2-Categorical Aspects of Quasi-Categories

Zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften

von der KIT-Fakultät für Mathematik des
Karlsruher Instituts für Technologie (KIT)
genehmigte

Dissertation

von

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Tag der mündlichen Prüfung: 28. Juni 2017
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Acknowledgements

First and foremost, I want to thank Fabian Januszewski, Frank Herrlich and Gabriela Weitze-Schmithüsen[1] for allowing me latitude in pursuing my research. During my time in Karlsruhe, I enjoyed a lot of freedom, which is not something to be taken for granted.

I am also indebted to Richard Garner who unhesitatingly accepted to serve as external reviewer for this thesis. I confess with embarrassement that those aspects of my work that are closest to his did not make it into this text.

Each member of the number theory and algebraic geometry group in Karlsruhe has had a stake in the completion of this work — be it by taking off some load of me during the last months, by discussing the very topic treated in this text or by simply having fun together during lunch or while having a coffee. From this group of my colleagues, I have to single out my officemate, Felix Wellen, with whom I had a lot of interesting discussions[2] and without

[1] in alphabetical order

[2] More often than not, I was spluttering callow thoughts and it was only with the help of Felix that I was able to arrange my ideas.

whom the last years would probably have been less fun.

Some of the ideas presented here came to light during my stay in Barcelona and I want to thank the organisers of the IRTATCA program at the CRM for giving me the opportunity to work and live in such a beautiful and friendly place. There, during insightful discussions with Hanno Becker, Anthony Blanc, Vincent Franjou, Moritz Groth, Martin Kalck and Greg Stevenson[3], I learned a lot about mathematics, doing mathematics and life in general.

It was the workshop on derivators in Prague in December 2016, however, that prompted me to make precise my intuition about pasting. This ultimately led to a manuscript that slowly transformed into this thesis and I thank the organisers, Moritz Groth and Jan Stovicek, for this impetus.

Finally, I thank my family for having supported me during all the years and I want to apologise to Anna, Jakob and Luise.

[3] I hope that all those that I forgot to mention will be forgiving.

Contents

Overview

This thesis is concerned with a pasting theorem for categories enriched over quasi-categories. The notion of pasting goes back to the theory of 2-categories, where one frequently encounters diagrams such as

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[dr, bend right, "u"'] & B \arrow[r, "b"] \arrow[d, "f"'] \arrow[l, phantom, shift right=4, "\psi\Swarrow"] & C \arrow[d, "g"] \arrow[dr, bend left, "c"] \arrow[l, phantom, shift right=4, "\phi\Swarrow"] \arrow[r, phantom, shift right=4, "\Swarrow\xi"] & \\
& X \arrow[r, "v"'] & Y \arrow[r, "w"'] & Z .
\end{tikzcd} \tag{0.1}$$

The axioms of a 2-category ensure that one can compose two cells like e. g. ψ and ϕ above. In this way, one obtains a composite

$$cba \xrightarrow{\xi \cdot ba} wgba \xrightarrow{w \cdot \phi \cdot a} wvfa \xrightarrow{wv \cdot \psi} wvu$$

of the whole diagram. Pasting refers to this cobbling of cells in 2-categories and a diagram such as (0.1) is consequently called a pasting diagram.

In the case of (0.1), the order in which one forms the composites of two such cells so as to obtain a composite of the whole diagram is actually unique.

However, there are other diagrams such as e. g.

$$\begin{tikzcd} A \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\phi\Downarrow"] & B \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\psi\Downarrow"] & C, \end{tikzcd}$$

where we have the two possibilities

$$fu \xrightarrow{\psi \cdot u} gu \xrightarrow{g \cdot \phi} gv \qquad \text{and} \qquad fu \xrightarrow{f \cdot \phi} fv \xrightarrow{\psi \cdot v} gv$$

to obtain a composite of the whole diagram. These two compositions actually do coincide by the axioms of a 2-category. A *pasting theorem* simply asserts that any well-formed diagram admits a composite that is independent from the order in which we form compositions of individual cells in the diagram. Such a theorem was conjectured for 2-categories in [KS74] and proven by John Power in [Pow90].

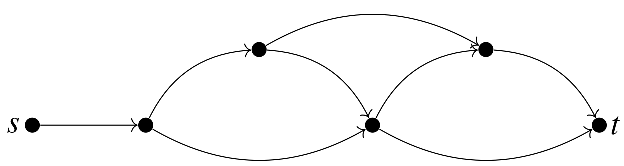
The work of Power in [Pow90] has two facets. First of all, the formulation of a pasting theorem requires a formal definition of what exactly a well-formed diagram actually is. Power chooses certain plane graphs as his model of a diagram in a 2-category. Building on this notion of a diagram and its basic combinatorial properties, Power then goes on to show that any labeling of such a diagram in a 2-category admits a uniquely determined composite.

In this thesis, we prove a pasting theorem for a specific model of $(\infty, 2)$-categories, namely categories enriched over quasi-categories. We reuse the model of diagrams introduced by Power and extend his work from 2-categories to $(\infty, 2)$-categories. However, we have to cope with serious technical difficulties due to the fact that composition in $(\infty, 2)$-categories is only unital and associative up to higher dimensional invertible cells. Moreover, we cannot even expect a diagram to have a unique composite but have to settle with composites that are unique up to higher dimensional invertible cells. More precisely, we show that the space of compositions of a given diagram is nonempty and contractible:

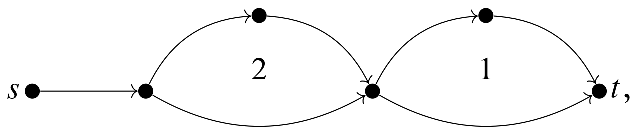
Theorem D *Consider a globular graph G and a category $\mathbb{A}$ enriched over quasi-categories. The space $C(\Lambda)$ of compositions of a given labeling Λ of G in $\mathbb{A}$ is a nonempty contractible Kan complex.*

A labeling as in the statement of the above theorem or the work of Power simply refers to a sensible assignment of cells in the category $\mathbb{A}$ to any cell in the graph G.

Let us say a few words about our model of diagrams before we comment on the proof of Theorem D. We model our diagrams on certain plane graphs that we call globular graphs for they typically look as follows:



Although the concept of globular graphs is due to [Pow90], we have to extend and adapt it to our model of $(\infty, 2)$-categories. This means in particular that we have to associate with any such graph a family $X_{a,b}$ of simplicial sets indexed by pairs (a, b) of vertices of the graph. To this end, we introduce the nerve of such a graph and develop a pictorial calculus to describe its simplices. A simplex in $\mathrm{N}(G)$ essentially corresponds to a picture such as



,

where the numbers in the faces should be thought of as specifying the order of composition. In order to describe partial composites in terms of these simplicial sets, we then go on to restrict the type of simplices that may appear in this nerve. This line of thought eventually leads to our notion of pasting diagram.

Let us now comment on the proof of Theorem D. There are three main aspects to the proof and each aspect can be handled individually. In fact, the three main chapters of this thesis are more or less independent from each other and deal with one issue at a time.

Global aspect If a pasting diagram Σ satisfies a certain technical assumption, then we are able to associate with Σ a simplicial category $\mathbb{C}[\Sigma]$. For the maximal pasting diagram with underlying graph G one obtains for example the free 2-category on this graph in the sense of Power. We use these simplicial categories $\mathbb{C}[\Sigma]$ to model partial composites of diagrams in some simplicial category $\mathbb{A}$ by simplicial functors $\mathbb{C}[\Sigma] \to \mathbb{A}$. Given such a partially composed diagram $u\colon \mathbb{C}[\Sigma] \to \mathbb{A}$, we may add all missing composite cells to the abstract diagram Σ and thus obtain a new diagram Π. The problem of finding a composition of the concrete diagram $u\colon \mathbb{C}[\Sigma] \to \mathbb{A}$ then amounts to the problem of finding an extension of u as in the diagram

$$\begin{tikzcd} \mathbb{C}[\Sigma] \arrow[r, "u"] \arrow[d, "\text{inclusion}"'] & \mathbb{A} \\ \mathbb{C}[\Pi]. \arrow[ur, dashed] & \end{tikzcd} \tag{0.2}$$

We attack the problem of constructing such an extension by considering the more general problem of finding a simplicial functor $\mathbb{C}[\Pi] \to \mathbb{A}$ that renders a diagram such as

$$\begin{tikzcd} \mathbb{C}[\Sigma] \arrow[r, "u"] \arrow[d, "\text{inclusion}"'] & \mathbb{A} \arrow[d, "p"] \\ \mathbb{C}[\Pi] \arrow[r] \arrow[ur, dashed] & \mathbb{B} \end{tikzcd} \tag{0.3}$$

commutative. This solves the original problem of finding a composition of $u\colon \mathbb{C}[\Sigma] \to \mathbb{A}$ as long as we allow the unique simplicial functor $\mathbb{A} \to *$ to feature as the functor p.

Consider a class $\mathcal{R}$ of maps of simplicial sets that contains all isomorphisms and let $\mathcal{L} = {}^{\pitchfork}\mathcal{R}$ be the class of maps that have the left lifting property against all maps in $\mathscr{R}$. We solve the problem (0.3) for those diagrams of simplicial categories, where p is a local $\mathscr{R}$-functor, that is, $p\colon \mathbb{A}(a, a') \to \mathbb{B}(pa, pa')$ is an $\mathscr{R}$-map for all $a, a' \in \mathbb{A}$. Our solution to (0.3) then takes the form of the following theorem:

Theorem B *The functor $\mathbb{C}[\Sigma] \to \mathbb{C}[\Pi]$ induced by an inclusion of complete pasting diagrams has the left lifting property against all local $\mathscr{R}$-functors if and only if the map*

$$\mathrm{N}(\Sigma_{x,y} \multimap \Pi_{x,y}) \to \mathrm{N}(\Pi_{x,y})$$

is an $\mathscr{L}$-map for all vertices $x, y \in \Sigma$.

The diagrams $\Sigma_{x,y} \multimap \Pi_{x,y}$ appearing in the statement of Theorem B are certain intermediate diagrams between Σ and Π that are introduced at the very end of § 2.5.

Local aspect Now suppose that the classes $\mathscr{L}$ and $\mathscr{R}$ from above are the classes of mid anodyne maps and mid fibrations, respectively. In this case, the unique functor $\mathbb{A} \to *$ from a simplicial category $\mathbb{A}$ to the terminal simplicial category is a local $\mathscr{R}$-functor if and only if $\mathbb{A}$ is enriched over quasi-categories. Theorem B hence reduces the problem (0.2) of finding a composition of a diagram $u\colon \mathbb{C}[\Sigma] \to \mathbb{A}$ in some category enriched over quasi-categories to the problem of showing that certain maps

$$\mathrm{N}(\Sigma_{x,y} \multimap \Pi_{x,y}) \to \mathrm{N}(\Pi_{x,y})$$

are mid anodyne. We identify a certain class of inclusions $\Sigma \to \Pi$ of pasting diagrams for which these maps are indeed mid anodyne. More precisely, we deduce that the maps in question are mid anodyne from the following theorem:

Theorem C *Let $\Sigma \to \Pi$ be an inclusion of complete pasting diagrams such that both Σ and Π contain all the interior faces of the underlying graph and are closed under taking subdivisions. Then*

$$\mathrm{N}(\Sigma) \to \mathrm{N}(\Pi)$$

is mid anodyne.

Bootstrapping There is still one problem left that did not yet appear in the above discussion. The simplicial categories $\mathbb{C}[\Sigma]$ featuring as the domain of diagrams $\mathbb{C}[\Sigma] \to \mathbb{A}$ exist only under certain technical assumptions. In order to complete the proof of Theorem D that we sketched so far, we have to make sure that we can associate with any labeling Λ of a diagram a simplicial functor $\mathbb{C}[\Sigma] \to \mathbb{A}$. This is achieved by

Theorem A *Suppose that Σ is the minimal complete pasting diagram on some globular graph G. The map*

$$\mathrm{Cat}_{\widehat{\mathbb{\Delta}}}(\mathbb{C}[\Sigma], \mathbb{A}) \to L(G, \mathbb{A}), \quad u \mapsto \Lambda_u,$$

that sends a simplicial functor u to its associated labeling Λ_u is a bijection.

The chapters of this thesis correspond to these three aspects of our main theorem. The mutual interdependence of the individual chapters can be seen in the following diagram:

$$\begin{tikzcd}
& \begin{array}{c}\text{chapter 1}\\ \text{Preliminaries}\end{array} \arrow[d] & \\
& \begin{array}{c}\text{chapter 2}\\ \text{Basic definitions}\end{array} \arrow[dl] \arrow[d] \arrow[dr] & \\
\begin{array}{c}\text{chapter 3}\\ \text{Theorem A}\end{array} \arrow[r, dashed, "\S\,3.1"] \arrow[dr] & \begin{array}{c}\text{chapter 4}\\ \text{Theorem B}\end{array} \arrow[d] & \begin{array}{c}\text{chapter 5}\\ \text{Theorem C}\end{array} \arrow[dl] \\
& \begin{array}{c}\text{chapter 6}\\ \text{Theorem D}\end{array} &
\end{tikzcd}$$

1 Preliminaries

1.1 Lifting properties

Fix a category $\mathcal{C}$. A map $i: A \to B$ is said to have the left lifting property against a map $p: X \to Y$ if one finds a lift ℓ in any commutative solid diagram such as

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

In this case, we also write $i \pitchfork p$. A map i as above has the left lifting property against a class $\mathcal{R}$ of maps in $\mathcal{C}$, if it has the left lifting property against any map $p \in \mathcal{R}$.

1.1.1. Proposition *The class $\mathcal{L}$ of all maps i that have the left lifting property against a given class $\mathcal{R}$ of maps is closed under pushouts and (transfinite) compositions.*

We learned the following proposition and its proof from Richard Garner:

1.1.2. Proposition *Let $\mathcal{R}$ be a class of maps in a category $\mathcal{C}$ with finite colimits and let $\mathcal{L}$ be the class of maps having the left lifting property against all $\mathcal{R}$-maps. Given a diagram*

$$\begin{tikzcd} X_2 \arrow[d, "f_2"'] & X_0 \arrow[l, "u"'] \arrow[r, "v"] \arrow[d, "f_0"'] & X_1 \arrow[d, "f_1"] \\ Y_2 & Y_0 \arrow[l, "u'"] \arrow[r, "v'"'] & Y_1 \end{tikzcd}$$

with f_0 and the induced maps $X_i \amalg_{X_0} Y_0 \to Y_i$ in $\mathcal{L}$, the induced map

$$f_2 \amalg_{f_0} f_1 \colon X_2 \amalg_{X_0} X_1 \to Y_2 \amalg_{Y_0} Y_1$$

is also in $\mathcal{L}$.

Proof. Consider the class

$$\mathcal{R}' = \left\{(\mathrm{id}_X, p) \colon \mathrm{id}_X \to p \mid p \colon X \to Y \in \mathcal{R}\right\}$$

of maps in the arrow category $\mathcal{C}^2$ and let $\mathcal{L}'$ be the class of maps with the left lifting property against $\mathcal{R}'$.

Let us consider an $\mathcal{R}'$-map $(\mathrm{id}_X, p) \colon \mathrm{id}_X \to p$ and a lifting problem of a map $(u, v) \colon f \to g$ against (id_X, p) in $\mathcal{C}^2$. Such a lifting problem corresponds to a diagram

$$\begin{tikzcd}
& A \arrow[rr, "\mathrm{id}"] \arrow[dd, "\mathrm{id}"'] & & A \arrow[dd, "v"] \\
X \arrow[ur] \arrow[rr, "f"'] \arrow[dd, "u"'] & & Y \arrow[ur] \arrow[dd, "v"] & \\
& A \arrow[rr, "u"'] & & B \\
X' \arrow[ur] \arrow[rr, "g"'] & & Y'. \arrow[ur] &
\end{tikzcd}$$

If the source f of (u, v) is the identity of the initial object $\varnothing$ in $\mathcal{C}$, this lifting problem is equivalent to the lifting problem in the right face of the cube. We therefore have $f \in \mathcal{L}$ if and only if $\mathrm{id}_\varnothing \to f \in \mathcal{L}'$.

Moreover, we have for any such lifting problem an induced square

$$\begin{tikzcd}
X' \amalg_X Y \arrow[r] \arrow[d] & A \arrow[d, "p"] \\
Y' \arrow[r] & B
\end{tikzcd}$$

in $\mathcal{C}$. It is easy to see that a solution to the latter lifting problem in $\mathcal{C}$ is equivalent to a solution to the lifting problem in $\mathcal{C}^2$.

The maps $f_0 \to f_1$ and $f_0 \to f_2$ occuring in the span

$$f_2 \leftarrow f_0 \to f_1$$

in the statement of the lemma thus have the left lifting property against all $\mathcal{R}'$-maps. This implies in particular that the inclusion $f_1 \to f_2 \amalg_{f_0} f_1$ has the left lifting property against all $\mathcal{R}'$-maps and so does the composition

$$\mathrm{id}_\varnothing \to f_0 \to f_1 \to f_2 \amalg_{f_0} f_1.$$

This finishes the proof as $f \in \mathcal{L}$ if and only if $\mathrm{id}_\varnothing \to f \in \mathcal{L}'$. $\square$

For the statement of an important corollary of this proposition, recall that a class $\mathcal{L}$ of maps in some category has the *right cancellation property* if for any diagram

$$\bullet \xrightarrow{i} \bullet \xrightarrow{j} \bullet$$

with $i \in \mathcal{L}$ and $ji \in \mathcal{L}$, we have $j \in \mathcal{L}$, too.

1.1.3. Corollary *Let $\mathcal{R}$ be a class of maps in a category $\mathcal{C}$ with finite colimits and let $\mathcal{L}$ be the class of maps having the left lifting property against all $\mathcal{R}$-maps. Further suppose that there is a diagram*

$$\begin{tikzcd}
X_2 \arrow[d, "f_2"'] & X_0 \arrow[l, "u"'] \arrow[r, "v"] \arrow[d, "f_0"'] & X_1 \arrow[d, "f_1"] \\
Y_2 & Y_0 \arrow[l, "u'"] \arrow[r, "v'"'] & Y_1
\end{tikzcd}$$

with $f_0, f_1, f_2 \in \mathcal{L}$. If $\mathcal{L}$ has the right cancellation property, then

$$f_2 \amalg_{f_0} f_1 \colon X_2 \amalg_{X_0} X_1 \to Y_2 \amalg_{Y_0} Y_1$$

is in $\mathcal{L}$.

Proof. We verify the hypotheses of Proposition 1.1.2. We have a commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r, "v"] \arrow[d, "f_0"'] & X_1 \arrow[d] \arrow[ddr, bend left, "f_1"] & \\
Y_0 \arrow[r] \arrow[drr, bend right, "v'"'] & X_1 \amalg_{X_0} Y_0 \arrow[dr] & \\
& & Y_1
\end{tikzcd}$$

in which both the map $X_1 \to X_1 \amalg_{X_0} Y_0$ and the map $X_1 \to Y_1$ are in $\mathcal{L}$. We thus conclude that $X_1 \amalg_{X_0} Y_0 \to Y_1$ is in $\mathcal{L}$ by the right cancellation property of $\mathcal{L}$. The other square is handled analogously. $\square$

1.2 Simplicial sets

We briefly recall some aspects of the theory of simplicial sets and quasi-categories. This serves merely the purpose of fixing notation and quoting some results from the literature, so that we can easily refer to them when needed. For more information on simplicial sets see e. g. [GJ09]. An introductory text on quasi-categories is [Gro10], for instance, while the original sources [Joy08a; Lur09] develop the theory in much greater detail.

Basic definitions Let $\mathbb{\Delta}$ denote the category of nonempty, finite ordinals $[n] = \{0, \dots, n\}$ and monotone maps. The category $\mathbb{\Delta}$ admits a presentation with generators cofaces $\delta^i \colon [n-1] \to [n]$ and codegeneracies $\sigma^i \colon [n+1] \to [n]$, $0 \leq i \leq n$, given by

$$\delta^i(j) = \begin{cases} j & \text{if } j < i, \\ j+1 & \text{else} \end{cases} \qquad \text{and} \qquad \sigma^i(j) = \begin{cases} j & \text{if } j \leq i, \\ j-1 & \text{else,} \end{cases}$$

and relations

$$\begin{aligned} \delta^i\delta^j &= \delta^j\delta^{i-1} \quad \text{for all } i > j, \\ \sigma^i\sigma^j &= \sigma^j\sigma^{i+1} \quad \text{for all } i \geq j, \\ \sigma^i\delta^j &= \begin{cases} \delta^j\sigma^{i-1} & \text{if } j < i, \\ 1 & \text{if } i = j \text{ or } j = i+1, \\ \delta^{j-1}\sigma^i & \text{if } j > i+1. \end{cases} \end{aligned} \tag{1.1}$$

The category $\widehat{\mathbb{\Delta}}$ of simplicial sets is the category of presheaves $X \colon \mathbb{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ and natural transformations. Simplicial sets are a complete and cocomplete category as is true for any category of set-valued presheaves. For $X \in \widehat{\mathbb{\Delta}}$

we write X_n instead of $X([n])$. Given the presentation of $\mathbb{\Delta}$ by generators and relations, it follows that a simplicial set X is completely determined by the sets X_n, $n \in \mathbb{N}$, together with maps $d_i\colon X_n \to X_{n-1}$ and $s_i\colon X_n \to X_{n+1}$ subject to relations that are dual to (1.1). Observe that simplices in a simplicial set are oriented as the existence of a 1-simplex σ with $d_1\sigma = a$ and $d_0\sigma = b$ does not guarantee the existence of a 1-simplex with $d_0\sigma = a$ and $d_1\sigma = b$.

The elements $x \in X_n$ are referred to as n-simplices. Such an n-simplex is *degenerate* if $x = s_i(x')$ for some $x' \in X_{n-1}$ and *nondegenerate* otherwise.

The representable presheaf $\mathbb{\Delta}(\bullet, [n])$ is denoted by Δ^n and called the n-simplex. There are two types of simplicial subsets of Δ^n that are of interest to us, namely the *boundary* $\partial\Delta^n \subseteq \Delta^n$ given by

$$\partial\Delta^n\big([m]\big) = \big\{\alpha\colon [m] \to [n] \;\big|\; \alpha \text{ not surjective}\big\}$$

and the *i-th horn* $\Lambda^n_i \subseteq \partial\Delta^n$ for $0 \le i \le n$ given by

$$\Lambda^n_i\big([m]\big) = \big\{\alpha\colon [m] \to [n] \;\big|\; [n] \smallsetminus \{i\} \not\subseteq \alpha\big([m]\big)\big\}.$$

A horn $\Lambda^n_i \subseteq \Delta^n$ is an *inner horn* if $0 < i < n$. One should think of a horn Λ^n_i as the boundary $\partial\Delta^n$ with the face opposite to the i-th vertex removed.

Nerves Consider a category $\mathcal{C}$ and a functor $f\colon \mathbb{\Delta} \to \mathcal{C}$.[1] This data gives rise to a *nerve functor* $\mathrm{N}\colon \mathcal{C} \to \widehat{\mathbb{\Delta}}$ by virtue of the composition

$$\mathcal{C} \xrightarrow{\text{Yoneda}} \widehat{\mathcal{C}} \xrightarrow{f^*} \widehat{\mathbb{\Delta}},$$

where f^* denotes the functor $X \mapsto X \circ f$. In less fancy language, this means nothing but that $\mathrm{N}(c) = \hom(f(*), c)$.

Important examples of this construction are the nerves of partially ordered sets and, more generally, of categories. The category $\mathbb{\Delta}$ embeds in the category Poset of partially ordered sets and monotone maps, hence there is a nerve $\mathrm{N}\colon \mathrm{Poset} \to \widehat{\mathbb{\Delta}}$. Unravelling the definitions, one sees that an n-simplex σ in $\mathrm{N}(P)$ is a chain $\sigma = (x_0 \leq \cdots \leq x_n)$ of elements in P. The boundary maps d_i omit the i-th element in this chain and the degeneracies s_i repeat it.

One similarly has an embedding of $\mathbb{\Delta}$ into the category Cat of small categories that sends the ordinal $[n]$ to the category $[n]$ having $n+1$ objects $0, \dots, n$ and a unique morphism $i \to j$ if and only if $i \leq j$. The nerve $\mathrm{N}(\mathcal{A})$ of a category is thus the simplicial set whose set $\mathrm{N}(\mathcal{A})_n$ of n-simplices is the set of functors $[n] \to \mathcal{A}$.

Kan fibrations and anodyne maps Let us consider the sets

$$\mathbb{\Lambda} = \{\Lambda^n_i \to \Delta^n \mid n \in \mathbb{N} \text{ and } 0 \leq i \leq n\}$$

[1]Such functors are commonly called cosimplicial objects in $\mathcal{C}$.

of all canonical horn inclusions and the set

$$\partial\mathbb{\Delta} = \{\partial\Delta^n \to \Delta^n \mid n \in \mathbb{N}\}.$$

of all canonical boundary inclusions.

A map $p\colon X \to Y$ between simplicial sets is a *Kan fibration* if it possesses the right lifting property against all maps in $\mathbb{\Lambda}$, that is, if for all n and all $0 \leq i \leq n$ and each diagram

$$\begin{tikzcd}
\Lambda_i^n \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & Y,
\end{tikzcd}$$

where the left vertical map is a map in $\mathbb{\Lambda}$, there exists a diagonal lift as indicated. A simplicial set X is a *Kan complex* if the unique map $X \to \Delta^0$ is a Kan fibration. The map p is a *trivial Kan fibration* if it possesses the right lifting property against all maps in $\partial\mathbb{\Delta}$. In fact, trivial Kan fibrations have the right lifting property against all monomorphisms in $\widehat{\mathbb{\Delta}}$.

Kan fibrations have the right lifting property against all retracts of transfinite compositions of pushouts of maps in $\mathbb{\Lambda}$. These maps are often called anodyne and they actually comprise the class of all maps having the left lifting property against all Kan fibrations.

1.3 Quasi-categories

Quasi-categories are certain simplicial sets that first showed up in the work of Boardman and Vogt [BV73]. Joyal seems to have been the first one to realise that quasi-categories behave like categories with compositions defined only up to infinitely many higher coherence cells. It seems important to the author to mention that quasi-categories are by far not the only models available for such generalised categories. In fact, there are many such models linked by an intricate net of comparison theorems.

Mid fibrations and quasi-categories With the advent of quasi-categories came a whole family of different fibrations in $\widehat{\mathbb{\Delta}}$ that serve different purposes. As we need so-called mid fibrations only, we refer the reader to [Joy08a] and [Lur09, Chapter 2] for further notions of fibrations relevant to the theory of quasi-categories.

A map $p\colon X \to Y$ is a *mid fibration* if it possesses the right lifting property against all maps in the set

$$\mathbb{\Lambda}^{\mathrm{i}} = \{\Lambda^n_i \to \Delta^n \mid n \in \mathbb{N} \text{ and } 0 < i < n\}$$

of the canonical inclusions of inner horns. A simplicial set X is a *quasi-category* if the unique map $X \to \Delta^0$ is a mid fibration.

Let us give a quick example that illustrates why it might make sense to consider a quasi-category as a kind of generalised category. We picture the

0-simplices of a quasi-category X as its set of objects and the 1-simplices σ with $d_1\sigma = a$ and $d_0\sigma = b$ as the morphisms $a \to b$. Given two composable 1-simplices σ and τ, we fabricate a map $\Lambda^2_1 \to X$ that sends the edge $0 \to 1$ in Λ^2_1 to σ and the edge $1 \to 2$ to τ. The fact that this map is well-defined is equivalent to the condition that σ and τ be composable. As X is a quasi-category, we find a lift in the diagram

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^2 \arrow[r] \arrow[ur, dashed] & \Delta^0. \end{tikzcd}$$

This lift yields the composition of σ and τ as the face of Δ^2 missing in the inner horn Λ^2_1. Moreover, it also yields a 2-simplex witnessing that the 1-simplex in X selected by this new edge is indeed a composition of σ and τ. The composition is not uniquely determined but one can show with a similar argument that any two compositions can be compared by a non-unique 3-simplex. This continues ad infinitum. In order to be able to compare higher and higher cells in the quasi-category and talk about suitable notions of uniqueness, one needs to develop some techniques – mid fibrations and mid anodyne maps are one of these.

A *mid anodyne* map is a retract of a transfinite composition of pushouts of maps in $\mathbb{\Lambda}^{\mathrm{i}}$. Again, mid anodyne maps are precisely the class of maps having the left lifting property against all mid fibrations. The mid anodyne maps, however, are not the trivial cofibrations in Joyal's model structure for quasi-categories. In fact, no set of generating trivial cofibrations for the model structure with fibrant objects the quasi-categories and cofibrations the monomorphisms is known.

Closure properties of mid anodyne maps We have already seen that mid anodyne maps are closed under transfinite compositions, pushouts and retracts. Here, we collect some closure properties of mid anodyne maps that are harder to obtain. The following propositions are due to Joyal, see [Joy08b, Theorem 2.17 and 2.18] or [Lur09, Corollary 2.3.2.4 and 2.3.2.5].

1.3.1. Proposition *Let $i: A \to A'$ be a mid anodyne map and $j: B \to B'$ be an arbitrary monomorphism of simplicial sets. Then the canonical map*

$$A \times B' \coprod_{A \times B} A' \times B \to A' \times B'$$

is mid anodyne.

1.3.2. Corollary *Mid anodyne maps are closed under products with arbitrary simplicial sets X.*

1.3.3. Proposition *Let $p: X \to Y$ be a mid fibration and $j: A \to B$ be an arbitrary monomorphism of simplicial sets. Then the canonical map*

$$X^B \to X^A \underset{Y^A}{\times} Y^B \tag{1.2}$$

is a mid fibration. Moreover, if j *is mid anodyne, then* (1.2) *is a trivial Kan fibration.*

1.3.4. Remark We cannot resist to point out that these propositions are formally dual to each other, see [Rie14, Lemma 11.1.10].

For the statement of the last proposition, recall that a class $\mathcal{C}$ of monomorphisms in some category has the *right cancellation property* if for any diagram

$$\bullet \xrightarrow{\ i\ } \bullet \xrightarrow{\ j\ } \bullet$$

with $i \in \mathcal{C}$ and $ji \in \mathcal{C}$, we have $j \in \mathcal{C}$, too. The following result is due to Stevenson [Ste16, Theorem E].

1.3.5. Proposition *The class of mid anodyne maps has the right cancellation property.*

1.4 Graphs

This section serves the purpose of fixing notation and terminology for graphs. All our graphs are directed and have possibly multiple distinct edges between two vertices.

Graphs A *graph* G consists of a set G_0 of *vertices* and a set G_1 of *edges* together with two maps $s, t\colon G_1 \to G_0$ called *source* and *target*, respectively. A vertex $v \in G_0$ and an edge $e \in G_1$ are *incident* if $v = s(e)$ or $v = t(e)$. A graph

G is *finite* if both G_0 and G_1 are finite sets. We usually abbreviate $v \in G_0$ to $v \in G$.

A graph is nothing but a set-valued presheaf G on the category

$$0 \rightrightarrows 1.$$

This point of view has the advantage that it does not only yield the notion of a single graph but rather the category of graphs, which is complete and cocomplete as is any category of set-valued presheaves. We denote this category by Graph. Explicitly, a morphism $f : G \to H$ of graphs is given by two maps $f_0 \colon G_0 \to H_0$ and $f_1 \colon G_1 \to H_1$ with the property that $s(f_1(e)) = f_0(s_1(e))$ and $t(f_1(e)) = f_0(t(e))$ for all edges e of G.

