Central Limit Theorems for Geometric Functionals of Gaussian Excursion Sets

Zur Erlangung des akademischen Grades eines

DOKTORS DER NATURWISSENSCHAFTEN

von der KIT-Fakultät für Mathematik des
Karlsruher Instituts für Technologie (KIT)
genehmigte

DISSERTATION

von

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Tag der mündlichen Prüfung: 20. Juni 2018

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Publications

Variants of the results of Section 3.1 and Section 3.3 of this thesis are the subject of the following publication:

D. Müller, *A central limit theorem for Lipschitz–Killing curvatures of Gaussian excursions*, J. Math. Anal. Appl. **452** (2017), no. 2, 1040 – 1081.

Acknowledgements

This PhD thesis was written during my time as a teaching and research assistant at the Institute of Stochastics at Karlsruhe Institute of Technology, financed in parts by the German Research Foundation DFG (grant number FOR 1548) through the research unit "Geometry and Physics of Spatial Random Systems" and the Institute of Stochastics.

My sincere thanks go to my advisor Prof. Dr. Daniel Hug not only for introducing me to the realms of spatial random systems as an undergraduate student, but also for the valuable input and for the many discussions during my time as a PhD student. His inimitable keen eye for detail always led to the right questions. At least as many thanks go to my other—not second—advisor Prof. Dr. Günter Last, who no matter how many mistakes I made was always optimistic about my work, for his valuable input, for many discussions and for the general guidance in the last years. I also benefited from several discussions with Michael Klatt.

Furthermore, I thank all former and current members of the Institut of Stochastics for making everyday life at work so pleasant. Special thanks go to the two secretaries for keeping track of all the administrative challenges accompanying the daily routine.

Moreover, I owe huge thanks to the "Muschetta" collective, without whom studying mathematics in Karlsruhe would not have been such an enjoyable and fruitful endeavour.

I am deeply indebted to my great friends and beloved family for all the wonderful moments we shared and for coping with the mathematician in me. You are the one thing in life that truly matters—thanks for being part of my journey. Special thanks go to my parents for always teaching me "the sky's the limit".

Finally, I want to thank my wife Anika for her constant support. There aren't any words to describe how you enchant my life.

Contents

CHAPTER 1

Introduction

The normal or Gaussian distribution on the real line is undeniably one of the most basic and essential distributions in probability theory. It was first mentioned in the works of the French mathematician Abraham de Moivre (1667–1754) in the approximation of binomial probabilities, cf. [14]. Its importance and widespread use throughout all applied sciences is a consequence of the central limit theorem, which was regarded as the central, if not the only problem of probability theory for a long time, cf. [40, Theorem 15.37] in a modern formulation. It states that the standardisation of a sum of independent random variables with the same and existing expectation and variance can be approximated by a normal distribution to arbitrary precision with increasing length of the summation.

In applications, however, reality is sometimes more complex than just a single number or vector and needs to be modelled by the versatility of a whole function. This requires the generalization of the normal distribution from the Euclidean space to a "normal distribution" in a suitable function space and was a subject in mathematical research in the first half of the twentieth century. The question of existence of such an object was settled in the groundbreaking work "Grundbegriffe der Wahrscheinlichkeitstheorie" by Andrej N. Kolmogorov (1903–1987), cf. [41]. It turns out that the key to the generalisation of a normally distributed random variable to a "normally distributed" random function are the finite-dimensional distributions, i.e. the distributions of the vectors consisting of the function values of arbitrary points in the domain. The normality is encoded into the random function by requiring that the finite-dimensional distributions are given by "consistent" families of normal distributions. Because of this property, random functions satisfying this condition are called Gaussian processes. For a rigorous explanation of the latter, we refer the reader to Section 2.1.

The basic objects of randomness for this thesis are Gaussian random processes with domain

$\mathbb{R}^d$ and codomain $\mathbb{R}$, and to highlight the geometric viewpoint, these processes are then typically called Gaussian (random) fields. Figure 1.1 shows a realisation of a specific Gaussian field from $\mathbb{R}^2$ to $\mathbb{R}$. Gaussian fields and their excursion sets, i.e. all points in the domain, where the field exceeds a certain threshold, are widely used models in applied sciences, for instance for data analysis in medicine, cf. [77], in machine learning, cf. [65], in chaotic quantum systems, cf. [8, 9], in the modelling of sea waves, cf. [52], in cosmology, cf. [51], and in materials science, cf. [55] and references therein. Moreover, in probability theory the excursion sets of Gaussian fields are one of the basic models for random sets and are still an active area of research, cf. [1], [4], [46], [11], [20] among others.

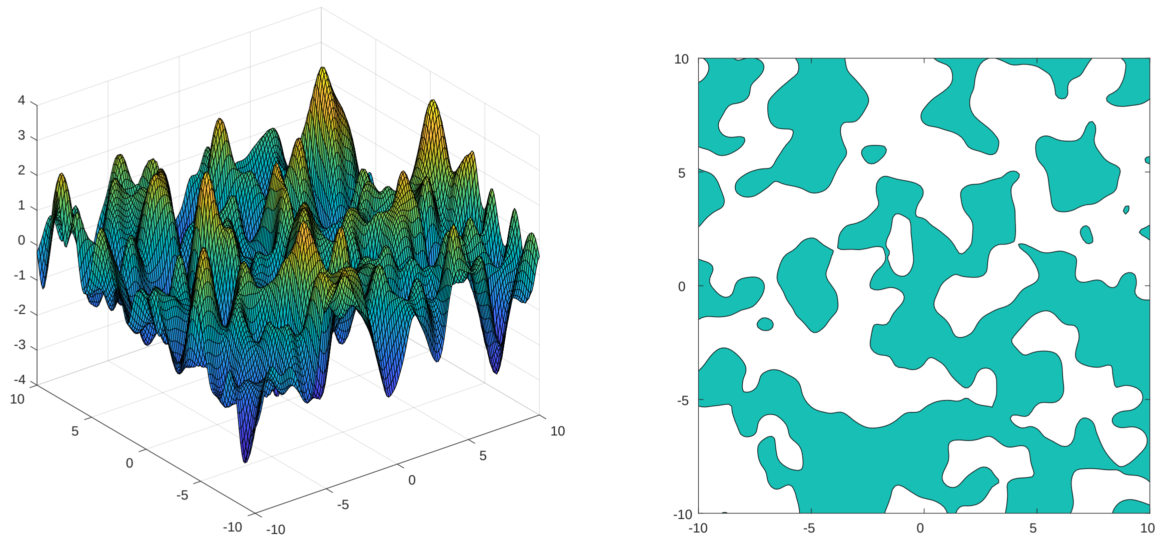

Figure 1.1.: Left: Realisation of a centered stationary and isotropic Gaussian random field with covariance given by [27, (21)]. Right: The excursion set of the realisation from the left for the threshold 0.

This thesis is a contribution to the ongoing research with the aim to deepen the understanding of the geometry of Gaussian excursion sets. For this purpose many geometric characteristics can be used to retrieve information about the nature of these random sets and the ones we are going to use in the first part of this work are the so-called Lipschitz–Killing curvatures $\mathcal{L}_m$, $m = 0, \dots, d-1$, cf. Section 2.3 for a definition. The reason for the special interest in these functionals lies in a generalisation of Hadwiger's famous characterisation theorem (cf. [29], [68, Theorem 6.4.14]) due to Zähle (cf. [82, Theorem 3]), which states that any motion invariant, additive and continuous functional on the set of compact sets of positive reach is a linear combination of the Lipschitz–Killing curvatures and the volume.

In an ideal world, we would now go on and proclaim that in this work we calculate the distribution of the random variables given by

$$\mathcal{L}_m(X^{-1}((u,\infty]) \cap A), \quad A \subset \mathbb{R}^d \text{ convex}, m = 0, \dots, d-1, \tag{1.1}$$

where $X\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ denotes a Gaussian field and $u \in \mathbb{R}$. Unfortunately, we do not live in an ideal world and a proof for such a result is out of reach with the methods developed up to now. Instead, we pursue a very common strategy in mathematics and aim for asymptotic results. We analyse the scenario of an ever-growing observation window—given by A in the above formulation—and derive the asymptotic normality of the suitably standardised Lipschitz–Killing curvatures given in (1.1). Predecessors of this result are the works [44] in the case $d = 2$ and $m = 1$, and [22] for $m = 0$ and general d.

In the second part of this thesis, we use the developed approach to establish the normal approximation of integrated level functionals of the type

$$\int_{\mathbb{R}} \int_{X^{-1}(\{u\})\cap A} h(\nabla X(t), D^2 X(t), X(t))\, \mathcal{H}^{d-1}(dt)\, du,$$

where $X\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ denotes a Gaussian field, see Section 4.1 for more nomenclature, as the observation window A grows to the whole Euclidean space. The general case will be specialised to integrated Minkowski surface tensors, which is a worthwhile undertaking because of the geometric information contained in the tensors. At last, we specialise the general theorem to some of Federer's curvature measures and thereby learn about the limits of the applicability of our method of proof, as not all curvature measures are tractable due to integrability issues.

As a basic tool in our approach, we use the so-called Wiener chaos expansion, which was already an essential technique in the works [13], [72] and [44]. We combine this technique with recent results in the normal approximation based on Malliavin calculus, cf. [59], which is a popular method to derive central limit theorems, cf. [57], where the Euler integration of random functions is analysed, cf. [66], which collects some of the recent results on the nodal geometry of random eigenfunctions on Riemannian surfaces, cf. [58], where critical points of random Fourier series on the m-dimensional torus are studied, or cf. [3], where the number of real roots of Kostlan–Shub–Smale random polynomial systems is investigated.

Although less explicit, the results of the first part of this thesis might be compared with recent progress in the second order analysis of the Boolean model, another fundamental model of stochastic geometry, cf. [31], [30], [49]. In contrast to the present work, this progress is largely based on the Malliavin calculus for general Poisson processes.

Future research continuing the work of this thesis could be manifold. One natural generalisation of the established central limit theorems is to aim for versions including rates for the speed of convergence. The underlying theory of Malliavin calculus is rich enough to allow for results of this type, cf. [26]. Even more generality could be achieved by dropping the isotropy assumption in (A1) in Chapter 3. First results, which indicate that the approximation procedure in Section 3.2.1 is still valid, can be found in [21]. Generalizing the results of this thesis to non-Gaussian random fields seems to be a challenging goal, cf. [47], [37] and references therein for recent results in this direction. Another direction of further research could be the investigation of limit theorems in the regime of long range dependence, in the spirit of [45], in contrast to short range dependence analysed in this thesis, cf. (A3) in Chapter 3 and (AF3) in Chapter 4. A further interesting direction could be to change the asymptotic scenario of a growing observation window to an additionally increasing threshold parameter u (in (1.1)) depending on the window size, cf. [33, Theorem 2.8.1], [15]. Applications in applied sciences could profit from the development of a test on Gaussianity of the underlying field based on the central limit theorems in this thesis, cf. [16] and [12] for results in this direction based on the central limit theorem in Chapter 3.

This thesis is organised as follows: In Chapter 2 we provide the reader with the necessary background of the relevant parts in probability theory and geometry to read and understand the results and proofs contained in this thesis. The first section recalls basic facts and tools from the theory of Gaussian random fields, which are the basic building blocks in this work. The next sections provide a very brief introduction to the required geometric tools applied in Chapter 3. In the last section, we provide background information about the theory of isonormal Gaussian processes, including a central limit theorem, which lies at the heart of every normal approximation in this work.

Chapter 3 contains the main result of this thesis, namely the central limit theorem for the standardised Lipschitz–Killing curvatures of the intersection of the excursion set of a Gaussian field with an observation window as the window grows to the whole Euclidean space. We also derive a lower bound for the asymptotic variance, hence ensuring that the limit distribution is nontrivial. At last, by using the already established results, we prove a central limit theorem in the multivariate case, i.e. for the vector containing all Lipschitz–Killing curvatures.

We then proceed in Chapter 4 with the exploitation of the developed techniques and prove a quite general central limit theorem for integrated level functionals. The general result will be specialised to the case of integrated Minkowski surface tensors and integrated curvature measures. In the first case a simulation study is conducted to illustrate the theoretical result.

At last, Appendix A contains results concerning Gaussian fields, which hold almost surely and are indispensable for this thesis. In Appendix B, we give the tedious but important proof of Lemma 3.2, which is the basis of the approximation procedure used in Chapter 3.

CHAPTER 2

Basics

In this chapter we recall basic results and fix the notation used in this thesis.

We denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the positive integers, the nonnegative integers, the integers, the real numbers and the complex numbers, respectively. The d-dimensional Euclidean space $\mathbb{R}^d$, $d \in \mathbb{N}$, is equipped with the standard inner product $\langle \cdot , \cdot \rangle$, which induces the Euclidean norm $\| \cdot \|$. For a set $A \subset \mathbb{R}^d$, we denote the interior, closure and boundary of A by $\operatorname{int} A$, $\operatorname{cl} A$ and $\operatorname{bd} A$, respectively. The dimension $\dim A$ of A is defined by the dimension of the affine hull $\operatorname{aff}(A)$ of A. The Minkowski sum $A + B$ of two sets $A, B \subset \mathbb{R}^d$ is given by the set $\{a + b \mid a \in A, b \in B\}$ and we use the notation $A + t$ for $A + \{t\}$, $t \in \mathbb{R}^d$. By I_d, we denote the identity matrix in dimension d. For $t_1, \dots, t_n \in \mathbb{R}^d$, we write $(t_1, \dots, t_n)$ for the (nd)-dimensional vector given by $(t_1^\top, \dots, t_n^\top)^\top$ in contrast to the $(d \times n)$-matrix $(t_1^i, \dots, t_n^i)_{i=1}^d$, for which we write $(t_1 | \cdots | t_n)$.

For a topological space T, we denote the Borel σ-algebra by $\mathcal{B}(T)$. We write $\mathcal{H}^s$, $s \geq 0$, for the s-dimensional Hausdorff measure on $\mathcal{B}(\mathbb{R}^d)$. By $B_r^d \subset \mathbb{R}^d$ we denote the open ball of radius $r \geq 0$ with center 0 and we write $C_N^d \subset \mathbb{R}^d$ for the centered cube of side length $2N$, i.e. $C_N^d := [-N, N]^d$. The d-dimensional volume of B_1^d is abbreviated by κ_d and the $(d-1)$-dimensional Hausdorff measure of its boundary S^{d-1} by ω_d. A function $C \colon T \times T \to \mathbb{C}^{d \times d}$ is called positive semidefinite, if

$$\sum_{i,j=1}^{n} c_i^\top C(t_i, t_j) \overline{c_j} \geq 0, \qquad \text{for all } c_1, \dots, c_n \in \mathbb{C}^d, t_1, \dots, t_n \in T, n \in \mathbb{N}. \tag{2.1}$$

If the function C is only defined on T, that is $C \colon T \to \mathbb{C}^{d \times d}$, it is positive semidefinite if the condition in (2.1) holds with $C(t_i, t_j)$ replaced by $C(t_i - t_j)$. We use the symbol π for various kinds of projections, where the most frequent is the orthogonal projection. All other usages will be defined when the need arises. For a manifold M, the notation T_tM denotes the tangent space of M at the point $t \in M$.

The symbol $\mathcal{C}^k$ denotes the set of all functions $f\colon \mathbb{R}^d \to \mathbb{R}$ which are k-times continuously differentiable. In the following, let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a mapping of class $\mathcal{C}^2$. For $t \in \mathbb{R}^d$, we denote by $\nabla f(t)$ the gradient and by $D^2 f(t)$ the $d \times d$-matrix $(\frac{\partial^2}{\partial t_i \partial t_j} f(t))_{1\le i,j\le d}$ of second partial derivatives of f. For $v \in S^{d-1}$, we write $\frac{\partial}{\partial v} f(t)$ for the directional derivative of f in direction v.

Let A_m^d, $m = 0,\dots,d$, denote the affine Grassmannian consisting of m-dimensional affine subspaces of $\mathbb{R}^d$ and let G_m^d, $m = 0,\dots,d$, denote the (linear) Grassmannian consisting of m-dimensional linear subspaces of $\mathbb{R}^d$. For $F \in A_m^d$, we write F° for the directional space of F, which is an element in G_m^d. We endow the spaces A_m^d and G_m^d with the trace topology of the Fell topology for the space $\mathrm{Cl}(\mathbb{R}^d)$ of closed subsets of $\mathbb{R}^d$, cf. [69, Section 12.2]. A more explicit and equivalent way to define the topology on these spaces is described in [69, Section 13.2]. The rotation invariant measure ν on G_m^d, cf. [69, Theorem 13.2.11], is normalized such that $\nu(G_m^d) = \left[{d \atop m}\right]$, where the flag coefficients are defined for $m = 1,\dots,d-1$ by

$$\begin{bmatrix} d \\ m \end{bmatrix} := \binom{d}{m} \frac{\omega_d}{\omega_m \omega_{d-m}} \quad \text{and} \quad \begin{bmatrix} d \\ d \end{bmatrix} := 1.$$

By μ we denote the rigid motion invariant measure on the affine Grassmannian A_m^d, which satisfies

$$\int_{A_m^d} f \, d\mu = \int_{G_m^d} \int_{L^\perp} f(L+y)\, \mathcal{H}^{d-m}(dy)\, \nu(dL) \tag{2.2}$$

for every μ-integrable function f on A_m^d, cf. [69, Theorem 13.2.12] where ν, and therefore μ, are normalized differently but the equality holds nevertheless. In order to avoid long terms in the calculations to come, we do not indicate the dependence of μ and ν on d and m. Two linear subspaces L, L' are said to be in general position if

$$\dim(L \cap L') = \max\{0, \dim L + \dim L' - d\}.$$

Let $F \in A_m^d$ and let $W \subset \mathbb{R}^d$ be an open convex subset of an affine subspace of $\mathbb{R}^d$ such that $\dim \operatorname{aff} W = l > d - m$ and moreover, F° and $(\operatorname{aff} W)^\circ$ are in general position. Then, we denote by $b(W,F) := b_F^W := (v_1,\dots,v_{m+l-d})$ an orthonormal basis of $(\operatorname{aff}(W) \cap F)^\circ$ and define the gradient of $f|_{W\cap F}$ as the vector field given by

$$\nabla(f|_{W\cap F})(t) := \sum_{i=1}^{m+l-d} \frac{\partial}{\partial v_i} f(t) v_i,$$

for $t \in W \cap F$, where $\frac{\partial}{\partial v_i}$ denotes the directional derivative in direction v_i. The second derivative of $f|_{W\cap F}$ in $t \in W \cap F$ is defined as the linear mapping on $(\operatorname{aff}(W) \cap F)^\circ$ given by

$$D^2(f|_{W\cap F})(t)(v) := \Big(v_1 \,\Big|\, \cdots \,\Big|\, v_{m+l-d}\Big) \left(\frac{\partial^2}{\partial v_i \partial v_j} f(t)\right)_{i,j=1}^{m+l-d} \Big(v_1 \,\Big|\, \cdots \,\Big|\, v_{m+l-d}\Big)^\top v,$$

for $v \in \operatorname{aff}(W \cap F)^\circ$. We note that these definitions coincide with the Riemannian ones and therefore do not depend on the choice of b_F^W. Indeed, if we choose the coordinate map $\varphi\colon W \cap F \to \mathbb{R}^{m+l-d}$ given by $t \mapsto (v_1|\cdots|v_{m+l-d})^\top t$ for the submanifold $W \cap F \subset \mathbb{R}^d$, equipped with the induced inner product, the Riemannian definitions of the gradient and the Hessian specialise to the ones already given. Moreover, we define for $t \in \operatorname{cl}(W \cap F)$

$$\nabla_{b_F^W} f\colon \mathbb{R}^d \to \mathbb{R}^{m+l-d}, \quad t \mapsto \left(\frac{\partial}{\partial v_i} f(t)\right)_{i=1}^{m+l-d}, \tag{2.3}$$

whose components are the coefficients of $\nabla(f|_{W\cap F})(t)$ in the basis b_F^W if $t \in W \cap F$, as well as

$$D^2_{b_F^W} f\colon \mathbb{R}^d \to \mathbb{R}^{(m+l-d)\times(m+l-d)}, \quad t \mapsto \left(\frac{\partial^2}{\partial v_i \partial v_j} f(t)\right)_{i,j=1}^{m+l-d}, \tag{2.4}$$

which is the transformation matrix of the mapping $D^2(f|_{W\cap F})(t)$ in the basis b_F^W if $t \in W \cap F$.

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ always denotes a probability space, where the probability measure $\mathbb{P}$ is assumed to be a complete measure on the σ-algebra $\mathcal{F}$. A real random variable N is said to have a normal distribution with parameters $\mu \in \mathbb{R}$ and $\sigma^2 \geq 0$ if its distribution admits the Lebesgue density

$$\phi(x) := (2\pi\sigma^2)^{-\frac{1}{2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R},$$

for $\sigma^2 > 0$, and in the case $\sigma^2 = 0$ its distribution is given by the Dirac measure δ_μ. A random variable N in $\mathbb{R}^d$ is said to have a d-dimensional normal distribution if for all $c \in \mathbb{R}^d$ the real random variable $\langle c, N\rangle$ is normally distributed. Then its distribution is denoted by $\mathcal{N}_d(\mu, \Sigma)$, where $\mu := \mathbb{E}[N]$ and $\Sigma := \mathbb{E}\left[(N-\mu)(N-\mu)^\top\right]$. Moreover, N is called nondegenerate if $\det \Sigma > 0$, and in this case $\mathcal{N}_d(\mu, \Sigma)$ has the Lebesgue density

$$\phi_d(x) := (2\pi)^{-\frac{d}{2}} \det \Sigma^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)\right), \quad x \in \mathbb{R}^d.$$

Given a measure space $(\Omega', \mathcal{A}, \mu)$, we write $L^2(\Omega', \mathcal{A}, \mu)$, or $L^2(\mu)$ for short, if there is no risk of ambiguity, for the set of all measurable functions $f\colon \Omega' \to \mathbb{R}$ with $\int_{\Omega'} f^2\, d\mu < \infty$, and identify functions that agree almost everywhere. Then the mapping $\langle\cdot\,,\cdot\rangle\colon L^2(\Omega', \mathcal{A}, \mu) \times L^2(\Omega', \mathcal{A}, \mu) \to \mathbb{R}$ given by

$$(f, g) \mapsto \int_{\Omega'} f \cdot g\, d\mu$$

defines an inner product on $L^2(\Omega', \mathcal{A}, \mu)$. Finally, given a measurable function $f\colon \mathbb{R}^d \to \mathbb{R}$, we define the measure $f\lambda^d$ by

$$\left[f\lambda^d\right](A) := \int_{\mathbb{R}^d} \mathbb{1}_A(x) f(x)\, \lambda^d(dx), \quad A \in \mathcal{B}(\mathbb{R}^d),$$

where dx denotes integration with respect to the Lebesgue measure λ^d.

2.1. Gaussian fields

In this section, we recall some basic definitions and results on random fields, especially Gaussian random fields, which lie at the heart of this thesis.

Let T be an index set and $(E, \mathcal{E})$ be a measurable space. Then E^I, $I \subset T$, denotes the space of all mappings $f\colon I \to E$. We equip E^I with the smallest σ-field $\otimes_{i\in I}\mathcal{E}$ such that the projections $\pi_t\colon E^I \to E$ given by $\pi_t(f) := f(t)$ are measurable, i.e.

$$\otimes_{i\in I}\mathcal{E} := \sigma\left(\left\{\pi_t^{-1}(B) \mid t \in I, B \in \mathcal{E}\right\}\right). \tag{2.5}$$

Then $(E^I, \otimes_{i\in I}\mathcal{E})$ is a measurable space.

Definition. A measurable mapping $X\colon (\Omega, \mathcal{F}, \mathbb{P}) \to (E^T, \otimes_{i\in T}\mathcal{E})$ is called an E-valued random field on T.

A random field X is called a version of the random field Y, if they satisfy the condition

$$\mathbb{P}(X(t) = Y(t)) = 1, \qquad \text{for any } t \in T.$$

For fixed $\omega \in \Omega$, any realisation, which is given by the mapping $X(\omega)\colon T \to E$, is called trajectory or path of X, and for $(t_1, \dots, t_n) \in T$, $n \in \mathbb{N}$, the distribution of

$$(X(t_1), \dots, X(t_n))$$

is called a finite-dimensional distribution of X at the points $t_1, \dots, t_n$. In practice we are often interested in random fields with specific finite-dimensional distributions. The question, which has to be asked is, whether there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random field $X\colon (\Omega, \mathcal{F}, \mathbb{P}) \to (E^T, \otimes_{i\in T}\mathcal{E})$ such that X has the prescribed finite-dimensional distributions. To give an answer, we introduce the following concepts.

A measurable space $(E, \mathcal{E})$ is a called Borel space, if there is a bijective, Borel-measurable mapping $\varphi\colon E \to D$, $D \subset [0, 1]$ and $D \in \mathcal{B}(\mathbb{R}^d)$, with Borel-measurable inverse. We note here that every Borel subspace of a Polish space (a separable topological space with a complete metrization) is a Borel space, cf. [36, Theorem A1.2]. Furthermore, for $J \subset I \subset T$, the restriction map

$$\pi_J^I\colon E^I \to E^J, \quad f \mapsto f|_J,$$

is called the canonical projection from I to J. We abbreviate π_I^T by π_I. A family of probability measures $(\mu_I,\ \text{finite } I \subset T)$ on the space $(E^I, \otimes_{i\in I}\mathcal{E})$ is called consistent if

$$\mu_I \circ \left(\pi_J^I\right)^{-1} = \mu_J, \qquad \text{for any finite } J \subset I \subset T.$$

Then the following existence theorem due to Kolmogorov, cf. [41], in the modern formulation of [40, Theorem 14.36] gives an answer to the above posed question.

Theorem 2.1. *Let T be an arbitrary index set and let E be a Borel space. Let $(\mu_I,\ \text{finite } I \subset T)$ be a consistent family of probability measures. Then there exists an unique probability measure μ on $(E^T, \otimes_{i\in T}\mathcal{E})$ with $\mu_I = \mu \circ \pi_I^{-1}$ for every finite set $I \subset T$.*

Thus, taking $(\Omega, \mathcal{F}, \mathbb{P}) := (E^T, \otimes_{i\in T}\mathcal{E}, \mu)$ and choosing the random field $X := \mathrm{id}_{E^T}$, settles the question of existence.

The general nature of the state space T can lead to problems concerning the measurability of events, in which we are definitely interested. For example, the preimage of a supremum of a real-valued random field X on $\mathbb{R}^d$ over the set I can be written in the following manner

$$\left\{\omega \in \Omega \mid \sup_{t\in I} X(\omega,t) \le u\right\} = \bigcap_{t\in I}\{\omega \in \Omega \mid X(\omega,t) \le u\}, \quad u \in \mathbb{R},$$

which is measurable if I is countable but leads to problems if the set I is uncountable. To ensure the measurability of events which depend on uncountably many points, we demand that all random fields appearing in this work satisfy the property called separability, originating from Doob [17, 2.§2].

Definition. Let $d \in \mathbb{N}$. An $\mathbb{R}^d$-valued random field X on a topological space T is called separable if there exists a countable dense subset $D \subset T$ and an event $N \in \mathcal{F}$ of probability 0, such that for any closed $B \subset \mathbb{R}^d$ and open $I \subset T$

$$\{\omega \in \Omega \mid X(\omega,t) \in B\, \forall t \in I\} \Delta \{\omega \in \Omega \mid X(\omega,t) \in B\, \forall t \in D \cap I\} \subset N,$$

where Δ denotes the symmetric difference, i.e. $A\Delta B = (A \cap B^c) \cup (A^c \cap B)$.

The existence of a separable version of a real valued field X on a separable metric space T is established e.g. in [62, Theorem 2.6].

We now define the central object of this thesis, namely the notion of a Gaussian random field. Let T be an arbitrary topological space and let $I = \{t_1, \dots, t_{|I|}\} \subset T$ be finite. With every subset I we associate a vector $a^I = (a_i^I)_{i=1}^{|I|} \in \mathbb{R}^{d|I|}$, $a_i^I \in \mathbb{R}^d$, and a symmetric and positive semidefinite matrix $\Sigma^I = (\Sigma_{ij}^I)_{i,j=1}^{|I|} \in \mathbb{R}^{d|I|\times d|I|}$, $\Sigma_{ij}^I \in \mathbb{R}^{d\times d}$. Then

$$\mu_I := \mathcal{N}_{|I|}(a^I, \Sigma^I), \quad I \subset T \text{ finite},$$

defines a family of measures, where each measure μ_I is defined on $\mathbb{R}^{d|I|} \cong (\mathbb{R}^d)^I$. We note that a bijection is given by the mapping $(f\colon I \to \mathbb{R}^d) \mapsto (f(t_1), \dots, f(t_{|I|}))$. Now, let $J = \{t_{j_1}, \dots, t_{j_{|J|}}\} \subset I$ such that $j_1 < \ldots < j_{|J|}$. By properties of the normal distribution

$$\mu_I \circ (\pi_J^I)^{-1} = \mathcal{N}_{|I|}(a^I, \Sigma^I) \circ (\pi_J^I)^{-1} = \mathcal{N}_{|J|}(a^I(J), \Sigma^I(J)),$$

where $a^I(J) := (a_{j_i}^I)_{1\le i\le |J|}$ and $\Sigma^I(J) := (\Sigma_{j_r j_s}^I)_{1\le r,s\le |J|}$. Therefore, the family of probability measures $(\mu_I,\ \text{finite } I \subset T)$ is consistent, if

$$a^I(J) = a^J \text{ and } \Sigma^I(J) = \Sigma^J,$$

for all finite $J \subset I \subset T$. This is the case if and only if there are functions

$$\begin{aligned}&a\colon T \to \mathbb{R}^d \quad \text{(mean function)}\\ &C\colon T \times T \to \mathbb{R}^{d\times d} \quad \text{symmetric and positive semidefinite (covariance function)}\end{aligned}$$

such that $a^I = (a(t))_{t\in I}$ and $\Sigma^I = (C(t_1,t_2))_{t_1,t_2\in I}$. Then by Kolmogorov's existence theorem, cf. Theorem 2.1, there exists a random field in $\mathbb{R}^T$, whose finite-dimensional distributions are given by μ_I, $I \subset T$ finite.

Definition. An $\mathbb{R}^d$-valued Gaussian random field X on the topological space T with mean function $m\colon T \to \mathbb{R}^d$ and symmetric and positive semidefinite covariance function $C\colon T\times T \to \mathbb{R}^{d\times d}$ is a random field $X\colon (\Omega,\mathcal{F},\mathbb{P}) \to ((\mathbb{R}^d)^T, \otimes_{t\in T}\mathcal{B}(\mathbb{R}^d))$ such that the finite-dimensional distributions $(X(t_1),\dots,X(t_n))$ are given by $\mathcal{N}_n\left((a(t_i)_{i=1}^n), (C(t_i,t_j))_{i,j=1}^n\right)$, $t_1,\dots,t_n \in T$, $n \in \mathbb{N}$. A Gaussian field with everywhere vanishing mean function is called a centered Gaussian field.

As an immediate consequence of the definition of a Gaussian field, we obtain the fact that its finite dimensional distributions are uniquely determined once we specify its mean and covariance function. We note that the mean and covariance functions satisfy the relations

$$\begin{aligned}m(t) &= \mathbb{E}\left[X(t)\right], \quad t\in T,\\ C(t_1,t_2) &= \mathbb{E}\left[(X(t_1) - \mathbb{E}\left[X(t_1)\right])(X(t_2)-\mathbb{E}\left[X(t_2)\right])^\top\right], \quad t_1,t_2\in T,\end{aligned}$$

and therefore can be defined for any random field. Since the two functions determine the finite dimensional distributions of a Gaussian field, it is natural to ask, which properties of these functions imply specific pathwise properties, such as differentiability of the paths. The investigation of differentiability of a real-valued Gaussian field $\{X_t\colon \Omega \to \mathbb{R} \mid t \in \mathbb{R}^d\}$ requires knowledge about the existence of limits given by

$$\lim_{h\to 0} \frac{X(t+hu) - X(t)}{h}, \quad t \in \mathbb{R}^d, u \in S^{d-1}, \tag{2.6}$$

in the case of directional derivatives, or about the condition that there exists a random vector $\nabla X(t_0) \in \mathbb{R}^d$, $t_0 \in \mathbb{R}^d$, such that

$$\lim_{t\to t_0} \frac{X(t) - X(t_0) - \langle \nabla X(t_0), t - t_0\rangle}{\|t-t_0\|} = 0, \tag{2.7}$$

in the case of (total) differentiability. But the above expressions in the limits are random variables and therefore we can examine their existence in several ways. Two specific ones are important for us. The Gaussian field X is said to possess mean square directional derivatives at $t_0 \in \mathbb{R}^d$ in direction $u \in S^{d-1}$ if the limit in (2.6) exists in the L^2-sense. If the limit in (2.7) exists in the L^2-sense, then it is called mean square differentiable in $t_0 \in \mathbb{R}^d$. Analogously, we define the two properties in the almost sure sense. The following criterion is known in the case of almost sure differentiability, cf. [63, Corollary 4.4].

Theorem 2.2. *Let X be a real-valued, centered Gaussian field on $\mathbb{R}^d$ such that for all $i,j \in \{1,\dots,d\}$ the partial derivative $\frac{\partial^2}{\partial s_i \partial t_j} C(s,t)$ exists for all $s,t \in \mathbb{R}^d$ and is continuous as a function on $\mathbb{R}^d \times \mathbb{R}^d$. If there exist $K,\rho,\gamma > 0$ such that for all $i \in \{1,\dots,d\}$, and all $x,y \in \mathbb{R}^d$ with $\|x-y\| < \rho$,*

$$\left(\frac{\partial^2}{\partial s_i \partial t_i} C(x,x) - 2\frac{\partial^2}{\partial s_i \partial t_i} C(x,y) + \frac{\partial^2}{\partial s_i \partial t_i} C(y,y) \right)^2 \leq K\|x-y\|^\gamma$$

holds, then there is a version of X, which is almost surely differentiable. Moreover ∇X is a centered Gaussian field with covariance function $\left(\frac{\partial^2}{\partial s_i \partial t_j} C\right)_{i,j=1}^d$.

For differentiability of higher order this theorem can be applied repeatedly. Results giving less specific answers with accordingly less restrictive conditions, e.g. the existence of mean square directional derivatives, can also be found in the article [63]. For the converse statement of Theorem 2.2 it is enough to assume that the Gaussian field possesses mean square partial derivatives and the partial derivatives are mean square continuous. That is, a centered Gaussian field with mean square partial derivatives and mean square continuous mean square partial derivatives has a covariance function with existing and continuous partial derivatives $\frac{\partial^2}{\partial s_i \partial t_j} C(s,t)$, $i,j = 1,\dots,d$. The existence of the derivatives can be seen directly, since

$$\begin{aligned}\infty > \mathbb{E}\left[\frac{\partial}{\partial t_i} X(t) \frac{\partial}{\partial t_j} X(s)\right] &= \lim_{h,h' \to 0} \mathbb{E}\left[\left(\frac{X(t+he_i) - X(t)}{h}\right)\left(\frac{X(s+h'e_j) - X(s)}{h'}\right)\right] \\ &= \frac{\partial^2}{\partial s_j \partial t_i} C(t,t'),\end{aligned}$$

for $t,s \in \mathbb{R}^d$, whereas the continuity of the derivatives $\frac{\partial^2}{\partial s_i \partial t_j} C(s,t)$, $i,j = 1,\dots,d$, is derived in [67, Theorem 2.3.2].

Another basic assumption on the fields examined in this thesis is that of homogeneity.

Definition. A random field X on $\mathbb{R}^d$ is called stationary if for all $h, t_1,\dots,t_n \in \mathbb{R}^d$ and $n \in \mathbb{N}$

$$(X(t_1+h),\dots,X(t_n+h)) \stackrel{\mathcal{D}}{=} (X(t_1),\dots,X(t_n)).$$

Furthermore, X is said to be isotropic, if for all $\rho \in SO(d)$, $t_1,\dots,t_n \in \mathbb{R}^d$ and $n \in \mathbb{N}$

$$(X(\rho(t_1)),\dots,X(\rho(t_n))) \stackrel{\mathcal{D}}{=} (X(t_1),\dots,X(t_n)).$$

In the Gaussian case, a field is stationary if and only if the mean function is constant and the covariance function $C(s,t)$, $s,t \in \mathbb{R}^d$, is only a function of the difference $s-t$. We note that stationarity, defined as above, implies for any random field that its covariance function is a function of $s-t$ only, whereas the converse implication is not true for any field. Isotropy, in the case of a stationary process, can be characterised by the fact that the mean function is constant and the covariance function depends only on the norm of its argument.

If the field is stationary, Gaussian and centered the covariance function, i.e. $C(s-t) := C(s,t)$, carries all relevant information to determine the distribution. Then Bochner's theorem becomes important, cf. [67, Theorem 1.7.4].

Theorem 2.3. *A continuous complex-valued function f on $\mathbb{R}^d$ is positive semidefinite if and only if it can be represented in the form*

$$f(t) = \int_{\mathbb{R}^d} e^{i\langle t,x\rangle}\, \mu(dx), \quad t \in \mathbb{R}^d,$$

with some nonnegative finite Borel measure μ on $\mathbb{R}^d$. The measure μ is uniquely determined by f.

Theorem 2.3 written in terms of random fields takes the following form, cf. [67, Theorem 2.9.3].

Theorem 2.4. *A continuous real-valued function C on $\mathbb{R}^d$ is the covariance function of a continuous, centered, stationary random field on $\mathbb{R}^d$ if and only if it allows the representation*

$$C(t) = \int_{\mathbb{R}^d} e^{i\langle t,x\rangle}\, \mu(dx), \quad t \in \mathbb{R}^d,$$

with some nonnegative finite Borel measure μ on $\mathbb{R}^d$.

If the measure μ admits a Lebesgue density f, then f is called the spectral density of X. In the setting of a stationary random field X, we have $C(t) = C(-t)$, $t \in \mathbb{R}^d$, and thus obtain $\mu(A) = \mu(-A)$, $A \in \mathcal{B}(\mathbb{R}^d)$. This yields

$$\int_{\mathbb{R}^d} x_1^{i_1} \cdots x_d^{i_d}\, \mu(dx) = 0 \quad \text{if} \quad \sum_{j=1}^{d} i_j \text{ is odd,} \tag{2.8}$$

and if we further assume that X is almost surely smooth, then equation (2.8) together with

$$\mathbb{E}\left[\frac{\partial^{\alpha+\beta}}{\partial^\alpha t_i \partial^\beta t_j} X(t) \frac{\partial^{\gamma+\delta}}{\partial^\gamma t_k \partial^\delta t_l} X(t)\right] = (-1)^{\alpha+\beta} i^{\alpha+\beta+\gamma+\delta} \int_{\mathbb{R}^d} x_i^\alpha x_j^\beta x_k^\gamma x_l^\delta\, \mu(dx), \quad t \in \mathbb{R}^d, \tag{2.9}$$

cf. [1, (5.5.5.)], implies that X and its first derivatives as well as the first derivatives and the second derivatives are uncorrelated at equal times. Moreover, if we additionally assume isotropy, we obtain $\mu(A) = \mu(\rho(A))$, $A \in \mathcal{B}(\mathbb{R}^d)$, $\rho \in SO(d)$, and therefore

$$\mathbb{E}\left[\frac{\partial}{\partial t_i} X(t) \frac{\partial}{\partial t_j} X(t)\right] = \mathbb{1}\{i = j\}\mathbb{E}\left[\left(\frac{\partial}{\partial t_1} X(0)\right)^2\right]. \tag{2.10}$$

We note that in the important case of a stationary Gaussian random field X the fields given by the partial derivatives of X are again stationary and Gaussian yielding that the preceding random variables, e.g. $X(t)$, $\frac{\partial}{\partial t_i} X(t)$, $\frac{\partial^2}{\partial t_i \partial t_j} X(t)$, are centered and normally distributed and therefore, if they are uncorrelated they are independent.

As a basic tool, in the analysis of moments of random counting variables depending on random fields, we will make use of the famous Rice formulas. In the following, we state them as they appear in the book of Azaïs and Wschebor, cf. [4, Chapter 6].

Theorem 2.5. *Let $X\colon U \to \mathbb{R}^d$ be a random field, U an open subset of $\mathbb{R}^d$, and $y \in \mathbb{R}^d$. We assume that*

(i) *X is a centered Gaussian field,*

(ii) *almost surely the trajectories of X are of class $\mathcal{C}^1$,*

(iii) *for any $t \in U$, the matrix $\mathbb{E}\left[X(t)X(t)^\top\right]$ is positive definite,*

(iv) $\mathbb{P}(\exists t \in U : X(t) = y, \det DX(t) = 0) = 0.$

Then, for every Borel set B contained in U, we have

$$\mathbb{E}\left[\#\{t \in B \mid X(t) = y\}\right] = \int_B \mathbb{E}\left[|\det DX(t)| \mid X(t) = y\right] p_{X(t)}(y)\, dt,$$

where $p_{X(t)}(x) := (2\pi C(t)^2)^{-\frac{1}{2}} \exp(-\frac{x^2}{2C(t)^2})$ denotes the density of $X(t)$. Moreover, if B is compact, then both sides are finite.

The next formula, also known under the name of Rice formula, gives an expression for higher factorial moments. By using this version and the previous one, it is possible to analyse second moments, which will play a crucial role in this thesis.

Theorem 2.6. *Let $k \geq 2$ be an integer and assume* (i), (ii) *and* (iv) *as in Theorem 2.5 together with*

(iii') *for pairwise different $t_1, \dots, t_k \in U$, the distribution of*

$$(X(t_1), \dots, X(t_k))$$

is nondegenerate in $(\mathbb{R}^d)^k$.

Then, for every Borel set B contained in U, we have

$$\begin{aligned}&\mathbb{E}\left[\#\{t \in B \mid X(t) = y\}(\#\{t \in B \mid X(t) = y\} - 1) \cdots (\#\{t \in B \mid X(t) = y\} - k + 1)\right] \\ &= \int_{B^k} \mathbb{E}\left[\prod_{i=1}^k |\det DX(t_i)| \mid X(t_1) = \ldots = X(t_k) = y\right] p_{X(t_1),\dots,X(t_k)}(y, \dots, y)\, d(t_1, \dots, t_k),\end{aligned}$$

where $p_{X(t_1),\dots,X(t_k)}$ denotes the density of $(X(t_1), \dots, X(t_k))$. Both sides may be infinite.

For verification of assumption (iv) in Theorem 2.5 and 2.6 we state the following Lemma, cf. [4, Proposition 6.5].

Lemma 2.7. *Let $y \in \mathbb{R}^d$, let $U \subset \mathbb{R}^d$ be compact and let $X\colon \mathbb{R}^d \to \mathbb{R}^d$ be a random field with paths of class $\mathcal{C}^2$. We assume that $p_{X(t)}(x) \leq c$ for a constant $c > 0$, for all $t \in U$ and for x in some neighborhood of y. Then assumption* (iv) *in Theorem 2.5 and Theorem 2.6 is satisfied.*

2.2. Morse theory

In this section, we summarize the critical point theory on stratified spaces introduced in Chapters 8 and 9 by Adler and Taylor [1]. The aim of this theory is to describe global characteristics of a given stratified space via the local behaviour of functions defined on it. We start with the definition of stratified spaces.

Definition. Let $k \in \mathbb{N}$. A $\mathcal{C}^k$ stratified space or stratified manifold $M \subset \mathbb{R}^d$ is a subset of $\mathbb{R}^d$ together with a finite partition $\mathcal{Z}$ of M such that

(i) each stratum $S \in \mathcal{Z}$ is an embedded $\mathcal{C}^k$ submanifold of $\mathbb{R}^d$,

(ii) if $R, S \in \mathcal{Z}$ where $R \cap \operatorname{cl} S \neq \emptyset$, then $R \subset \operatorname{cl} S$.

We note that a generalisation of this concept to locally finite partitions is possible but not needed in this work. By $\partial_l M$, $l = 0, \ldots, \dim M$, where $\dim M := \max_{S \in \mathcal{Z}} \dim S$, we denote the collection of all l-dimensional strata in $\mathcal{Z}$.

As a generic example, we stratify the cube C_N^d into its relatively open faces. In contrast to the viewpoint of convex geometry, where faces are always closed, the strata here do not have a boundary. Thus for $J_N \in \partial_l C_N^d$ there exists a set $\sigma(J_N) \subset \{1, \ldots, d\}$ with $|\sigma(J_N)| = l$ and a sequence $(\epsilon_j)_{j \in \{0,\ldots,d\} \setminus \sigma(J_N)}$ in $\{1, -1\}^{d-l}$ so that

$$J_N = \Big\{ t \in C_N^d \mid -N < t_i < N \text{ for } i \in \sigma(J_N), t_i = \epsilon_i N \text{ for } i \notin \sigma(J_N) \Big\}. \tag{2.11}$$

Moreover, the strata of this stratification lie nicely with respect to each other in the ambient space $\mathbb{R}^d$ (in the sense of the next definition), which is a property we would always like to have. Therefore, we formulate the following conditions, which are known as Whitney's conditions (A) and (B).

Definition. Let $M \subset \mathbb{R}^d$ be a stratified space with stratification $\mathcal{Z}$. Then M satisfies Whitney's condition (A) if for any strata $R, S \in \mathcal{Z}$, where $R \subset \operatorname{cl} S$ the following holds: For any $x \in R$ and any sequence $(x_n)_{n \in \mathbb{N}}$ in S such that $x_n \to x$ and $T_{x_n} S \to T$ in $G^d_{\dim S}$, we have $T_x R \subset T$. Moreover, M is said to satisfy Whitney's condition (B), if for any strata $R, S \in \mathcal{Z}$, with $R \subset \operatorname{cl} S$ the following holds: For any sequence $(x_n)_{n \in \mathbb{N}}$ as above, which satisfies the additional constraint that the line $\overline{x x_n}$ converges in G_1^d to G, we have $G \subset T$. A stratified space satisfying Whitney's conditions (A) and (B) is called a Whitney stratified space.

We are now in the position to define a characteristic of stratified manifolds, which lies at the heart of Chapter 3, namely the Euler characteristic.

Definition. A triangulation of a compact Whitney stratified space M is a covering of M by diffeomorphic images of simplices of dimension smaller or equal to the dimension of M such that if two images are not disjoint, then the preimages of the intersection must be faces of the corresponding simplices. By identifying the faces of the simplices, whose images have nonempty intersection, we obtain a simplicial complex, cf. [54, Definition 2.3.5], induced by the triangulation of M, say S_M. Then the Euler characteristic χ is defined by

$$\chi(M) := \sum_{j=0}^{\dim M} (-1)^j \alpha_j(S_M),$$

where $\alpha_j(S_M)$ is the number of j-dimensional faces in S_M.

We note that per se it is not clear, whether a compact Whitney stratification admits a triangulation corresponding to a finite simplicial complex. However, this is the main result in [35, Theorem 2.1].

In the following, we reduce the set of possible stratified manifolds even further by requiring the property of local convexity, which is defined as follows:

Definition. A Whitney stratified space $M \subset \mathbb{R}^d$ is called locally convex if the support cone S_tM is convex for any $t \in M$, where

$$\begin{aligned} S_tM := \{x \in T_t\mathbb{R}^d \mid \exists \delta > 0, c\colon \mathbb{R} \to \mathbb{R}^d \text{ with } c \in \mathcal{C}^1 \\ \text{such that } c(0) = t, c'(t) = x, c(s) \in M \text{ for all } s \in [0, \delta)\}. \end{aligned}$$

This definition excludes sets with concave cusps and therefore with no positive reach, cf. Definition (2.16). If S denotes a stratum of the stratified space $M \subset \mathbb{R}^d$, the support cone in the point $t \in S$ contains as a subset the tangent space of S in point t. We would end up with the tangent space, if we changed the condition $s \in [0, \delta)$ in the preceding definition to $s \in (-\delta, \delta)$. We note that the support cone defined in this way coincides with the support cone of convex geometry in the case, in which M is a convex subset of $\mathbb{R}^d$, cf. [68, Section 2.2]. The dual cone of the support cone is called the normal cone N_tM, i.e.

$$N_tM := \{X \in T_t\mathbb{R}^d \mid \langle X, Y\rangle \leq 0 \text{ for all } Y \in S_tM\}, \quad t \in M,$$

which plays a crucial role in the formulation of the Morse lemma.

In this thesis the stratified manifold $C_N^d \cap F$, for $F \in A_{d-m}^d$, with the stratification given by the strata $J_N \cap F$, $J_N \in \partial_l C_N^d$, $m \leq l \leq d$, will be of utmost importance. Therefore, we will have a look at its normal cones and in a special case at the condition $\nabla X(t) \in N_t(C_N^d \cap F)$, where $X\colon \mathbb{R}^d \to \mathbb{R}$ and $t \in J_N \cap F$, such that for $F \in A_{d-m}^d$ the linear spaces $\operatorname{aff}(J_N)^\circ$ and F° are in general position. By [68, (2.5)], we obtain

$$N_t(C_N^d \cap F) = N_{F^\circ}((C_N^d \cap F) - t, 0) + (F^\circ)^\perp \subset (\operatorname{aff}(J_N)^\circ \cap F^\circ)^\perp,$$

where $N_L(K, t)$ denotes the normal cone of the convex subset $K \subset L$ in the linear subspace L at the point $t \in K$. The subset relation holds since $\operatorname{aff}(J_N)^\circ \cap F^\circ \subset S_t(C_N^d \cap F)$ yields

$$N_t(C_N^d \cap F) = \operatorname{dual}(S_t(C_N^d \cap F)) \subset \operatorname{dual}(\operatorname{aff}(J_N)^\circ \cap F^\circ) = (\operatorname{aff}(J_N)^\circ \cap F^\circ)^\perp, \tag{2.12}$$

where $\operatorname{dual}(K)$ denotes the dual convex cone of the convex set K, cf. [68, page 35]. Moreover by [68, (2.25)]

$$\begin{aligned}\dim(N_t(C_N^d \cap F)) &= d - \dim(J_N \cap F) = d - (l - m) \\ &= d - \dim(\operatorname{aff}(J_N)^\circ \cap F^\circ) = \dim((\operatorname{aff}(J_N)^\circ \cap F^\circ)^\perp),\end{aligned} \tag{2.13}$$

thus $N_t(C_N^d \cap F)$ is full dimensional in $(\operatorname{aff}(J_N)^\circ \cap F^\circ)^\perp$. The result [68, Theorem 2.4.9] allows for the following more explicit representation

$$\begin{aligned}N_{F^\circ}((C_N^d \cap F) - t, 0) = \operatorname{pos}\{n_i^{J_N}(F) \mid n_i^{J_N}(F) \text{ outer unit normal vector of facet } S_i \text{ of the set}\\ C_N^d \cap F \text{ in } F \text{ with } J_N \cap F \subset S_i\},\end{aligned} \tag{2.14}$$

from which we deduce that the normal cone $N_t(C_N^d \cap F)$ is independent of the specific choice of $t \in J_N$. Using the decomposition $\nabla X(t) = \nabla(X|_{J_N \cap F})(t) + \pi_{(\operatorname{aff}(J_N)^\circ \cap F^\circ)^\perp}(\nabla X(t))$, we obtain the equivalence

$$\begin{aligned}&\nabla(X|_{J_N \cap F})(t) = 0 \quad \text{and} \quad \nabla X(t) \in N_t(C_N^d \cap F)\\ &\Leftrightarrow \nabla(X|_{J_N \cap F})(t) = 0 \quad \text{and} \quad \pi_{(\operatorname{aff}(J_N)^\circ \cap F^\circ)^\perp}(\nabla X(t)) \in N_t(C_N^d \cap F).\end{aligned} \tag{2.15}$$

The next topic introduced is known under the name of cone spaces, cf. [61, Section 3.10] for further reference, and is required for Adler and Taylor's proof of the Morse lemma for excursion sets.

Definition. Let $M \subset \mathbb{R}^d$ be a C^l, $l \in \mathbb{N}$, stratified space with stratification $\mathcal{Z}$. Then M is said to be a cone space of class C^l and depth 0 if it is the topological sum of countably many connected C^l manifolds, the strata S of which are the unions of connected components of equal dimension. A stratified space $M \subset \mathbb{R}^d$ is said to be a cone space of class $C^{l,m}$, $m \geq 0$, and depth $d+1$ $(d \geq 0)$, if every $t \in S \in \mathcal{Z}$ has a neighborhood $U \subset \mathbb{R}^d$ such that $U \cap M$ is C^m diffeomorphic to $(U \cap S) \times \operatorname{Cone}(L_S)$, where L_S is a compact C^l cone space of depth d, and $\operatorname{Cone}(L_S)$ denotes the cone generated by L_S. When $m = 0$, "C^m diffeomorphic" means homeomorphic.

To get acquainted with this Definition, consider the example of a polytope P with a stratification given by the open faces. Then P is a cone space of class $\mathcal{C}^\infty$ and depth 0, since the open faces are connected $\mathcal{C}^\infty$ submanifolds of $\mathbb{R}^d$. To see that P is also a $C^{2,1}$ cone space of depth 1, let $U \subset \mathbb{R}^d$ be a neighborhood of $t \in F$, where F is an arbitrary face of P. Then $U \cap P$ is diffeomorphic to $(U \cap F) \times \operatorname{Cone}(S^{d-1} \cap N_t P)$, where $S^{d-1} \cap N_t P$ is itself a cone space of class $\mathcal{C}^\infty$ and depth 0, since it is a spherically convex set.

Definition. A closed, locally convex Whitney stratified manifold $M \subset \mathbb{R}^d$ is said to be tame, if for any stratum S, where $\dim S > 0$, the set

$$\left\{\lim_{t_n \to t} T_{t_n} S \mid (t_n)_{n \in \mathbb{N}} \text{ in } S \text{ such that } t_n \to t \in \operatorname{bd} S\right\}$$

has Hausdorff dimension $< \dim S$ in the affine Grassmannian $A^d_{\dim S}$.

Again, we examine the special case of a polytope P, which is stratified into its open faces. Then for any open face F the linearity of F implies that the set $\{\lim_{t_n \to t} T_{t_n} F \mid t \in \operatorname{bd} F\}$ is exactly $\operatorname{aff}(F)^\circ$, yielding that P is tame.

We summarize the introduced terminology in one notion.

Definition. Let M be a locally convex C^2 Whitney stratified manifold. We assume further that M is a $C^{2,1}$ cone space of arbitrary depth and that M is tame. Then M is called a regular stratified manifold.

The class of regular stratified manifolds is the class of spaces for which we formulate the Morse theorem for excursion sets. For this thesis the most important example for such a set is the intersection $C_N^d \cap F$, for $F \in A_{d-m}^d$, which fits into the theory described above, since it is a polytope, cf. [1, Chapter 8.3].

However, we first need to define the concept of Morse functions and start with recalling some basic definitions. Let $M \subset \mathbb{R}^d$ be a C^2 stratified manifold, let $S \subset M$ be a stratum and let $f\colon \mathbb{R}^d \to \mathbb{R}$ of class $\mathcal{C}^2$ be given. Then a point $t \in S$ is a critical point of $f|_S$ if $\nabla(f|_S)(t) = \pi_{T_t S}(\nabla f(t)) = 0$. Therefore, if $\dim S = 0$ then every point $t \in S$ is a critical point, since in this case $T_t S = \{0\}$. Moreover, a critical point $t \in S$ of $f|_S$ is said to be nondegenerate, if the Hessian $D^2 f|_{T_t S}(t)$ is nondegenerate, considered as a bilinear mapping from $T_t S$ into $T_t S$. Furthermore, the function f is said to be nondegenerate on M, if all critical points of the mappings $f|_S$, $S \in \cup_{i=0}^{\dim M} \partial_i M$, are nondegenerate. A point $u \in \mathbb{R}$ is called a regular value of $f|_S$ if for all $t \in (f|_S)^{-1}(\{u\})$ the point t is not a critical point of $f|_S$.

Definition. Let M be a compact, regular stratified manifold. A function $f\colon \mathbb{R}^d \to \mathbb{R}$ of class $\mathcal{C}^2$ is called a Morse function on M, if it satisfies the following conditions:

(i) f is nondegenerate on M.

(ii) For any stratum $S \subset M$, $\dim S > 0$, and any point $t \in \operatorname{bd} S$, we have $\pi_{T_{\lim} S}(\nabla f(t)) \neq 0$, where $T_{\lim} S$ is any limit of sequences $T_{t_n} S$ in $G_{\dim S}^d$ with $(t_n)_{n \in \mathbb{N}}$ a sequence in S such that $t_n \to t$.

Moreover, for a stratum S of M and $t \in S$, we denote by $\iota_S^f(t)$ the dimension of the largest subspace L of $T_t S$ such that $D^2(f|_L)(t)$ is negative definite.

Finally we are able to formulate the Morse lemma for excursion sets on regular stratified spaces, cf. [1, Corollary 9.3.5]. We note that $M \cap f^{-1}[u, \infty)$ in the More lemma is Whitney stratified since $u \in \mathbb{R}$ is a regular value of $f|_S$, $S \in \cup_{i=1}^{\dim M} \partial_i M$, cf. [28, Definition 1.3.1].

Theorem 2.8. *Let $M \subset \mathbb{R}^d$ be a compact, regular stratified manifold and let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a Morse function on M. Moreover, let $u \in \mathbb{R}$ be a regular value of $f|_S$, $S \in \cup_{i=1}^{\dim M} \partial_i M$. Then*

$$\begin{aligned}
&\chi(M \cap f^{-1}[u, \infty)) \\
&= \sum_{i=0}^{\dim M} \sum_{S \in \partial_i M} \#\{t \in S \mid f(t) \geq u, \nabla(f|_S)(t) = 0, \iota_S^{-f}(t) \text{ even}, \nabla f(t) \in N_t(M)\} \\
&\qquad - \#\{t \in S \mid f(t) \geq u, \nabla(f|_S)(t) = 0, \iota_S^{-f}(t) \text{ odd}, \nabla f(t) \in N_t(M)\}.
\end{aligned}$$

2.3. Lipschitz–Killing curvatures

In this section, we introduce the geometric functionals known as Lipschitz–Killing curvatures, which we will use to study the excursion sets of Gaussian fields.

Let $A \subset \mathbb{R}^d$ be given. Then by $\mathrm{Unp}(A)$, we denote the set of all points $x \in \mathbb{R}^d$ for which there is a unique point $\pi_A(x)$ in A such that

$$\inf\{\|x-a\| \mid a \in A\} = \|\pi_A(x) - x\|.$$

We call $\pi_A\colon \mathrm{Unp}(A) \to \mathbb{R}^d$ the metric projection onto A. For any $a \in A$, we define

$$\mathrm{reach}(A, a) := \sup\{r \geq 0 \mid B_r^d + a \subset \mathrm{Unp}(A)\}$$

as the reach of the set A in the point a. The corresponding global notion, the reach of the set A, is then defined by

$$\mathrm{reach}(A) := \inf\{\mathrm{reach}(A, a) \mid a \in A\}.$$

The set A is said to have positive reach, if

$$\mathrm{reach}(A) > 0. \tag{2.16}$$

The most important example of a set of positive reach for this work is the set $C_N^d \cap X^{-1}([u,\infty))$, which has almost surely positive reach if the Gaussian field X satisfies some regularity conditions, specified in Lemma A.1.

We now define the Lipschitz–Killing curvatures of a set by means of the Steiner formula in [24, 5.6 Theorem], where Federer proved the existence of curvature measures for sets of positive reach.

Definition. Let $A \subset \mathbb{R}^d$ satisfy $\mathrm{reach}(A) > 0$. Then the Lipschitz–Killing curvatures $\mathcal{L}_i$, $i = 0, \ldots, d$, of A are defined as the coefficients in the polynomial expansion

$$\mathcal{H}^d(A + \varepsilon \,\mathrm{cl}\, B_1^d) = \sum_{i=0}^{d} \alpha_{d-i} \mathcal{L}_i(A) \varepsilon^{d-i}, \tag{2.17}$$

where $0 \leq \varepsilon < \mathrm{reach}(A)$ and $\alpha_k := \mathcal{H}^k(B_1^k)$.

It was also Federer, who showed in the same paper that

$$\mathcal{L}_0(A) = \chi(A) \quad \text{for compact sets } A \subset \mathbb{R}^d \text{ with } \mathrm{reach}(A) > 0, \tag{2.18}$$

cf. [24, 5.19 Federer]. This allows us to use the Morse theory described in Section 2.2 to determine $\mathcal{L}_0\left(C_N^d \cap X^{-1}([u,\infty))\right)$.

Moreover, Federer did not only prove the existence of these curvature notions, but also established a principal kinematic formula, cf. [24, 6.11 Theorem]. A special case, namely the

Crofton formula, will be a basic ingredient in the derivation of the central limit theorem in Chapter 3. This Crofton formula can be stated as follows, cf. [24, 6.13 Theorem].

Theorem 2.9. *Let* $A \subset \mathbb{R}^d$ *be compact and assume that* $\operatorname{reach}(A) > 0$*. Then for* $i = 0, \dots, d$ *and* $m = 0, \dots, d-i$

$$\mathcal{L}_{m+i}(A) = \int_{A^d_{d-m}} \mathcal{L}_i(A \cap F)\, \mu(dF).$$

In the setting of Theorem 2.9 the intersection $A \cap F$ is a set of positive reach for almost all $F \in A^d_{d-m}$, hence, implying that the integrand on the right side of the Crofton formula is well defined for almost all $F \in A^d_{d-m}$, cf. [24, 6.11 Theorem (i)].

