Voting and Discounting in Simple Growth Models

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Erklärung

gemäß §6 Abs. 1 Ziff. 4 der Promotionsordnung des Karlsruher Instituts für Technologie für die Fakultät für Wirtschaftswissenschaften vom 30. November 2012

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Abstract (German Version)

Diese Dissertation verbindet die Sozialwahltheorie mit der Wachstumstheorie und behandelt das Problem der Aggregation heterogener Zeitpräferenzen. Ein Überblick über die Literatur zur sozialen Wahl in Wachstumsmodellen mit vielen Agenten zeigt, dass es fundamentale Schwierigkeiten bei der Aggregation heterogener Zeitpräferenzen in dynamischen Wachstumsmodellen gibt. Es wird insbesondere gezeigt, dass es aufgrund der Hochdimensionalität des Wahlmöglichkeitenraumes in diesen Modellen keine Abstimmungsgleichgewichte gibt.

Um diese Schwierigkeiten zu überwinden wird ein einfaches und intuitives Abstimmungsverfahren (die intertemporale Mehrheitswahl) vorgeschlagen. Dieses Verfahren beruht auf zwei Prinzipien: (*i*) die Abstimmung erfolgt Schritt für Schritt in jeder Periode neu, und (*ii*) es wird dabei nicht über die absoluten Größen (z.B. das Konsumniveau), sondern über deren relativen Wert (z.B. die Konsumrate) entschieden. Die intertemporale Mehrheitswahl wird auf ein einfaches Wachstumsmodell mit einem öffentlichen Gut angewendet, in dem die Agenten unter verschiedenen momentanen Nutzenfunktionen und verschiedenen Zeitpräferenzraten einen gemeinsamen Konsumpfad wählen. Das vorgeschlagene Abstimmungsverfahren stellt eine mikroökonomische Fundierung der Wahl des optimalen Konsumpfades des Medianwähler dar.

Die intertemporale Mehrheitswahl wird dann auf ein allgemeines Gleichgewichtsmodell mit einer nicht erneuerbaren Ressource angewendet. Die Agenten haben wiederum verschiedene Zeitpräferenzraten und wählen die Extraktionsraten, um eine sozial optimale Nutzung der Ressource zu bestimmen. Das vorgeschlagene Abstimmungsverfahren ergibt sich in diesem Zusammenhang besonders natürlich, da es sich um eine kollektive Entscheidung über einen relativen Wert (die Extraktionsrate) handelt. Auch hier stellt sich heraus, dass das Abstimmungsgleichgewicht der intertemporalen Mehrheitswahl von der Median-Zeitpräferenzrate abhängt. Das vorgeschlagene Abstimmungsverfahren lässt sich daher sehr einfach auf dynamische allgemeine Gleichgewichtsmodelle anwenden.

Schlagworte: Theorie kollektiver Entscheidungen, Wachstumsmodelle, heterogene Agenten, Mehrheitswahl.

JEL-Klassifikation: D11, D71, D91, O13, O43.

Abstract (English Version)

My thesis combines social choice theory with economic growth theory and is concerned with the problem of aggregating heterogeneous time preferences. A survey of the literature on the problem of social choice in growth models with many agents indicates that there are serious difficulties with the aggregation of heterogeneous time preferences in many-agent dynamic growth models. In particular, it is shown that due to the multi-dimensional setting, there is no reason to expect that voting in such models yields a stable outcome.

To overcome these difficulties, we propose a simple and intuitive voting procedure (intertemporal majority voting). This procedure is based on the two principles: (*i*) voting is done step by step, and (*ii*) voting is not over the absolute value (e.g., consumption level), but over the relative value (e.g., consumption rate). We apply intertemporal majority voting to a simple optimal growth model with common consumption in which agents who differ in their felicity functions and discount factors choose a common consumption path. We show that our procedure provides a microfoundation for the choice of the optimal path of the "median" agent (whenever this notion is well defined).

We also apply intertemporal majority voting to a general equilibrium growth model with exhaustible natural resources. Agents who differ in their discount factors vote over extraction rates. Our procedure naturally arises in this context, because voting is precisely over the relative values. We show that the outcome of intertemporal majority voting is determined by the agent with the median discount factor. Hence our voting procedure can be applied also in a dynamic general equilibrium framework.

Keywords: collective choice, economic growth, heterogeneous agents, voting.

JEL Classification: D11, D71, D91, O13, O43.

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Contents

1. Introduction

In dynamic models of economic growth, time preference (patience) appears as the fundamental factor which governs aggregate accumulation processes. Many such models typically rely on the representative agent assumption, so that economic growth in these models (e.g., the long-run level or the growth rate of per capita output) depends on the rate of time preference of the representative agent. In particular, the widely known Ramsey optimal growth model implies that the more patient is the representative agent, the higher is the steady-state output in the economy.

However, models with a representative agent are subject to serious criticism, as individuals in the real world differ in many characteristics that affect their economic decisions. The qualitative results obtained from models with heterogeneous agents are completely different from those of models with a representative agent. For instance, many standard economic policies that increase aggregate income in the representative agent models, influence only the distribution of income in the heterogeneous agents models.

It is generally acknowledged that individuals are not equally patient. Recent empirical evidence shows that different people value the future differently, and this heterogeneity in time preferences plays a crucial role in the process of economic development. When individuals are heterogeneous in their time preferences, it is not clear at all whether there is any "representative" individual, what is the rate of time preference from the society's perspective, and how this heterogeneity affects economic growth.

Thus the recognition of the fact that individuals in the society discount their future differently leads to a very important problem of aggregating heterogeneous time preferences. This problem is at the junction of economic growth theory and social choice theory, and has lately received significant attention. It is reasonable to assume that each individual plays a role in collective action, and the question is, what is the relation between individual and collective behavior. This is where the present thesis aims to contribute.

In Chapter 2 we provide a survey of the literature on the aggregation of heterogeneous time preferences. Recently there appeared a number of technically sophisticated papers, which treat the problem of social choice in a purely abstract and axiomatic manner, creating the impression of complexity of the problem. We want to show that this complexity is somewhat artificial, and the problem of social choice in growth models with many agents is more understandable that it may seem. To achieve this goal, we illustrate contributions to the literature in the simplest framework of a one-sector deterministic Ramsey model with agents who are identical except for their time preferences. Agents derive utility from lifetime consumption and face a trade-off between present and future consumptions. The simplicity of this framework is sufficient for our purposes and allows us to explain the main results in an instructive manner.

There are two principal ways of aggregation considered in the literature. The first is to construct a social welfare function, which somehow takes into account individual utility functions of different agents. The second is to aggregate heterogeneous preferences via some social choice procedure (majority voting being the most important example).

It follows from the surveyed literature that both principal ways of aggregating heterogeneous time preferences in many-agent growth models face serious difficulties. In particular, a social welfare function which is a weighted sum of different individual utility functions satisfies certain reasonable conditions (e.g., Pareto-efficiency and time consistency) if and only if it coincides with the utility function of some agent, so that the preferences of all other agents are completely ignored. Moreover, a natural attempt to determine the decisions of the society via some voting procedure also does not lead to an unambiguous outcome. A Condorcet winner in voting over multi-dimensional choice space fails to exist even if agents are heterogeneous only in one dimension, and any non-dictatorial voting rule appears to be inherently intransitive.

In the remaining chapters of the thesis we propose and study a simple and natural voting procedure (intertemporal majority voting) which is based on the two principles: (*i*) voting is done step by step, and (*ii*) voting is not over the absolute value (e.g., consumption level), but over the relative value (e.g., consumption rate). These two principles allow us to avoid all the difficulties with majority voting in multi-dimensional settings and obtain a stable voting outcome (intertemporal voting equilibrium) under reasonable assumptions.

In Chapter 3 we apply our voting procedure to the many-agent Ramsey model with common consumption.[1] In this framework, agents who may differ in their felicity functions and time preferences (discount factors) share a common consumption stream which arises from a collectively consumed public good or a common property resource. Agents' personal utilities are based on their collective decisions, and the main question is how a common consumption stream is chosen.

Our voting procedure can be described as follows. At each point in time agents vote over the current consumption rate, given the expectations about future consumption rates. We show that in this one-dimensional voting problem agents' preferences over the current consumption rate are single-peaked, and the median voter theorem applies. Therefore, at each point in time there exists an "instantaneous" Condorcet winner, which generically

[1] Chapter 3 is an adapted version of the paper "On Discounting and Voting in a Simple Growth Model" (Borissov, Pakhnin and Puppe, 2017).

depends on all expected future consumption rates. An intertemporal voting equilibrium is the sequence of instantaneous Condorcet winners under perfect foresight about outcomes of future votes.

We show that if agents have the same felicity function and differ only in their discount factors, then there is a unique intertemporal voting equilibrium which coincides with the optimal consumption path for the agent with the median discount factor. We also consider the multi-dimensional heterogeneity case in which agents differ both in their felicity functions and discount factors. In this general case we characterize steady-state and balanced-growth voting equilibria. The steady-state voting equilibrium is fully determined by the median discount factor, and the balanced-growth voting equilibrium is determined by the preferences of the agent with the median growth rate. Thus, in some sense, our procedure provides a microfoundation for the assumption that the median agent (whenever this notion is well defined) is the representative of society.

In Chapter 4 we apply our voting procedure to the general equilibrium Ramsey-type model with exhaustible natural resources.[2] The goal of this chapter is to compare private and public property regimes over natural resources in terms of economic growth.

In the private property regime, the resource stock is an asset to its owner. Agents can invest in natural resources as well as in physical capital. Extraction rates are determined in an equilibrium by the market forces of supply and demand. We show that every competitive equilibrium converges to a balanced-growth equilibrium, and the long-run growth rate depends on the discount factor of the most patient agent.

In the public property regime, the resource stock is controlled by a government that acts in the interest of the agents. Resource income is equally distributed among agents. There are no market forces to determine extraction rates, so they are chosen by majority voting. Here our voting procedure is naturally applied because agents vote precisely over the relative values. We show that the sequence of winners in one-dimensional votes over the current extraction rate under perfect foresight is determined by the agent with the median discount factor. We define an intertemporal voting equilibrium (in this model it consists of the voting equilibrium sequence of extraction rates along with the corresponding competitive equilibrium) and prove that it also converges to a balanced-growth equilibrium. In this case the long-run growth rate depends on the median discount factor. Thus the application of our voting procedure allows us to reasonably model the public property regime over natural resources in the general equilibrium framework.

Each chapter of this thesis has a slightly different focus and can be read independently. To facilitate readability and maintain a consistent presentation, the conclusions and introductions are connecting passages between the chapters.

[2] Chapter 4 is an adapted version of the paper "Economic Growth and Property Rights on Natural Resources" (Borissov and Pakhnin, 2018).

2. Social Choice in Growth Models with Many Agents: An Overview

Time preference — the intrinsic propensity of the individual decision-maker to postpone immediate gratification in exchange for larger but delayed rewards — have been at the core of economic analysis for a very long time. Without any exaggeration it can be argued that questions of time preferences and intertemporal choice have occupied the minds of economists since the development of the discipline. It was Adam Smith who first pointed out the connection between intertemporal choice and economic growth. He saw thrift, i.e., the propensity to save and invest some amount of capital instead of spending it immediately, as an important virtue which leads to the accumulation of capital and increases the wealth of a nation. His ideas were developed further by many other scholars, and nowadays time preference (patience) is widely regarded as the fundamental factor influencing economic growth. The rate of time preference is tied to the growth rate of per capita output in many dynamic models of economic growth.

A vast amount of literature on time preferences is devoted to the analysis of representative agent models. The role of time preferences in economic models with a representative agent as well as theoretical contributions in this field are reviewed by Hamada and Takeda (2009). However, it is becoming increasingly recognized that models with a representative agent are subject to serious criticism, as individuals in the real world differ in many characteristics that may affect their economic decisions. For instance, Alan Kirman argues that the representative agent assumption "is not simply an analytical convenience as often explained, but is both unjustified and leads to conclusions which are usually misleading and often wrong" (Kirman, 1992, p. 117).

This is particularly evident when comparing policy implications from models with heterogeneous agents and models with a representative agent. Many standard economic policies that increase aggregate income in the representative agent models do influence only the distribution of income in the heterogeneous agents models. This effect is apparent already in models with only two types of agents (see, e.g., Smetters, 1999; Mankiw, 2000, for the discussion of fiscal policy and Palivos, 2005, for the discussion of monetary policy under agents' heterogeneity).

It is reasonable to assume that in the presence of heterogeneous agents the policy parameters are chosen by the agents themselves. Policy should be a collective decision, and hence policy cannot be pursued without taking into account heterogeneous preferences of individuals.

The same argument applies to the growth models, and becomes especially important in this framework. It is generally acknowledged that individuals are not equally patient, and there is no convergence toward an agreed-on or unique rate of time preference. It is not clear at all, what are the rate of time preference and the growth rate in the economy where agents are heterogeneous in their time preferences.

Thus the recognition of the fact that individuals in the society discount their future differently leads to a very important problem of collective choice and aggregation of heterogeneous time preferences. This is the topic of social choice theory.[1] How a society can make a collective decision when all its members have different time preferences? Which rate of time preference determines the growth rate in the heterogeneous society? These questions are at the junction of economic growth theory and social choice theory, and have lately received considerable attention.

In this chapter we will review the main results related to the aggregation of heterogeneous time preferences and discuss the problem of social choice in growth models with many agents. In recent years there appeared a number of technically sophisticated and obscure papers that create the impression of complexity of the problem. One of the goals of this chapter is to show that this complexity is somewhat artificial, while the problem is more understandable that it may seem.

To make our presentation as clear as possible, we focus on one-sector deterministic growth models with two agents whose preferences exhibit constant exponential discounting. Though there is a vast literature on hyperbolic and quasi-hyperbolic discounting, including some empirical evidence (see, e.g., Frederick et al., 2002), we do not discuss departures from exponential discounting. The problem of collective choice when individual preferences exhibit non-exponential discounting is studied, e.g., by Lizzeri and Yariv (2015) and Drugeon and Wigniolle (2017). There is also a substantial amount of literature on discounting under uncertainty (see, e.g., Gollier and Weitzman, 2010; Traeger, 2013). Though this is a promising line of research which remains a topic of current interest, the introduction of uncertainty complicates matters quite dramatically, so this chapter deals only with deterministic models. The simplicity of our framework is actually an advantage here, as this allows us to explain the main difficulties in an instructive manner and to highlight the role of discounting.

This chapter is organized as follows. Section 2.1 reviews some novel empirical evidence which shows that people in the real world differ in their time preferences, and that their heterogeneity plays an important role in the process of economic development. In Section

[1] For the recent contributions in the field of social choice and welfare which seem to be close to the subject of this chapter, see Anand et al. (2009).

2.2 we briefly describe the Ramsey model with many agents that are heterogeneous in their time preferences. Sections 2.3 and 2.4 are devoted to the properties of social optima in the many-agent Ramsey model with private and common consumption correspondingly. In Section 2.5 we study majority voting over the common consumption streams. Section 2.6 reviews and discusses different answers to the normative question of how a social discount rate should be determined. Section 2.7 concludes.

2.1. Empirical studies

The empirical side of the discounting literature is excellently and extensively reviewed by Frederick et al. (2002) and Cohen et al. (2016). In this section we want to emphasize the most recent empirical evidence which shows that different individuals value the future differently, and this heterogeneity in time preferences plays an important role in the process of economic development.

Falk et al. (2015) provide the results of the Global Preference Survey implemented through the Gallup World Poll in 2012. Their data on time preferences, collected for 80 000 individuals from 76 countries, is representative within country as well as across countries. The measure of patience is based on respondents' hypothetical binary choices between receiving a payment now or a larger payment in a year. It was found that while average patience across countries varies by 1.7 of a standard deviation, the within-country variation is much larger than the between-country variation. In the total individual-level variation in patience, the variance of the average patience across countries amounts to 13.5%, while the remaining 86.5% is due to the within-country variance. It is also documented that in the world population as a whole, time preferences vary significantly with individual characteristics such as gender or age. For instance, patience has a hump-shaped age profile: the middle-aged are more patient than young people and the elderly.

The rich and comprehensive data from the Global Preference Survey allowed the same group of researchers to study how time preferences are related to income and capital accumulation both at the country and individual levels. Dohmen et al. (2016) report that average degree of patience in a country is strongly correlated with this country's level of economic and institutional development. The correlation between the patience measure and log GDP per capita is 0.63, i.e., patience alone explains about 40% of the variation in income per capita. This result is robust to including other explanatory variables (patience continues to have strong explanatory power even in the specification with geographic, climatic, colonial, and diversity covariates), changing the dependent variable (the observed positive relationship holds for log GDP per worker and the United Nations Human Development Index as measures of development), and controlling for inflation, deposit interest rates and borrowing constraints. It also turns out that greater patience is significantly associated with higher annual growth rates. A one standard deviation increase in patience leads to 1.1 percentage points increase in annual growth rate from 1975 to 2010.

Patience is found to be positively related to both the stocks of and investments into physical capital, human capital, and productivity. Patience alone explains about a third of the variation in the capital stock per capita, and more than 40% of the variation in the average years of schooling. Authors' estimates also suggest that patience is a significant correlate of democracy, property rights, social infrastructure, and long-term credit ratings. What is more, the same relationship between time preferences and income holds at the individual level. Within countries, more patient people tend to be richer and have higher educational attainment.

Though the causality here can flow in either direction, there is also some tentative evidence that it is exactly patience that affects aggregate or individual income. Using the share of protestants in a country as an instrument for patience, Dohmen et al. (2016) estimate the corresponding instrumental variable regressions and find that the reported effect of patience on income remains large and significant.

Wang et al. (2016) provide the results of another international survey on time preferences, comprising about 7000 university students in 53 advanced and developing countries. Their main measure of patience is the share of participants in each country who preferred a higher future reward over immediate payoff in the answer to the hypothetical binary choice question. They also find that the measured level of patience is heterogeneous both at individual and country levels, and this heterogeneity on a cross-country level cannot be explained by differences in interest or inflation rates. It is reported that time preferences are systematically correlated with the economic growth: participants from countries with higher GDP per capita tend to be more patient. Their results suggest that people seem to be more patient in countries that are politically more stable, hold more public trust to politicians or have more efficient markets.

Hübner and Vannoorenberghe (2015) use three different proxies for patience (the measure taken from Wang et al. (2016), the Index of Long-Term Orientation and the Future Orientation Index) and analyze a panel of 89 countries. The average degree of patience appears to have a strong positive impact on income per worker, total factor productivity and the capital stock. Their regressions suggest that patience explains about 40% of the cross-country variation in income, which is very close to the results of Dohmen et al. (2016). It is also found that increasing patience by one standard deviation raises per-capita income by between 43% and 78%.

Apart from hypothetical binary choice questions, the individuals' rates of time preference can be measured directly from some real-life decisions. In the discrete-time framework, time preferences are usually expressed in terms of the (one-period) *discount factor*, β, which is the relative weight that individual attaches to her next period utility from the current-period perspective. A closely related notion is the *discount rate*, ρ, which determines the present value of a next period utility. The discount factor and the discount rate are related by $\beta = 1/(1+\rho)$. It follows that greater patience implies higher discount factors and lower discount rates.[2]

In the empirical literature on discounting, typical measures of the rates of time preference are discount rates, since they are easier to estimate and are directly comparable to interest rates. In particular, Simon et al. (2015) provide estimates of the U.S. military personnel's discount rates based on their actual choices of retirement plans. The estimated discount rates range from 3.1% to 9.5% per year. It is documented that individual discount rates vary significantly depending on the race, gender, income and education. For instance, white personnel are more patient than black, and female personnel are more patient than male.[3] More educated and cognitively more able personnel appear to be more patient. Statistically as well as economically significant is the fact that less patient individuals tend to save less, are more likely to experience financial difficulties, and face higher average credit card and car loan interest rates.

As we have seen, empirical research not only supports "the internal consistency of a dynamic development framework in which time preferences play a critical role" (Dohmen et al., 2016, p. 30), but also clearly indicates that people have heterogeneous time preferences. Thus empirical evidence strongly suggests that heterogeneity in individuals' time preferences should be explicitly taken into account in economic modeling.

2.2. Equilibria and optima in growth models with many agents

As we have mentioned, Adam Smith (1776) was the first who noted that the propensity to save (i.e., thrift) leads to the accumulation of capital. However, he did not address the question of what determines this propensity. This problem was explored by another classical Scottish economist, John Rae (1834), who argued that there are differences in the strength of the desire to accumulate among different members of society. He stated that people whose desire to accumulate is low become poor, while people whose desire to accumulate is high become rich. Hence funds are gradually redistributed from the impatient consumers to the patient ones. Further, Irving Fisher (1907) developed a more

[2] These notions also have continuous-time generalizations. In continuous-time framework, future utility is discounted by the instantaneous discount rate ρ, which is related to the instantaneous discount factor: $\beta = \exp(-\rho)$.

[3] Castillo et al. (2011) obtain qualitatively similar results in a field experiment with almost 900 eighth graders in a school district in Georgia, U.S. They show that white children are more patient than black children and that girls are more patient than boys.

precise notion of time preference and argued that it is ultimately differences in rates of time preference that drive the redistribution of income and wealth.

This idea was formalized in a seminal paper by Frank Ramsey (1928). He proposed a model of optimal capital accumulation in which a consumer acting over an infinite horizon maximizes discounted utility from consumption at different moments in time subject to the resource constraints and given the initial capital stock. This model is now widely known as the optimal growth model, or the Ramsey model. Apart from this, in the latter part of his paper Ramsey considered a model with many infinitely lived consumers who differ in their time preferences. He conjectured that in a stationary equilibrium, i.e., an equilibrium where all variables are constant over time, the whole capital stock belongs to the most patient individual in the society whose consumption is the largest. All other, less patient individuals, consume only at the subsistence level necessary to support their lives. This property of an equilibrium is known in the literature as the Ramsey conjecture (though it is more correct to label it the "Rae–Fisher–Ramsey conjecture").

Though Ramsey himself neither spelled out the details of his many-agent model nor gave a definition of equilibrium, his insights were developed further by many other scholars. In particular, Bewley (1982) proposed an interpretation of the many-agent Ramsey model as a general equilibrium model with infinitely many commodities. He considered an economy with complete markets populated by consumers who have private consumption streams and differ in their discount factors. Extending the Arrow–Debreu theory, Bewley proved that there exists a competitive equilibrium and any equilibrium allocation is Pareto-optimal. Bewley (1982) also established a link between general equilibrium theory and turnpike theory by showing that in every equilibrium eventually only the most patient consumer, i.e. the individual with the highest discount factor, has positive consumption levels. All the less patient consumers after some finite time consume nothing.[4] The impatient individuals borrow against their wealth in order to consume more early and repay their loans later, thus driving their future consumption levels towards zero, which can be seen as a justification of the Ramsey conjecture.

The latter property of an equilibrium (and optimal) allocation was considered somewhat unsatisfactory, because despite the fact that the less patient agents consume nothing from some time onward thereby leaving the economy's demand side, they still continue to supply their labor services. This objection can be overcome following Becker (1980). He considered a many-agent Ramsey model with borrowing constraints. In each period, consumers can sell or accumulate capital, but cannot borrow, which implies that nobody consumes zero or even approaches zero asymptotically: even the less patient individuals

[4] This result was obtained under the assumption that a felicity function of consumption, $u(c)$, is such that $u'(0) < \infty$. In the case of the more general felicity functions, consumption levels of the less patient agents converge to zero as time goes to infinity.

will always consume at least part of their labor income. Becker (1980) showed that under appropriate assumptions a stationary equilibrium (steady state) in this model exists, is unique and verifies the Ramsey conjecture: all the capital is owned by the most patient consumer. Models of this kind received reasonable attention, and many properties of equilibria were established (see Becker, 2006, for an excellent survey).

Since the many-agent Ramsey model with borrowing constraints is an incomplete markets model, there is no reason to expect that the first welfare theorem holds, i.e., that equilibria are Pareto-optimal. Nevertheless, Becker and Mitra (2012) proved that the equilibrium sequences of aggregate capital and consumption in this kind of models are technologically efficient.[5] Recent studies (see Borissov and Dubey, 2015; Becker et al., 2015b) show that the extreme no-borrowing constraints introduced initially can be relaxed to allow more liberal borrowing by agents. Equilibria in the many-agent Ramsey model under alternative borrowing regimes are proved to exist, and it is shown that if an equilibrium converges to the steady state, then this equilibrium is also efficient.

2.3. Social optima in growth models with private consumption

Let us take a closer look at the general equilibrium version of the many-agent Ramsey model proposed by Bewley (1982). Our goal is to explore this model from the social welfare perspective, and discuss certain difficulties that arise with the notion of a social optimum under heterogeneous time preferences (for details, see Le Van and Vailakis, 2003; Becker, 2012). For ease of exposition, we restrict our consideration to a one-sector two-agent Ramsey model with private consumption. To highlight the role of discounting, we assume that the two consumers in our economy differ only in their discount factors and are otherwise identical.

Let the preferences of agent $i = 1, 2$ over infinite consumption streams $\mathcal{C}^i = \{c_t^i\}_{t=0}^{\infty}$ be given by the additively time-separable intertemporal utility function of the form

$$U^i(\mathcal{C}^i) = \sum_{t=0}^{\infty} \beta_i^t u(c_t^i),$$

where β_i is her discount factor and $u(c)$ is her felicity (instantaneous utility) function.[6] Suppose that agent 1 is the most patient: $1 > \beta_1 > \beta_2 > 0$.

[5] Becker and Mitra (2012) prove that the aggregate capital and consumption sequences are intertemporally efficient if the most patient consumer's capital stock is positive from some time onward — a condition satisfied in all currently known examples.

[6] This form of preference representation, the exponential discounting model, was introduced by Samuelson (1937) and axiomatized later by Koopmans (1960). Nowadays, exponential discounting is the most convenient and popular framework for analyzing intertemporal decisions.

A single homogeneous good is produced. In each period t the available amount of good is allocated between aggregate consumption $C_t = c_t^1 + c_t^2$ and capital k_{t+1} for use in the next period production: $C_t + k_{t+1} = f(k_t)$, where $f(k)$ is a neoclassical production function. Capital is assumed to depreciate completely within the period.

If there is only one consumer ("representative agent") with the discount factor β, the felicity function $u(c)$, and the intertemporal utility function

$$\sum_{t=0}^{\infty} \beta^t u(C_t) = u(C_0) + \beta u(C_1) + \beta^2 u(C_2) + \dots, \tag{2.1}$$

then the problem in question is the standard optimal growth problem. The optimal consumption stream in the economy is obtained as a solution to the optimization problem of this representative agent:

$$\begin{aligned} \max\ & \sum_{t=0}^{\infty} \beta^t u(C_t), \\ \text{s. t.}\ & C_t + k_{t+1} = f(k_t), \\ & C_t \geq 0, \quad k_{t+1} \geq 0, \quad t = 0, 1, \dots, \\ & k_0 = \hat{k}_0. \end{aligned} \tag{2.2}$$

Under some reasonable assumptions on the felicity function $u(c)$ and the production function $f(k)$, problem (2.2) has a unique solution $\{C_t^*, k_{t+1}^*\}_{t=0}^{\infty}$.

In the optimal growth problem, the representative agent with the discount factor β determines the optimal consumption stream. However, in our economy there are two different consumers, each with her own discount factor β_i. It is not clear who should "represent" the society and determine the aggregate consumption stream, so it is reasonable to take into account the preferences of both consumers. This is typically done by introducing a social welfare function $W(\mathcal{C}^1, \mathcal{C}^2)$ which evaluates different consumption streams from the perspective of the society as a whole.

A very natural and widely accepted property of a social welfare function $W(\mathcal{C}^1, \mathcal{C}^2)$ is *Pareto-efficiency*, i.e., the requirement that *for any consumption bundles* $\mathcal{C} = \{\mathcal{C}^1, \mathcal{C}^2\}$ *and* $\tilde{\mathcal{C}} = \{\tilde{\mathcal{C}}^1, \tilde{\mathcal{C}}^2\}$, $W(\mathcal{C}^1, \mathcal{C}^2) \geq W(\tilde{\mathcal{C}}^1, \tilde{\mathcal{C}}^2)$ *whenever* $U^i(\mathcal{C}^i) \geq U^i(\tilde{\mathcal{C}}^i)$ *for both* $i = 1, 2$, *and the first inequality is strict whenever the second is strict for both* $i = 1, 2$. Hence a Pareto-efficient (Paretian) social welfare function must respect the preferences of individuals.

