$$\begin{aligned}\|K^n_{stoch} u\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,E_r)} \lesssim & \|B_1\left(\varphi_n(u)|u|^{\gamma-1}u\right) + B_2(u)\|_{L^p(\Omega,L^2(0,r;\mathrm{HS}(Y,L^2)))} \\ \leq & \left(\sum_{m=1}^{\infty} \|e_m\|^2_{L^\infty(\mathbb{R}^d)}\right)^{\frac{1}{2}} \|\varphi_n(u)|u|^{\gamma-1}u\|_{L^p(\Omega,L^2(0,r;L^2))} \\ & + \left(\sum_{m=1}^{\infty} \|B_m\|^2_{\mathcal{L}(L^2)}\right)^{\frac{1}{2}} \|u\|_{L^p(\Omega,L^2(0,r;L^2))}\end{aligned}$$

$$\lesssim \|\varphi_n(u)|u|^{\gamma-1}u\|_{L^p(\Omega,L^2(0,r;L^2))} + r^{\frac{1}{2}}\|u\|_{L^p(\Omega,L^\infty(0,r;L^2))}$$

From the pathwise inequality

$$\|\varphi_n(u)|u|^{\gamma-1}u\|_{L^2(0,r;L^2)} \leq \|u\|^\gamma_{L^{2\gamma}(0,\tau;L^{2\gamma})} \leq \tau^{\frac{\tilde{\delta}}{2}}\|u\|^\gamma_{L^{\tilde{q}}(0,\tau;L^{2\gamma})} \leq r^{\frac{\tilde{\delta}}{2}}(2n)^\gamma$$

we conclude

$$\|K^n_{stoch}u\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,E_r)} \lesssim r^{\frac{\tilde{\delta}}{2}}(2n)^\gamma + r^{\frac{1}{2}}\|u\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,E_r)}$$

and altogether,

$$\|K^n u\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,E_r)} \lesssim \|u_0\|_{L^2(\mathbb{R}^d)} + (2n)^\alpha r^\delta + (2n)^{2\gamma-1}r^{\tilde{\delta}} + r^{\frac{\tilde{\delta}}{2}}(2n)^\gamma + \left(r + r^{\frac{1}{2}}\right)\|u\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,E_r)} < \infty$$

for $u \in \mathbb{M}^p_{\mathbb{F}}(\Omega,E_r)$. In particular, $\mathbb{M}^p_{\mathbb{F}}(\Omega,E_r)$ is invariant under K^n. To show the contractivity of K^n, we take $u_1, u_2 \in \mathbb{M}^p_{\mathbb{F}}(\Omega,E_r)$ and define stopping times

$$\tau_j := \inf\left\{t \geq 0 : \|u_j\|_{L^q(0,t;L^{\alpha+1})} + \|u_j\|_{L^{\tilde{q}}(0,t;L^{2\gamma})} \geq 2n\right\} \wedge r, \qquad j = 1,2,$$

and fix $\omega \in \Omega$. Without loss of generality, we assume $\tau_1(\omega) \leq \tau_2(\omega)$. We use the deterministic Strichartz estimates from Proposition 2.14

$$\begin{aligned}\|K^n_{det}(u_1) - K^n_{det}(u_2)\|_{E_r} \lesssim& \|\varphi_n(u_1)|u_1|^{\alpha-1}u_1 - \varphi_n(u_2)|u_2|^{\alpha-1}u_2\|_{L^{q'}(0,r;L^{\frac{\alpha+1}{\alpha}})} \\ \leq& \|\varphi_n(u_1)\left(|u_1|^{\alpha-1}u_1 - |u_2|^{\alpha-1}u_2\right)\|_{L^{q'}(0,r;L^{\frac{\alpha+1}{\alpha}})} \\ &+ \|\left[\varphi_n(u_1) - \varphi_n(u_2)\right]|u_2|^{\alpha-1}u_2\|_{L^{q'}(0,r;L^{\frac{\alpha+1}{\alpha}})}.\end{aligned}$$

By (3.10) and Lemma 3.4, we get

$$\begin{aligned}&|\varphi_n(u_1,s) - \varphi_n(u_2,s)| \\ &\quad\leq \frac{1}{n}\left|\|u_1\|_{L^q(0,s;L^{\alpha+1})} + \|u_1\|_{L^{\tilde{q}}(0,s;L^{2\gamma})} - \|u_2\|_{L^q(0,s;L^{\alpha+1})} - \|u_2\|_{L^{\tilde{q}}(0,s;L^{2\gamma})}\right| \\ &\quad\leq \frac{1}{n}\left(\|u_1 - u_2\|_{L^q(0,s;L^{\alpha+1})} + \|u_1 - u_2\|_{L^{\tilde{q}}(0,s;L^{2\gamma})}\right) \leq \frac{2}{n}\|u_1 - u_2\|_{E_s}\end{aligned} \tag{3.19}$$

and we can use this as well as Lemma 3.5 with $p = \alpha + 1$ and $\sigma = \alpha$ to derive

$$\begin{aligned}\|\varphi_n(u_1)\left(|u_1|^{\alpha-1}u_1 - |u_2|^{\alpha-1}u_2\right)\|_{L^{q'}(0,r;L^{\frac{\alpha+1}{\alpha}})} &\leq \||u_1|^{\alpha-1}u_1 - |u_2|^{\alpha-1}u_2\|_{L^{q'}(0,\tau_1;L^{\frac{\alpha+1}{\alpha}})} \\ &\leq \tau_1^\delta\left(\|u_1\|_{L^q(0,\tau_1,L^{\alpha+1})} + \|u_2\|_{L^q(0,\tau_1,L^{\alpha+1})}\right)^{\alpha-1}\|u_1 - u_2\|_{L^q(0,\tau_1,L^{\alpha+1})} \\ &\leq r^\delta(4n)^{\alpha-1}\|u_1 - u_2\|_{L^q(0,\tau_1,L^{\alpha+1})} \leq r^\delta(4n)^{\alpha-1}\|u_1 - u_2\|_{E_r}\end{aligned}$$

and

$$\begin{aligned}\|\left[\varphi_n(u_1) - \varphi_n(u_2)\right]|u_2|^{\alpha-1}u_2\|_{L^{q'}(0,r;L^{\frac{\alpha+1}{\alpha}})} &\leq \frac{2}{n}\left\|\|u_1 - u_2\|_{E_\cdot}|u_2|^{\alpha-1}u_2\right\|_{L^{q'}(0,\tau_2;L^{\frac{\alpha+1}{\alpha}})} \\ &\leq \frac{2}{n}\|u_1 - u_2\|_{E_r}\||u_2|^{\alpha-1}u_2\|_{L^{q'}(0,\tau_2;L^{\frac{\alpha+1}{\alpha}})} \\ &\leq \frac{2}{n}\|u_1 - u_2\|_{E_r}\tau_2^\delta\|u_2\|^\alpha_{L^q(0,\tau_2;L^{\alpha+1})} \leq \frac{2}{n}\|u_1 - u_2\|_{E_r}r^\delta(2n)^\alpha.\end{aligned}$$

We obtain

$$\|K^n_{det}(u_1) - K^n_{det}(u_2)\|_{E_r} \lesssim \left(2^{\alpha+1} + 4^{\alpha-1}\right)r^\delta n^{\alpha-1}\|u_1 - u_2\|_{E_r}.$$

Analogously, we get

$$\left|[\varphi_n(u_1,s)]^2-[\varphi_n(u_2,s)]^2\right| \le \frac{4}{n}\|u_1-u_2\|_{E_s} \tag{3.20}$$

for the squared cut-off function and deduce the inequality

$$\begin{aligned}\|K^n_{Strat}(u_1)-K^n_{Strat}(u_2)\|_{E_r} &\lesssim \left\|[\varphi_n(u_1)]^2|u_1|^{2(\gamma-1)}u_1-[\varphi_n(u_2)]^2|u_2|^{2(\gamma-1)}u_2\right\|_{L^{\tilde{q}'}(0,r;L^{\frac{2\gamma}{2\gamma-1}})}\\ &\quad+\|u_1-u_2\|_{L^1(0,r;L^2)}\\ &\lesssim \left(2^{2\gamma+1}+4^{2(\gamma-1)}\right)r^{\tilde{\delta}}n^{2(\gamma-1)}\|u_1-u_2\|_{L^{\tilde{q}}(0,r;L^{2\gamma})}\\ &\quad+r\|u_1-u_2\|_{L^\infty(0,r;L^2)}\\ &\le \left[\left(2^{2\gamma+1}+4^{2(\gamma-1)}\right)r^{\tilde{\delta}}n^{2(\gamma-1)}+r\right]\|u_1-u_2\|_{E_r}.\end{aligned}$$

for the Stratonovich correction term. For the stochastic convolution, we estimate

$$\begin{aligned}\|\varphi_n(u_1)|u_1|^{\gamma-1}u_1-\varphi_n(u_2)|u_2|^{\gamma-1}u_2\|_{L^2(0,r;L^2)} &\lesssim \|\varphi_n(u_1)\left(|u_1|^{\gamma-1}u_1-|u_2|^{\gamma-1}u_2\right)\|_{L^2(0,r;L^2)}\\ &\quad+\|(\varphi_n(u_1)-\varphi_n(u_2))|u_2|^{\gamma-1}u_2\|_{L^2(0,r;L^2)}.\end{aligned}$$

The terms on the RHS can be treated by Lemma 3.5 with $r=2\gamma$ and $\sigma=\gamma$ and Lemma 3.4

$$\begin{aligned}&\|\varphi_n(u_1)\left(|u_1|^{\gamma-1}u_1-|u_2|^{\gamma-1}u_2\right)\|_{L^2(0,r;L^2)}\\ &\qquad\lesssim \left(\|u_1\|_{L^{2\gamma}(0,\tau_1;L^{2\gamma})}+\|u_2\|_{L^{2\gamma}(0,\tau_1;L^{2\gamma})}\right)^{\gamma-1}\|u_1-u_2\|_{L^{2\gamma}(0,\tau_1;L^{2\gamma})}\\ &\qquad\lesssim \tau_1^{\frac{\tilde{\delta}}{2}}\left(\|u_1\|_{L^{\tilde{q}}(0,\tau_1;L^{2\gamma})}+\|u_2\|_{L^{\tilde{q}}(0,\tau_1;L^{2\gamma})}\right)^{\gamma-1}\|u_1-u_2\|_{L^{\tilde{q}}(0,\tau_1;L^{2\gamma})}\\ &\qquad\lesssim r^{\frac{\tilde{\delta}}{2}}\left(4n\right)^{\gamma-1}\|u_1-u_2\|_{E_r}\end{aligned}$$

and by the estimate (3.20)

$$\begin{aligned}\|(\varphi_n(u_1)-\varphi_n(u_2))|u_2|^{\gamma-1}u_2\|_{L^2(0,r;L^2)} &\le \frac{2}{n}\|u_1-u_2\|_{E_r}\||u_2|^{\gamma-1}u_2\|_{L^2(0,\tau_2;L^2)}\\ &\le \frac{2}{n}\|u_1-u_2\|_{E_r}\tau_2^{\frac{\tilde{\delta}}{2}}\|u_2\|^\gamma_{L^{\tilde{q}}(0,\tau_2;L^{2\gamma})}\\ &\le \frac{2}{n}\|u_1-u_2\|_{E_r}r^{\frac{\tilde{\delta}}{2}}\left(2n\right)^\gamma.\end{aligned}$$

By Corollary 2.23, this yields

$$\begin{aligned}&\|K^n_{stoch}(u_1)-K^n_{stoch}(u_2)\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,E_r)}\\ &\qquad\lesssim \|B_1(\varphi_n(u_1)|u_1|^{\gamma-1}u_1-\varphi_n(u_2)|u_2|^{\gamma-1}u_2)\|_{L^p(\Omega,L^2(0,r;\mathrm{HS}(Y,L^2)))}\\ &\qquad\quad+\|B_2(u_1-u_2)\|_{L^p(\Omega,L^2(0,r;\mathrm{HS}(Y,L^2)))}\\ &\qquad\le \left(\sum_{m=1}^\infty\|e_m\|^2_{L^\infty}\right)^{\frac{1}{2}}r^{\frac{\tilde{\delta}}{2}}n^{\gamma-1}\left(4^{\gamma-1}+2^{\gamma+1}\right)\|u_1-u_2\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,E_r)}\\ &\qquad\quad+\left(\sum_{m=1}^\infty\|B_m\|^2_{\mathcal{L}(L^2)}\right)^{\frac{1}{2}}r^{\frac{1}{2}}\|u_1-u_2\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,E_r)}\\ &\qquad\lesssim \left[r^{\frac{\tilde{\delta}}{2}}n^{\gamma-1}\left(4^{\gamma-1}+2^{\gamma+1}\right)+r^{\frac{1}{2}}\right]\|u_1-u_2\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,E_r)}.\end{aligned}$$

Collecting the estimates for the other terms leads to

$$\|K^n(u_1)-K^n(u_2)\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,E_r)} \lesssim\Big[\left(2^{\alpha+1}+4^{\alpha-1}\right)r^{\delta}n^{\alpha-1}+\left(2^{2\gamma+1}+4^{2(\gamma-1)}\right)r^{\tilde{\delta}}n^{2(\gamma-1)} + r+\left(4^{\gamma-1}+2^{\gamma+1}\right)r^{\frac{\tilde{\delta}}{2}}n^{\gamma-1}+r^{\frac{1}{2}}\Big]\|u_1-u_2\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,E_r)}. \tag{3.21}$$

Hence, there is a small time $r=r(n,\alpha,\gamma)>0$ such that K^n is a strict contraction in $\mathbb{M}^p_{\mathbb{F}}(\Omega,E_r)$ with Lipschitz constant $\leq \frac{1}{2}$ and Banach's Fixed Point Theorem yields $u^{n,1}\in \mathbb{M}^p_{\mathbb{F}}(\Omega,E_r)$ with $K^n(u_1^n)=u_1^n$.

Step 2: Let us start with preliminary comments. For some $T_0>0$, we denote the shifted filtration $(\mathcal{F}_{t+T_0})_{t\geq 0}$ by $\mathbb{F}^{T_0}$. Then, the process given by $W^{T_0}(t):=W(T_0+t)-W(T_0)$, for $t\geq 0$ is a cylindrical Wiener process with respect to $\mathbb{F}^{T_0}$ as we have proved in Proposition A.4. Note that for an $\mathbb{F}$-predictable process Φ, we have

$$\int_0^t e^{\mathrm{i}(t-s)\Delta}\Phi(T_0+s)\mathrm{d}W^{T_0}(s)=\int_{T_0}^{T_0+t}e^{\mathrm{i}(T_0+t-s)\Delta}\Phi(s)\mathrm{d}W(s) \tag{3.22}$$

almost surely for all t.

We choose $r>0$ as in the first step and assume that we have $k\in\mathbb{N}$ and $u_k^n\in \mathbb{M}^p_{\mathbb{F}}(\Omega,E_{kr})$ with

$$u_k^n=e^{\mathrm{i}\cdot\Delta}u_0+K^n_{det}u_k^n+K^n_{Strat}u_k^n+K_{stoch}u_k^n$$

on the interval $[0,kr]$. In order to extend u_k^n to $[kr,(k+1)r]$, we define a new cutoff function by $\varphi_{n,k}(u,t):=\theta_n\left(Z_t(u)\right)$, where $(Z_t(u))_{t\in[0,r]}$ is a continuous, $\mathbb{F}^{kr}$-adapted process given by

$$Z_t(u):=(\|u_k^n\|^q_{L^q(0,kr;L^{\alpha+1})}+\|u\|^q_{L^q(0,t;L^{\alpha+1})})^{\frac{1}{q}}+(\|u_k^n\|^{\tilde{q}}_{L^{\tilde{q}}(0,kr;L^{2\gamma})}+\|u\|^{\tilde{q}}_{L^{\tilde{q}}(0,t;L^{2\gamma})})^{\frac{1}{\tilde{q}}}$$

for $t\in[0,r]$ and $u\in\mathbb{M}^p_{\mathbb{F}^{kr}}(\Omega,E_r)$. Moreover, we set

$$K^n_{det,k}u(t):=-\mathrm{i}\lambda\int_0^t e^{\mathrm{i}(t-s)\Delta}\left[\varphi_{n,k}(u,s)|u(s)|^{\alpha-1}u(s)\right]\mathrm{d}s,$$

$$K^n_{Strat,k}u(t):=\int_0^t e^{\mathrm{i}(t-s)\Delta}\left[\varphi_{n,k}(u,s)\mu_1\left(|u(s)|^{2(\gamma-1)}u(s)\right)+\mu_2\left(u(s)\right)\right]\mathrm{d}s,$$

$$K^n_{stoch,k}u(t):=-\mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}\left[\varphi_{n,k}(u,s)B_1\left(|u(s)|^{\gamma-1}u(s)\right)+B_2u(s)\right]\mathrm{d}W^{kr}(s)$$

for $t\in[0,r]$ and $u\in\mathbb{M}^p_{\mathbb{F}^{kr}}(\Omega,E_r)$ and

$$K^n_k u:=e^{\mathrm{i}\cdot\Delta}u_k^n(kr)+K^n_{det,k}u+K^n_{Strat,k}u+K^n_{stoch,k}u,\qquad u\in\mathbb{M}^p_{\mathbb{F}^{kr}}(\Omega,E_r).$$

We take $v_1,v_2\in\mathbb{M}^p_{\mathbb{F}^{kr}}(\Omega,E_r)$ and define the $\mathcal{F}^{kr}$-stopping times

$$\tau_j:=\inf\left\{t\geq 0: Z_t(v_j)\geq 2n\right\}\wedge r,\qquad j=1,2. \tag{3.23}$$

Without loss of generality, we assume $\tau_1(\omega)\leq\tau_2(\omega)$ and follow the lines of the initial step where we replace u_j by v_j and $\varphi_n(u_j)$ by $\varphi_{n,k}(v_j)$ for $j=1,2$. We obtain

$$\|K^n_{det,k}v_1-K^n_{det,k}v_2\|_{E_r}\leq\tau_1^{\delta}\left(\|v_1\|_{L^q(0,\tau_1,L^{\alpha+1})}+\|v_2\|_{L^q(0,\tau_1,L^{\alpha+1})}\right)^{\alpha-1}\|v_1-v_2\|_{E_r}$$

$$+\frac{2}{n}\|v_1-v_2\|_{E_r}\tau_2^\delta\|v_2\|_{L^q(0,\tau_2;L^{\alpha+1})}^\alpha$$

and by $\|v_j\|_{L^q(0,\tau_1,L^{\alpha+1})}\le Z_{\tau_1}(v_j)\le 2n$ for $j=1,2$, we conclude

$$\begin{aligned}\|K_{det,k}^n v_1-K_{det,k}^n v_2\|_{E_r}&\le \tau_1^\delta\,(4n)^{\alpha-1}\,\|v_1-v_2\|_{E_r}+\frac{2}{n}\|v_1-v_2\|_{E_r}\tau_2^\delta\,(2n)^\alpha\\&\le r^\delta\left((4n)^{\alpha-1}+\frac{2}{n}\,(2n)^\alpha\right)\|v_1-v_2\|_{E_r}.\end{aligned}$$

Analogously, the estimates for K_{Strat}^n and K_{stoch}^n from the first step can be adapted to get

$$\|K_{Strat,k}^n(v_1)-K_{Strat,k}^n(v_2)\|_{E_r}\lesssim\left(\left(2^{2\gamma+1}+4^{2(\gamma-1)}\right)r^{\tilde{\delta}}n^{2(\gamma-1)}+r\right)\|v_1-v_2\|_{E_r},$$

$$\|K_{stoch,k}^n(v_1)-K_{stoch,k}^n(v_2)\|_{\mathbb{M}^p_{\mathbb{F}^{kr}}(\Omega,E_r)}\lesssim\left(r^{\frac{\tilde{\delta}}{2}}n^{\gamma-1}\left(4^{\gamma-1}+2^{\gamma+1}\right)+r^{\frac{1}{2}}\right)\|v_1-v_2\|_{\mathbb{M}^p_{\mathbb{F}^{kr}}(\Omega,E_r)}$$

and thus

$$\begin{aligned}\|K_k^n(v_1)-K_k^n(v_2)\|_{\mathbb{M}^p_{\mathbb{F}^{kr}}(\Omega,E_r)}\lesssim&\Big[\left(2^{\alpha+1}+4^{\alpha-1}\right)r^\delta n^{\alpha-1}+\left(2^{2\gamma+1}+4^{2(\gamma-1)}\right)r^{\tilde{\delta}}n^{2(\gamma-1)}\\&+r+\left(4^{\gamma-1}+2^{\gamma+1}\right)r^{\frac{\tilde{\delta}}{2}}n^{\gamma-1}+r^{\frac{1}{2}}\Big]\|v_1-v_2\|_{\mathbb{M}^p_{\mathbb{F}^{kr}}(\Omega,E_r)}.\end{aligned}\tag{3.24}$$

Since the constant is the same as in the initial step, the definition of $r>0$ yields that K_k^n is a strict contraction in $\mathbb{M}^p_{\mathbb{F}^{kr}}(\Omega,E_r)$. We call the unique fixed point v_{k+1}^n and set

$$u_{k+1}^n(t):=\begin{cases}u_k^n(t), & t\in[0,kr],\\ v_{k+1}^n(t-kr), & t\in[kr,(k+1)r].\end{cases}$$

Obviously, u_{k+1}^n is a process which is continuous and $\mathbb{F}$-adapted in $L^2(\mathbb{R}^d)$ and $\mathbb{F}$-predictable in $L^{2\gamma}(\mathbb{R}^d)$ and satisfies $\|u_{k+1}^n\|_{L^p(\Omega,E_{(k+1)r})}<\infty$. Therefore $u_{k+1}^n\in\mathbb{M}^p_{\mathbb{F}}(\Omega,E_{(k+1)r})$. Let $t\in[kr,(k+1)r]$ and define $\tilde{t}:=t-kr$. Then, the definition of K_k^n and the induction assumption yield

$$\begin{aligned}u_{k+1}^n(t)=&v_{k+1}^n(\tilde{t})=K_k^n v_{k+1}^n(\tilde{t})\\=&e^{\mathrm{i}\tilde{t}\Delta}u_k^n(kr)+K_{det,k}^n v_{k+1}^n(\tilde{t})+K_{Strat,k}^n v_{k+1}^n(\tilde{t})+K_{stoch,k}^n v_{k+1}^n(\tilde{t})\\=&e^{\mathrm{i}t\Delta}u_0+\left[e^{\mathrm{i}\tilde{t}\Delta}K_{det}^n u_k^n(kr)+K_{det,k}^n v_{k+1}^n(\tilde{t})\right]+\left[e^{\mathrm{i}\tilde{t}\Delta}K_{Strat,k}^n u_k^n(kr)+K_{Strat,k}^n v_{k+1}^n(\tilde{t})\right]\\&+\left[e^{\mathrm{i}\tilde{t}\Delta}K_{stoch}u_k^n(r)+K_{stoch,k}^n v_{k+1}^n(\tilde{t})\right].\end{aligned}$$

Using the identities

$$\varphi_n(u_k^n,s)=\varphi_n(u_{k+1}^n,s),\qquad \varphi_{n,k}(v_{k+1}^n,\tilde{s})=\varphi_n(u_{k+1}^n,kr+\tilde{s})$$

for $s\in[0,kr]$ and $\tilde{s}\in[0,r]$, we compute

$$\begin{aligned}e^{\mathrm{i}\tilde{t}\Delta}K_{det}^n u_k^n(kr)+K_{det,k}^n v_{k+1}^n(\tilde{t})=&-\mathrm{i}\lambda e^{\mathrm{i}\tilde{t}\Delta}\int_0^{kr}e^{\mathrm{i}(kr-s)\Delta}\left[\varphi_n(u_k^n,s)|u_k^n(s)|^{\alpha-1}u_k^n(s)\right]\mathrm{d}s\\&-\mathrm{i}\lambda\int_0^{\tilde{t}}e^{\mathrm{i}(\tilde{t}-\tilde{s})\Delta}\left[\varphi_{n,k}(v_{k+1}^n,\tilde{s})|v_{k+1}^n(\tilde{s})|^{\alpha-1}v_{k+1}^n(\tilde{s})\right]\mathrm{d}\tilde{s}\end{aligned}$$

$$\begin{aligned}
&= - \mathrm{i}\lambda \int_0^{kr} e^{\mathrm{i}(t-s)\Delta} \left[\varphi_n(u_{k+1}^n, s)|u_{k+1}^n(s)|^{\alpha-1} u_{k+1}^n(s)\right] \mathrm{d}s \\
&\quad - \mathrm{i}\lambda \int_0^{\tilde{t}} e^{\mathrm{i}(\tilde{t}-\tilde{s})\Delta} \left[\varphi_n(u_{k+1}^n, kr+\tilde{s})|u_{k+1}^n(kr+\tilde{s})|^{\alpha-1} u_{k+1}^n(kr+\tilde{s})\right] \mathrm{d}\tilde{s} \\
&= - \mathrm{i}\lambda \int_0^{t} e^{\mathrm{i}(t-s)\Delta} \left[\varphi_n(u_{k+1}^n, s)|u_{k+1}^n(s)|^{\alpha-1} u_{k+1}^n(s)\right] \mathrm{d}s = K_{det}^n u_{k+1}^n(t),
\end{aligned}$$

where we used the substitution $s = kr + \tilde{s}$ in the second integral for the last step. Analogously,

$$e^{\mathrm{i}\tilde{t}\Delta} K_{Strat,k}^n u_k^n(kr) + K_{Strat,k}^n v_{k+1}^n(\tilde{t}) = K_{Strat}^n u_{k+1}^n(t),$$

$$e^{\mathrm{i}\tilde{t}\Delta} K_{stoch}^n u_k^n(kr) + K_{Stoch,k} v_{k+1}^n(\tilde{t}) = K_{stoch}^n u_{k+1}^n(t),$$

where one uses (3.22) for the stochastic convolutions. Hence, we get

$$u_{k+1}^n(t) = e^{\mathrm{i}t\Delta}u_0 + K_{det}^n u_{k+1}^n(t) + K_{Strat}^n u_{k+1}^n(t) + K_{stoch}^n u_{k+1}^n(t) = K^n u_{k+1}^n(t)$$

for $t \in [kr, (k+1)r]$ and therefore, u_{k+1}^n is a fixed point of K^n in $\mathbb{M}_{\mathbb{F}}^p(\Omega, E_{(k+1)r})$. Define $k := \lfloor \frac{T}{r} + 1 \rfloor$. Then, $u^n := u_k^n$ is the process from the assertion.

Step 3: Now, we turn our attention to uniqueness. Let $(\tilde{u}, \tau)$ be another strong solution of (3.7). As in (3.21), we get

$$\begin{aligned}
\|u - \tilde{u}\|_{\mathbb{M}_{\mathbb{F}}^p(\Omega, E_{\tau\wedge r})} =& \|K^n(u) - K^n(\tilde{u})\|_{\mathbb{M}_{\mathbb{F}}^p(\Omega, E_{\tau\wedge r})} \\
\leq& C\Big[\left(2^{\alpha+1} + 4^{\alpha-1}\right) r^{\delta} n^{\alpha-1} + \left(2^{2\gamma+1} + 4^{2(\gamma-1)}\right) r^{\tilde{\delta}} n^{2(\gamma-1)} + r \\
&\quad + \left(4^{\gamma-1} + 2^{\gamma+1}\right) r^{\frac{\tilde{\delta}}{2}} n^{\gamma-1} + r^{\frac{1}{2}} \Big] \|u - \tilde{u}\|_{\mathbb{M}_{\mathbb{F}}^p(\Omega, E_{\tau\wedge r})} \\
\leq& \frac{1}{2} \|u - \tilde{u}\|_{\mathbb{M}_{\mathbb{F}}^p(\Omega, E_{\tau\wedge r})},
\end{aligned}$$

which leads to $u(t) = \tilde{u}(t)$ in $\mathbb{M}_{\mathbb{F}}^p(\Omega, E_{\tau\wedge r})$, i.e. $u = \tilde{u}$ almost surely on $\{t \leq \tau \wedge r\}$. This can be iterated to see that $u(t) = \tilde{u}(t)$ almost surely on $\{t \leq \sigma_k\}$ with $\sigma_k := \tau \wedge (kr)$ for $k \in \mathbb{N}$. The assertion follows from $\sigma_k = \tau$ for k large enough. □

In the following two Propositions, we use the results on the truncated equation (3.7) to derive existence and uniqueness for the original problem (3.1). The proofs are quite standard and in the literature, analogous arguments have been used in various contexts for extensions of existence and uniqueness results from integrable to non-integrable initial values and from globally to locally Lipschitz nonlinearities, see for example [123], Theorem 7.1, [26], Theorem 4.10, and [110], Theorem 1.5.

Proposition 3.8. *Let $\alpha \in (1, 1 + \frac{4}{d})$, $\gamma \in (1, 1 + \frac{2}{d})$ and $(u^n)_{n\in\mathbb{N}} \subset \mathbb{M}_{\mathbb{F}}^p(\Omega, E_T)$ be the sequence constructed in Proposition 3.7. For $n \in \mathbb{N}$, we define the stopping time τ_n by*

$$\tau_n := \inf\left\{t \in [0,T] : \|u^n\|_{L^q(0,t;L^{\alpha+1})} + \|u^n\|_{L^{\tilde{q}}(0,t;L^{2\gamma})} \geq n\right\} \wedge T.$$

Then, the following assertions hold:

a) We have $0 < \tau_n \leq \tau_k$ almost surely for $n \leq k$ and $u^n(t) = u^k(t)$ almost surely on $\{t \leq \tau_n\}$.

b) The triple $\left(u, (\tau_n)_{n\in\mathbb{N}}, \tau_\infty\right)$ with $u(t) := u^n(t)$ for $t \in [0, \tau_n]$ and $\tau_\infty := \sup_{n\in\mathbb{N}} \tau_n$ is an analytically and stochastically strong solution of (3.1) *in $L^2(\mathbb{R}^d)$ in the sense of Definition 2.1.*

Proof. ad a): We note that τ_n is a welldefined stopping time with $\tau_n > 0$ almost surely, since

$$Z^n(t) := \|u^n\|_{L^q(0,t;L^{\alpha+1})} + \|u^n\|_{L^{\tilde{q}}(0,t;L^{2\gamma})} \le 2\|u^n\|_{E_t} \le 2\|u^n\|_{E_T} < \infty, \qquad t \in [0,T],$$

defines an increasing, continuous and $\mathbb{F}$-adapted process $Z^n : \Omega \times [0,T] \to [0,\infty)$ with $Z^n(0) = 0$. For $n \le k$, we set

$$\tau_{k,n} := \inf \left\{ t \in [0,T] : Z^k(t) \ge n \right\} \wedge T.$$

Then, we have $\tau_{k,n} \le \tau_k$ and $\varphi_n(u^k,t) = 1 = \varphi_k(u^k,t)$ on $\{t \le \tau_{k,n}\}$. Hence, $(u^k, \tau_{k,n})$ is a solution of (3.7) and by the uniqueness part of Proposition 3.7, we obtain $u^k(t) = u^n(t)$ almost surely on $\{t \le \tau_{k,n}\}$. But this leads to $Z^k(t) = Z^n(t)$ on $\{t \le \tau_{k,n}\}$ and $\tau_{k,n} = \tau_n$ almost surely which implies the assertion.

ad b): By part a), u is well-defined up to a null set, where we define $u := 0$ and $\tau_\infty = T$. The monotonicity of $(\tau_n)_{n\in\mathbb{N}}$ yields $\tau_n \to \tau_\infty$ almost surely. Moreover, $u \in \mathbb{M}^p_{\mathbb{F}}(\Omega, E_{\tau_n})$ by Proposition 3.7 and therefore $u \in \mathbb{M}^p_{\mathbb{F}}(\Omega, E_{[0,\tau)})$. Since u_n is a global strong solution of (3.7), the identity

$$\varphi_n(u,t) = \varphi_n(u^n,t) = 1 \qquad \text{a.s on } \{t \wedge \tau_n\},$$

yields

$$\begin{aligned} u(t) =& u_0 + \int_0^t \left[\mathrm{i}\Delta u(s) - \mathrm{i}\lambda |u(s)|^{\alpha-1} u(s) + \mu_1 \left(|u(s)|^{2(\gamma-1)} u(s) \right) + \mu_2(u(s)) \right] \mathrm{d}s \\ & - \mathrm{i} \int_0^t \left[B_1 \left(|u(s)|^{\gamma-1} u(s) \right) + B_2 u(s) \right] \mathrm{d}W(s) \end{aligned}$$

almost surely on $\{t \le \tau_n\}$ for all $n \in \mathbb{N}$. Therefore, the triple $\left(u, (\tau_n)_{n\in\mathbb{N}}, \tau_\infty\right)$ is a strong solution of (3.4) in $L^2(\mathbb{R}^d)$. □

Proposition 3.9. *Let $\alpha \in (1, 1+\frac{4}{d})$, $\gamma \in (1, 1+\frac{2}{d})$ and $\left(u_1, (\sigma_n)_{n\in\mathbb{N}}, \sigma\right)$, $\left(u_2, (\tau_n)_{n\in\mathbb{N}}, \tau\right)$ be strong solutions to* (3.4) *in $L^2(\mathbb{R}^d)$. Then,*

$$u_1(t) = u_2(t) \qquad \textit{a.s. on } \{t < \sigma \wedge \tau\},$$

i.e. the solution of (3.4) *is unique.*

3.93.8

Proof. We fix $k, n \in \mathbb{N}$ and define a stopping time by

$$\begin{aligned} \nu_{k,n} :=& \inf \Big\{ t \in [0,T] : \left(\|u_1\|_{L^q(0,t;L^{\alpha+1})} + \|u_1\|_{L^{\tilde{q}}(0,t;L^{2\gamma})} \right) \\ & \qquad \vee \left(\|u_2\|_{L^q(0,t;L^{\alpha+1})} + \|u_2\|_{L^{\tilde{q}}(0,t;L^{2\gamma})} \right) \ge n \Big\} \wedge \sigma_k \wedge \tau_k. \end{aligned}$$

Hence, $\varphi_n(u_1,t) = \varphi_n(u_2,t) = 1$ on $\{t \le \nu_{k,n}\}$ and therefore, $(u_1, \nu_{k,n})$ and $(u_2, \nu_{k,n})$ are strong solutions of (3.7). By the uniqueness part of Proposition 3.7, we get

$$u_1(t) = u_2(t) \qquad \text{a.s. on } \{t \le \nu_{k,n}\},$$

which yields the assertion since $\nu_{k,n} \to \sigma \wedge \tau$ almost surely for $n, k \to \infty$. □

In the Propositions 3.8 and 3.9, we have proved Theorem 3.2 a) in the subcritical case, i.e. $\alpha \in (1, 1 + \frac{4}{d})$, $\gamma \in (1, 1 + \frac{2}{d})$. We continue with the critical setting. In contrast to the proof of Proposition 3.7, the argument to construct the solution of the truncated equation already involves stopping times. Thus, we have to make some preparations.

Definition 3.10. Let Λ be a family of non-negative random variables on Ω. Then, we define the *essential supremum* $\operatorname{ess\,sup} \Lambda$ of Λ by the following properties:

a) For all $X \in \Lambda$, we have $X \leq \operatorname{ess\,sup} \Lambda$ almost surely.

b) If Y is a random variable with $X \leq Y$ almost surely for all $X \in \Lambda$, then we also have $\operatorname{ess\,sup} \Lambda \leq Y$ almost surely.

Obviously, the essential supremum is unique in the sense that two essential suprema coincide almost surely. The following Lemma is devoted to the existence of the essential supremum. The proof can be found in [73], Theorem A.3.

Lemma 3.11. *Let Λ be a family of non-negative and bounded random variables on Ω.*

a) The essential supremum $\operatorname{ess\,sup} \Lambda$ exists.

b) Suppose that Λ is additionally closed under pairwise maximization, i.e.

$$X_1, X_2 \in \Lambda \quad \text{implies} \quad X_1 \vee X_2 \in \Lambda.$$

Then, there is an increasing sequence $(X_n)_{n\in\mathbb{N}} \subset \Lambda$ with $\operatorname{ess\,sup} \Lambda = \lim_{n\to\infty} X_n$ almost surely.

Let us formulate our local existence and uniqueness result in the critical setting.

Proposition 3.12. *Let $\alpha \in (1, 1 + \frac{4}{d}]$, $\gamma \in (1, 1 + \frac{2}{d}]$ with $\alpha = 1 + \frac{4}{d}$ or $\gamma = 1 + \frac{2}{d}$.*

a) There is a unique maximal mild solution $(u, (\tau_n)_{n\in\mathbb{N}}, \tau_\infty)$ of (3.4).

b) We set

$$p_1 := \begin{cases} 2\gamma, & \alpha = 1 + \dfrac{4}{d}, \\ \alpha + 1, & \gamma = 1 + \dfrac{2}{d}, \end{cases} \tag{3.25}$$

and choose q_2 such (p_1, q_1) is a Strichartz pair. Then, we have the blow-up alternative

$$\mathbb{P}\left(\tau_\infty < T, \quad \|u\|_{L^{2+\frac{4}{d}}(0,\tau_\infty,L^{2+\frac{4}{d}})} < \infty, \quad \|u\|_{L^{q_1}(0,\tau_\infty,L^{p_1})} < \infty\right) = 0.$$

Proof. Step 1. We remark that $(2 + \frac{4}{d}, 2 + \frac{4}{d})$ is a Strichartz pair. For $r > 0$, we define

$$Y_r := L^{2+\frac{4}{d}}(0, r; L^{2+\frac{4}{d}}(\mathbb{R}^d)), \qquad E_r := C([0, r], L^2(\mathbb{R}^d)) \cap Y_r.$$

We proceed as in the proof of Proposition 3.7 with the difference that $n \in \mathbb{N}$ is now substituted by a possibly small $\nu > 0$, which does not change the estimates at all. We set

$$K_1^\nu u := e^{\mathrm{i}\cdot\Delta} u_0 + K_{det}^\nu u + K_{Strat}^\nu u + K_{stoch}^\nu u$$

with the convolution operators from (3.15), (3.16) and (3.17) and obtain the estimates

$$\|K_1^\nu u\|_{\mathbb{M}_{\mathbb{F}}^p(\Omega, E_r)} \lesssim \|u_0\|_{L^2(\mathbb{R}^d)} + (2\nu)^\alpha r^\delta + (2\nu)^{2\gamma-1} r^{\tilde{\delta}} + r^{\frac{\tilde{\delta}}{2}} (2\nu)^\gamma + \left(r + r^{\frac{1}{2}}\right) \|u\|_{\mathbb{M}_{\mathbb{F}}^p(\Omega, E_r)}$$

$$\|K_1^\nu(u_1) - K_1^\nu(u_2)\|_{\mathbb{M}_{\mathbb{F}}^p(\Omega,E_r)} \lesssim \Big[\left(2^{\alpha+1} + 4^{\alpha-1}\right) r^\delta \nu^{\alpha-1} + \left(2^{2\gamma+1} + 4^{2(\gamma-1)}\right) r^{\tilde{\delta}} \nu^{2(\gamma-1)} + r + \left(4^{\gamma-1} + 2^{\gamma+1}\right) r^{\frac{\tilde{\delta}}{2}} \nu^{\gamma-1} + r^{\frac{1}{2}} \Big] \|u_1 - u_2\|_{\mathbb{M}_{\mathbb{F}}^p(\Omega,E_r)}$$

for $u, u_1, u_2 \in \mathbb{M}_{\mathbb{F}}^p(\Omega, E_r)$, where we set $\delta := 1 + \frac{d}{4}(1-\alpha)$ and $\tilde{\delta} := 1 + \frac{d}{2}(1-\gamma)$ as before. Since we have $\delta = 0$ or $\tilde{\delta} = 0$ by the assumption, we cannot ensure that K_1^ν is a contraction by taking r small enough. But if we choose ν and r sufficiently small, we get a unique fixed point $u_1 \in \mathbb{M}_{\mathbb{F}}^p(\Omega, E_r)$ of K_1^ν.
By the definition of the truncation function φ_ν in (3.9), u_1 is a solution of the original equation, as long as $\|u_1\|_{L^{2+\frac{4}{d}}(0,t;L^{2+\frac{4}{d}})} + \|u_1\|_{L^{q_1}(0,t;L^{p_1})} \leq \nu$, where (p_1, q_1) is the Strichartz pair defined in (3.25). In particular, the pair (u_1, τ_1) with

$$\tau_1 := \inf \left\{ t \geq 0 : \|u_1\|_{L^{2+\frac{4}{d}}(0,t;L^{2+\frac{4}{d}})} + \|u_1\|_{L^{q_1}(0,t;L^{p_1})} \geq \nu \right\} \wedge r$$

is a local solution of (3.4).

Step 2. We want to extend the solution from the first step up to a maximal stopping time following [70], Theorem 14.21. We define the set

$$\mathcal{S} := \Big\{ \tau : \Omega \to [0,T] \quad \mathbb{F}\text{-stopping time} \Big| \ \exists u \in \mathbb{M}_{\mathbb{F}}^p(\Omega, E_{[0,\tau]}) : (u,\tau) \text{ is the unique solution to (3.4)} \Big\}$$

which is non-empty by Step 1. Moreover, one can show that $\mathcal{S}$ is stable under the maximum-operation. Indeed, given stopping times τ_j and processes $u_j \in \mathbb{M}_{\mathbb{F}}^p(\Omega, E_{[0,\tau_j]})$ such that (u_j, τ_j) are solutions for $j = 1, 2$, we set

$$u(t) := u_1(t \wedge \tau_1) + u_2(t \wedge \tau_2) - u_1(t \wedge \tau_1 \wedge \tau_2), \qquad t \in [0,T].$$

Uniqueness implies $u_1(t) = u_2(t)$ almost surely on $\{t < \tau_1 \wedge \tau_2\}$ and therefore, we have $u = u_1$ on $\{\tau_1 > \tau_2\} \times [0, \tau_1)$ and $u = u_1$ on $\{\tau_1 \leq \tau_2\} \times [0, \tau_2)$. It is easily checked that $(u, \tau_1 \vee \tau_2)$ is a solution to (3.4) and uniqueness is inherited from u_1 and u_2.

Thus, we can apply Lemma 3.11 and obtain $\tau_\infty := \operatorname{esssup} \mathcal{S}$ as well as a nondecreasing sequence $(\tau_n)_{n\in\mathbb{N}} \subset \mathcal{S}$ with $\tau_\infty = \lim_{n\to\infty} \tau_n$ almost surely. In particular, τ_∞ is a stopping time by Lemma 2.11 in [74]. We denote the solutions associated to τ_n by u_n and define $u \in \mathbb{M}_{\mathbb{F}}^p(\Omega, E_{[0,\tau_\infty)})$ by

$$u(t) := \mathbf{1}_{\{t=0\}} u_0 + \sum_{n=1}^{\infty} u_n(t) \mathbf{1}_{(\tau_{n-1},\tau_n]}(t) \quad \text{on } \{t < \tau_\infty\}.$$

The triple $\left(u, (\tau_n)_{n\in\mathbb{N}}, \tau_\infty\right)$ is a unique local mild solution.

Step 3. In order to show the blow-up criterion, we set

$$\tilde{\Omega} := \left\{ \tau_\infty < T, \quad \|u\|_{L^{2+\frac{4}{d}}(0,\tau_\infty,L^{2+\frac{4}{d}})} < \infty, \quad \|u\|_{L^{q_1}(0,\tau_\infty,L^{p_1})} < \infty \right\}$$

and assume $\mathbb{P}(\tilde{\Omega}) > 0$. This implies, that we have

$$\|u\|_{L^{2+\frac{4}{d}}(\tau_n,\tau_\infty,L^{2+\frac{4}{d}})} + \|u\|_{L^{q_1}(\tau_n,\tau_\infty,L^{p_1})} \xrightarrow{n\to\infty} 0$$

on $\tilde{\Omega}$ and by Egoroff's Theorem, we get $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) > 0$ such that the limit is uniform on Ω_0. In particular, there is $n \in \mathbb{N}$ with

$$\|u\|_{L^{2+\frac{4}{d}}(\tau_n,\tau_\infty,L^{2+\frac{4}{d}})} + \|u\|_{L^{q_1}(\tau_n,\tau_\infty,L^{p_1})} \leq \frac{\nu}{2} \tag{3.26}$$

for all $\omega \in \Omega_0$, where $\nu > 0$ is chosen similarly to the first step. Let us recall from Proposition A.4 that $W^{\tau_n} := W(\cdot + \tau_n) - W(\tau_n)$ defines a cylindrical Wiener process relative to the shifted filtration $\mathbb{F}_{\tau_n} := (\mathcal{F}_{\tau_n+t})_t$. As above, we can construct a unique mild solution v_n solution of

$$\begin{cases} \mathrm{d}v_n(t) = \Big[\mathrm{i}\Delta v_n(t) - \mathrm{i}\lambda|v_n(t)|^{\alpha-1}v_n(t) + \mu_1\left(|v_n(t)|^{2(\gamma-1)}v_n(t)\right) + \mu_2(v_n(t))\Big]\,dt \\ \qquad\qquad - \mathrm{i}\left[B_1\left(|v_n(t)|^{\gamma-1}v_n(t)\right) + B_2 v_n(t)\right]\mathrm{d}W^{\tau_n}(t), \\ v_n(0) = u(\tau_n), \end{cases} \tag{3.27}$$

with existence time

$$\sigma_n := \inf\left\{t \in [0,T] : \|v_n\|_{L^q(0,t;L^q)} + \|v_n\|_{L^{q_1}(0,t;L^{p_1})} \geq \nu\right\} \wedge r.$$

We would like to show that $\tau_n + \sigma_n$ is an $\mathcal{F}$-stopping time. In view of the right-continuity of $\mathbb{F}$, it is sufficient to verify $\{\tau_n + \sigma_n < s\} \in \mathcal{F}_s$ for all $s \in (0,T]$. Given $s \in (0,T]$, we have

$$\{\tau_n + \sigma_n < s\} = \bigcup_{q \in (0,T]\cap\mathbb{Q}} \{\sigma_n < q\} \cap \{\tau_n + q < s\}\,. \tag{3.28}$$

Now, we fix $q \in (0,T]\cap\mathbb{Q}$. Since σ_n is an $\mathbb{F}_{\tau_n}$-stopping time by construction, we have $\{\sigma_n < q\} \in \mathcal{F}_{\tau_n+q}$. From the definition of $\mathbb{F}_{\tau_n}$, we infer $\{\sigma_n < q\} \cap \{\tau_n + q < s\} \in \mathcal{F}_s$. In combination with (3.28), we obtain $\{\tau_n + \sigma_n < s\} \in \mathcal{F}_s$.

As in the proof of Proposition 3.7, we can glue the solutions u and v_n together and get a unique solution $(\tilde{u}, \tau_n + \sigma_n)$ with $\tilde{u} := u\mathbf{1}_{[0,\tau_n)} + v_n\mathbf{1}_{[\tau_n,\tau_n+\sigma_n)}$. In particular, we infer $\tau_n + \sigma_n \in \mathcal{S}$. The definition of σ_n yields

$$\|\tilde{u}\|_{L^q(\tau_n,\tau_n+\sigma_n;L^q)} + \|\tilde{u}\|_{L^{q_1}(\tau_n,\tau_n+\sigma_n;L^{p_1})} = \nu$$

and by uniqueness, we obtain $\tilde{u} = u$ on $[0,(\tau_n+\sigma_n)\wedge\tau_\infty)$. Due to (3.26), we have $\tau_n + \sigma_n > \tau_\infty$ on Ω_0 contradicting the definition of the essential supremum. Hence, we conclude $\mathbb{P}(\tilde{\Omega}) = 0$.

Step 4. In order to prove that the solution $(u, (\tau_n)_{n\in\mathbb{N}}, \tau_\infty)$ is maximal, we take another local mild solution $\left(w, (\tilde{\tau}_n)_{n\in\mathbb{N}}, \tau\right)$ and assume that there is $\Lambda \in \mathcal{F}$ with $\mathbb{P}(\Lambda) > 0$ and $\tau > \tau_\infty$ on Λ. In particular, for all $\omega \in \Lambda$, we can choose $n = n(\omega) \in \mathbb{N}$ with $\tilde{\tau}_n(\omega) > \tau_\infty(\omega)$ which implies

$$\|u\|_{L^{2+\frac{4}{d}}(0,\tau_\infty;L^{2+\frac{4}{d}})} < \infty, \qquad \|u\|_{L^{q_1}(0,\tau_\infty;L^{p_1})} < \infty$$

on Λ. By the blow-up alternative, we conclude $\tau_\infty = T$ almost surely on Λ. This is a contradiction to the assumption since τ is also bounded by T. □

We would like to remark that the proof of the blow-up alternative and the elegant iteration procedure based on the essential supremum of stopping times is inspired by [65], Theorem 4.3. In this article, the author used similar techniques in the context of quasilinear stochastic evolution equations. We close this section with remarks on possible slight generalizations of Theorem 3.2 a) and continuous dependence of the initial data and comment on the transfer of our method to the energy space $H^1(\mathbb{R}^d)$.

Remark 3.13. In the proof of the local result, we did not use the special structure of the terms B_1, B_2 from (3.5) and μ_1, μ_2 from (3.6). In fact, we only used $B_1, B_2 \in \mathcal{L}(L^2(\mathbb{R}^d), \mathrm{HS}(Y, L^2(\mathbb{R}^d)))$, $\mu_1 \in \mathcal{L}(L^2(\mathbb{R}^d)) \cap \mathcal{L}(L^{2\gamma}(\mathbb{R}^d))$ and $\mu_2 \in \mathcal{L}(L^2(\mathbb{R}^d))$. But since the definition of μ_1, μ_2 is motivated by the Stratonovich product and will be important for the global existence in the following section, we decided to start with the special case from the beginning.

A generalization of the result from Theorem 3.2 from determistic initial values $u_0 \in L^2(\mathbb{R}^d)$ to $u_0 \in L^q(\Omega, \mathcal{F}_0; L^2(\mathbb{R}^d))$ is straightforward. By the standard localization technique (see e.g. [123]), a further generalization to $\mathcal{F}_0$-measurable $u_0 : \Omega \to L^2(\mathbb{R}^d)$ can be done if one relaxes the condition $u \in \mathbb{M}^p_{\mathbb{F}}(\Omega, E_{[0,\tau)})$ to $u \in \mathbb{M}^0_{\mathbb{F}}(\Omega, E_{[0,\tau)})$, i.e. u is a continuous $\mathbb{F}$-adapted process in $L^2(\mathbb{R}^d)$ and $\mathbb{F}$-predictable in $L^{2\gamma}(\mathbb{R}^d)$ with

$$\sup_{t\in[0,\tau]} \|u(t)\|^p_{L^2} + \|u\|^p_{Y_\tau} < \infty \qquad \text{a.s.}$$

For the sake of simplicity, we decided to restrict ourselves to deterministic initial values.

Remark 3.14. Using the estimates of the fixed point argument in Proposition 3.7, it is straightforward to show that we have the following Lipschitz continuous dependence on the initial data: For two solutions $(u_1, (\tau_{n,1})_n, \tau_1)$ and $(u_2, (\tau_{n,2})_n, \tau_2)$ of (3.4) corresponding to initial data $u_{0,1}$ and $u_{0,2}$ constructed as in Proposition 3.8, we have

$$\|u_1 - u_2\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega, E_{\tau_n})} \le C(n) \|u_{0,1} - u_{0,2}\|_{L^2}, \tag{3.29}$$

where $\tau_n := \tau_{n,1} \wedge \tau_{n,2}$. We can compute the constant

$$C(n) = \sum_{l=1}^{\lfloor \frac{T}{r(n)} \rfloor + 1} \left(\frac{C_{Str}}{1-L}\right)^{l-1}$$

explicitly. This yields $C(n) \to \infty$ for $n \to \infty$ as a consequence of $C_{Str} \ge 1$ and $L \in (0,1)$. In particular, the estimate is not strong enough to imply Lipschitz continuous dependence on $[0, \tau)$ with $\tau := \tau_1 \wedge \tau_2$.

3.2. Global existence in $L^2(\mathbb{R}^d)$

The goal of this section is to prove part b) and c) of Theorem 3.2. To this end, we study global existence of the solution to the subcritical stochastic NLS

$$\begin{cases} \mathrm{d}u(t) = \left[\mathrm{i}\Delta u(t) - \mathrm{i}\lambda |u(t)|^{\alpha-1} u(t) + \mu_1\left(|u(t)|^{2(\gamma-1)} u(t)\right) + \mu_2(u(t))\right] \mathrm{d}t \\ \qquad\qquad - \mathrm{i}\left[B_1\left(|u(t)|^{\gamma-1} u(t)\right) + B_2 u(t)\right] \mathrm{d}W(t), \\ u(0) = u_0, \end{cases} \tag{3.30}$$

with $\alpha \in (1, 1 + \frac{4}{d})$ and $\gamma \in (1, 1 + \frac{2}{d})$. Let us recall that the local solution $\left(u, (\tau_n)_{n\in\mathbb{N}}, \tau_\infty\right)$ is given by $u = u_n$ on $[0, \tau_n]$, where

$$\tau_n := \inf\left\{t \in [0,T] : \|u_n\|_{L^q(0,t;L^{\alpha+1})} + \|u_n\|_{L^{\tilde q}(0,t;L^{2\gamma})} \ge n\right\} \wedge T, \qquad n \in \mathbb{N}, \tag{3.31}$$

for exponents $q, \tilde q \in (2, \infty)$ satisfying the Strichartz conditions

$$\frac{2}{q} + \frac{d}{\alpha+1} = \frac{d}{2}, \qquad \frac{2}{\tilde q} + \frac{d}{2\gamma} = \frac{d}{2}.$$

Moreover, $\tau_\infty = \sup_{n\in\mathbb{N}} \tau_n$ and u_n is the solution of the truncated problem

$$\begin{cases} \mathrm{d}u_n = \Big(\mathrm{i}\Delta u_n - \mathrm{i}\varphi_n(u_n,\cdot)|u_n|^{\alpha-1}u_n + [\varphi_n(u_n,\cdot)]^2\mu_1(|u_n|^{2(\gamma-1)}u_n) + \mu_2\left(u_n\right)\Big)\,\mathrm{d}t \\ \qquad\quad - \mathrm{i}\left(\varphi_n(u_n,\cdot)B_1\left(|u_n|^{\gamma-1}u_n\right) + B_2 u_n\right)\mathrm{d}W, \\ u(0) = u_0. \end{cases} \tag{3.32}$$

The strategy to prove global existence is determined by the definition of the existence times in (3.31): We need to find uniform bounds for u_n in the space $L^q(0,T;L^{\alpha+1}) \cap L^{\tilde{q}}(0,T;L^{2\gamma})$. Note that this is a drawback of our approach based on the the truncation of the nonlinearities and can be avoided in the deterministic case, where the local existence result comes with a natural blow-up alternative in $L^2(\mathbb{R}^d)$ and the mass conservation directly yields global existence. However, we overcome this problem by applying the deterministic and stochastic Strichartz estimates once again. The strategy of the proofs presented below is essentially due to de Bouard and Debussche, [41], Proposition 4.1. We start with global bounds for the mass of the solutions u_n for $n \in \mathbb{N}$ in the case of linear noise.

Proposition 3.15. *Let $\alpha \in (1, 1+\frac{4}{d})$, $B_1 = 0$ and $\mu_1 = 0$. Let $n \in \mathbb{N}$ and u_n be the global mild solution of* (3.7) *from Proposition 3.7. Then, we have*

$$\|u_n(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 - 2\int_0^t \operatorname{Re}\left(u_n(s), \mathrm{i}B_2 u_n(s)\mathrm{d}W(s)\right)_{L^2}, \qquad t \in [0,T], \tag{3.33}$$

almost surely. Moreover, for all $p \in [1,\infty)$, there is a constant $D_p = D_p(T, \|u_0\|_{L^2}) > 0$ independent of n with

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|u_n(t)\|_{L^2}^p\Big] \le D_p. \tag{3.34}$$

Note that the estimate (3.47) for $p = 2$ previously occurred in [11] and in the special case of Stratonovich noise with selfadjoint operators B_m, $m \in \mathbb{N}$, (3.47) simplifies to $\|u_n(t)\|_{L^2} = \|u_0\|_{L^2}$ almost surely for all $t \in [0,T]$. This generalizes the L^2-conservation of the NLS, see [88], equation (6.2), to the stochastic setting.

Proof. Step 1. To prove (3.33), we set $M = \mathbb{R}^d$, $A = \Delta$, $F(t,u) := \lambda\varphi_n(u,t)|u|^{\alpha-1}u$ for $u \in L^{\alpha+1}(\mathbb{R}^d)$ and $t \in [0,T]$ and obtain

$$\|F(t,u)\|_{L^{\frac{\alpha+1}{\alpha}}} \lesssim \|u\|_{L^{\alpha+1}}^\alpha, \qquad \operatorname{Re}\langle \mathrm{i}u, F(t,u)\rangle_{L^{\alpha+1}, L^{\frac{\alpha+1}{\alpha}}} = 0, \qquad u \in L^{\alpha+1}.$$

By construction, we have $u_n \in C([0,T], L^2(\mathbb{R}^d)) \cap L^q(0,T;L^{\alpha+1}(\mathbb{R}^d))$ almost surely and $q > \alpha+1$, since α is in the subcritical range $(1, 1+\frac{4}{d})$. Hence, Corollary 2.9 yields (3.33).

Step 2. First, let $p \in [2,\infty)$ and fix $t \in [0,T]$. Applying the $L^{\frac{p}{2}}(\Omega, C([0,t]))$-norm to (3.33), we get

$$\left(\mathbb{E}\Big[\sup_{s\in[0,t]} \|u_n(s)\|_{L^2}^p\Big]\right)^{\frac{2}{p}} \le \|u_0\|_{L^2}^2 + 2\left(\mathbb{E}\left[\sup_{s\in[0,t]}\left|\int_0^s \left(u_n(r), \mathrm{i}Bu_n(r)\mathrm{d}W(r)\right)_{L^2}\right|^{\frac{p}{2}}\right]\right)^{\frac{2}{p}}$$

The second term can be estimated by the Burkholder-Davis-Gundy inequality

$$\mathbb{E}\left[\sup_{s\in[0,t]}\left|\int_0^s \left(u_n(r), \mathrm{i}Bu_n(r)\mathrm{d}W(r)\right)_{L^2}\right|^{\frac{p}{2}}\right] \lesssim \mathbb{E}\left[\left(\sum_{m=1}^\infty \int_0^t |\left(u_n(r), \mathrm{i}B_m u_n(r)\right)_{L^2}|^2 \mathrm{d}r\right)^{\frac{p}{4}}\right]$$

$$\leq \mathbb{E}\left[\left(\sum_{m=1}^{\infty}\|B_m\|_{\mathcal{L}(L^2)}^2\int_0^t\|u_n(r)\|_{L^2}^4\mathrm{d}r\right)^{\frac{p}{4}}\right]\lesssim \mathbb{E}\left[\left(\int_0^t\|u_n(r)\|_{L^2}^4\mathrm{d}r\right)^{\frac{p}{4}}\right]$$

such that we obtain

$$\left(\mathbb{E}\Big[\sup_{s\in[0,t]}\|u_n(s)\|_{L^2}^p\Big]\right)^{\frac{2}{p}}\lesssim\|u_0\|_{L^2}^2+\varepsilon\left(\mathbb{E}\Big[\sup_{s\in[0,t]}\|u_n(s)\|_{L^2}^p\Big]\right)^{\frac{2}{p}}$$
$$+\frac{1}{4\varepsilon}\int_0^t\left(\mathbb{E}\Big[\sup_{r\in[0,s]}\|u_n(r)\|_{L^2}^p\Big]\right)^{\frac{2}{p}}\mathrm{d}s,$$

for $\varepsilon>0$ by an application of Lemma 2.11 with $Y(s)=\|u_n(s)\|_{L^2}^2$. If we choose $\varepsilon>0$ small enough, the last estimate implies

$$\left(\mathbb{E}\Big[\sup_{s\in[0,t]}\|u_n(s)\|_{L^2}^p\Big]\right)^{\frac{2}{p}}\lesssim\|u_0\|_{L^2}^2+\int_0^t\left(\mathbb{E}\Big[\sup_{r\in[0,s]}\|u_n(r)\|_{L^2}^p\Big]\right)^{\frac{2}{p}}\mathrm{d}s,$$

and by Gronwall's Lemma, there is a $C>0$ with

$$\left(\mathbb{E}\Big[\sup_{s\in[0,t]}\|u_n(s)\|_{L^2}^p\Big]\right)^{\frac{2}{p}}\leq C\|u_0\|_{L^2}^2e^{Ct},\qquad t\in[0,T].$$

For $p\in[1,2)$, the assertion is an immediate consequence of Hölder's inequality. □

Unfortunately, the Gronwall argument from the previous Proposition cannot be transfered to nonlinear noise. For real-valued coefficients, however, this is not necessary since we even get conservation of the mass of u_n, $n\in\mathbb{N}$. At this point, we employ that the approximated equation has Stratonovich structure due to the use of squared cut-off functions in the correction term.

Proposition 3.16. *Let $\alpha\in(1,1+\frac{4}{d})$, $\gamma\in(1,1+\frac{2}{d})$, $\mu_2=0$, $B_2=0$ and $e_m\in L^\infty(\mathbb{R}^d,\mathbb{R})$ for each $m\in\mathbb{N}$ with $\sum_{m=1}^\infty\|e_m\|_{L^\infty}^2<\infty$. Let $n\in\mathbb{N}$ and u_n be the global mild solution of* (3.7) *from Proposition 3.7. Then, we have*

$$\|u_n(t)\|_{L^2}=\|u_0\|_{L^2}\tag{3.35}$$

almost surely for all $t\in[0,T]$.

Proof. As in the previous proof, we set $M=\mathbb{R}^d$, $A=\Delta$ and $F(t,u):=\lambda\varphi_n(u,t)|u|^{\alpha-1}u$ for $u\in L^{\alpha+1}(\mathbb{R}^d)$. Let $g(t,x):=\varphi_n(u_n,t)x^{\frac{\gamma-1}{2}}$. With these definitions, u_n satisfies (2.22) and therefore, we have

$$\|u_n(t)\|_{L^2}^2=\|u_0\|_{L^2}^2-2\int_0^t\operatorname{Re}\big(u_n(s),\mathrm{i}B(s)\mathrm{d}W(s)\big)_{L^2},$$

where $B(s)f_m:=e_m\varphi_n(u_n,s)|u_n(s)|^{\gamma-1}u_n(s)$ for $m\in\mathbb{N}$ and $s\in[0,T]$. Finally, the stochastic integral cancels due to

$$\operatorname{Re}\big(u_n(s),\mathrm{i}B(s)f_m\big)_{L^2}=\operatorname{Im}\int_{\mathbb{R}^d}e_m\varphi_n(u_n,s)|u_n(s)|^{\gamma+1}\mathrm{d}s=0,\qquad m\in\mathbb{N},\quad s\in[0,T].$$

□

Before we continue with the proof of global existence for linear noise, we introduce the abbreviation

$$Y_r := L^q(0,r;L^{\alpha+1}(\mathbb{R}^d)), \qquad r>0,$$

which will be frequently used below.

Theorem 3.17. *Let $\alpha \in (1, 1+\frac{4}{d})$, $B_1 = 0$ and $\mu_1 = 0$. Let $(u_n)_{n\in\mathbb{N}}$ be the sequence of global mild solutions to* (3.7) *from Proposition 3.7 and $(\tau_n)_{n\in\mathbb{N}}$ be the sequence of stopping times from* (3.31). *Then, we have*

$$\mathbb{P}\Big(\bigcup_{n\in\mathbb{N}}\{\tau_n = T\}\Big) = 1.$$

In particular, $\tau_\infty = T$ almost surely and the pair (u,T) is a global strong solution of (3.30) *in $L^2(\mathbb{R}^d)$.*

Proof. Step 1. We want to prove that there is a constant $C = C(\|u_0\|_{L^2}, T) > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\|u_n\|_{Y_T} \le C. \tag{3.36}$$

We fix $n\in\mathbb{N}$ and recall that u_n has the representation

$$u_n = e^{\mathrm{i}\cdot\Delta}u_0 + K_{det}^n u_n + K_{Strat}^n u_n + K_{Stoch}^n u_n \quad \text{in } M_{\mathbb{F}}^p(\Omega, E_T). \tag{3.37}$$

We fix a path $\omega\in\Omega$ and $\sigma_n(\omega)\in[0,T]$ to be chosen later. Let $\delta := 1+\frac{d}{4}(1-\alpha)$. Then, we apply the deterministic Strichartz inequalities from Proposition 2.14 to estimate K_{det} and K_{Strat} (compare the proof of Proposition 3.7) and obtain

$$\begin{aligned}\|u_n\|_{Y_{\sigma_n}} &\le C\|u_0\|_{L^2} + C\sigma_n^\delta \|u_n\|_{Y_{\sigma_n}}^\alpha + C\|u_n\|_{L^1(0,\sigma_n,L^2)}\sum_{m=1}^\infty \|B_m\|_{\mathcal{L}(L^2)}^2 + \|K_{Stoch}u_n\|_{Y_{\sigma_n}}\\ &\le K_n + C\sigma_n^\delta\|u_n\|_{Y_{\sigma_n}}^\alpha\end{aligned} \tag{3.38}$$

where K_n is defined by

$$K_n := C\|u_n\|_{L^\infty(0,T;L^2)}\left(1+T\sum_{m=1}^\infty \|B_m\|_{\mathcal{L}(L^2)}^2\right) + \|K_{Stoch}u_n\|_{Y_T}.$$

W.l.o.g we assume $u_0\neq 0$ and thus $K_n>0$. We conclude

$$\frac{\|u_n\|_{Y_{\sigma_n}}}{K_n} \le 1 + C\sigma_n^\delta K_n^{\alpha-1}\left(\frac{\|u_n\|_{Y_{\sigma_n}}}{K_n}\right)^\alpha$$

Now, the following fact

$$\forall x\ge 0\,\exists c_1\le 2,\, c_2>c_1: \quad x\le 1+\frac{x^\alpha}{2^{\alpha+1}} \quad\Rightarrow\quad x\le c_1 \quad \text{or} \quad x\ge c_2 \tag{3.39}$$

from elementary calculus yields

$$\|u_n\|_{Y_{\sigma_n}} \le c_1 K_n \le 2K_n,$$

if we choose σ_n according to $C\sigma_n^\delta K_n^{\alpha-1}\le \frac{1}{2^{\alpha+1}}$. This condition is fulfilled by

$$\sigma_n = C^{-\frac{1}{\delta}}\left(2^{\alpha+1}K_n^{\alpha-1}\right)^{-\frac{1}{\delta}} \wedge T.$$

Note that the second alternative in (3.39) can be excluded because of $\|u_n\|_{Y_0} = 0$ and the continuity of the map $t \mapsto \|u_n\|_{Y_t}$. Next, we decompose $\Omega = \Omega_1 \cup \Omega_2$ with

$$\Omega_1 := \left\{C^{-\frac{1}{\delta}}\left(2^{\alpha+1}K_n^{\alpha-1}\right)^{-\frac{1}{\delta}} < T\right\}, \qquad \Omega_2 := \left\{C^{-\frac{1}{\delta}}\left(2^{\alpha+1}K_n^{\alpha-1}\right)^{-\frac{1}{\delta}} \geq T\right\}.$$

Fix $\omega \in \Omega_1$ and define $N := \lfloor \frac{T}{\sigma_n} \rfloor$. Using the abbreviation

$$Y_j := L^q(j\sigma_n, (j+1)\sigma_n; L^{\alpha+1}(\mathbb{R}^d)), \qquad j = 0, \dots, N,$$

we get

$$\begin{aligned}\|u_n\|_{Y_j} &\leq C\|u_n(j\sigma_n)\|_{L^2} + C\sigma_n^{\delta}\|u_n\|_{Y_j}^{\alpha} \\ &\quad + CT\|u_n\|_{L^\infty(j\sigma_n,(j+1)\sigma_n;L^2)}\sum_{m=1}^{\infty}\|B_m\|_{\mathcal{L}(L^2)}^2 + \|K_{Stoch}u_n\|_{Y_j} \\ &\leq K_n + C\sigma_n^{\delta}\|u_n\|_{Y_j}^{\alpha}\end{aligned}$$

for all $j = 0, \dots, N$ by analogous estimates as in (3.38) and thus again $\|u_n\|_{Y_j} \leq 2K_n$. We conclude

$$\|u_n\|_{Y_T} \leq \sum_{j=0}^{N}\|u_n\|_{Y_j} \leq 2\,(N+1)\,K_n \leq 2\left(\frac{T}{\sigma_n}+1\right)K_n \leq 2K_n + 2^{\frac{\alpha+1}{\delta}+1}C^{\frac{1}{\delta}}TK_n^{\frac{\alpha-1}{\delta}+1}. \quad (3.40)$$

Since we have $\|u_n\|_{Y_T} \leq 2K_n$ on Ω_2, the estimate (3.40) holds almost surely. Then, we integrate over Ω to obtain

$$\|u_n\|_{L^1(\Omega,Y_T)} \lesssim 2\mathbb{E}\Big[K_n\Big] + 2T\mathbb{E}\Big[K_n^{\frac{\alpha-1}{\delta}+1}\Big].$$

By Corollary 2.23 and Proposition 3.15, we get for each $p \in (1,\infty)$

$$\mathbb{E}\Big[\|K_{Stoch}u_n\|_{Y_T}^p\Big] \lesssim \mathbb{E}\Big[\|u_n\|_{L^2(0,T;L^2)}^p\Big] \leq T^{\frac{p}{2}}\mathbb{E}\Big[\|u_n\|_{L^\infty(0,T;L^2)}^{\frac{\alpha-1}{\delta}+1}\Big] \leq T^{\frac{p}{2}}D_p. \quad (3.41)$$

This yields

$$\begin{aligned}\mathbb{E}\left[K_n\right] &\leq C\mathbb{E}\|u_n\|_{L^\infty(0,T,L^2)}\left(1 + T\sum_{m=1}^{\infty}\|B_m\|_{\mathcal{L}(L^2)}^2\right) + \left(\mathbb{E}\|K_{Stoch}u_n\|_{Y_T}^2\right)^{\frac{1}{2}} \\ &\leq CD_2\left(1 + T\sum_{m=1}^{\infty}\|B_m\|_{\mathcal{L}(L^2)}^2\right) + T^{\frac{1}{2}}D_2^{\frac{1}{2}} \qquad (3.42)\end{aligned}$$

for the first term, whereas for the second one, we write

$$\mathbb{E}\Big[K_n^{\frac{\alpha-1}{\delta}+1}\Big] \lesssim \mathbb{E}\Big[\|u_n\|_{L^\infty(0,T;L^2)}^{\frac{\alpha-1}{\delta}+1}\Big] + \mathbb{E}\Big[\|K_{Stoch}u_n\|_{Y_T}^{\frac{\alpha-1}{\delta}+1}\Big] \lesssim D_{\frac{\alpha-1}{\delta}+1}\big(1 + T^{\frac{\alpha-1}{2\delta}+\frac{1}{2}}\big).$$

Hence, we have proved

$$\sup_{n\in\mathbb{N}}\mathbb{E}\|u_n\|_{Y_T} \leq C(\|u_0\|_{L^2}, T, \alpha, d).$$

Step 2. Recall $\tau_\infty := \sup_{n\in\mathbb{N}}\tau_n$. The exponent γ appears in τ_n as well as in the truncation function φ_n, but it is arbitrary since the nonlinear part of the noise vanishes due to $\mu_1 = B_1 = 0$. It is pragmatic to set $\gamma = \frac{\alpha+1}{2}$ in order to get

$$\tau_n = \inf\{t \in [0,T] : 2\|u_n\|_{Y_t} \geq n\} \wedge T$$

and by the Tschebyscheff inequality and (3.36)

$$\mathbb{P}(\tau_n = T) = \mathbb{P}\left(\|u_n\|_{Y_{\tau_n}} \leq \frac{n}{2}\right) \geq 1 - \frac{2C}{n}.$$

Using the continuity of the measure, we conclude

$$\mathbb{P}(\tau_\infty = T) \geq \mathbb{P}\Big(\bigcup_{n\in\mathbb{N}} \{\tau_n = T\}\Big) = \lim_{n\to\infty} \mathbb{P}(\tau_n = T) = 1.$$

□

In the previous result, we used the linearity of the noise to estimate the stochastic convolution via the stochastic Strichartz estimate from Corollary 2.23 and the mass estimate from Proposition 3.16. To cover nonlinear noise, we combine the techniques we have seen above with an interpolation argument between $L^\infty(0,T;L^2(\mathbb{R}^d))$ and $L^q(0,T;L^{\alpha+1}(\mathbb{R}^d))$.