Remark. We want to emphasize that although the definition of the Lipschitz–Killing curvatures in the book of Adler and Taylor [1] differs from the one given here, both curvature notions coincide if the set in question is regular enough, for instance if it is a set of positive reach. This can be deduced from the fact that both curvatures are determined as coefficients in the Steiner formula, where the one for the Lipschitz–Killing curvatures of Adler and Taylor can be found in [1, Theorem 10.5.6].

2.4. Isonormal Gaussian processes

In this section, we provide background information about the theory of isonormal Gaussian processes, including a central limit theorem, which lies at the heart of every normal approximation in this work.

2.4.1. Hermite polynomials

In the following we introduce the Hermite polynomials, which form the backbone of the L^2-theory for Gaussian processes.

Definition. Let $n \geq 0$ be an integer. Then the n-th Hermite polynomial is defined by

$$H_n\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto (-1)^n e^{\frac{x^2}{2}} \frac{\partial^n}{\partial x^n} e^{-\frac{x^2}{2}}.$$

For $x \in \mathbb{R}$, the first three polynomials are explicitly given by the expressions $H_0(x) = 1$, $H_1(x) = x$, $H_2(x) = x^2 - 1$. In the multivariate case, we define:

Definition. Let $n \in \mathbb{N}_0^d$ and moreover $d \in \mathbb{N}$. We define the multivariate Hermite polynomial $\widetilde{H}_n\colon \mathbb{R}^d \to \mathbb{R}$ by

$$\widetilde{H}_n(x) := \prod_{i=1}^d H_{n_i}(x_i), \quad x \in \mathbb{R}^d.$$

Moreover, we use the notation $|n| := \sum_{i=1}^d n_i$ as well as $n! := \prod_{i=1}^d n_i!$.

The Hermite polynomials satisfy the following properties. Proofs can be found for instance in [75, Chapter 5.5], [78, Chapter 6], [59, Proposition 1.4.2] and [34, Example E.9].

Lemma 2.10. (i) *For any* $n \geq 0$*:* $H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)$, $x \in \mathbb{R}$.

(ii) *For any* $n, m \geq 0$

$$\int_{\mathbb{R}} H_n(x)H_m(x)\phi(x)\,dx = \begin{cases} n!, & \text{if } n = m, \\ 0, & \text{otherwise.} \end{cases}$$

(iii) *The family* $\left\{\frac{1}{\sqrt{n!}}H_n \mid n \geq 0\right\}$ *is an orthonormal basis of* $L^2(\mathbb{R}, \mathcal{N}_1(0,1))$.

(iv) *For all* $c, x \in \mathbb{R}$, *we have pointwise* $e^{cx-c^2/2} = \sum_{n=0}^{\infty} \frac{c^n}{n!} H_n(x)$.

(v) *For any* $n \geq 0$ *and* $x \in \mathbb{R}$*:* $H_n(-x) = (-1)^n H_n(x)$.

(vi) *The family* $\left\{\frac{1}{\sqrt{n!}}\widetilde{H}_n \mid n \in \mathbb{N}_0^d\right\}$ *is an orthonormal basis of* $L^2(\mathbb{R}^d, \mathcal{N}_d(0, I_d))$.

2.4.2. Isonormal Gaussian processes

In this section, we summarize the notations and definitions of the relevant parts in stochastic analysis used in this thesis and therefore retrace the steps in the monograph by Olav Kallenberg, cf. [36, Chapter 13]. We start with the definition of isonormal Gaussian processes on Hilbert spaces and close with the famous Wiener chaos expansion of integrable functionals.

Definition. Let $\mathfrak{H}$ denote a real, separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$. An isonormal Gaussian process on $\mathfrak{H}$ is a real valued, centered Gaussian field W on $\mathfrak{H}$ such that

$$\mathbb{E}\left[W(g)W(h)\right] = \langle g, h \rangle_{\mathfrak{H}}.$$

For an explicit construction of such a process, we can proceed as follows. Let $e_1, e_2, \ldots \in \mathfrak{H}$ denote an orthonormal basis and let $Z_1, Z_2, \ldots$ denote independent standard normally distributed random variables. Then for any element $h = \sum_i \langle h, e_i \rangle_{\mathfrak{H}} e_i \in \mathfrak{H}$, we set $W(h) := \sum_i \langle h, e_i \rangle_{\mathfrak{H}} Z_i$, where the series converges in $L^2(\mathbb{P})$, since $\sum_i \langle h, e_i \rangle_{\mathfrak{H}}^2 = \|h\|_{\mathfrak{H}}^2 < \infty$. Moreover, by Levy's equivalence theorem, cf. [18, Theorem 9.7.1], the sum converges also in the almost sure sense. Then W is centered Gaussian by definition, cf. [67, Theorem 1.10.7], and we obtain for its covariance

$$\mathbb{E}\left[W(g)W(h)\right] = \sum_{i,j} \langle g, e_i \rangle_{\mathfrak{H}} \langle h, e_j \rangle_{\mathfrak{H}} \mathbb{E}\left[Z_i Z_j\right] = \sum_i \langle g, e_i \rangle_{\mathfrak{H}} \langle h, e_i \rangle_{\mathfrak{H}} = \langle g, h \rangle_{\mathfrak{H}},$$

for $g, h \in \mathfrak{H}$, establishing the desired result.

We also note that an isonormal process is unique in the sense that any other isonormal process has the same law.

Remark. An isonormal process W, as a mapping $h \mapsto W(h)$ is linear, which can be seen by the equation for $a, b \in \mathbb{R}$ and $g, h \in \mathfrak{H}$

$$\mathbb{E}\left[(aW(h) + bW(g) - W(ah + bg))^2\right] = 0.$$

We proceed with the introduction of multiple Wiener–Itô integrals I_n, also known as multiple stochastic integrals, with respect to an isonormal Gaussian process W on a real, separable Hilbert space $\mathfrak{H}$. Following Kallenberg [36, Chapter 13], we repeat the basic notions of tensor products of Hilbert spaces, without loss of generality, in the case $\mathfrak{H} := L^2(S, \mu)$, where (S, μ) is a measure space with μ an atom-free measure. Then $\mathfrak{H}^{\otimes n}$, $n \in \mathbb{N}$, the n-fold tensor product, can be identified with $L^2(S^n, \mu^{\otimes n})$, where $\mu^{\otimes n}$ denotes the n-fold product measure of μ, and the tensor product $h_1 \otimes \cdots \otimes h_n$ is then identified with the mapping $(a_1, \ldots, a_n) \mapsto h_1(a_1) \cdots h_n(a_n)$ for $(a_1, \ldots, a_n) \in S^n$. Moreover, for an orthonormal basis (e_i) of $\mathfrak{H}$ the tensor products $e_{k_1} \otimes \cdots \otimes e_{k_n}$, where $k_1, \ldots, k_n \in \mathbb{N}$, form an orthonormal basis in $\mathfrak{H}^{\otimes n}$. Furthermore, for $0 \leq r \leq \min\{p, q\}$, $p, q \in \mathbb{N}$, the r-th contraction of $g \in \mathfrak{H}^{\otimes q}$ and $h \in \mathfrak{H}^{\otimes p}$ is defined by

$$\begin{aligned} g \otimes_r h(a_1, \ldots, a_{q+p-2r}) = \int_{S^r} & g(x_1, \ldots, x_r, a_1, \ldots, a_{q-r}) \\ & \times h(x_1, \ldots, x_r, a_{q-r+1}, \ldots, a_{q+p-2r})\, \mu^r(dx_1, \ldots, dx_r), \end{aligned}$$

where $a_1, \ldots, a_{q+p-2r} \in S$, thus yielding an element in $\mathfrak{H}^{\otimes(q+p-2r)}$. To be consistent in the formulations to come, we define $\mathfrak{H}^{\otimes 0} := \mathbb{R}$. For more background information about tensor products of Hilbert spaces with a focus on probability theory, we recommend appendix E in the book [34] by Svante Janson.

The following existence and uniqueness result simultaneously defines the multiple stochastic integrals, cf. [36, Theorem 13.21].

Theorem 2.11. *Let W be an isonormal Gaussian process on a separable Hilbert space $\mathfrak{H}$. Then, for $n \in \mathbb{N}$ there exists a unique continuous linear mapping $I_n \colon \mathfrak{H}^{\otimes n} \to L^2(\mathbb{P})$ such that for pairwise orthogonal $h_1, \ldots, h_n \in \mathfrak{H}$*

$$I_n(h_1 \otimes \cdots \otimes h_n) = \prod_{i=1}^{n} W(h_i) \qquad \textit{almost surely,}$$

yielding the invariance of the mapping I_n with respect to permutations of the tensor products.

For consistency, we define I_0 as the identity mapping on $\mathbb{R}$. An important subspace of $\mathfrak{H}^{\otimes n}$ is given by the space $\mathfrak{H}^{\odot n}$ of symmetric tensor products, consisting of symmetric functions in the case of an L^2-space, i.e. of functions $f \in L^2(S^n, \mu^{\otimes n})$, such that $f = \widetilde{f}$, where

$$\widetilde{f}(a_1, \ldots, a_n) := \frac{1}{n!} \sum_{\sigma \in S_n} f(a_{\sigma(1)}, \ldots, a_{\sigma(n)}), \qquad a_1, \ldots, a_n \in S.$$

It is this subspace of symmetric tensor products, for which the multiple stochastic integrals are an isometry (up to a constant), as shown in the following Lemma, cf. [59, Proposition 2.7.5].

We note that the stochastic integrals just defined are the same on $\mathfrak{H}^{\odot n}$ as the ones in the book of Nourdin and Peccati [59] as is obvious by Section 2.7 of that monograph.

Lemma 2.12. *Let $\mathfrak{H}$ be a separable Hilbert space and let I_n, $n \geq 0$, denote the stochastic integrals introduced in Theorem 2.11. Then for $g \in \mathfrak{H}^{\odot p}$ and $h \in \mathfrak{H}^{\odot q}$*

$$\mathbb{E}\left[I_p(g)I_q(h)\right] = \begin{cases} p!\langle g, h\rangle_{\mathfrak{H}^{\otimes p}}, & \text{if } p = q, \\ 0, & \text{otherwise.} \end{cases}$$

The following connection to the Hermite polynomials, which is a generalisation of Theorem 2.11, makes the multiple stochastic integrals so valuable for us.

Theorem 2.13. *On a separable Hilbert space $\mathfrak{H}$, let W be an isonormal Gaussian process with associated multiple Wiener-Itô integrals $I_1, I_2, \ldots$ Then for any orthonormal elements $e_1, \ldots, e_m \in \mathfrak{H}$ and integers $n_1, \ldots, n_m \geq 1$ with $\sum_{i=1}^m n_i = q$, we have*

$$I_q(e_1^{\otimes n_1} \otimes \cdots \otimes e_m^{\otimes n_m}) = \prod_{i=1}^m H_{n_i}(W(e_i)).$$

Finally, we state the classical Wiener chaos expansion, which is a crucial ingredient in the technique applied in this thesis to derive central limit theorems. For this purpose, we define the n-th homogeneous chaos $\mathcal{H}_n$ as the closed subspace of $L^2(\mathbb{P}) = L^2(\Omega, \sigma(W), \mathbb{P})$ consisting of all integrals $I_n(h)$, $h \in \mathfrak{H}^{\otimes n}$.

Theorem 2.14. *On a separable Hilbert space $\mathfrak{H}$, let W be an isonormal Gaussian process with associated homogeneous chaos $\mathcal{H}_n$. Then the subspaces $\mathcal{H}_n$ are orthogonal, closed, linear subspaces of $L^2(\mathbb{P})$, satisfying*

$$L^2(\mathbb{P}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$

Furthermore, every $F \in L^2(\mathbb{P})$ has a unique almost sure representation

$$F = \sum_{i=0}^{\infty} I_n(g_n),$$

with symmetric kernels $g_n \in \mathfrak{H}^{\odot n}$, $n \geq 0$.

2.4.3. A multivariate central limit theorem for isonormal Gaussian processes

In this section, we provide a multivariate version of Theorem 6.3.1 in [59]. We note that in the monograph [59] condition (iv) is stated slightly differently but the main ideas of the proof given there remain the same.

Theorem 2.15. *Let $(F_N^1)_{N\in\mathbb{N}}, \dots, (F_N^k)_{N\in\mathbb{N}}$, $k \in \mathbb{N}$, be sequences in $L^2(\mathbb{P})$ such that*

$$\mathbb{E}\left[F_N^i\right] = 0 \quad \text{and} \quad F_N^i = \sum_{q\geq 1} I_q(g_{N,q}^i)$$

for $i = 1, \dots, k$ and $g_{N,q}^i \in \mathfrak{H}^{\odot q}$. For all $i, j = 1, \dots, k$ we assume that

(i) *for every $q \geq 1$ there exists $\sigma_q^{ij} \in \mathbb{R}$ such that $q!\langle g_{N,q}^i, g_{N,q}^j\rangle_{\mathfrak{H}^{\otimes q}} \overset{N\to\infty}{\longrightarrow} \sigma_q^{ij}$,*

(ii) *the series $\sum_{q=1}^{\infty} \sigma_q^{ij}$ converges,*

(iii) *for every $q \geq 2$ and every $r = 1, \dots, q-1$ we have $\|g_{N,q}^i \otimes_r g_{N,q}^i\|_{\mathfrak{H}^{\otimes 2q-2r}} \overset{N\to\infty}{\longrightarrow} 0$,*

(iv) $\lim_{Q\to\infty} \limsup_{N\to\infty} \sum_{q=Q+1}^{\infty} q!\|g_{N,q}^i\|_{\mathfrak{H}^{\otimes q}}^2 = 0$.

Then

$$\left(F_N^1, \dots, F_N^k\right) \overset{\mathcal{D}}{\longrightarrow} \mathcal{N}_k(0, \Sigma) \quad \text{as } N \to \infty,$$

where $\Sigma := (\sigma^{ij})_{i,j=1}^k$ and $\sigma^{ij} := \sum_{q=1}^{\infty} \sigma_q^{ij}$.

Proof. We follow the proof of [59, Theorem 6.3.1]. For $i = 1, \dots, k$ and $N, Q \geq 1$ we set $F_{N,Q}^i := \sum_{q=1}^{Q} I_q(g_{N,q}^i)$, $G_Q \sim \mathcal{N}_k(0, \Sigma_Q)$, where $\Sigma_Q := (\sum_{q=1}^{Q} \sigma_q^{ij})_{i,j=1}^k$, and $G \sim \mathcal{N}_k(0, \Sigma)$. We note that these distributions exist since Σ_Q and Σ are symmetric and positive semidefinite. Indeed, since $(\sigma_q^{ij})_{i,j=1}^k$ is positive semidefinite for all q, the matrices Σ_Q and, as their limit, Σ is positive semidefinite. To obtain the former, we take $c \in \mathbb{R}^k$ and observe

$$c^\top(\sigma_q^{ij})_{i,j=1}^k c = \sum_{i=1}^k c_i \sum_{j=1}^k \lim_{N\to\infty} q!\langle g_{N,q}^i, g_{N,q}^j\rangle_{\mathfrak{H}^{\otimes q}} c_j = \lim_{N\to\infty} q! \left\| \sum_{i=1}^k c_i g_{N,q}^i \right\|_{\mathfrak{H}^{\otimes q}}^2 \geq 0.$$

For $t \in \mathbb{R}^k$, we will show that

$$\begin{aligned}
&\left|\mathbb{E}\left[\exp(i\langle t, (F_N^1, \dots, F_N^k)\rangle)\right] - \mathbb{E}\left[\exp(i\langle t, G\rangle)\right]\right| \\
&\quad \leq \left|\mathbb{E}\left[e^{i\langle t, (F_N^i)_{i=1}^k\rangle}\right] - \mathbb{E}\left[e^{i\langle t, (F_{N,Q}^i)_{i=1}^k\rangle}\right]\right| + \left|\mathbb{E}\left[e^{i\langle t, (F_{N,Q}^i)_{i=1}^k\rangle}\right] - \mathbb{E}\left[e^{i\langle t, G_Q\rangle}\right]\right| \\
&\qquad + \left|\mathbb{E}\left[e^{i\langle t, G_Q\rangle}\right] - \mathbb{E}\left[e^{i\langle t, G\rangle}\right]\right| =: a_{N,Q} + b_{N,Q} + c_Q
\end{aligned}$$

tends to zero as $N \to \infty$, which will prove the assertion by Lévy's continuity theorem, cf. [36, Theorem 5.3]. More precisely, we prove that

$$\limsup_{N\to\infty} \left|\mathbb{E}\left[\exp(i\langle t, (F_N^1, \dots, F_N^k)\rangle)\right] - \mathbb{E}\left[\exp(i\langle t, G\rangle)\right]\right| = 0,$$

implying, with the non negativity of the sequence, the existence of the limit, which then has to be zero.

We start with establishing $\lim_{Q\to\infty} c_Q = 0$. By the mean value theorem applied to the map $x \mapsto e^{-x}$, $x \geq 0$, and the specific form of the characteristic function of normal distributions

$$c_Q = |\exp(-1/2t^\top\Sigma_Q t) - \exp(-1/2t^\top\Sigma t)| \leq 1/2|t^\top(\Sigma_Q - \Sigma)t|.$$

By applying the submultiplicativity of the Frobenius norm twice, we obtain

$$|t^\top(\Sigma_Q - \Sigma)t| \le \|t\|^2 \left(\sum_{i,j=1}^{k} \left(\sum_{q=Q+1}^{\infty} \sigma_q^{ij} \right)^2 \right)^{\frac{1}{2}} \overset{Q\to\infty}{\longrightarrow} 0,$$

where condition (ii) yields the convergence.

We proceed with the analysis of the term $a_{N,Q}$ and show $\lim_{Q\to\infty} \limsup_{N\to\infty} a_{N,Q} = 0$. The observation that in Euclidean space segments are the shortest connections implies the inequality $|\exp(ix) - \exp(iy)| \le |x-y|$, for $x, y \in \mathbb{R}$, which yields

$$\limsup_{N\to\infty} a_{N,Q} \le \limsup_{N\to\infty} \mathbb{E}\left[|\langle t, (F_N^i - F_{N,Q}^i)_{i=1}^k \rangle| \right].$$

The Cauchy–Schwarz inequality and Jensen's inequality, cf. [36, Lemma 3.5], bound this by

$$\|t\| \limsup_{N\to\infty} \mathbb{E}\left[\|(F_N^i - F_{N,Q}^i)_{i=1}^k\| \right] \le \|t\| \limsup_{N\to\infty} \mathbb{E}\left[\sum_{i=1}^{k} \left(F_N^i - F_{N,Q}^i \right)^2 \right]^{\frac{1}{2}},$$

The fact $\limsup_{n\to\infty} f(x_n) \le f(\limsup_{n\to\infty} x_n)$, for $x_n \ge 0$ and $f\colon \mathbb{R}\to\mathbb{R}$ continuous and increasing, and the orthogonality of the stochastic integrals, yield the upper bound

$$\|t\| \left(\limsup_{N\to\infty} \sum_{i=1}^{k} \mathbb{E}\left[(F_N^i - F_{N,Q}^i)^2 \right] \right)^{\frac{1}{2}} \le \|t\| \left(\sum_{i=1}^{k} \limsup_{N\to\infty} \sum_{q=Q+1}^{\infty} q! \|g_{N,q}^i\|_{\mathfrak{H}^{\otimes q}}^2 \right)^{\frac{1}{2}} \overset{Q\to\infty}{\longrightarrow} 0,$$

where we used assumption (iv).

Finally, we deduce $\lim_{N\to\infty} b_{N,Q} = 0$. Assumptions (i) and (iii) imply with [59, Theorem 5.2.7] (if $\lim_{N\to\infty} \mathbb{E}\left[I_q(g_{N,q}^i)^2 \right] = 0$ the assertion holds trivially) that

$$I_q(g_{N,q}^i) \overset{\mathcal{D}}{\longrightarrow} \mathcal{N}(0, \sigma_q^{ii}) \quad \text{as } N\to\infty \text{ for every fixed } i \in \{1,\dots,k\} \text{ and } q \ge 2.$$

Assumption (i) also implies $I_1(g_{N,1}^i) \overset{\mathcal{D}}{\longrightarrow} \mathcal{N}(0, \sigma_1^{ii})$, since $I_1(g_{N,1}^i) = W(g_{N,1}^i)$, where W denotes the underlying isonormal Gaussian process, and moreover

$$\mathbb{E}\left[I_q(g_{N,q}^i) I_p(g_{N,p}^j) \right] = \mathbb{1}\{q=p\} q! \langle g_{N,q}^i, g_{N,q}^j \rangle \overset{N\to\infty}{\longrightarrow} \mathbb{1}\{q=p\} \sigma_q^{ij}.$$

Hence, we conclude with [59, Theorem 6.2.3]

$$(I_1(g_{N,1}^1), \dots, I_Q(g_{N,Q}^1), \dots, I_1(g_{N,1}^k), \dots, I_Q(g_{N,Q}^k)) \overset{\mathcal{D}}{\longrightarrow} \mathcal{N}_{kQ}(0, C_Q^N),$$

as $N\to\infty$, where

$$C_Q^N := \begin{pmatrix} B_{11} & \cdots & B_{1k} \\ \vdots & \ddots & \vdots \\ B_{k1} & \cdots & B_{kk} \end{pmatrix} \in \mathbb{R}^{kQ\times kQ} \text{ and } B_{ij} := \operatorname{diag}(\sigma_1^{ij}, \dots, \sigma_Q^{ij}),$$

for $i, j = 1, \ldots, k$, is positive semidefinite, due to the structure of σ_k^{ij}. Then the continuous mapping theorem, cf. [36, Theorem 4.27], applied to the function $h\colon \mathbb{R}^{kQ} \to \mathbb{R}^k$ defined as

$$x \mapsto Ax, \text{ where } A := \left(v_1 \,\middle|\, \cdots \,\middle|\, v_k\right)^\top \text{ and } v_i := \left(0_{1\times(i-1)Q} \quad 1_{1\times Q} \quad 0_{1\times(k-i)Q}\right) \in \mathbb{R}^{kQ}$$

yields

$$(F_{N,Q}^1, \ldots, F_{N,Q}^k) \xrightarrow{\mathcal{D}} \mathcal{N}_k\left(0, \left(\sum_{q=1}^{Q} \sigma_q^{ij}\right)_{i,j=1}^{k}\right) = G_Q, \qquad \text{as } N \to \infty,$$

since $c^\top h(N) = \left(A^\top c\right)^\top N$, for $N \sim \mathcal{N}_{kQ}(0, C_Q^N)$ and $c \in \mathbb{R}^k$, and moreover $AC_Q^N A^\top = \left(\sum_{q=1}^Q \sigma_q^{ij}\right)_{i,j=1}^k$. Thus we obtain $\lim_{N\to\infty} b_{N,Q} = 0$ by [36, Theorem 5.3] and we finally conclude

$$\begin{aligned} &\limsup_{N\to\infty} \left|\mathbb{E}\left[\exp(i\langle t, (F_N^1, \ldots, F_N^k)\rangle)\right] - \mathbb{E}\left[\exp(i\langle t, G\rangle)\right]\right| \\ &\qquad \leq \lim_{Q\to\infty} \limsup_{N\to\infty} a_{N,Q} + \lim_{Q\to\infty} \limsup_{N\to\infty} b_{N,Q} + \lim_{Q\to\infty} c_Q = 0, \end{aligned}$$

which shows the assertion. □

Chapter 3

A Central Limit Theorem for Lipschitz–Killing Curvatures

In this chapter, we study the excursion set of a real stationary isotropic Gaussian random field X above a fixed level $u \in \mathbb{R}$. We present a proof for the asymptotic normality of the standardised Lipschitz–Killing curvatures of the intersection of the excursion set with an observation window as the window grows to the d-dimensional Euclidean space. Moreover a lower bound for the asymptotic variance is derived.

The result generalizes the work of [22], where a central limit theorem is established for the Euler characteristic. In the case $d = 2$, the surface is treated in [44]. For the volume of the excursion set, the central limit theorem holds under weaker requirements than Gaussianity, for instance, for quasi-associated random fields, PA- or NA-random fields, Max- or α-stable fields, cf. the survey [73] and the references therein. For this reason we concentrate on the Lipschitz–Killing curvatures of degree $m = 0, \dots, d-1$ in this work. We note that Theorem 3.1 was independently derived by Marie Kratz and Sreekar Vadlamani in [43].

We pursue the following strategy of proof. We apply the Crofton formula (cf. Theorem 2.9) from integral geometry to express the m-th Lipschitz–Killing curvature $\mathcal{L}_m\left(C_N^d \cap X^{-1}([u,\infty))\right)$ as an integral average of the Euler characteristics of the intersections of $X^{-1}([u,\infty))$ with affine $(d-m)$-flats, where the integration is with respect to the motion invariant measure μ over the affine Grassmannian A_{d-m}^d. By Morse Theory (cf. Theorem 2.8), this characteristic can be expressed as a difference of counting variables. From these variables the ones depending on the interior of the intersection of the affine flat and the cube C_N^d are asymptotically dominating and the ones depending on the boundary of the intersection are asymptotically negligible, where the last statement is shown in Section 3.2.5. We then use a refinement of the approach in [22] to control the dependence of the dominating counting variables on the affine flat. That is, we use Rice's formulas, cf. Section 2.1, in the affine flat to obtain a Hermite expansion of the m-th Lipschitz–Killing curvature via an approximation argument, cf. Section 3.2.2. This Hermite expansion leads to a representation of the relevant part of $\mathcal{L}_m\left(C_N^d \cap X^{-1}([u,\infty))\right)$ by stochastic integrals, cf. Section 3.2.3, to which we apply Theorem 2.15, which is a result from the theory of normal approximation based on Stein's method and Malliavin calculus, in order to obtain a central limit theorem, cf. Section 3.2.4. In Section 3.2.5 we derive Hermite expansions for the counting variables on the boundary and establish that they are asymptotically negligible.

3.1. Main Theorem

We impose the following conditions on a given real random field $X = \{X(t) \mid t \in \mathbb{R}^d\}$.

(A1) X is a centered, stationary, isotropic Gaussian field. The trajectories are almost surely of class $\mathcal{C}^3$. The covariance function $C^X(t) = \mathbb{E}\left[X(t)X(0)\right]$, $t \in \mathbb{R}^d$, of X satisfies $C^X(0) = 1$ and $D^2 C^X(0) = -I_d$.

(A2) For all $0 \neq t \in \mathbb{R}^d$ the covariance matrices of the vectors

$$\left(X(0), \left(\frac{\partial^2}{\partial t_i \partial t_j} X(0)\right)_{1 \leq i \leq j \leq d}\right) \quad \text{and} \quad \left(\left(\frac{\partial}{\partial t_i} X(0)\right)_{i=1}^d, \left(\frac{\partial}{\partial t_i} X(t)\right)_{i=1}^d\right)$$

have full rank.

(A3) The mapping defined by

$$\psi(t) := \max\left\{\left|\frac{\partial^k}{\partial t_{j_1} \dots \partial t_{j_k}} C^X(t)\right| : k \in \{0, \dots, 4\}, 1 \leq j_1, \dots, j_k \leq d\right\}$$

for $t \in \mathbb{R}^d$, satisfies

$$\psi(t) \overset{\|t\| \to \infty}{\longrightarrow} 0 \quad \text{and} \quad \psi \in L^1(\mathbb{R}^d, \lambda^d).$$

We heavily rely on (A1) in several places, for instance in the proof of Lemma 3.5 and in the calculations in the appendix for Lemma 3.2. If (A2) holds, then the conditions on the covariance from (A1) are always satisfied after normalizing the Gaussian field. We believe that it is enough to assume $\mathcal{C}^2$ regularity and an integrability condition on C^X, cf. [21], but stick to the $\mathcal{C}^3$ assumption to smoothen the computations in the appendix. Under the differentiability assumptions of (A1) and stationarity, the condition (A2) ensures that the paths of X are almost surely Morse functions and allows us to perform calculations involving Gaussian regressions. Condition (A3) ensures that we are in the regime of short range dependence and moreover that the decay of the covariances of the field and its derivatives up to degree four are fast enough for a central limit theorem to hold. We note that from (A3) we obtain that $\psi \in L^q(\mathbb{R}^d)$, $q \in \mathbb{N}$, and moreover that X admits a continuous spectral density, cf. [70, Theorem 2.§12.3 (Inversion

Formula)]. Furthermore the mapping defined by

$$\widetilde{\psi}(t) := \sup\left\{\left|\frac{\partial^k}{\partial v_1 \dots \partial v_k} C^X(t)\right| : k \in \{0, \dots, 4\}, v_1, \dots, v_k \in S^{d-1}\right\}, \quad t \in \mathbb{R}^d$$

satisfies $\widetilde{\psi}(t) \leq d^2\psi(t)$, for $t \in \mathbb{R}^d$, and therefore is also in $L^q(\mathbb{R}^d)$, $q \in \mathbb{N}$.

Let $u \in \mathbb{R}$ be the level of the considered excursion set and denote by $C_N^d := [-N, N]^d \subset \mathbb{R}^d$ the cube of side length $N > 0$ centered at the origin. We prove the following central limit theorem.

Theorem 3.1. *Let X be a real Gaussian field on $\mathbb{R}^d$, which satisfies the assumptions (A1)–(A3), and let $m \in \{0, \dots, d-1\}$. Then the m-th Lipschitz–Killing curvature $\mathcal{L}_m$ of the excursion set for the level $u \in \mathbb{R}$ satisfies*

$$\frac{\mathcal{L}_m\left(C_N^d \cap X^{-1}([u, \infty))\right) - \mathbb{E}\left[\mathcal{L}_m\left(C_N^d \cap X^{-1}([u, \infty))\right)\right]}{\mathcal{H}^d(C_N^d)^{\frac{1}{2}}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_m^2)$$

for $N \to \infty$ and some $\sigma_m^2 \geq 0$.

A lower bound for the asymptotic variance σ_m^2 will be derived in Lemma 3.23. The fact that $\mathcal{L}_m\left(C_N^d \cap X^{-1}([u, \infty))\right)$ is indeed a random variable is established in Lemma A.8.

3.2. Proof of the main theorem

In this section we explain the details of the proof for the central limit theorem stated in Theorem 3.1.

3.2.1. Approximation of Lipschitz–Killing curvatures

We start the proof by the derivation of a more convenient representation of the m-th Lipschitz–Killing curvature $\mathcal{L}_m$ of the excursion set in C_N^d. For this purpose, we apply the Crofton formula, cf. Theorem 2.9, which is applicable for sets of positive reach. Hence, we recall that almost surely $\operatorname{reach}(C_N^d \cap X^{-1}([u, \infty))) > 0$ by Lemma A.1, and obtain

$$\mathcal{L}_m\left(C_N^d \cap X^{-1}([u, \infty))\right) = \int_{A_{d-m}^d} \mathcal{L}_0\left(C_N^d \cap X^{-1}([u, \infty)) \cap F\right) \mu(dF). \tag{3.1}$$

By (2.18)

$$L_0(C_N^d \cap X^{-1}([u, \infty)) \cap F) = \chi(C_N^d \cap X^{-1}([u, \infty)) \cap F),$$

for almost all F. Moreover, by the assumptions made, we know that the trajectories of X are almost surely and for μ almost all F Morse functions on $C_N^d \cap F$, cf. [1, Definition 9.3.1] and Lemma A.5. Therefore, restricting the integration to a suitable measurable subset $A' \subset A_{d-m}^d$, we can apply Theorem 2.8 to the above integrand and obtain

$$\begin{aligned}\mathcal{L}_0(C_N^d \cap F \cap X^{-1}([u,\infty))) &= \#\{t \in \operatorname{int} C_N^d \cap F : X(t) \geq u, \nabla(X|_F)(t) = 0, \iota_F^{-X}(t) \text{ even}\} \\ &\quad - \#\{t \in \operatorname{int} C_N^d \cap F : X(t) \geq u, \nabla(X|_F)(t) = 0, \iota_F^{-X}(t) \text{ odd}\} \\ &\quad + \sum_{j=0}^{d-m-1} \sum_{J_N \in \partial_{j+m} C_N^d} e(X, F, J_N), \end{aligned} \tag{3.2}$$

where $e(X, F, J_N)$ is given by

$$\begin{aligned}&\#\{t \in J_N \cap F : X(t) \geq u, \nabla(X|_{J_N \cap F})(t) = 0, \iota_{J_N \cap F}^{-X}(t) \text{ even}, \nabla X(t) \in N_t(C_N^d \cap F)\} \\ &- \#\{t \in J_N \cap F : X(t) \geq u, \nabla(X|_{J_N \cap F})(t) = 0, \iota_{J_N \cap F}^{-X}(t) \text{ odd}, \nabla X(t) \in N_t(C_N^d \cap F)\}. \end{aligned} \tag{3.3}$$

We note that the counting variables in (3.3) are structurally the same as the ones in the first two lines of equation (3.2). For $t \in \operatorname{int} C_N^d \cap F$, we have $N_t(C_N^d \cap F) = (F^\circ)^\perp$. Hence, the condition $\nabla X(t) \in N_t(C_N^d \cap F)$ is true for every $t \in \operatorname{int} C_N^d \cap F$ with $\nabla(X|_F)(t) = 0$ and is therefore omitted. The measurability of these counting variables is established in Lemma A.11.

For further reference, we define the following variables. Let $l \geq m$ and $J_N \in \partial_l C_N^d$. If $l = d$ we have $J_N = \operatorname{int} C_N^d$ and we define

$$\begin{aligned}\zeta_N^m := \int_{A_{d-m}^d} &\#\{t \in \operatorname{int} C_N^d \cap F : X(t) \geq u, \nabla(X|_F)(t) = 0, \iota_F^{-X}(t) \text{ even}\} \\ &- \#\{t \in \operatorname{int} C_N^d \cap F : X(t) \geq u, \nabla(X|_F)(t) = 0, \iota_F^{-X}(t) \text{ odd}\}\, \mu(dF). \end{aligned} \tag{3.4}$$

In the cases $l < d$ we define

$$\begin{aligned}&\epsilon_{J_N}^m \\ &:= \int_{A_{d-m}^d} \#\{t \in J_N \cap F : X(t) \geq u, \nabla(X|_{J_N \cap F})(t) = 0, \iota_{J_N \cap F}^{-X}(t) \text{ even}, \nabla X(t) \in N_t(C_N^d \cap F)\} \\ &- \#\{t \in J_N \cap F : X(t) \geq u, \nabla(X|_{J_N \cap F})(t) = 0, \iota_{J_N \cap F}^{-X}(t) \text{ odd}, \nabla X(t) \in N_t(C_N^d \cap F)\}\, \mu(dF), \end{aligned} \tag{3.5}$$

where the definitions in (3.4) and (3.5) are tailored to satisfy

$$\mathcal{L}_m\left(C_N^d \cap X^{-1}([u,\infty))\right) = \zeta_N^m + \sum_{j=0}^{d-m-1} \sum_{J_N \in \partial_{j+m} C_N^d} \epsilon_{J_N}^m.$$

In the following, we show a central limit theorem for the standardized random variable ζ_N^m, whereas the boundary terms are treated in Section 3.2.5. By the statement (3.41) and Slutzky's theorem, we deduce that after standardizing, the term

$$\sum_{j=0}^{d-m-1} \sum_{J_N \in \partial_{j+m} C_N^d} \epsilon_{J_N}^m$$

is asymptotically negligible and only the integrated counting variables in the $(d-m)$-dimensional set $\operatorname{int} C_N^d \cap F$ contribute to the central limit theorem.

For the approximations to come, we define for $\varepsilon > 0$ and $l \in \{1, \dots, d\}$ the mapping

$$\delta_\varepsilon^l \colon \mathbb{R}^d \to \mathbb{R}, \quad x \mapsto \frac{1}{\varepsilon^{l-m}\kappa_{l-m}} \mathbb{1}_{B_\varepsilon^d}(x),$$

which is a Dirac sequence for $\varepsilon \to 0$ on every $(l-m)$-dimensional linear subspace E of $\mathbb{R}^d$, that is, for each continuous mapping $f \colon E \to \mathbb{R}$, we have

$$\lim_{\varepsilon \to 0} \int_E \delta_\varepsilon^l(x) f(x) \, \mathcal{H}^{l-m}(dx) = f(0).$$

We note that although the mapping δ_ε^l depends on m, we do not indicate this dependence in favor of a shorter notation.

The following lemma lays the foundation of the approximation of the random variables in (3.4) and (3.5). We postpone its proof to the Appendix B.

Lemma 3.2. *Let $G \subset \mathbb{R}^d$ be compact and assume the conditions (A1) and (A2). Furthermore let $J_N \in \partial_l C_N^d$ and $l > m$. Then the following is true:*

(i) *There is a constant $c = c(X, d, m, l, N, G) > 0$ such that for almost all $F \in A_{d-m}^d$ and all $y \in G$*

$$\mathbb{E}\left[\#\{t \in J_N \cap F \colon \nabla(X|_{J_N \cap F})(t) = y\}^2\right] < c.$$

(ii) *For almost all $F \in A_{d-m}^d$ the mapping*

$$y \mapsto \mathbb{E}\left[\#\{t \in J_N \cap F \colon \nabla(X|_{J_N \cap F})(t) = y\}^2\right]$$

is continuous on $(\operatorname{aff}(J_N) \cap F)^\circ \cap G$.

(iii) *For almost all $F \in A_{d-m}^d$*

$$\xi_N(F, \varepsilon) \xrightarrow{L^2(\mathbb{P})} \xi_N(F), \qquad \text{as } \varepsilon \to 0,$$

where

$$\xi_N(F, \varepsilon) := (-1)^{d-m} \int_{C_N^d \cap F} \delta_\varepsilon^d(\nabla(X|_F)(t)) \mathbb{1}\{X(t) \geq u\} \det(D^2(X|_F)(t)) \, \mathcal{H}^{d-m}(dt),$$

$$\begin{aligned}\xi_N(F) := \#\{t \in \operatorname{int} C_N^d \cap F : X(t) \geq u, \nabla(X|_F)(t) = 0, \iota_F^{-X}(t) \text{ even}\} \\ - \#\{t \in \operatorname{int} C_N^d \cap F : X(t) \geq u, \nabla(X|_F)(t) = 0, \iota_F^{-X}(t) \text{ odd}\}.\end{aligned}$$

Motivated by the frequent use of a Dirac sequence to approximate the counting variables in

(3.2), e.g. [1, Lemma 11.2.10], we introduce the approximations

$$\zeta_{N,\varepsilon}^m := (-1)^{d-m} \int_{A_{d-m}^d} \int_{C_N^d \cap F} \delta_\varepsilon^d(\nabla(X|_F)(t)) \mathbb{1}\{X(t) \geq u\} \det(D^2(X|_F)(t)) \, \mathcal{H}^{d-m}(dt) \, \mu(dF). \tag{3.6}$$

Using the results of the Lemma 3.2, we show that $\zeta_{N,\varepsilon}^m$ is indeed an approximation of the variable ζ_N^m in the sense of $L^2(\mathbb{P})$ convergence.

Lemma 3.3. *Let $(X_t)_{t\in\mathbb{R}^d}$ be a real-valued Gaussian field satisfying (A1) and (A2). Then*

$$\zeta_{N,\varepsilon}^m \overset{L^2(\mathbb{P})}{\longrightarrow} \zeta_N^m$$

as $\varepsilon \to 0$, where ζ_N^m and $\zeta_{N,\varepsilon}^m$ are defined by (3.4) *and* (3.6)*, respectively.*

Proof. By Jensen's inequality and Fubini's theorem

$$\begin{aligned} \mathbb{E}\left[\left(\zeta_N^m - \zeta_{N,\varepsilon}^m\right)^2\right] &\leq c\mathbb{E}\left[\int_{A_{d-m}^d} \left(\xi_N(F) - \xi_N(F,\varepsilon)\right)^2 \, \mu(dF)\right] \\ &= c\int_{A_{d-m}^d} \mathbb{E}\left[\left(\xi_N(F) - \xi_N(F,\varepsilon)\right)^2\right] \, \mu(dF), \end{aligned}$$

where $c = \mu(\{F : F \cap C_N^d \neq \emptyset\}) \leq \nu(G_{d-m}^d) \operatorname{diam}(C_N^d)^m \kappa_m$. Thus, if we justify changing the order of the limit $\lim_{\varepsilon\to 0}$ and the integral $\int_{A_{d-m}^d}$, we are done by Lemma 3.2 (iii). In order to apply the dominated convergence theorem, we bound the integrand by an integrable function, not depending on ε. Observe that

$$\begin{aligned} \mathbb{E}\left[\left(\xi_N(F) - \xi_N(F,\varepsilon)\right)^2\right] &\leq 2\mathbb{E}\left[\#\{t \in \operatorname{int} C_N^d \cap F \mid \nabla(X|_F)(t) = 0\}^2\right] \\ &\quad + 2\mathbb{E}\left[\left(\int_{\operatorname{int} C_N^d \cap F} \delta_\varepsilon^d(\nabla(X|_F)(t)) |\det(D^2(X|_F)(t))| \, \mathcal{H}^{d-m}(dt)\right)^2\right]. \end{aligned}$$

For the first term, Lemma 3.2 (i) yields

$$\mathbb{E}\left[\#\{t \in \operatorname{int} C_N^d \cap F : \nabla(X|_F)(t) = 0\}^2\right] \leq c\mathbb{1}\{\operatorname{int} C_N^d \cap F \neq \emptyset\},$$

where $c > 0$ is a constant depending on X,d, m and N. For the second term, we first apply the coarea formula to $\nabla(X|_F)$, cf. [25, Theorem 3.2.12], which yields

$$\begin{aligned} &\mathbb{E}\left[\left(\int_{\operatorname{int} C_N^d \cap F} \delta_\varepsilon^d(\nabla(X|_F)(t)) |\det(D^2(X|_F)(t))| \, \mathcal{H}^{d-m}(dt)\right)^2\right] \\ &= \mathbb{E}\left[\left(\int_{F^\circ} \#\{t \in \operatorname{int} C_N^d \cap F \mid \nabla(X|_F)(t) = y\} \delta_\varepsilon^d(y) \, \mathcal{H}^{d-m}(dy)\right)^2\right]. \end{aligned}$$

Then by Jensen's inequality applied to the measure $\mathbb{1}\{y \in F^\circ\}\delta_\varepsilon^d(y)\mathcal{H}^{d-m}(dy)$ followed by

Fubini's theorem, we bound this by

$$\int_{F^\circ} \mathbb{E}\left[\#\{t \in \operatorname{int} C_N^d \cap F \mid \nabla(X|_F)(t) = y\}^2\right] \delta_\varepsilon^d(y)\, \mathcal{H}^{d-m}(dy).$$

Again by Lemma 3.2 (i), we can bound this for all $\varepsilon \le 1$ by the expression

$$c\mathbb{1}\{\operatorname{int} C_N^d \cap F \ne \emptyset\} \int_{F^\circ} \delta_\varepsilon^d(y)\, \mathcal{H}^{d-m}(dy) = c\mathbb{1}\{\operatorname{int} C_N^d \cap F \ne \emptyset\},$$

for a constant $c > 0$. Both bounds are independent of ε and integrable with respect to μ, which shows the assertion. □

Before we move on with the main proof, we show the following lemma to obtain a more concrete representation of $\zeta_{N,\varepsilon}^m$. We introduce the following notation. For $F \in A_{d-m}^d$, let $b_F = (v_i)_{i=1}^{d-m}$ denote an orthonormal basis of F°. In the formulation of the following lemma, the specific choice of this basis is irrelevant.

Lemma 3.4. *Let $\varepsilon > 0$ and assume (A1). Then*

$$\zeta_{N,\varepsilon}^m = (-1)^{d-m} \int_{G_{d-m}^d} \int_{C_N^d} \delta_\varepsilon^d(\nabla_{b_L} X(t)) \mathbb{1}\{X(t) \ge u\} \det\left(D_{b_L}^2 X(t)\right) dt\, \nu(dL),$$

where ∇_{b_F} and $D_{b_F}^2$ are defined in (2.3) and (2.4), respectively.

Proof. Recall that by definition $\nabla(f|_F)(t) = \sum_{i=1}^{d-m} \frac{\partial}{\partial v_i} f(t) v_i$ and therefore the rotation invariance of δ_ε^d yields

$$\delta_\varepsilon^d(\nabla(X|_F)) = \frac{1}{\varepsilon^{d-m}\kappa_{d-m}} \mathbb{1}_{B_\varepsilon^d}(\nabla(X|_F)) = \delta_\varepsilon^d(\nabla_{b_F} X).$$

Also by definition $D^2(X|_F)(t) = \left(v_1 \,\middle|\, \cdots \,\middle|\, v_{d-m}\right) \left(\frac{\partial^2}{\partial v_i \partial v_j} X(t)\right)_{i,j=1}^{d-m} \left(v_1 \,\middle|\, \cdots \,\middle|\, v_{d-m}\right)^\top$ so that, as a linear mapping from F° into F°, it has the transformation matrix $\left(\frac{\partial^2}{\partial v_i \partial v_j} X(t)\right)_{i,j=1}^{d-m}$ with respect to the chosen basis, and therefore we have

$$\det(D^2(X|_F)) = \det\left(D_{b_F}^2 X\right).$$

This yields with definition (3.6)

$$\zeta_{N,\varepsilon}^m = (-1)^{d-m} \int_{A_{d-m}^d} \int_{C_N^d \cap F} \delta_\varepsilon^d(\nabla_{b_F} X(t)) \mathbb{1}\{X(t) \ge u\} \det\left(D_{b_F}^2 X(t)\right) \mathcal{H}^{d-m}(dt)\, \mu(dF)$$

and we conclude by representation of the measure μ, cf. (2.2),

$$\zeta_{N,\varepsilon}^m = (-1)^{d-m} \int_{G_{d-m}^d} \int_{L^\perp} \int_{C_N^d \cap (L+y)} \delta_\varepsilon^d(\nabla_{b_{L+y}} X(t)) \mathbb{1}\{X(t) \ge u\} \\ \times \det\left(D_{b_{L+y}}^2 X(t)\right) \mathcal{H}^{d-m}(dt)\, \mathcal{H}^m(dy)\, \nu(dL).$$

Hence, by Fubini's theorem

$$\begin{aligned}(-1)^{d-m} \int_{G^d_{d-m}} \int_{L^\perp} \int_L \mathbb{1}\{t+y \in C^d_N\} \delta^d_\varepsilon(\nabla_{b_L} X(t+y)) \mathbb{1}\{X(t+y) \geq u\} \\ \times \det\left(D^2_{b_L} X(t+y)\right) \mathcal{H}^{d-m}(dt)\, \mathcal{H}^m(dy)\, \nu(dL) \\ = (-1)^{d-m} \int_{G^d_{d-m}} \int_{C^d_N} \delta^d_\varepsilon(\nabla_{b_L} X(t)) \mathbb{1}\{X(t) \geq u\} \det\left(D^2_{b_L} X(t)\right) \mathcal{H}^d(dt)\, \nu(dL),\end{aligned}$$

which establishes the assertion. □

3.2.2. Hermite type expansion

From now on, let the field X satisfy the assumptions (A1)–(A3). We start this subsection by defining for $D := d-m+(d-m)(d-m+1)/2+1$ the $\mathbb{R}^D$-valued Gaussian random field $\{Z(L,t) \colon \Omega \to \mathbb{R}^D \mid (L,t) \in G^d_{d-m} \times \mathbb{R}^d\}$ by

$$Z(L,t) := \left(\left(\frac{\partial}{\partial v_i} X(t)\right)_{i=1}^{d-m}, \left(\frac{\partial^2}{\partial v_i \partial v_j} X(t)\right)_{1 \leq i \leq j \leq d-m}, X(t)\right)$$

and denote by Σ the covariance matrix of $Z(L,t)$, $(L,t) \in G^d_{d-m} \times \mathbb{R}^d$. We note that the definition depends on the choice of b_L, but considering Lemma 3.4, this does not matter. We formulate the following lemma.

Lemma 3.5. *The matrix Σ is independent of $t \in \mathbb{R}^d$ and $L \in G^d_{d-m}$. Moreover, we have $\Sigma = \Lambda\Lambda^\top$, where $\Lambda \in \mathbb{R}^{D \times D}$ is invertible and given by $\Lambda = \begin{pmatrix} I_{d-m} & 0 \\ 0 & \Lambda_2 \end{pmatrix}$, for some invertible, lower triangular matrix $\Lambda_2 \in \mathbb{R}^{(D-d+m)\times(D-d+m)}$.*

Proof. By assumption (A1) on the random field X, we obtain from (2.8), (2.10) and isotropy

$$\begin{aligned}\mathbb{E}\left[\frac{\partial}{\partial v_i} X(t) \frac{\partial}{\partial v_j} X(t)\right] &= \mathbb{E}\left[\frac{\partial}{\partial t_i} X(0) \frac{\partial}{\partial t_j} X(0)\right] = \delta_{ij}, \\ \mathbb{E}\left[\frac{\partial}{\partial v_i} X(t) \frac{\partial^2}{\partial v_k \partial v_l} X(t)\right] &= \mathbb{E}\left[\frac{\partial}{\partial t_i} X(0) \frac{\partial^2}{\partial t_k \partial t_l} X(0)\right] = 0, \\ \mathbb{E}\left[\frac{\partial}{\partial v_i} X(t) X(t)\right] &= \mathbb{E}\left[\frac{\partial}{\partial t_i} X(0) X(0)\right] = 0,\end{aligned} \tag{3.7}$$

as well as

$$\begin{aligned}\mathbb{E}\left[\frac{\partial^2}{\partial v_i \partial v_j} X(t) \frac{\partial^2}{\partial v_k \partial v_l} X(t)\right] &= \mathbb{E}\left[\frac{\partial^2}{\partial t_i \partial t_j} X(0) \frac{\partial^2}{\partial t_k \partial t_l} X(0)\right], \\ \mathbb{E}\left[\frac{\partial^2}{\partial v_i \partial v_j} X(t) X(t)\right] &= \mathbb{E}\left[\frac{\partial^2}{\partial t_i \partial t_j} X(0) X(0)\right], \\ \mathbb{E}\left[X(t)X(t)\right] &= \mathbb{E}\left[X(0)X(0)\right].\end{aligned}$$

Assumption (A2) and stationarity yield that Σ is positive definite. Hence, the well-known Cholesky decomposition, cf. [7, Fact 8.9.37], yields the assertion. $\square$

Using Λ, we define the decorrelated process

$$Y(L,t) := \Lambda^{-1} Z(L,t), \quad t \in \mathbb{R}^d, L \in G^d_{d-m}. \tag{3.8}$$

For fixed $t \in \mathbb{R}^d$ and $L \in G^d_{d-m}$, the random vector $Y(L,t)$ is standard normal, i.e. $Y(L,t) \sim \mathcal{N}_D(0, I_D)$. However, note that for different $t, s \in \mathbb{R}^d$ the vectors $Y(L,t)$ and $Y(L,s)$ are in general not independent. In what follows we will be using the stationarity

$$(Y(L,t), Y(L',t')) \overset{\mathcal{D}}{=} (Y(L,t+h), Y(L',t'+h)),$$

where $t, t', h \in \mathbb{R}^d$ and $L, L' \in G^d_{d-m}$. Indeed, by the stationarity of X and its derivatives, we have for suitable mappings f_L and $f_{L'}$ that

$$\begin{aligned}
&(Y(L,t), Y(L',t')) \\
&\quad = (f_L(\nabla X(t), D^2 X(t), X(t)), f_{L'}(\nabla X(t'), D^2 X(t'), X(t'))) \\
&\quad \overset{\mathcal{D}}{=} (f_L(\nabla X(t+h), D^2 X(t+h), X(t+h)), f_{L'}(\nabla X(t'+h), D^2 X(t'+h), X(t'+h))) \\
&\quad = (Y(L,t+h), Y(L',t'+h)).
\end{aligned}$$

We will now use the approximation of Lemma 3.3 to obtain a Hermite decomposition of the random variable of interest. We thus define the mapping $G_\varepsilon \colon \mathbb{R}^{d-m} \times \mathbb{R}^{(d-m)(d-m+1)/2+1} \to \mathbb{R}$ by

$$G_\varepsilon(x,y) := (-1)^{d-m} \delta^d_\varepsilon(x) \det\Big((\Lambda_2 y)_{1,\dots,(d-m)(d-m+1)/2} \Big) \mathbb{1}\{(\Lambda_2 y)_{D-(d-m)} \geq u\},$$

where we use the shorthand notation $(x)_{i_1,\dots,i_k} := (x_{i_1}, \dots, x_{i_k})$ for the projection onto the specified coordinates. In this definition the vector $(\Lambda_2 y)_{1,\dots,(d-m)(d-m+1)/2}$ is identified with the symmetric $(d-m) \times (d-m)$-matrix, whose diagonal and upper diagonal entries are given by $(\Lambda_2 y)_{1,\dots,(d-m)(d-m+1)/2}$, according to the way one identifies $\Big(\frac{\partial^2}{\partial v_i \partial v_j} X(t)\Big)_{1 \leq i \leq j \leq d-m}$ with a vector.

Thus by Lemma 3.4 we obtain

$$\zeta^m_{N,\varepsilon} = \int_{G^d_{d-m}} \int_{C^d_N} G_\varepsilon(Y(L,t))\, dt\, \nu(dL),$$

where now the randomness enters the random variable $\zeta^m_{N,\varepsilon}$ through the decorrelated process Y instead of the process Z. We note now that the mapping G_ε is an element of $L^2(\mathbb{R}^D, \mathcal{N}_D(0, I_D))$. Therefore, G_ε can be expanded in the orthonormal basis $\{n!^{-1/2} \widetilde{H}_n : n \in \mathbb{N}^D_0\}$, cf. Section 2.4.1. Thus we obtain

$$G_\varepsilon = \sum_{q=0}^{\infty} \sum_{n \in \mathbb{N}^D_0, |n|=q} c(G_\varepsilon, n) \widetilde{H}_n, \tag{3.9}$$

in $L^2(\mathbb{R}^D, \mathcal{N}_D(0, I_D))$, where

$$\begin{aligned}
c(G_\varepsilon, n) :=& n!^{-1} \int_{\mathbb{R}^D} G_\varepsilon(x) \widetilde{H}_n(x) \phi_D(x)\, dx \\
=& \frac{(-1)^{d-m}}{\prod_{i=1}^D n_i!} \int_{\mathbb{R}^{d-m}} \delta_\varepsilon^d(x) \prod_{i=1}^{d-m} H_{n_i}(x) \phi_{d-m}(x)\, dx \int_{\mathbb{R}^{D-(d-m)}} \mathbb{1}\{(\Lambda_2 y)_{D-(d-m)} \geq u\} \\
& \times \det\left((\Lambda_2 y)_{1,\dots,(d-m)(d-m+1)/2}\right) \prod_{i=d-m+1}^{D} H_{n_i}(y) \phi_{D-(d-m)}(y)\, dy. \qquad (3.10)
\end{aligned}$$

It is this Hermite expansion of the mapping G_ε, which helps to establish an expansion of the random variable $\zeta_{N,\varepsilon}^m$, as is shown in the next lemma. We already note here that the limit $\varepsilon \to 0$ of the coefficients exists, due to the continuity of the Hermite polynomials and the density of the normal distribution.

Lemma 3.6. *Let $\varepsilon > 0$ and let X satisfy (A1) - (A3). Then*

$$\zeta_{N,\varepsilon}^m = \sum_{q \geq 0} \sum_{n \in \mathbb{N}_0^D, |n| = q} \int_{G_{d-m}^d} c(G_\varepsilon, n) \int_{C_N^d} \widetilde{H}_n(Y(L,t))\, dt\, \nu(dL),$$

where the convergence is in $L^2(\mathbb{P})$.

Proof. The right side is an element in $L^2(\mathbb{P})$ since it is the limit of a Cauchy sequence, which can be seen by Jensen's inequality and (3.9). Indeed, let $k_1 < k_2$ be integers. Then

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{G_{d-m}^d} \int_{C_N^d} \sum_{q=k_1+1}^{k_2} \sum_{n \in \mathbb{N}_0^D, |n|=q} c(G_\varepsilon, n) \widetilde{H}_n(Y(L,t))\, dt\, \nu(dL)\right)^2\right] \\
&\leq \nu(G_{d-m}^d) \mathcal{H}^d(C_N^d) \mathbb{E}\left[\int_{G_{d-m}^d} \int_{C_N^d} \left(\sum_{q=k_1+1}^{k_2} \sum_{n \in \mathbb{N}_0^D, |n|=q} c(G_\varepsilon, n) \widetilde{H}_n(Y(L,t))\right)^2 dt\, \nu(dL)\right],
\end{aligned}$$

and by Fubini's theorem the preceding term equals

$$\nu(G_{d-m}^d) \mathcal{H}^d(C_N^d) \int_{G_{d-m}^d} \int_{C_N^d} \mathbb{E}\left[\left(\sum_{q=k_1+1}^{k_2} \sum_{n \in \mathbb{N}_0^D, |n|=q} c(G_\varepsilon, n) \widetilde{H}_n(Y(L,t))\right)^2\right] dt\, \nu(dL). \qquad (3.11)$$

Properties of the Hermite polynomials, cf. Lemma 2.10 (ii), yield $\mathbb{E}[\widetilde{H}_n(Y(L,t))\widetilde{H}_{n'}(Y(L,t))] = \mathbb{1}\{n = n'\} n!$, since the process is standard normal for given L and t. Thus the term in (3.11) equals

$$\begin{aligned}
&\nu(G_{d-m}^d) \mathcal{H}^d(C_N^d) \int_{G_{d-m}^d} \int_{C_N^d} \sum_{q=k_1+1}^{k_2} \sum_{n \in \mathbb{N}_0^D, |n|=q} c(G_\varepsilon, n)^2 n!\, dt\, \nu(dL) \\
&= \nu(G_{d-m}^d)^2 \mathcal{H}^d(C_N^d)^2 \sum_{q=k_1+1}^{k_2} \sum_{n \in \mathbb{N}_0^D, |n|=q} c(G_\varepsilon, n)^2 n!.
\end{aligned}$$

By adding positive terms, we bound this by

$$\nu(G^d_{d-m})^2\mathcal{H}^d(C^d_N)^2 \sum_{q=k_1+1}^{\infty} \sum_{n\in\mathbb{N}_0^D,|n|=q} c(G_\varepsilon,n)^2 n!. \tag{3.12}$$

Parseval's equality yields the bound $\nu(G^d_{d-m})^2\mathcal{H}^d(C^d_N)^2\|G_\varepsilon\|^2_{L^2(\mathcal{N}_D(0,I_D))} < \infty$, and we deduce that for k_1 large enough the term in (3.12) tends to zero, showing the Cauchy property.

To establish the asserted equality, we recall that by Lemma 3.4

$$\zeta^m_{N,\varepsilon} = \int_{G^d_{d-m}} \int_{C^d_N} G_\varepsilon(Y(L,t))\,dt\,\nu(dL),$$

thus for $k \in \mathbb{N}$, we have that

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\zeta^\varepsilon_{m,N} - \int_{G^d_{d-m}} \int_{C^d_N} \sum_{q=0}^{k} \sum_{n\in\mathbb{N}_0^D,|n|=q} c(G_\varepsilon,n)\widetilde{H}_n(Y(L,t))\,dt\,\nu(dL)\Big)^2\Big]\\
&\quad= \mathbb{E}\Big[\Big(\int_{G^d_{d-m}} \int_{C^d_N} G_\varepsilon(Y(L,t)) - \sum_{q=0}^{k} \sum_{n\in\mathbb{N}_0^D,|n|=q} c(G_\varepsilon,n)\widetilde{H}_n(Y(L,t))\,dt\,\nu(dL)\Big)^2\Big].
\end{aligned}$$

By two applications of Jensen's inequality, this can be bounded by

$$c\mathbb{E}\Big[\int_{G^d_{d-m}} \int_{C^d_N} \Big(G_\varepsilon(Y(L,t)) - \sum_{q=0}^{k} \sum_{n\in\mathbb{N}_0^D,|n|=q} c(G_\varepsilon,n)\widetilde{H}_n(Y(L,t))\Big)^2\,dt\,\nu(dL)\Big],$$

where $c = \nu(G^d_{d-m})\mathcal{H}^d(C^d_N)$. By Fubini's theorem, we obtain equality to

$$\begin{aligned}
&c\int_{G^d_{d-m}} \int_{C^d_N} \mathbb{E}\Big[\Big(G_\varepsilon(Y(L,t)) - \sum_{q=0}^{k} \sum_{n\in\mathbb{N}_0^D,|n|=q} c(G_\varepsilon,n)\widetilde{H}_n(Y(L,t))\Big)^2\Big]\,dt\,\nu(dL)\\
&= c\int_{G^d_{d-m}} \int_{C^d_N} \int_{\mathbb{R}^D} \Big(G_\varepsilon(x) - \sum_{q=0}^{k} \sum_{n\in\mathbb{N}_0^D,|n|=q} c(G_\varepsilon,n)\widetilde{H}_n(x)\Big)^2 \phi_D(x)\,dx\,dt\,\nu(dL)\\
&= c^2\int_{\mathbb{R}^D} \Big(G_\varepsilon(x) - \sum_{q=0}^{k} \sum_{n\in\mathbb{N}_0^D,|n|=q} c(G_\varepsilon,n)\widetilde{H}_n(x)\Big)^2 \phi_D(x)\,dx.
\end{aligned}$$

Hence, by (3.9), we conclude

$$c^2\int_{\mathbb{R}^D} \Big(G_\varepsilon(x) - \sum_{q=0}^{k} \sum_{n\in\mathbb{N}_0^D,|n|=q} c(G_\varepsilon,n)\widetilde{H}_n(x)\Big)^2 \phi_D(x)\,dx \overset{k\to\infty}{\longrightarrow} 0,$$

which shows the assertion. □

The following lemma is needed to deduce that the established Hermite expansion is orthogonal. It is a special case of [76, Lemma 3.2] and we give a proof for completeness.