Paths Fix a graph G. An *undirected path* in G is a non-empty sequence $p = (v_0, e_1, v_1, e_2, \dots, e_k, v_k)$ alternating between vertices $v_0, \dots, v_k$ and edges $e_1, \dots, e_k$ of G, such that e_i is incident to both v_{i-1} and v_i. We often denote a path simply by its sequence of vertices and consider the edges understood. A path is *trivial* if $p = (v_0)$. If $v_{i-1} = s(e_i)$ and $v_i = t(e_i)$ for all $0 \leq i \leq k$, we say that p is a *directed path*. The *source* and *target* – denoted $s(p)$ and $t(p)$ – of an undirected path p are its first vertex v_0 and its last vertex v_k, respectively. All other vertices are *interior*. An undirected path p with $s = s(p)$ and $t = t(p)$ is also referred to as a *path from s to t* or an *st-path*. The *reverse path* p^{op} of a path p is obtained by reversing the sequence of vertices and edges, that is, $p^{\mathrm{op}} = (v_k, e_k, \dots, e_1, v_0)$. A path p is *simple* if each interior vertex $v \in p$ appears exactly once in p. It is a *cycle* if it is nontrivial and $t(p) = s(p)$. A graph G is *acyclic* if there is no directed cycle in G. Given paths p and q with $t(p) = s(q)$, their *concatenation* $p \cdot q$ has as its sequence of vertices and edges the sequence of p without $t(p)$ followed by the sequence of q.

Induced subgraphs Let G be a graph and let X be a subset of the vertices of G. The *vertex-induced subgraph* $G[X]$ is the maximal subgraph of G with vertices X. In concrete terms, $G[X]$ has vertices X and edges determined by the pullback

$$\begin{tikzcd} G[X]_1 \arrow[r] \arrow[d] & G_1 \arrow[d, "(s,t)"] \\ X \times X \arrow[r] & G_0 \times G_0 \end{tikzcd}$$

We also say that $G[X]$ is obtained from G by *removing* the complement $Y = G_0 \setminus X$ of X from G and write $G \setminus Y$.

Connectivity A graph G is *connected* if there is an undirected path between any two vertices $u, v \in G$. It is *n-connected* if $G \setminus X$ is connected for all $X \subseteq G_0$ with $|X| \leq n - 1$. Any vertex, whose removal makes a connected but not 2-connected graph G disconnected, is called a *cut vertex.*

1.5 Plane graphs

In this section, we give a bit of intuition for plane graphs and cite some results from the literature that justify this intuition.

Topological Realisation The *topological realisation* $|G|$ of a graph G is the topological space that may be obtained as follows: One equips G_0 with the discrete topology and glues for each edge e of G a copy of the unit interval I along its endpoints to $s(e)$ and $t(e)$, respectively. Formally, $|G|$ is the pushout

$$\begin{tikzcd} \coprod_{e \in G_1} \partial I \arrow[r, "\coprod s \amalg t"] \arrow[d] & G_0 \arrow[d] \\ \coprod_{e \in G_1} I \arrow[r] & |G|, \end{tikzcd}$$

in which the left hand vertical map is just the coproduct of all the inclusions $\partial I \subseteq I$ and in which the top horizontal map is the coproduct of all the maps that send the points 0 and 1 in the copy of $\partial I = \{0, 1\}$ corresponding to the edge e to $s(e)$ and $t(e)$, respectively. We do not distinguish between vertices of G and those points of $|G|$ that correspond to them. Similarly, we often identify an edge $e \in G_1$ with the copy of the unit interval in $|G|$ corresponding to e. Using the universal property of pushouts, topological realisation is easily enhanced to a functor $|\bullet| \colon \mathrm{Graph} \to \mathrm{Top}$, where Top denotes the category of topological spaces and continuous functions.

Curves A *curve* α in a topological space X is a continuous map $\alpha\colon I \to X$. The curve α is said to *join* or *connect* its *endpoints* $\alpha(0)$ and $\alpha(1)$. The *inverse* α^{op} of α is given by $t \mapsto \alpha(1-t)$. A topological space X is *pathwise connected* if for any two points $x, y \in X$ there exists a curve in X with endpoints x and y. A curve α is *closed* if $\alpha(0) = \alpha(1)$. As is common, we view closed curves as continuous maps $\alpha\colon S^1 \to X$. The restrictions of α to closed connected subsets $J \subseteq I$ – or $J \subseteq S^1$ if α is closed – are called *segments* of α. A (closed) curve is *simple* if it is injective as a map $I \to X$ ($S^1 \to X$). We often identify a simple curve α with its image, so that we can speak of e. g. the complement $X \setminus \alpha$ of a simple curve or the union $\alpha \cup \beta$ of two simple curves in X.

We are primarily interested in simple curves in the plane $\mathbb{R}^2$. Note that since the unit interval I and the sphere S^1 are compact and the plane is Hausdorff, any simple (closed) curve is actually an embedding. If $C \subseteq \mathbb{R}^2$ is any set, we call the connected components of its complement $\mathbb{R}^2 \setminus C$ the *faces* of C. Any non-empty, open and connected subset F of the plane is pathwise connected. A classic theorem concerning simple closed curves in the plane is the following theorem of Jordan that we cite without proof.

1.5.1. Theorem A simple closed curve α in the plane has exactly one bounded face $\mathrm{int}(\alpha)$ and one unbounded face $\mathrm{ext}(\alpha)$. The boundary of both faces is α.

Proof. See e. g. [Tho92]. $\square$

We will also have occasion to use the following strengthening of the Jordan curve theorem due to Schönflies.

1.5.2. Theorem Let $\alpha, \beta \subseteq \mathbb{R}^2$ be simple closed curves in the plane. Any homeomorphism $f : \alpha \to \beta$ can be extended to a homeomorphism of the entire plane.

Proof. See e. g. [Tho92]. □

One should note that if the homeomorphism f between the curves in Schönflies' theorem is orientation preserving, then so is its extension to the plane.

Embeddings Let G be a graph and let $i : |G| \to \mathbb{R}^2$ be a topological embedding of its realisation in the plane. A point $x \in \mathbb{R}^2$ is a vertex (edge) accumulation point of i if each neighbourhood U of x contains infinitely many vertices (intersects infinitely many edges) of G. An *embedding* of a graph G is a topological embedding of its realisation without any vertex or edge accumulation points. Embeddings are rather well-behaved topological gadgets and can only exist for graphs with countably many vertices with locally finite valency, see e. g. [Moh88].

Actually, we are not interested in concrete embeddings but rather in topological equivalence classes. Two embeddings i and j of the same graph G are *topologically* equivalent if there is an orientation preserving homeomorphism f of the plane such that $j = fi$.

A *plane graph* is a graph G together with a chosen embedding up to topological equivalence. A map $f: G \to H$ between plane graphs consists of a map $f: G \to H$ of abstract graphs such that $i_H \circ |f|$ and i_G are topologically equivalent. Observe that any map of plane graphs necessarily is a monomorphism of abstract graphs.

Again, we deliberately conceal the distinction between the abstract graph G, its realisation $|G|$ and the image of the latter in the plane. Such abuse of notation and terminology in conjunction with the natural orientation of the plane supplies us with notions as e. g. the clockwise order on the set of edges with a prescribed source or target. Fix a graph G and an embedding i of G in the plane. A *face* of this embedding is a connected component of $\mathbb{R}^2 \smallsetminus i(|G|)$. There are exactly one unbounded face, the *exterior face* of G, and probably many bounded *interior faces*. We denote the set of interior faces of a plane graph G by $\Phi(G)$. The Jordan Curve Theorem tells us that an interior face is homeomorphic to the open unit disk. Moreover, there exists an undirected cycle in G such that the boundary of ϕ is precisely this cycle. We usually consider the boundary $\partial\phi$ of a face as an undirected cycle in the abstract graph G such that the face is enclosed in clockwise orientation. We warn the reader that this convention has its disadvantages when it comes to the exterior face.

Most of the time, we will work with plane graphs on the intuitive level. However, the following lemma will be used very often and we feel obliged to state it, at the very least.

1.5.3. Lemma *Let ϕ be a face of the given embedding and let e be an edge of G.*

1. *Either e is contained in the boundary $\partial\phi$ of ϕ or its interior is disjoint from $\partial\phi$.*

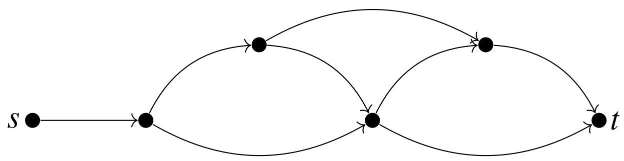
2. *If e lies on a simple cycle or an infinite path of G, then e is contained in the boundary of exactly two faces.*

3. *If e does not lie on any simple cycle of G and if G is finite, then e is contained in the boundary of exactly one face.*

Proof. See e. g. [Die10, Lemma 4.2.1] $\square$

2 Globular Graphs and Pasting Diagrams

This chapter has essentially two parts. The first part comprises § 2.1–2.3 and is concerned with globular graphs. We review the definition of a globular graph from [Pow90] and discuss some technical observations on globular graphs due to Power. Globular graphs are essentially diagrams such as



As we eventually want to interpret these diagrams in categories enriched over simplicial sets or quasi-categories, we then go on to define the nerve of a globular graph and establish some of its technical properties. It turns out, that the nerve of a globular graph appears as the nerve of the category $F_2G(s,t)$, where F_2G is the free 2-category on the graph G. This means in particular that $\mathrm{N}(G)$ contains the composite of any of its 1-simplices and is therefore inadequate as a model of the input data of a vertical composition in a category enriched over quasi-categories. In the second part of this chapter we thus introduce pasting diagrams in § 2.4 and their nerves in § 2.5. Our notion of pasting diagram is built on the notion of a globular graph but allows for a specification of which vertical composites are present in its nerve.

2.1 Globular Graphs

In this section, we review relevant parts of the work [Pow90] of Power and extend it by some definitions and mostly trivial observations of our own.

Globular graphs Let G be a graph. A vertex $s \in G$ is called a *source* if there is a directed path from s to any vertex $u \neq s$ of G. A *target* in G is a source in G^{op}. An *st-graph* is a nontrivial plane graph G with unique source and target that are both incident to the exterior face.

A face ϕ of a plane graph is *globular* if its clockwise oriented boundary $\partial\phi$ decomposes as $\partial\phi = p \cdot q^{\mathrm{op}}$ for two nontrivial directed paths p and q. If ϕ is an interior face, we call $\operatorname{dom}\phi = p$ the *domain* and $\operatorname{cod}\phi = q$ the codomain of ϕ. However, we use the exact opposite convention for the exterior face ε, i. e. $\operatorname{dom}\varepsilon = q$ and $\operatorname{cod}\varepsilon = p$ if $\partial\varepsilon = p \cdot q^{\mathrm{op}}$.

As $s(p) = s(q)$ and $t(p) = t(q)$, we simply denote these vertices by $s(\phi)$ and $t(\phi)$, respectively. A *globular graph* is an st-graph in which all faces are globular.

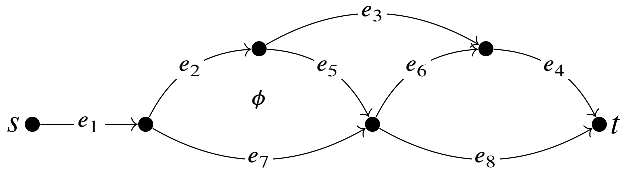


Figure 2.1: An example of a globular graph.

2.1.1. Example Consider the graph shown in Figure 2.1. It certainly is an st-graph for the vertices marked s and t are its source and target, respectively. Moreover, the clockwise boundary $\partial\phi$ of any face ϕ of G decomposes as $\partial\phi = p \cdot q^{\mathrm{op}}$ for two directed paths p and q, that is, all faces of G are globular. Let us check this for the interior face ϕ marked in Figure 2.1 and the exterior face ε of G. The clockwise boundary of ϕ is $e_2 \cdot e_5 \cdot e_7^{\mathrm{op}} = p \cdot q^{\mathrm{op}}$ with $p = e_2 \cdot e_5$ and $q = e_7$. The face ϕ thus has $\operatorname{dom}\phi = e_2 \cdot e_5$ and $\operatorname{cod}\phi = e_7$. The exterior face ε has clockwise boundary $\partial\varepsilon = e_4^{\mathrm{op}} \cdot e_3^{\mathrm{op}} \cdot e_2^{\mathrm{op}} \cdot e_1^{\mathrm{op}} \cdot e_1 \cdot e_7 \cdot e_8$ and this clockwise boundary decomposes up to rotation as $\partial\varepsilon = p \cdot q^{\mathrm{op}}$ with $p = e_1 \cdot e_7 \cdot e_8$ and $q = e_1 \cdot e_2 \cdot e_3 \cdot e_4$. We thus have $\operatorname{dom}\varepsilon = e_1 \cdot e_2 \cdot e_3 \cdot e_4$ and $\operatorname{cod}\varepsilon = e_1 \cdot e_7 \cdot e_8$.

The following characterisation of globular graphs is taken from [Pow90, Proposition 2.6].

2.1.2. Proposition *A nontrivial st-graph G is globular if and only if it is acyclic, i. e. contains no directed cycles.*

Proof. Let us first show that an acyclic st-graph G is globular. To this end, consider an arbitrary face ϕ of G. As G is nontrivial and has no cycles, $\partial\phi$

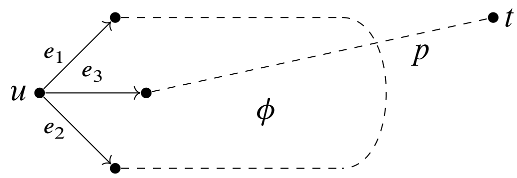


Figure 2.2: Three edges e_1, e_2, e_3 with source u in the boundary of a face ϕ of some st-graph G.

contains two distinct edges e_1 and e_2 with common source u. Observe that there cannot exist a third edge $e_3 \subseteq \partial\phi$ with source u, for then the path p from $t(e_3)$ to the target t of G would necessarily cut ϕ, see Figure 2.2. It therefore suffices to show that e_1 and e_2 are the only such edges in $\partial\phi$, for the maximal directed paths p_1 and p_2 in $\partial\phi$ starting with e_1 and e_2, respectively, then meet at a common target and our claim follows. Let us assume the contrary, i. e. that there are edges $d_1 \neq d_2$ in $\partial\phi$ that have a common source $v \neq u$. We may suppose without loss of generality that $v \neq s$. As G is an st-graph, we find paths q_i from $t(d_i)$ to t and a path p from s to v. Observe that s is not contained in the undirected cycle $c = d_1 \cdot q_1 \cdot q_2^{\mathrm{op}} \cdot d_2^{\mathrm{op}}$ as s does not occur as target of any edge of G and $v \neq s$. Therefore, s lies in the exterior of the cycle c and the path p from s to v has to cross q_1 or q_2. This, however, is the desired contradiction, for we find a directed cycle consisting of parts of p, q_i and d_i. This can also be seen in Figure 2.3 in the case that ϕ is interior. The reader is invited to adapt the picture to the case that ϕ is the exterior face of G.

Let us now check that any globular graph G is acyclic. Consider a directed

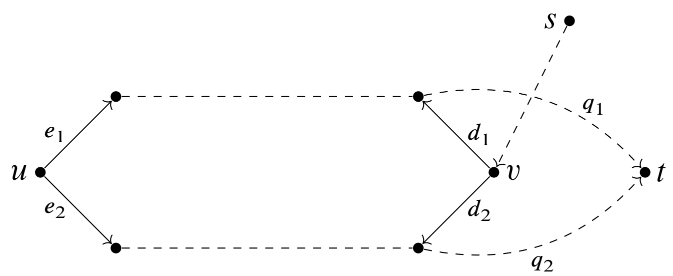


Figure 2.3: The paths and vertices occurring in the first part of the proof of Proposition 2.1.2.

cycle c in G that encloses a minimal number of faces. Choose any edge e of c and the face ϕ in the interior of c that is incident to e. Without loss of generality assume that e lies in $\mathrm{dom}(\phi)$. Observe that this implies that $\mathrm{cod}(\phi)$ intersects the directed cycle c at most at its endpoints. Choose paths p and q from s to $s(\phi)$ and from $t(\phi)$ to t, respectively. These paths have to cross the cycle c and we may thus decompose $p = p_0 \cdot p_1$ and $q = q_1 \cdot q_0$ in such a way that both p_1 and q_1 lie in the interior of c and p_1 connects a vertex of c to $s(\phi)$ and q_1 connects $t(\phi)$ to a vertex of c. Further let r denote the part of c that has source $t(q_1)$ and target $s(p_1)$. As can also be seen in Figure 2.4, $p_1 \cdot \mathrm{cod}(\phi) \cdot q_1 \cdot r$ is a directed cycle in G that encloses fewer faces than c — a contradiction. □

The boundary, domain and codomain of a globular graph are the boundary, domain and codomain of its exterior face. Note that our conventions concerning the domain and codomain of the exterior face now ensure that

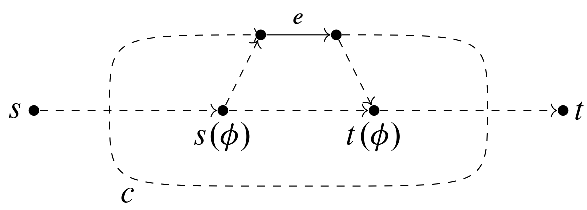


Figure 2.4: The paths and vertices occurring in the second part of the proof of Proposition 2.1.2.

the domain of a face does not depend upon whether we consider it as a subgraph or as a face.

2.1.3. Definition *A globular subgraph $H \subseteq G$ is* wide *if $s(H) = s(G)$ and $t(H) = t(G)$.*

2.1.4. Definition *If $H \subseteq G$ is a globular subgraph of G, we call a subgraph $K \subseteq G$ a* subdivision *of H if $H \subseteq K$ and $\partial H = \partial K$.*

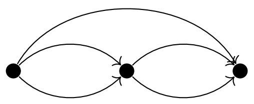
2.1.5. Definition *A* glob *γ in a globular graph G is a globular subgraph of G with the property that any of its edges is incident with the exterior face of γ. A glob is* nondegenerate *if it has at least one interior face and* degenerate *otherwise. A glob is* proper *if it is 2-connected.*

2.1.6. Example

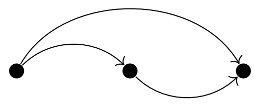
(a) Any path p in a globular graph G is a degenerate glob.

(b) Any face ϕ of a globular graph G is a nondegenerate, proper glob.

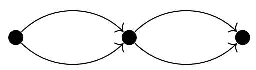
(c) Let $H \subseteq G$ be a globular subgraph of some globular graph G. The boundary ∂H of H is a glob in G that is degenerate if and only H has



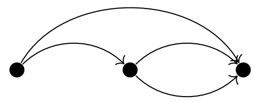
(a) A globular graph G.



(b) A proper glob $\gamma_1 \subseteq G$.



(c) A nonproper glob $\gamma_2 \subseteq G$.



(d) A globular subgraph $H \subseteq G$ that is not a glob.

Figure 2.5: Example of a globular graph G and two of its globs γ_1 and γ_2.

no interior faces and proper if and only if H is 2-connected.

(d) Consider the graph G shown in Figure 2.5a and its subgraphs γ_1, γ_2 and H shown in Figure 2.5b, 2.5c and 2.5d, respectively. Both γ_1 and γ_2 are globs, but H is not since it contains an edge that is not incident with the exterior face of H.

2.1.7. Lemma *The following are equivalent for a glob γ:*

(i) γ *is degenerate.*

(ii) γ *is a directed path in G.*

(iii) $\partial\gamma^{\mathrm{op}}$ *is oriented clockwise.*

Proof. Let γ be a degenerate glob with source s and target t in G. If γ is no directed path, then there are two st-paths $p \neq q$ in γ. The cycle $p \cdot q^{\mathrm{op}}$

encloses at least one interior face of γ — a contradiction. As directed paths are certainly degenerate globs, this proves the equivalence of (i) and (ii).

The oriented boundary of a directed path p is simply $\partial p = p \cdot p^{\mathrm{op}}$ and this cycle clearly satisfies $\partial p^{\mathrm{op}} = \partial p$, which proves the implication "(ii) $\Rightarrow$ (iii)".

Now assume that both $\partial\gamma$ and $\partial\gamma^{\mathrm{op}}$ are clockwise orientations of $\partial\gamma$. Write $\partial\gamma = p \cdot q^{\mathrm{op}}$ and $\partial\gamma^{\mathrm{op}} = q \cdot p^{\mathrm{op}}$. As clockwise orientations are unique up to cyclic rotation, we conclude that $p = q$, i. e. that γ is a simple directed path in G. □

2.1.8. Remark If u, v are two vertices of a globular graph G, the subgraph $G_{u,v}$ consisting of all paths with source u and target v is either empty or itself a globular graph. If $\{u, v\} \neq \{s, t\}$, then $G_{u,v}$ is strictly smaller than G in terms of arrows and vertices.

The following two lemmata are due to Power [Pow90].

2.1.9. Lemma *Let v be a vertex of a globular graph G. The clockwise cyclic order of edges around v is $e_1 \prec \cdots \prec e_r \prec d_1 \prec \cdots \prec d_s$, where the edges e_i are the edges with source v and the edges d_i are the edges with target v.*

Proof. If the edges with source v do no appear consecutively in the cyclic order around v, we find edges e_1 and e_2 with source v and edges d_1 and d_2 with target v such that the cyclic order around v is $e_1 \prec d_1 \prec e_2 \prec d_2$. We find paths q_i from the target of e_i to t. We may suppose without loss of generality that the source u of d_1 lies in the interior of the cycle $c = e_1 \cdot q_1 \cdot q_2^{\mathrm{op}} \cdot e_2^{\mathrm{op}}$.

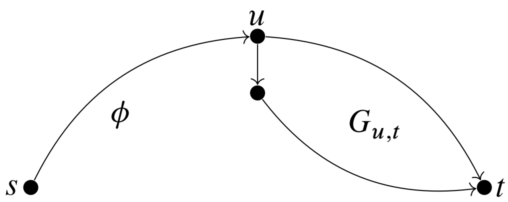


Figure 2.6: An illustration of the argument in the proof of Lemma 2.1.10.

Any path from s to u has to cross the cycle c and we find a directed cycle just as in the proof of Proposition 2.1.2. $\square$

2.1.10. Lemma *Let G be a globular graph with exterior face ε and at least one interior face. There then exists an interior face ϕ of G with* $\mathrm{dom}(\phi)$ *lying entirely in* $\mathrm{dom}(G)$.

Proof. We proceed by induction on the number of vertices. Possibly removing a path that starts in s and whose edges are incident with the exterior face only, we may suppose that there is an interior face ϕ of G such that $\mathrm{dom}\,\phi$ and $\mathrm{dom}\,G$ both start at s and have at least one edge in common. If ϕ is not a face of the desired type, choose the last vertex u of the common subpath of $\mathrm{dom}\,G$ and $\mathrm{dom}\,\phi$ that starts in s. Then, $G_{u,t}$ is a globular graph with fewer vertices and at least one interior face lying in the interior of the cycle consisting of the paths from u to t in $\mathrm{dom}(G)$ and via $t(\phi)$ to t. Note that $\mathrm{dom}(G_{u,t}) \subseteq \mathrm{dom}\,G$, see Figure 2.6. Thus, the claim follows by induction. $\square$

Joins of globular graphs The operation of glueing two globular graphs at their respective source and target vertices very much looks like horizontal composition in 2-categories. In fact, this is precisely the role that this operation is going to play in the later chapters of this work. In this paragraph, we just define this operation and prove one technical lemma.

2.1.11. Definition *The* join $G \vee H$ *of two globular graphs is the globular graph obtained by gluing $t(G)$ to $s(H)$. This is a well-defined globular graph as we consider topological equivalence classes of actual embeddings.*

2.1.12. Remark Note that the domain and codomain of $G \vee H$ are given by $\mathrm{dom}(G \vee H) = \mathrm{dom}(G) \cdot \mathrm{dom}(H)$ and $\mathrm{cod}(G \vee H) = \mathrm{cod}(G) \cdot \mathrm{cod}(H)$.

2.1.13. Remark Consider a globular graph G and some vertex $x \in G$. Recall that $G_{s,x}$ and $G_{x,t}$ are the subgraphs of G consisting of all the paths from s to x and from x to t, respectively. As $t(G_{s,x}) = x = s(G_{x,t})$, we can form their join $G_{s,x} \vee G_{x,t}$ and obtain a wide globular subgraph of G.

We end this paragraph with a technical lemma on the interplay between joins and intersections of globular subgraphs of the form $G_{u,v}$.

2.1.14. Lemma *Let G be a globular graph with source s and target t. Further let $x, y \in G$ be two vertices. The intersection*

$$\left(G_{s,x} \vee G_{x,t}\right) \cap \left(G_{s,y} \vee G_{y,t}\right)$$

contains a directed path from s to t if and only if G contains a directed path from x to y or from y to x. Moreover, if G contains a directed path, say, from x to y, *then*

$$\left(G_{s,x} \vee G_{x,t}\right) \cap \left(G_{s,y} \vee G_{y,t}\right) = G_{s,x} \vee G_{x,y} \vee G_{y,t}.$$

Proof. Let $H_x = G_{s,x} \vee G_{x,t}$ and $H_y = G_{s,y} \vee G_{y,t}$. If $H_x \cap H_y$ contains a directed path p from s to t, this path p passes through x as $p \subseteq H_x$ and it passes through y as $p \subseteq H_y$. Taking the subpath of p between x and y, we have found a directed path between x and y.

Now suppose that there exists some directed path p, say, from x to y in G. Then $G_{x,y} \neq \varnothing$ and we have $G_{s,x} \vee G_{x,y} \subseteq G_{s,y}$ and $G_{x,y} \vee G_{y,t} \subseteq G_{x,t}$. Thus

$$G_{s,x} \vee G_{x,y} \vee G_{y,t} \subseteq H_x \cap H_y.$$

In order to show the converse inclusion, let us consider some path p in $H_x \cap H_y$ that consists of at most one edge. We then find directed paths q_x and q_y from s to t such that $x \in q_x$, $p \subseteq q_x$, $y \in q_y$ and $p \subseteq q_y$. We distinguish the following cases:

1. The vertex x does not precede $t(p)$ on q_x. Note that p then lies on a path from s to x, i. e. in $G_{s,x} \subseteq G_{s,x} \vee G_{x,y} \vee G_{y,t}$.

2. The vertex $s(p)$ does not precede y on q_y. The path p then lies on a path from y to t and hence in $G_{y,t} \subseteq G_{s,x} \vee G_{x,y} \vee G_{y,t}$.

3. The vertex x precedes $t(p)$ on q_x and the vertex $s(p)$ precedes y on q_y. Write $q_x = a_x \cdot b_x \cdot p \cdot c_x$ with a_x a path from s to x and b_x a possibly empty path from x to $s(p)$. This is possible as p consists of at most one

edge. Similarly, write $q_y = a_y \cdot p \cdot b_y \cdot c_y$ with a_y a path from s to $s(p)$ and b_y a path from $t(p)$ to y. The concatenation $b_x \cdot p \cdot b_y$ is a path from x to y that contains p and we conclude that $p \subseteq G_{x,y} \subseteq G_{s,x} \vee G_{x,y} \vee G_{y,t}$.

□

2.2 Nerves of globular graphs

In this section, we first give a formal definition of the nerve $\mathrm{N}(G)$ of a globular graph as the nerve of a certain partially ordered set. After some basic examples, we then show that the simplices of $\mathrm{N}(G)$ are in bijection with certain marked subgraphs of G. This ultimately leads to a quite intuitive description of the nerve of a globular graph and to a pictorial calculus for the action of simplicial operators on $\mathrm{N}(G)$.

Definition and examples Let G be a globular graph and consider two st-paths p and q. Let us write $p \leq q$ if there exists a glob $\gamma \subseteq G$ and possibly trivial paths a and b in G such that $p = a \cdot \operatorname{dom} \gamma \cdot b$ and $q = a \cdot \operatorname{cod} \gamma \cdot b$. We call any such glob γ a ***witness*** for the relation $p \leq q$. Observe that minimal witnesses for a relation $p < q$ are unique.

2.2.1. Lemma *The relation "$\leq$" defines a partial order on the set PG of st-paths in G.*

Proof. We have $p \leq p$ as p itself is a glob. Suppose $p \leq q$ and $q \leq r$ for st-paths p, q and r in G. Choose globs $\gamma, \delta \subseteq G$ and decompositions $p = a \cdot \operatorname{dom} \gamma \cdot b$, $q = a \cdot \operatorname{cod} \gamma \cdot b = a' \cdot \operatorname{dom} \delta \cdot b'$ and $r = a' \cdot \operatorname{cod} \delta \cdot b'$. We may assume without loss of generality that $a = a'$ and $b = b'$, for we could otherwise decompose $a = a_0 \cdot a_1$, $a' = a_0 \cdot a_1'$, $b = b_1 \cdot b_0$ and $b' = b_1' \cdot b_0$ and choose $\gamma = a_1 \cup \gamma \cup b_1$ and $\delta = a_1' \cup \delta \cup b_1'$ as witnesses for $p \leq q$ and $q \leq r$. But if $a = a'$ and $b = b'$, then $\partial(\gamma \cup \delta)$ is a glob witnessing $p \leq r$.

Now suppose $p \leq q$ and $q \leq p$. As above, we find witnesses γ and δ such that $p = a \cdot \operatorname{dom} \gamma \cdot b = a \cdot \operatorname{cod} \delta \cdot b$ and $q = a \cdot \operatorname{cod} \gamma \cdot b = a \cdot \operatorname{dom} \gamma \cdot b$. We thus have

$$\partial\gamma = \operatorname{dom} \gamma \cdot \operatorname{cod} \gamma^{\mathrm{op}} = \operatorname{cod} \delta \cdot \operatorname{dom} \delta^{\mathrm{op}} = (\operatorname{dom} \delta \cdot \operatorname{cod} \delta)^{\mathrm{op}} = \partial\delta^{\mathrm{op}}$$

for the clockwise directed boundaries of γ and δ. This implies that both γ and δ are degenerate. □

2.2.2. Definition *The* nerve $\mathrm{N}(G)$ *of a globular graph G is the nerve of the partially ordered set $(PG, \leq)$. An n-simplex $\sigma \in \mathrm{N}(G)_n$ is thus an $(n+1)$-chain*

$$\sigma = (p_0 \leq \cdots \leq p_n)$$

of st-paths $p_i \in PG$ and the action of a simplicial operator $\alpha\colon [m] \to [n]$ on this simplex is determined by $\sigma\alpha = (q_0 \leq \cdots \leq q_m)$ with $q_i = p_{\alpha(i)}$.

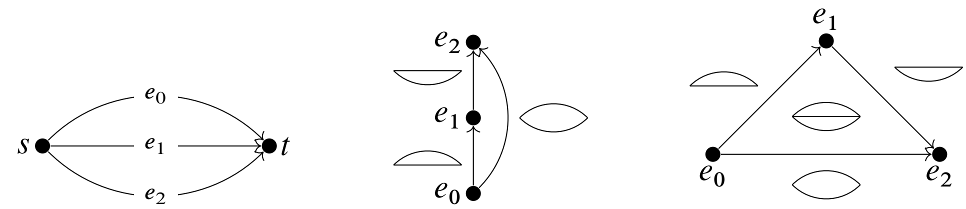
The reader might object that the purely combinatorial Definition 2.2.2 is not satisfactory, for any nerve $\mathrm{N}\colon \mathfrak{B} \to \widehat{\mathfrak{A}}$ taking values in a category $\widehat{\mathfrak{B}}$ of presheaves should arise as a composition $\mathrm{N} = i^* \circ y$, where $y\colon \mathfrak{B} \to \widehat{\mathfrak{B}}$ denotes the Yoneda embedding and $i^*\colon \widehat{\mathfrak{B}} \to \widehat{\mathfrak{A}}$ is restriction along a preferably dense functor $i\colon \mathfrak{A} \to \mathfrak{B}$. It is in fact easy to show, though, that the nerve of globular graphs arises in exactly this manner and we will come back to this topic in § 2.3.