Theorem 3.18. *Let $\alpha \in (1, 1+\frac{4}{d})$, $\mu_2 = 0$, $B_2 = 0$ and $e_m \in L^\infty(\mathbb{R}^d,\mathbb{R})$ for each $m \in \mathbb{N}$ with $\sum_{m=1}^\infty \|e_m\|_{L^\infty}^2 < \infty$. Let $(u_n)_{n\in\mathbb{N}}$ be the sequence of global mild solutions of* (3.7) *from Proposition 3.7. Suppose that γ satisfies*

$$1 < \gamma < \frac{\alpha-1}{\alpha+1}\frac{4+d(1-\alpha)}{4\alpha+d(1-\alpha)} + 1. \tag{3.43}$$

Then, we have

$$\mathbb{P}\Big(\bigcup_{n\in\mathbb{N}} \{\tau_n = T\}\Big) = 1.$$

In particular, $\tau_\infty = T$ almost surely and the pair (u,T) is a unique global strong solution of (3.1).

In Figure 3.2, we illustrate the condition (3.43) by plotting the set

$$\Big\{(\alpha,\gamma) \in (1,\infty)^2 : \alpha \in \big(1, 1+\frac{4}{d}\big) \text{ and (3.43) holds}\Big\}$$

for the dimensions $d = 1, 2, 3$. We observe that the condition is fairly restrictive, in particular close to $\alpha = 1$ and $\alpha = 1 + \frac{4}{d}$.

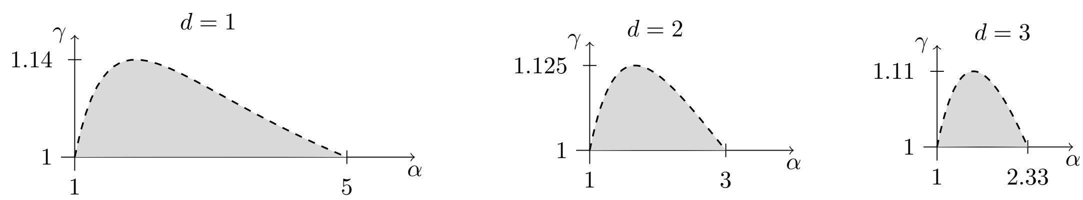

Figure 3.2.: Values of α and γ leading to global wellposedness in $L^2(\mathbb{R}^d)$.

Let us proceed with the proof of Theorem 3.18.

Proof of Theorem 3.18. Step 1. For all $\alpha \in (1, 1 + \frac{4}{d})$, we have

$$1 < \gamma < \frac{\alpha - 1}{\alpha + 1} \frac{4 + d(1 - \alpha)}{4\alpha + d(1 - \alpha)} + 1 < \frac{\alpha - 1}{\alpha + 1} + 1 < \frac{\alpha - 1}{2} < \frac{\alpha + 1}{2}.$$

Thus, the distinction of the cases in (3.12) vanishes and we have $Y_r := L^q(0, r; L^{\alpha+1}(\mathbb{R}^d))$. As in the previous proof, we want to prove that there is a uniform constant $C > 0$ such that

$$\sup_{n \in \mathbb{N}} \mathbb{E}\|u_n\|_{Y_T} \leq C. \tag{3.44}$$

Let us fix $n \in \mathbb{N}$ as well as

$$\delta := 1 + \frac{d}{4}(1 - \alpha), \qquad \tilde{\delta} := 1 + \frac{d}{2}(1 - \gamma), \qquad \theta = \frac{\alpha + 1 - 2\gamma}{(\alpha - 1)\gamma}.$$

Here, θ is chosen according to $\frac{1}{2\gamma} = \frac{\theta}{2} + \frac{1-\theta}{\alpha+1}$. In particular, we have $\theta \in (0, 1)$ and from Lemma 2.13 and Proposition 3.16, we infer

$$\|u_n\|_{L^{\tilde{q}}(0,\sigma_n,L^{2\gamma})} \leq \|u_n\|^{\theta}_{L^\infty(0,\sigma_n;L^2)} \|u_n\|^{1-\theta}_{L^q(0,\sigma_n;L^{\alpha+1})} \leq \|u_0\|^{\theta}_{L^2} \|u_n\|^{1-\theta}_{L^q(0,\sigma_n;L^{\alpha+1})} \tag{3.45}$$

almost surely for all $t \in [0, T]$. As in the previous proof, we fix $\omega \in \Omega$ and $\sigma_n(\omega) \in (0, T]$ and use the fixed point representation (3.37) of u_n and Strichartz estimates to deduce

$$\begin{aligned} \|u_n\|_{Y_{\sigma_n}} &\leq C\|u_0\|_{L^2} + C\sigma_n^{\delta}\|u_n\|^{\alpha}_{Y_{\sigma_n}} + C\sigma_n^{\tilde{\delta}}\|u_n\|^{2\gamma-1}_{L^{\tilde{q}}(0,\sigma_n,L^{2\gamma})} \sum_{m=1}^{\infty} \|e_m\|^2_{L^\infty} + \|K_{Stoch}u_n\|_{Y_{\sigma_n}} \\ &\leq C\|u_0\|_{L^2} + C\sigma_n^{\delta}\|u_n\|^{\alpha}_{Y_{\sigma_n}} + C\sigma_n^{\tilde{\delta}}\|u_0\|^{(2\gamma-1)\theta}_{L^2}\|u_n\|^{(2\gamma-1)(1-\theta)}_{Y_{\sigma_n}} \sum_{m=1}^{\infty} \|e_m\|^2_{L^\infty} \\ &\quad + \|K_{Stoch}u_n\|_{Y_{\sigma_n}}. \end{aligned} \tag{3.46}$$

We denote

$$\begin{aligned} K_n :=& C\|u_0\|_{L^2} + CT^{\tilde{\delta}}\|u_0\|^{(2\gamma-1)\theta}_{L^2} \sum_{m=1}^{\infty} \|e_m\|^2_{L^\infty} + \|K_{Stoch}u_n\|_{Y_T}, \\ C_1 :=& C\left[1 + T^{\tilde{\delta}-\delta}\|u_0\|^{(2\gamma-1)\theta}_{L^2} \sum_{m=1}^{\infty} \|e_m\|^2_{L^\infty}\right]. \end{aligned}$$

Due to $1 < \gamma < \frac{\alpha+1}{2}$ and $\theta \in (0, 1)$ we have $(2\gamma - 1)(1 - \theta) < \alpha$, which leads to

$$\|u_n\|^{(2\gamma-1)(1-\theta)}_{Y_{\sigma_n}} \leq \|u_n\|^{\alpha}_{Y_{\sigma_n}} + 1.$$

Using this estimate and $\tilde{\delta} > \delta$ in (3.46), we deduce

$$\begin{aligned} \|u_n\|_{Y_{\sigma_n}} &\leq C\|u_0\|_{L^2} + C\sigma_n^{\delta}\left[1 + \sigma_n^{\tilde{\delta}-\delta}\|u_0\|^{(2\gamma-1)\theta}_{L^2} \sum_{m=1}^{\infty} \|e_m\|^2_{L^\infty}\right]\|u_n\|^{\alpha}_{Y_{\sigma_n}} \\ &\qquad + C\sigma_n^{\tilde{\delta}}\|u_0\|^{(2\gamma-1)\theta}_{L^2} \sum_{m=1}^{\infty} \|e_m\|^2_{L^\infty} + \|K_{Stoch}u_n\|_{Y_{\sigma_n}} \\ &\leq K_n + C_1\sigma_n^{\delta}\|u_n\|^{\alpha}_{Y_{\sigma_n}}. \end{aligned}$$

We choose

$$\sigma_n = C_1^{-\frac{1}{\delta}} \left(2^{\alpha+1} K_n^{\alpha-1}\right)^{-\frac{1}{\delta}} \wedge T,$$

which leads to $\|u_n\|_{Y_{\sigma_n}} \leq 2K_n$ by analogous arguments as in Theorem 3.17. Iterating this argument as in (3.40), we end up with

$$\|u_n\|_{Y_T} \leq 2\left(\frac{T}{\sigma_n}+1\right) K_n \leq 2K_n + 2^{\frac{\alpha+1}{\delta}+1} C_1^{\frac{1}{\delta}} T K_n^{\frac{\alpha-1}{\delta}+1} \qquad \text{a.s.}$$

We set $p := \frac{\alpha-1}{\delta}+1$ and integrate over Ω to obtain

$$\begin{aligned}\|u_n\|_{L^1(\Omega,Y_T)} &\leq 2\mathbb{E}\Big[K_n\Big] + 2^{\frac{\alpha+1}{\delta}+1} C_1^{\frac{1}{\delta}} T \mathbb{E}\Big[K_n^{\frac{\alpha-1}{\delta}+1}\Big] \lesssim 1 + \mathbb{E}\|K_{Stoch} u_n\|_{Y_T} + \mathbb{E}\|K_{Stoch} u_n\|_{Y_T}^p \\ &\leq 1 + \|K_{Stoch} u_n\|_{L^p(\Omega,Y_T)} + \|K_{Stoch} u_n\|_{L^p(\Omega,Y_T)}^p.\end{aligned}$$

Now, we choose $\tilde{p} \in (p\gamma, \infty)$ according to $\frac{1}{p\gamma} = \frac{\theta}{\tilde{p}} + \frac{1-\theta}{1}$. Using Lemma 2.13 and Proposition 3.16, we estimate

$$\begin{aligned}\|K_{Stoch} u_n\|_{L^p(\Omega,Y_T)} &\lesssim \|\varphi_n(u,s)|u_n|^{\gamma-1} u_n\|_{L^p(\Omega,L^2(0,T;L^2))} \leq T^{\frac{\tilde{\delta}}{2}} \|u_n\|_{L^{p\gamma}(\Omega,L^{\tilde{q}}(0,T;L^{2\gamma}))}^{\gamma} \\ &\leq T^{\frac{\tilde{\delta}}{2}} \|u_n\|_{L^{\tilde{p}}(\Omega,L^\infty(0,T;L^2))}^{\gamma\theta} \|u_n\|_{L^1(\Omega;Y_T)}^{\gamma(1-\theta)} = T^{\frac{\tilde{\delta}}{2}} \|u_0\|_{L^2}^{\gamma\theta} \|u_n\|_{L^1(\Omega;Y_T)}^{\gamma(1-\theta)} \\ &\lesssim \|u_n\|_{L^1(\Omega;Y_T)}^{\gamma(1-\theta)}.\end{aligned}$$

Finally, we end up with

$$\|u_n\|_{L^1(\Omega,Y_T)} \lesssim 1 + \|u_n\|_{L^1(\Omega;Y_T)}^{\gamma(1-\theta)} + \|u_n\|_{L^1(\Omega;Y_T)}^{p\gamma(1-\theta)} \lesssim 1 + \|u_n\|_{L^1(\Omega;Y_T)}^{p\gamma(1-\theta)}.$$

In particular, there is $C = C(\|u_0\|_{L^2}, \|e_m\|_{\ell^2(\mathbb{N},L^\infty)}, T, \alpha, \gamma) > 0$ with

$$\sup_{n\in\mathbb{N}} \|u_n\|_{L^1(\Omega,Y_T)} \leq C, \qquad n \in \mathbb{N},$$

if we have

$$p\gamma(1-\theta) < 1 \qquad \Leftrightarrow \qquad \gamma < \frac{\alpha-1}{\alpha+1}\frac{4+d(1-\alpha)}{4\alpha+d(1-\alpha)} + 1.$$

Step 2. Using the result of the first step and taking the expectation in (3.45), we obtain

$$\|u_n\|_{L^1(\Omega,L^{\tilde{q}}(0,T;L^{2\gamma}))} \leq \|u_0\|_{L^2}^{\theta} C^{1-\theta}$$

and the definition of τ_n followed by the Tschebyscheff inequality and (3.44) yield

$$\begin{aligned}\mathbb{P}(\tau_n = T) &= \mathbb{P}\left(\|u_n\|_{Y_T} + \|u_n\|_{L^{\tilde{q}}(0,T;L^{2\gamma})} \leq n\right) \geq 1 - \frac{\|u_n\|_{L^1(\Omega,Y_T)} + \|u_n\|_{L^1(\Omega,L^{\tilde{q}}(0,T;L^{2\gamma}))}}{n} \\ &\geq 1 - \frac{C + \|u_0\|_{L^2}^{\theta} C^{1-\theta}}{n}.\end{aligned}$$

By the continuity of the measure, we conclude

$$\mathbb{P}(\tau_\infty = T) \geq \mathbb{P}\left(\bigcup_{n\in\mathbb{N}} \{\tau_n = T\}\right) = \lim_{n\to\infty} \mathbb{P}(\tau_n = T) = 1.$$

□

Let us comment on the critical case $\alpha = 1 + \frac{4}{d}$ which has been excluded for the global existence results in Theorem 3.17 and Theorem 3.18. Our proof cannot be transfered to this setting since we have $\delta = 0$ and the strategy crucially relies on $\delta > 0$ to apply (3.39). But global existence for general L^2-initial data cannot be expected in this case, anyway, since there are blow-up examples in the deterministic setting for the focusing nonlinearity, see [96].

However, it is easily possible to apply similar arguments to prove global existence of the local solutions from Section 3.3 for initial values in $H^1(\mathbb{R}^d)$, as soon as one has an analogue of Proposition 3.15 in $H^1(\mathbb{R}^d)$, i.e. a uniform estimate

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|u_n(t)\|_{H^1}^p\Big] \le D_p. \tag{3.47}$$

for the solutions $(u_n)_{n\in\mathbb{N}}$ from the Propositions 3.21 and 3.22, respectively. However, we decided to skip this part since it is rather extensive to rigorously justify evolution formulae for the energy. Instead, we refer to [12], Theorem 3.1 for a formula of this type.

3.3. Local existence and uniqueness in $H^1(\mathbb{R}^d)$

In this section, we prove Theorem 3.3 by a similar strategy as in the L^2-case based on the truncation of the nonlinear terms. In this way, we overcome the problem that Strichartz estimates do not gain integrability in Ω. Once we have the solutions of the truncated problems, existence and uniqueness can be shown analogously as in $L^2(\mathbb{R}^d)$.

Throughout the whole section, we consider a fixed cylindrical Wiener process W on a real Hilbert space Y with ONB $(f_m)_{m\in\mathbb{N}}$. We assume

$$\sum_{m=1}^\infty \left(\|e_m\|_{L^\infty} + \|e_m\|_F\right)^2 < \infty, \qquad F := \begin{cases} L^d(\mathbb{R}^d), & d \ge 3, \\ L^{2+\varepsilon}(\mathbb{R}^d), & d = 2, \\ L^2(\mathbb{R}^d), & d = 1. \end{cases}$$

Thus $B(u)f_m := e_m u$ for $u \in H^1(\mathbb{R}^d)$ and $m \in \mathbb{N}$ defines a linear bounded operator $B : H^1(\mathbb{R}^d) \to \operatorname{HS}(Y, H^1(\mathbb{R}^d))$. Moreover, we denote

$$\mu := -\frac{1}{2}\sum_{m=1}^\infty |e_m|^2.$$

Comparing Theorem 3.2 a) and Theorem 3.3, we observe that there is a gap in the range of exponents α and γ occurring in the H^1-setting. As we will see below, this is due to technical difficulties in extending the fixed point argument from the deterministic case to the stochastic setting via the truncation argument from Section 3.1. For the deterministic NLS, local well-posedness for all energy-subcritical exponents $\alpha \in (1, 1 + \frac{4}{(d-2)_+})$ is usually proved by a fixed point argument in the ball

$$\tilde{X}_{r,R} := \left\{u \in L^\infty(0,r; H^1(\mathbb{R}^d)) \cap L^q(0,r; W^{1,\alpha+1}(\mathbb{R}^d)) : \quad \|u\|_{L^\infty H^1 \cap L^q W^{1,\alpha+1}} \le R\right\}$$

with $q \in (2, \infty)$ such that $(\alpha + 1, q)$ is a Strichartz pair. As a consequence of the Banach-Alaoglu Theorem, the ball $\tilde{X}_R$ is complete with respect to the metric induced by the norm in $L^\infty(0,T; L^2(\mathbb{R}^d)) \cap L^q(0,T; L^{\alpha+1}(\mathbb{R}^d))$ which significantly simplifies the contraction estimate. For further details on this argument which employs the Strichartz estimates from Proposition

2.14 and the Sobolev embedding $H^1(\mathbb{R}^d) \hookrightarrow L^{\alpha+1}(M)$, we refer to the monographs [114], [36] and [88].

To prepare a similar reasoning for the stochastic NLS, we introduce some notations and show that a stochastic version of the ball $\tilde{X}_{r,R}$ is also a complete metric space. We choose $\alpha \in (1, 1 + \frac{4}{(d-2)_+})$ and $\gamma \in (1, 1 + \frac{2}{(d-2)_+})$, which ensures

$$H^1(\mathbb{R}^d) \hookrightarrow L^{\alpha+1}(\mathbb{R}^d), \qquad H^1(\mathbb{R}^d) \hookrightarrow L^{2\gamma}(\mathbb{R}^d).$$

Moreover, there are $q, \tilde{q} \in (2, \infty)$ such that $(\alpha+1, q)$ and $(2\gamma, \tilde{q})$ are Strichartz pairs. We set

$$Y^k_{[a,b]} := \begin{cases} L^q(a, b; W^{k,\alpha+1}(\mathbb{R}^d)), & \alpha + 1 \geq 2\gamma, \\ L^{\tilde{q}}(a, b; W^{k,2\gamma}(\mathbb{R}^d)), & \alpha + 1 < 2\gamma, \end{cases}$$

and

$$E^k_{[a,b]} := Y^k_{[a,b]} \cap L^\infty(a, b; H^k(\mathbb{R}^d)), \qquad F_{[a,b]} := Y^1_{[a,b]} \cap C([a, b], H^1(\mathbb{R}^d))$$

for $0 \leq a < b \leq T$ and $k = 0, 1$. Let $r > 0$. We abbreviate $Y^k_r := Y^k_{[0,r]}$, $E^k_r := E^k_{[0,r]}$ and $F_r := F_{[0,r]}$. For $p \in (1, \infty)$, we denote by $\mathbb{M}^p_{\mathbb{F}}(\Omega, E^k_{[0,r]})$ the space of predictable processes $u : [0, r] \times \Omega \to W^{k,\alpha+1}(\mathbb{R}^d) \cap W^{k,2\gamma}(\mathbb{R}^d)$ with

$$\|u\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega, E^k_{[0,r]})} := \max\left\{\|u\|_{L^p(\Omega, L^\infty(0,r;H^k))}, \|u\|_{L^p(\Omega, Y^k_r)}\right\} < \infty.$$

Moreover, we use the notation $\mathbb{M}^p_{\mathbb{F}}(\Omega, F_r)$ for the space of predictable processes $u : [0, r] \times \Omega \to W^{1,\alpha+1}(\mathbb{R}^d) \cap W^{1,2\gamma}(\mathbb{R}^d)$ such that

$$\|u\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega, F_r)} := \max\left\{\|u\|_{L^p(\Omega, L^\infty(0,r;H^1))}, \|u\|_{L^p(\Omega, Y^1_r)}\right\} < \infty.$$

Similarly to Lemma 3.4, we deduce the embedding

$$E^k_{[a,b]} \hookrightarrow L^q(a, b; W^{k,\alpha+1}(\mathbb{R}^d)) \cap L^{\tilde{q}}(a, b; W^{k,2\gamma}(\mathbb{R}^d)), \qquad k = 0, 1. \tag{3.48}$$

Lemma 3.19. *Let $R > 0$ and $r > 0$. Then, the set*

$$X_{r,R} := \left\{u \in \mathbb{M}^p_{\mathbb{F}}(\Omega, E^1_r) : \|u\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega, E^1_r)} \leq R\right\}$$

equipped with $d(u, v) := \|u - v\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega, E^0_r)}$ for $u, v \in X_{r,R}$ is a complete metric space.

Proof. To show that $X_{r,R}$ is complete, let $(u_n)_{n\in\mathbb{N}} \subset X_{r,R}$ be a Cauchy sequence. As $\mathbb{M}^p_{\mathbb{F}}(\Omega, E^0_r)$ is a Banach space, there is $u \in \mathbb{M}^p_{\mathbb{F}}(\Omega, E^0_r)$ with $u_n \to u$ for $n \to \infty$. We obtain the assertion, if we show $\|u\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega, E^1_r)} \leq R$.
The sequence $(u_n)_{n\in\mathbb{N}}$ is contained in the balls

$$\begin{aligned} B_{L^p(\Omega, L^\infty(0,r,H^1))} &= \{v \in L^p(\Omega, L^\infty(0, r, H^1)) : \|v\|_{L^p(\Omega, L^\infty(0,r,H^1))} \leq R\}, \\ B_{L^p(\Omega, Y^1_r)} &= \{v \in L^p(\Omega, Y^1_r) : \|v\|_{L^p(\Omega, Y^1_r)} \leq R\}. \end{aligned}$$

Hence, the Banach-Alaoglu Theorem implies that there are a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and elements $v \in B_{L^p(\Omega, L^\infty(0,r,H^1))}$ and $w \in B_{L^p(\Omega, Y^1_r)}$ with

$$u_{n_k} \rightharpoonup^* v \quad \text{in} \quad L^p(\Omega, L^\infty(0, r, H^1(\mathbb{R}^d))), \qquad u_{n_k} \rightharpoonup w \quad \text{in} \quad L^p(\Omega, Y^1_r) \tag{3.49}$$

for $k \to \infty$. By the embeddings

$$L^{p'}(\Omega, (Y_r^0)^*) \hookrightarrow L^{p'}(\Omega, (Y_r^1)^*)$$

and

$$L^{p'}(\Omega, L^1(0, r; L^2(\mathbb{R}^d))) \hookrightarrow L^{p'}(\Omega, L^1(0, r; H^{-1}(\mathbb{R}^d)))$$

we conclude

$$u_{n_k} \rightharpoonup^* v \quad \text{in} \quad L^p(\Omega, L^\infty(0, r; L^2(\mathbb{R}^d))), \quad u_{n_k} \rightharpoonup w \quad \text{in} \quad L^p(\Omega, Y_r^0).$$

By the uniqueness of limits, we get $u = v = w$. Therefore, we have $u \in \mathbb{M}_{\mathbb{F}}^p(\Omega, E_r^1)$ with $\|u\|_{\mathbb{M}_{\mathbb{F}}^p(\Omega, E_r^1)} \leq R$. □

Next, we state a Lemma about the mapping properties of the gradient of the power nonlinearity.

Lemma 3.20. *Let $p > \sigma > 1$ and $w \in W^{1,p}(\mathbb{R}^d)$. Then, we have $|w|^{\sigma-1}w \in W^{1,\frac{p}{\sigma}}(\mathbb{R}^d)$ and*

$$\|\nabla\left[|w|^{\sigma-1}w\right]\|_{L^{\frac{p}{\sigma}}} \lesssim \|w\|_{L^p}^{\sigma-1}\|\nabla w\|_{L^p}. \tag{3.50}$$

If we assume $p > \sigma > 2$, we get

$$\begin{aligned}\left\|\nabla\left[|w_1|^{\sigma-1}w_1\right] - \nabla\left[|w_2|^{\sigma-1}w_2\right]\right\|_{L^{\frac{p}{\sigma}}} \lesssim &\|w_1\|_{L^p}^{\sigma-1}\|\nabla w_1 - \nabla w_2\|_{L^p} \\ &+ \left(\|w_1\|_{L^p}^{\sigma-2} + \|w_2\|_{L^p}^{\sigma-2}\right)\|\nabla w_2\|_{L^p}\|w_1 - w_2\|_{L^p}\end{aligned} \tag{3.51}$$

for $w_1, w_2 \in W^{1,p}(\mathbb{R}^d)$.

We sketch the proof for convenience since it is not easy to find a reference for the assertion although it seems to be a classical result. We need some preliminaries. Below, we identify $\mathbb{C}$ with $\mathbb{R}^2$ and differentiability is always understood in the real sense. For a continuously differentiable function $f : \mathbb{C} \to \mathbb{C}$, we denote

$$\partial_z f(z) := \frac{1}{2}\left(\partial_x f(z) - \mathrm{i}\partial_y f(z)\right), \qquad \partial_{\bar{z}} f(z) := \frac{1}{2}\left(\partial_x f(z) + \mathrm{i}\partial_y f(z)\right), \qquad z = x + \mathrm{i}y \in \mathbb{C}.$$

Then, the chain rule can be formulated as

$$\nabla f(u) = \partial_z f(u)\nabla u + \partial_{\bar{z}} f(u)\nabla\bar{u}, \qquad u \in C_c^\infty(\mathbb{R}^d) \tag{3.52}$$

and consequently, we get the integral inequality

$$f(z_2) - f(z_1) = \int_0^1 \left[\partial_z f\big(sz_1 + (1-s)z_2\big)(z_1 - z_2) + \partial_{\bar{z}} f\big(sz_1 + (1-s)z_2\big)\overline{(z_1 - z_2)}\right]\mathrm{d}s \tag{3.53}$$

for $z_1, z_2 \in \mathbb{C}$.

Proof. Step 1. First, we prove the estimates

$$\begin{aligned}\left|\nabla[|w|^{\sigma-1}w]\right| &\lesssim |w|^{\sigma-1}|\nabla w|, \\ \left|\nabla\left[|w_1|^{\sigma-1}w_1\right] - \nabla\left[|w_2|^{\sigma-1}w_2\right]\right| &\lesssim |w_1|^{\sigma-1}|\nabla w_1 - \nabla w_2| + \left(|w_1|^{\sigma-2} + |w_2|^{\sigma-2}\right)|\nabla w_2||w_1 - w_2|\end{aligned} \tag{3.54}$$

for $w, w_1, w_2 \in C_c^\infty(\mathbb{R}^d)$. We define $\Phi : \mathbb{C} \to \mathbb{C}$ by $\Phi(z) = |z|^{\sigma-1}z$. For $\sigma > 1$, Φ is continuously differentiable with

$$\partial_z \Phi(z) = \frac{1}{2}(\sigma+1)|z|^{\sigma-1}, \qquad \partial_{\bar{z}}\Phi(z) = \frac{1}{2}(\sigma-1)|z|^{\sigma-3}z^2.$$

In particular, we have

$$\nabla[|w|^{\sigma-1}w] = \frac{1}{2}(\sigma+1)|w|^{\sigma-1}\nabla w + \frac{1}{2}(\sigma-1)|w|^{\sigma-3}w^2\nabla\bar{w}, \qquad w \in C_c^\infty(\mathbb{R}^d), \tag{3.55}$$

and thus

$$\left|\nabla[|w|^{\sigma-1}w]\right| \leq \sigma|w|^{\sigma-1}|\nabla w|, \qquad w \in C_c^\infty(\mathbb{R}^d).$$

For $\sigma > 2$, Φ is twice continuously differentiable with

$$\begin{aligned} \partial_z^2\Phi(z) &= \frac{1}{2}(\sigma+1)(\sigma-1)|z|^{\sigma-3}\bar{z}, \\ \partial_z\partial_{\bar{z}}\Phi(z) &= \frac{1}{2}(\sigma+1)(\sigma-1)|z|^{\sigma-3}z, \\ \partial_{\bar{z}}^2\Phi(z) &= \frac{1}{4}(\sigma-1)(\sigma-3)|z|^{\sigma-5}z^3. \end{aligned} \tag{3.56}$$

From (3.52) and the triangle inequality, we infer

$$\begin{aligned} \left|\nabla\left[|w_1|^{\sigma-1}w_1\right] - \nabla\left[|w_2|^{\sigma-1}w_2\right]\right| \leq{}& |\partial_z\Phi(w_1)||\nabla w_1 - \nabla w_2| + |\partial_z\Phi(w_1) - \partial_z\Phi(w_2)||\nabla w_2| \\ &+ |\partial_{\bar{z}}\Phi(w_1)||\overline{\nabla w_1} - \overline{\nabla w_2}| + |\partial_{\bar{z}}\Phi(w_1) - \partial_{\bar{z}}\Phi(w_2)||\overline{\nabla w_2}|. \end{aligned}$$

The integral identity (3.53) and (3.56) yield

$$\begin{aligned} |\partial_z\Phi(w_1) - \partial_z\Phi(w_2)| &\leq \int_0^1 \left[|\partial_z^2\Phi(sw_1 + (1-s)w_2)| + |\partial_z\partial_{\bar{z}}\Phi(sw_1 + (1-s)w_2)|\right]\mathrm{d}s|w_1 - w_2| \\ &\lesssim \left(|w_1|^{\sigma-2} + |w_2|^{\sigma-2}\right)|w_1 - w_2|. \end{aligned}$$

Together with a similar reasoning for $|\partial_{\bar{z}}\Phi(w_1) - \partial_{\bar{z}}\Phi(w_2)|$, we obtain the second inequality in (3.54).

Step 2. By the estimates (3.54) from the first step and the Hölder inequality based on the exponent identities $\frac{\sigma}{p} = \frac{1}{p} + \frac{\sigma-1}{p}$ and $\frac{\sigma}{p} = \frac{1}{p} + \frac{1}{p} + \frac{\sigma-2}{p}$, we deduce (3.50) and (3.51) for $w, w_1, w_2 \in C_c^\infty(\mathbb{R}^d)$. The assertion for general $w, w_1, w_2 \in W^{1,p}(\mathbb{R}^d)$ can be obtained by an approximation argument. □

We continue with the notations for the approximation of (3.1). As in Section 3.1, we employ the cut-off function

$$\theta_n(x) := \begin{cases} 1, & x \in [0,n], \\ 2 - \dfrac{x}{n}, & x \in [n, 2n], \\ 0, & x \in [2n, \infty), \end{cases}$$

and the process

$$Z_t(v) := \|v\|_{L^q(0,t;L^{\alpha+1})} + \|v\|_{L^{\tilde{q}}(0,t;L^{2\gamma})}, \qquad t \in [0,T],\ v \in E_T^0.$$

Moreover, we set

$$\varphi_n(v,t)=\theta_n(Z_t(v)),\qquad \psi_n(v,t)=\theta_n(\|v\|_{E_t^1}),\qquad t\in[0,T],\ t\in E_T^1, \tag{3.57}$$

and consider the following two different ways of truncating (3.1):

$$\begin{cases} \mathrm{d}u_n=\Big(\mathrm{i}\Delta u_n-\mathrm{i}\lambda\varphi_n(u_n,\cdot)|u_n|^{\alpha-1}u_n+[\varphi_n(u_n,\cdot)]^2\mu(|u_n|^{2(\gamma-1)}u_n)\Big)\,\mathrm{d}t\\ \qquad\qquad -\mathrm{i}\varphi_n(u_n,\cdot)B\left(|u_n|^{\gamma-1}u_n\right)\mathrm{d}W,\\ u_n(0)=u_0, \end{cases} \tag{3.58}$$

and

$$\begin{cases} \mathrm{d}v_n=\Big(\mathrm{i}\Delta v_n-\mathrm{i}\lambda\psi_n(v_n,\cdot)|v_n|^{\alpha-1}v_n+[\psi_n(v_n,\cdot)]^2\mu(|v_n|^{2(\gamma-1)}v_n)\Big)\,\mathrm{d}t\\ \qquad\qquad -\mathrm{i}\psi_n(v_n,\cdot)B\left(|v_n|^{\gamma-1}v_n\right)\mathrm{d}W,\\ v_n(0)=u_0. \end{cases} \tag{3.59}$$

Let us compare (3.58) and (3.59) and outline our strategy to solve these problems. In Proposition 3.21, (3.58) will be tackled with a fixed point argument in the ball $X_{r,R}$ from Lemma 3.19. For two reasons, this is not possible for (3.59). On the one hand, the cut-off with ψ_n is not strong enough to get a contraction estimate in $\mathbb{M}_{\mathbb{F}}^p(\Omega,E_r^0)$. On the other hand, the truncation argument needs the continuity of $t\mapsto\|v\|_t$ to guarantee that the existence times are stopping times. This is only true for $v\in F_r$, which forbids the use of the Banach-Alaoglu argument in Lemma 3.19.

Under the restrictions $\alpha>2$ and $\gamma>2$, however, Lemma 3.20 provides Lipschitz estimates for

$$v\mapsto\nabla[|v|^{\alpha-1}v],\qquad v\mapsto\nabla[|v|^{2(\gamma-1)}v],\qquad v\mapsto\nabla[|v|^{\gamma-1}v]$$

Hence, we can apply Banach's fixed point theorem in $\mathbb{M}_{\mathbb{F}}^p(\Omega,F_r)$ without the restriction to a ball. This will be the content of Proposition 3.22.

Proposition 3.21. *Let $u_0\in H^1(\mathbb{R}^d)$, $\alpha\in(1,1+\frac{4}{d})$, $\gamma\in(1,1+\frac{2}{d})$ and $p\in(1,\infty)$. For fixed $n\in\mathbb{N}$, there is a unique global strong solution (u_n,T) of (3.58) in $H^1(\mathbb{R}^d)$ satisfying $u_n\in\mathbb{M}_{\mathbb{F}}^p(\Omega,E_T^1)$.*

Proof. We define

$$K_{det}^n u(t):=-\mathrm{i}\lambda\int_0^t e^{\mathrm{i}(t-s)\Delta}\left[\varphi_n(u,s)|u(s)|^{\alpha-1}u(s)\right]\mathrm{d}s,$$

$$K_{Strat}^n u(t):=\int_0^t e^{\mathrm{i}(t-s)\Delta}\mu\left(\left[\varphi_n(u_n,t)\right]^2|u(s)|^{2(\gamma-1)}u(s)\right)\mathrm{d}s,$$

$$K_{stoch}^n u(t):=-\mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}B\left(\varphi_n(u,s)|u(s)|^{\gamma-1}u(s)\right)\mathrm{d}W(s)$$

and construct a unique solution $u_n\in\mathbb{M}_{\mathbb{F}}^p(\Omega,E_T^1)$ of the mild equation

$$u_n=e^{\mathrm{i}\cdot\Delta}u_0+K_{det}^n u_n+K_{Strat}^n u_n+K_{stoch}^n u_n.$$

We remark that by Proposition 2.14 and Corollary 2.23, a solution of this equation has continuous paths in $H^1(\mathbb{R}^d)$ and as in Lemma 3.6, one can show that u_n is a strong solution of (3.58) in $H^1(\mathbb{R}^d)$.

Step 1. We take $u \in X_{r,R}$ for some $r > 0$ and $R > 0$ to be specified later and define a stopping time by

$$\tau := \inf\left\{t \geq 0 : \|u\|_{L^q(0,t;L^{\alpha+1})} + \|u\|_{L^{\tilde{q}}(0,t;L^{2\gamma})} \geq 2n\right\} \wedge r.$$

Moreover, we set

$$\delta := 1 + \frac{d}{4}(1-\alpha) \in (0,1), \qquad \tilde{\delta} = 1 + \frac{d}{2}(1-\gamma) \in (0,1).$$

A pathwise application of Proposition 2.14 and integration over Ω yield

$$\|e^{\mathrm{i}\cdot\Delta}u_0\|_{\mathbb{M}_{\mathbb{F}}^p(\Omega,E_r^1)} \lesssim \|u_0\|_{H^1}.$$

Using Proposition 2.14, Lemma 3.20 for $p = \alpha+1$ and $\sigma = \alpha$, Hölder's inequality in time based on the identity $\frac{1}{q'} = \frac{1}{q} + \frac{\alpha-1}{q} + \delta$ and finally (3.48), we estimate

$$\begin{aligned}
\|K_{det}^n u\|_{E_r} \lesssim & \|\varphi_n(u)|u|^{\alpha-1}u\|_{L^{q'}(0,r;W^{1,\frac{\alpha+1}{\alpha}})} \\
\lesssim & \||u|^{\alpha-1}u\|_{L^{q'}(0,\tau;W^{1,\frac{\alpha+1}{\alpha}})} \\
\leq & \|u\|_{L^q(0,\tau;L^{\alpha+1})}^{\alpha}\tau^{\delta} + \|u\|_{L^q(0,\tau;L^{\alpha+1})}^{\alpha-1}\|\nabla u\|_{L^q(0,\tau;L^{\alpha+1})}\tau^{\delta} \\
\leq & (2n)^{\alpha-1}\|u\|_{L^q(0,r;W^{1,\alpha+1})}r^{\delta} \leq (2n)^{\alpha-1}\|u\|_{E_r^1}r^{\delta}.
\end{aligned}$$

Similarly, we get

$$\|K_{Strat}^n u\|_{E_r} \lesssim (2n)^{2\gamma-2}\|u\|_{E_r^1}r^{\tilde{\delta}}.$$

Integrating over Ω yields

$$\|K_{det}^n u\|_{\mathbb{M}_{\mathbb{F}}^p(\Omega,E_r^1)} \lesssim (2n)^{\alpha-1}Rr^{\delta}, \qquad \|K_{Strat}^n u\|_{\mathbb{M}_{\mathbb{F}}^p(\Omega,E_r^1)} \lesssim (2n)^{2\gamma-2}Rr^{\tilde{\delta}}.$$

By Corollary 2.23 and the boundedness of B, we obtain

$$\begin{aligned}
\|K_{stoch}^n u\|_{\mathbb{M}_{\mathbb{F}}^p(\Omega,E_r^1)} \lesssim & \|B\left(\varphi_n(u)|u|^{\gamma-1}u\right)\|_{L^p(\Omega,L^2(0,r;\mathrm{HS}(Y,H^1)))} \\
\lesssim & \|\varphi_n(u)|u|^{\gamma-1}u\|_{L^p(\Omega,L^2(0,r;H^1))}.
\end{aligned}$$

From Lemma 3.20 with $p = 2\gamma$ and $\sigma = \gamma$, the Hölder inequality and (3.48), we infer the pathwise inequality

$$\begin{aligned}
\|\varphi_n(u)|u|^{\gamma-1}u\|_{L^2(0,\tau;H^1)} \lesssim & \|u\|_{L^{2\gamma}(0,\tau;L^{2\gamma})}^{\gamma} + \|u\|_{L^{2\gamma}(0,\tau;L^{2\gamma})}^{\gamma-1}\|\nabla u\|_{L^{2\gamma}(0,\tau;L^{2\gamma})} \\
\lesssim & \tau^{\frac{\tilde{\delta}}{2}}\|u\|_{L^{\tilde{q}}(0,\tau;L^{2\gamma})}^{\gamma-1}\|u\|_{L^{\tilde{q}}(0,\tau;W^{1,2\gamma})} \\
\leq & r^{\frac{\tilde{\delta}}{2}}(2n)^{\gamma-1}\|u\|_{E_r^1}
\end{aligned}$$

and therefore

$$\|K_{stoch}^n u\|_{\mathbb{M}_{\mathbb{F}}^p(\Omega,E_r^1)} \lesssim r^{\frac{\tilde{\delta}}{2}}(2n)^{\gamma-1}R.$$

Altogether, there are constants $C_1 > 0$ and $C_2 = C_2(r,n) > 0$ with $C_2(r,n) \to 0$ for $r \to 0$ such that

$$\|K^n u\|_{\mathbb{M}_{\mathbb{F}}^p(\Omega,E_r^1)} \leq C_1\left[\|u_0\|_{H^1} + RC_2(r,n)\right], \qquad u \in X_{r,R}. \tag{3.60}$$

The same arguments as in the proof of Proposition 3.7 lead to the estimate

$$\|K^n(u_1)-K^n(u_2)\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,E^0_r)} \lesssim \Big[\left(2^{\alpha+1}+4^{\alpha-1}\right) r^{\delta} n^{\alpha-1} + \left(2^{2\gamma+1}+4^{2(\gamma-1)}\right) r^{\tilde{\delta}} n^{2(\gamma-1)} + \left(4^{\gamma-1}+2^{\gamma+1}\right) r^{\frac{\tilde{\delta}}{2}} n^{\gamma-1}\Big] \|u_1-u_2\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,E^0_r)}.$$

Hence, there is a constant $C_3 = C_3(r,n) > 0$ with $C_3(r,n) \to 0$ for $r \to 0$ and

$$\|K^n(u_1)-K^n(u_2)\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,E^0_r)} \le C_3(r,n)\|u_1-u_2\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,E^0_r)}. \tag{3.61}$$

Now, we choose $r > 0$ small enough to ensure $C_2(r,n) \le \frac{1}{2}$ and $C_3(r,n) \le \frac{1}{2}$ and take $R = 2C_1\|u_0\|_{L^2}$. Then, K^n is contractive and leaves $X_{r,R}$ invariant and Banach's Fixed Point Theorem yields $u^{n,1} \in X_{r,R}$ with $K^n(u^n_1) = u^n_1$. This argument can be iterated to get a global mild solution $u^n \in \mathbb{M}^p_{\mathbb{F}}(\Omega, E^1_T)$ of (3.58) since the existence time $r > 0$ is independent of $\|u_0\|_{L^2}$. □

We continue with the proof of existence and uniqueness for (3.22) under the restrictions $\alpha > 2$ and $\gamma > 2$.

Proposition 3.22. *Let $u_0 \in H^1(\mathbb{R}^d)$, $\alpha \in (2, 1+\frac{4}{(d-2)_+})$, $\gamma \in (2, 1+\frac{2}{(d-2)_+})$ and $p \in (1,\infty)$. For fixed $n \in \mathbb{N}$, there is a unique global strong solution (v^n, T) of* (3.59) *in $H^1(\mathbb{R}^d)$ with $v_n \in \mathbb{M}^p_{\mathbb{F}}(\Omega, F_T)$.*

Proof. We define

$$K^n_{det}v(t) := -\,\mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-s)\Delta}\left[\psi_n(v,s)|v(s)|^{\alpha-1}v(s)\right]\mathrm{d}s,$$

$$K^n_{Strat}v(t) := \int_0^t e^{\mathrm{i}(t-s)\Delta}\mu\left(\left[\psi_n(v,t)\right]^2|v(s)|^{2(\gamma-1)}v(s)\right)\mathrm{d}s,$$

$$K^n_{stoch}v(t) := -\,\mathrm{i}\int_0^t e^{\mathrm{i}(t-s)\Delta}B\left(\psi_n(v,s)|v(s)|^{\gamma-1}v(s)\right)\mathrm{d}W(s)$$

and construct a unique solution $v_n \in \mathbb{M}^p_{\mathbb{F}}(\Omega, F_T)$ of the mild equation

$$v_n = e^{\mathrm{i}\cdot\Delta}u_0 + K^n_{det}v_n + K^n_{Strat}v_n + K^n_{stoch}v_n.$$

As in Lemma 3.6, one can show that v_n is a strong solution of (3.59) in $H^1(\mathbb{R}^d)$.

Step 1. Let us fix $r > 0$ to be specified later and take $v \in \mathbb{M}^p_{\mathbb{F}}(\Omega, F_r)$. We define the stopping time τ by

$$\tau := \inf\{t \ge 0 : \|v\|_{F_t} \ge 2n\} \wedge r.$$

A pathwise application of Proposition 2.14 and integration over Ω yields

$$\|e^{\mathrm{i}\cdot\Delta}u_0\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,F_r)} \lesssim \|u_0\|_{H^1}.$$

Using Proposition 2.14, Lemma 3.20 with $p = \alpha+1$ and $\sigma = \alpha$ and the Hölder inequality with $\frac{1}{q'} = \frac{1}{\infty} + \frac{1}{q} + \left(\frac{1}{q'} - \frac{1}{q}\right)$, we estimate

$$\|K^n_{det}v\|_{F_r} \lesssim \|\psi_n(v)|v|^{\alpha-1}v\|_{L^{q'}(0,r;W^{1,\frac{\alpha+1}{\alpha}})}$$

$$
\begin{aligned}
&\lesssim \||v|^{\alpha-1}v\|_{L^{q'}(0,\tau;L^{\frac{\alpha+1}{\alpha}})} + \left\|\|v\|_{L^{\alpha+1}}^{\alpha-1}\|\nabla v\|_{L^{\alpha+1}}\right\|_{L^{q'}(0,\tau)}\\
&\le \|v\|_{L^\infty(0,\tau;L^{\alpha+1})}^{\alpha-1}\tau^{\frac{1}{q'}-\frac{1}{q}}\|v\|_{L^q(0,\tau;L^{\alpha+1})} + \|v\|_{L^\infty(0,\tau;L^{\alpha+1})}^{\alpha-1}\tau^{\frac{1}{q'}-\frac{1}{q}}\|\nabla v\|_{L^q(0,\tau;L^{\alpha+1})}.
\end{aligned}
$$

The Sobolev embedding $H^1(\mathbb{R}^d)\hookrightarrow L^{\alpha+1}(\mathbb{R}^d)$ and (3.48) yield

$$
\begin{aligned}
\|K_{det}^n v\|_{F_r} &\lesssim \|v\|_{L^\infty(0,\tau;H^1)}^{\alpha-1} r^{\frac{1}{q'}-\frac{1}{q}}\|v\|_{L^q(0,\tau;W^{1,\alpha+1})}\\
&\le (2n)^{\alpha-1} r^{\frac{1}{q'}-\frac{1}{q}}\|v\|_{F_r}
\end{aligned}
$$

and similarly, we get

$$
\|K_{Strat}^n v\|_{F_r} \lesssim (2n)^{2\gamma-2}\|v\|_{F_r} r^{\frac{1}{\tilde{q}'}-\frac{1}{\tilde{q}}}.
$$

Integrating over Ω, we obtain

$$
\begin{aligned}
\|K_{det}^n v\|_{\mathbb{M}_{\mathbb{F}}^p(\Omega,F_r)} &\lesssim (2n)^{\alpha-1}\|v\|_{\mathbb{M}_{\mathbb{F}}^p(\Omega,F_r)} r^{\frac{1}{q'}-\frac{1}{q}},\\
\|K_{Strat}^n v\|_{\mathbb{M}_{\mathbb{F}}^p(\Omega,F_r)} &\lesssim (2n)^{2\gamma-2}\|v\|_{\mathbb{M}_{\mathbb{F}}^p(\Omega,F_r)} r^{\frac{1}{\tilde{q}'}-\frac{1}{\tilde{q}}}.
\end{aligned}
$$

For the stochastic convolution, we deduce

$$
\begin{aligned}
\|K_{stoch}^n v\|_{\mathbb{M}_{\mathbb{F}}^p(\Omega,F_r)} &\lesssim \|B\left(\psi_n(v)|v|^{\gamma-1}v\right)\|_{L^p(\Omega,L^2(0,r;\mathrm{HS}(Y,H^1)))}\\
&\lesssim \|\psi_n(v)|v|^{\gamma-1}v\|_{L^p(\Omega,L^2(0,r;H^1))}.
\end{aligned}
$$

From Lemma 3.20 with $p=2\gamma$ and $\sigma=\gamma$ and the Hölder inequality, we infer

$$
\begin{aligned}
\|\psi_n(v)|v|^{\gamma-1}v\|_{L^2(0,\tau;H^1)} &\lesssim \|\psi_n(v)|v|^{\gamma-1}v\|_{L^2(0,\tau;L^2)} + \|\psi_n(v)\nabla[|v|^{\gamma-1}v]\|_{L^2(0,\tau;L^2)}\\
&\le \left\|\|v\|_{L^{2\gamma}}^{\gamma}\right\|_{L^2(0,\tau)} + \left\|\|v\|_{L^{2\gamma}}^{\gamma-1}\|\nabla v\|_{L^{2\gamma}}\right\|_{L^2(0,\tau)}\\
&\le \|v\|_{L^\infty(0,\tau;L^{2\gamma})}^{\gamma-1}\|v\|_{L^2(0,\tau;W^{1,2\gamma})}\\
&\le (2n)^{\gamma-1} r^{\frac{1}{2}-\frac{1}{\tilde{q}}}\|v\|_{L^{\tilde{q}}(0,\tau;W^{1,2\gamma})}.
\end{aligned}
$$

This leads to

$$
\|K_{stoch}^n v\|_{\mathbb{M}_{\mathbb{F}}^p(\Omega,F_r)} \lesssim (2n)^{\gamma-1} r^{\frac{1}{2}-\frac{1}{\tilde{q}}}\|v\|_{\mathbb{M}_{\mathbb{F}}^p(\Omega,F_r)}.
$$

The previous estimates yield the invariance of $\mathbb{M}_{\mathbb{F}}^p(\Omega,F_r)$ under K^n.

Step 2. To check that K^n is a contraction in $\mathbb{M}_{\mathbb{F}}^p(\Omega,F_r)$ for sufficiently small $r>0$, we take $v_1,v_2\in\mathbb{M}_{\mathbb{F}}^p(\Omega,F_r)$ and define stopping times τ_1 and τ_2 by

$$
\tau_j := \inf\{t\ge 0 : \|v_j\|_{F_t}\ge 2n\}\wedge r
$$

for $j=1,2$. We fix $\omega\in\Omega$ and w.l.o.g., we assume $\tau_1(\omega)\le\tau_2(\omega)$. We use the deterministic Strichartz inequalities from Proposition 2.14 and $\psi_n(v_1)\equiv 0$ on $[\tau_1,\tau_2]$ to estimate

$$
\begin{aligned}
\|K_{det}^n(v_1)-K_{det}^n(v_2)\|_{F_r} &\lesssim \|\psi_n(v_1)|v_1|^{\alpha-1}v_1 - \psi_n(v_2)|v_2|^{\alpha-1}v_2\|_{L^{q'}(0,T;W^{1,\frac{\alpha+1}{\alpha}})}\\
&\le \|\psi_n(v_1)\left(|v_1|^{\alpha-1}v_1 - |v_2|^{\alpha-1}v_2\right)\|_{L^{q'}(0,\tau_1;W^{1,\frac{\alpha+1}{\alpha}})}\\
&\quad + \|\left[\psi_n(v_1)-\psi_n(v_2)\right]|v_2|^{\alpha-1}v_2\|_{L^{q'}(0,\tau_1;W^{1,\frac{\alpha+1}{\alpha}})}\\
&\quad + \|\psi_n(v_2)|v_2|^{\alpha-1}v_2\|_{L^{q'}(\tau_1,\tau_2;W^{1,\frac{\alpha+1}{\alpha}})} := I + II + III
\end{aligned}
$$

As in (3.20), we deduce

$$|\psi_n(v_1,s)-\psi_n(v_2,s)| \le \frac{1}{n}\|v_1-v_2\|_{F_s}, \qquad s\in[0,r],$$

which leads to

$$\begin{aligned} II \le& \frac{1}{n}\|v_1-v_2\|_{F_r}\||v_2|^{\alpha-1}v_2\|_{L^{q'}(0,\tau_1;W^{1,\frac{\alpha+1}{\alpha}})}\\ \lesssim& \frac{1}{n}\|v_1-v_2\|_{F_r}\|v_2\|^{\alpha-1}_{L^\infty(0,\tau_1;L^{\alpha+1})}\|v_2\|_{L^{q'}(0,\tau_1;W^{1,\alpha+1})}\\ \lesssim& \frac{1}{n}\|v_1-v_2\|_{F_r}\|v_2\|^{\alpha-1}_{L^\infty(0,\tau_1;H^1)}\|v_2\|_{L^{q}(0,\tau_1;W^{1,\alpha+1})}\tau_1^{\frac{1}{q'}-\frac{1}{q}}\\ \lesssim& \|v_1-v_2\|_{F_r}n^{\alpha-1}r^{\frac{1}{q'}-\frac{1}{q}} \end{aligned}$$

by similar arguments as in the first step. By $\psi_n(v_1)|v_2|^{\alpha-1}v_2 \equiv 0$ on $[\tau_1,\tau_2]$ followed by the same estimates as above, we obtain

$$\begin{aligned} III =& \| [\psi_n(v_1)-\psi_n(v_2)]\,|v_2|^{\alpha-1}v_2\|_{L^{q'}(\tau_1,\tau_2;W^{1,\frac{\alpha+1}{\alpha}})}\\ \lesssim& \frac{1}{n}\|v_1-v_2\|_{F_r}\||v_2|^{\alpha-1}v_2\|_{L^{q'}(\tau_1,\tau_2;W^{1,\frac{\alpha+1}{\alpha}})}\\ \lesssim& \|v_1-v_2\|_{F_r}n^{\alpha-1}r^{\frac{1}{q'}-\frac{1}{q}}. \end{aligned}$$

Let us continue with the estimate of the first term. We start with

$$\begin{aligned} I \lesssim& \|\psi_n(v_1)\left(|v_1|^{\alpha-1}v_1-|v_2|^{\alpha-1}v_2\right)\|_{L^{q'}(0,\tau_1;L^{\frac{\alpha+1}{\alpha}})}\\ &+\|\psi_n(v_1)\left(\nabla\left[|v_1|^{\alpha-1}v_1\right]-\nabla\left[|v_2|^{\alpha-1}v_2\right]\right)\|_{L^{q'}(0,\tau_1;L^{\frac{\alpha+1}{\alpha}})} := I_1+I_2. \end{aligned}$$

The local Lipschitz-property of $\mathbb{C}\ni z\mapsto |z|^{\alpha-1}z$ and the Hölder inequality yield

$$\begin{aligned} I_1 \lesssim& \left(\|v_1\|_{L^\infty(0,\tau_1;L^{\alpha+1}(\mathbb{R}^d))}+\|v_2\|_{L^\infty(0,\tau_1;L^{\alpha+1}(\mathbb{R}^d))}\right)^{\alpha-1}\|v_1-v_2\|_{L^{q'}(0,\tau_1;L^{\alpha+1}(\mathbb{R}^d))}\\ \lesssim& (4n)^{\alpha-1}\|v_1-v_2\|_{F_r}r^{\frac{1}{q'}-\frac{1}{q}}. \end{aligned}$$

From the second assertion in Lemma 3.20 (recall $\alpha>2$), the Hölder inequality and (3.48), we infer

$$\begin{aligned} I_2 \lesssim& \|v_1\|^{\alpha-1}_{L^\infty(0,\tau_1;L^{\alpha+1})}\|\nabla v_1-\nabla v_2\|_{L^{q'}(0,\tau_1;L^{\alpha+1})}\\ &+\left(\|v_1\|^{\alpha-2}_{L^\infty(0,\tau_1;L^{\alpha+1})}+\|v_2\|^{\alpha-2}_{L^\infty(0,\tau_1;L^{\alpha+1})}\right)\|\nabla v_2\|_{L^{q'}(0,\tau_1;L^{\alpha+1})}\|v_1-v_2\|_{L^\infty(0,\tau_1;L^{\alpha+1})}\\ \lesssim& (2n)^{\alpha-1}r^{\frac{1}{q'}-\frac{1}{q}}\|\nabla v_1-\nabla v_2\|_{L^{q}(0,\tau_1;L^{\alpha+1})}\\ &+2(2n)^{\alpha-1}r^{\frac{1}{q'}-\frac{1}{q}}\|v_1-v_2\|_{L^\infty(0,\tau_1;L^{\alpha+1})}\\ \lesssim& n^{\alpha-1}r^{\frac{1}{q'}-\frac{1}{q}}\|v_1-v_2\|_{F_r}. \end{aligned}$$

Putting together all the estimates for I, II and III, we obtain

$$\|K^n_{det}(v_1)-K^n_{det}(v_2)\|_{F_r} \lesssim r^{\frac{1}{q'}-\frac{1}{q}}n^{\alpha-1}\|v_1-v_2\|_{F_r}$$

and by integrating over Ω, we end up with

$$\|K^n_{det}(v_1)-K^n_{det}(v_2)\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,F_r)} \lesssim r^{\frac{1}{q'}-\frac{1}{q}}n^{\alpha-1}\|v_1-v_2\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,F_r)}.$$

With the same techniques, we deduce

$$\|K^n_{Strat}(v_1)-K^n_{Strat}(v_2)\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,F_r)}\lesssim r^{\frac{1}{\tilde{q}'}-\frac{1}{\tilde{q}}}n^{2(\gamma-1)}\|v_1-v_2\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,F_r)}$$

and

$$\|K^n_{Stoch}(v_1)-K^n_{Stoch}(v_2)\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,F_r)}\lesssim r^{\frac{1}{2}-\frac{1}{\tilde{q}}}n^{\gamma-1}\|v_1-v_2\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,F_r)},$$

which finally leads to

$$\|K^n v_1-K^n v_2\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,F_r)}\lesssim\left[r^{\frac{1}{q'}-\frac{1}{q}}n^{\alpha-1}+r^{\frac{1}{\tilde{q}'}-\frac{1}{\tilde{q}}}n^{2(\gamma-1)}+r^{\frac{1}{2}-\frac{1}{\tilde{q}}}n^{\gamma-1}\right]\|v_1-v_2\|_{\mathbb{M}^p_{\mathbb{F}}(\Omega,F_r)}.$$

Hence, K^n is a strict contraction in $\mathbb{M}^p_{\mathbb{F}}(\Omega,F_r)$ for sufficiently small $r=r(n)>0$. With the same arguments as in Proposition 3.7, we can iterate the procedure and get the assertion. □

In the following two Propositions, we use the results for the truncated problems to get existence and uniqueness for (3.1). We omit the proofs since they are similar to Propositions 3.8 and 3.9. Combining both Propositions yields Theorem 3.3.

Proposition 3.23. *Assume that either i) or ii) holds.*

i) Let $\alpha\in(1,1+\frac{4}{d})$, $\gamma\in(1,1+\frac{2}{d})$ and $(u_n)_{n\in\mathbb{N}}\subset\mathbb{M}^p_{\mathbb{F}}(\Omega,E^1_T)$ be the sequence constructed in Proposition 3.21. For $n\in\mathbb{N}$, we define the stopping time τ_n by

$$\tau_n:=\inf\left\{t\in[0,T]:\|u_n\|_{L^q(0,t;L^{\alpha+1})}+\|u_n\|_{L^{\tilde{q}}(0,t;L^{2\gamma})}\geq n\right\}\wedge T.$$

ii) Let $\alpha\in(2,1+\frac{4}{(d-2)_+})$, $\gamma\in(2,1+\frac{2}{(d-2)_+})$ and $(v_n)_{n\in\mathbb{N}}\subset\mathbb{M}^p_{\mathbb{F}}(\Omega,E^1_T)$ be the sequence constructed in Proposition 3.22. For $n\in\mathbb{N}$, we define the stopping time τ_n by

$$\tau_n:=\inf\{t\in[0,T]:\|v_n\|_{F_t}\geq n\}\wedge T.$$

If i) holds, we denote u_n by u^n and otherwise, we write u^n for v_n. Then, the following assertions hold:

a) We have $0<\tau_n\leq\tau_k$ almost surely for $n\leq k$ and $u^n(t)=u^k(t)$ almost surely on $\{t\leq\tau_n\}$.

b) The triple $\left(u,(\tau_n)_{n\in\mathbb{N}},\tau_\infty\right)$ with $u(t):=u^n(t)$ for $t\in[0,\tau_n]$ and $\tau_\infty:=\sup_{n\in\mathbb{N}}\tau_n$ is an analytically and stochastically strong solution of (3.1) in $H^1(\mathbb{R}^d)$ in the sense of Definition 2.1.

Proposition 3.24. *Let $\alpha\in(1,1+\frac{4}{d})\cup(2,1+\frac{4}{(d-2)_+})$, $\gamma\in(1,1+\frac{2}{d})\cup(2,1+\frac{2}{(d-2)_+})$ and $\left(u_1,(\sigma_n)_{n\in\mathbb{N}},\sigma\right)$, $\left(u_2,(\tau_n)_{n\in\mathbb{N}},\tau\right)$ be strong solutions to (3.4) in $H^1(\mathbb{R}^d)$. Then,*

$$u_1(t)=u_2(t)\qquad a.s.\ on\ \{t<\sigma\wedge\tau\}.$$

We close this chapter with a remark on the critical setting and classify Theorem 3.3 in the context of the results by Barbu, Röckner and Zhang, [12], and de Bouard and Debussche, [43], for the stochastic NLS in $H^1(\mathbb{R}^d)$.

Remark 3.25. a) The statement of Theorem 3.3 contains the critical values $\alpha\in\left\{1+\frac{4}{d},1+\frac{4}{(d-2)_+}\right\}$ and $\gamma\in\left\{1,1+\frac{2}{d},1+\frac{2}{(d-2)_+}\right\}$ which have not been treated so far. The case of linear noise is simpler and could be treated simultaneously as in Section 3.1. In the critical setting, it is not hard to combine the estimates from this section with the argument from Proposition 3.12 to prove local existence and uniqueness in this setting. Although the exponent $\alpha=1+\frac{4}{d}$ is energy-subcritical, the local result result cannot be used for global existence since there is no blow-up criterium which is strong enough.

b) The comparison of Theorem 3.3 with the local results from [12] and [43] is similar as in the L^2-case we described briefly in the introduction. The main advantage of the present result is the fact that nonlinear noise is allowed. Moreover, the assumptions on the coefficients e_m are significantly weaker compared to [12] and the range of exponents α is larger compared to [43]. In contrast to the L^2-case, however, the rescaling approach has an advantage here since it allows to adapt the deterministic fixed point argument pathwise in a ball of $L^\infty H^1 \cap L^q W^{1,\alpha+1}$ equipped with the metric from $L^\infty L^2 \cap L^q L^{\alpha+1}$. Therefore, the authors of [12] obtain local wellposedness for linear noise and all expected exponents $\alpha \in (1, 1 + \frac{4}{(d-2)_+}]$ without the unsatisfactory gap in $(1, 1 + \frac{4}{d}] \cup (2, 1 + \frac{4}{(d-2)_+}]$ from above that is restrictive for dimensions $d \geq 4$. Similarly to the L^2-setting, [12] also contains pathwise continuous dependence of the initial value up to the maximal existence time.

4. A general framework for existence results

In the following chapter, we derive an existence result for the general nonlinear stochastic Schrödinger equation

$$\begin{cases} \mathrm{d}u(t) = \big(-\mathrm{i}Au(t) - \mathrm{i}F(u(t)) + \mu(u(t))\big)\mathrm{d}t - \mathrm{i}B(u(t))\mathrm{d}W(t), \qquad t \in [0,T], \\ u(0) = u_0 \in E_A, \end{cases} \tag{4.1}$$

in the energy space $E_A := X_{\frac{1}{2}} := \mathcal{D}\big((\mathrm{Id}+A)^{\frac{1}{2}}\big)$, where A is a selfadjoint, non-negative operator with a compact resolvent in an L^2-space H and W is a cylindrical Wiener process on some real Hilbert space Y with ONB $(f_m)_{m\in\mathbb{N}}$. Moreover, $F: E_A \to E_A^*$ is a nonlinearity we will specify later and the nonlinear noise is defined by

$$B(u)f_m := e_m g(|u|^2)u, \qquad u \in L^2(M), \quad m \in \mathbb{N},$$

for certain $g: [0,\infty) \to \mathbb{R}$ and a sequence $(e_m)_{m\in\mathbb{N}}$ of complex valued functions. The correction term μ is given by

$$\mu(u) := -\frac{1}{2}\sum_{m=1}^{\infty} |e_m|^2 g(|u|^2)^2 u, \qquad u \in L^2(M).$$

We construct a martingale solution of the problem (4.1) by a modified Faedo-Galerkin approximation

$$\begin{cases} \mathrm{d}u_n(t) = \big(-\mathrm{i}Au_n(t) - \mathrm{i}P_nF(u_n(t)) + S_n\mu(u_n(t))\big)\mathrm{d}t - \mathrm{i}S_nB(u_n(t))\mathrm{d}W(t), \\ u_n(0) = S_n u_0, \end{cases} \tag{4.2}$$

in finite dimensional subspaces H_n of H spanned by some eigenvectors of A. Here, P_n are the standard orthogonal projections onto H_n and $S_n: H \to H_n$ are selfadjoint operators derived from the Littlewood-Paley decomposition associated to A. The reason for using the operators $(S_n)_{n\in\mathbb{N}}$ lies in the uniform estimate

$$\sup_{n\in\mathbb{N}} \|S_n\|_{L^p\to L^p} < \infty, \quad 1 < p < \infty,$$

which turns out to be necessary in the estimates of the noise and which would be false in general if one replaced S_n by P_n.

The chapter is organized as follows. Section 4.1 is devoted to the relevant assumptions on the operator A, the nonlinearity F and the noise B. Moreover, we formulate the main result of this chapter. In Section 4.2, we study the Galerkin equation (4.2) and obtain its global wellposedness as well as uniform estimates for the mass and the energy of the solutions u_n, $n \in \mathbb{N}$. In Section 4.3, we prove the main result by a limit argument based on the Martingale-Representation Theorem A.12. In Section 4.4, we present some concrete examples of our theory.

4.1. Assumptions and main result

In this section, we formulate the abstract framework for the stochastic nonlinear Schrödinger equation and the main result of this chapter. Let $(\tilde{M},\Sigma,\mu)$ be a σ-finite metric measure space with metric ρ satisfying the *doubling property*, i.e. $\mu(B(x,r))<\infty$ for all $x\in\tilde{M}$ and $r>0$ and

$$\mu(B(x,2r))\lesssim\mu(B(x,r)). \tag{4.3}$$

This estimate implies

$$\mu(B(x,tr))\lesssim t^d\mu(B(x,r)),\qquad x\in\tilde{M},\quad r>0,\quad t\geq 1 \tag{4.4}$$

and the number $d\in\mathbb{N}$ is called *doubling dimension.* Let $M\subset\tilde{M}$ be an open subset with finite measure and abbreviate $H:=L^2(M)$. The standard complex inner product on H is denoted by

$$\big(u,v\big)_H=\int_M u\bar{v}\,\mu(\mathrm{d}x),\qquad u,v\in H.$$

Let A be a $\mathbb{C}$-linear non-negative selfadjoint operator on H with domain $\mathcal{D}(A)$ and denote the scale of fractional domains of A by $(X_\theta)_{\theta\in\mathbb{R}}$. In the context of the NLS, it is necessary that all our function spaces consist of $\mathbb{C}$-valued functions. However, in view of the stochastic integration theory, the compactness results from Section 2.4 and the computations below, it is more convenient to interpret these spaces as real Hilbert or Banach spaces. Hence, we often interpret H as as real a Hilbert spaces with the inner product $\operatorname{Re}\big(u,v\big)_H$ for $u,v\in H$. Obviously, both products introduce the same norms and hence, both spaces are topologically the equivalent. The Hilbert space $E_A:=X_{\frac{1}{2}}$ with

$$\big(u,v\big)_{E_A}:=\big((\mathrm{Id}+A)^{\frac{1}{2}}u,(\mathrm{Id}+A)^{\frac{1}{2}}v\big)_H,\qquad u,v\in E_A,$$

is called the *energy space* and $\|\cdot\|_{E_A}$ the *energy norm* associated to A. We further use the notation $E_A^*:=X_{-\frac{1}{2}}$ which is justified since E_A and $X_{-\frac{1}{2}}$ are dual by Appendix A.3. We remark that (E_A,H,E_A^*) is a Gelfand triple, i.e.

$$E_A\hookrightarrow H\cong H^*\hookrightarrow E_A^*$$

and recall from Proposition A.41 that $A_{-\frac{1}{2}}$ is a non-negative selfadjoint operator on E_A^* with domain E_A. For simplicity, we also denote $A_{-\frac{1}{2}}$ by A. Similarly to H, the spaces E_A and E_A^* can also be interpreted as real Hilbert spaces.

Assumption 4.1. We assume the following:

i) There is a selfadjoint operator S on the complex Hilbert space $\big(H,\big(\cdot,\cdot\big)_H\big)$ which is strictly positive, has a compact resolvent, commutes with A and fulfills $\mathcal{D}(S^k)\hookrightarrow E_A$ for sufficiently large k. Moreover, we assume that S has *generalized Gaussian* (p_0,p_0')*-bounds* for some $p_0\in[1,2)$, i.e.

$$\|\mathbf{1}_{B(x,t^{\frac{1}{m}})}e^{-tS}\mathbf{1}_{B(y,t^{\frac{1}{m}})}\|_{\mathcal{L}(L^{p_0},L^{p_0'})}\leq C\mu(B(x,t^{\frac{1}{m}}))^{\frac{1}{p_0'}-\frac{1}{p_0}}\exp\left\{-c\left(\frac{\rho(x,y)^m}{t}\right)^{\frac{1}{m-1}}\right\} \tag{4.5}$$

for all $t>0$ and $(x,y)\in M\times M$ with constants $c,C>0$ and $m\geq 2$.

ii) Let $\alpha \in (1, p_0' - 1)$ be such that E_A is compactly embedded in $L^{\alpha+1}(M)$. We set

$$p_{\max} := \sup\{p \in (1,\infty] : E_A \hookrightarrow L^p(M) \quad \text{is continuous}\}$$

and note that $p_{\max} \in [\alpha+1, \infty]$. In the case $p_{\max} < \infty$, we assume that $E_A \hookrightarrow L^{p_{\max}}(M)$ is continuous, but not necessarily compact.

In the following, we abbreviate the real duality in E_A with $\mathrm{Re}\langle\cdot,\cdot\rangle := \mathrm{Re}\langle\cdot,\cdot\rangle_{\frac{1}{2},-\frac{1}{2}}$. Note that the duality between $L^{\alpha+1}(M)$ and $L^{\frac{\alpha+1}{\alpha}}(M)$ given by

$$\langle u, v\rangle_{L^{\alpha+1},L^{\frac{\alpha+1}{\alpha}}} := \int_M u\bar{v}\,\mathrm{d}\mu, \qquad u \in L^{\alpha+1}(M), \quad v \in L^{\frac{\alpha+1}{\alpha}}(M),$$

extends $\langle\cdot,\cdot\rangle$ in the sense that we have

$$\langle u, v\rangle = \langle u, v\rangle_{L^{\alpha+1},L^{\frac{\alpha+1}{\alpha}}}, \qquad u \in E_A, \quad v \in L^{\frac{\alpha+1}{\alpha}}(M).$$

Let us comment on Assumption 4.1 i).

Remark 4.2. If $p_0 = 1$, then it is proved in [20] that (4.5) is equivalent to the usual upper Gaussian estimate, i.e. for all $t > 0$ there is a measurable function $p(t,\cdot,\cdot) : M \times M \to \mathbb{R}$ with

$$e^{-tS}f(x) = \int_M p(t,x,y)f(y)\mu(dy), \quad t > 0, \quad \text{a.e. } x \in M$$

for all $f \in H$ and

$$|p(t,x,y)| \le \frac{C}{\mu(B(x,t^{\frac{1}{m}}))}\exp\left\{-c\left(\frac{\rho(x,y)^m}{t}\right)^{\frac{1}{m-1}}\right\}, \tag{4.6}$$

for all $t > 0$ and almost all $(x,y) \in M \times M$ with constants $c, C > 0$ and $m \ge 2$.