In the two-agent Ramsey model with private consumption, a Paretian social welfare function naturally appears as a weighted sum of the intertemporal utilities of both agents:

$$\begin{aligned} W(\mathcal{C}^1, \mathcal{C}^2) = \lambda \sum_{t=0}^{\infty} \beta_1^t u(c_t^1) + (1-\lambda) \sum_{t=0}^{\infty} \beta_2^t u(c_t^2) \\ = \lambda u(c_0^1) + (1-\lambda) u(c_0^2) + \lambda \beta_1 u(c_1^1) + (1-\lambda)\beta_2 u(c_1^2) \\ + \lambda \beta_1^2 u(c_2^1) + (1-\lambda)\beta_2^2 u(c_2^2) + \dots, \end{aligned} \tag{2.3}$$

where $\lambda \geq 0$ and $1 - \lambda \geq 0$ are the constant non-negative Pareto weights that sum to 1.

A Pareto-optimal allocation is a solution to the optimization problem of a social planner who maximizes social welfare function (2.3):

$$\begin{aligned} &\max \left\{ \lambda \sum_{t=0}^{\infty} \beta_1^t u(c_t^1) + (1-\lambda) \sum_{t=0}^{\infty} \beta_2^t u(c_t^2) \right\}, \\ &\text{s. t. } c_t^1 + c_t^2 + k_{t+1} = f(k_t), \\ &\quad c_t^1 \geq 0, \quad c_t^2 \geq 0, \quad k_{t+1} \geq 0, \quad t = 0, 1, \dots, \\ &\quad k_0 = \hat{k}_0. \end{aligned} \tag{2.4}$$

All Pareto-optimal allocations can be found by varying $0 \leq \lambda \leq 1$. Under some mild assumptions, problem (2.4) also has a unique solution, $\{c_t^{1*}, c_t^{2*}, k_{t+1}^*\}_{t=0}^{\infty}$ (see, e.g., Le Van and Vailakis, 2003). Clearly, this solution depends on λ.

The properties of the Paretian social welfare function of the form (2.3) in a continuous-time framework are studied by Gollier and Zeckhauser (2005). In their model, agents have additively time-separable utility functions and differ in their discount rates as well as felicity functions. A social planner determines a Pareto-optimal allocation of an exogenously given flow of the single consumption good among agents. Gollier and Zeckhauser construct the Paretian social welfare function (i.e., a weighted sum of the utility functions of all agents), show that it is also additively time-separable and naturally define the instantaneous discount rate of the social planner. They prove that the discount rate of the social planner is a weighted mean of the agents' discount rates with weights being proportional to the corresponding individual tolerances for consumption fluctuations (i.e., the reciprocals of the Arrow–Pratt measure of absolute risk-aversion). Whenever agents are heterogeneous in their rates of time preference, the Paretian social planner generically has the time-varying discount rate. Moreover, under the usual assumption that felicity functions of all agents exhibit increasing tolerance for consumption fluctuations (i.e., decreasing absolute risk aversion), the weights of the more patient consumers are growing in the discount rate of the social planner as time goes forward. It is shown that the discount rate of the social planner tends to the discount rate of the most patient agent.[7]

Note that in both problems (2.2) and (2.4) the sum of discounted utilities is maximized "once-and-for-all" at date 0. This is an atemporal view on the problems in question. However, choices in the real world can hardly be regarded as once-and-for-all choices among specific plans of actions. Instead, they are sequential step-by-step choices based on the presently available opportunities. To illustrate this point, Koopmans (1967) used a metaphor of ascending a mountain covered with fog. In these circumstances, "Rather than searching for a largely invisible optimal path, one may have to look for a good rule for choosing the next stretch of the path with the help of all information available at the time" (Koopmans, 1967, p. 12).

A typical difficulty with the once-and-for-all choices is a problem of precommitment. The optimality criteria in both problems (2.2) and (2.4) presume that the decision maker ("representative agent" and "social planner" correspondingly) can credibly commit to implement her decisions in the future. But what happens if a decision maker cannot precommit her future behavior?[8] Then at each date τ ("today") she has to reconsider the optimal solution implemented at date $\tau - 1$ ("yesterday") and to solve a new problem. Thus each decision maker actually faces a **sequence** of problems, each problem at different date. This view on growth models, emphasized by Koopmans (1967), is essentially intertemporal.

Problem (2.2) in our terms is the date 0 problem. At date 1 the representative agent faces with the new optimization problem:

$$\begin{aligned} \max\ & \sum_{t=1}^{\infty} \beta^{t-1} u(C_t), \\ \text{s. t.}\ & C_t + k_{t+1} = f(k_t), \\ & C_t \geq 0, \quad k_{t+1} \geq 0, \quad t = 1, 2, \dots, \\ & k_1 = k_1^*, \end{aligned} \tag{2.5}$$

[7] In the model of Gollier and Zeckhauser (2005), the amount of consumption good at each date is fixed and non-storable. Heal and Millner (2013) generalize this model by endogenizing agents' consumption. In their model agents derive consumption from a common managed resource which can be thought of as physical or natural capital. They also show that the discount rate of the social planner asymptotically approaches the discount rate of the most patient member of the society.

[8] It is well-known (see, e.g., Kydland and Prescott, 1977; Fischer, 1980) that if the current decision explicitly affects future opportunity sets, then the currently chosen plan of actions typically will not be followed in the future by the rational individual. The problems of precommitment and dynamic inconsistency in the framework of the optimal growth model were studied by a number of prominent economists (see, e.g., Strotz, 1956; Pollak, 1968; Phelps and Pollak, 1968; Peleg and Yaari, 1973), and many of their insights are also helpful in the social choice framework.

where k_1^* is the optimal capital stock at date 1 from the time 0 perspective (i.e., the capital stock in the economy at the beginning of period 1 in the optimal solution determined at date 0).

The date 1 utility function,

$$\sum_{t=1}^{\infty} \beta^{t-1} u(C_t) = u(C_1) + \beta u(C_2) + \beta^2 u(C_3) + \dots,$$

has the same form as the date 0 utility function (2.1), because from the perspective of the decision maker time is restarted. However, being multiplied by β, the date 1 utility function coincides with the "tail" of the date 0 utility function (2.1) which starts from the moment of time 1:

$$\beta \sum_{t=1}^{\infty} \beta^{t-1} u(C_t) = \beta u(C_1) + \beta^2 u(C_2) + \beta^3 u(C_3) + \dots .$$

Hence the date 1 problem (2.5) is equivalent to the following problem:

$$\begin{aligned} \max \ & \sum_{t=1}^{\infty} \beta^t u(C_t), \\ \text{s. t.} \ & C_t + k_{t+1} = f(k_t), \\ & C_t \geq 0, \quad k_{t+1} \geq 0, \quad t = 1, 2, \dots, \\ & k_1 = k_1^*. \end{aligned}$$

Clearly, the solution to the latter problem, and hence to problem (2.5), coincides with the "tail" of the solution determined at date 0, $\{C_t^*, k_{t+1}^*\}_{t=1}^{\infty}$.

Thus for the representative agent whose utility function at date 0 is given by (2.1), i.e., whose preferences exhibit constant exponential discounting, it makes no difference whether she can commit or not: the optimal solution chosen at date 0 once and for all remains optimal at any future date.

This well-known observation creates a wrong impression that the preferences of the decision maker can be described by a single utility function and obscures the fact that the decision maker actually has to solve a **sequence of problems**, so that her preferences are described by a **sequence of utility functions**.

This becomes clear when we consider the problem of precommitment for the social planner. At date 1 the social planner faces with the new optimization problem:

$$\begin{aligned} &\max \left\{ \lambda_1 \sum_{t=1}^{\infty} \beta_1^{t-1} u(c_t^1) + (1-\lambda_1) \sum_{t=1}^{\infty} \beta_2^{t-1} u(c_t^2) \right\}, \\ &\text{s. t. } \; c_t^1 + c_t^2 + k_{t+1} = f(k_t), \\ &\qquad c_t^1 \geq 0, \quad c_t^2 \geq 0, \quad k_{t+1} \geq 0, \quad t = 1, 2, \ldots, \\ &\qquad k_1 = k_1^*, \end{aligned} \tag{2.6}$$

where $\lambda_1 \geq 0$ and $1 - \lambda_1 \geq 0$ are the constant non-negative Pareto weights at date 1. Since this is a new problem, posed at different date, *a priori* there is no particular reason to assume that the planner generically weighs the utilities of agents with the same Pareto weights as at date 0.

The planner's date 1 social welfare function is given by

$$\lambda_1 u(c_1^1) + (1-\lambda_1) u(c_1^2) + \lambda_1 \beta_1 u(c_2^1) + (1-\lambda_1) \beta_2 u(c_2^2) + \ldots,$$

and it does not necessarily coincide with the "tail" of the date 0 social welfare function (2.3):

$$\lambda \beta_1 u(c_1^1) + (1-\lambda) \beta_2 u(c_1^2) + \lambda \beta_1^2 u(c_2^1) + (1-\lambda) \beta_2^2 u(c_2^2) + \ldots.$$

Note that different cases may arise. If at date 1 the planner uses the same Pareto weights as at date 0, $\lambda_1 = \lambda$, then the date 1 social welfare function has the same form as the date 0 social welfare function, but does not coincide with the "tail" of the date 0 social welfare function. It follows that the "tail" of the optimal solution at date 0 is not a solution to the date 1 optimization problem. On the contrary, if at date 1 the planner uses the properly adjusted weights, namely,

$$\lambda_1 = \frac{\lambda \beta_1}{\lambda \beta_1 + (1-\lambda)\beta_2}, \quad 1 - \lambda_1 = \frac{\lambda \beta_2}{\lambda \beta_1 + (1-\lambda)\beta_2},$$

then the date 1 social welfare function coincides with the "tail" of the date 0 social welfare function (up to the constant factor $1/\left(\lambda\beta_1 + (1-\lambda)\beta_2\right)$). In this case, the optimal solution determined at date 0 remains optimal at date 1.

Hence the problem of precommitment, i.e., whether the decision maker can precommit to the initially chosen optimal plan, is actually the problem of which sequence of utility functions the decision maker has. The answer to the question of whether the currently chosen plan of actions is followed in the future by the rational decision maker, is determined by the property of the sequence of preferences which is called **time consistency**.

In order to clarify the notion of time consistency, we have to consider **sequences of preferences** and distinguish between three different properties: time consistency, time invariance and stationarity. Later we will formally define these properties for the sequences of social welfare functions, but let us begin with three simple examples which illustrate time consistency, time invariance and stationarity in terms of the sequences of individual utility functions and shed light to the differences between these properties.

First, consider the sequence of utility functions $\{U_\tau^{TC}\}_{\tau=0}^\infty$ over consumption streams $\{c_t\}_{t=0}^\infty$ which is given by

$$\begin{array}{lcl}
U_0^{TC} & = & u(c_0) + \alpha\beta u(c_1) + \alpha\beta^2 u(c_2) + \dots, \\
U_1^{TC} & = & u(c_1) + \beta u(c_2) + \beta^2 u(c_3) + \dots, \\
U_2^{TC} & = & u(c_2) + \beta u(c_3) + \beta^2 u(c_4) + \dots, \\
 & & \dots.
\end{array}$$

It is easily checked that up to the constant factor $\alpha\beta^\tau$ the utility function U_τ^{TC} at any date τ coincides with the "tail" of the utility function U_0^{TC}. However, due to the presence of α, U_0^{TC} does not have the same form as U_τ^{TC}. Moreover, the discount factor in the utility function U_0^{TC} is not constant: the discount factor between periods 1 and 0 is $\alpha\beta$, while it is β between periods $t+1$ and t for all $t \geq 1$. It is said that the sequence $\{U_\tau^{TC}\}_{\tau=0}^\infty$ is time-consistent, but non-stationary and not time-invariant.

Second, consider the sequence of utility functions $\{U_\tau^{TI}\}_{\tau=0}^\infty$ given by

$$\begin{array}{lcl}
U_0^{TI} & = & u(c_0) + \alpha\beta u(c_1) + \alpha\beta^2 u(c_2) + \dots, \\
U_1^{TI} & = & u(c_1) + \alpha\beta u(c_2) + \alpha\beta^2 u(c_3) + \dots, \\
U_2^{TI} & = & u(c_2) + \alpha\beta u(c_3) + \alpha\beta^2 u(c_4) + \dots, \\
 & & \dots.
\end{array}$$

The utility function U_0^{TI} has precisely the same form as the utility function U_τ^{TI} at any date τ. However, U_τ^{TI} does not coincide with the corresponding "tail" of U_0^{TI}, and again the discount factor in U_τ^{TI} is not constant. It is said that the sequence $\{U_\tau^{TI}\}_{\tau=0}^\infty$ is time-invariant, but non-stationary and not time-consistent.

Finally, consider the sequence of utility functions $\{U_\tau^{ST}\}_{\tau=0}^\infty$ given by

$$\begin{array}{lcl}
U_0^{ST} & = & u(c_0) + \beta_0 u(c_1) + \beta_0^2 u(c_2) + \dots, \\
U_1^{ST} & = & u(c_1) + \beta_1 u(c_2) + \beta_1^2 u(c_3) + \dots, \\
U_2^{ST} & = & u(c_2) + \beta_2 u(c_3) + \beta_2^2 u(c_4) + \dots, \\
 & & \dots.
\end{array}$$

for $\beta_0 \neq \beta_1 \neq \beta_2 \neq \ldots$. For any τ, the discount factor in U_τ^{ST} is constant, and U_τ^{ST} is the exponentially discounted utility function. However, the function U_τ^{ST} depends on τ and is not related to any other function in this sequence. It is said that the sequence $\{U_\tau^{ST}\}_{\tau=0}^{\infty}$ is stationary, but not time-consistent and not time-invariant.

Now let us give formal definitions of time consistency, time invariance and stationarity for the sequences of social welfare functions. In order to link these properties directly to discounting, we assume that any social welfare function in this sequence is a weighted sum of additively time-separable individual utility functions.[9]

Let $\mathcal{C}^i = \{c_t^i\}_{t=0}^{\infty}$ be a private consumption stream for agent i. Suppose that the sequence of preferences over bundles of consumption streams $\mathcal{C} = \{\mathcal{C}^1, \mathcal{C}^2\}$ at different decision dates τ is given by the following sequence of social welfare functions:

$$W_\tau = \sum_{i=1}^{2} \lambda_\tau^i \sum_{t=\tau}^{\infty} \gamma_{\tau,t}^i \; u(c_t^i) = \lambda_\tau^1 \sum_{t=\tau}^{\infty} \gamma_{\tau,t}^1 \; u(c_t^1) + \lambda_\tau^2 \sum_{t=\tau}^{\infty} \gamma_{\tau,t}^2 \; u(c_t^2), \quad \tau = 0, 1, \ldots,$$

where $\lambda_\tau^1 \geq 0$ and $\lambda_\tau^2 \geq 0$ are non-negative Pareto weights at date τ, and

$$\Gamma_\tau^i = \{\gamma_{\tau,t}^i\}_{t=\tau}^{\infty}$$

is a *discount function* of agent i at date τ.[10]

In other words, we assume that the sequence of preferences is such that every element of this sequence (the planner's social welfare function at some decision date) is a weighted sum of additively time-separable utility functions (individual utility functions at the same date). The sequence of individual utility functions is of the form

$$U_\tau^i = \sum_{t=\tau}^{\infty} \gamma_{\tau,t}^i \; u(c_t^i), \quad \tau = 0, 1, \ldots,$$

i.e., in this sequence the felicity function of agent i, $u(c)$, is the same in all U_τ^i (at all dates and for both agents), but discount functions of agent i, Γ_τ^i, may differ for different dates τ.

Note that any discount function Γ_τ^i is defined up to some constant factor: the sequences $\{\gamma_{\tau,\tau}^i, \gamma_{\tau,\tau+1}^i, \gamma_{\tau,\tau+2}^i, \ldots\}$ and $\{1, \frac{\gamma_{\tau,\tau+1}^i}{\gamma_{\tau,\tau}^i}, \frac{\gamma_{\tau,\tau+2}^i}{\gamma_{\tau,\tau}^i}, \ldots\}$ determine the same utility function U_τ^i. Similarly, the Pareto weights $\{\lambda_\tau^1, \lambda_\tau^2\}$ and the same weights multiplied by some constant

[9] General definitions in terms of sequences of preference relations are given in Halevy (2015). See also definitions in terms of history-dependent intertemporal utility functions in Millner and Heal (2016). Once it is understood that these properties should be defined in terms of sequences, the ideas which are behind these notions become clear and transparent.

[10] The term "discount function" originally appeared in continuous-time models, and may be somewhat misleading in the discrete-time framework. Perhaps it would be more correct to call Γ_τ^i a "discount sequence", but we will continue to use the more familiar term "discount function".

factor determine the same social welfare function W_τ. Without any loss of generality we will apply the following convenient and standard normalizations:

$$\gamma^i_{\tau,\tau} = 1, \qquad \tau = 0, 1, \ldots, \qquad i = 1, 2,$$
$$\lambda^1_\tau + \lambda^2_\tau = 1, \qquad \tau = 0, 1, \ldots.$$

Clearly, the sequence of social welfare functions $\{W_\tau\}_{\tau=0}^\infty$ is fully determined by the sequences of discount functions and Pareto weights $\{\Gamma^1_\tau, \Gamma^2_\tau; \lambda^1_\tau, \lambda^2_\tau\}_{\tau=0}^\infty$.

Now let us define three properties of the sequences of social welfare functions. **Time consistency**, introduced by Strotz (1956), requires that the different elements of the sequence of preferences are *consistent*, in the sense that social welfare functions at different dates preserve the preference order between any two consumption streams. Time consistency is a rationality criterion which implies that the choice for any future date is independent of the decision date. If a sequence of the planner's preferences is not time-consistent, she may reverse a decision made at the earlier date. For instance, she may decide to undertake certain expensive project this year, but cancel it halfway next year. Formally, *a sequence* $\{W_\tau\}_{\tau=0}^\infty$ *is time consistent if for any* τ, $\tau' > \tau$, *and any two bundles of consumption streams* $\mathcal{C}$ *and* $\tilde{\mathcal{C}}$,

$$\begin{aligned} &\sum_{i=1}^{2} \lambda^i_\tau \sum_{t=\tau'}^{\infty} \gamma^i_{\tau,t}\, u(c^i_t) \geq \sum_{i=1}^{2} \lambda^i_\tau \sum_{t=\tau'}^{\infty} \gamma^i_{\tau,t}\, u(\tilde{c}^i_t) \\ &\qquad \Longleftrightarrow \sum_{i=1}^{2} \lambda^i_{\tau'} \sum_{t=\tau'}^{\infty} \gamma^i_{\tau',t}\, u(c^i_t) \geq \sum_{i=1}^{2} \lambda^i_{\tau'} \sum_{t=\tau'}^{\infty} \gamma^i_{\tau',t}\, u(\tilde{c}^i_t). \end{aligned} \tag{2.7}$$

Time invariance, introduced recently by Halevy (2015), requires that different elements of the sequence of preferences are *the same*. In other words, time invariance "by itself ... does not impose restrictions on the structure of preferences at any given time, but only implies that preferences are not a function of calendar time" (Halevy, 2015, p. 341). Formally, *a sequence* $\{W_\tau\}_{\tau=0}^\infty$ *is time invariant if for any* τ, $\tau' > \tau$, $\mathcal{C}$ *and* $\tilde{\mathcal{C}}$,

$$\begin{aligned} &\sum_{i=1}^{2} \lambda^i_\tau \sum_{t=\tau}^{\infty} \gamma^i_{\tau,t}\, u(c^i_t) \geq \sum_{i=1}^{2} \lambda^i_\tau \sum_{t=\tau}^{\infty} \gamma^i_{\tau,t}\, u(\tilde{c}^i_t) \\ &\qquad \Longleftrightarrow \sum_{i=1}^{2} \lambda^i_{\tau'} \sum_{t=\tau}^{\infty} \gamma^i_{\tau',\tau'+t-\tau}\, u(c^i_t) \geq \sum_{i=1}^{2} \lambda^i_{\tau'} \sum_{t=\tau}^{\infty} \gamma^i_{\tau',\tau'+t-\tau}\, u(\tilde{c}^i_t). \end{aligned} \tag{2.8}$$

Stationarity, introduced by Koopmans (1960), requires that for a given decision date the preference order between any two consumption streams is preserved when the streams are postponed by the same amount of time. This property plays a key role in the Koopmans' axiomatization of discounted utilitarian time preferences. Formally, *a social welfare function W_τ is stationary if for any $\tau' > \tau$, $\mathcal{C}$ and $\tilde{\mathcal{C}}$,*

$$\begin{aligned} &\sum_{i=1}^{2} \lambda_\tau^i \sum_{t=\tau}^{\infty} \gamma_{\tau,t}^i \; u(c_t^i) \geq \sum_{i=1}^{2} \lambda_\tau^i \sum_{t=\tau}^{\infty} \gamma_{\tau,t}^i \; u(\tilde{c}_t^i) \\ &\qquad \iff \sum_{i=1}^{2} \lambda_\tau^i \sum_{t=\tau}^{\infty} \gamma_{\tau,\tau'+t-\tau}^i \; u(c_t^i) \geq \sum_{i=1}^{2} \lambda_\tau^i \sum_{t=\tau}^{\infty} \gamma_{\tau,\tau'+t-\tau}^i \; u(\tilde{c}_t^i). \end{aligned} \tag{2.9}$$

A sequence $\{W_\tau\}_{\tau=0}^\infty$ is stationary if W_τ is stationary for each τ.

Note that stationarity is a property of the preferences at a given decision date. Time consistency and time invariance deals with decisions made at different dates, and hence they are applied to the sequences of preferences.[11]

The above definitions look somewhat complicated and cumbersome. They are usually given in a general axiomatic context, where they look even more obscure. However, these complicated definitions are actually related to simple properties of discounting. The following three propositions show that time consistency, time invariance and stationarity of the sequence of social welfare functions imply special forms of the individuals' discount functions. While these results are by no means new or surprising, they deserve to be explicitly stated and proved in simple terms of sequences of social welfare functions, as they clarify many important issues regarding collective intertemporal choice.

The following proposition characterizes a time-consistent sequence of social welfare functions.

Proposition 2.1. *A sequence of discount functions and positive Pareto weights $\{\Gamma_\tau^1, \Gamma_\tau^2; \lambda_\tau^1, \lambda_\tau^2\}_{\tau=0}^\infty$ determines a time-consistent sequence $\{W_\tau\}_{\tau=0}^\infty$ if and only if for all τ, $\tau' > \tau$ and $i = 1, 2$,*

$$\frac{\lambda_\tau^i \gamma_{\tau,t}^i}{\lambda_\tau^1 \gamma_{\tau,\tau'}^1 + \lambda_\tau^2 \gamma_{\tau,\tau'}^2} = \lambda_{\tau'}^i \gamma_{\tau',t}^i, \quad t = \tau', \tau'+1, \ldots. \tag{2.10}$$

Proof. Suppose that (2.10) holds. Then the condition in (2.7) is trivially satisfied, as for all τ, $\tau' > \tau$, $\mathcal{C}^i = \{c_t^i\}_{t=0}^\infty$ and $i = 1, 2$,

$$\frac{1}{\lambda_\tau^1 \gamma_{\tau,\tau'}^1 + \lambda_\tau^2 \gamma_{\tau,\tau'}^2} \lambda_\tau^i \sum_{t=\tau'}^{\infty} \gamma_{\tau,t}^i \; u(c_t^i) = \lambda_{\tau'}^i \sum_{t=\tau'}^{\infty} \gamma_{\tau',t}^i \; u(c_t^i).$$

[11] It is clear that the above definitions can be easily generalized to the case where any number of agents with arbitrary felicity functions enjoy private consumption streams, as well as to the case where an arbitrary number of agents have common consumption stream.

Conversely, suppose that (2.7) holds for all bundles $\mathcal{C}$ and $\tilde{\mathcal{C}}$. Denote for simplicity $\mu_\tau^i = \frac{\lambda_\tau^i}{\lambda_\tau^1 \gamma_{\tau,\tau'}^1 + \lambda_\tau^2 \gamma_{\tau,\tau'}^2}$, and suppose further that (2.10) fails to hold, i.e., for some τ, $\tau' > \tau$ and some i^* there exists $T \geq \tau'$ such that

$$\mu_\tau^{i^*} \gamma_{\tau,T}^{i^*} \neq \lambda_{\tau'}^{i^*} \gamma_{\tau',T}^{i^*}, \qquad \mu_\tau^i \gamma_{\tau,T}^i = \lambda_{\tau'}^i \gamma_{\tau',T}^i, \quad i \neq i^*,$$
$$\mu_\tau^i \gamma_{\tau,t}^i = \lambda_{\tau'}^i \gamma_{\tau',t}^i, \qquad t \neq T, \quad i = 1, 2.$$

Let for definiteness $i^* = 1$, and $\mu_\tau^1 \gamma_{\tau,T}^1 > \lambda_{\tau'}^1 \gamma_{\tau',T}^1$ (all other cases can be considered similarly). Then it is possible to construct two consumption bundles, $\mathcal{C} = \{\mathcal{C}^1, \mathcal{C}^2\}$ and $\tilde{\mathcal{C}} = \{\tilde{\mathcal{C}^1}, \tilde{\mathcal{C}^2}\}$, which satisfy the following conditions:

$$u(c_T^1) - u(\tilde{c}_T^1) = 1, \qquad \lambda_{\tau'}^1 \gamma_{\tau',T}^1 < \mu_\tau^1 \sum_{t \geq \tau',\ t \neq T} \gamma_{\tau,t}^1 \left(u(\tilde{c}_t^1) - u(c_t^1) \right) \leq \mu_\tau^1 \gamma_{\tau,T}^1,$$
$$c_t^2 = \tilde{c}_t^2, \quad t = \tau', \tau' + 1, \ldots .$$

It follows that

$$\mu_\tau^1 \gamma_{\tau,T}^1 \left(u(c_T^1) - u(\tilde{c}_T^1) \right) = \mu_\tau^1 \gamma_{\tau,T}^1 \geq \mu_\tau^1 \sum_{t \geq \tau',\ t \neq T} \gamma_{\tau,t}^1 \left(u(\tilde{c}_t^1) - u(c_t^1) \right),$$

which is equivalent to

$$\lambda_\tau^1 \sum_{t=\tau'}^{\infty} \gamma_{\tau,t}^1\, u(c_t^1) \geq \lambda_\tau^1 \sum_{t=\tau'}^{\infty} \gamma_{\tau,t}^1\, u(\tilde{c}_t^1).$$

Moreover,

$$\lambda_\tau^2 \sum_{t=\tau'}^{\infty} \gamma_{\tau,t}^2\, u(c_t^2) = \lambda_\tau^2 \sum_{t=\tau'}^{\infty} \gamma_{\tau,t}^2\, u(\tilde{c}_t^2),$$

and hence

$$\sum_{i=1}^{2} \lambda_\tau^i \sum_{t=\tau'}^{\infty} \gamma_{\tau,t}^i\, u(c_t^i) \geq \sum_{i=1}^{2} \lambda_\tau^i \sum_{t=\tau'}^{\infty} \gamma_{\tau,t}^i\, u(\tilde{c}_t^i). \tag{2.11}$$

At the same time,

$$\lambda_{\tau'}^1 \gamma_{\tau',T}^1 \left(u(c_T^1) - u(\tilde{c}_T^1) \right) = \lambda_{\tau'}^1 \gamma_{\tau',T}^1$$
$$< \mu_\tau^1 \sum_{t \geq \tau',\ t \neq T} \gamma_{\tau,t}^1 \left(u(\tilde{c}_t^1) - u(c_t^1) \right) = \lambda_{\tau'}^1 \sum_{t \geq \tau',\ t \neq T} \gamma_{\tau',t}^1 \left(u(\tilde{c}_t^1) - u(c_t^1) \right),$$

from which follows that

$$\lambda_{\tau'}^1 \sum_{t=\tau'}^{\infty} \gamma_{\tau',t}^1\, u(c_t^1) < \lambda_{\tau'}^1 \sum_{t=\tau'}^{\infty} \gamma_{\tau',t}^1\, u(\tilde{c}_t^1).$$

Furthermore,

$$\lambda^2_{\tau'} \sum_{t=\tau'}^{\infty} \gamma^2_{\tau',t} \; u(c^2_t) = \lambda^2_{\tau'} \sum_{t=\tau'}^{\infty} \gamma^2_{\tau',t} \; u(\tilde{c}^2_t),$$

and hence

$$\sum_{i=1}^{2} \lambda^i_{\tau'} \sum_{t=\tau'}^{\infty} \gamma^i_{\tau',t} \; u(c^i_t) < \sum_{i=1}^{2} \lambda^i_{\tau'} \sum_{t=\tau'}^{\infty} \gamma^i_{\tau',t} \; u(\tilde{c}^i_t). \tag{2.12}$$

Clearly, (2.11) and (2.12) contradict (2.7), which proves the proposition. □

It immediately follows from Proposition 2.1 that for a sequence of social welfare functions to be time-consistent, the Pareto weights at different dates should be such that for all τ, $\tau' > \tau$ and $i = 1, 2$,

$$\lambda^i_{\tau'} = \frac{\lambda^i_\tau \gamma^i_{\tau,\tau'}}{\lambda^1_\tau \gamma^1_{\tau,\tau'} + \lambda^2_\tau \gamma^2_{\tau,\tau'}}.$$

The following proposition provides a characterization of a time-invariant sequence of social welfare functions.