2.2.3. Example Consider the graph B_n with two vertices s and t and $n+1$ distinct edges $e_0, \ldots, e_n$ between them. The set PB_n of st-paths in B_n is the set $\{e_0, \ldots, e_n\}$. We embed B_n such that for all $i \in \{1, \ldots, n\}$ there is an interior face ϕ_i with oriented boundary $\partial\phi_i = e_{i-1} \cdot e_i^{\mathrm{op}}$. A glob γ in B_n is then nothing but a pair (e_i, e_j) of edges with $i \leq j$. The boundary of such a $\gamma = (e_i, e_j)$ is given by $\partial\gamma = e_i \cdot e_j^{\mathrm{op}}$ and we thus have $\operatorname{dom}\gamma = e_i$ and $\operatorname{cod}\gamma = e_j$. This implies that $e_i \leq e_j$ if and only if $i \leq j$. Altogether, we see that PB_n is isomorphic to the ordinal $[n]$ and $\mathrm{N}(B_n)$ is isomorphic to Δ^n. Note that a nondegenerate 1-simplex $(p_0 < p_1)$ of $\mathrm{N}(B_n)$ is contained in the spine of $\mathrm{N}(B_n)$ if and only if there is a single face of B_n witnessing the relation $p_0 < p_1$. This observation along with our computation of $\mathrm{N}(B_n)$ is also illustrated in Figure 2.7 for the case $n = 2$.

2.2.4. Example Let us next consider the join $G = B_n \vee B_m$ of two such graphs B_n and B_m. We denote the edges of B_n and B_m by $e_0, \ldots, e_n$ and $d_0, \ldots, d_m$, respectively. The set of st-paths in G is then given by

$$PG = \{e_i \cdot d_k \mid 0 \leq i \leq n \text{ and } 0 \leq k \leq m\}.$$

Any glob γ in G is the join of two globs (e_i, e_j) in B_n and (d_k, d_l) in B_m. We thus have $e_i \cdot d_k \leq e_j \cdot d_l$ if and only if $i \leq j$ and $k \leq l$. Summing up, we

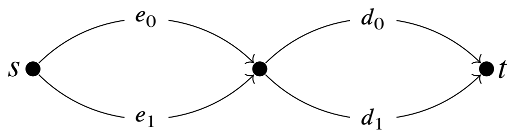


(a) B_2 (b) PB_2 (c) $\mathrm{N}(B_2) \simeq \Delta^2$

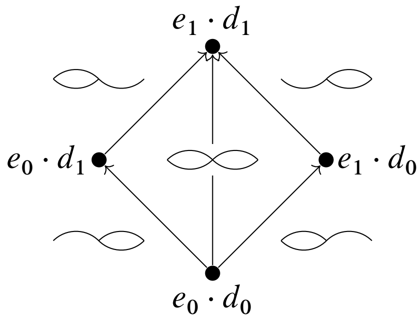
Figure 2.7: Computation of $\mathrm{N}(B_2)$.

have PG isomorphic to the product $[n] \times [m]$ and hence $\mathrm{N}(G) \simeq \Delta^n \times \Delta^m$. This observation on joins and products actually holds true for all globular graphs as we discuss in a more general setting in Proposition 2.5.6 below. An explicit computation of $\mathrm{N}(B_1 \vee B_1)$ along the lines of the computation of $\mathrm{N}(B_2)$ in Figure 2.7 can be seen in Figure 2.8.

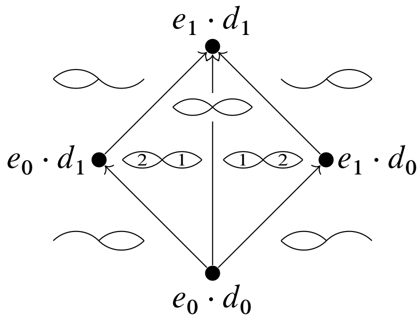
A pictorial representation of $\mathrm{N}(G)$ Even though the definition of $\mathrm{N}(G)$ in Definition 2.2.2 is concise and sufficient for most technical purposes, it leaves a lot to be desired from an intuitional point of view. Therefore, in this paragraph, we give another description of $\mathrm{N}(G)$ in terms of certain marked globular subgraphs of G such as those that we already used in the computation of $\mathrm{N}(B_1 \vee B_1)$ in Figure 2.8. We hope that seeing $\mathrm{N}(G)$ from a slightly different angle might help the reader on her or his way through the remaining parts of this work. In fact, in § 2.5 below, we will refine the material presented here so as to provide the reader with some intuition for further definitions that are based on the notion of the nerve of a globular graph. Moreover, in chapter 3, we will actually need some



(a) $B_1 \vee B_1$



(b) $P(B_1 \vee B_1)$



(c) $\mathrm{N}(B_1 \vee B_1) \simeq \Delta^1 \times \Delta^1$

Figure 2.8: Computation of $\mathrm{N}(B_1 \vee B_1)$.

facets of the description of $\mathrm{N}(G)$ obtained in this paragraph for the proof of Theorem A.

Let us fix a globular graph G with source s and target t and consider an n-simplex $\sigma = (p_0 \leq \cdots \leq p_n) \in \mathrm{N}(G)$. Observe that Proposition 2.1.2 implies that $P = \bigcup_{i=0}^{n} p_i$ is a wide globular subgraph of G, for it is acyclic and contains both a source and target. The following lemma is crucial to the description of $\mathrm{N}(G)$ that we are about to give:

2.2.5. Lemma *Let* $\sigma = (p_0 \leq \cdots \leq p_n)$ *be an* n*-simplex in* $\mathrm{N}(G)$ *and let* ϕ *be an interior face of* $P = \bigcup_{i=0}^{n} p_i$. *There then exists some* $i \in \{1, \ldots, n\}$ *such that* $\operatorname{cod} \phi \subseteq p_i$. *Moreover,* $\operatorname{dom} \phi \subseteq p_{i-1}$ *if and only if* i *is chosen minimal.*

Proof. The proof is by induction on n and the claim is obvious for $n = 0$ and $n = 1$. Now let $n \geq 2$. The interior faces of P are the bounded connected components of $(\mathbb{R}^2 \smallsetminus P') \smallsetminus p_n$, where $P' = \bigcup_{i=0}^{n-1} p_i$. Observe that p_n and P' intersect only in $p_{n-1} \subseteq \partial P'$ since $p_{n-1} \leq p_n$. This implies in particular that p_n does not intersect any interior face of P' and that any interior face of P that is not already an interior face of P' is necessarily bounded by subpaths of p_{n-1} and p_n. The claim follows.

□

Lemma 2.2.5 implies in particular that we may associate with a given n-simplex $\sigma = (p_0 \leq \cdots \leq p_n)$ of $\mathrm{N}(G)$ a tuple $(P_\sigma, \lambda_\sigma)$, where $P_\sigma = \bigcup_{i=0}^{n} p_i$ is a wide globular subgraph of G and where $\lambda_\sigma \colon \Phi(P_\sigma) \to \{1, \ldots, n\}$ is the

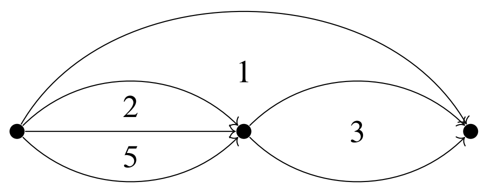


Figure 2.9: A 5-marked subgraph (P, λ).

map that associates with any interior face $\phi \in \Phi(P_\sigma)$ the unique $i \in \{1, \dots, n\}$ such that $\phi \subseteq p_{i-1} \cup p_i$. This observation motivates the following definition:

2.2.6. Definition *An n-*marked subgraph (P, λ) *of a globular graph G consists of a wide globular subgraph $P \subseteq G$ and a map $\lambda \colon \Phi(P) \to \{1, \dots, n\}$ from the set of its interior faces into $\{1, \dots, n\}$.*

We picture n-marked subgraphs by drawing the graph P and labelling the interior faces ϕ of P with $\lambda(\phi)$, see Figure 2.9. The reader should note that this pictorial representation of (P, λ) does not determine the number n, though.

It is clear that not all n-marked globular subgraphs (P, λ) arise from n-simplices σ of $\mathrm{N}(G)$. One condition that all the maps λ_σ satisfy, though, is the following:

2.2.7. Lemma *Let $(P_\sigma, \lambda_\sigma)$ be the n-marked subgraph associated with some n-simplex $\sigma \in \mathrm{N}(G)$. Suppose that ϕ and ψ are two faces of P_σ such that there exists some edge e of P_σ such that $e \subseteq \operatorname{cod} \phi$ and $e \subseteq \operatorname{dom} \psi$. Then $\lambda_\sigma(\phi) < \lambda_\sigma(\psi)$.*

Proof. We know that $j = \lambda_\sigma(\psi)$ is minimal with $\operatorname{cod}\psi \subseteq p_j$. Any p_k that contains the edge e cannot contain $\operatorname{cod}\psi$ simply because $e \subseteq \operatorname{dom}\psi$. We thus have $j > k$ for all k with $e \subseteq p_k$ and this implies in particular $j > \lambda_\sigma(\phi)$ as $e \subseteq \operatorname{cod}(\phi)$ by assumption. □

2.2.8. Definition *An n-marked subgraph (P, λ) is* admissible *if $\lambda(\phi) < \lambda(\psi)$ whenever $\operatorname{cod}\phi \cap \operatorname{dom}\psi$ contains an edge.*

2.2.9. Remark Let (P, λ) be admissible and let p be some st-path in P. Any face ϕ on the left of p then satisfies

$$\lambda(\phi) \leq \max\{\lambda(\xi) \mid \operatorname{cod}\xi \cap p \text{ contains an edge}\}$$

and any face ψ on the right of p satisfies

$$\lambda(\psi) \geq \min\{\lambda(\xi) \mid \operatorname{dom}\xi \cap p \text{ contains an edge}\}.$$

Indeed, if $\phi = \phi_0$ is an interior face on the left of p such that $\operatorname{cod}\phi_0 \cap p$ does not contain an edge, we find an interior face ϕ_1 to the left of p such that $\operatorname{cod}\phi_0 \cap \operatorname{dom}\phi_1$ contains an edge, i. e. $\lambda(\phi_0) < \lambda(\phi_1)$. If $\operatorname{cod}\phi_1 \cap p$ contains an edge, we are done. Otherwise, we can continue in this manner to obtain a sequence of faces $\phi_0, \dots, \phi_k$ that are on the left of p and satisfy $\lambda(\phi_0) < \cdots < \lambda(\phi_k)$. This procedure terminates after a finite number of steps, for P is a globular graph. We thus find a face ϕ_k such that $\lambda(\phi_0) < \lambda(\phi_k)$ and $\operatorname{cod}\phi_k \cap p$ contains an edge.

2.2.10. Lemma *Let $\lambda\colon P \to \{1, \dots, n\}$ be admissible. For each $k \in \{0, \dots, n\}$ there exists a unique st-path $p_k \subseteq P$ such that all the interior faces ϕ on the left of p_k satisfy $\lambda(\phi) \le k$ and all the interior faces ϕ on the right of p_k satisfy $\lambda(\phi) > k$.*

Proof. We obviously have $p_0 = \operatorname{dom} P$. Given such a path p_k, we can construct p_{k+1} by replacing each subpath of p_k of the form $\operatorname{dom}\phi$ for some interior face ϕ with $\lambda(\phi) = k + 1$ by $\operatorname{cod}\phi$. This is indeed the sought-for path p_k by Remark 2.2.9. Let us now suppose that we have two such paths $p_k \neq q_k$. As both p_k and q_k start in s, there is then some vertex u and edges $e \neq d$ with source u such that $e \subseteq p_k$ and $d \subseteq q_k$. We know from Lemma 2.1.9 that the edges with source u appear consecutively in the cyclic order of edges around u and after possibly changing the role of p_k and q_k we therefore find interior faces $\phi_1, \dots, \phi_r$ such that $e \subseteq \operatorname{dom}\phi_1$, $d \subseteq \operatorname{cod}\phi_r$ and such that $\operatorname{cod}\phi \cap \operatorname{dom}\phi_{i+1}$ contains an edge for all $1 \le i < r$, see Figure 2.10. We then have $k < \lambda(\phi_1) \le \lambda(\phi_r) \le k$ by admissibility of λ and the definitions of p_k and q_k. This is a contradiction and we conclude $p_k = q_k$.

□

2.2.11. Remark One consequence of Lemma 2.2.10 is that the path p_i occuring in some simplex $\sigma = (p_0 \le \dots \le p_n)$ of $\mathrm{N}(G)$ can be characterised as the unique st-path p in P_σ with the property that all interior faces ϕ of P_σ with $\lambda_\sigma(\phi) \le i$ are on the left hand side of p.

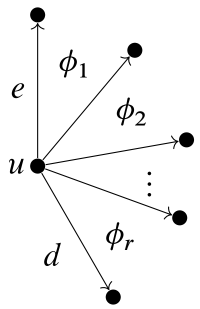


Figure 2.10: An illustration of an argument in the proof of Lemma 2.2.10.

The following proposition gives us the promised description of the simplices of $\mathrm{N}(G)$ in terms of n-marked graphs:

2.2.12. Proposition *Let G be a globular graph. The assignment $\sigma \mapsto (P_\sigma, \lambda_\sigma)$ defines a bijection between the set of n-simplices σ of $\mathrm{N}(G)$ and the set of n-marked subgraphs of G.*

Proof. Given some admissible n-marked subgraph (P, λ) of G, we obtain by Lemma 2.2.10 a set of st-paths $p_0, \dots, p_n$ in the wide subgraph P of G such that p_i has precisely the interior faces of P with $\lambda(p_i) \leq i$ on its left. The graph $p_{i-1} \cup p_i$ is a glob witnessing $p_{i-1} \leq p_i$ and we thus obtain an n-simplex $\sigma = \sigma(P, \lambda) = (p_0 \leq \cdots \leq p_n)$ of $\mathrm{N}(G)$. We obviously have $P = P_\sigma$. Furthermore, the interior faces ϕ of P with $\lambda(\phi) = i$ are precisely those interior faces of P that lie on the right of p_{i-1} and on the left of p_i, i. e. in $p_{i-1} \cup p_i$. It now follows that $\lambda = \lambda_\sigma$.

Conversely, given a simplex σ in $\mathrm{N}(G)$, it follows immediately from Remark 2.2.11 that the simplex $\sigma(P_\sigma, \lambda_\sigma)$ is σ again. $\square$

It remains to desribe the action of simplicial operators in terms of n-marked graphs. This is achieved in the following proposition.

2.2.13. Proposition *Let $\alpha\colon [m] \to [n]$ be a simplicial operator and let $(P_\sigma, \lambda_\sigma)$ be the tuple associated with some n-simplex $\sigma \in \mathrm{N}(G)$. The tuple $(P_{\sigma\alpha}, \lambda_{\sigma\alpha})$ associated with $\sigma\alpha$ can then be obtained from $(P_\sigma, \lambda_\sigma)$ by the following steps:*

1. *Remove the edges and interior vertices of all paths* $\mathrm{dom}(\phi)$, *where ϕ is some interior face of P_σ with $\lambda_\sigma(\phi) \leq \alpha(0)$.*
2. *Remove the edges and interior vertices of all paths* $\mathrm{cod}(\phi)$, *where ϕ is some interior face of P_σ with $\lambda_\sigma(\phi) > \alpha(m)$.*
3. *Define $\widehat{\lambda}$ on the remaining graph by*
$$\widehat{\lambda}(\phi) = \min\bigl\{k \in \{1, \ldots, m\} \mid \alpha(k) \geq \lambda_\sigma(\phi)\bigr\}.$$
4. *In each maximal globular subgraph $Q \subseteq P$ on which $\widehat{\lambda}$ takes some constant value c, remove all edges and vertices of Q that are not incident with the exterior face of Q and define $\lambda_{\sigma\alpha}(\phi) = c$ on the resulting interior faces.*

Proof. Let us write $\sigma = (p_0 \leq \cdots \leq p_n)$. The simplex $\sigma\alpha$ is then given by $\sigma\alpha = (q_0 \leq \cdots \leq q_m)$ with $q_j = p_{\alpha(j)}$. Observe that $Q = \bigcup q_j$ is a subgraph of $P = \bigcup p_i$. We fix some embedding of P in $\mathbb{R}^2$ and consider the induced embedding of Q. The closure $\overline{\psi}$ of any face ψ of Q can be written as a union of the closures of a certain set of faces of P and each face of P occurs in at most one such union. Now consider a path $q_j = p_{\alpha(j)}$ occuring in $\sigma\alpha$. By Remark 2.2.11, it is the unique st-path in P such that an interior face $\phi \in \Phi(P)$ lies to the left of $p_{\alpha(j)}$ if and only if $\lambda_\sigma(\phi) \leq \alpha(j)$. It follows that the interior faces ϕ in P between q_{j-1} and q_j are precisely those interior faces of P with $\alpha(j-1) < \lambda_\sigma(\phi) \leq \alpha(j)$. This implies that all interior faces ϕ of P with $\lambda_\sigma(\phi) \leq \alpha(0)$ or $\lambda_\sigma(\phi) > \alpha(m)$ are not contained in any interior face ψ of Q and justifies the correctness of the first two steps. Moreover, the closure of any interior face ψ of $q_{j-1} \cup q_j$ is the union of the closure of interior faces ϕ of P with $\alpha(j-1) < \lambda_\sigma(\phi) \leq \alpha(j)$. But for any of these faces ϕ we certainly have

$$\lambda_{\sigma\alpha}(\psi) = j = \min\big\{k \in \{1, \dots, m\} \mid \alpha(k) \geq \lambda_\sigma(\phi)\big\}$$

and this justifies the remaining two steps.

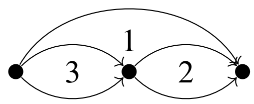
□

2.2.14. Example Let us compute the action of $d_1, d_2 \colon [2] \to [3]$ on the admissible 3-marked globular subgraph (P, λ) shown in Figure 2.11a. Let us first consider the action of d_1. As $d_1(0) = 0$ and $d_1(2) = 3$, the first two steps of the procedure given in Proposition 2.2.13 do not change the graph P. The values of the map $\widehat{\lambda}$ of step 3 are easily computed and can be seen in Figure 2.11b. There is then one edge between two faces ϕ and ψ of P with $1 = \widehat{\lambda}(\phi) = \widehat{\lambda}(\psi)$. Removing this edge then results in $d_1(P, \lambda)$ as can be seen in Figure 2.11c and 2.11d.

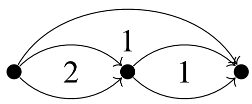
Let us now consider the action of d_2. Again, there are no edges to be removed in the first two steps and the map $\widehat{\lambda}$ can easily seen to be the one in Figure 2.11e. It turns out that $(P, \widehat{\lambda})$ is already the result of the algorithm given in Proposition 2.2.13.

2.3 Globular graphs and 2-computads

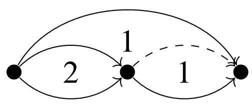
It is well-known that the category Cat of small categories is monadic over the category Graph of graphs. The monad T_1 for categories sends a graph G to the graph $T_1 G$ having the same vertices but possibly empty directed paths p as edges between $s(p)$ and $t(p)$. The monad T_1 is in fact strongly cartesian, meaning that $\widehat{S}\colon \mathrm{Graph} \to \mathrm{Graph}/T_1 \mathbb{1}$ has a right adjoint and that all the naturality squares of $\mu\colon T_1^2 \to T_1$ and $\eta\colon 1 \to T_1$ are cartesian. Similar results hold true for higher categories if one is willing to substitute n-globular sets for graphs. In the case of 2-categories this was first observed in [Str76] and has since been studied in various contexts, see e. g. [Bat98] for a construction of computads for a general finitary monad on globular sets or [Mét16] for a slick proof of the monadicity of strict ω-categories over computads. In this paragraph, following [Lei04], we sketch the relation between Street's 2-computads, 2-categories and globular graphs. Note that this section is kept a bit vague because the material is still work in progress.



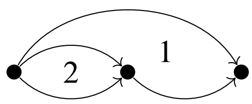
(a) An admissible 3-marked subgraph (P, λ).



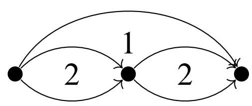
(b) The map $\widehat{\lambda}$ in the computation of $d_1(P, \lambda)$.



(c) Removal of the edges of P that are not present in $d_1(P, \lambda)$.



(d) The 2-marked subgraph $d_1(P, \lambda)$.



(e) The map $\widehat{\lambda}$ in the computation of $d_2(P, \lambda)$.

Figure 2.11: Examples of the procedure given in Proposition 2.2.13 for the computation of the action of simplicial operators on n-marked subgraphs.

2-globular sets and 2-categories Consider the category $\mathbb{G}_2$ generated by the graph

$$\bullet \overset{\sigma_0}{\underset{\tau_0}{\rightrightarrows}} \bullet \overset{\sigma_1}{\underset{\tau_1}{\rightrightarrows}} \bullet$$

subject to relations $\sigma_1\sigma_0 = \tau_1\sigma_0$ and $\sigma_1\tau_0 = \tau_1\tau_0$. A presheaf $X \in \widehat{\mathbb{G}}_2$ consists of three sets X_0, X_1 and X_2 together with maps

$$X_0 \overset{s_0}{\underset{t_0}{\leftleftarrows}} X_1 \overset{s_1}{\underset{t_1}{\leftleftarrows}} X_2$$

such that $s_0s_1 = s_0t_1$ and $t_0s_1 = t_0t_1$. Such a presheaf X is also known as 2-*globular set*.

We will have the occasion to use a slightly different description of $\widehat{\mathbb{G}}_2$. Towards this description let us define the category Graph_2 of 2-graphs. A 2-graph G consists of a set $V(G)$ of vertices together with graphs $G_{x,y}$ for each pair $x, y \in V(G)$ of vertices. A map $G \to H$ of 2-graphs is a map $f : V(G) \to V(H)$ together with compatible maps $G_{x,y} \to H_{fx,fy}$ of graphs.

Given such a 2-graph G, we associate with it the globular set

$$X_G = \left(\quad V(G) \overset{s_0}{\underset{t_0}{\leftleftarrows}} \coprod_{x,y}(G_{x,y})_0 \overset{s_1}{\underset{t_1}{\leftleftarrows}} \coprod_{x,y}(G_{x,y})_1 \quad \right),$$

where $s_0(u) = x$ and $t_0(u) = y$ for all $u \in (G_{x,y})_0$ and where the maps s_1 and t_1 are induced by the source and target maps of the graphs $G_{x,y}$. The assignment $G \mapsto X_G$ can easily be seen to be functorial in G.

Conversely, if X is a 2-globular set, we can associate with it a 2-graph as follows: For each pair $x, y \in X_0$ take the pullbacks

$$\begin{tikzcd} (G_{x,y})_0 \arrow[r] \arrow[d] & \mathbb{1} \arrow[d, "{(x,y)}"] \\ X_1 \arrow[r, "{(s_0,t_0)}"'] & X_0 \times X_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} (G_{x,y})_1 \arrow[r] \arrow[d] & \mathbb{1} \arrow[d, "{(x,y)}"] \\ X_2 \arrow[r, "{(s_0,t_0)\circ s_1}"'] & X_0 \times X_0. \end{tikzcd}$$

The bottom map in the right hand square satisfies $(s_0, t_0) \circ s_1 = (s_0, t_0) \circ t_1$ as X is a 2-globular set by assumption. We therefore get two factorisations

$$\begin{tikzcd} (G_{x,y})_1 \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] \arrow[d] & (G_{x,y})_0 \arrow[r] \arrow[d] & \{(x,y)\} \arrow[d, "{(x,y)}"] \\ X_2 \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] & X_1 \arrow[r, "{(s_0,t_0)}"'] & X_0 \times X_0 \end{tikzcd}$$

of the right hand square through the left hand square and hence graphs $(G_{x,y})_1 \rightrightarrows (G_{x,y})_0$ for all pairs $x, y \in X_0$. The set X_0 together with the graphs $G_{x,y}$ obviously form a 2-graph G_X and it is easy to check that the construction $G \mapsto G_X$ is functorial and inverse to the functor $X \mapsto X_G$ constructed above. We have thus proven the following well-known proposition:

2.3.1. Proposition *The category $\widehat{\mathbb{G}}_2$ of 2-globular sets and the category* Graph_2 *of 2-graphs are equivalent.*

Our interest in 2-globular sets stems from the fact that the category of small 2-categories is monadic over $\widehat{\mathbb{G}}_2$, see e. g. the end of section 2 in [Str76] or [Lei04, Appendix F]. Let us describe the monadic adjunction $F_2\colon \mathrm{Graph}_2 \rightleftarrows \mathrm{Cat}_2 : U_2$ in terms of the category of 2-graphs and the monadic adjunction $F_1\colon \mathrm{Graph} \rightleftarrows \mathrm{Cat} : U_1$ between graphs and small categories. The forgetful functor $U_2\colon \mathrm{Cat}_2 \to \mathrm{Graph}_2$ maps a 2-category $\mathfrak{A}$ to the 2-graph that has the objects of $\mathfrak{A}$ as vertices and associated with any two vertices $a, b \in \mathrm{Ob}\,\mathfrak{A}$ the graph $U_1\mathfrak{A}(a, b)$. The left adjoint F_2 takes a 2-graph G to the 2-category F_2G with objects $V(G)$ and

$$F_2G(x, y) = \coprod_{x=x_0,\dots,x_n=y} F_1G_{x_0,x_1} \times \cdots \times F_1G_{x_{n-1},x_n},$$

where the coproduct ranges over all positive integers n and all sequences $(x_0, \dots, x_n)$ of elements of $V(G)$ with $x_0 = x$ and $x_n = y$.

Using the fact that U_1 preserves coproducts and arbitrary limits, we obtain the following description of the monad $T_2 = U_2F_2$ on Graph_2:

2.3.2. Proposition *The monad T_2 for 2-categories takes a 2-graph G to the 2-graph T_2G on the same objects but with*

$$T_2G_{x,y} = \coprod_{x=x_0,\dots,x_n=y} T_1G_{x_0,x_1} \times \cdots \times T_1G_{x_{n-1},x_n},$$

where T_1 denotes the monad on Graph *for categories. One has to be careful to add another summand $\mathbb{1}$ in the description of T_2 in case that $x = y$ to cater for the identities of F_2G.*

In fact, Leinster shows in [Lei04, Appendix F] that T_2 (and even T_n for any $n \in \mathbb{N}$) is a cartesian monad.

2-computads The category of 2-globular sets and the monad T_2 suffer from one serious drawback: The shapes allowed to generate a 2-category are rather restrictive. Each 2-cell in a 2-globular set has only one source and one target 1-cell. However, it occurs frequently that one has more complicated diagrams that should generate a free 2-category. This defect can be remedied by passing from 2-globular sets to Street's category of computads at the expense of more involved combinatorics occuring in the base category.

The category of computads is a category of presheaves, see [CJ95], and one can either guess or compute the index category $\mathcal{C}$ for 2-computads from the proof of Carboni and Johnstone. The index category is given as the collage and hence explicitly computable. We circumvent this technicality and define the category of 2-computads as the category of presheaves on the category $\mathcal{C}$ that can be described as follows: The objects of $\mathcal{C}$ are 0, 1 and $\gamma_{n,m}$ for all pairs $n, m \in \mathbb{N}$. The morphisms are generated by

1. $\mathcal{C}(0,1) = \{\sigma, \tau\}$,
2. $\mathcal{C}(1, \gamma_{n,m}) = \{\sigma_1, \dots, \sigma_n\} \cup \{\tau_1, \dots, \tau_m\}$

subject to the relations

1. $\sigma_1 \sigma = \tau_1 \sigma$ and $\sigma_n \tau = \tau_m \tau$,

2. $\sigma_{i+1}\sigma = \sigma_i\tau$ for all $0 \leq i < n$,

3. $\tau_{i+1}\sigma = \tau_i\tau$ for all $0 \leq i < m$.

A 2-computad thus consists of a graph G together with a family $B_{n,m}$ of globs with dom $B_{n,m}$ a path of length n and cod $B_{n,m}$ a path of length m. It follows from this description that the terminal computad $\mathbb{1}$ has one vertex $*$ with one loop $t \colon * \to *$ and one glob $B_{n,m}$ of each size. We can moreover consider any globular graph as a 2-computad, the set $B_{n,m}$ of globs given by the interior faces with appropriate domains and codomains.

It was proven in [Str76] that 2-categories are monadic over $\widehat{\mathfrak{C}}$ and it follows from the explicit description of the monad T_2 on $\widehat{\mathfrak{C}}$ for 2-categories given there and our description of the terminal computad $\mathbb{1}$ that the cells in $T_2\mathbb{1}$ certainly include our globular graphs. One might thus hope for a characterisation of globular graphs as the *canonical arities* of the monad T_2 in the sense of [Web04; Web07; BMW12].

Nerves of globular graphs revisited So far, we did not speak of morphisms between globular graphs. It turns out that there are several choices and all of them have their merits. However, if one wants to give a presentation of the nerve of globular graphs in terms of a cosimplicial object, 2-functors between the free 2-categories on them are the correct choice.

By virtue of the preceding paragraph, we consider globular graphs as certain types of 2-computads. Let B_n denote the globular graph from Example 2.2.3. The free 2-category F_2B_n on B_n has 2 objects s, t and $n+1$ 1-cells

$e_0, \ldots, e_n \colon s \to t$ and a 2-cell $\lambda_{i,j} \colon e_i \to e_j$ whenever $i \leq j$.[1] It is immediate from this description that $B_\bullet$ is a cosimplicial object in the category whose objects are globular graphs and whose morphisms are 2-functors between the free categories.

Consider a 2-functor $f \colon F_2B_n \to F_2G$ into the free 2-category on some globular graph G that preserves source and target. It maps each edge e_i of B_n to a morphism in F_2G, i. e. to an st-path in G. Moreover, each 2-cell $\lambda_{i,j}$ is mapped to a 2-cell in F_2G, i. e. a pasting of globs of G. Taking the boundary of this pasting, we obtain a glob γ in G witnessing $f(e_i) \leq f(e_j)$. Summing up, we thus have the following proposition:

2.3.3. Proposition *The nerve* $\mathrm{N}(G)$ *of a globular graph* G *is isomorphic to the simplicial set* $[F_2B_\bullet, F_2G]_{s,t}$ *of source and target preserving* 2*-functors from the cosimplicial object* $F_2B_\bullet$ *into* F_2G.

Note that Proposition 2.3.3 is equivalent to the assertion that $\mathrm{N}(G)$ is naturally isomorphic to the nerve of the category $F_2G(s,t)$.

[1] Observe that one obtains essentially the nerve of the globular graph B_n upon applying the nerve functor $\mathrm{N} \colon \mathrm{Cat} \to \widehat{\mathbb{\Delta}}$ to each hom-set in this free 2-category. This is, of course, no coincidence but true more generally. However, we do not touch upon this topic in this text.

2.4 Pasting diagrams

2.4.1. Definition *A* pasting diagram $(G, \mathcal{S})$ *consists of a globular graph G and a set $\mathcal{S}$ of globular subgraphs of G that contains all paths and is closed under taking subgraphs.*

2.4.2. Remark The condition that $\mathcal{S}$ be closed under taking subgraphs in Definition 2.4.1 is not essential to the notion of pasting diagram but merely convenient for some considerations below, see e. g. the formulation of Definition 2.4.10. Moreover, if $\mathcal{S}$ is an arbitrary set of globular subgraphs of G, then there is a unique smallest set $\langle\mathcal{S}\rangle \supseteq \mathcal{S}$ of globular subgraphs of G that is closed under taking subgraphs and contains all paths in G. In this situation, we call $(G, \langle\mathcal{S}\rangle)$ the pasting diagram *generated* by $(G, \mathcal{S})$. In fact, we often give only $(G, \mathcal{S})$ and understand that we actually mean the pasting diagram $(G, \langle\mathcal{S}\rangle)$ generated by it.

2.4.3. Definition *Let $\Sigma = (G, \mathcal{S})$ and $\Pi = (G, \mathcal{T})$ be two pasting diagrams on the same underlying graph G. We then say that $\Sigma \to \Pi$ is an* inclusion of pasting diagrams *if $\mathcal{S} \subseteq \mathcal{T}$.*

2.4.4. Remark We warn the reader that any inclusion $\Sigma \to \Pi$ in the sense of Definition 2.4.3 is necessarily the identity on the underlying graphs.