We continue with the assumptions on the nonlinear part of our problem.

Assumption 4.3. Let $\alpha \in (1, p_0' - 1)$ be chosen as in Assumption 4.1. Then, we assume the following:

i) Let $F : L^{\alpha+1}(M) \to L^{\frac{\alpha+1}{\alpha}}(M)$ be a function satisfying the following estimate

$$\|F(u)\|_{L^{\frac{\alpha+1}{\alpha}}(M)} \lesssim \|u\|_{L^{\alpha+1}(M)}^{\alpha}, \quad u \in L^{\alpha+1}(M). \tag{4.7}$$

Note that this leads to $F : E_A \to E_A^*$ by Assumption 4.1 ii), because $E_A \hookrightarrow L^{\alpha+1}(M)$ implies $(L^{\alpha+1}(M))^* = L^{\frac{\alpha+1}{\alpha}}(M) \hookrightarrow E_A^*$. We further assume $F(0) = 0$ and

$$\mathrm{Re}\langle \mathrm{i}u, F(u)\rangle = 0, \quad u \in L^{\alpha+1}(M). \tag{4.8}$$

ii) The map $F : L^{\alpha+1}(M) \to L^{\frac{\alpha+1}{\alpha}}(M)$ is continuously real Fréchet differentiable with

$$\|F'[u]\|_{L^{\alpha+1}\to L^{\frac{\alpha+1}{\alpha}}} \lesssim \|u\|_{L^{\alpha+1}(M)}^{\alpha-1}, \quad u \in L^{\alpha+1}(M). \tag{4.9}$$

iii) The map F has a real antiderivative $\hat{F}$, i.e. there exists a Fréchet-differentiable map $\hat{F} : L^{\alpha+1}(M) \to \mathbb{R}$ with

$$\hat{F}'[u]h = \mathrm{Re}\langle F(u), h\rangle, \quad u, h \in L^{\alpha+1}(M). \tag{4.10}$$

By Assumption 4.3 ii) and the mean value theorem for Fréchet differentiable maps, we get

$$\begin{aligned}\|F(x)-F(y)\|_{L^{\frac{\alpha+1}{\alpha}}(M)} &\le \sup_{t\in[0,1]}\|F'[tx+(1-t)y]\|_{\mathcal{L}(L^{\alpha+1})}\|x-y\|_{L^{\alpha+1}(M)}\\ &\lesssim \left(\|x\|_{L^{\alpha+1}(M)}+\|y\|_{L^{\alpha+1}(M)}\right)^{\alpha-1}\|x-y\|_{L^{\alpha+1}(M)},\quad x,y\in L^{\alpha+1}(M).\end{aligned} \tag{4.11}$$

In particular, the nonlinearity is Lipschitz on bounded sets of $L^{\alpha+1}(M)$. We will cover the following two standard types of nonlinearities.

Definition 4.4. Let F satisfy Assumption 4.3. Then, F is called *defocusing*, if $\hat{F}(u)\ge 0$ and *focusing*, if $\hat{F}(u)\le 0$ for all $u\in L^{\alpha+1}(M)$.

Assumption 4.5. We assume either i) or i'):

i) Let F be defocusing and satisfy

$$\|u\|_{L^{\alpha+1}(M)}^{\alpha+1}\lesssim \hat{F}(u),\quad u\in L^{\alpha+1}(M). \tag{4.12}$$

i') Let F be focusing and satisfy

$$-\hat{F}(u)\lesssim \|u\|_{L^{\alpha+1}(M)}^{\alpha+1},\quad u\in L^{\alpha+1}(M). \tag{4.13}$$

Suppose that there is $\theta\in(0,\frac{2}{\alpha+1})$ with

$$(H,E_A)_{\theta,1}\hookrightarrow L^{\alpha+1}(M). \tag{4.14}$$

Here, $(\cdot,\cdot)_{\theta,1}$ denotes the real interpolation space and we remark that by [118], Lemma 1.10.1, (4.14) is equivalent to

$$\|u\|_{L^{\alpha+1}(M)}^{\alpha+1}\lesssim \|u\|_H^{\beta_1}\|u\|_{E_A}^{\beta_2},\quad u\in E_A, \tag{4.15}$$

for some $\beta_1>0$ and $\beta_2\in(0,2)$ with $\alpha+1=\beta_1+\beta_2$. Let us continue with the definitions and assumptions for the stochastic part. The type of nonlinearity which we allow here is often called *saturated*.

Assumption 4.6. We assume the following:

i) Let Y be a separable real Hilbert space with ONB $(f_m)_{m\in\mathbb{N}}$ and W a Y-cylindrical Wiener process adapted to the filtration $\mathbb{F}$.

ii) Let $p\in\{\alpha+1,2\}$ and suppose that $g:[0,\infty)\to\mathbb{R}$ is a function such that the linear growth conditions

$$\|g(|v|^2)^j v\|_{E_A}\lesssim\|v\|_{E_A},\qquad \|g(|u|^2)^j u\|_{L^p}\lesssim\|u\|_{L^p},\qquad v\in E_A,\, u\in L^p(M), \tag{4.16}$$

and Lipschitz conditions

$$\|g(|u|^2)^j u-g(|v|^2)^j v\|_{L^p}\lesssim\|u-v\|_{L^p},\qquad u,v\in L^p(M), \tag{4.17}$$

are satisfied for $j=1,2$.

iii) For $m \in \mathbb{N}$, take $e_m \in L^\infty(M,\mathbb{C})$ such that the associated multiplication operator defined by $M_{e_m}u = e_m u$ for $u \in E_A$ is bounded on E_A. Assume

$$\sum_{m=1}^{\infty} \|M_{e_m}\|^2_{\mathcal{L}(E_A)} < \infty, \qquad \sum_{m=1}^{\infty} \|e_m\|^2_{L^\infty} < \infty. \tag{4.18}$$

iv) Let $B : H \to \mathrm{HS}(Y,H)$ be the nonlinear operator given by

$$B(u)f_m := e_m g(|u|^2)u, \qquad m \in \mathbb{N}, \quad u \in H.$$

Obviously, part ii) of the previous assumption is fulfilled for the constant function $g \equiv 1$ which leads to linear multiplicative noise. In Section 4.4.4, we will present other choices of g which satisfy Assumption 4.6.

Remark 4.7. Choose $E \in \{H, L^{\alpha+1}(M)\}$ and let $u \in L^r(\Omega, L^2(0,T;E))$ be a random variable represented by a strongly measurable and adapted process. From the estimates (4.16) and (4.18), we get

$$\|B(u)\|_{L^2(0,T;\mathrm{HS}(Y,H))} \lesssim \|u\|_{L^2(0,T;H)},$$
$$\left\| \left(\sum_{m=1}^{\infty} \int_0^T |B(u)f_m|^2 \mathrm{d}s \right)^{\frac{1}{2}} \right\|_{L^{\alpha+1}} \lesssim \|u\|_{L^2(0,T;L^{\alpha+1})}.$$

Moreover, the process

$$B(u)y = \sum_{m=1}^{\infty} \left(y, f_m\right)_Y e_m g(|u|^2)u, \qquad y \in Y,$$

is strongly measurable and adapted since the estimate

$$\begin{aligned} &\left\| \sum_{m=1}^{\infty} \left(y, f_m\right)_Y e_m \left[g(|w_1|^2)w_1 - g(|w_2|^2)w_2\right] \right\|_E \\ &\qquad \le \left(\sum_{m=1}^{\infty} |\left(y, f_m\right)_Y|^2 \right)^{\frac{1}{2}} \left(\sum_{m=1}^{\infty} \|e_m\|^2_{L^\infty} \right)^{\frac{1}{2}} \|g(|w_1|^2)w_1 - g(|w_2|^2)w_2\|_E \\ &\qquad \lesssim \|w_1 - w_2\|_E \end{aligned}$$

for $w_1, w_2 \in E$ implies that the map

$$E \ni w \mapsto \sum_{m=1}^{\infty} \left(y, f_m\right)_Y e_m g(|w|^2)w \in E$$

is Lipschitz continuous for fixed $y \in Y$. In view of Appendix A.1, $B(u)$ is stochastically integrable in E and this allows to define the stochastic integral $\int_0^t B(u(s))\mathrm{d}W(s)$ in E.

Finally, we have sufficient background in order to formulate and study the problem

$$\begin{cases} \mathrm{d}u(t) = \left(-\mathrm{i}Au(t) - \mathrm{i}F(u(t)) + \mu(u(t))\right)\mathrm{d}t - \mathrm{i}B(u(t))\mathrm{d}W(t), \\ u(0) = u_0 \in E_A, \end{cases} \tag{4.19}$$

where

$$B(u)f_m := e_m g(|u|^2)u, \qquad \mu(u) := -\frac{1}{2}\sum_{m=1}^{\infty}|e_m|^2 g(|u|^2)^2 u, \qquad u \in L^2(M), \quad m \in \mathbb{N}.$$

In the following, we would like to motivate the choice of the correction term μ. This is prepared by the next Lemma.

Lemma 4.8. *a) Take $p \in [1,\infty)$ and a continuously real differentiable function $\Phi : \mathbb{C} \to \mathbb{C}$ with*

$$|\Phi(z)| \le C|z|, \qquad |\Phi'(z)| \le C, \qquad z \in \mathbb{C}.$$

Then, the map $G : L^p(M) \to L^p(M)$ defined by $G(u) := \Phi(u)$ for $u \in L^p(M)$ is Gâteaux differentiable with $G'[u]h := \Phi'(u)h$ for $u, h \in L^p(M)$.

b) Suppose that $g : [0,\infty) \to \mathbb{R}$ is continuously differentiable with

$$\sup_{r>0}|g(r)| < \infty, \qquad \sup_{r>0} r|g'(r)| < \infty. \tag{4.20}$$

Then, the operator $B : H \to \mathrm{HS}(Y,H)$ defined by

$$B(u)f_m := e_m g(|u|^2)u, \qquad m \in \mathbb{N},\ u \in H.$$

is Gâteaux differentiable and the derivative $B'[u] \in \mathcal{L}(H, \mathrm{HS}(Y,H))$ for $u \in H$ is given by

$$(B'[u]h)f_m := e_m g'(|u|^2) 2\,\mathrm{Re}\langle u, h\rangle_{\mathbb{C}} u + e_m g(|u|^2)h, \qquad h \in H, \quad m \in \mathbb{N}.$$

In particular,

$$-\mathrm{i}B'[u]\left(-\mathrm{i}B(u)f_m\right)f_m := -e_m^2 g(|u|^2)^2 u, \qquad u \in H, \quad m \in \mathbb{N},$$

if e_m is real-valued for each $m \in \mathbb{N}$.

*Proof. ad a).*Let $u, h \in L^p(M)$. Since Φ is continuously differentiable, we have

$$\frac{1}{t}\left[G(u+th) - G(u)\right] \xrightarrow{t\to 0} \Phi'(u)h \tag{4.21}$$

almost everywhere in M. Moreover, we can estimate

$$\left|\frac{1}{t}\left[G(u+th) - G(u)\right]\right| \le \sup_{s\in[0,t]} |\Phi'(u+sh)||h| \le C|h|$$

and obtain (4.21) in $L^p(M)$ by Lebesgue's convergence theorem.

ad b). In view of Assumption 4.6 iii), $(B_0 v)f_m = e_m v$ for $v \in H$ and $m \in \mathbb{N}$ defines a linear operator $B_0 \in \mathcal{L}(H, \mathrm{HS}(Y,H))$. We set $\Phi(z) := g(|z|^2)z$ for $z \in \mathbb{C}$ and compute

$$\Phi'(z_1)z_2 = 2g'(|z_1|^2)\,\mathrm{Re}\langle z_1, z_2\rangle_{\mathbb{C}} z_1 + g(|z_1|^2)z_2, \qquad z_1, z_2 \in \mathbb{C},$$

From (4.20) and the boundedness of g, we infer

$$|\Phi'(z_1)z_2| \le 2|g'(|z_1|^2)||z_1|^2|z_2| + |g(|z_1|^2)||z_2| \lesssim |z_2|.$$

In particular, we have $|\Phi(z)| \lesssim |z|$ and $|\Phi'(z)| \lesssim 1$ for $z \in \mathbb{C}$. By part a), $G(u) := \Phi(u)$ for $u \in H$ defines a Gâteaux differentiable map $G : L^2(M) \to L^2(M)$ with

$$G'[u]h = g'(|u|^2)2\operatorname{Re}\langle u, h\rangle_{\mathbb{C}} u + g(|u|^2)h, \qquad u \in H, \quad h \in H.$$

Then, the first assertion is a consequence of the fact that the composition of a Gâteaux differentiable map with a bounded linear operator is still Gâteaux differentiable. The second assertion follows from

$$\operatorname{Re}\langle u, -\mathrm{i}B(u)f_m\rangle_{\mathbb{C}} = \operatorname{Re}\left[\mathrm{i}\overline{e_m} g(|u|^2)|u|^2\right] = 0$$

for real-valued e_m. □

Remark 4.9. The choice of the correction term $\mu(u)$ is motivated by the following. On the one hand, Corollary 2.8 yields the simple formula

$$\|u(t)\|_H^2 = \|u_0\|_H^2 - 2\int_0^t \operatorname{Re}\left(u(s), \mathrm{i}B(s)\mathrm{d}W(s)\right)_H, \qquad t \in [0,T],$$

for the evolution of the mass of solutions to (4.19). In particular, $t \mapsto \|u(t)\|_H^2$ is a martingale and in the special case of real valued coefficients e_m, $m \in \mathbb{N}$, it is almost surely constant.

On the other hand, there is the following relationship to the Stratonovich noise defined by

$$-\mathrm{i}B(u(t)) \circ \mathrm{d}W(t) = -\mathrm{i}B(u(t))\mathrm{d}W(t) + \frac{1}{2}\sum_{m=1}^{\infty} \mathcal{M}[u(t)](f_m, f_m)\mathrm{d}t \tag{4.22}$$

with

$$\mathcal{M}[u](y_1, y_2) := -\mathrm{i}B'[u](-\mathrm{i}B(u)y_1)y_2, \quad u \in H, \quad y_1, y_2 \in Y.$$

For real-valued coefficients e_m, $m \in \mathbb{N}$, Lemma 4.8 yields

$$\begin{aligned}
-\mathrm{i}B(u(t)) \circ \mathrm{d}W(t) &= -\mathrm{i}B(u(t))\mathrm{d}W(t) - \frac{1}{2}\sum_{m=1}^{\infty} e_m^2 g(|u(t)|^2)^2 u(t)\mathrm{d}t \\
&= -\mathrm{i}B(u(t))\mathrm{d}W(t) + \mu\left(u(t)\right)\mathrm{d}t.
\end{aligned}$$

Thus, (4.19) coincides with the NLS with Stratonovich noise.

The main content of this chapter is the proof of the following Theorem.

Theorem 4.10. *Let $T > 0$ and $u_0 \in E_A$. Under the Assumptions* 4.1, 4.3, 4.5, 4.6, *there exists an analytically weak global martingale solution* $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbb{F}}, u\right)$ *of* (4.1) *in E_A^* which satisfies $u \in C_w([0,T], E_A)$ almost surely and $u \in L^q(\tilde{\Omega}, L^\infty(0,T;E_A))$ for all $q \in [1,\infty)$.*

This theorem can be viewed as the first step in our study of the stochastic NLS for other settings than $\mathbb{R}^d$. Because the result is rather general, we will illustrate it in Section 4.4 by various examples. One might say that the disadvantage of Theorem 4.10 lies in the fact that it only provides a martingale, i.e. stochastically weak solution. In the special cases where we can also prove pathwise uniqueness, however, the Yamada-Watanabe Theorem 2.4 leads to the existence of a stochastically strong solution and consequently, this disadvantage can be compensated. Let us close this section with a remark on the case of linear noise.

Remark 4.11. Following the lines of the proof of Theorem 4.10 below, one can check that in the case $g \equiv 1$, it is not necessary that the coefficients of the noise are multiplication operators. We can also consider a sequence $(B_m)_{m\in\mathbb{N}}$ of operators with

$$\sum_{m=1}^{\infty} \|B_m\|^2_{\mathcal{L}(L^2)} < \infty, \qquad \sum_{m=1}^{\infty} \|B_m\|^2_{\mathcal{L}(E_A)} < \infty, \qquad \sum_{m=1}^{\infty} \|B_m\|^2_{\mathcal{L}(L^{\alpha+1})} < \infty$$

and set

$$B(u)f_m := B_m u, \qquad \mu(u) := -\frac{1}{2}\sum_{m=1}^{\infty} B_m B_m^* u, \qquad u \in L^2(M).$$

4.2. Galerkin approximation

In this section, we introduce the Galerkin approximation which will be used for the proof of Theorem 4.10. Moreover, we prove the wellposedness of the approximated equation and uniform estimates for the solutions that are sufficient to apply Corollary 2.40.

We start with some immediate conclusions from the assumptions.

Lemma 4.12. *a) The embedding $E_A \hookrightarrow H$ is compact.*

b) There is an orthonormal basis $(h_n)_{n\in\mathbb{N}}$ of the complex Hilbert space $\big(H, (\cdot,\cdot)_H\big)$ and a nondecreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n > 0$ and $\lambda_n \to \infty$ as $n \to \infty$ and

$$Sx = \sum_{n=1}^{\infty} \lambda_n \big(x, h_n\big)_H h_n, \quad x \in \mathcal{D}(S) = \Big\{x \in H : \sum_{n=1}^{\infty} \lambda_n^2 |\big(x, h_n\big)_H|^2 < \infty\Big\}.$$

Proof. ad a). The embedding $E_A \hookrightarrow L^{\alpha+1}(M)$ is compact by Assumption 4.1 ii) and $L^{\alpha+1}(M) \hookrightarrow H$ is continuous due to $\mu(M) < \infty$. Hence, $E_A \hookrightarrow H$ is compact.
ad b). This is an immediate consequence of the spectral theorem since S has a compact resolvent. □

For $n \in \mathbb{N}_0$, we set

$$H_n := \operatorname{span}\big\{h_m : m \in \mathbb{N}, \quad \lambda_m < 2^{n+1}\big\}$$

and denote the orthogonal projection from H to H_n by P_n, i.e.

$$P_n x = \sum_{\lambda_m < 2^{n+1}} \big(x, h_m\big)_H h_m, \qquad x \in H.$$

Although all norms on H_n are equivalent, it is natural to equip H_n with the restriction of the H-norm, i.e.

$$\|u\|^2_{H_n} = \sum_{\lambda_m < 2^{n+1}} |\big(x, h_m\big)_H|^2, \qquad u \in H_n.$$

Lemma 4.13. *We fix $n \in \mathbb{N}_0$.*

a) P_n is an orthogonal projection in H with range $H_n \subset E_A$ and $\|P_n\|_{\mathcal{L}(E_A)} \le 1$.

b) P_n *can be extended to an operator* $P_n : E_A^* \to E_A^*$ *with* $\|P_n\|_{E_A^* \to E_A^*} \le 1$, $P_n(E_A^*) = H_n$ *and*

$$\langle v, P_n v\rangle \in \mathbb{R}, \qquad \langle v, P_n w\rangle = \big(P_n v, w\big)_H, \qquad v \in E_A^*, \quad w \in H. \tag{4.23}$$

Proof. As an eigenvector of S, each h_m satisfies $h_m \in \bigcap_{k\in\mathbb{N}} \mathcal{D}(S^k)$ and thus, we obtain by Assumption 4.1 that H_n is a closed subspace of E_A for $n \in \mathbb{N}_0$. In particular, H_n is a closed subspace of E_A^*. Moreover, we have $P_n = \mathbf{1}_{(0,2^{n+1})}(S)$ and hence, P_n commutes with $(\operatorname{Id}+A)^{\frac{1}{2}}$ and $(\operatorname{Id}+A)^{-\frac{1}{2}}$ since S and A commute by Assumption 4.1. We obtain

$$\|P_n x\|_{E_A} = \|\,(\operatorname{Id}+A)^{\frac{1}{2}}\, P_n x\|_H \le \|\,(\operatorname{Id}+A)^{\frac{1}{2}}\, x\|_H = \|x\|_{E_A}, \qquad x \in E_A,$$

and

$$\|P_n x\|_{E_A^*} = \|\,(\operatorname{Id}+A)^{-\frac{1}{2}}\, P_n x\|_H \le \|\,(\operatorname{Id}+A)^{-\frac{1}{2}}\, x\|_H = \|x\|_{E_A^*}, \qquad x \in H.$$

By density, we can extend P_n to an operator $P_n : E_A^* \to E_A^*$ with $\|P_n\|_{E_A^* \to E_A^*} \le 1$ and we have $P_n(E_A^*) = H_n \subset E_A$. For $w \in H$ and $v \in E_A^*$ with $H \ni v_k \to v$ as $k \to \infty$, we conclude

$$\langle v, P_n v\rangle = \lim_{k\to\infty} \big(v_k, P_n v_k\big)_H \in \mathbb{R}$$

and

$$\langle v, P_n w\rangle = \lim_{k\to\infty} \big(v_k, P_n w\big)_H = \lim_{k\to\infty} \big(P_n v_k, w\big)_H = \big(P_n v, w\big)_H.$$

□

Despite their nice behavior as orthogonal projections, it turns out that the operators P_n, $n \in \mathbb{N}_0$, lack the crucial property needed in the proof of the a priori estimates of the stochastic terms: In general, they are not uniformly bounded from $L^{\alpha+1}(M)$ to $L^{\alpha+1}(M)$. To overcome this deficit, we construct another sequence $(S_n)_{n\in\mathbb{N}_0}$ of operators $S_n : H \to H_n$.

Proposition 4.14. *There exists a sequence* $(S_n)_{n\in\mathbb{N}_0}$ *of selfadjoint operators* $S_n : H \to H_n$ *for* $n \in \mathbb{N}_0$ *with* $S_n\psi \to \psi$ *in* E_A *for* $n \to \infty$ *and* $\psi \in E_A$ *and the uniform norm estimates*

$$\sup_{n\in\mathbb{N}_0} \|S_n\|_{\mathcal{L}(H)} \le 1, \quad \sup_{n\in\mathbb{N}_0} \|S_n\|_{\mathcal{L}(E_A)} \le 1, \quad \sup_{n\in\mathbb{N}_0} \|S_n\|_{\mathcal{L}(L^{\alpha+1})} < \infty. \tag{4.24}$$

In Figure 4.1, we display the functions $p_n, s_n : (0,\infty) \to [0,1]$ with $P_n = p_n(S)$ and $S_n = s_n(S)$. Somehow, s_n represents a smoothed version of the indicator function p_n. This allows to use spectral multiplier theorems to prove the uniform $L^{\alpha+1}$-boundedness of the sequence $(S_n)_{n\in\mathbb{N}_0}$.

Proof. Step 1. We take a function $\rho \in C_c^\infty(0,\infty)$ with $\operatorname{supp}\rho \subset [\frac{1}{2}, 2]$ and $\sum_{m\in\mathbb{Z}} \rho(2^{-m}t) = 1$ for all $t > 0$. For the existence of ρ with these properties, we refer to [15], Lemma 6.1.7. Then, we fix $n \in \mathbb{N}_0$ and define

$$s_n : (0,\infty) \to \mathbb{C}, \qquad s_n(\lambda) := \sum_{m=-\infty}^{n} \rho(2^{-m}\lambda).$$

Let $k \in \mathbb{Z}$ and $\lambda \in [2^{k-1}, 2^k)$. From $\operatorname{supp}\rho \subset [\frac{1}{2}, 2]$, we infer

$$1 = \sum_{m=-\infty}^{\infty} \rho(2^{-m}\lambda) = \rho(2^{-k}\lambda) + \rho(2^{-(k+1)}\lambda).$$

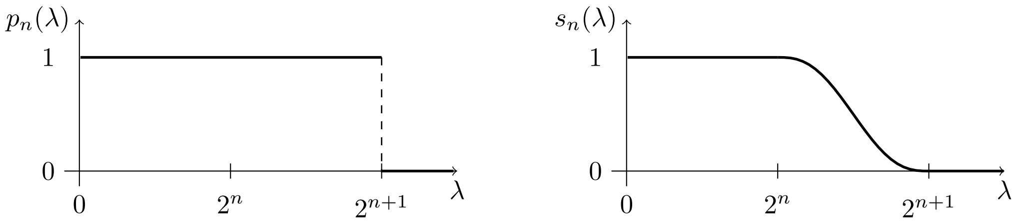

Figure 4.1.: Plot of the functions p_n and s_n with $P_n = p_n(S)$ and $S_n = s_n(S)$.

In particular

$$s_n(\lambda) = \begin{cases} 1, & \lambda \in (0, 2^n), \\ \rho(2^{-n}\lambda), & \lambda \in [2^n, 2^{n+1}), \\ 0, & \lambda \geq 2^{n+1}. \end{cases} \tag{4.25}$$

We define $S_n := s_n(S)$. Since s_n is real-valued and bounded by 1, the operator S_n is selfadjoint with $\|S_n\|_{\mathcal{L}(H)} \leq 1$. Furthermore, S_n and A commute due to the assumption that S and A commute. In particular, this implies $\|S_n\|_{\mathcal{L}(E_A)} \leq 1$ and $S_n\psi \to \psi$ for all $\psi \in E_A$ by the convergence property of the Borel functional calculus. Moreover, the range of S_n is contained in H_n since we have the representation

$$S_n x = \sum_{\lambda_m < 2^n} \big(x, h_m\big)_H h_m + \sum_{\lambda_m \in [2^n, 2^{n+1})} \rho(2^{-n}\lambda_m)\big(x, h_m\big)_H h_m, \quad x \in H,$$

as a consequence of(4.25).

Step 2. Next, we show the uniform estimate in $L^{\alpha+1}(M)$ based on a spectral multiplier theorem by Kunstmann and Uhl, [83], for operators with generalized Gaussian bounds. In view of Theorem 5.3 in [83], Lemma 2.19 and Fact 2.20 in [120], it is sufficient to show that s_n satisfies the Mihlin condition

$$\sup_{\lambda>0} |\lambda^k s_n^{(k)}(\lambda)| \leq C_k, \qquad k = 0, \ldots, \gamma, \tag{4.26}$$

for some $\gamma \in \mathbb{N}$ uniformly in $n \in \mathbb{N}_0$. This can be verified by the calculation

$$\sup_{\lambda>0} |\lambda^k s_n^{(k)}(\lambda)| = \sup_{\lambda \in [2^n, 2^{n+1})} |\lambda^k s_n^{(k)}(\lambda)| = \sup_{\lambda \in [2^n, 2^{n+1})} \left| \lambda^k \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} \rho(2^{-n}\lambda) \right| \leq 2^k \|\rho^{(k)}\|_\infty.$$

for all $k \in \mathbb{N}_0$. □

Remark 4.15. In view of the examples we treat in this thesis, it would be sufficient to assume that the heat semigroup associated to A has the upper Gaussian bounds from Remark 4.2 and use the spectral multiplier theorem 7.23 from the monograph by Ouhabaz, [104]. On the other hand, it would also be possible to use the weaker assumption that A is a 0-sectorial operator on $L^{\alpha+1}(M)$ with a Mihlin functional calculus, i.e. a bounded functional calculus for functions satisfying the Mihlin condition

$$\sup_{\lambda>0} |\lambda^k F^{(k)}(\lambda)| \leq C_k, \qquad k = 0, \ldots, \gamma, \tag{4.27}$$

for some $\gamma \in \mathbb{N}$. Then, Lemma 4.1 in [80] implies

$$\|x\|_{L^{\alpha+1}} \eqsim \sup_{\|a\|_{\ell^\infty(\mathbb{N}_0)}\le 1} \left\| a_0 \sum_{m=-\infty}^{0} \rho(2^{-m}S)x + \sum_{m=1}^{\infty} a_m \rho(2^{-m}S)x \right\|_{L^{\alpha+1}} \tag{4.28}$$

if we choose ρ as in the previous proof. Thus, the boundedness of the sequence $(S_n)_{n\in\mathbb{N}_0}$ in $\mathcal{L}(L^{\alpha+1}(M))$ can be obtained by the particular choice $\tilde{a}_m = 1$ for $0 \le m \le n$ and $\tilde{a}_m = 0$ for $m > n$ in (4.28). Indeed,

$$\begin{aligned} \|S_n x\|_{L^{\alpha+1}} &= \left\| \tilde{a}_0 \sum_{m=-\infty}^{0} \rho(2^{-m}S)x + \sum_{m=1}^{\infty} \tilde{a}_m \rho(2^{-m}S)x \right\|_{L^{\alpha+1}} \\ &\le \sup_{\|a\|_{\ell^\infty(\mathbb{N}_0)}\le 1} \left\| a_0 \sum_{m=-\infty}^{0} \rho(2^{-m}S)x + \sum_{m=1}^{\infty} a_m \rho(2^{-m}S)x \right\|_{L^{\alpha+1}} \lesssim \|x\|_{L^{\alpha+1}}. \end{aligned}$$

We further remark that a similar construction with $S = \mathrm{Id} - \Delta_H$, where Δ_H denotes the Hodge-Laplacian, has been used in [63] and [64] for the approximation of a semilinear Maxwell equation.

Using the operators P_n and S_n, $n \in \mathbb{N}_0$, we approximate our original problem (4.1) by the following stochastic differential equation

$$\begin{cases} \mathrm{d}u_n(t) = \big(-\mathrm{i}Au_n(t) - \mathrm{i}P_nF(u_n(t)) + S_n\mu(u_n(t))\big)\mathrm{d}t - \mathrm{i}S_nB(u_n(t))\mathrm{d}W(t), \\ u_n(0) = S_nu_0. \end{cases} \tag{4.29}$$

in the finite dimensional real Hilbert space $\big(H_n, \mathrm{Re}\,(\cdot,\cdot)_H\big)$. Here, we need that A leaves the space H_n invariant since A commutes with P_n and $H_n \subset E_A$. By the well known theory for finite dimensional stochastic differential equations with locally Lipschitz coefficients, we get a local wellposedness result for (4.29).

Proposition 4.16. *For each $n \in \mathbb{N}_0$, there is a unique maximal solution $\big(u_n, (\tau_{n,k})_{k\in\mathbb{N}}, \tau_n\big)$ of* (4.29) *with continuous paths in H_n, i.e. there is an increasing sequence $(\tau_{n,k})_{k\in\mathbb{N}}$ of stopping times with $\tau_n = \sup_{k\in\mathbb{N}} \tau_{n,k}$ and*

$$u_n(t) = S_nu_0 + \int_0^t \big[-\mathrm{i}Au_n(s) - \mathrm{i}P_nF(u_n(s)) + S_n\mu(u_n(s))\big]\mathrm{d}s - \mathrm{i}\int_0^t S_nB(u_n(s))\mathrm{d}W(s) \tag{4.30}$$

almost surely on $\{t \le \tau_{n,k}\}$ for all $k \in \mathbb{N}$. Moreover, we have the blow-up criterion

$$\mathbb{P}(\tau_{n,k} < T \quad \forall k \in \mathbb{N}, \quad \sup_{t\in[0,\tau_n)} \|u_n(t)\|_{H_n} < \infty) = 0. \tag{4.31}$$

Proof. Let $n \in \mathbb{N}_0$. The assertion follows, if we can show that the functions $f_n : H_n \to H_n$ and $\sigma_n : H_n \to \mathrm{HS}(Y, H_n)$ defined by

$$f_n(x) := -\mathrm{i}Ax - iP_nF(x) + S_n\mu(x), \qquad \sigma_n(x) := -\mathrm{i}S_nB(x), \qquad x \in H_n,$$

are Lipschitz on balls in H_n. Given $R > 0$ and $x, y \in H_n$ with $\|x\|_{H_n} \le R$ and $\|y\|_{H_n} \le R$, we estimate

$$\|S_n\mu(x) - S_n\mu(y)\|_H \le \left\| \sum_{m=1}^{\infty} |e_m|^2 \left(g(|x|^2)^2x - g(|y|^2)^2y\right) \right\|_H$$

$$\leq \sum_{m=1}^{\infty} \|e_m\|_{L^\infty}^2 \|g(|x|^2)^2 x - g(|y|^2)^2 y\|_H \lesssim \|x-y\|_H$$

where we used (4.18) and (4.17). Since all norms on H_n are equivalent, (4.11) yields

$$\begin{aligned}\|P_nF(x)-P_nF(y)\|_H &\lesssim_n \|P_nF(x)-P_nF(y)\|_{E_A^*} \lesssim \|F(x)-F(y)\|_{L^{\frac{\alpha+1}{\alpha}}}\\ &\lesssim (\|x\|_{L^{\alpha+1}}+\|y\|_{L^{\alpha+1}})^{\alpha-1}\|x-y\|_{L^{\alpha+1}}\\ &\lesssim_n (\|x\|_H+\|y\|_H)^{\alpha-1}\|x-y\|_H \leq \alpha 2^{\alpha-1}R^{\alpha-1}\|x-y\|_H.\end{aligned}$$

Hence, we obtain

$$\|f_n(x)-f_n(y)\|_H \lesssim_{n,R} \|x-y\|_H$$

since $A|_{H_n}$ is a bounded operator. From (4.18) and (4.17), we infer

$$\begin{aligned}\|\sigma_n(x)-\sigma_n(y)\|_{\mathrm{HS}(Y,H_n)}^2 &= \sum_{m=1}^{\infty} \|S_n\left(e_m g(|x|^2)x\right) - S_n\left(e_m g(|y|^2)y\right)\|_H^2\\ &\leq \left(\sum_{m=1}^{\infty}\|e_m\|_{L^\infty}^2\right)\|g(|x|^2)x-g(|y|^2)y\|_H^2 \lesssim \|x-y\|_H.\end{aligned}$$

□

The global existence for equation (4.29) is based on the boundedness of the L^2-norm of solutions.

Proposition 4.17. *For each $n \in \mathbb{N}_0$, there is a unique global solution u_n of* (4.29) *with continuous paths in H_n and for each $q \in [1,\infty)$, there is a constant $C>0$ such that*

$$\mathbb{E}\Big[\sup_{s\in[0,T]}\|u_n(s)\|_H^{2q}\Big] \leq C^q \|u_0\|_H^{2q} e^{CqT}. \tag{4.32}$$

Proof. Step 1: We fix $n \in \mathbb{N}_0$ and take the unique maximal solution (u_n,τ_n) from Proposition 4.16. First, we show that the estimate (4.32) holds almost surely on $\{t \leq \tau_n\}$. From Theorem 2.6, we infer

$$\begin{aligned}\|u_n(t)\|_H^2 =& \|S_n u_0\|_H^2 - 2\sum_{m=1}^{\infty}\int_0^t \operatorname{Re}\left(|e_m|^2 S_n u_n(s), g(|u_n(s)|^2)^2 u_n(s)\right)_H \mathrm{d}s\\ &+ 2\int_0^t \operatorname{Re}\left(u_n(s), -\mathrm{i}S_n B(u_n(s))\mathrm{d}W(s)\right)_H + \sum_{m=1}^{\infty}\int_0^t \|S_n\left[e_m g(|u_n(s)|^2)u_n(s)\right]\|_H^2 \mathrm{d}s\end{aligned}$$

almost surely in $\{t \leq \tau_{n,k}\}$. Note that one can also obtain this identity from a direct application of the finite dimensional Itô formula without the regularization procedure from Theorem 2.6.

Next, we would like to apply the norm in $L^q(\Omega, L^\infty(0, t\wedge\tau_{n,k}))$ to this identity for $t \in [0,T]$ and start with the estimates of the terms on the RHS. By Assumption 4.6, Proposition 4.14 and the Minkowski inequality, we obtain

$$\left\|\sum_{m=1}^{\infty}\int_0^\cdot \operatorname{Re}\left(|e_m|^2 S_n u_n(s), g(|u_n(s)|^2)^2 u_n(s)\right)_H \mathrm{d}s\right\|_{L^q(\Omega,L^\infty(0,t\wedge\tau_{n,k}))}$$

$$\leq \left\| \int_0^{t\wedge\tau_{n,k}} \left(\sum_{m=1}^{\infty} \|e_m\|_{L^\infty}^2 \right) \|u_n(s)\|_H \|g(|u_n(s)|^2)^2 u_n(s)\|_H \mathrm{d}s \right\|_{L^q(\Omega)}$$
$$\lesssim \left\| \int_0^{t\wedge\tau_{n,k}} \|u_n(s)\|_H^2 \mathrm{d}s \right\|_{L^q(\Omega)} \leq \int_0^t \left\| \|u_n\|_H^2 \right\|_{L^q(\Omega, L^\infty(s\wedge\tau_{n,k}))} \mathrm{d}s.$$

Similarly, we conclude

$$\left\| \sum_{m=1}^{\infty} \int_0^{\cdot} \|S_n \left[e_m g(|u_n(s)|^2) u_n(s) \right] \|_H^2 \mathrm{d}s \right\|_{L^q(\Omega, L^\infty(0,t\wedge\tau_{n,k}))}$$
$$\lesssim \left\| \int_0^{t\wedge\tau_{n,k}} \|g(|u_n(s)|^2)^2 u_n(s)\|_H^2 \mathrm{d}s \right\|_{L^q(\Omega)} \lesssim \int_0^t \left\| \|u_n\|_H^2 \right\|_{L^q(\Omega, L^\infty(s\wedge\tau_{n,k}))} \mathrm{d}s.$$

Fix $\varepsilon > 0$ to be specified later. For the stochastic term, we additionally use the BDG-inequality and Lemma 2.11 for the estimate

$$\left\| \int_0^{\cdot} \operatorname{Re}\left(u_n(s), -\mathrm{i} S_n B(u_n(s)) \mathrm{d}W(s) \right)_H \right\|_{L^q(\Omega, L^\infty(0,t\wedge\tau_{n,k}))}$$
$$\lesssim \left\| \left(\sum_{m=1}^{\infty} \left| \left(u_n, -\mathrm{i} S_n \left[e_m g(|u_n|^2) u_n \right] \right)_H \right|^2 \right)^{\frac{1}{2}} \right\|_{L^q(\Omega, L^2(0,t\wedge\tau_{n,k}))}$$
$$\leq \left\| \left(\sum_{m=1}^{\infty} \|e_m\|_{L^\infty}^2 \right)^{\frac{1}{2}} \|u_n\|_H \|g(|u_n|^2) u_n\|_H \right\|_{L^q(\Omega, L^2(0,t\wedge\tau_{n,k}))}$$
$$\leq \left\| \|u_n\|_H^2 \right\|_{L^q(\Omega, L^2(0,t\wedge\tau_{n,k}))}$$
$$\leq \varepsilon \left\| \|u_n\|_H^2 \right\|_{L^q(\Omega, L^\infty(0,t\wedge\tau_{n,k}))} + \frac{1}{4\varepsilon} \int_0^t \left\| \|u_n\|_H^2 \right\|_{L^q(\Omega, L^\infty(s\wedge\tau_{n,k}))} \mathrm{d}s.$$

Hence, we get

$$\left\| \|u_n\|_H^2 \right\|_{L^q(\Omega, L^\infty(0,t\wedge\tau_{n,k}))} \lesssim \|u_0\|_H^2 + \varepsilon \left\| \|u_n\|_H^2 \right\|_{L^q(\Omega, L^\infty(0,t\wedge\tau_{n,k}))} + \int_0^t \left\| \|u_n\|_H^2 \right\|_{L^q(\Omega, L^\infty(s\wedge\tau_{n,k}))} \mathrm{d}s.$$

If we choose $\varepsilon > 0$ small enough, we can apply the Gronwall Lemma and infer

$$\left\| \|u_n\|_H^2 \right\|_{L^q(\Omega, L^\infty(0,t\wedge\tau_{n,k}))} \leq C \|u_0\|_H^2 e^{Ct} \tag{4.33}$$

with a constant independent of $n \in \mathbb{N}_0$.

Step 2. To show $\tau_n = T$ almost surely, we proceed as follows. We decompose

$$\Omega = \left(\bigcup_{k\in\mathbb{N}} \{T = \tau_{n,k}\} \right) \cup \{\tau_{n,k} < T \quad \forall k \in \mathbb{N}, \quad \sup_{t\in[0,\tau_n)} \|u_n(t)\|_{H_n} < \infty\}$$
$$\cup \{\tau_{n,k} < T \quad \forall k \in \mathbb{N}, \quad \sup_{t\in[0,\tau_n)} \|u_n(t)\|_{H_n} = \infty\}$$

and use that the second and the third set have measure zero by the blow-up-criterion (4.31) and the first step, respectively. Thus, the first set has measure one and in particular, we have $\tau_n = T$ almost surely. Then, the estimate (4.32) is a consequence of (4.33). □

The next goal is to find uniform energy estimates for the global solutions of the equation (4.29). Recall that by Assumption 4.3, the nonlinearity F has a real antiderivative defined on $L^{\alpha+1}(M)$ and denoted by $\hat{F}$.

Definition 4.18. We define the energy $\mathcal{E}: E_A \to \mathbb{R}$ by

$$\mathcal{E}(u) := \frac{1}{2}\|A^{\frac{1}{2}}u\|_H^2 + \hat{F}(u), \qquad u \in E_A.$$

Note that $\mathcal{E}(u)$ is welldefined by the embedding $E_A \hookrightarrow L^{\alpha+1}(M)$. The next Proposition is the key step to show that we can apply Corollary 2.40 to the sequence of solutions $(u_n)_{n\in\mathbb{N}_0}$ to the equation (4.29) in the defocusing case.

Proposition 4.19. *Under Assumption 4.5 i), the following assertions hold.*

a) For all $q \in [1,\infty)$ there is a constant $C = C(q, \|u_0\|_{L^2}, \mathcal{E}(u_0), \alpha, F, (e_m)_{m\in\mathbb{N}}, T) > 0$ with

$$\sup_{n\in\mathbb{N}_0} \mathbb{E}\Big[\sup_{t\in[0,T]} \left[\|u_n(t)\|_H^2 + \mathcal{E}(u_n(t))\right]^q\Big] \le C.$$

In particular, for all $r \in [1,\infty)$ there is $C_1 = C_1(r, \|u_0\|_{L^2}, \mathcal{E}(u_0), \alpha, F, (e_m)_{m\in\mathbb{N}}, T) > 0$ such that

$$\sup_{n\in\mathbb{N}_0} \mathbb{E}\Big[\sup_{t\in[0,T]} \|u_n(t)\|_{E_A}^r\Big] \le C_1.$$

b) The sequence $(u_n)_{n\in\mathbb{N}_0}$ satisfies the Aldous condition $[A]$ in E_A^*.

Proof. ad a): By Assumption 4.3 ii) and iii), the restriction of the energy $\mathcal{E}: H_n \to \mathbb{R}$ is twice continuously Fréchet-differentiable with

$$\mathcal{E}'[v]h_1 = \operatorname{Re}\langle Av + F(v), h_1\rangle;$$
$$\mathcal{E}''[v][h_1,h_2] = \operatorname{Re}\left(A^{\frac{1}{2}}h_1, A^{\frac{1}{2}}h_2\right)_H + \operatorname{Re}\langle F'[v]h_2, h_1\rangle$$

for $v, h_1, h_2 \in H_n$. We compute

$$\begin{aligned}
\operatorname{tr}&\Big(\mathcal{E}''[u_n(s)]\left(-\mathrm{i}S_n B(u_n(s)), -\mathrm{i}S_n B(u_n(s))\right)\Big)\\
&= \sum_{m=1}^\infty \mathcal{E}''[u_n(s)]\left(-\mathrm{i}S_n B(u_n(s)) f_m, -\mathrm{i}S_n B(u_n(s)) f_m\right)\\
&= \sum_{m=1}^\infty \left\|A^{\frac{1}{2}} S_n\left[e_m g(|u_n(s)|^2)u_n(s)\right]\right\|_H^2\\
&\quad + \sum_{m=1}^\infty \operatorname{Re}\left\langle F'[u_n(s)]\left(S_n\left[e_m g(|u_n(s)|^2)u_n(s)\right]\right), S_n\left[e_m g(|u_n(s)|^2)u_n(s)\right]\right\rangle
\end{aligned}$$

and therefore, Itô's formula leads to the identity

$$\begin{aligned}
\|u_n(t)\|_H^2 + \mathcal{E}(u_n(t)) &= \|u_n(t)\|_H^2 + \mathcal{E}(S_n u_0)\\
&\quad + \int_0^t \operatorname{Re}\left\langle Au_n(s) + F(u_n(s)), -\mathrm{i}Au_n(s) - \mathrm{i}P_n F(u_n(s))\right\rangle \mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \operatorname{Re}\left\langle Au_n(s)+F(u_n(s)),S_n\mu(u_n(s))\right\rangle \mathrm{d}s\\
&+\int_0^t \operatorname{Re}\left\langle Au_n(s)+F(u_n(s)),-\mathrm{i}S_nB\left(u_n(s)\right)\mathrm{d}W(s)\right\rangle\\
&+\frac{1}{2}\sum_{m=1}^\infty\int_0^t \|A^{\frac{1}{2}}S_n\left[e_mg(|u_n(s)|^2)u_n(s)\right]\|_H^2\mathrm{d}s\\
&+\frac{1}{2}\int_0^t\sum_{m=1}^\infty \operatorname{Re}\left\langle F'[u_n(s)]\left(S_n\left[e_mg(|u_n(s)|^2)u_n(s)\right]\right),S_n\left[e_mg(|u_n(s)|^2)u_n(s)\right]\right\rangle \mathrm{d}s
\end{aligned}
\tag{4.34}
$$

almost surely for all $t\in[0,T]$. We can use Lemma 4.13 b) for

$$\operatorname{Re}\left\langle F(v),-\mathrm{i}P_nF(v)\right\rangle=\operatorname{Re}\left[\mathrm{i}\left\langle F(v),P_nF(v)\right\rangle\right]=0;$$

$$\operatorname{Re}\left[\left\langle Av,-\mathrm{i}P_nF(v)\right\rangle+\left\langle F(v),-\mathrm{i}Av\right\rangle\right]=\operatorname{Re}\left[-\left\langle Av,\mathrm{i}F(v)\right\rangle+\overline{\left\langle Av,\mathrm{i}F(v)\right\rangle}\right]=0;$$

$$\operatorname{Re}\left(Av,-\mathrm{i}Av\right)_H=\operatorname{Re}\left[\mathrm{i}\|Av\|_H^2\right]=0$$

for all $v\in H_n$ to simplify (4.34) and get

$$
\begin{aligned}
\|u_n(t)\|_H^2+\mathcal{E}\left(u_n(t)\right)=\|u_n(t)\|_H^2+\mathcal{E}\left(S_nu_0\right)&+\int_0^t \operatorname{Re}\left\langle Au_n(s)+F(u_n(s)),S_n\mu(u_n(s))\right\rangle \mathrm{d}s\\
&+\int_0^t \operatorname{Re}\left\langle Au_n(s)+F(u_n(s)),-\mathrm{i}S_nB\left(u_n(s)\right)\mathrm{d}W(s)\right\rangle\\
&+\frac{1}{2}\sum_{m=1}^\infty\int_0^t \|A^{\frac{1}{2}}S_n\left[e_mg(|u_n(s)|^2)u_n(s)\right]\|_H^2\mathrm{d}s\\
&+\frac{1}{2}\int_0^t\sum_{m=1}^\infty \operatorname{Re}\left\langle F'[u_n(s)]\left(S_n\left[e_mg(|u_n(s)|^2)u_n(s)\right]\right),S_n\left[e_mg(|u_n(s)|^2)u_n(s)\right]\right\rangle \mathrm{d}s
\end{aligned}
\tag{4.35}
$$

almost surely for all $t\in[0,T]$. We introduce the short notation

$$Y(s):=\|u_n(s)\|_H^2+\mathcal{E}\left(u_n(s)\right),\qquad s\in[0,T],$$

and would like to take the $L^q(\Omega,L^\infty(0,t))$-norm in the identity (4.35). As a preparation, we start with the estimates for the integrands for fixed $s\in[0,T]$ and $m\in\mathbb{N}$. Note that $A^{\frac{1}{2}}\left(\mathrm{Id}+A\right)^{-\frac{1}{2}}$ is a bounded operator on H due to the functional calculus for selfadjoint operators. This fact and Proposition 4.14 will be frequently used without further reference. We use (4.16) to estimate

$$
\begin{aligned}
\left|\left(Au_n(s),-\mathrm{i}S_nB\left(u_n(s)\right)f_m\right)_H\right|&\le\|A^{\frac{1}{2}}u_n(s)\|_H\|A^{\frac{1}{2}}S_n\left[e_mg(|u_n(s)|^2)u_n(s)\right]\|_H\\
&\le\|A^{\frac{1}{2}}u_n(s)\|_H\|S_n\left[e_mg(|u_n(s)|^2)u_n(s)\right]\|_{E_A}\\
&\le\|A^{\frac{1}{2}}u_n(s)\|_H\|S_n\|_{\mathcal{L}(E_A)}\|M_{e_m}\|_{\mathcal{L}(E_A)}\|g(|u_n(s)|^2)u_n(s)\|_{E_A}\\
&\le\left(\|u_n(s)\|_H^2+\|A^{\frac{1}{2}}u_n(s)\|_H^2\right)\|M_{e_m}\|_{\mathcal{L}(E_A)}\\
&\lesssim Y(s)\|M_{e_m}\|_{\mathcal{L}(E_A)}
\end{aligned}
\tag{4.36}
$$

and (4.7) as well as (4.12) to obtain

$$\begin{aligned}\left|\left\langle F(u_n(s)),-\mathrm{i}S_nB\left(u_n(s)\right)f_m\right\rangle\right| &\le \|F(u_n(s))\|_{L^{\frac{\alpha+1}{\alpha}}(M)}\|S_n\left[e_mg(|u_n(s)|^2)u_n(s)\right]\|_{L^{\alpha+1}}\\ &\le \|u_n(s)\|_{L^{\alpha+1}}^{\alpha}\|S_n\|_{\mathcal{L}(L^{\alpha+1})}\|e_m\|_{L^\infty}\|g(|u_n(s)|^2)u_n(s)\|_{L^{\alpha+1}}\\ &\lesssim \|u_n(s)\|_{L^{\alpha+1}}^{\alpha+1}\|S_n\|_{\mathcal{L}(L^{\alpha+1})}\|e_m\|_{L^\infty}\\ &\lesssim \hat{F}(u_n(s))\|e_m\|_{L^\infty}\\ &\lesssim Y(s)\|e_m\|_{L^\infty}. \end{aligned} \tag{4.37}$$

Again by (4.16), we get

$$\begin{aligned}&\mathrm{Re}\left(Au_n(s),S_n\left[|e_m|^2g(|u_n(s)|^2)^2u_n(s)\right]\right)_H\\ &\quad\le \|A^{\frac{1}{2}}u_n(s)\|_H\|A^{\frac{1}{2}}S_n\left[|e_m|^2g(|u_n(s)|^2)^2u_n(s)\right]\|_H\\ &\quad\le \|A^{\frac{1}{2}}u_n(s)\|_H\|S_n\|_{\mathcal{L}(E_A)}\|M_{|e_m|^2}\|_{\mathcal{L}(E_A)}\|g(|u_n(s)|^2)^2u_n(s)\|_{E_A}\\ &\quad\le \|A^{\frac{1}{2}}u_n(s)\|_H\|S_n\|_{\mathcal{L}(E_A)}\|M_{|e_m|^2}\|_{\mathcal{L}(E_A)}\|u_n(s)\|_{E_A}\\ &\quad\le \left(\|u_n(s)\|_H^2+\|A^{\frac{1}{2}}u_n(s)\|_H^2\right)\|M_{|e_m|^2}\|_{\mathcal{L}(E_A)}\\ &\quad\lesssim Y(s)\|M_{|e_m|^2}\|_{\mathcal{L}(E_A)} \end{aligned} \tag{4.38}$$

and (4.7), (4.12) and (4.16) yield

$$\begin{aligned}&\mathrm{Re}\left\langle F(u_n(s)),S_n\left[|e_m|^2g(|u_n(s)|^2)^2u_n(s)\right]\right\rangle\\ &\quad\le \|F(u_n(s))\|_{L^{\frac{\alpha+1}{\alpha}}(M)}\|S_n\left[|e_m|^2g(|u_n(s)|^2)^2u_n(s)\right]\|_{L^{\alpha+1}}\\ &\quad\lesssim \|u_n(s)\|_{L^{\alpha+1}}^{\alpha}\|S_n\|_{\mathcal{L}(L^{\alpha+1})}\|M_{|e_m|^2}\|_{\mathcal{L}(L^{\alpha+1})}\|g(|u_n(s)|^2)^2u_n(s)\|_{L^{\alpha+1}}\\ &\quad\lesssim \|u_n(s)\|_{L^{\alpha+1}}^{\alpha+1}\|S_n\|_{\mathcal{L}(L^{\alpha+1})}\|e_m\|_{L^\infty}^2\\ &\quad\lesssim \hat{F}(u_n(s))\|e_m\|_{L^\infty}^2\lesssim Y(s)\|e_m\|_{L^\infty}^2 \end{aligned} \tag{4.39}$$

and

$$\begin{aligned}\|A^{\frac{1}{2}}S_n\left[e_mg(|u_n(s)|^2)u_n(s)\right]\|_H^2 &\le \|S_n\|_{\mathcal{L}(E_A)}^2\|M_{e_m}\|_{\mathcal{L}(E_A)}^2\|g(|u_n(s)|^2)u_n(s)\|_{E_A}^2\\ &\le \|M_{e_m}\|_{\mathcal{L}(E_A)}^2\left(\|u_n(s)\|_H^2+\|A^{\frac{1}{2}}u_n(s)\|_H^2\right)\\ &\lesssim \|M_{e_m}\|_{\mathcal{L}(E_A)}^2Y(s). \end{aligned} \tag{4.40}$$

From (4.9), (4.12) and (4.16), we infer

$$\begin{aligned}&\mathrm{Re}\left\langle F'[u_n(s)]S_n\left[e_mg(|u_n(s)|^2)u_n(s)\right],S_n\left[e_mg(|u_n(s)|^2)u_n(s)\right]\right\rangle\\ &\quad\lesssim \|F'[u_n(s)]\|_{\mathcal{L}(L^{\alpha+1},L^{\frac{\alpha+1}{\alpha}})}\|S_n\left[e_mg(|u_n(s)|^2)u_n(s)\right]\|_{L^{\alpha+1}}^2\\ &\quad\le \|u_n(s)\|_{L^{\alpha+1}}^{\alpha-1}\|S_n\|_{\mathcal{L}(L^{\alpha+1})}^2\|e_m\|_{L^\infty}^2\|g(|u_n(s)|^2)u_n(s)\|_{L^{\alpha+1}}^2\\ &\quad\le \|u_n(s)\|_{L^{\alpha+1}}^{\alpha+1}\|S_n\|_{\mathcal{L}(L^{\alpha+1})}^2\|e_m\|_{L^\infty}^2\\ &\quad\lesssim \hat{F}(u_n(s))\|e_m\|_{L^\infty}^2\lesssim Y(s)\|e_m\|_{L^\infty}^2. \end{aligned} \tag{4.41}$$

After these preparations, we can estimate the terms on the RHS of (4.35) in the $L^q(\Omega,L^\infty(0,t))$-norm, where the summations over $m\in\mathbb{N}$ are handled with (4.18). Applying (4.38), (4.39) and

$$\|M_{|e_m|^2}\|_{\mathcal{L}(E_A)}=\|M_{e_m}M_{e_m}^*\|_{\mathcal{L}(E_A)}=\|M_{e_m}\|_{\mathcal{L}(E_A)}^2,$$

we obtain

$$\begin{aligned}
&\left\|\int_0^{\cdot} \operatorname{Re}\left\langle Au_n(s)+F(u_n(s)), S_n\mu(u_n(s))\right\rangle \mathrm{d}s\right\|_{L^q(\Omega,L^\infty(0,t))}\\
&\qquad\le \left\|\int_0^t \sum_{m=1}^\infty \left|\operatorname{Re}\left\langle Au_n(s)+F(u_n(s)), S_n\left[|e_m|^2 g(|u_n(s)|^2)^2 u_n(s)\right]\right\rangle\right| \mathrm{d}s\right\|_{L^q(\Omega)}\\
&\qquad\lesssim \left\|\int_0^t Y(s)\sum_{m=1}^\infty \left[\|M_{|e_m|^2}\|_{\mathcal{L}(E_A)}+\|e_m\|_{L^\infty}^2\right]\mathrm{d}s\right\|_{L^q(\Omega)} \lesssim \int_0^t \|Y\|_{L^q(\Omega,L^\infty(0,s))}\mathrm{d}s.
\end{aligned} \tag{4.42}$$

The BDG-inequality, (4.36), (4.37) and Lemma 2.11 yield

$$\begin{aligned}
&\left\|\int_0^{\cdot} \operatorname{Re}\left\langle Au_n(s)+F(u_n(s)), -\mathrm{i}S_n B\left(u_n(s)\right)\mathrm{d}W(s)\right\rangle\right\|_{L^q(\Omega,L^\infty(0,t))}\\
&\qquad\lesssim \left\|\left(\sum_{m=1}^\infty \left|\operatorname{Re}\left\langle Au_n+F(u_n), -\mathrm{i}S_n\left[e_m g(|u_n|^2)u_n\right]\right\rangle\right|^2\right)^{\frac12}\right\|_{L^q(\Omega,L^2(0,t))}\\
&\qquad\lesssim \left\|Y\left(\sum_{m=1}^\infty \left[\|M_{e_m}\|_{\mathcal{L}(E_A)}+\|e_m\|_{L^\infty}\right]^2\right)^{\frac12}\right\|_{L^q(\Omega,L^2(0,t))}\\
&\qquad\lesssim \|Y\|_{L^q(\Omega,L^2(0,t))} \le \varepsilon\|Y\|_{L^q(\Omega,L^\infty(0,t))}+\frac{1}{4\varepsilon}\int_0^t \|Y\|_{L^q(\Omega,L^\infty(0,s))}\mathrm{d}s
\end{aligned} \tag{4.43}$$

where $\varepsilon>0$ is arbitrary. Moreover, we employ (4.40) to get

$$\begin{aligned}
&\left\|\sum_{m=1}^\infty \int_0^{\cdot} \|A^{\frac12}S_n\left[e_m g(|u_n(s)|^2)u_n(s)\right]\|_H^2\mathrm{d}s\right\|_{L^q(\Omega,L^\infty(0,t))}\\
&\qquad\le \left\|\sum_{m=1}^\infty \int_0^t \|A^{\frac12}S_n\left[e_m g(|u_n(s)|^2)u_n(s)\right]\|_H^2\mathrm{d}s\right\|_{L^q(\Omega)}\\
&\qquad\le \left\|\int_0^t Y(s)\mathrm{d}s\sum_{m=1}^\infty \|M_{e_m}\|_{\mathcal{L}(E_A)}^2\right\|_{L^q(\Omega)} \lesssim \int_0^t \|Y\|_{L^q(\Omega,L^\infty(0,s))}\mathrm{d}s
\end{aligned} \tag{4.44}$$

and (4.41) to estimate

$$\begin{aligned}
&\left\|\int_0^{\cdot}\sum_{m=1}^\infty \operatorname{Re}\left\langle F'[u_n(s)]\left(S_n\left[e_m g(|u_n(s)|^2)u_n(s)\right]\right), S_n\left[e_m g(|u_n(s)|^2)u_n(s)\right]\right\rangle\mathrm{d}s\right\|_{L^q(\Omega,L^\infty(0,t))}\\
&\quad\le \left\|\int_0^t\sum_{m=1}^\infty \left|\operatorname{Re}\left\langle F'[u_n(s)]\left(S_n\left[e_m g(|u_n(s)|^2)u_n(s)\right]\right), S_n\left[e_m g(|u_n(s)|^2)u_n(s)\right]\right\rangle\right|\mathrm{d}s\right\|_{L^q(\Omega)}\\
&\quad\le \left\|\int_0^t Y(s)\mathrm{d}s\sum_{m=1}^\infty \|e_m\|_{L^\infty}^2\right\|_{L^q(\Omega)} \lesssim \int_0^t \|Y\|_{L^q(\Omega,L^\infty(0,s))}\mathrm{d}s.
\end{aligned} \tag{4.45}$$

From (4.35) and the estimates (4.42)-(4.45), we infer

$$\|Y\|_{L^q(\Omega,L^\infty(0,t))} \lesssim \|\|u_n\|_H^2\|_{L^q(\Omega,L^\infty(0,t))}+\mathcal{E}\left(S_n u_0\right)+\left(1+\frac{1}{\varepsilon}\right)\int_0^t \|Y\|_{L^q(\Omega,L^\infty(0,s))}\mathrm{d}s$$

$$+\varepsilon\|Y\|_{L^q(\Omega,L^\infty(0,t))}$$

and if we choose $\varepsilon>0$ sufficiently small and employ Proposition 4.17, we obtain

$$\|Y\|_{L^q(\Omega,L^\infty(0,t))}\lesssim C\|u_0\|_H^2e^{Ct}+\mathcal{E}\left(S_nu_0\right)+\int_0^t\|Y\|_{L^q(\Omega,L^\infty(0,s))}\mathrm{d}s.$$

Proposition 4.14 yields

$$\mathcal{E}\left(S_nu_0\right)\le\max\{1,\|S_n\|_{\mathcal{L}(L^{\alpha+1})}^{\alpha+1}\}\mathcal{E}\left(u_0\right)$$

and thus, we can deduce from the Gronwall Lemma that there is a $C_1>0$ such that

$$\|Y\|_{L^q(\Omega,L^\infty(0,t))}\le C_1\left(C\|u_0\|_H^2e^{CT}+\mathcal{E}\left(u_0\right)\right)e^{C_1t},\qquad t\in[0,T].$$

ad b): We continue with the proof of the Aldous condition. We have

$$\begin{aligned}u_n(t)-S_nu_0=&-\mathrm{i}\int_0^tAu_n(s)\mathrm{d}s-\mathrm{i}\int_0^tP_nF(u_n(s))\mathrm{d}s+\int_0^tS_n\mu(u_n(s))\mathrm{d}s\\&-\mathrm{i}\int_0^tS_nB(u_n(s))\mathrm{d}W(s)\\=:&\,J_1(t)+J_2(t)+J_3(t)+J_4(t)\end{aligned}$$

in H_n almost surely for all $t\in[0,T]$ and therefore

$$\|u_n((\tau_n+\theta)\wedge T)-u_n(\tau_n)\|_{E_A^*}\le\sum_{k=1}^4\|J_k((\tau_n+\theta)\wedge T)-J_k(\tau_n)\|_{E_A^*}$$

for each sequence $(\tau_n)_{n\in\mathbb{N}_0}$ of stopping times and $\theta>0$. Hence, we get

$$\mathbb{P}\left\{\|u_n((\tau_n+\theta)\wedge T)-u_n(\tau_n)\|_{E_A^*}\ge\eta\right\}\le\sum_{k=1}^4\mathbb{P}\left\{\|J_k((\tau_n+\theta)\wedge T)-J_k(\tau_n)\|_{E_A^*}\ge\frac{\eta}{4}\right\}\tag{4.46}$$

for a fixed $\eta>0$. We aim to apply Tschebyscheff's inequality and estimate the expected value of each term in the sum. From the functional calculus for selfadjoint operators, we infer

$$\|Av\|_{E_A^*}=\|\left(\mathrm{Id}+A\right)^{-\frac12}Av\|_H\le\sup_{\lambda\ge0}\left\{\left(1+\lambda\right)^{-\frac12}\lambda^{\frac12}\right\}\|A^{\frac12}v\|_H\le\|A^{\frac12}v\|_H,\qquad v\in H_n,$$

and the uniform estimates from part a) yield

$$\begin{aligned}\mathbb{E}\|J_1((\tau_n+\theta)\wedge T)-J_1(\tau_n)\|_{E_A^*}&\le\mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\|Au_n(s)\|_{E_A^*}\mathrm{d}s\le\mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\|A^{\frac12}u_n(s)\|_H\mathrm{d}s\\&\lesssim\theta\mathbb{E}\big[\sup_{s\in[0,T]}\|u_n(s)\|_{E_A}\big]\le\theta\mathbb{E}\big[\sup_{s\in[0,T]}\|u_n(s)\|_{E_A}^2\big]^{\frac12}\le\theta C_1.\end{aligned}$$

Moreover, Lemma 4.13 b), the embedding $L^{\frac{\alpha+1}{\alpha}}(M)\hookrightarrow E_A^*$ and the estimates (4.7) and (4.12) of the nonlinearity lead to

$$\mathbb{E}\|J_2((\tau_n+\theta)\wedge T)-J_2(\tau_n)\|_{E_A^*}\le\mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\|P_nF(u_n(s))\|_{E_A^*}\mathrm{d}s$$

$$\leq \mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T} \|F(u_n(s))\|_{E_A^*}\mathrm{d}s \lesssim \mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T} \|F(u_n(s))\|_{L^{\frac{\alpha+1}{\alpha}}}\mathrm{d}s$$
$$\lesssim \mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T} \|u_n(s)\|_{L^{\alpha+1}}^{\alpha}\mathrm{d}s \lesssim \theta\mathbb{E}\big[\sup_{s\in[0,T]}\|u_n(s)\|_{E_A}^{\alpha}\big] \leq \theta C_2.$$

By Propositions 4.14 and 4.17, we get

$$\begin{aligned}
\mathbb{E}\|J_3((\tau_n+\theta)\wedge T)-J_3(\tau_n)\|_{E_A^*} &= \frac{1}{2}\mathbb{E}\left\|\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\sum_{m=1}^{\infty} S_n\left[|e_m|^2 g(|u_n(s)|^2)^2 u_n(s)\right]\mathrm{d}s\right\|_{E_A^*}\\
&\leq \frac{1}{2}\mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\sum_{m=1}^{\infty}\left\|S_n\left[|e_m|^2 g(|u_n(s)|^2)^2 u_n(s)\right]\right\|_{E_A^*}\mathrm{d}s\\
&\lesssim \mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\sum_{m=1}^{\infty}\left\|S_n\left[|e_m|^2 g(|u_n(s)|^2)^2 u_n(s)\right]\right\|_{H}\mathrm{d}s\\
&\lesssim \mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\|u_n(s)\|_H\mathrm{d}s\\
&\leq \theta\mathbb{E}\big[\sup_{s\in[0,T]}\|u_n(s)\|_H\big] = C_3\theta.
\end{aligned}$$

Finally, we use the Itô isometry and again the Propositions 4.14 and 4.17 for

$$\begin{aligned}
\mathbb{E}\|J_4((\tau_n+\theta)\wedge T)-J_4(\tau_n)\|_{E_A^*}^2 &\leq \mathbb{E}\left\|\int_{\tau_n}^{(\tau_n+\theta)\wedge T} S_nB\left(u_n(s)\right)\mathrm{d}W(s)\right\|_H^2\\
&= \mathbb{E}\left[\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\|S_nB\left(u_n(s)\right)\|_{\mathrm{HS}(Y,H)}^2\mathrm{d}s\right]\\
&= \mathbb{E}\left[\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\sum_{m=1}^{\infty}\left\|S_n\left[e_m g(|u_n(s)|^2)u_n(s)\right]\right\|_H^2\mathrm{d}s\right]\\
&\lesssim \mathbb{E}\left[\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\|u_n(s)\|_H^2\mathrm{d}s\right]\\
&\leq \theta\mathbb{E}\big[\sup_{s\in[0,T]}\|u_n(s)\|_H^2\big] = \theta C_4.
\end{aligned}$$

By the Tschebyscheff inequality, we obtain for a given $\eta > 0$

$$\mathbb{P}\left\{\|J_k((\tau_n+\theta)\wedge T)-J_k(\tau_n)\|_{E_A^*}\geq\frac{\eta}{4}\right\}\leq\frac{4}{\eta}\mathbb{E}\|J_k((\tau_n+\theta)\wedge T)-J_k(\tau_n)\|_{E_A^*}\leq\frac{4C_k\theta}{\eta} \tag{4.47}$$

for $k\in\{1,2,3\}$ and

$$\mathbb{P}\left\{\|J_4((\tau_n+\theta)\wedge T)-J_4(\tau_n)\|_{E_A^*}\geq\frac{\eta}{4}\right\}\leq\frac{16}{\eta^2}\mathbb{E}\|J_4((\tau_n+\theta)\wedge T)-J_4(\tau_n)\|_{E_A^*}^2\leq\frac{16C_4\theta}{\eta^2}. \tag{4.48}$$

Let us fix $\varepsilon > 0$. Due to the estimates (4.47) and (4.48) we can choose $\delta_1,\dots,\delta_4 > 0$ such that

$$\mathbb{P}\left\{\|J_k((\tau_n+\theta)\wedge T)-J_k(\tau_n)\|_{E_A^*}\geq\frac{\eta}{4}\right\}\leq\frac{\varepsilon}{4}$$

for $0 < \theta \le \delta_k$ and $k = 1, \dots, 4$. With $\delta := \min\{\delta_1, \dots, \delta_4\}$ and (4.46) we get

$$\mathbb{P}\left\{\|u_n((\tau_n+\theta)\wedge T) - u_n(\tau_n)\|_{E_A^*} \ge \eta\right\} \le \varepsilon$$

for all $n \in \mathbb{N}_0$ and $0 < \theta \le \delta$ and therefore, the Aldous condition $[A]$ holds in E_A^*. □

We continue with the a priori estimate for solutions of (4.29) with a focusing nonlinearity. Note that this case is harder since the expression

$$\frac{1}{2}\|v\|_H^2 + \mathcal{E}(v) := \frac{1}{2}\|v\|_{E_A}^2 + \hat{F}(v), \qquad v \in H_n,$$

does not dominate $\|v\|_{E_A}^2$, because $\hat{F}$ is negative.

Proposition 4.20. *Under Assumption* 4.5 *i'), the following assertions hold:*

a) For all $r \in [1,\infty)$, there is a constant $C = C(r, \|u_0\|_{E_A}, \alpha, F, (e_m)_{m\in\mathbb{N}}, T) > 0$ with

$$\sup_{n\in\mathbb{N}_0} \mathbb{E}\Big[\sup_{t\in[0,T]} \|u_n(t)\|_{E_A}^r\Big] \le C.$$

b) The sequence $(u_n)_{n\in\mathbb{N}_0}$ satisfies the Aldous condition $[A]$ in E_A^*.