Proposition 2.2. *A sequence of discount functions and positive Pareto weights* $\{\Gamma^1_\tau, \Gamma^2_\tau; \lambda^1_\tau, \lambda^2_\tau\}_{\tau=0}^\infty$ *determines a time-invariant sequence* $\{W_\tau\}_{\tau=0}^\infty$ *if and only if for all* τ, $\tau' > \tau$ *and* $i = 1, 2$,

$$\lambda^i_\tau \gamma^i_{\tau,t} \; = \lambda^i_{\tau'} \gamma^i_{\tau',\tau'+t-\tau} \; , \quad t = \tau, \tau+1, \ldots . \tag{2.13}$$

Proof. The "if" part is trivial: it follows from (2.13) that for all τ, $\tau' > \tau$, $\mathcal{C}^i = \{c^i_t\}_{t=0}^\infty$ and $i = 1, 2$,

$$\lambda^i_\tau \sum_{t=\tau}^{\infty} \gamma^i_{\tau,t} \; u(c^i_t) = \lambda^i_{\tau'} \sum_{t=\tau}^{\infty} \gamma^i_{\tau',\tau'+t-\tau} \; u(c^i_t).$$

To prove the "only if" part, suppose that (2.8) holds for all bundles $\mathcal{C}$ and $\tilde{\mathcal{C}}$, while (2.13) fails to hold, i.e., for some τ, $\tau' > \tau$ there exists $T \geq \tau$ such that

$$\lambda^1_\tau \gamma^1_{\tau,T} \; > \lambda^1_{\tau'} \gamma^1_{\tau',\tau'+T-\tau} \; , \qquad \lambda^2_\tau \gamma^i_{\tau,T} \; = \lambda^2_{\tau'} \gamma^2_{\tau',\tau'+T-\tau} \; ,$$
$$\lambda^i_\tau \gamma^i_{\tau,t} \; = \lambda^i_{\tau'} \gamma^i_{\tau',\tau'+t-\tau} \; , \qquad t \neq T, \quad i = 1, 2.$$

Again, all other cases can be considered similarly. It is possible to construct two consumption bundles, $\mathcal{C} = \{\mathcal{C}^1, \mathcal{C}^2\}$ and $\tilde{\mathcal{C}} = \{\tilde{\mathcal{C}}^1, \tilde{\mathcal{C}}^2\}$, which satisfy the following conditions:

$$u(c^1_T) - u(\tilde{c}^1_T) = 1, \qquad \lambda^1_{\tau'} \gamma^1_{\tau',\tau'+T-\tau} \; < \lambda^1_\tau \sum_{t \geq \tau, \; t \neq T} \gamma^1_{\tau,t} \; \left(u(\tilde{c}^1_t) - u(c^1_t)\right) \leq \lambda^1_\tau \gamma^1_{\tau,T} \; ,$$
$$c^2_t = \tilde{c}^2_t, \quad t = \tau, \tau+1, \ldots .$$

Therefore,

$$\lambda^1_\tau \gamma^1_{\tau,T} \ \left(u(c^1_T) - u(\tilde{c}^1_T)\right) = \lambda^1_\tau \gamma^1_{\tau,T} \geq \lambda^1_\tau \sum_{t\geq\tau,\ t\neq T} \gamma^1_{\tau,t} \ \left(u(\tilde{c}^1_t) - u(c^1_t)\right),$$

while

$$\begin{aligned}\lambda^1_{\tau'} \gamma^1_{\tau',\tau'+T-\tau} \ \left(u(c^1_T) - u(\tilde{c}^1_T)\right) &= \lambda^1_{\tau'} \gamma^1_{\tau',\tau'+T-\tau} \\ &< \lambda^1_\tau \sum_{t\geq\tau,\ t\neq T} \gamma^1_{\tau,t} \ \left(u(\tilde{c}^1_t) - u(c^1_t)\right) = \lambda^1_{\tau'} \sum_{t\geq\tau,\ t\neq T} \gamma^1_{\tau',\tau'+t-\tau} \ \left(u(\tilde{c}^1_t) - u(c^1_t)\right).\end{aligned}$$

Since

$$\begin{aligned}\lambda^2_\tau \sum_{t=\tau}^{\infty} \gamma^2_{\tau,t} \ u(c^2_t) &= \lambda^2_\tau \sum_{t=\tau}^{\infty} \gamma^2_{\tau,t} \ u(\tilde{c}^2_t), \quad \text{and} \\ \lambda^2_{\tau'} \sum_{t=\tau}^{\infty} \gamma^2_{\tau',\tau'+t-\tau} \ u(c^2_t) &= \lambda^2_{\tau'} \sum_{t=\tau}^{\infty} \gamma^2_{\tau',\tau'+t-\tau} \ u(\tilde{c}^2_t),\end{aligned}$$

it follows that for these $\mathcal{C}$ and $\tilde{\mathcal{C}}$,

$$\begin{aligned}\sum_{i=1}^{2} \lambda^i_\tau \sum_{t=\tau}^{\infty} \gamma^i_{\tau,t} \ u(c^i_t) &\geq \sum_{i=1}^{2} \lambda^i_\tau \sum_{t=\tau}^{\infty} \gamma^i_{\tau,t} \ u(\tilde{c}^i_t), \quad \text{but} \\ \sum_{i=1}^{2} \lambda^i_{\tau'} \sum_{t=\tau}^{\infty} \gamma^i_{\tau',\tau'+t-\tau} \ u(c^i_t) &< \sum_{i=1}^{2} \lambda^i_{\tau'} \sum_{t=\tau}^{\infty} \gamma^i_{\tau',\tau'+t-\tau} \ u(\tilde{c}^i_t),\end{aligned}$$

which contradicts (2.8). □

It follows from Proposition 2.2 that a time-invariant sequence of social welfare functions $\{W_\tau\}_{\tau=0}^{\infty}$ is characterized by the same Pareto weights and the same discount functions at each date: for all τ, $\tau' > \tau$ and $i = 1, 2$, $\lambda^i_\tau = \lambda^i_{\tau'}$ and $\Gamma^i_\tau = \Gamma^i_{\tau'}$. This is the only way to ensure that the social welfare function W_τ in this sequence does not depend on τ.

The following proposition fully characterizes a stationary social welfare function.

Proposition 2.3. *Discount functions and positive Pareto weights* $\{\Gamma^1_\tau, \Gamma^2_\tau; \lambda^1_\tau, \lambda^2_\tau\}$ *determine a stationary social welfare function* W_τ *if and only if*

$$\gamma^i_{\tau,t} = \beta_\tau^{t-\tau}, \qquad t = \tau, \tau+1, \ldots, \quad i = 1, 2, \tag{2.14}$$

for some constant $0 < \beta_\tau < 1$.

Proof. Suppose that (2.14) holds. Then for all $\tau' > \tau$, $\mathcal{C}^i = \{c_t^i\}_{t=0}^{\infty}$ and $i = 1, 2$,

$$\sum_{t=\tau}^{\infty} \gamma_{\tau,t}^i \; u(c_t^i) = \sum_{t=\tau}^{\infty} \beta_\tau^{t-\tau} u(c_t^i) = \frac{1}{\beta_\tau^{\tau'-\tau}} \sum_{t=\tau}^{\infty} \beta_\tau^{\tau'+t-2\tau} u(c_t^i) = \frac{1}{\beta_\tau^{\tau'-\tau}} \sum_{t=\tau}^{\infty} \gamma_{\tau,\tau'+t-\tau}^i \; u(c_t^i),$$

which ensures (2.9) for any λ_τ^1 and λ_τ^2.

Conversely, suppose that (2.9) holds for all bundles $\mathcal{C}$ and $\tilde{\mathcal{C}}$, while (2.14) fails to hold, i.e., there exists $T \geq \tau$ such that

$$\gamma_{\tau,T}^1 \; > \beta_\tau^{T-\tau}, \qquad \gamma_{\tau,T}^2 \; = \beta_\tau^{T-\tau}, \qquad \gamma_{\tau,t}^i \; = \beta_\tau^{t-\tau}, \quad t \neq T, \quad i = 1, 2.$$

Let us construct two consumption bundles, $\mathcal{C} = \{\mathcal{C}^1, \mathcal{C}^2\}$ and $\tilde{\mathcal{C}} = \{\tilde{\mathcal{C}}^1, \tilde{\mathcal{C}}^2\}$, such that

$$u(c_T^1) - u(\tilde{c}_T^1) = 1, \qquad \beta_\tau^{T-\tau} < \sum_{t \geq \tau, \; t \neq T} \beta_\tau^{t-\tau} \left(u(\tilde{c}_t^1) - u(c_t^1) \right) \leq \gamma_{\tau,T}^1,$$
$$c_t^2 = \tilde{c}_t^2, \quad t = \tau, \tau+1, \ldots.$$

It follows that for these consumption bundles,

$$\gamma_{\tau,T}^1 \left(u(c_T^1) - u(\tilde{c}_T^1) \right) = \gamma_{\tau,T}^1 \geq \sum_{t \geq \tau, \; t \neq T}^{\infty} \beta_\tau^{t-\tau} \left(u(\tilde{c}_t^1) - u(c_t^1) \right),$$

and hence

$$\sum_{t=\tau}^{\infty} \gamma_{\tau,t}^1 \; u(c_t^1) \geq \sum_{t=\tau}^{\infty} \gamma_{\tau,t}^1 \; u(\tilde{c}_t^1).$$

At the same time,

$$\beta_\tau^{T-\tau} \left(u(c_T^1) - u(\tilde{c}_T^1) \right) = \beta_\tau^{T-\tau} < \sum_{t \geq \tau, \; t \neq T} \beta_\tau^{t-\tau} \left(u(\tilde{c}_t^1) - u(c_t^1) \right),$$

from which follows that

$$\sum_{t=\tau}^{\infty} \beta_\tau^{t-\tau} u(c_t^1) < \sum_{t=\tau}^{\infty} \beta_\tau^{t-\tau} u(\tilde{c}_t^1),$$

and hence for any $\tau' > T$,

$$\sum_{t=\tau}^{\infty} \gamma_{\tau,\tau'+t-\tau}^1 \; u(c_t^1) < \sum_{t=\tau}^{\infty} \gamma_{\tau,\tau'+t-\tau}^1 \; u(\tilde{c}_t^1).$$

Since

$$\sum_{t=\tau}^{\infty} \beta_\tau^{t-\tau} u(c_t^2) = \sum_{t=\tau}^{\infty} \beta_\tau^{t-\tau} u(\tilde{c}_t^2),$$

for any $\lambda_\tau^1 > 0$, $\lambda_\tau^2 > 0$ and $\tau' > T$ we have

$$\sum_{i=1}^{2} \lambda_\tau^i \sum_{t=\tau}^{\infty} \gamma_{\tau,t}^i \; u(c_t^i) \geq \sum_{i=1}^{2} \lambda_\tau^i \sum_{t=\tau}^{\infty} \gamma_{\tau,t}^i \; u(\tilde{c}_t^i), \quad \text{but}$$
$$\sum_{i=1}^{2} \lambda_\tau^i \sum_{t=\tau}^{\infty} \gamma_{\tau,\tau'+t-\tau}^i \; u(c_t^i) < \sum_{i=1}^{2} \lambda_\tau^i \sum_{t=\tau}^{\infty} \gamma_{\tau,\tau'+t-\tau}^i \; u(\tilde{c}_t^i),$$

a contradiction with (2.9). □

Proposition 2.3 implies that the planner's social welfare function W_τ, which is a weighted sum of individual utility functions, is stationary if and only if both individual utility functions exhibit constant exponential discounting with the same discount factor. This is a famous result of Koopmans (1960), formulated in our terms. Whenever the discount factors of individuals are different, the social welfare function which takes into account both of them is non-stationary.

Now let us show that time consistency, time invariance and stationarity are closely interrelated. Consider the sequence of discount functions and Pareto weights $\{\Gamma_\tau^1, \Gamma_\tau^2; \lambda_\tau^1, \lambda_\tau^2\}_{\tau=0}^{\infty}$ such that discount functions of both agents are exponential and at each date are generated by **the same** discount factor $0 < \beta < 1$:

$$\Gamma_\tau^i = \{1, \beta, \beta^2, \ldots\}, \quad \tau = 0, 1, \ldots, \quad i = 1, 2,$$

and positive Pareto weights are the same at each date:

$$\lambda_\tau^i = \lambda^i, \quad \tau = 0, 1, \ldots, \quad i = 1, 2.$$

The corresponding sequence of social welfare functions $\{W_\tau\}_{\tau=0}^{\infty}$ is given by

$$W_\tau = \sum_{i=1}^{2} \lambda^i \sum_{t=\tau}^{\infty} \beta^{t-\tau} u(c_t^i) = \sum_{t=\tau}^{\infty} \beta^{t-\tau} \sum_{i=1}^{2} \lambda^i u(c_t^i), \quad \tau = 0, 1, \ldots. \tag{2.15}$$

We show that a sequence of social welfare functions satisfies any two of the considered properties if and only if it is given by (2.15).

Proposition 2.4. *A sequence of social welfare functions $\{W_\tau\}_{\tau=0}^{\infty}$ satisfies any two of the three properties: time consistency, time invariance and stationarity, if and only if it is of the form (2.15).*

Proof. First, $\{W_\tau\}_{\tau=0}^{\infty}$ is stationary and time-invariant if and only if for all τ and $i = 1, 2$, the date τ discount functions of both agents are of the same form $\Gamma_\tau^i = \{1, \beta_\tau, \beta_\tau^2, \ldots\}$, and for all $\tau' > \tau$, $\lambda^i_\tau = \lambda^i_{\tau'}$ and $\Gamma^i_\tau = \Gamma^i_{\tau'}$. This is possible if and only if for all τ, $\tau' > \tau$, and $i = 1,2$, $\lambda^i_\tau = \lambda^i_{\tau'} = \lambda^i$ and $\beta_\tau = \beta_{\tau'} = \beta$.

Second, $\{W_\tau\}_{\tau=0}^\infty$ is stationary and time-consistent if and only if for all τ and $i = 1,2$, $\Gamma^i_\tau = \{1, \beta_\tau, \beta_\tau^2, \ldots\}$ and for all $\tau' > \tau$, (2.10) holds, i.e.,

$$\begin{aligned}
\lambda^i_{\tau'} &= \frac{\lambda^i_\tau \beta_\tau^{\tau'}}{\lambda^1_\tau \beta_\tau^{\tau'} + \lambda^2_\tau \beta_\tau^{\tau'}} = \lambda^i_\tau, \\
\lambda^i_{\tau'}\beta_{\tau'} &= \frac{\lambda^i_\tau \beta_\tau^{\tau'+1}}{\lambda^1_\tau \beta_\tau^{\tau'} + \lambda^2_\tau \beta_\tau^{\tau'}} = \lambda^i_{\tau'}\beta_\tau, \\
&\ldots .
\end{aligned}$$

Obviously, this is also equivalent to (2.15).

Finally, $\{W_\tau\}_{\tau=0}^\infty$ is time-consistent and time-invariant if and only if for all τ, $\tau' > \tau$ and $i = 1,2$, $\lambda^i_\tau = \lambda^i_{\tau'}$, $\Gamma^i_\tau = \Gamma^i_{\tau'}$ and (2.10) holds:

$$\begin{aligned}
\lambda^i_{\tau'} &= \frac{\lambda^i_\tau \gamma^i_{\tau,\tau'}}{\lambda^1_\tau \gamma^1_{\tau,\tau'} + \lambda^2_\tau \gamma^2_{\tau,\tau'}}, \\
\lambda^i_{\tau'}\gamma^i_{\tau',\tau'+1} &= \frac{\lambda^i_\tau \gamma^i_{\tau,\tau'+1}}{\lambda^1_\tau \gamma^1_{\tau,\tau'} + \lambda^2_\tau \gamma^2_{\tau,\tau'}} = \lambda^i_{\tau'}\frac{\gamma^i_{\tau,\tau'+1}}{\gamma^i_{\tau,\tau'}}, \\
&\ldots .
\end{aligned}$$

Since $\lambda^i_\tau = \lambda^i_{\tau'}$, it is clear that $\gamma^1_{\tau,\tau'} = \gamma^2_{\tau,\tau'}$. Using the fact that $\Gamma^i_\tau = \Gamma^i_{\tau'}$, we also get

$$\frac{\gamma^i_{\tau,\tau'+t}}{\gamma^i_{\tau,\tau'}} = \gamma^i_{\tau',\tau'+t} = \gamma^i_{\tau,\tau+t}\ , \quad t = 1,2,\ldots .$$

Hence for all τ and $i = 1,2$,

$$\gamma^i_{\tau,\tau+1+t} = \left(\gamma^i_{\tau,\tau+1}\right)^t, \quad t = 1,2,\ldots .$$

Denoting $\gamma^1_{\tau,\tau+1} = \gamma^2_{\tau,\tau+1} = \beta$ and $\lambda^i_\tau = \lambda^i_{\tau'} = \lambda_i$, again yields (2.15). □

Our results indicate that while stationarity, time consistency and time invariance are pairwise independent, any two properties imply the third.[12] Moreover, Proposition 2.4 shows that a sequence of social welfare functions satisfies any two of the considered properties if and only if the corresponding individual utility functions at each date exhibit constant exponential discounting with **the same discount factor**.

[12] This illustrates a general result of Halevy (2015), proved in terms of preference relations over temporal payments.

Now let us illustrate these definitions and properties in our two-agent Ramsey model. The sequence of individual preferences in the two-agent Ramsey model with private consumption naturally has the form

$$U_\tau^i = \sum_{t=\tau}^{\infty} \beta_i^{t-\tau} u(c_t^i), \quad \tau = 0, 1, \ldots.$$

For each individual, this sequence is time-consistent, time-invariant and stationary. Which of these properties are inherited by the sequence of the planner's social welfare functions? The first and the most evident result is that if a Paretian social welfare function at some date takes into account heterogeneous preferences of both agents, then it is necessarily non-stationary.

Indeed, it follows from Proposition 2.3 that a Paretian social welfare function at date τ, i.e., a weighted sum of functions U_τ^i,

$$W_\tau = \lambda_\tau \sum_{t=\tau}^{\infty} \beta_1^{t-\tau} u(c_t^1) + (1-\lambda_\tau) \sum_{t=\tau}^{\infty} \beta_2^{t-\tau} u(c_t^2),$$

is non-stationary unless $\lambda_\tau = 1$, in which case W_τ coincides with U_τ^1, or $\lambda_\tau = 0$, in which case W_τ coincides with U_τ^2.

Non-stationarity of the social welfare function W_τ leads to the following consequence. Rewrite the function W_τ as

$$\sum_{t=\tau}^{\infty} \beta_1^{t-\tau} \left\{ \lambda_\tau u(c_t^1) + \left(\frac{\beta_2}{\beta_1}\right)^{t-\tau} (1-\lambda_\tau) u(c_t^2) \right\}.$$

Since agent 1 is the most patient agent, the factor $(\beta_2/\beta_1)^{t-\tau}$ converges to zero as time goes to infinity. Hence for $0 < \lambda_\tau < 1$, the relative weight associated with consumption of the less patient agent decreases and become arbitrarily small in the long run. The most patient agent eventually dominates in the date 0 social welfare function, and increasingly influences consumption decisions of the society.

Moreover, since the most patient agent emerges as the dominant consumer in the social welfare function, her socially optimal level of consumption is always positive, and its share in aggregate consumption converges to 1. At the same time, for the less patient agent both the socially optimal level of consumption and its share in aggregate consumption converge to 0. Recall that this effect was already mentioned in Section 2.2, where we have discussed the properties of a competitive equilibrium in the many-agent Ramsey model. The above observation illustrates this property of a competitive equilibrium from the social welfare perspective.

Another interesting result is that a non-stationary sequence of social welfare functions can be either time consistent or time invariant. It is seen by comparing Propositions 2.1 and 2.2. Consider the following sequence of social welfare functions:

$$
\begin{aligned}
W_\tau^{TI} &= \lambda \sum_{t=\tau}^{\infty} \beta_1^{t-\tau} u(c_t^1) + (1-\lambda) \sum_{t=\tau}^{\infty} \beta_2^{t-\tau} u(c_t^2) \\
&= \lambda u(c_\tau^1) + (1-\lambda) u(c_\tau^2) + \lambda \beta_1 u(c_{\tau+1}^1) + (1-\lambda)\beta_2 u(c_{\tau+1}^2) + \ldots, \quad \tau = 0, 1, \ldots.
\end{aligned}
$$

The planner uses the same Pareto weights λ and $1-\lambda$ at each date. It follows from Propositions 2.1 and 2.2 that for $0 < \lambda < 1$, the sequence $\{W_\tau^{TI}\}_{\tau=0}^{\infty}$ is time-invariant but not time-consistent.

Suppose now that at date τ the planner assigns to agents the weights

$$
\lambda_\tau^1 = \frac{\lambda \beta_1^\tau}{\lambda \beta_1^\tau + (1-\lambda)\beta_2^\tau}, \quad 1 - \lambda_\tau^1 = \frac{(1-\lambda)\beta_2^\tau}{\lambda \beta_1^\tau + (1-\lambda)\beta_2^\tau}, \qquad 0 \le \lambda \le 1,
$$

and consider the following sequence of social welfare functions:

$$
W_\tau^{TC} = \lambda_\tau^1 \sum_{t=\tau}^{\infty} \beta_1^{t-\tau} u(c_t^1) + (1-\lambda_\tau^1) \sum_{t=\tau}^{\infty} \beta_2^{t-\tau} u(c_t^2), \quad \tau = 0, 1, \ldots.
$$

Clearly, up to the constant factor $1/\left(\lambda \beta_1^\tau + (1-\lambda)\beta_2^\tau\right)$, the date τ social welfare function has the form

$$
W_\tau^{TC} = \lambda \beta_1^\tau u(c_\tau^1) + (1-\lambda)\beta_2^\tau u(c_\tau^2) + \lambda \beta_1^{\tau+1} u(c_{\tau+1}^1) + (1-\lambda)\beta_2^{\tau+1} u(c_\tau^2) + \ldots, \quad \tau = 0, 1, \ldots.
$$

Hence at any date τ, W_τ^{TC} coincides with the corresponding "tails" of the functions W_t^{TC} at all earlier dates, $0 < t < \tau$. It follows from Propositions 2.1 and 2.2 that the sequence $\{W_\tau^{TC}\}_{\tau=0}^{\infty}$ is time-consistent, but not time-invariant (again, unless $\lambda = 0$ or $\lambda = 1$).[13]

Note that when $\lambda = 0$, $\lambda = 1$ or $\beta_1 = \beta_2$, the social welfare functions W_τ^{TI} and W_τ^{TC} at each date τ coincide. Hence in these special cases, where the planner's preferences essentially depend on a single discount factor, the sequences $\{W_\tau^{TI}\}_{\tau=0}^{\infty}$ and $\{W_\tau^{TC}\}_{\tau=0}^{\infty}$ define the same sequence of social welfare functions which is both time-consistent and time-invariant. Moreover, this sequence is also stationary, as Proposition 2.3 indicates.

It follows from Proposition 2.4 that these special cases are actually the only examples in which the sequence of planner's preferences simultaneously satisfies these three properties. Thus our simple two-agent model shows that it is precisely heterogeneity in discount factors that plays a key role here. This conclusion is a particular case of a very

[13] The general case of the time-consistent sequences of social welfare functions with time-varying Pareto weights is studied in Alcalá (2016).

general result proved by Zuber (2011). In his framework, agents with arbitrary utility functions choose arbitrary consumption streams. He studies the properties of utilitarian aggregation of individual preferences, i.e., social welfare functions in which the individual utility functions get equal weights. In our terms, utilitarian aggregation combines Pareto-efficiency and time invariance. Zuber proves that the sequence of planner's preferences is Paretian, time-invariant, stationary and time-consistent if and only if all sequences of individual preferences exhibit constant exponential discounting with the same discount factor. As he puts it, "Although some people seem to be more patient than others, any departure from the homogeneous patience case would introduce non-stationarity in the planner's objective" (Zuber, 2011, p. 375).

An important way to deal with heterogeneous discount factors in the one-sector many-agent Ramsey model with private consumption was recently proposed by Drugeon and Wigniolle (2016). They consider a slightly different concept of social optimum which is based on the "strategy of consistent planning" introduced by Strotz (1956). At each date the planner chooses the "today's" optimal consumption level for each agent as a function of the current capital stock, under the assumption that the future consumption levels are also chosen optimally using the same function. This procedure ensures time consistency in the optimal choices of the planner at each date. Actually, the proposed sequence of consumption levels is an equilibrium in the Nash sense and has the following consistency property: in any period there is no motivation for the planner to change or to regret her action, given her actions in all other periods.

Drugeon and Wigniolle show that their time-consistent solution can be obtained as a once-and-for-all solution to a standard discounted optimization problem with some time-varying discount function. In our two-agent case, this problem would take the form

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \gamma_{0,t} \left(\lambda u(c_t^1) + (1-\lambda) u(c_t^2) \right), \\ \text{s. t.} \quad & c_t^1 + c_t^2 + k_{t+1} = f(k_t), \quad c_t^1 \geq 0, \quad c_t^2 \geq 0, \quad k_{t+1} \geq 0, \quad t = 0, 1, \dots, \end{aligned}$$

for a properly chosen discount function $\{\gamma_{0,t}\}_{t=0}^{\infty}$ and given $k_0 = \hat{k}_0$.

Drugeon and Wigniolle (2016) study the properties of this solution and show that if agents differ in their discount factors, but have the same felicity function and get equal weights in the social welfare function (in our case, $\lambda = 1/2$), then in the time-consistent solution both agents have identical consumption levels at any date ($c_t^{*1} = c_t^{*2}$ for all t). Here the distribution of discount factors in the population determines only the aggregate consumption level at different dates, and does not influence the allocation of consumption between heterogeneous agents.

The general equilibrium version of the many-agent Ramsey model with private consumption, studied in this section, is usually not paid much attention in social choice theory. The reason is that, as we have mentioned, any equilibrium allocation is Pareto-optimal, and hence all the discussed difficulties with social optima can be circumvented by decentralization and application of the first welfare theorem. Establishing complete system of financial markets in which consumers can lend or borrow against their wealth and letting the markets do their job, leads to an optimal allocation. However, in the next section we consider the many-agent Ramsey model with common consumption, which is very much like the model studied above, but in which social choice plays a central role.

2.4. Social optima in growth models with common consumption

Now let us consider the many-agent Ramsey model with common consumption. In this framework, instead of independent private consumption streams, agents share a common consumption stream which arises from a collectively consumed public good or a common property resource. Hence agents' personal utilities are based on their collective decisions, and an important question is how the society can make a collective decision, i.e., how a common consumption stream can be chosen.

Problems of this kind naturally arise in a wide range of settings related to common property resources. Examples include hunting for animals or grazing of cattle in a common land, pollution of the atmosphere, or drilling for oil in the common underground reservoir.

Consider a village situated near a fishing ground. The fishing ground is self-managed by village citizens who differ in their time preferences. What could be said about the fish stock exploitation, i.e., what is the harvest rate of the fish stock, collectively set by heterogeneous agents? If all village citizens were identical, then the rate of the fish stock exploitation depends on their common discount factor. However, it is not immediately clear, how to determine the harvest rate when there are many different discount factors.

The discussion of the common property resource management was initiated by Hardin (1968), who noted that an open-access resource tends to be overexploited and labeled this situation the "tragedy of the commons". The absence of property rights or difficulties with establishing them lead to the exploitation of the resource at an excessive rate (compared to the socially optimal one). A free access market equilibrium is not Pareto optimal: in the aforementioned different settings there are always excessive fishing, overgrazing, or redundantly rapid depletion of oil.