Lemma 3.7. *Let V, W be two D-dimensional random vectors where*

$$(V,W) \sim \mathcal{N}_{2D}\left(0, \begin{pmatrix} I_D & (\mathbb{E}[V_iW_j])_{1\le i,j\le D} \\ (\mathbb{E}[W_iV_j])_{1\le i,j\le D} & I_D \end{pmatrix}\right)$$

and let $n, n' \in \mathbb{N}_0^D$. Then

$$\mathbb{E}\left[\widetilde{H}_n(V)\widetilde{H}_{n'}(W)\right] = \mathbb{1}\{|n| = |n'|\} \sum_{\substack{d\in\mathbb{N}_0^{D\times D} \\ \sum_{i=1}^D d_{ij}=n_j,\, \sum_{j=1}^D d_{ij}=n'_i}} n!n'! \prod_{1\le i,j\le D} \frac{\mathbb{E}[V_iW_j]^{d_{ij}}}{d_{ij}!}.$$

Proof. Observe that via the moment generating function of a multivariate normal distribution, we obtain for $t \in \mathbb{R}^{2D}$

$$\mathbb{E}\left[\prod_{i=1}^D \exp(t_iV_i - \frac{1}{2}t_i^2) \prod_{i=D+1}^{2D} \exp(t_iW_{i-D} - \frac{1}{2}t_i^2)\right] = \exp\left(\sum_{i,j=1}^D t_it_{D+j}\mathbb{E}[V_iW_j]\right). \tag{3.13}$$

We use Lemma 2.10 (iv) to see the equality of the left side in (3.13) to

$$\sum_{n_1,\ldots,n_D,n'_1,\ldots,n'_D=0}^{\infty} \frac{t_1^{n_1}\cdots t_D^{n_D}t_1^{n'_1}\cdots t_D^{n'_D}}{n!n'!}\mathbb{E}\left[\widetilde{H}_n(V)\widetilde{H}_{n'}(W)\right],$$

where we used [76, Lemma 3.1] to change the order of summation and expectation. The right side in (3.13) equals

$$\begin{aligned}
&\sum_{r=0}^{\infty} \frac{1}{r!}\left(\sum_{i,j=1}^D t_it_{D+j}\mathbb{E}[V_iW_j]\right)^r \\
&= \sum_{r=0}^{\infty} \sum_{d\in\mathbb{N}_0^{D\times D}, \sum_{i,j=1}^D d_{ij}=r} \prod_{1\le i,j\le D} \frac{1}{d_{ij}!}(t_it_{D+j})^{d_{ij}}\mathbb{E}[V_iW_j]^{d_{ij}} \\
&= \sum_{r=0}^{\infty} \sum_{d\in\mathbb{N}_0^{D\times D}, \sum_{i,j=1}^D d_{ij}=r} \prod_{1\le i,j\le D}\left(\frac{\mathbb{E}[V_iW_j]^{d_{ij}}}{d_{ij}!}\right) t_1^{\sum_{k=1}^D d_{1k}}\cdots t_D^{\sum_{k=1}^D d_{Dk}} t_{D+1}^{\sum_{k=1}^D d_{k1}}\cdots t_{2D}^{\sum_{k=1}^D d_{kD}},
\end{aligned}$$

by the multinomial theorem in the first line. Note that the sum over the exponents of the variables $t_1, \ldots, t_D$ equals the one over the exponents of variables $t_{D+1}, \ldots, t_{2D}$, i.e. $\sum_{i=1}^D\sum_{j=1}^D d_{ji} = \sum_{i=1}^D\sum_{j=1}^D d_{ij} = r$. Hence by comparing the coefficients of the resulting equality between power series in t, we obtain for $|n| \ne |n'|$ that

$$\mathbb{E}[\widetilde{H}_n(V)\widetilde{H}_{n'}(W)] = 0.$$

Furthermore for $|n| = |n'|$, the monomial of degree (n, n') corresponds to $r = \frac{1}{2}(|n| + |n'|)$ and can therefore be found in a unique term of the sum over r, which yields the assertion. □

To apply this lemma in our situation, we recall that the process $(Y(L,t))_{(L,t)\in G(d,d-m)\times\mathbb{R}^d}$ is Gaussian and the vector $Y(L,t)$ is standard normal for fixed $(L,t) \in G(d,d-m)\times\mathbb{R}^d$.

Using the preceding lemmas, we can now give a Hermite type expansion of the random variable ζ_N^m. We first define

$$c(n) := (2\pi)^{-(d-m)/2} \prod_{i=1}^{d-m} H_{n_i}(0) \frac{(-1)^{d-m}}{\prod_{i=1}^{D} n_i!} \int_{\mathbb{R}^{D-(d-m)}} \det\left((\Lambda_2 y)_{1,\ldots,(d-m)(d-m+1)/2}\right)$$
$$\times \mathbb{1}\{(\Lambda_2 y)_{D-(d-m)} \geq u\} \prod_{i=d-m+1}^{D} H_{n_i}(y) \phi_{D-(d-m)}(y)\, dy. \tag{3.14}$$

The continuity of the Hermite polynomials and the Gaussian density yield

$$\int_{\mathbb{R}^{d-m}} \delta_\varepsilon^d(x) \prod_{i=1}^{d-m} H_{n_i}(x)\phi_{d-m}(x)\, dx \to (2\pi)^{-(d-m)/2} \prod_{i=1}^{d-m} H_{n_i}(0), \qquad \text{as } \varepsilon \to 0,$$

and we obtain

$$c(n) = \lim_{\varepsilon\to 0} c(G_\varepsilon, n).$$

These are the Hermite coefficients in the Hermite expansion of ζ_N^m as we see in the next lemma. We note that the following expansion is orthogonal due to Lemma 3.7.

Theorem 3.8. *Let X satisfy (A1) – (A3). Then*

$$\zeta_N^m \overset{L^2(\mathbb{P})}{=} \sum_{q\geq 0} \sum_{n\in\mathbb{N}_0^D, |n|=q} \int_{G_{d-m}^d} c(n) \int_{C_N^d} \widetilde{H}_n(Y(L,t))\, dt\, \nu(dL). \tag{3.15}$$

Proof. We show that the right side of the asserted equality is the limit of a Cauchy sequence, which implies that it is a well-defined element in $L^2(\mathbb{P})$. For integers $k_1 < k_2$ we have

$$\mathbb{E}\left[\left(\sum_{q=k_1+1}^{k_2} \sum_{|n|=q} \int_{G_{d-m}^d} c(n) \int_{C_N^d} \widetilde{H}_n(Y(L,t))\, dt\, \nu(dL)\right)^2\right]$$
$$= \sum_{q=k_1+1}^{k_2} \mathbb{E}\left[\left(\sum_{|n|=q} \int_{G_{d-m}^d} c(n) \int_{C_N^d} \widetilde{H}_n(Y(L,t))\, dt\, \nu(dL)\right)^2\right]$$

by the orthogonality established in Lemma 3.7. We note that in order to use the orthogonality, we need Fubini's theorem, which is applicable as a consequence of [76, Lemma 3.1]. An application of Fatou's lemma yields the upper bound

$$\sum_{q=k_1+1}^{k_2} \liminf_{\varepsilon\to 0} \mathbb{E}\left[\left(\sum_{|n|=q} \int_{G_{d-m}^d} c(G_\varepsilon, n) \int_{C_N^d} \widetilde{H}_n(Y(L,t))\, dt\, \nu(dL)\right)^2\right]$$
$$\leq \sum_{q=k_1+1}^{\infty} \liminf_{\varepsilon\to 0} \mathbb{E}\left[\left(\sum_{|n|=q} \int_{G_{d-m}^d} c(G_\varepsilon, n) \int_{C_N^d} \widetilde{H}_n(Y(L,t))\, dt\, \nu(dL)\right)^2\right], \tag{3.16}$$

where we added positive terms in the second line. By adding even more positive terms, another application of Fatou's lemma and the orthogonality together with the continuity of the inner product, the expression in (3.16) is bounded from above by

$$\begin{aligned}\liminf_{\varepsilon\to 0}\mathbb{E}[(\sum_{q=0}^{\infty}\sum_{|n|=q}\int_{G^d_{d-m}} c(G_\varepsilon,n)\int_{C^d_N}\widetilde{H}_n(Y(L,t))\,dt\,\nu(dL))^2] &= \liminf_{\varepsilon\to 0}\mathbb{E}[(\zeta^m_{N,\varepsilon})^2]\\ &= \mathbb{E}[(\zeta^m_N)^2]<\infty,\end{aligned}$$

where we have used Lemma 3.6 and finally Lemma 3.3. Thus (3.16) is the tail of a convergent series, which yields that the sequence is Cauchy.

To show the asserted equality, we define

$$I_q := \sum_{|n|=q}\int_{G^d_{d-m}} c(n)\int_{C^d_N}\widetilde{H}_n(Y(L,t))\,dt\,\nu(dL)$$

and write $\pi^k\colon L^2(\mathbb{P})\to L^2(\mathbb{P})$ for the projection onto the homogeneous chaos of degree 0 up to k, that is onto $\cup_{i=0}^k\mathcal{H}_i$ (cf. Theorem 2.14), and $\pi_k\colon L^2(\mathbb{P})\to L^2(\mathbb{P})$ for the projection onto the homogeneous chaos greater than k, that is $\cup_{i\geq k+1}\mathcal{H}_i$, $k\in\mathbb{N}_0$. We observe that

$$\begin{aligned}&\|\zeta^m_N-\sum_{q=0}^{\infty}I_q\|_{L^2(\mathbb{P})}\\ &\quad\leq \|\pi_k(\zeta^m_N)-\sum_{q=k}^{\infty}I_q\|_{L^2(\mathbb{P})}+\|\pi^k(\zeta^m_N-\zeta^m_{N,\varepsilon})\|_{L^2(\mathbb{P})}+\|\pi^k(\zeta^m_{N,\varepsilon})-\sum_{q=0}^{k}I_q\|_{L^2(\mathbb{P})}\\ &\quad\leq \|\pi_k(\zeta^m_N)\|_{L^2(\mathbb{P})}+\|\sum_{q=k}^{\infty}I_q\|_{L^2(\mathbb{P})}+\|\zeta^m_N-\zeta^m_{N,\varepsilon}\|_{L^2(\mathbb{P})}+\|\pi^k(\zeta^m_{N,\varepsilon})-\sum_{q=0}^{k}I_q\|_{L^2(\mathbb{P})}.\end{aligned}$$

The first two terms tend to 0 for $k\to\infty$, since both functions belong to $L^2(\mathbb{P})$, as does the third one for $\varepsilon\to 0$, due to Lemma 3.3. For the last one we have

$$\begin{aligned}\|\pi^k(\zeta^m_{N,\varepsilon})-\sum_{q=0}^{k}I_q\|_{L^2(\mathbb{P})}=\mathbb{E}\Bigg[\Bigg(&\sum_{q=0}^{k}\sum_{|n|=q}\int_{G^d_{d-m}} c(G_\varepsilon,n)\int_{C^d_N}\widetilde{H}_n(Y(L,t))\,dt\,\nu(dL)\\ &-\sum_{q=0}^{k}\sum_{|n|=q}\int_{G^d_{d-m}}\lim_{\varepsilon\to 0}c(G_\varepsilon,n)\int_{C^d_N}\widetilde{H}_n(Y(L,t))\,dt\,\nu(dL)\Bigg)^2\Bigg],\end{aligned}$$

which equals

$$\begin{aligned}&\sum_{q,q'=0}^{k}\sum_{|n|=q}\sum_{|n'|=q'}\Big(c(G_\varepsilon,n)-\lim_{\varepsilon\to 0}c(G_\varepsilon,n)\Big)\Big(c(G_\varepsilon,n')-\lim_{\varepsilon\to 0}c(G_\varepsilon,n')\Big)\\ &\quad\times\mathbb{E}\left[\int_{G^d_{d-m}}\int_{C^d_N}\widetilde{H}_n(Y(L,t))\,dt\,\nu(dL)\int_{G^d_{d-m}}\int_{C^d_N}\widetilde{H}_{n'}(Y(L,t))\,dt\,\nu(dL)\right].\end{aligned}$$

Hence, the assertion follows by first taking the limit $\varepsilon\to 0$ and then $k\to\infty$. □

3.2.3. Embedding into an isonormal Gaussian process

In this section, we derive a representation of the standardized random variable ζ_N^m in terms of multiple stochastic integrals with respect to a suitable isonormal process. The motivation comes from the theory developed in [59], in which a powerful central limit theorem for isonormal Gaussian processes on Hilbert spaces is established, cf. Theorem 2.15.

Lemma 3.9. *Let X satisfy (A1) – (A3) and let $N > 0$. Then*

$$\frac{\zeta_N^m - \mathbb{E}[\zeta_N^m]}{\mathcal{H}^d(C_N^d)^{1/2}} \stackrel{\mathcal{D}}{=} \sum_{q=1}^{\infty} I_q(g_{N,q}),$$

with I_q denoting the q-th multiple Wiener-Itô integral with respect to the underlying isonormal Gaussian process defined in (3.19) *and*

$$g_{N,q} := \frac{1}{\mathcal{H}^d(C_N^d)^{1/2}} \sum_{k\in\{1,\dots,D\}^q} \int_{G_{d-m}^d} b(k) \int_{C_N^d} \varphi_{t,k_1}^L \otimes \cdots \otimes \varphi_{t,k_q}^L \, dt\, \nu(dL), \tag{3.17}$$

where φ_{t,k_j}^L and $b(\cdot)$ are defined in (3.20) *and* (3.22)*, respectively.*

Proof. We first embed the Gaussian field $\{Y(L,t)\colon \Omega \to \mathbb{R}^D \mid (L,t) \in G_{d-m}^d \times \mathbb{R}^d\}$ into an isonormal process. By standard theory, cf. Theorem 2.4, we obtain for $s,t \in \mathbb{R}^d$

$$\mathbb{E}\left[X(t)X(s)\right] = \int_{\mathbb{R}^d} e^{i\langle t-s,x\rangle} f(x)\, dx,$$

where f denotes the spectral density of X. Recall that the spectral density exists due to (A3). Moreover, Theorem 2.2 combined with the latter equality and a subsequent change in order of differentiation and integration, which is allowed by [67, Theorem 1.2.9], lead to

$$\mathbb{E}\left[\frac{\partial^k}{\partial v_1 \dots \partial v_k} X(t) \frac{\partial^l}{\partial v_1' \dots \partial v_l'} X(s)\right] = (-1)^l \int_{\mathbb{R}^d} g_x(t-s) f(x)\, dx, \tag{3.18}$$

where $g_x := \frac{\partial^{(k+l)}}{\partial v_1 \dots \partial v_k \partial v_1' \dots \partial v_l'} e^{i\langle \cdot, x\rangle}$, $x \in \mathbb{R}^d$, and $k,l \in \{0,1,2\}$, $v_1,\dots,v_k,v_1',\dots,v_l' \in S^{d-1}$. Following [60, Section 9.1] adapted to our setting, we define the real Hilbert space of complex valued functions

$$\mathfrak{H} := \left\{ h\colon \mathbb{R}^d \to \mathbb{C} \mid h(-x) = \overline{h(x)},\ \int_{\mathbb{R}^d} |h(x)|^2 f(x)\, dx < \infty \right\}$$

equipped with the inner product

$$\langle g,h\rangle_{L^2(f\lambda^d)} := \int_{\mathbb{R}^d} g(x)\overline{h(x)} f(x)\, dx,$$

which is real since the functions are Hermitian and the measure $f\lambda^d$ is symmetric. By [59, Prop. 2.1.1], there exists an isonormal Gaussian process W on $\mathfrak{H}$, so that for $g,h \in \mathfrak{H}$

$$\mathbb{E}\left[W(g)W(h)\right] = \langle g,h\rangle_{L^2(f\lambda^d)}. \tag{3.19}$$

Moreover we define for $L \in G^d_{d-m}$, $t \in \mathbb{R}^d$ and $j = 1, \dots, D$ the mapping

$$\varphi^L_{t,j} \colon \mathbb{R}^d \to \mathbb{C}, \quad x \mapsto \sum_{\alpha=1}^{D} \Lambda^{-1}_{j\alpha} \nu_\alpha(L,x) e^{i\langle t,x\rangle} \in \mathfrak{H}, \tag{3.20}$$

where

$$\nu(L,\cdot) \colon \mathbb{R}^d \to \mathbb{C}^D, \quad x \mapsto \big((i\langle v_\alpha, x\rangle)_{1\le\alpha\le d-m}, (-\langle v_\alpha, x\rangle\langle v_\beta, x\rangle)_{1\le\alpha\le\beta\le d-m}, 1\big)$$

and $v_1, \dots, v_{d-m}$ denotes the chosen orthonormal basis b_F of L. We note that $\nu_k(L,x)e^{i\langle\cdot,x\rangle}$ is the directional derivative of $e^{i\langle\cdot,x\rangle}$ of the same order and in the same direction as the derivative of X in the k-th component of $Z(L,\cdot)$. Then we obtain

$$Y(L,t) \overset{\mathcal{D}}{=} \Big(W(\varphi^L_{t,1}), \dots, W(\varphi^L_{t,D})\Big)$$

as processes on $G^d_{d-m} \times \mathbb{R}^d$. To see this, it suffices to show that their covariance structures coincide, since both processes are centered Gaussian processes. By the definition of Y, cf. (3.8), and (3.18), we have for $(L,t), (L',t') \in G^d_{d-m} \times \mathbb{R}^d$ and $i, j \in \{1, \dots, D\}$

$$\begin{aligned}
\mathbb{E}[Y_i(L,t)Y_j(L',t')] &= \sum_{r,s=1}^{D} \Lambda^{-1}_{ir}\Lambda^{-1}_{js}\mathbb{E}[Z_r(L,t)Z_s(L',t')] \\
&= \sum_{r,s=1}^{D} \Lambda^{-1}_{ir}\Lambda^{-1}_{js} \int_{\mathbb{R}^d} \nu_r(L,x)e^{i\langle t,x\rangle}\overline{\nu_s(L',x)e^{i\langle t',x\rangle}} f(x)\, dx \\
&= \langle \varphi^L_{t,i}, \varphi^{L'}_{t',j}\rangle_{L^2(f\lambda^d)}.
\end{aligned} \tag{3.21}$$

By (3.19) we obtain

$$\langle \varphi^L_{t,i}, \varphi^{L'}_{t',j}\rangle_{L^2(f\lambda^d)} = \mathbb{E}[W(\varphi^L_{t,i})W(\varphi^{L'}_{t',j})]$$

and therefore the assertion. Moreover, we observe that

$$\langle \varphi^L_{t,i}, \varphi^L_{t,j}\rangle_{L^2(f\lambda^d)} = \mathbb{E}[Y_i(L,t)Y_j(L,t)] = \delta_{ij},$$

for $i, j = 1, \dots, D$ and $(L,t) \in G^d_{d-m} \times \mathbb{R}^d$. Hence, Theorem 2.13 implies the second equality in

$$\prod_{i=1}^{D} H_{n_i}(Y_i(\cdot,\cdot)) \overset{\mathcal{D}}{=} \prod_{i=1}^{D} H_{n_i}(W(\varphi^{\cdot}_{\cdot,i})) = I_q((\varphi^{\cdot}_{\cdot,1})^{\otimes n_1} \otimes \cdots \otimes (\varphi^{\cdot}_{\cdot,D})^{\otimes n_D}),$$

where $q \in \mathbb{N}$ and $n = (n_1, \dots, n_D) \in \mathbb{N}_0^D$ such that $|n| = q$. The last equation and Theorem 3.8 yield

$$\frac{\zeta^m_N - \mathbb{E}[\zeta^m_N]}{\mathcal{H}^d(C^d_N)^{1/2}} \overset{\mathcal{D}}{=} \sum_{q=1}^{\infty} \frac{1}{\mathcal{H}^d(C^d_N)^{1/2}} \sum_{\substack{n\in\mathbb{N}_0^D \\ |n|=q}} \int_{G^d_{d-m}} c(n) \int_{C^d_N} I_q((\varphi^L_{t,1})^{\otimes n_1} \otimes \cdots \otimes (\varphi^L_{t,D})^{\otimes n_D})\, dt\, \nu(dL),$$

where the right side converges in $L^2(\mathbb{P})$. We now symmetrise the arguments of the stochastic integral. To this end define for $q, D \in \mathbb{N}$ and $n \in \mathbb{N}_0^D$ with $|n| = q$ the set

$$\mathcal{A}_n := \{k \in \{1,\dots,D\}^q : \sum_{j=1}^q \mathbb{1}_{\{i\}}(k_j) = n_i, \forall i = 1,\dots,D\}$$

of multiindices, which contain the number i exactly n_i times, $i = 1,\dots,D$. Note that for $k \in \mathcal{A}_n$ all permutations of k are also in $\mathcal{A}_n$, and moreover, these sets form a partition of the set $\{1,\dots,D\}^q$, i.e. $\{1,\dots,D\}^q = \dot\cup_{n\in\mathbb{N}_0^D,|n|=q}\mathcal{A}_n$. We further define for $k \in \{1,\dots,D\}^q$

$$b(k) := \sum_{n\in\mathbb{N}_0^D,|n|=q} \mathbb{1}\{k \in \mathcal{A}_n\}\frac{c(n)}{|\mathcal{A}_n|}, \tag{3.22}$$

which is symmetric in the components of k. Since the Wiener-Itô integrals are invariant with respect to permutations, cf. Theorem 2.11, we obtain for $n \in \mathbb{N}_0^D$ with $|n| = q$

$$I_q(\varphi_{t,1}^{L\otimes n_1} \otimes \cdots \otimes \varphi_{t,D}^{L\otimes n_D}) = \frac{1}{|\mathcal{A}_n|}\sum_{k\in\mathcal{A}_n} I_q(\varphi_{t,k_1}^L \otimes \cdots \otimes \varphi_{t,k_q}^L)$$

and thus

$$\begin{aligned}\sum_{n\in\mathbb{N}_0^D,|n|=q} c(n)I_q(\varphi_{t,1}^{L\otimes n_1} \otimes \cdots \otimes \varphi_{t,D}^{L\otimes n_D}) &= \sum_{n\in\mathbb{N}_0^D,|n|=q}\sum_{k\in\mathcal{A}_n}\frac{c(n)}{|\mathcal{A}_n|}I_q(\varphi_{t,k_1}^L \otimes \cdots \otimes \varphi_{t,k_q}^L)\\ &= \sum_{k\in\{1,\dots,D\}^q} b(k)I_q(\varphi_{t,k_1}^L \otimes \cdots \otimes \varphi_{t,k_q}^L).\end{aligned}$$

Hence by Fubini's theorem for stochastic integrals (cf. [60, Theorem 5.13.1]) the assertion follows. Finally, we note that the functions $g_{N,q}$ are symmetric, since the coefficients $b(\cdot)$ are symmetric, and furthermore state the following equality

$$I_q(g_{N,q}) \overset{\mathcal{D}}{=} \sum_{n\in\mathbb{N}_0^D,|n|=q}\int_{G_{d-m}^d} c(n)\int_{C_N^d}\widetilde{H}_n(Y(L,t))\,dt\,\nu(dL), \tag{3.23}$$

which can be deduced by the preceding arguments, for later reference. □

3.2.4. Application of a central limit theorem for isonormal processes

In this section, we verify the conditions of Theorem 2.15 in the univariate case, which yields the central limit for the standardized random variable ζ_N^m. Before we start, we need to prove the following lemma, which will be needed for condition (ii) and (iv).

Lemma 3.10. *There exists $K = K(X,d,m) > 0$ such that*

$$\sum_{n\in\mathbb{N}_0^D,|n|=q} c(n)^2 n! \le Kq^D, \qquad \text{for } q \ge 1.$$

Proof. In the following the constant K may change from appearance to appearance. We start by recalling the definition (3.14)

$$c(n) = (2\pi)^{-(d-m)/2} \prod_{i=1}^{d-m} \frac{H_{n_i}(0)}{n_i!} \frac{(-1)^{d-m}}{\prod_{i=d-m+1}^{D} n_i!} z(n),$$

where

$$z(n) := \int_{\mathbb{R}^{D-d+m}} \det\left((\Lambda_2 y)_{1,\dots,\frac{(d-m)(d-m+1)}{2}}\right) \mathbb{1}\{(\Lambda_2 y)_{D-d+m} \geq u\} \prod_{i=d-m+1}^{D} H_{n_i}(y) \phi_{D-d+m}(y)\, dy.$$

Proposition 3 in [32] yields $\prod_{i=1}^{d-m} \frac{|H_{n_i}(0)|}{\sqrt{n_i!}} \leq K$, for a constant $K > 0$, and thus

$$\left((2\pi)^{-(d-m)/2} \prod_{i=1}^{d-m} \frac{H_{n_i}(0)}{n_i!}\right)^2 \leq \frac{K}{\prod_{i=1}^{d-m} n_i!}.$$

By Hölder's inequality and Lemma 2.10 (iii), we obtain

$$\begin{aligned} z(n)^2 \leq & \int_{\mathbb{R}^{D-d+m}} \det\left((\Lambda_2 y)_{1,\dots,\frac{(d-m)(d-m+1)}{2}}\right)^2 \mathbb{1}\{(\Lambda_2 y)_{D-d+m} \geq u\} \phi_{D-d+m}(y)\, dy \\ & \times \int_{\mathbb{R}^{D-d+m}} \left(\prod_{i=d-m+1}^{D} H_{n_i}(y)\right)^2 \phi_{D-d+m}(y)\, dy \\ = & K \prod_{i=d-m+1}^{D} n_i!. \end{aligned}$$

The previous inequalities yield for $q \geq 1$

$$\sum_{n \in \mathbb{N}_0^D, |n|=q} c(n)^2 n! \leq K \sum_{n \in \mathbb{N}_0^D, |n|=q} 1 \leq K \sum_{0 \leq n_1,\dots,n_D \leq q} 1 \leq K(q+1)^D \leq K q^D,$$

which shows the assertion. □

We verify the conditions of Theorem 2.15 and proceed in numerical order. For **condition (i)**, we calculate the norm of $g_{N,q}$, cf. (3.17), by an application of Fubini's theorem and obtain

$$\begin{aligned} & q! \|g_{N,q}\|^2_{\mathfrak{H}^{\otimes q}} \\ & = \frac{q!}{\mathcal{H}^d(C_N^d)} \sum_{k,l \in \{1,\dots,D\}^q} b(k) b(l) \int_{\mathbb{R}^{dq}} \int_{G_{d-m}^d} \int_{G_{d-m}^d} \int_{C_N^d} \int_{C_N^d} \varphi_{t,k_1}^L \otimes \dots \otimes \varphi_{t,k_q}^L (x_1,\dots,x_q) \\ & \quad \times \overline{\varphi_{t',l_1}^{L'}} \otimes \dots \otimes \overline{\varphi_{t',l_q}^{L'}} (x_1,\dots,x_q)\, dt\, dt'\, \nu(dL)\, \nu(dL') \prod_{i=1}^{q} f(x_i)\, d(x_1,\dots,x_q) \\ & = \frac{q!}{\mathcal{H}^d(C_N^d)} \sum_{k,l \in \{1,\dots,D\}^q} b(k) b(l) \int_{G_{d-m}^d} \int_{G_{d-m}^d} \int_{C_N^d} \int_{C_N^d} \prod_{i=1}^{q} \int_{\mathbb{R}^d} \varphi_{t,k_i}^L(x) \overline{\varphi_{t',l_i}^{L'}(x)} \\ & \qquad \times f(x)\, dx\, dt\, dt'\, \nu(dL)\, \nu(dL') \end{aligned}$$

By (3.21) the above equals

$$\frac{q!}{\mathcal{H}^d(C_N^d)} \sum_{k,l \in \{1,\dots,D\}^q} b(k)b(l) \int_{G_{d-m}^d} \int_{G_{d-m}^d} \int_{C_N^d} \int_{C_N^d} \prod_{i=1}^q \mathbb{E}\left[Y_{k_i}(L,t)Y_{l_i}(L',t')\right]\, dt\, dt'\, \nu(dL)\, \nu(dL'). \tag{3.24}$$

Stationarity and Fubini's theorem imply the following equalities

$$\begin{aligned}\int_{C_N^d}\int_{C_N^d} \prod_{i=1}^q \mathbb{E}\left[Y_{k_i}(L,t)Y_{l_i}(L',t')\right]\, dt\, dt' &= \int_{C_N^d}\int_{C_N^d-t'} \prod_{i=1}^q \mathbb{E}\left[Y_{k_i}(L,t+t')Y_{l_i}(L',t')\right]\, dt\, dt' \\ &= \int_{C_{2N}^d} \prod_{i=1}^q \mathbb{E}\left[Y_{k_i}(L,t)Y_{l_i}(L',0)\right] \mathcal{H}^d(C_N^d \cap (C_N^d - t))\, dt.\end{aligned}$$

Thus, by Fubini's theorem, (3.24) equals

$$\begin{aligned} q! \sum_{k,l\in\{1,\dots,D\}^q} b(k)b(l) \int_{G_{d-m}^d}\int_{G_{d-m}^d}\int_{C_{2N}^d} \prod_{i=1}^q \mathbb{E}\left[Y_{k_i}(L,t)Y_{l_i}(L',0)\right] \\ \times \frac{\mathcal{H}^d(C_N^d \cap (C_N^d - t))}{\mathcal{H}^d(C_N^d)}\, dt\, \nu(dL)\, \nu(dL').\end{aligned}$$

The definition of Y, cf. (3.8), yields $Y_i(L,t) = \sum_{k=1}^D \Lambda_{ik}^{-1} Z_k(L,t)$, for $i = 1,\dots,D$ and $t \in \mathbb{R}^d$. Hence, by assumption (A3) there exists a constant $c = c(X,D,q) \geq 0$ such that

$$\left|\prod_{i=1}^q \mathbb{E}\left[Y_i(L,t)Y_j(L',0)\right]\right| \leq c\psi(t),$$

which is an integrable upper bound on $\mathbb{R}^d$. Therefore, the dominated convergence theorem yields

$$\begin{aligned} &q!\|g_{N,q}\|^2_{\mathfrak{H}^{\otimes q}} \\ &\quad = q! \sum_{k,l\in\{1,\dots,D\}^q} b(k)b(l) \\ &\qquad \times \int_{C_{2N}^d}\int_{G_{d-m}^d}\int_{G_{d-m}^d} \frac{\mathcal{H}^d((C_N^d - t)\cap C_N^d)}{\mathcal{H}^d(C_N^d)} \prod_{i=1}^q \mathbb{E}\left[Y_{k_i}(L,t)Y_{l_i}(L',0)\right]\, \nu(dF)\, \nu(dF')\, dt \\ &\quad \overset{N\to\infty}{\longrightarrow} q! \sum_{k,l\in\{1,\dots,D\}^q} b(k)b(l) \int_{\mathbb{R}^d}\int_{G_{d-m}^d}\int_{G_{d-m}^d} \prod_{i=1}^q \mathbb{E}\left[Y_{k_i}(L,t)Y_{l_i}(L',0)\right]\, \nu(dL)\, \nu(dL')\, dt, \end{aligned} \tag{3.25}$$

where we define the limit as $\sigma_{m,q}^2$. Note that we implicitly used $\mathcal{H}^d((C_N^d - t)\cap C_N^d)/\mathcal{H}^d(C_N^d) \to 1$ for $N \to \infty$ and $t \in \mathbb{R}^d$. To see this, consider the discussion following equation (3.22) in [31], where the assertion is shown in a far more general case. This establishes the first condition.

We now verify **condition (ii)**. We first observe that

$$\sum_{q=1}^{\infty} \lim_{N\to\infty} q!\|g_{N,q}\|^2_{\mathfrak{H}^{\otimes q}} = \sum_{q=1}^{\infty} \lim_{N\to\infty} \mathbb{E}\left[I_q(g_{N,q})^2\right]$$

by Lemma 2.12. Reversing some of the earlier manipulations, cf. (3.23), this series equals

$$\sum_{q=1}^{\infty} \lim_{N\to\infty} \frac{1}{\mathcal{H}^d(C_N^d)} \mathbb{E}\left[\left(\sum_{n\in\mathbb{N}_0^D, |n|=q} \int_{G_{d-m}^d} c(n) \int_{C_N^d} \widetilde{H}_n(Y(L,t))\, dt\, \nu(dL)\right)^2\right].$$

Fatou's lemma and orthogonality together with the continuity of the inner product yield the upper bound

$$\liminf_{N\to\infty} \frac{1}{\mathcal{H}^d(C_N^d)} \mathbb{E}\left[\left(\sum_{q=1}^{\infty} \sum_{n\in\mathbb{N}_0^D, |n|=q} \int_{G_{d-m}^d} c(n) \int_{C_N^d} \widetilde{H}_n(Y(L,t))\, dt\, \nu(dL)\right)^2\right]. \tag{3.26}$$

Partitioning the space $\mathbb{R}^d$ into translates of the unit cube $[0,1)^d$, expression (3.26) without the limit inferior equals

$$\begin{aligned} \frac{1}{\mathcal{H}^d(C_N^d)} \sum_{z_1,z_2\in\mathbb{Z}^d} \mathbb{E}\Bigg[&\sum_{q=1}^{\infty} \sum_{|n|=q} \int_{G_{d-m}^d} c(n) \int_{[0,1)^d+z_1} \mathbb{1}\{t\in C_N^d\} \widetilde{H}_n(Y(L,t))\, dt\, \nu(dL) \\ &\times \sum_{q=1}^{\infty} \sum_{|n|=q} \int_{G_{d-m}^d} c(n) \int_{[0,1)^d+z_2} \mathbb{1}\{t\in C_N^d\} \widetilde{H}_n(Y(L,t))\, dt\, \nu(dL)\Bigg]. \end{aligned} \tag{3.27}$$

We define

$$\tau(L,L',t) := \max\{\max_i \sum_{k=1}^{D} |\mathbb{E}\left[Y_i(L,0)Y_k(L',t)\right]|, \max_k \sum_{i=1}^{D} |\mathbb{E}\left[Y_i(L,0)Y_k(L',t)\right]|\}$$

for $t\in\mathbb{R}^d$, $L,L'\in G_{d-m}^d$, and note that due to (A3) there is a constant $c>0$, such that $\tau(L,L',t)\le c\psi(t)$. Moreover (A3) implies that for $\rho\in(0,1)$ and $\rho<1/c$ there is a constant $s>0$ such that

$$\psi(t)\le\rho, \text{ for } \|t\|\ge s.$$

Using s, we split the above summation into one over $I_1 := \{(z_1,z_2)\in(\mathbb{Z}^d)^2 \mid \|z_1-z_2\|_\infty \ge s+1\}$ and $I_2 := \{(z_1,z_2)\in(\mathbb{Z}^d)^2 \mid \|z_1-z_2\|_\infty \le s\}$. By Fubini's theorem and orthogonality the first sum equals

$$\begin{aligned} \frac{1}{\mathcal{H}^d(C_N^d)} \sum_{(z_1,z_2)\in I_1} \sum_{q=1}^{\infty} \int_{G_{d-m}^d} \int_{G_{d-m}^d} \int_{[0,1)^d+z_2} \int_{[0,1)^d+z_1} \mathbb{1}\{t\in C_N^d\}\mathbb{1}\{t'\in C_N^d\} \\ \times \mathbb{E}\left[\sum_{|n|=q} c(n)\widetilde{H}_n(Y(L,t)) \sum_{|n|=q} c(n)\widetilde{H}_n(Y(L',t'))\right] dt\, dt'\, \nu(dL)\, \nu(dL'). \end{aligned}$$

Then the translation invariance of the Lebesgue measure and stationarity imply equality of the inner integrals to

$$\int_{[0,1)^d+z_2}\int_{[0,1)^d+z_1}\mathbb{1}\{t\in C_N^d\}\mathbb{1}\{t'\in C_N^d\}\mathbb{E}\left[\sum_{|n|=q}c(n)\widetilde{H}_n(Y(L,t))\sum_{|n|=q}c(n)\widetilde{H}_n(Y(L',t'))\right]dt\,dt'$$
$$=\int_{[0,1)^d}\int_{[0,1)^d-t'}\mathbb{1}\{t+t'+z_1\in C_N^d\}\mathbb{1}\{t'+z_2\in C_N^d\}$$
$$\times\mathbb{E}\left[\sum_{|n|=q}c(n)\widetilde{H}_n(Y(L,t+t'+z_1))\sum_{|n|=q}c(n)\widetilde{H}_n(Y(L',t'+z_2))\right]dt\,dt'$$
$$=\int_{[0,1)^d}\int_{[-1,1)^d}\mathbb{1}\{t\in[0,1)^d-t'\}\mathbb{1}\{t+t'+z_1\in C_N^d\}\mathbb{1}\{t'+z_2\in C_N^d\}$$
$$\times\mathbb{E}\left[\sum_{|n|=q}c(n)\widetilde{H}_n(Y(L,t+z_1))\sum_{|n|=q}c(n)\widetilde{H}_n(Y(L',z_2))\right]dt\,dt',$$

which equals by Fubini's theorem

$$\int_{[-1,1)^d}\mathbb{E}\left[\sum_{|n|=q}c(n)\widetilde{H}_n(Y(L,t+z_1))\sum_{|n|=q}c(n)\widetilde{H}_n(Y(L',z_2))\right]$$
$$\times\mathcal{H}^d\left(([0,1)^d-t)\cap(C_N^d-t-z_1)\cap(C_N^d-z_2)\cap[0,1)^d\right)dt.$$

Thus, the first summand of (3.27) equals

$$\sum_{(z_1,z_2)\in I_1}\sum_{q=1}^{\infty}\int_{G_{d-m}^d}\int_{G_{d-m}^d}\int_{[-1,1)^d}\mathbb{E}\left[\sum_{|n|=q}c(n)\widetilde{H}_n(Y(L,t+z_1))\sum_{|n|=q}c(n)\widetilde{H}_n(Y(L',z_2))\right]$$
$$\times\frac{1}{\mathcal{H}^d(C_N^d)}\mathcal{H}^d\left(([0,1)^d-t)\cap(C_N^d-t-z_1)\cap(C_N^d-z_2)\cap[0,1)^d\right)dt\,\nu(dL)\,\nu(dL'). \quad (3.28)$$

Now, we use Lemma 1 in [2], which we state for completeness. Alternatively, an approach using Lemma 3.18 and a variant of Lemma 3.19 could also be used. We stick to the one using Arcones Lemma for diversity reasons.

Lemma 3.11 (Arcones 1994)**.** *Let V,W be centered d-dimensional Gaussian random vectors such that $\mathbb{E}[V_iV_j]=\mathbb{E}[W_iW_j]=\delta_{ij}$ and let $h\colon\mathbb{R}^d\to\mathbb{R}\in L^2(\mathcal{N}_d(0,I_d))$ have Hermite rank $r\in\mathbb{N}$ (i.e. $r=\inf\{k\in\mathbb{N}:\exists l_j$ such that $\sum_{j=1}^d l_j=k$ and $\mathbb{E}[(h(N)-\mathbb{E}[h(N)])\widetilde{H}_l(N)]\neq 0\}$ where $N\sim\mathcal{N}_d(0,I_{d\times d})$). Moreover, we define*

$$\tau:=\max\{\max_{1\le j\le d}\sum_{k=1}^d|\mathbb{E}[V_jW_k]|,\max_{1\le k\le d}\sum_{j=1}^d|\mathbb{E}[V_jW_k]|\},$$

which is assumed to be less than 1. Then we have

$$|\mathbb{E}[(h(V)-\mathbb{E}[h(V)])(h(W)-\mathbb{E}[h(W)])]|\le\tau^r\mathbb{E}[h(V)^2].$$

To apply the Lemma, we choose $V=Y(L',z_2)$, $W=Y(L,t+z_1)$ and $h_q\colon\mathbb{R}^D\to\mathbb{R}$ given by

$h_q(x) := \sum_{n\in\mathbb{N}_0^D,|n|=q} c(n)\widetilde{H}_n(x)$. Then, we have $r = q$ by Lemma 2.10 (ii) and since h_q can be assumed to be nonzero. Furthermore $\tau(L, L', t + z_1 - z_2) \le c\psi(t + z_1 - z_2) < 1$ for $t \in [-1, 1)^d$ and $z_1, z_2 \in I_1$. Moreover we have

$$\begin{aligned}
\mathbb{E}\left[h_q(Y(L', z_2))^2\right] &= \sum_{n,n'\in\mathbb{N}_0^D,|n|=|n'|=q} c(n')c(n)\mathbb{E}[\widetilde{H}_n(Y(L',0))\widetilde{H}_{n'}(Y(L',0))] \\
&= \sum_{n,n'\in\mathbb{N}_0^D,|n|=|n'|=q} c(n')c(n)\prod_{i=1}^{D}\mathbb{E}[H_{n_i}(Y_i(L',0))H_{n_i'}(Y_i(L',0))] \\
&= \sum_{n\in\mathbb{N}_0^D,|n|=q} c(n)^2 n!
\end{aligned}$$

and for $q \ge 1$ we obtain $\mathbb{E}\left[h_q(Y(L', z_2))\right] = \mathbb{E}\left[h_q(Y(L, t + z_1))\right] = 0$. Hence we bound (3.28) by

$$\nu(G_{d-m}^d)^2\mathcal{H}^d(C_N^d)^{-1} \sum_{\substack{(z_1,z_2)\in I_1 \\ (z_2+[0,1)^d)\cap C_N^d\neq\emptyset}} \sum_{q=1}^{\infty}\int_{[-1,1)^d} c^q\psi(t+z_1-z_2)^q\,dt \sum_{n\in\mathbb{N}_0^D,|n|=q} c(n)^2 n!.$$

Lemma 3.10 and $\psi(t + z_1 - z_2)^q \le \rho^{q-1}\psi(t + z_1 - z_2)$ yield

$$\begin{aligned}
K\nu(G_{d-m}^d)^2(\rho\mathcal{H}^d(C_N^d))^{-1} &\sum_{(z_1,z_2)\in I_1} \mathbb{1}\{(z_2+[0,1)^d)\cap C_N^d\neq\emptyset\} \\
&\times \int_{[-1,1)^d}\psi(t+z_1-z_2)\,dt\sum_{q=1}^{\infty} q^D(c\rho)^q,
\end{aligned} \tag{3.29}$$

as an upper bound, where K is the constant coming from Lemma 3.10. By the estimate

$$\sum_{z_1\in\mathbb{Z}^d}\int_{(-1,1)^d+z_1-z_2}\psi(t)\,dt \le 2^d\int_{\mathbb{R}^d}\psi(t)\,dt,$$

for fixed $z_2 \in \mathbb{Z}^d$, we obtain

$$\begin{aligned}
\sum_{\substack{(z_1,z_2)\in I_1 \\ z_2+[0,1)^d\cap C_N^d\neq\emptyset}} \int_{[-1,1)^d}\psi(t+z_1-z_2)\,dt &\le \sum_{\substack{z_2\in\mathbb{Z}^d \\ z_2+[0,1)^d\cap C_N^d\neq\emptyset}} \sum_{z_1\in\mathbb{Z}^d}\int_{[-1,1)^d+z_1-z_2}\psi(t)\,dt \\
&\le (2N+1)^d 2^d\int_{\mathbb{R}^d}\psi(t)\,dt.
\end{aligned}$$

Hence, we get for the term (3.29) the upper bound

$$\frac{2^dK\nu(G_{d-m}^d)^2}{\rho}\frac{(2N+1)^d}{\mathcal{H}^d(C_N^d)}\int_{\mathbb{R}^d}\psi(t)\,dt\sum_{q=1}^{\infty}q^D(c\rho)^q.$$

This expression is finite in the limit inferior since

$$\lim_{N\to\infty}\frac{(2N+1)^d}{\mathcal{H}^d(C_N^d)} = 1$$

and since the series converges by the ratio test.

We now analyse the sum over I_2 and start by using the inequality $ab \leq a^2 + b^2$, $a, b \in \mathbb{R}$, to obtain the upper bound

$$\frac{1}{\mathcal{H}^d(C_N^d)} \sum_{z_1 \in \mathbb{Z}^d} \sum_{\substack{z_2 \in \mathbb{Z}^d \\ \|z_2 - z_1\|_\infty \leq s}} \mathbb{E}\left[\left(\sum_{q=1}^{\infty} \sum_{|n|=q} \int_{G_{d-m}^d} c(n) \int_{[0,1)^d + z_1} \mathbb{1}\{t \in C_N^d\} \widetilde{H}_n(Y(L,t))\, dt\, \nu(dL)\right)^2\right]$$
$$+ \frac{1}{\mathcal{H}^d(C_N^d)} \sum_{z_2 \in \mathbb{Z}^d} \sum_{\substack{z_1 \in \mathbb{Z}^d \\ \|z_2 - z_1\|_\infty \leq s}} \mathbb{E}\left[\left(\sum_{q=1}^{\infty} \sum_{|n|=q} \int_{G_{d-m}^d} c(n) \int_{[0,1)^d + z_2} \mathbb{1}\{t \in C_N^d\} \widetilde{H}_n(Y(L,t))\, dt\, \nu(dL)\right)^2\right],$$

wich equals

$$\frac{2(2s+1)^d}{\mathcal{H}^d(C_N^d)} \sum_{z \in \mathbb{Z}^d} \mathbb{E}\left[\left(\sum_{q=1}^{\infty} \sum_{|n|=q} \int_{G_{d-m}^d} c(n) \int_{[0,1)^d + z} \mathbb{1}\{t \in C_N^d\} \widetilde{H}_n(Y(L,t))\, dt\, \nu(dL)\right)^2\right]. \quad (3.30)$$

We define the vector $x := (\frac{1}{2}, \dots, \frac{1}{2}) \in \mathbb{R}^d$ such that $[0,1)^d - x = \left[-\frac{1}{2}, \frac{1}{2}\right)^d$. Then, by the translation invariance of the Lebesgue measure the expectation in expression (3.30) is given by expectation

$$\mathbb{E}\left[\left(\sum_{q=1}^{\infty} \sum_{|n|=q} \int_{G_{d-m}^d} c(n) \int_{[0,1)^d - x} \mathbb{1}\{t + x + z \in C_N^d\} \widetilde{H}_n(Y(L,t+z+x))\, dt\, \nu(dL)\right)^2\right]$$
$$= \mathbb{E}\left[\left(\sum_{q=1}^{\infty} \sum_{|n|=q} \int_{G_{d-m}^d} c(n) \int_{[-\frac{1}{2},\frac{1}{2})^d} \mathbb{1}\{t + x + z \in C_N^d\} \widetilde{H}_n(Y(L,t))\, dt\, \nu(dL)\right)^2\right],$$

where we used Fubini's theorem and the stationarity of the process $Y(L,\cdot)$ in the equality. Since $\mathbb{1}\{t+x+z \in C_N^d\} = 1$ for all $t \in [-\frac{1}{2}, \frac{1}{2})^d$ and $z \in [-N,N)^d \cap \mathbb{Z}^d$ as well as $\mathbb{1}\{t+x+z \in C_N^d\} = 0$ for almost all $t \in [-\frac{1}{2}, \frac{1}{2})^d$ and $z \in ([-N,N)^d)^c \cap \mathbb{Z}^d$, we obtain that the expectation is independent of the sum over $z \in \mathbb{Z}^d$ if it is nonzero. Hence, the term in expression (3.30) is given by

$$2(2s+1)^d \frac{(2N)^d}{\mathcal{H}^d(C_N^d)} \mathbb{E}\left[\left(\sum_{q=1}^{\infty} \sum_{|n|=q} \int_{G_{d-m}^d} c(n) \int_{[-\frac{1}{2},\frac{1}{2})^d} \widetilde{H}_n(Y(L,t))\, dt\, \nu(dL)\right)^2\right]$$

which is independent of N, since $(2N)^d / \mathcal{H}^d(C_N^d) = 1$ by definition of C_N^d as the hypercube of side length $2N$, and moreover finite by Theorem 3.8.

We start the verification of **condition (iii)** with the calculation of the r-th contraction of

$g_{N,q}$, cf. (3.17), with itself. This is given by

$$\begin{aligned}
&g_{N,q} \otimes_r g_{N,q}(a_1, \dots, a_{2q-2r}) \\
&= \int_{\mathbb{R}^{dr}} \frac{1}{\mathcal{H}^d(C_N^d)} \sum_{k \in \{1,\dots,D\}^q} b(k) \int_{G_{d-m}^d} \int_{C_N^d} \varphi_{t,k_1}^L(x_1) \cdots \varphi_{t,k_r}^L(x_r) \\
&\quad \times \varphi_{t,k_{r+1}}^L(a_1) \cdots \varphi_{t,k_q}^L(a_{q-r})\, dt\, \nu(dL) \sum_{l \in \{1,\dots,D\}^q} b(l) \int_{G_{d-m}^d} \int_{C_N^d} \overline{\varphi_{t',l_1}^{L'}(x_1)} \cdots \overline{\varphi_{t',l_r}^{L'}(x_r)} \\
&\quad \times \varphi_{t',l_{r+1}}^{L'}(a_{q-r+1}) \cdots \varphi_{t',l_q}^{L'}(a_{2q-2r})\, dt'\, \nu(dL') \prod_{i=1}^r f(x_i)\, d(x_1, \dots, x_r),
\end{aligned}$$

for $a_1, \dots, a_{2q-2r} \in \mathbb{R}^d$. By Fubini's theorem and (3.21) the above equals

$$\begin{aligned}
&\frac{1}{\mathcal{H}^d(C_N^d)} \sum_{k,l \in \{1,\dots,D\}^q} b(k)b(l) \int_{G_{d-m}^d} \int_{G_{d-m}^d} \int_{C_N^d} \int_{C_N^d} \prod_{i=1}^r \int_{\mathbb{R}^d} \varphi_{t,k_i}^L(x) \overline{\varphi_{t',l_i}^{L'}(x)} f(x)\, dx \\
&\qquad \times \prod_{i=r+1}^q \varphi_{t,k_i}^L(a_{i-r}) \varphi_{t',l_i}^{L'}(a_{q-2r+i})\, dt\, dt'\, \nu(dL)\, \nu(dL') \\
&\quad = \frac{1}{\mathcal{H}^d(C_N^d)} \sum_{k,l \in \{1,\dots,D\}^q} b(k)b(l) \int_{G_{d-m}^d} \int_{G_{d-m}^d} \int_{C_N^d} \int_{C_N^d} \prod_{i=1}^r \mathbb{E}[Y_{k_i}(L,t) Y_{l_i}(L',t')] \\
&\qquad \times \prod_{i=r+1}^q \varphi_{t,k_i}^L(a_{i-r}) \varphi_{t',l_i}^{L'}(a_{q-2r+i})\, dt\, dt'\, \nu(dL)\, \nu(dL').
\end{aligned}$$

Thus we obtain for the norm

$$\begin{aligned}
&\|g_{N,q} \otimes_r g_{N,q}\|^2_{\mathfrak{H}^{\otimes(2q-2r)}} \\
&= \int_{\mathbb{R}^{d(2q-2r)}} \frac{1}{\mathcal{H}^d(C_N^d)} \sum_{k,l \in \{1,\dots,D\}^q} b(k)b(l) \int_{G_{d-m}^d} \int_{G_{d-m}^d} \int_{C_N^d} \int_{C_N^d} \prod_{i=1}^r \mathbb{E}[Y_{k_i}(L,t) Y_{l_i}(L',t')] \\
&\quad \times \prod_{i=r+1}^q \varphi_{t,k_i}^L(a_{i-r}) \varphi_{t',l_i}^{L'}(a_{q-2r+i})\, dt\, dt'\, \nu(dL)\, \nu(dL') \\
&\quad \times \frac{1}{\mathcal{H}^d(C_N^d)} \sum_{k,l \in \{1,\dots,D\}^q} b(k)b(l) \int_{G_{d-m}^d} \int_{G_{d-m}^d} \int_{C_N^d} \int_{C_N^d} \prod_{i=1}^r \mathbb{E}[Y_{k_i}(L,t) Y_{l_i}(L',t')] \\
&\quad \times \prod_{i=r+1}^q \overline{\varphi_{t,k_i}^L(a_{i-r}) \varphi_{t',l_i}^{L'}(a_{q-2r+i})}\, dt\, dt'\, \nu(dL)\, \nu(dL') \prod_{i=1}^{2q-2r} f(a_i)\, d(a_1, \dots, a_{2q-2r}).
\end{aligned}$$

And again Fubini's theorem yields equality to

$$\begin{aligned}
&\frac{1}{\mathcal{H}^d(C_N^d)^2} \sum_{k,l,k',l' \in \{1,\dots,D\}^q} b(k)b(l)b(k')b(l') \int_{(G_{d-m}^d)^4} \int_{(C_N^d)^4} \\
&\qquad \times \prod_{i=r+1}^q \int_{\mathbb{R}^d} \varphi_{t_1,k_i}^{L_1}(x) \overline{\varphi_{t_3,k_i'}^{L_3}(x)} f(x)\, dx \int_{\mathbb{R}^d} \varphi_{t_2,l_i}^{L_2}(x) \overline{\varphi_{t_4,l_i'}^{L_4}(x)} f(x)\, dx \\
&\qquad \times \prod_{i=1}^r \mathbb{E}[Y_{k_i}(L_1,t_1) Y_{l_i}(L_2,t_2)] \mathbb{E}[Y_{k_i'}(L_3,t_3) Y_{l_i'}(L_4,t_4)]\, d(t_1, \dots, t_4)\, \nu^4(d(L_1, \dots, L_4),
\end{aligned}$$

which by (3.21) equals

$$\frac{1}{\mathcal{H}^d(C_N^d)^2} \sum_{k,l,k',l' \in \{1,\dots,D\}^q} b(k)b(l)b(k')b(l')$$
$$\times \int_{(G_{d-m}^d)^4} \int_{(C_N^d)^4} \prod_{i=1}^{r} \mathbb{E}[Y_{k_i}(L_1,t_1)Y_{l_i}(L_2,t_2)]\mathbb{E}[Y_{k_i'}(L_3,t_3)Y_{l_i'}(L_4,t_4)]$$
$$\times \prod_{i=r+1}^{q} \mathbb{E}[Y_{k_i}(L_1,t_1)Y_{k_i'}(L_3,t_3)]\mathbb{E}[Y_{l_i}(L_2,t_2)Y_{l_i'}(L_4,t_4)]\, d(t_1,\dots,t_4)\, \nu^4(d(L_1,\dots,L_4).$$

By (A3) and stationarity there exists a constant $c > 0$ such that for all $t \in \mathbb{R}^d$ and $L, L' \in G_{d-m}^d$

$$\max_{1 \le i,j \le D} \left| \mathbb{E}\left[Y_i(L,t)Y_j(L',s) \right] \right| \le c\psi(t-s)$$

and hence

$$\begin{aligned}
&\|g_{N,q} \otimes_r g_{N,q}\|^2_{\mathfrak{H}^{\otimes(2q-2r)}} \\
&\quad \le \frac{c^{2q}}{\mathcal{H}^d(C_N^d)^2} \sum_{k,l,k',l' \in \{1,\dots,D\}^q} b(k)b(l)b(k')b(l') \int_{(G_{d-m}^d)^4} \int_{(C_N^d)^4} \psi(t_1-t_2)^r \\
&\quad\quad \times \psi(t_3-t_4)^r \psi(t_1-t_3)^{q-r} \psi(t_2-t_4)^{q-r}\, d(t_1,\dots,t_4)\, \nu^4(d(L_1,\dots,dL_4)) \\
&\quad = c^{2q} \nu(G_{d-m}^d)^4 \sum_{k,l,k',l' \in \{1,\dots,D\}^q} b(k)b(l)b(k')b(l') z(N),
\end{aligned}$$

where

$$z(N) := \frac{1}{\mathcal{H}^d(C_N^d)^2} \int_{(C_N^d)^4} \psi(t_1-t_2)^r \psi(t_3-t_4)^r \psi(t_1-t_3)^{q-r} \psi(t_2-t_4)^{q-r}\, d(t_1,\dots,t_4).$$

Now, the inequality $1 \le (a/b)^{q-r} + (b/a)^r$, for $a, b > 0$, implies $a^r b^{q-r} \le a^q + b^q$ for $a, b \ge 0$, since for $a = 0$ or $b = 0$ the second inequality holds trivially. An application of the second inequality for $a = \psi(t_3 - t_4)$ and $b = \psi(t_1 - t_3)$ yields

$$\begin{aligned}
z(N) &\le \frac{1}{\mathcal{H}^d(C_N^d)^2} \int_{(C_N^d)^4} \psi(t_1-t_2)^r \psi(t_3-t_4)^q \psi(t_2-t_4)^{q-r}\, d(t_1,\dots,t_4) \\
&\quad + \frac{1}{\mathcal{H}^d(C_N^d)^2} \int_{(C_N^d)^4} \psi(t_1-t_2)^r \psi(t_1-t_3)^q \psi(t_2-t_4)^{q-r}\, d(t_1,\dots,t_4). \qquad (3.31)
\end{aligned}$$

Assumption (A3) implies that

$$\infty > c_n := \int_{\mathbb{R}^d} \psi(x)^n\, dx \ge \int_{C_N^d} \psi(x)^n\, dx,$$

for $n \in \mathbb{N}$, and by Fubini's theorem the first summand in (3.31) equals

$$\frac{1}{\mathcal{H}^d(C_N^d)^2} \int_{(C_N^d)^3} \psi(t_1-t_2)^r \psi(t_2-t_4)^{q-r} \int_{\mathbb{R}^d} \psi(t_3-t_4)^q\, dt_3\, d(t_1,t_2,t_4).$$

This is then bounded from above by

$$\frac{c_q}{\mathcal{H}^d(C_N^d)^2}\int_{C_N^d}\int_{C_N^d}\psi(t_1-t_2)^r\int_{\mathbb{R}^d}\psi(t_2-t_4)^{q-r}\,dt_4\,dt_1\,dt_2.$$

Repeating this argument yields the upper bound

$$\frac{c_q c_{q-r}}{\mathcal{H}^d(C_N^d)^2}\int_{C_N^d}\int_{\mathbb{R}^d}\psi(t_1-t_2)^r\,dt_1\,dt_2=\frac{c_q c_{q-r}c_r}{\mathcal{H}^d(C_N^d)^2}\mathcal{H}^d(C_N^d)\overset{N\to\infty}{\longrightarrow}0.$$

Proceeding analogously for the second summand yields

$$\|g_{N,q}\otimes_r g_{N,q}\|^2_{\mathfrak{H}^{\otimes(2q-2r)}}\le c^{2q}\sum_{k,l,k',l'\in\{1,\dots,D\}^q}b(k)b(l)b(k')b(l')z(N)\overset{N\to\infty}{\longrightarrow}0$$

and we established the validity of condition (iii).

To verify **condition (iv)**, we start analogously as in the verification of condition (ii), by applying [59, Prop 2.7.5] to see the identity

$$\sum_{q=Q+1}^{\infty}q!\|g_{N,q}\|^2_{\mathfrak{H}^{\otimes q}}=\sum_{q=Q+1}^{\infty}\mathbb{E}\left[I_q(g_{N,q})^2\right].$$

Now, in order to work with the original structure of the random variable, we reverse some of the earlier manipulations leading to the definition of the function $g_{N,q}$ and obtain the equality of the series to

$$\mathcal{H}^d(C_N^d)^{-1}\mathbb{E}\left[\left(\sum_{q=Q+1}^{\infty}\sum_{|n|=q}\int_{G^d_{d-m}}c(n)\int_{C_N^d}\widetilde{H}_n(Y(L,t))\,dt\,\nu(dL)\right)^2\right].$$

A repetition of the arguments in the verification of condition (ii) yields the upper bound

$$\begin{aligned}&\frac{2^dK\nu(G^d_{d-m})^2}{\rho}\frac{(2N+1)^d}{\mathcal{H}^d(C_N^d)}\int_{\mathbb{R}^d}\psi(t)\,dt\sum_{q=Q+1}^{\infty}q^D(c\rho)^q\\&\qquad+2(2s+1)^d\mathbb{E}\left[\left(\sum_{q=Q+1}^{\infty}\sum_{|n|=q}\int_{G^d_{d-m}}c(n)\int_{[-\frac{1}{2},\frac{1}{2})^d}\widetilde{H}_n(Y(L,t))\,dt\,\nu(dL)\right)^2\right].\end{aligned}$$

The first term vanishes in the limit $N\to\infty$ and then $Q\to\infty$, since $\frac{(2N+1)^d}{\mathcal{H}^d(C_N^d)}\to 1$ as $N\to\infty$ and the series is the tail of a convergent series. The second term is independent of N and vanishes for $Q\to\infty$ by Theorem 3.8. This establishes condition (iv).

3.2.5. The Boundary terms

In this section, we exploit the scaling behaviour of the centered integrated counting variables on the boundary of the window defined in (3.3) to show that they are asymptotically negligible if suitably normalised, cf. equation (3.41) for the formal statement.

The strategy of proving the latter is to establish a Hermite type expansion for $\epsilon^m_{J_N}$, closely resembling the approach in Section 3.2.2. Although the following proof of the Hermite type expansion is similar to the previous one, the more complex structure of the involved counting variables leads to more complicated details, which is why we are going to spell out these details. Once we have obtained the Hermite type expansion, we use calculations, similar to the ones performed in the verification of the conditions of Theorem 2.15 in Section 3.2.4, to show the asserted asymptotic behaviour. Due to the different structure of the set $J_N \cap F$ in the cases $\dim J_N = m = d - \dim F$ and $\dim J_N > m = d - \dim F$, we will distinguish these and derive the Hermite expansion only in the second case.