Proof. Let $\varepsilon > 0$. By the same calculations as in the proof of Proposition 4.19 we get

$$\begin{aligned}
\frac{1}{2}\|A^{\frac{1}{2}}u_n(s)\|_H^2 &= \mathcal{E}(u_n(s)) - \hat{F}(u_n(s)) \\
&= -\hat{F}(u_n(s)) + \mathcal{E}(S_n u_0) + \int_0^s \mathrm{Re}\left\langle Au_n(r) + F(u_n(r)), S_n\mu(u_n(r))\right\rangle \mathrm{d}r \\
&\quad + \int_0^s \mathrm{Re}\left\langle Au_n(r) + F(u_n(r)), -\mathrm{i}S_n B(u_n(r))\,\mathrm{d}W(r)\right\rangle \\
&\quad + \frac{1}{2}\sum_{m=1}^\infty \int_0^s \|A^{\frac{1}{2}}S_n\left[e_m g(|u_n(r)|^2)u_n(r)\right]\|_H^2 \mathrm{d}r \\
&\quad + \frac{1}{2}\int_0^s \sum_{m=1}^\infty \mathrm{Re}\left\langle F'[u_n(r)]\left(S_n\left[e_m g(|u_n(r)|^2)u_n(r)\right]\right), S_n\left[e_m g(|u_n(r)|^2)u_n(r)\right]\right\rangle \mathrm{d}r
\end{aligned} \tag{4.49}$$

almost surely for all $s \in [0,T]$. In the following, we fix $q \in [1,\infty)$ and $t \in (0,T]$ and want to apply the $L^q(\Omega, L^\infty(0,t))$-norm to the identity (4.49). We will use the notation

$$X(s) := \left[\|u(s)\|_H^2 + \|A^{\frac{1}{2}}u_n(s)\|_H^2 + \|u_n(s)\|_{L^{\alpha+1}}^{\alpha+1}\right], \qquad s \in [0,T], \tag{4.50}$$

and estimate the stochastic integral by the Burkholder-Davis-Gundy inequality, the estimates (4.36) and (4.37), the Assumption (4.18) as well as Lemma 2.11

$$\begin{aligned}
&\left\|\int_0^\cdot \mathrm{Re}\left\langle Au_n(r) + F(u_n(r)), -\mathrm{i}S_n B(u_n(r))\,\mathrm{d}W(r)\right\rangle\right\|_{L^q(\Omega, L^\infty(0,t))} \\
&\qquad \lesssim \left\|\left(\sum_{m=1}^\infty |\left\langle Au_n(r) + F(u_n(r)), -\mathrm{i}S_n B(u_n(r))f_m\right\rangle|^2\right)^{\frac{1}{2}}\right\|_{L^q(\Omega, L^2(0,t))}
\end{aligned}$$

$$\lesssim \left\| X \left(\sum_{m=1}^{\infty} \left[\|M_{e_m}\|_{\mathcal{L}(E_A)} + \|e_m\|_{L^\infty} \right]^2 \right)^{\frac{1}{2}} \right\|_{L^q(\Omega, L^2(0,t))}$$

$$\lesssim \|X\|_{L^q(\Omega,L^2(0,t))} \leq \varepsilon \|X\|_{L^q(\Omega,L^\infty(0,t))} + \frac{1}{4\varepsilon} \int_0^t \|X\|_{L^q(\Omega,L^\infty(0,s))} \mathrm{d}s. \tag{4.51}$$

To control the term $-\hat{F}(u_n)$, we use that Assumption 4.5 i') and Young's inequality imply the existence of $\gamma > 0$ and $C_\varepsilon > 0$ such that

$$-\hat{F}(u) \lesssim \|u\|_{L^{\alpha+1}}^{\alpha+1} \leq \varepsilon \|u\|_{E_A}^2 + C_\varepsilon \|u\|_H^\gamma, \qquad u \in E_A.$$

Thus, Proposition 4.17 applied with exponents q and $\frac{\gamma q}{2}$ yields

$$\begin{aligned}
\| -\hat{F}(u_n)\|_{L^q(\Omega,L^\infty(0,t))} \lesssim & \left\| \|u\|_{L^{\alpha+1}}^{\alpha+1} \right\|_{L^q(\Omega,L^\infty(0,t))} \\
\leq & \varepsilon \left\| \|A^{\frac{1}{2}} u_n\|_H^2 \right\|_{L^q(\Omega,L^\infty(0,t))} + \varepsilon \left\| \|u_n\|_H^2 \right\|_{L^q(\Omega,L^\infty(0,t))} \\
& + C_\varepsilon \left\| \|u_n\|_H^\gamma \right\|_{L^q(\Omega,L^\infty(0,t))} \\
\leq & \varepsilon \left\| \|A^{\frac{1}{2}} u_n\|_H^2 \right\|_{L^q(\Omega,L^\infty(0,t))} + C(\varepsilon, q, \gamma, T, \|u_0\|_H).
\end{aligned} \tag{4.52}$$

The following estimates are quite similar as in the defocusing case. We use the estimates (4.38)-(4.41), the Assumption (4.18) on the noise coefficients and the Minkowski inequality to deduce

$$\begin{aligned}
\left\| \int_0^\cdot \operatorname{Re} \langle A u_n(s) + F(u_n(s)), S_n \mu(u_n(s)) \rangle \mathrm{d}s \right\|_{L^q(\Omega,L^\infty(0,t))} & \lesssim \left\| \int_0^t X(s) \mathrm{d}s \right\|_{L^q(\Omega)} \\
& \lesssim \int_0^t \|X(s)\|_{L^q(\Omega)} \mathrm{d}s;
\end{aligned} \tag{4.53}$$

$$\left\| \sum_{m=1}^{\infty} \int_0^\cdot \|A^{\frac{1}{2}} S_n B(u_n(s)) f_m\|_H^2 \mathrm{d}s \right\|_{L^q(\Omega,L^\infty(0,t))} \lesssim \left\| \int_0^t X(s) \mathrm{d}s \right\|_{L^q(\Omega)} \lesssim \int_0^t \|X(s)\|_{L^q(\Omega)} \mathrm{d}s; \tag{4.54}$$

$$\begin{aligned}
& \left\| \int_0^\cdot \sum_{m=1}^{\infty} \operatorname{Re} \langle F'[u_n(s)] \left(S_n \left[e_m g(|u_n(s)|^2) u_n(s) \right] \right), S_n \left[e_m g(|u_n(s)|^2) u_n(s) \right] \rangle \mathrm{d}s \right\|_{L^q(\Omega,L^\infty(0,t))} \\
& \qquad \lesssim \left\| \int_0^t X(s) \mathrm{d}s \right\|_{L^q(\Omega)} \lesssim \int_0^t \|X(s)\|_{L^q(\Omega)} \mathrm{d}s.
\end{aligned} \tag{4.55}$$

By the Itô representation (4.49) and the estimates (4.51)-(4.55), we get

$$\begin{aligned}
\left\| \|A^{\frac{1}{2}} u_n\|_H^2 \right\|_{L^q(\Omega,L^\infty(0,t))} \lesssim & \left\| \|A^{\frac{1}{2}} u_n(t)\|_H^2 \right\|_{L^q(\Omega,L^\infty(0,t))} \varepsilon + C(\varepsilon, q, \gamma, T, \|u_0\|_H) + \mathcal{E}(S_n u_0) \\
& + \int_0^t \|X(s)\|_{L^q(\Omega)} \mathrm{d}s + \varepsilon \|X\|_{L^q(\Omega,L^\infty(0,t))} \\
& + \frac{1}{4\varepsilon} \int_0^t \|X\|_{L^q(\Omega,L^\infty(0,s))} \mathrm{d}s
\end{aligned} \tag{4.56}$$

In order to estimate the terms with X by the LHS of (4.56), we exploit Proposition 4.17 and (4.52) to get

$$\begin{aligned}\|X\|_{L^q(\Omega,L^\infty(0,t))} &\leq C\|u_0\|_H^2 e^{CT} + \left\| \|A^{\frac{1}{2}}u_n\|_H^2 \right\|_{L^q(\Omega,L^\infty(0,t))} + \left\| \|u_n\|_{L^{\alpha+1}}^{\alpha+1} \right\|_{L^q(\Omega,L^\infty(0,t))} \\ &\leq (1+\varepsilon) \left\| \|A^{\frac{1}{2}}u_n\|_H^2 \right\|_{L^q(\Omega,L^\infty(0,t))} + C(\varepsilon,q,\gamma,T,\|u_0\|_H).\end{aligned}$$

Hence, by (4.50), we obtain

$$\begin{aligned}\left\| \|A^{\frac{1}{2}}u_n\|_H^2 \right\|_{L^q(\Omega,L^\infty(0,t))} \lesssim \left\| \|A^{\frac{1}{2}}u_n(t)\|_H^2 \right\|_{L^q(\Omega,L^\infty(0,t))} \varepsilon + \tilde{C}(\varepsilon,q,\gamma,T,\|u_0\|_H) + \mathcal{E}(S_n u_0) \\ + \int_0^t \left\| \|A^{\frac{1}{2}}u_n\|_H^2 \right\|_{L^q(\Omega,L^\infty(0,s))} \mathrm{d}s.\end{aligned}$$

Choosing $\varepsilon > 0$ small enough and exploiting Lemma 4.14 for

$$\mathcal{E}\left(S_n u_0\right) \leq \max\{1, \|S_n\|_{\mathcal{L}(L^{\alpha+1})}^{\alpha+1}\} \mathcal{E}\left(u_0\right) \lesssim \mathcal{E}\left(u_0\right),$$

we end up with

$$\left\| \|A^{\frac{1}{2}}u_n\|_H^2 \right\|_{L^q(\Omega,L^\infty(0,t))} \leq C_1(q,\gamma\|u_0\|_H, \mathcal{E}\left(u_0\right), T) + \int_0^t C_2(q) \left\| \|A^{\frac{1}{2}}u_n\|_H^2 \right\|_{L^q(\Omega,L^\infty(0,s))} \mathrm{d}s,$$

for $t \in [0,T]$. Finally, the Gronwall Lemma yields

$$\left\| \|A^{\frac{1}{2}}u_n(s)\|_H^2 \right\|_{L^q(\Omega,L^\infty(0,t))} \leq C_1(q,\gamma\|u_0\|_H, \mathcal{E}\left(u_0\right), T) e^{C_2(q)t}, \qquad t \in [0,T].$$

In view of Proposition 4.17, this implies that there is $C > 0$ with

$$\sup_{n\in\mathbb{N}_0} \mathbb{E}\Big[\sup_{t\in[0,T]} \|u_n(t)\|_{E_A}^{2q}\Big] \leq C.$$

Therefore, we obtain the assertion for $r \geq 2$. The case $r \in [1,2)$ is an application of Hölder's inequality.

ad b): Analogous to the proof of Proposition 4.19 b). □

4.3. Construction of a martingale solution

The aim of this section is to construct a solution of equation (4.1) by a suitable limiting process in the Galerkin equation (4.29) based on the uniform estimates from the previous section and the compactness results from Section 2.4. Let us start with a classical convergence Theorem which we will use very often throughout this section.

Lemma 4.21 (Vitali). *Let E be a Banach space, $(S,\mathcal{A},\nu)$ be a finite measure space, $p \in [1,\infty)$, $(f_n)_{n\in\mathbb{N}} \subset L^p(S,E)$ and $f: S \to E$ strongly measurable such that*

i) $f_n \to f$ in measure;

ii) $(f_n)_{n\in\mathbb{N}}$ *is uniformly integrable, i.e. for each* $\varepsilon > 0$ *there is* $\delta > 0$ *such that for all* $A \in \mathcal{A}$ *with* $\nu(A) \leq \delta$, *we have*

$$\sup_{n\in\mathbb{N}} \int_A \|f_n\|^p \mathrm{d}\nu \leq \varepsilon.$$

Then, $f \in L^p(S,E)$ *and* $f_n \to f$ *in* $L^p(S,E)$ *for* $n \to \infty$.

For a proof of this Lemma in the scalar-valued case, we refer to [51], Theorem VI, 5.6. However, the proof can be transfered to the Bochner integral without any changes. The uniform integrability will often be checked via the following observation.

Remark 4.22. Assume that there is $r \in (p, \infty]$ with

$$\sup_{n\in\mathbb{N}} \|f_n\|_{L^r(S,E)} < \infty.$$

Then, the sequence $(f_n)_{n\in\mathbb{N}}$ is uniformly integrable, since the Hölder inequality with exponents $\frac{r}{p}$ and $\left(1-\frac{p}{r}\right)^{-1}$ yields

$$\int_A \|f_n\|^p \mathrm{d}\nu \leq (\nu(A))^{1-\frac{p}{r}} \|f_n\|^p_{L^r(S,E)}, \qquad A \in \mathcal{A}.$$

As a first step in the proof of existence, we apply the results of Section 2.4 to the solutions u_n of the Galerkin equation and obtain a sequence $(v_n)_{n\in\mathbb{N}}$ on an enlarged probability space $\tilde{\Omega}$ that converges almost surely in

$$Z_T := C([0,T], E_A^*) \cap L^{\alpha+1}(0,T; L^{\alpha+1}(M)) \cap C_w([0,T], E_A).$$

Proposition 4.23. *Let* $(u_n)_{n\in\mathbb{N}}$ *be the sequence of solutions to the Galerkin equation* (4.29).

a) There are a subsequence $(u_{n_k})_{k\in\mathbb{N}}$, *a probability space* $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\right)$ *and* Z_T*-valued random variables* v_k, $k \in \mathbb{N}$, *and* v *on* $\tilde{\Omega}$ *with* $\tilde{\mathbb{P}}^{v_k} = \mathbb{P}^{u_{n_k}}$ *such that* $v_k \to v$ $\tilde{\mathbb{P}}$*-a.s. in* Z_T *for* $k \to \infty$.

b) We have $v_k \in C([0,T], H_k)$ $\tilde{\mathbb{P}}$*-a.s. and for all* $r \in [1,\infty)$, *there is* $C > 0$ *with*

$$\sup_{k\in\mathbb{N}} \tilde{\mathbb{E}}\left[\|v_k\|^r_{L^\infty(0,T;E_A)}\right] \leq C.$$

c) For all $r \in [1,\infty)$, *we have*

$$\tilde{\mathbb{E}}\left[\|v\|^r_{L^\infty(0,T;E_A)}\right] \leq C$$

with the same constant $C > 0$ *as in b).*

d) Let $\theta < \frac{1}{2}$. *For almost all* $\omega \in \tilde{\Omega}$, *we have* $v(\omega) \in C([0,T], X_\theta)$.

For the precise dependence of the constants, we refer to the Propositions 4.19 and 4.20.

Proof. ad a): The estimates to apply Corollary 2.40 are provided by Propositions 4.19 and 4.20.

ad b): Since we have $u_{n_k} \in C([0,T],H_k)$ $\mathbb{P}$-a.s. and $C([0,T],H_k)$ is closed in $C([0,T],E_A^*)$ and therefore a Borel set, we conclude $v_k \in C([0,T],H_k)$ $\tilde{\mathbb{P}}$-a.s. by the identity of the laws. Furthermore, the map $C([0,T],H_k) \ni u \mapsto \|u\|^2_{L^\infty(0,T;E_A)} \in [0,\infty)$ is continuous and therefore measurable. Hence, we can conclude that

$$\begin{aligned}\tilde{\mathbb{E}}\left[\|v_k\|^2_{L^\infty(0,T;E_A)}\right] &= \int_{C([0,T],H_k)} \|u\|^r_{L^\infty(0,T;E_A)} \mathrm{d}\tilde{\mathbb{P}}^{v_k}(u) = \int_{C([0,T],H_k)} \|u\|^r_{L^\infty(0,T;E_A)} \mathrm{d}\mathbb{P}^{u_{n_k}}(u)\\ &= \mathbb{E}\left[\|u_{n_k}\|^r_{L^\infty(0,T;E_A)}\right].\end{aligned}$$

Use the Propositions 4.19 in the defocusing case respectively 4.20 in the focusing case to get the assertion.

ad c): We have $v_n \to v$ almost surely in $L^{\alpha+1}(0,T;L^{\alpha+1}(M))$ by part a). From part b) and the embedding $L^\infty(0,T;E_A) \hookrightarrow L^{\alpha+1}(0,T;L^{\alpha+1}(M))$, we obtain that the sequence $(v_n)_{n\in\mathbb{N}}$ is bounded in $L^{\alpha+1}(\tilde{\Omega}\times[0,T]\times M)$. By Lemma 4.21, we conclude

$$v_n \xrightarrow{n\to\infty} v \quad \text{in} \quad L^2(\tilde{\Omega}, L^{\alpha+1}(0,T;L^{\alpha+1}(M))).$$

On the other hand, part b) yields $\tilde{v} \in L^r(\tilde{\Omega}, L^\infty(0,T;E_A))$ for all $r\in[1,\infty)$ and a subsequence $(v_{n_k})_{k\in\mathbb{N}}$, such that $v_{n_k} \rightharpoonup^* \tilde{v}$ for $k\to\infty$. Especially, $v_{n_k} \rightharpoonup^* \tilde{v}$ in $L^2(\tilde{\Omega}, L^{\alpha+1}(0,T;L^{\alpha+1}(M)))$ for $k\to\infty$ and hence,

$$v = \tilde{v} \in L^r(\tilde{\Omega}, L^\infty(0,T;E_A)).$$

ad d). We only have to prove the assertion for $\theta \in (-\frac{1}{2},\frac{1}{2})$. For almost all $\omega\in\tilde{\Omega}$, we have $v(\omega) \in Z_T \cap L^\infty(0,T;E_A)$. We shortly write $v = v(\omega)$. There is a nullset $N \subset [0,T]$ with $\|v(t)\|_{E_A} \le \|v\|_{L^\infty(0,T;E_A)}$ for $t\in[0,T]\setminus N$. From Proposition A.41, we infer

$$\begin{aligned}\|v(t)-v(s)\|_\theta &\lesssim \|v(t)-v(s)\|^{\frac{1}{2}-\theta}_{E_A^*}\|v(t)-v(s)\|^{\frac{1}{2}+\theta}_{E_A}\\ &\le \|v(t)-v(s)\|^{\frac{1}{2}-\theta}_{E_A^*}\left(2\|v\|_{L^\infty(0,T;E_A)}\right)^{\frac{1}{2}+\theta}, \qquad s,t\in[0,T]\setminus N. \end{aligned} \tag{4.57}$$

Let us define $\tilde{v}$ by $\tilde{v}(t) = v(t)$ for $t\in[0,T]\setminus N$ and $\tilde{v}(t) := \lim_{n\to\infty} v(t_n)$ for $t\in N$, where $[0,T]\setminus N \ni t_n \to t$ as $n\to\infty$. Note that the limit is well-defined in X_θ since $(v(t_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in X_θ by (4.57). It is straightforward to show that $\tilde{v}$ is continuous in X_θ.

Next, we show that $v(t) = \tilde{v}(t)$ for all $t\in[0,T]$. For $t\in[0,T]\setminus N$, this is the definition of $\tilde{v}$. For $t\in N$, we choose $[0,T]\setminus N \ni t_n \to t$ as $n\to\infty$. Since $X_\theta \hookrightarrow E_A^*$, we get $v(t_n)\to\tilde{v}(t)$ in E_A^*. By $v\in C([0,T],E_A^*)$, however, we also have $v(t_n)\to v(t)$ in E_A^*. Hence, $v(t) = \tilde{v}(t) \in X_\theta$. □

The next Lemma shows how convergence in Z_T can be used for the convergence of the terms appearing in the Galerkin equation.

Lemma 4.24. *Let $z_n \in C([0,T],H_n)$ for $n\in\mathbb{N}$ and $z\in Z_T$. Assume $z_n\to z$ for $n\to\infty$ in Z_T. Then, for $t\in[0,T]$ and $\psi\in E_A$, we have*

$$\left(z_n(t),\psi\right)_H \xrightarrow{n\to\infty} \langle z(t),\psi\rangle,$$

$$\int_0^t \left(Az_n(s),\psi\right)_H \mathrm{d}s \xrightarrow{n\to\infty} \int_0^t \langle Az(s),\psi\rangle \mathrm{d}s,$$

$$\int_0^t \big(\mu_n\left(z_n(s)\right),\psi\big)_H \mathrm{d}s \xrightarrow{n\to\infty} \int_0^t \langle \mu\left(z(s)\right),\psi\rangle \mathrm{d}s,$$

$$\int_0^t \big(P_n F(z_n(s)),\psi\big)_H \mathrm{d}s \xrightarrow{n\to\infty} \int_0^t \langle F(z(s)),\psi\rangle \mathrm{d}s.$$

Proof. Step 1: We fix $\psi \in E_A$ and $t\in[0,T]$. Recall that the assumption implies $z_n \to z$ for $n\to\infty$ in $C([0,T],E_A^*)$. This can be used to deduce

$$\left|\big(z_n(t),\psi\big)_H - \langle z(t),\psi\rangle\right| \le \|z_n - z\|_{C([0,T],E_A^*)}\|\psi\|_{E_A} \xrightarrow{n\to\infty} 0.$$

By $z_n\to z$ in $C_w([0,T],E_A)$ we get $\sup_{s\in[0,T]}|\langle z_n(s)-z(s),\varphi\rangle| \to 0$ for $n\to\infty$ and all $\varphi\in E_A^*$. We plug in $\varphi = A\psi$ and use the identity $\langle Az_n(s),\psi\rangle = \langle z_n(s),A\psi\rangle$ from Proposition A.41 to get

$$\begin{aligned}\int_0^t \left|\big(Az_n(s),\psi\big)_H - \langle z(s),A\psi\rangle\right| \mathrm{d}s &= \int_0^t |\langle z_n(s)-z(s),A\psi\rangle| \,\mathrm{d}s\\ &\le T \sup_{s\in[0,T]} |\langle z_n(s)-z(s),A\psi\rangle| \xrightarrow{n\to\infty} 0.\end{aligned}$$

Step 2: First, we fix $m\in\mathbb{N}$. Using the selfadjointness of S_n and the properties of g from Assumption 4.6, we get

$$\begin{aligned}&\Big|\int_0^t \Big(S_n\left[|e_m|^2 g(|z_n(s)|^2)^2 z_n(s)\right],\psi\Big)_H \mathrm{d}s - \int_0^t \big\langle |e_m|^2 g(|z(s)|^2)^2 z(s),\psi\big\rangle \mathrm{d}s\Big|\\
&\quad\le \int_0^t \Big|\Big(S_n\left[|e_m|^2 g(|z_n(s)|^2)^2 z_n(s)\right],\psi\Big)_H - \big\langle |e_m|^2 g(|z(s)|^2)^2 z(s),\psi\big\rangle\Big| \mathrm{d}s\\
&\quad\le \int_0^t \Big|\Big((S_n - I)\left[|e_m|^2 g(|z_n(s)|^2)^2 z_n(s)\right],\psi\Big)_H\Big| \mathrm{d}s\\
&\qquad+ \int_0^t \Big|\Big(|e_m|^2\left(g(|z_n(s)|^2)^2 z_n(s) - g(|z(s)|^2)^2 z(s)\right),\psi\Big)_H\Big| \mathrm{d}s\\
&\quad\lesssim \|S_n\psi-\psi\|_H \|e_m\|_{L^\infty}^2 \int_0^t \|z_n(s)\|_H \mathrm{d}s + \|\psi\|_H\|e_m\|_{L^\infty}^2 \int_0^t \|z_n(s)-z(s)\|_H \mathrm{d}s\\
&\quad\xrightarrow{n\to\infty} 0\end{aligned}$$

since we have the embedding

$$L^{\alpha+1}(0,T;L^{\alpha+1}(M)) \hookrightarrow L^1(0,T;L^2(M))$$

and $z_n\to z$ in $L^{\alpha+1}(0,T;L^{\alpha+1}(M))$ as well as $S_n\psi\to\psi$ for $n\to\infty$. By the estimate

$$\begin{aligned}\Big|\int_0^t \Big(S_n\left[|e_m|^2 g(|z_n(s)|^2)^2 z_n(s)\right],\psi\Big)_H \mathrm{d}s\Big| &\lesssim \|\psi\|_H \|e_m\|_{L^\infty}^2 \int_0^t \|z_n(s)\|_H \mathrm{d}s\\ &\lesssim \|e_m\|_{L^\infty}^2 \in \ell^1(\mathbb{N})\end{aligned}$$

and Lebesgue's convergence Theorem, we obtain

$$\sum_{m=1}^\infty \Big|\int_0^t \Big(S_n\left[|e_m|^2 g(|z_n(s)|^2)^2 z_n(s)\right],\psi\Big)_H \mathrm{d}s - \int_0^t \langle |e_m|^2 g(|z(s)|^2)^2 z(s),\psi\rangle \mathrm{d}s\Big| \xrightarrow{n\to\infty} 0$$

and therefore, we can employ the triangle inequality to deduce

$$\int_0^t \left(\mu_n\left(z_n(s)\right), \psi\right)_H \mathrm{d}s \xrightarrow{n\to\infty} \int_0^t \langle \mu\left(z(s)\right), \psi\rangle \mathrm{d}s.$$

Step 3. In order to prove the last assertion, we estimate

$$\begin{aligned}
&\int_0^t \left|\left(P_n F(z_n(s)), \psi\right)_H - \langle F(z(s)), \psi\rangle\right| \mathrm{d}s \\
&\leq \int_0^t |\langle F(z_n(s)), (P_n - I)\psi\rangle| \,\mathrm{d}s + \int_0^t |\langle F(z_n(s)) - F(z(s)), \psi\rangle| \,\mathrm{d}s
\end{aligned} \tag{4.58}$$

where we used (4.23). For the first term in (4.58), we look at

$$\begin{aligned}
\int_0^t |\langle F(z_n(s)), (P_n - I)\psi\rangle| \,\mathrm{d}s &\leq \|F(z_n)\|_{L^1(0,T;E_A^*)} \|(P_n - I)\psi\|_{E_A} \\
&\lesssim \|F(z_n)\|_{L^1(0,T;L^{\frac{\alpha+1}{\alpha}})} \|(P_n - I)\psi\|_{E_A} \\
&\lesssim \|z_n\|_{L^\alpha(0,T;L^{\alpha+1})}^\alpha \|(P_n - I)\psi\|_{E_A} \\
&\lesssim \|z_n\|_{L^{\alpha+1}(0,T;L^{\alpha+1})}^\alpha \|(P_n - I)\psi\|_{E_A} \xrightarrow{n\to\infty} 0.
\end{aligned}$$

By Assumption (4.3), we get

$$\|F(z_n(s)) - F(z(s))\|_{L^{\frac{\alpha+1}{\alpha}}(M)} \lesssim \left(\|z_n(s)\|_{L^{\alpha+1}} + \|z(s)\|_{L^{\alpha+1}}\right)^{\alpha-1} \|z_n(s) - z(s)\|_{L^{\alpha+1}}$$

for $s \in [0,T]$. Now, we apply Hölder's inequality in time with $\frac{1}{\alpha+1} + \frac{1}{\alpha+1} + \frac{\alpha-1}{\alpha+1} = 1$ and obtain

$$\begin{aligned}
\|F(z_n) - F(z)\|_{L^1(0,T;L^{\frac{\alpha+1}{\alpha}}(M))} &\leq T^{\frac{1}{\alpha+1}} \left(\|z_n\|_{L^{\alpha+1}(0,T;L^{\alpha+1})} + \|z\|_{L^{\alpha+1}(0,T;L^{\alpha+1})}\right)^{\alpha-1} \\
&\qquad \|z_n - z\|_{L^{\alpha+1}(0,T;L^{\alpha+1})} \to 0, \qquad n \to \infty.
\end{aligned}$$

This leads to the last claim. □

Remark 4.25. We identify the random variables $v_n, v : \tilde{\Omega} \to Z_T$ with stochastic processes $\bar{v}_n, \bar{v} : \tilde{\Omega} \times [0,T] \to E_A$ by

$$v_k(\omega) = \bar{v}_k(\omega, \cdot), \qquad v(\omega) = \bar{v}(\omega, \cdot), \qquad \omega \in \tilde{\Omega}.$$

For sake of readability, we also denote $\bar{v}_k = v_k$ and $\bar{v} = v$. For the details, we refer to [33], Proposition B.4.

So far, we have replaced the Galerkin solutions u_n by the processes v_n on $\tilde{\Omega}$ via the Skorohod-Jakubowski Theorem. In the next step, we want to transfer the properties given by the Galerkin equation (4.29). Therefore, we define the process $N_n : \tilde{\Omega} \times [0,T] \to H_n$ by

$$N_n(t) := -v_n(t) + S_n u_0 + \int_0^t \left[-\mathrm{i}Av_n(s) - \mathrm{i}P_n F(v_n(s)) + \mu_n(v_n(s))\right] \mathrm{d}s$$

for $n \in \mathbb{N}$ and $t \in [0,T]$. In the following lemma, we prove its martingale property.

Lemma 4.26. *For each $n \in \mathbb{N}$, the process N_n is an H-valued continuous square integrable martingale w.r.t. the filtration $\tilde{\mathcal{F}}_{n,t} := \sigma\left(v_n(s) : s \leq t\right)$. The quadratic variation of N_n is given by*

$$\langle\langle N_n \rangle\rangle_t \psi = \int_0^t \sum_{m=1}^\infty \operatorname{Re}\left(\mathrm{i}S_n\left[e_m g(|v_n(s)|^2)v_n(s)\right],\psi\right)_H \mathrm{i}S_n\left[e_m g(|v_n(s)|^2)v_n(s)\right]\mathrm{d}s, \qquad \psi \in H.$$

Proof. Fix $n \in \mathbb{N}$. We define $K_n : C([0,T],H_n) \to C([0,T],H_n)$ by

$$K_n(u)(t) := -u(t) + S_n u_0 + \int_0^t \left[-\mathrm{i}Au(s) - \mathrm{i}P_n F(u(s)) + \mu_n(u(s))\right]\mathrm{d}s, \qquad u \in H_n, \quad t \in [0,T],$$

and a process $M_n : \Omega \times [0,T] \to H_n$ by $M_n(\omega,t) := K(u_n(\omega))(t)$. Since u_n is a solution of the Galerkin equation (4.29), we obtain the representation

$$M_n(t) = \mathrm{i}\int_0^t S_n B(u_n(s))\mathrm{d}W(s)$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$. The estimate

$$\begin{aligned}\mathbb{E}\left[\sum_{m=1}^\infty \int_0^T \|S_n\left[e_m g(|u_n(s)|^2)u_n(s)\right]\|_H^2 \mathrm{d}s\right] &\leq \sum_{m=1}^\infty \|e_m\|_{L^\infty}^2 \mathbb{E}\left[\int_0^T \|g(|u_n(s)|^2)u_n(s)\|_H^2 \mathrm{d}s\right] \\ &\lesssim \mathbb{E}\left[\sup_{t\in[0,T]} \|u_n(s)\|_H^2\right] < \infty\end{aligned}$$

yields that M_n is a square integrable continuous martingale w.r.t. the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. The adjoint of the operator $\Phi_n(s) := \mathrm{i}S_n B(u_n(s)) : Y \to H$ for $s \in [0,T]$ is given by

$$\Phi_n^*(s)\psi = \sum_{m=1}^\infty \operatorname{Re}\left(\mathrm{i}S_n\left[e_m g(|u_n(s)|^2)u_n(s)\right],\psi\right)_H f_m, \qquad \psi \in H, \quad s \in [0,T].$$

Therefore,

$$\Phi_n(s)\Phi_n^*(s)\psi = \sum_{m=1}^\infty \operatorname{Re}\left(\mathrm{i}S_n\left[e_m g(|u_n(s)|^2)u_n(s)\right],\psi\right)_H \mathrm{i}S_n\left[e_m g(|u_n(s)|^2)u_n(s)\right].$$

By Theorem A.11, M_n is a square integrable $\mathbb{F}$-martingale with quadratic variation

$$\langle\langle M_n \rangle\rangle_t \psi = \int_0^t \sum_{m=1}^\infty \operatorname{Re}\left(\mathrm{i}S_n\left[e_m g(|u_n(s)|^2)u_n(s)\right],\psi\right)_H \mathrm{i}S_n\left[e_m g(|u_n(s)|^2)u_n(s)\right]\mathrm{d}s, \qquad \psi \in H.$$

Since the operator K_n is Lipschitz on balls with constant depending on n, M_n is even adapted to the smaller σ-field $\mathcal{F}_{n,t} := \sigma\left(u_n(s) : s \leq t\right)$ and therefore a square integrable martingale w.r.t. $\left(\mathcal{F}_{n,t}\right)_{t\in[0,T]}$. From Lemma A.16, we infer

$$\mathbb{E}\left[\operatorname{Re}\left(M_n(t) - M_n(s),\psi\right)_H h(u_n|_{[0,s]})\right] = 0$$

and

$$0 = \mathbb{E}\Bigg[h(u_n|_{[0,s]})\Bigg(\operatorname{Re}\left(M_n(t),\psi\right)_H \operatorname{Re}\left(M_n(t),\varphi\right)_H - \operatorname{Re}\left(M_n(s),\psi\right)_H \operatorname{Re}\left(M_n(s),\varphi\right)_H$$

$$- \sum_{m=1}^{\infty} \int_0^t \operatorname{Re}\left(\mathrm{i}S_n\left[e_m g(|u_n(s)|^2)u_n(s)\right],\psi\right)_H \operatorname{Re}\left(\mathrm{i}S_n\left[e_m g(|u_n(s)|^2)u_n(s)\right],\varphi\right)_H \mathrm{d}s\Bigg)\Bigg]$$

for all $\psi, \varphi \in H$, $0 \leq s \leq t \leq T$ and bounded, continuous functions h on $C([0,T], H_n)$. We use the identity of the laws of u_n and v_n on $C([0,T], H_n)$ to obtain

$$\tilde{\mathbb{E}}\left[\operatorname{Re}\left(N_n(t) - N_n(s), \psi\right)_H h(v_n|_{[0,s]})\right] = 0$$

and

$$0 = \mathbb{E}\Bigg[h(v_n|_{[0,s]})\Bigg(\operatorname{Re}\left(N_n(t),\psi\right)_H \operatorname{Re}\left(N_n(t),\varphi\right)_H - \operatorname{Re}\left(N_n(s),\psi\right)_H \operatorname{Re}\left(N_n(s),\varphi\right)_H$$
$$- \sum_{m=1}^{\infty} \int_0^t \operatorname{Re}\left(\mathrm{i}S_n\left[e_m g(|v_n(s)|^2)v_n(s)\right],\psi\right)_H \operatorname{Re}\left(\mathrm{i}S_n\left[e_m g(|v_n(s)|^2)v_n(s)\right],\varphi\right)_H \mathrm{d}s\Bigg)\Bigg]$$

for all $\psi, \varphi \in H$ and bounded, continuous functions h on $C([0,T], H_n)$. As a consequence of Lemma A.16, N_n is a continuous square integrable martingale w.r.t $\tilde{\mathcal{F}}_{n,t} := \sigma\left(v_n(s) : s \leq t\right)$ and the quadratic variation is given as we have claimed in the lemma. □

From Proposition 4.23, we infer that $v \in Z_T$ almost surely and

$$\|F(v)\|_{L^1(0,T;E_A^*)} \lesssim \|F(v)\|_{L^\infty(0,T;L^{\frac{\alpha+1}{\alpha}})} = \|v\|_{L^\infty(0,T;L^{\alpha+1})}^\alpha < \infty \quad \text{a.s.};$$

$$\|Av\|_{L^1(0,T;E_A^*)} \lesssim \|v\|_{L^\infty(0,T;E_A)} < \infty \quad \text{a.s.};$$

$$\|\mu(v)\|_{L^1(0,T;E_A^*)} \lesssim \|\mu(v)\|_{L^1(0,T;H)} \leq \frac{1}{2}\sum_{m=1}^{\infty} \|e_m\|_{L^\infty}^2 \|g(|v|^2)^2 v\|_{L^1(0,T;H)}$$
$$\lesssim \|v\|_{L^1(0,T;H)} \lesssim \|v\|_{L^{\alpha+1}(0,T;L^{\alpha+1}(M))} < \infty \quad \text{a.s.}$$

Hence, we can define a process $N : \tilde{\Omega} \times [0,T] \to E_A^*$ with continuous paths by

$$N(t) := -v(t) + u_0 + \int_0^t \left[-\mathrm{i}Av(s) - \mathrm{i}F(v(s)) + \mu(v(s))\right] \mathrm{d}s, \quad t \in [0,T].$$

Let $\iota : E_A \hookrightarrow H$ be the usual embedding and $\iota^* : H \to E_A$ its Hilbert-space adjoint operator. Further, we set $L := (\iota^*)' : E_A^* \to H$ as the dual operator of ι^* with respect to the Gelfand triple $E_A \hookrightarrow H \approx H^* \hookrightarrow E_A^*$. The definition of ι^* and L can be rephrased by the identities

$$\operatorname{Re}\left(\iota\psi,\varphi\right)_H = \operatorname{Re}\left(\psi,\iota^*\varphi\right)_{E_A}, \qquad \operatorname{Re}\langle\iota^*\varphi, w\rangle_{E_A,E_A^*} = \operatorname{Re}\left(\varphi, Lw\right)_H \tag{4.59}$$

for $\psi \in E_A$ and $\varphi \in H$ and $w \in E_A^*$. We remark that the range of ι^* is dense in E_A since ι is injective. Hence, L is injective by the second identity in (4.59). Moreover, L is a bounded operator with $\|L\|_{\mathcal{L}(E_A^*,H)} \leq \|\iota\|_{\mathcal{L}(E_A,H)} \leq 1$. In the next Lemma, we use the martingale property of N_n for $n \in \mathbb{N}$ and a limiting process based on Proposition 4.23 and Lemma 4.24. to conclude that LN is also an H-valued martingale.

Lemma 4.27. *The process LN is an H-valued continuous square integrable martingale with respect to the filtration $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t\in[0,T]}$, where $\tilde{\mathcal{F}}_t := \sigma\left(v(s) : s \le t\right)$. The quadratic variation is given by*

$$\langle\langle LN\rangle\rangle_t\zeta = \sum_{m=1}^{\infty}\int_0^t \mathrm{i}L\left[e_m g(|v(s)|^2)v(s)\right]\operatorname{Re}\left(\mathrm{i}L\left[e_m g(|v(s)|^2)v(s)\right],\zeta\right)_H \mathrm{d}s$$

for all $\zeta \in H$.

Proof. Step 1: Let $t \in [0,T]$. We will first show that $\tilde{\mathbb{E}}\left[\|N(t)\|_{E_A^*}^2\right] < \infty$. By Lemma 4.24, we have $N_n(t) \to N(t)$ almost surely in E_A^* for $n \to \infty$. By the Davis inequality for continuous martingales (see [105]), Lemma 4.26 and Proposition 4.23 , we conclude

$$\begin{aligned}
\tilde{\mathbb{E}}\left[\sup_{t\in[0,T]}\|N_n(t)\|_H^{\alpha+1}\right] &\lesssim \tilde{\mathbb{E}}\left[\left(\sum_{m=1}^{\infty}\int_0^T \|S_n\left[e_m g(|v_n(s)|^2)v_n(s)\right]\|_H^2\mathrm{d}s\right)^{\frac{\alpha+1}{2}}\right]\\
&\le \left(\sum_{m=1}^{\infty}\|e_m\|_{L^\infty}^2\right)^{\frac{\alpha+1}{2}}\tilde{\mathbb{E}}\left[\left(\int_0^T \|g(|v_n(s)|^2)v_n(s)\|_H^2\mathrm{d}s\right)^{\frac{\alpha+1}{2}}\right]\\
&\lesssim \tilde{\mathbb{E}}\left[\left(\int_0^T \|v_n(s)\|_H^2\mathrm{d}s\right)^{\frac{\alpha+1}{2}}\right] \lesssim \tilde{\mathbb{E}}\left[\|v_n\|_{L^\infty(0,T;H)}^{\alpha+1}\right] \lesssim 1. \qquad (4.60)
\end{aligned}$$

Since $\alpha + 1 > 2$, we deduce $N(t) \in L^2(\tilde{\Omega}, E_A^*)$ by Lemma 4.21 and $N_n(t) \to N(t)$ in $L^2(\tilde{\Omega}, E_A^*)$ for $n \to \infty$.

Step 2: Let $\psi, \varphi \in E_A$ and h be a bounded continuous function on $C([0,T], E_A^*)$.
For $0 \le s \le t \le T$, we define the random variables

$$f_n(t,s) := \operatorname{Re}\left(N_n(t) - N_n(s), \psi\right)_H h(v_n|_{[0,s]}), \qquad f(t,s) := \operatorname{Re}\langle N(t) - N(s), \psi\rangle h(v|_{[0,s]}).$$

The $\tilde{\mathbb{P}}$-a.s.-convergence $v_n \to v$ in Z_T for $n \to \infty$ yields $f_n(t,s) \to f(t,s)$ $\tilde{\mathbb{P}}$-a.s. for all $0 \le s \le t \le T$ by Lemma 4.24. We use $(a+b)^p \le 2^{p-1}\left(a^p + b^p\right)$ for $a, b \ge 0$ and $p \ge 1$ and the estimate (4.60) for

$$\begin{aligned}
\tilde{\mathbb{E}}|f_n(t,s)|^{\alpha+1} &\le 2^{\alpha}\|h\|_\infty^{\alpha+1}\|\psi\|_H^{\alpha+1}\tilde{\mathbb{E}}\left[\|N_n(t)\|_H^{\alpha+1} + \|N_n(s)\|_H^{\alpha+1}\right]\\
&\lesssim \|h\|_\infty^{\alpha+1}\|\psi\|_H^{\alpha+1}.
\end{aligned}$$

In view of Lemma 4.21, we get

$$0 = \lim_{n\to\infty}\tilde{\mathbb{E}}\big[f_n(t,s)\big] = \tilde{\mathbb{E}}\big[f(t,s)\big], \qquad 0 \le s \le t \le T.$$

Step 3: For $0 \le s \le t \le T$, we define

$$g_{1,n}(t,s) := \Big(\operatorname{Re}\left(N_n(t),\psi\right)_H \operatorname{Re}\left(N_n(t),\varphi\right)_H - \operatorname{Re}\left(N_n(s),\psi\right)_H \operatorname{Re}\left(N_n(s),\varphi\right)_H\Big)h(v_n|_{[0,s]})$$

and

$$g_1(t,s) := \Big(\operatorname{Re}\langle N(t),\psi\rangle\operatorname{Re}\langle N(t),\varphi\rangle - \operatorname{Re}\langle N(s),\psi\rangle\operatorname{Re}\langle N(s),\varphi\rangle\Big)h(v|_{[0,s]}).$$

By Lemma 4.24, we obtain $g_{1,n}(t,s) \to g_1(t,s)$ $\tilde{\mathbb{P}}$-a.s. for all $0 \le s \le t \le T$. In order to get uniform integrability, we set $r := \frac{\alpha+1}{2} > 1$ and estimate

$$\begin{aligned}\tilde{\mathbb{E}}|g_{1,n}(t,s)|^r \le & 2^{r-1}\|h\|_\infty^r \tilde{\mathbb{E}}\left[|\operatorname{Re}\left(N_n(t),\psi\right)_H \operatorname{Re}\left(N_n(t),\varphi\right)_H|^r\right] \\ & + 2^{r-1}\|h\|_\infty^r \tilde{\mathbb{E}}\left[|\operatorname{Re}\left(N_n(s),\psi\right)_H \operatorname{Re}\left(N_n(s),\varphi\right)_H|^r\right] \\ \le & 2^{r-1}\|h\|_\infty^r \|\psi\|_H^r \|\varphi\|_H^r \tilde{\mathbb{E}}\left[\|N_n(t)\|_H^{\alpha+1} + \|N_n(s)\|_H^{\alpha+1}\right] \lesssim \|h\|_\infty^r \|\psi\|_H^r \|\varphi\|_H^r,\end{aligned}$$

where we used (4.60) again. As above, Lemma 4.21 yields

$$0 = \lim_{n\to\infty} \tilde{\mathbb{E}}\big[g_{1,n}(t,s)\big] = \tilde{\mathbb{E}}\big[g_1(t,s)\big], \qquad 0 \le s \le t \le T.$$

Step 4: For $0 \le s \le t \le T$, we define

$$\begin{aligned} g_{2,n}(t,s) := h(v_n|_{[0,s]}) \sum_{m=1}^\infty \int_s^t \Big[& \operatorname{Re}\left(\mathrm{i}S_n\left[e_m g(|v_n(\tau)|^2)v_n(\tau)\right],\psi\right)_H \\ & \operatorname{Re}\left(\mathrm{i}S_n\left[e_m g(|v_n(\tau)|^2)v_n(\tau)\right],\varphi\right)_H\Big]\mathrm{d}\tau, \\ g_2(t,s) := h(v|_{[0,s]}) \sum_{m=1}^\infty \int_s^t & \operatorname{Re}\left(\mathrm{i}e_m g(|v(\tau)|^2)v(\tau),\psi\right)_H \operatorname{Re}\left(\mathrm{i}e_m g(|v(\tau)|^2)v(\tau),\varphi\right)_H \mathrm{d}\tau.\end{aligned}$$

Because of $h(v_n|_{[0,s]}) \to h(v|_{[0,s]})$ $\tilde{\mathbb{P}}$-a.s. and the continuity of the inner product $L^2([s,t]\times\mathbb{N})$, the convergence

$$\operatorname{Re}\left(\mathrm{i}S_n\left[e_m g(|v_n|^2)v_n\right],\psi\right)_H \to \operatorname{Re}\left(\mathrm{i}e_m g(|v|^2)v,\psi\right)_H$$

$\tilde{\mathbb{P}}$-a.s. in $L^2([s,t]\times\mathbb{N})$ already implies $g_{2,n}(t,s) \to g_2(t,s)$ $\tilde{\mathbb{P}}$-a.s. Therefore, we have to estimate

$$\begin{aligned}&\|\operatorname{Re}\left(\mathrm{i}S_n\left[e_m g(|v_n|^2)v_n\right],\psi\right)_H - \operatorname{Re}\left(\mathrm{i}e_m g(|v|^2)v,\psi\right)_H\|_{L^2([s,t]\times\mathbb{N})} \\ &\quad\le \|\operatorname{Re}\left(\mathrm{i}e_m g(|v_n|^2)v_n, (S_n - I)\psi\right)_H\|_{L^2([s,t]\times\mathbb{N})} \\ &\qquad + \|\operatorname{Re}\left(\mathrm{i}e_m\left(g(|v_n|^2)v_n - g(|v|^2)v\right),\psi\right)_H\|_{L^2([s,t]\times\mathbb{N})} \\ &\quad\le \left(\sum_{m=1}^\infty \|e_m\|_{L^\infty}^2\right)^{\frac12} \|v_n\|_{L^2(s,t;H)} \|(S_n-I)\psi\|_H + \left(\sum_{m=1}^\infty \|e_m\|_{L^\infty}^2\right)^{\frac12} \|v_n - v\|_{L^2(s,t;H)}\|\psi\|_H \\ &\quad\lesssim \|v_n\|_{L^{\alpha+1}(0,T;L^{\alpha+1}(M))} \|(S_n-I)\psi\|_H + \|v_n - v\|_{L^{\alpha+1}(0,T;L^{\alpha+1}(M))}\|\psi\|_H.\end{aligned}$$

Hence, we conclude

$$\|\operatorname{Re}\left(\mathrm{i}S_n\left[e_m g(|v_n|^2)v_n\right],\psi\right)_H - \operatorname{Re}\left(\mathrm{i}e_m g(|v|^2)v,\psi\right)_H\|_{L^2([s,t]\times\mathbb{N})} \xrightarrow{n\to\infty} 0$$

almost surely. Furthermore, we estimate

$$\begin{aligned}\sum_{m=1}^\infty \int_s^t |\operatorname{Re}\left(\mathrm{i}S_n\left[e_m g(|v_n(\tau)|^2)v_n(\tau)\right],\psi\right)_H|^2 \mathrm{d}\tau &\le \int_0^T \|g(|v_n(\tau)|^2)v_n(\tau)\|_H^2 \mathrm{d}\tau \|\psi\|_H^2 \sum_{m=1}^\infty \|e_m\|_{L^\infty}^2 \\ &\lesssim \int_0^T \|v_n(\tau)\|_H^2 \mathrm{d}\tau \lesssim \|v_n\|_{L^{\alpha+1}(0,T;L^{\alpha+1}(M))}^2\end{aligned}$$

and continue with $r := \frac{\alpha+1}{2} > 1$ and

$$\tilde{\mathbb{E}}|g_{2,n}(t,s)|^r \le \tilde{\mathbb{E}}\Big[\|\operatorname{Re}\left(\mathrm{i}S_n\left[e_m g(|v_n|^2)v_n\right],\psi\right)_H\|_{L^2([s,t]\times\mathbb{N})}^r$$

$$\left. \| \operatorname{Re}\left(\mathrm{i} S_n\left[e_m g(|v_n|^2) v_n\right], \varphi\right)_H \|_{L^2([s,t]\times\mathbb{N})}^r |h(v_n|_{[0,s]})|^r \right]$$
$$\lesssim \sup_{n\in\mathbb{N}} \tilde{\mathbb{E}}\left[\|v_n\|_{L^{\alpha+1}(0,T;L^{\alpha+1})}^{\alpha+1}\right] \lesssim 1.$$

From Lemma 4.21, we infer

$$\lim_{n\to\infty} \tilde{\mathbb{E}}\left[g_{2,n}(t,s)\right] = \tilde{\mathbb{E}}\left[g_2(t,s)\right], \qquad 0 \le s \le t \le T.$$

Step 5: From Step 2, we have

$$\tilde{\mathbb{E}}\left[\operatorname{Re}\langle N(t)-N(s),\psi\rangle h(v|_{[0,s]})\right] = 0 \tag{4.61}$$

and Step 3, Step 4 and Lemma 4.26 yield

$$\tilde{\mathbb{E}}\Bigg[\Bigg(\operatorname{Re}\langle N(t),\psi\rangle \operatorname{Re}\langle N(t),\varphi\rangle - \operatorname{Re}\langle N(s),\psi\rangle \operatorname{Re}\langle N(s),\varphi\rangle$$
$$+\sum_{m=1}^{\infty}\int_s^t \operatorname{Re}\left(\mathrm{i} e_m g(|v(\tau)|^2)v(\tau),\psi\right)_H \operatorname{Re}\left(\mathrm{i} e_m g(|v(\tau)|^2)v(\tau),\varphi\right)_H \mathrm{d}\tau\Bigg) h(v|_{[0,s]})\Bigg] = 0. \tag{4.62}$$

Now, let $\eta,\zeta \in H$. Then, we have $\iota^*\eta, \iota^*\zeta \in E_A$ and the identities (4.59), (4.61) and (4.62) imply

$$\tilde{\mathbb{E}}\left[\operatorname{Re}\left(LN(t)-LN(s),\eta\right)_H h(u|_{[0,s]})\right] = 0$$

and

$$\tilde{\mathbb{E}}\Bigg[\Bigg(\operatorname{Re}\left(LN(t),\eta\right)_H \operatorname{Re}\left(LN(t),\zeta\right)_H - \operatorname{Re}\left(LN(s),\eta\right)_H \operatorname{Re}\left(LN(s),\zeta\right)_H$$
$$+\sum_{m=1}^{\infty}\int_s^t \operatorname{Re}\left(\mathrm{i} L\left[e_m g(|v(\tau)|^2)v(\tau)\right],\eta\right)_H \operatorname{Re}\left(\mathrm{i} L\left[e_m g(|v(\tau)|^2)v(\tau)\right],\zeta\right)_H \mathrm{d}\tau\Bigg) h(v|_{[0,s]})\Bigg] = 0$$

for all bounded and continuous functions h on $C([0,T],E_A^*)$. Hence, LN is a continuous, square integrable martingale in H with respect to $\tilde{\mathcal{F}}_t := \sigma\left(v|_{[0,t]}\right)$, where v is viewed as a random element of $C([0,T],E_A^*)$. The quadratic variation is given by

$$\langle\langle LN\rangle\rangle_t \zeta = \sum_{m=1}^{\infty}\int_0^t \mathrm{i} L\left[e_m g(|v(\tau)|^2)v(\tau)\right] \operatorname{Re}\left(\mathrm{i} L\left[e_m g(|v(\tau)|^2)v(\tau)\right],\zeta\right)_H \mathrm{d}\tau, \qquad \zeta\in H.$$

Since we showed in Proposition 4.23 that v has in fact continuous paths in X_θ for each $\theta < \frac{1}{2}$, one can also regard $\tilde{\mathcal{F}}_t$ as the smallest σ-algebra such that $v|_{[0,t]}$ is strongly measurable in X_θ. □

Finally, we can prove our main result Theorem 4.10 using Theorem A.12.

Proof of Theorem 4.10. We choose $H = L^2(M)$, and $\Phi(s) := \mathrm{i} LB\left(v(s)\right)$ for all $s\in[0,T]$. The adjoint $\Phi(s)^*$ is given by $\Phi(s)^*\zeta := \sum_{m=1}^{\infty} \operatorname{Re}\left(\mathrm{i} L\left[e_m g(|v(s)|^2)v(s)\right],\zeta\right)_H f_m$ and hence,

$$\Phi(s)\Phi(s)^*\zeta = \sum_{m=1}^{\infty} \operatorname{Re}\left(\mathrm{i} L\left[e_m g(|v(s)|^2)v(s)\right],\zeta\right)_H \mathrm{i} L\left[e_m g(|v(s)|^2)v(s)\right], \qquad \zeta\in H.$$

By Proposition 4.23, v is continuous in X_θ for $\theta < \frac{1}{2}$ and obviously, v is adapted to

$$\tilde{\mathcal{F}}_t = \sigma\left(v(s) : 0 \le s \le t\right).$$

From Assumption 4.6, we infer that the process $[0,T] \ni t \mapsto \mathrm{i}B(v(t)) \in \mathrm{HS}(Y,H)$ is continuous and adapted to $\tilde{\mathbb{F}}$ and therefore progressively measurable. Since L is a bounded operator from E_A^* to H, this property transfers to Φ. By an application of Theorem A.12 to the process LN from Lemma 4.27, we obtain a Y-cylindrical Wiener process $\tilde{W}$ defined on a probability space

$$(\Omega', \mathcal{F}', \mathbb{P}') := \left(\tilde{\Omega} \times \tilde{\tilde{\Omega}}, \tilde{\mathcal{F}} \otimes \tilde{\tilde{\mathcal{F}}}, \tilde{\mathbb{P}} \otimes \tilde{\tilde{\mathbb{P}}}\right)$$

with

$$LN(t) = \int_0^t \Phi(s) \mathrm{d}\tilde{W}(s) = \int_0^t \mathrm{i}LB\left(v(s)\right) \mathrm{d}\tilde{W}(s), \qquad t \in [0,T],$$

if we can show $B(v) \in \mathcal{M}^2_{\mathbb{F},Y}(0,T;H)$. This is a consequence of

$$\begin{aligned}\|Bv\|^2_{L^2([0,T]\times\Omega,\mathrm{HS}(Y,H))} =& \mathbb{E}\int_0^T \sum_{m=1}^\infty \|e_m g(|v(s)|^2)v(s)\|_H^2 \mathrm{d}s \lesssim \mathbb{E}\int_0^T \|g(|v(s)|^2)v(s)\|_H^2 \mathrm{d}s \\ \lesssim& \mathbb{E}\int_0^T \|v(s)\|_H^2 \mathrm{d}s \lesssim 1.\end{aligned}$$

Using the continuity of the linear operator L and Proposition $A.14$, we get

$$\int_0^t \mathrm{i}LB\left(v(s)\right) \mathrm{d}\tilde{W}(s) = L\left(\int_0^t \mathrm{i}B\left(v(s)\right) \mathrm{d}\tilde{W}(s)\right)$$

almost surely for all $t \in [0,T]$. The definition of N and the injectivity of L yield the equality

$$\int_0^t \mathrm{i}Bv(s)\mathrm{d}\tilde{W}(s) = -v(t) + u_0 + \int_0^t \left[-\mathrm{i}Av(s) - \mathrm{i}F(v(s)) + \mu(v(s))\right] \mathrm{d}s \tag{4.63}$$

in E_A^* almost surely for all $t \in [0,T]$. The weak continuity of the paths of v in E_A and the estimates for $u \in L^q(\tilde{\Omega}, L^\infty(0,T;E_A))$ have already been shown in Proposition 4.23. Hence, the system $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbb{F}}, v\right)$ is a martingale solution of equation (4.1) with the properties we claimed. □

4.4. Examples

In this section, we consider concrete situations and verify that they are covered by the general framework presented in Section 4.1. As a result, we obtain several Corollaries of Theorem 4.10.

First, we show that the class of the general nonlinearities from Assumption 4.3 covers the standard power type nonlinearity.

Proposition 4.28. *Let $\alpha \in (1,\infty)$ be chosen as in Assumption 4.1. Define the following function*

$$F_\alpha^\pm(u) := \pm|u|^{\alpha-1}u, \qquad \hat{F}_\alpha^\pm(u) := \pm\frac{1}{\alpha+1}\|u\|^{\alpha+1}_{L^{\alpha+1}(M)}, \qquad u \in L^{\alpha+1}(M).$$

Then, $F_\alpha^\pm$ satisfies Assumption 4.3 with antiderivative $\hat{F}_\alpha^\pm$.

Proof. Obviously, $F_\alpha^\pm : L^{\alpha+1}(M) \to L^{\frac{\alpha+1}{\alpha}}(M)$ and

$$\|F_\alpha^\pm(u)\|_{L^{\frac{\alpha+1}{\alpha}}(M)} = \|u\|_{L^{\alpha+1}(M)}^\alpha, \qquad u \in L^{\alpha+1}(M).$$

Furthermore,

$$\operatorname{Re}\langle \mathrm{i}u, F_\alpha^\pm(u)\rangle = \pm \operatorname{Re} \int_M \mathrm{i}u|u|^{\alpha-1}\bar{u}\,\mathrm{d}\mu = \pm \operatorname{Re}\left[\mathrm{i}\|u\|_{L^{\alpha+1}(M)}^{\alpha+1}\right] = 0.$$

We can apply the following Lemma 3.5 with $p = \alpha + 1$ and

$$\Phi(a,b) = \left(a^2+b^2\right)^{\frac{\alpha-1}{2}} \begin{pmatrix} a \\ b \end{pmatrix}, \qquad a, b \in \mathbb{R},$$

to obtain part ii) and iii) of Assumption 4.3. □

4.4.1. The Laplace-Beltrami Operator on compact manifolds

In this subsection, we deduce the following Corollary from Theorem 4.10.

Corollary 4.29. *Let (M,g) be a compact d-dimensional Riemannian manifold without boundary and $A := -\Delta_g$ be the Laplace-Beltrami operator on M. Under Assumption 4.6 and either i) or ii)*

i) $F(u) = |u|^{\alpha-1}u$ with $\alpha \in \left(1, 1+\frac{4}{(d-2)_+}\right)$,

ii) $F(u) = -|u|^{\alpha-1}u$ with $\alpha \in \left(1, 1+\frac{4}{d}\right)$,

the equation

$$\begin{cases} \mathrm{d}u(t) = \left(\mathrm{i}\Delta_g u(t) - \mathrm{i}F(u(t)) + \mu(u(t))\right)\mathrm{d}t - \mathrm{i}B(u(t))\mathrm{d}W(t) \quad \text{in } H^{-1}(M), \\ u(0) = u_0 \in H^1(M), \end{cases} \tag{4.64}$$

has an analytically weak martingale solution which satisfies $u \in C_w([0,T], H^1(M))$ almost surely and $u \in L^q(\tilde{\Omega}, L^\infty(0,T;H^1(M)))$ for all $q \in [1,\infty)$.

In order to fulfill the assumptions of Theorem 4.10, we choose $S := I - \Delta_g$. Then, S is selfadjoint, strictly positive with compact resolvent and commutes with A. The manifold M has the doubling property and S has upper Gaussian bounds by [61], Corollary 5.5 and Theorem 6.1, since these results imply

$$|p(t,x,y)| \le \frac{C}{t^{\frac{d}{2}}} e^{-t} \exp\left\{-c\frac{\rho(x,y)^2}{t}\right\}, \qquad t > 0, \quad (x,y) \in M \times M$$

for the kernel p of the semigroup $\left(e^{-tS}\right)_{t\ge 0}$. In view of the doubling property (4.4), this is sufficient for (4.6). In particular, S has generalized Gaussian bounds with $p_0 = 1$.

We have the following relation between the scale of Sobolev spaces from Appendix B and the fractional domains of the Laplace-Beltrami operator. By Proposition $A.53$ a), the scale of Sobolev spaces on M is given by

$$H^s(M) = \operatorname{range}\left(S^{-\frac{s}{2}}\right) = \mathcal{D}\left(S^{\frac{s}{2}}\right) = \mathcal{D}\left((\operatorname{Id} - \Delta_g)^{\frac{s}{2}}\right), \quad s > 0.$$

In particular, we have $E_A = H^1(M)$. Let $1 < \alpha < 1 + \frac{4}{(d-2)_+}$. Then, by Proposition $A.53$ d) and Lemma 4.12, the embeddings

$$E_A = H^1(M) \hookrightarrow H^{-1}(M) = E_A^*, \qquad E_A = H^1(M) \hookrightarrow L^{\alpha+1}(M)$$

are compact. Hence, Assumption 4.1 holds with our choice of A, S and α. The range of admissible powers in the focusing case is the content of the following Lemma.

Lemma 4.30. *F_α^+ satisfies Assumption 4.5 i) and F_α^- satisfies i') under the restriction $\alpha \in \left(1, 1 + \frac{4}{d}\right)$.*

Proof. Obviously, the assertion for F_α^+ is true. We consider F_α^-.
Case 1. Let $d \geq 3$. Then, $p_{\max} := \frac{2d}{d-2}$ is the maximal exponent with $H^1(M) \hookrightarrow L^{p_{\max}}(M)$. Since $\alpha \in (1, p_{\max} - 1)$, we can interpolate $L^{\alpha+1}(M)$ between H and $L^{p_{\max}}(M)$ and get

$$\|u\|_{L^{\alpha+1}(M)} \leq \|u\|_{L^2}^{1-\theta} \|u\|_{L^{p_{\max}}(M)}^{\theta} \lesssim \|u\|_{L^2}^{1-\theta} \|u\|_{H^1(M)}^{\theta}.$$

with $\theta = \frac{d(\alpha-1)}{2(\alpha+1)} \in (0,1)$. The restriction $\beta_2 := \theta(\alpha+1) < 2$ from Assumption 4.5 i') leads to $\alpha < 1 + \frac{4}{d}$.
Case 2. In the case $d = 2$, Assumption i') is guaranteed for $\alpha \in (1,3)$. To see this, take $p > \frac{4}{3-\alpha}$ which is equivalent to $\theta(\alpha+1) < 2$ when $\theta \in (0,1)$ is chosen as

$$\theta = \frac{(\alpha-1)p}{(\alpha+1)(p-2)}.$$

We have $H^1(M) \hookrightarrow L^p(M)$ and as above, interpolation between H and $L^p(M)$ yields

$$\|u\|_{L^{\alpha+1}(M)}^{\alpha+1} \lesssim \|u\|_{L^2}^{(\alpha+1)(1-\theta)} \|u\|_{E_A}^{(\alpha+1)\theta}.$$

Case 3. Let $d = 1$ and fix $\varepsilon \in (0, \frac{1}{2})$. Proposition $A.53$ yields

$$H^{\frac{1}{2}+\varepsilon}(M) \hookrightarrow L^\infty(M), \qquad H^{\frac{1}{2}+\varepsilon}(M) = \left[L^2(M), H^1(M)\right]_{\frac{1}{2}+\varepsilon}.$$

Hence,

$$\|v\|_{L^{\alpha+1}}^{\alpha+1} \leq \|v\|_{L^2}^2 \|v\|_{L^\infty}^{\alpha-1} \lesssim \|v\|_{L^2}^2 \|v\|_{H^{\frac{1}{2}+\varepsilon}}^{\alpha-1} \lesssim \|v\|_{L^2}^{2+(\frac{1}{2}-\varepsilon)(\alpha-1)} \|v\|_{H^1}^{(\frac{1}{2}+\varepsilon)(\alpha-1)}$$

The condition $(\frac{1}{2}+\varepsilon)(\alpha-1) < 2$ is equivalent to $\alpha < 1 + \frac{4}{1+2\varepsilon}$. Choosing ε small enough, we see that Assumption 4.5 i') is true for $\alpha \in (1,5)$. ☐

4.4.2. Laplacians on bounded domains

We can apply Theorem 4.10 to the stochastic NLS on bounded domains.

Corollary 4.31. *Let $M \subset \mathbb{R}^d$ be a bounded domain and choose between*

a) the Dirichlet Laplacian $A := -\Delta_D$ and $E_A := H_0^1(M)$;

b) the Neumann Laplacian $A := -\Delta_N$ and $E_A := H^1(M)$, where we additionally assume that ∂M is Lipschitz.

Under Assumption 4.6 and either i) or ii) *i)* $F(u) = |u|^{\alpha-1}u$ *with* $\alpha \in \left(1, 1+\frac{4}{(d-2)_+}\right)$;

ii) $F(u) = -|u|^{\alpha-1}u$ *with* $\alpha \in \left(1, 1+\frac{4}{d}\right)$;

the equation

$$\begin{cases} \mathrm{d}u(t) = \big(-\mathrm{i}Au(t) - \mathrm{i}F(u(t)) + \mu(u(t))\big)dt - \mathrm{i}B(u(t))\mathrm{d}W(t) \quad \textit{in } E_A^*, \\ u(0) = u_0 \in E_A, \end{cases} \tag{4.65}$$

has an analytically weak martingale solution which satisfies $u \in C_w([0,T], E_A)$ *almost surely and* $u \in L^q(\tilde{\Omega}, L^\infty(0,T; E_A))$ *for all* $q \in [1, \infty)$.

We remark that one could consider uniformly elliptic operators and more general boundary conditions, but for the sake of simplicity, we concentrate on the two most prominent examples.

Proof. We consider the Dirichlet form $a_V : V \times V \to \mathbb{C}$,

$$a_V(u,v) = \int_M \nabla u \cdot \nabla v \,\mathrm{d}x, \quad u, v \in V,$$

with associated operator $(A_V, \mathcal{D}(A_V))$ in the following two situations:

i) $V = H_0^1(M)$;

ii) $V = H^1(M)$ and M has Lipschitz-boundary.

The operator $A_{H_0^1(M)} = \Delta_D$ is the Dirichlet Laplacian and $A_{H^1(M)} = \Delta_N$ is the Neumann Laplacian. In both cases, $V = E_{A_V}$ by the square root property (see [104], Theorem 8.1) and the embedding $E_{A_V} \hookrightarrow L^{\alpha+1}(M)$ is compact if and only if $1 < \alpha < p_{\max} - 1$ with $p_{\max} := 2 + \frac{4}{(d-2)_+}$. Hence, we obtain the same range of admissible powers α for the focusing and the defocusing nonlinearity as in the case of the Riemannian manifold without boundary.

In the Dirichlet case, we choose $S := A = -\Delta_D$ which is a strictly positive operator and [104], Theorem 6.10, yields the Gaussian estimate for the associated semigroup. Hence, we can directly apply Theorem 4.10 to construct a martingale solution of problem (4.65).

In the Neumann case, we have $0 \in \sigma(\Delta_N)$ and the kernel p of the semigroup $\left(e^{-t\Delta_N}\right)_{t \geq 0}$ only satisfies the estimate

$$|p(t,x,y)| \leq \frac{C_\varepsilon}{\mu(B(x, t^{\frac{1}{m}}))} e^{\varepsilon t} \exp\left\{ -c \left(\frac{\rho(x,y)^m}{t} \right)^{\frac{1}{m-1}} \right\}$$

for all $t > 0$ and almost all $(x,y) \in M \times M$ with an arbitrary $\varepsilon > 0$, see [104], Theorem 6.10. In order to get a strictly positive operator with the Gaussian bound from Remark 4.2, we fix $\varepsilon > 0$ and choose $S := \varepsilon I - \Delta_N$. □

4.4.3. The fractional NLS

In this subsection, we prove an existence result for the fractional stochastic NLS. In particular, we show how the range of admissible nonlinearities changes when the Laplacians in the previous examples are replaced by their fractional powers $(-\Delta)^\beta$ for $\beta \in (0,1)$. Exemplary, we treat the case of a compact Riemannian manifold without boundary. Similar results are also true for the Dirichlet and the Neumann Laplacian on a bounded domain.

In the setting of Corollary 4.29, we look at the fractional Laplace-Beltrami operator given by $A := (-\Delta_g)^\beta$ for $\beta > 0$ which is also a selfadjoint nonnegative operator by the functional calculus. Once again, we choose $S := I - \Delta_g$. We apply Theorem 4.10 with $E_A = H^\beta(M)$. Therefore, the range of admissible pairs (α, β) in the defocusing case is given by

$$\beta > \frac{d}{2} - \frac{d}{\alpha+1} \quad \Leftrightarrow \quad \alpha \in \left(1, 1 + \frac{4\beta}{(d-2\beta)_+}\right),$$

since Proposition $A.53$ d) implies that this is exactly the range of α and β with a compact embedding $E_A \hookrightarrow L^{\alpha+1}(M)$. In the focusing case, analogous calculations as in the proof of Lemma 4.30 (with the distinction of $\beta > \frac{d}{2}$, $\beta = \frac{d}{2}$ and $\beta < \frac{d}{2}$) imply that the range of exponents reduces to

$$\alpha \in \left(1, 1 + \frac{4\beta}{d}\right).$$

Hence, we get the following Corollary.

Corollary 4.32. *Let (M,g) be a compact d-dimensional Riemannian manifold without boundary, $\beta \in (0,1)$ and $u_0 \in H^\beta(M)$. Under Assumption 4.6 and either i) or ii)*

i) $F(u) = |u|^{\alpha-1}u$ *with* $\alpha \in \left(1, 1 + \frac{4\beta}{(d-2\beta)_+}\right)$,

ii) $F(u) = -|u|^{\alpha-1}u$ *with* $\alpha \in \left(1, 1 + \frac{4\beta}{d}\right)$,

the equation

$$\begin{cases} \mathrm{d}u(t) = \big(-\mathrm{i}\,(-\Delta_g)^\beta\, u(t) - \mathrm{i}F(u(t)) + \mu(u(t))\big)dt - \mathrm{i}Bu(t)\mathrm{d}W(t), \\ u(0) = u_0 \in H^\beta(M), \end{cases}$$

has an analytically weak martingale solution $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbb{F}}, u\right)$ *in* $H^\beta(M)$ *which satisfies* $u \in C_w([0,T]; H^\beta(M))$ *almost surely and* $u \in L^q(\tilde{\Omega}, L^\infty(0,T;H^\beta(M)))$ *for all* $q \in [1,\infty)$.

4.4.4. Concrete examples for the multiplicative noise

In Corollaries 4.29 and 4.31, we considered the general nonlinear noise from Assumption 4.6. We would like to illustrate this class of noises by concrete examples. For presentation purposes, we only treat the case that M is a bounded domain. Similar arguments work in the Riemannian setting. For the fractional NLS, however, our argument based on the fact that $E_A = H^1(M)$ breaks down.

Proposition 4.33. *In the setting of Corollary 4.31, we assume that* $g : [0,\infty) \to \mathbb{R}$ *is continuously differentiable and satisfies*

$$\sup_{r>0} |g(r)| < \infty, \qquad \sup_{r>0} r|g'(r)| < \infty. \tag{4.66}$$

In particular, we can choose

$$g(r) = \frac{r}{1+\sigma r}, \qquad g(r) = \frac{r(2+\sigma r)}{(1+\sigma r)^2}, \qquad g(r) = \frac{\log(1+\sigma r)}{1+\log(1+\sigma r)}, \qquad r \in [0,\infty), \tag{4.67}$$

for a constant $\sigma > 0$*. Moreover, we suppose that the coefficient functions* e_m*,* $m \in \mathbb{N}$*, fulfill*

$$e_m \in F := \begin{cases} H^{1,d}(M) \cap L^\infty(M), & d \geq 3, \\ H^{1,q}(M), & d = 2, \\ H^1(M), & d = 1, \end{cases} \tag{4.68}$$

for some $q > 2$ *in the case* $d = 2$ *and*

$$\sum_{m=1}^\infty \|e_m\|_F^2 < \infty.$$

Then, the nonlinear operator $B : H \to \mathrm{HS}(Y, H)$ *given by*

$$B(u)f_m := e_m g(|u|^2)u, \qquad m \in \mathbb{N}, u \in H,$$

fulfills Assumption 4.6.