2.4.5. Example For any globular graph G there exist minimal and maximal pasting diagrams $\Sigma_{\min} = (G, \mathcal{S}_{\min})$ and $\Pi_{\max} = (G, \mathcal{S}_{\max})$ with underlying graph G. The collections $\mathcal{S}_{\min}$ and $\mathcal{S}_{\max}$ of globular subgraphs of G are given by

$$\mathcal{S}_{\min} = \{A \subseteq G \mid A \text{ is a face of } G \text{ or a path in } G\}$$

and

$$\mathcal{S}_{\max} = \{A \subseteq G \mid A \text{ an arbitrary globular subgraph of } G\},$$

respectively. The maximal pasting diagram on G is generated by $\{G\}$, of course. Moreover, we have an obvious inclusion $\Sigma_{\min} \to \Pi_{\max}$.

2.4.6. Definition *Let $\Sigma = (G, \mathcal{S})$ be a pasting diagram.*

(a) We call Σ generated by wide subgraphs *if there is a set $\mathcal{R}$ of wide subgraphs of G such that $(G, \mathcal{S}) = (G, \langle \mathcal{R} \rangle)$.*

(b) We call Σ complete *if $\mathcal{S}$ is closed under taking joins, that is, if $H_1, H_2 \in \mathcal{S}$ implies $H_1 \vee H_2 \in \mathcal{S}$ whenever the join $H_1 \vee H_2$ is defined.*

(c) We call Σ closed under taking subdivisions *if for any subdivision K of some $H \in \mathcal{S}$ we have $K \in \mathcal{S}$.*

2.4.7. Remark Given a pasting diagram $\Sigma = (G, \mathcal{S})$ on some globular graph G, there exists a minimal complete pasting diagram $\Sigma^c = (G, \mathcal{S}^c)$ on G such that $\mathcal{S} \subseteq \mathcal{S}^c$.

2.4.8. Example Both the minimal pasting diagram $\Sigma_{\min}$ and the maximal pasting diagram $\Pi_{\max}$ on some globular graph G are closed under taking subdivisions. Moreover, $\Pi_{\max}$ is always complete and generated by wide subgraphs. However, $\Sigma_{\min}$ need neither be complete nor generated by wide subgraphs as can be seen for e. g. $G = B_1 \vee B_1$, where $\Sigma^c_{\min} = \Pi_{\max}$.

2.4.9. Lemma *Any complete pasting diagram is generated by wide subgraphs.*

Proof. Let $\Sigma = (G, \mathcal{S})$ be a complete pasting diagram and consider any

$A \in \mathcal{S}$. There are paths p and q from s to $s(A)$ and from $t(A)$ to t, respectively. We then have $B(A) = p \vee A \vee q \in \mathcal{S}$ as $\mathcal{S}$ is closed under taking joins and the set $\{B(A) \mid A \in \mathcal{S}\}$ is a generating set of wide subgraphs. □

2.4.10. Definition *Let $\Sigma = (G, \mathcal{S})$ be a pasting diagram and let $H \subseteq G$ be a globular subgraph. We call the set*

$$\mathcal{S}_H = \{A \in \mathcal{S} \mid A \subseteq H\}.$$

the restriction *of $\mathcal{S}$ to H and refer to the pasting diagram $\Sigma_H = (H, \mathcal{S}_H)$ as the* restriction *of Σ to H. In the case that $H = G_{x,y}$ for two vertices $x, y \in G$, we also write $\Sigma_{x,y}$ for the restriction of Σ to H.*

2.4.11. Remark Observe that if Σ is complete, closed under taking subdivisions, or generated by wide subgraphs, then so are all of its restrictions. This implies in particular that $(\Sigma_H)_c = (\Sigma^c)_H$.

2.4.12. Definition *Let $(G_1, \mathcal{S})$ and $(G_2, \mathcal{T})$ be pasting diagrams generated by collections $\mathcal{S}_0 \subseteq \mathcal{S}$ and $\mathcal{T}_0 \subseteq \mathcal{T}$ of wide subgraphs. Their* join *$(G_1, \mathcal{S}) \vee (G_2, \mathcal{T})$ is the pasting diagram with underlying graph $G_1 \vee G_2$ that is generated by the set*

$$\mathcal{S}_0 \vee \mathcal{T}_0 = \{A \vee B \mid A \in \mathcal{S}_0 \text{ and } B \in \mathcal{S}_1\}.$$

2.4.13. Remark We have to justify that the join of two pasting diagrams as in Definition 2.4.12 is well-defined, i. e. independent of the choice of the generating sets $\mathcal{S}_0$ and $\mathcal{T}_0$. To this end, consider another pair $\mathcal{S}_0'$ and $\mathcal{T}_0'$ of collections of wide subgraphs that generate $\mathcal{S}$ and $\mathcal{T}$, respectively. For each

$A \in \mathcal{S}_0$ and $B \in \mathcal{T}_0$ we then find $A' \in \mathcal{S}'_0$ and $B' \in \mathcal{T}'_0$ such that $A \subseteq A'$ and $B \subseteq B'$. But this implies $A \vee B \subseteq A' \vee B'$. As the argument is symmetric in $\mathcal{S}_0, \mathcal{T}_0$ and $\mathcal{S}'_0, \mathcal{T}'_0$, we may conclude that the pasting diagrams generated by $\mathcal{S}_0 \vee \mathcal{T}_0$ and $\mathcal{S}'_0 \vee \mathcal{T}'_0$, respectively, coincide.

2.4.14. Remark Suppose that Σ is some pasting diagram on $G_1 \vee G_2$ and let Σ_i be the restriction of Σ to G_i. We warn the reader that the obvious inclusion $\Sigma \subseteq \Sigma_1 \vee \Sigma_2$ is generally strict as demonstrated by the minimal pasting diagram $\Sigma_{\min}$ on $B_1 \vee B_1$. In the case that Σ is complete, however, we always have an equality $\Sigma = \Sigma_1 \vee \Sigma_2$.

2.5 Nerves of Pasting Diagrams

In this section we define the nerve $\mathrm{N}(\Sigma)$ of a pasting diagram $\Sigma = (G, \mathcal{S})$ as a certain simplicial subset of $\mathrm{N}(G)$ and record some elementary properties. We also observe that the pictorial calculus for $\mathrm{N}(G)$ from Proposition 2.2.12 and 2.2.13 carries over to nerves of complete pasting diagrams. The larger part of this section, however, consists of technical lemmata concerning the interplay of the nerve with operations such as restriction of pasting diagrams or the join of two pasting diagrams that we introduced above. These basic results will be used throughout the remaining chapters.

Definition and Elementary Properties We now give the promised definition of the nerve of a pasting diagram as a subset of the nerve of the underlying globular graph.

2.5.1. Definition *Let* $\Sigma = (G, \mathcal{S})$ *be a pasting diagram. Its nerve* $\mathrm{N}(\Sigma)$ *is the simplicial subset of* $\mathrm{N}(G)$ *consisting of those simplices* $\sigma = (p_0 \leq \cdots \leq p_n)$ *with the property that there exists some* $A \in \mathcal{S}$ *such that all the relations* $p_{i-1} \leq p_i$ *for* $1 \leq i \leq n$ *have witnesses* $\gamma_i \subseteq A$.

2.5.2. Example Let us consider a pasting diagram on the graph B_n from Example 2.2.3. We then have $\mathrm{N}(B_n, \mathcal{S}) = \Delta^1 \vee \ldots \vee \Delta^1$ for $\mathcal{S}$ the set of all interior faces of B_n.

2.5.3. Lemma *Let* $(G, \mathcal{S}) \to (G, \mathcal{T})$ *be an inclusion and suppose that* $\mathcal{S}$ *is closed under taking subdivisions. If an inner face of a simplex of* $\mathrm{N}(G, \mathcal{T})$ *is contained in* $\mathrm{N}(G, \mathcal{S})$*, then so is the simplex itself.*

Proof. Consider a simplex $\sigma \subseteq \mathrm{N}(G, \mathcal{T})$ with $d_i\sigma$ in $\mathrm{N}(G, \mathcal{S})$. Write

$$\sigma = (p_0 \leq \cdots \leq p_n)$$

and consider the face $d_i\sigma$ along with a witness $\gamma \subseteq A$ for $p_{i-1} \leq p_{i+1}$ and some $A \in \mathcal{S}$. Cutting γ along p_i provides us with witnesses γ_i, γ_{i+1} for $p_{i-1} \leq p_i \leq p_{i+1}$. Possibly passing to a subdivision of A we may assume $\gamma_i, \gamma_{i+1} \subseteq A$ and thus find $\sigma \subseteq \mathrm{N}(G, \mathcal{S})$. $\square$

In the case that Σ is complete, we have the following lemma that allows us to describe $\mathrm{N}(\Sigma)$ in terms of admissible n-marked subgraphs of G.

2.5.4. Lemma *Consider a complete pasting diagram* $\Sigma = (G, \mathcal{S})$*. An* n*-simplex* σ *with associated* n*-marked subgraph* $(P_\sigma, \lambda_\sigma)$ *of* $\mathrm{N}(G)$ *is contained in* $\mathrm{N}(\Sigma)$ *if and only if* $P_\sigma \in \mathcal{S}$.

Proof. Write $\sigma = (p_0 \leq \cdots \leq p_n)$. Recall that $P_\sigma = \bigcup_i p_i$. It is immediate that $\sigma \in \mathrm{N}(\Sigma)$ if $P_\sigma \in \mathcal{S}$. Now suppose that $\sigma \in \mathrm{N}(\Sigma)$. We then find some $A \in \mathcal{S}$ that contains all the witnesses γ_i for $p_{i-1} \leq p_i$. We also find possibly trivial paths q_1 from $s(G)$ to $s(A)$ and q_2 from $t(A)$ to $t(G)$ such that all the paths p_i can be written as $p_i = q_1 \cdot r_i \cdot q_2$ for some path r_i from $s(A)$ to $t(A)$ in A. It now follows that $R = \bigcup_i r_i \in \mathcal{S}$ as $\mathcal{S}$ is closed under taking subgraphs and this in turn implies $P = q_1 \vee R \vee q_2 \in \mathcal{S}$ as $\mathcal{S}$ is closed under taking joins. □

2.5.5. Corollary *Let $\Sigma = (G, \mathcal{S})$ be a complete pasting diagram. The n-simplices of $\mathrm{N}(\Sigma)$ are in bijection with admissible n-marked subgraphs (P, λ) of G such that $P \in \mathcal{S}$. Moreover, the action of simplicial operators on these n-marked subgraphs may be computed as described in Proposition 2.2.13.*

Nerves and the join operation In this paragraph we compute the nerve $\mathrm{N}(\Sigma_1 \vee \Sigma_2)$ of the join of two pasting diagrams that are generated by wide subgraphs. More precisely, we show that N is a strong monoidal functor from the monoidal category of pasting diagrams that are generated by wide subgraphs with the join operation to the cartesian closed category of simplicial sets. This observation will be used in § 3.1 to construct composition laws for the simplicial category that we associate with a pasting diagram.

2.5.6. Proposition *Let Σ and Π be pasting diagrams that are generated by wide subgraphs. There then exists an isomorphism*

$$\phi_{\Sigma,\Pi}\colon \mathrm{N}(\Sigma) \times \mathrm{N}(\Pi) \to \mathrm{N}(\Sigma \vee \Pi)$$

and these isomorphisms $\phi_{\Sigma,\Pi}$ equip the nerve with the structure of a strong monoidal functor from the monoidal category of pasting diagrams that are generated by wide subgraphs with the join operation to the cartesian closed category of simplicial sets

Proof. Given n-simplices $\sigma^k = (p_0^k \leq \cdots \leq p_n^k)$ of $\mathrm{N}(\Sigma_k)$ we have an n-simplex $\sigma^1 \cdot \sigma^2 = (p_0^1 \cdot p_0^2 \leq \cdots \leq p_n^1 \cdot p_n^2)$ in $\mathrm{N}(\Sigma_1 \vee \Sigma_2)$. One easily verifies that the assignment $(\sigma^1, \sigma^2) \mapsto \sigma^1 \cdot \sigma^2$ defines a map

$$\mathrm{N}(\Sigma_1) \times \mathrm{N}(\Sigma_2) \to \mathrm{N}(\Sigma_1 \vee \Sigma_2).$$

Let us show that this map is an isomorphism. To this end write $\Sigma_k = (G_k, \mathcal{S}_k)$ and denote by s_k and t_k the source and target of G_k. Consider an n-simplex $\sigma = (p_0 \leq \cdots \leq p_n)$ of $\mathrm{N}(\Sigma_1 \vee \Sigma_2)$. Each path p_i necessarily contains the cut vertex $t_1 = s_2$ of $G_1 \vee G_2$ and thus decomposes uniquely as $p_i = q_i^1 \cdot q_i^2$ for $s_k t_k$-paths q_i^k in G_k. We have thus found a unique pair (σ^1, σ^2) of n-simplices $\sigma^k = (q_0^k \leq \cdots \leq q_n^k)$ of $\mathrm{N}(\Sigma_k)$ such that $\sigma = \sigma^1 \cdot \sigma^2$.

Note that the pasting diagram Σ_0 on the trivial globular graph serves as unit in our monoidal category of pasting diagrams and we obviously have $\mathrm{N}(\Sigma_0) = \Delta^0$. Moreover, one of the axioms of a strong monoidal functor requires the diagram

$$\begin{tikzcd}
\mathrm{N}(\Sigma_1) \times \mathrm{N}(\Sigma_2) \times \mathrm{N}(\Sigma_3) \arrow[r, "\mathrm{N}(\Sigma_1) \times \phi"] \arrow[d, "\phi \times \mathrm{N}(\Sigma_2)"'] & \mathrm{N}(\Sigma_1) \times \mathrm{N}(\Sigma_2 \vee \Sigma_3) \arrow[d, "\phi"] \\
\mathrm{N}(\Sigma_1 \vee \Sigma_2) \times \mathrm{N}(\Sigma_3) \arrow[r, "\phi"'] & \mathrm{N}(\Sigma_1 \vee \Sigma_2 \vee \Sigma_3)
\end{tikzcd}$$

to be commutative. Given our definition of ϕ this is immediate, though. The remaining axioms of a strong monoidal functor are left to the reader. □

2.5.7. Corollary *If Σ is a complete pasting diagram on $G = G_1 \vee G_2$, then $\Sigma = \Sigma_1 \vee \Sigma_2$ and hence*

$$\mathrm{N}(\Sigma) \simeq \mathrm{N}(\Sigma_1) \times \mathrm{N}(\Sigma_2),$$

where Σ_i denotes the restriction of Σ to G_i.

2.5.8. Remark We did not specify the maps in the category that is to be the domain of the strong monoidal functor N. One could choose e. g. maps $(H, \mathcal{S}) \to (G, \mathcal{T})$ with $H \subseteq G$ a globular subgraph and $\mathcal{S} \subseteq \mathcal{T}$. The nerve is certainly functorial with respect to these maps and one easily verifies that the isomorphisms ϕ in Proposition 2.5.6 are natural.

2.5.9. Remark The hypothesis that the pasting diagrams in Proposition 2.5.6 are generated by wide subgraphs is essential to the construction of the isomorphism ϕ, for otherwise $\phi(\sigma^1, \sigma^2) = \sigma^1 \cdot \sigma^2$ need not even be a simplex of $\mathrm{N}(\Sigma_1 \vee \Sigma_2)$.

2.5.10. Remark The isomorphisms ϕ constructed in Proposition 2.5.6 can also be described in terms of n-marked subgraphs. The image $\phi(\sigma, \tau)$ of an n-simplex $(\sigma, \tau) \in \mathrm{N}(\Sigma_1) \times \mathrm{N}(\Sigma_2)$ with admissible n-marked subgraphs $(P_\sigma, \lambda_\sigma)$ and (P_τ, λ_τ) has $(P_\sigma \vee P_\tau, \lambda_\sigma \vee \lambda_\tau)$ as its associated n-marked subgraph. Put differently, the isomorphisms ϕ are given by the join of the respective n-marked subgraphs.

Unions and intersections In this paragraph we collect some technical yet easy lemmata concerning intersections and unions of nerves of pasting diagrams.

2.5.11. Lemma *Consider a family* $(G, \mathcal{S}_j)$, $j \in J$, *of pasting diagrams on the same underlying graph* G. *We then have an equality*

$$\bigcup_{j \in J} \mathrm{N}(G, \mathcal{S}_j) = \mathrm{N}\Big(G, \bigcup_{j \in J} \mathcal{S}_j\Big)$$

of simplicial subsets of $\mathrm{N}(G)$.

Proof. Consider some simplex $\sigma = (p_0 \leq \cdots \leq p_n)$ in $\mathrm{N}(G)$. We have $\sigma \in \bigcup_j \mathrm{N}(G, \mathcal{S}_j)$ if and only if there exists some $j \in J$ and some $A \in \mathcal{S}_j$ such that all the witnesses of $p_{i-1} \leq p_i$ are contained in A. This is obviously equivalent to the condition that there exists some $A \in \bigcup_j \mathcal{S}_j$ such that all the witnesses of $p_{i-1} \leq p_i$ are contained in A, i. e. to the statement $\sigma \in \mathrm{N}(G, \bigcup_j \mathcal{S}_j)$. $\square$

2.5.12. Lemma *Let $(G, \mathcal{S})$ be a pasting diagram, $H \subseteq G$ a wide globular subgraph and $\mathcal{S}_H$ the restriction of $\mathcal{S}$ to H. Then*

$$\begin{tikzcd} \mathrm{N}(H, \mathcal{S}_H) \arrow[r] \arrow[d] & \mathrm{N}(G, \mathcal{S}_G) \arrow[d] \\ \mathrm{N}(H) \arrow[r] & \mathrm{N}(G) \end{tikzcd}$$

is cartesian.

Proof. Immediate from the definitions. □

2.5.13. Corollary *Let $(G, \mathcal{S}) \subseteq (G, \mathcal{T})$ be an inclusion of pasting diagrams, $H \subseteq G$ a wide globular subgraph and $\mathcal{S}_H$ and $\mathcal{T}_H$ the restrictions of $\mathcal{S}$ and $\mathcal{T}$ to H. Then*

$$\begin{tikzcd} \mathrm{N}(H, \mathcal{S}_H) \arrow[r] \arrow[d] & \mathrm{N}(G, \mathcal{S}) \arrow[d] \\ \mathrm{N}(H, \mathcal{T}_H) \arrow[r] & \mathrm{N}(G, \mathcal{T}) \end{tikzcd}$$

is cartesian.

2.5.14. Lemma *Let $(G, \mathcal{S})$ and $(H, \mathcal{T})$ be pasting diagrams such that H is a wide subgraph of G. Further let $\mathcal{S}_H$ be the restriction of $\mathcal{S}$ to H and suppose $\mathcal{S}_H \subseteq \mathcal{T}$, i. e. $\mathcal{S}_H = \mathcal{S} \cap \mathcal{T}$. Then*

$$\begin{tikzcd} \mathrm{N}(H, \mathcal{S}_H) \arrow[r] \arrow[d] & \mathrm{N}(G, \mathcal{S}) \arrow[d] \\ \mathrm{N}(H, \mathcal{T}) \arrow[r] & \mathrm{N}(G, \mathcal{S} \cup \mathcal{T}) \end{tikzcd}$$

is bicartesian.

Proof. Let us first show that the square is cocartesian. We have jointly surjective inclusions $\mathrm{N}(G, \mathcal{S}) \to \mathrm{N}(G, \mathcal{S} \cup \mathcal{T})$ and $\mathrm{N}(H, \mathcal{T}) \to \mathrm{N}(G, \mathcal{S} \cup \mathcal{T})$ since $H \subseteq G$ is wide by assumption. It is furthermore easy to see that the intersection of $\mathrm{N}(H, \mathcal{T})$ and $\mathrm{N}(G, \mathcal{S})$ in $\mathrm{N}(G, \mathcal{S} \cup \mathcal{T})$ is precisely $\mathrm{N}(H, \mathcal{S}_H)$. In fact, if $\sigma = (p_0 \leq \cdots \leq p_n)$ is a simplex in this intersection, then $p_i \subseteq H$ for all i and we consequently find some $A \in \mathcal{S}$, $A \subseteq H$, that contains witnesses for all the relations $p_{i-1} \leq p_i$. This implies $\sigma \in \mathrm{N}(H, \mathcal{S}_H)$.

It now either follows from general properties of coherent categories or from Corollary 2.5.13 that the square is cartesian, too. $\square$

2.5.15. Lemma *Consider a complete pasting diagram $(G, \mathcal{S})$ on some globular graph G with source s and target t. Let H_1 and H_2 be wide subgraphs of G and let $H_0 = H_1 \cap H_2$. Denote by $\mathcal{S}_i$ the restriction of $\mathcal{S}$ to H_i.*

(a) *The intersection of* $\mathrm{N}(H_1, \mathcal{S}_1)$ *and* $\mathrm{N}(H_2, \mathcal{S}_2)$ *in* $\mathrm{N}(G, \mathcal{S})$ *is nonempty if and only if* H_0 *contains a path from* s *to* t.

(b) *If* H_0 *is a wide globular subgraph, then*

$$\begin{tikzcd} \mathrm{N}(H_0, \mathcal{S}_0) \arrow[r] \arrow[d] & \mathrm{N}(H_1, \mathcal{S}_1) \arrow[d] \\ \mathrm{N}(H_2, \mathcal{S}_2) \arrow[r] & \mathrm{N}(G, \mathcal{S}) \end{tikzcd}$$

is cartesian.

Proof. Part (a) is obvious, for st-paths p from s to t in H_0 are in bijective correspondence with 0-simplices (p) in $\mathrm{N}(\Sigma_1) \cap \mathrm{N}(\Sigma_2)$.

Let us now suppose that H_0 is a wide globular subgraph of G. It is clear that $\mathrm{N}(G_0, \mathcal{S}_0)$ is a simplicial subset of the intersection of $\mathrm{N}(H_1, \mathcal{S}_1)$ and $\mathrm{N}(H_2, \mathcal{S}_2)$. It thus suffices to show $\mathrm{N}(H_1, \mathcal{S}_1) \cap \mathrm{N}(H_2, \mathcal{S}_2) \subseteq \mathrm{N}(H_0, \mathcal{S}_0)$. To this end, consider any simplex $\sigma = (p_0 \leq \cdots \leq p_n)$ in the intersection of $\mathrm{N}(H_1, \mathcal{S}_1)$ and $\mathrm{N}(H_2, \mathcal{S}_2)$. Observe that all the paths p_i are paths in $H_0 = H_1 \cap H_2$ and that $\bigcup_i p_i \in \mathcal{S}_k$ for $k \in \{1, 2\}$ by virtue of Lemma 2.5.4. We thus have $\bigcup_i p_i \in \mathcal{S}_1 \cap \mathcal{S}_2 = \mathcal{S}_0$ and conclude $\sigma \in \mathrm{N}(H_0, \mathcal{S}_0)$ by Lemma 2.5.4. $\square$

2.5.16. Corollary *Let Σ be a complete pasting diagram on a globular graph G with source s and target t. Further let $x, y \in G$. If there exists a directed path from x to y in G, then*

$$\begin{tikzcd} \mathrm{N}(\Sigma_{s,x} \vee \Sigma_{x,y} \vee \Sigma_{y,t}) \arrow[r] \arrow[d] & \mathrm{N}(\Sigma_{s,x} \vee \Sigma_{x,t}) \arrow[d] \\ \mathrm{N}(\Sigma_{s,y} \vee \Sigma_{y,t}) \arrow[r] & \mathrm{N}(\Sigma) \end{tikzcd}$$

is a cartesian square of simplicial sets, i. e.

$$\mathrm{N}(\Sigma_{s,x} \vee \Sigma_{x,t}) \cap \mathrm{N}(\Sigma_{s,y} \vee \Sigma_{y,t}) = \mathrm{N}(\Sigma_{s,x} \vee \Sigma_{x,y} \vee \Sigma_{y,t}).$$

Moreover, the intersection of $\mathrm{N}(\Sigma_{s,x} \vee \Sigma_{x,t})$ *and* $\mathrm{N}(\Sigma_{s,y} \vee \Sigma_{y,t})$ *in* $\mathrm{N}(\Sigma)$ *is empty whenever G contains neither a directed path from x to y nor a directed path from y to x.*

Proof. Immediate from Lemma 2.5.15 and Lemma 2.1.14. $\square$

Formal partial composites In this short final paragraph, we define an operation "$\multimap$" that associates with an inclusion $\Sigma \to \Pi$ of pasting digarams a new pasting diagram $\Sigma \multimap \Pi$. In case that Π is complete, $\Sigma \multimap \Pi$ has the property that the inclusion $\Sigma \to \Pi$ factors as

$$\begin{tikzcd} \Sigma \arrow[rr] \arrow[dr] & & \Pi \\ & \Sigma \multimap \Pi. \arrow[ur] & \end{tikzcd}$$

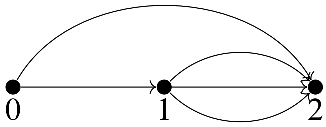
The pasting diagram $\Sigma \multimap \Pi$ should hence be thought of as a formal partial composite of the inclusion $\Sigma \to \Pi$. In fact, $\Sigma \multimap \Pi$ features in exactly this role in chapter 4 and we advise the reader to skip this section on a first reading and only come back to it after having had a first look at the material in chapter 4.

2.5.17. Definition *Let $\Sigma \to \Pi$ be an inclusion of pasting diagrams $\Sigma = (G, \mathcal{S})$ and $\Pi = (G, \mathcal{T})$. We define a pasting diagram $\Sigma \multimap \Pi = (G, \mathcal{S} \multimap \mathcal{T})$ on G by*

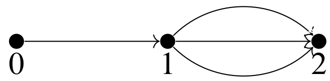
$$\mathcal{S} \multimap \mathcal{T} = \mathcal{S} \cup \bigcup_{x \in G \smallsetminus \{s,t\}} \mathcal{T}_{s,x} \vee \mathcal{T}_{x,t}.$$

2.5.18. Remark We warn the reader that generally $\mathcal{S} \multimap \mathcal{T} \nsubseteq \mathcal{T}$ as can already be seen for $\Sigma = \Pi$ some non-complete pasting diagram. However, $\mathcal{S} \multimap \mathcal{T} \subseteq \mathcal{T}$ whenever $\mathcal{T}$ is complete, see Lemma 2.5.22.

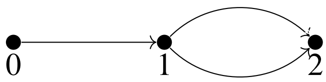
2.5.19. Example Let us compute $\Sigma \multimap \Pi$ for $\Sigma = (G, \mathcal{S}^c_{\min})$ the minimal complete and $\Pi = (G, \mathcal{S}_{\max})$ the maximal pasting diagram on the globular



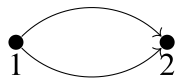
(a) A globular graph G



(b) The subgraph $G_{0,1} \vee G_{1,2}$ of G.



(c) A wide subgraph of $G_{0,1} \vee G_{1,2}$.



(d) A glob γ of $G_{0,1} \vee G_{1,2}$.

Figure 2.12: The graphs occuring in Example 2.5.19.

graph G shown in Figure 2.12a. The definition of $\mathcal{S}^c_{\min} \multimap \mathcal{S}_{\max}$ gives us

$$\mathcal{S}^c_{\min} \multimap \mathcal{S}_{\max} = \mathcal{S}^c_{\min} \cup \big(\mathcal{S}_{\max,0,1} \vee \mathcal{S}_{\max,1,2}\big).$$

The set $\mathcal{S}_{\max,0,1} \vee \mathcal{S}_{\max,1,2}$ is the set of all globular subgraphs of $G_{0,1} \vee G_{1,2}$, i. e. of the graph shown in Figure 2.12b. One now easily sees that the only subgraphs in $(\mathcal{S} \multimap \mathcal{T}) \smallsetminus \mathcal{S}$ are $G_{0,1} \vee G_{1,2}$ itself and the two of its subgraphs shown in Figure 2.12c and 2.12d.

2.5.20. Remark Consider an inclusion $\Sigma \to \Pi$ of pasting diagrams on some graph G. For any globular subgraph $H \subseteq G$, there exists a canonical inclusion

$$(\Sigma_H \multimap \Pi_H) \to (\Sigma \multimap \Pi)_H.$$

This inclusion is strict in general, even for subgraphs H of the form $H = G_{x,y}$.

This can already be seen in Example 2.5.19, where the glob shown in Figure 2.12d is an element of $(\mathcal{S}_{\min} \multimap \mathcal{S}_{\max})_{1,2}$ but $\mathcal{S}_{\min,1,2} \multimap \mathcal{S}_{\max,1,2}$ coincides with $\mathcal{S}_{\min,1,2}$ simply because there are no vertices $x \in G_{1,2} \setminus \{1,2\}$.

We end this paragraph with some technical but easy observations on "$\multimap$" that will be of good use in chapter 4.

2.5.21. Lemma *Let $\Sigma \to \Pi$ be an inclusion of pasting digarams. If Π is complete, then so is $\Sigma \multimap \Pi$.*

Proof. Let us write $\Pi = (G, \mathcal{T})$ and $\Sigma = (G, \mathcal{S})$ and consider $A, B \in \mathcal{S} \multimap \mathcal{T}$ with $t(A) = s(B) = x$. Observe that $A \subseteq G_{s,x}$ and $B \subseteq G_{x,t}$, i. e. $A \in \mathcal{T}_{s,y} \vee \mathcal{T}_{y,x}$ and $B \in \mathcal{T}_{x,,z} \vee \mathcal{T}_{z,t}$. But this implies $A \vee B \in \mathcal{T}_{s,x} \vee \mathcal{T}_{x,t} \subseteq \mathcal{S} \multimap \mathcal{T}$ by completeness of Π. $\square$

2.5.22. Lemma *Consider an inclusion $\Sigma \to \Pi$ of pasting diagrams on some globular graph G with source s and target t. If Π is complete and x and y are two vertices of G such that $(x, y) \neq (s, t)$, then $(\Sigma \multimap \Pi)_{x,y} = \Pi_{x,y}$.*

Proof. Let us write $\Pi = (G, \mathcal{T})$ and $\Sigma = (G, \mathcal{S})$. Let us first show that $\mathcal{T}_{x,y} \subseteq (\mathcal{S} \multimap \mathcal{T})_{x,y}$. To this end, consider any $A \in \mathcal{T}$ with $A \subseteq G_{x,y}$. If $x \neq s$ then $A \subseteq G_{x,y} \subseteq G_{x,t}$ and hence $A \in \mathcal{T}_{x,t} \subseteq \mathcal{T}_{s,x} \vee \mathcal{T}_{x,t} \subseteq \mathcal{S} \multimap \mathcal{T}$. The case $y \neq t$ is handled analogously.

Conversely, if $A \in (\mathcal{S} \multimap \mathcal{T})_{x,y}$, then $A \in \mathcal{S}_{x,y} \subseteq \mathcal{T}_{x,y}$ or $A \subseteq G_{x,y}$ with $A \in \mathcal{T}_{s,z} \vee \mathcal{T}_{z,t}$ for some vertex $z \in G \setminus \{x, y\}$. In the latter case, we conclude $A \in \mathcal{T}_{x,y}$ by completeness of Π. $\square$

2.5.23. Lemma *Let $\Sigma \to \Pi$ be an inclusion of pasting diagrams on the underlying graph G with source s and target t. We then have an equality*

$$\mathrm{N}(\Sigma \multimap \Pi) = \mathrm{N}(\Sigma) \cup \bigcup_{x \in G \smallsetminus \{s,t\}} \mathrm{N}(\Pi_{s,x} \vee \Pi_{x,t})$$

of simplicial subsets of $\mathrm{N}(G)$.