We start with introducing the approximation

$$\epsilon^m_{J_N,\varepsilon} := (-1)^{l-m} \int_{A^d_{d-m}} \int_{J_N \cap F} \delta^l_\varepsilon(\nabla(X|_{J_N\cap F})(t)) \det(D^2(X|_{J_N\cap F})(t)) \\ \times \mathbb{1}\{X(t) \geq u, \pi_{(\mathrm{aff}(J_N)^\circ \cap F^\circ)^\perp}(\nabla X(t)) \in N_t(C^d_N \cap F)\} \mathcal{H}^{l-m}(dt)\, \mu(dF) \quad (3.32)$$

for $m \leq l < d$ and $J_N \in \partial^l C^d_N$ and proceed with proving the counterpart to Lemma 3.2 (iii).

Lemma 3.12. *Let $J_N \in \partial^l C^d_N$ and $m < l < d$. Then for almost all $F \in A^d_{d-m}$*

$$\xi_{J_N}(F,\varepsilon) \overset{L^2(\mathbb{P})}{\longrightarrow} \xi_{J_N}(F), \qquad \text{as } \varepsilon \to 0,$$

where

$$\xi_{J_N}(F,\varepsilon) := (-1)^{l-m} \int_{J_N\cap F} \delta^l_\varepsilon(\nabla(X|_{J_N\cap F})(t)) \det(D^2(X|_{J_N\cap F})(t)) \\ \times \mathbb{1}\{X(t) \geq u, \pi_{(\mathrm{aff}(J_N)^\circ \cap F^\circ)^\perp}(\nabla X(t)) \in N_t(C^d_N \cap F)\} \mathcal{H}^{l-m}(dt),$$

and

$$\xi_{J_N}(F) := \#\{t \in J_N \cap F \mid X(t) \geq u, \nabla(X|_{J_N\cap F})(t) = 0, \\ \iota^{-X}_{J_N\cap F}(t)\,\text{even}, \pi_{(\mathrm{aff}(J_N)^\circ \cap F^\circ)^\perp}(\nabla X(t)) \in N_t(C^d_N \cap F)\} \\ - \#\{t \in J_N \cap F \mid X(t) \geq u, \nabla(X|_{J_N\cap F})(t) = 0, \\ \iota^{-X}_{J_N\cap F}(t)\,\text{odd}, \pi_{(\mathrm{aff}(J_N)^\circ \cap F^\circ)^\perp}(\nabla X(t)) \in N_t(C^d_N \cap F)\}.$$

Proof. The assertion can be deduced analogously to Lemma 3.2 (iii), except that we use Lemma A.4 instead of Lemma A.3, whose conditions are true by [1, Lemma 11.2.10 - 11.2.12]. □

With the preceding lemma, we are able to show that the claimed approximation of the error terms coming from the boundary of the observation window is a proper approximation in $L^2(\mathbb{P})$.

Lemma 3.13. *Let $J_N \in \partial_l C^d_N$ and $m < l < d$. Then*

$$\epsilon^m_{J_N,\varepsilon} \overset{L^2(\mathbb{P})}{\longrightarrow} \epsilon^m_{J_N}, \qquad \text{as } \varepsilon \to 0,$$

where $\epsilon^m_{J_N,\varepsilon}$ is defined in (3.32) and $\epsilon^m_{J_N}$ in (3.5).

Proof. First, we note that by definitions (3.5) and (3.32) as well as (2.15)

$$\mathbb{E}\left[(\epsilon^m_{J_N} - \epsilon^m_{J_N,\varepsilon})^2\right] = \mathbb{E}\left[\left(\int_{A^d_{d-m}} \xi_{J_N}(F) - \xi_{J_N}(F,\varepsilon)\,\mu(dF)\right)^2\right].$$

Then, after realising that

$$\mathbb{E}\left[\xi_{J_N}(F)^2\right] \leq \mathbb{E}\left[\#\{t \in J_N \cap F \mid \nabla(X|_{J_N \cap F})(t) = 0\}^2\right]$$

and

$$\mathbb{E}\left[\xi_{J_N}(F,\varepsilon)^2\right] \leq \mathbb{E}\left[\left(\int_{J_N \cap F} \delta^l_\varepsilon(\nabla(X|_{J_N \cap F})(t))|\det D^2(X|_{J_N\cap F})(t)|\,\mathcal{H}^{l-m}(dt)\right)^2\right],$$

we can prove the assertion analogously to the proof of Lemma 3.3. □

We fix for every $J_N \in \partial_l C^d_N$, $m < l < d$, and every $F \in A^d_{d-m}$ such that $J_N \cap F \neq \emptyset$ and $\mathrm{aff}(J_N)^\circ$ and F° are in general position, an orthonormal basis $b(J_N, F) := b^F_{J_N} := (v_i(F))^d_{i=1}$ of $\mathbb{R}^d$ such that

$$c(J_N, F) := c^F_{J_N} := (v_1(F), \ldots, v_{l-m}(F))$$

is a basis of the vector space $(\mathrm{aff}(J_N) \cap F)^\circ = \mathrm{aff}(J_N)^\circ \cap F^\circ$ and that

$$d(J_N, F) := d^F_{J_N} := (v_{l-m+1}(F), \ldots, v_d(F))$$

is a basis of $(\mathrm{aff}(J_N)^\circ \cap F^\circ)^\perp$. We note that the basis only depends on the directional space of F and furthermore does not depend on N. Using this basis we show the following more explicit representation of the random variable $\epsilon^m_{J_N,\varepsilon}$.

Lemma 3.14. *Let $J_N \in \partial_l C^d_N$ and $m < l < d$. Then*

$$\begin{aligned}\epsilon^m_{J_N,\varepsilon} = (-1)^{l-m} \int_{A^d_{d-m}} \int_{J_N\cap F} &\delta^l_\varepsilon(\nabla_{c^F_{J_N}} X(t)) \det(D^2_{c^F_{J_N}} X(t)) \\ &\times \mathbb{1}\left\{X(t) \geq u, \sum_{j=1}^{d+m-l} (\nabla_{d^F_{J_N}} X(t))_j d^F_{J_N,j} \in N_t(C^d_N \cap F)\right\} \mathcal{H}^{l-m}(dt)\,\mu(dF).\end{aligned}$$

Proof. The fact that $\pi_{(\mathrm{aff}(J_N)^\circ\cap F^\circ)^\perp}(\nabla X(t)) = \sum_{j=1}^{d+m-l} (\nabla_{d^F_{J_N}} X(t))_j d^F_{J_N,j}$ and δ^l_ε is rotational invariant, and the fact that $D^2_{c^F_{J_N}} X(t)$ is the transformation matrix of $D^2(X|_{J_N\cap F})$, imply the result (cf. Lemma 3.4). □

In the following, we use a Hermite expansion of the integrand of the latter representation to obtain a Hermite expansion of the random variable $\epsilon^m_{J_N,\varepsilon}$, which carries over to one for the random variable $\epsilon^m_{J_N}$. We therefore define the $D := d + (l - m)(l - m + 1)/2 + 1$-dimensional

Gaussian random field $Z^b(L,t)$ for $(L,t) \in G^d_{d-m} \times \mathbb{R}^d$ by

$$Z^b(L,t) := \left(\nabla_{c^L_{J_N}} X(t), \nabla_{d^L_{J_N}} X(t), \left(\frac{\partial^2}{\partial v_i(L) \partial v_j(L)} X(t) \right)_{1 \le i \le j \le l-m}, X(t) \right)$$

and denote its covariance matrix at the point (L,t) by Σ. Analogously to Lemma 3.5, we show that the covariance matrix is indeed independent of the chosen basis, that is:

Lemma 3.15. *The matrix Σ is independent of $t \in \mathbb{R}^d$ and $L \in G^d_{d-m}$. Moreover, we have $\Sigma = \Lambda\Lambda^\top$, where $\Lambda \in \mathbb{R}^{D \times D}$ is invertible and given by $\Lambda = \begin{pmatrix} I_d & 0 \\ 0 & \Lambda_2 \end{pmatrix}$, for some invertible, lower triangular matrix $\Lambda_2 \in \mathbb{R}^{(D-d)\times(D-d)}$.*

Using Λ, we define the decorrelated process

$$Y^b(L,t) := \Lambda^{-1} Z^b(L,t), \quad t \in \mathbb{R}^d, L \in G^d_{d-m}. \tag{3.33}$$

For fixed $t \in \mathbb{R}^d$ and $L \in G^d_{d-m}$, the random vector $Y^b(L,t)$ is standard normal, i.e. $Y^b(L,t) \sim \mathcal{N}_D(0, I_{D\times D})$. However, note that for different $t, s \in \mathbb{R}^d$ the vectors $Y^b(L,t)$ and $Y^b(L,s)$ are in general not independent. In the same manner as before, we will be using the stationarity

$$\left(Y^b(L,t), Y^b(L',t') \right) \stackrel{\mathcal{D}}{=} \left(Y^b(L,t+h), Y^b(L',t'+h) \right),$$

where $t, t', h \in \mathbb{R}^d$ and $L, L' \in G^d_{d-m}$, in the arguments to come.

We now define the mapping $G^b_\varepsilon \colon A^d_{d-m} \times \mathbb{R}^{l-m} \times \mathbb{R}^{d+m-l} \times \mathbb{R}^{D-d} \to \mathbb{R}$ by

$$\begin{aligned} G^b_\varepsilon(F,a,b,c) := {} & (-1)^{l-m} \delta^l_\varepsilon(a) \det(\Lambda_2 c)_{1,\dots,D-d-1} \\ & \times \mathbb{1}\left\{ (\Lambda_2 c)_{D-d} \ge u, \sum_{j=1}^{d+m-l} b_j d^F_{J_N,j} \in N_t(C^d_N \cap F) \right\}, \end{aligned}$$

where the notation does not reflect a dependence in t, which is correct since $N_t(C^d_N \cap F)$ depends on J_N, which is fixed, and not on $t \in J_N$, cf. (2.14). Hence, we obtain by Lemma 3.14

$$\epsilon^m_{J_N,\varepsilon} = \int_{A^d_{d-m}} \int_{J_N \cap F} G^b_\varepsilon(F, Y^b(F^\circ, t))\, \mathcal{H}^{l-m}(dt)\, \mu(dF). \tag{3.34}$$

We note that the mapping G^b_ε is only defined for almost all $F \in A^d_{d-m}$, but since G^b_ε is only needed when it is integrated over F, this is not a problem. However, the mere fact that this mapping depends on F is the main difference to the arguments in the full dimensional case and forces us to refine the approach used there.

For $F \in A^d_{d-m}$, we note that $G^b_\varepsilon(F,\cdot) \in L^2(\mathbb{R}^D, \mathcal{N}_D(0, I_D))$ and therefore

$$G^b_\varepsilon(F,x) = \sum_{q=0}^\infty \sum_{n \in \mathbb{N}_0^D, |n|=q} c^b(F,\varepsilon,n) \widetilde{H}_n(x)$$

in $L^2(\mathbb{R}^D, \mathcal{N}_D(0, I_D))$, where

$$c^b(F, \varepsilon, n) := n!^{-1} \int_{\mathbb{R}^D} G^b_\varepsilon(F, x) \widetilde{H}_n(x) \phi_D(x)\, dx.$$

We note that due to Fubini's theorem and the definition of G^b_ε this coefficient factors into

$$c^b(F, \varepsilon, n) = c^b_1(\varepsilon, n) c^b_2(F, n), \tag{3.35}$$

where

$$c^b_1(\varepsilon, n) := \frac{1}{\prod_{i=1}^{l-m} n_i!} \int_{\mathbb{R}^{l-m}} (-1)^{l-m} \delta^l_\varepsilon(a) \widetilde{H}_{(n_1,\ldots,n_{l-m})}(a) \phi_{l-m}(a)\, da$$

and

$$\begin{aligned} c^b_2(F, n) := \frac{1}{\prod_{i=l-m+1}^{D} n_i!} \int_{\mathbb{R}^{d+m-l} \times \mathbb{R}^{D-d}} \mathbb{1}\Big\{ (\Lambda_2 c)_{D-d} \ge u,\, \sum_{j=1}^{d+m-l} b_j d^F_{J_N,j} \in N_t(C^d_N \cap F) \Big\} \\ \times \det(\Lambda_2 c)_{1,\ldots,D-d-1} \widetilde{H}_{(n_{l-m+1},\ldots,n_D)}(b, c) \phi_{D-(l-m)}(b, c)\, d(b, c). \end{aligned}$$

Using this expansion we are able to show a Hermite expansion of the random variable $\epsilon^m_{J_N,\varepsilon}$, which in turn will lead to an expansion of $\epsilon^m_{J_N}$.

Lemma 3.16. *Let $J_N \in \partial_l C^d_N$ and $m < l < d$. Then*

$$\epsilon^m_{J_N,\varepsilon} \overset{L^2(\mathbb{P})}{=} \sum_{q \ge 0} \sum_{n \in \mathbb{N}_0^D, |n| = q} \int_{A^d_{d-m}} c^b(F, \varepsilon, n) \int_{J_N \cap F} \widetilde{H}_n(Y^b(F^\circ, t))\, \mathcal{H}^{l-m}(dt)\, \mu(dF). \tag{3.36}$$

Proof. We start by showing that the right side is well defined, meaning that it is an element in $L^2(\mathbb{P})$. Using the completeness of this space, it is enough to show that it is a Cauchy sequence. Thus, let $k_1, k_2 \in \mathbb{N}$ and $k_1 < k_2$. Then by two applications of Jensen's inequality

$$\begin{aligned} &\mathbb{E}\left[\left(\sum_{q=k_1+1}^{k_2} \sum_{|n|=q} \int_{A^d_{d-m}} c^b(F, \varepsilon, n) \int_{J_N \cap F} \widetilde{H}_n(Y^b(F^\circ, t))\, \mathcal{H}^{l-m}(dt)\, \mu(dF) \right)^2 \right] \\ &\le \mathbb{E}\left[\int_{A^d_{d-m}} \mathcal{H}^{l-m}(J_N \cap F) \int_{J_N \cap F} \left(\sum_{q=k_1+1}^{k_2} \sum_{|n|=q} c^b(F, \varepsilon, n) \widetilde{H}_n(Y^b(F^\circ, t)) \right)^2 \mathcal{H}^{l-m}(dt)\, \mu(dF) \right] \\ &\quad \times \mu(\{F \in A^d_{d-m} \mid F \cap J_N \neq 0\}). \end{aligned}$$

It is [76, Lemma 3.1] that allows us to apply Fubini's theorem, yielding equality of the expectation to

$$\begin{aligned} \int_{A^d_{d-m}} \mathcal{H}^{l-m}(F \cap J_N) \int_{J_N \cap F} \sum_{q, q' = k_1+1}^{k_2} \sum_{|n|=q} \sum_{|n'|=q'} c^b(F, \varepsilon, n) c^b(F, \varepsilon, n') \\ \times \mathbb{E}\left[\widetilde{H}_n(Y^b(F^\circ, t)) \widetilde{H}_{n'}(Y^b(F^\circ, t)) \right] \mathcal{H}^{l-m}(dt)\, \mu(dF). \end{aligned}$$

Properties of the Hermite polynomials yield (cf. Lemma 2.10 (ii))

$$\mathbb{E}[\widetilde{H}_n(Y^b(F^\circ,t))\widetilde{H}_{n'}(Y^b(F^\circ,t))] = \mathbb{1}\{n=n'\}n!,$$

so that the $L^2(\mathbb{P})$ norm equals

$$\begin{aligned}&\mu(\{F \mid F\cap J_N \neq 0\})\sum_{q=k_1+1}^{k_2}\int_{A^d_{d-m}}\mathcal{H}^{l-m}(F\cap J_N)^2\sum_{|n|=q}c^b(F,\varepsilon,n)^2 n!\,\mu(dF)\\ &\quad\le \mu(\{F \mid F\cap J_N \neq 0\})\sum_{q=k_1+1}^{\infty}\int_{A^d_{d-m}}\mathcal{H}^{l-m}(F\cap J_N)^2\sum_{|n|=q}c^b(F,\varepsilon,n)^2 n!\,\mu(dF).\end{aligned}$$

It is now enough to show the convergence of this series to obtain that the right side of the asserted equality is a Cauchy sequence. The monotone convergence theorem implies that the series is bounded from above by

$$\int_{A^d_{d-m}}\mathcal{H}^{l-m}(F\cap J_N)^2\sum_{q=0}^{\infty}\sum_{|n|=q}c^b(F,\varepsilon,n)^2 n!\,\mu(dF),$$

where we added some terms to start the summation by 0. By Parseval's identity

$$\sum_{q=0}^{\infty}\sum_{|n|=q}c^b(F,\varepsilon,n)^2 n! = \|G^b_\varepsilon(F,\cdot)\|^2_{L^2(\mathcal{N}_D(0,I_D))} = \int_{\mathbb{R}^d}G^b_\varepsilon(F,x)^2\phi_D(x)\,dx$$

and this can be bounded by

$$\int_{\mathbb{R}^D}\left(\delta^l_\varepsilon(x)\det(\Lambda_2 z)_{1,\dots,D-d-1}\right)^2\phi_D(x,y,z)\,d(x,y,z), \tag{3.37}$$

which is independent of F and finite, since we are integrating polynomials of a bounded degree and ϕ_D. We conclude that the right-hand side of (3.36) is an element in $L^2(\mathbb{P})$.

We now show the equality of the assertion and start by observing with the aid of equation (3.34)

$$\begin{aligned}&\mathbb{E}\left[\left(\epsilon^m_{J_N,\varepsilon}-\sum_{q=0}^{k}\sum_{|n|=q}\int_{A^d_{d-m}}c^b(F,\varepsilon,n)\int_{J_N\cap F}\widetilde{H}_n(Y^b(F^\circ,t))\,\mathcal{H}^{l-m}(dt)\,\mu(dF)\right)^2\right]\\ &=\mathbb{E}\left[\left(\int_{A^d_{d-m}}\int_{J_N\cap F}G^b_\varepsilon(F,Y^b(F^\circ,t))-\sum_{q=0}^{k}\sum_{|n|=q}c^b(F,\varepsilon,n)\widetilde{H}_n(Y^b(F^\circ,t))\,\mathcal{H}^{l-m}(dt)\,\mu(dF)\right)^2\right],\end{aligned}$$

which we bound from above by Jensen's inequality by

$$\begin{aligned}&\mu(\{F\in A^d_{d-m} \mid F\cap J_N\neq 0\})\mathbb{E}\Bigg[\int_{A^d_{d-m}}\mathcal{H}^{l-m}(J_N\cap F)\\ &\quad\times\int_{J_N\cap F}\left(G^b_\varepsilon(F,Y^b(F^\circ,t))-\sum_{q=0}^{k}\sum_{|n|=q}c^b(F,\varepsilon,n)\widetilde{H}_n(Y^b(F^\circ,t))\right)^2\mathcal{H}^{l-m}(dt)\,\mu(dF)\Bigg].\end{aligned}$$

By Fubini's theorem, the expectation equals

$$\int_{A^d_{d-m}} \mathcal{H}^{l-m}(J_N \cap F)$$
$$\times \int_{J_N\cap F} \mathbb{E}\Big[\Big(G^b_\varepsilon(F, Y^b(F^\circ, t)) - \sum_{q=0}^{k}\sum_{|n|=q} c^b(F,\varepsilon,n)\widetilde{H}_n(Y^b(F^\circ,t))\Big)^2\Big]\,\mathcal{H}^{l-m}(dt)\,\mu(dF)$$
$$= \int_{A^d_{d-m}} \mathcal{H}^{l-m}(J_N\cap F)^2 \int_{\mathbb{R}^D} \Big(G^b_\varepsilon(F,x) - \sum_{q=0}^{k}\sum_{|n|=q} c^b(F,\varepsilon,n)\widetilde{H}_n(x)\Big)^2 \phi_D(x)\,dx\,\mu(dF).$$

Hence, if the dominated convergence theorem is applicable to interchange the limit $k\to\infty$ and the integral with respect to μ, the assertion follows. In order to find an integrable dominating function, it is enough to bound the term

$$\int_{\mathbb{R}^D} \left(G^b_\varepsilon(F,x) - \sum_{q=0}^{k}\sum_{|n|=q} c^b(F,\varepsilon,n)\widetilde{H}_n(x)\right)^2 \phi_D(x)\,dx \tag{3.38}$$

by a constant independent of F, since the factor $\mathcal{H}^{l-m}(J_N\cap F)$ is only nonzero for a set of finite measure and moreover independent of k. For this purpose, we apply the inequality $(a-b)^2 \le 2(a^2+b^2)$, $a,b\in\mathbb{R}$, to bound the expression in (3.38) by twice the term

$$\int_{\mathbb{R}^D} G^{J_N}_\varepsilon(F,x)^2\phi_D(x)\,dx + \int_{\mathbb{R}^D}\left(\sum_{q=0}^{k}\sum_{|n|=q} c^b(F,\varepsilon,n)\widetilde{H}_n(x)\right)^2 \phi_D(x)\,dx.$$

The first term can again be bounded by (3.37) and using properties of the Hermite polynomials the second equals

$$\sum_{q=0}^{k}\sum_{|n|=q} c^b(F,\varepsilon,n)^2 n!,$$

which can be bounded by (3.37). The assertion follows. □

In the next lemma we establish the desired Hermite expansion of the random variable $\epsilon^m_{J_N}$ by using the previous lemma and the $L^2(\mathbb{P})$ convergence of $\epsilon^m_{J_N,\varepsilon}$ to $\epsilon^m_{J_N}$, as $\varepsilon\to 0$. We first define

$$c^b(F,n) := c^b_1(n)c^b_2(F,n), \qquad \text{where} \qquad c^b_1(n) := \frac{(-1)^{l-m}(2\pi)^{-(l-m)/2}}{\prod_{i=1}^{l-m} n_i!}\prod_{i=1}^{l-m} H_{n_i}(0), \tag{3.39}$$

which satisfies

$$c^b(F,n) = \lim_{\varepsilon\to 0} c^b(F,\varepsilon,n) = \lim_{\varepsilon\to 0} c^b_1(\varepsilon,n)c^b_2(F,n),$$

by the continuity of the Hermite polynomials and the Gaussian density. The terms $c^b(F,n)$ are the coefficients in the Hermite expansion of $\epsilon^m_{J_N}$ as we will see in the next lemma.

Lemma 3.17. *Let* $J_N \in \partial_l C_N^d$ *and* $m < l < d$. *Then*

$$\epsilon^m_{J_N} \overset{L^2(\mathbb{P})}{=} \sum_{q\geq 0} \sum_{n\in\mathbb{N}_0^D, |n|=q} \int_{A^d_{d-m}} c^b(F,n) \int_{J_N\cap F} \widetilde{H}_n(Y^b(F^\circ,t))\, \mathcal{H}^{l-m}(dt)\, \mu(dF).$$

Proof. We first show that the right side is in $L^2(\mathbb{P})$. Let $k_2 > k_1 \in \mathbb{N}$ be integers. Then by Fubini's theorem, which is applicable by [76, Lemma 3.1], we use the orthogonality established in Lemma 3.7 to obtain

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{q=k_1+1}^{k_2} \sum_{n\in\mathbb{N}_0^D, |n|=q} \int_{A^d_{d-m}} c^b(F,n) \int_{J_N\cap F} \widetilde{H}_n(Y^b(F^\circ,t))\, \mathcal{H}^{l-m}(dt)\, \mu(dF)\right)^2\right] \\
&= \sum_{q=k_1+1}^{k_2} \mathbb{E}\left[\left(\sum_{n\in\mathbb{N}_0^D, |n|=q} \int_{A^d_{d-m}} c^b(F,n) \int_{J_N\cap F} \widetilde{H}_n(Y^b(F^\circ,t))\, \mathcal{H}^{l-m}(dt)\, \mu(dF)\right)^2\right] \\
&\leq \sum_{q=k_1+1}^{k_2} \liminf_{\varepsilon\to\infty} \mathbb{E}\left[\left(\sum_{n\in\mathbb{N}_0^D, |n|=q} \int_{A^d_{d-m}} c^b(F,\varepsilon,n) \int_{J_N\cap F} \widetilde{H}_n(Y^b(F^\circ,t))\, \mathcal{H}^{l-m}(dt)\, \mu(dF)\right)^2\right],
\end{aligned}$$

where we used Fatou's lemma in the last line. We note that we do not have to worry about interchanging the integral over the affine Grassmannian and the limit $\varepsilon \to 0$. This is the case, since by (3.35) the dependence of $c^b(F,\varepsilon,n)$ on F and ε factors in two terms, one depending solely on F and the other depending solely on ε. Now, by adding positive terms, we bound the previous term by

$$\sum_{q=k_1+1}^{\infty} \liminf_{\varepsilon\to 0} \mathbb{E}\left[\left(\sum_{n\in\mathbb{N}_0^D, |n|=q} \int_{A^d_{d-m}} c^b(F,\varepsilon,n) \int_{J_N\cap F} \widetilde{H}_n(Y^b(F^\circ,t))\, \mathcal{H}^{l-m}(dt)\, \mu(dF)\right)^2\right].$$

Showing that this series converges will also establish that the series on the right-hand side of the assertion is a Cauchy sequence and therefore is an element in $L^2(\mathbb{P})$. Again Fatou's lemma and the orthogonality combined with the continuity of the inner product bound the preceding term by

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \mathbb{E}\left[\left(\sum_{q=k_1+1}^{\infty} \sum_{n\in\mathbb{N}_0^D, |n|=q} \int_{A^d_{d-m}} c^b(F,\varepsilon,n) \int_{J_N\cap F} \widetilde{H}_n(Y^b(F^\circ,t))\, \mathcal{H}^{l-m}(dt)\, \mu(dF)\right)^2\right] \\
&\qquad = \liminf_{\varepsilon\to 0} \mathbb{E}\left[(\epsilon^m_{J_N,\varepsilon})^2\right] \\
&\qquad = \mathbb{E}\left[(\epsilon^m_{J_N})^2\right] < \infty,
\end{aligned}$$

where we used Lemma 3.16 and Lemma 3.13. Thus the right side of the assertion is a well-defined element in $L^2(\mathbb{P})$.

We define the abbreviation

$$I_q := \sum_{|n|=q} \int_{A^d_{d-m}} c^b(F,n) \int_{J_N \cap F} \widetilde{H}_n(Y^b(F^\circ,t))\, \mathcal{H}^{l-m}(dt)\, \mu(dF)$$

and denote by $\pi^k \colon L^2(\mathbb{P}) \to L^2(\mathbb{P})$ the projection onto the homogeneous chaos of degree 0 up to k, that is $\cup_{i=0}^k \mathcal{H}_i$ (cf. Theorem 2.14), and by $\pi_k \colon L^2(\mathbb{P}) \to L^2(\mathbb{P})$ the projection onto the homogeneous chaos greater than k, that is $\cup_{i \geq k+1} \mathcal{H}_i$, $k \in \mathbb{N}_0$. Then

$$\begin{aligned} &\left\| \epsilon^m_{J_N} - \sum_{q \geq 0} I_q \right\|_{L^2(\mathbb{P})} \\ &\quad = \left\| \pi_k(\epsilon^m_{J_N}) + \pi^k(\epsilon^m_{J_N}) - \sum_{q=0}^{k} I_q - \sum_{q=k+1}^{\infty} I_q + \pi^k(\epsilon^m_{J_N,\varepsilon}) - \pi^k(\epsilon^m_{J_N,\varepsilon}) \right\|_{L^2(\mathbb{P})}, \end{aligned}$$

which can be bounded by

$$\left\| \pi_k(\epsilon^m_{J_N}) \right\|_{L^2(\mathbb{P})} + \left\| \sum_{q=k+1}^{\infty} I_q \right\|_{L^2(\mathbb{P})} + \left\| \pi^k(\epsilon^m_{J_N,\varepsilon}) - \sum_{q=0}^{k} I_q \right\|_{L^2(\mathbb{P})} + \left\| \pi^k\left(\epsilon^m_{J_N} - \epsilon^m_{J_N,\varepsilon} \right) \right\|_{L^2(\mathbb{P})}.$$

The first two terms vanish for $k \to \infty$, since both arguments are elements in $L^2(\mathbb{P})$. The last one can be bounded by

$$\left\| \epsilon^m_{J_N} - \epsilon^m_{J_N,\varepsilon} \right\|_{L^2(\mathbb{P})}$$

which vanishes in the limit $\varepsilon \to 0$, due to Lemma 3.13. By Lemma 3.16, the third term equals

$$\mathbb{E}\left[\left(\sum_{q=0}^{k} \sum_{|n|=q} \int_{A^d_{d-m}} (c^b(F,n) - c^b(F,\varepsilon,n)) \int_{J_N \cap F} \widetilde{H}_n(Y^b(F^\circ,t))\, \mathcal{H}^{l-m}(dt)\, \mu(dF) \right)^2 \right]$$

and Fubini's theorem allows us to rewrite this term as

$$\begin{aligned} &\sum_{q,q'=0}^{k} \sum_{|n|=q} \sum_{|n|=q'} \int_{A^d_{d-m}} \int_{A^d_{d-m}} (c^b(F,n) - c^b(F,\varepsilon,n))(c^b(F',n') - c^b(F',\varepsilon,n')) \\ &\quad \times \mathbb{E}\left[\int_{J_N \cap F} H_n(Y^b(F^\circ,t))\, \mathcal{H}^{l-m}(dt) \int_{J_N \cap F} H_{n'}(Y^b(F'^\circ,t))\, \mathcal{H}^{l-m}(dt) \right] \mu(dF)\, \mu(dF'). \end{aligned}$$

By (3.35) and (3.39) this equals

$$\begin{aligned} &\sum_{q,q'=0}^{k} \sum_{|n|=q} \sum_{|n|=q'} (c_1^b(n) - c_1^b(\varepsilon,n))(c_1^b(n') - c_1^b(\varepsilon,n')) \int_{A^d_{d-m}} \int_{A^d_{d-m}} c_2^b(F,n) c_2^b(F',n') \\ &\quad \times \mathbb{E}\left[\int_{J_N \cap F} H_n(Y^b(F^\circ,t))\, \mathcal{H}^{l-m}(dt) \int_{J_N \cap F} H_{n'}(Y^b(F'^\circ,t))\, \mathcal{H}^{l-m}(dt) \right] \mu(dF)\, \mu(dF'), \end{aligned}$$

which vanishes for $\varepsilon \to 0$. The assertion follows by taking the limit $\varepsilon \to \infty$ and then $k \to \infty$. □

Before we can give a proof of the scaling property of the boundary terms, we need to establish the following auxiliary lemmas.

Lemma 3.18. *Let N and N' be centered and jointly Gaussian random variables in $\mathbb{R}^D$ with*

$$\operatorname{Cov}(N,N') = \begin{pmatrix} I_D & \operatorname{Cov}(N,N') \\ \operatorname{Cov}(N',N) & I_D \end{pmatrix}$$

and $n, n' \in \mathbb{N}_0^D$ such that $|n| = |n'| = q \in \mathbb{N}$. Then

$$\mathbb{E}\left[\widetilde{H}_n(N)\widetilde{H}_{n'}(N')\right] \le (n!n'!)^{\frac{1}{2}} D^q \max_{i,j=1,\dots,D} \left|\mathbb{E}\left[N_i N_j'\right]\right|^q.$$

Proof. By Lemma 3.7

$$\begin{aligned}\mathbb{E}\left[\widetilde{H}_n(N)\widetilde{H}_{n'}(N')\right] &= n!n'! \sum_{\substack{d\in\mathbb{N}_0^{D\times D} \\ \sum_{i=1}^D d_{ij}=n_j,\, \sum_{j=1}^D d_{ij}=n_i'}} \prod_{1\le i,j\le D} \frac{\mathbb{E}\left[N_i N_j'\right]^{d_{ij}}}{d_{ij}!} \\ &\le \max_{i,j=1,\dots,D} \left|\mathbb{E}\left[N_i N_j'\right]\right|^q n!n'! \sum_{\substack{d\in\mathbb{N}_0^{D\times D} \\ \sum_{i=1}^D d_{ij}=n_j,\, \sum_{j=1}^D d_{ij}=n_i'}} \prod_{1\le i,j\le D} \frac{1}{d_{ij}!}\end{aligned}$$

Now, let $v := (1,\dots,1) \in \mathbb{R}^D$ and let W, W' be centered and jointly Gaussian in $\mathbb{R}^D$, such that

$$\operatorname{Cov}(W,W') = \begin{pmatrix} I_D & \frac{1}{D}vv^\top \\ \frac{1}{D}vv^\top & I_D \end{pmatrix}.$$

We note that (W, W') exists, since the covariance matrix is positive semidefinite. To see this, we use [7, Fact 8.11.13], which states that for this special choice of the covariance matrix being positive semidefinite is equivalent to $I_D - (\frac{1}{D}vv^\top)^2$ having nonnegative eigenvalues. This is the case since for $x \in \mathbb{R}^d$

$$\left(I_D - \left(\frac{1}{D}vv^\top\right)^2\right)x = x - \left(\frac{1}{D}vv^\top\right)^2 x = \left(1 - \lambda^2_{\frac{1}{D}vv^\top}\right)x,$$

where $\lambda_{\frac{1}{D}vv^\top}$ denotes an eigenvalue of $\frac{1}{D}vv^\top$, and $\lambda_{\frac{1}{D}vv^\top} \in \{0,1\}$. Thus

$$\begin{aligned} n!n'! \sum_{\substack{d\in\mathbb{N}_0^{D\times D} \\ \sum_{i=1}^D d_{ij}=n_j,\, \sum_{j=1}^D d_{ij}=n_i'}} \prod_{1\le i,j\le D} \frac{1}{d_{ij}!} &\le D^q n!n'! \sum_{\substack{d\in\mathbb{N}_0^{D\times D} \\ \sum_{i=1}^D d_{ij}=n_j,\, \sum_{j=1}^D d_{ij}=n_i'}} \prod_{1\le i,j\le D} \frac{D^{-d_{ij}}}{d_{ij}!} \\ &= D^q n!n'! \sum_{\substack{d\in\mathbb{N}_0^{D\times D} \\ \sum_{i=1}^D d_{ij}=n_j,\, \sum_{j=1}^D d_{ij}=n_i'}} \prod_{1\le i,j\le D} \frac{\mathbb{E}\left[W_i W_j'\right]^{d_{ij}}}{d_{ij}!},\end{aligned}$$

which by Lemma 3.7 equals $D^q\mathbb{E}\left[\widetilde{H}_n(W)\widetilde{H}_{n'}(W')\right]$. Finally, the Cauchy–Schwarz inequality yields

$$\mathbb{E}\left[\widetilde{H}_n(W)\widetilde{H}_{n'}(W')\right] \leq \left(\mathbb{E}\left[\widetilde{H}_n(W)^2\right]\mathbb{E}\left[\widetilde{H}_{n'}(W')^2\right]\right)^{\frac{1}{2}} = (n!n'!)^{\frac{1}{2}},$$

which proves the assertion. □

Lemma 3.19. *Let $q \in \mathbb{N}$ and $F \in A^d_{d-m}$. Then there exists a constant $c = c(X, d, m, l) \geq 0$ such that*

$$\sum_{n\in\mathbb{N}_0^D, |n|=q} n!^{\frac{1}{2}} c^b(F,n) \leq cq^D.$$

Proof. In the following the constant c may change from appearance to appearance. We recall the definition

$$c^b(F,n) = (2\pi)^{-(l-m)/2} \prod_{i=1}^{l-m} \frac{H_{n_i}(0)}{n_i!} \frac{(-1)^{l-m}}{\prod_{i=l-m+1}^{D} n_i!} z^b(F,n),$$

where

$$\begin{aligned} z^b(F,n) := \int_{\mathbb{R}^{D-(l-m)}} &\det(\Lambda_2 z)_{1,\dots,D-d-1} \mathbb{1}\{(\Lambda_2 z)_{D-d} \geq u\} \\ &\times \mathbb{1}\left\{\sum_{j=1}^{d+m-l} y_j d^F_{J_N,j} \in N_t(C^d_N \cap F)\right\} \left(\prod_{i=l-m+1}^{D} H_{n_i}\right)(y,z)\phi_{D-(l-m)}(y,z)\, d(y,z). \end{aligned}$$

Proposition 3 in [32] yields $\prod_{i=1}^{l-m} \frac{|H_{n_i}(0)|}{\sqrt{n_i!}} \leq c$, and thus

$$(2\pi)^{-(l-m)/2} \prod_{i=1}^{l-m} \frac{H_{n_i}(0)}{n_i!} \leq c \left(\prod_{i=1}^{l-m} n_i!\right)^{-\frac{1}{2}}.$$

By the Cauchy–Schwarz inequality and bounding the indicator functions by 1, we get

$$\begin{aligned} z^b(F,n) &\leq \Bigg(\int_{\mathbb{R}^{D-(l-m)}} \det(\Lambda_2 z)^2_{1,\dots,D-d-1} \phi_{D-(l-m)}(y,z)\, d(y,z) \\ &\qquad \times \int_{\mathbb{R}^{D-(l-m)}} \left(\prod_{i=l-m+1}^{D} H_{n_i}\right)(y,z)^2 \phi_{D-(l-m)}(y,z)\, d(y,z)\Bigg)^{\frac{1}{2}} \\ &= c\left(\prod_{i=l-m+1}^{D} n_i!\right)^{\frac{1}{2}}. \end{aligned}$$

Thus we obtain $c^b(F,n) \leq cn!^{-\frac{1}{2}}$ and therefore

$$\sum_{|n|=q} n!^{\frac{1}{2}} c^b(F,n) \leq c \sum_{|n|=q} 1 \leq c \sum_{|n|\leq q} 1 \leq cq^D.$$

□

Lemma 3.20. *Let* $J_N = \{t \in \mathbb{R}^d \mid -N \leq t_i \leq n, i \in \sigma(J_N), t_i = \epsilon_i N, i \notin \sigma(J_N)\} \in \partial_l C_N^d$*, where* $m < l < d$*, and* $F \in A_{d-m}^d$ *such that* $F \cap J_N \neq 0$*. Furthermore, let* $x \in \mathbb{R}^d$ *be such that* $(F + x) \cap J_{1/2} \neq 0$*, where* $J_{1/2} := \{t \in \mathbb{R}^d \mid -\frac{1}{2} \leq t_i \leq \frac{1}{2}, i \in \sigma(J_N), t_i = \epsilon_i \frac{1}{2}, i \notin \sigma(J_N)\}$*. Then*

$$c^{b_{J_N}^F}(F, n) = c^{b_{J_{1/2}}^F}(F + x, n).$$

Proof. The dependence on J_N and F enters the definition of $c^{b_{J_N}^F}(F, n)$, cf. (3.39), in the choice of the basis $d_{J_N}^F$ and the normal cone $N_t(C_N^d \cap F)$, which is determined by the vectors $n_i^{J_N}(F)$, cf. (2.14). We recall that $d_{J_N}^F$ denotes a basis of the space $(\text{aff}(J_N)^\circ \cap F^\circ)^\perp$ and therefore only depends on the directional space of F and the directional space of the chosen stratum of the cube. Therefore it is invariant under translation of F and changes in N, yielding $d_{J_N}^F = d_{J_{1/2}}^{F+x}$.

The normal cone $N_t(C_N^d \cap F)$ is determined by the outer unit normal vectors $n_i^{J_N}(F)$ of the facets of $C_N^d \cap F$ in F° containing $J_N \cap F$. Note that these facets are given by $J_N^{d-1} \cap F$, where $J_N^{d-1} \in \partial_{d-1} C_N^d$ such that $J_N \subset J_N^{d-1}$. Thus the outer unit normal vectors are given as the vectors with norm one and pointing outwards in the spaces

$$F^\circ \cap \left(\left(\text{aff}(J_N^{d-1}) \cap F\right)^\circ\right)^\perp = F^\circ \cap \left(\text{aff}(J_N^{d-1})^\circ \cap F^\circ\right)^\perp, \quad J_N^{d-1} \in \partial_{d-1} C_N^d \text{ s.t. } J_N \subset J_N^{d-1}.$$

The same definition applied for $c^{b_{J_{1/2}}^F}(F + x, n)$, yields that n_i^{F+x} are the outer unit normal vectors of the facets of $C_{1/2}^d \cap (F + x)$ in $(F + x)^\circ$ containing the set $J_{1/2} \cap (F + x)$. Again, these facets are given by $J_{1/2}^{d-1} \cap (F + x)$, where $J_{1/2}^{d-1} \in \partial_{d-1} C_N^d$ such that $J_{1/2} \subset J_{1/2}^{d-1}$. Thus, in this case, the outer unit normal vectors are given as the unit vectors pointing outwards in the spaces

$$\begin{aligned}(F + x)^\circ \cap \left(\left(\text{aff}(J_{1/2}^{d-1}) \cap (F + x)\right)^\circ\right)^\perp &= (F + x)^\circ \cap \left(\text{aff}(J_{1/2}^{d-1})^\circ \cap (F + x)^\circ\right)^\perp \\ &= F^\circ \cap \left(\text{aff}(J_N^{d-1})^\circ \cap F^\circ\right)^\perp.\end{aligned}$$

Since outward has the same meaning for $C_N^d \cap F$ and $C_{1/2}^d \cap (F + x)$ and moreover their facets are one-to-one, we obtain that $\{n_1^{J_N}(F), \ldots, n_{d-l}^{J_N}(F)\} = \{n_1^{J_{1/2}}(F+x), \ldots, n_{d-l}^{J_{1/2}}(F+x)\}$. This shows the assertion. □

The last auxiliary lemma concerns the integrability of the maximum of derivatives of the covariance function, namely ψ, on linear subspaces of $\mathbb{R}^d$. By S_d we denote the symmetric group on d elements.

Lemma 3.21. *Let* ψ *be defined as in (A3) and satisfy the assumptions made there. If* $k \in \{1, \ldots, d-1\}$ *and* $\sigma \in S_d$*, then*

$$\int_{L_k^\sigma} \psi(t)\, \mathcal{H}^k(dt) < \infty,$$

where $L_k^\sigma := \{t \in \mathbb{R}^d \mid t_{\sigma(i)} = 0, i = k+1, \ldots, d\}$*.*

Proof. We first observe that by the integrability assumptions on ψ made in (A3)

$$\infty > \int_{\mathbb{R}^d} \psi(t)\,dt = \int_0^\infty \int_{S^{d-1}} \psi(ru) r^{d-1}\,du\,dr$$

and thus by Fubini's theorem $\int_0^\infty \psi(ru)r^{d-1}\,dr < \infty$ for almost all $u \in S^{d-1}$. Therefore, we may choose $v \in \{u \in S^{d-1} \mid \int_0^\infty \psi(ru)r^{d-1}\,dr < \infty\}$.

We note that $\frac{\partial}{\partial t_i} C^X(t) = \frac{\partial}{\partial \rho(e_i)} C^X(\rho(t))$ for $i = 1, \dots, d$, $t \in \mathbb{R}^d$ and $\rho \in SO(d)$. Indeed, by definition of the partial derivative

$$\begin{aligned}\frac{\partial}{\partial t_i} C^X(t) &= \lim_{h\to 0} \frac{C^X(t + he_i) - C^X(t)}{h} = \lim_{h\to 0} \frac{C^X(\rho(t) + h\rho(e_i)) - C^X(\rho(t))}{h} \\ &= \frac{\partial}{\partial \rho(e_i)} C^X(\rho(t)),\end{aligned}$$

where we used the isotropy of the field X. This implies

$$\left|\frac{\partial}{\partial t_i} C^X(t)\right| = \left|\sum_{j=1}^d \rho(e_i)_j \frac{\partial}{\partial t_j} C^X(\rho(t))\right| \le \sum_{j=1}^d \left|\frac{\partial}{\partial t_j} C^X(\rho(t))\right|.$$

Analogous reasoning for $m \in \{0, \dots, 4\}$ and $i_1, \dots, i_m \in \{1, \dots, d\}$ yields

$$\left|\frac{\partial^m}{\partial t_{i_1} \dots \partial t_{i_m}} C^X(t)\right| \le \sum_{j_1,\dots,j_m=1}^d \left|\frac{\partial^m}{\partial t_{j_1} \dots \partial t_{j_m}} (C^X)(\rho(t))\right|.$$

Thus, if we choose for fixed but arbitrary $t \in \mathbb{R}^d$ the rotation $\rho^{tv} \in SO(d)$ such that $\rho^{tv}(t) = \|t\|v$, we obtain

$$\begin{aligned}\psi(t) = \max_{\substack{r=0,\dots,4 \\ i_1,\dots,i_r=\{1,\dots,d\}}} \left|\frac{\partial^r}{\partial t_{i_1} \dots \partial t_{i_r}} C^X(t)\right| &\le \max_{r=0,\dots,4} \sum_{j_1,\dots,j_r=1}^d \left|\frac{\partial^r}{\partial t_{j_1} \dots \partial t_{j_r}} C^X(\|t\|v)\right| \\ &\le 4^d \psi(\|t\|v). \end{aligned} \tag{3.40}$$

Then by spherical coordinates and the estimate (3.40)

$$\int_{L_k^\sigma} \psi(t)\,dt = \int_0^\infty \int_{S_\sigma^{k-1}} \psi(ru) r^{k-1}\,du\,dr \le 4^d \omega_k \int_0^\infty \psi(rv) r^{k-1}\,dr,$$

where S_σ^{k-1} denotes the sphere in the linear subspace L_k^σ. This can be bounded by

$$4^d \omega_k \int_0^1 \psi(rv) r^{k-1}\,dr + 4^d \omega_k \int_1^\infty \psi(rv) r^{d-1}\,dr,$$

which is finite by the continuity of ψ and the choice of v. This shows the assertion. □

Using the Chaos expansion and the auxiliary lemmas, we can finally show the following scaling behaviour of the boundary terms $\epsilon_{J_N}^m$.

Lemma 3.22. *Let $J_N \in \partial_l C_N^d$ and $m \le l < d$. Then*

$$\mathbb{E}\left[(\epsilon_{J_N}^m - \mathbb{E}\left[\epsilon_{J_N}^m\right])^2\right] = o(N^{l+1}),$$

where $\epsilon_{J_N}^m$ is defined in (3.5).

We conclude from Lemma 3.22 that

$$\lim_{N\to\infty} N^{-d/2}\mathbb{E}\left[\left(\epsilon_{J_N}^m - \mathbb{E}\left[\epsilon_{J_N}^m\right]\right)^2\right] = 0,$$

and since convergence in the quadratic mean implies convergence in probability, cf. [70, II.4 Theorem 2], we conclude the desired result

$$\frac{\epsilon_{J_N}^m - \mathbb{E}\left[\epsilon_{J_N}^m\right]}{N^{d/2}} \xrightarrow{\mathbb{P}} 0, \qquad \text{as } N\to\infty, \text{ for } m \le l < d. \tag{3.41}$$

Proof. We distinguish the case in which $J_N \cap F$ is countable ($l = m$) and the case in which it is a continuum ($m < l \le d$).

Case: $l = m$. If $l = m$, then $\#(J_N \cap F) \in \{0,1\}$ for almost all $F \in A_{d-m}^d$. This implies $\dim T_t(J_N \cap F) = 0$, $t \in J_N \cap F$, and therefore

$$\nabla(X|_{J_N\cap F})(t) = \pi_{T_t(J_N\cap F)}(\nabla X(t)) = 0$$

as well as $\iota_{J_N\cap F}^{-X}(t) = 0$. Thus, we obtain

$$\epsilon_{J_N}^m = \int_{A_{d-m}^d} \#\{t \in J_N \cap F \mid X(t) \ge u, \nabla X(t) \in N_t(C_N^d \cap F)\}\,\mu(dF).$$

The integrand equals $\int_{J_N\cap F} \mathbb{1}\{X(t) \ge u, \nabla X(t) \in N_t(C_N^d \cap F)\}\,\mathcal{H}^0(dt)$, and, in contrast to the cases $m < l$, the measure $\mathcal{H}^0$ is σ-finite on $J_N \cap F$. This allows us to apply Fubini's theorem to deduce

$$\begin{aligned}
&\mathbb{E}\left[\left(\epsilon_{J_N}^m - \mathbb{E}\left[\epsilon_{J_N}^m\right]\right)^2\right] \\
&= \mathbb{E}\Bigg[\Bigg(\int_{A_{d-m}^d}\int_{J_N\cap F} \mathbb{1}\{X(t) \ge u, \nabla X(t) \in N_t(C_N^d \cap F)\}\,\mathcal{H}^0(dt)\,\mu(dF) \\
&\qquad - \int_{A_{d-m}^d}\int_{J_N\cap F} \mathbb{E}\left[\mathbb{1}\{X(t) \ge u, \nabla X(t) \in N_t(C_N^d \cap F)\}\right]\,\mathcal{H}^0(dt)\,\mu(dF)\Bigg)^2\Bigg],
\end{aligned}$$

which equals by another application of Fubini's theorem

$$\begin{aligned}
&\int_{A_{d-m}^d}\int_{J_N\cap F_2}\int_{A_{d-m}^d}\int_{J_N\cap F_1} \mathbb{E}\left[(I(F_1,t_1) - \mathbb{E}\left[I(F_1,t_1)\right])\right. \\
&\qquad\qquad \left.\times (I(F_2,t_2) - \mathbb{E}\left[I(F_2,t_2)\right])\right]\,\mathcal{H}^0(dt_1)\,\mu(dF_1)\,\mathcal{H}^0(dt_2)\,\mu(dF_2),
\end{aligned} \tag{3.42}$$

where $I(F,t) := \mathbb{1}\{X(t) \ge u, \nabla X(t) \in N_t(C_N^d \cap F)\}$. By Assumption (A3) there exists $s > 0$

such that

$$\psi(t_1 - t_2) < \frac{1}{d+1} \quad \text{for } \|t_1 - t_2\| \geq s.$$

We use this number s to split the integration in (3.42) into one integration over the domain $\|t_1 - t_2\| \geq s$ and one over the domain $\|t_1 - t_2\| < s$. In the first case, we use Lemma 3.11 with the choices $V := (X(t_1), \nabla X(t_1))$, $W := (X(t_2), \nabla X(t_2))$ and $h \colon \mathbb{R}^{d+1} \to \mathbb{R}$ given by $(x, y) \mapsto \mathbb{1}\{x \geq u, y \in N_{J_N}(C_N^d \cap F)\}$. We note that writing $N_{J_N}(C_N^d \cap F)$ for $N_t(C_N^d \cap F)$ emphasizes the fact that this normal cone does not depend on the location of t in $J_N \cap F$. Then, since $X(t)$ and $\nabla X(t)$ are independent due to the stationarity and moreover $l = m$ and (2.13)

$$\mathbb{E}\left[(h(V) - \mathbb{E}\left[h(V)\right])H_1(V_1)\right] = \mathbb{E}\left[\mathbb{1}\{X(0) \geq u\}X(0)\right] \mathbb{P}\left(\nabla X(0) \in N_{J_N}(C_N^d \cap F)\right) \neq 0,$$

yielding $r = 1$. Moreover, by definition of ψ, we get $\tau \leq (d+1)\psi(t_1 - t_2) < 1$, for $\|t_1 - t_2\| \geq s$, and therefore obtain by Lemma 3.11

$$\left|\mathbb{E}\left[\left(I(F_1, t_1) - \mathbb{E}\left[I(F_1, t_1)\right]\right)\left(I(F_2, t_2) - \mathbb{E}\left[I(F_2, t_2)\right]\right)\right]\right| \leq (d+1)\psi(t_1 - t_2),$$

for $\|t_1 - t_2\| \geq s$. Hence, we split the integration in (3.42) in the one over $\|t_1 - t_2\| \geq s$ and its complement, and bound the first one by

$$\int_{A_{d-m}^d} \int_{J_N \cap F_2} \int_{A_{d-m}^d} \int_{J_N \cap F_1} \mathbb{1}\{\|t_1 - t_2\| \geq s\}(d+1)\psi(t_1 - t_2)\, \mathcal{H}^0(dt_1)\, \mu(dF_1)\, \mathcal{H}^0(dt_2)\, \mu(dF_2).$$

With the aid of [69, Theorem 5.4.3] and Fubini's theorem the latter equals

$$\begin{aligned} &\int_{J_N} \int_{J_N} \mathbb{1}\{\|t_1 - t_2\| \geq s\}(d+1)\psi(t_1 - t_2)\, \mathcal{H}^m(dt_1)\, \mathcal{H}^m(dt_2) \\ &\qquad = \int_{J_N} \int_{J_N - J_N} \mathbb{1}\{\|t_1\| \geq s, t_1 \in J_N - t_2\}(d+1)\psi(t_1)\, \mathcal{H}^m(dt_1)\, \mathcal{H}^m(dt_2), \end{aligned}$$

where we used the translation invariance of the Hausdorff measure. We bound the indicator with the condition $t_1 \in J_n - t_2$ by 1 and thus obtain the upper bound

$$\int_{J_N - J_N} \mathbb{1}\{\|t_1\| \geq s\}(d+1)\psi(t_1)\, \mathcal{H}^m(dt_1)\mathcal{H}^m(J_N) \leq \int_{\{t \in \mathbb{R}^d | t_i = 0, i \notin \sigma(J_N)\}} \psi(t)\, dt N^m \mathcal{H}^m(J_1).$$

Lemma 3.21 implies the finiteness of the integral and we conclude the assertion in the first case. In the second case, we consider

$$\begin{aligned} \int_{A_{d-m}^d} \int_{J_N \cap F_2} \int_{A_{d-m}^d} \int_{J_N \cap F_1} &\mathbb{1}\{\|t_1 - t_2\| < s\}\mathbb{E}\left[\left(I(F_1, t_1) - \mathbb{E}\left[I(F_1, t_1)\right]\right)\right. \\ &\left.\times \left(I(F_2, t_2) - \mathbb{E}\left[I(F_2, t_2)\right]\right)\right]\, \mathcal{H}^0(dt_1)\, \mu(dF_1)\, \mathcal{H}^0(dt_2)\, \mu(dF_2), \end{aligned}$$

and again bound some the indicator functions by 1. This yields the upper bound

$$4\int_{A^d_{d-m}}\int_{J_N\cap F_2}\int_{A^d_{d-m}}\int_{J_N\cap F_1}\mathbb{1}\{\|t_1-t_2\|<s\}\,\mathcal{H}^0(dt_1)\,\mu(dF_1)\,\mathcal{H}^0(dt_2)\,\mu(dF_2)$$
$$=4\int_{J_N}\int_{J_N}\mathbb{1}\{\|t_1-t_2\|<s\}\,\mathcal{H}^m(dt_1)\,\mathcal{H}^m(dt_2),$$

where we applied Fubini's theorem and [69, Theorem 5.4.3] twice. Again the translation invariance allows us to write the integrals as

$$\int_{J_N}\int_{J_N-J_N}\mathbb{1}\{\|t_1\|<s,t_1\in J_N-t_2\}\,\mathcal{H}^m(dt_1)\,\mathcal{H}^m(dt_2)$$
$$\le\int_{\{t\in\mathbb{R}^d|t_i=0,i\notin\sigma(J_N)\}}\mathbb{1}\{\|t\|<s\}\,\mathcal{H}^m(dt)\mathcal{H}^m(J_N)$$
$$=\mathcal{H}^m(\{t\in\mathbb{R}^m\mid\|t\|<s\})N^m\mathcal{H}^m(J_1),$$

which shows the assertion in the case $l=m$.

Case: $l\in\{m+1,\dots,d-1\}$**.** This case is more involved, due to the fact that Fubini's theorem is not applicable in the argument leading to (3.42). Instead, we use the already established Hermite type expansion of $\epsilon^m_{J_N}$. By this expansion, cf. Lemma 3.17, we obtain

$$\mathbb{E}\left[\left(\epsilon^m_{J_N}-\mathbb{E}\left[\epsilon^m_{J_N}\right]\right)^2\right]$$
$$=\mathbb{E}\left[\left(\sum_{q\ge1}\sum_{n\in\mathbb{N}_0^D,|n|=q}\int_{A^d_{d-m}}c^b(F,n)\int_{J_N\cap F}\widetilde{H}_n(Y^b(F^\circ,t))\,\mathcal{H}^{l-m}(dt)\,\mu(dF)\right)^2\right].\tag{3.43}$$

Partitioning the l-dimensional, affine subspace $\{t\in\mathbb{R}^d\mid t_i=\epsilon_iN,i\notin\sigma(J_N)\}$, where σ is defined in (2.11), into translates of the l-dimensional unit cube

$$[0,1)^l:=\{t\in\mathbb{R}^d\mid 0\le t_i\le1,i\in\sigma(J_N)\text{ and }t_i=0,i\notin\sigma(J_N)\}$$

yields equality of the term (3.43) to

$$\mathbb{E}\Bigg[\Bigg(\sum_{q\ge1}\sum_{n\in\mathbb{N}_0^D,|n|=q}\int_{A^d_{d-m}}c^b(F,n)$$
$$\times\sum_{z\in\mathbb{Z}^d}\int_{J_N\cap F}\mathbb{1}\{t\in[0,1)^l+z\}\widetilde{H}_n(Y^b(F^\circ,t))\,\mathcal{H}^{l-m}(dt)\,\mu(dF)\Bigg)^2\Bigg],$$

which equals

$$\sum_{z_1,z_2\in\mathbb{Z}^d}\mathbb{E}\Bigg[\sum_{q\ge1}\sum_{n\in\mathbb{N}_0^D,|n|=q}\int_{A^d_{d-m}}c^b(F,n)\int_{J_N\cap F}\mathbb{1}\{t\in[0,1)^l+z_1\}\widetilde{H}_n(Y^b(F^\circ,t))\,\mathcal{H}^{l-m}(dt)\,\mu(dF)$$
$$\times\sum_{q\ge1}\sum_{n\in\mathbb{N}_0^D,|n|=q}\int_{A^d_{d-m}}c^b(F,n)\int_{J_N\cap F}\mathbb{1}\{t\in[0,1)^l+z_2\}\widetilde{H}_n(Y^b(F^\circ,t))\,\mathcal{H}^{l-m}(dt)\,\mu(dF)\Bigg].$$

We note that due to the definition of ψ, cf. (A3), there is a constant $c = c(X, d, m, l) > 0$ so that

$$D \max_{i,j} |\mathbb{E}\left[Y^b(F_1^\circ, t_1)_i Y^b(F_2^\circ, t_2)_j\right]| \le c\psi(t_1 - t_2) \tag{3.44}$$

for $F_1, F_2 \in A^d_{d-m}$ and $t_1, t_2 \in \mathbb{R}^d$. Moreover (A3) implies that for $\rho \in (0,1)$ and $\rho < 1/c$ there is a constant $s > 0$ such that

$$\psi(t) \le \rho, \text{ for } \|t\| \ge s. \tag{3.45}$$

Using s, we split the above summation into one over $I_1 := \{(z_1, z_2) \in (\mathbb{Z}^d)^2 \mid \|z_1 - z_2\|_\infty \ge s+1\}$ and $I_2 := \{(z_1, z_2) \in (\mathbb{Z}^d)^2 \mid \|z_1 - z_2\|_\infty \le s\}$. By orthogonality the first sum equals

$$\begin{aligned} &\sum_{(z_1,z_2)\in I_1} \sum_{q\ge 1} \sum_{|n_1|=q=|n_2|} \int_{A^d_{d-m}} \int_{A^d_{d-m}} c^b(F_1, n_1) c^b(F_2, n_2) \\ &\quad \times \int_{J_N \cap F_2} \int_{J_N\cap F_1} \mathbb{1}\{t_2 \in [0,1)^l + z_2\} \mathbb{1}\{t_1 \in [0,1)^l + z_1\} \\ &\quad \times \mathbb{E}\left[\widetilde{H}_{n_1}(Y^b(F_1^\circ, t_1)) \widetilde{H}_{n_2}(Y^b(F_2^\circ, t_2))\right] \mathcal{H}^{l-m}(dt_1)\, \mathcal{H}^{l-m}(dt_2)\, \mu(dF_1)\, \mu(dF_2). \end{aligned} \tag{3.46}$$

By first applying Lemma 3.18 and then equation (3.44), we bound this expression by

$$\begin{aligned} &\sum_{(z_1,z_2)\in I_1} \sum_{q\ge 1} \sum_{|n_1|=q=|n_2|} \int_{A^d_{d-m}} \int_{A^d_{d-m}} c^b(F_1, n_1) c^b(F_2, n_2) \int_{J_N \cap F_2} \int_{J_N\cap F_1} \mathbb{1}\{t_2 \in [0,1)^l + z_2\} \\ &\quad \times \mathbb{1}\{t_1 \in [0,1)^l + z_1\} (n_1! n_2!)^{\frac{1}{2}} c^q \psi(t_1 - t_2)^q\ \mathcal{H}^{l-m}(dt_1)\, \mathcal{H}^{l-m}(dt_2)\, \mu(dF_1)\, \mu(dF_2), \end{aligned}$$

which, by rearranging the terms, equals

$$\begin{aligned} &\sum_{(z_1,z_2)\in I_1} \sum_{q\ge 1} \int_{A^d_{d-m}} \int_{A^d_{d-m}} \int_{J_N \cap F_2} \int_{J_N\cap F_1} \sum_{|n_1|=q} n_1!^{\frac{1}{2}} c^b(F_1, n_1) \sum_{|n_2|=q} n_2!^{\frac{1}{2}} c^b(F_2, n_2) \\ &\quad \times \mathbb{1}\{t_2 \in [0,1)^l + z_2\} \mathbb{1}\{t_1 \in [0,1)^l + z_1\} c^q \psi(t_1 - t_2)^q\ \mathcal{H}^{l-m}(dt_1)\, \mathcal{H}^{l-m}(dt_2)\, \mu(dF_1)\, \mu(dF_2). \end{aligned}$$

Lemma 3.19 bounds this by

$$\begin{aligned} &c^2 \sum_{(z_1,z_2)\in I_1} \sum_{q\ge 1} c^q q^{2D} \int_{A^d_{d-m}} \int_{J_N\cap F_2} \mathbb{1}\{t_2 \in [0,1)^l + z_2\} \\ &\quad \times \int_{A^d_{d-m}} \int_{J_N\cap F_1} \mathbb{1}\{t_1 \in [0,1)^l + z_1\} \psi(t_1 - t_2)^q\ \mathcal{H}^{l-m}(dt_1)\, \mathcal{H}^{l-m}(dt_2)\, \mu(dF_1)\, \mu(dF_2). \end{aligned}$$

Applying [69, Theorem 5.4.3] twice, we see that this term equals

$$c_1^2 \sum_{(z_1,z_2)\in I_1} \sum_{q\ge 1} c^q q^{2D} \int_{J_N} \mathbb{1}\{t_2 \in [0,1)^l + z_2\} \int_{J_N} \mathbb{1}\{t_1 \in [0,1)^l + z_1\} \psi(t_1 - t_2)^q\, \mathcal{H}^l(dt_1)\, \mathcal{H}^l(dt_2),$$

for a constant $c_1 > 0$. Now, the choice of the constant s implies $\psi(t_1 - t_2) \le \rho$, for $t_1 \in [0,1)^l + z_1$,

$t_2 \in [0,1)^l + z_2$ and $(z_1, z_2) \in I_1$, and hence the upper bound

$$c_1^2 \rho^{-1} \sum_{(z_1,z_2)\in I_1} \int_{J_N} \mathbb{1}\{t_2 \in [0,1)^l + z_2\} \times \int_{J_N} \mathbb{1}\{t_1 \in [0,1)^l + z_1\} \psi(t_1 - t_2)\, \mathcal{H}^l(dt_1)\, \mathcal{H}^l(dt_2) \sum_{q\geq 1} (\rho c)^q q^{2D}.$$

For the sum over the integrals, we observe

$$\begin{aligned}
&\sum_{(z_1,z_2)\in I_1} \int_{J_N} \mathbb{1}\{t_2 \in [0,1)^l + z_2\} \int_{J_N} \mathbb{1}\{t_1 \in [0,1)^l + z_1\} \psi(t_1 - t_2)\, \mathcal{H}^l(dt_1)\, \mathcal{H}^l(dt_2) \\
&= \sum_{z_2 \in \mathbb{Z}^d} \int_{J_N} \mathbb{1}\{t_2 \in [0,1)^l + z_2\} \sum_{\substack{z_1 \in \mathbb{Z}^d \\ \|z_1 - z_2\|_\infty \geq s+1}} \int_{J_N} \mathbb{1}\{t_1 \in [0,1)^l + z_1\} \psi(t_1 - t_2)\, \mathcal{H}^l(dt_1)\, \mathcal{H}^l(dt_2) \\
&\leq \sum_{z_2 \in \mathbb{Z}^d} \int_{J_N} \mathbb{1}\{t_2 \in [0,1)^l + z_2\} \int_{J_N} \psi(t_1 - t_2)\, \mathcal{H}^l(dt_1)\, \mathcal{H}^l(dt_2),
\end{aligned}$$

which equals by Fubini's theorem

$$\begin{aligned}
&\sum_{z_2 \in \mathbb{Z}^d} \int_{J_N} \mathbb{1}\{t_2 \in [0,1)^l + z_2\} \int_{J_N - J_N} \mathbb{1}\{t_1 \in J_N - t_2\} \psi(t_1)\, \mathcal{H}^l(dt_1)\, \mathcal{H}^l(dt_2) \\
&= \int_{J_N - J_N} \psi(t_1) \sum_{z_2 \in \mathbb{Z}^d} \int_{J_N} \mathbb{1}\{t_2 \in [0,1)^l + z_2, t_2 \in J_N - t_1\} \mathcal{H}^l(dt_2)\, \mathcal{H}^l(dt_1) \\
&\leq \mathcal{H}^l(J_N) \int_{\{t \in \mathbb{R}^d | t_i = 0, i \in \sigma(J_N)\}} \psi(t)\, \mathcal{H}^l(dt).
\end{aligned}$$

To summarize, we obtain the following upper bound for the first summand

$$c_1^2 \rho^{-1} \mathcal{H}^l(J_N) \int_{\{t \in \mathbb{R}^d | t_i = 0, i \in \sigma(J_N)\}} \psi(t)\, \mathcal{H}^l(dt) \sum_{q \geq 1} (\rho c)^q q^{2D},$$

where the series converges by the ratio test, since $c\rho < 1$, and the integral is finite by Lemma 3.21. Finally, the equality $\mathcal{H}^l(J_N) = N^l \mathcal{H}^l(J_1)$ establishes the assertion for the first summand.