Proof. Let us fix $p \in \{\alpha+1, 2\}$ and $j = 1, 2$. By the boundedness of g, we immediately obtain

$$\|g(|u|^2)^j u\|_{L^p} \lesssim \|u\|_{L^p}, \qquad u \in L^p(M). \tag{4.69}$$

To show the Lipschitz condition (4.17), we set $\Phi_j(z) := g(|z|^2)^j z$ for $z \in \mathbb{C}$. We take $z_1, z_2 \in \mathbb{C}$ and compute

$$\begin{aligned} \Phi_1'(z_1)z_2 &= 2g'(|z_1|^2)\operatorname{Re}\langle z_1, z_2\rangle_{\mathbb{C}} z_1 + g(|z_1|^2)z_2, \\ \Phi_2'(z_1)z_2 &= 4g(|z_1|^2)g'(|z_1|^2)\operatorname{Re}\langle z_1, z_2\rangle_{\mathbb{C}} z_1 + g(|z_1|^2)^2 z_2. \end{aligned}$$

As in part b) of the proof of Lemma 4.8, we obtain $|\Phi_1(z)| \lesssim |z|$ and $|\Phi_1'(z)| \lesssim 1$ for $z \in \mathbb{C}$. The corresponding estimates for Φ_2 can be shown analogously. Hence, we can apply Lemma 4.8 to deduce that the map $L^p(M) \ni u \mapsto g(|u|^2)^j u \in L^p(M)$ is Gâteaux differentiable. From the mean value theorem, we infer

$$\|g(|u|^2)^j u - g(|v|^2)^j v\|_{L^p} \leq \sup_{t\in[0,1]} \|\Phi_j'\big(tu + (1-t)v\big)(u-v)\|_{L^p} \lesssim \|u-v\|_{L^p},$$

which proves (4.17). To show the remaining estimate

$$\|g(|u|^2)^j u\|_{E_A} \lesssim \|u\|_{E_A}, \qquad u \in E_A, \tag{4.70}$$

we use $E_A = H^1(M)$ in the Neumann case and $E_A = H_0^1(M)$ in the Dirichlet case. From the weak chain rule, see [56], Theorem 7.8, we obtain that $g(|u|^2)^j u \in H^1(M)$ for $u \in H^1(M)$ and

$$\|\nabla[g(|u|^2)^j u]\|_{L^2} = \|\Phi_j'(u)\nabla u\|_{L^2} \lesssim \|\nabla u\|_{L^2}.$$

In view of (4.69), we have proved (4.70) and it is not hard to check that the particular choices for g from (4.67) satisfy (4.66).

We continue with the conditions on the coefficients e_m, $m \in \mathbb{N}$. We get

$$\|e_m u\|_{L^p} \leq \|e_m\|_{L^\infty(M)} \|u\|_{L^p}, \qquad u \in L^p(M),$$

for $p \in [1,\infty]$. First, let $d \geq 3$. The Sobolev embedding $H^1(M) \hookrightarrow L^{p_{\max}}(M)$ for $p_{\max} = \frac{2d}{d-2}$ and the Hölder inequality with $\frac{1}{2} = \frac{1}{d} + \frac{1}{p_{\max}}$ yield

$$\begin{aligned}\|\nabla(e_m u)\|_{L^2} \leq& \|u\nabla e_m\|_{L^2} + \|e_m \nabla u\|_{L^2} \leq \|\nabla e_m\|_{L^d} \|u\|_{L^{p_{\max}}} + \|e_m\|_{L^\infty(M)} \|\nabla u\|_{L^2} \\ \lesssim& \left(\|\nabla e_m\|_{L^d} + \|e_m\|_{L^\infty(M)}\right) \|u\|_{H^1}, \qquad u \in H^1(M).\end{aligned}$$

Now, let $d = 2$ and $q > 2$ as in (4.68). Then, we have $F \hookrightarrow L^\infty(M)$. Furthermore, we choose $p > 2$ according to $\frac{1}{2} = \frac{1}{q} + \frac{1}{p}$ and observe $H^1(M) \hookrightarrow L^p(M)$. As above, we obtain

$$\|\nabla(e_m u)\|_{L^2} \lesssim \left(\|\nabla e_m\|_{L^q} + \|e_m\|_{L^\infty(M)}\right) \|u\|_{H^1} \lesssim \|e_m\|_{H^{1,q}} \|u\|_{H^1}, \qquad u \in H^1(M).$$

Hence, we conclude in both cases

$$\|e_m u\|_{H^1} \lesssim \|e_m\|_F \|u\|_{H^1}, \qquad m \in \mathbb{N}, \quad u \in H^1(M).$$

For $d = 1$, this inequality directly follows from the embedding $H^1(M) \hookrightarrow L^\infty(M)$. Therefore, we obtain

$$\sum_{m=1}^\infty \|M_{e_m}\|^2_{\mathcal{L}(E_A)} < \infty.$$

for arbitrary dimension d. The properties of M_{e_m} as operator in $\mathcal{L}(L^{\alpha+1}(M))$ and in $\mathcal{L}(L^2(M))$ can be deduced from the embedding $F \hookrightarrow L^\infty(M)$. □

5. Uniqueness results for the stochastic NLS

In the previous chapter, we proved existence of a martingale solution to the stochastic NLS in a general framework. However, the construction of the solution via an approximation argument does not guarantee that the solution is unique. In this chapter, we address the question of pathwise uniqueness for our problem on different geometries and dimensions.

In the field of stochastic PDE, pathwise uniqueness has the particular significance that it even improves existence results. Roughly speaking, the Yamada-Watanabe theory states

existence of a martingale solution and pathwise uniqueness
$\Rightarrow$ existence of a strong solution.

Hence, uniqueness solves the problem that approximation arguments typically only lead to martingale solutions. For a mathematically rigorous statement of this result, we refer to Theorem 2.4.

In contrast to the existence proof in the previous chapter which was only based on the conservation laws of the NLS and certain compactness properties of the underlying geometry, uniqueness results only hold in special situations. Our proof will be based on the formula

$$\|u_1(t)-u_2(t)\|_{L^2}^2 = 2\int_0^t \operatorname{Re}\langle u_1(s)-u_2(s), -\mathrm{i}F(u_1(s))+\mathrm{i}F(u_2(s))\rangle_{L^{\alpha+1},L^{\frac{\alpha+1}{\alpha}}}\,\mathrm{d}s \tag{5.1}$$

almost surely for all $t\in[0,T]$ for two solutions u_1,u_2 of the stochastic NLS

$$\begin{cases} \mathrm{d}u(t)=(-\mathrm{i}Au(t)-\mathrm{i}F(u(t)))\,\mathrm{d}t-\mathrm{i}Bu(t)\circ\mathrm{d}W(t),\\ u(0)=u_0. \end{cases} \tag{5.2}$$

This formula has been proved in Corollary 2.10. It can be viewed as an extension of mass conservation and is only true for *linear conservative noise*, i.e. the operators B_m defined by $B_m u = B(u)f_m$ for $u\in L^2(M)$ and $m\in\mathbb{N}$ are linear and symmetric. Note that the absence of stochastic integrals in (5.1) is crucial since pathwise estimates of Itô integrals are not available. The identity (5.1) leads to sufficient criteria for pathwise uniqueness that will be proved in Lemma 5.2 and Lemma 5.3. Due to the nonlinear structure of the integrand in (5.1), one needs a control of the L^∞-norm or at least the L^p-norms for large p of u_1 and u_2 to prove $u_1=u_2$.

For $d=2$, we can use the *Moser-Trudinger-inequality* to get a precise dependence of the embedding constant in $H^1(M)\hookrightarrow L^p(M)$ and prove pathwise uniqueness since the general existence theory provides $u_1,u_2\in L^\infty(0,T;H^1(M))$. This argument works in a wide range of our existence results, i.e. M can be chosen to be a Riemannian manifold with the Laplace-Beltrami operator and it also possible that M is a bounded domain with Dirichlet or Neumann Laplacian.

If we restrict ourselves to the Riemannian setting, we can also employ the integrability gain induced by *Strichartz estimates* to prove pathwise uniqueness for $d=2,3$. Note that for $d=$

2, we can even work in a more general setting than in the previous chapter and allow not necessarily compact manifolds M. Moreover, we can lower the regularity level to $H^s(M)$ for

$$s \in \begin{cases} (1-\frac{1}{2\alpha},1] & \text{for } \alpha \in (1,3], \\ (1-\frac{1}{\alpha(\alpha-1)},1] & \text{for } \alpha > 3. \end{cases}$$

However, our arguments do not work in high dimensions $d \geq 4$, since in this setting, the improvement of Strichartz estimates with respect to Sobolev embeddings is not large enough any more.

The chapter is organized as follows. The first section is devoted to two Lemmata which identify the properties solutions should have in order to be pathwise unique. In the sections 5.2 and 5.3, we prove pathwise uniqueness for the stochastic NLS on various two and three dimensional geometries.

Throughout this chapter, we consider the problem (5.2) under the following assumptions.

Assumption 5.1. Let M be a σ-finite metric measure space and $s \in (0,1]$, $\alpha \in (1,\infty)$. Suppose that A is a non-negative selfadjoint operator $L^2(M)$ with the scale $(X_\theta)_{\theta\in\mathbb{R}}$ of fractional domains. We assume the following:

a) Let Y be a separable real Hilbert space and $B : X_{\frac{s}{2}} \to \operatorname{HS}(Y, X_{\frac{s}{2}})$ a linear operator. For an ONB $(f_m)_{m\in\mathbb{N}}$ of Y and $m \in \mathbb{N}$, we set $B_m := B(\cdot)f_m$. Additionally, we assume that B_m, $m \in \mathbb{N}$, are bounded operators on $L^2(M)$ and $X_{\frac{s}{2}}$ with

$$\sum_{m=1}^{\infty} \|B_m\|^2_{\mathcal{L}(L^2)} < \infty, \qquad \sum_{m=1}^{\infty} \|B_m\|^2_{\mathcal{L}(X_{\frac{s}{2}})} < \infty \tag{5.3}$$

and that B_m is symmetric as operator on $L^2(M)$, i.e.

$$\big(B_m u, v\big)_{L^2} = \big(u, B_m v\big)_{L^2}, \qquad u, v \in X_{\frac{s}{2}}. \tag{5.4}$$

b) Let $u_0 \in X_{\frac{s}{2}}$ and $F(u) = F_\alpha^{\pm}(u) = \pm|u|^{\alpha-1}u$ for $u \in X_{\frac{s}{2}}$.

5.1. Model proofs for Uniqueness

To clarify which integrability solutions should have in order to be pathwise unique, we state the following two deterministic Lemmata. In the subsequent sections, they will be used to prove pathwise uniqueness in the special situations mentioned above. The first Lemma contains the classical Gronwall argument.

Lemma 5.2. *Let $T > 0$, $\alpha \in (1,\infty)$ and $q \geq 1 \vee (\alpha-1)$. Then, for all*

$$u_1, u_2 \in C([0,T], L^2(M)) \cap L^q(0,T;L^\infty)$$

with

$$\|u_1(t) - u_2(t)\|^2_{L^2} \lesssim \int_0^t \int_M |u_1(\tau,x) - u_2(\tau,x)|^2 \left[|u_1(\tau,x)|^{\alpha-1} + |u_2(\tau,x)|^{\alpha-1}\right] \mathrm{d}x\mathrm{d}\tau \tag{5.5}$$

for $t \in [0,T]$, we have $u_1 = u_2$.

Proof. We have

$$\|u_1(t)-u_2(t)\|_{L^2}^2 \lesssim \int_0^t \int_M |u_1(\tau,x)-u_2(\tau,x)|^2 \left[1+|u_1(\tau,x)|^q+|u_2(\tau,x)|^q\right] \mathrm{d}x\mathrm{d}\tau$$
$$\leq \int_0^t \|u_1(\tau)-u_2(\tau)\|_{L^2}^2 \left[1+\|u_1(\tau)\|_{L^\infty}^q+\|u_2(\tau)\|_{L^\infty}^q\right] \mathrm{d}\tau$$

By the assumption, the function b defined by

$$b(\tau) := \left[\|u_1(\tau)\|_{L^\infty}^{\alpha-1}+\|u_2(\tau)\|_{L^\infty}^{\alpha-1}\right], \qquad \tau\in[0,T],$$

is in $L^1(0,T)$ and therefore, the Gronwall inequality yields $u_1(t)=u_2(t)$ for all $t\in[0,T]$. □

Next, we present a refinement of Lemma 5.2 developed by Yudovitch, [131], for the Euler equation. It has also been frequently used to show uniqueness for the deterministic NLS, see Vladimirov [127], Ogawa and Ozawa [100], [101], Burq, Gérard and Tzvetkov [35] and Blair, Smith and Sogge in [19].

Lemma 5.3. *Let $T>0$, $\alpha\in(1,3]$ and $q\geq \max\{1,\alpha-1\}$. Let*

$$u_1,u_2\in C([0,T],L^2(M))\cap L^\infty(0,T;L^6)\cap L^2(0,T;L^{p_n}), \tag{5.6}$$

where $(p_n)_{n\in\mathbb{N}}\in[6,\infty)^{\mathbb{N}}$ with $p_n\to\infty$ as $n\to\infty$. Let us suppose that

$$\|u_j\|_{L^2(J,L^{p_n})}\lesssim 1+\left(|J|p_n\right)^{\frac12}, \qquad n\in\mathbb{N}, \tag{5.7}$$

for all intervals $J\subset[0,T]$ and $j=1,2$. Furthermore, we assume that the map

$$(0,T)\ni t\mapsto G(t):=\|u_1(t)-u_2(t)\|_{L^2}^2$$

is weakly differentiable with

$$|G'(t)|\lesssim \int_M |u_1(t,x)-u_2(t,x)|^2\left[|u_1(t,x)|^{\alpha-1}+|u_2(t,x)|^{\alpha-1}\right]\mathrm{d}x, \quad t\in[0,T]. \tag{5.8}$$

Then, we have $u_1=u_2$.

Proof. Step 1. We fix $n\in\mathbb{N}$ and define $q_n:=\frac{p_n}{\alpha-1}$. By the estimate (5.8) and Hölder's inequality with exponents $\frac{1}{q_n'}+\frac{1}{q_n}=1$, we get

$$|G'(t)|\lesssim \|u_1(t)-u_2(t)\|_{L^{2q_n'}}^2 \left\||u_1(t)|^{\alpha-1}+|u_2(t)|^{\alpha-1}\right\|_{L^{q_n}}, \qquad t\in[0,T].$$

The choice of q_n yields $2q_n'\in[2,6]$ and for $\theta:=\frac{3}{2q_n}\in(0,1)$, we have $\frac{1}{2q_n'}=\frac{1-\theta}{2}+\frac{\theta}{6}$. Hence, we obtain

$$\|u_1-u_2\|_{L^{2q_n'}}^2\leq \|u_1-u_2\|_{L^2}^{2-\frac{3}{q_n}}\|u_1-u_2\|_{L^6}^{\frac{3}{q_n}}\leq \|u_1-u_2\|_{L^2}^{2-\frac{3}{q_n}}\|u_1-u_2\|_{L^\infty(0,T;L^6)}^{\frac{3}{q_n}}$$

by interpolation. We choose a constant $C_1>0$ such that

$$\|u_1\|_{L^\infty(0,T;L^6)}+\|u_2\|_{L^\infty(0,T;L^6)}\leq C_1,$$

which leads to the estimate

$$|G'(t)|\lesssim C_1^{\frac{3}{q_n}} G(t)^{1-\frac{3}{2q_n}}\left[\|u_1(t)\|_{L^{p_n}}^{\alpha-1}+\|u_2(t)\|_{L^{p_n}}^{\alpha-1}\right]. \tag{5.9}$$

Step 2. We argue by contradiction and assume that there is $t_2 \in [0,T]$ with $G(t_2) > 0$. By the continuity of G, we get

$$\exists t_1 \in [0,t_2) : G(t_1) = 0 \quad \text{and} \quad \forall t \in (t_1,t_2) : G(t) > 0. \tag{5.10}$$

We set $J_\varepsilon := (t_1, t_1+\varepsilon)$ with $\varepsilon \in (0, t_2 - t_1)$ to be chosen later. By the weak chain rule (see [56], Theorem 7.8) and (5.9), we get

$$G(t)^{\frac{3}{2q_n}} = \frac{3}{2q_n}\int_{t_1}^t G'(s)G(s)^{\frac{3}{2q_n}-1}\mathrm{d}s \lesssim \frac{3}{2q_n}C_1^{\frac{3}{q_n}}\int_{t_1}^t \left[\|u_1(s)\|_{L^{p_n}}^{\alpha-1} + \|u_2(s)\|_{L^{p_n}}^{\alpha-1}\right]\mathrm{d}s$$

for $t \in J_\varepsilon$. From another application of Hölder's inequality with exponents $\frac{2}{\alpha-1}$ and $\frac{2}{3-\alpha}$, we infer

$$G(t)^{\frac{3}{2q_n}} \lesssim \frac{3}{2q_n}C_1^{\frac{3}{q_n}}\left[\|u_1\|_{L^2(t_1,t;L^{p_n})}^{\alpha-1} + \|u_2\|_{L^2(t_1,t;L^{p_n})}^{\alpha-1}\right]\varepsilon^{\frac{3-\alpha}{2}}.$$

Now, we are in the position to apply (5.7) and we obtain

$$G(t)^{\frac{3}{2q_n}} \lesssim \frac{3}{2q_n}C_1^{\frac{3}{q_n}}\left(1 + (\varepsilon p_n)^{\frac{\alpha-1}{2}}\right)\varepsilon^{\frac{3-\alpha}{2}}.$$

In particular, there is a constant $C > 0$ such that

$$\begin{aligned} G(t) &\leq C_1^2\left(\frac{3C}{2q_n}\left(1 + (\varepsilon(\alpha-1)q_n)^{\frac{\alpha-1}{2}}\right)\varepsilon^{\frac{3-\alpha}{2}}\right)^{\frac{2q_n}{3}} \\ &\leq C_1^2\left(\frac{3C}{2q_n}\left(1 + \varepsilon^{\frac{\alpha-1}{2}}(\alpha-1)q_n\right)\varepsilon^{\frac{3-\alpha}{2}}\right)^{\frac{2q_n}{3}} =: b_n, \end{aligned} \tag{5.11}$$

where we used $p_n := q_n(\alpha-1)$ and $\frac{\alpha-1}{2} \in (0,1]$.

Step 3. We aim to show that the sequence $(b_n)_{n\in\mathbb{N}}$ on the RHS of (5.11) converges to 0 for ε sufficiently small. Then, we have proved $G(t) = 0$ for $t \in J_\varepsilon$ which contradicts (5.10). Hence, we have $u_1(t) = u_2(t)$ for all $t \in [0,T]$.

To this end, we choose $\varepsilon \in (0, \min\{t_2 - t_1, \frac{2}{3C(\alpha-1)}\})$. Then,

$$\begin{aligned} b_n =& C_1^2\left(\frac{3C}{2q_n}\left(1 + \varepsilon^{\frac{\alpha-1}{2}}(\alpha-1)q_n\right)\varepsilon^{\frac{3-\alpha}{2}}\right)^{\frac{2q_n}{3}} \\ =& C_1^2\left(\frac{3C\varepsilon(\alpha-1)}{2}\right)^{\frac{2q_n}{3}}\left(\frac{1}{\varepsilon^{\frac{\alpha-1}{2}}(\alpha-1)q_n} + 1\right)^{\frac{2q_n}{3}} \xrightarrow{n\to\infty} 0. \end{aligned}$$

□

5.2. Uniqueness in two dimensions

In this section, we consider the problem of pathwise uniqueness for solutions of the stochastic NLS in two dimensions. The proofs are based on the Moser-Trudinger inequality in the first subsection and on Strichartz estimates in the second one.

5.2.1. Uniqueness via the Moser-Trudinger inequality

In this section, we prove uniqueness for the stochastic NLS on various geometries in dimension $d = 2$ by an application of Lemma 5.3. Our result is based on the Moser-Trudinger equality and inspired by [36], [100] and [101] who gave a similar proof in the deterministic setting. As in chapter 4, we fix the notation $E_A := X_{\frac{1}{2}}$.

Theorem 5.4. *Let Assumption 5.1 be fulfilled with $s = 1$, $\alpha \in (1, 3]$ and an open subset M of a σ-finite metric measure space $\tilde{M}$ with doubling-property and dimension $d = 2$. We assume that A supports a Moser-Trudinger-inequality, i.e. for all $R > 0$, there exists $\beta > 0$ and $K = K(R) > 0$ such that*

$$\int_M \left(e^{\beta|u|^2} - 1\right) \mathrm{d}x \le K(R), \qquad u \in E_A, \quad \|u\|_{E_A} \le R. \tag{5.12}$$

For $p \in \left\{\alpha + 1, \frac{\alpha+1}{\alpha}\right\}$, we suppose that there is a C_0-semigroup $(T_p(t))_{t\ge 0}$ on $L^p(M)$ which is consistent with $\left(e^{-tA}\right)_{t\ge 0}$. Then, solutions of problem (5.2) *are pathwise unique in $L^0_\omega L^\infty(0, T; E_A)$.*

Proof. In Corollary 2.10, we computed

$$\|u_1(t) - u_2(t)\|_{L^2}^2 = 2\int_0^t \operatorname{Re}\left(u_1(s) - u_2(s), -\mathrm{i}\lambda|u_1(s)|^{\alpha-1}u_1(s) + \mathrm{i}\lambda|u_2(s)|^{\alpha-1}u_2(s)\right)_{L^2} \mathrm{d}s \tag{5.13}$$

almost surely for all $t \in [0, T]$. Let $\tilde{\Omega}_1$ be a set of full probability such that we have (5.13) for $\omega \in \tilde{\Omega}_1$ as well as $u_j(\cdot, \omega) \in L^\infty(0, T; E_A)$.
We fix $\omega \in \tilde{\Omega}_1$ and check the assumptions of Lemma 5.3 for $u_j(\cdot, \omega)$, $j = 1, 2$. In the following, we drop the dependence on ω. We use the elementary fact

$$x^p \le \left(\frac{p}{2\beta}\right)^{\frac{p}{2}} \left(e^{\beta x^2} - 1\right), \qquad x \ge 0,$$

and Trudinger's inequality to get

$$\|u_j\|_{L^\infty(0,T;L^p(M))} \le \left(\frac{p}{2\beta}\right)^{\frac{1}{2}} K(\|u_j\|_{L^\infty(0,T;E_A)})$$

for $p \in [6, \infty)$. In particular, $u_j \in L^\infty(0, T; L^6(M)) \cap L^2(0, T; L^p(M))$ with

$$\|u_j\|_{L^2(J,L^p(M))} \lesssim (|J|p)^{\frac{1}{2}}$$

for an arbitrary interval $J \subset [0, T]$. The solutions u_j are continuous in $L^2(M)$ due to the mild formulation of (5.2). Indeed, Lemma 2.5 yields

$$u_j(t) = e^{-\mathrm{i}tA}u_0 + \int_0^t e^{-\mathrm{i}(t-s)A} F_\alpha^{\pm}(u_j(s))\mathrm{d}s + \int_0^t e^{-\mathrm{i}(t-s)A} B(u_j(s))\mathrm{d}W(s)$$

almost surely in E_A^* for all $t \in [0, T]$ and $j = 1, 2$. Since each term on the RHS of this identity is almost surely in $C([0, T], L^2(M))$, we deduce $u_j \in C([0, T], L^2(M))$ almost surely. The formula (5.13) leads to the weak differentiability of $G := \|u_1 - u_2\|_{L^2}^2$ and to (5.8) by the estimate

$$|F_\alpha^{\pm}(z_1) - F_\alpha^{\pm}(z_2)| \lesssim \left(|z_1|^{\alpha-1} + |z_2|^{\alpha-1}\right)|z_1 - z_2|, \qquad z_1, z_2 \in \mathbb{C}.$$

Hence, the assertion follows from Lemma 5.3. □

Remark 5.5. a) The Moser-Trudinger inequality can be viewed as the embedding $E_A \hookrightarrow L_B(M)$, where

$$L_B(M) := \operatorname{span}\left\{u : M \to \mathbb{C} \quad \text{measurable:} \quad \int_M B(|u(x)|)\mathrm{d}x < \infty\right\}$$

denotes the *Orlicz space* given by the weight $B(t) = \exp(t^2) - 1$, $t \geq 0$. $L_B(M)$ is a Banach space equipped with the norm

$$\|u\|_{L_B(M)} := \inf\left\{k > 0 : \int_M B\left(k^{-1}|u(x)|\right)\mathrm{d}x \leq 1\right\}, \qquad u \in L_B(M).$$

In general dimensions, the Moser-Trudinger inequality typically holds for $u \in W^{k,p}(M)$ with $kp = d$. Hence, it is an improvement of Sobolev's embedding in the limit case.

b) In the proof of Theorem 5.4, we showed that (5.12) implies

$$\|u\|_{L^p} \leq C(R)p^{\frac{1}{2}}, \qquad u \in E_A, \quad \|u\|_{E_A} \leq R. \tag{5.14}$$

In fact, (5.12) and (5.14) are equivalent since we get

$$\int_M \left(e^{\beta|u|^2} - 1\right)\mathrm{d}x = \sum_{k=1}^{\infty} \frac{\beta^k}{k!}\|u\|_{L^{2k}}^{2k} \leq \sum_{k=1}^{\infty} \frac{\left(2\beta C(R)^2 k\right)^k}{k!} < \infty$$

if we choose $\beta < \left(2C(R)^2 e\right)^{-1}$.

In the following Corollary, we apply the abstract uniqueness result from above to different special geometries.

Corollary 5.6. *Let $F(u) = F_\alpha^{\pm}(u) = \pm|u|^{\alpha-1}u$ with $\alpha \in (1,3]$. Let M and A satisfy one of the following assumptions:*

a) M is a compact $2D$ Riemannian manifold and $A = -\Delta_g$ with $E_A = H^1(M)$.

b) $M \subset \mathbb{R}^2$ is a bounded C^2-domain and $A = -\Delta_N$ is the Neumann Laplacian with $E_A = H^1(M)$.

c) $M \subset \mathbb{R}^2$ is a domain and $A = -\Delta_D$ is the Dirichlet Laplacian with $E_A = H_0^1(M)$.

Suppose that Assumption 5.1 holds with $s = 1$. Then, solutions of problem (5.2) *are pathwise unique in $L_\omega^0 L^\infty(0,T;E_A)$.*

Proof. In view of Theorem 5.4, it is sufficient to give references for the Moser-Trudinger inequality in the three settings. Note that the assumption on the consistency of the semigroups is true due to the Gaussian bounds of the operators from a), b) and c), see Remark 2.7.
ad a): See [132], Theorem 1.2.
ad b): See [1], Theorem 8.27. For simplicity, we assumed C^2-regularity of the boundary. In the reference, it is only assumed that the domain satisfies the cone condition. For further details, see [1], Chapter 4.
ad c): See [109], Theorem 1.1. □

The pretense of Corollary 5.6 is rather to emphasize the typical range of applications of Theorem 5.4 than to be complete. For example, there is a recent result by Kristaly, [81] for non-compact manifolds with curvature bounds from below which we omit to avoid new notations and definitions. As a consequence of the uniqueness result from above and the existence results from the previous chapter, the Yamada-Watanabe Theorem 2.4 yields the existence of a stochastically strong solution.

Corollary 5.7. *Let M and A satisfy one of the conditions a), b) and c) from Corollary 5.6. and suppose that Assumption 5.1 holds with $s = 1$ and either i) or ii)*

i) $F(u) = |u|^{\alpha-1}u$ *with* $\alpha \in (1, 3]$,

ii) $F(u) = -|u|^{\alpha-1}u$ *with* $\alpha \in (1, 3)$.

Additionally, we assume $\sum_{m=1}^{\infty} \|B_m\|^2_{\mathcal{L}(L^{\alpha+1})} < \infty$. *Then, the equation*

$$\begin{cases} \mathrm{d}u(t) = (-\mathrm{i}Au(t) - \mathrm{i}F(u(t))\,dt - \mathrm{i}Bu(t) \circ \mathrm{d}W(t) \quad \text{in } E_A, \\ u(0) = u_0 \in E_A, \end{cases} \tag{5.15}$$

has a stochastically strong and analytically weak solution which satisfies $u \in C_w([0,T], E_A)$ *almost surely and* $u \in L^q(\Omega, L^\infty(0,T; E_A))$ *for all* $q \in [1, \infty)$. *Moreover, we have*

$$\|u(t)\|_{L^2(M)} = \|u_0\|_{L^2(M)}$$

almost surely for all $t \in [0, T]$.

Proof. We apply Theorem 2.4 with the operator $A : E_A \to E_A^*$ and $U = L^\infty(0,T; E_A)$. We set

$$\tilde{F}(u) := -\mathrm{i}F(u) - \frac{1}{2}\sum_{m=1}^{\infty} B_m^2 u$$

and choose $\rho := 1 - \frac{1}{\alpha+1} \in (0, \frac{3}{4})$. Then, we have the embeddings

$$X_\rho \hookrightarrow L^{\alpha+1}(M), \qquad L^{\frac{\alpha+1}{\alpha}}(M) \hookrightarrow X_{-\rho} \hookrightarrow E_A^*$$

and hence,

$$\|\tilde{F}(u)\|_{E_A^*} \lesssim \|F(u)\|_{L^{\frac{\alpha+1}{\alpha}}(M)} + \left\| \sum_{m=1}^{\infty} B_m^2 u \right\|_{L^2} \le \|u\|^\alpha_{L^{\alpha+1}(M)} + \|u\|_{L^2} \lesssim \|u\|^\alpha_{X_\rho} + \|u\|_{X_\rho}$$

for $u \in X_\rho$. Hence, $\tilde{F} : X_\rho \to X$ is bounded on bounded subsets of X_ρ. Similarly, one can check that $B : X_\rho \to \mathcal{L}(Y, X)$ is bounded on bounded subsets of X_ρ. In the setting a), we use the stochastically weak existence result from Corollary 4.29 and in b) and c), we use Corollary 4.31. Concerning the more general coefficients of the linear noise, we refer to Remark 4.11. The pathwise uniqueness is provided by Corollary 5.6. Consequently, the assumptions of Theorem 2.4 are satisfied and we obtain the existence of a stochastically strong solution. □

5.2.2. Uniqueness via Strichartz estimates

The uniqueness result from the last section has the benefit to hold on various geometries. On the other hand, there is a restriction to nonlinearities with $\alpha \in (1, 3]$, which is rather inconvenient in dimension two, where existence is typically true for all $\alpha \in (1, \infty)$.

The goal of this section is to significantly improve Theorem 5.4 by employing the smoothing effect of Strichartz estimates. We restrict ourselves to manifolds M with bounded geometry in order to apply the Strichartz estimates by Bernicot and Samoyeau from Proposition 2.15. In this setting, we will prove pathwise uniqueness of solutions in $L^r(\tilde{\Omega}, L^\beta(0,T;H^s(M)))$ for all $\alpha \in (1,\infty)$, $r > \alpha$, $\beta \geq \max\{\alpha, 2\}$ and

$$s \in \begin{cases} (1 - \frac{1}{2\alpha}, 1] & \text{for } \alpha \in (1,3], \\ (1 - \frac{1}{\alpha(\alpha-1)}, 1] & \text{for } \alpha > 3. \end{cases}$$

In particular, we drop the assumption that M is compact and replace it by

$$M \text{ is complete and connected, has a positive injectivity radius and a bounded geometry.} \tag{5.16}$$

We refer to Appendix A.4 for the definitions of the notions above and background references on differential geometry. We equip M with the canonical volume μ and suppose that M satisfies the doubling property: For all $x \in \tilde{M}$ and $r > 0$, we have $\mu(B(x,r)) < \infty$ and

$$\mu(B(x,2r)) \lesssim \mu(B(x,r)). \tag{5.17}$$

We emphasize that by Proposition A.50, our assumption (5.16) is satisfied by compact manifolds. Examples for manifolds with the property (5.17) are given by compact manifolds and manifolds with non-negative Ricci-curvature, see [39]. Let $A = -\Delta_g$ be the Laplace-Beltrami operator $F_\alpha^\pm(u) = |u|^{\alpha-1}u$ be the power-type model nonlinearity.

We start with a Lemma on the mapping properties of the nonlinearity between fractional Sobolev spaces.

Lemma 5.8. *Let $d = 2$, $\alpha > 1$ and $s \in (\frac{\alpha-1}{\alpha}, 1]$. Then, we have $F_\alpha^\pm : H^s(M) \to H^{\tilde{s}}(M)$ for all $\tilde{s} \in (0, 1 - \alpha + s\alpha)$ and*

$$\|F_\alpha^\pm(u)\|_{H^{\tilde{s}}} \lesssim \|u\|_{H^s}^\alpha, \qquad u \in H^s(M).$$

Proof. We prove the assertion in the special case $M = \mathbb{R}^2$. For a general M, the estimate follows by the definition of fractional Sobolev spaces via charts, see Definition A.52. We refer to [21], proof of Lemma III.1.4 for the details.

We start with the proof of

$$\|F_\alpha^\pm(u)\|_{H^{s,r}(\mathbb{R}^2)} \lesssim \|u\|_{H^s(\mathbb{R}^2)}^\alpha \tag{5.18}$$

for $r \in (1, \frac{2}{(1-s)\alpha+s})$. To show (5.18), we employ

$$\||\nabla|^s F_\alpha^\pm(u)\|_{L^r} \lesssim \|u\|_{L^q}^{\alpha-1} \||\nabla|^s u\|_{L^2}, \qquad \frac{1}{r} = \frac{1}{2} + \frac{\alpha-1}{q},$$

from [38], Proposition 3.1. Furthermore, we have $\|F_\alpha^\pm(u)\|_{L^r} = \|u\|_{L^{r\alpha}}^\alpha$ and thus, (5.18) follows from the Sobolev embeddings

$$H^s(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2), \qquad H^s(\mathbb{R}^2) \hookrightarrow L^{r\alpha}(\mathbb{R}^2)$$

for $r \in (1, \frac{2}{(1-s)\alpha+s}]$. The assertion follows from (5.18) and the embedding $H^{s,r}(\mathbb{R}^2) \hookrightarrow H^{\tilde{s}}(\mathbb{R}^2)$. □

In the following Proposition, we reformulate problem (5.2) in a mild form and use this to show additional regularity properties of solutions of (5.2). Let us therefore recall the notation

$$\mu = -\frac{1}{2} \sum_{m=1}^\infty B_m^2.$$

Proposition 5.9. *Let $\alpha \in (1,\infty)$ and M be a 2D Riemannian manifold satisfying* (5.16) *and* (5.17). *Choose $s \in (\frac{\alpha-1}{\alpha}, 1]$, $\alpha > 1$, $r > 1$, $\beta := \max\{\alpha, 2\}$ and $2 < p, q < \infty$ with*

$$\frac{2}{p} + \frac{2}{q} = 1.$$

Suppose that Assumption 5.1 holds with $A := -\Delta_g$ and that $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbb{F}}, u\right)$ is a martingale solution to (5.2) *such that*

$$u \in L^{r\alpha}(\tilde{\Omega}, L^{\beta}(0,T; H^s(M))). \tag{5.19}$$

Then, for each $\tilde{s} \in (0, 1-\alpha+s\alpha)$ and $\varepsilon \in (0,1)$, we have

$$u \in L^r\left(\tilde{\Omega}, C([0,T], H^{\tilde{s}}(M)) \cap L^q(0,T; H^{\tilde{s}-\frac{1+\varepsilon}{q},p}(M))\right) \tag{5.20}$$

and almost surely in $H^{\tilde{s}}(M)$ for all $t \in [0,T]$

$$\begin{aligned} \mathrm{i}u(t) = \mathrm{i}e^{-\mathrm{i}tA}u_0 &+ \int_0^t e^{-\mathrm{i}(t-\tau)A} F_\alpha^{\pm}(u(\tau))\mathrm{d}\tau + \int_0^t e^{-\mathrm{i}(t-\tau)A}\mu(u(\tau))\mathrm{d}\tau \\ &+ \int_0^t e^{-\mathrm{i}(t-\tau)A} B(u(\tau))\mathrm{d}W(\tau). \end{aligned} \tag{5.21}$$

Remark 5.10. Of course, (5.20) also holds for $\varepsilon \geq 1$, but then $u \in L^r(\Omega, L^q(0,T; H^{s-\frac{1+\varepsilon}{q},p}(M)))$ would be trivial by the Sobolev embedding $H^{\tilde{s}}(M) \hookrightarrow H^{\tilde{s}-\frac{1+\varepsilon}{q},p}(M)$. Being able to choose $\varepsilon \in (0,1)$ means a gain of regularity which will be used below via $H^{\tilde{s}-\frac{1+\varepsilon}{q},p}(M) \hookrightarrow L^\infty(M)$ for an appropriate choice of the parameters.

Proof of Proposition 5.9. Step 1. We fix $X = H^{\tilde{s}-2}(M)$. By Proposition A.41, $-\Delta_g$ is a selfadjoint operator on X with domain $H^{\tilde{s}}(M)$. Thus, we can apply Lemma 2.5 and obtain (5.21) almost surely in $H^{\tilde{s}-2}(M)$ for all $t \in [0,T]$.

Step 2. Using the Strichartz estimates from Lemma 2.16 and Corollary 2.23 b), we deal with the free term and each convolution term on the RHS of (5.21) to get (5.20) and the identity (5.21) in $H^{\tilde{s}}(M)$. For this purpose, we define

$$Y_T := L^q(0,T; H^{\tilde{s}-\frac{1+\varepsilon}{q},p}(M)) \cap L^\infty(0,T; H^{\tilde{s}}(M)).$$

By Lemma (2.16) a), we obtain

$$\|e^{-\mathrm{i}tA}u_0\|_{L^r(\tilde{\Omega},Y_T)} \lesssim \|u_0\|_{H^{\tilde{s}}} \lesssim \|u_0\|_{H^s} < \infty$$

and from Lemma (2.16) b) and Lemma 5.8, we infer

$$\left\|\int_0^t e^{-\mathrm{i}(t-\tau)A} F_\alpha^{\pm}(u(\tau))\,\mathrm{d}\tau\right\|_{Y_T} \lesssim \|F_\alpha^{\pm}(u)\|_{L^1(0,T;H^{\tilde{s}})} \lesssim \|u\|_{L^\alpha(0,T;H^s)}^\alpha.$$

Integration over $\tilde{\Omega}$ and (5.19) yields

$$\left\|\int_0^t e^{-\mathrm{i}(t-\tau)A} F_\alpha^{\pm}(u(\tau))\,\mathrm{d}\tau\right\|_{L^r(\tilde{\Omega},Y_T)} \lesssim \|u\|_{L^{r\alpha}(\tilde{\Omega},L^\alpha(0,T;H^s))}^\alpha < \infty.$$

To estimate the other convolutions, we need that μ is bounded in $H^{\tilde{s}}(M)$ and B is bounded from $H^{\tilde{s}}(M)$ to $\mathrm{HS}(Y, H^{\tilde{s}}(M))$. This can be deduced as a consequence of complex interpolation (see [91], Theorem 2.1.6), Hölder's inequality and Assumption 5.1. With $\theta := \frac{\tilde{s}}{s}$, we get

$$\begin{aligned}\sum_{m=1}^{\infty} \|B_m\|_{\mathcal{L}(H^{\tilde{s}})}^2 &\leq \sum_{m=1}^{\infty} \|B_m\|_{\mathcal{L}(H^s)}^{2\theta} \|B_m\|_{\mathcal{L}(L^2)}^{2(1-\theta)} \\ &\leq \left(\sum_{m=1}^{\infty} \|B_m\|_{\mathcal{L}(H^s)}^2\right)^{\theta} \left(\sum_{m=1}^{\infty} \|B_m\|_{\mathcal{L}(L^2)}^2\right)^{1-\theta} < \infty. \end{aligned} \tag{5.22}$$

Therefore, by Lemma 2.16, (5.22) and (5.19)

$$\begin{aligned}\left\|\int_0^t e^{-\mathrm{i}(t-\tau)A}\mu(u(\tau))\,\mathrm{d}\tau\right\|_{L^r(\tilde{\Omega},Y_T)} &\lesssim \|\mu(u)\|_{L^r(\tilde{\Omega},L^1(0,T;H^{\tilde{s}}))} \lesssim \|u\|_{L^r(\tilde{\Omega},L^1(0,T;H^{\tilde{s}}))} \\ &\lesssim \|u\|_{L^{r\alpha}(\tilde{\Omega},L^\beta(0,T;H^s))} < \infty.\end{aligned}$$

Corollary 2.23 b) and the estimates (5.22) and (5.19) imply

$$\begin{aligned}\left\|\int_0^t e^{-\mathrm{i}(t-\tau)A}B(u(\tau))\,\mathrm{d}W(\tau)\right\|_{L^r(\tilde{\Omega},Y_T)} &\lesssim \|B(u)\|_{L^r\left(\tilde{\Omega},L^2(0,T;\mathrm{HS}(Y,H^{\tilde{s}}))\right)} \lesssim \|u\|_{L^r(\tilde{\Omega},L^2(0,T;H^{\tilde{s}}))} \\ &\lesssim \|u\|_{L^{r\alpha}(\tilde{\Omega},L^\beta(0,T;H^s))} < \infty.\end{aligned}$$

Hence, the mild equation (5.21) holds almost surely in $H^{\tilde{s}}(M)$ for each $t \in [0,T]$ and thus, we get (5.20) by the pathwise continuity of deterministic and stochastic integrals. □

Finally, we are ready to prove the pathwise uniqueness of solutions to (5.2) in the present setting.

Theorem 5.11. *Let $\alpha \in (1,\infty)$ and M be a 2D Riemannian manifold satisfying* (5.16) *and* (5.17). *Let $r > \alpha$, $\beta \geq \max\{\alpha, 2\}$ and*

$$s \in \begin{cases} (1 - \frac{1}{2\alpha}, 1] & \text{for } \alpha \in (1,3], \\ (1 - \frac{1}{\alpha(\alpha-1)}, 1] & \text{for } \alpha > 3. \end{cases}$$

Suppose that Assumption 5.1 holds with $A := -\Delta_g$. Then, solutions of problem (5.2) *are pathwise unique in $L^r_\omega L^\beta(0,T;H^s(M))$.*

Proof. Take two solutions $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbb{F}}, u_j\right)$ of (5.2) with $u_j \in L^r(\tilde{\Omega}, L^\infty(0,T;H^s(M)))$ for $j = 1,2$, and define $w := u_1 - u_2$. From Proposition 5.9, we get

$$u \in L^r\big(\tilde{\Omega}, C([0,T], H^{\tilde{s}}(M)) \cap L^q(0,T;H^{\tilde{s}-\frac{1+\varepsilon}{q},p}(M))\big).$$

Similar to the proof of Theorem 5.4, we get

$$\|w(t)\|_{L^2}^2 \lesssim \int_0^t \int_M |w(\tau,x)|^2 \left[|u_1(\tau,x)|^{\alpha-1} + |u_2(\tau,x)|^{\alpha-1}\right] \mathrm{d}x\mathrm{d}\tau.$$

almost surely for all $t \in [0,T]$. In view of Lemma 5.2, we need to check $u_j \in L^q(0,T;L^\infty(M))$ almost surely for some $q \geq 1 \vee (\alpha - 1)$ and distinguish the cases $\alpha \in (1,3]$ and $\alpha > 3$.

Let $\alpha \in (1,3]$. By $s > 1 - \frac{1}{2\alpha}$, we can choose $q > 2$ and $\varepsilon \in (0,1)$ with

$$1 - \frac{1}{2\alpha} < 1 - \frac{1}{q\alpha} + \frac{\varepsilon}{q\alpha} < s.$$

Hence, there is $\tilde{s} \in (0, 1-\alpha+s\alpha)$ with $\tilde{s} > 1 - \frac{1}{q} + \frac{\varepsilon}{q}$. If we choose $p > 2$ according to $\frac{2}{p} + \frac{2}{q} = 1$, Proposition $A.53$ leads to $H^{\tilde{s}-\frac{1+\varepsilon}{q},p}(M) \hookrightarrow L^\infty(M)$ because of

$$\left(\tilde{s} - \frac{1+\varepsilon}{q}\right) - \frac{2}{p} = \tilde{s} - \frac{\varepsilon}{q} + \frac{1}{q} - 1 > 0.$$

Moreover, we have $u_j \in L^q(0,T; H^{\tilde{s}-\frac{1+\varepsilon}{q},p}(M))$ almost surely for $j = 1,2$ by Proposition 5.9.

Next, we treat the case $\alpha > 3$. We set $q := \alpha - 1$ and choose $p > 2$ with $\frac{2}{p} + \frac{2}{q} = 1$. Using $s > 1 - \frac{1}{\alpha(\alpha-1)}$, we fix $\varepsilon \in (0,1)$ with

$$1 - \frac{1}{\alpha(\alpha-1)} < 1 - \frac{1}{q\alpha} + \frac{\varepsilon}{q\alpha} < s.$$

As above, we can choose $\tilde{s} \in (0, 1-\alpha+s\alpha)$ with $H^{\tilde{s}-\frac{1+\varepsilon}{q},p}(M) \hookrightarrow L^\infty(M)$ and $u_j \in L^q(0,T; H^{\tilde{s}-\frac{1+\varepsilon}{q},p}(M))$ almost surely for $j = 1,2$. □

Remark 5.12. In [30], Brzeźniak and Millet proved pathwise uniqueness of solutions in the space $L^q(\Omega, C([0,T], H^1(M)) \cap L^q\big([0,T], H^{1-\frac{1}{q},p}(M)\big))$ with $\frac{2}{q} + \frac{2}{p} = 1$ and $q > \alpha+1$. Since they used the deterministic Strichartz estimates from [35] instead of [16], their result is restricted to compact manifolds M. Comparing the two results, we see that the assumptions of Theorem 5.11 are weaker with respect to space and time. On the other hand, the assumptions on the required moments is slightly weaker in [30].

We close this section by applying the Yamada-Watanabe Theorem 2.4 based on the uniqueness from Theorem 5.11 and the existence from Corollary 4.29.

Corollary 5.13. *Let (M,g) be a compact two dimensional Riemannian manifold and suppose that Assumption 5.1 holds with $A := -\Delta_g$ and $s = 1$ and either i) or ii)*

i) $F(u) = |u|^{\alpha-1}u$ with $\alpha \in (1,\infty)$,

ii) $F(u) = -|u|^{\alpha-1}u$ with $\alpha \in (1,3)$.

Additionally, we assume $\sum_{m=1}^\infty \|B_m\|^2_{\mathcal{L}(L^{\alpha+1})} < \infty$. Then, the equation

$$\begin{cases} \mathrm{d}u(t) = (\mathrm{i}\Delta_g u(t) - \mathrm{i}F(u(t))\,dt - \mathrm{i}Bu(t) \circ \mathrm{d}W(t) & \text{in } H^1(M), \\ u(0) = u_0 \in H^1(M), \end{cases} \tag{5.23}$$

has a stochastically strong and analytically weak solution which satisfies $u \in C_w([0,T], H^1(M))$ almost surely and $u \in L^q(\Omega, L^\infty(0,T; H^1(M)))$ for all $q \in [1,\infty)$. Moreover, we have

$$\|u(t)\|_{L^2(M)} = \|u_0\|_{L^2(M)}$$

almost surely for all $t \in [0,T]$.

Proof. We repeat the argument from Corollary 5.7 for $U = L^\beta(0,T;H^1(M))$ with $\beta := \max\{2,\alpha\}$ in order to show that the assumptions of Theorem 2.4 are satisfied. The existence of a stochastically weak solution is provided by Corollary 4.29 and pathwise uniqueness by Theorem 5.11. Concerning the more general coefficients of the linear noise compared to the existence result, we refer to Remark 4.11. □

In the previous results, we supposed that Assumption 5.1 holds. In particular, the operators B_m, $m \in \mathbb{N}$, have to satisfy

$$\sum_{m=1}^\infty \|B_m\|^2_{\mathcal{L}(L^2)} < \infty, \qquad \sum_{m=1}^\infty \|B_m\|^2_{\mathcal{L}(H^s(M))} < \infty. \tag{5.24}$$

We close this section with a Lemma that illustrates what this actually means if B_m, $m \in \mathbb{N}$, are multiplication operators. In combination with Theorem 5.11, this yields the assertion of Theorem 4 a) stated in the introduction of this thesis.

Lemma 5.14. *Let M be a 2D Riemannian manifold satisfying* (5.16) *and* (5.17). *Let $s \in (0,1)$, $q = \frac{2}{s}$ and $e_m \in L^\infty(M) \cap H^{s,q}(M)$ with*

$$\sum_{m=1}^\infty \|e_m\|^2_{L^\infty \cap H^{s,q}} < \infty.$$

Then, the operators defined by $B_m u := e_m u$ for $u \in H^s(M)$ and $m \in \mathbb{N}$ satisfy (5.24).

Proof. Let us fix $m \in \mathbb{N}$ and choose $p = \frac{2d}{d-2s}$ as well as $q = \frac{2}{s}$. This implies $H^s(M) \hookrightarrow L^p(M)$ and $\frac{1}{2} = \frac{1}{p} + \frac{1}{q}$. From Theorem 27 in [39], we infer

$$\begin{aligned}\|e_m u\|_{H^s} &\lesssim \|u\|_{L^p}\|e_m\|_{H^{s,q}} + \|e_m\|_{L^\infty}\|u\|_{H^s}\\ &\lesssim \left(\|e_m\|_{H^{s,q}} + \|e_m\|_{L^\infty}\right)\|u\|_{H^s}.\end{aligned}$$

□

5.3. Uniqueness for 3D compact manifolds

In this section, we consider the question of pathwise uniqueness of H^1-solutions to the stochastic NLS with $\alpha \in (1,3]$ in three dimensions. By the role of the dimension in Sobolev embeddings, this problem is significantly harder than the 2D-situation. The previous section, however, already suggests that Strichartz estimates might be good enough to prove the estimates to apply Lemma 5.3. Below, we need the sharp spectrally localized Strichartz estimates by Burq, Gérard and Tzvetkov, see Proposition 2.17, and thus, we have to restrict ourselves to compact manifolds.

Throughout this section, (M,g) is a compact three-dimensional Riemannian manifold without boundary and $A := -\Delta_g$ denotes the Laplace-Beltrami operator on M. The main result is the following Theorem.

Theorem 5.15. *Let (M,g) be a compact three dimensional Riemannian manifold without boundary and $A := -\Delta_g$ be the Laplace-Beltrami operator on M. Suppose that Assumption 5.1 holds with $s = 1$ and $F(u) = \pm|u|^{\alpha-1}u$ for $\alpha \in (1,3]$.*

a) *Let* $(\Omega, \mathcal{F}, \mathbb{P}, W, \mathbb{F}, u)$ *be a martingale solution of* (5.2). *Then, there is a measurable set* $\Omega_1 \subset \Omega$ *with* $\mathbb{P}(\Omega_1) = 1$ *such that for all* $\omega \in \Omega_1$, $p \in [6, \infty)$ *and intervals* $J \subset [0, T]$, *we have* $u(\cdot, \omega) \in L^2(J; L^p(M))$ *with*

$$\|u(\cdot,\omega)\|_{L^2(J,L^p)} \lesssim_\omega 1 + (|J|p)^{\frac{1}{2}} .$$

b) *Solutions of* (5.2) *are pathwise unique in* $L^2_\omega L^\infty(0, T; H^1(M))$.

As in the previous sections, we obtain the stochastically strong existence as a consequence of pathwise uniqueness and the existence of a martingale solution proved in Corollary 4.29.

Corollary 5.16. *Let* (M, g) *be a compact three dimensional Riemannian manifold and suppose that Assumption 5.1 holds with* $A := -\Delta_g$ *and* $s = 1$ *and either i) or ii)*

i) $F(u) = |u|^{\alpha-1}u$ *with* $\alpha \in (1, 3]$,

ii) $F(u) = -|u|^{\alpha-1}u$ *with* $\alpha \in (1, \frac{7}{3})$.

Additionally, we assume $\sum_{m=1}^\infty \|B_m\|^2_{\mathcal{L}(L^{\alpha+1})} < \infty$. *Then, the equation*

$$\begin{cases} \mathrm{d}u(t) = (\mathrm{i}\Delta_g u(t) - \mathrm{i}F(u(t))\, dt - \mathrm{i}Bu(t) \circ \mathrm{d}W(t) \quad \text{in } H^1(M), \\ u(0) = u_0 \in H^1(M), \end{cases} \tag{5.25}$$

has a stochastically strong and analytically weak solution which satisfies $u \in C_w([0, T], H^1(M))$ *almost surely and* $u \in L^q(\Omega, L^\infty(0, T; H^1(M)))$ *for all* $q \in [1, \infty)$. *Moreover, we have*

$$\|u(t)\|_{L^2(M)} = \|u_0\|_{L^2(M)}$$

almost surely for all $t \in [0, T]$.

Proof. We repeat the argument from Corollary 5.7 with $H^{-1+2\rho}(M) \hookrightarrow L^{\alpha+1}(M)$ for $\rho = \frac{5}{4} - \frac{3}{2(\alpha+1)}$ in order to show that the assumptions of Theorem 2.4 are satisfied. The existence of a stochastically weak solution is provided by Corollary 4.29 and pathwise uniqueness by Theorem 5.15. Concerning the more general coefficients of the linear noise compared to the existence result, we refer to Remark 4.11. □

Before we give the proof of Theorem 5.15, we would like to illustrate the assumptions on the noise term in the special case of multiplication operators.

Example 5.17. We define multiplication operators B_m, $m \in \mathbb{N}$ by

$$B_m u = e_m u, \qquad u \in H^1(M).$$

with real valued functions e_m satisfying

$$\sum_{m=1}^\infty \left(\|\nabla e_m\|_{L^3} + \|e_m\|_{L^\infty}\right)^2 < \infty \tag{5.26}$$

and would like to justify that they fit in the assumptions of Theorem 5.15. Rephrasing Assumption 5.1, we only need to show

$$\sum_{m=1}^\infty \|B_m\|^2_{\mathcal{L}(L^2)} < \infty, \qquad \sum_{m=1}^\infty \|B_m\|^2_{\mathcal{L}(H^1} < \infty, \tag{5.27}$$

since the symmetry condition (5.4) is immediate for real-valued multipliers. For the first part of (5.27), we just have to recall $\|B_m\|_{\mathcal{L}(L^2)} = \|e_m\|_{L^\infty}$. The Sobolev embedding $H^1(M) \hookrightarrow L^6(M)$ and the Hölder inequality yield

$$\begin{aligned}\|\nabla(e_m u)\|_{L^2} \leq& \|u\nabla e_m\|_{L^2} + \|e_m \nabla u\|_{L^2} \leq \|\nabla e_m\|_{L^3}\|u\|_{L^6} + \|e_m\|_{L^\infty}\|\nabla u\|_{L^2}\\ \lesssim& \left(\|\nabla e_m\|_{L^3} + \|e_m\|_{L^\infty}\right)\|u\|_{H^1}, \qquad u \in H^1(M).\end{aligned}$$

Thus,

$$\|B_m u\|_{H^1} \eqsim \|e_m u\|_{L^2} + \|\nabla(e_m u)\|_{L^2} \lesssim \left(\|\nabla e_m\|_{L^3} + \|e_m\|_{L^\infty}\right)\|u\|_{H^1}, \qquad u \in H^1(M)$$

and summing over $m \in \mathbb{N}$, we obtain the second estimate of (5.27).

The following Lemma gives an estimate for the power type nonlinearity in the problem (5.2) which will be very useful in the proof of Theorem 5.15.

Lemma 5.18. *Let $q \in [2,6]$ and $r \in (1,\infty)$ with $\frac{1}{r'} = \frac{1}{2} + \frac{\alpha-1}{q}$. Then, we have*

$$\||u|^{\alpha-1}u\|_{H^{1,r'}} \lesssim \|u\|_{H^1}^\alpha, \qquad u \in H^1(M).$$

Proof. See [21], Lemma III.1.4. □

The proof of Theorem 5.15 will employ the following equidistant partition of the time interval.

Notation 5.19. Let $J = [a,b]$ with $0 < a < b < \infty$. For $\rho > 0$ and $N := \lfloor \frac{b-a}{\rho} \rfloor$, the family $(I_j)_{j=0}^N$ defined by

$$\begin{aligned}I_j :=& \left[a + j\rho, a + (j+1)\rho\right], \quad j \in \{0, \dots N-1\},\\ I_N :=& \left[a + N\rho, b\right]\end{aligned}$$

is called *ρ-partition of I*. Observe

$$|I_j| \leq \rho, \quad j = 0, \dots, N, \qquad J = \bigcup_{j=0}^N I_j, \qquad I_j^\circ \cap I_k^\circ = \emptyset, \quad j \neq k.$$

After these preparations, we are finally in the position to prove Theorem 5.15.

Proof of Theorem 5.15. We start with the proof of pathwise uniqueness provided a) holds. Let us take two solutions $u_1, u_2 \in L^2(\Omega, L^\infty(0,T;H^1(M)))$ and $(p_n)_{n\in\mathbb{N}} \in [6,\infty)^\mathbb{N}$ with $p_n \to \infty$ as $n \to \infty$. Using a), we choose a null set $N_1 \in \mathcal{F}$ with

$$\|u_j(\cdot,\omega)\|_{L^2(J,L^{p_n})} \lesssim_\omega 1 + (|J|p_n)^{\frac{1}{2}}, \qquad \omega \in \Omega \setminus N_1.$$

By Corollary 2.10, we choose a null set $N_2 \in F$ such that

$$\|u_1(t) - u_2(t)\|_{L^2}^2 = 2\int_0^t \operatorname{Re}\left(u_1(s) - u_2(s), -\mathrm{i}\lambda|u_1(s)|^{\alpha-1}u_1(s) + \mathrm{i}\lambda|u_2(s)|^{\alpha-1}u_2(s)\right)_{L^2} \mathrm{d}s \tag{5.28}$$

holds on $\Omega \setminus N_2$ for all $t \in [0,T]$. In particular, this leads to the weak differentiability of the map $G := \|u_1 - u_2\|_{L^2}^2$ on $\Omega \setminus N_2$ and to the estimate (5.8) on $\Omega \setminus N_2$ via

$$|F_\alpha^\pm(z_1) - F_\alpha^\pm(z_2)| \lesssim \left(|z_1|^{\alpha-1} + |z_2|^{\alpha-1}\right)|z_1 - z_2|, \qquad z_1, z_2 \in \mathbb{C}.$$

The Sobolev embedding $H^1(M) \hookrightarrow L^6(M)$ yields $u_j \in L^\infty(0,T;L^6(M))$ almost surely. Moreover, a similar argument as in Proposition 5.9 leads to the mild representation

$$\begin{aligned} \mathrm{i}u_j(t) =& \mathrm{i}e^{\mathrm{i}t\Delta_g}u_0 + \int_0^t e^{\mathrm{i}(t-\tau)\Delta_g}\lambda|u_j(\tau)|^{\alpha-1}u_j(\tau)\mathrm{d}\tau + \mathrm{i}\int_0^t e^{\mathrm{i}(t-\tau)\Delta_g}\mu(u_j(\tau))\mathrm{d}\tau \\ &+ \int_0^t e^{\mathrm{i}(t-\tau)\Delta_g}B(u_j(\tau))\mathrm{d}W(\tau) \end{aligned}$$

almost surely for all $t \in [0,T]$ in $H^{-1}(M)$ for $j = 1,2$. As a consequence of $\alpha \in (1,3]$ and $u_j \in L^\infty(0,T;L^6(M))$, each of the terms on the RHS is in $L^2(M)$. In particular, we obtain $u_j \in C([0,T],L^2(M))$ almost surely and thus, we can take another null set $N_3 \in F$ such that

$$u_j \in L^\infty(0,T;L^6(M)) \cap C([0,T],L^2(M)) \quad \text{on} \quad \Omega \setminus N_3.$$

Now, we define

$$\Omega_1 := \Omega \setminus (N_1 \cup N_2 \cup N_3)$$

and fix $\omega \in \Omega_1$. By Lemma 5.3, we get $u_1(\cdot,\omega) = u_2(\cdot,\omega)$. Due to $\Omega_1 \in \mathcal{F}$ with $\mathbb{P}(\Omega_1) = 1$, the assertion is proved.

We are left to show assertion a).

Step 1. We choose $\beta > 0$ and $h \in (0,1]$ as in Proposition 2.17 and take a $\frac{\beta h}{4}$-partition $(I_j)_{j=0}^{N_T}$ of $[0,T]$ in the sense of Notation 5.19. Furthermore, we define a cover $\left(I_j'\right)_{j=0}^{N_T}$ of $(I_j)_{j=0}^{N_T}$ by

$$I_j' := \left(I_j + \left[-\frac{\beta h}{8}, \frac{\beta h}{8}\right]\right) \cap [0,T], \qquad m_j := \frac{j\beta h}{4} + \frac{\beta h}{8}, \qquad j = 0,\dots,N_T,$$

and a sequence $\left(\chi_{I_j}\right)_{j=0}^{N_T} \subset C_c^\infty([0,\infty))$ by $\chi_{I_j} := \chi\left((\beta h)^{-1}(\cdot - m_j)\right)$ for some $\chi \in C_c^\infty(\mathbb{R})$ with $\chi = 1$ on $[-\frac{1}{8}, \frac{1}{8}]$ and $\operatorname{supp}(\chi) \subset [-\frac{1}{4}, \frac{1}{4}]$. Then, we have

$$\chi_{I_j} = 1 \quad \text{on } I_j, \qquad \operatorname{supp}(\chi_{I_j}) \subset I_j', \quad \|\chi_{I_j}'\|_{L^\infty(\mathbb{R})} \le (\beta h)^{-1}\|\chi'\|_{L^\infty(\mathbb{R})}. \tag{5.29}$$

Let $\varphi, \tilde{\varphi} \in C_c^\infty(\mathbb{R} \setminus \{0\})$ with $\tilde{\varphi} = 1$ on $\operatorname{supp}(\varphi)$. In order to localize the solution u spectrally and in time, we set

$$v_{I_j}(t) = \chi_{I_j}(t)\varphi(h^2\Delta_g)u(t), \qquad j = 0,\dots,N_T.$$

We recall that u has the representation

$$u(t) = u_0 + \int_0^t \left\{\mathrm{i}\Delta_g u(s) - \mathrm{i}\lambda|u(s)|^{\alpha-1}u(s) + \mu(u(s))\right\}\mathrm{d}s - \mathrm{i}\int_0^t B(u(s))\mathrm{d}W(s)$$

in $H^{-1}(M)$ almost surely for all $t \in [0,T]$ and from the Itô formula as well as $\chi_{I_j} = 0$ on $[0, \min I_j']$, we infer

$$v_{I_j}(t) = \int_{\min I_j'}^t \left\{\mathrm{i}\Delta_g v_{I_j}(s) + \chi_{I_j}'(s)\varphi(h^2\Delta_g)u(s) + \chi_{I_j}(s)\varphi(h^2\Delta_g)\left[-\mathrm{i}\lambda|u(s)|^{\alpha-1}u(s) + \mu(u(s))\right]\right\}\mathrm{d}s$$

$$- \mathrm{i} \int_{\min I_j'}^t \chi_{I_j}(s)\varphi(h^2\Delta_g)Bu(s)\mathrm{d}W(s) \tag{5.30}$$

in $H^{-1}(M)$ almost surely for all $t \in I_j'$. Next, we employ Lemma 2.5 with $X = H^{-1}(M)$ and $Af = -\Delta_g f$ for $f \in H^1(M) =: D(A)$ to rewrite (5.30) in the mild form

$$\begin{aligned} v_{I_j}(t) = &\int_{\min I_j'}^t e^{\mathrm{i}(t-s)\Delta_g}\chi_{I_j}'(s)\varphi(h^2\Delta_g)u(s)\mathrm{d}s \\ &+ \int_{\min I_j'}^t e^{\mathrm{i}(t-s)\Delta_g}\chi_{I_j}(s)\varphi(h^2\Delta_g)\left[-\mathrm{i}\lambda|u(s)|^{\alpha-1}u(s) + \mu(u(s))\right]\mathrm{d}s \\ &- \mathrm{i}\int_{\min I_j'}^t e^{\mathrm{i}(t-s)\Delta_g}\chi_{I_j}(s)\varphi(h^2\Delta_g)Bu(s)\mathrm{d}W(s) \end{aligned} \tag{5.31}$$

for $j = 1, \dots, N_T$ in $H^{-1}(M)$ almost surely for $t \in I_j'$. Because of $\alpha \le 3$, each term is so regular that this identity also holds in $L^2(M)$. Analogously, we get

$$\begin{aligned} v_{I_0}(t) = &e^{\mathrm{i}t\Delta_g}v_{I_0}(\min I_0') + \int_{\min I_0'}^t e^{\mathrm{i}(t-s)\Delta_g}\chi_{I_0}'(s)\varphi(h^2\Delta_g)u(s)\mathrm{d}s \\ &+ \int_{\min I_0'}^t e^{\mathrm{i}(t-s)\Delta_g}\chi_{I_0}(s)\varphi(h^2\Delta_g)\left[-\mathrm{i}\lambda|u(s)|^{\alpha-1}u(s) + \mu(u(s))\right]\mathrm{d}s \\ &- \mathrm{i}\int_{\min I_0'}^t e^{\mathrm{i}(t-s)\Delta_g}\chi_{I_0}(s)\varphi(h^2\Delta_g)Bu(s)\mathrm{d}W(s) \end{aligned} \tag{5.32}$$

in $L^2(M)$ almost surely for $t \in I_0'$. We abbreviate

$$G_{I_j}(t) := \int_{\min I_j'}^t e^{\mathrm{i}(t-s)\Delta_g}\chi_{I_j}(s)\varphi(h^2\Delta_g)Bu(s)\mathrm{d}W(s)$$

for $\min I_0' \le t \in [0,T]$. We use the stochastic Strichartz estimate from Corollary 2.23 c), the properties of $(I_j)_{j=0}^{N_T}$ and $\left(I_j'\right)_{j=0}^{N_T}$ and Lemma $A.55$ b) to estimate

$$\begin{aligned} \mathbb{E}\sum_{j=0}^{N_T}\|G_{I_j}\|_{L^2(I_j',L^6)}^2 \lesssim &\mathbb{E}\sum_{j=0}^{N_T}\int_{I_j'}\|\varphi(h^2\Delta_g)B(u(s))\|_{\mathrm{HS}(Y,L^2)}^2\mathrm{d}s \\ \le &2\mathbb{E}\sum_{j=0}^{N_T}\int_{I_j}\|\varphi(h^2\Delta_g)B(u(s))\|_{\mathrm{HS}(Y,L^2)}^2\mathrm{d}s \\ = &2\mathbb{E}\int_0^T\|\varphi(h^2\Delta_g)B(u(s))\|_{\mathrm{HS}(Y,L^2)}^2\mathrm{d}s \\ \lesssim &h^2\mathbb{E}\int_0^T\|\varphi(h^2\Delta_g)B(u(s))\|_{\mathrm{HS}(Y,H^1)}^2\mathrm{d}s. \end{aligned}$$

Since $\varphi(h^2\Delta_g)$ is a bounded operator from $H^1(M)$ to $H^1(M)$ and B is bounded from $H^1(M)$ to $\mathrm{HS}(Y, H^1(M))$ by Assumption 4.6, we conclude

$$\mathbb{E}\sum_{j=0}^{N_T}\|G_{I_j}\|_{L^2(I_j',L^6)}^2 \lesssim h^2\mathbb{E}\int_0^T\|u(s)\|_{H^1}^2\mathrm{d}s.$$

Hence, there is $C = C(\omega)$ with $C < \infty$ almost surely such that

$$\sum_{j=0}^{N_T} \|G_{I_j}\|^2_{L^2(I'_j, L^6)} \le h^2 C \qquad \text{a.s.} \tag{5.33}$$

Step 2. We fix a path $\omega \in \Omega_1$, where Ω_1 is the intersection of the full measure sets from (5.31), (5.32) and (5.33). In the rest of the argument, we skip the dependence of ω to keep the notation simple. Let us pick those intervals $J_0, \dots, J_N$ from the partition $(I_j)_{j=0}^{N_T}$ which cover the given interval J. The associated intervals in $(I'_j)_{j=0}^{N}$ will be denoted by $J'_0, \dots, J'_N$. From (5.33), we infer

$$\sum_{j=0}^{N} \|G_{J_j}\|^2_{L^2(J'_j, L^6)} \le h^2 C. \tag{5.34}$$

Applying the homogeneous and inhomogeneous Strichartz estimates from Proposition 2.17 in (5.31) and in (5.32), we obtain

$$\begin{aligned} \|v_{J_j}\|_{L^2(J_j, L^6)} \le \|v_{J_j}\|_{L^2(J'_j, L^6)} &\lesssim \|\chi'_{J_j}\varphi(h^2\Delta_g)u\|_{L^1(J'_j, L^2)} + \|\chi_{J_j}\varphi(h^2\Delta_g)|u|^{\alpha-1}u\|_{L^2(J'_j, L^{\frac{6}{5}})} \\ &\quad + \|\chi_{J_j}\varphi(h^2\Delta_g)\mu(u)\|_{L^1(J'_j, L^2)} + \|G_{J_j}\|_{L^2(J'_j, L^6)} \end{aligned} \tag{5.35}$$

for $j = 1, \dots, N$ and

$$\begin{aligned} \|v_{J_0}\|_{L^2(J_0, L^6)} \le \|v_{J_0}\|_{L^2(J'_0, L^6)} &\lesssim \|v_{J_0}(\min J'_0)\|_{L^2} + \|\chi'_{J_0}\varphi(h^2\Delta_g)u\|_{L^1(J'_0, L^2)} \\ &\quad + \|\chi_{J_0}\varphi(h^2\Delta_g)|u|^{\alpha-1}u\|_{L^2(J'_0, L^{\frac{6}{5}})} + \|\chi_{J_0}\varphi(h^2\Delta_g)\mu(u)\|_{L^1(J'_0, L^2)} \\ &\quad + \|G_{J_0}\|_{L^2(J'_0, L^6)}. \end{aligned} \tag{5.36}$$

Note that $v_{J_0}(\min J'_0) = 0$ if $I_0 \neq J_0$. Next, we estimate the terms on the right hand side of (5.35) and (5.36). By (5.29), Lemma $A.55$ b) and Hölder's inequality, we get

$$\begin{aligned} \|\chi'_{J_j}\varphi(h^2\Delta_g)u\|_{L^1(J'_j, L^2)} &\lesssim h^{-1}\|\varphi(h^2\Delta_g)u\|_{L^1(J'_j, L^2)} \lesssim \|\varphi(h^2\Delta_g)u\|_{L^1(J'_j, H^1)} \\ &\lesssim h^{\frac{1}{2}}\|\varphi(h^2\Delta_g)u\|_{L^2(J'_j, H^1)}. \end{aligned}$$

Hölder's inequality with $|J'_j| \lesssim h$, Lemma $A.55$ b) and the boundedness of the operators $\varphi(h^2\Delta_g)$ and μ in $H^1(M)$ yield

$$\begin{aligned} \|\chi_{J_j}\varphi(h^2\Delta_g)\mu(u)\|_{L^1(J'_j, L^2)} &\lesssim h\|\chi_{J_j}\varphi(h^2\Delta_g)\mu(u)\|_{L^\infty(J'_j, L^2)} \le h\|\varphi(h^2\Delta_g)\mu(u)\|_{L^\infty(0,T;L^2)} \\ &\lesssim h^2\|\varphi(h^2\Delta_g)\mu(u)\|_{L^\infty(0,T;H^1)} \lesssim h^2\|u\|_{L^\infty(0,T;H^1)}. \end{aligned}$$