Proof. This follows from the definition of $\Sigma \multimap \Pi$ and Lemma 2.5.11. $\square$

3 Pasting Diagrams and Simplicial Categories

In this chapter we associate with any complete pasting diagram Σ a simplicial category $\mathbb{C}[\Sigma]$. It turns out that $\mathbb{C}[\Pi_{\max}]$, where $\Pi_{\max}$ denotes the maximal pasting diagram on some globular graph G, is nothing but the simplicial category obtained from the free 2-category on G by local application of the nerve functor, see Example 3.1.5. We are therefore led to think of functors $u\colon \mathbb{C}[\Sigma] \to \mathbb{A}$ with a pasting diagram $\Sigma_{\min} \subseteq \Sigma \subseteq \Pi_{\max}$ as mediating between the mere specification of maps and cells in $\mathbb{A}$ at the level of $\Sigma_{\min}$ and a fully coherent composition $\mathbb{C}[\Pi_{\max}] \to \mathbb{A}$. There is one problem in this picture, though. In general, $\Sigma_{\min}$ is not complete and our definition of $\mathbb{C}[\Sigma]$ does not work. In order to overcome this difficulty, we introduce labelings of globular graphs in a simplicial category $\mathbb{A}$ so as to capture the idea of a compatible specification of maps and cells in $\mathbb{A}$. Our main theorem in this chapter then essentially closes the gap between labelings of a globular graph G in some simplicial category $\mathbb{A}$ and simplicial functors $\mathbb{C}[\Sigma^c_{\min}] \to \mathbb{A}$.

Theorem A *Suppose that Σ is the minimal complete pasting diagram on some globular graph G. The map*

$$\mathrm{Cat}_{\widehat{\mathbb{\Delta}}}(\mathbb{C}[\Sigma], \mathbb{A}) \to L(G, \mathbb{A}), \quad u \mapsto \Lambda_u,$$

that sends a simplicial functor u to its associated labeling Λ_u is a bijection.

The proof of this theorem relies on a description of $\mathbb{C}[\Sigma^c_{\min}]$ in terms of products of low-dimensional simplices in $\mathbb{C}[\Sigma^c_{\min}](x, y)$. After having given the basic definitions sketched above in § 3.1, the whole of § 3.2 is devoted to this description of $\mathbb{C}[\Sigma^c_{\min}]$. The chapter then ends with a proof of Theorem A in § 3.3.

3.1 The simplicial category associated with a complete pasting diagram

This short section introduces the simplicial category $\mathbb{C}[\Sigma]$ associated with a complete pasting diagram Σ. Moreover, we give the definition of a labeling of a globular graph in some simplicial category $\mathbb{A}$.

3.1.1. Definition *Let Σ be a complete pasting diagram. We associate with Σ a simplicial category $\mathbb{C}[\Sigma]$ with objects the vertices of Σ and mapping spaces $\mathbb{C}[\Sigma](x, y) = \mathrm{N}(\Sigma_{x,y})$. The identities are given by the isomorphisms*

$$\Delta^0 \simeq \mathrm{N}(\Sigma_{x,x}) = \mathbb{C}[\Sigma](x, x)$$

and the composition laws are given by

$$\begin{tikzcd}
\mathbb{C}[\Sigma](x,y) \times \mathbb{C}[\Sigma](y,z) \arrow[rr] \arrow[d, equal] & & \mathbb{C}[\Sigma](x,z) \arrow[d, equal] \\
\mathrm{N}(\Sigma_{x,y}) \times \mathrm{N}(\Sigma_{y,z}) \arrow[r, "\sim"] & \mathrm{N}(\Sigma_{x,y} \vee \Sigma_{y,z}) \arrow[r, "inclusion"'] & \mathrm{N}(\Sigma_{x,z}).
\end{tikzcd}$$

3.1.2. Remark The fact that $\mathbb{C}[\Sigma]$ is well-defined, i. e. that the composition laws are associative and unital, follows immediately from the fact that N is a strong monoidal functor, see Proposition 2.5.6.

3.1.3. Remark The necessity of the condition that Σ be complete in Definition 3.1.1 is somewhat subtle. In fact, the isomorphism

$$\mathrm{N}(\Sigma_{x,y}) \times \mathrm{N}(\Sigma_{y,z}) \to \mathrm{N}(\Sigma_{x,y} \vee \Sigma_{y,z})$$

exists for all Σ that are generated by wide subgraphs. However, as already pointed out in Remark 2.4.14, for non-complete Σ it might very well happen that $\Sigma_{x,y} \vee \Sigma_{y,z} \nsubseteq \Sigma_{x,z}$, i. e. that the composition laws of $\mathbb{C}[\Sigma]$ are not well-defined.

3.1.4. Remark The assignment $\Sigma \mapsto \mathbb{C}[\Sigma]$ is obviously functorial in arbitrary inclusions of complete pasting diagrams, that is, we have a functor from the category of complete pasting diagrams and inclusions into the category of simplicial categories.

3.1.5. Example Let us compute the category $\mathbb{C}[\Pi_{\max}]$ for the maximal pasting diagram on some globular graph G. The objects of $\mathbb{C} = \mathbb{C}[\Pi_{\max}]$ are the vertices of G and the mapping spaces are $\mathbb{C}(x, y) = \mathrm{N}(\Pi_{\max,x,y}) = \mathrm{N}(G_{x,y})$. According to Proposition 2.3.3, we can identify $\mathrm{N}(G_{x,y})$ with the nerve of the category $F_2G(x, y)$, where F_2G denotes the free 2-category on the globular graph G considered as a 2-computad. Altogether, we thus find that $\mathbb{C}[\Pi_{\max}]$ has objects the vertices of G and mapping spaces

$$\mathbb{C}[\Pi_{\max}](x, y) = \mathrm{N}\big(F_2G(x, y)\big).$$

We leave it to the reader to verify that the compositions in $\mathbb{C}[\Pi_{\max}]$ are induced by those of F_2G. Altogether, we thus find that $\mathbb{C}[\Pi_{\max}]$ is obtained from F_2G by applying the nerve functor $\mathrm{Cat} \to \widehat{\mathbb{\Delta}}$ locally to each of the categories $F_2G(x, y)$.

3.1.6. Remark The composition laws in the category $\mathbb{C}[\Sigma]$ for some complete pasting diagram $\Sigma = (G, \mathcal{S})$ admit a neat description in terms of n-marked subgraphs. We know from Corollary 2.5.5 that n-simplices in $\mathbb{C}[\Sigma](x, y)$ correspond to admissible n-marked subgraphs $(P_\sigma, \lambda_\sigma)$ with $P_\sigma \in \mathcal{S}_{x,y}$. Moreover, Remark 2.5.10 tells us that the composition $\sigma \circ \tau$ of two such simplices is nothing but $(P_\tau \vee P_\sigma, \lambda_\tau \vee \lambda_\sigma)$, where $\lambda_\tau \vee \lambda_\sigma$ is given by λ_τ on P_τ and by λ_σ on P_σ.

3.1.7. Remark The categories $\mathbb{C}[\Sigma]$ are simplicial computads in the sense of [RV16], that is, any n-simplex $\sigma \in \mathbb{C}[\Sigma](x, y)$ has a unique decomposition $\sigma = (\sigma_1\alpha_1) \circ \cdots \circ (\sigma_a\alpha_a)$, where the σ_i are nondegenerate atomic n_i-simplices and the α_i are degeneracy operators in $\mathbb{\Delta}$. Here, an atomic n-simplex is a simplex that cannot be written as a nontrivial composition in $\mathbb{C}[\Sigma]$. We will use the decomposition of σ into atomic n-simplices more or less explicitly in our proof of Theorem A.

The above decomposition of some simplex σ is easy to get hold of using the pictorial description of $\mathbb{C}[\Sigma]$ given in the preceding remark. It is clear that an n-simplex σ is atomic if and only if P_σ is 2-connected or a single edge. Given any admissible n-marked subgraph $(P_\sigma, \lambda_\sigma)$ in $\mathbb{C}[\Sigma](x, y)$, we therefore write $P_\sigma = P_1 \vee \ldots \vee P_a$ with each P_i a 2-connected globular subgraph and thus obtain $\sigma = \tau_a \circ \cdots \circ \tau_1$ with $(P_{\tau_i}, \lambda_{\tau_i}) = (P_i, \lambda_\sigma|_{P_i})$. The Eilenberg-Zilber lemma now yields nondegenerate simplices σ_i and degeneracy operators α_i with $\tau_i = \sigma_i \alpha_i$ and it is clear from the description of the simplicial operators in Proposition 2.2.13 that σ_i is atomic, too.

The following definition formalises the notion of a 2-dimensional diagram in a simplicial category.

3.1.8. Definition *Let G be a globular graph. A* labeling Λ *of G in some simplicial category $\mathbb{A}$ consists of the following data:*

1. *An object $\Lambda x \in \mathbb{A}$ for each vertex $x \in G$.*
2. *A 0-simplex $\Lambda e \in \mathbb{A}(\Lambda x, \Lambda y)$ for each edge e from x to y in G.*
3. *A 1-simplex $\Lambda\phi \in \mathbb{A}(\Lambda x, \Lambda y)$ for each interior face ϕ of G with $s(\phi) = x$ and $t(\phi) = y$.*

This data is subject to the condition that for each interior face ϕ of G we have *equalities*

$$d_0\Lambda\phi = \Lambda e_r \circ \cdots \circ \Lambda e_1 \quad \textit{and} \quad d_1\Lambda\phi = \Lambda f_s \circ \cdots \circ \Lambda f_1,$$

where $\mathrm{dom}(\phi) = f_1 \cdot \cdots \cdot f_s$ *and* $\mathrm{cod}(\phi) = e_1 \cdot \cdots \cdot e_r$.

3.1.9. Example (a) A labeling Λ of the graph B_n from Example 2.2.3 in some simplicial category $\mathbb{A}$ consists of the choice of two objects $\Lambda s, \Lambda t \in \mathbb{A}$ and a map $\Delta^1 \vee \ldots \vee \Delta^1 \to \mathbb{A}(\Lambda s, \Lambda t)$ of simplicial sets. In the case that $\mathbb{A}$ is enriched over quasi-categories, a labeling of B_n is therefore nothing but a string of n composable cells in $\mathbb{A}(\Lambda s, \Lambda t)$.

(b) A labeling Λ of the graph $B_1 \vee B_1$ in Figure 2.8 in some simplicial category $\mathbb{A}$ consists of the choice of three vertices Λx, Λy and Λz in $\mathbb{A}$ together with two maps $\Delta^1 \to \mathbb{A}(\Lambda x, \Lambda y)$ and $\Delta^1 \to \mathbb{A}(\Lambda y, \Lambda z)$. In anticipation of § 3.2, the reader might want to compare the notion of a labeling of $B_1 \vee B_1$ with that of a functor $\mathbb{C}[\Sigma^c_{\mathrm{min}}] \to \mathbb{A}$ from the simplicial category associated with the minimal complete pasting diagram on $B_1 \vee B_1$.

3.1.10. Remark Each functor $u\colon \mathbb{C}[\Sigma] \to \mathbb{A}$ determines a labeling Λ_u of the graph G underlying Σ in $\mathbb{A}$. The labeling is given by $\Lambda_u x = u(x)$, $\Lambda_u e = u(e)$ and $\Lambda_u \phi = u(\phi)$, where e and $\phi = (\mathrm{dom}\,\phi \leq \mathrm{cod}\,\phi)$ are considered as 0- and 1-simplices in $\mathbb{C}[\Sigma](s(e), t(e))$ and $\mathbb{C}[\Sigma](s(\phi), t(\phi))$, respectively.

3.2 The simplicial category associated with a minimal complete pasting diagram

In this section we prepare the ground for the proof of Theorem A in the following section and give an explicit description of the category $\mathbb{C}[\Sigma]$ in the case that Σ is the minimal complete pasting diagram on some globular graph G. In a certain sense, we show that $\mathbb{C}[\Sigma]$ is freely generated by G and additional witnesses of the Godement interchange law, that is, any simplex $\sigma \in \mathbb{C}[\Sigma](x, y)$ of dimension $n \geq 2$ sits inside some cube $(\Delta^1)^a \subseteq \mathbb{C}[\Sigma](x, y)$ that corresponds to the different orders of composition of a diagram such as

$$\begin{tikzcd} x \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet & \cdots & \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet\, y \end{tikzcd}$$

whose a globs are faces of G, see Proposition 3.2.2. The argument leading to this presentation relies on the description of simplices in $\mathrm{N}(\Sigma)$ in terms of admissible n-marked subgraphs $(P_\sigma, \lambda_\sigma)$ from Corollary 2.5.5.

This section has two parts. In the first part we prove the statement sketched above, i. e. that any n-simplex σ in $\mathbb{C}(x, y)$ is contained in some cube $(\Delta^1)^a$. In the second part we then determine the action of simplicial operators α on the simplices of $\mathbb{C}(x, y)$ in terms of the maps $\Delta^n \to (\Delta^1)^a$. Especially this latter part is rather technical and the reader might want to skip some of the proofs until she or he has had a look at the proof of Theorem A in the forthcoming section, where these technicalities are crucial.

Fix a globular graph G and let $\Sigma = \Sigma^c_{\min} = (G, \mathcal{S})$ be the minimal complete pasting diagram on G. Throughout this section we abbreviate $\mathbb{C} = \mathbb{C}[\Sigma]$. Recall from Lemma 2.5.4 that there is a one-to-one-correspondence between n-simplices $\sigma \in \mathrm{N}(\Sigma)$ and n-marked subgraphs $(P_\sigma, \lambda_\sigma)$, where $P_\sigma \in \mathcal{S}$ and $\lambda_\sigma \colon \Phi(P_\sigma) \to \{1, \ldots, n\}$ is admissible in the sense of Definition 2.2.8. However, any $P_\sigma \in \mathcal{S} = \mathcal{S}_{\min}$ is a join of edges and faces of G, so that admissibility of λ_σ turns out to be an empty condition in the case at hand, because there are no interior faces ϕ and ψ in P_σ with some edge e in both $\mathrm{cod}(\phi)$ and $\mathrm{dom}(\psi)$. We thus identify n-simplices in $\mathrm{N}(\Sigma)$ or $\mathrm{N}(\Sigma_{x,y}) = \mathbb{C}(x, y)$ with n-marked subgraphs $(P_\sigma, \lambda_\sigma)$, where $P_\sigma \in \mathcal{S}_{x,y}$ and $\lambda_\sigma \colon \Phi(P_\sigma) \to \{1, \ldots, n\}$ is an arbitrary map.

Let us now consider an n-simplex $\sigma = (p_0 \leq \cdots \leq p_n) \in \mathbb{C}(x, y)$ and let $(P_\sigma, \lambda_\sigma)$ be the corresponding n-marked subgraph. Observe that P_σ can be written uniquely as a join $P_\sigma = P_1 \vee \ldots \vee P_a$ with each P_i an edge or a face of P_σ since $\Sigma = (G, \mathcal{S})$ is the minimal complete pasting diagram on G. With

$$\varepsilon_i(\sigma) = \begin{cases} 0 & \text{if } P_i \text{ is an edge of } P, \\ 1 & \text{if } P_i \text{ is a face of } P, \end{cases}$$

each P_i corresponds to an $\varepsilon_i(\sigma)$-dimensional simplex $\sigma_i \in \mathbb{C}(sP_i, tP_i)$, namely $\sigma_i = (P_i)$ if P_i is an edge and $\sigma_i = (\mathrm{dom}\, P_i \leq \mathrm{cod}\, P_i)$ if P_i is a face. We thus obtain a map

$$\Delta^{\varepsilon_1(\sigma)} \times \Delta^{\varepsilon_2(\sigma)} \times \cdots \times \Delta^{\varepsilon_a(\sigma)} \xrightarrow{(\sigma_1, \ldots, \sigma_a)} \mathbb{C}(sP_1, tP_1) \times \cdots \times \mathbb{C}(sP_a, tP_a).$$

As this map is crucial to the rest of this paragraph, we introduce the abbreviations

$$\Delta^{\varepsilon(\sigma)} = \Delta^{\varepsilon_1(\sigma)} \times \Delta^{\varepsilon_2(\sigma)} \times \cdots \times \Delta^{\varepsilon_a(\sigma)} \tag{3.1}$$

and

$$\mathbb{C}(\sigma) = \mathbb{C}(sP_1, tP_1) \times \cdots \times \mathbb{C}(sP_a, tP_a). \tag{3.2}$$

3.2.1. Remark The reader should note that the above discussion is merely a special case of the decomposition $\sigma = (\sigma_a\alpha_a) \circ \cdots \circ (\sigma_1\alpha_1)$ of an n-simplex that we already sketched in Remark 3.1.7. In fact, the decomposition $P_\sigma = P_1 \vee \ldots \vee P_a$ of P_σ into faces and edges corresponds to a decomposition $\sigma = \tau_a \circ \cdots \circ \tau_1$ of σ into atomic n-simplices. Moreover, writing $\tau_i = \sigma_i\alpha_i$ with σ_i nondegenerate recovers the construction of the simplices σ_i given above.

With this notation at hand, we can already show that any simplex in $\mathbb{C}$ is contained in some cube $(\Delta^1)^n$.

3.2.2. Proposition *Let $\sigma \in \mathbb{C}(x, y)$ be an arbitrary n-simplex. There then exists a unique map $\widehat{\sigma} \colon \Delta^n \to \Delta^{\varepsilon(\sigma)}$ such that*

$$\begin{tikzcd}
\Delta^{\varepsilon(\sigma)} \arrow[r, "(\sigma_1, \ldots, \sigma_a)"] & \mathbb{C}(\sigma) \arrow[d, "composition"] \\
\Delta^n \arrow[u, "\widehat{\sigma}"] \arrow[r, "\sigma"'] & \mathbb{C}(x, y)
\end{tikzcd}$$

commutes.

Proof. We keep the notation fixed that we use throughout this section. That means in particular that $\sigma = (p_0 \leq \cdots \leq p_n)$ has $(P_\sigma, \lambda_\sigma)$ as its associated n-marked subgraph and that $P_\sigma = P_1 \vee \ldots \vee P_a$ is the decomposition of P_σ into edges and faces of P. Moreover, $\sigma_i = (P_i)$ if P_i is an edge and $\sigma_i = (\operatorname{dom} P_i \leq \operatorname{cod} P_i)$ if P_i is a face.

Let us consider an arbitrary map $\widehat{\sigma}\colon \Delta^n \to \Delta^{\varepsilon(\sigma)}$. By the universal property of products, the map $\widehat{\sigma}$ is uniquely determined by the compositions

$$\beta_i\colon \Delta^n \to \Delta^{\varepsilon(\sigma)} \to \Delta^{\varepsilon_i(\sigma)},$$

that is, by maps $\beta_i\colon [n] \to [\varepsilon_i(\sigma)]$. The composition

$$(\sigma_1, \ldots, \sigma_a) \circ \widehat{\sigma}$$

thus classifies the n-simplex $(\sigma_1\beta_1, \ldots, \sigma_a\beta_a)$ of $\mathbb{C}(\sigma)$. Given the explicit description of the simplices σ_i at the beginning of this proof, one now easily computes the n-simplices $\tau_i = \sigma_i\beta_i$ as $\tau_i = (q_0^i \leq \cdots \leq q_n^i)$ with

$$q_j^i = \begin{cases} P_i & \text{if } \varepsilon_i(\sigma) = 0, \\ \operatorname{dom} P_i & \text{if } \varepsilon_i(\sigma) = 1 \text{ and } \beta_i(j) = 0, \\ \operatorname{cod} P_i & \text{if } \varepsilon_i(\sigma) = 1 \text{ and } \beta_i(j) = 1. \end{cases}$$

The composition $\tau = \tau_r \circ \ldots \circ \tau_1 = (q_0 \leq \cdots \leq q_n) \in \mathbb{C}(x, y)$ of these simplices τ_i therefore has

$$q_j = q_j^1 \cdot \ldots \cdot q_j^r$$

by the definition of composition in $\mathbb{C}$. Note that $\widehat{\sigma}$ renders the diagram

$$\begin{tikzcd}
\Delta^{\varepsilon(\sigma)} \arrow[r, "(\sigma_1, \ldots, \sigma_r)"] & \mathbb{C}(\sigma) \arrow[d, "\text{composition}"] \\
\Delta^n \arrow[u, "\widehat{\sigma}"] \arrow[r, "\sigma"'] & \mathbb{C}(x, y)
\end{tikzcd} \tag{3.3}$$

commutative if and only if $\tau = \sigma$, that is, if and only if $q_j = p_j$ for all $0 \leq j \leq n$. According to Remark 2.2.11, the path p_j occuring in $\sigma = (p_0 \leq \cdots \leq p_n)$ can be characterised as the unique wide path in P with exactly the faces P_i with $\lambda_\sigma(P_i) \leq j$ to its left. However, given the above description of q_j it is easy to see that a face P_i lies to the left of q_j if and only if $\beta_i(j) = 1$. Altogether, we may thus conclude that the map $\widehat{\sigma}$ induced by the maps

$$\beta_i(j) = \begin{cases} 0 & \text{if } j < \lambda_\sigma(P_i) \\ 1 & \text{if } j \geq \lambda_\sigma(P_i) \end{cases}$$

is the unique map that renders (3.3) commutative. □

An immediate consequence of the uniqueness of the map $\widehat{\sigma}$ asserted in

Proposition 3.2.7 is the following corollary:

3.2.3. Corollary *Let $\sigma \in \mathbb{C}(x, y)$ and $\tau \in \mathbb{C}(y, z)$ be two composable n-simplices. The map $\widehat{\tau \circ \sigma}$ is the map*

$$(\widehat{\sigma}, \widehat{\tau})\colon \Delta^n \to \Delta^{\varepsilon(\sigma)} \times \Delta^{\varepsilon(\tau)} = \Delta^{\varepsilon(\tau\circ\sigma)}.$$

Proof. The composite $\tau \circ \sigma$ in $\mathbb{C}$ corresponds to the join $(P_\sigma \vee P_\tau, \lambda_\sigma \vee \lambda_\tau)$ of n-marked subgraphs, see Remark 3.1.6. This implies $\Delta^{\varepsilon(\tau\circ\sigma)} = \Delta^{\varepsilon(\sigma)} \times \Delta^{\varepsilon(\tau)}$, $\mathbb{C}(\tau \circ \sigma) = \mathbb{C}(\sigma) \times \mathbb{C}(\tau)$ and $((\tau\sigma)_1, \dots, (\tau\sigma)_c) = (\sigma_1, \dots, \sigma_a, \tau_1, \dots, \tau_b)$. The diagram

$$\begin{tikzcd}
\Delta^{\varepsilon(\sigma)} \times \Delta^{\varepsilon(\tau)} \arrow[r, "{(\sigma_1,\dots,\sigma_a,\tau_1,\dots,\tau_b)}"] & \mathbb{C}(\sigma) \times \mathbb{C}(\tau) \arrow[d, "\text{composition} \times \text{composition}"'] \arrow[dr, "\text{composition}"] & \\
\Delta^n \arrow[u, "{(\widehat{\sigma}, \widehat{\tau})}"] \arrow[r, "{(\sigma,\tau)}"'] & \mathbb{C}(x, y) \times \mathbb{C}(y, z) \arrow[r, "\text{composition}"'] & \mathbb{C}(x, z)
\end{tikzcd}$$

commutes by associativity of composition in $\mathbb{C}$ and the definition of $\widehat{\sigma}$ and $\widehat{\tau}$. It now follows from the uniqueness asserted in Proposition 3.2.7 that $\widehat{\tau \circ \sigma} = (\widehat{\tau}, \widehat{\sigma})$. □

Let us now consider a simplicial operator $\alpha\colon [m] \to [n]$ and the n-marked subgraph $(P_{\sigma\alpha}, \lambda_{\sigma\alpha})$ associated with $\sigma\alpha$. The steps given in Proposition 2.2.13 to compute $(P_{\sigma\alpha}, \lambda_{\sigma\alpha})$ from $(P_\sigma, \lambda_\sigma)$ then admit some simplifications because any edge of $P_\sigma, P_{\sigma\alpha} \in \mathcal{S} = \mathcal{S}^c_{\min}$ is incident with the exterior face. More precisely, one obtains the following procedure to determine $(P_{\sigma\alpha}, \lambda_{\sigma\alpha})$ in terms of $(P_\sigma, \lambda_\sigma)$:

1. Remove the edges and interior vertices of all paths $\mathrm{dom}(\phi)$, where ϕ is some interior face of P_σ with $\lambda_\sigma(\phi) \leq \alpha(0)$.

2. Remove the edges and interior vertices of all paths $\mathrm{cod}(\phi)$, where ϕ is some interior face of P_σ with $\lambda_\sigma(\phi) > \alpha(m)$.

3. Define $\lambda_{\sigma\alpha}$ on the remaining graph by

$$\lambda_{\sigma\alpha}(\phi) = \min\big\{ k \in \{1, \ldots, m\} \mid \alpha(k) \geq \lambda_\sigma(\phi) \big\}.$$

As both P_σ and $P_{\sigma\alpha}$ are wide subgraphs of $G_{x,y}$, the following description of the decomposition $P_{\sigma\alpha} = Q_1 \vee \ldots \vee Q_b$ into edges and faces of $P_{\sigma\alpha}$ in terms of $P_\sigma = P_1 \vee \ldots \vee P_a$ is immediate.

3.2.4. Lemma *Let σ be an n-simplex in $\mathbb{C}(x, y)$ and let $P_\sigma = P_1 \vee \ldots \vee P_a$ be the decomposition of its associated n-marked subgraph into edges and faces. Further let $\alpha \colon [m] \to [n]$ be a simplicial operator in $\mathbb{\Delta}$ and let $P_{\sigma\alpha} = Q_1 \vee \ldots \vee Q_b$ be the decomposition of $P_{\sigma\alpha}$ into edges and faces. For any $i \in \{1, \ldots, a\}$ we then have the following relations between these decompositions:*

1. *If P_i is an edge, then $P_i = Q_j$ for some suitable j.*

2. *If P_i is an interior face ϕ with $\alpha(0) < \lambda_\sigma(\phi) \leq \alpha(m)$, then $P_i = Q_j$ for some suitable j. Moreover, $k < \lambda_{\sigma\alpha}(\phi)$ if and only if $\alpha(k) < \lambda_\sigma(\phi)$.*

3. *If* P_i *is an interior face with* $\lambda_\sigma(P_i) \leq \alpha(0)$*, then* $\operatorname{cod} P_i = Q_j \vee \ldots \vee Q_{j+k}$ *for suitable* j *and* k.

4. *If* P_i *is an interior face with* $\lambda_\sigma(P_i) > \alpha(m)$*, then* $\operatorname{dom} P_i = Q_j \vee \ldots \vee Q_{j+k}$ *for suitable* j *and* k.

Let us fix the notation of Lemma 3.2.4 for the remainder of this section. In order to be able to describe the action of simplicial operators on the simplices of $\mathbb{C}(x, y)$ in terms of the maps $\widehat{\sigma}$ and $\widehat{\sigma\alpha}$, we need to relate $\Delta^{\varepsilon(\sigma)}$ and $\Delta^{\varepsilon(\sigma\alpha)}$ in some sensible way. Recall from (3.1) that

$$\Delta^{\varepsilon(\alpha)} = \Delta^{\varepsilon_1(\alpha)} \times \cdots \times \Delta^{\varepsilon_a(\alpha)}.$$

From the description of $(P_{\sigma\alpha}, \lambda_{\sigma\alpha})$ given in Lemma 3.2.4 we fabricate a map $\varepsilon(\alpha)\colon \Delta^{\varepsilon(\sigma\alpha)} \to \Delta^{\varepsilon(\sigma)}$ whose component $\varepsilon(\alpha)_i$ at $\Delta^{\varepsilon_i(\sigma)}$ is the identity $\Delta^{\varepsilon_j(\sigma\alpha)} = \Delta^{\varepsilon_i(\sigma)}$ if $P_i = Q_j$ for some j and whose components $\varepsilon(\alpha)_i$ at $\Delta^{\varepsilon_i(\sigma)}$ with $P_i \not\subseteq Q$ are given by

$$\begin{cases} \Delta^{\varepsilon_j(\sigma\alpha)} \times \cdots \times \Delta^{\varepsilon_{j+k}(\sigma\alpha)} \simeq \Delta^0 \xrightarrow{d_0} \Delta^{\varepsilon_i(\sigma)} & \text{if } Q_j \vee \ldots \vee Q_{j+k} = \operatorname{cod} P_i, \\ \Delta^{\varepsilon_j(\sigma\alpha)} \times \cdots \times \Delta^{\varepsilon_{j+k}(\sigma\alpha)} \simeq \Delta^0 \xrightarrow{d_1} \Delta^{\varepsilon_i(\sigma)} & \text{if } Q_j \vee \ldots \vee Q_{j+k} = \operatorname{dom} P_i. \end{cases}$$

3.2.5. Remark The maps $\varepsilon(\alpha)$ are functorial in α in the sense that we have $\varepsilon(\alpha \circ \beta) = \varepsilon(\alpha) \circ \varepsilon(\beta)$ for all composable simplicial operators α and β.

We record the following technical remark in order to isolate all technical properties of the maps $\varepsilon(\alpha)$ from the actual proof of Theorem A in the forthcoming section. However, this remark might also provide the reader with some intuition for the whys and wherefores of the maps $\varepsilon(\alpha)$.

3.2.6. Remark Suppose that Λ is a labeling of $\Sigma = \Sigma^c_{\text{min}}$ in some simplicial category $\mathbb{A}$. Further suppose that σ is some n-simplex in $\mathbb{C}(x, y)$ and that $\alpha\colon [m] \to [n]$ is a simplicial operator in $\mathbb{\Delta}$. If we let

$$\mathbb{A}(\sigma) = \mathbb{A}(\Lambda sP_1, \Lambda tP_1) \times \cdots \times \mathbb{A}(\Lambda sP_a, \Lambda tP_a),$$

then there are obvious partial composition maps

$$\varpi\colon \mathbb{A}(\sigma\alpha) \to \mathbb{A}(\sigma)$$

given by the identities on those $\mathbb{A}(\Lambda sP_i, \Lambda tP_i)$ with $P_i = Q_j$ for some j and by the compositions

$$\mathbb{A}(\Lambda sQ_j, \Lambda tQ_j) \times \cdots \times \mathbb{A}(\Lambda sQ_{j+k}, \Lambda tQ_{j+k}) \to \mathbb{A}(\Lambda sQ_j, \Lambda tQ_{j+k})$$

whenever P_i is a face and $Q_j \vee \ldots \vee Q_{j+k}$ is either dom P_i or cod P_i.

Given the relation between $P_\sigma = P_1 \vee \ldots \vee P_a$ and $P_{\sigma\alpha} = Q_1 \vee \ldots \vee Q_b$ in Lemma 3.2.4, it is immediate that

$$\varpi\big(\Lambda(\sigma\alpha)_j\big)_i = \begin{cases} \Lambda\sigma_i & \text{if } P_i = Q_j \text{ for some } j, \\ \Lambda Q_{j+k} \circ \cdots \circ \Lambda Q_j & \text{if } \operatorname{dom} P_i = Q_j \vee \ldots \vee Q_{j+k} \\ & \text{or } \operatorname{cod} P_i = Q_j \vee \ldots \vee Q_{j+k}. \end{cases}$$

With our definition $\sigma_i = (\mathrm{dom}\, P_i \leq \mathrm{cod}\, P_i)$ and the definition of a labeling Λ, one then sees that

$$\varpi\big(\Lambda(\sigma\alpha)_j\big)_i = \begin{cases} \Lambda\sigma_i & \text{if } P_i = Q_j \text{ for some } j, \\ d_1(\Lambda\sigma_i) & \text{if } \mathrm{dom}\, P_i = Q_j \vee \ldots \vee Q_{j+k}, \\ d_0(\Lambda\sigma_i) & \text{if } \mathrm{cod}\, P_i = Q_j \vee \ldots \vee Q_{j+k}. \end{cases}$$

Moreover, it follows from our construction of $\varepsilon(\alpha)$ from above that $\sigma_i \circ \varepsilon(\alpha)$ classifies the simplex given by

$$\Lambda\sigma_i \cdot \varepsilon(\alpha)_i = \begin{cases} \Lambda\sigma_i & \text{if } P_i = Q_j \text{ for some } j, \\ d_1(\Lambda\sigma_i) & \text{if } \mathrm{dom}\, P_i = Q_j \vee \ldots \vee Q_{j+k}, \\ d_0(\Lambda\sigma_i) & \text{if } \mathrm{cod}\, P_i = Q_j \vee \ldots \vee Q_{j+k}. \end{cases}$$

Altogether, this proves that the diagram

$$\begin{tikzcd} \Delta^{\varepsilon(\sigma\alpha)} \arrow[r, "{(\Lambda(\sigma\alpha)_1, \ldots, \Lambda(\sigma\alpha)_b)}"] \arrow[d, "\varepsilon(\alpha)"'] & \mathbb{A}(\sigma\alpha) \arrow[d, "\varpi"] \\ \Delta^{\varepsilon(\sigma)} \arrow[r, "{(\Lambda\sigma_1, \ldots, \Lambda\sigma_a)}"'] & \mathbb{A}(\sigma) \end{tikzcd}$$

commutes.