We now analyse the second sum and start by applying the inequality $ab \leq a^2 + b^2$, $a, b \in \mathbb{R}$, to obtain for the expression

$$\begin{aligned}
\sum_{(z_1,z_2)\in I_2} \mathbb{E} \Bigg[&\sum_{q\geq 1} \sum_{|n|=q} \int_{A^d_{d-m}} c^b(F,n) \int_{J_N \cap F} \mathbb{1}\{t \in [0,1)^l + z_1\} \widetilde{H}_n(Y^b(F^\circ, t))\, \mathcal{H}^{l-m}(dt)\, \mu(dF) \\
&\times \sum_{q\geq 1} \sum_{|n|=q} \int_{A^d_{d-m}} c^b(F,n) \int_{J_N \cap F} \mathbb{1}\{t \in [0,1)^l + z_2\} \widetilde{H}_n(Y^b(F^\circ, t))\, \mathcal{H}^{l-m}(dt)\, \mu(dF) \Bigg]
\end{aligned}$$

the upper bound

$$\sum_{(z_1,z_2)\in I_2}\Bigg\{\mathbb{E}\Bigg[\Bigg(\sum_{q\geq 1}\sum_{|n|=q}\int_{A^d_{d-m}} c^b(F,n)\int_{J_N\cap F}\mathbb{1}\{t\in[0,1)^l+z_1\}\widetilde{H}_n(Y^b(F^\circ,t))\,\mathcal{H}^{l-m}(dt)\,\mu(dF)\Bigg)^2\Bigg]$$
$$+\mathbb{E}\Bigg[\Bigg(\sum_{q\geq 1}\sum_{|n|=q}\int_{A^d_{d-m}} c^b(F,n)\int_{J_N\cap F}\mathbb{1}\{t\in[0,1)^l+z_1\}\widetilde{H}_n(Y^b(F^\circ,t))\,\mathcal{H}^{l-m}(dt)\,\mu(dF)\Bigg)^2\Bigg]\Bigg\},$$

which equals

$$2\sum_{(z_1,z_2)\in I_2}\mathbb{E}\Bigg[\Bigg(\sum_{q\geq 1}\sum_{|n|=q}\int_{A^d_{d-m}} c^b(F,n)$$
$$\times\int_{J_N\cap F}\mathbb{1}\{t\in[0,1)^l+z_1\}\widetilde{H}_n(Y^b(F^\circ,t))\,\mathcal{H}^{l-m}(dt)\,\mu(dF)\Bigg)^2\Bigg].$$

Since the summand does not depend on z_2 and we sum over all $z_1,z_2\in\mathbb{Z}^d$ where $\|z_1-z_2\|_\infty\leq s$ the latter equals $2(2s+1)^d$-times the term

$$\sum_{z\in\mathbb{Z}^d}\mathbb{E}\Bigg[\Bigg(\sum_{q\geq 1}\sum_{|n|=q}\int_{A^d_{d-m}} c^{b^F_{J_N}}(F,n)\int_{J_N\cap F}\mathbb{1}\{t\in[0,1)^l+z\}\widetilde{H}_n(Y^b(F^\circ,t))\,\mathcal{H}^{l-m}(dt)\,\mu(dF)\Bigg)^2\Bigg]. \tag{3.47}$$

We now exploit the stationarity to see that the expectation is actually independent of N and we only need to count how often this expectation is nonzero. This happens to be the case $(2N+1)^l$ times, since otherwise the intersection $J_N\cap([0,1)^l+z)$ is empty.

We first define $x\in\mathbb{R}^d$ by $x_i=-\frac{1}{2}$ for $i\in\sigma(J_N)$ and $x_i=\varepsilon_i\frac{1}{2}$ for $i\notin\sigma(J_N)$ such that $[0,1]^l+x=J_{1/2}$. Then for $F\in A^d_{d-m}$ and $z\in\mathbb{Z}^d$ the condition $F\cap J_N\cap([0,1)^l+z)\neq\emptyset$ implies $(F-z+x)\cap J_{1/2}\neq\emptyset$. Thus by Lemma 3.20 the expectation in (3.47) equals

$$\mathbb{E}\Bigg[\Bigg(\sum_{q\geq 1}\sum_{|n|=q}\int_{A^d_{d-m}} c^{b^F_{J_{1/2}}}(F+x-z,n)$$
$$\times\int_{J_N\cap F}\mathbb{1}\{t\in[0,1)^l+z\}\widetilde{H}_n(Y^b(F^\circ,t))\,\mathcal{H}^{l-m}(dt)\,\mu(dF)\Bigg)^2\Bigg].$$

Now, by orthogonality and Fubini's theorem, we obtain equality to

$$\sum_{q\geq 1}\sum_{|n_1|=q=|n_2|} c^{b^F_{J_{\frac{1}{2}}}}(F_1+x-z,n_1)c^{b^F_{J_{\frac{1}{2}}}}(F_2+x-z,n_2)$$
$$\times\int_{A^d_{d-m}}\int_{A^d_{d-m}}\int_{J_N\cap F_2}\mathbb{1}\{t_2\in[0,1)^l+z\}\int_{J_N\cap F_1}\mathbb{1}\{t_1\in[0,1)^l+z\}$$
$$\times\mathbb{E}\Big[H_{n_1}(Y^b(F_1^\circ,t_1))H_{n_2}(Y^b(F_2^\circ,t_2))\Big]\,\mathcal{H}^{l-m}(dt_1)\,\mathcal{H}^{l-m}(dt_2)\,\mu(dF_1)\mu(dF_2). \tag{3.48}$$

Stationarity implies for $z \in \mathbb{Z}^d$

$$\mathbb{E}\left[H_{n_1}(Y^b(F_1^\circ, t_1))H_{n_2}(Y^b(F_2^\circ, t_2))\right] = \mathbb{E}\left[H_{n_1}(Y^b(F_1^\circ, t_1 + x - z))H_{n_2}(Y^b(F_2^\circ, t_2 + x - z))\right]$$

and therefore (3.48) equals

$$\mathbb{E}\left[\left(\sum_{q\geq 1}\sum_{|n|=q}\int_{A^d_{d-m}} c^{b^F_{J_{1/2}}}(F + x - z, n) \right.\right.$$
$$\left.\left. \times \int_{J_N\cap F} \mathbb{1}\{t \in [0,1)^l + z\}\widetilde{H}_n(Y^b(F^\circ, t + x - z))\,\mathcal{H}^{l-m}(dt)\,\mu(dF)\right)^2\right].$$

The translation invariance of the Hausdorff measure implies for $z \in \mathbb{Z}^d$

$$\begin{aligned}
&\int_{J_N\cap F} \mathbb{1}\{t \in [0,1)^l + z\}H_n(Y^b(F^\circ, t + x - z))\,\mathcal{H}^{l-m}(dt)\\
&\quad = \int_{(J_N\cap F)-z+x} \mathbb{1}\{t + z - x \in [0,1)^l + z\}H_n(Y^b(F^\circ, t))\,\mathcal{H}^{l-m}(dt)\\
&\quad = \int_{(J_N\cap F)-z+x} \mathbb{1}\{t \in [0,1)^l + x\}H_n(Y^b(F^\circ, t))\,\mathcal{H}^{l-m}(dt),
\end{aligned}$$

and taking into account that $(J_N - z + x) \cap ([0,1]^l + x) = J_{1/2}$, in the cases in which the intersection is nonempty, as well as $Y^{b^F_{J_N}}(F^\circ, t) = Y^{b^F_{J_{1/2}}}(F^\circ, t)$, we obtain

$$\begin{aligned}
&\int_{(J_N\cap F)-z+x} \mathbb{1}\{t \in [0,1)^l + x\}H_n(Y^{b^F_{J_N}}(F^\circ, t))\,\mathcal{H}^{l-m}(dt)\\
&\quad = \int_{J_{1/2}\cap(F-z+x)} \widetilde{H}_n(Y^{b^F_{J_{1/2}}}(F^\circ, t))\,\mathcal{H}^{l-m}(dt).
\end{aligned}$$

To summarize the last steps, we established

$$\mathbb{E}\left[\left(\sum_{q\geq 1}\sum_{n\in\mathbb{N}_0^D,|n|=q}\int_{A^d_{d-m}} c^{b^F_{J_N}}(F, n)\int_{J_N\cap F}\mathbb{1}\{t \in [0,1)^l + z\}\widetilde{H}_n(Y^{b^F_{J_N}}(F^\circ, t))\,\mathcal{H}^{l-m}(dt)\,\mu(dF)\right)^2\right]$$
$$= \mathbb{E}\left[\left(\sum_{q\geq 1}\sum_{n\in\mathbb{N}_0^D,|n|=q}\int_{A^d_{d-m}} c^{b^F_{J_{1/2}}}(F + x - z, n)\right.\right.$$
$$\left.\left.\times \int_{J_{1/2}\cap(F+x-z)} \widetilde{H}_n(Y^{b^F_{J_{1/2}}}(F^\circ, t))\,\mathcal{H}^{l-m}(dt)\,\mu(dF)\right)^2\right].$$

By the translation invariance of the measure μ this equals

$$\mathbb{E}\left[\left(\sum_{q\geq 1}\sum_{n\in\mathbb{N}_0^D,|n|=q}\int_{A^d_{d-m}} c^b(F, n)\int_{J_{1/2}\cap F}\widetilde{H}_n(Y^b(F^\circ, t))\,\mathcal{H}^{l-m}(dt)\,\mu(dF)\right)^2\right],$$

which is independent of N. Hence, we deduce for the second summand the upper bound

$$2(2s+1)^d(2N+1)^l\mathbb{E}\left[\left(\sum_{q\geq 1}\sum_{n\in\mathbb{N}_0^D,|n|=q}\int_{A^d_{d-m}} c^b(F,n)\right.\right.$$
$$\left.\left.\times\int_{J_{1/2}\cap F}\widetilde{H}_n(Y^b(F^\circ,t))\,\mathcal{H}^{l-m}(dt)\,\mu(dF)\right)^2\right],$$

which establishes the assertion. □

3.3. A lower bound for the asymptotic variance

We outline in this section where the arguments of [22, Lemma 2.2] have to be altered to establish a lower bound for the asymptotic variance.

Lemma 3.23. *Let X be a real Gaussian field on $\mathbb{R}^d$, which satisfies the assumptions (A1)–(A3). Then for σ_m^2 and u given as in Theorem 3.1, and $m=0,\dots,d-1$*

$$\sigma_m^2\geq\begin{bmatrix} d\\ d-m\end{bmatrix}^2(2\pi)^m f(0)H_{d-m}(u)^2\phi(u)^2.$$

Proof. Recall that according to Theorem 2.15 the asymptotic variance is given by $\sum_{q\geq 1}\sigma^2_{m,q}$, where $\sigma^2_{m,q}$ is defined as the limit in condition (i) of that theorem. Hence, we obtain a lower bound for the asymptotic variance by computing $\sigma^2_{m,1}$. By (3.25)

$$\sigma^2_{m,1}=\sum_{k,l\in\{1,\dots,D\}}b(k)b(l)\int_{\mathbb{R}^d}\int_{G^d_{d-m}}\int_{G^d_{d-m}}\mathbb{E}\left[Y_k(L,t)Y_l(L',0)\right]\,\nu(dL)\,\nu(dL')\,dt,$$

where the coefficients $b(\cdot)$ are given by

$$b(k)=\sum_{n\in\mathbb{N}_0^D,|n|=1}\mathbb{1}\{k\in\mathcal{A}_n\}\frac{c(n)}{|\mathcal{A}_n|}.$$

The sets $\mathcal{A}_n$ consist of only one element, namely the number of the component of n, which contains the 1. Thus if we write $e_i\in\mathbb{R}^D$ for the vector, whose components are 0 except for the i-th component, which is 1, we obtain

$$b(k)=c(e_k).$$

By the definition of the coefficients $c(\cdot)$, cf. (3.14), we see that $c(e_k)=0$ for $k=1,\dots,d-m$ and therefore obtain

$$\sigma^2_{m,1}=\sum_{k,l=d-m+1}^{D}c(e_k)c(e_l)\int_{\mathbb{R}^d}\int_{G^d_{d-m}}\int_{G^d_{d-m}}\mathbb{E}\left[Y_k(L,t)Y_l(L',0)\right]\,\nu(dL)\,\nu(dL')\,dt.$$

We now show that

$$\int_{\mathbb{R}^d} \mathbb{E}\left[Z_k(L,t)Z_l(L',0)\right]\, dt = (2\pi)^d f(0)\mathbb{1}\{(k,l)=(D,D)\},$$

for $L, L' \in G^d_{d-m}$ and $k,l = 1,\ldots,D$. We first consider the case $(k,l)=(D,D)$. Then the equality $\mathbb{E}\left[Z_D(L,t)Z_D(L',0)\right] = \mathbb{E}\left[X(t)X(0)\right] = (2\pi)^{d/2}\mathcal{L}(f)(t)$ holds, where $\mathcal{F}$ denotes the Fourier transformation. By (A3) the spectral density f is continuous and $\mathbb{E}\left[X(t)X(0)\right]$ is integrable, which yields that $\int_{\mathbb{R}^d} \mathbb{E}\left[X(t)X(0)\right] dt = (2\pi)^d f(0)$, via the Fourier cotransformation. In the cases where $(k,l)\neq(D,D)$, at least one of the factors $Z_D(L,\cdot)$ or $Z_D(L',\cdot)$ is a directional derivative of the field X of order greater or equal than 1, say in direction $u \in S^{d-1}$. This yields that $\mathbb{E}\left[Z_k(L,t)Z_l(L',0)\right]$ equals, up to a power of -1, the function $\frac{\partial}{\partial u} g$, where g is either the covariance function or a derivative of it. Thus by Fubini's theorem, we conclude that

$$\int_{\mathbb{R}^d} \mathbb{E}\left[Z_k(L,t)Z_l(L',0)\right] dt = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \frac{\partial}{\partial u} g(t_1,\ldots,t_d)\, dt_1 \ldots dt_d.$$

By writing the directional derivative as the inner product of the direction and the gradient, this equals

$$\begin{aligned}\sum_{i=1}^d u^{(i)} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \frac{\partial}{\partial t_i} g(t_1,\ldots,t_d)\, dt_i\, dt_1 \ldots \overline{dt_i} \ldots dt_d \\ = \sum_{i=1}^d u^{(i)} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g(t_1,\ldots,t_d)|_{t_i=-\infty}^{\infty}\, dt_1 \ldots \overline{dt_i} \ldots dt_d = 0,\end{aligned}$$

where we used assumption (A3) in the last line.

The definition of Y, cf. (3.8), implies

$$\mathbb{E}\left[Y_k(L,t)Y_l(L',0)\right] = \sum_{r=1}^D \sum_{s=1}^D \Lambda^{-1}_{l,r}\Lambda^{-1}_{k,s}\mathbb{E}\left[Z_r(L,t)Z_s(L',0)\right],$$

which yields

$$\int_{\mathbb{R}^d} \mathbb{E}\left[Y_k(L,t)Y_l(L',0)\right]\, dt = \Lambda^{-1}_{l,D}\Lambda^{-1}_{k,D}(2\pi)^d f(0)$$

and we conclude with Fubini's theorem that

$$\sigma^2_{m,1} = \begin{bmatrix} d \\ d-m \end{bmatrix}^2 \sum_{k,l=d-m+1}^{D} c(e_k)c(e_l)\Lambda^{-1}_{l,D}\Lambda^{-1}_{k,D}(2\pi)^d f(0) = \begin{bmatrix} d \\ d-m \end{bmatrix}^2 c(e_D)^2(\Lambda^{-1}_{D,D})^2(2\pi)^d f(0),$$

where the last equality holds since Λ is lower triangular. In order to calculate the coefficients $c(e_D)$, we have to analyse the covariance matrix of $Z(L,0)$. We first write the $K+1 := (d-m)(d-m+1)/2+1$ last coordinates of $Z(L,0)$ in the order

$$\left(\left(\frac{\partial^2}{\partial v_i \partial v_j} X(0)\right)_{1\le i<j\le d-m}, \left(\frac{\partial^2}{\partial v_i \partial v_i} X(0)\right)_{i=1}^{d-m}, X(0)\right).$$

Thus, using stationarity, isotropy and $C^X(0) = 1$, the covariance matrix of this vector at 0 is given by the matrix

$$\begin{pmatrix} \mathrm{Cov}\left(\left(\frac{\partial^2}{\partial t_i \partial t_j} X(0)\right)_{i<j}\right) & \mathrm{Cov}\left(\left(\frac{\partial^2}{\partial t_i \partial t_j} X(0)\right)_{i<j}, \left(\frac{\partial^2}{\partial t_i \partial t_i} X(0)\right)_{i=1}^{d-m}\right) & 0 \\ \mathrm{Cov}\left(\left(\frac{\partial^2}{\partial t_i \partial t_i} X(0)\right)_{i=1}^{d-m}, \left(\frac{\partial^2}{\partial t_i \partial t_j} X(0)\right)_{i<j}\right) & \mathrm{Cov}\left(\left(\frac{\partial^2}{\partial t_i \partial t_i} X(0)\right)_{i=1}^{d-m}\right) & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

which equals the product $\Lambda_2 \Lambda_2^\top$, where $\Lambda_2 \in \mathbb{R}^{K+1}$ is the lower triangular matrix, given in Lemma 3.5. We choose the matrix $L \in \mathbb{R}^{K \times K}$, the vector $l \in \mathbb{R}^K$ and $\alpha > 0$ such that

$$\Lambda_2 = \begin{pmatrix} L & 0 \\ l^\top & \alpha \end{pmatrix}.$$

Then the relation $\|l\|^2 + \alpha^2 = 1$ holds as well as $Ll = (0_{1\times K}, -1_{1\times d-m})$. With this specific representation of Λ_2 we have

$$\begin{aligned} c(e_D) &= (2\pi)^{-(d-m)/2}(-1)^{d-m} \int_{\mathbb{R}^K \times \mathbb{R}} \det(Ly) \mathbb{1}\{\langle l, y\rangle + \alpha z \geq u\} z \phi_K(y) \phi(z) \, d(y,z) \\ &= -(2\pi)^{-(d-m)/2}(-1)^{d-m} \int_{\mathbb{R}^K \times \mathbb{R}} \det(Ly) \mathbb{1}\{\langle l, y\rangle + \alpha z \geq u\} \phi_K(y) \phi'(z) \, d(y,z) \\ &= (2\pi)^{-(d-m)/2}(-1)^{d-m} \int_{\mathbb{R}^K} \det(Ly) \phi_K(y) \phi\left(\alpha^{-1}(u - \langle l, y\rangle)\right) dy, \end{aligned}$$

where we used that $z\phi(z) = -\phi'(z)$ in the second line and Fubini's theorem in the second. We note that the K-dimensional vector Ly is identified with the symmetric $(d-m)\times(d-m)$-matrix, whose nondiagonal entries are given by the first $(d-m)(d-m-1)/2$ entries of Ly and whose diagonal is given by the $d-m$ last entries of Ly. Using the Hermite expansion of $y \mapsto \det(Ly)$ given in [22, Lemma A.2], we obtain

$$\begin{aligned} c(e_D) &= (2\pi)^{-(d-m)/2}(-1)^{d-m} \sum_{m \in \mathbb{N}_0^K, |m| = d-m} \beta_m \int_{\mathbb{R}^K} \widetilde{H}_m(y) \phi_K(y) \phi\left(\alpha^{-1}(u - \langle l, y\rangle)\right) dy \\ &= (2\pi)^{-(d-m)/2} \sum_{m \in \mathbb{N}_0^K, |m| = d-m} \beta_m \int_{\mathbb{R}^K} D^m \phi_K(y) \phi\left(\alpha^{-1}(u - \langle l, y\rangle)\right) dy, \end{aligned}$$

where $D^m\phi$ denotes $\frac{\partial^{|m|}}{\partial t_{m_1} \ldots \partial t_{m_K}} \phi$ and β_m are real coefficients. Following the argument in [22], we define $h\colon \mathbb{R}^K \to \mathbb{R}, x \mapsto \phi(\alpha^{-1}\langle l, y\rangle)$ and choose l' such that $\langle l, l'\rangle = 1$. We then obtain

$$\int_{\mathbb{R}^K} D^m \phi_K(y) \phi\left(\alpha^{-1}(u - \langle l, y\rangle)\right) dy = (h * D^m \phi_K)(ul') = D^m(h * \phi_K)(ul')$$

by properties of the convolution operation $*$. Using [22, Remark A.4], which reads $(h*\phi_K)(y) = \alpha\phi(\langle l, y\rangle)$ for $y \in \mathbb{R}^K$, we obtain

$$D^m(h * \phi_K)(y) = \alpha l^{(m)} \phi^{(d-m)}(\langle l, y\rangle) = (-1)^{d-m} \alpha l^{(m)} H_{d-m}(\langle l, y\rangle) \phi(\langle l, y\rangle).$$

Thus by [22, Lemma A.2], which establishes the Hermite expansion of $y \mapsto \det(Ly)$, in the second equality

$$c(e_D) = (2\pi)^{-(d-m)/2} \sum_{\substack{m \in \mathbb{N}_0^K \\ |m|=d-m}} \beta_m l^{(m)} (-1)^{d-m} \alpha H_{d-m}(u)\phi(u)$$
$$= (2\pi)^{-(d-m)/2} \det(Ll)(-1)^{d-m} \alpha H_{d-m}(u)\phi(u).$$

We recall that the K-dimensional vector Ll corresponds to the symmetric $(d-m) \times (d-m)$-matrix, whose nondiagonal entries are given by the first $(d-m)(d-m-1)/2$ entries of Ll and whose diagonal is given by the $d-m$ last entries of Ll, thus $\det(Ll) = (-1)^{d-m}$, since $Ll = (0_{1\times K}, -1_{1\times d-m})$. Hence, we obtain

$$c(e_D) = (2\pi)^{-(d-m)/2} \alpha H_{d-m}(u)\phi(u)$$

and therefore conclude as asserted

$$\sigma_{m,1}^2 = \begin{bmatrix} d \\ d-m \end{bmatrix}^2 (2\pi)^m f(0) H_{d-m}(u)^2 \phi(u)^2. \qquad \square$$

3.4. The multivariate case

In this section, we establish a multivariate central limit theorem for all Lipschitz–Killing curvatures of an excursion set of a Gaussian field in the asymptotic scenario of an ever-growing observation window. Furthermore, different choices for the thresholds of the excursion sets are possible. We define for $N \in \mathbb{N}$, $u \in \mathbb{R}$ and $m \in \{0, \dots, d-1\}$

$$\Psi_N^{u,m} := \frac{\mathcal{L}_m\left(C_N^d \cap X^{-1}([u,\infty))\right) - \mathbb{E}\left[\mathcal{L}_m\left(C_N^d \cap X^{-1}([u,\infty))\right)\right]}{\mathcal{H}^d(C_N^d)^{\frac{1}{2}}}$$

and show the following theorem.

Theorem 3.24. *Let X be a real Gaussian field on $\mathbb{R}^d$, which satisfies the assumptions (A1)–(A3) and let $u_0, \dots, u_{d-1} \in \mathbb{R}$. Then*

$$(\Psi_N^{u_0,0}, \dots, \Psi_N^{u_{d-1},d-1}) \xrightarrow{\mathcal{D}} \mathcal{N}_d(0, \Sigma^u)$$

as $N \to \infty$ and $\Sigma^u \in \mathbb{R}^{d\times d}$ is positive semidefinite.

Proof. We first note that, as in the univariate case, there are terms which dominate the asymptotic behaviour of the vector $(\Psi_N^{u_0,0}, \dots, \Psi_N^{u_{d-1},d-1})$. Indeed, by using the definitions of Section 3.2.1, we have for $c \in \mathbb{R}^d$

$$c^\top (\Psi_N^{u_0,0}, \dots, \Psi_N^{u_{d-1},d-1})$$
$$= \sum_{m=0}^{d-1} c_{m+1} \frac{\zeta_N^{u_m,m} - \mathbb{E}\left[\zeta_N^{u_m,m}\right]}{\mathcal{H}^d(C_N^d)^{\frac{1}{2}}} + \sum_{m=0}^{d-1} c_{m+1} \sum_{j=0}^{d-m-1} \sum_{J_N \in \partial_{j+m} C_N^d} \frac{\epsilon_{J_N}^{u_m,m} - \mathbb{E}\left[\epsilon_{J_N}^{u_m,m}\right]}{\mathcal{H}^d(C_N^d)^{\frac{1}{2}}},$$

where the second sum converges to 0 in probability, by (3.41), and the fact that convergence in probability is additive. We note that we changed the notation to reflect the dependence on u_m and m. Now, suppose for a moment that the random vector given by

$$\left(\frac{\zeta_N^{u_0,0} - \mathbb{E}\left[\zeta_N^{u_0,0}\right]}{\mathcal{H}^d(C_N^d)^{\frac{1}{2}}}, \ldots, \frac{\zeta_N^{u_{d-1},d-1} - \mathbb{E}\left[\zeta_N^{u_{d-1},d-1}\right]}{\mathcal{H}^d(C_N^d)^{\frac{1}{2}}} \right)$$

satisfies a multivariate central limit theorem as $N \to \infty$, then the theorem of Cramér-Wold, cf. [36, Corollary 5.5], applied twice, and Slutzky's lemma yield the asserted central limit theorem. Hence, all that is left to show is a multivariate central limit theorem for the above random vector. We would like to apply Theorem 2.15, which forces us to develop a representation of the random vector in terms of multiple stochastic integrals with respect to an isonormal Gaussian process on a suitable Hilbert space. For this purpose, we recall Theorem 3.8

$$\zeta_N^{u_m,m} \overset{L^2(\mathbb{P})}{=} \sum_{q \geq 0} \sum_{n \in \mathbb{N}_0^{D_m}, |n|=q} \int_{G_{d-m}^d} c(n, u_m, m) \int_{C_N^d} \widetilde{H}_n(Y^m(L,t))\, dt\, \nu(dL). \tag{3.49}$$

The Gaussian field $Y^m(L,t)$ was defined by

$$Y^m(L,t) = \Lambda^{-1} G^m(L, \nabla X(t), D^2X(t), X(t)),$$

where $\Lambda \in \mathbb{R}^{D_m \times D_m}$ is defined as in Lemma 3.5 and $G^m \colon G_{d-m}^d \times \mathbb{R}^d \times \mathbb{R}^{d \times d} \times \mathbb{R} \to \mathbb{R}^{D_m}$ is explicitly given by

$$G^m(L,x,A,y) := \left((\langle v_i, x\rangle)_{i=1}^{d-m}, \left(\left((v_1|\cdots|v_{d-m})^\top A (v_1|\cdots|v_{d-m}) \right)_{i,j} \right)_{1 \leq i \leq j \leq d-m}, y \right)$$

with $(v_i)_{i=1}^{d-m}$ denoting an orthonormal basis of L. The only way in which randomness enters the right side of equation (3.49), is through the field $(\nabla X, D^2X, X)$. Thus, it is this field we need to embed into an isonormal Gaussian process. The real Hilbert space $\mathfrak{H}$, which suits our needs, is again given by

$$\mathfrak{H} := \left\{ h \colon \mathbb{R}^d \to \mathbb{C} \mid h(-x) = \overline{h(x)}, \int_{\mathbb{R}^d} |h(x)|^2 f(x)\, dx < \infty \right\}$$

equipped with the inner product

$$\langle g, h \rangle_{L^2(f\lambda^d)} := \int_{\mathbb{R}^d} g(x)\overline{h(x)} f(x)\, dx$$

and accompanied by the isonormal Gaussian process W. We recall that f denotes the spectral density of f. Then, with $\varphi_{t,j} \in \mathfrak{H}$, $t \in \mathbb{R}^d$, $j = 1, \ldots, d + d^2 + 1$, given by $\varphi_{t,j}(x) := \nu_j(x) e^{i\langle t,x\rangle}$, $\nu(x) := ((ix_j)_{j=1}^d, (-x_r x_s)_{r,s=1}^d, 1)$, we obtain

$$(\nabla X(\cdot), D^2X(\cdot), X(\cdot)) \overset{\mathcal{D}}{=} (W(\varphi_{\cdot,1}), \ldots, W(\varphi_{\cdot,d+d^2+1}))$$

as processes on $\mathbb{R}^d$. Indeed, the equality of the covariance functions of both processes yields the equality in distribution, since both processes are centered Gaussian fields. Then

$$\Big(\mathcal{H}^d(C_N^d)^{-\frac{1}{2}}(\zeta_N^{u_m,m} - \mathbb{E}\,[\zeta_N^{u_m,m}])\Big)_{m=0}^{d-1}$$
$$\overset{\mathcal{D}}{=} \mathcal{H}^d(C_N^d)^{-\frac{1}{2}} \left(\sum_{q\geq 1} \sum_{n\in\mathbb{N}_0^{D_m},|n|=q} \int_{G_{d-m}^d} c(n,u_m,m) \right.$$
$$\left. \times \int_{C_N^d} \widetilde{H}_n(\Lambda^{-1}G^m(L,(W(\varphi_{t,j}))_{j=1}^{d+d^2+1}))\,dt\,\nu(dL) \right)_{m=0}^{d-1}.$$

We note now that $\Lambda^{-1}G^m(L,\cdot)$ is a linear mapping. Hence, by the linearity of the isonormal process

$$\Lambda^{-1}G^m(L,(W(\varphi_{t,j}))_{j=1}^{d+d^2+1}) = (W(\varphi_{t,1}^L),\dots,W(\varphi_{t,D_m}^L)),$$

where $\varphi_{t,i}^L$ is defined in (3.20). Then by Theorem 2.13

$$\Big(\mathcal{H}^d(C_N^d)^{-\frac{1}{2}}(\zeta_N^{u_m,m} - \mathbb{E}\,[\zeta_N^{u_m,m}])\Big)_{m=0}^{d-1}$$
$$\overset{\mathcal{D}}{=} \mathcal{H}^d(C_N^d)^{-\frac{1}{2}} \left(\sum_{q\geq 1} \sum_{n\in\mathbb{N}_0^{D_m},|n|=q} \int_{G_{d-m}^d} c(n,u_m,m) \right.$$
$$\left. \times \int_{C_N^d} I_q((\varphi_{t,1}^L)^{\otimes n_1}\otimes\cdots\otimes(\varphi_{t,D_m}^L)^{\otimes n_{D_m}})\,dt\,\nu(dL) \right)_{m=0}^{d-1}.$$

Fubini's theorem for Wiener-Itô integrals and the same combinatorial manipulations as in the proof of Lemma 3.9 yield

$$\Big(\mathcal{H}^d(C_N^d)^{-\frac{1}{2}}(\zeta_N^{u_m,m} - \mathbb{E}\,[\zeta_N^{u_m,m}])\Big)_{m=0}^{d-1} \overset{\mathcal{D}}{=} \left(\sum_{q\geq 1} I_q(g_{N,q}^{u_m,m})\right)_{m=0}^{d-1},$$

where

$$g_{N,q}^{u_m,m} := \frac{1}{\mathcal{H}^d(C_N^d)^{1/2}} \sum_{k\in\{1,\dots,D_m\}^q} \int_{G_{d-m}^d} b(k,u_m,m) \int_{C_N^d} \varphi_{t,k_1}^L\otimes\cdots\otimes\varphi_{t,k_q}^L\,dt\,\nu(dL).$$

We note that $g_{N,q}^{u_m,m}$ equals $g_{N,q}$ in Lemma 3.9, we merely changed the notation to reflect the dependence on u_m and m.

The last step in the proof of the asserted central limit theorem is to check the conditions of Theorem 2.15. Condition (iii) and condition (iv) do not use the interplay of the different coordinates and therefore the same approaches as in the univariate case hold, cf. Section 3.2.4. The verification of condition (ii) also strongly relies on the univariate case after realising that for $m,n = 0,\dots,d-1$

$$\sum_{q\geq 1}\left|\sigma_q^{mn}\right| = \sum_{q\geq 1}\lim_{N\to\infty} q!\langle g_{N,q}^{u_m,m}, g_{N,q}^{u_n,n}\rangle_{\mathfrak{H}^{\otimes q}} \leq \sum_{q\geq 1}\lim_{N\to\infty}\|g_{N,q}^{u_m,m}\|_{\mathfrak{H}^{\otimes q}}\|g_{N,q}^{u_n,n}\|_{\mathfrak{H}^{\otimes q}},$$

which can be bounded from above by

$$\sum_{q\geq 1} \lim_{N\to\infty} \|g_{N,q}^{u_m,m}\|^2_{\mathfrak{H}^{\otimes q}} + \sum_{q\geq 1} \lim_{N\to\infty} \|g_{N,q}^{u_n,n}\|^2_{\mathfrak{H}^{\otimes q}} = \sum_{q\geq 1} \sigma_q^{mm} + \sum_{q\geq 1} \sigma_q^{nn} < \infty,$$

where the finiteness is derived in Section 3.2.4 while verifying condition (ii). Since absolute convergence implies convergence of a series, we obtain condition (ii). For condition (i), the same reasoning as in the one-dimensional case yields

$$q!\langle g_{N,q}^{u_m,m}, g_{N,q}^{u_n,n}\rangle_{\mathfrak{H}^{\otimes q}} \overset{N\to\infty}{\longrightarrow} q! \sum_{k\in\{1,\dots,D_m\}^q} \sum_{l\in\{1,\dots,D_n\}^q} b(k,u_m,m)b(l,u_n,n) \\ \times \int_{\mathbb{R}^d}\int_{G^d_{d-m}}\int_{G^d_{d-n}} \prod_{s=1}^q \mathbb{E}\left[Y^m_{k_s}(L,t)Y^n_{l_s}(L',0)\right]\, \nu(dL)\,\nu(dL')\,dt,$$

and therefore the assertion. □

Remark. Theorem 2.15 does not only provide sufficient conditions for a multivariate central limit theorem, but also contains a, to some extent, explicit representation of the asymptotic covariance matrix Σ. In the specific case of Theorem 3.24, we obtain

$$\Sigma_{m,n} = \sum_{q\geq 1} q! \sum_{k\in\{1,\dots,D_m\}^q} \sum_{l\in\{1,\dots,D_n\}^q} b(k,u_m,m)b(l,u_n,n) \\ \times \int_{\mathbb{R}^d}\int_{G^d_{d-m}}\int_{G^d_{d-n}} \prod_{s=1}^q \mathbb{E}\left[Y^m_{k_s}(L,t)Y^n_{l_s}(L',0)\right]\, \nu(dL)\,\nu(dL')\,dt.$$

This representation could be the starting point for the investigation of criteria for the positive definiteness of the asymptotic covariance matrix.

CHAPTER 4

A Central Limit Theorem for Integrated Functionals

In this chapter, we provide a general multivariate central limit theorem for integrated level functionals, defined in (4.1), of a Gaussian excursion set. The asymptotic scenario is again that of an ever-growing observation window. Furthermore, we specialise the general case to integrated Minkowski surface tensors and integrated curvature measures.

4.1. The general case

In the following, we state and prove the normal approximation of integrated level functionals of a Gaussian excursion set. The proof relies on the same stochastic methods as the proof in Section 3.2, that is on the central limit theorem originating from Stein's method and Malliavin calculus, cf. Theorem 2.15.

Let $X = \{X_t\colon \Omega \to \mathbb{R} \mid t \in \mathbb{R}^d\}$ be a real Gaussian field defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Moreover let $k \in \mathbb{N}$ and the mapping $h\colon \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{R} \to \mathbb{R}^k$ be given. We impose the following conditions on X and h.

(AF1) X is a centered, stationary Gaussian field. The trajectories are almost surely of class $\mathcal{C}^2$. The covariance function $C^X(t) = \mathbb{E}\,[X(t)X(0)]$, $t \in \mathbb{R}^d$, of X satisfies $C^X(0) = 1$.

(AF2) The covariance matrices of the vectors

$$\left(X(0), \left(\frac{\partial^2}{\partial t_i \partial t_j} X(0)\right)_{1\le i\le j\le d}\right) \quad \text{and} \quad \left(\frac{\partial}{\partial t_i} X(0)\right)_{i=1}^{d}$$

have full rank.

(AF3) The mapping defined by

$$\psi(t) := \max\left\{\left|\frac{\partial^k}{\partial t_{j_1}\dots\partial t_{j_k}} C^X(t)\right| : k \in \{0,\dots,4\}, 1 \le j_1,\dots,j_k \le d\right\}$$

for $t \in \mathbb{R}^d$, satisfies

$$\psi(t) \overset{\|t\|\to\infty}{\longrightarrow} 0 \text{ and } \psi \in L^1(\mathbb{R}^d, \lambda^d).$$

(AF4) For all invertible linear mappings $A\colon \mathbb{R}^d \to \mathbb{R}^d$ and $B\colon \mathbb{R}^{d(d+1)/2+1} \to \mathbb{R}^{d(d+1)/2+1}$ the mapping h satisfies

$$\int_{\mathbb{R}^d \times \mathbb{R}^{d(d+1)/2+1}} h(Ax, \mathrm{m}((By)_{1,\dots,d(d+1)/2}), (By)_{d(d+1)/2+1})^2 \times \|x\|^2 \phi_{d+d(d+1)/2+1}(x,y)\, d(x,y) < \infty$$

coordinatewise, where $(x)_{k_1,\dots,k_j}$ abbreviates the projection onto the coordinates $k_1,\dots,k_j$ of $x \in \mathbb{R}^l$ and $\mathrm{m}\colon \mathbb{R}^{d(d+1)/2} \to \mathbb{R}^{d\times d}$ maps the upper half of a symmetric matrix to the matrix itself.

We note that under (AF1) the set $X^{-1}(\{u\})$ carries the structure of a $(d-1)$-dimensional, $\mathcal{C}^2$ submanifold of $\mathbb{R}^d$. In assumption (AF3) sufficient properties for a central limit theorem are formulated. Assumption (AF2) and assumption (AF4) are needed for calculations in our method of proof. In contrast to the assumption (A2) of Theorem 3.1 the complexity of assumption (AF2) is reduced, which is a result of the specific definition of the investigated functional $\Psi^h(X,\cdot)$. The integration over the threshold parameter u makes the approximation with a Dirac sequence (cf. Section 3.2.1) and the accompanying calculations involving the Rice formulas obsolete.

We define for $A \in \mathcal{B}(\mathbb{R}^d)$ the integrated level functional Ψ^h of X by

$$\Psi^h(X,A) := \int_{\mathbb{R}} \int_{X^{-1}(\{u\})\cap A} h(\nabla X(t), D^2X(t), X(t))\, \mathcal{H}^{d-1}(dt)\, du, \tag{4.1}$$

where the integration is defined coordinatewise, and establish a central limit theorem for the standardized functional $\Psi^h(X,\cdot)$ in the asymptotic scenario of an ever-growing observation window. The notation $A_N \nearrow \mathbb{R}^d$ used below is a shorthand notation for the assumption that the inradius of the $\mathcal{H}^d$-measurable set A_N and the inradius of the set $A_N - A_N$ tend to ∞ and moreover $\mathcal{H}^d((A_N - t)\cap A_N)/\mathcal{H}^d(A_N) \to 1$ as $N \to \infty$ for fixed $t \in \mathbb{R}^d$.

Theorem 4.1. *Let X be a real Gaussian field on $\mathbb{R}^d$ which satisfies the assumptions (AF1)–(AF3), and let $h\colon \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{R} \to \mathbb{R}^k$ satisfy assumption (AF4). Then*

$$\frac{\Psi^h(X,A_N) - \mathbb{E}\left[\Psi^h(X,A_N)\right]}{\mathcal{H}^d(A_N)^{1/2}} \xrightarrow{\mathcal{D}} \mathcal{N}_k(0,\Sigma^h), \qquad \text{as } A_N \nearrow \mathbb{R}^d,$$

where $\Sigma^h \in \mathbb{R}^{k\times k}$ is positive semidefinite.

Remark. (i) The presented setting is general enough to allow for weighted integrals in the threshold u, since $X(t)$ is exactly u for $t \in X^{-1}(\{u\})$. Thus, if we were interested in the level functional for a specific level u, we could pursue an approach as in Chapter 3 and approximate the level functional with the integrated level functionals by choosing a Dirac sequence for the dependence in u. However, such an endeavour makes lengthy calculations, as in the proof of Lemma 3.2, necessary and will not be pursued in this thesis. In a less general setting than the one presented here, this procedure is carried out in [10].

(ii) If the functional h does not depend on the second derivatives of the field X, then there is no need to assume that the field is twice continuously differentiable. The assumptions (AF1)–(AF4) change accordingly.

(iii) In [37] a multivariate central limit theorem for the standardisation of a functional of the type

$$\int_{A_N} h(X(t))\,dt, \qquad \text{as } A_N \nearrow \mathbb{R}^d,$$

where the $\mathbb{R}^s$-valued random field X is $BL(\theta)$-dependent and $h\colon \mathbb{R}^s \to \mathbb{R}^k$ is Lipschitz continuous, is derived. By Lemma 4.2 the functional Ψ^h also admits a representation of this type. The differences however are that in the setting presented in this work, the field X has to be a Gaussian field, which is more restrictive, and the function h has to satisfy the integrability condition (AF4). Because of the different condition on the function h, one result can not be derived from the other.

(iv) At first sight, Condition (AF4) seems to be very difficult to verify. A second more detailed inspection reveals that although (AF4) is restricting generality, it allows quite general functions h, for example polynomials of bounded degree. Special choices of h are treated in Section 4.2 and Section 4.3.

4.1.1. Hermite type expansion

We begin the proof of Theorem 4.1 with the following Lemma.

Lemma 4.2. *Let $A \in \mathcal{B}(\mathbb{R}^d)$ be compact. Then*

$$\Psi^h(X,A) = \int_A h(\nabla X(t), D^2X(t), X(t))\|\nabla X(t)\|\,dt.$$

Proof. We note that since A is compact and the field X is pathwise continuously differentiable, X is also Lipschitz on A. Then Federer's coarea formula, cf. [25, Theorem 3.2.12], yields the assertion. □

Lemma 4.2 explains why in this chapter an approximation involving Dirac sequences (cf. Section 3.2.1) is unnecessary. The basic application of the coarea formula yields a simpler representation of the integrated level functional, which leads to a Hermite expansion without the necessity of approximating the functional.

In the following, we describe the Hermite expansion and define for $D := d + d(d+1)/2 + 1$ the $\mathbb{R}^D$-valued Gaussian field $Z = \{Z(t) \mid t \in \mathbb{R}^d\}$ by

$$Z(t) := \left(\left(\frac{\partial}{\partial t_i} X(t) \right)_{i=1}^{d}, \left(\frac{\partial^2}{\partial t_i \partial t_j} X(t) \right)_{1 \le i \le j \le d}, X(t) \right), \quad t \in \mathbb{R}^d,$$

and denote its covariance matrix by Σ. Note that due to stationarity, Σ is independent of t. Assumption (AF2) and the well-known Cholesky decomposition, cf. [7, Fact 8.9.37], imply the decomposition $\Sigma = \Lambda\Lambda^\top$. Furthermore, stationarity implies that at a given point in space the first partial derivatives of X are independent of the second partial derivatives of X and the field X itself, cf. Section 2.1. Therefore $\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}$. With the aid of Λ, we define the decorrelated Gaussian field $Y = \{Y(t) \mid t \in \mathbb{R}^d\}$ by

$$Y(t) := \Lambda^{-1} Z(t), \quad t \in \mathbb{R}^d.$$

For fixed $t \in \mathbb{R}^d$, the random vector $Y(t)$ is standard normal, i.e. $Y(t) \sim \mathcal{N}_D(0, I_D)$. However, we note that for different $t, s \in \mathbb{R}^d$ the vectors $Y(t)$ and $Y(s)$ are in general not independent.

With the definition of the function $G^h \colon \mathbb{R}^d \times \mathbb{R}^{d(d+1)/2+1} \to \mathbb{R}^k$ by

$$(x_1, x_2) \mapsto h(\Lambda_1 x_1, \mathrm{m}((\Lambda_2 x_2)_{1,\dots,d(d+1)/2}), (\Lambda_2 x_2)_{d(d+1)/2+1}) \|\Lambda_1 x_1\|,$$

we obtain by Lemma 4.2

$$\Psi^h(X, A) = \int_A G^h(Y(t))\, dt. \tag{4.2}$$

Now, by (AF4), we observe that $G_i^h \in L^2(\mathcal{N}_D(0, I_D))$, $i = 1, \dots, k$, which yields the representation

$$G_i^h = \sum_{q \ge 0} \sum_{n \in \mathbb{N}_0^D, |n| = q} c(n, h, i) \widetilde{H}_n, \tag{4.3}$$

in $L^2(\mathcal{N}_D(0, I_D))$, where $\widetilde{H}_n$ denotes the n-th multivariate Hermite polynomial and $c(n, h, i)$ is given by

$$c(n, h, i) := 1/n! \int_{\mathbb{R}^D} G_i^h(x) \widetilde{H}_n(x) \phi_D(x)\, dx.$$

This expansion leads to the Hermite type expansion of the random variable Ψ^h described in the next lemma.

Lemma 4.3. *Let X satisfy assumptions (AF1) and (AF2), and let h satisfy (AF4). Then for*

$A \in \mathcal{B}(\mathbb{R}^d)$ compact and $i = 1, \dots, k$

$$\Psi^h(X,A)_i = \sum_{q\geq 0} \sum_{n\in\mathbb{N}_0^D, |n|=q} c(n,h,i) \int_A \widetilde{H}_n(Y(t))\,dt,$$

where the convergence is in $L^2(\mathbb{P})$.

Proof. We first show that both sides are elements in $L^2(\mathbb{P})$. By (4.2) and Jensen's inequality

$$\mathbb{E}\left[(\Psi^h(X,A)_i)^2\right] \leq \mathcal{H}^d(A)\mathbb{E}\left[\int_A G_i^h(Y(t))^2\,dt\right] = \mathcal{H}^d(A)^2 \int_{\mathbb{R}^D} G_i^h(x)^2\phi_D(x)\,dx,$$

where we used Fubini's theorem as well as the stationarity. Assumption (AF4) yields that this expression is finite.

The series on the right side of the asserted equality belongs to $L^2(\mathbb{P})$ if it is a Cauchy sequence. In order to show this, let $k_1, k_2 \in \mathbb{N}$ be such that $k_1 < k_2$. Then by Jensen's inequality

$$\begin{aligned} &\mathbb{E}\left[\left(\sum_{q=k_1+1}^{k_2} \sum_{n\in\mathbb{N}_0^D, |n|=q} c(n,h,i)\int_A \widetilde{H}_n(Y(t))\,dt\right)^2\right] \qquad (4.4)\\ &\leq \mathcal{H}^d(A)\mathbb{E}\left[\int_A \left(\sum_{q=k_1+1}^{k_2} \sum_{n\in\mathbb{N}_0^D, |n|=q} c(n,h,i)\widetilde{H}_n(Y(t))\right)^2 dt\right]. \end{aligned}$$

Fubini's theorem and stationarity yield equality to

$$\mathcal{H}^d(A)^2 \sum_{q_1,q_2=k_1+1}^{k_2} \sum_{|n_1|=q_1, |n_2|=q_2} c(n_1,s,i)c(n_2,s,i)\mathbb{E}\left[\widetilde{H}_{n_1}(Y(0))\widetilde{H}_{n_2}(Y(0))\right].$$

Since $Y(0)$ is standard normally distributed, by Lemma 2.10 (ii), this expression equals

$$\mathcal{H}^d(A)^2 \sum_{q=k_1+1}^{k_2} \sum_{n\in\mathbb{N}_0^D, |n|=q} c(n,h,i)^2 n! \leq \mathcal{H}^d(A)^2 \sum_{q=k_1+1}^{\infty} \sum_{n\in\mathbb{N}_0^D, |n|=q} c(n,h,i)^2 n!. \qquad (4.5)$$

By Bessel's inequality we obtain the finite upper bound

$$\mathcal{H}^d(A)^2 \|G_i^h\|^2_{L^2(\mathcal{N}_D(0,I_D))}. \qquad (4.6)$$

Hence, the series in (4.5) converges and we conclude that for k_1 large enough (4.4) will become arbitrarily small, which shows the assertion.

Finally observe that by (4.2), Jensen's inequality and Fubini's theorem the term

$$\mathbb{E}\left[\left(\Psi^h(X,A)_i - \sum_{q=0}^{k} \sum_{n\in\mathbb{N}_0^D, |n|=q} c(n,h,i)\int_A \widetilde{H}_n(Y(t))\,dt\right)^2\right]$$

can be bounded by the expression

$$\mathcal{H}^d(A)\int_A \mathbb{E}\left[\left(G_i^h(Y(t)) - \sum_{q=0}^{k}\sum_{n\in\mathbb{N}_0^D,|n|=q} c(n,h,i)\widetilde{H}_n(Y(t))\right)^2\right] dt.$$

By stationarity and since the one dimensional distributions of Y are standard normal, we obtain equality to

$$\mathcal{H}^d(A)^2 \left\| G_i^h - \sum_{q=0}^{k}\sum_{n\in\mathbb{N}_0^D,|n|=q} c(n,h,i)\widetilde{H}_n \right\|^2_{L^2(\mathcal{N}_D(0,I_D))},$$

which tends to zero for $k\to\infty$ by (4.3). □

4.1.2. Embedding into an isonormal Gaussian process

We, again, define the real Hilbert space of complex valued functions

$$\mathfrak{H} := \left\{ h\colon \mathbb{R}^d \to \mathbb{C} \mid h(-x) = \overline{h(x)},\ \int_{\mathbb{R}^d} |h(x)|^2 f(x)\,dx < \infty \right\},$$

where f denotes the spectral density of the field X, equipped with the inner product

$$\langle g,h\rangle_{L^2(f\lambda^d)} := \int_{\mathbb{R}^d} g(x)\overline{h(x)}f(x)\,dx,$$

which is real since the functions are Hermitian and the measure $f\lambda^d$ is symmetric. By [59, Prop. 2.1.1], we know that there exists an isonormal Gaussian process W on $\mathfrak{H}$, such that for $g,h\in\mathfrak{H}$

$$\mathbb{E}\left[W(g)W(h)\right] = \langle g,h\rangle_{L^2(f\lambda^d)}.$$

This time, we define for $j=1,\dots,D$ the mappings

$$\varphi_{t,j}\colon \mathbb{R}^d \to \mathbb{C},\quad x\mapsto \sum_{k=1}^{D} \Lambda_{jk}^{-1}\nu_k(x)e^{i\langle t,x\rangle} \quad \in \mathfrak{H},$$

where

$$\nu\colon \mathbb{R}^d \to \mathbb{C}^D,\quad x\mapsto \big((ix_l)_{1\le l\le d}, (-x_l x_s)_{1\le l\le s\le d}, 1\big).$$

We note that $\nu_k(x)e^{i\langle\cdot,x\rangle}$ is the derivative of $e^{i\langle\cdot,x\rangle}$ of the same order and in the same direction as the derivative of X in the k-th component of Z. Then we obtain for $k,l\in\{1,\dots,D\}$ and $t,s\in\mathbb{R}^d$

$$\mathbb{E}\left[Y_k(t)Y_l(s)\right] = \sum_{n,m=1}^{D} \Lambda_{km}^{-1}\Lambda_{ln}^{-1}\mathbb{E}\left[Z_m(t)Z_n(s)\right],$$

which by (3.18) equals

$$\sum_{n,m=1}^{D} \Lambda_{km}^{-1}\Lambda_{ln}^{-1} \int_{\mathbb{R}^d} \nu_m(x) e^{i\langle t,x\rangle} \overline{\nu_n(x) e^{i\langle s,x\rangle}} f(x)\, dx$$
$$= \langle \varphi_{t,k}, \varphi_{s,l}\rangle_{L^2(f\lambda^d)} = \mathbb{E}\left[W(\varphi_{t,k})W(\varphi_{s,l})\right], \tag{4.7}$$

yielding the equality in distribution of the process $Y(\cdot)$ and $(W(\varphi_{\cdot,i}))_{i=1}^D$, since both are centered Gaussian fields. Hence, the Hermite type expansion in Lemma 4.3 and the same combinatorial arguments as in Section 3.2.3 yield

$$\left(\frac{\Psi^h(X,A)_i - \mathbb{E}[\Psi^h(X,A)_i]}{\mathcal{H}^d(A)^{1/2}}\right)_{i=1}^k \overset{\mathcal{D}}{=} \left(\sum_{q=1}^{\infty} I_q(g_{A,q}^{h,i})\right)_{i=1}^k, \tag{4.8}$$

where

$$g_{A,q}^{h,i} := \frac{1}{\mathcal{H}^d(A)^{1/2}} \sum_{k\in\{1,\dots,D\}^q} b(k,h,i) \int_A \varphi_{t,k_1}\otimes\cdots\otimes\varphi_{t,k_q}\, dt \tag{4.9}$$

is symmetric, since the coefficients $b(\cdot,h,i) = \sum_{n\in\mathbb{N}_0^D, |n|=q} \mathbb{1}\{\cdot\in\mathcal{A}_n\} \frac{c(n,h,i)}{|\mathcal{A}_n|}$ are symmetric.

4.1.3. Applying a central limit theorem for isonormal processes

To prove the central limit theorem for the standardised $\Psi^h(X,A)$ as $A \nearrow \mathbb{R}^d$, we show that the conditions of Theorem 2.15 are satisfied for the representation in (4.8).

We start with the verification of **condition (i)**. By the definition of the inner product

$$q!\langle g_{A_N,q}^{h,i}, g_{A_N,q}^{h,j}\rangle_{\mathfrak{H}^{\otimes q}} = \frac{q!}{\mathcal{H}^d(A_N)} \int_{\mathbb{R}^{dq}} \sum_{k\in\{1,\dots,D\}^q} b(k,h,i) \int_{A_N} \varphi_{t,k_1}\otimes\cdots\otimes\varphi_{t,k_q}(x_1,\dots,x_q)\, dt$$
$$\times \sum_{l\in\{1,\dots,D\}^q} b(l,h,j) \int_{A_N} \overline{\varphi_{t,l_1}}\otimes\cdots\otimes\overline{\varphi_{t,l_q}}(x_1,\dots,x_q)\, dt \prod_{i=1}^q f(x_i)\, d(x_1,\dots,x_q).$$

By Fubini' theorem this expression equals

$$\frac{q!}{\mathcal{H}^d(A_N)} \sum_{k,l\in\{1,\dots,D\}^q} b(k,h,i)b(l,h,j) \int_{A_N}\int_{A_N} \prod_{m=1}^q \int_{\mathbb{R}^d} \varphi_{t,k_m}(x)\overline{\varphi_{s,l_m}}(x) f(x)\, dx\, dt\, ds.$$

An application of (4.7) yields the equality to

$$\frac{q!}{\mathcal{H}^d(A_N)} \sum_{k,l\in\{1,\dots,D\}^q} b(k,h,i)b(l,h,j) \int_{A_N}\int_{A_N} \prod_{m=1}^q \mathbb{E}\left[Y_{k_m}(t)Y_{l_m}(s)\right]\, dt\, ds.$$

By Fubini's theorem and stationarity, the integrals can be written as

$$\int_{A_N - A_N} \prod_{m=1}^q \mathbb{E}\left[Y_{k_m}(t)Y_{l_m}(0)\right] \mathcal{H}^d((A_N - t)\cap A_N)\, dt.$$

Then assumption (AF3) yields that

$$\mathcal{H}^d(A_N)^{-1}\left|\prod_{m=1}^{q}\mathbb{E}\left[Y_{k_m}(t)Y_{l_m}(0)\right]\right|\mathcal{H}^d((A_N-t)\cap A_N)\le c\psi(t)^q,$$

where $c=c(X,d,q)\ge 0$, so that the dominated convergence theorem is applicable. Hence

$$q!\langle g^{h,i}_{A_N,q},g^{h,j}_{A_N,q}\rangle \overset{N\to\infty}{\longrightarrow} q!\sum_{k,l\in\{1,\dots,D\}^q} b(k,h,i)b(l,h,j)\int_{\mathbb{R}^d}\prod_{m=1}^{q}\mathbb{E}\left[Y_{k_m}(t)Y_{l_m}(0)\right]dt.$$

Note that we implicitly used $\mathcal{H}^d((A_N-t)\cap A_N)/\mathcal{H}^d(A_N)\to 1$ for $N\to\infty$ and $t\in\mathbb{R}^d$ as well as the fact that A_N-A_N eventually contains every point in $\mathbb{R}^d$.