We apply Lemma 5.18 with $r' = \frac{6}{\alpha+2} \ge \frac{6}{5}$ and $q = 6$ and obtain the estimate

$$\||v|^{\alpha-1}v\|_{H^{1,\frac{6}{5}}} \lesssim \||v|^{\alpha-1}v\|_{H^{1,\frac{6}{\alpha+2}}} \lesssim \|v\|^{\alpha}_{H^1}, \qquad v \in H^1(M),$$

where we used $\alpha \le 3$. Together with Hölder's inequality, Lemma $A.55$ b) and the boundedness of $\varphi(h^2\Delta_g)$, this implies

$$\begin{aligned} \|\chi_{J_j}\varphi(h^2\Delta_g)|u|^{\alpha-1}u\|_{L^2(J'_j, L^{\frac{6}{5}})} &\lesssim h^{\frac{1}{2}}\|\varphi(h^2\Delta_g)|u|^{\alpha-1}u\|_{L^\infty(0,T;L^{\frac{6}{5}})} \\ &\lesssim h^{\frac{3}{2}}\|\varphi(h^2\Delta_g)|u|^{\alpha-1}u\|_{L^\infty(0,T;H^{1,\frac{6}{5}})} \end{aligned}$$

$$\lesssim h^{\frac{3}{2}} \| |u|^{\alpha-1} u \|_{L^\infty(0,T;H^{1,\frac{6}{5}})} \lesssim h^{\frac{3}{2}} \|u\|^{\alpha}_{L^\infty(0,T;H^1)}.$$

Inserting the last three estimates in (5.35) and (5.36) yields

$$\begin{aligned} \|v_{J_j}\|_{L^2(J_j,L^6)} \lesssim & h^{\frac{1}{2}} \|\varphi(h^2\Delta_g)u\|_{L^2(J_j',H^1)} + h^{\frac{3}{2}} \|u\|^{\alpha}_{L^\infty(0,T;H^1)} \\ & + h^2 \|u\|_{L^\infty(0,T;H^1)} + \|G_{J_j}\|_{L^2(J_j',L^6)}, \end{aligned} \tag{5.37}$$

$$\begin{aligned} \|v_{J_0}\|_{L^2(J_0,L^6)} \lesssim & h \|\varphi(h^2\Delta_g)u(\min J_0')\|_{H^1} + h^{\frac{1}{2}} \|\varphi(h^2\Delta_g)u\|_{L^2(J_0',H^1)} + h^{\frac{3}{2}} \|u\|^{\alpha}_{L^\infty(0,T;H^1)} \\ & + h^2 \|u\|_{L^\infty(0,T;H^1)} + \|G_{J_0}\|_{L^2(J_0',L^6)}. \end{aligned} \tag{5.38}$$

We square the estimates (5.37) and (5.38) and sum them up. Using $\chi_{J_j} = 1$ on J_j, (5.34) and $N \leq N_T = \left\lfloor \frac{4T}{\beta h} \right\rfloor$, we conclude

$$\begin{aligned} \|\varphi(h^2\Delta_g)u\|^q_{L^2(J,L^6)} \leq & \sum_{j=0}^{N} \|\chi_{J_j}\varphi(h^2\Delta_g)u\|^2_{L^2(J_j,L^6)} = \sum_{j=0}^{N} \|v_{J_j}\|^2_{L^2(J_j,L^6)} \\ \lesssim & h^2 \|\varphi(h^2\Delta_g)u(\min J_0')\|^2_{H^1} \\ & + \sum_{j=0}^{N} \left[h \|\varphi(h^2\Delta_g)u\|^2_{L^2(J_j',H^1)} + h^3 \|u\|^{2\alpha}_{L^\infty(0,T;H^1)} \right] \\ & + \sum_{j=0}^{N} \left[h^4 \|u\|^2_{L^\infty(0,T;H^1)} \right] + h^2 C \\ \lesssim & h^2 \|\varphi(h^2\Delta_g)u(\min J_0')\|^2_{H^1} + h \sum_{j=0}^{N} \|\varphi(h^2\Delta_g)u\|^2_{L^2(J_j',H^1)} \\ & + h^2 \|u\|^{2\alpha}_{L^\infty(0,T;H^1)} + h^3 \|u\|^2_{L^\infty(0,T;H^1)} + h^2 C. \end{aligned} \tag{5.39}$$

Below, we will use the notations

$$J_{N+1} := \left(\bigcup_{j=0}^{N} J_j' \right) \setminus \left(\bigcup_{j=0}^{N} J_j \right), \qquad J^h := \bigcup_{j=0}^{N+1} J_j.$$

By

$$\sum_{j=0}^{N} \|\varphi(h^2\Delta_g)u\|^2_{L^2(J_j',H^1)} \leq 2 \sum_{j=0}^{N+1} \|\varphi(h^2\Delta_g)u\|^2_{L^2(J_j,H^1)} = 2 \|\varphi(h^2\Delta_g)u\|^2_{L^2(J^h,H^1)}$$

we obtain

$$\begin{aligned} \|\varphi(h^2\Delta_g)u\|^2_{L^2(J,L^6)} \lesssim & h^2 \|\varphi(h^2\Delta_g)u(\min J_0')\|^2_{H^1} + h \|\varphi(h^2\Delta_g)u\|^2_{L^2(J^h,H^1)} \\ & + h^2 \|u\|^{2\alpha}_{L^\infty(0,T;H^1)} + h^3 \|u\|^2_{L^\infty(0,T;H^1)} + h^2 C. \end{aligned}$$

Let $p \geq 6$. Then, Lemma $A.55$ a) and $u \in L^\infty(0,T;H^1(M))$ imply

$$\begin{aligned} \|\varphi(h^2\Delta_g)u\|_{L^2(J,L^p)} \lesssim & h^{3\left(\frac{1}{p}-\frac{1}{6}\right)} \|\varphi(h^2\Delta_g)u\|_{L^2(J,L^6)} \\ \lesssim & h^{\frac{3}{p}+\frac{1}{2}} \|\varphi(h^2\Delta_g)u(\min J_0')\|_{H^1} + h^{\frac{3}{p}} \|\varphi(h^2\Delta_g)u\|_{L^2(J^h,H^1)} \end{aligned}$$

$$
\begin{aligned}
&+ h^{\frac{3}{p}+\frac{1}{2}}\|u\|^{\alpha}_{L^\infty(0,T;H^1)} + h^{\frac{3}{p}+1}\|u\|_{L^\infty(0,T;H^1)} + h^{\frac{3}{p}+\frac{1}{2}}C \\
&\lesssim h^{\frac{3}{p}+\frac{1}{2}} + h^{\frac{3}{p}}\|\varphi(h^2\Delta_g)u\|_{L^2(J^h,H^1)} + h^{\frac{3}{p}+\frac{1}{2}} + h^{\frac{3}{p}+1}. \qquad (5.40)
\end{aligned}
$$

Step 3. In the last step, we use (5.40) and Littlewood-Paley theory to derive the estimate stated in the Proposition. To this end, we set $h_k := 2^{-\frac{k}{2}}$ and $k_0 := \min\left\{k : |J| > \frac{\beta h_k}{4}\right\}$. Moreover, we choose $\psi \in C_c^\infty(\mathbb{R})$, $\varphi \in C_c^\infty(\mathbb{R}\setminus\{0\})$ such that

$$
1 = \psi(\lambda)u + \sum_{k=1}^{\infty}\varphi(2^{-k}\lambda), \qquad \lambda \in \mathbb{R}.
$$

Then, Lemma $A.54$, the embedding $\ell^1(\mathbb{N}) \hookrightarrow \ell^2(\mathbb{N})$ and (5.40) imply

$$
\begin{aligned}
\|u\|_{L^2(J,L^p)} &\lesssim \left\| \left(\|\psi(\Delta_g)u\|^2_{L^p} + \sum_{k=1}^{\infty}\|\varphi(2^{-k}\Delta_g)u\|^2_{L^p}\right)^{\frac{1}{2}}\right\|_{L^2(J)} \\
&= \left(\|\psi(\Delta_g)u\|^2_{L^2(J,L^p)} + \sum_{k=1}^{\infty}\|\varphi(2^{-k}\Delta_g)u\|^2_{L^2(J,L^p)}\right)^{\frac{1}{2}} \\
&\leq \|\psi(\Delta_g)u\|_{L^2(J,L^p)} + \sum_{k=1}^{\infty}\|\varphi(2^{-k}\Delta_g)u\|_{L^2(J,L^p)} \\
&\lesssim \|\psi(\Delta_g)u\|_{L^2(J,L^p)} + \sum_{k=1}^{k_0-1}\|\varphi(2^{-k}\Delta_g)u\|_{L^2(J,L^p)} \\
&\quad + \sum_{k=k_0}^{\infty} 2^{-\frac{3k}{2p}}\|\varphi(2^{-k}\Delta_g)u\|_{L^2(J^{h_k},H^1)} + \sum_{k=k_0}^{\infty}\left[2^{-\frac{k}{2}\left(\frac{3}{p}+\frac{1}{2}\right)} + 2^{-\frac{k}{2}\left(\frac{3}{p}+1\right)} + 2^{-\frac{k}{2}\left(\frac{3}{p}+\frac{1}{2}\right)}\right] \\
&\leq \|\psi(\Delta_g)u\|_{L^2(J,L^p)} + \sum_{k=1}^{k_0-1}\|\varphi(2^{-k}\Delta_g)u\|_{L^2(J,L^p)} \\
&\quad + \sum_{k=k_0}^{\infty} 2^{-\frac{3k}{2p}}\|\varphi(2^{-k}\Delta_g)u\|_{L^2(J^{h_k},H^1)} + \sum_{k=k_0}^{\infty}\left[2^{-\frac{k}{4}} + 2^{-\frac{k}{2}} + 2^{-\frac{k}{4}}\right] \\
&\lesssim \|\psi(\Delta_g)u\|_{L^2(J,L^p)} + \sum_{k=1}^{k_0-1}\|\varphi(2^{-k}\Delta_g)u\|_{L^2(J,L^p)} \\
&\quad + \left(\sum_{k=k_0}^{\infty} 2^{-\frac{3k}{p}}\right)^{\frac{1}{2}}\left(\sum_{k=k_0}^{\infty}\|\varphi(2^{-k}\Delta_g)u\|^2_{L^2(J^{h_k},H^1)}\right)^{\frac{1}{2}} + 1. \qquad (5.41)
\end{aligned}
$$

From Lemma $A.55$ a) with $h = 1$, we conclude

$$
\|\psi(\Delta_g)u\|_{L^2(J,L^p)} \lesssim \|\psi(\Delta_g)u\|_{L^2(J,L^2)} \lesssim \|u\|_{L^2(J,L^2)} \lesssim 1. \qquad (5.42)
$$

From Lemma $A.55$ a) and the Sobolev embedding, we infer

$$
\begin{aligned}
\|\varphi(2^{-k}\Delta_g)u\|_{L^2(J,L^p)} &\lesssim 2^{-k(\frac{3}{2p}-\frac{1}{4})}\|\varphi(2^{-k}\Delta_g)u\|_{L^2(J,L^6)} \\
&\lesssim 2^{\frac{k}{4}}\|\varphi(2^{-k}\Delta_g)u\|_{L^2(J,H^1)}
\end{aligned}
$$

for $k \in \{1, \ldots, k_0 - 1\}$. From the definition of k_0, we have $|J| \eqsim 2^{-\frac{k_0}{2}}$. Thus, we get

$$\sum_{k=1}^{k_0-1} \|\varphi(2^{-k}\Delta_g)u\|_{L^2(J,L^p)} \lesssim \left(\sum_{k=1}^{k_0-1} 2^{\frac{k}{2}}\right)^{\frac{1}{2}} \left(\sum_{k=1}^{k_0-1} \|\varphi(2^{-k}\Delta_g)u\|^2_{L^2(J,H^1)}\right)^{\frac{1}{2}}$$
$$\lesssim 2^{\frac{k_0}{4}} \|u\|_{L^2(J,H^1)} \lesssim |J|^{-\frac{1}{2}}|J|^{\frac{1}{2}} \lesssim 1. \tag{5.43}$$

We proceed with the estimate of the sums over $k \geq k_0$. The fact that we have $J^{h_{k+1}} \subset J^{h_k}$ for all $k \in \mathbb{N}$, leads to

$$\begin{aligned}\sum_{k=k_0}^{\infty} \|\varphi(2^{-k}\Delta_g)u\|^2_{L^2(J^{h_k},H^1)} &= \sum_{k:|J|>\frac{\beta h_k}{4}} \|\varphi(2^{-k}\Delta_g)u\|^2_{L^2(J^{h_k},H^1)} \\ &\leq \sum_{k:|J|>\frac{\beta h_k}{4}} \|\varphi(2^{-k}\Delta_g)u\|^2_{L^2(J^{h_{k_0}},H^1)} \\ &\lesssim \|u\|^2_{L^2(J^{h_{k_0}},H^1)} \leq |J^{h_{k_0}}| \, \|u\|^2_{L^\infty(0,T;H^1)}.\end{aligned}$$

Using $|J^{h_{k_0}}| \leq 3\frac{\beta h_{k_0}}{4} + |J| \leq 4|J|$ and $u \in L^\infty(0,T;H^1(M))$ almost surely, we obtain

$$\sum_{k=k_0}^{\infty} \|\varphi(2^{-k}\Delta_g)u\|^2_{L^2(J^{h_k},H^1)} \lesssim |J|. \tag{5.44}$$

Finally, the calculation

$$\lim_{p\to\infty} \frac{1}{p}\sum_{k=1}^{\infty} 2^{-\frac{3k}{p}} = \lim_{p\to\infty} \frac{1}{p}\left(\frac{1}{1-2^{-\frac{3}{p}}} - 1\right) = \lim_{p\to\infty} \frac{1}{p\left(2^{\frac{3}{p}}-1\right)} = \frac{1}{3\log(2)}$$

yields the boundedness of the function defined by $[6,\infty) \ni p \mapsto \frac{1}{p}\sum_{k=1}^{\infty} 2^{-\frac{3k}{p}}$ and hence,

$$\sum_{k=1}^{\infty} 2^{-\frac{3k}{p}} \lesssim p. \tag{5.45}$$

Using the estimates (5.42) (5.43), (5.44), and (5.45) in (5.41), we get

$$\|u\|_{L^2(J,L^p)} \lesssim 1 + (|J|p)^{\frac{1}{2}}, \qquad p \in [6,\infty),$$

which implies the assertion. □

We close this chapter with some remarks on the failure of seemingly natural extensions of the previous result to higher dimensions, nonlinear noise and non-compact manifolds.

Remark 5.20. We would like to comment on the case of higher dimensions $d > 3$. The Strichartz-endpoint is $(2, \frac{2d}{d-2})$ and the use of Lemma 5.18 leads to the restriction $\alpha \leq 1 + \frac{2}{d-2}$. The final estimate (5.41) has to be replaced by

$$\|u\|_{L^q(J,L^p)} \lesssim \|\psi(\Delta_g)u\|_{L^q(J,L^p)} + \sum_{k=1}^{\infty} 2^{-\frac{k}{2}\left(\frac{d}{p}-\nu(d)\right)} \|\varphi(2^{-k}\Delta_g)u\|_{L^2(J,H^1)}$$
$$+ \sum_{k=1}^{\infty}\left[2^{-\frac{k}{2}\left(\frac{d}{p}-\nu(d)+\frac{1}{q}\right)} + 2^{-\frac{k}{2}\left(\frac{d}{p}-\nu(d)+1\right)} + 2^{-\frac{k}{2}\left(\frac{d}{p}-\nu(d)+\frac{1}{2}\right)}\right]$$

for $p \geq \frac{2d}{d-2}$, where we set $\nu(d) := \frac{d-3}{2}$. Hence, the convergence of the sums requires an upper bound on p, which destroys the uniqueness proof from above. In fact, this problem occurs since the scaling condition for Strichartz exponents, Sobolev embeddings and Bernstein inequalities are more restrictive in higher dimensions. In particular, the restriction to $d = 3$ is of purely deterministic nature.

Remark 5.21. In the proof of Theorem 5.15, we did not need the optimal estimates for the correction term μ and the stochastic integral. In fact, it is possible to generalize the proof and show the estimate

$$\|u\|_{L^q(J,L^p)} \lesssim 1 + (|J|p)^{\frac{1}{2}} \qquad \text{a.s.,} \quad p \in [6,\infty), \quad q \in [1,2]$$

also for martingale solutions of the equation

$$\begin{cases} \mathrm{d}u(t) = \left(\mathrm{i}\Delta_g u(t) - \mathrm{i}\lambda|u(t)|^{\alpha-1}u(t) + \mu\left(|u(t)|^{2(\gamma-1)}u(t)\right)\right)\mathrm{d}t - \mathrm{i}B\left(|u(t)|^{\gamma-1}u(t)\right)\mathrm{d}W(t), \\ u(0) = u_0. \end{cases} \tag{5.46}$$

with nonlinear noise of power $\gamma \in [1,2)$. However, we do not know if this equation has a solution, since the existence theory developed in Chapter 4 only applies for $\gamma = 1$. Moreover, we do not know, how we can apply these estimates to prove pathwise uniqueness since there is no cancellation of the stochastic integrals in the Itô-formula for the L^2-norm of the difference of two solutions to (5.46). Thus, there is no analogue of Corollary 2.10 and a pathwise application of Lemma 5.3 is no longer possible.

Remark 5.22. Let us comment on the case of possibly non-compact manifolds with bounded geometry. In the two dimensional setting, the Strichartz estimates from Lemma 2.15 with an additional loss of ε regularity were sufficient for our proof of uniqueness. In fact, these estimates are derived from localized Strichartz estimates of the form

$$\|t \mapsto e^{\mathrm{i}t\Delta_g}\psi_{m,\frac{1}{2}}(-h^2\Delta_g)x\|_{L^q(J,L^p)} \leq C_\varepsilon\|x\|_{L^2}, \qquad |J| \leq \beta_\varepsilon h^{1+\varepsilon}, \tag{5.47}$$

for all $\varepsilon > 0$ and some $C_\varepsilon > 0$ and $\beta_\varepsilon > 0$, where we denote $\psi_{m,a}(\lambda) := \lambda^m e^{-a\lambda}$ for $m \in \mathbb{N}$ and $a > 0$. A continuous version of the Littlewood-Paley inequality which can substitute (A.36) is given by

$$\|f\|_{L^p} \eqsim \|\varphi_{m,a}(-\Delta_g)f\|_{L^p} + \left\|\left(\int_0^1 |\psi_{m,a}(-h^2\Delta_g)f|^2 \frac{\mathrm{d}h}{h}\right)^{\frac{1}{2}}\right\|_{L^p}, \qquad f \in L^p(M), \tag{5.48}$$

for $\varphi_{m,a}(\lambda) := \int_\lambda^\infty \psi_{m,a}(t)\frac{\mathrm{d}t}{t}$, see [16], Theorem 2.8. Based on (5.47) and (5.48), we can argue similarly as in the proof of Theorem 5.15 and end up with the estimate

$$\|u\|_{L^q(J,L^p)} \lesssim 1 + |J|^{\frac{1}{2}}\left(\frac{qp}{6q-2\varepsilon p}\right)^{\frac{1}{2}} \qquad \text{a.s.}$$

for each $\varepsilon > 0$, $q \in [1,2]$ and $p \in [6, 3q\varepsilon^{-1})$ with an implicit constant which goes to infinity for $\varepsilon \to 0$. The upper bound on p is due to the fact that the additional ε in (5.47) weakens the estimates of the critical term containing the derivative χ_j' of the temporal cut-off and enlarges the number of summands in (5.39). As in the case of higher dimensions than $d = 3$, the uniqueness argument breaks down since a limit process $p \to \infty$ is no longer possible.

6. The NLS driven by a jump process

In the last chapter of this thesis, we would like to transfer the existence result we derived in the fourth chapter for Gaussian noise to stochastic perturbations induced by a jump process. We consider the Marcus stochastic NLS

$$\begin{cases} \mathrm{d}u(t) = \big(-\mathrm{i}Au(t) - \mathrm{i}F(u(t))\big)\mathrm{d}t - \mathrm{i}\sum_{m=1}^{N} B_m u(t-) \diamond \mathrm{d}L_m(t), \qquad t \in [0,T], \\ u(0) = u_0 \in E_A, \end{cases} \tag{6.1}$$

in the energy space $E_A := X_{\frac{1}{2}}$, where A is a selfadjoint, non-negative operator A with a compact resolvent in an L^2-space H, $F : E_A \to E_A^*$ is a nonlinear map and $B_m \in \mathcal{L}(E_A)$ are linear operators for $m = 1, \dots, M$. Moreover, $L(t) := (L_1(t), \cdots, L_M(t))$ a $\mathbb{R}^M-$ valued Lévy process of pure jump type

$$L(t) = \int_0^t \int_{\{|l| \le 1\}} l \, \tilde{\eta}(\mathrm{d}s, \mathrm{d}l),$$

where η denotes a time homogeneous Poisson random measure on $\mathbb{R}^M$ with intensity measure ν and $\tilde{\eta} := \eta -$ Leb $\otimes \nu$ is the corresponding compensated time homogeneous Poisson random measure.

The goal is to construct a martingale solution of (6.1) and similarly to Chapter 4, the proof employs a Galerkin-type approximation and a priori estimates derived by the Itô formula and the Gronwall Lemma. However, we have to use more sophisticated methods to obtain tightness since we are faced with spaces of càdlàg functions instead of continuous ones.

6.1. General Framework and Assumptions

Let $(\tilde{M}, \Sigma, \mu)$ be a σ-finite metric measure space with metric ρ satisfying the *doubling property*, i.e. $\mu(B(x,r)) < \infty$ for all $x \in \tilde{M}$ and $r > 0$ and

$$\mu(B(x,2r)) \lesssim \mu(B(x,r)). \tag{6.2}$$

Let $M \subset \tilde{M}$ be an open subset with finite measure. We further abbreviate $H := L^2(M, \mathbb{C})$ and denote the standard complex L^2-inner product by $\left(\cdot,\cdot\right)_H$. Let A be a non-negative selfadjoint operator on H with the scale $(X_\theta)_{\theta \in \mathbb{R}}$ of fractional domains. The space $E_A := X_{\frac{1}{2}}$ is called *energy space* and its dual is denoted by $E_A^* := X_{-\frac{1}{2}}$.

As in the fourth chapter, it is appropriate to treat H, E_A and $X_{-\frac{1}{2}}$ as real Hilbert spaces with the real scalar products $\mathrm{Re}\left(\cdot,\cdot\right)_H$, $\mathrm{Re}\left(\cdot,\cdot\right)_{E_A}$ and $\mathrm{Re}\left(\cdot,\cdot\right)_{-\frac{1}{2}}$, respectively. Our assumptions on the functional analytic setting are identical to Chapter 4, but we recall them for the reader's convenience.

Assumption 6.1. We assume the following:

i) There is a strictly positive selfadjoint operator S on H with compact resolvent commuting with A and $\mathcal{D}(S^k) \hookrightarrow E_A$ for some $k \in \mathbb{N}$. Moreover, we assume that S has *generalized Gaussian* (p_0, p_0')*-bounds* for some $p_0 \in [1,2)$, i.e.

$$\|\mathbf{1}_{B(x,t^{\frac{1}{m}})} e^{-tS} \mathbf{1}_{B(y,t^{\frac{1}{m}})}\|_{\mathcal{L}(L^{p_0},L^{p_0'})} \le C\mu(B(x,t^{\frac{1}{m}}))^{\frac{1}{p_0'}-\frac{1}{p_0}} \exp\left\{-c\left(\frac{\rho(x,y)^m}{t}\right)^{\frac{1}{m-1}}\right\}, \tag{6.3}$$

for all $t > 0$ and $(x,y) \in M \times M$ with constants $c, C > 0$ and $m \ge 2$.

ii) Let $\alpha \in (1, p_0' - 1)$ be such that E_A is compactly embedded in $L^{\alpha+1}(M)$. We set

$$p_{\max} := \sup\left\{p \in (1,\infty] : E_A \hookrightarrow L^p(M) \quad \text{is continuous}\right\}.$$

In the case $p_{\max} < \infty$, we assume that $E_A \hookrightarrow L^{p_{\max}}(M)$ is continuous, but not necessarily compact.

Assumption 6.2. Let $\alpha \in (1, p_0' - 1)$ be chosen as in Assumption 6.1. Then, we assume the following:

i) Suppose that $F : L^{\alpha+1}(M) \to L^{\frac{\alpha+1}{\alpha}}(M)$ satisfies the following estimate

$$\|F(u)\|_{L^{\frac{\alpha+1}{\alpha}}} \le C_{F,1}\|u\|_{L^{\alpha+1}}^{\alpha}, \quad u \in L^{\alpha+1}(M). \tag{6.4}$$

We further assume and $F(0) = 0$ and

$$\operatorname{Re}\langle \mathrm{i}u, F(u)\rangle = 0, \quad u \in L^{\alpha+1}(M). \tag{6.5}$$

ii) The map $F : L^{\alpha+1}(M) \to L^{\frac{\alpha+1}{\alpha}}(M)$ is continuously real Fréchet differentiable with

$$\|F'[u]\|_{\mathcal{L}(L^{\alpha+1},L^{\frac{\alpha+1}{\alpha}})} \le C_{F,2}\|u\|_{L^{\alpha+1}}^{\alpha-1}, \quad u \in L^{\alpha+1}(M).$$

iii) The map F has a real antiderivative $\hat{F}$.

Assumption 6.3. We assume either i) or i'):

i) Let F be defocusing and satisfy

$$\frac{1}{C_{F,3}}\|u\|_{L^{\alpha+1}}^{\alpha+1} \le \hat{F}(u) \le C_{F,3}\|u\|_{L^{\alpha+1}}^{\alpha+1}, \quad u \in L^{\alpha+1}(M). \tag{6.6}$$

i') Let F be focusing and satisfy

$$-\hat{F}(u) \le C_{F,4}\|u\|_{L^{\alpha+1}}^{\alpha+1}, \quad u \in L^{\alpha+1}(M).$$

Assume that there is $\theta \in (0, \frac{2}{\alpha+1})$ with

$$(H, E_A)_{\theta,1} \hookrightarrow L^{\alpha+1}(M).$$

The only difference between the previous assumptions and the corresponding ones from the fourth chapter lies in the fact that we need a two-sided estimate in (6.6). This is a minor restriction, however, since the standard power nonlinearity is obviously still covered. We continue with the assumptions on the stochastic part of equation (6.1).

Assumption 6.4. (a) Assume that $(L(t))_{t\geq 0}$ is an $\mathbb{R}^N$-valued, $\mathbb{F}$-adapted Lévy process of pure jump type with the corresponding time homogeneous Poisson random measure η from the Lévy-Itô decomposition in Theorem A.28.

(b) Assume that the intensity measure ν of η is supported in the closed unit ball of $\mathbb{R}^N$. In particular, it satisfies

$$\int_{\{|l|\leq 1\}} |l|^2 \nu(\mathrm{d}l) < \infty. \tag{6.7}$$

c) Let $B_1, \dots, B_N \in \mathcal{L}(H)$ be selfadjoint operators on H with $B_m|_{E_A} \in \mathcal{L}(E_A)$ and $B_m|_{L^{\alpha+1}} \in \mathcal{L}(L^{\alpha+1}(M))$.

In view of the Lévy-Itô decomposition from Theorem A.28, the previous assumption implies

$$L(t) = \int_0^t \int_{\{|l|\leq 1\}} l \, \tilde{\eta}(\mathrm{d}s, \mathrm{d}l),$$

since we have $\eta([0,t] \times B) = 0$ for all Borel sets $B \subset \{|l| > 1\}$ and $t \geq 0$ as a consequence of the Poisson distribution of η. We abbreviate

$$b_{E_A} := \sum_{m=1}^N \|B_m\|^2_{\mathcal{L}(E_A)}, \qquad b_{L^{\alpha+1}} := \sum_{m=1}^N \|B_m\|^2_{\mathcal{L}(L^{\alpha+1})}, \qquad b_{L^2} := \sum_{m=1}^N \|B_m\|^2_{\mathcal{L}(L^2)} \tag{6.8}$$

and for $l \in \mathbb{R}^N$, we introduce the notation

$$\mathcal{B}(l) := \sum_{m=1}^N l_m B_m.$$

As in Appendix A.2.2, we reformulate (6.1) as an Itô stochastic evolution equation based on the stochastic integral driven by a compensated Poisson random measure. To this end, we note that the Marcus mapping $\Phi : [0,1] \times \mathbb{R}^N \times H \to H$, i.e. the continuously differentiable solution of the differential equation

$$\frac{dy}{dt}(t) = -\mathrm{i} \sum_{m=1}^N l_m B_m y(t), \qquad t \in [0,1], \tag{6.9}$$

with $y(0) = x \in H$, and $l = (l_1, l_2, \dots, l_N) \in \mathbb{R}^N$, is given by $\Phi(t,l,x) = e^{-\mathrm{i}t\mathcal{B}(l)}x$. Then, the equation (6.1) with the notation $\diamond$ is defined in the integral form

$$\begin{aligned} u(t) = u_0 - \mathrm{i} \int_0^t (Au(s) + F(u(s))) \, \mathrm{d}s + \int_0^t \int_{\{|l|\leq 1\}} \left[e^{-\mathrm{i}\mathcal{B}(l)} u(s-) - u(s-) \right] \tilde{\eta}(\mathrm{d}s, \mathrm{d}l) \\ + \int_0^t \int_{\{|l|\leq 1\}} \left\{ e^{-\mathrm{i}\mathcal{B}(l)} u(s) - u(s) + \mathrm{i} \sum_{m=1}^N l_m B_m u(s) \right\} \nu(\mathrm{d}l)\mathrm{d}s. \end{aligned} \tag{6.10}$$

In the next definition, we fix our notion of solution.

Definition 6.5. Let $u_0 \in E_A$. A *martingale solution* of the equation (6.1) is a system $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{\eta}, \tilde{\mathbb{F}}, u\right)$ consisting of

- a complete probability space $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\right)$;
- a filtration $\tilde{\mathbb{F}} = \left(\tilde{\mathcal{F}}_t\right)_{t\in[0,T]}$ with the usual conditions;
- a time homogeneous Poisson random measure η on $\mathbb{R}^N$ over $\tilde{\Omega}$ with intensity measure ν,
- an $\tilde{\mathbb{F}}$-adapted, E_A^*-valued càdlàg process such that $u \in L^2(\Omega \times [0,T], E_A^*)$ and almost all paths are in $\mathbb{D}_w([0,T], E_A)$

such that the equation (6.10) holds almost surely in E_A^* for all $t \in [0,T]$.

The main result of this chapter is the existence of a martingale solution of (6.1).

Theorem 6.6. *Choose the operator A and the energy space E_A according to Assumption 4.1, the nonlinearity F according to Assumptions 6.2 and 6.3 and the noise according to Assumption 6.4. Then, the problem* (6.1) *has a martingale solution $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{\eta}, \tilde{\mathbb{F}}, u\right)$ which satisfies $u \in \mathbb{D}_w\left([0,T], E_A\right)$ almost surely and*

$$u \in L^q(\tilde{\Omega}, L^\infty(0,T; E_A))$$

for all $q \in [1, \infty)$.

6.2. Energy Estimates for the Galerkin solutions

In this section, we consider the Galerkin approximation of (6.1). We prove its wellposedness and mass conservation as well as uniform energy estimates. The results of this section can be viewed as ingredients to apply Corollary 2.54 and get the tightness of the approximated solutions.

Recall from Lemma 4.12 that S has the representation

$$Sx = \sum_{m=1}^{\infty} \lambda_m \big(x, h_m\big)_H h_m, \quad x \in \mathcal{D}(S) = \left\{ x \in H : \sum_{m=1}^{\infty} \lambda_m^2 |\big(x, h_m\big)_H|^2 < \infty \right\},$$

with an orthonormal basis $(h_m)_{m\in\mathbb{N}}$ of the complex Hilbert space $\big(H, (\cdot,\cdot)_H\big)$, eigenvalues $\lambda_m > 0$ such that $\lambda_m \to \infty$ as $m \to \infty$. For $n \in \mathbb{N}_0$, we set

$$H_n := \operatorname{span}\left\{h_m : m \in \mathbb{N},\, \lambda_m < 2^{n+1}\right\}$$

and denote the orthogonal projection from H to H_n by P_n. Moreover, we use the sequence $(S_n)_{n\in\mathbb{N}} \subset \mathcal{L}(L^2(M))$ constructed in Proposition 4.14 and set

$$\mathcal{B}_n(l) = \sum_{m=1}^{N} l_m S_n B_m S_n, \qquad n \in \mathbb{N}, \quad l \in \mathbb{R}^N.$$

As an approximation of (6.10), we consider the Galerkin equation

$$u_n(t) = P_n u_0 - \mathrm{i} \int_0^t \left(A u_n(s) + P_n F(u_n(s))\right) \mathrm{d}s$$

$$+\int_0^t \int_{\{|l|\le 1\}} \left[e^{-\mathrm{i}\mathcal{B}_n(l)} u_n(s-) - u_n(s-)\right] \tilde{\eta}(\mathrm{d}s,\mathrm{d}l)$$
$$+\int_0^t \int_{\{|l|\le 1\}} \left\{e^{-\mathrm{i}\mathcal{B}_n(l)} u_n(s) - u_n(s) + \mathrm{i}\mathcal{B}_n(l) u_n(s)\right\} \nu(\mathrm{d}l)\mathrm{d}s. \tag{6.11}$$

We emphasize that the symmetric truncation by S_n in the definition of $\mathcal{B}_n(l)$ is useful since it leads to a similar structure of the noise in the approximated equation as in the original one and therefore to mass conservation as we will observe below. In order to prove global wellposedness of (6.11) and estimates for the solution u_n uniformly in $n \in \mathbb{N}$, we need some Lemmata. We start with properties of the operators $\mathcal{B}_n(l)$.

Lemma 6.7. *Let $n \in \mathbb{N}$ and $l \in \mathbb{R}^N$. Then, we have*

$$\|\mathcal{B}_n(l)\|_{\mathcal{L}(L^2)} \le |l| b_{L^2}^{\frac{1}{2}}, \quad \|\mathcal{B}_n(l)\|_{\mathcal{L}(E_A)} \le |l| b_{E_A}^{\frac{1}{2}}, \quad \|\mathcal{B}_n(l)\|_{\mathcal{L}(L^{\alpha+1})} \le |l| b_{\alpha+1}^{\frac{1}{2}} \sup_{n\in\mathbb{N}} \|S_n\|_{\mathcal{L}(L^{\alpha+1})}^2.$$

Moreover, $\left(e^{-\mathrm{i}t\mathcal{B}_n(l)}\right)_{t\in\mathbb{R}}$ is a group of unitary operators on $L^2(M)$ with

$$\|e^{-\mathrm{i}t\mathcal{B}_n(l)}\|_{\mathcal{L}(E_A)} \le e^{|t||l| b_{E_A}^{\frac{1}{2}}}, \qquad \|e^{-\mathrm{i}t\mathcal{B}_n(l)}\|_{\mathcal{L}(L^{\alpha+1})} \le e^{|t||l| b_{\alpha+1}^{\frac{1}{2}} \sup_{n\in\mathbb{N}} \|S_n\|_{\mathcal{L}(L^{\alpha+1})}^2}, \qquad t \in \mathbb{R}.$$

Proof. By the boundedness of $(S_n)_{n\in\mathbb{N}} \subset \mathcal{L}(L^{\alpha+1}(M))$, we obtain

$$\begin{aligned}\|\mathcal{B}_n(l)\|_{\mathcal{L}(L^{\alpha+1})} &\le \sum_{m=1}^N |l_m| \|S_n B_m S_n\|_{\mathcal{L}(L^{\alpha+1})} \le |l| \left(\sum_{m=1}^N \|B_m\|_{\mathcal{L}(L^{\alpha+1})}^2\right)^{\frac{1}{2}} \sup_{n\in\mathbb{N}} \|S_n\|_{\mathcal{L}(L^{\alpha+1})}^2 \\ &= |l| b_{\alpha+1}^{\frac{1}{2}} \sup_{n\in\mathbb{N}} \|S_n\|_{\mathcal{L}(L^{\alpha+1})}^2. \end{aligned} \tag{6.12}$$

The estimate of $\mathcal{B}_n(l)$ in E_A can be shown analogously using $\|S_n\|_{\mathcal{L}(E_A)} = 1$. Since S_n and B_m are selfadjoint on $L^2(M)$ for $n \in \mathbb{N}$ and $m \in \{1,\dots,M\}$, Stone's Theorem yields that $\left(e^{-\mathrm{i}t\mathcal{B}_n(l)}\right)_{t\in\mathbb{R}}$ is a unitary group on $L^2(M)$. Moreover,

$$\|e^{-\mathrm{i}t\mathcal{B}_n(l)} x\|_{E_A} \le e^{|t| \|\mathcal{B}_n(l)\|_{\mathcal{L}(E_A)}} \|x\|_{E_A} \le e^{|t||l| b_{E_A}^{\frac{1}{2}}} \|x\|_{E_A}, \qquad x \in E_A, \quad t \in \mathbb{R},$$

$$\begin{aligned}\|e^{-\mathrm{i}t\mathcal{B}_n(l)} x\|_{L^{\alpha+1}} &\le e^{|t| \|\mathcal{B}_n(l)\|_{\mathcal{L}(L^{\alpha+1})}} \|x\|_{L^{\alpha+1}} \\ &\le e^{|t||l| b_{\alpha+1}^{\frac{1}{2}} \sup_{n\in\mathbb{N}} \|S_n\|_{\mathcal{L}(L^{\alpha+1})}^2} \|x\|_{L^{\alpha+1}}, \qquad x \in L^{\alpha+1}(M), \quad t \in \mathbb{R}.\end{aligned}$$

□

In the next Lemma, we show how to control the integrands appearing in (6.11) in the L^2-norm.

Lemma 6.8. *For every $n \in \mathbb{N}$, $l \in \mathbb{R}^N$ and $x \in L^2(M)$, the following inequalities hold:*

$$\|e^{-\mathrm{i}\mathcal{B}_n(l)} x - x\|_{L^2} \le b_{L^2}^{\frac{1}{2}} |l| \|x\|_{L^2},$$

$$\|e^{-\mathrm{i}\mathcal{B}_n(l)} x - x + \mathrm{i}\mathcal{B}_n(l) x\|_{L^2} \le \frac{1}{2} b_{L^2} |l|^2 \|x\|_{L^2}.$$

Proof. The identities

$$e^{-\mathrm{i}\mathcal{B}_n(l)}x - x = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} e^{-\mathrm{i}t\mathcal{B}_n(l)}x \,\mathrm{d}t = -\mathrm{i}\mathcal{B}_n(l) \int_0^1 e^{-\mathrm{i}t\mathcal{B}_n(l)}x \,\mathrm{d}t$$

and

$$e^{-\mathrm{i}\mathcal{B}_n(l)}x - x + \mathrm{i}\mathcal{B}_n(l)x = \int_0^1 \int_0^s \frac{\mathrm{d}^2}{\mathrm{d}t^2} e^{-\mathrm{i}t\mathcal{B}_n(l)}x \,\mathrm{d}t\mathrm{d}s = -\mathcal{B}_n(l)^2 \int_0^1 \int_0^s e^{-\mathrm{i}t\mathcal{B}_n(l)}x \,\mathrm{d}t\mathrm{d}s$$

and Lemma 6.7 lead to

$$\|e^{-\mathrm{i}\mathcal{B}_n(l)}x - x\|_{L^2} \le \|\mathcal{B}_n(l)\|_{\mathcal{L}(L^2)} \int_0^1 \|e^{-\mathrm{i}t\mathcal{B}_n(l)}x\|_{L^2}\mathrm{d}t \le b_{L^2}^{\frac{1}{2}}|l|\|x\|_{L^2},$$

$$\begin{aligned}\|e^{-\mathrm{i}\mathcal{B}_n(l)}x - x + \mathrm{i}\mathcal{B}_n(l)x\|_{L^2} &\le \|\mathcal{B}_n(l)\|_{\mathcal{L}(L^2)}^2 \int_0^1 \int_0^s \|e^{-\mathrm{i}t\mathcal{B}_n(l)}x\|_{L^2}\mathrm{d}t\mathrm{d}s \\ &\le \frac{1}{2} b_{L^2}|l|^2\|x\|_{L^2}.\end{aligned}$$

□

Next, we prove the wellposedness of the Galerkin equation. Moreover, we show that the Marcus noise and the approximation do not destroy the mass conservation of the deterministic NLS.

Proposition 6.9. *For each $n \in \mathbb{N}$, there is a unique global strong solution $u_n \in \mathbb{D}([0,T], H_n)$ of* (6.11) *and we have the estimate*

$$\|u_n(t)\|_{L^2} = \|P_n u_0\|_{L^2} \le \|u_0\|_{L^2} \tag{6.13}$$

almost surely for all $t \in [0,T]$.

Proof. Step 1. We fix $n \in \mathbb{N}$. To obtain a global solution, we regard H_n as a finite dimensional real Hilbert space equipped with the scalar product $\left(u,v\right)_{H_n} := \operatorname{Re}\left(u,v\right)_H$ and check the assumptions of [2], Theorem 3.1 for the coefficients defined by

$$\begin{aligned}\xi &= P_n u_0, \qquad \sigma(u) = 0, \\ b(u) &= -\mathrm{i}Au - \mathrm{i}P_n F(u) + \int_{\{|l|\le 1\}} \left\{ e^{-\mathrm{i}\mathcal{B}_n(l)}u - u + \mathrm{i}\mathcal{B}_n(l)u \right\} \nu(\mathrm{d}l), \\ g(u,l) &= \left[e^{-\mathrm{i}\mathcal{B}_n(l)}u - u \right]\end{aligned}$$

for $u \in H_n$ and $l \in \mathbb{R}^N$. Let $R > 0$. We take $u, v \in H_n$ and estimate

$$\begin{aligned}\|b(u) - b(v)\|_{L^2} \le& \|A|_{H_n}\|_{\mathcal{L}(H)}\|u - v\|_{L^2} + \|F(u) - F(v)\|_{L^2} \\ &+ \int_{\{|l|\le 1\}} \|e^{-\mathrm{i}\mathcal{B}_n(l)}(u-v) - (u-v) + \mathrm{i}\mathcal{B}_n(l)(u-v)\|_{L^2} \,\nu(\mathrm{d}l).\end{aligned} \tag{6.14}$$

By Lemma 6.8 and (6.7)

$$\int_{\{|l|\le 1\}} \|e^{-\mathrm{i}\mathcal{B}_n(l)}(u-v) - (u-v) + \mathrm{i}\mathcal{B}_n(l)(u-v)\|_{L^2} \,\nu(\mathrm{d}l) \le \frac{1}{2} b_{L^2} \int_{\{|l|\le 1\}} |l|^2 \nu(\mathrm{d}l) \|u - v\|_{L^2}$$

$$\lesssim \|u-v\|_{L^2}. \tag{6.15}$$

To estimate the nonlinearity, we use the equivalence of all norms in H_n and (4.11)

$$\begin{aligned}\|F(u)-F(v)\|_{L^2} &\lesssim_n \|F(u)-F(v)\|_{L^{\frac{\alpha+1}{\alpha}}} \lesssim \left(\|u\|_{L^{\alpha+1}}+\|v\|_{L^{\alpha+1}}\right)^{\alpha-1}\|u-v\|_{L^{\alpha+1}}\\ &\lesssim \left(\|u\|_{L^2}+\|v\|_{L^2}\right)^{\alpha-1}\|u-v\|_{L^2} \lesssim_R \|u-v\|_{L^2}.\end{aligned} \tag{6.16}$$

We insert (6.16) and (6.15) in (6.14) to get a constant $C=C(R)$ such that

$$\|b(u)-b(v)\|_{L^2} \le C\|u-v\|_{L^2}. \tag{6.17}$$

Moreover, we have

$$\int_{\{|l|\le 1\}} \|g(u,l)-g(v,l)\|_{L^2}^2 \nu(\mathrm{d}l) \le b_{L^2}\int_{\{|l|\le 1\}} |l|^2\nu(\mathrm{d}l)\|u-v\|_{L^2}^2 \lesssim \|u-v\|_{L^2}^2 \tag{6.18}$$

where we used Lemma 6.8 and (6.7). To check the one-sided linear growth condition, we use (6.5) and (6.15) for $v=0$ and get a constant $K_1>0$ with

$$\begin{aligned}2\big(u,b(u)\big)_{H_n}+\int_{\{|l|\le 1\}}\|g(u,l)\|_{L^2}^2\nu(\mathrm{d}l) \le& 2\|A|_{H_n}\|_{\mathcal{L}(H)}\|u\|_{L^2}^2+2\operatorname{Re}\big(u,-\mathrm{i}F(u)\big)_H\\ &+2\|u\|_{L^2}\int_{\{|l|\le 1\}}\|e^{-\mathrm{i}\mathcal{B}_n(l)}u-u+\mathrm{i}\mathcal{B}_n(l)u\|_{L^2}\,\nu(\mathrm{d}l)\\ \le& K_1\|u\|_{L^2}^2.\end{aligned} \tag{6.19}$$

In view of (6.17), (6.18) and (6.19), we can apply Theorem 3.1 of [2] and get a unique global strong solution of (6.11) for each $n\in\mathbb{N}$.

Step 2. It remains to show (6.13). The function $\mathcal{M}: H_n\to\mathbb{R}$ defined by $\mathcal{M}(v):=\|v\|_{L^2}^2$ for $v\in H_n$ is continuously Fréchet-differentiable with

$$\mathcal{M}'[v]h_1 = 2\operatorname{Re}\big(v,h_1\big)_{L^2},$$

for $v,h_1,h_2\in H_n$. By Itô's formula, see Theorem A.40, and (6.10), we get

$$\begin{aligned}\|u_n(t)\|_{L^2}^2 =& \|P_nu_0\|_{L^2}^2+2\int_0^t \operatorname{Re}\big(u_n(s),-\mathrm{i}Au_n(s)-\mathrm{i}P_nF\left(u_n(s)\right)\big)_{L^2}\mathrm{d}s\\ &+\int_0^t\int_{\{|l|\le 1\}}\left[\|e^{-\mathrm{i}\mathcal{B}_n(l)}u_n(s-)\|_{L^2}^2-\|u_n(s-)\|_{L^2}^2\right]\tilde{\eta}(\mathrm{d}l,\mathrm{d}s)\\ &+\int_0^t\int_{\{|l|\le 1\}}\left[\|e^{-\mathrm{i}\mathcal{B}_n(l)}u_n(s)\|_{L^2}^2-\|u_n(s)\|_{L^2}^2\right]\nu(\mathrm{d}l)\mathrm{d}s\\ &-2\int_0^t\int_{\{|l|\le 1\}}\operatorname{Re}\big(u_n(s),-\mathrm{i}\sum_{m=1}^N l_mS_nB_mS_nu_n(s)\big)_{L^2}\nu(\mathrm{d}l)\mathrm{d}s\end{aligned}$$

almost surely for all $t\in[0,T]$. By

$$\operatorname{Re}\big(v,-\mathrm{i}Av\big)_{L^2}=\operatorname{Re}\left[\mathrm{i}\|A^{\frac12}v\|_{L^2}^2\right]=0,\qquad \operatorname{Re}\big(v,-\mathrm{i}P_nF\left(v\right)\big)_{L^2}=0,\qquad \operatorname{Re}\big(v,\mathrm{i}B_mv\big)_{L^2}=0$$

for $v\in H_n$ and the fact that $S_n\mathcal{B}(l)S_n$ is selfadjoint and hence, $e^{-\mathrm{i}\mathcal{B}_n(l)}$ is unitary, this simplifies to

$$\|u_n(t)\|_{L^2}^2=\|P_nu_0\|_{L^2}^2$$

almost surely for all $t\in[0,T]$. □

Let us recall that by Assumption 6.2, the nonlinearity F has a real antiderivative denoted by $\hat{F}$. This helps us to associate an energy $\mathcal{E}: E_A \to \mathbb{R}$ to the NLS which is given by

$$\mathcal{E}(u) := \frac{1}{2}\|A^{\frac{1}{2}}u\|_H^2 + \hat{F}(u), \qquad u \in E_A.$$

The main ingredient for uniform estimates in E_A besides the mass conservation is to control the energy. As a preparation, we need estimates of the differences which occur in the representation of $\mathcal{E}(u_n)$ based on the Itô formula.

Lemma 6.10. *a) There is a constant $C = C(b_{E_A}, b_{\alpha+1}, \alpha, F) > 0$ such that for every $n \in \mathbb{N}$, we have*

$$|\mathcal{E}(e^{-\mathrm{i}\mathcal{B}_n(l)}x) - \mathcal{E}(x)| \leq C|l| \left(\|x\|_{E_A}^2 + \|x\|_{L^{\alpha+1}}^{\alpha+1}\right)$$

for all $x \in H_n$, and $l \in \mathbb{R}^N$ with $|l| \leq 1$.

b) There is a constant $C = C(b_{E_A}, b_{\alpha+1}, q, \alpha, F) > 0$ such that for every $n \in \mathbb{N}$, we have

$$|\mathcal{E}(e^{-\mathrm{i}\mathcal{B}_n(l)}x) - \mathcal{E}(x) + \mathcal{E}'[x](\mathrm{i}\mathcal{B}_n(l)x)| \leq C|l|^2 \left(\|x\|_{E_A}^2 + \|x\|_{L^{\alpha+1}}^{\alpha+1}\right)$$

for all $x \in H_n$, and $l \in \mathbb{R}^N$ with $|l| \leq 1$.

Proof. ad a): The map $\mathcal{E}$ is twice continuously Fréchet-differentiable with

$$\mathcal{E}'[v]h = \mathrm{Re}\langle Av + F(v), h\rangle,$$
$$\mathcal{E}''[v](h_1, h_2) = \mathrm{Re}\left(A^{\frac{1}{2}}h_1, A^{\frac{1}{2}}h_2\right)_{L^2} + \mathrm{Re}\langle F'[v]h_1, h_2\rangle$$

for $v, h_1, h_2 \in H_n$. Hence, we get

$$\begin{aligned}\mathcal{E}(e^{-\mathrm{i}\mathcal{B}_n(l)}x) - \mathcal{E}(x) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(e^{-\mathrm{i}t\mathcal{B}_n(l)}x)\mathrm{d}t = \int_0^1 \mathcal{E}'[e^{-\mathrm{i}t\mathcal{B}_n(l)}x]\left(-\mathrm{i}\mathcal{B}_n(l)e^{-\mathrm{i}t\mathcal{B}_n(l)}x\right)\mathrm{d}t \\ &= \int_0^1 \mathrm{Re}\left\langle Ae^{-\mathrm{i}t\mathcal{B}_n(l)}x + F(e^{-\mathrm{i}t\mathcal{B}_n(l)}x), -\mathrm{i}\mathcal{B}_n(l)e^{-\mathrm{i}t\mathcal{B}_n(l)}x\right\rangle\mathrm{d}t. \qquad (6.20)\end{aligned}$$

We define $f: [0,1] \times \mathbb{R}^N \to [0,\infty)$ by

$$f(t,l) := \max\left\{1, e^{2t|l|b_{E_A}^{\frac{1}{2}}} + e^{(\alpha+1)t|l|b_{\alpha+1}^{\frac{1}{2}}\sup_{n\in\mathbb{N}}\|S_n\|_{\mathcal{L}(L^{\alpha+1})}^2}\right\}, \qquad t \in [0,1], \quad l \in \mathbb{R}^N,$$

and by the properties of $\mathcal{B}_n(l)$ from Lemma 6.7, we estimate the integrand of (6.20):

$$\begin{aligned}|\left(Ae^{-\mathrm{i}t\mathcal{B}_n(l)}x, -\mathrm{i}\mathcal{B}_n(l)e^{-\mathrm{i}t\mathcal{B}_n(l)}x\right)_{L^2}| &\leq \|A^{\frac{1}{2}}e^{-\mathrm{i}t\mathcal{B}_n(l)}x\|_{L^2}\|A^{\frac{1}{2}}\mathcal{B}_n(l)e^{-\mathrm{i}t\mathcal{B}_n(l)}x\|_{L^2} \\ &\leq e^{t|l|b_{E_A}^{\frac{1}{2}}}\|x\|_{E_A}|l|b_{E_A}^{\frac{1}{2}}\|e^{-\mathrm{i}t\mathcal{B}_n(l)}x\|_{E_A} \\ &\leq e^{2t|l|b_{E_A}^{\frac{1}{2}}}|l|b_{E_A}^{\frac{1}{2}}\|x\|_{E_A}^2 \qquad (6.21)\end{aligned}$$

and

$$\begin{aligned}\left|\left\langle F(e^{-\mathrm{i}t\mathcal{B}_n(l)}x), -\mathrm{i}\mathcal{B}_n(l)e^{-\mathrm{i}t\mathcal{B}_n(l)}x\right\rangle\right| &\leq \|F(e^{-\mathrm{i}t\mathcal{B}_n(l)}x)\|_{L^{\frac{\alpha+1}{\alpha}}}\|\mathcal{B}_n(l)e^{-\mathrm{i}t\mathcal{B}_n(l)}x\|_{L^{\alpha+1}} \\ &\leq C_{F,1}\|\mathcal{B}_n(l)\|_{\mathcal{L}(L^{\alpha+1})}\|e^{-\mathrm{i}t\mathcal{B}_n(l)}x\|_{L^{\alpha+1}}^{\alpha+1}\end{aligned}$$

$$\leq C_{F,1}|l|b_{\alpha+1}^{\frac{1}{2}} \sup_{n\in\mathbb{N}} \|S_n\|^2_{\mathcal{L}(L^{\alpha+1})} \|x\|^{\alpha+1}_{L^{\alpha+1}}$$
$$e^{(\alpha+1)t|l|b_{\alpha+1}^{\frac{1}{2}} \sup_{n\in\mathbb{N}} \|S_n\|^2_{\mathcal{L}(L^{\alpha+1})}}. \qquad (6.22)$$

We obtain

$$|\mathcal{E}(e^{-\mathrm{i}\mathcal{B}_n(l)}x) - \mathcal{E}(x)| \leq |l| \max\left\{b_{E_A}^{\frac{1}{2}}, C_{F,1} b_{\alpha+1}^{\frac{1}{2}} \sup_{n\in\mathbb{N}} \|S_n\|^2_{\mathcal{L}(L^{\alpha+1})}\right\} \left(\|x\|^2_{E_A} + \|x\|^{\alpha+1}_{L^{\alpha+1}}\right) \int_0^1 f(t,l)\mathrm{d}t$$

and the assertion follows from

$$\int_0^1 f(t,l)\mathrm{d}t = \int_0^1 \max\left\{1, e^{2t|l|b_{E_A}^{\frac{1}{2}}} + e^{(\alpha+1)t|l|b_{\alpha+1}^{\frac{1}{2}} \sup_{n\in\mathbb{N}} \|S_n\|^2_{\mathcal{L}(L^{\alpha+1})}}\right\} \mathrm{d}t$$
$$\leq \max\left\{1, e^{2b_{E_A}^{\frac{1}{2}}} + e^{(\alpha+1)b_{\alpha+1}^{\frac{1}{2}} \sup_{n\in\mathbb{N}} \|S_n\|^2_{\mathcal{L}(L^{\alpha+1})}}\right\} < \infty, \qquad |l| \leq 1. \qquad (6.23)$$

ad b): We start with the identity

$$\mathcal{E}(e^{-\mathrm{i}\mathcal{B}_n(l)}x) - \mathcal{E}(x) + \mathcal{E}'[x](\mathrm{i}\mathcal{B}_n(l))x = \int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(e^{-\mathrm{i}s\mathcal{B}_n(l)}x) - \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(e^{-\mathrm{i}s\mathcal{B}_n(l)}x)\Big|_{s=0}\right) \mathrm{d}s$$
$$= \int_0^1 \int_0^s \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{E}(e^{-\mathrm{i}t\mathcal{B}_n(l)}x)\mathrm{d}t\mathrm{d}s$$
$$= \int_0^1 \int_0^s \mathcal{E}'[e^{-\mathrm{i}t\mathcal{B}_n(l)}x]\left(-\mathcal{B}_n(l)^2 e^{-\mathrm{i}t\mathcal{B}_n(l)}x\right)\mathrm{d}t\mathrm{d}s$$
$$+ \int_0^1 \int_0^s \mathcal{E}''[e^{-\mathrm{i}t\mathcal{B}_n(l)}x]\left(-\mathrm{i}\mathcal{B}_n(l)e^{-\mathrm{i}t\mathcal{B}_n(l)}x, -\mathrm{i}\mathcal{B}_n(l)e^{-\mathrm{i}t\mathcal{B}_n(l)}x\right)\mathrm{d}t\mathrm{d}s$$
$$=: I_1 + I_2.$$

As above

$$|I_1| \leq |l|^2 \max\left\{b_{E_A}, C_{F,1} b_{\alpha+1} \sup_{n\in\mathbb{N}} \|S_n\|^4_{\mathcal{L}(L^{\alpha+1})}\right\} \left(\|x\|^2_{E_A} + \|x\|^{\alpha+1}_{L^{\alpha+1}}\right) \int_0^1 f(t,l)\mathrm{d}t.$$

We further decompose $I_2 = I_{2,1} + I_{2,2}$ with

$$I_{2,1} = \int_0^1 \int_0^s \|A^{\frac{1}{2}}\mathcal{B}_n(l)e^{-\mathrm{i}t\mathcal{B}_n(l)}x\|^2_{L^2}\mathrm{d}t\mathrm{d}s,$$

$$I_{2,2} = \int_0^1 \int_0^s \mathrm{Re}\left\langle F'[e^{-\mathrm{i}t\mathcal{B}_n(l)}x]\mathcal{B}_n(l)e^{-\mathrm{i}t\mathcal{B}_n(l)}x, \mathcal{B}_n(l)e^{-\mathrm{i}t\mathcal{B}_n(l)}x\right\rangle \mathrm{d}t\mathrm{d}s.$$

By Lemma 6.7,

$$|I_{2,1}| \leq \int_0^1 \int_0^s |l|^2 b_{E_A} \|e^{-\mathrm{i}t\mathcal{B}_n(l)}x\|^2_{E_A}\mathrm{d}t\mathrm{d}s \leq \|x\|^2_{E_A}|l|^2 b_{E_A} \int_0^1 f(t,l)\mathrm{d}t.$$

Moreover, the estimate

$$\left|\left\langle F'[e^{-\mathrm{i}t\mathcal{B}_n(l)}x]\mathcal{B}_n(l)e^{-\mathrm{i}t\mathcal{B}_n(l)}x, \mathcal{B}_n(l)e^{-\mathrm{i}t\mathcal{B}_n(l)}x\right\rangle\right|$$

$$\leq \|F'[e^{-\mathrm{i}t\mathcal{B}_n(l)}x]\mathcal{B}_n(l)e^{-\mathrm{i}t\mathcal{B}_n(l)}x\|_{L^{\frac{\alpha+1}{\alpha}}}\|\mathcal{B}_n(l)e^{-\mathrm{i}t\mathcal{B}_n(l)}x\|_{L^{\alpha+1}}$$
$$\leq C_{F,2}\|\mathcal{B}_n(l)\|^2_{\mathcal{L}(L^{\alpha+1})}\|e^{-\mathrm{i}t\mathcal{B}_n(l)}x\|^{\alpha+1}_{L^{\alpha+1}}$$
$$\leq C_{F,2}|l|^2 b_{\alpha+1}\sup_{n\in\mathbb{N}}\|S_n\|^4_{\mathcal{L}(L^{\alpha+1})}f(t,l)\|x\|^{\alpha+1}_{L^{\alpha+1}}$$

yields

$$|I_{2,2}| \leq \int_0^1\int_0^s \left|\left\langle F'[e^{-\mathrm{i}t\mathcal{B}_n(l)}x]\mathcal{B}_n(l)e^{-\mathrm{i}t\mathcal{B}_n(l)}x, \mathcal{B}_n(l)e^{-\mathrm{i}t\mathcal{B}_n(l)}x\right\rangle\right| \mathrm{d}t\mathrm{d}s$$
$$\leq C_{F,2}|l|^2 b_{\alpha+1}\sup_{n\in\mathbb{N}}\|S_n\|^4_{\mathcal{L}(L^{\alpha+1})}\|x\|^{\alpha+1}_{L^{\alpha+1}}\int_0^1 f(t,l)\mathrm{d}t$$

and finally, we find a constant $C = C(b_{\alpha+1}, b_{E_A}, \sup_{n\in\mathbb{N}}\|S_n\|_{\mathcal{L}(L^{\alpha+1})}, F)$ such that

$$\left|\mathcal{E}(e^{-\mathrm{i}\mathcal{B}_n(l)}x) - \mathcal{E}(x) + \mathcal{E}'[x](\mathrm{i}\mathcal{B}_n(l)x)\right| \leq C|l|^2\left(\|x\|^2_{E_A} + \|x\|^{\alpha+1}_{L^{\alpha+1}}\right)\int_0^1 f(t,l)\mathrm{d}t$$

and the second assertion also follows from (6.23). □

Now, we are ready prove that the solutions of (6.11) have uniform energy estimates and satisfy the Aldous condition.

Proposition 6.11. *Let $q\in[1,\infty)$. Then, the following assertions hold:*

a) There is $C = C(\|u_0\|_{E_A}, T, b_{E_A}, b_{\alpha+1}, q, \alpha, F) > 0$ with

$$\sup_{n\in\mathbb{N}}\mathbb{E}\Big[\sup_{t\in[0,T]}\left[\|u_n(t)\|^2_{L^2} + \mathcal{E}(u_n(t))\right]^q\Big] \leq C.$$

b) The sequence $(u_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition $[A]$ in E_A^*.

Proof. ad a): We only prove the assertion for $q>2$. The case $q\in[1,2]$ is a simple consequence of the Hölder inequality. Recall that the energy $\mathcal{E}$ is twice Frchet differentiable. In particular, $\mathcal{E}'$ is Hölder continuous. Hence, we can use Proposition 6.9 and Itô's formula A.40 to deduce

$$\begin{aligned}
\frac{1}{2}\|u_n(s)\|^2_{L^2} + \mathcal{E}(u_n(s)) &= \frac{1}{2}\|P_n u_0\|^2_{L^2} + \mathcal{E}(P_n u_0) \\
&+ \int_0^s \mathrm{Re}\langle Au_n(r) + F(u_n(r)), -\mathrm{i}Au_n(r) - \mathrm{i}P_nF(u_n(r))\rangle \mathrm{d}r \\
&+ \int_0^s\int_{\{|l|\leq 1\}}\left[\mathcal{E}(e^{-\mathrm{i}\mathcal{B}_n(l)}u_n(r-)) - \mathcal{E}(u_n(r-))\right]\tilde{\eta}(\mathrm{d}l,\mathrm{d}r) \\
&+ \int_0^s\int_{\{|l|\leq 1\}}\left[\mathcal{E}(e^{-\mathrm{i}\mathcal{B}_n(l)}u_n(r)) - \mathcal{E}(u_n(r)) + \mathcal{E}'[u_n(r)](\mathrm{i}\mathcal{B}_n(l)u_n(r))\right]\nu(\mathrm{d}l)\mathrm{d}r \\
&=: \frac{1}{2}\|P_nu_0\|^2_{L^2} + \mathcal{E}(P_nu_0) + I_1(s) + I_2(s) + I_3(s)
\end{aligned} \tag{6.24}$$

almost surely for all $s\in[0,T]$. The first integral $I_1(s)$ cancels for the same reasons as in the Gaussian case. We refer to the proof of Proposition 4.19. By the maximal inequality for the Poisson stochastic integral, see Theorem A.35, and Lemma 6.10, we get

$$\left(\mathbb{E}\left[\sup_{s\in[0,t]}|I_2(s)|^q\right]\right)^{\frac{1}{q}} \lesssim \left(\mathbb{E}\left(\int_0^t\int_{\{|l|\leq 1\}}\left|\mathcal{E}(e^{-\mathrm{i}\mathcal{B}_n(l)}u_n(s)) - \mathcal{E}(u_n(s))\right|^2\nu(\mathrm{d}l)\mathrm{d}s\right)^{\frac{q}{2}}\right)^{\frac{1}{q}}$$

$$+\left(\mathbb{E}\int_0^t\int_{\{|l|\le 1\}}\left|\mathcal{E}(e^{-\mathrm{i}\mathcal{B}_n(l)}u_n(s))-\mathcal{E}(u_n(s))\right|^q\nu(\mathrm{d}l)\mathrm{d}s\right)^{\frac{1}{q}}$$
$$\lesssim\left(\mathbb{E}\left(\int_0^t\int_{\{|l|\le 1\}}|l|^2\left(\|u_n(s)\|_{E_A}^2+\|u_n(s)\|_{L^{\alpha+1}}^{\alpha+1}\right)^2\nu(\mathrm{d}l)\mathrm{d}s\right)^{\frac{q}{2}}\right)^{\frac{1}{q}}$$
$$+\left(\mathbb{E}\int_0^t\int_{\{|l|\le 1\}}|l|^q\left(\|u_n(s)\|_{E_A}^2+\|u_n(s)\|_{L^{\alpha+1}}^{\alpha+1}\right)^q\nu(\mathrm{d}l)\mathrm{d}s\right)^{\frac{1}{q}}. \tag{6.25}$$

We introduce the abbreviation

$$X:=\frac{1}{2}\|u_n\|_{L^2}^2+\mathcal{E}(u_n)$$

and observe

$$\|u_n\|_{E_A}^2+\|u_n\|_{L^{\alpha+1}}^{\alpha+1}\lesssim X. \tag{6.26}$$

Moreover, we have

$$\int_{\{|l|\le 1\}}|l|^q\,\nu(\mathrm{d}l)\le\int_{\{|l|\le 1\}}|l|^2\,\nu(\mathrm{d}l)<\infty,\qquad q\ge 2. \tag{6.27}$$

Thus, we can conclude

$$\left(\mathbb{E}\left[\sup_{s\in[0,t]}|I_2(s)|^q\right]\right)^{\frac{1}{q}}\lesssim\left(\mathbb{E}\left(\int_0^t X(s)^2\mathrm{d}s\right)^{\frac{q}{2}}\right)^{\frac{1}{q}}+\left(\mathbb{E}\int_0^t X(s)^q\mathrm{d}s\right)^{\frac{1}{q}}$$
$$=\|X\|_{L^q(\Omega,L^2(0,t))}+\|X\|_{L^q(\Omega,L^q(0,t))}. \tag{6.28}$$

By Lemma 6.10 b), (6.26) and the Minkowski inequality

$$\left(\mathbb{E}\Big[\sup_{s\in[0,t]}|I_3(s)|^q\Big]\right)^{\frac{1}{q}}\lesssim\int_{\{|l|\le 1\}}|l|^2\nu(\mathrm{d}l)\left(\mathbb{E}\left(\int_0^t\left(\|u_n(r)\|_{E_A}^2+\|u_n(r)\|_{L^{\alpha+1}}^{\alpha+1}\right)\mathrm{d}r\right)^q\right)^{\frac{1}{q}}$$
$$\lesssim\int_{\{|l|\le 1\}}|l|^2\nu(\mathrm{d}l)\int_0^t\|X(r)\|_{L^q(\Omega)}\,\mathrm{d}r$$
$$\lesssim\int_0^t\|X\|_{L^q(\Omega,L^\infty(0,r))}\,\mathrm{d}r$$

and from (6.24) and the previous estimates, we get

$$\|X\|_{L^q(\Omega,L^\infty(0,t))}\le\frac{1}{2}\|P_nu_0\|_{L^2}^2+\mathcal{E}(P_nu_0)+\left(\mathbb{E}\Big[\sup_{s\in[0,t]}|I_2(s)|^q\Big]\right)^{\frac{1}{q}}+\left(\mathbb{E}\Big[\sup_{s\in[0,t]}|I_3(s)|^q\Big]\right)^{\frac{1}{q}}$$
$$\lesssim\frac{1}{2}\|P_nu_0\|_{L^2}^2+\mathcal{E}(P_nu_0)+\|X\|_{L^q(\Omega,L^2(0,t))}+\|X\|_{L^q(\Omega,L^q(0,t))}$$
$$+\int_0^t\|X\|_{L^q(\Omega,L^\infty(0,s))}\mathrm{d}s. \tag{6.29}$$

Using Lemma 2.11 with sufficiently small $\varepsilon > 0$ to estimate $\|X\|_{L^q(\Omega,L^2(0,t))}$ and $\|X\|_{L^q(\Omega,L^q(0,t))}$, we get

$$\|X\|_{L^q(\Omega,L^\infty(0,t))} \lesssim \frac{1}{2}\|P_n u_0\|_{L^2}^2 + \mathcal{E}(P_n u_0) + \varepsilon \|X\|_{L^q(\Omega,L^\infty(0,t))} + \int_0^t \|X\|_{L^q(\Omega,L^\infty(0,s))} \mathrm{d}s$$

and end up with

$$\|X\|_{L^q(\Omega,L^\infty(0,t))} \lesssim \frac{1}{2}\|P_n u_0\|_{L^2}^2 + \mathcal{E}(P_n u_0) + \int_0^t \|X\|_{L^q(\Omega,L^\infty(0,s))} \mathrm{d}s.$$

Finally, the Gronwall lemma yields

$$\|X\|_{L^q(\Omega,L^\infty(0,t))} \leq C \left(\frac{1}{2}\|P_n u_0\|_{L^2}^2 + \mathcal{E}(P_n u_0)\right) e^{Ct}, \qquad t \in [0,T],$$

where the constant $C = C(b_{E_A}, b_{\alpha+1}, q, \alpha, F) > 0$ is uniform in $n \in \mathbb{N}$. The assertion is a consequence of

$$\mathcal{E}(P_n u_0) \leq \frac{1}{2}\|P_n A^{\frac{1}{2}} u_0\|_{L^2}^2 + C_{F,3} C_{So}^{\alpha+1} \|P_n u_0\|_{E_A}^{\alpha+1} \tag{6.30}$$

and the uniform boundedness of $(P_n)_{n\in\mathbb{N}}$ as operators in $L^2(M)$ and E_A.

ad b): Now, we continue with the proof of the Aldous condition. We have

$$\begin{aligned} u_n(t) - P_n u_0 =& -\mathrm{i} \int_0^t A u_n(s) \mathrm{d}s - \mathrm{i} \int_0^t P_n F(u_n(s)) \mathrm{d}s \\ &+ \int_0^t \int_{\{|l|\leq 1\}} \left[e^{-\mathrm{i}\mathcal{B}_n(l)} u_n(s-) - u_n(s-) \right] \tilde{\eta}(\mathrm{d}s, \mathrm{d}l) \\ &+ \int_0^t \int_{\{|l|\leq 1\}} \left\{ e^{-\mathrm{i}\mathcal{B}_n(l)} u(s) - u(s) + \mathrm{i}\mathcal{B}_n(l) u(s) \right\} \nu(\mathrm{d}l) \mathrm{d}s \\ =&: J_1(t) + J_2(t) + J_3(t) + J_4(t) \end{aligned}$$

in H_n almost surely for all $t \in [0,T]$ and therefore

$$\|u_n((\tau_n+\theta)\wedge T) - u_n(\tau_n)\|_{E_A^*} \leq \sum_{k=1}^4 \|J_k((\tau_n+\theta)\wedge T) - J_k(\tau_n)\|_{E_A^*}$$

for each sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times and $\theta > 0$. Hence, we get

$$\mathbb{P}\left\{\|u_n((\tau_n+\theta)\wedge T) - u_n(\tau_n)\|_{E_A^*} \geq \eta\right\} \leq \sum_{k=1}^4 \mathbb{P}\left\{\|J_k((\tau_n+\theta)\wedge T) - J_k(\tau_n)\|_{E_A^*} \geq \frac{\eta}{4}\right\} \tag{6.31}$$

for a fixed $\eta > 0$. We aim to apply Tschebyscheff's inequality and estimate the expected value of each term in the sum. Similarly to Proposition 4.19, we derive

$$\mathbb{E}\|J_1((\tau_n+\theta)\wedge T) - J_1(\tau_n)\|_{E_A^*} \leq \theta \mathbb{E}\big[\sup_{s\in[0,T]} \|u_n(s)\|_{E_A}^2\big]^{\frac{1}{2}} \leq \theta C_1$$

as well as

$$\mathbb{E}\|J_2((\tau_n+\theta)\wedge T)-J_2(\tau_n)\|_{E_A^*}\lesssim \theta\mathbb{E}\big[\sup_{s\in[0,T]}\|u_n(s)\|_{E_A}^{\alpha}\big]\leq \theta C_2.$$

By the Levy-Itô-isometry, Lemma 6.8, (6.7) and Proposition 6.9

$$\begin{aligned}
&\mathbb{E}\|J_3((\tau_n+\theta)\wedge T)-J_3(\tau_n)\|_{E_A^*}^2\\
&\quad\leq \mathbb{E}\left\|\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\int_{\{|l|\leq 1\}}\Big[e^{-\mathrm{i}\mathcal{B}_n(l)}u_n(s-)-u_n(s-)\Big]\,\tilde{\eta}(\mathrm{d}s,\mathrm{d}l)\right\|_{L^2}^2\\
&\quad= \mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\int_{\{|l|\leq 1\}}\|e^{-\mathrm{i}\mathcal{B}_n(l)}u_n(s)-u_n(s)\|_{L^2}^2\,\nu(\mathrm{d}l)\mathrm{d}s\\
&\quad\leq b_{L^2}\int_{\{|l|\leq 1\}}|l|^2\nu(\mathrm{d}l)\mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\|u_n(s)\|_{L^2}^2\mathrm{d}s\\
&\quad\lesssim \theta\|P_nu_0\|_{L^2}^2\leq \theta\|u_0\|_{L^2}^2,
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\|J_4((\tau_n+\theta)\wedge T)-J_4(\tau_n)\|_{E_A^*}\\
&\quad= \mathbb{E}\left\|\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\int_{\{|l|\leq 1\}}\Big\{e^{-\mathrm{i}\mathcal{B}_n(l)}u_n(s)-u_n(s)+\mathrm{i}\mathcal{B}_n(l)u_n(s)\Big\}\,\nu(\mathrm{d}l)\mathrm{d}s\right\|_{E_A^*}\\
&\quad\lesssim \mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\int_{\{|l|\leq 1\}}\Big\|e^{-\mathrm{i}\mathcal{B}_n(l)}u_n(s)-u_n(s)+\mathrm{i}\mathcal{B}_n(l)u_n(s)\Big\|_{L^2}\,\nu(\mathrm{d}l)\mathrm{d}s\\
&\quad\leq \frac{1}{2}b_{L^2}\int_{\{|l|\leq 1\}}|l|^2\nu(\mathrm{d}l)\mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T}\|u_n(s)\|_{L^2}\mathrm{d}s\lesssim \theta\|u_0\|_{L^2}.
\end{aligned}$$

We follow the lines of the proof of Proposition 4.19 to combine the previous estimates with the Tschebyscheff inequality and (6.31) to show the Aldous condition in E_A^*. □

We continue with the a priori estimate for solutions of (6.11) with a focusing nonlinearity. By the additional restriction to the exponents α in 6.3 i'), we overcome the deficit that the expression

$$\frac{1}{2}\|v\|_H^2+\mathcal{E}(v)=\frac{1}{2}\|v\|_{E_A}^2+\hat{F}(v),\qquad v\in H_n,$$

does not dominate $\|v\|_{E_A}^2$ in this case.