A typical solution to the "tragedy of the commons" is to establish private property rights. Suppose that instead of the fishing ground, there is a meadow near the village where the citizens graze their cattle. This meadow can be divided into equal plots, and each citizen can be assigned proprietary rights over one of these plots. The introduction of private ownership decentralizes the decision-making process, and the resource exploitation becomes optimal from the perspective of the society.[14] When citizens make individually rational decisions on how many cattle to graze, each citizen's choice does not affect the ability of others to graze, and there is no damage to the commons. Once the property rights are established, each owner acts optimally according to her own rate of time preference. This is quite similar to the case of the many-agent Ramsey model with private consumption considered in the previous section: decentralization leads to a socially optimal outcome.

The situation is different in the case of common property resources or public goods, e.g., underground oil reservoir, the fishing ground or the so-called "global commons". The non-excludability of these goods prevents the enforcement of suitable private property rights (or makes the introduction of private ownership extremely costly). For instance, an obvious tendency of fish to migrate makes it almost impossible to define property rights over the fish stock. At the same time it is pointless to parcel the fishing ground into the different plots, which is anyway a complex and costly technical issue.

One might argue that in these cases a solution may be to introduce a governmental or community resource ownership, and then use quota and licensing systems. There are reasons to believe that if the socially optimal level of total catch is set equal to the sum of quotas, then the competitive price established in the market for quotas ensures the optimal resource exploitation. However, there is immediately another question: what is the level of total catch? There are no market forces to determine the socially optimal level of total catch; it depends on the harvest rate collectively set by the village citizens. Therefore, we came to the point where we have started our discussion: the real question is, what is the rate of the resource exploitation in a heterogeneous society. Thus the difficulties with the common property resource management provide a clear incentive to study the decision-making process under heterogeneous preferences over common consumption streams.

As a natural framework for the analysis of a collective intertemporal decision problem, we employ the one-sector two-agent Ramsey model with common consumption. Again, to simplify the presentation, two consumers in the economy differ only in their discount factors. As in the previous section, in order to discuss the difficulties with collective intertemporal choice, we need to analyze the sequences of preferences.

[14] Of course, this is true only under additional assumption that the property rights are established and enforced costlessly.

Let the preferences of agent i over consumption streams $C = \{c_t\}_{t=0}^{\infty}$ at decision date τ be given by the additively time-separable intertemporal utility function with constant exponential discounting:

$$U_\tau^i(C) = \sum_{t=\tau}^{\infty} \beta_i^{t-\tau} u(c_t), \quad \tau = 0, 1, \dots, \tag{2.16}$$

where β_i is the discount factor and $u(c)$ is the felicity function of agent i. As before, agent 1 is the most patient: $1 > \beta_1 > \beta_2 > 0$.

We assume that $C = \{c_t\}_{t=0}^{\infty}$ is a common consumption stream. In this model, consumption c_t in each period t can be thought as the amount of the extracted renewable or exhaustible natural resource (public good). The increase in the resource stock, $k_{t+1} - k_t$, i.e., its regenerative capacity, is described by a regeneration function $g(k_t)$ which depends on the current size of the stock. If we denote $f(k) = g(k) + k$, then the dynamics of the resource stock becomes $c_t + k_{t+1} = f(k_t)$, which is the familiar resource constraint. If the resource is exhaustible, then $f(k) = k$. If the resource is renewable, it is assumed that $f(k)$ satisfies the same properties as a neoclassical production function. This model is very much like the model with private consumption considered in the previous section, but conveniently interpreted as a common property resource model.

Each agent $i = 1, 2$ at each date τ can determine a consumption stream that is optimal from her perspective, by solving the following problem:

$$\begin{aligned} \max \ & \sum_{t=\tau}^{\infty} \beta_i^{t-\tau} u(c_t), \\ \text{s. t.} \ & c_t + k_{t+1} = f(k_t), \\ & c_t \geq 0, \quad k_{t+1} \geq 0, \quad t = \tau, \tau+1, \dots, \\ & k_\tau = \hat{k}_\tau. \end{aligned} \tag{2.17}$$

Since there are two agents that are heterogeneous in their time preferences, at each date there are two different "optimal" consumption streams, each stream is optimal for different agent. Which consumption stream will be chosen by the society consisting of these heterogeneous agents?

It is natural to assume that each agent plays a role in social decision making, and thus the question is how to aggregate heterogeneous preferences at each single date. This is actually a problem of constructing an appropriate social welfare function. There are two principal ways in which this can be done. The first is to construct a social welfare function *ex ante*, and determine the common consumption stream as a result of the maximization of this function by a social planner. The second is to determine the common consumption stream directly by some social choice procedure (such as majority voting). In the latter case, a social welfare function appears *ex post* being induced by the outcome of this procedure. Now we will focus on the first of these ways. For the discussion of the second way, see Section 2.5.

Suppose that a common consumption stream is determined by a benevolent social planner who maximizes a social welfare function. The social welfare function (sometimes referred to as the collective utility function) at each particular date is constructed from individual utility functions, and somehow takes into account and aggregates the preferences of different agents. Thus the social welfare function (at date τ) $W_\tau(C)$ evaluates different consumption streams from the perspective of the society.

As in the case of private consumption, a very important property of a social welfare function is *Pareto-efficiency. A social welfare function $W_\tau(C)$ is Paretian (Pareto-efficient) if for any consumption streams C and $\tilde{C}$, $U^i_\tau(C) \geq U^i_\tau(\tilde{C})$ for both $i = 1, 2$ implies $W_\tau(C) \geq W_\tau(\tilde{C})$, and the second inequality is strict whenever the first is strict for both $i = 1, 2$.* Pareto efficiency means that if all agents prefer consumption stream C to consumption stream $\tilde{C}$, then the social planner should also prefer C to $\tilde{C}$, i.e., the Paretian social welfare function respects unanimous preference of individuals. Hence this property is sometimes referred to as unanimity and is considered as the minimum reasonable requirement for the social welfare function.

In the two-agent Ramsey model with common as well as with private consumption, the Paretian social welfare function at an arbitrary date τ naturally appears as a weighted sum of the individual date τ utility functions of both agents:

$$W_\tau = \lambda^1_\tau \sum_{t=\tau}^{\infty} \beta_1^{t-\tau} u(c_t) + \lambda^2_\tau \sum_{t=\tau}^{\infty} \beta_2^{t-\tau} u(c_t) = \sum_{t=\tau}^{\infty} \left(\lambda^1_\tau \beta_1^{t-\tau} + \lambda^2_\tau \beta_2^{t-\tau}\right) u(c_t), \tag{2.18}$$

where the Pareto weights $\lambda^1_\tau \geq 0$ and $\lambda^2_\tau \geq 0$ are non-negative and sum to one.

The consumption stream for the society at date τ is obtained as the result of the maximization of the Paretian social welfare function by the social planner, who solves the following problem:

$$\begin{aligned} \max \ & \sum_{t=\tau}^{\infty} \left(\lambda^1_\tau \beta_1^{t-\tau} + \lambda^2_\tau \beta_2^{t-\tau}\right) u(c_t), \\ \text{s. t.} \ & c_t + k_{t+1} = f(k_t), \\ & c_t \geq 0, \quad k_{t+1} \geq 0, \quad t = \tau, \tau+1, \dots, \\ & k_\tau = \hat{k}_\tau. \end{aligned}$$

Note that Paretian social welfare function (2.18) at each date τ resembles the date τ individual utility function from the sequence (2.16), and is clearly additively time-separable. The sequence of social welfare functions has the form

$$W_\tau = \sum_{t=\tau}^{\infty} \gamma_{\tau,t} u(c_t), \quad \tau = 0, 1, \ldots, \tag{2.19}$$

where the planner's discount function at each date τ, $\Gamma_\tau = \{\gamma_{\tau,t}\}_{t=\tau}^{\infty}$, is the weighted average of the individual discount functions with the corresponding Pareto weights λ_τ^1 and λ_τ^2:

$$\gamma_{\tau,t} = \lambda_\tau^1 \beta_1^{t-\tau} + \lambda_\tau^2 \beta_2^{t-\tau}, \quad t = 0, 1, \ldots.$$

Due to the heterogeneity in the discount factors, the planner's discount function at date τ is not a geometric progression unless $\lambda_\tau = 1$ (in which case $\gamma_{\tau,t} = \beta_1^t$) or $\lambda_\tau = 0$ (in which case $\gamma_{\tau,t} = \beta_2^t$). Hence the problem of the social planner is in general a time varying discounted optimal growth problem. Nevertheless, under some reasonable assumptions on the felicity function $u(c)$ and the production function $f(k)$, this problem also has a solution (see, e.g., Mitra, 1979).

Let us show that in the model with common consumption there arise the same difficulties with time consistency, time invariance and stationarity as in the model with private consumption. The definitions given in Section 2.3 can be easily adapted to the sequences of the social welfare functions of the form (2.19). The following propositions are restatements of Propositions 2.1–2.4, and can be proved analogously.

Proposition 2.1′. *A sequence of discount functions $\{\Gamma_\tau\}_{\tau=0}^{\infty}$ determines a time consistent sequence $\{W_\tau\}_{\tau=0}^{\infty}$ if and only if for all τ and $\tau' > \tau$ the discount function $\Gamma_{\tau'}$ up to the constant factor coincides with the "tail" of the discount function Γ_τ starting from τ', i.e.,*

$$\frac{\gamma_{\tau,t}}{\gamma_{\tau,\tau'}} = \gamma_{\tau',t} \ , \quad t = \tau', \tau' + 1, \ldots.$$

Proposition 2.2′. *A sequence of discount functions $\{\Gamma_\tau\}_{\tau=0}^{\infty}$ determines a time invariant sequence $\{W_\tau\}_{\tau=0}^{\infty}$ if and only if for all τ and $\tau' > \tau$, the discount functions $\Gamma_{\tau'}$ and Γ_τ are the same, i.e.,*

$$\gamma_{\tau,t} = \gamma_{\tau',\tau'+t-\tau}, \quad t = \tau, \tau + 1, \ldots.$$

Proposition 2.3′. *A discount function Γ_τ determines a stationary utility function W_τ if and only if*

$$\gamma_{\tau,t} = \beta_\tau^{t-\tau}, \quad t = \tau, \tau + 1, \ldots,$$

for some constant $0 < \beta_\tau < 1$.

Proposition 2.4′. *A sequence of social welfare functions* $\{W_\tau\}_{\tau=0}^{\infty}$ *satisfies any two of the three properties: time consistency, time invariance and stationarity, if and only if it is of the form*

$$W_\tau = \sum_{t=\tau}^{\infty} \beta^{t-\tau} u(c_t), \quad \tau = 0, 1, \ldots, \tag{2.20}$$

for some $0 < \beta < 1$.

Again we find that time consistency, time invariance and stationarity are interrelated. Any two of these properties necessarily imply the third, and lead to the sequence of social welfare functions, each element of which exhibits constant exponential discounting with **the same discount factor**.

It also follows that aggregation of heterogeneous time preferences in the case of common consumption is subject to the same difficulties as in the case of private consumption. The sequences of individual intertemporal utility functions (2.16) are time-consistent, time-invariant and stationary, while the sequence of social welfare functions (2.19) is in general not.

Indeed, it follows from Proposition 2.4′ that the sequence of social welfare functions $\{W_\tau\}_{\tau=0}^{\infty}$ is time-consistent, time-invariant and stationary if and only if one of the following three conditions holds:

1. $\{W_\tau\}_{\tau=0}^{\infty}$ coincides with $\{U_\tau^1\}_{\tau=0}^{\infty}$ (i.e., $\lambda_\tau^1 = 1$ for all τ);
2. $\{W_\tau\}_{\tau=0}^{\infty}$ coincides with $\{U_\tau^2\}_{\tau=0}^{\infty}$ (i.e., $\lambda_\tau^1 = 0$ for all τ);
3. Both agents have the same discount factor, $\beta_1 = \beta_2$.

This observation is easily generalized to the case with an arbitrary number of agents who differ both in their discount factors and felicity functions, as shown by Jackson and Yariv (2015). Their result can be expressed in our terms as follows: a sequence of time-invariant Paretian social welfare functions is time-consistent if and only if each element of this sequence exhibits constant exponential discounting with the same discount factor which is **exactly that of some agent** i.[15] In other words, the only well-behaved sequence of social welfare functions is the one which coincides with the sequence of individual utility functions of some agent. If Pareto weights of at least two agents with different discount factors are positive, then the time-invariant sequence of social welfare functions is non-stationary and not time-consistent.

This "impossibility result" in the spirit of Arrow (1950) deserves some comments and discussion. Jackson and Yariv actually formulate their result as follows: "If there is any

[15] The result of Jackson and Yariv (2015) is basically the same as that of Zuber (2011), but obtained in a slightly different setting. While Zuber considers independent and private consumption streams, Jackson and Yariv focus on common consumption streams.

heterogeneity in temporal preferences by way of differing discount factors, then the only well-behaved collective utility functions that are time consistent and respect unanimity are dictatorial: they ignore the preferences of all but one agent (or a group of agents who share the same exact preferences)" (Jackson and Yariv, 2015, p. 161).

It should be emphasized that Jackson and Yariv consider only time-invariant sequences of social preferences, though not explicitly acknowledging it. Millner and Heal (2016) note that there is actually a trade-off between time consistency and time invariance in a non-stationary sequence of social welfare functions. They show that if the planner assigns different Pareto weights at different dates, then it is possible to achieve time consistency (at the expense of time invariance).

Their observation can be illustrated in our two-agent case. Suppose that at each date τ the planner assigns to agents the same positive weights, i.e., $\lambda_\tau^1 = \lambda$ and $\lambda_\tau^2 = 1 - \lambda$ for some $0 < \lambda < 1$. Then the sequence of social welfare functions (2.19) has the form

$$W_\tau = \sum_{t=\tau}^{\infty} \left(\lambda \beta_1^{t-\tau} + (1-\lambda)\beta_2^{t-\tau}\right) u(c_t), \quad \tau = 0, 1, \ldots,$$

and Propositions 2.1′–2.3′ imply that this sequence is time-invariant, though not time-consistent and non-stationary. Instead, suppose that at each date τ the planner assigns to agents different weights, namely:

$$\lambda_\tau^1 = \frac{\lambda \beta_1^\tau}{\lambda \beta_1^\tau + (1-\lambda)\beta_2^\tau}, \quad \lambda_\tau^2 = \frac{(1-\lambda)\beta_2^\tau}{\lambda \beta_1^\tau + (1-\lambda)\beta_2^\tau},$$

for some $0 < \lambda < 1$. Then the sequence of social welfare functions is given by

$$W_\tau = \frac{1}{\lambda \beta_1^\tau + (1-\lambda)\beta_2^\tau} \sum_{t=\tau}^{\infty} \left(\lambda \beta_1^t + (1-\lambda)\beta_2^t\right) u(c_t), \quad \tau = 0, 1, \ldots,$$

and it follows from Propositions 2.1′–2.3′ that this sequence is time-consistent, though not time-invariant and non-stationary.

This fact has led Millner and Heal (2016) to the conclusion that the choice between time consistency and time invariance of the planner's preferences is purely normative. They argue that time consistency may be more attractive for intragenerational choices, while time invariance is more suitable for intergenerational choice.

In a recent contribution to the field, Feng and Ke (2017) suggest that in the context of intergenerational choice, the notion of Paretio-efficiency seems to be too strong, which may be the key to the problem. Note that the two-agent Ramsey model with common consumption admits a natural interpretation as an intergenerational model. The sequence of individual utility functions (2.16) is interpreted as a sequence of intertemporal utility functions of different individuals from successive generations. Individual i from generation τ lives for one period, and her intertemporal utility function is given by U^i_τ. Her offspring, individual i from generation $\tau+1$, inherits her discount factor and felicity function, and has the intertemporal utility function $U^i_{\tau+1}$.

Feng and Ke (2017) note that the standard Pareto property is essentially "current-generation", in the sense that at each date the planner takes into account only the preferences of the current generation. However, future generations are also affected by the planner's decision, and hence the planner at each date should take into account their preferences as well. Thus it is reasonable to introduce the weaker "intergenerational Pareto" property: if one consumption stream is preferred to another by every individual from every generation, then the planner should also prefer the former. Clearly, if the planner is current-generation Paretian, she is also intergenerationally Paretian, but not vice versa.

Recall that in our two-agent case the current-generation Paretian social welfare function at date τ is given by (2.18), for some non-negative weights λ^1_τ and λ^2_τ. The intergenerationally Paretian social welfare function at date τ has the form

$$
\begin{aligned}
W_\tau &= \sum_{t=\tau}^{\infty}\left\{\lambda^1_{\tau,t}\sum_{s=t}^{\infty}\beta_1^{s-t}u(c_s)+\lambda^2_{\tau,t}\sum_{s=t}^{\infty}\beta_2^{s-t}u(c_s)\right\}\\
&=\left(\lambda^1_{\tau,\tau}+\lambda^2_{\tau,\tau}\right)u(c_\tau)+\left(\lambda^1_{\tau,\tau+1}+\lambda^2_{\tau,\tau+1}+\lambda^1_{\tau,\tau}\beta_1+\lambda^2_{\tau,\tau}\beta_2\right)u(c_{\tau+1})+\dots,
\end{aligned}
$$

where the sequence of Pareto weights is such that $0<\sum_{t=\tau}^{\infty}\left\{\lambda^1_{\tau,t}+\lambda^2_{\tau,t}\right\}<\infty$.

It follows from the results of Feng and Ke (2017) that for any $\beta>\beta_1$, each element in the sequence of the social welfare functions

$$
W_\tau=\sum_{t=\tau}^{\infty}\beta^{t-\tau}u(c_t),\quad \tau=0,1,\dots,
$$

is intergenerationally Paretian and strongly non-dictatorial, i.e., the planner at each date τ does not ignore the preferences of any individual from any generation. In other words, there always exist appropriate and strictly positive Pareto weights such that for each τ,

$$
\sum_{t=\tau}^{\infty}\beta^{t-\tau}u(c_t)=\sum_{t=\tau}^{\infty}\left\{\lambda^1_{\tau,t}\sum_{s=t}^{\infty}\beta_1^{s-t}u(c_s)+\lambda^2_{\tau,t}\sum_{s=t}^{\infty}\beta_2^{s-t}u(c_s)\right\}.
$$

Thus if the current-generation Pareto property is replaced by the intergenerational Pareto property, then the sequence of planner's preferences is stationary, time-consistent and time-invariant whenever the planner is more patient than the most patient individual.

Another subtlety in Jackson and Yariv (2015) concerns the use of the term "dictatorial", which is slightly misleading. By "dictatorial", they mean social welfare functions that coincide with the individual utility function of one of the agents. However, this is not the same as the usual definition of dictatorship in social choice theory: an agent is a dictator if she is chosen to represent the preferences of the society regardless of the agents' preferences. This notion can be labeled as *ex ante* dictatorship. The definition of dictatorship given by Jackson and Yariv is different: they call an agent a dictator if she appears to represent the preferences of the society for some fixed profile of preferences. This notion is essentially *ex post* dictatorship. The difference between these two notions is substantial: for instance, an aggregation rule that chooses the preferences of the median agent in the society is "ex post dictatorial" (in the sense of Jackson and Yariv), while it is not "ex ante dictatorial" (in the usual sense of social choice theory), because the median agent clearly differs across profiles of preferences.

Nevertheless, the discussed result of Jackson and Yariv (2015) stresses a very important problem for aggregation of heterogeneous time preferences. Even though each individual with her own discount factor is time-consistent, the utilitarian planner (in the standard Pareto sense) who assigns equal weights to different agents with different discount factors, is not.[16] The only case in which the sequence of social preferences satisfies time consistency, time invariance and stationarity is when only one individual represents the society as a whole. This case may be realized not only by selecting some agent as a dictator, but also as a result of some social choice procedure. One of these procedures, namely, majority voting, we will discuss in the next section.

2.5. Voting over common consumption streams

Another possible way to obtain a social preference ordering is to abandon the idea of a social planner and instead explicitly model the fact that decisions of the society are determined through some political process.

A natural way of aggregating heterogeneous preferences is voting: if a feasible plan is preferred to all other plans according to some voting procedure, then this is the plan a society chooses. As rightfully noted by Beck (1978), we can thus get rid of the troublesome terms "social welfare function" and "social discount rate", and work instead with the more familiar terms "individual utility" and "individual discount rate".

Evidently, there exist a lot of different voting procedures, which have been studied in theory and used in practice. In our discussion we will focus only on the majority voting rule. First of all, it is the most common in practice as well as in the literature. Second, the majority rule is known to be the most robust voting procedure in the sense that it

[16] See also Anchugina et al. (2016) who generalize the results of Jackson and Yariv (2015) and study the properties of utilitarian aggregations of heterogeneous discount functions from certain equivalence classes.

satisfies a number of desirable normative properties over a larger domain than any other voting procedure (see Dasgupta and Maskin, 2008).

Suppose that in the many-agent Ramsey model with common consumption, agents choose their common consumption stream by voting over all feasible streams. A natural idea is to find a Condorcet winner, i.e., a consumption stream that would be preferred by a majority of agents when pairwise compared with every other feasible consumption stream. In voting over one-dimensional choice space, under certain plausible assumptions a Condorcet winner exists and coincides with the optimal plan of the "median voter". This outcome holds much favor in economic and political contexts, because the median voter effectively appears as the representative of the population whose preferences determine all decisions of the society.

However, there immediately arises an important difficulty. Agents choose their common consumption stream over an infinite horizon, hence the choice space is made up of infinite sequences of consumption and is clearly multi-dimensional. It is known that due to the high dimensionality of the choice space, a Condorcet winner in general does not exist. A majority rule in this case is generically intransitive, and one should expect the emergence of cycles: a majority of agents would prefer consumption stream C_2 to C_1, C_3 to C_2 and C_1 to C_3, which is known as the Condorcet (1785) paradox. Bernheim and Slavov (2009) characterize this situation as the "curse of dimensionality".

Indeed, there are a number of classical theoretical papers (e.g., Davis et al., 1972; Kramer, 1973; Bucovetsky, 1990) in which different forms of necessary and/or sufficient conditions for the existence of a Condorcet winner in a general context are derived. It turns out that when a choice space is multi-dimensional, almost any departure from homogeneity of tastes would immediately lead to the intransitivity of majority rule. Moreover, De Donder et al. (2012) show that there is in general no Condorcet winner in voting over multi-dimensional choice space even if agents are heterogeneous only in one dimension. They also argue that in most cases and for any feasible proposal, it is possible to find another feasible proposal that is favored by all voters except one. These results are nicely summarized in a quote by Gerald Kramer (1973, p. 296):

> when the preferences of individual voters over a multi-dimensional space of commodity or policy vectors are representable by quasi-concave, differentiable utility functions, all exclusion conditions for equilibrium under majority rule are extremely restrictive in the sense that they are incompatible with even a very modest degree of heterogeneity of tastes. ... For most purposes such conditions are probably not significantly less restrictive than the condition of complete unanimity, and it seems unlikely that they will be of much prac-

tical help in making social welfare judgements or in understanding political processes based on majority rule, for multi-dimensional choice problems.

Let us illustrate these difficulties with a simple three-agent three-period example of the many-agent Ramsey model with common consumption.[17] Let the production function be given by $f(k) = k$; hence we are faced with the cake eating problem described by Gale (1967). Suppose that agents have the same logarithmic felicity function $u(c) = \ln c$, and are heterogeneous only in their discount factors: $1 > \beta_1 > \beta_2 > \beta_3 > 0$. Then the utility function of agent i is given by

$$U^i = \ln c_0 + \beta_i \ln c_1 + \beta_i^2 \ln c_2.$$

Note also that in this case the median voter is the agent with the median discount factor, i.e., agent 2.

Problem (2.17) for agent i at date 0 becomes as follows:

$$\begin{aligned} \max \ & \ln c_0 + \beta_i \ln c_1 + \beta_i^2 \ln c_2, \\ \text{s. t.} \ & c_0 + c_1 + c_2 = \hat{k}_0, \\ & c_t \geq 0, \quad t = 0, 1, 2. \end{aligned}$$

The solution to this problem, *the optimal consumption path for agent i*, $C^{i*} = \{c_0^{i*}, c_1^{i*}, c_2^{i*}\}$, is given by

$$c_0^{i*} = \frac{\hat{k}_0}{1 + \beta_i + \beta_i^2}, \quad c_1^{i*} = \frac{\beta_i \hat{k}_0}{1 + \beta_i + \beta_i^2}, \quad c_2^{i*} = \frac{\beta_i^2 \hat{k}_0}{1 + \beta_i + \beta_i^2}.$$

Suppose that agents vote for the whole consumption stream at time 0. Clearly, the set of alternatives over which they vote is

$$\mathcal{C} = \left\{ (c_0, c_1, c_2) \in \mathbb{R}_+^3 \mid c_0 + c_1 + c_2 = \hat{k}_0 \right\}.$$

Let us show that if a Condorcet winner exists, then it should coincide with the optimal consumption path for agent 2. Indeed, suppose the opposite, i.e., that a Condorcet winner is a consumption stream $C^* = \{c_0^*, c_1^*, c_2^*\}$ which lies in $\mathcal{C}$ and does not coincide with C^{2*}. It is easily seen that C^* and C^{2*} differ at least in two elements if and only if one of the following inequalities holds:

$$c_1^* \neq \beta_2 c_0^*, \quad \text{or} \quad c_2^* \neq \beta_2 c_1^*.$$

[17] It is convenient to have an odd number of voters, so from now on our simplest example includes three agents who differ in their time preferences. Note also that the finite horizon does not reduce the generality of the analysis, as this example can be easily generalized to the case of infinite time horizon.

Let for definiteness $c_1^* > \beta_2 c_0^*$ (all other cases can be considered similarly). Note that in this case also $c_1^* > \beta_3 c_0^*$.

Consider the gradient of U^i at the point (c_0^*, c_1^*, c_2^*):

$$\nabla U^i(c_0^*, c_1^*, c_2^*) = \left(\frac{1}{c_0^*}, \frac{\beta_i}{c_1^*}, \frac{\beta_i^2}{c_2^*}\right).$$

Compute the inner product of $\nabla U^i(c_0^*, c_1^*, c_2^*)$ and the vector $z = (1, -1, 0)$:

$$\nabla U^i(c_0^*, c_1^*, c_2^*) \cdot z = \frac{1}{c_0^*} - \frac{\beta_i}{c_1^*} = \frac{c_1^* - \beta_i c_0^*}{c_0^* c_1^*}.$$

It is positive for β_2 and β_3. It follows that for a sufficiently small perturbation of (c_0^*, c_1^*, c_2^*) in the direction z (i.e., slightly increasing consumption at time 0 and decreasing consumption at time 1), there exists a consumption stream $C' = \{c_0', c_1', c_2'\}$ which lies in $\mathcal{C}$ and which agents 2 and 3 prefer to C^*. Since two agents out of three prefer C' to C^*, the latter consumption stream is not a Condorcet winner. Thus a Condorcet winner, if it exists, should coincide with the optimal consumption path for agent 2, C^{2*}.

However, let us now show that C^{2*} is **not** a Condorcet winner. Indeed, the gradient of U^i at the point $(c_0^{2*}, c_1^{2*}, c_2^{2*})$ is

$$\nabla U^i(c_0^{2*}, c_1^{2*}, c_2^{2*}) = \frac{1 + \beta_2 + \beta_2^2}{\hat{k}_0}\left(1, \frac{\beta_i}{\beta_2}, \frac{\beta_i^2}{\beta_2^2}\right).$$

Compute the inner product of $\nabla U^i(c_0^{2*}, c_1^{2*}, c_2^{2*})$ and the vector $z = (1, -2, 1)$:

$$\nabla U^i(c_0^{2*}, c_1^{2*}, c_2^{2*}) \cdot z = \left(1 - 2\frac{\beta_i}{\beta_2} + \frac{\beta_i^2}{\beta_2^2}\right)\frac{1 + \beta_2 + \beta_2^2}{\hat{k}_0} = \left(1 - \frac{\beta_i}{\beta_2}\right)^2 \frac{1 + \beta_2 + \beta_2^2}{\hat{k}_0}.$$

It is positive for β_1 and β_3. Hence for a sufficiently small perturbation of (c_0^*, c_1^*, c_2^*) in the direction z, there is a consumption stream $\tilde{C}$ which lies in $\mathcal{C}$ and which agents 1 and 3 prefer to C^{2*}. This consumption stream has more consumption at time 0 (to satisfy the relatively impatient agent 3), more consumption at time 2 (to satisfy the relatively patient agent 1) and less consumption at time 1 (to make it feasible). Again, since two agents out of three prefer $\tilde{C}$ to C^{2*}, the latter consumption stream is not a Condorcet winner. This contradiction leads to the conclusion that a Condorcet winner does not exist.