Let us now give a description of the action of simplicial operators on the simplices of $\mathbb{C}(x, y)$ in terms of the maps $\widehat{\sigma}$.

3.2.7. Proposition *The diagram*

$$\begin{tikzcd}
\Delta^m \arrow[r, "\widehat{\sigma\alpha}"] \arrow[d, "\alpha"'] & \Delta^{\varepsilon(\sigma\alpha)} \arrow[d, "\varepsilon(\alpha)"] \\
\Delta^n \arrow[r, "\hat{\sigma}"'] & \Delta^{\varepsilon(\sigma)}
\end{tikzcd}$$

commutes for any simplicial operator $\alpha \colon [m] \to [n]$ *in* $\mathbb{\Delta}$ *and any* n*-simplex* $\sigma \in \mathbb{C}(x, y)$.

Proof. We still keep the notation from Lemma 3.2.4. According to the proof of Proposition 3.2.2, the composition

$$\Delta^m \xrightarrow{\alpha} \Delta^n \xrightarrow{\hat{\sigma}} \Delta^{\varepsilon(\sigma)} \to \Delta^{\varepsilon_i(\sigma)}$$

corresponds to the map $[m] \to [0]$ if $\varepsilon_i(\sigma) = 0$ and to $\beta_i \colon [m] \to [1]$ given by

$$\beta_i(j) = \begin{cases} 0 & \text{if } \alpha(j) < \lambda_\sigma(P_i), \\ 1 & \text{if } \alpha(j) \geq \lambda_\sigma(P_i), \end{cases}$$

if $\varepsilon_i(\sigma) = 1$. In order to show that the diagram in the lemma commutes, it suffices to show that these maps coincide with the maps $\gamma_i \colon [m] \to [\varepsilon_i(\sigma)]$ corresponding to the compositions

$$\Delta^m \xrightarrow{\widehat{\alpha\sigma}} \Delta^{\varepsilon(\sigma\alpha)} \xrightarrow{\varepsilon(\alpha)} \Delta^{\varepsilon(\alpha)} \to \Delta^{\varepsilon_i(\alpha)}. \tag{3.4}$$

To this end, we distinguish the following cases that correspond to the different cases in Lemma 3.2.4:

1. If $\varepsilon_i(\sigma) = 0$, i. e. if P_i is an edge, then $[\varepsilon_i(\alpha)] = [0]$ is terminal and we are done.

2. If $\varepsilon_i(\sigma) = 1$, i. e. if P_i is an interior face and $\alpha(0) < \lambda_\sigma(P_i) \leq \alpha(m)$, then $P_i = Q_k$ for some k. The component $\varepsilon(\alpha)_i$ is then the identity on $\Delta^{\varepsilon_k(\sigma\alpha)} = \Delta^{\varepsilon_i(\alpha)}$. The map $[m] \to [1]$ corresponding to the composition (3.4) is therefore given by

$$\gamma_i(j) = \begin{cases} 0 & \text{if } j < \lambda_{\sigma\alpha}(P_i) \\ 1 & \text{if } j \geq \lambda_{\sigma\alpha}(P_i). \end{cases}$$

   We conclude by the equivalence

$$j < \lambda_{\sigma\alpha(P_i)} \quad \text{if and only if} \quad \alpha(j) < \lambda_\sigma(P_i)$$

   stated in Lemma 3.2.4.

3. If $\varepsilon_i(\sigma) = 1$ and $\lambda_\sigma(P_i) \leq \alpha(0)$, then $\operatorname{cod} P_i = Q_k \vee \ldots \vee Q_{k+l}$ and $\varepsilon(\alpha)_i$ is the map

$$\Delta^{\varepsilon_k(\sigma\alpha)} \times \cdots \times \Delta^{\varepsilon_{k+l}(\sigma\alpha)} \xrightarrow{d_0} \Delta^{\varepsilon_i(\sigma)}$$

   The map $[m] \to [1]$ corresponding to the composition (3.4) is hence the constant map 1. This coincides with β_i since $\lambda_\sigma(P_i) \leq \alpha(0)$, i. e.

$\alpha(j) \geq \lambda_\sigma(P_i)$ for all $j \in [m]$.

4. The remaining case that $\varepsilon_i(\sigma) = 1$ and $\lambda_\sigma(P_i) > \alpha(m)$ can be handled as the preceding case.

This finishes the proof of Proposition 3.2.7.

□

3.3 Labelings of Globular Graphs

This section solely consists of a proof of

Theorem A *Suppose that Σ is the minimal complete pasting diagram on some globular graph G. The map*

$$\mathrm{Cat}_{\widehat{\mathbb{\Delta}}}(\mathbb{C}[\Sigma], \mathbb{A}) \to L(G, \mathbb{A}), \quad u \mapsto \Lambda_u,$$

that sends a simplicial functor u to its associated labeling Λ_u is a bijection.

Proof. Let us begin by introducing some notation and recalling some facts from § 3.2. Given a simplex σ in $\mathbb{C}[\Sigma](x, y)$ with associated n-marked subgraph $(P_\sigma, \lambda_\sigma)$, we decompose

$$P_\sigma = P_1 \vee \ldots \vee P_a$$

with each P_i an edge or an interior face of P_σ, see § 3.2. We then have simplices $\sigma_i = (P_i)$ if i is an edge and $\sigma_i = (\operatorname{dom} P_i \leq P_i)$ if P_i is an interior face. Similar to the abbreviation

$$\mathbb{C}[\Sigma](\sigma) = \mathbb{C}[\Sigma](sP_1, tP_1) \times \cdots \times \mathbb{C}[\Sigma](sP_a, tP_a)$$

introduced in (3.2) above, we write

$$\mathbb{A}(\sigma) = \mathbb{A}(\Lambda sP_1, \Lambda tP_1) \times \cdots \times \mathbb{A}(\Lambda sP_a, \Lambda tP_a).$$

Let us now show that a simplicial functor $u\colon \mathbb{C}[\Sigma] \to \mathbb{A}$ is uniquely determined by its associated labeling Λ_u. It is obvious that the labeling Λ_u completely determines the action of u on objects. Moreover, Λ_u also determines the action of u an all the simplices $\sigma \in \mathbb{C}[\Sigma](x, y)$ of the form $\sigma = (e)$ for an edge e or $\sigma = (\operatorname{dom}\phi \leq \operatorname{cod}\phi)$ for an interior face ϕ. Now consider some arbitrary n-simplex $\sigma \in \mathbb{C}[\Sigma](x, y)$. We then have a commutative diagram

$$\begin{tikzcd}
\Delta^{\varepsilon(\sigma)} \arrow[r, "(\sigma_1, \dots, \sigma_r)"] & \mathbb{C}[\Sigma](\sigma) \arrow[r, "(u, \dots, u)"] \arrow[d, "\text{composition}"'] & \mathbb{A}(\sigma) \arrow[d, "\text{composition}"] \\
\Delta^n \arrow[u, "\widehat{\sigma}"] \arrow[r, "\sigma"'] & \mathbb{C}[\Sigma](x, y) \arrow[r, "u"'] & \mathbb{A}(ux, uy)
\end{tikzcd}$$

by Proposition 3.2.2 and functoriality of u. Observe that the top row in this diagram is nothing but

$$(u, \dots, u) \circ (\sigma_1, \dots, \sigma_r) = (u\sigma_1, \dots, u\sigma_r) = \big(\Lambda_u P_1, \dots, \Lambda_u P_r\big).$$

The image $u(\sigma)$ of σ is hence the n-simplex in $\mathbb{A}(ux, uy)$ classified by the composition

$$\Delta^n \xrightarrow{\hat{\sigma}} \Delta^{\varepsilon(\sigma)} \xrightarrow{(\Lambda_u P_1, \dots, \Lambda_u P_r)} \mathbb{A}(\sigma) \xrightarrow{\text{composition}} \mathbb{A}(ux, uy)$$

and this composition is uniquely determined by Λ_u. Altogether, this proves that $u \mapsto \Lambda_u$ is injective.

In order to prove surjectivity of the assignment $u \mapsto \Lambda_u$, we have to construct for each labelling Λ of G in $\mathbb{A}$ a simplicial functor $u\colon \mathbb{C}[\Sigma] \to \mathbb{A}$ such that $\Lambda = \Lambda_u$. The argument so far already hints at the correct definition of u. In fact, we have no choice but to define $u(x) = \Lambda x$ on objects and to define the image $u(\sigma)$ of some n-simplex $\sigma \in \mathbb{C}[\Sigma](x, y)$ as the simplex classified by the map

$$\Delta^n \xrightarrow{\hat{\sigma}} \Delta^{\varepsilon(\sigma)} \xrightarrow{(\Lambda P_1, \dots, \Lambda P_r)} \mathbb{A}(\sigma) \xrightarrow{\text{composition}} \mathbb{A}(ux, uy).$$

Observe that if these assignments indeed define a functor u, then $\Lambda = \Lambda_u$. Let us verify that these assignments define a simplicial functor. For two composable n-simplices $\sigma \in \mathbb{C}[\Sigma](x, y)$ and $\tau \in \mathbb{C}[\Sigma](y, z)$ we have a commutative diagram

$$\begin{tikzcd}
\Delta^{\varepsilon(\tau\circ\sigma)} \arrow[r, "(\Lambda(\tau\circ\sigma)_1,\dots,\Lambda(\tau\circ\sigma)_{a+b})"] \arrow[d, equal] & \mathbb{A}(\tau\circ\sigma) \arrow[d, equal] \\
\Delta^{\varepsilon(\sigma)} \times \Delta^{\varepsilon(\tau)} \arrow[r, "(\Lambda\sigma_1,\dots,\Lambda\sigma_a,\Lambda\tau_1,\dots,\Lambda\tau_b)"] & \mathbb{A}(\sigma) \times \mathbb{A}(\tau) \arrow[d, "\text{composition} \times \text{composition}"] \\
& \mathbb{A}(x,y) \times \mathbb{A}(y,z) \arrow[d, "\text{composition}"] \\
\Delta^n \arrow[uu, "(\hat{\sigma},\hat{\tau}) = \widehat{\tau\circ\sigma}"] \arrow[ur, "(u(\sigma),u(\tau))"] \arrow[r, "u(\tau\circ\sigma)"'] & \mathbb{A}(ux,uz)
\end{tikzcd}$$

by Corollary 3.2.3. We thus conclude $u(\tau\circ\sigma) = u(\tau)\circ u(\sigma)$ by associativity of composition in $\mathbb{A}$. Moreover, for any simplicial operator α we have a diagram

$$\begin{tikzcd}
\Delta^m \arrow[r, "\widehat{\sigma\alpha}"] \arrow[d, "\alpha"] & \Delta^{\varepsilon(\sigma\alpha)} \arrow[r, "(\Lambda(\sigma\alpha)_1,\dots,\Lambda(\sigma\alpha)_b)"] \arrow[d, "\varepsilon(\alpha)"] & \mathbb{A}(\sigma\alpha) \arrow[d, "\text{composition}"'] \arrow[dr, "\text{composition}"] & \\
\Delta^n \arrow[r, "\hat{\sigma}"'] & \Delta^{\varepsilon(\sigma)} \arrow[r, "(\Lambda\sigma_1,\dots,\Lambda\sigma_a)"'] & \mathbb{A}(\sigma) \arrow[r, "\text{composition}"'] & \mathbb{A}(x,y)
\end{tikzcd}$$

in which the triangle commutes by associativity of composition in $\mathbb{A}$ and in which the two squares on the left hand side commute by Remark 3.2.6 and Proposition 3.2.7. □

4 Global Lifting Properties of Pasting Diagrams

Throughout this chapter we fix a class $\mathcal{R}$ of maps in $\widehat{\mathbb{\Delta}}$ that contains all isomorphisms and let $\mathcal{L}$ denote the class of maps that have the left lifting property against all maps in $\mathcal{R}$. We call the elements of $\mathcal{L}$ and $\mathcal{R}$ $\mathcal{L}$-maps and $\mathcal{R}$-maps, respectively. A simplicial functor $u\colon \mathbb{A} \to \mathbb{B}$ is a local $\mathcal{R}$-functor if all the maps $u\colon \mathbb{A}(a, a') \to \mathbb{B}(ua, ua')$, $a \in \mathbb{A}$, are $\mathcal{R}$-maps. The main result of this chapter is the following characterisation of those inclusions between pasting diagrams that induce simplicial functors that have the left lifting property against all local $\mathcal{R}$-functors:

Theorem B *The functor $\mathbb{C}[\Sigma] \to \mathbb{C}[\Pi]$ induced by an inclusion of complete pasting diagrams has the left lifting property against all local $\mathcal{R}$-functors if and only if the map*

$$\mathrm{N}(\Sigma_{x,y} \multimap \Pi_{x,y}) \to \mathrm{N}(\Pi_{x,y})$$

is an $\mathcal{L}$-map for all vertices $x, y \in \Sigma$.

The proof of Theorem B relies on three independent steps. As a first step, in § 4.1, we prove that the hypothesis on the inclusion $\Sigma \to \Pi$ in Theorem B is sufficient:

4.1.1. Proposition *Let* $\Sigma \to \Pi$ *be an inclusion of complete pasting diagrams such that* $\mathrm{N}(\Sigma_{x,y} \multimap \Pi_{x,y}) \to \mathrm{N}(\Pi_{x,y})$ *is an* $\mathcal{L}$*-map for all vertices* x, y *of* Σ*. The functor* $\mathbb{C}[\Sigma] \to \mathbb{C}[\Pi]$ *then has the left lifting property against all local* $\mathcal{R}$*-functors.*

The other two steps are necessary to show that the hypothesis on $\Sigma \to \Pi$ in Theorem B is necessary. To this end, we verify in § 4.2 that the hypothesis is necessary at least for the map $\mathrm{N}(\Sigma \multimap \Pi) \to \mathrm{N}(\Pi)$:

4.2.8. Proposition *Let* $\Sigma \to \Pi$ *be an inclusion of complete pasting diagrams. If* $\mathbb{C}[\Sigma] \to \mathbb{C}[\Pi]$ *has the left lifting property against all local* $\mathcal{R}$*-functors, then* $\mathrm{N}(\Sigma \multimap \Pi) \to \mathrm{N}(\Pi)$ *is an* $\mathcal{L}$*-map.*

The final step in the proof of Theorem B is then to descend from the maps $\mathrm{N}(\Sigma \multimap \Pi) \to \mathrm{N}(\Pi)$ handled by Proposition 4.2.8 to all the maps $\mathrm{N}(\Sigma_{x,y} \multimap \Pi_{x,y}) \to \mathrm{N}(\Pi_{x,y})$ with $x, y \in G$. This is achieved by the following proposition whose proof we give in § 4.3.

4.3.5. Proposition *If* $\Sigma \to \Pi$ *is an inclusion of complete pasting diagrams such that* $\mathbb{C}[\Sigma] \to \mathbb{C}[\Pi]$ *has the left lifting property against all local* $\mathcal{R}$*-functors, then so do the functors* $\mathbb{C}[\Sigma_{x,y}] \to \mathbb{C}[\Pi_{x,y}]$ *for all vertices* x, y *of* Σ*.*

In the final section § 4.4 we then give the proof of Theorem B that we sketched in this introduction.

4.1 Sufficiency of the hypothesis

We keep the classes $\mathscr{L}$ and $\mathscr{R}$ of maps in $\widehat{\mathbb{\Delta}}$ with $\mathscr{L} = {}^{\pitchfork}\mathscr{R}$ fixed. This section is devoted to a proof of the following proposition that we already stated above.

4.1.1. Proposition *Let* $\Sigma \to \Pi$ *be an inclusion of complete pasting diagrams such that* $\mathrm{N}(\Sigma_{x,y} \multimap \Pi_{x,y}) \to \mathrm{N}(\Pi_{x,y})$ *is an* $\mathscr{L}$*-map for all vertices* x, y *of* Σ*. The functor* $\mathbb{C}[\Sigma] \to \mathbb{C}[\Pi]$ *then has the left lifting property against all local* $\mathscr{R}$*-functors.*

In order to prove Proposition 4.1.1, we thus have to show that for each square

$$\begin{tikzcd} \mathbb{C}[\Sigma] \arrow[r, "u"] \arrow[d] & \mathbb{B} \arrow[d, "p"] \\ \mathbb{C}[\Pi] \arrow[r, "v"'] & \mathbb{A} \end{tikzcd} \tag{4.1}$$

in which p is a local $\mathscr{R}$-functor, there exists a simplicial functor $\ell \colon \mathbb{C}[\Pi] \to \mathbb{B}$ such that

$$\begin{tikzcd} \mathbb{C}[\Sigma] \arrow[r, "u"] \arrow[d] & \mathbb{B} \arrow[d, "p"] \\ \mathbb{C}[\Pi] \arrow[r, "v"'] \arrow[ur, "\ell"] & \mathbb{A} \end{tikzcd} \tag{4.2}$$

commutes. Throughout this section, we keep an inclusion $\Sigma \to \Pi$ of pasting diagrams satisfying the hypothesis of Proposition 4.1.1 and a square such as (4.1) fixed and assume p to be a local $\mathcal{R}$-functor. We let G denote the graph underlying both Σ and Π. Observe that $\mathbb{C}[\Sigma] \to \mathbb{C}[\Pi]$ is the identity on objects, so that $pu(x) = v(x)$ for any vertex $x \in G$. We thus define $\ell(x) = u(x)$ on objects and this definition lets (4.2) commute on the level of objects. It remains to construct for each two vertices $x, z \in G$ suitably functorial lifts $\ell_{x,z}$ as in the diagram

$$\begin{tikzcd} \mathrm{N}(\Sigma_{x,z}) \arrow[d, "\text{inclusion}"'] & = & \mathbb{C}[\Sigma](x,z) \arrow[r, "u"] \arrow[d] & \mathbb{B}(ux, uz) \arrow[d, "p"] \\ \mathrm{N}(\Pi_{x,z}) & = & \mathbb{C}[\Pi](x,z) \arrow[r, "v"'] \arrow[ur, "\ell_{x,z}"] & \mathbb{A}(vx, vz). \end{tikzcd} \tag{4.3}$$

Here, functoriality means nothing but that for all vertices x of G the map $\ell_{x,x} \colon \mathrm{N}(\Pi_{x,x}) = \Delta^0 \to \mathbb{B}(ux, ux)$ classifies the identity $\mathrm{id}_{ux} \in \mathbb{B}(ux, ux)_0$, and that the square

$$\begin{tikzcd} \mathrm{N}(\Pi_{x,y}) \times \mathrm{N}(\Pi_{y,z}) \arrow[r, "\ell_{x,y} \times \ell_{y,z}"] \arrow[d, "\text{composition}"'] & \mathbb{B}(ux, uy) \times \mathbb{B}(uy, uz) \arrow[d, "\text{composition}"] \\ \mathrm{N}(\Pi_{x,z}) \arrow[r, "\ell_{x,z}"'] & \mathbb{B}(ux, uz) \end{tikzcd} \tag{4.4}$$

commutes for all vertices $x, y, z \in G$. The constraint on $\ell_{x,x}$ is easily satisfied by simply defining $\ell_{x,x}$ to be the map classifying the identity in $\mathbb{B}(ux, ux)_0$.

It is then clear that (4.4) commutes for all $x, y, z \in G$ with $x = y$ or $y = z$. Moreover, the following lemma identifies a further case, where (4.4) commutes for trivial reasons.

4.1.2. Lemma *The square* (4.4) *commutes whenever* $y \notin G_{x,z}$.

Proof. Suppose $y \notin G_{x,z}$. There is then no directed path from x to y or no directed path from y to z in G. This implies in particular that $\mathrm{N}(\Pi_{x,y}) = \varnothing$ or $\mathrm{N}(\Pi_{y,z}) = \varnothing$ and thus $\mathrm{N}(\Pi_{x,y}) \times \mathrm{N}(\Pi_{y,z}) = \varnothing$. Hence, (4.4) commutes simply because its upper left corner is an initial object. □

Let us sum up our discussion so far: In order to solve the lifting problem (4.1) it suffices to devise maps $\ell_{x,z} \colon \mathrm{N}(\Pi_{x,z}) \to \mathbb{B}(ux, uz)$ as in (4.3) for all pairs x, z of distinct vertices of G such that diagram (4.4) commutes for all vertices $x, z \in G$ and $y \in G_{x,z} \smallsetminus \{x, z\}$. We construct these maps $\ell_{x,z}$ by recursion over the partial order "$\preceq$" on the set of pairs (x, z) of vertices of G given by

$$(x, z) \preceq (x', z') \quad \text{if and only if} \quad G_{x,z} \subseteq G_{x',z'}.$$

To this end, let us fix two vertices $x, z \in G$ and assume that all $\ell_{a,b}$ with $(a, b) \prec (x, z)$ have already been constructed. Further assume that these $\ell_{a,b}$ are functorial in the sense that (4.4) with a, b, c instead of x, y, z commutes for all $a, c \in G$ with $(a, c) \prec (x, z)$ and $b \in G_{a,c}$. Given this data, we have to furnish a map $\ell_{x,z}\colon\ \mathrm{N}(\Pi_{x,z}) \to \mathbb{B}(ux, uz)$ such that (4.3) commutes and such that (4.4) commutes for all $y \in G_{x,z}$.

Observe that for any vertex $y \in G_{x,z} \smallsetminus \{x, z\}$ we have $(x, y), (y, z) \prec (x, z)$. This means in particular that we have for any such vertex y the maps $\ell_{x,y}$ and $\ell_{y,z}$ at our disposal. We thus define for any such vertex y the map

$$h_y\colon\ \mathrm{N}(\Pi_{x,y} \vee \Pi_{y,z}) \to \mathbb{B}(ux, uz)$$

as the composition

$$\begin{tikzcd}
\mathrm{N}(\Pi_{x,y} \vee \Pi_{y,z}) \simeq \mathrm{N}(\Pi_{x,y}) \times \mathrm{N}(\Pi_{y,z}) \arrow[r, "\ell_{x,y} \times \ell_{y,z}"] \arrow[dr, "h_y"'] & \mathbb{B}(ux, uy) \times \mathbb{B}(uy, uz) \arrow[d, "\text{composition}"] \\
& \mathbb{B}(ux, uz).
\end{tikzcd}$$

We know from Lemma 2.5.23 that

$$\mathrm{N}(\Sigma_{x,y} \multimap \Pi_{x,y}) = \mathrm{N}(\Sigma_{x,z}) \ \cup \bigcup_{y \in G_{x,z} \smallsetminus \{x,z\}} \mathrm{N}(\Pi_{x,y} \vee \Pi_{y,z}),$$

where the unions are taken in $\mathrm{N}(\Pi_{x,z})$. Observe that the domain of h_y is a simplicial subset of $\mathrm{N}(\Sigma_{x,z} \multimap \Pi_{x,z})$. The following two lemmata therefore guarantee that these maps h_y together with $u\colon \mathbb{C}[\Sigma](x, z) \to \mathbb{B}(ux, uz)$ induce a map $h\colon\ \mathrm{N}(\Sigma_{x,z} \multimap \Pi_{x,z}) \to \mathbb{B}(ux, uz)$ and this permits us to eventually utilise the hypothesis on the inclusion $\Sigma \to \Pi$ in the statement of

Proposition 4.1.1.

4.1.3. Lemma *Let $y \in G_{x,z} \smallsetminus \{x,z\}$. The map h_y and*

$$u\colon\ \mathrm{N}(\Sigma_{x,z}) = \mathbb{C}[\Sigma](x,z) \to \mathbb{B}(ux,uz)$$

coincide on the intersection of their domains.

4.1.4. Lemma *Let $y, y' \in G_{x,z} \smallsetminus \{x,z\}$. The maps h_y and $h_{y'}$ coincide on the intersection of their domains.*

We defer the proof of these lemmata until the end of this section so as not to distract from the main argument. In order to understand the composition $p \circ h$, where $h\colon\ \mathrm{N}(\Sigma_{x,z} \multimap \Pi_{x,z}) \to \mathbb{B}(ux,uz)$ is the map whose existence is guaranteed by the above two lemmata, it suffices to understand the composition $p \circ u$ and all the compositions $p \circ h_y$, where $y \in G_{x,z} \smallsetminus \{x,z\}$. The former composition is known from (4.1) while the latter composition can easily be computed as in the following remark:

4.1.5. Remark The diagram

$$\begin{tikzcd}
\mathrm{N}(\Pi_{x,y}) \times \mathrm{N}(\Pi_{y,z}) \arrow[r, "\ell_{x,y} \times \ell_{y,z}"] \arrow[dr, "v \times v"'] & \mathbb{B}(ux,uy) \times \mathbb{B}(uy,uz) \arrow[r, "\text{comp.}"] \arrow[d, "p \times p"'] & \mathbb{B}(ux,uz) \arrow[d, "p"] \\
& \mathbb{A}(ux,uy) \times \mathbb{A}(uy,uz) \arrow[r, "\text{comp.}"'] & \mathbb{A}(ux,uz)
\end{tikzcd}$$

commutes by functoriality of p and (4.3) for $\ell_{x,y}$ and $\ell_{y,z}$. The composition

$p \circ h_y$ therefore appears in the commutative diagram

$$\begin{tikzcd}
\mathrm{N}(\Pi_{x,y}) \times \mathrm{N}(\Pi_{y,z}) \arrow[r, phantom, "\simeq"] \arrow[d, "v \times v"'] \arrow[rr, bend left, "\text{composition in } \mathbb{C}[\Pi]"] & \mathrm{N}(\Pi_{x,y} \vee \Pi_{y,z}) \arrow[r, "\text{inclusion}"] \arrow[dr, "p \circ h_y"'] & \mathrm{N}(\Pi_{x,z}) \arrow[d, "v"] \\
\mathbb{A}(ux, uy) \times \mathbb{A}(uy, uz) \arrow[rr, "\text{composition}"'] & & \mathbb{A}(ux, uz)
\end{tikzcd}$$

and we conclude that $p \circ h_y$ is nothing but the composition of v and the canonical inclusion $\mathrm{N}(\Pi_{x,y} \vee \Pi_{y,z}) \to \mathrm{N}(\Pi_{x,z})$.

Altogether, we see that the diagram

$$\begin{tikzcd}
\mathrm{N}(\Sigma_{x,z}) \arrow[r] \arrow[dr] \arrow[rr, bend left, "u"] & \mathrm{N}(\Sigma_{x,z} \multimap \Pi_{x,z}) \arrow[r, "h"] \arrow[d] & \mathbb{B}(ux, uz) \arrow[d, "p"] \\
& \mathrm{N}(\Pi_{x,z}) \arrow[r, "v"'] & \mathbb{A}(vx, vy)
\end{tikzcd} \tag{4.5}$$

commutes. Moreover, p is an $\mathcal{R}$-map by assumption and the left vertical map $\mathrm{N}(\Sigma_{x,z} \multimap \Pi_{x,z}) \to \mathrm{N}(\Pi_{x,z})$ is an $\mathcal{L}$-map by our hypothesis on the inclusion $\Sigma \to \Pi$ in the statement of Proposition 4.1.1. We therefore find a lift $\ell_{x,z} \colon \mathrm{N}(\Pi_{x,z}) \to \mathbb{B}(ux, uz)$ in (4.5) and this lift renders (4.3) commutative, too. It only remains to check that the lift $\ell_{x,z}$ in (4.5) is functorial in the sense that all the diagrams (4.4) with $y \in G_{x,z} \smallsetminus \{x,z\}$ commute. To this end, let us consider the diagram

$$\begin{tikzcd}
\mathrm{N}(\Pi_{x,y}) \times \mathrm{N}(\Pi_{y,z}) \simeq \mathrm{N}(\Pi_{x,y} \vee \Pi_{y,z}) \arrow[r] \arrow[d, "\ell_{x,y} \times \ell_{y,z}"'] & \mathrm{N}(\Sigma_{x,z} \multimap \Pi_{x,z}) \arrow[r] \arrow[dr, "h"'] & \mathrm{N}(\Pi_{x,z}) \arrow[d, "\ell_{x,z}"] \\
\mathbb{B}(ux,uy) \times \mathbb{B}(uy,uz) \arrow[rr, "\text{composition}"'] & & \mathbb{B}(ux,uz).
\end{tikzcd}$$

The triangle commutes by our construction of $\ell_{x,z}$ as a lift in (4.5) and the quadriliteral commutes because both compositions from $\mathrm{N}(\Pi_{x,y} \vee \Pi_{y,z})$ to $\mathbb{B}(ux,uz)$ are nothing but the map h_y. Functoriality of $\ell_{x,z}$ therefore follows from the simple observation that the top row in this diagram is composition in $\mathbb{C}[\Pi]$. This finishes the proof of Proposition 4.1.1 except for the fact that we owe the reader proofs for Lemma 4.1.3 and 4.1.4.

4.1.3. Lemma *Let $y \in G_{x,z} \smallsetminus \{x,z\}$. The map h_y and*

$$u\colon\ \mathrm{N}(\Sigma_{x,z}) = \mathbb{C}[\Sigma](x,z) \to \mathbb{B}(ux,uz)$$

coincide on the intersection of their domains.