Proposition 6.12. *Suppose that Assumption 6.3 i') is true and let $r\in[1,\infty)$. Then, the following assertions hold:*

a) There is a constant $C=C(\|u_0\|_{L^2},\|u_0\|_{E_A},\gamma,\alpha,T,F,b_{E_A},b_{\alpha+1},r)>0$ with

$$\sup_{n\in\mathbb{N}}\mathbb{E}\Big[\sup_{t\in[0,T]}\|u_n(t)\|_{E_A}^r\Big]\leq C.$$

b) The sequence $(u_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition [A] in E_A^*.

Proof. ad a): Let $\varepsilon>0$. Assumption 6.3 i') and Young's inequality imply that there are $\gamma>0$ and $C_\varepsilon>0$ such that

$$\|u\|_{L^{\alpha+1}}^{\alpha+1}\lesssim \varepsilon\|u\|_{E_A}^2+C_\varepsilon\|u\|_{L^2}^{\gamma},\qquad u\in E_A, \tag{6.32}$$

and therefore by Proposition 6.9, we infer that

$$\begin{aligned}-\hat{F}(u_n(s)) &\lesssim \|u_n(s)\|_{L^{\alpha+1}}^{\alpha+1} \lesssim \varepsilon\|u_n(s)\|_{E_A}^2 + C_\varepsilon\|u_n(s)\|_{L^2}^{\gamma} \\ &\lesssim \varepsilon\|A^{\frac{1}{2}}u_n(s)\|_H^2 + \varepsilon\|u_0\|_{L^2}^2 + C_\varepsilon\|u_0\|_{L^2}^{\gamma}, \qquad s\in[0,T]. \end{aligned} \tag{6.33}$$

By analogous calculations as in the proof of Proposition 6.11 we get

$$\begin{aligned}\frac{1}{2}\|A^{\frac{1}{2}}u_n(s)\|_{L^2}^2 &= -\hat{F}(u_n(s)) + \mathcal{E}\left(u_n(s)\right) \\ &= -\hat{F}(u_n(s)) + \mathcal{E}\left(P_n u_0\right) \\ &\quad + \int_0^s\int_{\{|l|\le 1\}} \Big[\mathcal{E}(e^{-\mathrm{i}\mathcal{B}_n(l)}u_n(r-)) - \mathcal{E}(u_n(r-))\Big]\tilde{\eta}(\mathrm{d}l,\mathrm{d}r) \\ &\quad + \int_0^s\int_{\{|l|\le 1\}} \Big[\mathcal{E}(e^{-\mathrm{i}\mathcal{B}_n(l)}u_n(r)) - \mathcal{E}(u_n(r)) + \mathcal{E}'[u_n(r)]\left(\mathrm{i}\mathcal{B}_n(l)u_n(s)\right)\Big]\nu(\mathrm{d}l)\mathrm{d}r \\ &=: -\hat{F}(u_n(s)) + \mathcal{E}\left(P_n u_0\right) + I_1(s) + I_2(s) \end{aligned} \tag{6.34}$$

almost surely for all $t\in[0,T]$. We abbreviate

$$X(s) := \|u_0\|_{L^2}^2 + \|A^{\frac{1}{2}}u_n(s)\|_{L^2}^2 + \|u_n(s)\|_{L^{\alpha+1}}^{\alpha+1}, \qquad s\in[0,T].$$

Let $q>2$ and recall (6.27) as well as the mass conservation from Proposition 6.9. As in the proof of Proposition 6.11, we estimate

$$\begin{aligned}\left(\mathbb{E}\left[\sup_{s\in[0,t]}|I_1(s)|^q\right]\right)^{\frac{1}{q}} &\lesssim \left(\int_{\{|l|\le 1\}}|l|^2\nu(\mathrm{d}l)\right)^{\frac{1}{2}}\left(\mathbb{E}\left(\int_0^t\left(\|u_n(s)\|_{E_A}^2 + \|u_n(s)\|_{L^{\alpha+1}}^{\alpha+1}\right)^2\mathrm{d}s\right)^{\frac{q}{2}}\right)^{\frac{1}{q}} \\ &\quad + \left(\int_{\{|l|\le 1\}}|l|^q\nu(\mathrm{d}l)\right)^{\frac{1}{q}}\left(\mathbb{E}\int_0^t\left(\|u_n(s)\|_{E_A}^2 + \|u_n(s)\|_{L^{\alpha+1}}^{\alpha+1}\right)^q\mathrm{d}s\right)^{\frac{1}{q}} \\ &\lesssim \|X\|_{L^q(\Omega,L^2(0,t))} + \|X\|_{L^q(\Omega,L^q(0,t))}; \end{aligned} \tag{6.35}$$

$$\begin{aligned}\left(\mathbb{E}\Big[\sup_{s\in[0,t]}|I_2(s)|^q\Big]\right)^{\frac{1}{q}} &\lesssim \int_{\{|l|\le 1\}}|l|^2\nu(\mathrm{d}l)\int_0^t\Big\|\|u_n\|_{E_A}^2 + \|u_n\|_{L^{\alpha+1}}^{\alpha+1}\Big\|_{L^q(\Omega,L^\infty(0,r))}\mathrm{d}r \\ &\lesssim \int_0^t\|X\|_{L^q(\Omega,L^\infty(0,r))}\mathrm{d}r. \end{aligned} \tag{6.36}$$

Using (6.30), (6.33), (6.35) and (6.36) in (6.34), we obtain

$$\begin{aligned}\Big\|\|A^{\frac{1}{2}}u_n\|_{L^2}^2\Big\|_{L^q(\Omega,L^\infty(0,t))} &\lesssim \Big\|\|A^{\frac{1}{2}}u_n\|_{L^2}^2\Big\|_{L^q(\Omega,L^\infty(0,t))}\varepsilon + \varepsilon\|u_0\|_{L^2}^2 + C_\varepsilon\|u_0\|_{L^2}^{\gamma} \\ &\quad + \|P_nA^{\frac{1}{2}}u_0\|_{L^2}^2 + \|P_nu_0\|_{E_A}^{\alpha+1} + \|X\|_{L^q(\Omega,L^2(0,t))} \\ &\quad + \|X\|_{L^q(\Omega,L^q(0,t))} + \int_0^t\|X\|_{L^q(\Omega,L^\infty(0,r))}\mathrm{d}r. \end{aligned}$$

If we employ Lemma 2.11 to estimate $\|X\|_{L^q(\Omega,L^2(0,t))}$ and $\|X\|_{L^q(\Omega,L^q(0,t))}$, we get

$$\Big\|\|A^{\frac{1}{2}}u_n\|_{L^2}^2\Big\|_{L^q(\Omega,L^\infty(0,t))} \lesssim \Big\|\|A^{\frac{1}{2}}u_n\|_{L^2}^2\Big\|_{L^q(\Omega,L^\infty(0,t))}\varepsilon + \varepsilon\|u_0\|_{L^2}^2 + C_\varepsilon\|u_0\|_{L^2}^{\gamma}$$

$$+ \|P_n A^{\frac{1}{2}} u_0\|_{L^2}^2 + \|P_n u_0\|_{E_A}^{\alpha+1} + \varepsilon \|X\|_{L^q(\Omega, L^\infty(0,t))}$$
$$+ \int_0^t \|X\|_{L^q(\Omega, L^\infty(0,r))} \mathrm{d}r.$$

Thanks to (6.33) it is possible to control $\|X\|_{L^q(\Omega,L^\infty(0,r))}$ by $\|\|A^{\frac{1}{2}} u_n\|_{L^2}^2\|_{L^q(\Omega,L^\infty(0,t))}$. Indeed,

$$\|X\|_{L^q(\Omega,L^\infty(0,t))} \leq (1+\varepsilon) \left\| \|A^{\frac{1}{2}} u_n\|_H^2 \right\|_{L^q(\Omega,L^\infty(0,t))} + C(\|u_0\|_{L^2})$$

follows from a similar reasoning as in the Gaussian case, see Proposition 4.20. Now, we choose $\varepsilon > 0$ sufficiently small and end up with

$$\left\| \|A^{\frac{1}{2}} u_n\|_{L^2}^2 \right\|_{L^q(\Omega,L^\infty(0,t))} \leq C \left(1 + \int_0^t \left\| \|A^{\frac{1}{2}} u_n\|_{L^2}^2 \right\|_{L^q(\Omega,L^\infty(0,r))} \mathrm{d}r \right)$$

for some $C = C(\|u_0\|_{L^2}, \|u_0\|_{E_A}, \gamma, \alpha, T, F, b_{E_A}, b_{\alpha+1}, q)$ independent of n. From the Gronwall Lemma, we infer

$$\left\| \|A^{\frac{1}{2}} u_n\|_{L^2}^2 \right\|_{L^q(\Omega,L^\infty(0,t))} \leq C e^{Ct}, \qquad t \in [0,T]. \tag{6.37}$$

In view of Proposition 6.9, we have proved the assertion for $r = 2q > 4$. The case $r \in [1,4]$ is an easy consequence of the Hölder inequality.

ad b). The proof of the Aldous condition is similar to the defocusing case, see Proposition 6.11 b). □

Corollary 6.13. *The sequence $(u_n)_{n\in\mathbb{N}}$ of Galerkin solutions is tight on*

$$Z_T^{\mathbb{D}} := \mathbb{D}([0,T], E_A^*) \cap L^{\alpha+1}(0,T; L^{\alpha+1}(M)) \cap \mathbb{D}_w([0,T], E_A).$$

Proof. This is an immediate consequence of the Propositions 2.39, 6.11 and 6.12. □

6.3. Construction of a martingale solution

In this section, we will use the compactness results from Section 2.4.2 and the uniform estimates from the previous section to complete the proof of Theorem 6.6. Let us recall

$$Z_T^{\mathbb{D}} := \mathbb{D}([0,T], E_A^*) \cap L^{\alpha+1}(0,T; L^{\alpha+1}(M)) \cap \mathbb{D}_w([0,T], E_A) =: Z_1 \cap Z_2 \cap Z_3.$$

By Lemma 6.13, we can apply Proposition 2.26 to the sequence $(u_n)_{n\in\mathbb{N}}$ of Galerkin solutions. As a result, we obtain a candidate v for the martingale solution.

Corollary 6.14. *Let $(u_n)_{n\in\mathbb{N}}$ be the sequence of solutions to the Galerkin equation* (6.11) *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *and $\mathcal{A}$ be the σ-algebra on $Z_T^{\mathbb{D}}$ defined in* (2.44).

a) There are a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, a subsequence $(u_{n_k})_{k\in\mathbb{N}}$, $Z_T^{\mathbb{D}}$-valued random variables v, v_k and $M_{\bar{\mathbb{N}}}^{\nu}([0,T] \times \mathbb{R}^N)$-valued random variables $\bar{\eta}_k, \bar{\eta}$ on $\bar{\Omega}$ such that

i) $\bar{\mathbb{P}}^{(\bar{\eta}_k, v_k)} = \mathbb{P}^{(\eta, u_{n_k})}$ for $k \in \mathbb{N}$,

ii) $(\bar{\eta}_k, v_k) \to (\bar{\eta}, v)$ in $M_{\bar{\mathbb{N}}}^{\nu}([0,T] \times \mathbb{R}^N) \times Z_T^{\mathbb{D}}$ almost surely for $k \to \infty$, *iii)* $\bar{\eta}_k = \bar{\eta}$ *almost surely.*

Moreover, $\bar{\eta}_k, \bar{\eta}$ *are time-homogeneous Poisson random measures on* $\mathbb{R}^N$ *with intensity measure* ν *which are adapted to the filtration* $\bar{\mathbb{F}}$ *defined by the augmentation of*

$$\bar{\mathcal{F}}_t := \sigma\left(\bar{\eta}_k(s), v_m(s), v(s) : k \in \mathbb{N}, m \in \mathbb{N}, s \in [0,t]\right).$$

b) We have $v_k \in \mathbb{D}\left([0,T], H_k\right)$ $\bar{\mathbb{P}}$*-a.s. and for all* $r \in [1,\infty)$*, there is* $C = C(T, \|u_0\|_{E_A}, r) > 0$ *with*

$$\sup_{k\in\mathbb{N}} \tilde{\mathbb{E}}\left[\|v_k\|_{L^\infty(0,T;E_A)}^r\right] \leq C.$$

c) For all $r \in [1,\infty)$*, we have*

$$\tilde{\mathbb{E}}\left[\|v\|_{L^\infty(0,T;E_A)}^r\right] \leq C$$

with the same constant $C > 0$ *as in b).*

Remark 6.15. We show that it is justified to view the process u_n, $n \in \mathbb{N}$, as a random variable in $(Z_T^{\mathbb{D}}, \mathcal{A})$. For $j = 1,2,3$, we have $\mathbb{D}([0,T], H_n) \subset Z_j$ with continuity of the canonical embedding. In particular, this implies

$$\begin{aligned}
\{B \cap \mathbb{D}([0,T], H_n) : B \in \mathcal{A}\} &\subset \left\{B \cap \mathbb{D}([0,T], H_n) : B \in \mathcal{B}(Z_T^{\mathbb{D}})\right\} \\
&= \sigma\left(\left\{B \cap \mathbb{D}([0,T], H_n) : B \text{ closed in } Z_T^{\mathbb{D}}\right\}\right) \\
&\subset \sigma(\left\{\tilde{B} : \tilde{B} \text{ closed in } \mathbb{D}([0,T], H_n)\right\}) \\
&= \mathcal{B}(\mathbb{D}([0,T], H_n)).
\end{aligned}$$

Since u_n is a random variable in $\mathbb{D}([0,T], H_n)$ equipped with the Borel σ-algebra, we infer for each $B \in \mathcal{A}$

$$\{u_n \in B\} = \{u_n \in B \cap \mathbb{D}([0,T], H_n)\} \in \mathcal{F}.$$

Proof of Corollary 6.14. ad a). As an immediate consequence of the Corollaries 6.13 and 2.54, we obtain the existence of the random variables $v_k, v, \bar{\eta}, \bar{\eta}_k$ for $k \in \mathbb{N}$ such that i),ii) and iii) are fulfilled. For the proof of the assertion that $\bar{\eta}$ and $\bar{\eta}_k$ for $k \in \mathbb{N}$ are adapted time-homogeneous Poisson random measures, we refer to [27], Section 8, Step III.

ad b). Since $\mathbb{D}\left([0,T], H_k\right)$ is contained in Z_j for $j = 1, \ldots, 3$, the definition of $\mathcal{A}$ yields that $\mathbb{D}\left([0,T], H_k\right) \in \mathcal{A}$. Hence, we obtain $v_k \in \mathbb{D}\left([0,T], H_k\right)$ $\bar{\mathbb{P}}$-a.s. by the identity of the laws of v_k and u_{n_k}. The uniform estimate follows from the respective estimates for $(u_{n_k})_{k\in\mathbb{N}}$, see Propositions 6.11 and 6.12, via the identity of laws, since $\mathbb{D}\left([0,T], H_k\right) \ni w \mapsto \sup_{t\in[0,T]} \|w(t)\|_{E_A}$ is a measurable function.

ad c). We can follow the lines of the proof of Proposition 4.23. □

It remains to verify that $\left(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \bar{\eta}, \bar{\mathbb{F}}, u\right)$ is indeed martingale solution. The compensated Poisson random measure induced by $\bar{\eta}$ is denoted by $\tilde{\bar{\eta}} := \bar{\eta} -$ Leb $\otimes\, \nu$. We need the following convergence results.

Lemma 6.16. *Let $\psi \in E_A$. Then, we have the following convergences in $L^2(\bar{\Omega} \times [0,T])$:*

$$\operatorname{Re}\left(v_n - P_n u_0, \psi\right)_H \xrightarrow{n\to\infty} \operatorname{Re}\left(v - u_0, \psi\right)_H; \tag{6.38}$$

$$\int_0^\cdot \operatorname{Re}\left(A v_n(s) + P_n F(v_n(s)), \psi\right)_H \mathrm{d}s \xrightarrow{n\to\infty} \int_0^\cdot \operatorname{Re}\langle A v(s) + F(v(s)), \psi\rangle \mathrm{d}s; \tag{6.39}$$

$$\begin{aligned}\int_0^\cdot \int_{\{|l|\le 1\}} &\operatorname{Re}\left(e^{-\mathrm{i}\mathcal{B}_n(l)} v_n(s-) - v_n(s-), \psi\right)_H \tilde{\bar{\eta}}(\mathrm{d}s, \mathrm{d}l) \\ &\xrightarrow{n\to\infty} \int_0^\cdot \int_{\{|l|\le 1\}} \operatorname{Re}\left(e^{-\mathrm{i}\mathcal{B}(l)} v(s-) - v(s-), \psi\right)_H \tilde{\bar{\eta}}(\mathrm{d}s, \mathrm{d}l);\end{aligned} \tag{6.40}$$

$$\begin{aligned}\int_0^\cdot \int_{\{|l|\le 1\}} &\operatorname{Re}\left(e^{-\mathrm{i}\mathcal{B}_n(l)} v_n(s) - v_n(s) + \mathrm{i}\mathcal{B}_n(l) v_n(s), \psi\right)_H \nu(\mathrm{d}l)\mathrm{d}s \\ &\xrightarrow{n\to\infty} \int_0^\cdot \int_{\{|l|\le 1\}} \operatorname{Re}\left(e^{-\mathrm{i}\mathcal{B}(l)} v(s) - v(s) + \mathrm{i}\mathcal{B}(l) v(s), \psi\right)_H \nu(\mathrm{d}l)\mathrm{d}s.\end{aligned} \tag{6.41}$$

Before we continue with the proof, we would like to remind the reader of Vitali's convergence result stated in Lemma 4.21 and the subsequent remark.

Proof. ad (6.38). We get (6.38) in $L^2(0,T)$ almost surely from $v_n \to v$ almost surely in $L^2(0,T;H)$. In view of

$$\begin{aligned}\tilde{\mathbb{E}} \int_0^T |\operatorname{Re}\left(v_n(t) - P_n u_0, \psi\right)_H|^r \mathrm{d}t &\le \|\psi\|_{L^2}^r \tilde{\mathbb{E}} \int_0^T \left(\|v_n(t)\|_{L^2} + \|u_0\|_{L^2}\right)^r \mathrm{d}t \\ &\le \|\psi\|_{L^2}^r T 2^r \|u_0\|_{L^2}^r < \infty\end{aligned}$$

for $r > 2$, Vitali's convergence Theorem yields the assertion.

ad (6.39). Let us fix $\omega \in \bar{\Omega}$ and $t \in [0,T]$. Then,

$$\int_0^t \operatorname{Re}\left(P_n F(v_n(s)), \psi\right)_H \mathrm{d}s \to \int_0^t \operatorname{Re}\langle F(v(s)), \psi\rangle \mathrm{d}s$$

follows from $v_n \to v$ in $L^{\alpha+1}(0,T;L^{\alpha+1}(M))$ in the same way as in Lemma 4.24. Moreover,

$$\operatorname{Re}\langle A(v_n(s) - v(s)), \psi\rangle = \operatorname{Re}\langle v_n(s) - v(s), A\psi\rangle \to 0$$

for all $s \in [0,T]$ by $v_n \to v$ in $\mathbb{D}_w([0,T], E_A)$. Via

$$\tilde{\mathbb{E}} \int_0^T \int_0^t |\operatorname{Re}\langle A v_n(s), \psi\rangle|^r \mathrm{d}s\mathrm{d}t \le \|\psi\|_{E_A}^r T^2 \tilde{\mathbb{E}}\Big[\sup_{s\in[0,T]} \|v_n(s)\|_{E_A}^r\Big] < \infty,$$

$$\begin{aligned}\tilde{\mathbb{E}} \int_0^T \left|\int_0^t \operatorname{Re}\left(P_n F(v_n(s)), \psi\right)_H \mathrm{d}s\right|^r \mathrm{d}t &\le T^{1+r} \|\psi\|_{E_A}^r \tilde{\mathbb{E}}\big[\sup_{s\in[0,T]} \|F(v_n(s))\|_{E_A^*}^r\big] \\ &\lesssim T^{1+r} \|\psi\|_{E_A}^r \tilde{\mathbb{E}}\big[\sup_{s\in[0,T]} \|v_n(s)\|_{E_A}^{r\alpha}\big] < \infty\end{aligned}$$

for $r > 2$, Vitali yields (6.39) in $L^2(\bar{\Omega} \times [0,T])$.

ad (6.40). In view of the Itô isometry, it is equivalent to prove

$$\int_0^{\cdot} \int_{\{|l|\leq 1\}} \left| \operatorname{Re}\left(e^{-\mathrm{i}\mathcal{B}_n(l)} v_n(s) - v_n(s) - \left[e^{-\mathrm{i}\mathcal{B}(l)} v(s) - v(s)\right], \psi\right)_H \right|^2 \nu(\mathrm{d}l)\mathrm{d}s \xrightarrow{n\to\infty} 0 \tag{6.42}$$

in $L^1(\bar{\Omega} \times [0,T])$. Before we proceed with this convergence, we remark that we have

$$\mathcal{B}_n(l)x \xrightarrow{n\to\infty} \mathcal{B}(l)x, \qquad x \in L^2(M), \quad l \in \mathbb{R}^N, \tag{6.43}$$

as a consequence of $S_n x \to x$ in $L^2(M)$ for $x \in L^2(M)$ which is included in Proposition 4.14. For $x \in H$, Lebesgue and (6.43) yield

$$\begin{aligned} \|e^{-\mathrm{i}\mathcal{B}_n(l)} x - e^{-\mathrm{i}\mathcal{B}(l)} x\|_{L^2} &= \left\| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} \left[e^{-\mathrm{i}s\mathcal{B}_n(l)} e^{-\mathrm{i}(1-s)\mathcal{B}(l)} x \right] \mathrm{d}s \right\|_{L^2} \\ &\leq \int_0^1 \| \left(\mathcal{B}_n(l) - \mathcal{B}(l)\right) e^{-\mathrm{i}s\mathcal{B}_n(l)} e^{-\mathrm{i}(1-s)\mathcal{B}(l)} x \|_{L^2} \mathrm{d}s \xrightarrow{n\to\infty} 0. \end{aligned}$$

From $v_n \to v$ almost surely in $L^2(0,T;H)$ and again Lebesgue, we infer

$$\begin{aligned} &\int_0^t |\operatorname{Re}\left(e^{-\mathrm{i}\mathcal{B}_n(l)} v_n - v_n - \left[e^{-\mathrm{i}\mathcal{B}(l)} v - v\right], \psi\right)_H|^2 \mathrm{d}s \\ &\quad \leq 2 \int_0^t \left(\|e^{-\mathrm{i}\mathcal{B}_n(l)} \left(v - v_n\right)\|_{L^2}^2 + \|v_n - v\|_{L^2}^2 + \| \left[e^{-\mathrm{i}\mathcal{B}_n(l)} - e^{-\mathrm{i}\mathcal{B}(l)}\right] v\|_{L^2} \right) \|\psi\|_{L^2}^2 \mathrm{d}s \\ &\quad \xrightarrow{n\to\infty} 0 \end{aligned} \tag{6.44}$$

almost surely for all $t \in [0,T]$ and $l \in \mathbb{R}^N$. Since we have

$$\begin{aligned} &\int_0^t |\operatorname{Re}\left(e^{-\mathrm{i}\mathcal{B}_n(l)} v_n - v_n - \left[e^{-\mathrm{i}\mathcal{B}(l)} v - v\right], \psi\right)_H|^2 \mathrm{d}s \\ &\quad \leq 2\|\psi\|_{L^2}^2 b_{L^2} |l|^2 \left(\|v_n\|_{L^2(0,t;H)}^2 + \|v\|_{L^2(0,t;H)}^2 \right) \lesssim |l|^2 \in L^1(\mathbb{R}^N; \nu) \end{aligned} \tag{6.45}$$

by Lemma 6.8 and (6.7), we get

$$\int_{\{|l|\leq 1\}} \int_0^t |\operatorname{Re}\left(e^{-\mathrm{i}\mathcal{B}_n(l)} v_n - v_n - \left[e^{-\mathrm{i}\mathcal{B}(l)} v - v\right], \psi\right)_H|^2 \mathrm{d}s\nu(\mathrm{d}l) \to 0$$

as $n \to \infty$ almost surely for all $t \in [0,T]$. For $r > 1$, we employ similar estimates as in (6.45) for

$$\begin{aligned} &\tilde{\mathbb{E}} \int_0^T \left(\int_{\{|l|\leq 1\}} \int_0^t |\operatorname{Re}\left(e^{-\mathrm{i}\mathcal{B}_n(l)} v_n - v_n - \left[e^{-\mathrm{i}\mathcal{B}(l)} v - v\right], \psi\right)_H|^2 \mathrm{d}s\nu(\mathrm{d}l) \right)^r \mathrm{d}r \\ &\quad \lesssim \|\psi\|_{L^2}^{2r} \tilde{\mathbb{E}} \int_0^T \left(\|v_n\|_{L^2(0,t;H)}^2 + \|v\|_{L^2(0,t;H)}^2 \right)^r \mathrm{d}r \\ &\quad \lesssim \|\psi\|_{L^2}^{2r} T^{1+r} \tilde{\mathbb{E}} \left[\sup_{s\in[0,T]} \left(\|v_n\|_H^2 + \|v\|_H^2 \right)^r \right] < \infty \end{aligned}$$

and thus, we get (6.40) by Vitali's Theorem.

ad (6.41). From (6.44),

$$\begin{aligned}\int_0^t &|\operatorname{Re}\left(\mathrm{i}\mathcal{B}_n(l)v_n-\mathrm{i}\mathcal{B}(l)v,\psi\right)_H|\mathrm{d}s\\ &\le\|\psi\|_{L^2}\left(\|\mathcal{B}_n(l)(v_n-v)\|_{L^1(0,t;H)}+\|\left[\mathcal{B}_n(l)-\mathcal{B}(l)\right]v\|_{L^1(0,t;H)}\right)\\ &\le\|\psi\|_{L^2}t^{\frac12}\left(\|\mathcal{B}(l)\|_{\mathcal{L}(H)}\|v_n-v\|_{L^2(0,t;H)}+\|\left[\mathcal{B}_n(l)-\mathcal{B}(l)\right]v\|_{L^2(0,t;H)}\right)\xrightarrow{n\to\infty}0\end{aligned}$$

and the bound

$$\begin{aligned}\int_0^t|\operatorname{Re}\left(e^{-\mathrm{i}\mathcal{B}_n(l)}v_n(s)-v_n(s)+\mathrm{i}\mathcal{B}_n(l)v_n(s),\psi\right)_H|\mathrm{d}s&\le\frac12 b_{L^2}t^{\frac12}\|\psi\|_{L^2}|l|^2\|v_n\|^2_{L^2(0,t;H)}\\ &\lesssim_{\omega,t}|l|^2\in L^1(\mathbb{R}^N;\nu)\end{aligned}$$

by Lemma 6.8, we infer (6.41) pointwise in $\bar\Omega\times[0,T]$. The $L^2(\bar\Omega\times[0,T])$-convergence follows similarly as in the previous step by a Vitali-argument based on the uniform bounds on v_n, $n\in\mathbb{N}$. □

Finally, we are ready to summarize our results and obtain the existence of a martingale solution.

Proof of Theorem 6.6. Let us define the maps

$$\begin{aligned}M_{n,\psi}(w,t)=&P_nu_0-\mathrm{i}\int_0^t\operatorname{Re}\langle Aw(s)+P_nF(w(s)),\psi\rangle\mathrm{d}s\\ &+\int_0^t\int_{\{|l|\le1\}}\operatorname{Re}\left(e^{-\mathrm{i}\mathcal{B}_n(l)}w(s-)-w(s-),\psi\right)_H\tilde{\bar\eta}(\mathrm{d}s,\mathrm{d}l)\\ &+\int_0^t\int_{\{|l|\le1\}}\operatorname{Re}\left(e^{-\mathrm{i}\mathcal{B}_n(l)}w(s)-w(s)+\mathrm{i}\mathcal{B}_n(l)w(s),\psi\right)_H\nu(\mathrm{d}l)\mathrm{d}s;\end{aligned}$$

$$\begin{aligned}M_{\psi}(w,t)=&u_0-\mathrm{i}\int_0^t\operatorname{Re}\langle Aw(s)+F(w(s)),\psi\rangle\mathrm{d}s\\ &+\int_0^t\int_{\{|l|\le1\}}\operatorname{Re}\left(e^{-\mathrm{i}\mathcal{B}(l)}w(s-)-w(s-),\psi\right)_H\tilde{\bar\eta}(\mathrm{d}s,\mathrm{d}l)\\ &+\int_0^t\int_{\{|l|\le1\}}\operatorname{Re}\left(e^{-\mathrm{i}\mathcal{B}(l)}w(s)-w(s)+\mathrm{i}\mathcal{B}(l)w(s),\psi\right)_H\nu(\mathrm{d}l)\mathrm{d}s.\end{aligned}$$

The results of Lemma 6.16 can be summarized as

$$\operatorname{Re}\left(v_n,\psi\right)_H-M_{n,\psi}(v_n,\cdot)\to\operatorname{Re}\left(v,\psi\right)_H-M_\psi(v,\cdot),\qquad n\to\infty,$$

in $L^2(\bar\Omega\times[0,T])$ for all $\psi\in E_A$ and from the definition of u_n via the Galerkin equation, we infer $\operatorname{Re}\left(u_n(t),\psi\right)_H=M_{n,\psi}(u_n,t)$ almost surely for all $t\in[0,T]$. Due to the identity

$$\mathrm{Leb}_{[0,T]}\otimes\mathbb{P}^{u_n}=\mathrm{Leb}_{[0,T]}\otimes\bar{\mathbb{P}}^{v_n},$$

we obtain

$$\tilde{\mathbb{E}}\int_0^T|\operatorname{Re}\left(v(t),\psi\right)_H-M_\psi(v,t)|^2\mathrm{d}t=\lim_{n\to\infty}\tilde{\mathbb{E}}\int_0^T|\operatorname{Re}\left(v_n(t),\psi\right)_H-M_{n,\psi}(v_n,t)|^2\mathrm{d}t$$

$$= \lim_{n\to\infty} \mathbb{E} \int_0^T |\operatorname{Re}\left(u_n(t),\psi\right)_H - M_{n,\psi}(u_n,t)|^2 \mathrm{d}t = 0$$

and thus,

$$\bar{\mathbb{P}}\left\{\operatorname{Re}\left(v(t),\psi\right)_H = M_\psi(v,t) \quad \text{f.a.a. } t\in[0,T]\right\} = 1.$$

Since both $\operatorname{Re}\left(v,\psi\right)_H$ and $M_\psi(v,\cdot)$ are almost surely in $\mathbb{D}([0,T])$, we obtain

$$\bar{\mathbb{P}}\left\{\operatorname{Re}\left(v(t),\psi\right)_H = M_\psi(v,t) \quad \forall t\in[0,T]\right\} = 1,$$

which means that $\left(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathbb{P}},\bar{\eta},\bar{\mathbb{F}},u\right)$ is a martingale solution to (6.10). □

6.4. Examples

In this section, we collect concrete settings which are covered by the general existence result from Theorem 6.6. We skip the proofs since checking the Assumptions 6.1, 6.2 and 6.3 is similar to Section 4.4.

Corollary 6.17. *Suppose that a) or b) or c) is true.*

a) M compact manifold, $A=-\Delta_g$, $E_A = H^1(M)$,

b) Let $M\subset\mathbb{R}^d$ be a bounded domain and $A=-\Delta_D$ be the Dirichlet-Laplacian, $E_A=H_0^1(M)$,

c) Let $M\subset\mathbb{R}^d$ be a bounded Lipschitz domain, $A=-\Delta_N$ be the Neumann-Laplacian and $E_A=H^1(M)$.

Choose the nonlinearity from $i)$ or $ii)$.

i) $F(u)=|u|^{\alpha-1}u$ with $\alpha\in\left(1,1+\frac{4}{(d-2)_+}\right)$,

ii) $F(u)=-|u|^{\alpha-1}u$ with $\alpha\in\left(1,1+\frac{4}{d}\right)$.

Set $B_m x = e_m x$ for $x\in H$ and $m=1,\dots,N$, with real-valued functions

$$e_m \in F := \begin{cases} H^{1,d}(M)\cap L^\infty(M), & d\geq 3,\\ H^{1,q}(M), & d=2,\\ H^1(M), & d=1,\end{cases} \tag{6.46}$$

for some $q>2$ in the case $d=2$. Then, the problem

$$\begin{cases} \mathrm{d}u(t) = \left(-\mathrm{i}Au(t)-\mathrm{i}F(u(t))\right)dt - \mathrm{i}\sum_{m=1}^N B_m u(t)\diamond dL_m(t),\\ u(0)=u_0\in E_A,\end{cases} \tag{6.47}$$

has a martingale solution which satisfies $u\in\mathbb{D}_w\left([0,T],E_A\right)$ almost surely and

$$u\in L^q(\tilde{\Omega},L^\infty(0,T;E_A))$$

for all $q\in[1,\infty)$.

Additionally to the stochastic NLS, we can also cover the fractional NLS with the Laplacians replaced by their fractional powers.

Corollary 6.18. *Choose one of the settings a), b) or c) in Corollary. Let $\beta > 0$ and suppose that we have either $i)$ or $ii)$ below.*

i) $F(u) = |u|^{\alpha-1}u$ *with* $\alpha \in \left(1, 1 + \frac{4\beta}{(d-2\beta)_+}\right)$,

ii) $F(u) = -|u|^{\alpha-1}u$ *with* $\alpha \in \left(1, 1 + \frac{4\beta}{d}\right)$.

Choose B_m for $m = 1, \dots, N$ as in Assumption 6.4. Then, the problem

$$\begin{cases} \mathrm{d}u(t) = \left(-\mathrm{i}A^\beta u(t) - \mathrm{i}F(u(t))\right) dt - \mathrm{i}\displaystyle\sum_{m=1}^N B_m u(t) \diamond dL_m(t), \\ u(0) = u_0 \in X^{\frac{\beta}{2}}, \end{cases} \tag{6.48}$$

has a martingale solution which satisfies $u \in \mathbb{D}_w([0,T], X^{\frac{\beta}{2}})$ almost surely and

$$u \in L^q(\tilde{\Omega}, L^\infty(0,T; X^{\frac{\beta}{2}}))$$

for all $q \in [1, \infty)$.

A. Appendix

In the appendix, we provide additional material which is frequently used throughout this thesis. We restrict ourselves to the results we actually need and do not aim for a complete presentation. For most of the proofs and further details, we give references to the literature.

A.1. Stochastic integration with respect to cylindrical Wiener processes

In this section, we introduce the stochastic integral with the properties we will need in this thesis. Because the results are classical, we omit most of the proofs and give references to the literature. Instead of presenting the theory in the most general case, namely UMD-Banach spaces, we restrict ourselves to prominent special cases: mixed L^p-spaces, i.e. spaces of the form $L^q(M_1, L^p(M_2))$, and Hilbert spaces. Their additional structure allows to build a stronger stochastic integration theory which will be useful later on.

For the sake of completeness, we start with the definition of the standard real-valued Brownian motion.

Definition A.1. An $\mathbb{F}$-adapted process $\beta : [0,\infty) \times \Omega \to \mathbb{R}$ is called *real-valued Brownian motion* relative to $\mathbb{F}$, if the following conditions are satisfied:

i) $\beta(0) = 0$ almost surely;

ii) for $0 \leq s < t$, the increment $\beta(t) - \beta(s)$ is Gaussian with mean 0 and variance $t - s$ and $\mathcal{F}_s-$independent;

iii) for almost all $\omega \in \Omega$, the path $[0,\infty) \in t \mapsto \beta(\omega, t)$ is continuous.

Throughout this section, we fix a real separable Hilbert space Y with ONB $(f_m)_{m\in\mathbb{N}}$. Next, we generalize the notion of a Brownian motion to Y.

Definition A.2. Let Y be a real separable Hilbert space. A family $W = (W(t))_{t\geq 0}$ of bounded linear operators from Y to $L^2(\Omega)$ is called *Y-cylindrical Wiener process* relative to $\mathbb{F}$ if the following conditions are satisfied.

i) For all $y \in Y$, the process $(W(t)y)_{t\geq 0}$ is a real-valued Brownian motion relative to $\mathbb{F}$.

ii) For all $s, t \geq 0$ and $y_1, y_2 \in Y$, we have

$$\mathbb{E}\left[W(s)y_1 W(t)y_2\right] = (s \wedge t)\left(y_1, y_2\right)_Y.$$

To illustrate the notion of a Y-cylindrical Wiener process, we recall the following example from [121], Example 6.12.

Example A.3. For independent real-valued Brownian motions β_m, $m \in \mathbb{N}$, a cylindrical Wiener process is given by

$$W(t)y = \sum_{m=1}^{\infty} \left(y, f_m\right)_Y \beta_m(t), \qquad t \geq 0, \quad y \in Y.$$

An important way to obtain a Y-cylindrical Wiener process is stated in the next Proposition. As a preparation, we set

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \quad \forall t \geq 0\}$$

for any stopping time τ and call $\mathcal{F}_\tau$ *σ-algebra of the τ-past.*

Proposition A.4. *Let $W = (W(t))_{t\geq 0}$ be a Y-cylindrical Wiener process relative to $\mathbb{F}$ and τ be an almost surely finite $\mathbb{F}$-stopping time. Then,*

$$W^\tau(t) := W(\tau + t) - W(\tau), \qquad t \geq 0,$$

defines a Y-cylindrical Wiener process relative to $\mathbb{F}^\tau := (\mathcal{F}_{t+\tau})_{t\geq 0}$ which is independent of $\mathcal{F}_\tau$.

Proof. Suppose that we are given two independent real-valued Brownian motions $(\beta_1(t))_{t\geq 0}$ and $(\beta_2(t))_{t\geq 0}$ relative to $\mathcal{F}$. Then, Theorem 6.16 in [74] implies that

$$\beta_1^\tau(t) := \beta_1(\tau + t) - \beta_1(\tau), \qquad \beta_2^\tau(t) := \beta_2(\tau + t) - \beta_2(\tau), \qquad t \geq 0,$$

define independent Brownian motions relative to $\mathcal{F}_\tau$. Let $(f_m)_{m\in\mathbb{N}}$ be an ONB of Y and $y_1, y_2 \in Y$. We compute

$$\mathbb{E}\left[W^\tau(s)y_1 W^\tau(t)y_2\right] = \sum_{m=1}^{\infty}\sum_{l=1}^{\infty} \left(y_1, f_m\right)_Y \left(y_2, f_l\right)_Y \mathbb{E}\left[W^\tau(s)f_m W^\tau(t)f_l\right].$$

For all $m \neq l \in \mathbb{N}$, Wf_m and Wf_l are independent real-valued Brownian motions, since W is a Y-cylindrical Wiener process. In particular, we get

$$\mathbb{E}\left[W^\tau(s)f_m W^\tau(t)f_l\right] = \delta_{ml}\,(t \wedge s)$$

as a consequence of the reasoning from above. This leads to

$$\mathbb{E}\left[W^\tau(s)y_1 W^\tau(t)y_2\right] = (s \wedge t)\left(y_1, y_2\right)_Y$$

and thus, we have proved that W^τ is a cylindrical Wiener process. □

Next, we define the following notions for Banach-space-valued and operator-valued stochastic processes.

Definition A.5. Let E be a real Banach space.

a) A process $X : [0,T] \times \Omega \to E$ is called $\mathbb{F}$-*adapted*, if $X(t)$ is strongly $\mathcal{F}_t$-measurable in E for all $t \in [0,T]$.

b) A stochastic process $X : [0,T] \times \Omega \to E$ is called $\mathbb{F}$-*predictable*, if the map

$$[0,t] \times \Omega \ni (s,\omega) \to X(s,\omega) \in E$$

is strongly $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable for all $t \in [0,T]$.

c) A process $B : [0, T] \times \Omega \to \mathcal{L}(Y, E)$ is called *elementary*, if it has the form

$$B(t, \omega) = \sum_{n=0}^{N} \sum_{m=1}^{M} \mathbf{1}_{(t_{n-1}, t_n] \times A_{m,n}}(t, \omega) \sum_{k=1}^{K} y_k \otimes x_{k,m,n} \tag{A.1}$$

with $0 \leq t_0 < \cdots < t_n \leq T$, disjoint sets $A_{1,n}, \ldots A_{M,n} \in \mathcal{F}_{t_{n-1}}$ for $n = 0, \ldots, N$, orthonormal vectors $y_1, \ldots, y_K \in Y$ and $x_{k,m,n} \in E$. Here, we denoted $(t_{-1}, t_0] := \{0\}$ and $\mathcal{F}_{t_{-1}} := \mathcal{F}_0$.

d) A process $B : [0, T] \times \Omega \to \mathcal{L}(Y, E)$ is called *Y-strongly measurable*, if By is strongly measurable in E for all $y \in Y$.

e) A Y-strongly measurable process $B : [0, T] \times \Omega \to \mathcal{L}(Y, E)$ is called *$\mathbb{F}$-adapted*, if By is $\mathbb{F}$-adapted.

We continue with a classical Lemma on the identity of two stochastic processes.

Lemma A.6. *Let E be a separable Banach space, $I \subset \mathbb{R}$ an interval and $X, Y : I \times \Omega \to E$ stochastic processes with almost surely right-continuous paths and X is a version of Y, i.e. $\mathbb{P}(X(t) = Y(t)) = 1$ for all $t \in I$. Then, X and Y are* indistinguishable, *i.e.*

$$\mathbb{P}(X(t) = Y(t) \quad \forall t \in I) = 1.$$

Proof. See [79], Lemma 21.5. □

The stochastic integral for elementary processes is the content of the next definition.

Definition A.7. For a Y-cylindrical Wiener process W and an elementary process B with representation (A.1), we define the *stochastic integral* as the E-valued random variable

$$\int_0^T B \mathrm{d}W := \sum_{n=0}^{N} \sum_{m=1}^{M} \mathbf{1}_{A_{m,n}} \sum_{k=1}^{K} \left(W(t_n) y_k - W(t_{n-1}) y_k\right) x_{k,m,n}.$$

Obviously, the stochastic integral defines a linear operator. To extend the integral to a class of integrands which is suitable for applications, we seek for estimates leading to a definition of $\int_0^T B \mathrm{d}W$ on the closure of the space of elementary processes by continuous extension. Of course, these estimates depend on the concrete Banach space E.

A.1.1. Stochastic integration in Hilbert spaces

Let H be a real separable Hilbert space with ONB $(h_m)_{m \in \mathbb{N}}$. In the following, we identify the right spaces to extend the H-valued stochastic integral for elementary processes from Definition A.7 and state some properties which will be relevant later on. The presentation is close to [40], where most of the proofs can be found.

We start with a short repetition on nuclear and Hilbert-Schmidt operators. Recall that an operator $A \in \mathcal{L}(H)$ is called *nuclear* or *trace class* if there are sequences $(x_m)_{m \in \mathbb{N}} \subset H$ and $(y_m)_{m \in \mathbb{N}} \subset H$ such that

$$\sum_{m=1}^{\infty} \|x_m\|_H \|y_m\|_H < \infty$$

and A can be written as

$$Ax = \sum_{m=1}^{\infty} \big(x, x_m\big)_H y_m, \qquad x \in H.$$

The space of all nuclear operator is a Banach space with the norm

$$\|A\|_{\mathcal{N}(H)} = \inf \left\{ \sum_{m=1}^{\infty} \|x_m\|_H \|y_m\|_H : Ax = \sum_{m=1}^{\infty} \big(x, x_m\big)_H y_m \right\}$$

and will be denoted by $\mathcal{N}(H)$. For $A \in \mathcal{N}(H)$,

$$H \times H \ni (x, y) \mapsto \big(Ax, y\big)_H \in \mathbb{K}$$

defines a continuous bilinear form and A is called *positive* if A is symmetric and $\big(Ax, x\big)_H \geq 0$ for all $x \in H$. In particular, a function $V : [0, T] \to \mathcal{N}(H)$ is called *increasing* if $V(t) - V(s)$ is positive for all $t \geq s \geq 0$. For $A \in \mathcal{N}(H)$, we define the *trace* by

$$\operatorname{tr}(A) := \sum_{m=1}^{\infty} \big(Ah_m, h_m\big)_H.$$

The trace is independent of the ONB and a nonnegative operator A is nuclear if and only if the trace is finite. In this case, we have $\operatorname{tr}(A) = \|A\|_{\mathcal{N}(H)}$. Given another separable Hilbert space E with ONB $(e_m)_{m\in\mathbb{N}}$, an operator $A \in \mathcal{L}(E, H)$ is called *Hilbert-Schmidt* if

$$\|A\|_{\mathrm{HS}(E,H)} := \left(\sum_{m=1}^{\infty} \|Ae_m\|_H^2 \right)^{\frac{1}{2}} < \infty.$$

The space $\mathrm{HS}(E, H)$ of all Hilbert-Schmidt operators is a separable Hilbert space with ONB $\big(h_j \otimes e_m\big)_{j,m\in\mathbb{N}}$ and inner product

$$\big(A, B\big)_{\mathrm{HS}} = \sum_{m=1}^{\infty} \big(Ae_m, Be_m\big)_H.$$

Now, we are ready for the stochastic integration theory. First, we fix the space of admissible integrands.

Definition A.8. Let $r \in (1, \infty)$. Then, a random variable $B \in L^r(\Omega, L^2(0, T; \mathrm{HS}(Y, H)))$ is called *L^r-stochastically integrable in H* if it is represented by a Y-strongly measurable and $\mathbb{F}$-adapted process $B : [0, T] \times \Omega \to \mathrm{HS}(Y, H)$. The space of stochastically integrable random variables is called $\mathcal{M}^r_{\mathbb{F},Y}(0, T; H)$.

As before, we fix a real separable Hilbert space Y with ONB $(f_m)_{m\in\mathbb{N}}$ and a Y-cylindrical Wiener process W.

Theorem A.9. *a) For all elementary processes $B : [0, T] \times \Omega \to \mathrm{HS}(Y, H)$, we have the isometry*

$$\mathbb{E} \left\| \int_0^T B \mathrm{d}W \right\|_H^2 = \mathbb{E} \int_0^T \|B(s)\|^2_{\mathrm{HS}(Y,H)} \mathrm{d}s.$$

b) *Let $r \in (1,\infty)$. The space $\mathcal{M}^r_{\mathbb{F},Y}(0,T;H)$ is a Banach space and the set of elementary processes is dense.*

Proof. See [40], Section 4.2.1 and in particular equation (4.30) for a). In view of the fact that $\gamma\big(L^2(0,T;Y),H\big)$ is isomorphic to $L^2(0,T;\mathrm{HS}(Y,H))$, the assertion b) is a consequence of Propositions 2.11 and 2.12 in [124]. □

By continuous extension, Theorem A.9 leads to the definition of the stochastic integral for $B \in \mathcal{M}^2_{\mathbb{F},Y}(0,T;H)$ and the *Itô isometry*

$$\mathbb{E}\left\|\int_0^T B\mathrm{d}W\right\|_H^2 = \mathbb{E}\int_0^T \|B(s)\|^2_{\mathrm{HS}(Y,H)}\mathrm{d}s, \qquad B \in \mathcal{M}^2_{\mathbb{F},Y}(0,T;H). \tag{A.2}$$

The next step is to explore the properties of the process

$$I(t) := \int_0^T \mathbf{1}_{[0,t]}B\mathrm{d}W, \qquad t \in [0,T], \tag{A.3}$$

associated to the stochastic integral. To prepare the following Theorem, we introduce the *quadratic variation* of an H-valued continuous square-integrable martingale.

Definition A.10. Let M be an H-valued continuous $L^2(\Omega)$-martingale with $M(0) = 0$. Then, an $\mathcal{N}(H)$-valued continuous adapted and increasing process $(V(t))_{t\in[0,T]}$ with $V(0) = 0$ is called *quadratic variation* if for all $h_1, h_2 \in H$, the process defined by

$$\big(M(t),h_1\big)_H\big(M(t),h_2\big)_H - \big(V(t)h_1,h_2\big)_H, \qquad t \in [0,T],$$

is an $\mathbb{F}$-martingale. We denote $\langle\langle M\rangle\rangle_t := V(t)$ for $t \in [0,T]$.

By [40], Proposition 3.13, the quadratic variation is well defined and for each $t \in [0,T]$, the quadratic variation $\langle\langle M\rangle\rangle_t$ is a symmetric operator. It can be constructed as

$$\langle\langle M\rangle\rangle_t = \sum_{i,j=1}^{\infty}\langle\langle M_i, M_j\rangle\rangle_t h_i \otimes h_j,$$

where $\langle\langle M_i, M_j\rangle\rangle_t$ denotes the classical scalar-valued cross quadratic variation of M_i and M_j for $M_i := \big(M,h_i\big)_H$. We continue with the properties of the process induced by the stochastic integral.

Theorem A.11 (Properties of the integral process). *a) For $B \in \mathcal{M}^2_{\mathbb{F},Y}(0,T;H)$, the stochastic process $(I(t))_{t\in[0,T]}$ from* (A.3) *is an $\mathbb{F}$-martingale with a continuous version $(M(t))_{t\in[0,T]}$ and quadratic variation*

$$\langle\langle M\rangle\rangle_t = \int_0^t B(s)^*B(s)\mathrm{d}s. \tag{A.4}$$

We denote $\int_0^t B\mathrm{d}W := M(t)$ for $t \in [0,T]$.

b) *Let $r \in [1,\infty)$. Then, we have the Burkholder-Davis-Gundy inequality*

$$\mathbb{E}\sup_{t\in[0,T]}\left\|\int_0^t B\mathrm{d}W\right\|_H^r \eqsim \mathbb{E}\left(\mathrm{tr}\langle\langle M\rangle\rangle_T\right)^{\frac{r}{2}} = \mathbb{E}\left(\int_0^T \|B(s)\|^2_{\mathrm{HS}(Y,H)}\mathrm{d}s\right)^{\frac{r}{2}} \tag{A.5}$$

for all $B \in \mathcal{M}^r_{\mathbb{F},Y}(0,T;H)$.

Proof. The martingale property and quadratic variation can be found in [40], Theorem 4.27 and consequently, the norm equivalence in b) follows from Doob's maximal inequality and (A.2) in the case $r = 2$. For general $r \in (1, \infty)$, the assertion is contained as a special case in [124], Theorem 4.4. For $r = 1$, we refer to [40], Theorem 3.15, and [105], p. 17-18, for the original source with proof. Finally, we use a) to calculate

$$\operatorname{tr}\langle\langle M\rangle\rangle_T = \sum_{m=1}^{\infty} \int_0^T \left(f_m, B^*(s)B(s)f_m\right)_Y \mathrm{d}s = \sum_{m=1}^{\infty} \int_0^T \|B(s)f_m\|_H^2 \mathrm{d}s = \int_0^T \|B(s)\|_{\mathrm{HS}(Y,H)}^2 \mathrm{d}s.$$

□

The next result is sort of a converse of the previous Theorem. It states that every square integrable continuous martingale with quadratic variation of the structure (A.4) can be represented as a stochastic integral.

Theorem A.12 (Martingale Representation Theorem)**.** *Let M be a square integrable continuous martingale with values in H on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$. Assume that there is $B \in \mathcal{M}^2_{\mathbb{F},Y}(0,T;H)$, such that the quadratic variation of M is given by*

$$\langle\langle M\rangle\rangle_t = \int_0^t B(s)B(s)^* \mathrm{d}s, \qquad t \in [0,T].$$

Then, there are a another stochastic basis $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{\mathbb{F}}\right)$ and a Y-cylindrical Wiener process W in Y defined on $\left(\Omega \times \tilde{\Omega}, \mathcal{F} \otimes \tilde{\mathcal{F}}, \mathbb{P} \otimes \tilde{\mathbb{P}}\right)$ adapted to $\left(\mathcal{F}_t \otimes \tilde{\mathcal{F}}_t\right)_{t\in[0,T]}$ with

$$M(t,\omega,\tilde{\omega}) = \left(\int_0^t B \mathrm{d}W\right)(\omega,\tilde{\omega})$$

for $t \in [0,T]$ and $(\omega,\tilde{\omega}) \in \Omega \times \tilde{\Omega}$, where we denote

$$M(t,\omega,\tilde{\omega}) := M(t,\omega), \qquad B(t,\omega,\tilde{\omega}) := B(t,\omega).$$

Proof. See [40], Theorem 8.2. □

In the next Proposition, we present the stochastic convolution with a contraction semigroup. In the special case of the Schrödinger group, the study of the stochastic convolution will be continued in the following sections.

Proposition A.13. *Assume that A generates a contraction semigroup $(U(t))_{t\geq 0}$ in H and let $B \in \mathcal{M}^r_{\mathbb{F},Y}(0,T;H)$ for some $r \geq 2$. Then, the process defined by*

$$K_{Stoch}B(t) := \int_0^t U(t-s)B(s)\mathrm{d}W(s), \qquad t \in [0,T],$$

has a continuous version which satisfies the estimate

$$\|K_{Stoch}B\|_{L^r(\Omega,C([0,T],H))} \lesssim \|B\|_{L^r(\Omega,L^2(0,T;\mathrm{HS}(Y,H)))}.$$

Proof. We refer to [40], Theorem 6.10, for the case $r = 2$. An extension to $r \geq 2$ is straightforward. □

By the Theorem of Hille, we can interchange the Bochner-integral with a closed operator A. A similar result is true for the stochastic integral.

Proposition A.14. *Let $r \in (1,\infty)$ and A be a closed operator on H with domain $\mathcal{D}(A)$ and $B \in \mathcal{M}^r_{\mathbb{F},Y}(0,T;H)$ such that $B(t) \in \mathcal{D}(A)$ almost surely for all $t \in [0,T]$ and $AB(\cdot) \in \mathcal{M}^r_{\mathbb{F},Y}(0,T;H)$. Then,*

$$A\int_0^t B(s)\mathrm{d}W(s) = \int_0^t AB(s)\mathrm{d}W(s) \tag{A.6}$$

almost surely for all $t \in [0,T]$.

Proof. By [40], Proposition 4.30, (A.6) holds for all $t \in [0,T]$ almost surely. In view of Lemma A.6, this is enough to prove the assertion if the processes on the LHS and RHS are continuous in H. This is true for the RHS by Theorem A.11 and due to the fact that $\int_0^\cdot B(s)\mathrm{d}W(s)$ is continuous in $\mathcal{D}(A)$ equipped with the graph norm $\|\cdot\|_A$ and $A:\mathcal{D}(A)\to H$ is a bounded operator, the LHS is also continuous. □

At some, but not many points in this thesis, it is important to extend the stochastic integral to a larger set of integrands which are not integrable in Ω. This is based on a localization argument and leads to the stochastic integral on $L^0_{\mathbb{F}}(\Omega, L^2(0,T;\mathrm{HS}(Y,H)))$. In the following definition, we explain this notion.

Definition A.15. Let E be a Banach space.

a) Then, we denote the space of all equivalence classes of strongly measurable random variables in E by $L^0(\Omega,E)$. Endowed with the metric

$$d(X,Y) := \mathbb{E}\left[\|X-Y\|_E \wedge 1\right],$$

$L^0(\Omega,E)$ is a complete metric space and convergence in this metric coincides with convergence in probability.

b) Let $p \in [1,\infty)$. The closure of the space of E-valued elementary processes in $L^0(\Omega, L^p(0,T;E))$ is called $L^0_{\mathbb{F}}(\Omega, L^p(0,T;E))$.

For the localization procedure, we refer to [40], page 99-100, in the Hilbert space case and to [124], Section 5, for a presentation with more details, but in a more general setting. We close this section with a Lemma which characterizes the martingale property if the filtration has a specific form. This will be useful in chapter 4.

Lemma A.16. *Let H_1, H_2 be separable real Hilbert spaces and $v:[0,T]\times\Omega\to H_1$ be a stochastic process with continuous paths. We set*

$$\mathcal{F}_{t,v} := \sigma\left(v(s) : s\in[0,t]\right), \qquad t\in[0,T].$$

Then, the following assertions hold:

a) We have $\mathcal{F}_{t,v} := \sigma\left(v|_{[0,t]}\right)$ for $t \in [0,T]$, where $v|_{[0,t]}$ is viewed as a random variable in $C([0,T],H_1)$.

b) Let $M : [0,T] \times \Omega \to H_2$ be a square integrable continuous process and $V : [0,T] \times \Omega \to \mathcal{N}(H_2)$ be a continuous, integrable and increasing process with $V(0) = 0$. Suppose that M and V are adapted to $(\mathcal{F}_{t,v})_{t\in[0,T]}$ and set

$$F(t,s,\psi,\varphi) := \big(M(t),\psi\big)_{H_2}\big(M(t),\varphi\big)_{H_2} - \big(M(s),\psi\big)_{H_2}\big(M(s),\varphi\big)_{H_2} - \big(V(t)\psi - V(s)\psi,\varphi\big)_{H_2}$$

for $s,t \in [0,T]$ and $\psi,\varphi \in H_2$. Then, M is a martingale w.r.t. $(\mathcal{F}_{t,v})_{t\in[0,T]}$ with quadratic variation V if and only if

$$\mathbb{E}\left[\big(M(t)-M(s),\psi\big)_{H_2} h(v|_{[0,s]})\right] = 0, \qquad \mathbb{E}\left[F(t,s,\psi,\varphi)h(v|_{[0,s]})\right] = 0 \tag{A.7}$$

for all $s,t \in [0,T]$ with $s \le t$ and $\psi,\varphi \in H_2$ and bounded, continuous functions h on $C([0,T],H_1)$.

Proof. ad a). Fix $t \in [0,T]$. Then, the linear span L of all functionals

$$H_1 \ni u \mapsto \big(u(s),x\big)_{H_1} \in \mathbb{K}, \qquad x \in H_1, \quad \|x\|_{H_1} \le 1, \qquad s \in [0,t],$$

is norming for the separable Banach space $C([0,t],H_1)$. In particular, the Pettis measurability Theorem yields that $v|_{[0,t]}$ is strongly $\mathcal{F}_{t,v}$-measurable in $C([0,t],H_1)$ if and only if $\big(v(s),x\big)_{H_1}$ is $\mathcal{F}_{t,v}$-measurable for all $x \in H_1$ with $\|x\|_{H_1} \le 1$ and $s \in [0,t]$. But this is equivalent to the strong $\mathcal{F}_{t,v}$-measurability in H_1 of $v(s)$ for all $s \in [0,t]$.

ad b) M is a martingale if and only if $\big(M,\psi\big)_{H_2}$ is a martingale for all $\psi \in H_2$ since one can interchange conditional expectations with bounded operators. In view of the definition of the quadratic variation and the conditional expectation, we have to show that (A.7) is equivalent to

$$\mathbb{E}\left[\big(M(t)-M(s),\psi\big)_{H_2}\mathbf{1}_{A_s}\right] = 0, \qquad \mathbb{E}\left[F(t,s,\psi,\varphi)\mathbf{1}_{A_s}\right] = 0 \tag{A.8}$$

for all $s,t \in [0,T]$ with $s \le t$ and $\psi,\varphi \in H_2$ and $A_s \in \mathcal{F}_{s,v}$. This reduces to the equivalence of

$$\mathbb{E}\left[X\mathbf{1}_B(v|_{[0,s]})\right] = 0, \qquad B \in \mathcal{B}\left(C([0,s],H_1)\right),$$

and

$$\mathbb{E}\left[Xh(v|_{[0,s]})\right] = 0, \qquad h \in C_b(C([0,s],H_1)),$$

where $X \in L^1(\Omega)$ is an arbitrary scalar valued random variable. Here, we used that $A_s \in \mathcal{F}_{s,v}$ if and only if there is a Borel set $B \subset C([0,t],H_1)$ such that $A_s = \{v|_{[0,s]} \in B\}$. This is an implication of part a) of the Lemma.

The first direction follows from the fact that continuous functions can be approximated pointwise by simple ones. For the second direction, let us first assume that B is closed. Then, Urysohn's Lemma implies that there is a sequence $(h_n)_{n\in\mathbb{N}}$ of uniformly bounded, continuous functions on $C([0,s],H_1)$ with $h_n(u) \to \mathbf{1}_B(u)$ for all $u \in C([0,s],H_1)$. Thus, the claim for closed B follows from Lebesgue's convergence Theorem.

For some system of sets $\tilde{\mathcal{D}}$, we denote the Dynkin system generated by $\tilde{\mathcal{D}}$ as $\delta(\tilde{\mathcal{D}})$. It is not hard to show that

$$\mathcal{D} := \left\{B \in \mathcal{B}(C([0,s],H_1)) : \quad \mathbb{E}\left[X\mathbf{1}_B(v|_{[0,s]})\right] = 0\right\}$$

is a Dynkin system. Hence, we get $\mathbb{E}\left[X\mathbf{1}_B(v|_{[0,s]})\right] = 0$ for all Borel sets $B \subset C([0,s],H_1)$, since

$$\begin{aligned}\mathcal{B}(C([0,s],H_1)) &= \sigma\left(\{B \subset C([0,s],H_1) : B \text{ closed}\}\right)\\ &= \delta\left(\{B \subset C([0,s],H_1) : B \text{ closed}\}\right)\\ &\subset \delta(\mathcal{D}) = \mathcal{D} \subset \mathcal{B}(C([0,s],H_1)).\end{aligned}$$

□

A.1.2. Stochastic integration in mixed L^p-spaces

Throughout this section, M_1 and M_2 are supposed to be σ-finite measure spaces and as above, we fix a real separable Hilbert space Y with ONB $(f_m)_{m\in\mathbb{N}}$ and a Y-cylindrical Wiener process W. We present the essential elements of the stochastic integration theory in the spaces $L^q(M_1, L^p(M_2))$ for $q,p \in (1,\infty)$. Often, we will abbreviate $L^qL^p := L^q(M_1, L^p(M_2))$. The particular feature of this theory is the fact that it contains a stronger version of the BDG-inequality which will be useful in the proof of Strichartz estimates for the stochastic convolution. For a more detailed presentation of stochastic integration in mixed L^p-spaces, we refer to the dissertation [6] by Antoni.

Once again, we start with the notion of *stochastic integrability* of an operator-valued process. It is motivated by a characterization of stochastic integrability in the more general case of UMD Banach function space, see Corollary 3.11 in [124].

Definition A.17. Let $p,q \in (1,\infty)$ and $r \in (1,\infty)$. Then, an Y-strongly measurable and $\mathbb{F}$-adapted $\mathcal{L}(Y, L^q(M_1, L^p(M_2,\mathbb{R})))$-valued process $B = (B(s))_{s\in[0,T]}$ is called *L^r-stochastically integrable* in $L^q(M_1, L^p(M_2,\mathbb{R}))$ if there is a strongly measurable function

$$\tilde{B} : [0,T] \times \Omega \times M_1 \times M_2 \to Y$$

with

$$(B(t)y)(\cdot) = \left(\tilde{B}(t,\cdot), y\right)_Y, \qquad y \in Y, \quad t \in [0,T], \tag{A.9}$$

and

$$\mathbb{E}\left\|\left(\int_0^T \|\tilde{B}(s,\cdot)\|_Y^2 \mathrm{d}s\right)^{\frac{1}{2}}\right\|_{L^qL^p}^r = \mathbb{E}\left\|\left(\sum_{m=1}^\infty \int_0^T |B(s)f_m|^2 \mathrm{d}s\right)^{\frac{1}{2}}\right\|_{L^qL^p}^r < \infty. \tag{A.10}$$

The space of stochastically integrable processes is called $\mathcal{M}^r_{\mathbb{F},Y}(0,T; L^q(M_1, L^p(M_2,\mathbb{R})))$.

Similar to the Hilbert space case, the following Theorem is the key to extend the stochastic integral.

Theorem A.18. *Let $p,q \in (1,\infty)$ and $r \in (1,\infty)$.*

a) For all elementary processes B, we have the two-sided norm estimate

$$\mathbb{E}\left\|\int_0^T B\mathrm{d}W\right\|_{L^qL^p}^r \eqsim \mathbb{E}\left\|\left(\sum_{m=1}^\infty \int_0^T |B(s)f_m|^2 \mathrm{d}s\right)^{\frac{1}{2}}\right\|_{L^qL^p}^r. \tag{A.11}$$

b) *The space* $\mathcal{M}^r_{\mathbb{F},Y}(0,T;L^q(M_1,L^p(M_2,\mathbb{R})))$ *is the closure of the elementary processes with respect to the norm the RHS of* (A.11).

Proof. Since $L^q(M_1,L^p(M_2,\mathbb{R}))$ is a UMD function space, (A.11) is a consequence of [124], Corollary 3.11. We show that each B is scalarly in $L^r(\Omega,L^2(0,T;Y))$, i.e.

$$B^*x^* = \langle \tilde{B}, x^*\rangle \in L^r(\Omega,L^2(0,T;Y))$$

for all $x^* \in L^{q'}(M_1,L^{p'}(M_2))$. In view of the characterizations of stochastic integrability in Corollary 3.11 and Theorem 3.6 in [124], Propositions 2.11 and 2.12 in [124] then yield assertion b). Indeed, by Minkowski's and Hölder's inequality, we obtain

$$\begin{aligned}\mathbb{E}\|\langle \tilde{B}, x^*\rangle\|^r_{L^2(0,T;Y)} =& \mathbb{E}\left\|\int_{M_1}\int_{M_2}\tilde{B}\overline{x^*}\mathrm{d}\mu_1\mathrm{d}\mu_2\right\|^r_{L^2(0,T;Y)} \le \mathbb{E}\left(\int_{M_1}\int_{M_2}\|\tilde{B}\|_{L^2(0,T;Y)}\overline{x^*}\mathrm{d}\mu_1\mathrm{d}\mu_2\right)^r\\ &\le \mathbb{E}\left\|\left(\int_0^T\|\tilde{B}(s,\cdot)\|^2_Y\mathrm{d}s\right)^{\frac{1}{2}}\right\|^r_{L^qL^p}\|x^*\|^r_{L^{q'}(M_1,L^{p'}(M_2))} < \infty\end{aligned}$$

for $x^* \in L^{q'}(M_1,L^{p'}(M_2))$. ☐

Corollary A.19 (Itô isomorphism). *a) The linear map* $B \mapsto \int_0^T B\mathrm{d}W$ *can be extended to an isomorphism from* $\mathcal{M}^r_{\mathbb{F},Y}(0,T;L^q(M_1,L^p(M_2,\mathbb{R})))$ *onto a closed subspace of* $L^r(\Omega;\mathcal{F}_T;L^q(M_1,L^p(M_2,\mathbb{R})))$ *with*

$$\mathbb{E}\left\|\int_0^T B\mathrm{d}W\right\|^r_{L^qL^p} \eqsim \mathbb{E}\left\|\left(\int_0^T\|\tilde{B}(s,\cdot)\|^2_Y\mathrm{d}s\right)^{\frac{1}{2}}\right\|^r_{L^qL^p} \tag{A.12}$$

for $B \in \mathcal{M}^r_{\mathbb{F},Y}(0,T;L^q(M_1,L^p(M_2,\mathbb{R})))$.

b) *If* $\mathbb{F}$ *is the Brownian filtration, the range of the isomorphism from a) is*

$$L^r(\Omega;\mathcal{F}_T;L^q(M_1,L^p(M_2,\mathbb{R}))).$$

Proof. Assertion a) is an immediate consequence of Theorem A.18. For a proof of b), we refer to [124], Theorem 3.5. ☐

In order to have a meaningful definition of stochastic integrability, it should coincide with Definition A.8 for $q=p=2$. Indeed, Fubini yields

$$\left\|\left(\sum_{m=1}^\infty\int_0^T|B(s)f_m|^2\mathrm{d}s\right)^{\frac{1}{2}}\right\|_{L^2L^2} = \left(\sum_{m=1}^\infty\int_0^T\|B(s)f_m\|^2_{L^2L^2}\mathrm{d}s\right)^{\frac{1}{2}} = \|B\|_{L^2(0,T;\mathrm{HS}(Y,L^2L^2))}. \tag{A.13}$$

Norms of the type (A.10) with interchanged time and space integration typically appear for $p,q\neq 2$, and they are called *square functions.* We also would like to remark that an assumption of the type (A.9) is not necessary for $p=q=2$, since a process B with (A.10) belongs to $L^r(\Omega,L^2(0,T;Y))$ scalarly by (A.13) and $\|B(s)^*\|_{\mathcal{L}(L^2L^2,Y)} = \|B(s)\|_{\mathcal{L}(Y,L^2L^2)}$ for all $s\in[0,T]$.