We continue with **condition (ii)**. First, we observe that by Fatou's Lemma and Lemma 2.12

$$\sum_{q=1}^{\infty}\lim_{N\to\infty}q!\|g^{h,i}_{A_N,q}\|^2_{\mathfrak{H}^{\otimes q}}\le\liminf_{N\to\infty}\sum_{q=1}^{\infty}q!\|g^{h,i}_{A_N,q}\|^2_{\mathfrak{H}^{\otimes q}}=\liminf_{N\to\infty}\sum_{q=1}^{\infty}\mathbb{E}\left[I_q(g^{h,i}_{A_N,q})^2\right].$$

Recall that by definition (4.9) and Fubini's theorem for stochastic integrals (cf. [60, Theorem 5.13.1])

$$I_q(g^{h,i}_{A_N,q})=\mathcal{H}^d(A_N)^{-1/2}\sum_{k\in\{1,\dots,D\}^q}b(k,h,i)\int_{A_N}I_q(\varphi_{t,k_1}\otimes\cdots\otimes\varphi_{t,k_q})\,dt$$

and therefore, after reversing earlier manipulations leading to 4.8, we obtain

$$I_q(g^{h,i}_{A_N,q})=\mathcal{H}^d(A_N)^{-1/2}\sum_{n\in\mathbb{N}_0^D,|n|=q}c(n,h,i)\int_{A_N}\widetilde{H}_n(Y(t))\,dt.$$

Then Fubini's theorem and stationarity imply

$$\begin{aligned}\sum_{q=1}^{\infty}\mathbb{E}\left[I_q(g^{h,i}_{A_N,q})^2\right]=\sum_{q=1}^{\infty}\int_{A_N-A_N}\mathbb{E}\left[\sum_{|n|=q}c(n,h,i)\widetilde{H}_n(Y(t))\sum_{|n|=q}c(n,h,i)\widetilde{H}_n(Y(0))\right]\\ \times\frac{\mathcal{H}^d((A_N-t)\cap A_N)}{\mathcal{H}^d(A_N)}\,dt.\end{aligned}\tag{4.10}$$

Now, we apply Lemma 3.11 and choose $V:=Y(0)$, $W:=Y(t)$ and $g^i_q\colon\mathbb{R}^d\to\mathbb{R}$ by $x\mapsto\sum_{|n|=q}c(n,h,i)\widetilde{H}_n(x)$. Then we have $r=q$ by Lemma 2.10 (ii) and since g^i_q can be assumed to be nonzero. By (AF3) there exists a constant $c>0$, such that $\tau(t)\le c\psi(t)$ for $t\in\mathbb{R}^d$. Let $\rho\in(0,1)$ be such that $\rho<1/c$. Then by assumption (AF3) there exists $s>0$ such that

$$\psi(t)\le\rho,\qquad\text{for }\|t\|\ge s.\tag{4.11}$$

Furthermore, we have

$$\mathbb{E}\left[g^i_q(Y(0))^2\right]=\sum_{|n|=q}\sum_{|n'|=q}c(n,h,i)c(n',h,i)\mathbb{E}\left[\widetilde{H}_n(Y(0))\widetilde{H}_{n'}(Y(0))\right],$$

wich by properties of the Hermite polynomials equals

$$\sum_{|n|=q}\sum_{|n'|=q} c(n,h,i)c(n',h,i)\mathbb{1}\{n=n'\}n! = \sum_{|n|=q} c(n,h,i)^2 n!.$$

Thus splitting the integration in (4.10) into

$$\begin{aligned}&\sum_{q=1}^{\infty}\int_{(A_N-A_N)\setminus B_s^d}\mathbb{E}\left[\sum_{|n|=q} c(n,h,i)\widetilde{H}_n(Y(t))\sum_{|n|=q} c(n,h,i)\widetilde{H}_n(Y(0))\right]\frac{\mathcal{H}^d((A_N-t)\cap A_N)}{\mathcal{H}^d(A_N)}\,dt\\ &\quad+\sum_{q=1}^{\infty}\int_{B_s^d}\mathbb{E}\left[\sum_{|n|=q} c(n,h,i)\widetilde{H}_n(Y(t))\sum_{|n|=q} c(n,h,i)\widetilde{H}_n(Y(0))\right]\frac{\mathcal{H}^d((A_N-t)\cap A_N)}{\mathcal{H}^d(A_N)}\,dt\end{aligned}$$

allows us to use Lemma 3.11 for the first series and we obtain for this term the upper bound

$$\sum_{q=1}^{\infty}\int_{(A_N-A_N)\setminus B_s^d} c^q\psi(t)^q\sum_{|n|=q} c(n,h,i)^2 n!\frac{\mathcal{H}^d((A_N-t)\cap A_N)}{\mathcal{H}^d(A_N)}\,dt. \tag{4.12}$$

By Bessel's inequality, cf. (4.6), and (4.11), we bound the expression (4.12) by

$$1/\rho\int_{\mathbb{R}^d}\psi(t)\,dt\|G_i^h\|_{L^2}^2\sum_{q=1}^{\infty}(c\rho)^q,$$

which is a convergent series and moreover independent of N.

An application of the inequality $ab\le a^2+b^2$, $a,b\in\mathbb{R}$, bounds the second summand by

$$\begin{aligned}&\sum_{q=1}^{\infty}\int_{B_s^d}\mathbb{E}\left[\left(\sum_{|n|=q} c(n,h,i)\widetilde{H}_n(Y(t))\right)^2+\left(\sum_{|n|=q} c(n,h,i)\widetilde{H}_n(Y(0))\right)^2\right]\frac{\mathcal{H}^d((A_N-t)\cap A_N)}{\mathcal{H}^d(A_N)}\,dt\\ &\quad\le 2\sum_{q=1}^{\infty}\mathbb{E}\left[\left(\sum_{|n|=q} c(n,h,i)\widetilde{H}_n(Y(0))\right)^2\right]\mathcal{H}^d(B_s^d)\\ &\quad= 2\mathcal{H}^d(B_s^d)\sum_{q=1}^{\infty}\sum_{|n|=q} c(n,h,i)^2 n!,\end{aligned} \tag{4.13}$$

which is again a convergent series, cf. (4.6), and independent of N. This shows the assertion.

In order to verify **condition (iii)**, we first calculate the r-th contraction

$$\begin{aligned}&g_{A_N,q}^{h,i}\otimes_r g_{A_N,q}^{h,i}(a_1,\dots,a_{2q-2r})\\ &\quad=\mathcal{H}^d(A_N)^{-1}\int_{\mathbb{R}^{dr}}\sum_{k\in\{1,\dots,D\}^q} b(k,h,i)\int_{A_N}\varphi_{t,k_1}(x_1)\cdots\varphi_{t,k_r}(x_r)\\ &\quad\times\varphi_{t,k_{r+1}}(a_1)\cdots\varphi_{t,k_q}(a_{q-r})\,dt\sum_{k\in\{1,\dots,D\}^q} b(k,h,i)\int_{A_N}\overline{\varphi_{t,k_1}}(x_1)\cdots\overline{\varphi_{t,k_r}}(x_r)\\ &\quad\times\varphi_{t,k_{r+1}}(a_{q-r+1})\cdots\varphi_{t,k_q}(a_{2q-2r})\,dt\prod_{i=1}^{r} f(x_i)\,d(x_1,\dots,x_r)\end{aligned}$$

for $(a_1, \dots, a_{2q-2r}) \in \mathbb{R}^d$. Fubini's theorem and (4.7) yield the equality to

$$\begin{aligned}\mathcal{H}^d(A_N)^{-1} \sum_{k,l\in\{1,\dots,D\}^q} b(k,h,i)b(l,h,i) \int_{A_N}\int_{A_N} \prod_{m=1}^{r} \mathbb{E}\left[Y_{k_m}(t)Y_{l_m}(s)\right] \\ \times \varphi_{t,k_{r+1}}(a_1)\cdots\varphi_{t,k_q}(a_{q-r})\varphi_{s,l_{r+1}}(a_{q-r+1})\cdots\varphi_{s,l_q}(a_{2q-2r})\,dt\,ds.\end{aligned}$$

Then we obtain for the norm

$$\begin{aligned}&\|g^{h,i}_{A_N,q}\otimes_r g^{h,i}_{A_N,q}\|^2_{\mathfrak{H}^{\otimes(2q-2r)}} \\ &= \mathcal{H}^d(A_N)^{-2}\int_{\mathbb{R}^{d(2q-2r)}} \sum_{k,l\in\{1,\dots,D\}^q} b(k,h,i)b(l,h,i)\int_{A_N}\int_{A_N}\prod_{m=1}^{r}\mathbb{E}\left[Y_{k_m}(t)Y_{l_m}(s)\right] \\ &\times \varphi_{t,k_{r+1}}(x_1)\cdots\varphi_{t,k_q}(x_{q-r})\varphi_{s,l_{r+1}}(x_{q-r+1})\cdots\varphi_{s,l_q}(x_{2q-2r})\,dt\,ds \\ &\times \sum_{k,l\in\{1,\dots,D\}^q} b(k,h,i)b(l,h,i)\int_{A_N}\int_{A_N}\prod_{m=1}^{r}\mathbb{E}\left[Y_{k_m}(t)Y_{l_m}(s)\right]\varphi_{t,k_{r+1}}(x_1)\cdots\varphi_{t,k_q}(x_{q-r}) \\ &\times \varphi_{s,l_{r+1}}(x_{q-r+1})\cdots\varphi_{s,l_q}(x_{2q-2r})\,dt\,ds\prod_{i=1}^{2q-2r} f(x_i)\,d(x_1,\dots,x_{2q-2r}).\end{aligned}$$

By Fubini's theorem and (4.7) the above equals

$$\begin{aligned}&\mathcal{H}^d(A_N)^{-2}\sum_{k,l,k',l'\in\{1,\dots,D\}^q} b(k,h,i)b(k',h,i)b(l,h,i)b(l',h,i)\int_{(A_N)^4}\prod_{m=1}^{r}\mathbb{E}\left[Y_{k_m}(t)Y_{l_m}(s)\right] \\ &\times \mathbb{E}\left[Y_{k'_m}(t')Y_{l'_m}(s')\right]\prod_{m=r+1}^{q}\mathbb{E}\left[Y_{k_m}(t)Y_{k'_m}(t')\right]\mathbb{E}\left[Y_{l_m}(s)Y_{l'_m}(s')\right]\,d(t,t',s,s').\end{aligned}$$

From assumption (AF3) and stationarity we deduce

$$\max_{i,j=1,\dots,D}\left|\mathbb{E}\left[Y_i(t)Y_j(s)\right]\right| \le c\psi(t-s),$$

for $s,t\in\mathbb{R}^d$ and $c=c(X,d)>0$. Hence

$$\|g^{h,i}_{A_N,q}\otimes_r g^{h,i}_{A_N,q}\|^2_{\mathfrak{H}^{\otimes(2q-2r)}} \le c^{2q}\sum_{k,l,k',l'\in\{1,\dots,D\}^q} b(k,h,i)b(k',h,i)b(l,h,i)b(l',h,i)z(N),$$

where

$$z(N) := \mathcal{H}^d(A_N)^{-2}\int_{(A_N)^4}\psi(t-s)^r\psi(t'-s')^r\psi(t-t')^{q-r}\psi(s-s')^{q-r}\,d(t,t',s,s').$$

Using $a^r b^{q-r}\le a^q+b^q$ for $a=\psi(t'-s')$ and $b=\psi(t-t')$, we obtain

$$\begin{aligned}z(N) \le \mathcal{H}^d(A_N)^{-2}&\int_{(A_N)^4}\psi(t-s)^r\psi(t'-s')^q\psi(s-s')^{q-r}\,d(t,t',s,s') \\ &+ \mathcal{H}^d(A_N)^{-2}\int_{(A_N)^4}\psi(t-s)^r\psi(t-t')^q\psi(s-s')^{q-r}\,d(t,t',s,s')\end{aligned}$$

and abbreviate the first summand by $z_1(N)$ and the second by $z_2(N)$. Assumption (AF3) yields $c_n =: \int_{\mathbb{R}^d} \psi(t)^n dt < \infty$ for $n \in \mathbb{N}$ and therefore

$$z_1(N) \leq \mathcal{H}^d(A_N)^{-2} \int_{(A_N)^3} \psi(t-s)^r \int_{\mathbb{R}^d} \psi(t'-s')^q \, dt' \, d(t,s,s').$$

Iterating this argument yields the upper bound

$$\begin{aligned} \frac{c_q}{\mathcal{H}^d(A_N)^2} \int_{A_N} \int_{A_N} \psi(s-s')^{q-r} \int_{\mathbb{R}^d} \psi(t-s)^r \, dt \, ds \, ds' &\leq \frac{c_q c_r}{\mathcal{H}^d(A_N)^2} \int_{A_N} \int_{A_N} \psi(s-s')^{q-r} \, ds \, ds' \\ &\leq c_q c_r c_{q-r} \mathcal{H}^d(A_N)^{-1}, \end{aligned}$$

which tends to 0 as $N \to \infty$. Analogously, $z_2(N) \to 0$ as $N \to \infty$, from which we conclude the desired condition.

To obtain **condition (iv)**, we observe that the same calculations as in condition (ii), cf. (4.12) and (4.13), lead to the estimate

$$\sum_{q=Q+1}^{\infty} q! \langle g_{A_N,q}^{h,i}, g_{A_N,q}^{h,i} \rangle \leq 1/\rho \int_{\mathbb{R}^d} \psi(t) \, dt \|G_i^h\|_{L^2}^2 \sum_{q=Q+1}^{\infty} (c\rho)^q + 2\mathcal{H}^d(B_s^d) \sum_{q=Q+1}^{\infty} \sum_{|n|=q} c(n,h,i)^2 n!.$$

Both series are the tail of a convergent series and moreover both expressions are independent of N. Hence in the limit superior $N \to \infty$ and the limit $Q \to \infty$, in that order, the terms vanish. This shows the assertion.

4.2. The special case of integrated surface tensors

In the first subsection of this section, we specialise the general result of the previous part to a central limit theorem for integrated Minkowski surface tensors. In the second part, we present a simulation study to illustrate the derived limit theorem.

4.2.1. The central limit theorem

Motivated by the use as a shape measure in physics, cf. [38, Chapter 2], [39] and the references therein, we define the integrated Minkowski surface tensor of rank $s \in \mathbb{N}$ by

$$\Phi^s(X,A) := \frac{2\pi}{s!\omega_{s+1}} \int_{\mathbb{R}} h(u) \int_{X^{-1}(\{u\}) \cap A} \|\nabla X(t)\|^{-s} \nabla X(t)^{\otimes s} \, \mathcal{H}^{d-1}(dt) \, du,$$

for $A \in \mathcal{B}(\mathbb{R}^d)$, $X \colon \mathbb{R}^d \to \mathbb{R}$ at least of class $\mathcal{C}^1$ with Lipschitz gradient and the weight function $h \colon \mathbb{R} \to \mathbb{R}$ in $L^2(\mathcal{N}_1(0,c))$, for all $c > 0$.

Our aim is to establish a central limit theorem for the standardised version of this functional. In order to speak about normal approximation in the space of tensors, we need to have a definition of a normally distributed element in this space. The following definition meets our needs.

Definition. Let $s \in \mathbb{N}$. The s-tensor-valued random variable $N\colon \Omega \to (\mathbb{R}^d)^{\otimes s}$ is said to have a normal distribution with expectation tensor $m\colon (\mathbb{R}^d)^s \to \mathbb{R}$ and covariance tensor $C\colon (\mathbb{R}^d)^s \times (\mathbb{R}^d)^s \to \mathbb{R}$ if for all $n \in \mathbb{N}$ and all $v_{1,1}, \dots, v_{1,s}, \dots, v_{n,1}, \dots, v_{n,s} \in \mathbb{R}^d$

$$(N(v_{1,1}, \dots, v_{1,s}), \dots, N(v_{n,1}, \dots, v_{n,s})) \\ \sim \mathcal{N}_n \left((m(v_{i,1}, \dots, v_{i,s}))_{i=1}^n, (C(v_{i,1}, \dots, v_{i,s}, v_{j,1}, \dots, v_{j,s}))_{i,j=1}^n \right).$$

The above definition is inspired by identifying tensors with multilinear mappings and thinking of random elements in the space of tensors as random fields in the space of multilinear mappings. We note that due to the multilinear nature of tensors, it is sufficient to require that

$$(N_{i_1,\dots,i_s})_{i_1,\dots,i_s=1}^d \sim \mathcal{N}_{d^s} \left((m(e_{i_1}, \dots, e_{i_s}))_{i_1,\dots,i_s=1}^d, (C(e_{i_1}, \dots, e_{i_s}, e_{j_1}, \dots, e_{j_s}))_{i_1,j_1\dots,i_s,j_s=1}^d \right),$$

where $a_{i_1,\dots,i_s}$ for $a \in (\mathbb{R}^d)^{\otimes s}$ denotes the $(i_1, \dots, i_s)$-th coordinate of a, i.e. $a(e_{i_1}, \dots, e_{i_s})$ where $e_1, \dots, e_d$ denotes the standard basis in $\mathbb{R}^d$.

Then, an application of Theorem 4.1 implies the following central limit theorem for integrated surface tensors.

Corollary 4.4. *Let X be a real Gaussian field on $\mathbb{R}^d$, which satisfies the assumptions (AF1)–(AF3). Then*

$$\left(\frac{\Phi^s(X, A_N)_{i_1,\dots,i_s} - \mathbb{E}\left[\Phi^s(X, A_N)_{i_1,\dots,i_s}\right]}{\mathcal{H}^d(A_N)^{1/2}} \right)_{i_1,\dots,i_s=1}^d \xrightarrow{\mathcal{D}} \mathcal{N}_{d^s}(0, C^s), \qquad \text{as } A_N \nearrow \mathbb{R}^d,$$

where $C^s \in \mathbb{R}^{d^s \times d^s}$ is positive semidefinite.

Proof. We note that

$$\Phi^s(X, A) = \frac{2\pi}{s!\omega_{s+1}} \int_{\mathbb{R}} \int_{X^{-1}(\{u\}) \cap A} h(X(t)) \|\nabla X(t)\|^{-s} \nabla X(t)^{\otimes s}\, \mathcal{H}^{d-1}(dt)\, du.$$

Therefore an application of Theorem 4.1 with the choice

$$h\colon \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{R} \to \mathbb{R}^{d^s}, \quad (x, y, z) \mapsto \left(\frac{2\pi}{s!\omega_{s+1}} h(z) \|x\|^{-s} (x^{\otimes s})_{i_1,\dots,i_s} \right)_{i_1,\dots,i_s=1}^d,$$

yields the assertion once we verified condition (AF4) for this specific mapping. For this purpose, we observe that for $i_1, \dots, i_s \in \{1, \dots, d\}$ and regular linear mappings $A\colon \mathbb{R}^d \to \mathbb{R}^d$ and $B\colon \mathbb{R}^{d(d+1)/2+1} \to \mathbb{R}^{d(d+1)/2+1}$ by Fubini's theorem

$$\int_{\mathbb{R}^d \times \mathbb{R}^{d(d+1)/2+1}} h((By)_{d(d+1)/2+1})^2 \|Ax\|^{-2s+2} \prod_{j=1}^s (Ax)_{i_j}^2 \phi_{d+d(d+1)/2+1}(x, y)\, d(x, y) \\ = \int_{\mathbb{R}^d} \|Ax\|^{-2s+2} \prod_{j=1}^s (Ax)_{i_j}^2 \phi_d(x)\, dx \int_{\mathbb{R}^{d(d+1)/2+1}} h((By)_{d(d+1)/2+1})^2 \phi_{d(d+1)/2+1}(y)\, dy. \tag{4.14}$$

We choose $b \in \mathbb{R}^{d(d+1)/2+1}$ such that $(By)_{d(d+1)/2+1} = \langle b, y \rangle$. We note that $b \neq 0$, since B is regular. An application of Fubini's theorem in the orthonormal basis $(b/\|b\|, u_1, \dots, u_{d(d+1)/2})$

yields

$$\begin{aligned}&\int_{\mathbb{R}^{d(d+1)/2+1}} h((By)_{d(d+1)/2+1})^2 \phi_{d(d+1)/2+1}(y)\,dy \\ &\qquad = \int_{\mathbb{R}^{d(d+1)/2}} \int_{\mathbb{R}} h(\|b\|x_1)^2 \phi(x_1)\,dx_1 \phi_{d(d+1)/2}(z)\,dz \\ &\qquad = \frac{1}{\|b\|} \int_{\mathbb{R}} h(x_1)^2 \phi\left(\frac{x_1}{\|b\|}\right)\,dx_1 < \infty, \end{aligned} \tag{4.15}$$

where the finiteness is an assumption on h. We now show that the first factor in the term (4.14) is finite. By the inequalities

$$\|Ax\|^{-s} \prod_{j=1}^{s} (Ax)_{i_j} \leq \|Ax\|^{-s} \prod_{j=1}^{s} \|Ax\| \leq 1,$$

we obtain

$$\int_{\mathbb{R}^d} \|Ax\|^{2(-s+1)} \prod_{j=1}^{s} (Ax)_{i_j}^2 \phi_d(x)\,dx \leq \int_{\mathbb{R}^d} \|Ax\|^2 \phi_d(x)\,dx.$$

Let $\|\cdot\|$ denote any with the Euclidean norm compatible matrix norm. Then $\|Ax\| \leq \|A\|\|x\|$ and this yields the upper bound

$$\|A\|^2 \int_{\mathbb{R}^d} \|x\|^2 \phi_d(x)\,dx < \infty,$$

which proves the assertion. □

4.2.2. A simulation study

In the following we present a simulation study in the setting of the previous section with the choice $h \equiv 1$. We simulate realisations of a specific Gaussian random field and calculate the integrated Minkowski surface tensor of rank 2. For the sake of nice pictures, we specialise to the parameter space dimension 2 and consider a stationary, isotropic and centered Gaussian field $\{X_t \colon \Omega \to \mathbb{R} \mid t \in \mathbb{R}^2\}$, where the covariance function is chosen from the Matérn class, cf. [71]. This class is given by functions of the form

$$C^X(t) := \frac{\sigma^2 2^{1-\nu}}{\Gamma(\nu)} \left(\frac{\sqrt{2\nu}\|t\|}{l}\right) K_\nu \left(\frac{\sqrt{2\nu}\|t\|}{l}\right), \quad t \in \mathbb{R}^2, \nu, l > 0,$$

where K_ν is the modified Bessel function of the second kind of order ν. The parameter σ^2 is the variance of $X(0)$ and will be chosen as 1. The constant l can be interpreted as the characteristic length-scale and ν determines the smoothness of the field X. That is, X is m times mean square differentiable if and only if $\nu > m$, cf. [74]. This class of covariance functions is widely used in applications, for instance in meteorology, cf. [27], and machine learning, cf. [65].

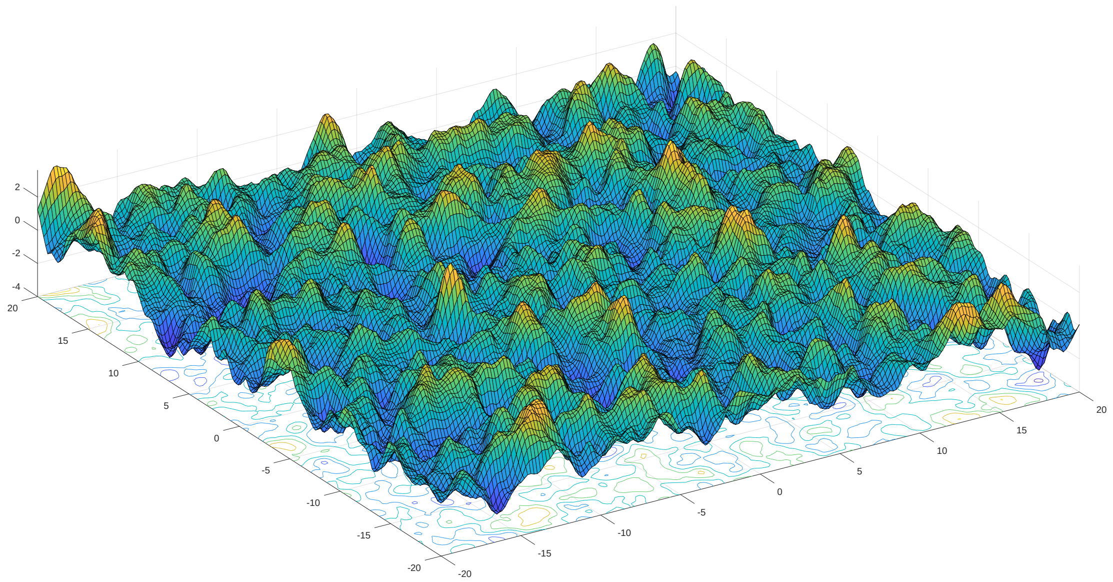

Figure 4.1.: Realisation of a centered isotropic Gaussian random field with covariance given by (4.16) simulated on a square grid of 200×200 points. Different level sets are plotted in the plane.

In the following we choose $\nu = 5/2$ and $l = 1$, such that

$$C^X(t) = \left(1 + \sqrt{5}\|t\| + \frac{5}{3}\|t\|^2\right) \exp\left(-\sqrt{5}\|t\|\right), \quad t \in \mathbb{R}^2, \tag{4.16}$$

where we exploited the fact that in the cases $\nu = n + \frac{1}{2}$, $n \in \mathbb{N}$, the modified Bessel function can be written in terms of elementary functions. Let $\{t_i \in \mathbb{R}^2 \mid i \in I\}$ denote the points in which we would like to simulate the field X. We choose the points as an equidistant grid but note that in general this is not necessary for the simulation of the field X. However, since we are also interested in the partial derivatives of the field X, simulating the field in grid points allows for a numerical approximation of the gradient. The property of X to be a centered Gaussian field implies that the vector $(X(t_i))_{i\in I}$ is normally distributed with expectation 0 and covariance matrix $(C^X(t_i - t_j))_{i,j=1}^{|I|}$. The positive definiteness of the covariance matrix allows for a Cholesky decomposition $(C^X(t_i - t_j))_{i,j=1}^{|I|} = LL^T$, where L is a regular, lower triangular matrix and we obtain

$$(X(t_i))_{i\in I} \stackrel{\mathcal{D}}{=} L\varepsilon, \qquad \text{where } \varepsilon \sim \mathcal{N}_{|I|}(0, I_{|I|}).$$

Hence, a realisation of $(X(t_i))_{i\in I}$ can be obtained by simulating ε and multiplying this realisation with the matrix L. The result of this procedure is illustrated in Figure 4.1. Once we simulated $(X(t_i))_{i\in I}$, we can calculate numerically the partial derivatives $\left(\frac{\partial}{\partial t_k} X(t_i)\right)_{i\in I}$, for instance by use of the function “gradient” of the software package MATLAB. With all the data generated so far, we determine the integrated Minkowski surface tensor of rank 2 by approximating the integral by Riemann sums. The results of repeating this procedure 10000 times can be seen in Figure 4.2, which shows the histogram of the values of the coordinates

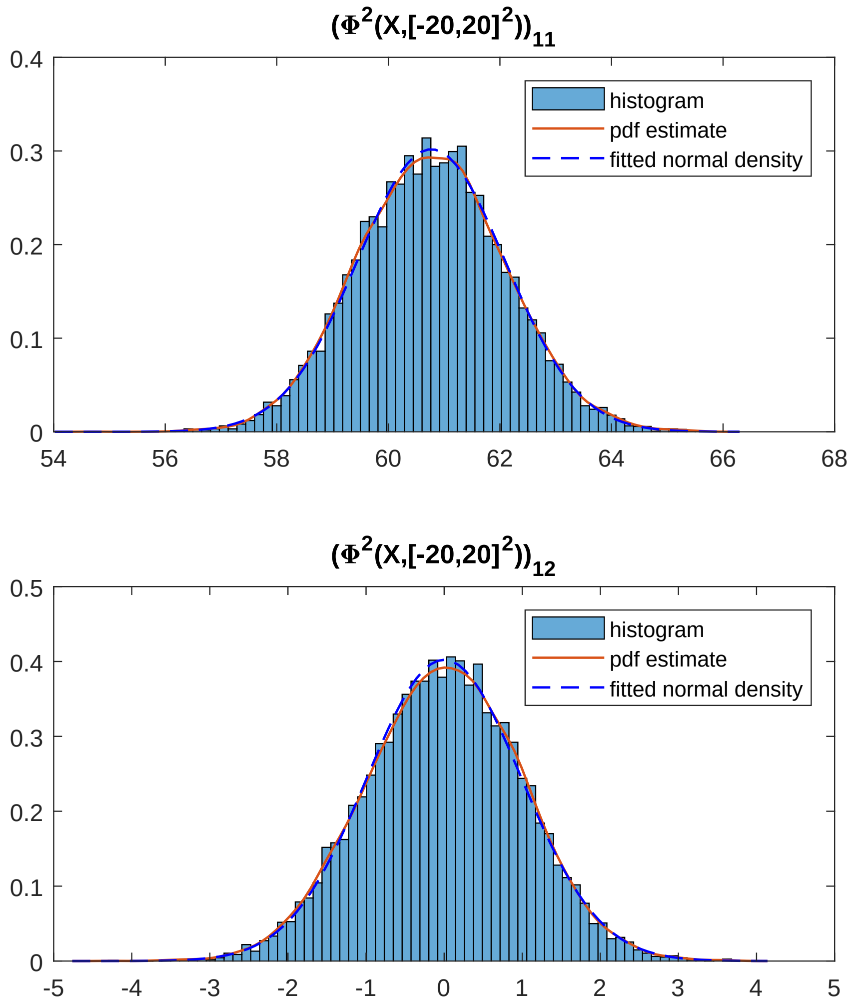

Figure 4.2.: Histogram of the values of the coordinates of the tensor with 10000 replications. Density of a fitted normal distribution (dashed). Kernel smoothing function estimate (solid).

of the tensor, the density of a normal distribution fitted to the data (based on maximum likelihood estimation) and a kernel smoothing function estimate (where Gaussian kernels are used, cf. the function "ksdensity" of the software package MATLAB) for the collected data.

The good agreement, considering that we approximate the integral and the partial derivatives, of the estimated probability density (Figure 4.2, solid line) and the density of a fitted normal distribution (Figure 4.2, dashed line), provides a numerical illustration of the theoretical central limit theorem of the previous section.

At last, we simulate an example in the anisotropic case. In order to obtain a stationary but anisotropic field, we modify the Matérn covariance function of equation (4.16) in the following way

$$C^X(t) := \left(1 + \sqrt{5\,(t_1^2 + ct_2^2)} + \frac{5}{3}\left(t_1^2 + ct_2^2\right)\right) \exp\left(-\sqrt{5\,(t_1^2 + ct_2^2)}\right), \quad t \in \mathbb{R}^2, \tag{4.17}$$

where $c > 0$ quantifies the anisotropy. Figure 4.3 illustrates the structurally different excursion sets of the isotropic and the anisotropic case. While it is challenging to identify the anisotropy

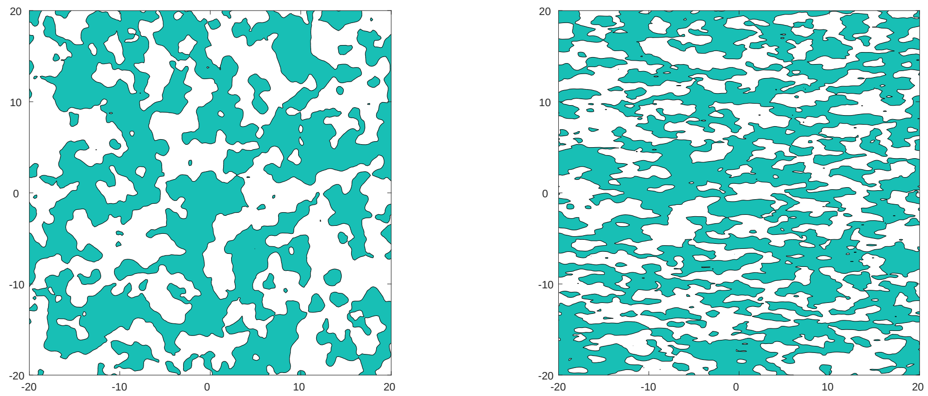

Figure 4.3.: Left: Excursion set for threshold 0 of a centered Gaussian field with covariance given by (4.16). Right: Excursion set for threshold 0 of a centered Gaussian field with covariance given by (4.17) with $c = 7$.

with the Lipschitz–Killing curvatures of Chapter 3, the integrated surface tensors of this section are perfectly suited to detect these structural differences, cf. [39]. One way to observe the anisotropy is illustrated in Figure 4.4, where the different expectations of the asymptotic normal distributions indicate the anisotropy in the case $c = 7$.

4.3. The special case of integrated curvature measures

In this section, we specialise the general central limit theorem in Theorem 4.1 to the case of integrated curvature measures. We first clarify rigorously what we mean by the latter term and derive a representation of this functional in terms of the first and second derivatives of the involved Gaussian field, so that the investigated scenario fits into the general setting of Section 4.1.

We start with noting that under assumptions (AF1) and (AF2) the set $X^{-1}(\{u\})$ carries almost surely the structure of a $(d-1)$-dimensional $\mathcal{C}^2$ submanifold of $\mathbb{R}^d$, since X is assumed to be almost surely of class $\mathcal{C}^2$ and $u \in \mathbb{R}$ is a regular value of X, cf. [4, Proposition 6.12]. By $H_j(x_1, \dots, x_k)$, we denote the j-th elementary symmetric function, which is given by

$$H_j(x_1, \dots, x_k) := \sum_{1 \le i_1 < \dots < i_j \le k} x_{i_1} \cdots x_{i_j},$$

for $x_1, \dots, x_k \in \mathbb{R}$, $j = 1, \dots, k$. We define the k-th curvature measure of $X^{-1}([u, \infty))$ for $A \in \mathcal{B}(\mathbb{R}^d)$ by

$$C_k^u(X, A) := \frac{1}{\omega_{d-k}} \int_{X^{-1}(\{u\})} \mathbb{1}_A(t) H_{d-1-k}(\kappa_1(t), \dots, \kappa_{d-1}(t))\, \mathcal{H}^{d-1}(dt),$$

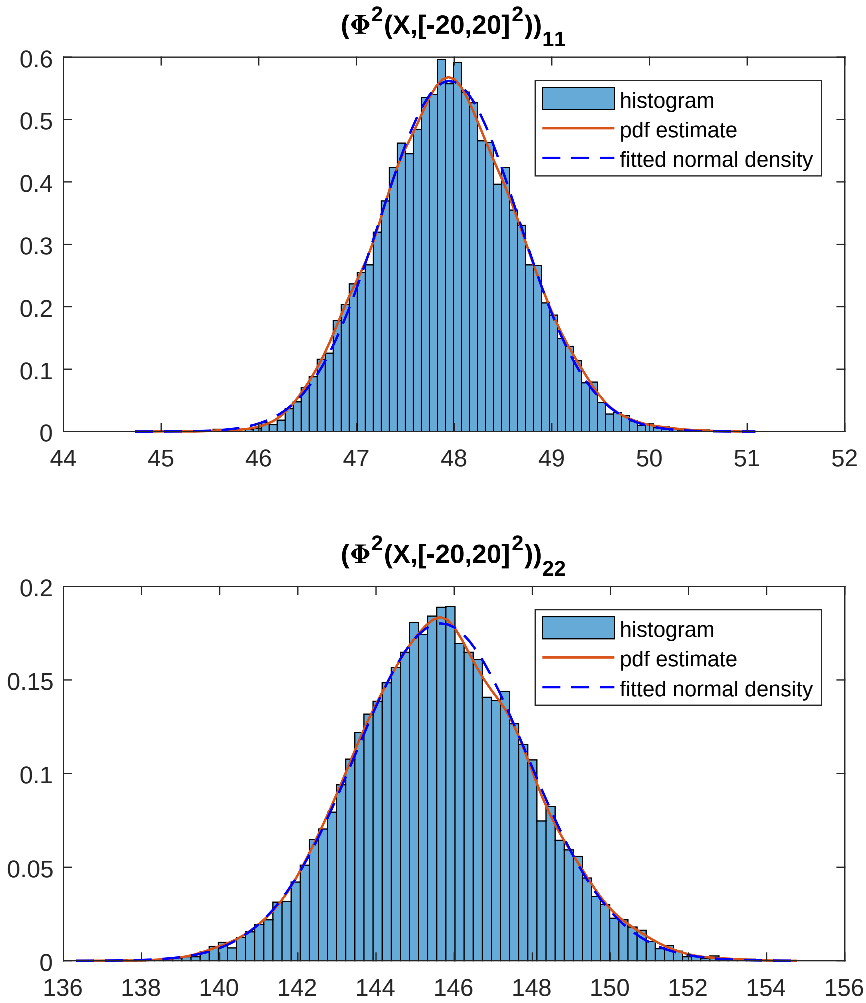

Figure 4.4.: Histogramm of the values of the coordinates of the tensor with 10000 replications. Density of a fitted normal distribution (dashed). Kernel smoothing function estimate (solid). The anisotropy causes different expectations of the asymptotic distributions.

where $k \in \{0, \ldots, d-1\}$ and $\kappa_1(t), \ldots, \kappa_{d-1}(t)$ denote the principal curvatures of $X^{-1}(\{u\})$ at the point t. We note that Federer's curvature measures specialise to the given definition in the setting of this section, cf. [81, Equation (13)]. We then define the k-th integrated curvature measure of X

$$\Phi_k^h(X, A) := \frac{1}{\omega_{d-k}} \int_{\mathbb{R}} h(u) \int_{X^{-1}(\{u\})} \mathbb{1}_A(t) H_{d-1-k}(\kappa_1(t), \ldots, \kappa_{d-1}(t))\, \mathcal{H}^{d-1}(dt)\, du$$

for $A \in \mathcal{B}(\mathbb{R}^d)$ and a weight function $h\colon \mathbb{R} \to \mathbb{R}$ in $L^4(\mathcal{N}_1(0, c))$, for all $c > 0$.

In the following we rewrite the elementary symmetric function of the principal curvatures as a function of the gradient ∇X and the second derivative $D^2 X$. We first note that by definition the principal curvatures $\kappa_1(t), \ldots, \kappa_{d-1}(t)$ are the eigenvalues of the shape operator $s \in \mathcal{T}_1^1(X^{-1}(\{u\}))$, where $\mathcal{T}_1^1(X^{-1}(\{u\}))$ denotes the space of 1-covariant, 1-contravariant tensor fields on $X^{-1}(\{u\})$. The shape operator can be thought of as a field of endomorphisms of $TX^{-1}(\{u\})$, the tangent bundle of $X^{-1}(\{u\})$, and is characterised by the Weingarten equation for Euclidean hypersurfaces, cf. [50, Equation (8.3)],

$$-sY = \overline{\nabla}_Y N$$

for $Y \in \mathcal{T}(X^{-1}(\{u\}))$, the space of vector fields, where $\overline{\nabla}$ denotes the Levi-Civita connection of $\mathbb{R}^d$ with the standard metric, thus the directional derivative, and N denotes the outer normal vector field of $X^{-1}(\{u\})$. However, since $X^{-1}(\{u\})$ is a level set, the outer normal in the point t is given by $-\|\nabla X(t)\|^{-1}\nabla X(t)$. Hence, for $Y = \sum_i Y^i \frac{\partial}{\partial x_i} \in \mathcal{T}(X^{-1}(\{u\}))$, where $\left(\frac{\partial}{\partial x_i}\Big|_t\right)$ denotes the standard basis of $T_t\mathbb{R}^d$, i.e. the one induced by the coordinate projections, and $t \in X^{-1}(\{u\})$

$$s(t)Y = \|\nabla X(t)\|^{-1} \sum_k Y|_t \left(\frac{\partial}{\partial x_k} X\right) \frac{\partial}{\partial x_k}\bigg|_t + Y|_t(\|\nabla X\|^{-1}) \sum_l \frac{\partial}{\partial x_l} X(t) \frac{\partial}{\partial x_l}\bigg|_t$$

by the Leibniz rule and the flatness of $\mathbb{R}^d$. Calculating the derivatives yields

$$\begin{aligned} s(t)Y =& \|\nabla X(t)\|^{-1} \sum_k \sum_i Y^i(t) \frac{\partial^2}{\partial x_i \partial x_k} X(t) \frac{\partial}{\partial x_k}\bigg|_t \\ &- \|\nabla X(t)\|^{-3} \sum_i Y^i(t) \sum_k \frac{\partial}{\partial x_k} X(t) \frac{\partial^2}{\partial x_i \partial x_k} X(t) \sum_l \frac{\partial}{\partial x_l} X(t) \frac{\partial}{\partial x_l}\bigg|_t, \end{aligned}$$

since we calculated

$$Y|_t(\|\nabla X\|^{-1}) = \sum_i Y^i(t) \frac{\partial}{\partial x_i}\bigg|_t \left(\|\nabla X\|^{-1}\right) = \sum_i Y^i(t) \sum_k -\|\nabla X(t)\|^{-3} \frac{\partial}{\partial x_k} X(t) \frac{\partial^2}{\partial x_k \partial x_i} X(t).$$

Identifying the basis $\left(\frac{\partial}{\partial x_i}\Big|_t\right)$ of $T_t\mathbb{R}^d$ with the standard basis (e_i) of $\mathbb{R}^d$, we obtain for $s(t)Y$

$$\begin{aligned} &\|\nabla X(t)\|^{-1} \left(\sum_i Y^i(t) \frac{\partial^2}{\partial x_i \partial x_k} X(t)\right)_{k=1}^d \\ &\quad - \|\nabla X(t)\|^{-3} \left(\sum_i Y^i(t) \sum_k \frac{\partial}{\partial x_k} X(t) \frac{\partial^2}{\partial x_i \partial x_k} X(t) \frac{\partial}{\partial x_l} X(t)\right)_{l=1}^d \\ &= \|\nabla X(t)\|^{-1} D^2 X(t) Y(t) - \|\nabla X(t)\|^{-3} \left(\sum_k \frac{\partial}{\partial x_l} X(t) \frac{\partial}{\partial x_k} X(t) \frac{\partial^2}{\partial x_k \partial x_i} X(t)\right)_{l,i=1}^d Y(t) \\ &= \|\nabla X(t)\|^{-1} (I_d - \|\nabla X(t)\|^{-2} \nabla X(t) \nabla X(t)^\top) D^2 X(t) Y(t). \end{aligned}$$

By concatenating the projection $\pi_{T_t X^{-1}(\{u\})} \colon T_t\mathbb{R}^d \to T_t X^{-1}(\{u\})$, with the matrix representation

$$M_\pi(\nabla X(t)) := I - \|\nabla X(t)\|^{-2} \nabla X(t) \nabla X(t)^\top,$$

and the mapping $s(t)$, we obtain the linear mapping $\tilde{s}(t) := s(t) \circ \pi_{T_t X^{-1}(\{u\})} \colon T_t\mathbb{R}^d \to T_t\mathbb{R}^d$

given by

$$v \mapsto \|\nabla X(t)\|^{-1} M_\pi(\nabla X(t)) D^2 X(t) M_\pi(\nabla X(t)) v.$$

Then, for $v \in T_t X^{-1}(\{u\})$, we obtain $\tilde{s}(t)v = s(t)v$ and for $v \in (T_t X^{-1}(\{u\}))^\perp$ we have $\tilde{s}(t)v = 0$. Therefore the eigenvalues of $\tilde{s}(t)$ are given by

$$\kappa_1(t), \dots, \kappa_{d-1}(t), 0.$$

Hence, the definition of the elementary symmetric functions and [7, Proposition 4.4.6] yield

$$H_{d-1-k}(\kappa_1(t), \dots, \kappa_{d-1}(t)) = H_{d-1-k}(\kappa_1(t), \dots, \kappa_{d-1}(t), 0) = \operatorname{detr}_{d-1-k}(\tilde{s}(t)),$$

where $\operatorname{detr}_i(A)$ denotes the sum over all $i \times i$ principal subdeterminants of $A \in \mathbb{R}^{d \times d}$, cf. [7, (4.4.15)]. We finally conclude

$$\begin{aligned}\Phi_k^h(X, A) = \frac{1}{\omega_{d-k}} \int_{\mathbb{R}} h(u) \int_{X^{-1}(\{u\})} & \mathbb{1}_A(t) \|\nabla X(t)\|^{-(d-1-k)} \\ & \times \operatorname{detr}_{d-1-k}(M_\pi(\nabla X(t)) D^2 X(t) M_\pi(\nabla X(t)))\, \mathcal{H}^{d-1}(dt)\, du,\end{aligned}$$

and formulate the following central limit theorem.

Corollary 4.5. *Let X be a real Gaussian field on $\mathbb{R}^d$, which satisfies the assumptions (AF1)–(AF3) and let $k > d/2 - 2$. Then*

$$\frac{\Phi_k^h(X, A_N) - \mathbb{E}\left[\Phi_k^h(X, A_N)\right]}{\mathcal{H}^d(A_N)^{1/2}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2) \quad \text{as } A_N \nearrow \mathbb{R}^d$$

where $\sigma^2 \geq 0$.

Proof. An application of Theorem 4.1 with the choice

$$h\colon \mathbb{R}^d \times \mathbb{R}^{d \times d} \times \mathbb{R} \to \mathbb{R}, \quad (x, y, z) \mapsto \frac{1}{\omega_{d-k}} h(z) \|x\|^{-(d-1-k)} \operatorname{detr}_{d-1-k}(M_\pi(x)\, \mathrm{m}(y) M_\pi(x))$$

yields the assertion once we checked condition (AF4) for this specific mapping. To this end, let $A\colon \mathbb{R}^d \to \mathbb{R}^d$ and $B\colon \mathbb{R}^{d(d+1)/2+1} \to \mathbb{R}^{d(d+1)/2+1}$ be regular linear mappings and we abbreviate $n_d := d(d+1)/2 + 1$. Then

$$\begin{aligned}\int_{\mathbb{R}^d \times \mathbb{R}^{n_d}} & h((By)_{n_d})^2 \|Ax\|^{-2(d-2-k)} \operatorname{detr}_{d-1-k}(M_\pi(Ax)\, \mathrm{m}((By)_{1,\dots,n_d-1}) M_\pi(Ax))^2 \\ & \quad \times \phi_{d+n_d}(x, y)\, d(x, y) \\ \leq \int_{\mathbb{R}^d} & \|Ax\|^{-2(d-2-k)} \left(\int_{\mathbb{R}^{n_d}} \operatorname{detr}_{d-1-k}(M_\pi(Ax)\, \mathrm{m}((By)_{1,\dots,n_d-1}) M_\pi(Ax))^4 \phi_{n_d}(y)\, dy \right)^{\frac{1}{2}} \\ & \quad \times \left(\int_{\mathbb{R}^{n_d}} h((By)_{n_d})^4 \phi_{n_d}(y)\, dy \right)^{\frac{1}{2}} \phi_d(x)\, dx, \end{aligned} \tag{4.18}$$

where we used Fubini's theorem and the Cauchy–Schwarz inequality. The integrability assumption on h and Fubini's theorem yield the finiteness of the third integral, cf. (4.15), in (4.18). We denote its value by c_h. Then, by use of spherical coordinates, the expression in (4.18) equals

$$cc_h^{\frac{1}{2}} \int_0^\infty \int_{S^{d-1}} \|rAu\|^{-2(d-2-k)} \left(\int_0^\infty \int_{S^{n_d-1}} \operatorname{detr}_{d-1-k} \left(M_\pi(Au) m((Br'u')_{1,\dots,n_d-1}) M_\pi(Au) \right)^4 \right. \\ \left. \times (r')^{n_d-1} e^{-\frac{(r')^2}{2}} \mathcal{H}^{n_d-1}(du')\, dr' \right)^{\frac{1}{2}} r^{d-1} e^{-\frac{r^2}{2}} \mathcal{H}^{d-1}(du)\, dr$$

rearranging the integrals and using the homogeneity of detr leads to

$$cc_h \int_0^\infty r^{2k+3-d} e^{-\frac{r^2}{2}}\, dr \left(\int_0^\infty r'^{4(d-1-k)+d(d+1)/2} e^{-\frac{r'^2}{2}}\, dr' \right)^{\frac{1}{2}} \int_{S^{d-1}} \|Au\|^{-2(d-2-k)} \\ \times \left(\int_{S^{n_d-1}} \operatorname{detr}_{d-1-k} (M_\pi(Au) m((Bu')_{1,\dots,n_d-1}) M_\pi(Au))^4\, \mathcal{H}^{n_d-1}(du') \right)^{\frac{1}{2}} \mathcal{H}^{d-1}(du). \quad (4.19)$$

We note that the first two integrals are finite for $k > d/2 - 2$ and moreover, since the integrand of the inner integral of the last factor is continuous in u and u', we can bound the integrand on $S^{d-1} \times S^{n_d-1}$, which yields the finiteness of the expression in (4.19). □

Remark. For the dimensions $d = 2$ and $d = 3$, the condition $k > d/2 - 2$ in the statement of Corollary 4.5 is satisfied. However, in general this condition is somewhat annoying, since we believe that the central limit theorem does also hold in the other cases. So where does this condition come from? By examining the proof, especially equation (4.19), we see that from both integrals, which may not exist, it is the first one which is more restrictive. Retracing the steps, which lead to the fateful equation (4.19), leads to the insight that it is the required square integrability of the functional, cf. (AF4), which results in the restriction on k. Hence, the condition on k is not a consequence of the calculations in this chapter but rather a fundamental issue of the technique used to prove the central limit theorems in this thesis.

APPENDIX A

Measurability and statements holding almost surely

In this chapter, we collect and prove various statements concerning properties of Gaussian fields holding almost surely. Moreover measurability results are shown, which ensure that several mappings we are investigating are indeed well-defined random variables.

We start with the derivation of the fact that the set $C_N^d \cap X^{-1}([u,\infty))$ is of sufficient regularity such that the Lipschitz–Killing curvatures, cf. Section 2.3, are well-defined.

Lemma A.1. *Let* $X\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ *be a stationary Gaussian field, which is almost surely of class* $\mathcal{C}^2$*. Then the set* $C_N^d \cap X^{-1}([u,\infty))$ *is almost surely a set of positive reach.*

Proof. We apply [24, Theorem 4.12] and repeat its relevant part here for the sake of completeness.

Lemma A.2. *Let* $f_1, \dots, f_m\colon \mathbb{R}^d \to \mathbb{R}$ *be continuously differentiable and let* ∇f_i *be Lipschitz,* $i = 1, \dots, m$*. Moreover let* $0 \le k \le m$*,*

$$A := \bigcap_{i=1}^{k} \{x \in \mathbb{R}^d \mid f_i(x) = 0\} \cap \bigcap_{i=k+1}^{m} \{x \in \mathbb{R}^d \mid f_i(x) \le 0\}$$

and $J_a := \{i \in \{1, \dots, m\} \mid f_i(a) = 0\}$ *for* $a \in A$*. Assume that there are no real numbers* c_i*,* $i \in J_a$*, such that* $c_i \neq 0$ *for some* $i \in J_a$*,* $c_i \ge 0$ *whenever* $i > k$*, and*

$$\sum_{i \in J_a} c_i \nabla f_i(a) = 0.$$

Then $reach(A, a) > 0$*.*

With the choices

$$f_i\colon \mathbb{R}^d \to \mathbb{R}, \quad t \mapsto t_i - N, \qquad \text{for } i = 1, \dots, d,$$
$$f_i\colon \mathbb{R}^d \to \mathbb{R}, \quad t \mapsto -t_{i-d} - N, \qquad \text{for } i = d+1, \dots, 2d$$

and

$$f_{2d+1}\colon \mathbb{R}^d \to \mathbb{R}, \quad t \mapsto -X(t) + u,$$

we obtain $A = C_N^d \cap X^{-1}([u,\infty))$ and $k = 0$. Hence, we obtain $\operatorname{reach}(C_N^d \cap X^{-1}([u,\infty)), a) > 0$ for $a \in C_N^d \cap X^{-1}([u,\infty))$, if we show that there are no numbers $c_j \geq 0$, $j \in J_a$, such that $c_j > 0$ for some j and

$$\sum_{j \in J_a} c_j \nabla f_j(a) = 0.$$

We first analyse the case $a \in \operatorname{int}(C_N^d) \cap X^{-1}([u,\infty))$. Then $J_a = \{2d+1\}$, since in the case of the empty index set, there is nothing to show. This implies $X(a) = u$. By [1, Lemma 11.2.10]

$$\mathbb{P}(\exists t \in C_N^d : X(t) = u, \nabla X(t) = 0) = 0$$

and therefore $c_{2d+1} \nabla X(a) \neq 0$ for all $c_{2d+1} > 0$. We note that the event of measure zero is independent of the point a, implying that almost surely

$$\operatorname{reach}(C_N^d \cap X^{-1}([u,\infty)), a) > 0 \qquad \text{for all } a \in \operatorname{int}(C_N^d) \cap X^{-1}([u,\infty)).$$

Now, we assume $a \in \operatorname{bd}(C_N^d) \cap X^{-1}([u,\infty))$, say $a \in J_N \cap X^{-1}([u,\infty))$, where $J_N \in \partial_l C_N^d$ and

$$J_N = \{t \in \mathbb{R}^d \mid -N < t_i < N, i \in \sigma(J_N), t_i = \epsilon_i N, i \notin \sigma(J_N)\}.$$

We assume first that $X(a) \neq u$, that is $2d+1 \notin J_a$. But then

$$\sum_{j \in J_a} c_j \nabla f_j(a) \neq 0$$

for $c_j \geq 0$ and at least one c_j nonzero, since the definition of the mappings yields that $f_i(a) = 0$ implies $f_{i+d}(a) \neq 0$ and vice versa $f_{i+d}(a) = 0$ implies $f_i(a) \neq 0$, $i = 1, \dots, d$. Thus, suppose $X(a) = u$ and therefore $2d+1 \in J_a$. Then $a \in J_N$ yields $J_a = \{2d+1\} \cup \{j + d\mathbb{1}\{\epsilon_j = -1\} \mid j \in \{1, \dots, d\} \setminus \sigma(J_N)\}$ and we need to check that

$$\sum_{j \in J_a} c_j \nabla f_j(a) = \sum_{j \in J_a \cap \{1,\dots,d\}} c_j e_j - \sum_{j+d \in J_a \cap \{d+1,\dots,2d\}} c_j e_j - c_{2d+1} \nabla X(a) = 0 \tag{A.1}$$

has no solutions for $c_j \geq 0$ and some c_j nonzero. We note that since $J_a \cap (J_a - d) = \emptyset$ a solution will always need c_{2d+1} nonzero. Then the crucial observation is that equation (A.1) implies

$$\frac{\partial}{\partial t_i} X(a) = 0 \quad \text{for } i \in \sigma(J_N).$$

But [1, Lemma 11.2.10] yields

$$\mathbb{P}\left(\exists t \in J_N : X(t) = u, \frac{\partial}{\partial t_i} X(t) = 0, i \in \sigma(J_N)\right) = 0,$$

and we deduce that equation (A.1) has no solution for $c_j \geq 0$ and some c_j nonzero. Hence, almost surely $\operatorname{reach}(C_N^d \cap X^{-1}([u,\infty)), a) > 0$ for $a \in J_N \cap X^{-1}([u,\infty))$.

The union of the events of measure zero in the first case, and in the second case over all j-faces of the cube C_N^d, $j = 0, \dots, d-1$, yields that almost surely

$$\operatorname{reach}(C_N^d \cap X^{-1}([u,\infty)), a) > 0 \quad \text{for all } a \in C_N^d \cap X^{-1}([u,\infty)).$$

Since $\operatorname{reach}(A, \cdot)$ is a continuous mapping for fixed $A \subset \mathbb{R}^d$, cf. [24, 4.2 Remark], and the set $C_N^d \cap X^{-1}([u,\infty))$ is compact, we conclude that almost surely

$$\operatorname{reach}(C_N^d \cap X^{-1}([u,\infty))) = \inf\left(\operatorname{reach}(C_N^d \cap X^{-1}([u,\infty)), a) \mid a \in C_N^d \cap X^{-1}([u,\infty))\right) > 0,$$

which shows the assertion. □

The following lemma is needed in the approximation of the counting variables appearing in the representation of the Euler characteristic via the Morse lemma, cf. (3.2).

Lemma A.3. *Let $F \in A_{d-m}^d$ and let $X\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a Gaussian field satisfying the assumptions:*

(i) *X has almost surely $\mathcal{C}^2$ paths.*

(ii) *There are almost surely no points $t \in C_N^d \cap F$*

  (a) *such that $\nabla(X|_F)(t) = 0$ and $X(t) = u$.*

  (b) *such that $\nabla(X|_F)(t) = 0$ and $\det(D^2(X|_F)(t)) = 0$.*

(iii) *There are almost surely no points $t \in \operatorname{bd}(\operatorname{int} C_N^d \cap F)$ with $\nabla(X|_F)(t) = 0$.*

Then almost surely

$$\begin{aligned}
&\#\{t \in \operatorname{int} C_N^d \cap F : X(t) \geq u, \nabla(X|_F)(t) = 0, \iota_F^{-X}(t) \text{ even}\} \\
&\qquad - \#\{t \in \operatorname{int} C_N^d \cap F : X(t) \geq u, \nabla(X|_F)(t) = 0, \iota_F^{-X}(t) \text{ odd}\} \\
&= (-1)^{d-m} \lim_{\varepsilon \to 0} \int_{\operatorname{int} C_N^d \cap F} \delta_\varepsilon^d(\nabla(X|_F)(t)) \mathbb{1}\{X(t) \geq u\} \det(D^2(X|_F)(t))\, \mathcal{H}^{d-m}(dt).
\end{aligned}$$

Proof. We follow the proof of [1, Theorem 11.2.3].

We consider the points $t_1, \dots, t_n \in C_N^d \cap F$ where $\nabla(X|_F)(t_i) = 0$ and $\iota_F^{-X}(t_i)$ even, for $i = 1, \dots, n$ and note that there are only finitely many because of (ii)(b), the fact that

$\mathrm{cl}(\mathrm{int}\, C_N^d \cap F)$ is compact and the inverse function theorem. Moreover, condition (iii) implies the existence of relatively open sets U_i in F such that $U_i \subset \mathrm{int}\, C_N^d \cap F$ and $t_i \in U_i$, $i = 1, \dots, n$. Shrinking the sets guarantees that $U_1, \dots, U_n$ are also pairwise disjoint. Furthermore, by condition (ii)(a), we may choose the open sets U_i, $i = 1, \dots, n$, small enough such that either for all $t \in U_i$ we have $X(t) \in (u, \infty)$ or for all $t \in U_i$ we have $X(t) \in (-\infty, u)$.

The same line of reasoning yields relatively open sets $U'_1, \dots, U'_{n'} \subset \mathrm{int}\, C_N^d \cap F$ containing the points $t'_1, \dots, t'_{n'}$ with $\nabla(X|_F)(t'_i) = 0$ and $\iota_F^{-X}(t'_i)$ odd, for $i = 1, \dots, n'$, and satisfying the same properties as $U_1, \dots, U_n$.

The continuity of the determinant and condition (ii)(b) imply that we can choose the sets $U_1, \dots, U_n$ and $U'_1, \dots, U'_{n'}$ small enough such that the sign of $\det(D^2(X|_F))$ stays constant on those sets. Moreover, by contradiction, we deduce the existence of a number $\varepsilon > 0$ small enough such that

$$\nabla(X|_F)^{-1}(B_\varepsilon^d) \cap \mathrm{int}\, C_N^d \cap F \subset \bigcup_{i=1}^{n} U_i \cup \bigcup_{i=1}^{n'} U'_i. \tag{A.2}$$

Indeed, suppose this is not the case. Then we find a sequence $(x_n)_{n\in\mathbb{N}}$ with

$$x_n \in \Big((\nabla(X|_F))^{-1}(B_{1/n}^d) \cap \mathrm{int}\, C_N^d \cap F\Big) \setminus (\bigcup_{i=1}^{n} U_i \cup \bigcup_{i=1}^{n'} U'_i)$$

and which has a convergent subsequence $(x_{n_j})_{j\in\mathbb{N}}$ with limit x_0 satisfying

$$x_0 \in \mathrm{cl}\Big(\mathrm{int}\, C_N^d \cap F\Big) \setminus (\bigcup_{i=1}^{n} U_i \cup \bigcup_{i=1}^{n'} U'_i).$$

Then, continuity implies $\nabla(X|_F)(x_0) = 0$, which leads to $x_0 \in \mathrm{int}\, C_N^d \cap F$ by condition (iii). This is the contradiction we seek, since then there must be an index i such that $x_0 = t_i \in \bigcup_{i=1}^{n} U_i \cup \bigcup_{i=1}^{n'} U'_i$ but by the last display $x_0 \notin \bigcup_{i=1}^{n} U_i \cup \bigcup_{i=1}^{n'} U'_i$.

Now, by assumption (ii)(b) and the inverse function theorem, we can choose the sets $U_1, \dots, U_n, U'_1, \dots, U'_{n'}$ and the number ε small enough to obtain $\nabla(X|_F)$ bijective on the set U_i respectively U'_j and onto $B_\varepsilon^{d-m} \subset F^\circ$. We note the abuse of notation in writing $\nabla(X|_F)^{-1}$ for every inverse. Hence we have

$$\begin{aligned}
&\#\{t \in \mathrm{int}\, C_N^d \cap F : X(t) \geq u, \nabla(X|_F)(t) = 0, \iota_F^{-X}(t) \text{ even}\} \\
&\qquad - \#\{t \in \mathrm{int}\, C_N^d \cap F : X(t) \geq u, \nabla(X|_F)(t) = 0, \iota_F^{-X}(t) \text{ odd}\} \\
&= \sum_{i=1}^{n} \int_{\nabla(X|_F)(U_i)} \delta_\varepsilon^d(y) \mathbb{1}\{X(\nabla(X|_F)^{-1}(y)) \geq u\}\, \mathcal{H}^{d-m}(dy) \\
&\qquad - \sum_{i=1}^{n'} \int_{\nabla(X|_F)(U'_i)} \delta_\varepsilon^d(y) \mathbb{1}\{X(\nabla(X|_F)^{-1}(y)) \geq u\}\, \mathcal{H}^{d-m}(dy).
\end{aligned}$$

We obtain with the substitution rule the equality to

$$\sum_{i=1}^{n}\int_{U_i}\delta_\varepsilon^d(\nabla(X|_F)(t))\mathbb{1}\{X(t)\geq u\}|\det(D^2(X|_F)(t))|\,\mathcal{H}^{d-m}(dt)$$
$$-\sum_{i=1}^{n'}\int_{U_i'}\delta_\varepsilon^d(\nabla(X|_F)(t))\mathbb{1}\{X(t)\geq u\}|\det(D^2(X|_F)(t))|\,\mathcal{H}^{d-m}(dt). \tag{A.3}$$

Since the sign of $\det(D^2(X|_F))$ is constant on the sets $U_1,\dots,U_n,U_1',\dots,U_{n'}'$ and furthermore, by the definition of ι, the equality $\operatorname{sign}(\det(D^2(X|_F(t_i)))) = (-1)^{d-m-\iota_F^{-X}(t_i)}$ holds as well as the same relation for the points t_i', we have

$$\begin{aligned}\operatorname{sign}(\det(D^2(X|_F(t)))) &= (-1)^{d-m}, \quad \text{for all } t\in U_i, i\in\{1,\dots,n\}\\ \operatorname{sign}(\det(D^2(X|_F(t)))) &= -(-1)^{d-m}, \quad \text{for all } t\in U_j', j\in\{1,\dots,n'\}.\end{aligned}$$

Therefore (A.3) equals

$$(-1)^{d-m}\Big(\sum_{i=1}^{n}\int_{U_i}\delta_\varepsilon^d(\nabla(X|_F)(t))\mathbb{1}\{X(t)\geq u\}\det(D^2(X|_F)(t))\,\mathcal{H}^{d-m}(dt)$$
$$+\sum_{i=1}^{n'}\int_{U_i'}\delta_\varepsilon^d(\nabla(X|_F)(t))\mathbb{1}\{X(t)\geq u\}\det(D^2(X|_F)(t))\,\mathcal{H}^{d-m}(dt)\Big),$$

which yields together with (A.2) the assertion. □

The preceding lemma phrased for the lower dimensional boundary terms takes the following form.