Proof of Lemma 4.1.3. The domain of h_y is $\mathrm{N}(\Pi_{x,y} \vee \Pi_{y,z})$ and the domain of u is $\mathrm{N}(\Sigma_{x,z})$. Their intersection in $\mathrm{N}(\Pi_{x,z})$ is nothing but $\mathrm{N}(\Sigma_{x,y} \vee \Sigma_{y,z})$ by Corollary 2.5.13. The restrictions of u and h_y to this intersection feature as the bottom-left and top-right compositions in the diagram

$$\begin{tikzcd}
\mathrm{N}(\Sigma_{x,y} \vee \Sigma_{y,z}) \arrow[r] \arrow[d, "\wr"'] & \mathrm{N}(\Pi_{x,y} \vee \Pi_{y,z}) \simeq \mathrm{N}(\Pi_{x,y}) \times \mathrm{N}(\Pi_{y,z}) \arrow[d, "\ell_{x,y} \times \ell_{y,z}"] \\
\mathrm{N}(\Sigma_{x,y}) \times \mathrm{N}(\Sigma_{y,z}) \arrow[r, "u \times u"'] \arrow[d, "\text{composition}"'] & \mathbb{B}(ux, uy) \times \mathbb{B}(uy, uz) \arrow[d, "\text{composition}"] \\
\mathrm{N}(\Sigma_{x,z}) \arrow[r, "u"'] & \mathbb{B}(ux, uz).
\end{tikzcd}$$

This is obvious for h_y and follows for u from the definition of composition in $\mathbb{C}[\Sigma]$. The bottom square in this diagram commutes by functoriality of u and top square commutes by naturality of the isomorphisms $\mathrm{N}(\Sigma_{x,y} \vee \Sigma_{y,z}) \simeq \mathrm{N}(\Sigma_{x,y}) \times \mathrm{N}(\Sigma_{y,z})$ together with the standing assumption that $\ell_{x,y}$ and $\ell_{y,z}$ render the diagram (4.3) commutative. In fact, the upper left triangle in (4.3) suffices for the diagram at hand. □

4.1.4. Lemma *Let $y, y' \in G_{x,z} \smallsetminus \{x, z\}$. The maps h_y and $h_{y'}$ coincide on the intersection of their domains.*

Proof of Lemma 4.1.4. Corollary 2.5.16 tells us that $\mathrm{dom}(h_y)$ and $\mathrm{dom}(h_{y'})$ intersect in $\mathrm{N}(\Pi_{x,z})$ if and only if G contains a directed path between y and y'. As the situation is symmetric, we may suppose that G contains a directed path from y to y'. According to Corollary 2.5.16, the intersection of $\mathrm{dom}(h_y)$ and $\mathrm{dom}(h_{y'})$ is then given by $\mathrm{N}(\Pi_{x,y} \vee \Pi_{y,y'} \vee \Pi_{y',z})$. By naturality of the isomorphisms $\mathrm{N}(\Pi_{x,y} \vee \Pi_{y,z}) \simeq \mathrm{N}(\Pi_{x,y}) \times \mathrm{N}(\Pi_{y,z})$ the statement of the lemma thus reduces to the commutativity of the diagram

$$\begin{tikzcd}
 & & \mathrm{N}(\Pi_{x,y}) \times \mathrm{N}(\Pi_{y,y'}) \times \mathrm{N}(\Pi_{y',z}) \arrow[dll, "\text{composition } \times \mathrm{N}(\Pi_{y',z})"'] \arrow[dd, "\ell_{x,y} \times \ell_{y,y'} \times \ell_{y',z}"] \arrow[drr, "\mathrm{N}(\Pi_{x,y}) \times \text{ composition}"] & & \\
\mathrm{N}(\Pi_{x,y'}) \times \mathrm{N}(\Pi_{y',z}) \arrow[dd, "\ell_{x,y'} \times \ell_{y',z}"'] & & & & \mathrm{N}(\Pi_{x,y}) \times \mathrm{N}(\Pi_{y,z}) \arrow[dd, "\ell_{x,y} \times \ell_{y,z}"] \\
 & & \mathbb{B}(ux, uy) \times \mathbb{B}(uy, uy') \times \mathbb{B}(uy', uz) \arrow[dll, "\text{composition } \times \mathbb{B}(uy', uz)"'] \arrow[drr, "\mathbb{B}(ux, uy) \times \text{ composition}"] & & \\
\mathbb{B}(ux, uy') \times \mathbb{B}(uy', uz) \arrow[drr, "\text{composition}"'] & & & & \mathbb{B}(ux, uy) \times \mathbb{B}(uy, uz) \arrow[dll, "\text{composition}"] \\
 & & \mathbb{B}(ux, uz), & &
\end{tikzcd}$$

as the compositions on the left and right hand side of this diagram are – up to coherent isomorphism – nothing but the restrictions of $h_{y'}$ and h_y to the intersection of their domains. However, the upper two squares in this diagram commute by functoriality of our lifts $\ell_{\bullet,\bullet}$ and the lower square commutes by associativity of composition in $\mathbb{B}$. This concludes the proof. $\square$

4.2 Necessity of the hypothesis, part 1

In this section, we give a proof of the following proposition:

4.2.8. Proposition *Let* $\Sigma \to \Pi$ *be an inclusion of complete pasting diagrams. If* $\mathbb{C}[\Sigma] \to \mathbb{C}[\Pi]$ *has the left lifting property against all local* $\mathcal{R}$*-functors, then* $\mathrm{N}(\Sigma \multimap \Pi) \to \mathrm{N}(\Pi)$ *is an* $\mathcal{L}$*-map.*

The proof of Proposition 4.2.8 works by exhibiting for each commutative square

$$\begin{tikzcd} \mathrm{N}(\Sigma \multimap \Pi) \arrow[r, "u"] \arrow[d] & X \arrow[d, "p"] \\ \mathrm{N}(\Pi) \arrow[r, "v"'] & Y, \end{tikzcd} \tag{4.6}$$

in which p is an $\mathcal{R}$-map, another commutative square

$$\begin{tikzcd} \mathbb{C}[\Sigma] \arrow[r] \arrow[d] & \mathbb{C}[\Sigma \multimap \Pi]_{/u} \arrow[d, "\mathbb{C}_{/p}"] \\ \mathbb{C}[\Pi] \arrow[r, "v/pu"'] & \mathbb{C}[\Sigma \multimap \Pi]_{/pu}, \end{tikzcd} \tag{4.7}$$

in which $\mathbb{C}_{/p}$ is a local $\mathcal{R}$-functor, that has the property that any solution to the lifting problem (4.7) gives a solution to the original lifting problem (4.6).

Even though the definitions and remarks of this section are only relevant to the proof of Proposition 4.2.8, we still prefer to give them in their general form.

4.2.1. Definition *Let Σ be a complete pasting diagram with source s and target t. Further let $u\colon \mathrm{N}(\Sigma) \to X$ be a map of simplicial sets. We define a simplicial category $\mathbb{C}[\Sigma]_{/u}$ as follows: The objects of $\mathbb{C}[\Sigma]_{/u}$ are the vertices of Σ and the mapping spaces of $\mathbb{C}[\Sigma]_{/u}$ are given by*

$$\mathbb{C}[\Sigma]_u(x,y) = \begin{cases} \mathbb{C}[\Sigma](x,y) & \textit{if } (x,y) \neq (s,t), \\ X & \textit{if } (x,y) = (s,t). \end{cases}$$

The composition laws in $\mathbb{C}[\Sigma]_{/u}$ are those of $\mathbb{C}[\Sigma]$ except for the compositions

$$\mathbb{C}[\Sigma]_{/u}(s,x) \times \mathbb{C}[\Sigma]_{/u}(x,t) \to \mathbb{C}[\Sigma]_{/u}(s,t),$$

which are given by

$$\begin{array}{ccccc} \mathbb{C}[\Sigma]_{/u}(s,x) \times \mathbb{C}[\Sigma]_{/u}(x,t) & \longrightarrow & & \longrightarrow & \mathbb{C}[\Sigma]_{/u}(s,t) \\ \| & & & & \| \\ \mathrm{N}(\Sigma_{s,x}) \times \mathrm{N}(\Sigma_{x,t}) & \xrightarrow[\textit{composition in } \mathbb{C}[\Sigma]]{} & \mathrm{N}(\Sigma_{s,t}) & \xrightarrow[u]{} & X. \end{array}$$

if $x \notin \{s,t\}$ and by the isomorphisms $\Delta^0 \times X \simeq X$ and $X \times \Delta^0 \simeq X$ if $x = s$ or $x = t$, respectively.

4.2.2. Remark We should justify that $\mathbb{C}[\Sigma]_{/u}$ is well-defined, i. e. that the composition laws for $\mathbb{C}[\Sigma]_{/u}$ are associative and unital. This, however, is immediate from the definition and left to the reader.

4.2.3. Remark Observe that we have a canonical functor $u\colon \mathbb{C}[\Sigma] \to \mathbb{C}[\Sigma]_{/u}$ that is the identity on objects and on all mapping spaces $\mathbb{C}[\Sigma](x,y)$ with $(x,y) \neq (s,t)$, and acts on $\mathbb{C}[\Sigma](s,t) = \mathrm{N}(\Sigma)$ by u.

4.2.4. Remark If $p\colon X \to Y$ is another map of simplicial sets, then there is a canonical functor $\mathbb{C}_{/p}\colon \mathbb{C}[\Sigma]_{/u} \to \mathbb{C}[\Sigma]_{/pu}$ that is the identity on objects and on all mapping spaces $\mathbb{C}[\Sigma]_{/u}(x,y)$ with $(x,y) \neq (s,t)$, and acts on $\mathbb{C}[\Sigma]_{/u}(s,t) = X$ by p. The reader should note that $\mathbb{C}_{/p}$ is a local $\mathcal{R}$-functor whenever $p\colon X \to Y$ is an $\mathcal{R}$-map.

4.2.5. Definition *Let* $v\colon\ \mathrm{N}(\Pi) \to Y$ *and* $w\colon\ \mathrm{N}(\Sigma \multimap \Pi) \to Y$ *be two maps of simplicial sets. We define a functor* $v/w\colon \mathbb{C}[\Pi] \to \mathbb{C}[\Sigma \multimap \Pi]_{/w}$ *as follows: The functor* v/w *is the identity on objects. The action of* v/w *on mapping spaces* $\mathbb{C}[\Pi](x,y)$ *with* $(x,y) \neq (s,t)$ *is given by the identity on*

$$\mathbb{C}[\Pi](x,y) = \mathrm{N}(\Pi_{x,y}) = \mathrm{N}\big((\Sigma \multimap \Pi)_{x,y}\big) = \mathbb{C}[\Sigma \multimap \Pi]_{/w}(x,y),$$

where the second equality follows from Lemma 2.5.22 and the action of v/w *on the mapping space* $\mathbb{C}[\Pi](s,t)$ *is given by the composition*

$$\mathbb{C}[\Pi](s,t) = \mathrm{N}(\Pi) \xrightarrow{v} Y = \mathbb{C}[\Sigma \multimap \Pi]_{/w}(s,t).$$

We leave it to the reader to verify that the functor v/w in Definition 4.2.5 is well-defined. Definition 4.2.5 was the last piece missing in order to be able to transform squares such as (4.6) into squares (4.7). This is accomplished by the following lemma:

4.2.6. Lemma *Suppose that a square such as* (4.6) *is given. The diagram*

$$\begin{tikzcd}
\mathbb{C}[\Sigma \multimap \Pi] \arrow[r, "u"] \arrow[d] & \mathbb{C}[\Sigma \multimap \Pi]_{/u} \arrow[d, "\mathbb{C}_{/p}"] \\
\mathbb{C}[\Pi] \arrow[r, "v/pu"'] & \mathbb{C}[\Sigma \multimap \Pi]_{/pu}
\end{tikzcd}$$

commutes.

Proof. Both compositions are the identity on objects and all mapping spaces $\mathbb{C}[\Sigma \multimap \Pi](x, y)$ with $(x, y) \neq (s, t)$. On the remaining mapping space $\mathbb{C}[\Sigma \multimap \Pi](s, t)$, the diagram given in the lemma is nothing but (4.6). □

4.2.7. Lemma *Suppose a square such as* (4.6) *to be given. Suppose further that the diagram*

$$\begin{tikzcd}
\mathbb{C}[\Sigma] \arrow[r] \arrow[d] & \mathbb{C}[\Sigma \multimap \Pi] \arrow[r, "u"] & \mathbb{C}[\Sigma \multimap \Pi]_{/u} \arrow[d, "\mathbb{C}_{/p}"] \\
\mathbb{C}[\Pi] \arrow[rru, "\ell"] \arrow[rr, "v/pu"'] & & \mathbb{C}[\Sigma \multimap \Pi]_{/pu}
\end{tikzcd}$$

commutes. The map $\ell_{s,t} \colon \mathrm{N}(\Pi) \to X$ *given by the action of* ℓ *on the mapping space* $\mathbb{C}[\Pi](s, t)$ *then solves the lifting problem* (4.6).

Proof. It is immediate that $p\ell_{s,t} = v$ from our definitions of $\mathbb{C}/p$ and v/w. Let us verify that the composition

$$\mathrm{N}(\Sigma \multimap \Pi) \to \mathrm{N}(\Pi) \xrightarrow{\ell_{s,t}} X$$

is equal to u. By Lemma 2.5.12, we may write the domain of this composition as

$$\mathrm{N}(\Sigma \multimap \Pi) = \mathrm{N}(\Sigma) \cup \bigcup_{x \in G \smallsetminus \{s,t\}} \mathrm{N}(\Pi_{s,x} \vee \Pi_{x,t})$$

and it thus suffices to show that the map

$$\mathrm{N}(\Sigma) \to \mathrm{N}(\Pi) \xrightarrow{\ell_{s,t}} X \tag{4.8}$$

and all the maps

$$\mathrm{N}(\Pi_{s,x} \vee \Pi_{x,t}) \to \mathrm{N}(\Pi) \xrightarrow{\ell_{s,t}} X \tag{4.9}$$

with $x \in G \smallsetminus \{s,t\}$ coincide with the restrictions of u to their domain. For the former map (4.8) this follows immediately from the commutative square given in the lemma. For the latter maps, we observe that $\ell_{s,x}$ and $\ell_{x,t}$ are necessarily identity maps and that functoriality of ℓ hence implies that the square

$$\begin{tikzcd}
\mathrm{N}(\Pi_{s,x}) \times \mathrm{N}(\Pi_{x,t}) \simeq \mathrm{N}(\Pi_{s,x} \vee \Pi_{x,t}) \arrow[rr, "\text{inclusion}"] \arrow[d, "\ell_{s,x} \times \ell_{x,t}"'] \arrow[d, shift left=20, "1"] & & \mathrm{N}(\Pi) \arrow[d, "\ell_{s,t}"] \\
\mathrm{N}(\Pi_{s,x}) \times \mathrm{N}(\Pi_{x,t}) \simeq \mathrm{N}(\Pi_{s,x} \vee \Pi_{x,t}) \arrow[r, "\text{inclusion}"'] & \mathrm{N}(\Sigma \multimap \Pi) \arrow[r, "u"'] & X
\end{tikzcd}$$

commutes, i. e. that the maps (4.9) coincide with the restriction of u to their domain. This finishes the proof. □

We close this section with the proof of Proposition 4.2.8 that we sketched at its beginning.

4.2.8. Proposition *Let* $\Sigma \to \Pi$ *be an inclusion of complete pasting diagrams. If* $\mathbb{C}[\Sigma] \to \mathbb{C}[\Pi]$ *has the left lifting property against all local* $\mathcal{R}$*-functors, then* $\mathrm{N}(\Sigma \multimap \Pi) \to \mathrm{N}(\Pi)$ *is an* $\mathcal{L}$*-map.*

Proof. Suppose that

$$\begin{tikzcd}
\mathrm{N}(\Sigma \multimap \Pi) \arrow[r, "u"] \arrow[d] & X \arrow[d, "p"] \\
\mathrm{N}(\Pi) \arrow[r, "v"'] & Y,
\end{tikzcd}$$

is a commutative diagram of simplicial sets in which p is an $\mathcal{R}$-map. We then have a commutative diagram

$$\begin{tikzcd}
\mathbb{C}[\Sigma] \arrow[r] \arrow[d] & \mathbb{C}[\Sigma \multimap \Pi]_{/u} \arrow[d, "\mathbb{C}_{/p}"] \\
\mathbb{C}[\Pi] \arrow[r, "v/pu"'] & \mathbb{C}[\Sigma \multimap \Pi]_{/pu},
\end{tikzcd}$$

by Lemma 4.2.6 and the functor $\mathbb{C}_{/p}$ is a local $\mathcal{R}$-functor by Remark 4.2.4. As $\mathbb{C}[\Sigma] \to \mathbb{C}[\Pi]$ has the left lifting property against all local $\mathcal{R}$-functors, we find a lift $\ell\colon \mathbb{C}[\Pi] \to \mathbb{C}[\Sigma \multimap \Pi]_{/u}$ in this diagram. However, any such lift ℓ induces a lift $\ell_{s,t}$ in the original diagram of simplicial sets by Lemma 4.2.7. □

4.3 Necessity of the hypothesis, part 2

This section is devoted to a proof of the following third proposition announced at the beginning of this chapter:

4.3.5. Proposition *If* $\Sigma \to \Pi$ *is an inclusion of complete pasting diagrams such that* $\mathbb{C}[\Sigma] \to \mathbb{C}[\Pi]$ *has the left lifting property against all local* $\mathcal{R}$*-functors, then so do the functors* $\mathbb{C}[\Sigma_{x,y}] \to \mathbb{C}[\Pi_{x,y}]$ *for all vertices* x, y *of* Σ.

The proof of Proposition 4.3.5 is very similar in spirit to the proof of Proposition 4.2.8 in the preceding section in the sense that we transform some given lifting problem of the form

$$\begin{tikzcd}
\mathbb{C}[\Sigma_{x,y}] \arrow[r, "u"] \arrow[d] & \mathbb{B} \arrow[d, "p"] \\
\mathbb{C}[\Pi_{x,y}] \arrow[r, "v"'] & \mathbb{A}
\end{tikzcd} \tag{4.10}$$

into a lifting problem

$$\begin{tikzcd}
\mathbb{C}[\Sigma] \arrow[r] \arrow[d] & \mathbb{B}_{\Diamond} \arrow[d, "p_{\Diamond}"] \\
\mathbb{C}[\Pi] \arrow[r] & \mathbb{A}_{\Diamond}
\end{tikzcd} \tag{4.11}$$

in such a way that $p_{\Diamond}$ is a local $\mathcal{R}$-functor whenever p is. Moreover, we show that any solution to the lifting problem (4.11) can be translated back into a solution to the lifting problem (4.10). However, the technical details are completely different.

4.3.1. Definition *Let $\mathbb{A}$ be a simplicial category. We define a simplicial category $\mathbb{A}_{\Diamond}$ as follows: The set of objects of $\mathbb{A}_{\Diamond}$ is the disjoint union of the set of objects of $\mathbb{A}$ and $\{s, t\}$. The mapping spaces of $\mathbb{A}_{\Diamond}$ are given by*

$$\mathbb{A}_{\Diamond}(x, y) = \begin{cases} \mathbb{A}(x, y) & \textit{if } x, y \in \mathbb{A} \\ \Delta^0 & \textit{if } x = s \textit{ or } y = t \\ \varnothing & \textit{otherwise} \end{cases}$$

and the compositions are those of $\mathbb{A}$ *and the maps determined by the universal properties of the initial object* $\varnothing$ *or the terminal object* Δ^0 *of* $\widehat{\mathbb{\Delta}}$.

4.3.2. Remark The category $\mathbb{A}_\Diamond$ should be pictured as the category $\mathbb{A}$ with an initial object s and a terminal object t freely adjoined. Consistent with this picture, the assignment $\mathbb{A} \mapsto \mathbb{A}_\Diamond$ is functorial in $\mathbb{A}$ and there are obvious inclusions $\mathbb{A} \to \mathbb{A}_\Diamond$ such that the diagrams

$$\begin{tikzcd} \mathbb{A} \arrow[r] \arrow[d, "f"'] & \mathbb{A}_\Diamond \arrow[d, "f_\Diamond"] \\ \mathbb{B} \arrow[r] & \mathbb{B}_\Diamond \end{tikzcd}$$

commute. Moreover, as $f_\Diamond$ is the identity on all mapping spaces $\mathbb{A}(x, y)$ with $\{x, y\} \not\subseteq \mathbb{A}$, it is immediate that $f \mapsto f_\Diamond$ takes local $\mathcal{R}$-functors to local $\mathcal{R}$-functors.

Let us now consider a span

$$\mathbb{A} \xleftarrow{u} \mathbb{B} \xrightarrow{i} \mathbb{C}$$

of simplicial categories with i fully faithful and injective on objects. We want to construct a functor $\widehat{u}\colon \mathbb{C} \to \mathbb{A}_\Diamond$ such that

$$\begin{tikzcd} \mathbb{B} \arrow[r, "u"] \arrow[d, "i"'] & \mathbb{A} \arrow[d] \\ \mathbb{C} \arrow[r, "\widehat{u}"'] & \mathbb{A}_\Diamond \end{tikzcd}$$

commutes. For this purpose, we consider a partition $\mathrm{Ob}\,\mathbb{C} = C_0 \cup C_1 \cup C_2$, $C_1 = \mathrm{Ob}\,\mathbb{B}$, of the objects of $\mathbb{C}$ such that $\mathbb{C}(c, c') = \varnothing$ whenever $x \in C_i$, $y \in C_j$ with $j < i$. We then define $\widehat{u}\colon \mathbb{C} \to \mathbb{A}_\Diamond$ by

$$\widehat{u}(c) = \begin{cases} s & \text{if } c \in C_0 \\ uc & \text{if } c \in \mathrm{Ob}\,\mathbb{B} \\ t & \text{if } c \in C_2 \end{cases}$$

on objects. We then have no choice but to define $\widehat{u}\colon \mathbb{C}(c, c') \to \mathbb{A}_\Diamond(\widehat{u}c, \widehat{u}c')$ on mapping spaces by

$$\begin{cases} \mathbb{C}(c, c') = \mathbb{B}(c, c') \xrightarrow{u} \mathbb{A}(uc, uc') = \mathbb{A}_\Diamond(\widehat{u}c, \widehat{u}c') & \text{if } c, c' \in \mathrm{Ob}\,\mathbb{B}, \\ \mathbb{C}(c, c') \to \Delta^0 = \mathbb{A}_\Diamond(\widehat{u}c, \widehat{u}c') & \text{if } c \in C_0 \text{ or } c' \in C_2, \\ \mathbb{C}(c, c') = \varnothing \to \mathbb{A}_\Diamond(\widehat{u}c, \widehat{u}c') & \text{otherwise.} \end{cases}$$

It is now easily verified that this assignment indeed defines a simplicial functor $\widehat{u}\colon \mathbb{C} \to \mathbb{A}_\Diamond$. Moreover, the construction of $\widehat{u}$ from a given partition

$\mathbb{C} = C_0 \cup C_1 \cup C_2$ is functorial in a certain sense:

4.3.3. Remark Consider a morphism

$$\begin{tikzcd} \mathbb{A} \arrow[d, "f"'] & \mathbb{B} \arrow[l, "u"'] \arrow[r, "i"] \arrow[d, "g"'] & \mathbb{C} \arrow[d, "h"] \\ \mathbb{A}' & \mathbb{B}' \arrow[l, "v"] \arrow[r, "j"'] & \mathbb{C}' \end{tikzcd}$$

of spans. Suppose that i and j are injective on objects and fully faithful. Further assume that g and h are bijective on objects. Finally, let

$$\operatorname{Ob} \mathbb{C}' = C_0' \cup C_1' \cup C_2', \quad C_1' = \operatorname{Ob} \mathbb{B}'$$

be a partition of the objects of $\mathbb{C}'$ such that $\mathbb{C}'(c, c') = \varnothing$ whenever $c \in C_i$, $c' \in C_j$ and $j < i$. By the preceding discussion, this partition gives rise to a functor $\widehat{v} \colon \mathbb{C}' \to \mathbb{A}'_{\Diamond}$.

But we also obtain a partition $\operatorname{Ob} \mathbb{C} = C_0 \cup C_1 \cup C_2$ with $C_i = h^{-1} C_i'$ as g and h are bijective on objects. It is now easy to see that this partition gives rise to a functor $\widehat{u} \colon \mathbb{C} \to \mathbb{A}_{\Diamond}$. Moreover, looking at the construction of $\widehat{u}$

and $\hat{v}$, one easily verifies that the square

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\hat{u}"] \arrow[d, "h"'] & \mathbb{A}_\diamondsuit \arrow[d, "f_\diamondsuit"] \\
\mathbb{C}' \arrow[r, "\hat{v}"'] & \mathbb{A}'_\diamondsuit
\end{tikzcd}$$

commutes.

In order to put this digression on the properties of the assignment $\mathbb{A} \mapsto \mathbb{A}_\diamondsuit$ to good use, we have to be able to produce certain partitions of the objects of categories like $\mathbb{C}[\Sigma]$. This is achieved by the following lemma:

4.3.4. Lemma *Let G be a globular graph. For any two vertices $x, y \in G$ there exists a partition $V = V_0 \cup V_1 \cup V_2$ such that V_1 is the set of vertices of $G_{x,y}$ and such that G contains no paths p from u to v with $u \in V_i$, $v \in V_j$ and $j < i$.*

Proof. Let V denote the set of vertices of G and let V_1 be the set of vertices of $G_{x,y}$. Let V_0 be the set of vertices v of $V \setminus V_1$ such that there is a directed path p from v to some vertex $w \in V_1$. Observe that there is no path from some vertex $w' \in V_1$ to some vertex $v \in V_0$, for one could then fabricate a path from x to y through v, see Figure 4.1.

Now let V_2 be the complement of $V_0 \cup V_1$ in V. Note that there cannot be any directed path from a vertex $v \in V_2$ to some vertex $w \in V_0 \cup V_1$ since this would imply $w \in V_0$.

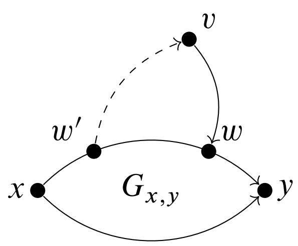


Figure 4.1: A vertex $v \in V_0$ with its path to some vertex $w \in G_{x,y}$. If the dashed path exists, then $v \in G_{x,y}$.

Altogether, we have thus found a partition $V_0 \cup V_1 \cup V_2$ with the desired properties. □

Let us now return to our actual concern, namely the promised proof of the following proposition:

4.3.5. Proposition *If* $\Sigma \to \Pi$ *is an inclusion of complete pasting diagrams such that* $\mathbb{C}[\Sigma] \to \mathbb{C}[\Pi]$ *has the left lifting property against all local* $\mathcal{R}$*-functors, then so do the functors* $\mathbb{C}[\Sigma_{x,y}] \to \mathbb{C}[\Pi_{x,y}]$ *for all vertices* x, y *of* Σ.

Proof. Consider a lifting problem

$$\begin{tikzcd}
\mathbb{C}[\Sigma_{x,y}] \arrow[r, "u"] \arrow[d] & \mathbb{B} \arrow[d, "p"] \\
\mathbb{C}[\Pi_{x,y}] \arrow[r, "v"'] & \mathbb{A}
\end{tikzcd}$$

with p a local $\mathcal{R}$-functor. We then get a morphism

$$\begin{tikzcd}
\mathbb{B} \arrow[d, "p"'] & \mathbb{C}[\Sigma_{x,y}] \arrow[l, "u"'] \arrow[r, "i"] \arrow[d] & \mathbb{C}[\Sigma] \arrow[d] \\
\mathbb{A} & \mathbb{C}[\Pi_{x,y}] \arrow[l, "v"] \arrow[r, "j"'] & \mathbb{C}[\Pi]
\end{tikzcd}$$

of spans in which the two vertical maps on the right hand side are bijective on objects and in which i and j are injective on objects and fully faithful. The partition of $\operatorname{Ob}\mathbb{C}[\Pi] = \operatorname{Ob}\mathbb{C}[\Sigma]$ constructed in Lemma 4.3.4 now gives rise to the commutative square on the right hand side of the diagram

$$\begin{tikzcd}
\mathbb{C}[\Sigma_{x,y}] \arrow[r] \arrow[d] & \mathbb{C}[\Sigma] \arrow[r, "\hat{u}"] \arrow[d] & \mathbb{B}_{\Diamond} \arrow[d, "p_{\Diamond}"] \\
\mathbb{C}[\Pi_{x,y}] \arrow[r] & \mathbb{C}[\Pi] \arrow[r, "\hat{v}"'] & \mathbb{A}_{\Diamond}.
\end{tikzcd}$$

Note that $p_{\Diamond}$ is a local $\mathcal{R}$-functor by Remark 4.3.2. Note further that the square on the left in this diagram commutes, too, and we thus find a solution to the original lifting problem that we started with. $\square$

4.4 The proof of Theorem B

Finally, we have all pieces together to give the proof of

Theorem B *The functor* $\mathbb{C}[\Sigma] \to \mathbb{C}[\Pi]$ *induced by an inclusion of complete pasting diagrams has the left lifting property against all local* $\mathcal{R}$*-functors if and only if the map*

$$\mathrm{N}(\Sigma_{x,y} \multimap \Pi_{x,y}) \to \mathrm{N}(\Pi_{x,y})$$

is an $\mathcal{L}$*-map for all vertices* $x, y \in \Sigma$.

Proof. Sufficiency of the condition in the theorem is Proposition 4.1.1 and necessity can be deduced as follows: Let $\Sigma \to \Pi$ be an inclusion between complete pasting diagrams such that $\mathbb{C}[\Sigma] \to \mathbb{C}[\Pi]$ has the left lifting property against all local $\mathcal{R}$-functors. According to Proposition 4.3.5, the functors $\mathbb{C}[\Sigma_{x,y}] \to \mathbb{C}[\Pi_{x,y}]$, where $x, y \in \Sigma$ are arbitrary vertices, then have the left lifting property against all local $\mathcal{R}$-functors, too. It now follows from Proposition 4.2.8 that the maps $\mathrm{N}(\Sigma_{x,y} \multimap \Pi_{x,y}) \to \mathrm{N}(\Pi_{x,y})$ are $\mathcal{L}$-maps for all vertices $x, y \in \Sigma$. □

5 Local Lifting Properties of Pasting Diagrams

This whole chapter is devoted to a proof of

Theorem C *Let $\Sigma \to \Pi$ be an inclusion of complete pasting diagrams such that both Σ and Π contain all the interior faces of the underlying graph and are closed under taking subdivisions. Then*

$$\mathrm{N}(\Sigma) \to \mathrm{N}(\Pi)$$

is mid anodyne.

Our proof of Theorem C closely parallels Power's proof of his 2-categorical pasting theorem in [Pow90]. More precisely, both proofs work by induction on the size of the underlying graph G and both inductive steps rest on Lemma 2.1.10. In our case, there are more technical difficulties to be overcome, though.

We explain the basic strategy and handle some trivial cases of the proof in § 5.1. Afterwards, in § 5.2 we reduce the problem to a combinatorial problem that we solve in § 5.3

5.1 The basic setup

Let us give a quick outline of the proof of Theorem C. We proceed by induction on the number of edges of the graph G underlying both Σ and Π. The base clause is basically trivial. For the induction step, we reduce to the case that G is 2-connected by decomposing complete pasting diagrams on non 2-connected graphs as

$$\mathrm{N}(\Sigma_1 \vee \Sigma_2) \simeq \mathrm{N}(\Sigma_1) \times \mathrm{N}(\Sigma_2).$$

The assumption that G is 2-connected then allows us to avail ourselves of Lemma 2.1.10, and construct certain globular subgraphs G_0, G_1, G_2 of G that are smaller than G. We may thus apply our induction hypothesis to the restriction of the pasting diagrams Σ and Π in Theorem C. The mid anodyne maps thus obtained can be glued so as to obtain a factorisation

$$\mathrm{N}(\Sigma) \to X_0 \to \mathrm{N}(\Pi)$$

in which $\mathrm{N}(\Sigma) \to X_0$ is mid anodyne. In order to finish the proof of Theorem C, we thus have to exhibit $X_0 \to \mathrm{N}(\Pi)$ as mid anodyne, too. We solve this problem in § 5.3 by a careful analysis of those simplices of $\mathrm{N}(\Pi)$ that are not contained in X_0. This analysis eventually leads to a filtration of $X_0 \to \mathrm{N}(\Pi)$

by pushouts of inclusions $\Lambda^n_i \to \Delta^n$ with $0 < i < n$. The map $X_0 \to \mathrm{N}(\Pi)$ is thus mid anodyne and this finishes the proof.

5.1.1. Remark Observe that the inclusion $\Sigma^c_{\mathrm{min}} \to \Pi_{\mathrm{max}}$ of the minimal complete pasting diagram into the maximal pasting diagram satisfies the hypothesis of Theorem C.

Moreover, if $\Sigma \to \Pi$ is an inclusion satisfying the hypothesis of Theorem C, then so do the inclusions $\Sigma \to \Sigma \multimap \Pi$, $\Sigma \multimap \Pi \to \Pi$ and all restrictions of $\Sigma \to \Pi$ to globular subgraphs of the form $G_{x,y}$ of the underlying graph G.

As announced above, we do induction on the size of G, the base clause being trivial:

5.1.2. Lemma *The map* $\mathrm{N}(\Sigma) \to \mathrm{N}(\Pi)$ *is mid anodyne for all inclusions* $\Sigma \to \Pi$ *satisfying the hypotheses of Theorem C on some globular graph G with at most one interior face.*

Proof. If G has no interior face, there is nothing to show as G is nothing but a possibly trivial directed path. Similarly, if G has only one interior face, then $\mathcal{S} = \mathcal{T}$ and the map in question is an identity. □

5.1.3. Lemma *Suppose that G decomposes as $G = G_1 \vee G_2$. For $i \in \{1,2\}$ denote the restrictions of Σ and Π to G_i by Σ_i and Π_i, respectively. If the maps*

$$\mathrm{N}(\Sigma_i) \to \mathrm{N}(\Pi_i), \quad i \in \{1,2\},$$

are mid anodyne, then so is the map $\mathrm{N}(\Sigma) \to \mathrm{N}(\Pi)$.