The reason for us to present the stochastic integration in mixed L^p-spaces rather than in general UMD-spaces is the following stronger version of the BDG-inequality with the supremum *inside* the mixed L^p-norm.

Theorem A.20 (Strong Burkholder-Davis-Gundy inequality). *Let* $p, q \in (1,\infty)$, $r \in [1,\infty)$ *and* $B \in \mathcal{M}^r_{\mathbb{F},Y}(0,T;L^q(M_1,L^p(M_2,\mathbb{R})))$. *Then, the integral process* $\left(\int_0^t B\mathrm{d}W\right)_{t\in[0,T]}$ *is an* $\mathbb{F}$*-martingale and has a continuous version which satisfies the maximal inequality*

$$\mathbb{E}\left\|\sup_{t\in[0,T]}\left|\int_0^t B\mathrm{d}W\right|\right\|^r_{L^qL^p} \lesssim \mathbb{E}\left\|\int_0^T B\mathrm{d}W\right\|^r_{L^qL^p}.$$

In particular, we get

$$\mathbb{E}\left\|\sup_{t\in[0,T]}\left|\int_0^t B\mathrm{d}W\right|\right\|^r_{L^qL^p} \eqsim \mathbb{E}\left\|\left(\int_0^T \|\tilde{B}(s,\cdot)\|^2_Y\mathrm{d}s\right)^{\frac{1}{2}}\right\|^r_{L^qL^p} \tag{A.14}$$

as a consequence of (A.12).

Proof. See [6], Theorem 1.3.7. □

Remark A.21. In contrast to the presentation above, the author in [6] develops the stochastic integration theory and in particular the strong BDG-inequality for the series

$$X(t) = \sum_{m=1}^\infty \int_0^t b_m(s)\mathrm{d}\beta_m(s), \qquad t\in[0,T],$$

in $L^r(\Omega, L^q(M_1, L^p(M_2,\mathbb{R}))$ with independent real-valued Brownian motions β_m, $m\in\mathbb{N}$. The series converges if and only if $(b_m)_{m\in\mathbb{N}} \in L^r_{\mathbb{F}}(\Omega, L^q(M_1, L^p(M_2, L^2([0,t]\times\mathbb{N}))))$. In the following sense, this approach is equivalent to the integration of operator-valued processes with respect to a cylindrical Wiener process.

a) Given a Y-cylindrical Wiener process and a process $B\in\mathcal{M}^r_{\mathbb{F},Y}(0,T;L^q(M_1,L^p(M_2,\mathbb{R})))$, we have the series representation

$$\int_0^t B\mathrm{d}W = \sum_{m=1}^\infty \int_0^t Bf_m\mathrm{d}(Wf_m)$$

in $L^q(M_1,L^p(M_2,\mathbb{R}))$ almost surely for all $t\in[0,T]$.

b) Given independent real-valued Brownian motions β_m, $m\in\mathbb{N}$, and a sequence

$$(b_m)_{m\in\mathbb{N}} \in L^r_{\mathbb{F}}(\Omega, L^q(M_1, L^p(M_2, L^2([0,T]\times\mathbb{N})))),$$

we can define a Y-cylindrical Wiener process and a process $B\in\mathcal{M}^r_{\mathbb{F},Y}(0,T;L^q(M_1,L^p(M_2,\mathbb{R})))$ by

$$W(y) = \sum_{m=1}^\infty \left(y,f_m\right)_Y\beta_m, \qquad B(t)(y) = \sum_{m=1}^\infty \left(y,f_m\right)_Y b_m(t), \qquad y\in Y. \tag{A.15}$$

In particular, we get $\tilde{B} = \sum_{m=1}^\infty b_m f_m$. Moreover,

$$\int_0^t B\mathrm{d}W = \sum_{m=1}^\infty \int_0^t b_m\mathrm{d}\beta_m$$

in $L^q(M_1,L^p(M_2))$ almost surely for all $t\in[0,T]$.

For the proof of $a)$, we refer to [124], Corollary 3.9. By Example A.3, we know that W is indeed a Y-cylindrical Wiener process and it can be checked that B is an element of $\mathcal{M}^r_{\mathbb{F},Y}(0,T;L^q(M_1,L^p(M_2,\mathbb{R})))$, see [5], Theorem 4.12. Hence part $b)$ is a consequence of $a)$. Obviously, a similar statement is also true for the stochastic integral in Hilbert spaces.

As in the previous section, we omit the localization procedure to extend the stochastic integral to non-integrable processes. Instead, we refer to [124], Section 5, and [6], Section 1.2, and just state that the resulting class of stochastically integrable processes is

$$L^0_{\mathbb{F}}\big(\Omega, L^q(M_1, L^p(M_2, L^2([0,T]\times\mathbb{N})))\big),$$

i.e. the closure of the space of elementary processes in $L^q(M_1, L^p(M_2, L^2([0,T]\times\mathbb{N})))$ in probability.

Remark A.22. In our application of the stochastic integration theory, it will be important to allow complex valued integrands. A process $B = B_1 + \mathrm{i}B_2$ will be called L^r-stochastically integrable in $L^q(M_1, L^p(M_2,\mathbb{C}))$ if B_1 and B_2 are L^r-stochastically integrable in the sense of Definition A.17. The stochastic integral in $L^q(M_1, L^p(M_2,\mathbb{C}))$ is defined as

$$\int_0^t B\mathrm{d}W := \int_0^t B_1\mathrm{d}W + \mathrm{i}\int_0^t B_2\mathrm{d}W, \qquad t\in[0,T].$$

Straightforward calculations using the equivalence of the norms in $\mathbb{C}\equiv\mathbb{R}^2$ yield the complex Itô isomorphism and the strong BDG-inequality in the same form as in the Theorems A.19 and A.20 if we replace the real absolute value by the complex one.

A.2. Stochastic integration with respect to Poisson random measures

In this appendix, we give a short introduction to the Hilbert space valued stochastic integral w.r.t. the compensated time-homogeneous Poisson random measure. Moreover, we define the noise of Marcus type and derive a corresponding Itô formula. We concentrate on the notions we will need in chapter 6 and do not aim for the most general results. For a more detailed treatment of the topics of this appendix, we refer to the monographs by Applebaum, [7], Ikeda and Watanabe, [67], Peszat and Zabczyk, [107] and the dissertation of Zhu, [135].

A.2.1. Time homogeneous Poisson random measure and stochastic integration

Let $\bar{\mathbb{N}} := \mathbb{N}\cup\{0\}\cup\{\infty\}$ denote the set of extended natural numbers. Let $(S,\mathcal{S})$ be a measurable space and as in Section 2.4.2, we employ the following notation. For a σ-finite measure ϑ on S and a sequence $(S_n)_{n\in\mathbb{N}}\subset\mathcal{S}$ such that $S_n\nearrow S$ and $\vartheta(S_n)<\infty$ for all $n\in\mathbb{N}$, we denote the set of all $\bar{\mathbb{N}}$-valued measures ξ on S with $\xi(S_n)<\infty$ for all $n\in\mathbb{N}$ by $M^\vartheta_{\bar{\mathbb{N}}}(S)$.

On the set $M^\vartheta_{\bar{\mathbb{N}}}(S)$, we consider the σ-field $\mathcal{M}^\vartheta_{\bar{\mathbb{N}}}(S)$ defined as the smallest σ-field such that for all $C\in\mathcal{S}$, the map

$$i_C : M^\vartheta_{\bar{\mathbb{N}}}(S)\ni\mu\mapsto\mu(C)\in\bar{\mathbb{N}}$$

is measurable. We start with the general definition of the Poisson random measure.

Definition A.23. Let $(S, \mathcal{S}, \mu)$ be a σ-finite measure space. A *Poisson random measure* π *on* $(S, \mathcal{S})$ *with intensity measure* μ is a random variable

$$\pi : (\Omega, \mathcal{F}) \to (M^{\mu}_{\bar{\mathbb{N}}}(S), \mathcal{M}^{\mu}_{\bar{\mathbb{N}}}(S))$$

such that

a) for each $C \in \mathcal{S}$, the random variable $\pi(C) := i_C \circ \pi : \Omega \to \bar{\mathbb{N}}$ is Poisson with parameter $\mu(C)$, i.e.

$$\mathbb{P}(\pi(C) = k) = \frac{\mu(C)^k}{k!} \exp(-\mu(C)), \qquad k \in \mathbb{N};$$

b) η is *independently scattered*, i.e., if the sets $C_1, C_2, \ldots, C_n \in \mathcal{S}$ are disjoint, then the random variables $\pi(C_1), \pi(C_2), \ldots, \pi(C_n)$ are independent.

Note that by the properties of Poisson variables, we obtain $\pi(C) < \infty$ almost surely for all $C \in \mathcal{S}$ with $\mu(C) < \infty$. For this reason π is welldefined as a map to $M^{\mu}_{\bar{\mathbb{N}}}(S)$. Moreover, we infer

$$\mathbb{E}[\pi(C)] = \mu(C), \quad C \in \mathcal{S}.$$

We continue with a special Poisson random measure on space-time used frequently in the sequel.

Definition A.24. Let $(Y, \mathcal{Y}, \nu)$ be a σ-finite measure space.

a) A *time homogeneous Poisson random measure* η *on* $(Y, \mathcal{Y})$ *with intensity measure* ν is a Poisson random measure on $([0, \infty) \times Y, \mathcal{B}([0, \infty)) \otimes \mathcal{Y})$ such that

   i) the intensity measure of η in the sense of Definition A.23 is given by Leb $\otimes\, \nu$, i.e.

   $$\mathbb{E}[\eta((0, t] \times U)] = t\nu(U), \quad U \in \mathcal{Y};$$

   ii) for all $U \in \mathcal{Y}$, the $\bar{\mathbb{N}}-$valued process $(N(t, U))_{t \geq 0}$ defined by

   $$N(t, U) := \eta((0, t] \times U), \quad t \geq 0,$$

   is $\mathbb{F}$-adapted and its increments are independent of the past, i.e., if $t > s \geq 0$, then $N(t, U) - N(s, U) = \eta((s, t] \times U)$ is independent of $\mathcal{F}_s$.

b) The difference $\tilde{\eta} := \eta -$ Leb $\otimes\, \nu$, is called *compensated time homogeneous Poisson random measure.*

We proceed with several classes of integrable processes. To this end, we recall that a process $\xi : [0, T] \times \Omega \times Y \to E$ is called *predictable,* if the map

$$[0, t] \times \Omega \times Y \ni (s, \omega, y) \to \xi(s, \omega, y) \in E$$

is strongly $\mathcal{B}([0, t]) \otimes \mathcal{F}_t \otimes \mathcal{Y}$-measurable for all $t \in [0, T]$.

Definition A.25. Let E be a real Banach space.

a) We denote the space of all predictable processes $\xi : [0,T] \times \Omega \times Y \to E$ such that

$$\mathbb{E}\bigg[\int_0^T \int_Y \|\xi(s,y)\|_H \, \eta(\mathrm{d}s, \mathrm{d}y)\bigg] < \infty.$$

by $\mathfrak{L}^1_{\eta,\mathbb{F}}([0,T] \times Y; E)$. Elements of $\mathfrak{L}^1_{\eta,\mathbb{F}}([0,T] \times Y; E)$ are called *η-Bochner integrable.*

b) We denote the space of all predictable processes $\xi : [0,T] \times \Omega \times Y \to E$ such that

$$\mathbb{E}\bigg[\int_0^T \int_Y \|\xi(s,y)\|_H \, \nu(\mathrm{d}y)\mathrm{d}s\bigg] < \infty.$$

by $\mathfrak{L}^1_{\nu,\mathbb{F}}([0,T] \times Y; E)$. Elements of $\mathfrak{L}^1_{\nu,\mathbb{F}}([0,T] \times Y; E)$ are called Leb $\otimes$ *ν-Bochner integrable.*

For an η-Bochner integrable process ξ_1 and an Leb $\otimes$ ν-Bochner integrable ξ_2, we interpret

$$\int_0^t \int_Y \xi(s,y) \, \eta(\mathrm{d}s, \mathrm{d}y), \qquad \int_0^t \int_Y \xi(s,y) \, \nu(\mathrm{d}y)\mathrm{d}s \qquad t \in [0,T], \tag{A.16}$$

pathwise as a Bochner integral. For an introduction to the theory of Bochner integration, we refer to [48], chapter 2.

Definition A.26. Let $\xi \in \mathfrak{L}^1_{\eta,\mathbb{F}}([0,T] \times Y; E) \cap \mathfrak{L}^1_{\nu,\mathbb{F}}([0,T] \times Y; E)$. Then, we define

$$(\mathrm{B}) \int_0^t \int_Y \xi(s,y) \, \tilde{\eta}(\mathrm{d}s, \mathrm{d}y) := \int_0^t \int_Y \xi(s,y) \, \eta(\mathrm{d}s, \mathrm{d}y) - \int_0^t \int_Y \xi(s,y) \, \nu(\mathrm{d}y)\mathrm{d}s \tag{A.17}$$

almost surely for all $t \in [0,T]$.

Note that we have equipped the Bochner integral w.r.t. to $\tilde{\eta}$ with the unusual prefix (B) to avoid confusion with the Itô-integral that will be declared below and used much more often in this thesis. Via the Bochner integrals we have just defined, one can connect Poisson random measures with general Lévy processes. This highlights the significance of Poisson random measures in the theory of stochastic processes.

Definition A.27. Let E be a real Banach space. An *E-valued Lévy process* is a stochastic process $L : [0,\infty) \times \Omega \to E$ with the following properties:

a) $L(0) = 0$ almost surely;

b) the increments of L are *stationary* and *independent*, i.e. for $0 \le s < t$, the law of $L(t) - L(s)$ depends only on $t - s$ and for $0 \le t_0 < t_1 < \cdots < t_n$, the random variables

$$L(t_n) - L(t_{n-1}), \quad L(t_{n-1}) - L(t_{n-2}), \ldots, \quad L(t_1) - L(t_0)$$

are independent;

c) L is *stochastically continuous*, i.e. for all $\varepsilon > 0$ and $t \ge 0$

$$\lim_{s \to t} \mathbb{P}\left\{\|L(t) - L(s)\|_E \ge \varepsilon\right\} = 0.$$

d) L has càdlàg paths.

Let us remark that part d) in Definition A.27 is minor since a)-c) already implies the existence of a càdlàg modification of L. The following celebrated Theorem which we quote in the version of Theorem 4.1 of [3], states that a deterministic drift term, the Banach space valued Brownian motion and the Poisson random measure are the only building blocks of general Lévy processes.

Theorem A.28 (Lévy-Itô decomposition). *Let E be a separable real Banach space.*

a) Let η be a time-homogeneous Poisson random measure on $E \setminus \{0\}$. Then, the formulae

$$L_1(t) = \int_0^t \int_{\{\|x\|_G < 1\}} x\, \tilde{\eta}(\mathrm{d}s, \mathrm{d}x), \qquad L_2(t) = \int_0^t \int_{\{\|x\|_G \geq 1\}} x\, \eta(\mathrm{d}s, \mathrm{d}x)$$

define E-valued Lévy processes.

b) For each E-valued Lévy process,

$$\eta([0,t], A) = \sum_{0 < s \leq t} \mathbf{1}_A \left(L(s) - L(s-)\right), \qquad t \geq 0,$$

defines a time homogeneous Poisson random measure with intensity measure ν. Suppose that one of the following conditions

i) E has type 2 and

$$\int_{E\setminus\{0\}} \left(1 \wedge \|x\|_E^2\right) \,\mathrm{d}\nu(l) < \infty,$$

ii) $\int_{E\setminus\{0\}} (1 \wedge \|x\|_E) \,\mathrm{d}\nu(l) < \infty$,

is true. Then, there are $b \in E$ and an E-valued Brownian motion B_Q with covariance Q independent of η such that

$$L(t) = bt + B_Q(t) + \int_0^t \int_{\{\|x\|_G < 1\}} x\, \tilde{\eta}(\mathrm{d}s, \mathrm{d}x) + \int_0^t \int_{\{\|x\|_G \geq 1\}} x\, \eta(\mathrm{d}s, \mathrm{d}x), \qquad t \geq 0.$$

If b and Q vanish, we say that the Lévy process L is of pure jump type.

To prepare the definition of the stochastic integral, we collect some martingale properties of the compensated Poisson random measure.

Lemma A.29. *Let $\tilde{\eta}$ be a compensated Poisson random measure.*

a) For all $T > 0$ and $U \in \mathcal{Y}$ with $\nu(U) < \infty$, the $\mathbb{R}-$valued process $\{\tilde{N}(t,U)\}_{t\in[0,T]}$ defined by

$$\tilde{N}(t,U) := \tilde{\eta}((0,t] \times U), \quad t \in [0,T],$$

is a square-integrable martingale on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Thus, $\tilde{\eta}$ is a martingale-valued measure.

b) For $U \in \mathcal{Y}$ and $0 \leq s \leq t$, we have

$$\mathbb{E}\Big[\big(\tilde{N}(t,U) - \tilde{N}(s,U)\big)^2 | \mathcal{F}_s\Big] = (t-s)\nu(U).$$

Proof. We refer [135], Lemma 3.1.13 and Proposition 3.1.16. □

Lemma A.29 suggests that it is possible to develop an Itô-type stochastic integral with the compensated time-homogeneous Poisson random measure as driving noise. Below, we will define this integral, deduce its main properties and compare it with the Bochner integral from Definition A.26. As in the Gaussian case from Appendix A.1, the integral will be defined for simple processes and extended to a more general class of integrands by a new type of Itô isometry. In the sequel, H always denotes a separable real Hilbert space.

Definition A.30. a) By $\mathfrak{L}^2_{\nu,\mathbb{F}}([0,T]\times Y;H)$, we denote the space of all predictable processes $\xi:[0,T]\times\Omega\times Y\to H$ such that

$$\mathbb{E}\bigg[\int_0^T\int_Y \|\xi(s,y)\|_H^2\,d\nu(y)\,\mathrm{d}s\bigg]<\infty.$$

Elements of $\mathfrak{L}^2_{\nu,\mathbb{F}}([0,T]\times Y;H)$ are called *$\tilde{\eta}$-stochastically integrable.*

b) A process $\xi:[0,T]\times\Omega\times Y\to H$ is called *simple process* if it has the representation

$$\xi(t,\omega,y)=\sum_{j=0}^{J}\sum_{k=1}^{K}\xi_{j-1}^k(\omega)\mathbf{1}_{(t_{j-1},t_j]}(t)\mathbf{1}_{A_{j-1}^k}(y),\qquad t\in[0,T],\,\omega\in\Omega,\,y\in Y,\tag{A.18}$$

for some $J,K\in\mathbb{N}$, $0=t_0<\cdots<t_{J-1}\leq T$ and square-integrable H-valued $\mathcal{F}_{t_{j-1}}$-measurable random variables $\left(\xi_{j-1}^k\right)_{k=1,\ldots,K}$ and disjoint sets $\left(A_{j-1}^k\right)_{k=1,\ldots,K}\subset\mathcal{Y}$ of finite ν-measure for each $j=0,\ldots J-1$.

From [135], Theorem 3.2.23, we get that the predictable processes can be approximated by simple ones.

Lemma A.31. *The space of all simple processes is dense in $\mathfrak{L}^2_{\nu,\mathbb{F}}([0,T]\times Y;H)$.*

For simple processes, the stochastic integral is defined in the natural way.

Definition A.32. Let $\tilde{\eta}:=\eta-$ Leb $\otimes\,\nu$ be a compensated time homogeneous Poisson random measure. For a simple process $\xi:[0,T]\times\Omega\times Y\to H$ of the form (A.18), we define the *stochastic integral* as

$$I_t(\xi):=\int_0^t\int_Y\xi(s,y)\tilde{\eta}(\mathrm{d}s,\mathrm{d}y):=\sum_{j=1}^{J}\sum_{k=1}^{K}\xi_{j-1}^k\tilde{\eta}\left((t_{j-1}\wedge t,t_j\wedge t]\times A_{j-1}^k\right),\qquad t\in[0,T].$$

The following identity is the core of the stochastic integration theory.

Lemma A.33 (Itô isometry). *Let $\xi:[0,T]\times\Omega\times Y\to H$ be a simple process. Then, we have*

$$\mathbb{E}\|I_t(\xi)\|_H^2=\mathbb{E}\int_0^t\int_Y\|\xi(s,y)\|_H^2\,d\nu(y)\,\mathrm{d}s,\qquad t\in[0,T].\tag{A.19}$$

Proof. The proof of the Itô isometry is quite similar to the classical Gaussian Itô isometry. A proof in the scalar case can be found in [7], Lemma 4.2.2. However, we sketch it to get used to the concepts introduced above. To simplify the notation, we restrict ourselves to $t=T$. We compute

$$\mathbb{E}\|I_T(\xi)\|_H^2=\sum_{j,k,l,m}\mathbb{E}\Big[\left(\xi_{j-1}^k,\xi_{l-1}^m\right)_H\tilde{\eta}\left((t_{j-1},t_j]\times A_{j-1}^k\right)\tilde{\eta}\left((t_{l-1},t_l]\times A_{l-1}^m\right)\Big].$$

Assume $j = l$ and $k = m$. Then

$$\mathbb{E}\Big[\big(\xi_{j-1}^k, \xi_{l-1}^m\big)_H \tilde{\eta}\left((t_{j-1}, t_j] \times A_{j-1}^k\right) \tilde{\eta}\left((t_{l-1}, t_l] \times A_{l-1}^m\right)\Big] = \mathbb{E}\Big[\|\xi_{j-1}^k\|_H^2 \tilde{\eta}\left((t_{j-1}, t_j] \times A_{j-1}^k\right)^2\Big]$$
$$= \mathbb{E}\Big[\|\xi_{j-1}^k\|_H^2 \mathbb{E}\big[\tilde{\eta}\left((t_{j-1}, t_j] \times A_{j-1}^k\right)^2 |\mathcal{F}_{t_{j-1}}\big]\Big] = \mathbb{E}\Big[\|\xi_{j-1}^k\|_H^2\Big](t_j - t_{j-1})\nu(A_{j-1}^k)$$

by the $\mathcal{F}_{t_{j-1}}$-measurability of ξ_{j-1}^k and Lemma A.29. For $j = l$ and $k \neq m$, we employ the property of independent scattering to compute

$$\mathbb{E}\Big[\big(\xi_{j-1}^k, \xi_{j-1}^m\big)_H \tilde{\eta}\left((t_{j-1}, t_j] \times A_{j-1}^k\right) \tilde{\eta}\left((t_{l-1}, t_l] \times A_{j-1}^m\right)\Big]$$
$$= \mathbb{E}\Big[\big(\xi_{j-1}^k, \xi_{j-1}^m\big)_H\Big]\mathbb{E}\Big[\tilde{\eta}\left((t_{j-1}, t_j] \times A_{j-1}^k\right)\Big]\mathbb{E}\Big[\tilde{\eta}\left((t_{j-1}, t_j] \times A_{j-1}^m\right)\Big] = 0.$$

For $j < l$, we get

$$\mathbb{E}\Big[\big(\xi_{j-1}^k, \xi_{l-1}^m\big)_H \tilde{\eta}\left((t_{j-1}, t_j] \times A_{j-1}^k\right) \tilde{\eta}\left((t_{l-1}, t_l] \times A_{l-1}^m\right)\Big]$$
$$= \mathbb{E}\Big[\big(\xi_{j-1}^k, \xi_{l-1}^m\big)_H \tilde{\eta}\left((t_{j-1}, t_j] \times A_{j-1}^k\right)\Big]\mathbb{E}\Big[\tilde{\eta}\left((t_{l-1}, t_l] \times A_{l-1}^m\right)\Big] = 0,$$

where we used that increments are independent of the past and the fact that the compensated Poisson random measure has mean 0 by definition. The case $j > l$ can be treated analogously. After all, we obtain

$$\mathbb{E}\|I_T(\xi)\|_H^2 = \sum_{j=1}^J \sum_{k=1}^K \mathbb{E}\Big[\|\xi_{j-1}^k\|_H^2\Big](t_j - t_{j-1})\nu(A_{j-1}^k) = \int_0^T \int_Y \mathbb{E}\Big[\|\xi(s,y)\|_H^2\Big]\nu(\mathrm{d}y)\mathrm{d}s.$$

□

In view of the Lemmata A.19 and A.31, it is natural to define the stochastic integral for integrands $\xi \in \mathfrak{L}^2_{\nu,\mathbb{F}}([0,T] \times Y; H)$ via continuous extension. Then, we get the following properties.

Theorem A.34. *Let $\xi \in \mathfrak{L}^2_{\nu,\mathbb{F}}([0,T] \times Y; H)$. Then, we have the isometry formula*

$$\mathbb{E}\|I_t(\xi)\|_H^2 = \mathbb{E}\int_0^t \int_Y \|\xi(s,y)\|_H^2 \, d\nu(y)\, \mathrm{d}s, \qquad t \in [0,T]. \tag{A.20}$$

Moreover, the integral process $(I_t(\xi))_{t\in[0,T]}$ is a square-integrable H-valued martingale and has a càdlàg modification.

Proof. The Itô isometry is an immediate consequence of the Lemmata A.19 and A.31. For the martingale and the càdlàg-property, we refer to [135], Theorem 3.3.2. □

Combining (A.20) with the Doob inequality, we get

$$\mathbb{E}\Big[\sup_{t\in[0,\tau]} \|I_t(\xi)\|_H^2\Big] \eqsim \mathbb{E}\int_0^\tau \int_Y \|\xi(s,y)\|_H^2 \, \nu(\mathrm{d}y)\, \mathrm{d}s.$$

In the Gaussian theory of stochastic integration, a similar type of equivalence can be strengthened to arbitrary moments by the Burkholder-Davis-Gundy inequality, see Theorems A.11 and A.20. Here, this role is played by the maximal inequality we present next.

Theorem A.35. *For $p \in (1,\infty)$ and a stopping time $\tau : \Omega \to [0,T]$, we have*

$$\left(\mathbb{E}\Big[\sup_{t\in[0,\tau]} \|I_t(\xi)\|_H^p\Big]\right)^{\frac{1}{p}} \eqsim_p \|\xi\|_{L^p(\Omega,\nu_p([0,T]\times Y,H))},$$

where $\nu_p([0,T]\times Y,H)$ is given by

$$\nu_p([0,T]\times Y,H) = \begin{cases} L^2([0,T]\times Y,H)\cap L^p([0,T]\times Y,H), & 2\le p<\infty, \\ L^2([0,T]\times Y,H)+L^p([0,T]\times Y,H), & 1\le p\le 2. \end{cases}$$

Proof. We refer to [49], Theorem 4.5 and Example 4.6. □

As we have introduced two types of stochastic integrals w.r.t the compensated time-homogeneous Poisson random measure, it is natural to ask the question under which assumption they coincide. This is the content of the following Proposition.

Proposition A.36. *a) We have the inclusion $\mathfrak{L}^1_{\nu,\mathbb{F}}([0,T]\times Y;H) \subset \mathfrak{L}^1_{\eta,\mathbb{F}}([0,T]\times Y;H)$ and*

$$\mathbb{E}\int_0^t\int_Y \xi(s,y)\,\eta(\mathrm{d}s,\mathrm{d}y) = \mathbb{E}\int_0^t\int_Y \xi(s,y)\,\nu(\mathrm{d}y)\mathrm{d}s$$

for $f\in\mathfrak{L}^1_{\nu,\mathbb{F}}([0,T]\times Y;H)$ and $t\ge 0$.

b) For $f\in\mathfrak{L}^1_{\eta,\mathbb{F}}([0,T]\times Y;H)\cap\mathfrak{L}^2_{\nu,\mathbb{F}}([0,T]\times Y;H)$, we have

$$\begin{aligned}\int_0^t\int_Y \xi(s,y)\,\tilde{\eta}(\mathrm{d}s,\mathrm{d}y) &= (\mathrm{B})\int_0^t\int_Y \xi(s,y)\,\tilde{\eta}(\mathrm{d}s,\mathrm{d}y) \\ &= \int_0^t\int_Y \xi(s,y)\,\eta(\mathrm{d}s,\mathrm{d}y) - \int_0^t\int_Y \xi(s,y)\,\nu(\mathrm{d}y)\mathrm{d}s\end{aligned}$$

almost surely for all $t\in[0,T]$.

Proof. We refer to [135], Proposition 3.4.7. □

In each stochastic integration theory, the Itô formula is one of the most important features. In the following Theorem, we state it in the form of [28], Theorem B.1.

Theorem A.37. *Let us define a process $X : [0,T]\times\Omega\to H$ by*

$$X(t) = X_0 + \int_0^t a(s)\mathrm{d}s + \int_0^t\int_Y f(s,y)\tilde{\eta}(\mathrm{d}s,\mathrm{d}y), \qquad t\in[0,T],$$

where $a:[0,T]\times\Omega\to H$ is a progressively measurable process with

$$\int_0^T \|a(s)\|_H\mathrm{d}s < \infty$$

almost surely and $f\in\mathfrak{L}^2_{\nu,\mathbb{F}}([0,T]\times Y;H)$. Let G be another separable real Hilbert space and $\varphi: H\to G$ be a C^1-function such that the first derivative $\varphi' : H\to\mathcal{L}(H,G)$ is Lipschitz. Then, we have

$$\varphi(X(t)) = \varphi(X_0) + \int_0^t \varphi'[X(s)]a(s)\mathrm{d}s + \int_0^t\int_Y \left\{\varphi(X(s-)+f(s,y)) - \varphi(X(s-))\right\}\tilde{\eta}(\mathrm{d}s,\mathrm{d}y)$$

$$+\int_0^t\int_Y \{\varphi(X(s-)+f(s,y))-\varphi(X(s-))-\varphi'[X(s-)]f(s,y)\}\,\nu(\mathrm{d}y)\mathrm{d}s \qquad \text{(A.21)}$$

almost surely in G for all $t\in[0,T]$.

Remark A.38. One can also write (A.21) as

$$\begin{aligned}\varphi(X(t)) =&\varphi(X_0)+\int_0^t \varphi'[X(s)]a(s)\mathrm{d}s+\int_0^t\int_Y \{\varphi(X(s-)+f(s,y))-\varphi(X(s-))\}\,\tilde{\eta}(\mathrm{d}s,\mathrm{d}y)\\ &+\int_0^t\int_Y \{\varphi(X(s)+f(s,y))-\varphi(X(s))-\varphi'[X(s)]f(s,y)\}\,\nu(\mathrm{d}y)\mathrm{d}s\end{aligned}$$

almost surely in G for all $t\in[0,T]$. This is due to the fact that X is càdlàg and thus $(X(t))_{t\in[0,T]}$ and $(X(t-))_{t\in[0,T]}$ only differ on a nullset w.r.t. the Lebesgue-measure in time.

A.2.2. Marcus stochastic evolution equations

In this thesis, we will always consider the noise induced by the compensated Poisson random measure in the Marcus form. In this section, we introduce this notion and deduce an Itô formula for this special type of noise.

We begin with an informal motivation of the Marcus product which will be denoted by $\diamond$. Let us explain its main properties: the *change of variables formula* without correction term and the consistency with *Wong-Zakai-type approximations*. In the case of a continuous driven process, these properties are considered to be the most important advantages of Stratonovich noise compared to Itô noise and lead to the fact that Stratonovich noise is often preferred in the modeling of noise phenomena in physics. However, in the case of a discontinuous driving process, Stratonovich noise does not have these favorable properties any more. This leads to a third type of stochastic differential equation introduced by Steven Marcus, see [93] and [94]. Roughly speaking, the Marcus form of discontinuous noise is the natural analogue to the Stratonovich form of continuous noise.
Let us explain the properties mentioned above in more detail. In this section, we consider the $\mathbb{R}^N$-valued Lévy process $L(t):=(L_1(t),\cdots,L_N(t))$ associated to the compensated Poisson random measure, i.e.

$$L(t)=\int_0^t\int_{\{\|l\|\le 1\}} l\,\tilde{\eta}(\mathrm{d}s,\mathrm{d}l) \qquad \text{(A.22)}$$

where $B:=\mathbb{B}(0,1)\subset\mathbb{R}^N$. The first property in the spirit of Stratonovich is a *change of variables formula*

$$\varphi(X(t))=\varphi(X_0)+\int_0^t \varphi'\big[X(s-)\big]v(X(s-))\diamond \mathrm{d}L(s)$$

as long as X solves the Marcus equation

$$X(t)=X_0+\int_0^t v(X(s-))\diamond \mathrm{d}L(s).$$

After we have given a precise meaning to $\diamond$, this will be the main result of this section. The second property in this direction is the consistency of the Marcus noise with a *Wong-Zakai-type approximation*, i.e. the solution of a Marcus SDE is the limit of solutions to the equations with L

being replaced by continuous approximations. For more details on this approximation result, we refer to [86] and [82]. We just want to give a short formulation of the main idea. Let $M = 1$ and $(X(t))_{t\geq 0}$ be scalar valued process with

$$X(t) = X_0 + \int_0^t X(s-) \diamond \mathrm{d}L(s).$$

We approximate L by $L_n(t) = n \int_{t-\frac{1}{n}}^t L(s)\mathrm{d}s$ for $n \in \mathbb{N}$ and $t \geq \frac{1}{n}$. Then, the solution to the equation

$$X_n(t) = X_0 + \int_0^t X(s)\mathrm{d}L_n(s).$$

converges almost surely to X.

Now, we would like to continue with a rigorous definition of the Marcus noise in a Hilbert space. Let H be a separable real Hilbert space and $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_N : H \to H$. Moreover, we define $\mathbf{v} : H \times \mathbb{R}^N \to H$ via

$$\mathbf{v}(y, l) := \sum_{m=1}^N \mathbf{v}_m(y) l_m, \qquad l \in \mathbb{R}^N,\, y \in H.$$

We assume that the evolution equation

$$\begin{cases} y'(t) = \mathbf{v}\left(y(t), l\right), \\ y(0) = y_0, \end{cases} \tag{A.23}$$

has a unique classical solution for any $l \in \mathbb{R}^m$ and $y_0 \in H$ on the time interval $[0, 1]$. The solution operator associated with (A.23) is called *Marcus mapping* and denoted by

$$\Phi : [0, 1] \times \mathbb{R}^N \times H \to H.$$

In this framework, we can give the definition of a Marcus stochastic evolution equation.

Definition A.39. Let $\tilde{\eta} := \eta -$ Leb $\otimes \nu$ be a compensated time homogeneous Poisson random measure and L be the Lévy process defined by (A.22). Then, a solution of the *Marcus stochastic evolution equation*

$$\begin{cases} \mathrm{d}X(t) = \mathbf{v}_0(X(t))\,\mathrm{d}t + \mathbf{v}(X(t-)) \diamond \mathrm{d}L(t), \\ X(0) = X_0, \end{cases} \tag{A.24}$$

is an adapted process $X : [0, T] \times \Omega \to H$ such that the integral equation

$$\begin{aligned} X(t) = X_0 + \int_0^t \mathbf{v}_0(X(s))\,\mathrm{d}s + \int_0^t \int_{\{\|l\|\leq 1\}} \Big[\Phi\left(1, l, X(s-)\right) - X(s-)\Big] \tilde{\eta}(\mathrm{d}s, \mathrm{d}l) \\ + \int_0^t \int_{\{\|l\|\leq 1\}} \Big[\Phi\left(1, l, X(s)\right) - X(s) - \mathbf{v}\left(X(s), l\right)\Big] \nu(\mathrm{d}l)\mathrm{d}s \end{aligned} \tag{A.25}$$

holds in H almost surely for all $t \in [0, T]$.

In the main result of this section, we formulate the change of variables formula announced in the motivation from above.

Theorem A.40 (Itô's formula). *Let G be a separable real Hilbert space and $\varphi : H \to G$ be a C^1-function such that the first derivative $\varphi' : H \to \mathcal{L}(H,G)$ is Lipschitz. If X is a solution of the* Marcus stochastic evolution equation (A.24), *then we have*

$$\begin{aligned}\varphi(X(t)) - \varphi(X_0) &= \int_0^t \varphi'\left[X(s)\right](\mathbf{v}_0(X(s)))\,\mathrm{d}s \\ &\quad + \int_0^t \int_{\{\|l\|\le 1\}} \Big[\varphi\big(\Phi\big(1,l,X(s-)\big)\big) - \varphi(X(s-))\Big]\tilde{\eta}(\mathrm{d}s,\mathrm{d}l) \\ &\quad + \int_0^t \int_{\{\|l\|\le 1\}} \Big[\varphi\big(\Phi\big(1,l,X(s)\big)\big) - \varphi(X(s)) - \sum_{j=1}^N l_m \varphi'\left[X(s)\right](\mathbf{v}_m(X(s)))\Big]\nu(\mathrm{d}l)\mathrm{d}s\end{aligned} \tag{A.26}$$

in G almost surely for $t \in [0,T]$. In particular, the process $(Y(t))_{t\in[0,T]}$ given by $Y(t) = \varphi(X(t))$ for $t \in [0,T]$ is a solution to the Marcus stochastic evolution equation

$$\begin{cases} \mathrm{d}Y(t) = \varphi'[X(s)]\mathbf{v}_0(X(t))\,\mathrm{d}t + \varphi'[X(t-)]\mathbf{v}(X(t-)) \diamond \mathrm{d}L(t), \\ Y(0) = \varphi(X_0). \end{cases}$$

Proof. For $h \in H$ and $l \in \mathbb{R}^N$ with $\|l\| \le 1$, we define

$$\begin{aligned} f(h,l) &:= \mathbf{1}_B(l)\left\{\Phi\left(1,l,h\right) - h\right\} \\ a(h) &:= \mathbf{v}_0(h) + \int_{\{\|l\|\le 1\}} \Big[\Phi\left(1,l,h\right) - h - \sum_{m=1}^N l_m \mathbf{v}_m(h)\Big]\nu(\mathrm{d}l). \end{aligned}$$

Then the H-valued process X given in (A.25) takes the form

$$X(t) = X_0 + \int_0^t a(X(s))\,\mathrm{d}s + \int_0^t \int_{\{\|l\|\le 1\}} f(X(s-),l)\tilde{\eta}(\mathrm{d}s,\mathrm{d}l). \tag{A.27}$$

From Theorem A.37, we infer

$$\begin{aligned} \varphi(X(t)) =& \varphi(X_0) + \int_0^t \varphi'[X(s)]a(s)\mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{R}^N} \left\{\varphi(X(s-) + f\big(X(s-),l\big)) - \varphi(X(s-))\right\}\,\tilde{\eta}(\mathrm{d}s,\mathrm{d}l) \\ &+ \int_0^t \int_{\mathbb{R}^N} \left\{\varphi(X(s-) + f\big(X(s-),l\big)) - \varphi(X(s-)) - \varphi'[X(s-)]f\big(X(s-),l\big)\right\}\nu(\mathrm{d}l)\mathrm{d}s \\ =& \varphi(X_0) + I_1(t) + I_2(t) + I_3(t) \end{aligned}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$, where I_1, I_2, I_3 can the simplified to

$$\begin{aligned} I_1(t) &= \int_0^t \varphi'\left[X(s)\right](a(X(s)))\,\mathrm{d}s \\ &= \int_0^t \varphi'\left[X(s)\right](\mathbf{v}_0(X(s)))\,\mathrm{d}s + \int_0^t \int_{\{\|l\|\le 1\}} \varphi'\left[X(s)\right]f(X(s),l)\,\nu(\mathrm{d}l)\,\mathrm{d}s \\ &\quad - \int_0^t \int_{\{\|l\|\le 1\}} \Big[\sum_{m=1}^N l_m \varphi'\left[X(s)\right](\mathbf{v}_m(X(s)))\Big]\nu(\mathrm{d}l)\,\mathrm{d}s; \end{aligned} \tag{A.28}$$

$$I_2(t) = \int_0^t \int_{\{\|l\|\le 1\}} \{\varphi(\Phi(1,l,X(s-))) - \varphi(X(s-))\}\, \tilde{\eta}(\mathrm{d}s,\mathrm{d}l);$$

$$I_3(t) = \int_0^t \int_{\{\|l\|\le 1\}} \{\Phi(1,l,X(s)) - X(s)\}\, \nu(\mathrm{d}l)\mathrm{d}s - \int_0^t \int_{\{\|l\|\le 1\}} \varphi'[X(s-)] f(X(s),l)\nu(\mathrm{d}l)\mathrm{d}s.$$

By the càdlàg-property of X, compare Remark A.38, the second terms in I_1 and I_3 cancel and we obtain

$$\begin{aligned}
\varphi(X(t)) =& \varphi(X_0) + \int_0^t \varphi'[X(s)](\mathbf{v}_0(X(s)))\,\mathrm{d}s \\
&- \int_0^t \int_{\{\|l\|\le 1\}} \Big[\sum_{m=1}^N l_m \varphi'[X(s)](\mathbf{v}_m(X(s)))\Big]\nu(\mathrm{d}l)\,\mathrm{d}s \\
&+ \int_0^t \int_{\{\|l\|\le 1\}} \{\varphi(\Phi(1,l,X(s-))) - \varphi(X(s-))\}\, \tilde{\eta}(\mathrm{d}s,\mathrm{d}l) \\
&+ \int_0^t \int_{\{\|l\|\le 1\}} \{\varphi(\Phi(1,l,X(s-))) - \varphi(X(s-))\}\, \nu(\mathrm{d}l)\mathrm{d}s \\
=& \varphi(X_0) + \int_0^t \varphi'[X(s)](\mathbf{v}_0(X(s)))\,\mathrm{d}s \\
&+ \int_0^t \int_{\{\|l\|\le 1\}} \{\varphi(\Phi(1,l,X(s-))) - \varphi(X(s-))\}\, \tilde{\eta}(\mathrm{d}s,\mathrm{d}l) \\
&+ \int_0^t \int_{\{\|l\|\le 1\}} \Big\{\varphi(\Phi(1,l,X(s))) - \varphi(X(s)) - \sum_{m=1}^N l_m \varphi'[X(s)](\mathbf{v}_m(X(s)))\Big\}\nu(\mathrm{d}l)\mathrm{d}s
\end{aligned}$$

almost surely for all $t \in [0,T]$.

The second assertion follows from the observation that $y(t,h,l) := \varphi(\Phi(t,h,l))$ for $t \in [0,1]$, $h \in H$ and $l \in \mathbb{R}^N$ fulfills

$$y'(t,h,l) = \varphi'[\Phi(t,h,l)]\,\Phi'(t,h,l) = \varphi'[y(t,h,l)]\,\mathbf{v}(y(t,h,l),l), \qquad y(0,h,l) = \varphi(h)$$

by the chain rule. Using y, the equation (A.26) can be rewritten in the form

$$\begin{aligned}
\varphi(X(t)) - \varphi(X_0) =& \int_0^t \varphi'[X(s)](\mathbf{v}_0(X(s)))\,\mathrm{d}s \\
&+ \int_0^t \int_{\{\|l\|\le 1\}} \Big[y(1,l,X(s-))) - \varphi(X(s-))\Big]\tilde{\eta}(\mathrm{d}s,\mathrm{d}l) \\
&+ \int_0^t \int_{\{\|l\|\le 1\}} \Big[y(1,l,X(s)) - \varphi(X(s)) - \varphi'[X(s)]\,\mathbf{v}(X(s),l)\Big]\nu(\mathrm{d}l)\mathrm{d}s,
\end{aligned}$$

which yields the assertion by Definition A.39. □

A.3. Fractional domains of a selfadjoint operator

In this section, we introduce the fractional domains of a selfadjoint operator on a complex Hilbert space H. These spaces are the key ingredient for our functional analytic formulation of the nonlinear Schrödinger equation on different levels of regularity.

Throughout this section, M is a σ-finite measure space and $A : L^2(M) \supset D(A) \to L^2(M)$ is a non-negative selfadjoint operator. We fix $\theta > 0$ and equip $X_\theta := \mathcal{D}((\mathrm{Id} + A)^\theta)$ with the norm

$$\|x\|_\theta := \|(\mathrm{Id} + A)^\theta x\|_{L^2}, \qquad x \in X_\theta,$$

where the fractional powers are defined via the Borel functional calculus for selfadjoint operators. Note that X_θ is a Hilbert space with the inner product

$$\big(x, y\big)_\theta = \big((\mathrm{Id} + A)^\theta x, (\mathrm{Id} + A)^\theta y\big)_{L^2},$$

since $(\mathrm{Id} + A)^\theta$ is a closed operator with $0 \in \rho((\mathrm{Id} + A)^\theta)$. Moreover, we define the extrapolation space $X_{-\theta}$ as the completion of $L^2(M)$ with respect to the norm

$$\|x\|_{-\theta} := \|(\mathrm{Id} + A)^{-\theta} x\|_{L^2}, \qquad x \in L^2(M),$$

and obtain a Hilbert space with the inner product

$$\big(x, y\big)_{-\theta} = \lim_{n,m\to\infty} \big((\mathrm{Id} + A)^{-\theta} x_n, (\mathrm{Id} + A)^{-\theta} y_m\big)_{L^2}, \qquad x, y \in X_{-\theta},$$

for sequences $(x_n)_{n\in\mathbb{N}}, (y_m)_{m\in\mathbb{N}} \subset L^2(M)$ with $x_n \to x$ and $y_m \to y$ in $X_{-\theta}$ as $n, m \to \infty$. Note that we have $X_{-\theta} = (X_\theta)^*$ and the duality is given by

$$\langle x, y\rangle_{\theta,-\theta} := \lim_{n\to\infty} \big(x, y_n\big)_{L^2}, \qquad x \in X_\theta, \quad y \in X_{-\theta},$$

with $(y_n)_{n\in\mathbb{N}} \subset L^2(M)$ such that $y_n \to y$ in $X_{-\theta}$ as $n \to \infty$.

We denote the closure of the operator A in $X_{-\theta}$ by $A_{-\theta}$ and the restriction of A to $X_{\theta+1}$ by A_θ. If there is no risk of ambiguity, we will drop the index θ and simply write A.

Proposition A.41. *a) For all $\theta \in \mathbb{R}$, A_θ is a non-negative selfadjoint operator on X_θ with domain $X_{\theta+1}$.*

b) We have

$$\langle x, A_{-\frac{1}{2}} y\rangle_{\frac{1}{2},-\frac{1}{2}} = \overline{\langle y, A_{-\frac{1}{2}} x\rangle}_{\frac{1}{2},-\frac{1}{2}}, \qquad x, y \in X_{\frac{1}{2}}.$$

c) For all $\alpha, \beta \in \mathbb{R}$ with $\beta > \alpha$ and $\theta \in [0,1]$, we have $[X_\alpha, X_\beta]_\theta = X_{(1-\theta)\alpha+\theta\beta}$. In particular,

$$\|x\|_{(1-\theta)\alpha+\theta\beta} \lesssim \|x\|_\alpha^{1-\theta} \|x\|_\beta^\theta, \qquad x \in X_\beta.$$

Proof. ad a). For $\theta > 0$, A_θ is obviously symmetric and non-negative since it commutes with $(\mathrm{Id} + A)^\theta$. Moreover, $\mathrm{Id} + A_\theta$ is a surjective isometry from $X_{\theta+1}$ to X_θ. Thus, $-1 \in \rho(A_\theta)$ and A_θ is selfadjoint.

Let $\theta \in [-1, 0)$. By the definition of X_θ and the density of $X_{\theta+1}$ in $L^2(M)$, X_θ and A_θ are the closures of $X_{\theta+1}$ and $A_{\theta+1}$ with respect to the norm $\|(\mathrm{Id} + A)^{-1} \cdot \|_{\theta+1}$. Let $(T_{\theta+1}(t))_{t\geq 0}$ be the C_0-semigroup generated by $-A_{\theta+1}$. By [52], Theorem 5.5, $(T_{\theta+1}(t))_{t\geq 0}$ can be extended to a C_0-semigroup $(T_\theta(t))_{t\geq 0}$ on X_θ with generator $-A_\theta$ and $\mathcal{D}(A_\theta) = X_{\theta+1}$.

To check the symmetry of A_θ, let $x, y \in \mathcal{D}(A_\theta)$. Then, there are $(x_n)_{n\in\mathbb{N}}, (y_m)_{m\in\mathbb{N}} \subset X_{\theta+1}$ with

$$x_n \to x, \qquad A_{\theta+1} x_n \to A_\theta x, \qquad y_m \to y, \qquad A_{\theta+1} y_n \to A_\theta y, \qquad n, m \to \infty,$$

in $\|(\mathrm{Id}+A)^{-1}\cdot\|_{\theta+1}$ and we obtain

$$\begin{aligned}\big(A_\theta x,y\big)_\theta &= \lim_{n,m\to\infty}\big((\mathrm{Id}+A)^{-1}A_{\theta+1}x_n,(\mathrm{Id}+A)^{-1}y_m\big)_{\theta+1}\\ &= \lim_{n,m\to\infty}\big((\mathrm{Id}+A)^{-1}x_n,(\mathrm{Id}+A)^{-1}A_{\theta+1}y_m\big)_{\theta+1}=\big(x,A_\theta y\big)_\theta\end{aligned}$$

as well as

$$\big(A_\theta x,x\big)_\theta=\lim_{n\to\infty}\big((\mathrm{Id}+A)^{-1}A_{\theta+1}x_n,(\mathrm{Id}+A)^{-1}x_n\big)_{\theta+1}\ge 0,$$

because $A_{\theta+1}$ is symmetric, non-negative and commutes with $(\mathrm{Id}+A)^{-1}$. Since $-A_\theta$ is generator of a C_0-semigroup on X_θ, the symmetry of A_θ directly implies selfadjointness. Inductively, we obtain the assertion for all $\theta<0$.

ad b). By the definition of $A_{-\frac{1}{2}}$, we can choose $(x_k)_{k\in\mathbb{N}}\subset X_1$ and $(y_l)_{l\in\mathbb{N}}\subset X_1$ such that

$$x_k\to x,\qquad Ax_k\to A_{-\frac{1}{2}}x,\qquad y_l\to y,\qquad Ay_l\to A_{-\frac{1}{2}}y$$

in $X_{-\frac{1}{2}}$ as $k,l\to\infty$ and obtain

$$\langle x,A_{-\frac{1}{2}}y\rangle_{\frac{1}{2},-\frac{1}{2}}=\lim_{k,l\to\infty}\big(y_l,Ax_k\big)_{L^2}=\lim_{k,l\to\infty}\overline{\big(x_k,Ay_l\big)}_{L^2}=\overline{\langle y,A_{-\frac{1}{2}}x\rangle}_{\frac{1}{2},-\frac{1}{2}}.$$

ad c). This is a consequence of [84], Theorem 15.28, and the fact that each non-negative selfadjoint operator A has bounded imaginary powers by the Borel functional calculus. □

A.4. Function spaces on Riemannian manifolds

In Chapter 5 and the examples in the Chapters 4 and 6, we frequently consider the stochastic nonlinear Schrödinger on Riemannian manifolds. For this reason, we would like to introduce fundamental notions from Riemannian geometry. Moreover, we present the Laplace-Beltrami operator which generalizes the classical Laplacian from $\mathbb{R}^d$ to Riemannian manifolds and connect it to Sobolev spaces. We are guided by the exposition of similar contents in [21], sections A.4 and III.1. For further details, we refer to the textbooks [8] and [87].

Let us start with some elementary definitions from differential geometry.

Definition A.42. a) A C^∞*-manifold without boundary* (M,τ) of dimension $d\in\mathbb{N}$ is a topological Hausdorff-space with countable basis such that for every $x\in M$, there are an open set $U\subset M$ with $x\in U$, an open set $V\subset\mathbb{R}^d$ and a C^∞-diffeomorphism $\varphi:U\to V$. Often, we shortly say manifold instead of C^∞-manifold without boundary and use the abbreviation M instead of (M,τ).

b) The pair (φ,U) from above is called *chart around* x and a collection of charts

$$\mathfrak{M}:=\{(\varphi_\alpha,U_\alpha):\alpha\in J\},\qquad M=\bigcup_{\alpha\in J}U_\alpha,$$

is called *atlas of* M. The *local coordinates* of $x\in U$ are given by the vector $(\varphi(x))_{n=1,\dots,d}$.

We continue with the notions of smooth functions, tangent spaces and vector fields.

Definition A.43. Let M and N be manifolds of dimensions d and d'.

a) A function $f : M \to N$ is called *smooth*, if for all $x \in M$, there are charts (U, φ) around x and (U', φ') around $f(x)$ such that

$$\varphi' \circ f \circ \varphi^{-1} \in C^\infty(U, U').$$

The space of all smooth functions $f : M \to N$ is denoted by $C^\infty(M, N)$ and we abbreviate $C^\infty(M) := C^\infty(M, \mathbb{R})$.

b) Let $x \in M$. Then, the space $T_x M$ of all linear maps $X_x : C^\infty(M) \to \mathbb{R}$ such that

$$X_x(fg) = X_x(f)g(x) + f(x)X_x(g), \qquad f, g \in C^\infty(M),$$

is called *tangent space of* M *in* p. The space of all bilinear forms $B : T_x M \times T_x M \to \mathbb{R}$ is denoted by $T_x^2 M$. Moreover, we set

$$TM := \bigcup_{x \in M} T_x M, \qquad T^2 M := \bigcup_{x \in M} T_x^2 M,$$

where the unions are understood in the disjoint sense.

c) A *vector field* is a map $V : M \to TM$ with $V(x) \in T_x M$ for all $x \in M$. The space of all vector fields is denoted by VM.

We remark that for all $x \in M$ and charts $\varphi : U \to V$, the tangent vectors $\partial_j|_x$, $j = 1, \dots, d$, defined by

$$\partial_j|_x(f) := \partial_j \left(f \circ \varphi^{-1}\right)(\varphi(x)) \tag{A.29}$$

form a basis of the tangent space $T_x M$. Analogous to (A.29), we denote

$$\partial^\alpha|_x(f) := \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d} \left(f \circ \varphi^{-1}\right)(\varphi(x))$$

for an arbitrary multi-index $\alpha \in \mathbb{N}_0^d$. In order to measure the distance of two points on a manifold and to generalize classical notions like the gradient of a function to the manifold setting, we need additional structure on the tangent space.

Definition A.44. A *Riemannian metric on* M is a smooth map $g : M \to T^2 M$ such that $g(x)$ is a scalar product on $T_x M$ for each $x \in M$. This scalar product is denoted by $\left(\cdot, \cdot\right)_x$ and we write $|\cdot|_x$ for the associated norm. A pair (M, g) consisting of a manifold M and a Riemannian metric g is called *Riemannian manifold.*

In local coordinates, the Riemannian metric g is uniquely determined by the matrix

$$G(x) := \left(g_{k,l}(x)\right)_{k,l=1,\dots,d}, \qquad g_{k,l}(x) := g(x)\left(\partial_k|_x, \partial_l|_x\right).$$

The inverse of G is written as $G(x)^{-1} = \left(g^{k,l}(x)\right)_{k,l=1,\dots,d}$. The gradient ∇f of a C^1-function $f : \mathbb{R}^d \to \mathbb{R}$ satisfies $Df[v] = \left(\nabla f, v\right)_{\mathbb{R}^d}$ for all $v \in \mathbb{R}^d$. An analogue of this identity can be used to define the gradient of a function $f : M \to \mathbb{R}$.

Definition A.45. Let (M, g) be a Riemannian manifold and $f \in C^\infty(M)$. The *gradient* ∇f is the unique vector field such that

$$\left(\nabla f(x), X_x\right)_x = X_x(f), \qquad x \in M, \quad X \in T_x M.$$

We state next, how the Riemannian metric g can be used to define a distance on the manifold M.

Proposition A.46. *Let (M, g) be a Riemannian manifold and set*

$$C^1_M(x_1, x_2) := \left\{\gamma \in C([a,b], M) : \gamma(a) = x_1, \quad \gamma(b) = x_2, \quad \gamma \textit{ is piecewise } C^1\right\}.$$

Then,

$$d_g(x_1, x_2) := \inf_{\gamma \in C^1_M(x_1,x_2)} \int_a^b |\gamma'(t)|_{\gamma(t)} \, \mathrm{d}t, \qquad x_1, x_2 \in M,$$

defines a metric on M.

Further properties of Riemannian manifolds which turn out to be important in the study of Sobolev spaces and Strichartz estimates are introduced in the following definition.

Definition A.47. Let (M, g) be a Riemannian manifold.

a) If the metric space (M, d_g) is complete, then (M, g) is called *complete.*

b) We say that (M, g) has *bounded geometry,* if for all multi-indices $\alpha \in \mathbb{N}_0^d$ and $k, l \in \{1, \ldots, d\}$ there is $C > 0$ such that $|\partial^\alpha g_{k,l}| \leq C$.

As a preparation for the definition of geodesics, we introduce the *Christoffel-symbol* $\Gamma^m_{k,l}(x)$ for $k, l, m = 1, \ldots, d$ by

$$\Gamma^m_{k,l}(x) = \frac{1}{2} \sum_{n=1}^{d} \left[\partial_k g_{n,l}(x) + \partial_l g_{n,k}(x) - \partial_n g_{k,l}(x)\right] g^{n,m}(x), \qquad x \in M.$$

Definition A.48. Let (M, g) be a Riemannian manifold. Let I be a compact interval, $\gamma : I \to M$ and denote the local coordinates of γ by $\gamma_1, \ldots \gamma_d$. Then, γ is called *geodesic* if we have

$$\gamma''_m(t) + \sum_{k,l=1}^{d} \Gamma^m_{k,l}(\gamma(t))\gamma'_k(t)\gamma'_l(t) = 0 \tag{A.30}$$

for $m \in \{1, \ldots, d\}$ and $t \in I$.

By the Hopf-Rinow Theorem, see [8], Theorem 1.37, we obtain a unique geodesic $\gamma_v : [0, 1] \to M$ with $\gamma(0) = v$ for each $x \in M$ and $v \in T_x M$. This motivates the following definition of the exponential map.

Definition A.49. Let (M, g) be a complete Riemannian manifold and $x \in M$.

a) The map

$$\exp_x : T_x M \to M, \qquad \exp_x(v) := \gamma_v(1),$$

is called *exponential map.*

b) The *injectivity radius* of (M, g) is defined as

$$\mathrm{inj}(M, g) := \inf_{x \in M} \sup \left\{\varepsilon > 0 : \quad \exp_x|_{B(x,\varepsilon)} \text{ is injective}\right\}.$$

In this thesis, the two most common assumptions on a Riemannian manifold (M, g) are either compactness or

$$M \text{ is complete, connected, has a positive injectivity radius and a bounded geometry.} \quad \text{(A.31)}$$

The following Proposition tells us that compactness is a stronger assumption.

Proposition A.50. *Let (M, g) be a connected and compact Riemannian manifold. Then, it satisfies* (A.31).

Proof. By the Hopf-Rinow Theorem 1.37 in [8], M is complete and Theorem 1.36 in [8] implies $\operatorname{inj}(M, g) > 0$. The boundedness of the geometry is an immediate consequence of compactness. □

After having introduced the basic notions from Riemannian geometry we will need in this thesis, we continue with the definition of function spaces on manifolds. We refer to [8], Section 3.4, and [87], Section 3.1.5, for a brief introduction to integration against the canonical volume measure $\mathcal{V}_g$ on manifolds. The spaces $L^p(M)$, $p \in [1, \infty]$ are defined in the usual way via the measure $\mathcal{V}_g$. For the sake of simplicity, we omit this measure in our notation and will just write $\mathrm{d}x$ instead of $\mathrm{d}\mathcal{V}_g(x)$. First, we turn our attention to the *Laplace-Beltrami operator*, which generalizes the standard Laplacian from $\mathbb{R}^d$ to the manifold setting.

Theorem A.51. *Let (M, g) be a complete Riemannian manifold.*

a) Then, the closure of the operator which is defined in local coordinates via

$$\Delta_g f := \frac{1}{\sqrt{\det g}} \sum_{j,k=1}^{d} \partial_j \left(g^{jk} \partial_k f \right), \qquad f \in C_c^\infty(M),$$

is a negative selfadjoint operator on $L^2(M)$. It is called Laplace-Beltrami operator and again denoted by Δ_g. In particular, Δ_g generates a contractive C_0-semigroup $\left(e^{t\Delta_g}\right)_{t\geq 0}$ on $L^2(M)$.

b) For each $p \in (1, \infty)$, the restriction of $\left(e^{t\Delta_g}\right)_{t\geq 0}$ to $L^2(M) \cap L^p(M)$ extends to contractive C_0-semigroup on $L^p(M)$. Its generator is called Laplace-Beltrami operator on $L^p(M)$ and is denoted by $\Delta_{g,p}$.

Proof. We refer to [112], Theorem 2.4, for part a) and Theorem 3.5 for b). □

We remark that we will often avoid the index p in $\Delta_{g,p}$ if there is no risk of confusion. Our next goal is to define Sobolev spaces on manifolds and relate them to the Laplace-Beltrami operator.

Definition A.52. Let (M, g) be a d-dimensional Riemannian manifold that satisfies (A.31).

a) Let $s \geq 0$, $p \in (1, \infty)$, $\mathfrak{M} := (U_i, \varphi_i)_{i\in I}$ be an atlas of M and $(\Psi_i)_{i\in I}$ a partition of unity subordinate to $\mathfrak{M}$. Then, we define the fractional Sobolev spaces $H^{s,p}(M)$ by

$$H^{s,p}(M) := \Big\{ f \in L^p(M) : \|f\|_{H^{s,p}(M)} := \Big(\sum_{i\in I} \|(\Psi_i f) \circ \varphi_i^{-1}\|^p_{H^{s,p}(\mathbb{R}^d)} \Big)^{\frac{1}{p}} < \infty \Big\}, \quad \text{(A.32)}$$

where $H^{s,p}(\mathbb{R}^d)$ is the Bessel potential space on $\mathbb{R}^d$. We write $H^s(M) := H^{s,2}(M)$.

b) For $p \in [1,\infty)$, we define $W^{1,p}(M)$ as the completion of $C_c^\infty(M)$ in the norm

$$\|f\|^p_{W^{1,p}(M)} := \int_M |f(x)|^p \mathrm{d}x + \int_M |\nabla f(x)|_x^p \mathrm{d}x, \qquad f \in C_c^\infty(M).$$

In the study of the Sobolev spaces from Definition A.52, the fractional powers of $I - \Delta_{g,p}$ turn out to be very useful. These operators are defined by

$$(I-\Delta_{g,p})^{-\alpha} f := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} e^{t\Delta_{g,p}} f \mathrm{d}t$$

for $\alpha > 0$. Note that in the case $p = 2$ this coincides with the definition via the Borel functional calculus because of the identity $\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-\lambda t} f \mathrm{d}t = \lambda^{-\alpha}$ for $\lambda > 0$. For further details on fractional powers of generators of C_0-semigroups, we refer to [98], chapter 6. In the following Proposition, we list characterizations and embedding properties of the Sobolev spaces from Definition $A.52$.

Proposition A.53. *Let (M,g) be a d-dimensional Riemannian manifold that satisfies* (A.31). *Let $s \geq 0$ and $p \in (1,\infty)$.*

a) We have $H^{s,p}(M) = R((I-\Delta_{g,p})^{-\frac{s}{2}})$ with $\|f\|_{H^{s,p}} \eqsim \|v\|_{L^p}$ for $f = (I-\Delta_{g,p})^{-\frac{s}{2}} v$. Furthermore, we have $H^{1,p}(M) = W^{1,p}(M)$.

b) For $s > \frac{d}{p}$, we have $H^{s,p}(M) \hookrightarrow L^\infty(M)$.

c) Let $s \geq 0$ and $p \in (1,\infty)$. Suppose $p \in [2, \frac{2d}{(d-2s)_+})$ or $p = \frac{2d}{d-2s}$ if $s < \frac{d}{2}$. Then, the embedding $H^s(M) \hookrightarrow L^p(M)$ is continuous.

d) If M is compact and we have $0 < s \leq 1$ as well as $p \in [1, \frac{2d}{(d-2s)_+})$, the embedding $H^s(M) \hookrightarrow L^p(M)$ is compact.

e) For $s, s_0, s_1 \geq 0$ and $p, p_0, p_1 \in (1,\infty)$ and $\theta \in (0,1)$ with

$$s = (1-\theta)s_0 + \theta s_1, \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

we have $[H^{s_0,p_0}(M), H^{s_1,p_1}(M)]_\theta = H^{s,p}(M)$.

Proof. ad a): See [117], Theorem 7.4.5. We remark that in the reference, $H^{s,p}$ is defined via the range identity from the Proposition and the identity from Definition A.52 is proved.
ad b): See [21], Theorem III.1.2. d1).
ad c): See [21], Theorem III.1.2. d1).
ad d): Since M is compact, we can choose a finite collection of charts and a finite partition of unity. Hence

$$\|f\|_{H^s(M)} := \Big(\sum_{i=1}^N \|(\Psi_i f) \circ \varphi_i^{-1}\|^2_{H^s(\mathbb{R}^d)}\Big)^{\frac{1}{2}} = \Big(\sum_{i=1}^N \|(\Psi_i f) \circ \varphi_i^{-1}\|^2_{H^s(\mathcal{O})}\Big)^{\frac{1}{2}} \tag{A.33}$$

for a sufficiently large smooth bounded domain $\mathcal{O} \subset \mathbb{R}^d$. By [47], Corollary 7.2 and Theorem 8.2, the embedding $H^s(\mathcal{O}) \hookrightarrow L^p(\mathcal{O})$ is compact for $s \in (0,1)$ with $s < \frac{d}{2}$ and $p \in [1, \frac{2d}{d-2s})$. Note that in the reference, the result is proved in terms of the Slobodetski space $W^{s,2}(\mathcal{O})$, but we can use the identity $W^{s,2}(\mathcal{O}) = H^s(\mathcal{O})$. The embedding result combined with (A.33) yields the assertion.

ad e): See [117], Section 7.4.5, Remark 2. □

In the theory of L^p-spaces on $\mathbb{R}^d$, Bernstein inequalities and the Littlewood-Paley Theorem are classical techniques to estimate functions. For example, one can find these results in [9], Lemma 2.1 and [60], Theorem 6.1.2, respectively. If one replaces the frequency analysis based on the Fourier transform by spectral theoretic methods, similar results can also be proved for the manifold case. In the following, we collect some of the results in this spirit which we will need in chapter 5. The following Lemma deals with a Littlewood-Paley type decomposition of $L^p(M)$ for $p \in [2, \infty)$.

Lemma A.54. *Let M be a compact Riemannian manifold and $\psi \in C_c^\infty(\mathbb{R})$, $\varphi \in C_c^\infty(\mathbb{R} \setminus \{0\})$ with*

$$1 = \psi(\lambda) + \sum_{k=1}^{\infty} \varphi(2^{-k}\lambda), \qquad \lambda \in \mathbb{R}. \tag{A.34}$$

Then, we have

$$\|f\|_{L^2} \eqsim \left(\|\psi(\Delta_g)f\|_{L^2}^2 + \sum_{k=1}^{\infty} \|\varphi(2^{-k}\Delta_g)f\|_{L^2}^2 \right)^{\frac{1}{2}}, \qquad f \in L^2(M), \tag{A.35}$$

and

$$\|f\|_{L^p} \lesssim_p \|\psi(\Delta_{g,p})f\|_{L^p} + \left(\sum_{k=1}^{\infty} \|\varphi(2^{-k}\Delta_{g,p})f\|_{L^p}^2 \right)^{\frac{1}{2}}, \qquad f \in L^p(M), \tag{A.36}$$

for $p \in [2, \infty)$.

Before we proceed with the proof, we refer to [9], Proposition 2.10, for certain ψ and φ which fulfill (A.34).

Proof. Let $p \in (1, \infty)$. From [22], page 2, we infer

$$\|f\|_{L^p} \eqsim \left\| \left(|\psi(\Delta_{g,p})f|^2 + \sum_{k=1}^{\infty} |\varphi(2^{-k}\Delta_{g,p})f|^2 \right)^{\frac{1}{2}} \right\|_{L^p}, \qquad f \in L^p(M). \tag{A.37}$$

Note that this equivalence can also be found in a more general setting in [80], Theorem 4.1 combined with estimate (2.9). The estimate (A.37) yields (A.35) by Fubini and (A.36) by Minkowski's inequality. □

The previous Lemma indicates the importance of estimating operators of the form $\varphi(h^2\Delta_g)$ for $h \in (0, 1]$. In the next Lemma, we state how they act in L^p-spaces and Sobolev spaces.

Lemma A.55. *Let M be a compact Riemannian manifold.*

a) Let $1 \le q \le r \le \infty$. For any $\varphi \in C_c^\infty(\mathbb{R})$, there is $C > 0$ such that for $h \in (0, 1]$

$$\|\varphi(h^2\Delta_g)\|_{\mathcal{L}(L^q, L^r)} \le C h^{d(\frac{1}{r} - \frac{1}{q})}.$$

b) Let $\varphi \in C_c^\infty(\mathbb{R} \setminus \{0\})$, $p \in (1, \infty)$ and $s \ge 0$. Then, there is $C > 0$ such that for $h \in (0, 1]$

$$\|\varphi(h^2\Delta_{g,p})f\|_{L^p} \le C h^s \|\varphi(h^2\Delta_{g,p})f\|_{H^{s,p}}, \qquad f \in H^{s,p}(M).$$

Note that these kind of estimates are usually called *Bernstein inequalities.*

Proof. ad a): See [35], Corollary 2.2.
ad b): We want to use the spectral multiplier Theorem from [104], Theorem 7.23. Compact manifolds are homogeneous spaces and the Laplace-Beltrami operator has upper Gaussian bounds by [61], Corollary 5.5 and Theorem 6.1. Take $\tilde{\varphi} \in C_c^\infty(\mathbb{R}\setminus\{0\})$ with $\tilde{\varphi} = 1$ on $\operatorname{supp}(\varphi)$ and define

$$f : (0,\infty) \to \mathbb{R}, \qquad f(t) := t^{-\frac{s}{2}}\tilde{\varphi}(-h^2 t),$$

Then, we have $\varphi(-h^2 t) = f(t)t^{\frac{s}{2}}\varphi(-h^2 t)$ and

$$\sup_{t>0} |t^k f^{(k)}(t)| \lesssim h^s.$$

By (7.69) in [104], we can apply Theorem 7.23 in [104] and obtain

$$\begin{aligned} \|\varphi(h^2\Delta_{g,p})f\|_{L^p} &= \|f(-\Delta_{g,p})\,(-\Delta_{g,p})^{\frac{s}{2}}\,\varphi(h^2\Delta_{g,p})f\|_{L^p} \lesssim h^s\|\,(-\Delta_{g,p})^{\frac{s}{2}}\,\varphi(h^2\Delta_{g,p})f\|_{L^p} \\ &\lesssim h^s\|\varphi(h^2\Delta_{g,p})f\|_{H^{s,p}}. \end{aligned}$$

□

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