Boylan et al. (1996) provide a generalization of our simple example and explicitly prove that there is no Condorcet winner in the many-agent Ramsey model with common consumption (public good) even when agents differ only in their discount factors. Announcing their own result, Boylan and McKelvey (1995, p. 863) provide the following comment:

> The above result may seem surprising at first glance. One might think that, when utility functions differ only by one parameter, the median voter theorem would apply, implying that the optimal plan for the voter with the median discount factor would be a majority core point. In fact, the optimal plan for the median-discount-factor voter is defeated by a plan supported by a coalition including patient and impatient voters.

Hence their result is actually well in line with the previously mentioned literature.

Furthermore, Jackson and Yariv (2015) in the already mentioned paper prove another general impossibility theorem, which implies that voting over consumption streams when agents have heterogeneous time preferences cannot lead to an unambiguous outcome. Any non-dictatorial voting rule (in particular, majority and weighted supermajority voting rules) leads to cycles in collective decisions, unless the set of potential consumption streams is severely restricted. It follows from their result that the preferences of any single agent, including the agent with the median discount factor, cannot determine a voting equilibrium (as it would have to be transitive).

Thus an attempt to incorporate political institutions into the many-agent Ramsey model faces a serious difficulty. Though the optimal plan of the median agent seems to be the stable and desirable outcome of majority voting, it is not clear whether there even exist political institutions that could support and justify the choice of the agent with the median discount factor.

However, it turns out that there are some ways to overcome all the above mentioned "impossibility results". One of these ways was considered by Beck (1978) and, more recently, by Heal and Millner (2014). Instead of the infinite-dimensional set of all feasible consumption streams, agents are allowed to vote only over the set of individually optimal consumption paths. This voting procedure can be actually interpreted as majority voting over the set of discount factors, and then implementing the optimal consumption path for the agent with the winning discount factor. Alternatively, Heal and Millner (2014) propose the following two-step procedure: each agent nominates a consumption stream for the society to follow, and then agents vote over each pair of nominated streams choosing a Condorcet winner. It is shown that in voting over all individually optimal paths there exists a Condorcet winner, which is the optimal consumption path for the agent with the median discount factor.

In our three-agent three-period example this procedure can be illustrated as follows. Instead of voting over the set of all feasible consumption streams, $\mathcal{C} = \left\{(c_0, c_1, c_2) \in \mathbb{R}^3_+ \mid c_0 + c_1 + c_2 = \hat{k}_0\right\}$, agents vote over the three-dimensional set $\{C^{1*}, C^{2*}, C^{3*}\}$, where C^{i*} is the optimal consumption path for agent i.

Let us show that C^{2*}, the optimal consumption path for the agent with the median discount factor, is a Condorcet winner in voting over this three-element set. Consider a pairwise contest between C^{2*} and C^{1*}. Clearly, agent 1 prefers her optimal consumption stream C^{1*}, while agent 2 prefers her optimal stream C^{2*}. The decisive vote belongs to agent 3, and it is easily checked that she prefers the stream C^{2*} to C^{1*}. Indeed, $c_0^{i*} = \hat{k}_0/(1 + \beta_i + \beta_i^2)$ is decreasing in β_i, and thus $c_0^{3*} > c_0^{2*} > c_0^{1*}$. Since agent 2 prefers C^{2*} to C^{1*}, the same should be true for the more impatient agent 3 who prefers more consumption at earlier dates even stronger. Similar argument can be applied to a pairwise contest between C^{2*} and C^{3*}: only agent 3 prefers the stream C^{3*}, while both agents 1 and 2 prefer the stream C^{2*}. It turns out that C^{2*} wins each of its pairwise contests, and is thus a Condorcet winner. It is easily seen that though this voting procedure ensures the existence of a stable outcome, this result comes at the expense of the severely restricted choice set.

A different voting procedure in the many-agent Ramsey model with common consumption is proposed by Boylan et al. (1996). They start with a finite horizon model and introduce two additional agents ("political candidates"). In each period the candidates propose a consumption level for the agents and care only about being elected. Agents vote for one of the candidates and take into account only their own intertemporal utility of consumption. The consumption level proposed by the winning candidate is implemented and becomes the actual consumption for this period. A specific noncooperative game is then constructed and a subgame perfect Nash equilibrium is studied. Boylan et al. (1996) prove that for any finite horizon, the optimal consumption path for the agent with the median discount factor is a unique subgame perfect Nash equilibrium. They also show that when the time horizon tends to infinity, the sequence of Nash equilibria converges to the optimal path for the agent with the median discount factor in the infinite horizon model. This voting procedure, though also yielding the intuitively appealing outcome, seems quite contrived.

As we have seen, all the considered ways to overcome the absence of a Condorcet winner are unnecessarily complex. There is, however, a much simpler procedure that avoids many of the difficulties mentioned above. A natural idea to overcome the "curse of dimensionality" is to convert an infinite-dimensional choice space, made up of consumption streams, into a series of one-dimensional choice spaces. The notion of "coordinate-wise" majority voting was proposed independently by Kramer (1972) and Shepsle (1979). The proposed procedure implies that agents vote separately in each period, and the resulting sequence of voting outcomes is an equilibrium in the Nash sense: each element is a one-dimensional Condorcet winner, given that all other elements are chosen in the same manner. The outcome of this procedure is known as a Kramer–Shepsle equilibrium.

A Kramer–Shepsle procedure has many advantages, especially in dynamic models. First, it has a convenient interpretation as intertemporal (step-by-step) majority voting under perfect foresight about outcomes of future votes. This is well in line with the Koopmans' intertemporal view on economic models discussed in Section 2.3.[18] Second, the impossibility result of Jackson and Yariv (2015) does not hold here. In their framework intransitivities arise only when individuals atemporally vote once and for all on the whole sequence, while in the Kramer–Shepsle procedure majority voting is done intertemporally, step by step. Finally, a Condorcet winner in the multi-dimensional problem is always a Kramer–Shepsle equilibrium, but the converse need not be true. This actually means that a Kramer–Shepsle equilibrium may exist in more general circumstances.

Unfortunately, the direct application of the Kramer–Shepsle procedure to the many-agent Ramsey model leads to a new difficulty. Let us return to the three-agent three-period example and find a Kramer–Shepsle equilibrium, i.e., a consumption stream each element of which coincides with the majority choice given the choices of all other elements. Suppose that at time 0 agents have some common expectations about future consumption, c_1^e and c_2^e, and vote over the time 0 consumption c_0. The preferred time 0 consumption for agent i is a solution to the following problem:

$$\max_{c_0} \ln c_0, \qquad \text{s. t.} \quad c_0 + c_1^e + c_2^e = \hat{k}_0.$$

Clearly, this optimization problem is degenerate. The overall resource constraint under given expectations fully predetermines the optimal value of c_0. Moreover, this value is the same for all agents: it is optimal for each agent to consume today as much as possible, given the future consumption profile and the initial amount of resource. The same argument applies to voting over c_1 and c_2. It follows that every consumption stream $\{c_0, c_1, c_2\}$ such that $c_t > 0$ for all t and $c_0+c_1+c_2 = \hat{k}_0$, can be obtained as the outcome of intertemporal voting over consumption levels under perfect foresight. While a Condorcet winner fails to exist in the considered framework, there is an uncountable number of Kramer–Shepsle equilibria.

However, Borissov et al. (2014b) propose a Ramsey-type model in which the Kramer–Shepsle procedure leads to an unambiguous outcome. In the model, agents who differ in their discount factors, in each period vote on the current shares of public goods in the aggregate output. It is shown that the equilibrium sequence of these shares is fully determined by the median discount factor. It follows that the voting outcome is stable and is determined by the preferences of the median agent — a property which seemed improbable according to our previous discussion. The two crucial principles in their framework

[18] Note also the similarity between the Kramer–Shepsle procedure and the concept of social optimum in the many-agent Ramsey model with private consumption proposed by Drugeon and Wigniolle (2016).

is that agents vote step by step, in accordance with the Kramer–Shepsle procedure, and agents vote not over the levels of public good, but over its shares in the aggregate output. It is the combination of these principles that appears to be very fruitful.

Indeed, these ideas can be developed and successfully applied to voting in dynamic settings, as shown by Borissov et al. (2017). In the framework of the many-agent Ramsey model with common consumption they consider the following voting procedure (intertemporal majority voting): agents vote step by step over the *consumption rates* (i.e., values $c_t/f(k_t)$) under perfect foresight about the future outcomes of votes. It is proved that the outcome of intertemporal majority voting coincides with the optimal consumption path for the agent with the median discount factor.

Note that the described procedure indirectly determines a sequence of social welfare functions,

$$W_\tau^* = \sum_{t=\tau}^{\infty} (\beta_{med})^{t-\tau} u(c_t), \quad \tau = 0, 1, \ldots.$$

In each social welfare function W_τ^*, the Pareto weight of the agent with the median discount factor is equal to 1, while the Pareto weights of all other agents are equal to zero. As we have seen in Section 2.4, this sequence is time-consistent, time-invariant and stationary. The outcome of intertemporal majority voting is both Pareto-optimal and time-consistent, and coincides with the result of the maximization of the social welfare function W_0. This observation can be interpreted in terms of Jackson and Yariv (2015): the sequence of Paretian social welfare functions $\{W_\tau^*\}_{\tau=0}^{\infty}$ is well-behaved (i.e., is time-consistent, time-invariant and stationary) and is "ex post dictatorial", in the sense that it coincides with the sequence of utility functions of one particular agent (i.e., the agent with the median discount factor).

2.6. Discussion

In this chapter we have considered the problem of aggregation of heterogeneous time preferences in one-sector many-agent Ramsey models. If these models are interpreted as intergenerational models, then the topic of this chapter is closely related to the important normative question, how should we discount the future as a society? This problem is widely discussed in the literature as the problem of determining an appropriate social discount rate.[19]

[19] Strictly speaking, the social discount rate does not necessarily coincide with the social rate of time preference. Using the Ramsey (1928) equation, the social discount rate can be written as $r = \rho + \eta g$, where ρ is the social rate of time preference, η is the consumption elasticity of marginal utility (i.e., a measure of relative risk aversion) and g is the growth rate of per capita consumption. However, where this does not lead to misinterpretation, we would use the term "social discount rate" meaning "social rate of time preference".

There is no consensus on how the social discount rate should be determined. Some theorists argue that the social discount rate should be chosen based on a set of ethical principles. For instance, Ramsey (1928) himself strongly advocated a social discount rate of zero: in order not to discriminate against future generations, individuals from different generations should be counted equally. He saw discounting as "a practice which is ethically indefensible and arises merely from the weakness of the imagination" (Ramsey, 1928, p. 543). However, Pearce et al. (2003) note that not discounting is in fact discounting at 0%, which also has its own ethical implications that may not be acceptable. This point of view is emphasized by Koopmans (1967) who coined the term "the paradox of the indefinitely postponed splurge": under zero discounting, current generation could never use the resources because reinvesting them will always do more good for future generations. According to him, "too much weight given to generations far into the future turns out to be self-defeating. It does nobody any good. How much weight is too much has to be determined in each case" (Koopmans, 1967, p. 9).

It seems that we should apply a strictly positive social discount rate. But should this rate be low or high? This question is especially important in the cost-benefit analysis of environmental projects, particularly in the models of climate change and global warming, which appear to be incredibly sensitive to the choice of the social discount rate. The disagreement about the correct value for the social discount rate resulted in the famous Stern–Nordhaus debate.

Stern (2007) in his review on the economics of climate change uses the ethical principle of "intergenerational equity" and applies a very low social rate of time preference of 0.1%, which results in the social discount rate of around 1.4%. This leads to the assertion that we should give strong and immediate response to global warming. Nordhaus (2007), in turn, uses "consumer sovereignty" as the ethical principle, i.e., believes that the social discount rate should reflect consumers' real decisions and be based on the revealed preferences of the members of society. He argues that the social discount rate should coincide with the real interest rate of 5.5%, and hence postulates the social rate of time preference of 1.5%. It turns out that Stern's conclusion no longer holds if costs and benefits from climate change are discounted at the market interest rate. Thus ethics by itself does not provide an unambiguous answer as to whether the social discount rate should be zero, small, or large.

By adopting certain elements of utilitarian ethics, one can apply the "economic" approach to the choice of the social discount rate. The social discount rate in this approach is interpreted as the rate of time preference of the social planner whose social welfare function is a utilitarian aggregation of individual utility functions. As we have seen in Sections 2.3 and 2.4, there arise certain difficulties with the construction of the appropriate social welfare function. Even when the preferences of individuals are time-consistent, time-invariant and stationary, the preferences of the social planner are in general not. Moreover, when individuals have constant and different discount rates, the planner's discount rate is non-constant — it declines as time goes on, and tends to the discount rate of the most patient agent.

The declining discount rates also emerge from some other approaches, for instance, when uncertainty about the future is taken into account (see, e.g., Pearce et al., 2003; Gollier and Weitzman, 2010). All these observations can be used to argue that instead of a single and constant social discount rate, we should use a declining discount rate in cost-benefit analysis from the perspective of the society (see also the discussion in Arrow et al., 2014). However, at least in the deterministic case, the declining discount rate of the social planner leads to the problem of time consistency.[20]

Yang (2003) suggests that economic modeling of climate change may use a dual-rate discounting approach. In this framework, social (environmental) discounting is separated from private (consumption) discounting. A social planner is assumed to have two different discount rates: a consumption discount rate used to discount utility from private consumption goods, and an environmental discount rate used to discount utility from public goods (e.g., environmental quality). It is assumed that the environmental discount rate is lower than the consumption discount rate. Clearly, the sequence of (utilitarian) social welfare functions here would take the form

$$W_\tau = \sum_{t=\tau}^{\infty} \left\{ \beta_1^{t-\tau} v(q_t) + \beta_2^{t-\tau} u(c_t) \right\}, \quad \tau = 0, 1, \dots,$$

where $v(q_t)$ is the utility from the environmental quality and $\beta_1 > \beta_2$ (see also Borissov and Shakhnov, 2011).

It follows from the results discussed in Section 2.3 that each social welfare function W_τ is non-stationary, and in the long run environmental quality determines the decisions of the society, while the weight of private consumption becomes negligible. Moreover, the sequence $\{W_\tau\}_{\tau=0}^{\infty}$ is time-inconsistent, which is another difficulty with this approach.[21]

Relating back to the Stern–Nordhaus debate, the consumer sovereignty principle is sometimes criticized on the grounds that the application of the revealed preference principle can be justified only under some strong and unrealistic conditions, which include both time consistency and time invariance (see, e.g., Caplin and Leahy, 2004). It is claimed that under more reasonable assumptions, the social planner should be more patient than

[20] Newell and Pizer (2003) argue that in the uncertainty case the use of the term "time inconsistency" is slightly incorrect, because it can be applied only to the cases where it is known with certainty that the today's optimal plan will be not followed in the future.

[21] It is argued that "As long as public goods and private goods are not substitutable, heterogenous discount rates can be time-consistent" (Yang, 2003, p. 942), but it does not seem to help in this case.

consumers. This conclusion is also supported by the results of Feng and Ke (2017) which we have discussed in Section 2.4. They show that the intergenerationally Paretian social planner is more patient than the most patient individual in the society. Note that in equilibrium models with agents that are heterogeneous in their time preferences, the long-run interest rate exactly coincides with the discount rate of the most patient agent. Therefore, this approach supports the choice of the social discount rate which is lower than the market interest rate.

At the same time, the results surveyed in Section 2.5 indicate that it is also possible to apply the "political" approach to the choice of the social discount rate, i.e., use political institutions to determine collective choices among heterogeneous agents. As we have seen, the common result in the literature is that in the presence of heterogeneous time preferences, multi-dimensional voting cannot lead to an unambiguous outcome. However, we argue that there exists a simple voting procedure (intertemporal majority voting) in which the agent with the median discount rate effectively appears as the representative of the population, whose preferences determine all decisions of the society.

In the case where the discount rates of consumers are heterogeneous, it seems reasonable that the preferences are revealed through voting decisions. It may be also speculated that consumer sovereignty is justified, since there is a voting procedure that aggregates well-behaved preferences, and the outcome of this procedure is Pareto-optimal and time-consistent, as we have noted in the end of Section 2.5. However, in this case the social discount rate coincides with the median discount rate in the population, which is presumably much higher than the real interest rate, i.e., the discount rate of the most patient agent.

2.7. Conclusion

The above discussion, inspired by a purely normative concern, clearly shows the importance of discounting in economic theory and practice. As we have seen, the question of discounting causes a lot of controversy even if heterogeneity in time preferences is ignored. However, an increasing number of empirical studies (see Section 2.1) show that different individuals discount the future differently, and this heterogeneity in individuals' time preferences should be taken into account in economic modeling.

Hence there arises the problem of aggregation of heterogeneous time preferences, which is especially relevant in growth models with many agents. Here economic growth theory meets social choice theory. In this chapter we have considered simple one-sector two-agent Ramsey models with private as well with common consumption. In order to highlight the role of discounting, we have assumed that consumers are identical and differ only in their discount factors. This allowed us to review the literature devoted to aggregation of heterogeneous time preferences, and to explain the main difficulties related to the problem of social choice in many-agent growth models as instructive as possible.

The main results here are in the spirit of Arrow's famous impossibility theorem. In particular, the preferences (the sequence of preferences) of the social planner satisfy certain reasonable conditions (time consistency, time invariance, stationarity) if and only if either all individuals have the same discount factor and are thus identical or the preferences of the social planner coincide with the preferences of some individual and hence the planner completely ignores the preferences of all but one individual.

Moreover, a natural attempt to aggregate heterogeneous discount factors via some voting procedure also faces serious difficulties. A Condorcet winner fails to exist in voting over multi-dimensional choice space, even though agents are heterogeneous only in one dimension, and any non-dictatorial voting rule appears to be inherently intransitive. At the same time, all the considered approaches to overcome the absence of a Condorcet winner seem quite contrived and complex.

This is where the present thesis fits in and aims to contribute. In Chapter 3 we propose a simple voting procedure (intertemporal majority voting) and apply it to the collective choice in the framework of the many-agent Ramsey model with common consumption. We prove that if agents vote step by step over consumption rates under perfect foresight about future outcomes of votes, then the outcome of this procedure coincides with the optimal consumption path for the agent with the median discount factor. This outcome holds much favor in economic and political contexts, but it was unclear whether there even exist political institutions that could support and justify this outcome. In some sense, we provide a microfoundation of the choice of the median agent as the representative of the society. In Chapter 4 we consider intertemporal majority voting over extraction rates in the general equilibrium Ramsey-type model with borrowing constraints and exhaustible natural resources. It is also proved that the equilibrium sequence of extraction rates is determined by the agent with the median discount factor.

3. On Discounting and Voting in a Simple Growth Model

As we have seen in Chapter 2, the aggregation of heterogeneous preferences via voting in dynamic resource allocation problems faces a serious difficulty. Due to the multi-dimensionality of the choice space, voting equilibria under majority rule fails to exist generically, even in cases where the agents' type space is one-dimensional. At the same time, all the proposed ways to overcome the absence of a Condorcet winner are unnecessarily complex.

We note that, despite all the "impossibility" results, at each point in time there may exist a "median voter" whose preferred choice of instantaneous consumption rate is supported by a majority of agents. Based on this observation, in this chapter we propose a simple institutional setup in the many-agent Ramsey model with common consumption (intertemporal majority voting) that does not suffer from the problem of generic non-existence of equilibrium. Importantly, in this setup the temporary voting is not over consumption *levels* but over consumption *rates*. The equilibrium concept that we employ is Kramer–Shepsle equilibrium with perfect foresight; that is, (*i*) each period's decision follows the majority vote under the assumption that agents maximize their utility given the decisions in all other periods, and (*ii*) agents' expectations about these decisions are correct in equilibrium ("perfect foresight"). It is worth emphasizing that the institutional framework is well-defined also without the assumption of perfect foresight.

We show that if agents have the same felicity function and differ only in their discount factors, the outcome of intertemporal majority voting (an intertemporal voting equilibrium) is unique and coincides with the optimal consumption stream for the agent with the median discount factor. In a sense, our voting procedure provides a microfoundation of the choice of the optimal consumption stream of the "median" agent.

As an important intermediate result we establish that, for each fixed agent, the step-by-step determination of the optimal consumption rate under perfect foresight yields the optimal intertemporal consumption stream. While this technical result probably belongs to the body of "folk wisdom" within the Ramsey model, we are not aware of a rigorous proof and include one in this chapter.

We also consider the multi-dimensional heterogeneity case in which agents differ both in their felicity functions and discount factors. For this general case we provide a characterization of steady-state and balanced-growth voting equilibria and show the important difference between them. The steady-state voting equilibrium is fully determined by the median discount factor, while the balanced-growth voting equilibrium depends not only on the agents' discount factors, but also on the agents' intertemporal elasticities of substitution.

This chapter is based on the published article "On Discounting and Voting in a Simple Growth Model" (Borissov, Pakhnin and Puppe, 2017) and is organized as follows. Section 3.1 provides a preliminary discussion of the topic. Section 3.2 introduces our model. In Section 3.3 two simple examples illustrate the idea of intertemporal voting and explain the role of consumption rate. In Section 3.4 we define temporary and intertemporal voting equilibria. In Section 3.5 we consider the step-by-step decision-making process for a single agent. Section 3.6 states our main results. In Section 3.7 we study the general case where agents differ both in their discount factors and felicity functions, and characterize steady-state and balanced-growth voting equilibria. Section 3.8 concludes. All the proofs are relegated to Section 3.9.

3.1. Introduction

The problem of aggregating heterogeneous time preferences arises in many contexts. The many-agent Ramsey model with common consumption that we employ in this chapter admits two interpretations. It can be viewed either as a model of growth and physical (man-made) capital accumulation, or as a model of renewable or exhaustible natural resource allocation over time.

Though we present our model in terms of the traditional theory of economic growth, it is instructive to consider the problem of aggregating heterogeneous time preferences also within the common property resource framework. Examples are the hunting for animals, the grazing of cattle on a common ground, the pollution of the atmosphere, or the drilling for oil in a common underground reservoir.

In these contexts, an issue of evident importance is the determination of the socially desirable harvest (extraction) rate. Consider a village situated near a fishing ground. The fishing ground is self-managed by village citizens, who differ in their time preferences. The question is: what is the harvest rate of the fish stock collectively set by heterogeneous agents? If all citizens in the village are identical, then the rate of the fish stock exploitation can be easily determined by their common discount factor. However, it is not clear how to determine the harvest rate when citizens have different discount factors.

One might try to argue that the introduction of property rights can (indirectly) solve the problem. Indeed, the typical and well-known solution to the "tragedy of the commons" is to establish private property rights. Once the property rights are enforced, each owner acts optimally according to her own time preference. This might circumvent the problem in cases where suitable property rights can be established.[1] However, often the non-excludability of public goods prevents the enforcement of suitable private property rights. This is likely to occur in the case of the underground oil reservoir, the fishing ground, or the so-called "global commons". For instance, the tendency of fish to migrate makes it impossible to define geographically determined property rights over the fish stock. In this case a solution may be to introduce a governmental or community resource ownership, but then it is necessary to find a non-market mechanism of determining the harvest rate.

The Ramsey (1928) optimal growth framework is often used to study general equilibrium models with heterogeneous agents who differ in their discount factors (see Becker, 2006, for an excellent survey). In this kind of models each agent separately solves her own optimization problem and thus has an independent private consumption stream.

However, the Ramsey framework also allows one to study how heterogeneous agents make joint decisions over common consumption streams. It does not matter whether "common consumption" is a collectively consumed public good or a private good that is consumed according to some fixed and commonly known sharing rule. What is important is that agents' personal utilities are based on their collective decisions, i.e., on the common consumption stream they choose. Here, economic growth theory meets social choice theory, and there is indeed a literature that analyzes how political institutions can be incorporated into growth models in order to determine collective choices among heterogeneous agents (see, e.g., Beck, 1978; Boylan, 1995; Boylan et al., 1996).

A natural way of aggregating heterogeneous preferences is voting. Suppose that agents vote over all feasible consumption streams by pairwise majority voting. Then, it is well-known that, due to the high dimensionality of the underlying choice space, there does not in general exist a Condorcet winner, i.e., for every feasible consumption stream there exists another feasible consumption stream that is preferred by a majority (see, e.g., Plott, 1967; Davis et al., 1972; Kramer, 1973; McKelvey, 1976; Bucovetsky, 1990).[2] Moreover, the fact that agents differ only in one parameter does not help: there still is no Condorcet winner in voting over a multi-dimensional choice space even if the agents' type space is one-dimensional (see, e.g., De Donder et al., 2012). Boylan et al. (1996) consider voting over feasible consumption paths in the Ramsey optimal growth model and prove that there is in general no Condorcet winner. In a more recent paper, Jackson and Yariv (2015) analyze general aggregation methods when agents differ only in their discount factors; they prove that, at any given profile of individual preferences, the collective preferences

[1] If the "owners" are *groups* of individuals with heterogeneous time preferences, the problem might of course persist within these groups.

[2] Bernheim and Slavov (2009) characterize this kind of situation as the "curse of dimensionality".

resulting from any Pareto efficient aggregation rule are either time-inconsistent or they coincide with the preferences of one individual in the given profile.[3]

Despite these negative results, it appears that in a model in which agents differ only in their discount factors, the optimal consumption path for the agent with the *median* discount factor has some claim to be a natural and appealing collective choice. But clearly, the mentioned impossibility results imply in particular that also the optimal consumption path for the "median" agent is in general not a Condorcet winner among all feasible paths.

One way to overcome this difficulty has been considered by Beck (1978) and, more recently, by Heal and Millner (2014). In these models, agents are only allowed to vote over the set of individually optimal paths. It can be shown that, among all individually optimal paths, the optimal path for the agent with the median discount factor is indeed a Condorcet winner. However, ensuring the existence of a stable voting outcome in this way is not very satisfactory since it is made possible only by severely restricting the choice set.

A different voting mechanism is proposed by Boylan et al. (1996) who introduce two additional agents ("political candidates") to the model. In each period the candidates propose a consumption level for the agents and care only about being elected. Agents vote for one of the candidates and care only about consumption. A specific noncooperative game is then constructed, and it is shown that the subgame perfect equilibrium coincides with the optimal path for the agent with the median discount factor. Although it yields the desired and intuitive outcome, this voting procedure seems quite contrived and complex.

The purpose of this chapter is to propose a more intuitive and tractable voting procedure that yields as outcome the optimal consumption path for the agent with the median discount factor if agents have the same felicity function, and that can be applied also in the general case in which agents have different discount factors and different felicity functions.[4]

We consider a Ramsey-type growth model with agents who may differ in their felicity functions and time preferences. Agents maximize their intertemporal discounted utilities by allocating at each point in time a given amount of a single good between consumption which provides instant utility, and investment which is used in production. The technology

[3] Note that the latter property, which may be labelled "ex post dictatorship", is weaker than the standard Arrowian notion of dictatorship since the individual whose preference coincides with the collective preference may differ across profiles.

[4] The idea to use dynamic voting in order to determine a stable outcome has been investigated in a number of specific models. In Borissov et al. (2014a) agents vote for a tax aimed at environmental maintenance, Borissov et al. (2014b) study voting over the shares of public goods in GDP, and Borissov and Pakhnin (2018) consider voting over extraction rates in a model with exhaustible natural resources. In all cases, the equilibrium policy is determined by the agent with the median discount factor as in this chapter. However, these models lack generality because they use specific forms of the felicity and production functions.

is described by a production function, which is assumed to be either strictly concave or linear.

We suppose that agents share a common consumption stream. The good is consumed either collectively or privately according to some fixed sharing rule. In the common property resource interpretation of our model, the capital stock is viewed as the renewable resource stock, the production function becomes the regeneration function, and the consumption level is the amount of the resource extracted (= the harvest rate times the available resource stock).