Lemma A.4. *Let $J_N\in\partial_l C_N^d$, $m<l<d$, and let $F\in A_{d-m}^d$ be such that $\operatorname{aff}(J_N)^\circ$ and F° are in general position. Moreover let $X\colon\Omega\times\mathbb{R}^d\to\mathbb{R}$ be a Gaussian field satisfying the assumptions:*

(i) *X has almost surely $\mathcal{C}^2$ paths.*

(ii) *There are almost surely no points $t\in\operatorname{cl}(J_N)\cap F$ such that*

  (a) *$\nabla(X|_{J_N\cap F})(t)=0$ and $X(t)=u$.*

  (b) *$\nabla(X|_{J_N\cap F})(t)=0$ and $\pi_{(\operatorname{aff}(J_N)^\circ\cap F^\circ)^\perp}(\nabla X(t))\in\operatorname{bd}(N_t(C_N^d\cap F))$*

  (c) *$\nabla(X|_{J_N\cap F})(t)=0$ and $\det(D^2(X|_{J_N\cap F})(t))=0$.*

(iii) *There are almost surely no points $t\in\operatorname{bd}(J_N\cap F)$ with $\nabla(X|_{J_N\cap F})(t)=0$.*

Then almost surely

$$\begin{aligned}
&\#\{t \in J_N \cap F : X(t) \geq u, \nabla(X|_{J_N\cap F})(t) = 0,\\
&\qquad\qquad \iota^{-X}_{J_N\cap F}(t)\,\text{even}, \pi_{(\text{aff}(J_N)^\circ\cap F^\circ)^\perp}(\nabla X(t)) \in N_t(C^d_N \cap F)\}\\
&- \#\{t \in J_N \cap F : X(t) \geq u, \nabla(X|_{J_N\cap F})(t) = 0,\\
&\qquad\qquad \iota^{-X}_{J_N\cap F}(t)\,\text{odd}, \pi_{(\text{aff}(J_N)^\circ\cap F^\circ)^\perp}(\nabla X(t)) \in N_t(C^d_N \cap F)\}\\
&= (-1)^{l-m} \lim_{\varepsilon\to 0} \int_{J_N\cap F} \delta^l_\varepsilon(\nabla(X|_{J_N\cap F})(t)) \det(D^2(X|_{J_N\cap F})(t))\\
&\qquad\qquad \times \mathbb{1}\{X(t) \geq u, \pi_{(\text{aff}(J_N)^\circ\cap F^\circ)^\perp}(\nabla X(t)) \in N_t(C^d_N \cap F)\}\, \mathcal{H}^{l-m}(dt).
\end{aligned}$$

Proof. In oder to guarantee that $\pi_{(\text{aff}(J_N)^\circ\cap F^\circ)^\perp}(\nabla X(t)) \in N_t(C^d_N \cap F)$ is true on a whole neighborhood of the zeros of $\nabla(X|_{J_N\cap F})$, we need assumption (ii) (b) and the fact that the normal cone $N_t(C^d_N \cap F)$ is full dimensional in the space $(\text{aff}(J_N)^\circ \cap F^\circ)^\perp$. This is established in the equations (2.12) and (2.13). Apart from that, the proof of Lemma A.4 is so similar to the one of Lemma A.3 that it can and will be omitted. □

The following lemma establishes the almost sure applicability of the Morse lemma by showing that all restrictions of nice enough Gaussian fields onto the intersection of affine subspaces with the observation window, are almost surely Morse functions.

Lemma A.5. *Let $X\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ be an almost surely of class C^2, stationary Gaussian field satisfying (A2) and let $J_N \in \partial_l C^d_N$, $m \leq l \leq d$. Then for almost all $\omega \in \Omega$ there is a μ-measurable set $A'(\omega) \subset A^d_{d-m}$, where $\mu(A'(\omega)^{\mathsf{c}}) = 0$, such that*

$$\mathbb{P}\left(\exists F \in A' \exists t \in J_N \cap F : \nabla(X|_{J_N\cap F})(t) = 0, X(t) = u\right) = 0$$

and if $m < l \leq d$

$$\begin{aligned}
&\mathbb{P}\Big(\exists F \in A' \exists t \in \text{bd}(J_N \cap F) : \pi_{\text{aff}(J_N\cap F)^\circ}(\nabla X(t)) = 0\Big) = 0,\\
&\mathbb{P}\Big(\exists F \in A' \exists t \in J_N \cap F : \nabla(X|_{J_N\cap F})(t) = 0, \det(D^2(X|_{J_N\cap F})(t)) = 0\Big) = 0.
\end{aligned}$$

Proof. We show the details for the third equality. First, since

$$\begin{aligned}
&\{\omega \in \Omega \mid \exists F \in A'(\omega) \exists t \in J_N \cap F : \nabla(X(\omega)|_{J_N\cap F})(t) = 0, \det(D^2(X(\omega)|_{J_N\cap F})(t)) = 0\}\\
&\subset \{\omega \in \Omega \mid \exists F \in A'(\omega) \exists t \in \text{cl}(J_N \cap F) : \pi_{\text{aff}(J_N)^\circ\cap F^\circ}(\nabla X(\omega, t)) = 0, \det(D^2_{b^{J_N}_F} X(\omega, t)) = 0\},
\end{aligned}$$

where $b^{J_N}_F$ denotes an orthonormal basis of $\text{aff}(J_N)^\circ \cap F^\circ$, the completeness of the probability measure implies that it is enough to show

$$\mathbb{P}\left(\exists F \in A' \exists t \in \text{cl}(J_N \cap F) : \pi_{\text{aff}(J_N)^\circ\cap F^\circ}(\nabla X(t)) = 0, \det(D^2_{b^{J_N}_F} X(t)) = 0\right) = 0,$$

where the closure is needed in the proof of Lemma A.6. We start with an application of [1, Lemma 11.2.11] yielding

$$\mathbb{P}\left(\exists t \in \mathrm{cl}(J_N \cap F) : \pi_{\mathrm{aff}(J_N)^\circ \cap F^\circ}(\nabla X(t)) = 0, \det(D^2_{b_F^{J_N}} X(t)) = 0\right) = 0$$

for a fixed $F \in A^{d,*}_{d-m}$, where $A^{d,*}_{d-m}$ is the set of all $F \in A^d_{d-m}$ such that F° and $\mathrm{aff}(I_N)^\circ$, for all $I_N \in \partial_i C^d_N$, $i = 1, \ldots, d$, are in general position, which is a measurable subset of A^d_{d-m} with full measure, cf. [69, Lemma 13.2.1]. This implies together with Fubini's theorem

$$\begin{aligned}
&\mathbb{E}\left[\int_{A^{d,*}_{d-m}} \mathbb{1}\{\exists t \in \mathrm{cl}(J_N \cap F) : \pi_{\mathrm{aff}(J_N)^\circ \cap F^\circ}(\nabla X(t)) = 0, \det(D^2_{b_F^{J_N}} X(t)) = 0\}\, \mu(dF)\right] \\
&\quad = \int_{A^{d,*}_{d-m}} \mathbb{P}\left(\exists t \in \mathrm{cl}(J_N \cap F) : \pi_{\mathrm{aff}(J_N)^\circ \cap F^\circ}(\nabla X(t)) = 0, \det(D^2_{b_F^{J_N}} X(t)) = 0\right) \mu(dF) = 0,
\end{aligned}$$

where the above integrand is $\mathbb{P} \otimes \mu$-measurable by Lemma A.6. Hence, we obtain the existence of a measurable set $B_3 \subset \Omega \times A^{d,*}_{d-m}$, such that $\mathbb{P} \otimes \mu(B_3^\complement) = 0$ and for all $(\omega, F) \in B_3$, we have that

$$\begin{aligned}
&\mathbb{1}\{\exists t \in \mathrm{cl}(J_N \cap F) : \pi_{\mathrm{aff}(J_N)^\circ \cap F^\circ}(\nabla X(\omega, t)) = 0, \det(D^2_{b_F^{J_N}} X(\omega, t)) = 0\} = 0 \\
&\Leftrightarrow \forall t \in \mathrm{cl}(J_N \cap F) : \neg(\pi_{\mathrm{aff}(J_N)^\circ \cap F^\circ}(\nabla X(\omega, t)) = 0, \det(D^2_{b_F^{J_N}} X(\omega, t)) = 0).
\end{aligned}$$

We now define for $\omega \in \Omega$ the ω-cross section of B_3 as $B_{3,\omega} := \{F \in A^{d,*}_{d-m} \mid (\omega, F) \in B_3\}$ and observe, cf. [23, Theorem 1.22], that for $\mathbb{P}$-almost all $\omega \in \Omega$ the set $B_{3,\omega}$ is μ-measurable and

$$\mu(B^\complement_{3,\omega}) = 0.$$

Similar reasoning, except that we use [1, Lemma 11.2.12] and [1, Lemma 11.2.10], yields sets $B_1, B_2 \in \mathcal{F} \otimes \mathcal{B}(A^{d,*}_{d-m})$ and cross sections $B_{1,\omega}, B_{2,\omega}$, whose complements have μ measure zero for $\mathbb{P}$-almost all $\omega \in \Omega$. Thus for $\mathbb{P}$-almost all ω the complement of $A'(\omega) := \cap_{i=1}^3 B_{i,\omega}$ has μ measure zero, and we conclude

$$\begin{aligned}
&\mathbb{P}(\exists F \in A' \exists t \in J_N \cap F : \nabla(X(\omega)|_{J_N \cap F})(t) = 0, \det(D^2(X(\omega)|_{J_N \cap F})(t)) = 0) \\
&\quad = \mathbb{P}(\omega \in \Omega \mid \exists F \in A^d_{d-m} : (\omega, F) \in \cap_{i=1}^3 B_i \text{ and} \\
&\qquad\qquad \exists t \in J_N \cap F : \nabla(X(\omega)|_{J_N \cap F})(t) = 0, \det(D^2(X(\omega)|_{J_N \cap F})(t)) = 0) \\
&\quad = \mathbb{P}(\emptyset) = 0.
\end{aligned}$$

And analogously

$$\begin{aligned}
&\mathbb{P}(\exists F \in A' \exists t \in J_N \cap F : \nabla(X|_{J_N \cap F})(t) = 0, X(t) = u) = 0, \\
&\mathbb{P}(\exists F \in A' \exists t \in \mathrm{bd}(J_N \cap F) : \pi_{\mathrm{aff}(J_N \cap F)^\circ}(\nabla X(t)) = 0) = 0.
\end{aligned}$$ □

In the following, we present a proof for the measurability of the indicator mapping used in the derivation of the preceding lemma.

Lemma A.6. *Let $X\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ be an almost surely of class C^2, stationary Gaussian field and let $J_N \in \partial_l C_N^d$, $m < l \leq d$. Then the mapping $\cdot\colon \Omega \times A_{d-m}^{d,*} \to \mathbb{R}$ given by*

$$(\omega, F) \mapsto \mathbb{1}\{\exists t \in \mathrm{cl}(J_N \cap F) : \pi_{\mathrm{aff}(J_N)^\circ \cap F^\circ}(\nabla X(\omega, t)) = 0, \det(D^2_{b_F^{J_N}} X(\omega, t)) = 0\}$$

is $\mathbb{P} \otimes \mu$-measurable.

Proof. The given mapping is an indicator function and as such it is measurable if the set

$$\{(\omega, F) \in \Omega \times A_{d-m}^{d,*} \mid \exists t \in \mathrm{cl}(J_N \cap F) : \pi_{\mathrm{aff}(J_N)^\circ \cap F^\circ}(\nabla X(\omega, t)) = 0, \det(D^2_{b_F^{J_N}} X(\omega, t)) = 0\}$$

is measurable in the product σ-algebra. The continuity in t of the mappings

$$t \mapsto f_1(t, \omega, F) := \pi_{\mathrm{aff}(J_N)^\circ \cap F^\circ}(\nabla X(\omega, t)) \quad \text{and} \quad t \mapsto f_2(t, \omega, F) := \det(D^2_{b_F^{J_N}} X(\omega, t))$$

together with the compactness of the set $\mathrm{cl}\, J_N$ yield

$$\begin{aligned}
&\{(\omega, F) \in \Omega \times A_{d-m}^{d,*} \mid \exists t \in \mathrm{cl}(J_N \cap F) : \pi_{\mathrm{aff}(J_N)^\circ \cap F^\circ}(\nabla X(\omega, t)) = 0, \det(D^2_{b_F^{J_N}} X(\omega, t)) = 0\} \\
&= \bigcap_{n \in \mathbb{N}} \bigcup_{t \in I} \Big(\Big\{(\omega, F) \in \Omega \times A_{d-m}^{d,*} \mid \pi_{\mathrm{aff}(J_N)^\circ \cap F^\circ}(\nabla X(\omega, t)) \in B^d_{1/n}\Big\} \\
&\qquad \cap \left\{(\omega, F) \in \Omega \times A_{d-m}^{d,*} \mid |\det(v_1|\cdots|v_{l-m})^\top D^2 X(\omega, t)(v_1|\cdots|v_{l-m})| < \frac{1}{n}\right\} \\
&\qquad \cap \Big\{(\omega, F) \in \Omega \times A_{d-m}^{d,*} \mid F \cap B^d_{1/n}(t) \neq \emptyset\Big\} \Big),
\end{aligned} \tag{A.4}$$

where I denotes a dense subset of $\mathrm{cl}\, J_N$ and $b_F^{J_N} = (v_i)_{i=1}^{l-m}$ an orthonormal basis of $\mathrm{aff}(J_N)^\circ \cap F^\circ$. Equation A.4 can be deduced by showing that

$$\begin{aligned}
&\{(\omega, F) \in \Omega \times A_{d-m}^{d,*} \mid \exists t \in \mathrm{cl}(J_N \cap F) : \pi_{\mathrm{aff}(J_N)^\circ \cap F^\circ}(\nabla X(\omega, t)) = 0, \det(D^2_{b_F^{J_N}} X(\omega, t)) = 0\} \\
&= \bigcap_{n \in \mathbb{N}} \Big\{(\omega, F) \in \Omega \times A_{d-m}^{d,*} \mid \exists t \in \mathrm{cl}\, J_N : F \cap B^d_{1/n}(t) \neq \emptyset, \\
&\qquad\qquad f_1(t, \omega, F) \in B^d_{1/n}, |f_2(t, \omega, F)| < \frac{1}{n}\Big\},
\end{aligned}$$

where the set on left side is trivially a subset of the intersection on the right side. To show the converse, let (ω, F) be an element in the intersection on the right side. Then there exists a sequence $(t_n)_{n \in \mathbb{N}}$ in $\mathrm{cl}\, J_N$ such that $F \cap B^d_{1/n}(t_n) \neq \emptyset$ and $f_1(t_n, \omega, F) \in B^d_{1/n}$ as well as $|f_2(t_n, \omega, F)| < \frac{1}{n}$. The compactness of $\mathrm{cl}\, J_N$ implies that there is a convergent subsequence $(t_{n_i})_{i \in \mathbb{N}}$ such that $t_{n_i} \overset{i \to \infty}{\longrightarrow} t_0 \in \mathrm{cl}\, J_N$. Then

$$d(F, t_0) \leq d(F, t_{n_i}) + d(t_{n_i}, t_0) \leq \frac{1}{n_i} + \|t_{n_i} - t_0\| \overset{i \to \infty}{\longrightarrow} 0,$$

implying $t_0 \in F$. Moreover by continuity

$$\|f_1(t_0, \omega, F)\| = \lim_{i \to \infty} \|f_1(t_{n_i}, \omega, F)\| \leq \lim_{i \to \infty} \frac{1}{n_i} = 0$$

and analogously $f_2(t_0,\omega,F)=0$, which yields that (ω,F) is in the left set. In the second step we establish

$$\Big\{(\omega,F)\in\Omega\times A^{d,*}_{d-m}\mid \exists t\in \operatorname{cl} J_N : F\cap B^d_{1/n}(t)\neq\emptyset, f_1(t,\omega,F)\in B^d_{1/n}, |f_2(t,\omega,F)|<\frac{1}{n}\Big\}$$
$$=\bigcup_{t\in I}\Big\{(\omega,F)\in\Omega\times A^{d,*}_{d-m}\mid F\cap B^d_{1/n}(t)\neq\emptyset, f_1(t,\omega,F)\in B^d_{1/n}, |f_2(t,\omega,F)|<\frac{1}{n}\Big\},$$

where the union on the right side is trivially a subset of the set on the left. To show the converse, let (ω,F) be an element in the set on the left side. Then there exists a point $t\in\operatorname{cl} J_N$ with the specified properties. Since I is dense in $\operatorname{cl} J_N$, there exists a sequence $(t_k)_{k\in\mathbb{N}}$ in I such that $t_k\overset{k\to\infty}{\longrightarrow} t$. If k is sufficiently large, we obtain $F\cap B^d_{1/n}(t_k)\neq\emptyset$, since $t_k\overset{k\to\infty}{\longrightarrow} t$ and $d(F,t)<\frac{1}{n}$. Moreover, continuity implies that $f_1(t_k,\omega,F)<\frac{1}{n}$ as well as $|f_2(t_k,\omega,F)|<\frac{1}{n}$, if k is large enough, which yields (ω,F) is an element in the set on the right side.

We note that the last set in the inner intersections in display (A.4) is measurable since it is by definition an open set in the Fell topology. For the measurability of the other two, we need to take a closer look. We start with the first one and define the mappings

$$\begin{aligned}
&f_1\colon(\Omega,A^{d,*}_{d-m})\to(\mathbb{R}^d,A^{d,*}_{d-m}),\quad (\omega,F)\mapsto(\nabla X(\omega,t),F),\\
&f_2\colon(\mathbb{R}^d,A^{d,*}_{d-m})\to(\mathbb{R}^d,G^d_{l-m}),\quad (x,F)\mapsto(x,\operatorname{aff}(J_N)^\circ\cap F^\circ),\\
&f_3\colon(\mathbb{R}^d,G^d_{l-m})\to\mathbb{R}^d,\quad (x,E)\mapsto\pi_E(x),
\end{aligned}$$

for $t\in\mathbb{R}^d$, such that

$$\Big\{(\omega,F)\in\Omega\times A^{d,*}_{d-m}\mid \pi_{\operatorname{aff}(J_N)^\circ\cap F^\circ}(\nabla X(\omega,t))\in B^d_{1/n}\Big\}=(f_3\circ f_2\circ f_1)^{-1}(B^d_{1/n}).$$

Then, $f_1^{-1}(A\times B)=\{\omega\in\Omega\mid\nabla X(\omega,t)\in A\}\times B$, for $t\in\mathbb{R}^d$, $A\in\mathcal{B}(\mathbb{R}^d)$ and $B\in\mathcal{B}(A^{d,*}_{d-m})$, which is measurable in the product σ-algebra by definition of random fields. Similarly, the measurability of f_2 follows by observing that the map $A^{d,*}_{d-m}\to A^{d,*}_{d-m}$, $F\mapsto F^\circ$ is continuous and moreover the map $G^{d,*}_{d-m}\to G^d_{l-m}$ defined by $E\mapsto\operatorname{aff}(J_N)^\circ\cap E$ is the restriction of a upper semicontinuous and thus measurable map, cf. [69, Thm 12.2.6 (a)]. At last, the measurability of the map f_3 follows with [6, Lemma 6.5.11] from the fact that f_3 is continuous in x for fixed $E\in G^d_{l-m}$, and continuous in E for fixed $x\in\mathbb{R}^d$, cf. [5, Theorem 3.1] since upper semicontinuity for multifunctions implies continuity in the single value case.

To establish the measurability of the remaining set of the above inner intersection, we define the mappings

$$\begin{aligned}
&f_1\colon\Omega\times A^{d,*}_{d-m}\to\mathbb{R}^{d\times(l-m)}\times\mathbb{R}^{d\times d},\quad (\omega,F)\mapsto(v_1,\dots,v_{l-m},D^2X(\omega,t))\\
&f_2\colon\mathbb{R}^{d\times(l-m)}\times\mathbb{R}^{d\times d}\to\mathbb{R},\quad (A,B)\mapsto|\det A^\top BA|,
\end{aligned}$$

for $t\in\mathbb{R}^d$, such that

$$\Big\{(\omega,F)\in\Omega\times A^{d,*}_{d-m}\mid |\det(v_1|\cdots|v_{l-m})^\top D^2X(\omega,t)(v_1|\cdots|v_{l-m})|<\frac{1}{n}\Big\}=(f_2\circ f_1)^{-1}(B^1_{1/n}).$$

Then, in order to obtain the measurability, we show that f_2 and f_1 are measurable mappings. For f_2 this is true since all involved manipulations are continuous and thus measurable. To establish the measurability of f_1, let $A \in \mathcal{B}(\mathbb{R}^{d\times(l-m)})$ and $B \in \mathcal{B}(\mathbb{R}^{d\times d})$. Then

$$f_1^{-1}(A \times B) = \{\omega \in \Omega \mid D^2X(\omega,t) \in B\} \times \{F \in A_{d-m}^{d,*} \mid (v_1,\dots,v_{l-m}) \in A\},$$

where the first set of the product is measurable by the properties of a random field and the second set is measurable if the choice of the orthonormal basis $v_1,\dots,v_{l-m}$ is measurable. To show that a measurable choice indeed exists, we invoke the measurable selection theorem 6.6.7 in [6], which we state for completeness.

Theorem A.7. *Let $(X,\mathcal{A})$ be a measurable space, Y be a Polish space and $T\colon X \rightrightarrows Y$ be a measurable multifunction with nonempty closed values. Then T admits a measurable selection $f\colon X \to Y$, i.e. for all $x \in X$ we have $f(x) \in T(x)$ and f is measurable.*

We choose $X := A_{d-m}^{d,*}$, $Y := \mathbb{R}^d$ and for $F \in A_{d-m}^{d,*}$

$$T_i(F) := \operatorname{aff}(J_N)^\circ \cap (\operatorname{lin}(f_1(F),\dots,f_{i-1}(F)))^\perp \cap S^{d-1} \cap F^\circ, \quad i = 1,\dots,l-m.$$

The multifunction T_i defined in this way has nonempty closed values by definition. The measurability of T_i as a multifunction can be obtained by showing that it is a measurable mapping when it is considered as a single valued mapping into the space $(\mathrm{Cl}(\mathbb{R}^d), \tau_{\mathrm{Fell}})$, the space of closed subset of $\mathbb{R}^d$ equipped with the Fell topology, cf. [6, Theorem 6.5.14], [6, Theorem 5.1.10]. After obtaining the measurability of the map

$$\{(v_i)_{i=1}^n \in \mathbb{R}^{d\times n} \mid v_1,\dots,v_n \text{ pairwise orthogonal}\} \to G_n^d, \quad (v_1,\dots,v_n) \mapsto (\operatorname{lin}(v_1,\dots,v_n))^\perp$$

with the aid of [69, Theorem 12.2.2] and [69, Theorem 12.2.5], we obtain the measurability of the multifunction by [69, Theorem 12.2.6 (a)], where the measurability of the intersection is established. Hence, the mapping

$$\cdot\colon A_{d-m}^{d,*} \to \mathbb{R}^{d\times(l-m)}, \quad F \mapsto (f_1(F),\dots,f_{l-m}(F))$$

yields the desired measurable choice of an orthonormal basis of $\operatorname{aff}(J_N)^\circ \cap F^\circ$, and we conclude the assertion. □

In the following lemma we establish the fact that the curvature notions of an excursion set in the cube of side length N investigated in Chapter 3 is a measurable mapping from the probability space into the extended reals. That is, it is a well defined random variable.

Lemma A.8. *Let $X\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a stationary Gaussian field, which is almost surely of class $\mathcal{C}^2$ and let $m \in \{0,\dots,d-1\}$. Then the mapping*

$$\Omega \to (\mathbb{R} \cup \{\infty\}), \quad \omega \mapsto \mathcal{L}_m\left(C_N^d \cap X(\omega)^{-1}([u,\infty))\right)$$

is $\mathcal{F}$-$\sigma(\mathcal{B}(\mathbb{R}) \cup \{\infty\})$-measurable, i.e. a random variable.

Proof. By Lemma A.1, we may assume that $C_N^d \cap X(\omega)^{-1}([u,\infty))$ is a set of positive reach for all $\omega \in \Omega$. Writing down the Steiner formula, cf. (2.17), d times with ε given by

$$\varepsilon_1 := \frac{\operatorname{reach}\left(C_N^d \cap X(\omega)^{-1}([u,\infty))\right)}{d}, \dots, \varepsilon_d := \frac{\operatorname{reach}\left(C_N^d \cap X(\omega)^{-1}([u,\infty))\right)}{1},$$

if $\operatorname{reach}\left(C_N^d \cap X(\omega)^{-1}([u,\infty))\right) < \infty$ and $\varepsilon_1 := 1, \dots, \varepsilon_d := d$ otherwise, yields a solvable system of linear equations, since its determinant is given by a Vandermonde determinant different from zero. We obtain a representation of $\mathcal{L}_m$ in terms of $\mathcal{H}^d$, i.e.

$$\mathcal{L}_m\left(C_N^d \cap X(\omega)^{-1}([u,\infty))\right) = \sum_{n=1}^{d} c_{mn}^{\varepsilon_n} \mathcal{H}^d\left(\left(C_N^d \cap X(\omega)^{-1}([u,\infty))\right) + \varepsilon_n \operatorname{cl} B_1^d\right),$$

where the coefficients $c_{m1}^{\varepsilon_1}, \dots, c_{md}^{\varepsilon_d} \in \mathbb{R}$ depend in a measurable way on the expression $\operatorname{reach}\left(C_N^d \cap X(\omega)^{-1}([u,\infty))\right)$. By [56, Proposition 1.1.16] the reach of a random closed set is a random variable, where the fact that $C_N^d \cap X(\omega)^{-1}([u,\infty))$ is indeed a random closed set is shown later in this proof. We recall [79, Theorem 2.1.3], where the Fell topology on the space $\mathrm{Cl}(\mathbb{R}^d)$ of closed subsets of $\mathbb{R}^d$ is introduced for instance in [69, Section 12.2] or [6, Section 5.1].

Theorem A.9. *Let $s \geq 0$ and let $B \subset \mathbb{R}^n$ be closed. Then the mapping*

$$(\mathrm{CL}(\mathbb{R}^d), \sigma(\tau_{Fell})) \to (\mathbb{R} \cup \{\infty\}, \sigma(\mathcal{B}(\mathbb{R}) \cup \{\infty\})), \quad F \mapsto \mathcal{H}^s(F \cap B)$$

is measurable.

Therefore, by taking $s = d$ and $B = \mathbb{R}^d$, it suffices to show the measurability of the mapping

$$\Omega \to \mathrm{Cl}(\mathbb{R}^\mathrm{d}), \quad \omega \mapsto \left(C_N^d \cap X(\omega)^{-1}([u,\infty))\right) + \varepsilon_n \operatorname{cl} B_1^d.$$

The intersection and the closure of Minkowski addition are measurable by [68, Theorem 12.2.6] and [68, Theorem 12.3.1], respectively. Thus, all that is left is to show the $\mathcal{F}$-$\sigma(\tau_{Fell})$-measurability of $\omega \mapsto X(\omega)^{-1}([u,\infty))$. To see that, it is enough to show that the sets

$$\{\omega \in \Omega \mid X(\omega)^{-1}([u,\infty)) \cap C = \emptyset\}, \quad C \subset \mathbb{R}^d \text{ compact,}$$
$$\{\omega \in \Omega \mid X(\omega)^{-1}([u,\infty)) \cap G \neq \emptyset\}, \quad G \subset \mathbb{R}^d \text{ open}$$

are measurable, by the existence of the countable base of the Fell topology, cf. [69, Chapter 12.2]. The sets in the first line are measurable since the compactness of C and the continuity of X allow the decomposition

$$\begin{aligned}\{\omega \in \Omega \mid X(\omega)^{-1}([u,\infty)) \cap C = \emptyset\}^\mathsf{c} &= \{\omega \in \Omega \mid \exists t \in C : X(\omega,t) \geq u\} \\ &= \bigcap_{n\in\mathbb{N}} \bigcup_{t\in I} \left\{\omega \in \Omega \mid X(\omega,t) > u - \frac{1}{n}\right\},\end{aligned}$$

where this is measurable since X is a random field, cf. (2.5). The sets involving a open set G are measurable since

$$\begin{aligned}\{\omega \in \Omega \mid X(\omega)^{-1}([u,\infty)) \cap G \neq \emptyset\} &= \{\omega \in \Omega \mid \exists t \in G : X(\omega,t) \geq u\} \\ &= \bigcup_B \{\omega \in \Omega \mid \exists t \in B : X(\omega,t) \geq u\},\end{aligned}$$

where the union is taken over all closed balls with rational radius and rational center contained in G. Then the measurability follows analogously to the preceding one and the assertion follows. □

The preceding lemma was proven by [79, Theorem 2.1.3] and basic principles. The arguments could be shortened by the following more advanced result from the literature, cf. [80, Theorem 2.1.2]. We denote by $\mathcal{P}$ the set of closed sets with positive reach, which is measurable in τ_{Fell}, cf. [80, Proposition 1.1.1], and equip it with the trace topology of the Fell topology τ_{Fell}.

Theorem A.10. *Let $m = 0, \dots, d$. Then the mapping*

$$(\mathcal{P}, \sigma(\tau_{Fell} \cap \mathcal{P})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})), \quad A \mapsto \mathcal{L}_m(A)$$

is measurable.

We proceed with the derivation of the fact that the counting variables defined in (3.2) are measurable mappings in ω and F.

Lemma A.11. *Let $X\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ be an almost surely of class C^2, stationary Gaussian field and let $J_N \in \partial_l C_N^d$, $m \leq l \leq d$. Then the mapping $\cdot\colon \Omega \times A_{d-m}^d \to \mathbb{R}$ given by*

$$(\omega, F) \mapsto \#\left\{t \in J_N \cap F \mid X(t) \geq u, \nabla(X|_{J_N \cap F})(t) = 0, \iota^{-X}_{J_N \cap F}(t) \text{ even}, \nabla X(t) \in N_t(C_N^d \cap F)\right\}$$

is $\mathcal{F} \otimes \mathcal{B}(A_{d-m}^d)$-$\sigma(\mathcal{B}(\mathbb{R}) \cup \{\infty\})$-measurable.

Proof. The measure $\mathcal{H}^0$ is given by the counting measure and therefore

$$\begin{aligned}&\#\left\{J_N \cap F \mid X(t) \geq u, \nabla(X|_{J_N \cap F})(t) = 0, \iota^{-X}_{J_N \cap F}(t) \text{ even}, \nabla X(t) \in N_t(C_N^d \cap F)\right\} \\ &\quad = \mathcal{H}^0\left(J_N \cap \left\{t \in F \mid X(t) \geq u, \nabla(X|_{J_N \cap F})(t) = 0, \iota^{-X}_{J_N \cap F}(t) \text{ even}, \nabla X(t) \in N_t(C_N^d \cap F)\right\}\right).\end{aligned}$$

By continuity from below of the measure $\mathcal{H}^0$ and Theorem A.9, the assertion of the lemma follows, if we show the measurability of the map $\cdot\colon \Omega \times A_{d-m}^{d,*} \to \mathrm{Cl}(\mathbb{R}^d)$, where $A_{d-m}^{d,*}$ denotes the set of all $F \in A_{d-m}^d$ with $F \cap J_N \neq \emptyset$ and $\mathrm{aff}(J_N)^\circ$ and F° are in general position, given by

$$\begin{aligned}(\omega, F) \to \{t \in F \mid &X(\omega,t) \geq u, \pi_{\mathrm{aff}(J_N)^\circ \cap F^\circ}(\nabla X(\omega,t)) = 0, \det D^2_{b^F_{J_N}} X(\omega,t) \geq 0, \\ &\nabla X(\omega,t) \in N_{J_N}(C_N^d \cap F)\}.\end{aligned}$$

We note that writing $N_{J_N}(C_N^d \cap F)$ for $N_t(C_N^d \cap F)$ emphasizes the fact that this normal cone does not depend on the location of t in $J_N \cap F$, cf. (2.14). To show this measurability, it is enough to establish that the following set for an arbitrary compact set $C \subset \mathbb{R}^d$ is measurable, cf. [53, Section 1.2],

$$\begin{aligned}
&\Big\{(\omega, F) \in \Omega \times A^{d,*}_{d-m} \mid \exists t \in F \cap C : X(\omega, t) \geq u, \pi_{\mathrm{aff}(J_N)^\circ \cap F^\circ}(\nabla X(\omega, t)) = 0, \\
&\qquad \det D^2_{b^F_{J_N}} X(\omega, t) \geq 0, \nabla X(\omega, t) \in N_{J_N}(C^d_N \cap F)\Big\} \\
&= \bigcap_{n\in\mathbb{N}} \Big\{(\omega, F) \in \Omega \times A^{d,*}_{d-m} \mid \exists t \in C : X(\omega, t) > u - \frac{1}{n}, \det D^2_{b^F_{J_N}} X(\omega, t) > -\frac{1}{n}, \\
&\qquad F \cap B^d_{\frac{1}{n}}(t) \neq \emptyset, \pi_{\mathrm{aff}(J_N)^\circ \cap F^\circ}(\nabla X(\omega, t)) \in B^d_{\frac{1}{n}}, \nabla X(\omega, t) \in (N_{J_N}(C^d_N \cap F) + B^d_{\frac{1}{n}})\Big\},
\end{aligned} \tag{A.5}$$

where the continuity of the involved functions and the compactness of C yields the equality, cf. the proof of Lemma A.6. Denoting by I a dense subset of the set C yields the equality of the set in (A.5) to

$$\begin{aligned}
&\bigcap_{n\in\mathbb{N}} \bigcup_{t\in I} \Big(\Big\{(\omega, F) \in \Omega \times A^{d,*}_{d-m} \mid X(\omega, t) > u - \frac{1}{n}\Big\} \\
&\quad \cap \Big\{(\omega, F) \in \Omega \times A^{d,*}_{d-m} \mid F \cap B^d_{\frac{1}{n}}(t) \neq \emptyset, \pi_{\mathrm{aff}(J_N)^\circ \cap F^\circ}(\nabla X(\omega, t)) \in B^d_{\frac{1}{n}}, \\
&\qquad \det D^2_{b^F_{J_N}} X(\omega, t) > -\frac{1}{n}\Big\} \\
&\quad \cap \Big\{(\omega, F) \in \Omega \times A^{d,*}_{d-m} \mid \nabla X(\omega, t) \in (N_{J_N}(C^d_N \cap F) + B^d_{\frac{1}{n}})\Big\} \Big).
\end{aligned}$$

The measurability of the inner set of the first line of the display is measurable by definition of a random field. The measurability of the set in the second and third line can be deduced as in the proof of Lemma A.6. To deduce the measurability of the set in the third line, we first note that

$$\begin{aligned}
&\Big\{(\omega, F) \in \Omega \times A^{d,*}_{d-m} \mid \nabla X(\omega, t) \in (N_{J_N}(C^d_N \cap F) + B^d_{\frac{1}{n}})\Big\} \\
&\quad = \bigcup_{m\in\mathbb{N}} \Big\{(\omega, F) \in \Omega \times A^{d,*}_{d-m} \mid \nabla X(\omega, t) \in (N_{J_N}(C^d_N \cap F) + \operatorname{cl} B^d_{\frac{1}{n}-\frac{1}{m}})\Big\}.
\end{aligned} \tag{A.6}$$

By defining

$$\begin{aligned}
&f_1 \colon \Omega \to \mathbb{R}, \quad \omega \mapsto \nabla X(\omega, t), \qquad f_2 \colon A^{d,*}_{d-m} \to \mathrm{Cl}(\mathbb{R}^{\mathrm{d}}), \quad F \mapsto N_{J_N}(C^d_N \cap F) + \operatorname{cl} B^d_{\frac{1}{n}-\frac{1}{m}}, \\
&f_3 \colon \Omega \times A^{d,*}_{d-m} \to \mathrm{Cl}(\mathbb{R}^{\mathrm{d}}) \times \mathbb{R}, \quad (\omega, F) \mapsto (f_2(F), f_1(\omega)), \\
&f_4 \colon \mathrm{Cl}(\mathbb{R}^{\mathrm{d}}) \times \mathbb{R}^d \to \mathbb{R}, \quad (A, x) \mapsto \mathbb{1}\{x \in A\}
\end{aligned}$$

the set in (A.6) equals $(f_4 \circ f_3)^{-1}(1)$. Thus, this is a measurable set if f_1, f_2 and f_4 are measurable. By [69, Theorem 12.2.7] the map f_4 is measurable and f_1 has this property since X is a random field. Since the closure of Minkowski addition is measurable, cf. [69, Theorem

12.3.1], it suffices to show the measurability of

$$A_{d-m}^{d,*} \to \mathrm{CL}(\mathbb{R}^d), \quad F \mapsto N_{J_N}(C_N^d \cap F) \tag{A.7}$$

in order to show that f_2 is measurable. This is the case, since (A.7) is an upper semicontinuous mapping on $A_{d-m}^{d,*}$, which can be seen with the aid of [69, Theorem 12.2.5], [5, Theorem3.1] and [69, Theorem 12.2.2], and the fact that $N_{J_N}(C_N^d \cap F) = (\pi_{C_N^d \cap F})^{-1}(t) - t$, for any $t \in J_N \cap F$. $\square$

APPENDIX B

Proof of Lemma 3.2

In the remaining part of the appendix we give a proof of Lemma 3.2. We state the lemma again for the convenience of the reader. The auxiliary Lemmas B.1 – B.6 are necessary for the proof of Lemma 3.2 (i) and (ii).

In order to prove part (i), we apply the Rice formulas, cf. Section 2.1, and find suitable bounds for the terms appearing in these formulas. To do so we define a mapping, which maps the set $J_N \cap F$ from the ambient space $\mathbb{R}^d$ into the ambient space $\mathbb{R}^{l-m}$ in which we can apply the Rice formulas. Then we exploit the differentiability of the covariance function to obtain the desired upper bounds in the Lemmas B.2– B.6. For part (ii) a Gaussian regression is used as well as the Lemmas B.2 and B.3. Part (iii) is established with the aid of Lemma A.3 and part (ii).

Lemma 3.2. *Let $G \subset \mathbb{R}^d$ be compact and assume the conditions (A1) and (A2). Furthermore let $J_N \in \partial_l C_N^d$ and $l > m$. Then the following is true:*

(i) *There is a constant $c = c(X, d, m, l, N, G) > 0$ such that for almost all $F \in A_{d-m}^d$ and all $y \in G$*

$$\mathbb{E}\left[\#\{t \in J_N \cap F \colon \nabla(X|_{J_N \cap F})(t) = y\}^2\right] < c.$$

(ii) *For almost all $F \in A_{d-m}^d$ the mapping*

$$y \mapsto \mathbb{E}\left[\#\{t \in J_N \cap F \colon \nabla(X|_{J_N \cap F})(t) = y\}^2\right]$$

is continuous on $(\mathrm{aff}(J_N) \cap F)^\circ \cap G$.

(iii) *For almost all* $F \in A^d_{d-m}$

$$\xi_N(F,\varepsilon) \overset{L^2(\mathbb{P})}{\longrightarrow} \xi_N(F), \qquad \textit{as } \varepsilon \to 0,$$

where

$$\begin{aligned} \xi_N(F,\varepsilon) &:= (-1)^{d-m} \int_{C^d_N \cap F} \delta^d_\varepsilon(\nabla(X|_F)(t)) \mathbb{1}\{X(t) \geq u\} \det(D^2(X|_F)(t))\, \mathcal{H}^{d-m}(dt), \\ \xi_N(F) &:= \#\{t \in \operatorname{int} C^d_N \cap F : X(t) \geq u, \nabla(X|_F)(t) = 0, \iota^{-X}_F(t) \,\text{even}\} \\ &\quad - \#\{t \in \operatorname{int} C^d_N \cap F : X(t) \geq u, \nabla(X|_F)(t) = 0, \iota^{-X}_F(t) \,\text{odd}\}. \end{aligned}$$

Proof. To prove (i), we refine the methods used to prove [22, Proposition 1.1 (1)]. First, we note that if $J_N \cap F \neq 0$, the assertion is trivially true. Moreover, let $F \in A^d_{d-m}$ be such that the linear subspaces F° and $\operatorname{aff}(J_N)^\circ$ are in general position, which is true for μ-almost all $F \in A^d_{d-m}$, cf. [69, Lemma 13.2.1]. Furthermore, let $b(J_N, F) := b^F_{J_N} := (v_1, \ldots, v_{l-m})$ be an orthonormal basis of $(\operatorname{aff}(J_N) \cap F)^\circ$. We define $y^{b^F_{J_N}} := (\langle y, v_1\rangle, \ldots, \langle y, v_{l-m}\rangle)$ as well as the affine linear mapping

$$\rho^{b^F_{J_N}} : \mathbb{R}^{l-m} \to \mathbb{R}^d, \quad s \mapsto \sigma^{b^F_{J_N}}(x) + \pi_{\operatorname{aff}(J_N)\cap F}(0),$$

where $\sigma^{b^F_{J_N}} : \mathbb{R}^{l-m} \to \mathbb{R}^d$ is given by $s \mapsto (v_1|\cdots|v_{l-m})s$ and $\pi_{\operatorname{aff}(J_N)\cap F}(0)$ is the metric projection of 0 onto $\operatorname{aff}(J_N) \cap F$. Using this mapping, we define the Gaussian field

$$X^{b^F_{J_N}} : \mathbb{R}^{l-m} \to \mathbb{R}^d, \quad s \mapsto X(\rho^{b^F_{J_N}}(s)),$$

which yields $\nabla X^{b^F_{J_N}}(s) = \nabla_{b^F_{J_N}} X(\rho^{b^F_{J_N}}(s))$ as well as $D^2 X^{b^F_{J_N}}(s) = D^2_{b^F_{J_N}} X(\rho^{b^F_{J_N}}(s))$, for $s \in \mathbb{R}^{l-m}$. Thus we obtain

$$\#\{t \in J_N \cap F \mid \nabla(X|_{J_N\cap F})(t) = y\} \leq \# \left\{ t \in J_N \cap F \mid \frac{\partial}{\partial v_i} X(t) = (y^{b^F_{J_N}})_i, i = 1, \ldots, l-m \right\}, \tag{B.1}$$

which equals

$$\# \left\{ s \in J_N^{b^F_{J_N}} \mid \frac{\partial}{\partial v_i} X(\rho^{b^F_{J_N}}(s)) = y_i^{b^F_{J_N}}, i = 1, \ldots, l-m \right\} = \# \left\{ s \in J_N^{b^F_{J_N}} \mid \nabla X^{b^F_{J_N}}(s) = y^{b^F_{J_N}} \right\},$$

where $J_N^{b^F_{J_N}} := (\rho^{b^F_{J_N}})^{-1}(J_N \cap F)$. We note that $\operatorname{diam} J_N^{b^F_{J_N}} \leq d^{1/2} N$.

In order to apply the Rice formula, cf. 2.5, we check its conditions first:

(i) $\nabla X^{b^F_{J_N}}(\cdot) = \left(\frac{\partial}{\partial v_i} X(\rho^{b^F_{J_N}}(\cdot)) \right)_{i=1}^{l-m}$ is a centered Gaussian field, since X satisfies this property.

(ii) Since X is almost surely of class $\mathcal{C}^3$ by assumption (A1), we obtain $\nabla X^{b^F_{J_N}}$ is almost surely of class $\mathcal{C}^2$.

(iii) Assumption (A1) implies $\mathrm{Cov}(\nabla X^{b^F_{J_N}}) = I_{l-m}$, which is positive definite.

(iv) This condition is satisfied by Lemma 2.7.

Then

$$\begin{aligned}
&\mathbb{E}\left[\#\left\{s \in J_N^{b^F_{J_N}} \mid \nabla X^{b^F_{J_N}}(s) = y^{b^F_{J_N}}\right\}\right] \\
&\qquad = \int_{J_N^{b^F_{J_N}}} \mathbb{E}\left[|\det D^2 X^{b^F_{J_N}}(s)| \mid \nabla X^{b^F_{J_N}}(s) = y^{b^F_{J_N}}\right] p_{\nabla X^{b^F_{J_N}}(s)}(y^{b^F_{J_N}})\, ds,
\end{aligned}$$

where $p_{\nabla X^{b^F_{J_N}}(s)}(y^{b^F_{J_N}})$ denotes the probability density of $\nabla X^{b^F_{J_N}}(s)$ at $y^{b^F_{J_N}}$. Stationarity and isotropy imply that $\left(\frac{\partial}{\partial v_i} X(t)\right)_{i=1}^{l-m} \overset{\mathcal{D}}{=} \left(\frac{\partial}{\partial t_i} X(0)\right)_{i=1}^{l-m}$, $t \in \mathbb{R}^d$, and that the first and second derivatives are independent at equal times. Thus the above equals

$$\begin{aligned}
&\int_{J_N^{b^F_{J_N}}} \mathbb{E}\left[|\det D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(s))| \mid \nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(s)) = y^{b^F_{J_N}}\right] p_{\nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(t))}(y^{b^F_{J_N}})\, ds \\
&\quad = \int_{J_N^{b^F_{J_N}}} \mathbb{E}\left[|\det D^2_{b^F_{J_N}} X(0)|\right] p_{\nabla_{b^F_{J_N}} X(0)}(y^{b^F_{J_N}})\, ds \\
&\quad = \mathbb{E}\left[|\det D^2_{b^F_{J_N}} X(0)|\right] p_{\frac{\partial}{\partial t_1} X(0) \dots \frac{\partial}{\partial t_{l-m}} X(0)}(y^{b^F_{J_N}}) \mathcal{H}^{l-m}(J_N \cap F), \qquad \text{(B.2)}
\end{aligned}$$

which can be bounded by

$$\mathbb{E}\left[|\det D^2_{b^F_{J_N}} X(0)|\right] p_{\frac{\partial}{\partial t_1} X(0) \dots \frac{\partial}{\partial t_{l-m}} X(0)}(0) \mathcal{H}^{l-m}(J_N \cap F),$$

since the density of a centered Gaussian random variable attains its maximum at 0. The density at 0 is explicitly known and the expectation of the determinant can be bounded by an application of Wick's formula independently of F, which yields the assertion for the first moment. Indeed, we observe that

$$\mathbb{E}\left[|\det D^2_{b^F_{J_N}} X(0)|\right] \leq \mathbb{E}\left[1 + \det(D^2_{b^F_{J_N}} X(0))^2\right]$$

and that by Hadamard's inequality, cf. [7, Fact 8.17.11],

$$\det(D^2_{b^F_{J_N}} X(0))^2 \leq \prod_{i=1}^{l-m} \sum_{k=1}^{l-m} \left(\frac{\partial^2}{\partial v_i \partial v_k} X(0)\right)^2 = \sum_{k_1=1}^{l-m} \dots \sum_{k_{l-m}=1}^{l-m} \prod_{i=1}^{l-m} \left(\frac{\partial^2}{\partial v_i \partial v_{k_i}} X(0)\right)^2.$$

Hence, we obtain with the definition $Y_j^k := \frac{\partial^2}{\partial v_{\lfloor (j+1)/2 \rfloor} \partial v_{k_{\lfloor (j+1)/2 \rfloor}}} X(0)$ for $j = 1, \dots, 2(l-m)$

$$\mathbb{E}\left[\det(D^2_{b^F_{J_N}} X(0))^2\right] \leq \sum_{k_1=1}^{l-m} \dots \sum_{k_{l-m}=1}^{l-m} \mathbb{E}\left[\prod_{j=1}^{2(l-m)} Y_j^k\right]$$

By Wick's formula, cf. [1, Lemma 11.6.1], this equals

$$\sum_{k_1=1}^{l-m} \cdots \sum_{k_{l-m}=1}^{l-m} \sum \mathbb{E}\left[Y_{j_1}^k Y_{j_2}^k\right] \ldots \mathbb{E}\left[Y_{j_{2(l-m)-1}}^k Y_{j_{2(l-m)}}^k\right],$$

where the inner sum is taken over the $(2(l-m))!/(2^{l-m}(l-m)!)$ possibilities of choosing $l-m$ pairs of $Y_1^k, \ldots, Y_{2(l-m)}^k$, where the order of the pairs does not matter. We conclude from $\mathbb{E}\left[Y_j^k Y_{j'}^k\right] \leq \widetilde{\psi}(0) \leq d^2\psi(0)$, cf. (A3), that

$$\mathbb{E}\left[\det(D^2_{b^F_{J_N}} X(0))^2\right] \leq c\psi^{l-m}(0),$$

where $c = c(X, d, m, l) > 0$, and therefore the expectation is finite independently of F.

However, we are interested in the second moment and therefore still need to apply the Rice formula for the second factorial of the counting variable, since $\mathbb{E}\left[X^2\right] = \mathbb{E}\left[X\right] + \mathbb{E}\left[X(X-1)\right]$, for any random variable X. We therefore check the conditions of Theorem 2.6, which differ from the previous ones only in condition (iii):

(iii) We note that $\left(\nabla X^{b^F_{J_N}}(s_1), \nabla X^{b^F_{J_N}}(s_2)\right) \overset{\mathcal{D}}{=} \left(\left(\frac{\partial}{\partial v_i} X(0)\right)_{i=1}^{l-m}, \left(\frac{\partial}{\partial v_i} X(t)\right)_{i=1}^{l-m}\right)$, where $t := \sigma^{b^F_{J_N}}(s_2 - s_1)$, and assume that this vector is degenerate, which will yield a contradiction. Then there exists a vector $c \in \mathbb{R}^{2(l-m)}$ nonzero and $\gamma \in \mathbb{R}$ such that

$$\mathbb{P}\left(\left\langle c, \left(\left(\frac{\partial}{\partial v_i} X(0)\right)_{i=1}^{l-m}, \left(\frac{\partial}{\partial v_i} X(t)\right)_{i=1}^{l-m}\right)\right\rangle = \gamma\right) = 1.$$

Moreover $\left(\frac{\partial}{\partial v_i} X(0)\right)_{i=1}^{l-m} = (v_1|\cdots|v_{l-m})^\top \nabla X(0)$ implies

$$\left\langle c, \left(\left(\frac{\partial}{\partial v_i} X(0)\right)_{i=1}^{l-m}, \left(\frac{\partial}{\partial v_i} X(t)\right)_{i=1}^{l-m}\right)\right\rangle = \left\langle c, \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} (\nabla X(0), \nabla X(t))\right\rangle,$$

where $A := (v_1|\cdots|v_{l-m})^\top$. Therefore, $c^\top (v_1|\cdots|v_{l-m})^\top = ((v_1|\cdots|v_{l-m})c)^\top \neq 0$, since $v_1, \ldots, v_{l-m}$ is a basis, which implies that there exists $c_1 \in \mathbb{R}^{2d}$ nonzero such that $\mathbb{P}(\langle c_1, (\nabla X(0), \nabla X(t))\rangle = \gamma) = 1$, a contradiction to assumption (A2). Thus $(\nabla X^{b^F_{J_N}}(s_1), \nabla X^{b^F_{J_N}}(s_2))$ is nondegenerate for $s_1 \neq s_2 \in \mathbb{R}^{l-m}$.

We therefore obtain

$$\begin{aligned}
&\mathbb{E}\left[\#\{s \in J_N^{b^F_{J_N}} \mid \nabla X^{b^F_{J_N}}(s) = y^{b^F_{J_N}}\} \left(\#\{s \in J_N^{b^F_{J_N}} \mid \nabla X^{b^F_{J_N}}(s) = y^{b^F_{J_N}}\} - 1\right)\right] \\
&= \int_{J_N^{b^F_{J_N}}} \int_{J_N^{b^F_{J_N}}} \mathbb{E}\left[|\det D^2 X^{b^F_{J_N}}(s_1) \det D^2 X^{b^F_{J_N}}(s_2)| \mid \nabla X^{b^F_{J_N}}(s_1) = \nabla X^{b^F_{J_N}}(s_2) = y^{b^F_{J_N}}\right] \\
&\qquad \times p_{\nabla X^{b^F_{J_N}}(s_1), \nabla X^{b^F_{J_N}}(s_2)}(y^{b^F_{J_N}}, y^{b^F_{J_N}})\, ds_1\, ds_2,
\end{aligned}$$

which equals

$$\int_{J_N^{b^F_{J_N}}} \int_{J_N^{b^F_{J_N}}} p_{\nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(s_1)),\nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(s_2))}(y^{b^F_{J_N}}, y^{b^F_{J_N}})$$
$$\times \mathbb{E}\left[|\det D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(s_1)) \det D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(s_2))| \mid \mathcal{E}(F, J_N, s_1, s_2, y)\right] ds_1\, ds_2,$$

where $p_{\nabla X^{b^F_{J_N}}(s_1),\nabla X^{b^F_{J_N}}(s_2)}(y^{b^F_{J_N}}, y^{b^F_{J_N}})$ denotes the density of the $2(l-m)$-dimensional Gaussian vector $(\nabla X^{b^F_{J_N}}(s_1), \nabla X^{b^F_{J_N}}(s_2))$ evaluated at the point $(y^{b^F_{J_N}}, y^{b^F_{J_N}})$ and where

$$\mathcal{E}(F, J_N, s, t, y) := \{\nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(s)) = \nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(t)) = y^{b^F_{J_N}}\}.$$

By stationarity and Fubini's theorem the above equals

$$\int_{J_N^{b^F_{J_N}} - J_N^{b^F_{J_N}}} p_{\nabla_{b^F_{J_N}} X(\rho^{b^F_{J_N}}(s)),\nabla_{b^F_{J_N}} X(\rho^{b^F_{J_N}}(0))}(y^{b^F_{J_N}}, y^{b^F_{J_N}})\mathcal{H}^{l-m}(J_N^{b^F_{J_N}} \cap (J_N^{b^F_{J_N}} - s))$$
$$\times \mathbb{E}\left[|\det D^2_{b^F_{J_N}} X(\rho^{b^F_{J_N}}(s)) \det D^2_{b^F_{J_N}} X(\rho^{b^F_{J_N}}(0))| \mid \mathcal{E}(F, J_N, s, 0, y)\right] ds. \tag{B.3}$$

We note that we can close the domain of integration, which leads to a compact set contained in $B^{l-m}_{2d^{1/2}N}$. This helps to exploit continuity arguments, when seeking bounds for the integrands. This and more is done in Lemmas B.2 and B.3, which provide an integrable upper bound for the integrand. The constants, appearing in these lemmata, are independent of $F \in A^d_{d-m}$ and we therefore obtain the assertion.

We now prove **part (ii)** of the assertion and start by mentioning that for $y \in (\operatorname{aff}(J_N)\cap F)^\circ \cap G$ the inequality (B.1) is an actual equality and we deduce by equation (B.2) that the first moment $\mathbb{E}[\#\{t \in J_N \cap F \colon \nabla(X|_{J_N\cap F})(t) = y\}]$ is continuous in y, since the density of a normal distribution is continuous. Thus it remains to establish the continuity of the second factorial moment, which by an application of the Rice formula equals, cf. (B.3),

$$\varphi(F, y) := \int_{J_N^{b^F_{J_N}} - J_N^{b^F_{J_N}}} G(F, s, y)\mathcal{H}^{l-m}\left(J_N^{b^F_{J_N}} \cap (J_N^{b^F_{J_N}} - s)\right) ds,$$

where

$$G(F, s, y) := \mathbb{E}\left[|\det D^2_{b^F_{J_N}} X(\rho^{b^F_{J_N}}(s)) \det D^2_{b^F_{J_N}} X(\rho^{b^F_{J_N}}(0))| \mid \mathcal{E}(F, J_N, s, 0, y)\right]$$
$$\times p_{\nabla_{b^F_{J_N}} X(\rho^{b^F_{J_N}}(s)),\nabla_{b^F_{J_N}} X(\rho^{b^F_{J_N}}(0))}(y^{b^F_{J_N}}, y^{b^F_{J_N}}).$$

By Lemma B.2 and Lemma B.3, we obtain for $y, y' \in (\operatorname{aff}(J_N) \cap F)^\circ \cap G$ by the triangle inequality

$$\begin{aligned}&|G(F,s,y)-G(F,s,y')|\\&\le 2c\left(\mathbb{1}\{s\in U\}\|s\|^{-(l-m)}+\mathbb{1}\{s\in B^{l-m}_{2d^{1/2}N}\setminus U\}\right)\|s\|^2\sup_{y\in G}\left(1+\|y\|^{4(l-m-1)}\right)^{\frac12}\left(1+\|y\|^4\right)^{\frac12},\end{aligned}$$

which is an integrable upper bound in s independent of y' and y. Thus by dominated convergence

$$\begin{aligned}&\lim_{y\to y'}|\varphi(F,y)-\varphi(F,y')|\\&\quad=\int_{J_N^{b^F_{J_N}}-J_N^{b^F_{J_N}}}\lim_{y\to y'}|G(F,s,y)-G(F,s,y')|\mathcal{H}^{l-m}(J_N^{b^F_{J_N}}\cap(J_N^{b^F_{J_N}}-s))\,ds.\end{aligned}$$

Therefore it remains to show the continuity of $G(F,s,\cdot)$ for all $y\in(\operatorname{aff}(J_N)\cap F)^\circ\cap G$. We note that the second factor in the definition of $G(F,s,\cdot)$ is continuous in y, since y enters this term as the argument of a Gaussian density, which is continuous. Hence all that is left, is to establish the continuity in y of the conditional expectation in the definition of $G(F,s,\cdot)$. We abbreviate

$$N_s:=\left(\frac{\partial^2}{\partial v_i\partial v_j}X(\rho^{b_N^{J_N}}(s))\right)_{1\le i\le j\le l-m}\in\mathbb{R}^{(l-m)(l-m+1)/2}$$

and

$$M_s:=\left(\nabla_{b_N^{J_N}}X(\rho^{b_N^{J_N}}(s)),\nabla_{b_N^{J_N}}X(\rho^{b_N^{J_N}}(0))\right)\in\mathbb{R}^{2(l-m)}.$$

Then by Gaussian regression, cf. [4, Proposition 1.2],

$$\begin{aligned}&\mathbb{E}\left[|\det m(N_s)\det m(N_0)|\mid M_s=(y,y)\right]\\&\quad=\mathbb{E}\Big[\Big|\det m\left(N_s-\operatorname{Cov}(N_s,M_s)\operatorname{Cov}(M_s)^{-1}(M_s-(y,y))\right)\\&\qquad\times\det m\left(N_0-\operatorname{Cov}(N_0,M_s)\operatorname{Cov}(M_s)^{-1}(M_s-(y,y))\right)\Big|\Big],\end{aligned}$$

where $\mathrm{m}\colon\mathbb{R}^{(l-m)(l-m+1)/2}\to\mathbb{R}^{(l-m)\times(l-m)}$ is the mapping that maps the upper half of a matrix, given in form of a vector, to the matrix itself. Then, with the abbreviations defined by $A(s_1,s_2):=N_{s_1}-\operatorname{Cov}(N_{s_1},M_{s_2})\operatorname{Cov}(M_{s_2})^{-1}M_{s_2}$ and $B(s_1,s_2):=\operatorname{Cov}(N_{s_1},M_{s_2})\operatorname{Cov}(M_{s_2})^{-1}$, this expectation equals

$$\begin{aligned}&\mathbb{E}\Bigg[\Bigg|\det m\left(\left(A(s,s)_\alpha-\sum_{i=1}^{l-m}(B(s,s)_{\alpha,i}+B(s,s)_{\alpha+l-m,i})y_i\right)_{\alpha=1}^{(l-m)(l-m+1)/2}\right)\\&\qquad\times\det m\left(\left(A(0,s)_\alpha-\sum_{i=1}^{l-m}(B(0,s)_{\alpha,i}+B(0,s)_{\alpha+l-m,i})y_i\right)_{\alpha=1}^{(l-m)(l-m+1)/2}\right)\Bigg|\Bigg].\end{aligned}$$

By the Leibniz formula for determinants, that is $\det A=\sum_{\sigma\in S_d}\operatorname{sgn}(\sigma)\prod_{i=1^d}a_{\sigma(i),i}$ for $A\in$

$\mathbb{R}^{d\times d}$, we deduce that the argument of this expectation is given by the absolute value of a multivariate polynomial in the components of y with random coefficients and as such it is continuous in y. Indeed, for a polynomial with random coefficients $P(y_1,\dots,y_{l-m}) = \sum_{i_1,\dots,i_{l-m}=1}^{2(l-m)} Z_{i_1,\dots,i_{l-m}} y_1^{i_1}\cdots y_{l-m}^{i_{l-m}}$, we obtain

$$\begin{aligned}\left|\mathbb{E}\left[|P(y)|\right]-\mathbb{E}\left[|P(y')|\right]\right| &\le \mathbb{E}\left[\left||P(y)|-|P(y')|\right|\right] \le \mathbb{E}\left[|P(y)-P(y')|\right]\\ &\le \sum_{i_1,\dots,i_{l-m}=1}^{2(l-m)} \mathbb{E}\left[|Z_{i_1,\dots,i_{l-m}}|\right]\left|y_1^{i_1}\cdots y_{l-m}^{i_{l-m}}-(y')_1^{i_1}\cdots(y')_{l-m}^{i_{l-m}}\right|,\end{aligned}$$

which vanishes in the limit $y'\to y$ if $\mathbb{E}\left[|Z_{i_1,\dots,i_{l-m}}|\right]<\infty$, which is the case since the underlying random variables are Gaussian. Thus, we conclude assertion (ii).