Proof. Follows from the assumption that Σ and Π are complete, Corollary 2.5.7 and Corollary 1.3.2. □

5.2 The inductive step

Given Lemma 5.1.3, we may suppose that G is 2-connected and has at least two faces. According to Lemma 2.1.10, there then exists a face ϕ of G such that $\operatorname{dom}\phi \subseteq \operatorname{dom} G$. Consider the following subgraphs G_i of G:

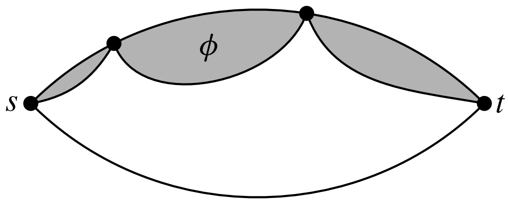
1. G_1 is the subgraph of G consisting of all directed paths containing either $\operatorname{dom}\phi$ or $\operatorname{codom}\phi$ as subpath.

2. G_2 is the subgraph of G consisting of all directed paths that contain no edge of $\operatorname{dom}\phi$, that is, G_2 is obtained from G by removing the interior vertices of $\operatorname{dom}(\phi)$.

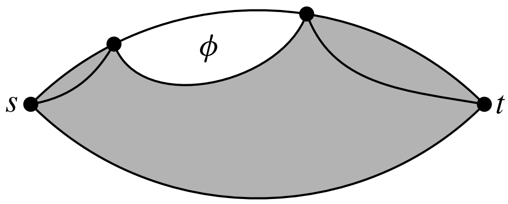
3. G_0 is the intersection of G_1 and G_2, i. e. the subgraph of G consisting of all directed paths containing $\operatorname{cod}\phi$ but no edge of $\operatorname{dom}\phi$.

In Figure 5.1 one can see a sketch of how these graphs roughly look like. Note that the graphs G_i are globular by Proposition 2.1.2. Moreover, they all have fewer edges than G — a fact that we record in the following lemma.

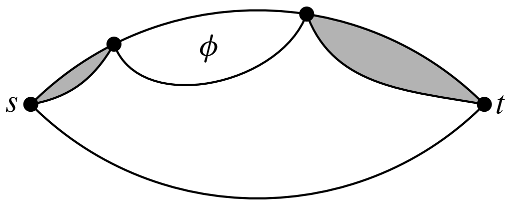
5.2.1. Lemma *The graphs G_i have fewer edges than G.*



(a) G_1



(b) G_2



(c) G_0

Figure 5.1: Schematic pictures of the graphs G_1, G_2 and G_0.

Proof. As $\operatorname{dom}\phi$ consists of at least one edge, both G_0 and G_2 have fewer edges than G. Finally, if G and G_1 had the same number of edges, then $s(\phi)$ or $t(\phi)$ would be a cut vertex of G as we assume G to have at least two faces. This contradicts our assumption that G is 2-connected, though. □

For $i \in \{0, 1, 2\}$ denote by Σ_i and Π_i the restrictions of Σ and Π to G_i. We gather the maps $\mathrm{N}(\Sigma_i) \to \mathrm{N}(\Pi_i)$ induced on the respective nerves in the commutative diagram

$$\begin{tikzcd} \mathrm{N}(\Sigma_1) \arrow[d] & \mathrm{N}(\Sigma_0) \arrow[l] \arrow[r] \arrow[d] & \mathrm{N}(\Sigma_2) \arrow[d] \\ \mathrm{N}(\Pi_1) & \mathrm{N}(\Pi_0) \arrow[l] \arrow[r] & \mathrm{N}(\Pi_2). \end{tikzcd} \tag{5.1}$$

5.2.2. Lemma *If the maps* $\mathrm{N}(\Sigma') \to \mathrm{N}(\Pi')$ *are mid anodyne for all inclusions* $\Sigma' \to \Pi'$ *satisfying the hypotheses of Theorem C on globular graphs* G' *with fewer edges than* G*, then the map*

$$\mathrm{N}(\Sigma_1) \coprod_{\mathrm{N}(\Sigma_0)} \mathrm{N}(\Sigma_2) \to \mathrm{N}(\Pi_1) \coprod_{\mathrm{N}(\Pi_0)} \mathrm{N}(\Pi_2)$$

induced by (5.1) *is mid anodyne, too.*

Proof. This is Corollary 1.1.3 and Proposition 1.3.5. □

Now consider the pushout

$$
\begin{tikzcd}
\operatorname{colim} \mathrm{N}(\Sigma_\bullet) \arrow[rr] \arrow[d] & & \mathrm{N}(\Sigma) \arrow[d] \\
\operatorname{colim} \mathrm{N}(\Pi_\bullet) \arrow[r] & \mathrm{N}(\Sigma) \coprod_{\operatorname{colim} \mathrm{N}(\Sigma_\bullet)} \operatorname{colim} \mathrm{N}(\Pi_\bullet) &
\end{tikzcd}
\tag{5.2}
$$

of the map in Lemma 5.2.2 along the inclusion $\operatorname{colim} \mathrm{N}(\Sigma_\bullet) \to \mathrm{N}(\Sigma)$. The vertical map on the right hand side in (5.2) is mid anodyne by the stability of mid anodyne maps under pushouts and Lemma 5.2.2. Moreover, the inclusion $\mathrm{N}(\Sigma) \to \mathrm{N}(\Pi)$ factors through this map and to finish the proof of Theorem C, it therefore suffices to show that the canonical map

$$
\mathrm{N}(\Sigma) \coprod_{\operatorname{colim} \mathrm{N}(\Sigma_\bullet)} \operatorname{colim} \mathrm{N}(\Pi_\bullet) \to \mathrm{N}(\Pi)
$$

is mid anodyne. This will be accomplished in the next section.

5.3 Fillable simplices

In order to finish the proof of Theorem C, we have to show that the map

$$
\mathrm{N}(\Sigma) \coprod_{\operatorname{colim} \mathrm{N}(\Sigma_\bullet)} \operatorname{colim} \mathrm{N}(\Pi_\bullet) \to \mathrm{N}(\Pi) \tag{5.3}
$$

induced by (5.1) is mid anodyne. We will achieve this by a direct inspection of the simplices of the domain and codomain of (5.3). For this purpose, let us first recollect the construction of G_1. We chose a face ϕ of G with $\operatorname{dom}\phi \subseteq \operatorname{dom} G$ and let G_1 be the subgraph of G that consists of all paths that contain either $\operatorname{dom}\phi$ or $\operatorname{cod}\phi$ as subpath. It is immediate from this description that we have the following characterisation of the simplices of $\mathrm{N}(\Pi_1)$ considered as a simplicial subset of $\mathrm{N}(\Pi)$:

5.3.1. Lemma *Consider an n-simplex $\sigma = (p_0 \leq \cdots \leq p_n)$ of* $\mathrm{N}(\Pi)$. *If all the paths p_i contain either* $\operatorname{dom}\phi$ *or* $\operatorname{cod}\phi$, *then $\sigma \in \mathrm{N}(\Pi_1)$.*

The graph G_2 was defined as the subgraph of G that consists of all paths that have no edge in common with $\operatorname{dom}\phi$. The following lemma and its corollary provide us with a description of the simplices of $\mathrm{N}(\Pi_2)$.

5.3.2. Lemma *Consider paths $p \leq q$ in G and suppose that* $\operatorname{dom}\phi$ *and p have no common edge. Then the same holds true for* $\operatorname{dom}\phi$ *and q.*

Proof. We may assume $p < q$. For any path q with $p < q$, we may write $p = a \cdot \operatorname{dom}\gamma \cdot b$ and $q = a \cdot \operatorname{cod}\gamma \cdot b$ for a nontrivial glob γ and suitable paths a and b. It follows that if q and $\operatorname{dom}\phi$ had a common edge, this edge would be an edge in the codomain of some nontrivial glob γ. This is impossible, though, as $\operatorname{dom}\phi \subseteq \operatorname{dom} G$ and no edge of $\operatorname{dom} G$ occurs in the codomain of a nontrivial glob. □

5.3.3. Corollary *Consider an n-simplex $\sigma = (p_0 \leq \cdots \leq p_n)$ of* $\mathrm{N}(\Pi)$. *If* $\operatorname{dom}\phi$ *and p_0 have no common edge, then $\sigma \in \mathrm{N}(\Pi_2)$.*

In fact, Lemma 5.3.1 and Lemma 5.3.2 tell us a bit more about the structure of the simplices of $\mathrm{N}(\Pi)$:

5.3.4. Corollary *Consider an n-simplex $\sigma = (p_0 \leq \cdots \leq p_n)$ of $\mathrm{N}(\Pi)$ with minimal witnesses γ_i for $p_{i-1} \leq p_i$. Then either $\sigma \in \operatorname{colim} \mathrm{N}(\Pi_\bullet)$ or $\operatorname{dom} \phi \subseteq p_0$ and $\gamma_i \nsubseteq G_1$ for some $i \in \{1, \ldots, n\}$.*

For a given n-simplex

$$\sigma = (p_0 \leq \cdots \leq p_n)$$

of $\mathrm{N}(\Pi)$ with minimal witnesses γ_i for $p_{i-1} \leq p_i$, we let $c(\sigma)$ be the minimal $c \in \{1, \ldots, n\}$ such that $\gamma_c \nsubseteq G_1$. If there is no such c, we let $c(\sigma) = n + 1$ by convention. We call the simplex *fillable* if $c(\sigma) = n + 1$ or $\gamma_{c(\sigma)} \cap G_1 \subseteq \partial G_1$. The following example illustrates the definition of fillable simplices:

5.3.5. Example In order to illustrate the definition of a fillable simplex, let us draw simplices σ in $\mathrm{N}(\Pi)$ as n-marked subgraphs of G.

(a) An n-simplex σ is certainly fillable, whenever $c(\sigma) = n+1$, i. e. whenever all $\gamma_i \subseteq G_1$. This does not necessarily imply that $P_\sigma = \bigcup_i p_i \subseteq G_1$ but it does imply that all the interior faces of P_σ lie in the interior of G_1, see Figure 5.2a.

(b) An n-simplex σ has $\gamma(c) \leq n$ if and only if there is some witness γ_i such that $\gamma_i \nsubseteq G_1$. The simplex shown in Figure 5.2b is fillable as the face of P_σ labeled 1 intersects G_1 only in its boundary.

(c) Finally, consider the face $d_1\sigma$ of the simplex σ of the previous example. Its 1-marked subgraph is shown in Figure 5.2c. We obviously have $c(\sigma) = 1$ but the minimal glob γ_1 witnessing $p_0 \leq p_1$ has nontrivial intersection with the interior of G_1. This simplex is thus non-fillable.

With this definition at hand, Corollary 5.3.4 admits the following reformulation:

5.3.6. Lemma *For a fillable simplex σ of $\mathrm{N}(\Pi)$ of dimension n, we either have $\sigma \in \operatorname{colim} \mathrm{N}(\Pi_\bullet)$ or $c(\sigma) \in \{2, \ldots, n\}$.*

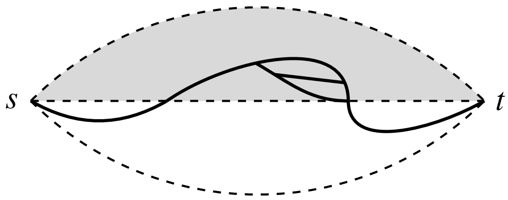
Proof. If $\sigma \notin \operatorname{colim} \mathrm{N}(\Pi_\bullet)$, then $\sigma \notin \mathrm{N}(\Pi_2)$ and thus $\operatorname{dom} \phi \subseteq p_0$. Assume $c(\sigma) = 1$, i. e. $\gamma_1 \nsubseteq G_1$. Since σ is fillable, γ_1 intersects G_1 only in its boundary. Let us consider the order of the vertices $s(\gamma_1), t(\gamma_1), s(\phi), t(\phi)$ on the path p_0. Observe that the orders

$$s(\phi) \rightsquigarrow t(\phi) \rightsquigarrow s(\gamma_1) \rightsquigarrow t(\gamma_1) \qquad \text{and} \qquad s(\gamma_1) \rightsquigarrow t(\gamma_1) \rightsquigarrow s(\phi) \rightsquigarrow t(\phi)$$

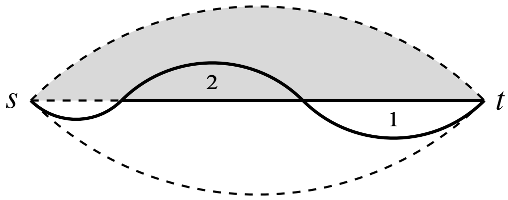
are impossible as they would imply $\gamma_1 \subseteq G_1$ by our definition of G_1. Moreover, the orders

$$\begin{aligned} &s(\phi) \rightsquigarrow s(\gamma_1) \rightsquigarrow t(\gamma_1) \rightsquigarrow t(\phi), \qquad && t(\gamma_1) \rightsquigarrow t(\phi) \rightsquigarrow s(\phi) \rightsquigarrow s(\gamma_1) \\ &s(\phi) \rightsquigarrow s(\gamma_1) \rightsquigarrow t(\phi) \rightsquigarrow t(\gamma_1), \qquad && s(\gamma_1) \rightsquigarrow s(\phi) \rightsquigarrow t(\gamma_1) \rightsquigarrow t(\phi) \end{aligned}$$

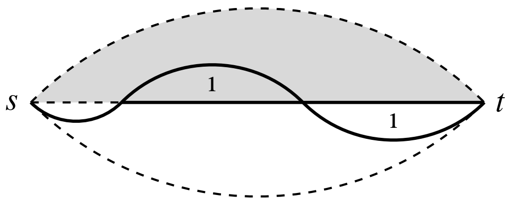
are also impossible, for γ_1 is minimal, intersects G_1 only in its boundary but is not contained in G_1 and we have the path $\operatorname{dom} \phi$ between $s(\phi)$ and $t(\phi)$.



(a) Sketch of the n-marked subgraph of a fillable n-simplex σ with $c(\sigma) = n + 1$.



(b) The 2-marked subgraph of a fillable 2-simplex σ with $c(\sigma) = 1$.



(c) The 1-marked subgraph of the non-fillable face $d_1\sigma$ of the simplex whose 2-marked subgraph is shown in Figure 5.2b.

Figure 5.2: Illustration of the definition of a fillable simplex. The shaded area of the shown globular graph represents G_1.

However, these are all possible orders and we conclude $c(\sigma) \geq 2$. Moreover, $c(\sigma) \leq n$ as $\sigma \notin \mathrm{N}(\Pi_1)$. $\square$

In order to finish the proof of Theorem C we want to express the map (5.3) as a composition of pushouts of inner horn inclusions $\Lambda_i^n \to \Delta^n$ in which each Δ^n maps to a fillable n-simplex of $\mathrm{N}(\Pi)$. We therefore need to understand the faces of fillable simplices in $\mathrm{N}(\Pi)$. To this end, we will employ the following remark.

5.3.7. Remark Consider a glob $\gamma \subseteq G$ with $\gamma \nsubseteq G_1$ and $\gamma \nsubseteq G_2$. Cutting the glob γ along the boundary ∂G_1 supplies us with two globs γ_1 and γ_2 such that (i) $\gamma = \partial(\gamma_1 \cup \gamma_2)$, (ii) $\gamma_1 \subseteq G_1$ and (iii) $\gamma_2 \cap G_1 \subseteq \partial G_1$. This is also illustrated in Figure 5.3. Observe that γ_1 corresponds to the part of γ that lies left of ∂G_1 while γ_2 lies right of ∂G_1. Given any relation $p < r$ witnessed by γ, we thus find a path q such that γ_1 and γ_2 witness $p < q < r$.

5.3.8. Remark The above Remark 5.3.7 immediately implies that each simplex of $\mathrm{N}(\Pi)$ is an inner face of some fillable simplex, for $\Pi = (G, \mathcal{T})$ is closed under taking subdivisions and the globs γ_1 and γ_2 constructed above therefore appear in $\mathcal{T}$ whenever $\gamma \in \mathcal{T}$, see Lemma 2.5.3.

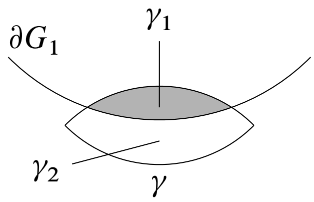


Figure 5.3: Cutting a glob γ along ∂G_1.

5.3.9. Lemma *Let σ be a nondegenerate, fillable n-simplex of $\mathrm{N}(\Pi)$ and suppose $c = c(\sigma) \in \{2, \ldots, n\}$.*

(a) The faces $d_i\sigma$ are fillable for all $i \notin \{c-1, c\}$.

(b) The face $d_{c-1}\sigma$ is not fillable and there is no fillable n-simplex $\tau \neq \sigma$ of $\mathrm{N}(\Pi)$ with $c(\tau) \geq c$ such that $d_{c-1}\sigma \subseteq \partial\tau$.

(c) If the face $d_c\sigma$ is not fillable, then there exists a fillable n-simplex τ of $\mathrm{N}(\Pi)$ such that $d_c\sigma \subseteq \partial\tau$ and $c(\tau) > c$.

Proof. Let us write $\sigma = (p_0 < \cdots < p_n)$ and choose minimal witnesses γ_j of $p_{j-1} < p_j$. Then $d_i\sigma = (p_0 < \cdots < \widehat{p_i} < \cdots < p_n)$, where the circumflex signals omission of p_i. The minimal witnesses for the relations $p_{j-1} < p_j$ that occur in $d_i\sigma$ are thus all the γ_j with $j \notin \{i, i+1\}$ and a minimal glob containing $\gamma_i \cup \gamma_{i+1}$.

Part (a) is now obvious and so is the assertion that $d_{c-1}\sigma$ is not fillable since $\gamma_{c-1} \cup \gamma_c$ is neither contained in G_1 nor in G_2 by the definition of fillable simplices. Towards the second claim in part (b), it suffices to observe that the decomposition $\gamma_i \cup \gamma_{i+1}$ is unique with the property that $\gamma_i \subseteq G_1$ and $\gamma_{i+1} \cap G_1 \subseteq \partial G_1$. There is thus no fillable simplex $\tau \neq \sigma$ with $c(\tau) \geq c$ and $d_{c-1}\sigma \subseteq \partial\tau$.

Finally, suppose that $d_c\sigma$ is not fillable. This is equivalent to the condition that $(\gamma_c \cup \gamma_{c+1}) \cap G_1 \not\subseteq \partial G_1$. We may thus decompose $\gamma_c \cup \gamma_{c+1}$ into $\gamma'_c \cup \gamma'_{c+1}$ with $\gamma'_c \subseteq G_1$ and $\gamma'_{c+1} \cap G_1 \subseteq \partial G_1$. There then exists a path q with γ'_c and γ'_{c+1} witnessing $p_{c-1} < q < p_{c+1}$ and the simplex

$$\tau = (p_0 < \dots p_{c-1} < q < p_{c+1} < \dots < p_n)$$

is a fillable simplex with $c(\tau) > c(\sigma)$ and $d_c\sigma \subseteq \partial\tau$. □

We have finally gathered all technical details to finish the proof of Theorem C.

5.3.10. Lemma *The map* (5.3) *admits a filtration by mid anodyne maps and is hence mid anodyne itself.*

Proof. Let us denote the domain of (5.3) by X_0 and let $Y_{n,c}$ be the simplicial subset of $\mathrm{N}(\Pi)$ that is generated by the fillable nondegenerate simplices σ of dimension n with $c(\sigma) \geq c$ together with the fillable nondegenerate simplices of dimension less than n. Note that

$$Y_{n-1,2} \cup X_0 = Y_{n-1,1} \cup X_0 = Y_{n,n+1} \cup X_0$$

by Lemma 5.3.6. The first equality follows from the fact that any fillable simplex σ of $\mathrm{N}(\Pi)$ with $c(\sigma) = 1$ is already contained in $\operatorname{colim} \mathrm{N}(\Pi_\bullet) \subseteq X_0$ and the second equality follows from the fact that any fillable n-simplex with $c(\sigma) = n + 1$ is contained in $\operatorname{colim} \mathrm{N}(\Pi_\bullet) \subseteq X_0$. We thus have a filtration

$$X_0 = Y_{2,3} \cup X_0 \subseteq Y_{2,2} \cup X_0 = Y_{2,1} \cup X_0 = Y_{3,4} \cup X_0 \subseteq Y_{3,3} \cup X_0 \subseteq \dots$$

of the inclusion $X_0 \to \mathrm{N}(\Pi)$ that is exhaustive by Remark 5.3.8. It therefore suffices to show that each of the inclusions $Y_{n,c+1} \cup X_0 \subseteq Y_{n,c} \cup X_0$ with $2 \leq c \leq n$ is mid anodyne. To this end, it suffices to prove the following statements:

(i) For any nondegenerate simplex $\sigma \in Y_{n,c}$ that is not contained in $Y_{n,c+1} \cup X_0$ there is an inner horn $\Lambda^n_{c-1} \subseteq Y_{n,c+1} \cup X_0$ such that σ is a filler for this horn.

(ii) For any simplex σ as in (i), the face $d_{c-1}\sigma$ is not contained in $Y_{n,c+1} \cup X_0$.

(iii) If $\sigma \neq \tau$ are two simplices as in (i), then $d_{c-1}\sigma \neq d_{c-1}\tau$.

Consider a nondegenerate simplex $\sigma \in Y_{n,c} \setminus (Y_{n,c+1} \cup X_0)$ for some $2 \leq c \leq n$. Note that σ is fillable with $c(\sigma) = c$. Part (a) of Lemma 5.3.9 implies that all the faces $d_i\sigma$ of σ with $i \notin \{c-1, c\}$ are fillable of dimension strictly less than n, i. e. contained in $Y_{n,c+1}$. Moreover, by part (c) of Lemma 5.3.9, $d_c\sigma$ is either fillable or a face of some $\tau \in Y_{n,c+1}$. In either case, it is contained in $Y_{n,c+1}$. The faces $d_i\sigma$ of σ with $i \neq c-1$ are therefore an inner horn $\Lambda^n_{c-1} \subseteq Y_{n,c+1}$ of the desired kind since $0 < c - 1 < n$ by assumption.

Moreover, according to part (b) of Lemma 5.3.9, $d_{c-1}\sigma$ is not contained in $Y_{n,c+1}$. Let us assume for the sake of contradiction that $d_{c-1}\sigma$ is contained in

$$X_0 = \mathrm{N}(\Sigma) \coprod_{\operatorname{colim} \mathrm{N}(\Sigma_\bullet)} \operatorname{colim} \mathrm{N}(\Pi_\bullet),$$

i. e. $d_{c-1}\sigma \in \mathrm{N}(\Sigma)$ or $d_{c-1}\sigma \in \mathrm{N}(\Pi_i)$. As Σ and Π_i are closed under taking subdivisions, this implies $\sigma \in X_0$ by Lemma 2.5.3.

Finally, if σ and τ are two distinct fillable simplices with $c(\sigma) = c(\tau) = c$, Lemma 5.3.9 (b) implies $d_{c-1}\sigma \neq d_{c-1}\tau$.

□

6 A Pasting Theorem for Riehl and Verity's Cosmoi

In this final chapter, we collect all the results obtained so far and deduce our pasting theorem. Let us commence by combining Theorem B and Theorem C:

6.1. Proposition *Let $\Sigma \to \Pi$ be an inclusion of complete pasting diagrams such that Σ and Π are both closed under taking subdivisions and contain all the interior faces of the underlying graph. The functor*

$$\mathbb{C}[\Sigma] \to \mathbb{C}[\Pi]$$

has the left lifting property against all local mid fibrations.

Proof. Recall that if $\Sigma \to \Pi$ satisfies the hypothesis of Theorem C, then so do all the restrictions $\Sigma_{x,y} \to \Pi_{x,y}$ and all the inclusions $\Sigma_{x,y} \multimap \Pi_{x,y} \to \Pi_{x,y}$. The proposition thus follows from Theorem C and Theorem B. □

Recall from [RV16] that the category of small simplicial categories is simplicially enriched with mapping spaces given by $\mathrm{icon}(\mathbb{A}, \mathbb{B})_n = \mathrm{Cat}_{\widehat{\mathbb{\Delta}}}(\mathbb{A}, \mathbb{B}^{\Delta^n})$, where $\mathbb{B}^{\Delta^n}$ denotes the simplicial category with $\mathbb{B}^{\Delta^n}(a,b) = \mathbb{B}(a,b)^{\Delta^n}$. The acronym icon stands for "identity component oplax natural transformations" and is due to Lack in the case of 2-categories, see [Lac10].

6.2. Proposition *Consider an inclusion $\Sigma \to \Pi$ of complete pasting diagrams such that both Σ and Π are closed under taking subdivisions and contain all the interior faces of the underlying graph. Further let $\mathbb{B} \to \mathbb{A}$ be a local mid fibration of simplicial categories. The canonical map*

$$\mathrm{icon}\big(\mathbb{C}[\Pi], \mathbb{B}\big) \to \mathrm{icon}\big(\mathbb{C}[\Sigma], \mathbb{B}\big) \underset{\mathrm{icon}(\mathbb{C}[\Sigma],\mathbb{A})}{\times} \mathrm{icon}\big(\mathbb{C}[\Pi], \mathbb{A}\big)$$

is a trivial Kan fibration.

Proof. Any lifting problem

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & \mathrm{icon}\big(\mathbb{C}[\Pi], \mathbb{B}\big) \arrow[d] \\
\Delta^n \arrow[r] & \mathrm{icon}\big(\mathbb{C}[\Sigma], \mathbb{B}\big) \underset{\mathrm{icon}(\mathbb{C}[\Sigma],\mathbb{A})}{\times} \mathrm{icon}\big(\mathbb{C}[\Pi], \mathbb{A}\big)
\end{tikzcd}$$

transposes to a lifting problem

$$\begin{tikzcd}
\mathbb{C}[\Sigma] \arrow[r] \arrow[d] & \mathbb{B}^{\Delta^n} \arrow[d] \\
\mathbb{C}[\Pi] \arrow[r] & \mathbb{B}^{\partial\Delta^n} \underset{\mathbb{A}^{\partial\Delta^n}}{\times} \mathbb{A}^{\Delta^n}.
\end{tikzcd}$$

Similar to the proof of [RV16, Lemma 4.4.2], we observe that the map on the right hand side in this latter diagram is locally a mid fibration by Proposition 1.3.3 and the claim thus follows from Proposition 6.1. □

Recall from Theorem A that any labeling Λ of a globular graph G in some simplicial category $\mathbb{A}$ determines a simplicial functor $u_\Lambda \colon \mathbb{C}[\Sigma^c_{\min}] \to \mathbb{A}$.

6.3. Definition *The space $C(\Lambda)$ of compositions of a labeling Λ of a globular graph G in some simplicial category $\mathbb{A}$ is obtained as the pullback in*

$$\begin{tikzcd} C(\Lambda) \arrow[r] \arrow[d] & \mathrm{icon}\big(\mathbb{C}[\Pi_{\max}], \mathbb{A}\big) \arrow[d] \\ \Delta^0 \arrow[r, "u_\Lambda"'] & \mathrm{icon}\big(\mathbb{C}[\Sigma^c_{\min}], \mathbb{A}\big). \end{tikzcd}$$

6.4. Remark Using the fact that u_Λ and Λ determine each other uniquely, we see that the 0-simplices in $C(\Lambda)$ are those functors $v \colon \mathbb{C}[\Pi_{\max}] \to \mathbb{A}$ such that $\Lambda_v = \Lambda$, i. e. extensions of Λ to the fully coherent $\mathbb{C}[\Pi_{\max}]$.

Theorem D *Consider a globular graph G and a category $\mathbb{A}$ enriched over quasi-categories. The space $C(\Lambda)$ of compositions of a given labeling Λ of G in $\mathbb{A}$ is a nonempty contractible Kan complex.*

Proof. Observe that $\mathbb{A} \to *$ is a mid fibration in the case that $\mathbb{A}$ is enriched over quasi-categories. Proposition 6.2 thus implies that the map

$$\mathrm{icon}\big(\mathbb{C}[\Pi_{\max}], \mathbb{A}\big) \to \mathrm{icon}\big(\mathbb{C}[\Sigma^{c}_{\min}], \mathbb{A}\big)$$

is a trivial Kan fibration. The space $C(\Lambda)$ of compositions of Λ is hence a contractible Kan complex. □

6.5. Remark Note that we can deduce Power's original result from [Pow90] from Theorem D. Indeed, given any labeling Λ of a globular graph G in some 2-category A, we obtain a labeling of G in the simplicial category $\mathbb{A}$ obtained by applying the nerve functor to all the categories $A(a, b)$. As $C(\Lambda)$ is nonempty there is at least one extension $v\colon \mathbb{C}[\Pi_{\max}] \to \mathbb{A}$ of Λ and v restricts to a map

$$v\colon\ \mathrm{N}(G) = \mathbb{C}[\Pi_{\max}](s, t) \to \mathbb{A}(s, t)$$

of simplicial sets. We now apply the left adjoint τ_1 of N to the inclusion

$$\Delta^1 \xrightarrow{\partial G} \mathrm{N}(G) \to \mathbb{A}(s, t)$$

and thus obtain a composite 2-cell $\phi\colon\ f \to g$ in $A(s, t) = \tau_1\,\mathrm{N}\,A(s, t)$.

Now consider two such extensions $v, v' \in \mathrm{icon}(\mathbb{C}[\Pi_{\max}], \mathbb{A})$ with associated composite 2-cells ϕ and ψ. We then find a lift h as in the diagram

$$\begin{tikzcd}
\partial\Delta^1 \arrow[rr, "v \amalg v'"] \arrow[d] & & \mathrm{icon}\big(\mathbb{C}[\Pi_{\max}], \mathbb{A}\big) \arrow[d] \\
\Delta^1 \arrow[r] \arrow[urr, "h"] & \Delta^0 \arrow[r, "u_\Lambda"'] & \mathrm{icon}\big(\mathbb{C}[\Sigma^c_{\min}], \mathbb{A}\big).
\end{tikzcd} \tag{6.1}$$

As above, this gives rise to a map $h\colon \Delta^1 \times \mathrm{N}(G) \to \mathbb{A}(s,t)$ of simplicial sets and hence to a square

$$\begin{tikzcd}
\bullet \arrow[r, "u"] \arrow[d, "\phi"'] & \bullet \arrow[d, "\psi"] \\
\bullet \arrow[r, "v"'] & \bullet
\end{tikzcd}$$

in $A(s,t)$. Commutativity of the lower triangle in (6.1) now implies that both u and v are the identity, i. e. $\phi = \psi$.

7 Concluding Remarks

Conducting the research for and especially writing down this thesis has left me behind with the very question that initially triggered my journey into the combinatorics of compositions in higher categories unanswered. Moreover, along the way, studying beautiful — yet sometimes daunting — mathematics, I came to know other issues of which more than one has been left unexplored not because of lack of ideas but rather because of the amount of technicalities that have to be sorted out. This thesis therefore closes with a short list of some of these questions. We begin with rather technical points and broaden our perspective gradually:

1. Can we strengthen Theorem C?

2. Pasting of categories enriched over quasi-categories should include a calculus of monoidal $(\infty, 1)$-categories. One can pass from a globular graph to a string diagram by taking (roughly) the dual graph. Can anything meaningful about monoidal $(\infty, 1)$-categories be said in this way?

3. There should exist a category of computads such that the categories $\mathbb{C}[\Pi]$ are free categories on computads.[1]
4. If there exists such a category of computads, can we describe it in terms of pasting diagrams? Moreover, the induced comonad on $\mathrm{Cat}_{\widehat{\mathbb{A}}}$ is a candidate for the construction of pseudo functors between categories enriched over quasi-categories, see [Gar10].
5. Can we use Theorem B for other weak factorisation systems on $\widehat{\mathbb{A}}$?

[1] I have a sketch of a construction but there are still some details missing.

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