Within our framework, we propose a simple and natural voting procedure according to which agents choose a consumption path from the set of all feasible consumption paths by "intertemporal majority voting". The two crucial principles in this institutional setup are that (i) voting is done "step-by-step", and (ii) voting is not over the consumption level itself, but over the consumption *rate*. We avoid the "curse of dimensionality" by transforming a multi-dimensional choice space into a series of one-dimensional choice spaces. Indeed, the dynamic intertemporal structure of the model itself naturally suggests to consider institutions that also allow for intertemporal choices of agents. The solution concept given the proposed intertemporal voting procedure is *Kramer–Shepsle* equilibrium (Kramer, 1972; Shepsle, 1979).[5]

Given the general idea to transform the multi-dimensional choice problem into a sequence of one-dimensional choice problems, an important issue that has to be addressed is how the expectations should be formed. The important feature of our approach is that expectations are formed precisely about future consumption *rates*. If this is the case, agents can vote today over the consumption level as well as over the consumption rate, the outcome of voting will be the same. On the other hand, one-dimensional voting over the current consumption level under given expectations about the consumption *levels* in all other periods is pointless, since consumption in each period is uniquely determined by consumption in all other periods via the overall resource constraint. Thus, if future consumption is given, there is no trade-off between consumption today and consumption in the future (we provide a simple example in Section 3.3.1 below to illustrate this point), and formally, *every* feasible consumption path is a Kramer–Shepsle equilibrium in our model.

To implement the idea of intertemporal majority voting in a fruitful manner, we look at the problem at hand from a slightly different perspective. Originally, the model is formulated in terms of consumption *levels*, and mathematically, it involves an optimal

[5] In general, a vector of policies is a Kramer–Shepsle equilibrium if for any single dimension the corresponding policy in this dimension coincides with the majority choice, given the equilibrium choices in all other dimensions. Clearly, if the multi-dimensional problem admits a Condorcet winner, then the Condorcet winner constitutes a Kramer–Shepsle equilibrium.

control problem with the consumption level as control variable. We make a change of variables and use instead the consumption *rate* (= 1 − savings rate) as control variable.[6] Using this change of variable, we define a voting equilibrium in two stages, following the traditions of dynamic macroeconomics and applying the Hicks–Grandmont temporary equilibrium approach (Hicks, 1939; Grandmont, 1977).

First, for any point in time agents vote by majority rule over the current consumption rate, given the current capital stock and *some* expectations about future consumption rates. This yields a one-dimensional decision problem, and we show that agents' preferences over the current consumption rate are single-peaked, and therefore the median voter theorem applies. If, in addition, agents have the same felicity function and the same expectations, at each given point in time the temporary voting equilibrium (i.e., the instantaneous Condorcet winner) is the preferred consumption rate for the agent with the median discount factor.

Second, an intertemporal majority voting equilibrium is defined as a sequence of temporary voting equilibria such that all agents have *perfect foresight* about outcomes of future votes. We prove that if agents have the same felicity function, there is a unique intertemporal voting equilibrium, which is the optimal consumption path for the agent with the median discount factor. The proof is based on the general result that the step-by-step determination of the consumption rate under perfect foresight for any given agent results in the ("once-and-for-all") optimum in terms of consumption levels for this agent. This result, though not surprising, is of interest in itself, and we present it in Section 3.5 below.

We thus view our analysis as providing an institutional "microfoundation" for the choice of the optimal consumption path for the agent with the median discount factor, a proposal that has been repeatedly put forward in the literature but without an ultimately appealing justification so far.

If agents differ both in their discount factors and felicity functions, the intertemporal voting equilibrium is clearly no longer determined by the discount factor alone. However, even with infinite-dimensional heterogeneity we are still able to obtain some results. In the case of a strictly concave production function, we define the notion of a steady-state voting equilibrium, and show that it is unique and again determined by the median discount factor. In the case of a linear production function, we define the notion of a balanced-growth voting equilibrium, prove its existence and uniqueness, and show that it is determined by the "median growth rate".

[6] The savings rate as control variable in the Ramsey model has been used by Phelps and Pollak (1968) and Peleg and Yaari (1973). These authors study agents' behavior under time-inconsistent preferences, and ask when the chosen plan of actions will be actually followed by rational individuals in the future. By contrast, we assume time-consistent preferences throughout.

Note that the outcome of intertemporal majority voting (i.e., the optimal consumption path for the "median" agent) is clearly both time-consistent and Pareto efficient. This is well in line with the aforementioned result of Jackson and Yariv (2015) since the collective choice in our setup indeed coincides with the preferences of one particular ("median") agent.

In this chapter we do not address questions related to uncertainty of future economic development. There is a lively and ongoing discussion on how to discount the future under uncertainty (see, e.g., Pearce et al., 2003; Gollier and Weitzman, 2010; Traeger, 2013). While the introduction of uncertainty seems to bring the problem of choosing a consumption path closer to real life decisions, it also complicates matters quite dramatically. Our hope is that, even though our analysis does not directly contribute to the literature of discounting the future under uncertainty, the idea of intertemporal majority voting might be also fruitfully applicable to this more general setting.

3.2. The model

We consider a Ramsey-type growth model with heterogeneous agents and common consumption. Suppose $T \in \mathbb{N} \cup \{\infty\}$ is the length of the time horizon, which can be finite or infinite. Let time be $\mathbb{T} = \{0, 1, \dots, T\}$ when $T < \infty$, and $\mathbb{T} = \{0, 1, \dots\}$ when $T = \infty$.

There is an odd number L of heterogeneous agents indexed by $i = \{1, 2, \dots, L\}$. The heterogeneity is captured by agents' discount factors and felicity functions. Agent i has a discount factor $\beta_i \in (0, 1)$, and a felicity function $u_i : \mathbb{R}_{++} \to \mathbb{R}$ which satisfies the following conditions:

$$u_i'(c) > 0, \quad u_i''(c) < 0, \quad \lim_{c \to 0} u_i'(c) = +\infty.$$

Her intertemporal utility function is of the form $\sum_{t \in \mathbb{T}} \beta_i^t u_i(c_t)$, where $C = \{c_t\}_{t \in \mathbb{T}}$ is the *common* consumption stream. It is not critical whether actual consumption is common or private. In the latter case there is a fixed and commonly known sharing rule. For example, if this rule is egalitarian, then $u_i(c)$ should be replaced with $u_i(c/L)$. What is important is that agents' personal utilities are based on their collective decisions.

A single homogeneous good is produced. In each period $t \in \mathbb{T}$ the available amount of good is allocated between consumption c_t and capital k_{t+1} for use in the next period production: $c_t + k_{t+1} = f(k_t)$, where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a production function.

As was noted above, our model can also be considered as a common property resource model. In this case k should be viewed as the stock of a renewable or exhaustible natural resource, $f(k)$ as a function describing regenerative capacity of the resource, and consumption as the amount of the resource extracted. In resource economics, the regenerative capacity of a resource is typically described in terms of a so-called regeneration (net growth) function $g(k)$, which gives the level of net growth of the resource stock depending on the size of the stock, k. If the resource stock at the beginning of period t is k_t and harvest during period t is c_t, then the resource stock at the beginning of period $t+1$ is $k_{t+1} = g(k_t) - c_t + k_t$. Thus, if we interpret our model as a common property resource model, then $f(k) = g(k) + k$. If the resource is exhaustible, then $f(k) = k$.

We assume that either

Case 1. ***Strictly concave production function***
The production function satisfies the following properties:

$$f(0) = 0, \quad f'(k) > 0, \quad f''(k) < 0, \quad \exists \bar{k} \ : \ f(\bar{k}) = \bar{k}, \quad \beta_{min} f'(0) > 1, \tag{3.1}$$

where β_{min} is the minimal discount factor in the set $\{\beta_i\}_{i=1}^{L}$

or

Case 2. ***Linear production function***
The production function is linear:

$$f(k) = Ak, \quad A > 0.$$

In this case we additionally assume that the felicity function of every agent is of the CIES (constant intertemporal elasticity of substitution) form:

$$u_i(c) = \begin{cases} \frac{c^{1-\rho_i}}{1-\rho_i}, & \text{if} \quad 0 < \rho_i < +\infty, \ \rho_i \neq 1, \\ \ln c, & \text{if} \quad \rho_i = 1. \end{cases} \tag{3.2}$$

For each agent i, consider the following optimization problem:

$$\max \sum_{t \in \mathbb{T}} \beta_i^t u_i(c_t), \quad \text{s. t.} \quad c_t + k_{t+1} = f(k_t), \quad c_t \geq 0, \quad k_{t+1} \geq 0, \quad t \in \mathbb{T}. \tag{3.3}$$

In Case 1 (strictly concave production function), maximization problem (3.3) has a unique solution (optimal path) for each agent i. The same holds true in Case 2 (linear production function) if $\beta_i A^{1-\rho_i} < 1$.

Definition. *The solution to problem (3.3), $\{c_t^{i*}, k_{t+1}^{i*}\}_{t \in \mathbb{T}}$, is called an optimum in terms of consumption levels for agent i.*

3.3. Intertemporal voting: examples

If all agents have the same felicity function and discount factor, they have the same optimal path, so there is no collective choice problem. Which consumption stream will be chosen by a society that consists of heterogeneous agents?

A natural way of aggregating heterogeneous preferences is voting, and one may hope that some form of majority voting would result in the outcome supported by the "median voter". For instance, in the case where agents differ only in their time preferences, the median voter is the agent with the median discount factor. However, Boylan et al. (1996) show that in the latter case the path optimal for the agent with the median discount factor is "blocked" by the coalition consisting of all other agents. Moreover, a Condorcet winner does not in general exist.

A natural idea to overcome the absence of a Condorcet winner is to convert a multi-dimensional choice space, made up of consumption streams, into a series of one-dimensional choice spaces. The basic notion of "coordinate-wise" majority voting was proposed independently by Kramer (1972) and Shepsle (1979). This approach in dynamic models can be interpreted as intertemporal (step-by-step) voting under perfect foresight about outcomes of future votes. In this section we consider two simple examples in order to gain intuition about intertemporal voting procedures and analyze the applicability of such procedures to our model.

The aim of the first example is to explain our approach within the simplest framework. It illustrates the absence of a Condorcet winner and shows that one-dimensional voting over current consumption under given expectations about the future consumption levels is pointless, but if the voting agents have perfect foresight about the future consumption *rates*, then the outcome of intertemporal voting coincides with the optimal consumption stream for the "median" agent. The second example relates intertemporal voting to well-known results for the Ramsey model with logarithmic preferences and Cobb–Douglas production function.

3.3.1. Finite horizon example

Consider the following three-period three-agent example: $T = 3$, $L = 3$, $u_i(c) = \ln c$, and $f(k) = k$ (thus we are dealing with an intertemporal cake-eating problem).

First, suppose that agents vote over the whole consumption stream only once, at time 0. It is easy to note that in this case, the set of alternatives over which they vote is $\mathcal{C} = \{(c_0, c_1, c_2) \in \mathbb{R}^3_+ \mid c_0+c_1+c_2 = k_0\}$, the objective function of agent i is $\mathcal{U}^i(c_0, c_1, c_2) = \ln c_0 + \beta_i \ln c_1 + \beta_i^2 \ln c_2$, and problem (3.3) becomes as follows:

$$\max\ \mathcal{U}^i(c_0, c_1, c_2) \quad \text{s. t.} \quad c_0 + c_1 + c_2 = k_0, \quad c_0 \geq 0, \quad c_1 \geq 0, \quad c_2 \geq 0.$$

The solution to this problem, $\{c_0^{i*}, c_1^{i*}, c_2^{i*}\}$, is given by

$$c_0^{i*} = \frac{k_0}{1+\beta_i+\beta_i^2}, \quad c_1^{i*} = \frac{\beta_i k_0}{1+\beta_i+\beta_i^2}, \quad c_2^{i*} = \frac{\beta_i^2 k_0}{1+\beta_i+\beta_i^2}.$$

As was noted above, it seems natural to conjecture that a Condorcet winner exists and coincides with the solution to problem (3.3) for the agent with the medial discount factor β_{med}, i.e., with the triple $\{c_0^*, c_1^*, c_2^*\}$ given by

$$c_0^* = \frac{k_0}{1+\beta_{med}+\beta_{med}^2}, \quad c_1^* = \frac{\beta_{med} k_0}{1+\beta_{med}+\beta_{med}^2}, \quad c_2^* = \frac{\beta_{med}^2 k_0}{1+\beta_{med}+\beta_{med}^2}.$$

However, this conjecture is false. Indeed, the gradient of $\mathcal{U}^i$ at the point (c_0^*, c_1^*, c_2^*) is

$$\nabla\mathcal{U}^i(c_0^*, c_1^*, c_2^*) = \frac{1+\beta_{med}+\beta_{med}^2}{k_0}\left(1, \frac{\beta_i}{\beta_{med}}, \frac{\beta_i^2}{\beta_{med}^2}\right).$$

Let us compute the inner product of $\nabla\mathcal{U}^i(c_0^*, c_1^*, c_2^*)$ and $z = (1, -2, 1)$:

$$\begin{aligned}\nabla\mathcal{U}^i(c_0^*, c_1^*, c_2^*)\cdot z &= \left(1 - 2\frac{\beta_i}{\beta_{med}} + \frac{\beta_i^2}{\beta_{med}^2}\right)\frac{1+\beta_{med}+\beta_{med}^2}{k_0} \\ &= \left(1 - \frac{\beta_i}{\beta_{med}}\right)^2 \frac{1+\beta_{med}+\beta_{med}^2}{k_0}.\end{aligned}$$

It is positive if $\beta_i \neq \beta_{med}$. It follows that for a sufficiently small perturbation of $\{c_0^*, c_1^*, c_2^*\}$ in the direction z, we can find a consumption stream $\{c_0', c_1', c_2'\}$ which lies in $\mathcal{C}$ and which the agents with $\beta_i \neq \beta_{med}$ prefer to $\{c_0^*, c_1^*, c_2^*\}$. This consumption stream has more consumption at time 0 (to satisfy the agent whose discount factor is lower than β_{med}), more consumption at time 2 (to satisfy the agent whose discount factor is higher than β_{med}) and less consumption at time 1 (to make it feasible). Since two agents out of three prefer $\{c_0', c_1', c_2'\}$ to $\{c_0^*, c_1^*, c_2^*\}$, the latter consumption stream is not a Condorcet winner. Moreover, it is possible to show (see Boylan et al., 1996) that no other consumption stream is a Condorcet winner and thus a Condorcet winner does not exist.

Let us now try to find a Kramer–Shepsle equilibrium, i.e., a consumption stream each element of which coincides with the majority choice, given the choices of all other elements.

Suppose that at time 0 agents have some common expectations about future consumption, c_1 and c_2, and vote over the time 0 consumption c_0. The preferred time 0 consumption for agent i is the solution to the following problem:

$$\max_{c_0} \ln c_0, \quad \text{s. t.} \quad c_0 + c_1 + c_2 = k_0.$$

However, this optimization problem is degenerate. The overall resource constraint under given expectations fully predetermines the optimal value of c_0. Moreover, this value is the same for all agents: it is optimal for all agents to consume today as much as possible, given the future consumption profile and the initial amount of capital. The same argument applies to voting over c_1 and c_2. It follows that every consumption stream $\{c_0, c_1, c_2\}$ such that $c_0 < k_0$, $c_1 < k_0 - c_0$, and $c_2 = k_0 - c_0 - c_1$, can be obtained as a result of intertemporal voting over consumption levels under perfect foresight. Such a voting procedure seems to be meaningless.

However, this observation does not invalidate the idea to transform a multi-dimensional choice problem into a sequence of one-dimensional choice problems. To look at the same example from a different perspective, let us formulate the initial problem in terms of *consumption rates* $e_0 = c_0/k_0$, $e_1 = c_1/k_1$, $e_2 = c_2/k_2$, instead of consumption levels $\{c_0, c_1, c_2\}$. Then the utility function of agent i takes the form

$$V^i(e_0, e_1, e_2) = \ln(e_0 k_0) + \beta_i \ln(e_1(1 - e_0)k_0) + \beta_i^2 \ln(e_2(1 - e_1)(1 - e_0)k_0),$$

the problem of utility maximization for agent i becomes

$$\max V^i(e_0, e_1, e_2), \quad \text{s. t.} \quad 0 \leq e_0 \leq 1, \quad 0 \leq e_1 \leq 1, \quad 0 \leq e_2 \leq 1,$$

and its solution $\{e_0^{i*}, e_1^{i*}, e_2^{i*}\}$ is given by

$$e_0^{i*} = \frac{1}{1 + \beta_i + \beta_i^2}, \quad e_1^{i*} = \frac{1}{1 + \beta_i}, \quad e_2^{i*} = 1.$$

Let us apply the intertemporal majority voting procedure to the problem formulated *in terms of consumption rates*. Agents vote over the current consumption rate under given past consumption rates and expectations about future consumption rates. Suppose that at time 0 expectations about future consumption rates are e_1 and e_2. Then the preferred time 0 consumption rate for agent i, e_0^i, is the solution to the following problem:

$$\max_{0 \leq e_0 \leq 1} V^i(e_0, e_1, e_2).$$

It is not difficult to check that it coincides with the first element of the optimum in terms of consumption rates for agent i: $e_0^i = e_0^{i*}$.[7]

Clearly, the preferences of agents in one-dimensional voting over e_0 are single-peaked, and the preferred values e_0^i, $i = 1, 2, 3$, are decreasing in β_i. By the median voter theorem,

[7] Due to the simplicity of the example, e_0^i does not depend on expectations about future consumption rates.

the Condorcet winner in this vote exists. It is equal to the preferred time 0 consumption rate for the agent with the median discount factor, $e_0^* = \frac{1}{1+\beta_{med}+\beta_{med}^2}$.

Now consider voting over the time 1 consumption rate. Agents already know that the time 0 consumption rate is equal to e_0^* and have expectations about the time 2 consumption rate, e_2. Then the preferred time 1 consumption rate for agent i, e_1^i, is the solution to the following problem:

$$\max_{0\le e_1\le 1} V^i(e_0^*, e_1, e_2).$$

Evidently, the preferred time 1 consumption rate for agent i coincides with the second element of her optimum in terms of consumption rates: $e_1^i = e_1^{i*}$. By the median voter theorem, a Condorcet winner exists and is equal to the preferred time 1 consumption rate for the agent with the median discount factor, $e_1^* = \frac{1}{1+\beta_{med}}$.

Finally, the problem of finding the preferred time 2 consumption rate for agent i takes the form:

$$\max_{0\le e_2\le 1} V^i(e_0^*, e_1^*, e_2).$$

The solution to this problem coincides with $e_2^{i*} = 1$. Since all agents vote unanimously, a Condorcet winner exists and is equal to $e_2^* = 1$.

Thus we obtain the sequence of consumption rates $E^* = \left\{\frac{1}{1+\beta_{med}+\beta_{med}^2}, \frac{1}{1+\beta_{med}}, 1\right\}$. Each element of E^* is the Condorcet winner in one-dimensional voting over the single consumption rate at the corresponding instant in time under known values of previous consumption rates and given expectations about future consumption rates. It is clear that the sequence E^* is the solution to the utility maximization problem in terms of consumption rates for the agent with the median discount factor.

3.3.2. Infinite horizon example

Consider now the infinite horizon problem with the same logarithmic felicity function for all agents and a Cobb–Douglas production function. Given $k_0 > 0$, the optimization problem *in terms of consumption levels* for agent i is as follows:

$$\max \sum_{t=0}^{\infty} \beta_i^t \ln c_t, \quad \text{s. t.} \quad c_t + k_{t+1} = k_t^\alpha, \quad c_t \ge 0, \quad k_{t+1} \ge 0, \quad t = 0, 1, \ldots,$$

where $0 < \alpha \le 1$. It is well-known that the solution to this problem is characterized by a constant over time savings rate, equal to $\alpha\beta_i$. Therefore the optimal consumption rate is also constant over time and equal to $1 - \alpha\beta_i$. It follows that the optimum in terms of consumption rates for agent i is given by the sequence $\{1 - \alpha\beta_i, 1 - \alpha\beta_i, \ldots\}$.

Let us introduce consumption rates: $e_t = c_t/k_t^\alpha$, $t = 0, 1, \ldots$. To describe intertemporal majority voting over consumption rates, suppose that at an arbitrarily chosen point in time, τ, the stock of capital, $k_\tau > 0$, is given and agents have some expectations about future consumption rates, $\{e_t\}_{t=\tau+1}^{\infty}$. Then the objective function of agent i in voting over e_τ is given by

$$\ln\left(e_\tau(k_\tau)^\alpha\right) + \beta_i \ln\left(e_{\tau+1}(1-e_\tau)^\alpha (k_\tau)^{\alpha^2}\right) + \beta_i^2 \ln\left(e_{\tau+2}(1-e_{\tau+1})^\alpha (1-e_\tau)^{\alpha^2} (k_\tau)^{\alpha^3}\right) + \ldots$$
$$= \ln e_\tau + \alpha\beta_i \ln(1-e_\tau) + \alpha^2\beta_i^2 \ln(1-e_\tau) + \ldots + \Gamma_\tau^i = \ln e_\tau + \frac{\alpha\beta_i}{1-\alpha\beta_i}\ln(1-e_\tau) + \Gamma_\tau^i,$$

where

$$\Gamma_\tau^i = \ln\left((k_\tau)^\alpha\right) + \beta_i \ln\left(e_{\tau+1}(k_\tau)^{\alpha^2}\right) + \beta_i^2 \ln\left(e_{\tau+2}(1-e_{\tau+1})^\alpha (k_\tau)^{\alpha^3}\right) + \ldots$$

is a term that depends on k_τ and expectations, but does not depend on the variable over which agents vote. If $0 < e_t < 1,\ t > \tau$, and $0 < \liminf_{t\to\infty} e_t \le \limsup_{t\to\infty} e_t < 1$, then $-\infty < \Gamma_\tau^i < +\infty$ and hence the objective function of each agent is well-defined. To find her preferred time τ consumption rate, agent i needs to solve the following equation:

$$\frac{d}{de_\tau}\left(\ln e_\tau + \frac{\alpha\beta_i}{1-\alpha\beta_i}\ln(1-e_\tau)\right) = 0.$$

Clearly, the solution to this equation is equal to the optimal consumption rate $1-\alpha\beta_i$.[8]

The preferences of agents in one-dimensional voting over the time τ consumption rate are single-peaked and the preferred consumption rates negatively depend on β_i. Therefore, by the median voter theorem, the winner in majority voting over the time τ consumption rate is the preferred consumption rate for the "median" agent (i.e., the agent with the median discount factor β_{med}), $1-\alpha\beta_{med}$. If voting takes place at each time, we obtain the sequence $\{1-\alpha\beta_{med}, 1-\alpha\beta_{med}, \ldots\}$, which is exactly the optimum in terms of consumption rates for the agent with the median discount factor.

The above examples illustrate two important aspects of using the consumption rate as the control variable. First, for each agent the sequence of the preferred consumption rates coincides with the optimum in terms of consumption rates. Second, intertemporal majority voting over consumption rates yields, as outcome, the optimum in terms of consumption rates for the agent with the median discount factor.

It is natural to ask whether these results can be generalized to a general Ramsey-type model. The main difficulty is that the preferred time τ consumption rate for each agent is generically a function of all expected future consumption rates. If agents form

[8] Due to the logarithmic felicity functions and the Cobb–Douglas production function, the preferred time τ consumption rate for each agent is independent of expectations. In a number of papers (see Borissov et al., 2014a,b; Borissov and Pakhnin, 2018), this fact is used to generalize the considered example to voting in a dynamic general equilibrium framework.

expectations arbitrarily, there is no reason to expect that a reasonable outcome of step-by-step voting procedure will be obtained. However, we shall show that if agents have perfect foresight about future decisions, then under appropriate assumptions the outcome of such a procedure indeed coincides with the optimum in terms of consumption rates for the "median" agent.

3.4. Intertemporal voting: definitions

As we have seen, the intertemporal majority voting procedure is based on the two crucial principles: (*i*) voting is done step-by-step, and (*ii*) voting is not over consumption levels, but over consumption rates. In this section we begin by presenting the initial optimization problem in terms of consumption rates, and then give a formal definition of an intertemporal voting equilibrium.

3.4.1. Optimization problem in terms of consumption rates

The control variable in problem (3.3) is the consumption level c_t. Let us make a change of variables and take the consumption rate

$$e_t = \frac{c_t}{f(k_t)}, \quad t \in \mathbb{T}, \tag{3.4}$$

as the control variable.[9] Clearly, to be feasible, the sequence of consumption rates must be such that $0 \leq e_t \leq 1, \ t \in \mathbb{T}$. If we interpret our model as a model of natural resource allocation over time, then the consumption rate e_t becomes the rate of extraction.

Taking into account (3.4) and the constraints in (3.3), we can express the time t consumption and capital stock in terms of k_0 and all previous consumption rates:

$$\begin{cases} c_t = e_t f\left((1-e_{t-1})f\left((1-e_{t-2})f(\cdots f(k_0))\right)\right), \quad t \in \mathbb{T}, \\ k_{t+1} = (1-e_t) f\left((1-e_{t-1})f\left((1-e_{t-2})f(\cdots f(k_0))\right)\right), \quad t \in \mathbb{T}. \end{cases} \tag{3.5}$$

Clearly, given k_0, there is a one-to-one correspondence between feasible consumption paths $\{c_t\}_{t\in\mathbb{T}}$ and feasible sequences of consumption rates $\{e_t\}_{t\in\mathbb{T}}$.

Substituting the resource constraints into the objective function and using (3.5), we rewrite the utility function of agent i in terms of consumption rates as follows:

$$u_i\left(e_0 f(k_0)\right) + \beta_i u_i\left(e_1 f\left((1-e_0)f(k_0)\right)\right) + \beta_i^2 u_i\left(e_2 f\left((1-e_1)f\left((1-e_0)f(k_0)\right)\right)\right) + \ldots .$$

[9] We apply the change of control variable similar to that of Phelps and Pollak (1968), and Peleg and Yaari (1973). They used as control variable savings rate, which in the Ramsey model is naturally related to consumption rate: $k_{t+1}/f(k_t) = s_t = 1 - e_t$. However, in the decision-making context it seems reasonable to use consumption rate instead of savings rate.

Then problem (3.3) becomes

$$\max \sum_{t\in\mathbb{T}} \beta_i^t u_i \left(e_t f \left((1-e_{t-1}) f \left((1-e_{t-2}) f(\cdots f(k_0))\right)\right)\right), \quad \text{s. t.} \quad 0 \le e_t \le 1, \ t \in \mathbb{T}. \tag{3.6}$$

Definition. *The solution to problem (3.6), $E^{i*} = \{e_t^{i*}\}_{t\in\mathbb{T}}$, is called an optimum in terms of consumption rates for agent i.*

It is clear that, given k_0, there is a one-to-one correspondence between optima in terms of consumption levels and optima in terms of consumption rates. It follows that there is a unique optimum in terms of consumption rates for each agent.

3.4.2. Intertemporal voting equilibria

We give the definition of an intertemporal voting equilibrium in two stages, following the Hicks–Grandmont approach (Hicks, 1939; Grandmont, 1977). First, for an arbitrary point in time τ we define a time τ temporary voting equilibrium as a Condorcet winner in voting over the current consumption rate given a current stock of capital and some expectations about future consumption rates. Secondly, we define an intertemporal voting equilibrium as a sequence, each element of which is a time τ temporary voting equilibrium provided agents have perfect foresight about future consumption rates.

Consider an arbitrary point in time τ. Suppose that the capital stock is $k_\tau > 0$ and agents have some expectations about future consumption rates represented by a sequence $E_{\tau+1,T} = \{e_t\}_{t=\tau+1}^T$.[10] Preferences of agent i in voting over the time τ consumption rate are given by the following objective function:

$$V_\tau^i(k_\tau, e_\tau, E_{\tau+1,T}) = \sum_{t=\tau}^{T} \beta_i^{t-\tau} u_i \left(e_t f \left((1-e_{t-1}) f \left((1-e_{t-2}) f(\cdots f(k_\tau))\right)\right)\right).$$

It is clear that the objective function is well-defined only if $V_\tau^i(k_\tau, e_\tau, E_{\tau+1,T}) \neq \pm\infty$. In order to ensure that the objective function is finite, it is necessary to impose certain restrictions on the sequence of expectations. We shall require that the sequence of expectations is ***non-degenerate***. The formal definition of a non-degenerate sequence can be found in Section 3.9.1. Here it is sufficient to say that if the sequence of expectations, $E_{\tau+1,T}$, is non-degenerate, then for any $k_\tau > 0$, the function $V_\tau^i(k_\tau, e_\tau, E_{\tau+1,T})$ is well-defined and differentiable with respect to e_τ on the interval $(0, 1)$.[11]

[10] For simplicity of presentation, we assume that all agents have the same expectations, though as a general rule each agent can have her own expectations.