In order to prove the remaining assertion, namely **point (iii)**, we first prove

$$\int_{\operatorname{int} C_N^d\cap F} \delta_\varepsilon^d(\nabla(X|_F)(t))|\det D^2(X|_F)(t)|\,\mathcal{H}^{d-m}(dt) \overset{L^2(\mathbb{P})}{\longrightarrow} \#\{t\in\operatorname{int} C_N^d\cap F \mid \nabla(X|_F)(t)=0\},$$

as $\varepsilon\to 0$. The first step is to note that the same proof as the one of Lemma A.3 yields the convergence in the almost sure sense. For fixed $F\in A_{d-m}^d$ the conditions of Lemma A.3 are verified in [1, Lemma 11.2.10 - 11.2.12]. Thus by Fatou's lemma

$$\begin{aligned}&\mathbb{E}\left[\#\{t\in\operatorname{int} C_N^d\cap F \mid \nabla(X|_F)(t)=0\}^2\right]\\ &\qquad\le \liminf_{\varepsilon\to0}\mathbb{E}\left[\left(\int_{\operatorname{int} C_N^d\cap F}\delta_\varepsilon^d(\nabla(X|_F)(t))|\det(D^2(X|_F)(t))|\,\mathcal{H}^{d-m}(dt)\right)^2\right]\\ &\qquad\le \limsup_{\varepsilon\to0}\mathbb{E}\left[\left(\int_{\operatorname{int} C_N^d\cap F}\delta_\varepsilon^d(\nabla(X|_F)(t))|\det(D^2(X|_F)(t))|\,\mathcal{H}^{d-m}(dt)\right)^2\right].\end{aligned} \tag{B.4}$$

An application of the coarea formula, cf. [25, Theorem 3.2.12], yields

$$\begin{aligned}&\int_{\operatorname{int} C_N^d\cap F}\delta_\varepsilon^d(\nabla(X|_F)(t))|\det(D^2(X|_F)(t))|\,\mathcal{H}^{d-m}(dt)\\ &\qquad=\int_{F^\circ}\#\{t\in\operatorname{int} C_N^d\cap F\mid\nabla(X|_F)(t)=y\}\delta_\varepsilon^d(y)\,\mathcal{H}^{d-m}(dy),\end{aligned}$$

which leads to the upper bound for the term in (B.4)

$$\begin{aligned}&\limsup_{\varepsilon\to0}\mathbb{E}\left[\left(\int_{F^\circ}\#\{t\in\operatorname{int} C_N^d\cap F\mid\nabla(X|_F)(t)=y\}\delta_\varepsilon^d(y)\,\mathcal{H}^{d-m}(dy)\right)^2\right]\\ &\qquad\le\limsup_{\varepsilon\to0}\mathbb{E}\left[\int_{F^\circ}\#\{t\in\operatorname{int} C_N^d\cap F\mid\nabla(X|_F)(t)=y\}^2\delta_\varepsilon^d(y)\,\mathcal{H}^{d-m}(dt)\right]\end{aligned}$$

where we used Jensen's inequality for the measure $\delta_\varepsilon^d(y)dy$. Then the already proven assertion

(ii) yields

$$\limsup_{\varepsilon\to 0} \int_{F^\circ} \mathbb{E}\left[\#\{t \in \operatorname{int} C_N^d \cap F \mid \nabla(X|_F)(t) = y\}^2\right] \delta_\varepsilon^d(y)\, \mathcal{H}^{d-m}(dy)$$
$$= \mathbb{E}\left[\#\{t \in \operatorname{int} C_N^d \cap F \mid \nabla(X|_F)(t) = 0\}^2\right],$$

This establishes the $L^2(\mathbb{P})$ convergence of the simplified counting variable and its approximation. Together with the fact that

$$|\xi_N(F,\varepsilon)| \le \int_{\operatorname{int} C_N^d \cap F} \delta_\varepsilon^d(\nabla(X|_F)(t))|\det(D^2(X|_F)(t))|\, \mathcal{H}^{d-m}(dt)$$

and Lemma A.3, whose assumptions are again checked in [1, Section 11.2], we conclude the Lemma by [19, VI.§5 5.3 Satz]. □

In the following lemma we calculate a basic determinate, which is frequently used in the remaining part of the appendix.

Lemma B.1. *For $c_1, c_2 \in \mathbb{R}$ and $v \in \mathbb{R}^d$ define the matrix $A := c_1 I_d + c_2 vv^\top$. Then*

$$\det A = c_1^d + c_1^{d-1} c_2 \|v\|^2.$$

Proof. Note that for $c \in \mathbb{R}$ and $u \in v^\perp$, we obtain

$$(I_d + cvv^\top)v = (1 + c\|v\|^2)v \quad \text{and} \quad (I_d + cvv^\top)u = u.$$

Thus the linear mapping associated with $I_d + cvv^\top$ has the eigenvalues $1 + c\|v\|^2, 1, \ldots, 1$, yielding

$$\det(I_d + cvv^\top) = 1 + c\|v\|^2.$$

Hence by choosing $c := c_2/c_1$ — for $c_1 = 0$ the lemma holds trivially — we obtain

$$\det(A) = c_1^d \det(I_d + cvv^\top) = c_1^d + c_1^{d-1} c_2 \|v\|^2. \qquad \square$$

The next lemma establishes upper bounds for the density of a normal distribution.

Lemma B.2. *Let $J_N \in \partial_l C_N^d$ and $l > m$. Then there exist an open neighborhood $U \subset \mathbb{R}^{l-m}$ of 0 and constants $c_1 = c_1(X,d,m,l,N) \ge 0$, $c_2 = c_2(X,d,m,l,N) \ge 0$, such that for almost all $F \in A_{d-m}^d$, $y \in \mathbb{R}^d$ and $s \in U \setminus \{0\}$*

$$p_{\nabla_{b_{J_N}^F}(X)(\rho^{b_{J_N}^F}(s))\nabla_{b_{J_N}^F}(X)(\rho^{b_{J_N}^F}(0))}(y^{b_{J_N}^F}, y^{b_{J_N}^F}) \le c_1 \|s\|^{-(l-m)}$$

and moreover for $s \in B_{2d^{1/2}N}^{l-m} \setminus U$

$$p_{\nabla_{b_{J_N}^F}(X)(\rho^{b_{J_N}^F}(s))\nabla_{b_{J_N}^F}(X)(\rho^{b_{J_N}^F}(0))}(y^{b_{J_N}^F}, y^{b_{J_N}^F}) \le c_2.$$

Proof. We first note that for $s \in \mathbb{R}^{l-m}$ nonzero

$$p_{\nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(s))\nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(0))}(y^{b^F_{J_N}}, y^{b^F_{J_N}}) \leq p_{\nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(s))\nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(0))}(0,0),$$

since the random vector $(\nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(s)), \nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(0)))$ follows a centered normal distribution, whose density attains the global maximum at 0. The right side is explicitly given by

$$(2\pi)^{-(l-m)} \det \operatorname{Cov}(\nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(s)), \nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(0)))^{-\frac{1}{2}}.$$

Therefore a detailed analysis of the covariance matrix is necessary. It is given by the matrix

$$\begin{pmatrix} I_{l-m} & \left(-\frac{\partial^2}{\partial v_i \partial v_j} C^X(\sigma^{b^F_{J_N}}(s))\right)_{i,j}^{l-m} \\ \left(-\frac{\partial^2}{\partial v_i \partial v_j} C^X(\sigma^{b^F_{J_N}}(s))\right)_{i,j}^{l-m} & I_{l-m} \end{pmatrix},$$

due to the imposed conditions in (A1) and the fact $\mathbb{E}\left[\frac{\partial}{\partial u} X(t) \frac{\partial}{\partial u'} X(t')\right] = -\frac{\partial^2}{\partial u \partial u'} C^X(t-t')$, for $u, u' \in S^{d-1}$ and $t, t' \in \mathbb{R}^d$. Using [7, 2.8.4], the determinant of this matrix equals

$$\det\left(I_{l-m} - \left(\frac{\partial^2}{\partial v_i \partial v_j}(C^X)(\sigma^{b^F_{J_N}}(s))\right)^2_{i,j=1,\dots,l-m}\right). \tag{B.5}$$

Furthermore, the stationarity and isotropy, assumed in (A1), imply that all information of the covariance of X can be captured by the mapping $R\colon [0,\infty) \to \mathbb{R}$ given by $r \mapsto C^X(re_1)$ of class $\mathcal{C}^6$, i.e. we have the equality $C^X(t) = R(\|t\|)$, $t \in \mathbb{R}^d$. Differentiating this identity yields for $t \in \mathbb{R}^d \setminus \{0\}$

$$D^2 C^X(t) = R'(\|t\|)\|t\|^{-1} I_d + (R''(\|t\|)\|t\|^{-2} - R'(\|t\|)\|t\|^{-3})(t_i t_j)_{i,j=1}^d$$

and we obtain

$$\begin{aligned} D^2_{b^F_{J_N}} C^X(t) &= \begin{pmatrix} v_1| & \cdots & |v_{l-m}\end{pmatrix}^\top D^2 C^X(t) \begin{pmatrix} v_1| & \cdots & |v_{l-m}\end{pmatrix} \\ &= R'(\|t\|)\|t\|^{-1} I_{l-m} + (R''(\|t\|)\|t\|^{-2} - R'(\|t\|)\|t\|^{-3})(\langle v_i, t\rangle \langle v_j, t\rangle)_{i,j=1}^{l-m}. \end{aligned}$$

Thus for $s \in \mathbb{R}^{l-m} \setminus \{0\}$ this implies

$$D^2_{b^{J_N}_F}(C^X)(\sigma^{b^{J_N}_F}(s)) = R'(\|s\|)\|s\|^{-1} I_{l-m} + (R''(\|s\|)\|s\|^{-2} - R'(\|s\|)\|s\|^{-3})(s_i s_j)_{i,j=1}^{l-m}. \tag{B.6}$$

Note that the right side is independent of the specific choice of affine space $\operatorname{aff}(J_N) \cap F$ as a result of the translation invariance and rotational invariance of C^X. Moreover, we note that

$(ss^\top)^2 = \|s\|^2 ss^\top$, and therefore

$$\begin{aligned}
&\left(D^2_{b_F^{J_N}}(C^X)(\sigma^{b_F^{J_N}}(s))\right)^2 \\
&\qquad = (R'(\|s\|)\|s\|^{-1})^2 I_{l-m} + (R''(\|s\|)\|s\|^{-2} - R'(\|s\|)\|s\|^{-3}) \\
&\qquad\quad \times (2R'(\|s\|)\|s\|^{-1} + \|s\|^2(R''(\|s\|)\|s\|^{-2} - R'(\|s\|)\|s\|^{-3}))(s_i s_j)_{i,j=1}^{l-m} \\
&\qquad = (R'(\|s\|)\|s\|^{-1})^2 I_{l-m} + \|s\|^{-2}(R''(\|s\|)^2 - (R'(\|s\|)\|s\|^{-1})^2)(s_i s_j)_{i,j=1}^{l-m}. \qquad \text{(B.7)}
\end{aligned}$$

Hence the determinant in (B.5) is independent of $F \in A^d_{d-m}$ and continuous in $s \in \mathbb{R}^{l-m} \setminus \{0\}$ and can therefore be bounded independently of F and y for $s \in B^{l-m}_{2d^{\frac{1}{2}}N} \setminus U$, where $U \subset \mathbb{R}^{l-m}$ is an open set containing the origin. We continue with the proof of the asserted estimate in a neighborhood of 0. First, by Taylor's theorem, we obtain the following expansions up to the fifth derivative

$$R'(r) = \sum_{k=0}^{4} \frac{R^{(k+1)}(0)}{k!} r^k + o(r^4) \quad \text{and} \quad R''(r) = \sum_{k=0}^{3} \frac{R^{(k+2)}(0)}{k!} r^k + o(r^3),$$

for $r \to 0$. Note that due to assumption (A1), namely stationarity and the normalisation of the second derivatives of C^X, we obtain $R''(0) = -1$ and odd derivatives of R vanish at 0 due to the stationarity of X, cf. the discussion following (2.8). We therefore obtain

$$R'(\|s\|) = -\|s\| + \frac{\mu}{3!}\|s\|^3 + o(\|s\|^4) \quad \text{and} \quad R''(\|s\|) = -1 + \frac{\mu}{2}\|s\|^2 + o(\|s\|^3) \qquad \text{(B.8)}$$

for $\|s\| \to 0$, where $\mu := \mathbb{E}\left[\frac{\partial^2}{\partial t_1 \partial t_1} X(0)^2\right] > 0$ by (A2). We calculate for $s \neq 0$ and $\|s\| \to 0$

$$\left(R'(\|s\|)\|s\|^{-1}\right)^2 = 1 - \frac{\mu}{3}\|s\|^2 + o(\|s\|^3)$$

and

$$\|s\|^{-2}\left(R''(\|s\|)^2 - \left(R'(\|s\|)\|s\|^{-1}\right)^2\right) s_i s_j = -\frac{2\mu}{3} s_i s_j + o(\|s\|^3)$$

where $i, j = 1, \dots, l-m$, which yields with (B.7)

$$I_{l-m} - \left(D^2_{b^F_{J_N}}(C^X)(\sigma^{b^F_{J_N}}(s))\right)^2 = \frac{\mu}{3}\|s\|^2 I_{l-m} + \frac{2}{3}\mu(s_i s_j)_{i,j=1}^{l-m} + o(\|s\|^3). \qquad \text{(B.9)}$$

Then the multilinearity of the determinant implies

$$\begin{aligned}
&\det\left(I_{l-m} - \left(D^2_{b^F_{J_N}}(C^X)(\sigma^{b^F_{J_N}}(s))\right)^2\right) \\
&\qquad = \|s\|^{2(l-m)}\left(\det\left(\frac{\mu}{3} I_{l-m} + \frac{2\mu}{3\|s\|^2}(s_i s_j)_{i,j=1}^{l-m}\right) + o(\|s\|)\right).
\end{aligned}$$

Hence we conclude with Lemma B.1

$$\det\left(I_{l-m} - \left(D^2_{b_F^{J_N}}(C^X)(\sigma^{b_F^{J_N}}(s))\right)^2\right) = 3\left(\frac{\mu}{3}\right)^{l-m} \|s\|^{2(l-m)} + o(\|s\|^{2(l-m)+1}), \tag{B.10}$$

for $\|s\| \to 0$ and uniformly in F. If we choose a constant $c > 0$ such that $\frac{3}{c}\left(\frac{\mu}{3}\right)^{l-m} > 1$, then equation B.10 implies

$$\begin{aligned}\frac{\det\left(I_{l-m} - \left(D^2_{b_F^{J_N}}(C^X)(\sigma^{b_F^{J_N}}(s))\right)^2\right)}{c\|s\|^{2(l-m)}} &= \frac{3}{c}\left(\frac{\mu}{3}\right)^{l-m} + \frac{\|s\|}{c}\frac{o(\|s\|^{2(l-m)+1})}{\|s\|^{2(l-m)+1}} \\ &\xrightarrow{\|s\|\to 0} \frac{3}{c}\left(\frac{\mu}{3}\right)^{l-m} > 1.\end{aligned}$$

Therefore, by continuity we find a neighborhood $U \subset \mathbb{R}^{l-m}$ of 0 such that for $s \in U$

$$\det\left(I_{l-m} - \left(D^2_{b_F^{J_N}}(C^X)(\sigma^{b_F^{J_N}}(s))\right)^2\right) \geq c\|s\|^{2(l-m)}.$$

From this estimate, the asserted bound

$$\begin{aligned} p_{\nabla_{b_F^{J_N}}(X)(\rho^F_{b_F^{J_N}}(s))\nabla_{b_F^{J_N}}(X)(\rho^F_{b_F^{J_N}}(0))}(0,0) &= (2\pi)^{-(l-m)}\det\left(I_{l-m} - \left(D^2_{b_F^{J_N}}(C^X)(\sigma^{b_F^{J_N}}(s))\right)^2\right)^{-\frac{1}{2}} \\ &\leq c_1\|s\|^{-(l-m)}, \end{aligned} \tag{B.11}$$

where $c_1 > 0$, follows. □

Lemma B.3. *Let $J_N \in \partial_l C_N^d$ and $l > m$. Then there is a constant $c = c(X, d, m, l, N) > 0$ such that for $s \in B^{l-m}_{2d^{1/2}N}$, almost all $F \in A^d_{d-m}$, where $F \cap J_N \neq \emptyset$, and $y \in \mathbb{R}^d$*

$$\begin{aligned}&\mathbb{E}\left[|\det D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(s)) \det D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(0))| \mid \mathcal{E}(F, J_N, s, y)\right] \\ &\quad \leq c\|s\|^2(1 + \|y\|^{4(l-m-1)})^{\frac{1}{2}}(1 + \|y\|^4)^{\frac{1}{2}},\end{aligned}$$

where $\mathcal{E}(F, J_N, s, y) := \{\nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(s)) = \nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(0)) = y^{b^F_{J_N}}\}$.

Proof. We start with an application of the Cauchy–Schwarz inequality for conditional expectations, cf. [64, Theorem 2.2.4], to obtain for $s \in \mathbb{R}^{l-m}$

$$\begin{aligned}&\mathbb{E}\left[|\det D^2_{b_F^{J_N}}(X)(\rho^{b^F_{J_N}}(s)) \det D^2_{b_F^{J_N}}(X)(\rho^{b^F_{J_N}}(0))| \mid \mathcal{E}(F, J_N, s, y)\right] \\ &\leq \mathbb{E}\left[\det\left(D^2_{b_F^{J_N}}(X)(\rho^{b^F_{J_N}}(s))\right)^2 \mid \mathcal{E}(F, J_N, s, y)\right]^{\frac{1}{2}} \\ &\quad \times \mathbb{E}\left[\det\left(D^2_{b_F^{J_N}}(X)(\rho^{b^F_{J_N}}(0))\right)^2 \mid \mathcal{E}(F, J_N, s, y)\right]^{\frac{1}{2}}\end{aligned}$$

which equals by stationarity

$$\mathbb{E}\left[\det\left(D^2_{b_F^{J_N}}(X)(\rho^{b^F_{J_N}}(0))\right)^2 \mid \mathcal{E}(F, J_N, -s, y)\right]^{\frac{1}{2}}$$
$$\times\, \mathbb{E}\left[\det\left(D^2_{b_F^{J_N}}(X)(\rho^{b^F_{J_N}}(0))\right)^2 \mid \mathcal{E}(F, J_N, s, y)\right]^{\frac{1}{2}}.$$

In the following we bound the right factor by a bound solely depending on the norm of s, hence giving a bound for the left one as well.

We first use Hadamard's inequality, cf. [7, Fact 8.17.11], which reads: For a symmetric and positive semidefinite matrix $A \in \mathbb{R}^{(l-m)\times(l-m)}$ and an orthonormal basis $(u_1, \dots, u_{l-m})$ of $\mathbb{R}^{l-m}$, we have that

$$\det(A) \leq \prod_{i=1}^{l-m} \langle Au_i, u_i\rangle.$$

Note that the square of a symmetric matrix is symmetric and positive semidefinite and we therefore obtain for $s \in \mathbb{R}^{l-m} \setminus \{0\}$ and a suitable choice of $(u_2, \dots, u_{l-m})$

$$\begin{aligned}
&\det\left(D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(0)\right)^2 \\
&\leq \left\langle \left(D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(0))\right)^2 \frac{s}{\|s\|}, \frac{s}{\|s\|}\right\rangle \prod_{i=2}^{l-m} \left\langle \left(D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(0))\right)^2 u_i, u_i\right\rangle \\
&\leq \|s\|^{-2} \|D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(0))s\|^2 \|D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(0))\|^{2(l-m-1)},
\end{aligned} \tag{B.12}$$

where we used the symmetry of $D^2_{b^F_{J_N}}(X)(\rho_{b^F_{J_N}}(0))$, the Cauchy–Schwarz inequality and a matrix norm that is compatible with the Euclidean norm and submultiplicative, e.g. the induced Euclidean norm. We now define the family of mappings

$$Y_s\colon [0,1] \to \mathbb{R}^{l-m}, \quad z \mapsto \nabla_{b_F^{J_N}}(X)(\rho^{b_F^{J_N}}(zs)), \quad \text{for } s \in \mathbb{R}^{l-m},$$

to obtain $Y_s(0) = \nabla_{b_F^{J_N}}(X)(\rho^{b_F^{J_N}}(0))$, $Y_s(1) = \nabla_{b_F^{J_N}} X(\rho^{b_F^{J_N}}(s))$ and

$$\frac{\partial}{\partial z} Y_s(z) = D^2_{b_F^{J_N}}(X)(\rho^{b_F^{J_N}}(zs))s,$$

thus $\frac{\partial}{\partial z} Y_s(0) = D^2_{b_F^{J_N}}(X)(\rho^{b_F^{J_N}}(0))s$. Calculating the second derivative of Y_s yields for the j-th component of $\frac{\partial^2}{\partial z \partial z} Y_s$, $j = 1, \dots, l-m$,

$$\frac{\partial^2}{\partial z \partial z} Y_{s,j}(z) = \sum_{i=1}^{d} \frac{\partial}{\partial z} \frac{\partial^2}{\partial v_j \partial v_i}(X)(\rho^{b_F^{J_N}}(zs))s_i = \left\langle \frac{\partial}{\partial v_j} D^2_{b_F^{J_N}}(X)(\rho^{b_F^{J_N}}(zs))s, s\right\rangle,$$

where $\frac{\partial}{\partial v_j} D^2_{b_F^{J_N}} X := \left(\frac{\partial}{\partial v_j}\frac{\partial^2}{\partial v_\alpha \partial v_\beta} X\right)_{\alpha,\beta=1}^{l-m}$. Using Taylor's theorem for the mapping Y at 0 and evaluating the expansion at 1, yields

$$Y_s(1) = Y_s(0) + \frac{\partial}{\partial z} Y_s(0) + \frac{1}{2}\left(\frac{\partial^2}{\partial z \partial z} Y_{s,1}(\xi_1), \ldots, \frac{\partial^2}{\partial z \partial z} Y_{s,l-m}(\xi_{l-m})\right),$$

for suitable points $\xi_1, \ldots, \xi_{l-m} \in [0,1]$. Conditioning of this equation on the event

$$\mathcal{E}(F, J_N, s, y) = \left\{\nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(s)) = \nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(0)) = y^{b^F_{J_N}}\right\}$$

leads to

$$D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(0))s = -\frac{1}{2}\left(\left\langle \frac{\partial}{\partial v_i} D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(\xi_i s))s, s\right\rangle\right)_{i=1}^{l-m}.$$

By taking norms, an application of the Cauchy–Schwarz inequality in every component and the compatibility of the matrix norm, we obtain conditioned on $\mathcal{E}(F, J_N, s, y)$

$$\begin{aligned}\left\|D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(0))s\right\|^2 &\leq \frac{1}{4}\|s\|^4 \sum_{i=1}^{l-m} \left\|\frac{\partial}{\partial v_i} D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(\xi_i s))\right\|^2 \\ &\leq \frac{1}{4}\|s\|^4 \sum_{i=1}^{l-m} \sup_{z\in[0,1]} \left\|\frac{\partial}{\partial v_i} D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(zs))\right\|^2.\end{aligned}$$

We note that the last term, conditioned on $\mathcal{E}(F, J_N, s, y)$, is almost surely finite by Lemma B.4. Hence, we conclude with (B.12)

$$\begin{aligned}&\mathbb{E}\left[\det\left(D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(0))\right)^2 \mid \mathcal{E}(F, J_N, s, y)\right] \\ &\leq c\|s\|^{-2}\mathbb{E}\left[\left\|D^2_{b^F_{J_N}} X(\rho^{b^F_{J_N}}(0))s\right\|^2 \left\|D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(0))\right\|^{2(l-m-1)} \mid \mathcal{E}(F, J_N, s, y)\right] \\ &\leq c\|s\|^2 \sum_{i=1}^{l-m} \mathbb{E}\left[\sup_{z\in[0,1]} \left\|\frac{\partial}{\partial v_i} D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(zs))\right\|^2 \left\|D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(0))\right\|^{2(l-m-1)} \mid \mathcal{E}(F, J_N, s, y)\right]\end{aligned}$$

and bound this term with the aid of the Cauchy–Schwarz inequality by

$$\begin{aligned}&c\|s\|^2\mathbb{E}\left[\left\|D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(0))\right\|^{4(l-m-1)} \mid \mathcal{E}(F, J_N, s, y)\right]^{\frac{1}{2}} \\ &\quad\times \sum_{i=1}^{l-m} \mathbb{E}\left[\sup_{z\in[0,1]} \left\|\frac{\partial}{\partial v_i} D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(zs))\right\|^4 \mid \mathcal{E}(F, J_N, s, y)\right]^{\frac{1}{2}}. \qquad \text{(B.13)}\end{aligned}$$

Invoking the Lemmata B.5 and B.4 concludes the proof. □

In the subsequent lemma we use the technique called Gaussian regression to derive an upper bound for the second factor in the expression in (B.13).

Lemma B.4. *Let $J_N \in \partial_l C^d_N$ and $l > m$. Then there is a constant $c = c(X, d, m, l, N) > 0$, such that for $s \in B^{l-m}_{2d^{1/2}N}$ nonzero, $i \in \{1, \ldots, l-m\}$, almost all $F \in A^d_{d-m}$, where $F \cap J_N \neq \emptyset$,* *and* $y \in \mathbb{R}^d$

$$\mathbb{E}\left[\sup_{z\in[0,1]}\left\|\frac{\partial}{\partial v_i}D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(zs))\right\|^4 \mid \mathcal{E}(F,J_N,s,y)\right] \le c(1+\|y\|^4),$$

where $\mathcal{E}(F,J_N,t,y) := \{\nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(s)) = \nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(0)) = y^{b^F_{J_N}}\}$.

Proof. We use for the matrix norm the Frobenius norm, that is

$$\sup_{z\in[0,1]}\left\|\frac{\partial}{\partial v_i}D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(zs))\right\|^4 = \sup_{z\in[0,1]}\left(\sum_{\alpha,\beta=1}^{l-m}\left(\frac{\partial^3}{\partial v_i\partial v_\alpha\partial v_\beta}(X)(\rho^{b^F_{J_N}}(zs))\right)^2\right)^2$$

and bound this by Jensen's inequality by

$$(l-m)\sup_{z\in[0,1]}\sum_{\alpha,\beta=1}^{l-m}\left(\frac{\partial^3}{\partial v_i\partial v_\alpha\partial v_\beta}(X)(\rho^{b^F_{J_N}}(zs))\right)^4.$$

The fact that $b^F_{J_N}$ is an orthonormal basis and Jensen's inequality imply the upper bound

$$(l-m)^3 d^9\sum_{\alpha,\beta,\gamma=1}^{d}\sup_{z\in[0,1]}\left(\frac{\partial^3}{\partial t_\alpha\partial t_\beta\partial t_\gamma}(X)(\rho^{b^F_{J_N}}(zs))\right)^4.$$

Gaussian regression, cf. [4, Proposition 1.2], allows us to express the conditional expectation

$$\mathbb{E}\left[\sup_{z\in[0,1]}\left|\frac{\partial^3}{\partial t_\alpha\partial t_\beta\partial t_\gamma}(X)(\rho^{b^F_{J_N}}(zs))\right|^4 \mid \mathcal{E}(F,L,s,y)\right]$$

through the unconditional expectation

$$\mathbb{E}\left[\sup_{z\in[0,1]}\left|\frac{\partial^3}{\partial t_\alpha\partial t_\beta\partial t_\gamma}(X)(\rho^{b^F_{J_N}}(zs)) + C_{12}^{\alpha,\beta,\gamma}(b^F_{J_N},z,s)C_2(b^F_{J_N},s)^{-1}\left(\left(y^{b^F_{J_N}},y^{b^F_{J_N}}\right)-X_2(b^F_{J_N},s)\right)\right|^4\right], \tag{B.14}$$

for $s \in \mathbb{R}^{l-m}$, where

$$X_2(b^F_{J_N},s) := \left(\nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(0)), \nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(s))\right) \in \mathbb{R}^{2(l-m)},$$

$$\begin{aligned}C_{12}^{\alpha,\beta,\gamma}(b^F_{J_N},z,s) &:= \mathrm{Cov}\left(\frac{\partial^3}{\partial t_\alpha\partial t_\beta\partial t_\gamma}(X)(\rho^{b^F_{J_N}}(zs)), X_2(b^F_{J_N},s)\right)\\ &= \left(K^{\alpha,\beta,\gamma}(b^F_{J_N},\sigma^{b^F_{J_N}}(zs)), K^{\alpha,\beta,\gamma}(b^F_{J_N},\sigma^{b^F_{J_N}}((z-1)s))\right) \in \mathbb{R}^{1\times 2(l-m)}\end{aligned}$$

with $K^{\alpha,\beta,\gamma}(b^F_{J_N},t) := \left(-\frac{\partial^4}{\partial t_\alpha\partial t_\beta\partial t_\gamma\partial v_i}C^X(t)\right)_{i=1}^{l-m}$, for $t \in \mathbb{R}^d$, by stationarity and interchanging the differentiation and expectation, cf. (2.9). Moreover

$$C_2(b^F_{J_N}, s) := \mathrm{Cov}\left(X_2(b^F_{J_N}, s)\right) \in \mathbb{R}^{2(l-m)\times 2(l-m)}.$$

Note that $C_2^{-1}(b^F_{J_N}, s)$, $0 \neq s \in \mathbb{R}^{l-m}$, exists due to (A2) and that by [7, Proposition 2.8.7]

$$C_2(b^F_{J_N}, s)^{-1} = \begin{pmatrix} A(b^F_{J_N}, s) & B(b^F_{J_N}, s) \\ B(b^F_{J_N}, s) & A(b^F_{J_N}, s) \end{pmatrix},$$

where

$$\begin{aligned} A(b^F_{J_N}, s) &:= \left(I_{l-m} - \left(D^2_{b^F_{J_N}}(C^X)(\sigma^{b^F_{J_N}}(s))\right)^2\right)^{-1}, \\ B(b^F_{J_N}, s) &:= -\left(I_{l-m} - D^2_{b^F_{J_N}}(C^X)(\sigma^{b^F_{J_N}}(s))^2\right)^{-1} D^2_{b^F_{J_N}}(C^X)(\sigma^{b^F_{J_N}}(s)). \end{aligned} \tag{B.15}$$

By the triangle inequality and Jensen's inequality

$$\begin{aligned} &\sup_{z\in[0,1]} \left| \frac{\partial^3}{\partial t_\alpha \partial t_\beta \partial t_\gamma}(X)(\rho^{b^F_{J_N}}(zs)) + C^{\alpha,\beta,\gamma}_{12}(b^F_{J_N}, z, s) C_2^{-1}(b^F_{J_N}, s) \left(\left(y^{b^F_{J_N}}, y^{b^F_{J_N}}\right) - X_2(b^F_{J_N}, s) \right) \right|^4 \\ &\quad \leq 3^3 \sup_{z\in[0,1]} \frac{\partial^3}{\partial t_\alpha \partial t_\beta \partial t_\gamma}(X)(\rho^{b^F_{J_N}}(zt))^4 + 3^3 \sup_{z\in[0,1]} |C^{\alpha,\beta,\gamma}_{12}(b^F_{J_N}, z, s) C_2^{-1}(b^F_{J_N}, s) X_2(b^F_{J_N}, s)|^4 \\ &\quad\quad + 3^3 \sup_{z\in[0,1]} \left| C^{\alpha,\beta,\gamma}_{12}(b^F_{J_N}, z, s) C_2^{-1}(b^F_{J_N}, s) \left(y^{b^F_{J_N}}, y^{b^F_{J_N}}\right) \right|^4. \end{aligned}$$

Again the submultiplicativity of the norm and Jensen's inequality yield

$$\begin{aligned} &\sup_{z\in[0,1]} \left| C^{\alpha,\beta,\gamma}_{12}(b^F_{J_N}, z, s) C_2^{-1}(b^F_{J_N}, s) X_2(b^F_{J_N}, s) \right|^4 \\ &\qquad \leq \sup_{z\in[0,1]} \left\| C^{\alpha,\beta,\gamma}_{12}(b^F_{J_N}, z, s) C_2^{-1}(b^F_{J_N}, s) \right\|^4 2(l-m)^2 d^3 \\ &\qquad\quad \times \sum_{j=1}^{d} \left(\sup_{\|t\|\leq d^{1/2}N} \frac{\partial}{\partial t_j} X(t)^4 + \sup_{\|t\|\leq 3d^{1/2}N} \frac{\partial}{\partial t_j} X(t)^4 \right), \end{aligned}$$

where we used in the last line that $F \cap J_N \neq \emptyset$ implies for $s \in B^{l-m}_{2d^{\frac{1}{2}}N}$ that $\|\rho^{b^F_{J_N}}(s)\| \leq 3d^{\frac{1}{2}}N$ as well as $\|\rho^{b^F_{J_N}}(0)\| \leq d^{\frac{1}{2}}N$. Using these facts again, and summarizing the estimates, we obtain for the term in (B.14) the upper bound

$$\begin{aligned} &3^3 \mathbb{E}\left[\sup_{\|t\|\leq 3d^{\frac{1}{2}}N} \frac{\partial^3}{\partial t_\alpha \partial t_\beta \partial t_\gamma} X(t)^4 \right] + 3^3 \sup_{z\in[0,1]} \left\| C^{\alpha,\beta,\gamma}_{12}(b^F_{J_N}, z, s) C_2(b^F_{J_N}, s)^{-1} \right\|^4 2(l-m)^2 d^3 \\ &\quad \times \sum_{j=1}^{d} \left(2\mathbb{E}\left[\sup_{\|t\|\leq 3d^{\frac{1}{2}}N} \frac{\partial}{\partial t_j} X(t)^4 \right] \right) + 3^3 \sup_{z\in[0,1]} \left| C^{\alpha,\beta,\gamma}_{12}(b^F_{J_N}, z, s) C_2^{-1}(b^F_{J_N}, s) \left(y^{b^F_{J_N}}, y^{b^F_{J_N}}\right) \right|^4. \end{aligned}$$

Note that the arguments of the expectations neither depend on F nor on s and moreover, the involved Gaussian fields are all continuous. The continuity implies that it is sufficient to bound the expectation of the supremum of a dense index set and moreover that the necessary conditions in [48, Theorem 5] are satisfied, which guarantees the finiteness of these expectations.

To prove the lemma, it remains to bound $\sup_{z\in[0,1]} \|C_{12}^{\alpha,\beta,\gamma}(b_{J_N}^F, z, s)C_2(b_{J_N}^F, s)^{-1}\|^4$ for $s \in B_{2d^{1/2}N}^{l-m}$, independently of F. Observe that

$$\begin{aligned}
&\left\|C_{12}^{\alpha,\beta,\gamma}(b_{J_N}^F, z, s)C_2^{-1}(b_{J_N}^F, s)\right\| \\
&\quad\le \left\|K^{\alpha,\beta,\gamma}(b_{J_N}^F, \sigma^{b_{J_N}^F}(zs))A(b_{J_N}^F, s) + K^{\alpha,\beta,\gamma}(b_{J_N}^F, \sigma^{b_{J_N}^F}((z-1)s))B(b_{J_N}^F, s)\right\| \\
&\quad\quad + \left\|K^{\alpha,\beta,\gamma}(b_{J_N}^F, \sigma^{b_{J_N}^F}(zs))B(b_{J_N}^F, s) + K^{\alpha,\beta,\gamma}(b_{J_N}^F, \sigma^{b_{J_N}^F}((z-1)s))A(b_{J_N}^F, s)\right\|
\end{aligned}$$

and moreover

$$\begin{aligned}
&K^{\alpha,\beta,\gamma}(b_{J_N}^F, \sigma^{b_{J_N}^F}(zs))A(b_{J_N}^F, s) + K^{\alpha,\beta,\gamma}(b_{J_N}^F, \sigma^{b_{J_N}^F}((z-1)s))B(b_{J_N}^F, s) \\
&\quad = K^{\alpha,\beta,\gamma}(b_{J_N}^F, \sigma^{b_{J_N}^F}((z-1)s))(A(b_{J_N}^F, s) + B(b_{J_N}^F, s)) \\
&\quad\quad + (K^{\alpha,\beta,\gamma}(b_{J_N}^F, \sigma^{b_{J_N}^F}(zs)) - K^{\alpha,\beta,\gamma}(b_{J_N}^F, \sigma^{b_{J_N}^F}((z-1)s)))A(b_{J_N}^F, s).
\end{aligned}$$

Since $A(b_{J_N}^F, s) + B(b_{J_N}^F, s) = \left(I_{l-m} - D^2_{b_{J_N}^F}(C^X)(\sigma^{b_{J_N}^F}(s))\right)^{-1}$, this equals

$$\begin{aligned}
&K^{\alpha,\beta,\gamma}(b_{J_N}^F, \sigma^{b_{J_N}^F}((z-1)s)) \left(I_{l-m} - D^2_{b_{J_N}^F}(C^X)(\sigma^{b_{J_N}^F}(t))\right)^{-1} \\
&\quad + \left(K^{\alpha,\beta,\gamma}(b_{J_N}^F, \sigma^{b_{J_N}^F}(zs)) - K^{\alpha,\beta,\gamma}(b_{J_N}^F, \sigma^{b_{J_N}^F}((z-1)s))\right) \\
&\quad \times \left(I_{l-m} + D^2_{b_{J_N}^F}(C^X)(\sigma^{b_{J_N}^F}(t))\right)^{-1} \left(I_{l-m} - D^2_{b_{J_N}^F}(C^X)(\sigma^{b_{J_N}^F}(s))\right)^{-1}.
\end{aligned}$$

By algebraic manipulations the above equals

$$\begin{aligned}
&\Bigg(K^{\alpha,\beta,\gamma}(b_{J_N}^F, \sigma^{b_{J_N}^F}((z-1)s)) + \left(K^{\alpha,\beta,\gamma}(b_{J_N}^F, \sigma^{b_{J_N}^F}(zs)) - K^{\alpha,\beta,\gamma}(b_{J_N}^F, \sigma^{b_{J_N}^F}((z-1)s))\right) \\
&\quad \times \left(I_{l-m} + D^2_{b_{J_N}^F}(C^X)(\sigma^{b_{J_N}^F}(s))\right)^{-1}\Bigg) \times \left(I_{l-m} - D^2_{b_{J_N}^F}(C^X)(\sigma^{b_{J_N}^F}(s))\right)^{-1}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&K^{\alpha,\beta,\gamma}(F, \sigma^{b_{J_N}^F}(zs))B(b_{J_N}^F, s) + K^{\alpha,\beta,\gamma}(b_{J_N}^F, \sigma^{b_{J_N}^F}((z-1)s))A(b_{J_N}^F, s) \\
&\quad = \Bigg(K^{\alpha,\beta,\gamma}(b_{J_N}^F, \sigma^{b_{J_N}^F}(zs)) - \left(K^{\alpha,\beta,\gamma}(b_{J_N}^F, \sigma^{b_{J_N}^F}(zs)) - K^{\alpha,\beta,\gamma}(b_{J_N}^F, \sigma^{b_{J_N}^F}((z-1)s))\right) \\
&\quad\quad \times \left(I_{l-m} + D^2_{b_{J_N}^F}(C^X)(\sigma^{b_{J_N}^F}(s))\right)^{-1}\Bigg) \times \left(I_{l-m} - D^2_{b_{J_N}^F}(C^X)(\sigma^{b_{J_N}^F}(s))\right)^{-1}.
\end{aligned}$$

We now use Lemma B.6 to bound $\|(I_{l-m} - D^2_{b^F_{J_N}} C^X(\sigma^{b^F_{J_N}}(s)))^{-1}\|$ and

$$\sup_{z\in[0,1]} \left\| \left(K^{\alpha,\beta,\gamma}(b^F_{J_N}, \sigma^{b^F_{J_N}}(zs)) - K^{\alpha,\beta,\gamma}(b^F_{J_N}, \sigma^{b^F_{J_N}}((z-1)s)) \right) \left(I_{l-m} + D^2_{b^F_{J_N}} C^X(\sigma^{b^F_{J_N}}(s)) \right)^{-1} \right\|$$

for $s \in B^{l-m}_{2d^{1/2}N}$, independently of F. For the term $\|K^{\alpha,\beta,\gamma}(b^F_{J_N}, \sigma^{b^F_{J_N}}(zs))\|$ and the term $\|K^{\alpha,\beta,\gamma}(b^F_{J_N}, \sigma^{b^F_{J_N}}((z-1)s))\|$, we bound the directional derivatives by the partial ones and use the continuity of C^X with the estimates $\|\sigma^{b^F_{J_N}}((z-1)s)\| \le 2d^{1/2}N$ and $\|\sigma^{b^F_{J_N}}(zs)\| \le 2d^{1/2}N$, to bound their norms for $z \in [0,1]$ and $s \in B^{l-m}_{2d^{1/2}N}$, independently of F, as is shown exemplarily in the following:

$$\begin{aligned} \left\| K^{\alpha,\beta,\gamma}(b^F_{J_N}, \sigma^{b^F_{J_N}}(zs)) \right\|^2 &= \sum_{i=1}^{l-m} \left(\frac{\partial^4}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial v_i} C^X(\sigma^{b^F_{J_N}}(zs)) \right)^2 \\ &= \sum_{i=1}^{l-m} \left(\sum_{\delta=1}^{d} v_i^{(\delta)} \frac{\partial^4}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial t_\delta} C^X(\sigma^{b^F_{J_N}}(zs)) \right)^2, \end{aligned}$$

which can be bounded by

$$(l-m) \left(\sum_{\delta=1}^{d} \sup_{t\in B^d_{2d^{1/2}N}} \left| \frac{\partial^4}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial t_\delta} C^X(t) \right| \right)^2 < \infty$$

independently of F and s. □

Lemma B.5. *Let $J_N \in \partial_l C^d_N$ and $l > m$. Then there is a constant $c = c(X,d,m,l,N) > 0$ such that for $s \in B^{l-m}_{2d^{1/2}N}$, almost all $F \in A^d_{d-m}$, where $F \cap J_N \neq \emptyset$, and $y \in \mathbb{R}^d$*

$$\mathbb{E}\left[\|D^2_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(0))\|^{4(l-m-1)} \mid \mathcal{E}(F, J_N, s, y) \right] \le c(1 + \|y\|^{4(l-m-1)}),$$

where $\mathcal{E}(F, J_N, t, y) := \{\nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(s)) = \nabla_{b^F_{J_N}}(X)(\rho^{b^F_{J_N}}(0)) = y^{b^F_{J_N}}\}$.

Since the proofs of Lemma B.5 and Lemma B.4 use the same ideas, the one of Lemma B.4 can be omitted. At last, we show a technical lemma, which is used in the proof of Lemmata B.5 and B.4.

Lemma B.6. *Let $J_N \in \partial_l C^d_N$ and $l > m$. Then there exists a constant $c = c(X,d,m,l,N) > 0$ such that for $s \in B^{l-m}_{2d^{1/2}N}$ nonzero, almost all $F \in A^d_{d-m}$, where $F \cap J_N \neq \emptyset$, and $\alpha, \beta, \gamma = 1, \ldots, d$*

$$\left\| \left(I_{l-m} - D^2_{b^F_{J_N}}(C^X)(\sigma^{b^F_{J_N}}(s)) \right)^{-1} \right\| \le c,$$

and

$$\left\| \left(\frac{\partial^3}{\partial t_\alpha \partial t_\beta \partial v_i} (C^X)(\sigma^{b^F_{J_N}}(s)) \right)_{i=1}^{l-m} \right\|^2 \left\| \left(I_{l-m} - \left(D^2_{b^F_{J_N}} (C^X)(\sigma^{b^F_{J_N}}(s)) \right)^2 \right)^{-1} \right\| \leq c,$$

$$\sup_{z\in[0,1]} \left\| \left(- \frac{\partial^4}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial v_i} (C^X)(\sigma^{b^F_{J_N}}(zs)) \right)_{i=1}^{l-m} \right.$$
$$\left. + \left(\frac{\partial^4}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial v_i} (C^X)(\sigma^{b^F_{J_N}}((z-1)s)) \right)_{i=1}^{l-m} \right\| \left\| \left(I_{l-m} + D^2_{b^F_{J_N}} (C^X)(\sigma^{b^F_{J_N}}(s)) \right)^{-1} \right\| \leq c.$$

Proof. We distinguish the case $s \in B^{l-m}_{2d^{1/2}N} \setminus U$, where U is a neighborhood of 0, and $s \in U$, in order to use continuity arguments in the first case. We note that for the different equalities the set U may be chosen differently and we think of the matrix norms as the norm, which suits us most, knowing that we can bound one by a multiple of the other.

Let $s \in B^{l-m}_{2d^{1/2}N} \setminus U$ and think of the matrix norm as the spectral norm. We observe that in this case

$$\|A^{-1}\| = |\lambda_{\min}(A)|^{-1},$$

where A is an invertible, symmetric matrix and $\lambda_{\min}(A)$ denotes the eigenvalue of A with smallest absolute value. Furthermore, we see by Lemma B.1 and equation (B.6), resp. equation (B.7), that the coefficients of the polynomials in λ

$$\det \left(I_{l-m} - D^2_{b^F_{J_N}} (C^X)(\sigma^{b^F_{J_N}}(s)) - \lambda I_{l-m} \right),$$
$$\det \left(I_{l-m} - \left(D^2_{b^F_{J_N}} (C^X)(\sigma^{b^F_{J_N}}(s)) \right)^2 - \lambda I_{l-m} \right),$$
$$\det \left(I_{l-m} + D^2_{b^F_{J_N}} (C^X)(\sigma^{b^F_{J_N}}(s)) - \lambda I_{l-m} \right),$$

are independent of F but continuous in $s \in B^{l-m}_{2d^{1/2}N} \setminus U$. Due to (B.5) and (A2), we know that

$$0 \neq \det \mathrm{Cov} \left(\nabla_{b^F_{J_N}} X(0), \nabla_{b^F_{J_N}} X(\sigma^{b^F_{J_N}}(s)) \right) = \det \left(I_{l-m} - \left(D^2_{b^F_{J_N}} (C^X)(\sigma^{b^F_{J_N}}(s)) \right)^2 \right)$$
$$= \det \left(I_{l-m} - D^2_{b^F_{J_N}} (C^X)(\sigma^{b^F_{J_N}}(s)) \right) \det \left(I_{l-m} + D^2_{b^F_{J_N}} (C^X)(\sigma^{b^F_{J_N}}(s)) \right),$$

for $s \neq 0$ and therefore none of the involved matrices has eigenvalue 0. And since the zeros of a polynomial are continuous in the coefficients, we conclude that the norms

$$\left\| \left(I_{l-m} - D^2_{b^F_{J_N}} (C^X)(\sigma^{b^F_{J_N}}(s)) \right)^{-1} \right\|,$$
$$\left\| \left(I_{l-m} - \left(D^2_{b^F_{J_N}} (C^X)(\sigma^{b^F_{J_N}}(s)) \right)^2 \right)^{-1} \right\|$$

as well as

$$\left\|\left(I_{l-m} + D^2_{b^F_{J_N}}(C^X)(\sigma^{b^F_{J_N}}(s))\right)^{-1}\right\|$$

are bounded for $s \in B^{l-m}_{2d^{1/2}N} \setminus U$, independently of F. In order to bound the supremum of the norm

$$\left\|\left(-\frac{\partial^4}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial v_i}(C^X)(\sigma^{b^F_{J_N}}(zs))\right)_{i=1}^{l-m} + \left(\frac{\partial^4}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial v_i}(C^X)(\sigma^{b^F_{J_N}}((z-1)s))\right)_{i=1}^{l-m}\right\|$$

for $z \in [0,1]$, as well as the norm $\left\|\left(\frac{\partial^3}{\partial t_\alpha \partial t_\beta \partial v_i}(C^X)(\sigma^{b^F_{J_N}}(s))\right)_{i=1}^{l-m}\right\|$, we bound the directional derivatives by the partial ones and use the continuity of the covariance function, as shown exemplarily in the following:

$$\begin{aligned}\left\|\left(\frac{\partial^3}{\partial t_\alpha \partial t_\beta \partial v_i}(C^X)(\sigma^{b^F_{J_N}}(s))\right)_{i=1}^{l-m}\right\|^2 &= \sum_{i=1}^{l-m}\left(\frac{\partial^3}{\partial t_\alpha \partial t_\beta \partial v_i}(C^X)(\sigma^{b^F_{J_N}}(s))\right)^2 \\ &= \sum_{i=1}^{l-m}\left(\sum_{\gamma=1}^{d} v_i^{(\gamma)} \frac{\partial^3}{\partial t_\alpha \partial t_\beta \partial t_\gamma}(C^X)(\sigma^{b^F_{J_N}}(s))\right)^2,\end{aligned}$$

which can be bounded by

$$(l-m)\left(\sum_{\gamma=1}^{d} \sup_{t \in B^d_{2d^{1/2}N}} \left|\frac{\partial^3}{\partial t_\alpha \partial t_\beta \partial t_\gamma} C^X(t)\right|\right)^2 < \infty,$$

independently of F and s.

To analyse the behaviour for s near 0, observe that $I_{l-m} - D^2_{b^F_{J_N}} C^X(s) \overset{\|s\|\to 0}{\longrightarrow} 2I_{l-m}$ and thus

$$\left\|\left(I_{l-m} - D^2_{b^F_{J_N}} C^X(s)\right)^{-1}\right\| \to \frac{1}{2}, \qquad \text{as } \|s\| \to 0.$$

Hence, there is no singularity at $s = 0$ and the norm can easily be bounded using continuity arguments as above. Since $I_{l-m} + D^2_{b^F_{J_N}} C^X(s) \overset{\|s\|\to 0}{\longrightarrow} 0$, this is different in the other cases. We proceed with the second inequality of the assertion. The identity (B.9) yields for $0 \neq s \in \mathbb{R}^{l-m}$ and $\|s\| \to 0$

$$I_{l-m} - \left(D^2_{b^F_{J_N}}(C^X)(\sigma^{b^F_{J_N}}(s))\right)^2 = \Theta(s) + o(\|s\|^3),$$

uniformly in F, where

$$\Theta(s) := \frac{\mu}{3}\|s\|^2 I_{l-m} + \frac{2}{3}\mu(s_i s_j)_{i,j=1}^{l-m}. \tag{B.16}$$

Since, cf. Lemma B.1,

$$\det \Theta(s) = \left(\mu/3\|s\|^2\right)^{l-m} + \left(\mu/3\|s\|^2\right)^{l-m-1} 2/3\mu\|s\|^2 \neq 0$$

for $s \neq 0$, we conclude that $\Theta(s)$ is invertible and we denote its inverse by $\Delta(s)$, for $s \neq 0$. Observe that for $\alpha \geq 0$ the identity $\Theta(\alpha s) = \alpha^2\Theta(s)$ holds and therefore $\Delta(\alpha s) = \alpha^{-2}\Delta(s)$. Thus we obtain

$$\left(I_{l-m} - \left(D^2_{b^F_{J_N}}(C^X)(\sigma^{b^F_{J_N}}(s))\right)^2\right)^{-1} = \Delta(s)\left(I_{l-m} - o(\|s\|^3)\Delta(s)\right)^{-1}.$$

Now, we can conclude from [7, Proposition 9.4.13] that for a given matrix A with $\|A(s)\| \to 0$ for $\|s\| \to 0$, we have $\|(I - A(s))^{-1}\| \leq 1 + \|A(s)\| + o(\|A(s)\|)$, since $\sum_{k=0}^{\infty} \|A(s)\|^k$ is a geometric series for s small enough. Before we apply this result, observe that

$$\sup_{u\in S^{l-m-1}} \|\Delta(u)\|$$

is actually a maximum and moreover independent of F. To see this, we think of the norm again as the spectral norm and observe by Lemma B.1 that the zeros of the polynomial in λ

$$\det(\Theta(u) - \lambda I_{l-m})$$

are independent of F but continuous in $u \in S^{l-m-1}$, from which we conclude the assertion. Thus we obtain

$$\begin{aligned}\left\|\left((I_{l-m} - o(\|s\|^3)\Delta(s)\right)^{-1}\right\| &\leq 1 + \left\|o(\|s\|^3)\Delta(s)\right\| + o\left(\left\|o(\|s\|^3)\Delta(s)\right\|\right)\\ &= 1 + o(\|s\|^1) + o(\|s\|^1)\\ &= 1 + o(\|s\|^1),\end{aligned}$$

for $\|s\| \to 0$, where we used that $\|o(\|s\|^3)\Delta(s)\| = \|o(\|s\|^3)\|s\|^{-2}\Delta(s/\|s\|))\| = o(\|s\|^1)$ and $g \in o(o(f))$ yields $g \in o(f)$. Hence, we conclude

$$\left\|\left(I_{l-m} - \left(D^2_{b^F_{J_N}}(C^X)(\sigma^{b^F_{J_N}}(s))\right)^2\right)^{-1}\right\| \leq \|s\|^{-2}\left\|\Delta\left(\frac{s}{\|s\|}\right)\right\|\left(1 + o(\|s\|^1)\right) = O(\|s\|^{-2})$$

for $\|s\| \to 0$ and uniformly in F. Taylor's theorem applied to $\frac{\partial^3}{\partial t_\alpha \partial t_\beta \partial v_i}(C^X)(\sigma^{b^F_{J_N}}(\cdot))$ yields for $i = 1, \ldots, l-m$, $0 \neq s \in \mathbb{R}^{l-m}$ and a suitable $\xi \in [0,1]$

$$\begin{aligned}\frac{\partial^3}{\partial t_\alpha \partial t_\beta \partial v_i}(C^X)(\sigma^{b^F_{J_N}}(s)) &= \frac{\partial^3}{\partial t_\alpha \partial t_\beta \partial v_i}C^X(0) + \sum_{j=1}^{l-m}\frac{\partial}{\partial t_j}\left(\frac{\partial^3}{\partial t_\alpha \partial t_\beta \partial v_i}C^X \circ \sigma^{b^F_{J_N}}\right)(\xi s)s_j\\ &= O(\|s\|),\end{aligned}$$

since $\frac{\partial^3}{\partial t_\alpha \partial t_\beta \partial v_i}C^X(0) = 0$ by stationarity, cf. [1, Equation (5.5.3)]. Note this equality holds uniformly in F, since $\frac{\partial}{\partial t_j}(\frac{\partial^3}{\partial t_\alpha \partial t_\beta \partial v_i} C^X \circ \sigma^{b^F_{J_N}})(s)$ can be bounded independently of F for $s \in B^{l-m}_{2d^{1/2}N}$. Therefore, we conclude

$$\left\| \left(\frac{\partial^3}{\partial t_\alpha \partial t_\beta \partial v_i} (C^X)(\sigma^{b^F_{J_N}}(s)) \right)_{i=1}^{l-m} \right\|^2 \left\| \left(I_{l-m} - \left(D^2_{b^F_{J_N}}(C^X)(\sigma^{b^F_{J_N}}(s)) \right)^2 \right)^{-1} \right\| = O(1),$$

for $\|s\| \to 0$ and uniformly in F. Hence, there is a neighborhood of 0 on which the left-hand side is bounded by a constant $c > 0$ not depending on F.

To show the last inequality of the assertion, we proceed similarly. First, we use identity (B.6) to obtain

$$\begin{aligned} &I_{l-m} + D^2_{b^F_{J_N}}(C^X)(\sigma^{b^F_{J_N}}(s)) \\ &\quad = (1 + R'(\|s\|)\|s\|^{-1}) I_{l-m} + \|s\|^{-2}(R''(\|s\|) - R'(\|s\|)\|s\|^{-1})(s_i s_j)_{i,j=1}^{l-m}. \end{aligned}$$

Then, the Taylor expansion in (B.8) yields for $0 \neq s \in \mathbb{R}^{l-m}$ and $\|s\| \to 0$

$$I_{l-m} + D^2_{b^F_{J_N}}(C^X)(\sigma^{b^F_{J_N}}(s)) = \frac{1}{2}\Theta(s) + o(\|s\|^3),$$

where Θ is defined in (B.16). The same approach as before, yields

$$\left\| \left(I_{l-m} + D^2_{b^F_{J_N}}(C^X)(\sigma^{b^F_{J_N}}(s)) \right)^{-1} \right\| = O(\|s\|^{-2}), \quad \text{as } \|s\| \to 0,$$

uniformly in F. Taylor's theorem, cf. [42, Section 2.4], applied to $-\frac{\partial^4}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial v_i}(C^X)(\sigma^{b^F_{J_N}}(z\cdot))$, yields for $s \in B^{l-m}_{2d^{1/2}N}$, $i = 1, \ldots, l-m$ and $z \in [0,1]$

$$\begin{aligned} &-\frac{\partial^4}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial v_i}(C^X)(\sigma^{b^F_{J_N}}(zs)) \\ &\quad = -\frac{\partial^4}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial v_i} C^X(0) - \sum_{i_1, i_2 = 1}^{l-m} \frac{\partial^2}{\partial s_{i_1} \partial s_{i_2}} \left(\frac{\partial^4}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial v_i}(C^X)(\sigma^{b^F_{J_N}}(z\cdot)) \right)(\xi) s_{i_1} s_{i_2}, \end{aligned}$$

where $\xi \in [0, s]$, since $\frac{\partial^5}{\partial t_j \partial t_\alpha \partial t_\beta \partial t_\gamma \partial v_i} C^X(0) = 0$, $j = 1, \ldots, d$, as X is stationary, cf. [1, Equation (5.5.3)]. Analogously, we obtain

$$\begin{aligned} &\frac{\partial^4}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial v_i}(C^X)(\sigma^{b^F_{J_N}}((z-1)s)) \\ &= \frac{\partial^4}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial v_i} C^X(0) + \sum_{i_1, i_2 = 1}^{l-m} \frac{\partial^2}{\partial s_{i_1} \partial s_{i_2}} \left(\frac{\partial^4}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial v_i}(C^X)(\sigma^{b^F_{J_N}}((z-1)\cdot)) \right)(\xi') s_{i_1} s_{i_2}, \end{aligned}$$

where $\xi' \in [0, s]$. Thus the term

$$\left| \frac{\partial^4}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial v_i}(C^X)(\sigma^{b^F_{J_N}}((z-1)s)) - \frac{\partial^4}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial v_i}(C^X)(\sigma^{b^F_{J_N}}(zs)) \right|$$

is given by

$$\left| \sum_{i_1,i_2=1}^{l-m} \left(\frac{\partial^2}{\partial s_{i_1} \partial s_{i_2}} \left(\frac{\partial^4}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial v_i} (C^X)(\sigma^{b^F_{J_N}}((z-1)\cdot)) \right) (\xi') \right. \right.$$
$$\left. \left. - \frac{\partial^2}{\partial s_{i_1} \partial s_{i_2}} \left(\frac{\partial^4}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial v_i} (C^X)(\sigma^{b^F_{J_N}}(z\cdot)) \right) (\xi) \right) s_{i_1} s_{i_2} \right| .$$

This expression can be bounded by

$$2 \sup_{t \in B^d_{2d^{1/2}N}} \sum_{i_1,i_2,i_3=1}^{d} \left| \frac{\partial^6}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial t_{i_1} \partial t_{i_2} \partial t_{i_3}} C^X(t) \right| \|s\|^2$$

for $z \in [0,1]$, $\xi, \xi' \in [0,s]$ and $s \in B^d_{2d^{1/2}N}$. Thus we obtain

$$\sup_{z \in [0,1]} \left\| \left(- \frac{\partial^4}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial v_i} (C^X)(\sigma^{b^F_{J_N}}(zs)) \right)_{i=1}^{l-m} - \left(\frac{\partial^4}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial v_i} (C^X)(\sigma^{b^F_{J_N}}((z-1)s)) \right)_{i=1}^{l-m} \right\|$$
$$\leq (l-m)^{\frac{1}{2}} 2 \sup_{t \in B^d_{2d^{1/2}N}} \sum_{i_1,i_2,i_3=1}^{d} \left| \frac{\partial^6}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial t_{i_1} \partial t_{i_2} \partial t_{i_3}} C^X(t) \right| \|s\|^2$$

and therefore

$$\left\| (I_{l-m} + D^2_{b^F_{J_N}} (C^X)(\sigma^{b^F_{J_N}}(s)))^{-1} \right\| \sup_{z \in [0,1]} \left\| \left(- \frac{\partial^4}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial v_i} (C^X)(\sigma^{b^F_{J_N}}(zs)) \right)_{i=1}^{l-m} \right.$$
$$\left. + \left(\frac{\partial^4}{\partial t_\alpha \partial t_\beta \partial t_\gamma \partial v_i} (C^X)(\sigma^{b^F_{J_N}}((z-1)s)) \right)_{i=1}^{l-m} \right\| = O(1)$$

uniformly in F and z, which shows the assertion. □

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