[11] In particular, in the case of a finite horizon, the sequence $E_{\tau+1,T} = \{e_t\}_{t=\tau+1}^T$ is called non-degenerate if $0 < e_t < 1,\ t = \tau+1, t = \tau+2, ..., T-1$, and $0 < e_T \le 1$. In the case of an infinite horizon, the definition is a little more complicated.

Definition. *Given the stock of capital at time $\tau < T$, $k_\tau > 0$, and non-degenerate expectations $E_{\tau+1,T}$, we call e^i_τ a preferred time τ consumption rate for agent i if it is a solution to the one-dimensional optimization problem*

$$\max_{0 \le e_\tau \le 1} V^i_\tau(k_\tau, e_\tau, E_{\tau+1,T}).$$

If $T < \infty$, the preferred time T consumption rate for agent i is $e^i_T = 1$.

It should be emphasized that a preferred time τ consumption rate depends on the current capital stock and expectations.

Definition. *Given the current stock of capital at time $\tau < T$, $k_\tau > 0$, and non-degenerate expectations $E_{\tau+1,T}$, we call e^*_τ a time τ (temporary) voting equilibrium if it is a Condorcet winner in one-dimensional voting over the time τ consumption rate. If $T < \infty$, the time T voting equilibrium is $e^*_T = 1$.*

Now let us explain what we mean by an intertemporal voting equilibrium. Suppose that agents vote step by step starting from time 0. They are given the initial capital stock k_0 and some non-degenerate expectations about future consumption rates, $E_{1,T}$. The winner in voting over the time 0 consumption rate, the time 0 voting equilibrium e^*_0, generically depends on expectations. At time 1, all relevant information about the decision made at time 0 is gathered in the new capital stock k_1. Agents vote over the time 1 consumption rate given k_1 and some non-degenerate expectations about future consumption rates, $E_{2,T}$. And so on. If agents have perfect foresight, then an outcome of this dynamic procedure is called an intertemporal voting equilibrium.

Formally, suppose that we are given an initial stock of capital, $k_0 > 0$, and a sequence of consumption rates, $E_{0,T} = \{e_\tau\}^T_{\tau=0}$. At every date τ, the current capital stock k_τ is unambiguously determined by the past consumption rates. Hence for every $\tau \in \mathbb{T}$, $k_{\tau+1}$ can be considered as a function of k_0 and the past values of consumption rates $E_{0,\tau} = \{e_0, e_1, \dots, e_\tau\}$. Denote $k_{0,0} = k_0$, and recursively define the functions $k_{0,\tau}(\cdot,\cdot)$ as

$$k_{0,1}(k_0, E_{0,0}) = (1 - e_0) f(k_{0,0}),$$
$$k_{0,\tau}(k_0, E_{0,\tau-1}) = (1 - e_{\tau-1}) f(k_{0,\tau-1}(k_0, E_{0,\tau-2})), \quad \tau = 2, 3, \dots.$$

Definition. *We call a non-degenerate sequence of consumption rates $E^*_{0,T} = \{e^*_\tau\}^T_{\tau=0}$ an intertemporal voting equilibrium starting from $k_0 > 0$ if for each τ, e^*_τ is a time τ voting equilibrium for the current stock of capital given by $k_\tau = k_{0,\tau}(k_0, E^*_{0,\tau-1})$[12] under perfect foresight about outcomes of future votes (i.e., under expectations given by $E^*_{\tau+1,T} = \{e^*_t\}^T_{t=\tau+1}$).*

[12] The sequence $E^*_{0,\tau-1}$ contains the first τ elements from an intertemporal voting equilibrium $E^*_{0,T}$, i.e., $E^*_{0,\tau-1} = \{e^*_t\}^{\tau-1}_{t=0}$.

Technically, an intertemporal voting equilibrium is a non-degenerate sequence, every element of which is chosen by a majority of agents provided all other consumption rates are already chosen according to the same procedure. Hence an intertemporal voting equilibrium is essentially a Kramer–Shepsle equilibrium.

3.5. Step-by-step intertemporal optimum

Before presenting our main results about an intertemporal voting equilibrium, let us gain more insight into the proposed voting procedure by analyzing the optimization problem in terms of consumption rates for a single agent. In this section we introduce the notion of a step-by-step intertemporal optimum, to which the notion of an intertemporal voting equilibrium is reduced in the case with only one agent, and show that it coincides with the optimum in terms of consumption rates. This simple result, which will be useful in what follows, seems not surprising and is in fact quite natural. However, to the best of our knowledge, it has not yet been explicitly stated and proved.

Consider problem (3.6) for an arbitrary agent, and omit the index i for the simplicity of notation:

$$\max \sum_{t\in\mathbb{T}} \beta^t u\left(e_t f\left((1-e_{t-1}) f\left((1-e_{t-2}) f(\cdots f(k_0))\right)\right)\right), \quad \text{s. t.} \quad 0 \le e_t \le 1, \quad t \in \mathbb{T}. \quad (3.7)$$

Suppose that instead of solving this problem "once-and-for-all" at time 0, the agent tries to solve it in a step-by-step manner. Namely, suppose that at each time τ she determines e_τ by solving the problem

$$\max_{0\le e_\tau \le 1} V_\tau(k_\tau, e_\tau, E_{\tau+1,T}), \quad (3.8)$$

where

$$V_\tau(k_\tau, e_\tau, E_{\tau+1,T}) = \sum_{t=\tau}^{T} \beta^{t-\tau} u\left(e_t f\left((1-e_{t-1}) f\left((1-e_{t-2}) f(\cdots f(k_\tau))\right)\right)\right), \quad (3.9)$$

$k_\tau > 0$ is the capital stock at time τ (it is determined in the previous step) and $E_{\tau+1,T} = \{e_t\}_{t=\tau+1}^T$ are expectations about future consumption rates. Clearly, the outcome of this procedure depends on the way expectations are formed. We are interested in the case where the agent has perfect foresight about her future decisions about consumption rate.

Definition. *Consider problem (3.7). We call a non-degenerate sequence of consumption rates* $E^*_{0,T} = \{e^*_\tau\}_{\tau=0}^T$ *a step-by-step intertemporal optimum if for each* $\tau \in \mathbb{T}$, e^*_τ *is a solution to the following problem:*

$$\max_{0\le e_\tau \le 1} V_\tau(k_\tau, e_\tau, E^*_{\tau+1,T}),$$

where $k_\tau = k_{0,\tau}(k_0, E^*_{0,\tau-1})$ *and* $E^*_{\tau+1,T} = \{e^*_t\}^T_{t=\tau+1}$.

If we call a solution to problem (3.8) *a time* τ *temporary optimum*, then a step-by-step intertemporal optimum is a sequence of temporary optima obtained under perfect foresight about future extractions rates. It should be noted that in an intertemporal voting equilibrium the agent has perfect foresight about future collective decisions (outcomes of voting), while in a step-by-step intertemporal optimum she has perfect foresight about her personal decisions.

Also it is noteworthy that if we adopt an atemporal point of view, then a step-by-step intertemporal optimum can simply be considered as a result of coordinate-wise maximization of the function $\sum_{t\in\mathbb{T}} \beta^t u\left(e_t f\left((1-e_{t-1})f\left((1-e_{t-2})f(\cdots f(k_0))\right)\right)\right)$.

Clearly, if some sequence is a "once-and-for-all" solution to problem (3.7), then this sequence is a step-by-step intertemporal optimum. There arises a question, whether the opposite is also true. Is it correct that any step-by-step intertemporal optimum is the optimum in terms of consumption rates?

The answer to this question is positive if either $T < \infty$ or $T = \infty$ and the felicity function $u(c)$ satisfies

Regularity condition. *There exists* $\gamma > 0$ *such that* $\lim_{c\to 0} c^\gamma u'(c) = 0$.

This condition means that $u'(c)$ tends to infinity at $c \to 0$ no faster than any power function. For instance, every CIES felicity function meets the regularity condition.[13] Moreover, every felicity function such that $u(0) > -\infty$ also satisfies this condition.

Proposition 3.1. *Consider problem (3.7). Suppose that* $T < \infty$, *or* $T = \infty$ *and the felicity function satisfies the regularity condition. A step-by-step intertemporal optimum exists and coincides with the optimum in terms of consumption rates.*

Proof. See Section 3.9.2. □

It follows that there exists a unique (non-degenerate) step-by-step intertemporal optimum for each agent. It coincides with the unique optimum in terms of consumption rates, and corresponds to the unique optimum in terms of consumption levels for this agent.

3.6. Main results

Now we are ready to characterize temporary and intertemporal voting equilibria. We begin by explicitly stating the existence of a time τ voting equilibrium.

[13] The class of felicity functions that satisfy the regularity condition is similar to that considered by Ekeland and Scheinkman (1986). In Case 2 (linear production function), this condition is redundant.

Proposition 3.2. *For any non-degenerate expectations, the preferences of each agent in voting over the time τ consumption rate are strictly concave and single-peaked; hence the median voter theorem applies and a time τ voting equilibrium exists.*

Proof. See Section 3.9.3. □

It follows that at each given point in time there exists an instantaneous Condorcet winner, which is by definition a time τ voting equilibrium. Proposition 3.2 states that our institutional framework (intertemporal majority voting) is well-defined even if expectations are not correct and differ for different agents. In any case, voting over the current consumption rate is a well-defined one-dimensional decision problem.

Now we can provide a characterization of an intertemporal voting equilibrium in the special but important case where all agents have the same felicity function, so that agents are heterogeneous only in their time preferences. First we characterize a time τ voting equilibrium.

Proposition 3.3. *Suppose all agents have the same felicity function and the same non-degenerate expectations. A time τ voting equilibrium exists, is unique, and coincides with the preferred time τ consumption rate for the agent with the median discount factor β_{med}.*

Proof. Here we give only a sketch the proof. For the details, see Section 3.9.3. We prove that if agents have the same felicity function and the same non-degenerate expectations, then higher values of the discount factor correspond to lower values of the preferred time τ consumption rate. Therefore it follows from Proposition 3.2 that a time τ voting equilibrium is the preferred consumption rate for the agent with the median discount factor. □

Combining Propositions 3.1 and 3.3, we can formulate the following theorem.

Theorem 3.1. *Suppose all agents have the same felicity function. Suppose further that $T < \infty$, or $T = \infty$ and the felicity function satisfies the regularity condition. An intertemporal voting equilibrium starting from any $k_0 > 0$ exists, is unique and coincides with the optimum in terms of consumption rates for the agent with the median discount factor β_{med}.*

Proof. Proposition 3.3 states that the time τ voting equilibrium is the time τ temporary optimum for the agent with the median discount factor. Taking into account the definition of a step-by-step intertemporal optimum, it follows from Proposition 3.1 that an intertemporal voting equilibrium is the optimum in terms of consumption rates for the agent with the median discount factor. □

Since there is a one-to-one correspondence between the optimum in terms of consumption rates and the optimum in terms of consumption levels, *there is a one-to-one correspondence between the intertemporal voting equilibrium and the optimum in terms of consumption levels for the agent with the median discount factor*. Thus in the case where all agents have the same felicity function, the proposed voting procedure yields as the outcome the optimum in terms of consumption levels for the agent with the median discount factor.

Note that when $T = \infty$ in Case 2 (linear production function) the optimum in terms of consumption levels is a balanced-growth path. If all agents have the same CIES felicity function with the parameter ρ, then the solution to problem (3.3) is

$$c_{t+1}^{i*} = (\beta_i A)^{\frac{1}{\rho}} c_t^{i*}, \quad k_{t+1}^{i*} = (\beta_i A)^{\frac{1}{\rho}} k_t^{i*}, \quad t = 0, 1, \ldots .$$

Therefore, regardless of the initial conditions, consumption and the capital stock for agent i grow at a constant rate $(\beta_i A)^{\frac{1}{\rho}}$. The corresponding intertemporal voting equilibrium is a constant sequence of consumption rates, $\{e^*, e^*, \ldots\}$, where e^* is determined by the median discount factor:

$$e^* = 1 - \frac{1}{A}\left(1 + (\beta_{med} A)^{\frac{1}{\rho}}\right).$$

3.7. Steady-state and balanced-growth voting equilibria

Now suppose that agents differ both in their time preferences and felicity functions. Clearly, there is no reason to expect that the result of Theorem 3.1 still holds in this case. Because of multi-dimensional heterogeneity, it is in principle impossible to claim that an intertemporal voting equilibrium is determined by the discount factor alone. However, we are able to obtain some results even with multi-dimensional heterogeneity. In Case 1 (strictly concave production function) these results concern steady-state voting equilibria, and in Case 2 (linear production function) — balanced-growth voting equilibria. We show the important difference between the two cases: in Case 1 (strictly concave production function), the steady-state voting equilibrium is fully determined by the median discount factor, whereas in Case 2 (linear production function), the balanced-growth voting equilibrium depends not only on the agents' discount factors, but also on the agents' intertemporal elasticities of substitution.[14]

[14] See also Nakamura (2014) who stresses the role of the intertemporal elasticity of substitution in determining the long-run growth in the Ramsey model with AK technology, as opposed to the standard Ramsey model.

3.7.1. Steady-state voting equilibrium

In the Case 1 (strictly concave production function), if $T = \infty$ and the sequence of consumption rates is constant over time, $E = \{e, e, \ldots\}$, then the considered model becomes the Solow model with a unique steady state. The capital stock in this steady state, $k(e)$, is the only positive solution to the following equation[15] in k:

$$k = (1 - e) f(k). \tag{3.10}$$

Definition. *Consider Case 1 (strictly concave production function). We call e^* a steady-state voting equilibrium if the sequence $\{e^*, e^*, \ldots\}$ is an intertemporal voting equilibrium starting from $k_0 = k(e^*) > 0$.*

Suppose first that all agents have the same felicity function that satisfies the regularity condition. It follows from Theorem 3.1 that for any k_0 there is a unique intertemporal voting equilibrium, which corresponds to the optimum for the agent with the median discount factor. Take as the initial capital stock the value k^* determined by the "modified golden rule" for the agent with the median discount factor β_{med}: $\beta_{med} f'(k^*) = 1$. The optimum for the agent with the median discount factor starting from $k_0 = k^*$ is her steady-state optimum. The corresponding optimum in terms of consumption rates is a constant sequence $E^* = \{e^*, e^*, \ldots\}$, and, by Theorem 3.1, is a unique intertemporal voting equilibrium. Clearly, k^* is the unique solution to equation (3.10) at $e = e^*$. Hence e^*, the optimal consumption rate for the agent with the median discount factor β_{med}, is the unique stationary voting equilibrium.

Since k^* is given by the "modified golden rule" and e^* depends only on the median discount factor, the stationary voting equilibrium does not depend on the felicity function of agents. This observation leads to the following theorem which holds in the general case where agents have different felicity functions.

Theorem 3.2. *In Case 1 (strictly concave production function), there is a unique steady-state voting equilibrium. It is given by*

$$e^* = 1 - \frac{k^*}{f(k^*)}. \tag{3.11}$$

Proof. See Section 3.9.4. □

Thus, even in the case with different felicity functions there is a unique steady-state voting equilibrium. It is completely determined by the "modified golden rule" for the agent with the median discount factor and independent of felicity functions.

[15] It exists if $(1 - e) f'(0) > 1$.

It is well-known that in a single-agent Ramsey model, the optimal capital stock converges to the modified golden rule path, which is fully determined by the discount factor of the agent, and is independent of her felicity function. In our model, any intertemporal voting equilibrium converges to the steady-state voting equilibrium if all agents have the same felicity function. Is the same result true in the case where agents have different felicity functions? This, as well as the proof of the existence of an intertemporal voting equilibrium in the general case, is a topic for further research.

3.7.2. Balanced-growth voting equilibrium

As we noted above, in Case 2 (linear production function) with $T = \infty$ if all agents have the same CIES felicity function, then the optimum in terms of consumption levels for each agent is a balanced-growth path, and the intertemporal voting equilibrium is characterized by a constant consumption rate.

If agents have different CIES felicity functions, then the optimum in terms of consumption levels for each agent i is also a balanced-growth path in which consumption and capital grow at a constant rate γ_i given by

$$1 + \gamma_i = (\beta_i A)^{\frac{1}{\rho_i}} . \tag{3.12}$$

Though agents differ both in the discount factors and in the elasticities of intertemporal substitution, their heterogeneity can in some sense be considered as one-dimensional. Due to the linear production function, agents are naturally characterized by their growth rates that aggregate both heterogeneity parameters.

It is natural to conjecture that any intertemporal voting equilibrium in this case is also characterized by a constant growth rate. At the moment, we cannot prove this conjecture. However, we shall prove now that a *balanced-growth voting equilibrium exists* and, what is important, *is determined not by the median discount factor, but by the median growth rate* γ_{med}.

If we are given a constant over time consumption rate e, then for any initial stock $k_0 > 0$ the corresponding capital stock and consumption grow at a constant rate:

$$k_{t+1} = (1 + \gamma)k_t, \quad c_{t+1} = (1 + \gamma)c_t, \quad t = 0, 1, \ldots,$$

where $\gamma = (1 - e)A - 1$.

Definition. *Consider Case 2 (linear production function). We call e^* a balanced-growth voting equilibrium starting from $k_0 > 0$ if the sequence $\{e^*, e^*, \ldots\}$ is an intertemporal voting equilibrium starting from k_0.*

The following theorem shows that a balanced-growth voting equilibrium is fully determined by the preferences of the agent with the median growth rate γ_{med}.

Theorem 3.3. *In Case 2 (linear production function), for any $k_0 > 0$, there is a unique balanced-growth voting equilibrium starting from k_0. It is given by*

$$e^* = 1 - \frac{1}{A}(1 + \gamma_{med}). \tag{3.13}$$

Proof. See Section 3.9.5. □

Clearly, the consumption path corresponding to the balanced-growth voting equilibrium $E^* = \{e^*, e^*, \ldots\}$ is the balanced-growth path for the agent with the median growth rate γ_{med}. However, it is a topic for further research whether there exist intertemporal voting equilibria that are not balanced-growth voting equilibria.

3.8. Conclusion

The problem of collective choice naturally arises in many economic applications with heterogeneous agents. In this chapter, we study a Ramsey-type growth model with common consumption and agents who may differ in their instantaneous utility ("felicity") functions and discount factors. It is well known that, in general, there is no Condorcet winner if agents vote over feasible consumption streams. This is true even if agents differ only in their discount factors and heterogeneity is one-dimensional. Notwithstanding these negative findings, we show in this chapter that the choice of the optimal consumption stream of the "median" agent in many important cases can be obtained as the result of a simple and natural institutional setup, intertemporal majority voting.

Our voting procedure is based on two principles. First, agents vote step-by-step at each point in time. Second, agents vote over the consumption *rate*, not over the consumption *level*. We define a temporary voting equilibrium, which is a Condorcet winner among all current consumption rates under some expectations about future consumption rates. Then, we define an intertemporal voting equilibrium as a sequence of temporary voting equilibria under the assumption that agents have perfect foresight about future consumption rates. From the technical point of view, an intertemporal voting equilibrium is a Kramer–Shepsle equilibrium in terms of consumption rates.

Our main result concerns the case where agents have identical felicity functions and differ only in their discount factors. We prove that an intertemporal voting equilibrium exists, is unique, and coincides with the optimum in terms of consumption rates for the agent with the median discount factor. We thus show that even in the absence of a Condorcet winner there is a stable outcome of intertemporal majority voting. Since this outcome is determined by the preferences of the agent with the median discount factor, it is both time-consistent and Pareto efficient.

We also consider the general framework where agents may differ also in their felicity functions, and analyze two special cases. In the case of a strictly concave production function and arbitrary felicity functions, we define a steady-state voting equilibrium, and show that it is unique and is fully determined by the median discount factor. Our analysis suggests that in the case of a strictly concave production function the analogy with the standard Ramsey model may fruitfully be applied. One may conjecture that every intertemporal voting equilibrium converges to the steady-state voting equilibrium, and thus the winner of the voting procedure eventually depends only on the discount factor. However, further research is needed to confirm or reject this conjecture.

In the case of a linear production function and CIES felicity functions, we define a balanced-growth voting equilibrium, prove its uniqueness, and show that it is determined by the median growth rate, where the growth rate for each agent depends not only on the discount factor, but also on the elasticity of intertemporal substitution.

It should be recognized that on the one hand, in our model agents are excessively sophisticated because in an intertemporal voting equilibrium they have perfect foresight about outcomes of future votes. On the other hand, they are sophisticated to a limited extent, because the set of their strategies is limited (consumption depends on production in a linear way). Introducing either less or more sophisticated agents into our framework might be a topic for further research.

Another possible research avenue is to apply the proposed procedure to voting in a dynamic general equilibrium framework. This is done in Chapter 4 of the thesis, where we consider intertemporal majority voting in the general equilibrium Ramsey-type model with exhaustible natural resources. The goal of the next chapter is to compare different property regimes over natural resources in terms of economic growth, and our voting procedure serves to determine the equilibrium sequence of extraction rates in the public property regime over natural resources.

3.9. Proofs

3.9.1. Non-degenerate sequences and properties of the objective functions

For an arbitrary τ and a sequence of consumption rates $E_{\tau,T} = \{e_\tau, e_{\tau+1}, \dots, e_T\}$, denote

$$E_{\tau,\tau} = \{e_\tau\}, \quad E_{\tau,t} = \{e_\tau, e_{\tau+1}, \dots, e_t\} = \{e_\tau, E_{\tau+1,t}\},\ t = \tau+1, \tau+2, \dots,$$
$$k_{\tau,\tau} = k_\tau, \quad k_{\tau,\tau+1}(k_\tau, E_{\tau,\tau}) = (1-e_\tau) f\left(k_{\tau,\tau}\right),$$
$$k_{\tau,t+1}(k_\tau, E_{\tau,t}) = (1-e_t) f\left(k_{\tau,t}(k_\tau, E_{\tau,t-1})\right),\ t = \tau+1, \tau+2, \dots,$$
$$f_{\tau,\tau} = f\left(k_\tau\right), \quad f_{\tau,\tau+1}(k_\tau, E_{\tau,\tau}) = f\left((1-e_\tau) f_{\tau,\tau}\right),$$
$$f_{\tau,t+1}(k_\tau, E_{\tau,t}) = f\left((1-e_t) f_{\tau,t}(k_\tau, E_{\tau,t-1})\right),\ t = \tau+1, \tau+2, \dots.$$

Thus for $t > \tau$,

$$k_{\tau,t+1}(k_\tau, E_{\tau,t}) = (1-e_t) f_{\tau,t}\left(k_\tau, E_{\tau,t-1}\right),\ f_{\tau,t+1}(k_\tau, E_{\tau,t}) = f\left(k_{\tau,t+1}(k_\tau, E_{\tau,t})\right).$$

For simplicity of notation, we often drop the arguments of these functions when they play no significant role. However, the reader should bear in mind that $f_{\tau,t+1}$ is a function of k_τ and the $t-\tau+1$ consumption rates $\{e_\tau, e_{\tau+1}, \dots, e_t\}$.

The derivatives of $f_{\tau,t+1}$ can be obtained using the chain rule of differentiation:

$$\frac{\partial f_{\tau,t+1}}{\partial e_t} = -f'(k_{\tau,t+1}) f_{\tau,t},\ \frac{\partial f_{\tau,t+1}}{\partial e_{t-1}} = -f'(k_{\tau,t+1})(1-e_t) f'(k_{\tau,t}) f_{\tau,t-1},$$
$$\frac{\partial f_{\tau,t+1}}{\partial e_{t-2}} = -f'(k_{\tau,t+1})(1-e_t) f'(k_{\tau,t})(1-e_{t-1}) f'(k_{\tau,t-1}) f_{\tau,t-2},\ \dots.$$

It is clear that the derivative of $f_{\tau,t+1}(k_\tau, E_{\tau,t})$ with respect to each consumption rate $\{e_\tau, e_{\tau+1}, \dots, e_t\}$ is negative.

Definition. *A) Suppose that $T < \infty$. We call a sequence $E_{\tau,T} = \{e_t\}_{t=\tau}^T$ non-degenerate if*

$$0 < e_t < 1,\ t = \tau, \tau+1, \dots, T-1; \quad 0 < e_T \le 1.$$

B) Suppose that $T = \infty$ and consider a sequence $E_{\tau,\infty} = \{e_t\}_{t=\tau}^\infty$ such that $0 < e_t < 1$ for all t.

1. *In Case 1 (strictly concave production function), the sequence $E_{\tau,\infty} = \{e_t\}_{t=\tau}^\infty$ is called non-degenerate if*

   1.1.

$$0 < \liminf_{t\to\infty} e_t \le \limsup_{t\to\infty} e_t < 1, \tag{3.14}$$

1.2. for some $\tilde{k}_\tau > 0$ *the sequence* $\{\tilde{k}_t\}_{t=\tau}^{\infty}$ *given by* $\tilde{k}_{t+1} = (1-e_t)f(\tilde{k}_t)$, $t = \tau, \tau+1, \ldots$, *satisfies*

$$\liminf_{t\to\infty} \tilde{k}_t > 0. \tag{3.15}$$

2. In Case 2 (linear production function), the sequence $E_{\tau,\infty} = \{e_t\}_{t=\tau}^{\infty}$ *is called non-degenerate if there exist* $\underline{e}$ *and* $\bar{e}$ *such that*

2.1.

$$0 \le \underline{e} < \liminf_{t\to\infty} e_t \le \limsup_{t\to\infty} e_t < \bar{e} \le 1, \tag{3.16}$$

2.2. for all i,[16]

$$\begin{cases} \beta_i \left(A(1-\underline{e})\right)^{1-\rho_i} < 1, & \text{if} \quad \rho_i \le 1, \\ \beta_i \left(A(1-\bar{e})\right)^{1-\rho_i} < 1, & \text{if} \quad \rho_i > 1. \end{cases} \tag{3.17}$$

We need the above definition to establish certain important properties of the agents' objective functions. Recall that the objective function of agent i in voting over the time τ consumption rate is given by

$$V_\tau^i(k_\tau, e_\tau, E_{\tau+1,T}) = u_i\left(e_\tau f_{\tau,\tau}\right) + \beta_i u_i\left(e_{\tau+1} f_{\tau,\tau+1}\right) + \beta_i^2 u_i\left(e_{\tau+2} f_{\tau,\tau+2}\right) + \ldots.$$

Let us show that if the sequence of expectations $E_{\tau+1,T}$ is non-degenerate, then for any $k_\tau > 0$ the objective function of agent i is well-defined.

Lemma 3.1. *Suppose that the sequence of expectations* $E_{\tau+1,T}$ *is non-degenerate. For any* $k_\tau > 0$, $-\infty < V_\tau^i(k_\tau, e_\tau, E_{\tau+1,T}) < +\infty$.

Proof. When $T < \infty$, the finiteness of $V_\tau^i(k_\tau, e_\tau, E_{\tau+1,T})$ is evident, since the expected consumption rates are bounded away from 0 and 1 (except for the time T).

Suppose $T = \infty$ and consider Case 1 (strictly concave production function). Since there is a maximum sustainable stock, $\bar{k} = f(\bar{k})$ (see (3.1)), condition (3.14) guarantees that $V_\tau^i(k_\tau, e_\tau, E_{\tau+1,\infty})$ is bounded from above. Conditions (3.14) and (3.15)[17] also ensure that for any $k_\tau > 0$ the path of the capital stock $\{k_t\}_{t=\tau}^{\infty}$ constructed by $k_{t+1} = (1-e_t)f(k_t)$ and the corresponding consumption path $\{c_t\}_{t=\tau}^{\infty}$ constructed by $c_t = e_t f(k_t)$ are bounded away from zero:

$$\liminf_{t\to\infty} k_t > 0, \quad \liminf_{t\to\infty} f(k_t) > 0, \quad \liminf_{t\to\infty} c_t > 0. \tag{3.18}$$

Indeed, if $k_\tau > \tilde{k}_\tau$, then $k_t > \tilde{k}_t$, $f(k_t) > f(\tilde{k}_t)$ and $c_t > \tilde{c}_t$ for all $t = \tau, \tau+1, \ldots$. If $k_\tau < \tilde{k}_\tau$, then for all $t = \tau, \tau+1, \ldots$, $k_t < \tilde{k}_t$ and $k_{t+1}/\tilde{k}_{t+1} > k_t/\tilde{k}_t$, because $f(k)/k$ is decreasing

[16] Recall that in Case 2 (linear production function) the felicity function is given by (3.2).
[17] Note that if $f'(0) = +\infty$, then condition (3.15) is redundant, because it follows from condition (3.14).

in k. Therefore $\lim_{t\to\infty} k_t/\tilde{k}_t > 0$, $\lim_{t\to\infty} f(k_t)/f(\tilde{k}_t) > 0$ and $\lim_{t\to\infty} c_t/\tilde{c}_t > 0$. Hence, in both cases the inequalities in (3.18) hold true and $V^i_\tau(k_\tau, e_\tau, E_{\tau+1,\infty})$ is also bounded from below.

Consider Case 2 (linear production function). The objective function of agent i takes the form[18]

$$V^i_\tau(k_\tau, e_\tau, E_{\tau+1,\infty}) = \sum_{t=\tau}^{\infty} \frac{\beta_i^{t-\tau}}{1-\rho_i} \left(A^{t-\tau+1} e_t (1-e_{t-1})(1-e_{t-2})\cdots(1-e_\tau)k_\tau\right)^{1-\rho_i}.$$

For i such that $0 < \rho_i < 1$ it follows from (3.16) and (3.17) that

$$\begin{aligned} 0 < V^i_\tau(k_\tau, e_\tau, E_{\tau+1,\infty}) &\le \frac{1}{1-\rho_i}\left(Ae_\tau k_\tau\right)^{1-\rho_i} + \frac{1}{1-\rho_i}\beta_i \left(A^2(1-e_\tau)k_\tau\right)^{1-\rho_i} \times \\ &\times \left[(\bar{e})^{1-\rho_i} + \beta_i A^{1-\rho_i}(\bar{e}(1-\underline{e}))^{1-\rho_i} + (\beta_i A^{1-\rho_i})^2(\bar{e}(1-\underline{e})^2)^{1-\rho_i} + \ldots\right] \\ &= \frac{1}{1-\rho_i}\left(Ae_\tau k_\tau\right)^{1-\rho_i} + \frac{1}{1-\rho_i}\beta_i \left(A^2(1-e_\tau)k_\tau\right)^{1-\rho_i}(\bar{e})^{1-\rho_i} \times \\ &\times \left[1 + \beta_i (A(1-\underline{e}))^{1-\rho_i} + (\beta_i (A(1-\underline{e}))^{1-\rho_i})^2 + \ldots\right] \\ &= \frac{1}{1-\rho_i}\left(Ae_\tau k_\tau\right)^{1-\rho_i} + \frac{\beta_i}{1-\rho_i} \frac{\left(A^2(1-e_\tau)k_\tau\right)^{1-\rho_i}(\bar{e})^{1-\rho_i}}{1-\beta_i\left(A(1-\underline{e})\right)^{1-\rho_i}} < +\infty. \end{aligned}$$

Slightly modifying the above argument, for i such that $\rho_i > 1$ we obtain that

$$0 > V^i_\tau(k_\tau, e_\tau, E_{\tau+1,\infty}) > \frac{1}{1-\rho_i}\left(Ae_\tau k_\tau\right)^{1-\rho_i} + \frac{\beta_i}{1-\rho_i} \frac{\left(A^2(1-e_\tau)k_\tau\right)^{1-\rho_i}(\underline{e})^{1-\rho_i}}{1-\beta_i\left(A(1-\bar{e})\right)^{1-\rho_i}} > -\infty,$$

which completes the proof. □

Thus we have shown that for a non-degenerate sequence of expectations $E_{\tau+1,T}$, the objective function of agent i exists and is well-defined. However, we also need to prove that $V^i_\tau(k_\tau, e_\tau, E_{\tau+1,T})$ is continuously differentiable with respect to e_τ. By differentiating the objective function term by term, we get

$$\frac{\partial V^i_\tau(k_\tau, e_\tau, E_{\tau+1,T})}{\partial e_\tau} = f(k_\tau)\left(\Phi^i_\tau(k_\tau, e_\tau) - \Psi^i_\tau(k_\tau, e_\tau, E_{\tau+1,T})\right), \tag{3.19}$$

where

$$\Phi^i_\tau(k_\tau, e_\tau) = u_i'\left(e_\tau f(k_\tau)\right), \tag{3.20}$$

[18] Here and hereafter we consider $\rho_i \neq 1$. The case with the logarithmic felicity function ($\rho_i = 1$) can be considered similarly.

and

$$\begin{aligned}\Psi^i_\tau(k_\tau, e_\tau, E_{\tau+1,T}) &= \beta_i u'_i\left(e_{\tau+1} f_{\tau,\tau+1}\right) e_{\tau+1} f'\left((1-e_\tau) f_{\tau,\tau}\right) \\ &+ \beta_i^2 u'_i\left(e_{\tau+2} f_{\tau,\tau+2}\right) e_{\tau+2} f'\left((1-e_{\tau+1}) f_{\tau,\tau+1}\right)(1-e_{\tau+1}) f'\left((1-e_\tau) f_{\tau,\tau}\right) + \dots \\ &+ \beta_i^s u'_i\left(e_{\tau+s} f_{\tau,\tau+s}\right) e_{\tau+s} \left[\prod_{t=1}^{s-1}\left[f'\left((1-e_{\tau+t}) f_{\tau,\tau+t}\right)(1-e_{\tau+t})\right]\right] f'\left((1-e_\tau) f_{\tau,\tau}\right) + \dots .\end{aligned} \tag{3.21}$$

The following lemma shows that term-by-term differentiation is valid.

Lemma 3.2. *Suppose that the sequence of expectations $E_{\tau+1,T}$ is non-degenerate. For any $k_\tau > 0$, $V^i_\tau(k_\tau, e_\tau, E_{\tau+1,T})$ is continuously differentiable with respect to e_τ on the interval $(0,1)$, and its derivative is given by (3.19).*

Proof. When $T < \infty$, the statement of the lemma is evident. When $T = \infty$, the proof is based on a well-known theorem of analysis (see, e.g., Zorich, 2015, p. 388). We need to show that $\Psi^i_\tau(k_\tau, e_\tau, E_{\tau+1,\infty})$, which is the infinite series of continuous functions, is uniformly convergent on the interval $\{e_\tau \mid \xi \le e_\tau \le 1-\xi\}$ for any $0 < \xi < 1$.

Consider Case 1 (strictly concave production function). It follows from (3.14) and (3.18) that

$$0 < \liminf_{t\to\infty} u'\left(e_{\tau+t} f_{\tau,\tau+t}\right) e_{\tau+t} \le \limsup_{t\to\infty} u'\left(e_{\tau+t} f_{\tau,\tau+t}\right) e_{\tau+t} < +\infty.$$

Using the fact that $f'(k) \le f(k)/k$ for any k, for $s = 2, 3, \dots$ we have

$$\begin{aligned}\prod_{t=1}^{s-1}\left[f'\left((1-e_{\tau+t}) f_{\tau,\tau+t}\right)(1-e_{\tau+t})\right] &\le \prod_{t=1}^{s-1}\left[\frac{f\left((1-e_{\tau+t}) f_{\tau,\tau+t}\right)}{(1-e_{\tau+t}) f_{\tau,\tau+t}}(1-e_{\tau+t})\right] \\ = \prod_{t=1}^{s-1}\left[\frac{f_{\tau,\tau+t+1}}{f_{\tau,\tau+t}}\right] = \frac{f_{\tau,\tau+s}}{f_{\tau,\tau+1}} &= \frac{f_{\tau,\tau+s}}{f\left((1-e_\tau) f_{\tau,\tau}\right)} \le \frac{\bar f}{f\left((1-e_\tau) f_{\tau,\tau}\right)} \le \frac{\bar f}{f\left(\xi f_{\tau,\tau}\right)},\end{aligned}$$

where $\bar f = \max\{f(\bar k), f(k_\tau)\}$. The penultimate inequality follows from the existence of a maximum sustainable stock $\bar k$, and from the fact that if $k_t > \bar k$, then $k_{t+1} < k_t$.

By the Weierstrass M-test (see, e.g., Zorich, 2015, p. 374), $\Psi^i_\tau(k_\tau, e_\tau, E_{\tau+1,\infty})$ converges uniformly on the interval $\{e_\tau \mid \xi \le e_\tau \le 1-\xi\}$ for any $0 < \xi < 1$, and hence, by the uniform limit theorem (see, e.g., Zorich, 2015, p. 383), is continuous in e_τ. It follows that for a non-degenerate sequence $E_{\tau+1,\infty}$, $\partial V^i_\tau(k_\tau, e_\tau, E_{\tau+1,\infty})/\partial e_\tau$ exists, is continuous in e_τ, and is given by (3.19).

Consider now Case 2 (linear production function). The function $\Psi^i_\tau(k_\tau, e_\tau, E_{\tau+1,\infty})$ defined by (3.21) takes the form

$$\begin{aligned}\Psi^i_\tau(k_\tau, e_\tau, E_{\tau+1,\infty}) &= \beta_i A e_{\tau+1}\left(e_{\tau+1}(1-e_\tau)A^2k_\tau\right)^{-\rho_i} \\ &\quad + \beta_i^2 e_{\tau+2}(1-e_{\tau+1})A^2\left(e_{\tau+2}(1-e_{\tau+1})(1-e_\tau)A^3k_\tau\right)^{-\rho_i} + \dots \\ &= \beta_i A(A^2k_\tau)^{-\rho_i}(1-e_\tau)^{-\rho_i}\Big[(e_{\tau+1})^{1-\rho_i} + \beta_i A^{1-\rho_i}\left(e_{\tau+2}(1-e_{\tau+1})\right)^{1-\rho_i} \\ &\quad + \left(\beta_i A^{1-\rho_i}\right)^2\left(e_{\tau+3}(1-e_{\tau+2})(1-e_{\tau+1})\right)^{1-\rho_i} + \dots\Big].\end{aligned}$$

Applying the same argument as in the proof of the finiteness of the objective function in Case 2, we obtain that for i such that $0 < \rho_i < 1$ it follows from (3.16) and (3.17) that

$$0 < \Psi^i_\tau(k_\tau, e_\tau, E_{\tau+1,\infty}) < \frac{\beta_i A(A^2k_\tau)^{-\rho_i}(\bar{e})^{1-\rho_i}(1-e_\tau)^{-\rho_i}}{1-\beta_i\left(A(1-\underline{e})\right)^{1-\rho_i}} < \frac{\beta_i A(A^2k_\tau)^{-\rho_i}(\bar{e})^{1-\rho_i}\,\xi^{-\rho_i}}{1-\beta_i\left(A(1-\underline{e})\right)^{1-\rho_i}}.$$

Analogously, it can be shown that for i such that $\rho_i > 1$,

$$0 < \Psi^i_\tau(k_\tau, e_\tau, E_{\tau+1,\infty}) < \frac{\beta_i A(A^2k_\tau)^{-\rho_i}(\underline{e})^{1-\rho_i}(1-e_\tau)^{-\rho_i}}{1-\beta_i\left(A(1-\bar{e})\right)^{1-\rho_i}} < \frac{\beta_i A(A^2k_\tau)^{-\rho_i}(\underline{e})^{1-\rho_i}\,\xi^{-\rho_i}}{1-\beta_i\left(A(1-\bar{e})\right)^{1-\rho_i}}.$$

It follows that $\Psi^i_\tau(k_\tau, e_\tau, E_{\tau+1,\infty})$ converges uniformly on the interval $\{e_\tau \mid \xi \le e_\tau \le 1-\xi\}$ for any $0 < \xi < 1$, and hence is continuous in e_τ. Therefore, for a non-degenerate sequence $E_{\tau+1,\infty}$, $\partial V^i_\tau(k_\tau, e_\tau, E_{\tau+1,\infty})/\partial e_\tau$ exists, is continuous in e_τ, and is given by (3.19). □

Remark 3.1. *Note that $\Phi^i_\tau(k_\tau, 0) = +\infty$, $\Phi^i_\tau(k_\tau, e_\tau)$ is continuous and strictly decreasing in e_τ for any $0 < e_\tau \le 1$. Furthermore, $\Psi^i_\tau(k_\tau, e_\tau, E_{\tau+1,T})$ is continuous in e_τ, and, due to the strict concavity of u_i and f, is strictly increasing in e_τ. Each term in $\Psi^i_\tau(k_\tau, e_\tau, E_{\tau+1,T})$ contains a multiplier of the form*

$$u_i'\left(e_t f_{\tau,t}\right) = u_i'\left(e_t f\left((1-e_{t-1})f\left((1-e_{t-2})f\left(\cdots f\left((1-e_\tau)f(k_\tau)\right)\right)\right)\right)\right).$$

If $e_\tau = 1$, due to the fact that $f(0) = 0$, $\Psi^i_\tau(k_\tau, 1, E_{\tau+1,T}) = +\infty$. Taking into account Lemma 3.2, we obtain that there exists a unique interior solution to the equation $\Phi^i_\tau(k_\tau, e_\tau) = \Psi^i_\tau(k_\tau, e_\tau, E_{\tau+1,T})$ in e_τ.

3.9.2. Proof of Proposition 3.1

Consider problem (3.3) with omitted index i:

$$\max \sum_{t\in\mathbb{T}} \beta^t u(c_t), \quad \text{s. t.} \quad c_t + k_{t+1} = f(k_t), \quad c_t \ge 0, \quad k_{t+1} \ge 0, \quad t \in \mathbb{T}.$$

The first-order conditions are given by

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta f'(k_{t+1}), \quad t < T. \tag{3.22}$$

The transversality condition in the case of a finite horizon is as follows:

$$k_{T+1} = 0. \tag{3.23}$$

In the infinite horizon case, the transversality condition is given by

$$\lim_{t\to\infty} \beta^t u'(c_t) k_{t+1} = 0. \tag{3.24}$$

Let us show that the sequence $\{c_t, k_{t+1}\}_{t=0}^{T}$ corresponding to a step-by-step intertemporal optimum, satisfies the first-order conditions (3.22).

Let $E_{0,T}$ be a non-degenerate sequence of consumption rates. For any $\tau < T$ consider the objective function $V_\tau(k_\tau, e_\tau, E_{\tau+1,T})$ given by (3.9). By Remark 3.1, a step-by-step intertemporal optimum is a solution to the following system of equations[19]:

$$\frac{\partial V_t\left(k_{0,t}(k_0, E_{0,t-1}), e_t, E_{t+1,T}\right)}{\partial e_t} = 0, \quad t < T. \tag{3.25}$$

Consider two adjacent equations of the system (3.25), for $t = \tau$ and for $t = \tau + 1$. It follows from (3.19)–(3.21) (again omitting index i) that the equation for $t = \tau$ is as follows:

$$\begin{aligned} u'(e_\tau f_{0,\tau}) = {} & \beta e_{\tau+1} u'(e_{\tau+1} f_{0,\tau+1}) f'(k_{0,\tau+1}) \\ & + \beta^2 (1 - e_{\tau+1}) e_{\tau+2} u'(e_{\tau+2} f_{0,\tau+2}) f'(k_{0,\tau+2}) f'(k_{0,\tau+1}) \\ & + \beta^3 (1 - e_{\tau+1})(1 - e_{\tau+2}) e_{\tau+3} u'(e_{\tau+3} f_{0,\tau+3}) f'(k_{0,\tau+3}) f'(k_{0,\tau+2}) f'(k_{0,\tau+1}) + \ldots . \end{aligned} \tag{3.26}$$

Note that the right-hand side of the above equation can be rewritten as

$$\begin{aligned} & \beta f'(k_{0,\tau+1}) e_{\tau+1} u'(e_{\tau+1} f_{0,\tau+1}) \\ & \quad + \beta f'(k_{0,\tau+1})(1 - e_{\tau+1}) \big[\beta e_{\tau+2} u'(e_{\tau+2} f_{0,\tau+2}) f'(k_{0,\tau+2}) \\ & \qquad + \beta^2 (1 - e_{\tau+2}) e_{\tau+3} u'(e_{\tau+3} f_{0,\tau+3}) f'(k_{0,\tau+3}) f'(k_{0,\tau+2}) + \ldots \big]. \end{aligned} \tag{3.27}$$

[19] If $T < +\infty$, then there is an additional equation $e_T = 1$.

The equation for $t = \tau + 1$ is as follows:

$$u'(e_{\tau+1} f_{0,\tau+1}) = \beta e_{\tau+2} u'(e_{\tau+2} f_{0,\tau+2}) f'(k_{0,\tau+2}) \\ + \beta^2 (1 - e_{\tau+2}) e_{\tau+3} u'(e_{\tau+3} f_{0,\tau+3}) f'(k_{0,\tau+3}) f'(k_{0,\tau+2}) + \dots .$$

Substituting the right-hand side of the above equation into (3.27), we infer from (3.26) that

$$u'(e_\tau f_{0,\tau}) = \beta f'(k_{0,\tau+1}) \left(e_{\tau+1} u'(e_{\tau+1} f_{0,\tau+1}) + (1 - e_{\tau+1}) u'(e_{\tau+1} f_{0,\tau+1}) \right) \\ = \beta f'(k_{0,\tau+1}) u'(e_{\tau+1} f_{0,\tau+1}).$$

Applying this argument for all $\tau < T$, we obtain that the system of equations (3.25) is equivalent to the system

$$u'(e_t f_{0,t}) = \beta f'(k_{0,t+1}) u'(e_{t+1} f_{0,t+1}), \quad t < T. \tag{3.28}$$

Now it is straightforward to see that the mapping defined by (3.5) converts the system of equations (3.28) to the system of the first-order conditions (3.22).

It remains to show that the sequence $\{c_t, k_{t+1}\}_{t=0}^T$ corresponding to a step-by-step intertemporal optimum satisfies the transversality condition.

Consider the case $T < \infty$. Then the transversality condition (3.23) follows from the fact that the optimal consumption rate at time T for every agent is $e_T = 1$.

Consider the case $T = \infty$. Iterating equation for $t = \tau$ from the system (3.28), we obtain

$$u'(e_\tau f_{0,\tau}) = \beta^{t-\tau} u'(e_t f_{0,t}) f'(k_{0,t}) f'(k_{0,t-1}) \cdots f'(k_{0,\tau+1}), \quad t = \tau + 1, \tau + 2, \dots .$$

Hence equation (3.26) can be rewritten as follows:

$$u'(e_\tau f_{0,\tau}) - e_{\tau+1} u'(e_\tau f_{0,\tau}) \\ - (1 - e_{\tau+1}) e_{\tau+2} u'(e_\tau f_{0,\tau}) - (1 - e_{\tau+1})(1 - e_{\tau+2}) e_{\tau+3} u'(e_\tau f_{0,\tau}) \\ - (1 - e_{\tau+1})(1 - e_{\tau+2})(1 - e_{\tau+3}) e_{\tau+4} u'(e_\tau f_{0,\tau}) - \dots = 0. \tag{3.29}$$

Regrouping the terms on the left-hand side of (3.29), we get

$$
\begin{aligned}
&u'(e_\tau f_{0,\tau})(1-e_{\tau+1}) - e_{\tau+2}(1-e_{\tau+1})u'(e_\tau f_{0,\tau}) \\
&- (1-e_{\tau+1})(1-e_{\tau+2})e_{\tau+3}u'(e_\tau f_{0,\tau}) - (1-e_{\tau+1})(1-e_{\tau+2})(1-e_{\tau+3})e_{\tau+4}u'(e_\tau f_{0,\tau}) - \dots \\
&\qquad = u'(e_\tau f_{0,\tau})(1-e_{\tau+1})(1-e_{\tau+2}) - (1-e_{\tau+1})(1-e_{\tau+2})e_{\tau+3}u'(e_\tau f_{0,\tau}) \\
&\qquad\qquad - (1-e_{\tau+1})(1-e_{\tau+2})(1-e_{\tau+3})e_{\tau+4}u'(e_\tau f_{0,\tau}) - \dots \\
&= u'(e_\tau f_{0,\tau})(1-e_{\tau+1})(1-e_{\tau+2})(1-e_{\tau+3}) - (1-e_{\tau+1})(1-e_{\tau+2})(1-e_{\tau+3})e_{\tau+4}u'(e_\tau f_{0,\tau}) - \dots .
\end{aligned}
$$

Repeating this argument, (3.29) finally can be rewritten as

$$u'(e_\tau f_{0,\tau})(1-e_{\tau+1})(1-e_{\tau+2})(1-e_{\tau+3})(1-e_{\tau+4})\cdots = 0.$$

Since τ is chosen arbitrarily, and $u'(e_\tau f_{0,\tau}) > 0$, it follows that $\prod_{t=1}^{\infty}(1-e_t) = 0$, which is equivalent to

$$\sum_{t=1}^{\infty} e_t = +\infty. \tag{3.30}$$

Now let us show that the sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ corresponding to the solution to the system (3.25) satisfies the transversality condition (3.24).

Consider Case 1 (strictly concave production function). Let us analyze the possible dynamics of the sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty}$. We begin with two lemmas.

Lemma 3.3. *Suppose that $\beta f'(k_{\Theta+1}) > 1$ for some Θ, and $k_{\Theta+1} \le k_\Theta$. Then the transversality condition (3.24) holds.*

Proof. Let us show that the conditions of the lemma imply $k_{t+1} < k_t$ for all $t \ge \Theta$. Indeed, by (3.22), $u'(c_\Theta) > u'(c_{\Theta+1})$, or $c_{\Theta+1} > c_\Theta$. Hence, $k_{\Theta+2} - k_{\Theta+1} = (f(k_{\Theta+1}) - f(k_\Theta)) + (c_\Theta - c_{\Theta+1}) < 0$. Thus $k_{\Theta+2} < k_{\Theta+1}$ and hence $\beta f'(k_{\Theta+2}) > 1$. Repeating the argument, we infer that for all $t > \Theta$, $k_{t+1} < k_\Theta$, and $u'(c_t) < u'(c_\Theta)$. Therefore, starting from $t = \Theta$, $\beta^t u'(c_t)k_{t+1} < \beta^t u'(c_\Theta)k_\Theta$, which implies (3.24). □

Lemma 3.4. *Suppose that there exists Θ such that $\beta f'(k_{t+1}) \le 1$ for all $t \ge \Theta$. Then the transversality condition (3.24) holds.*

Proof. Recall that there is $\bar{k}$ such that $0 < f(\bar{k}) = \bar{k} < +\infty$, and thus k_t is bounded from above. Since $\beta f'(0) > 1$, it follows that for $t > \Theta$, $k_t \ge (f')^{-1}(1/\beta) > 0$. Hence $f(k_t) \ge f((f')^{-1}(1/\beta)) > 0$, and from (3.4) we get $c_t = e_t f(k_t) \ge e_t f\left((f')^{-1}(1/\beta)\right)$, $t > \Theta$. Therefore, due to (3.30),

$$\sum_{t=\Theta}^{\infty} c_t = +\infty. \tag{3.31}$$

It follows from (3.22) that $u'(c_{t+1}) \geq u'(c_t)$, $t \geq \Theta$. Therefore, the sequence $\{c_t\}_{t=\Theta}^{\infty}$ is monotonically non-increasing and converges. Suppose that $c_t \to 0$ as $t \to \infty$. Then $k_t \to \bar{k} = f(\bar{k})$, and since $f'(\bar{k}) < 1$, for some $0 < \xi < 1$ there exists Θ' such that for all $t > \Theta'$, $\beta f'(k_{t+1}) < 1 - \xi$. Thus $u'(c_t) = \beta f'(k_{t+1}) u'(c_{t+1}) < (1-\xi)u'(c_{t+1})$. At the same time, it follows from the regularity condition that for some $\gamma > 0$, the sequence $u'(c_t)c_t^{\gamma}$ converges, and hence there exists Θ'' such that for all $t > \Theta''$,

$$\frac{u'(c_{t+1})c_{t+1}^{\gamma}}{u'(c_t)c_t^{\gamma}} < 1 + \xi.$$

Thus for all $t > \max\{\Theta', \Theta''\}$,

$$\frac{c_{t+1}}{c_t} < \left((1+\xi)\frac{u'(c_t)}{u'(c_{t+1})}\right)^{\frac{1}{\gamma}} < ((1+\xi)(1-\xi))^{\frac{1}{\gamma}} = (1-\xi^2)^{\frac{1}{\gamma}} < 1,$$

which contradicts (3.31).

It follows that c_t converges to a positive number, and so does $u'(c_t)$. Since k_t is bounded, (3.24) clearly holds. □

Let us consider different cases that may arise. If $\beta f'(k_{t+1}) \leq 1$ for all sufficiently large t, then the transversality condition (3.24) holds by Lemma 3.4.

Suppose there exists Θ such that $\beta f'(k_{\Theta+1}) > 1$. Then there are only two possibilities. Either there exists $\Theta_1 > \Theta$ such that $\beta f'(k_{\Theta_1+1}) \leq 1$ or $\beta f'(k_{t+1}) > 1$ for all $t > \Theta$. In the former case, either $\beta f'(k_{t+1}) \leq 1$ for all $t > \Theta_1$ so that Lemma 3.4 holds or there exists $\Theta_2 > \Theta_1$ such that $\beta f'(k_{\Theta_2+1}) > 1$ in which case we are in the conditions of Lemma 3.3. In the latter case, for all $t > \Theta$, $u'(c_t) < u'(c_\Theta)$ and $k_{t+1} \leq (f')^{-1}(1/\beta)$. Therefore, $\beta^t u'(c_t)k_{t+1} < \beta^t u'(c_\Theta)(f')^{-1}(1/\beta)$, which implies (3.24).

Now consider Case 2 (linear production function). The first-order conditions in this case state that $A^t \beta^t c_t^{-\rho_i} = c_0^{-\rho_i}$, $t = 1, 2, \ldots$.

Assume that the transversality condition (3.24) fails. Then there exist Θ and $N > 0$ such that $\beta^t c_t^{-\rho_i} k_{t+1} \geq N$, for all $t > \Theta$, and hence $k_{t+1}/A^t \geq N/c_0^{-\rho_i}$. It follows that for all $t > \Theta$, $e_{t+1} = c_{t+1}/Ak_{t+1} \leq \left(c_0^{-\rho_i}/N\right)(c_{t+1}/A^{t+1})$. Therefore, by (3.30),

$$\sum_{t=\Theta}^{\infty} \frac{c_{t+1}}{A^{t+1}} = +\infty.$$

However, iterating the equation $c_t + k_{t+1} = Ak_t$, we easily get $c_0 + c_1/A + c_2/A^2 + \ldots \leq Ak_0$, a contradiction.

3.9.3. Proof of Propositions 3.2 and 3.3

Consider agent i with the discount factor β_i and the felicity function $u_i(c)$. Note that for all τ (except $\tau = T$ in the finite horizon case) and any non-degenerate expectations $E_{\tau+1,T}$, her preferred time τ consumption rate is a unique solution to the following equation:

$$\frac{\partial V_\tau^i(k_\tau, e_\tau, E_{\tau+1,T})}{\partial e_\tau} = 0, \quad 0 \le \tau < T.$$

Indeed, the above equation can be rewritten as

$$\Phi_\tau^i(k_\tau, e_\tau) = \Psi_\tau^i(k_\tau, e_\tau, E_{\tau+1,T}). \tag{3.32}$$

where $\Phi_\tau^i(k_\tau, e_\tau)$ is defined by (3.20), and $\Psi_\tau^i(k_\tau, e_\tau, E_{\tau+1,T})$ is defined by (3.21). It follows from Remark 3.1 that there exists a unique solution e_τ^* to equation (3.32), and $0 < e_\tau^* < 1$. Thus there is a unique time τ preferred consumption rate for agent i.

Using (3.19) and Remark 3.1, we infer that $\partial V_\tau^i(k_\tau, e_\tau, E_{\tau+1,T})/\partial e_\tau$ is strictly decreasing in e_τ, and hence the preferences of agent i in voting over the time τ consumption rate are strictly concave. Thus the preferences of each agent in voting over any consumption rate are single-peaked, which proves Proposition 3.2.

Now suppose that all agents have the same felicity function $u(c)$ and the same non-degenerate expectations $E_{\tau+1,T}$. Then for all τ (except $\tau = T$ in the finite horizon case) higher values of the discount factor β_i correspond to lower values of the preferred time τ consumption rate e_τ^{i*}. Indeed, $\Phi_\tau^i(k_\tau, e_\tau)$ is independent of β_i and strictly decreasing in e_τ, while $\Psi_\tau^i(k_\tau, e_\tau, E_{\tau+1,T})$ is strictly increasing both in β_i and e_τ. Therefore, the solution to equation (3.32), e_τ^*, is strictly decreasing in β_i.

It follows from the median voter theorem that the time τ voting equilibrium is the preferred consumption rate for the agent with the median discount factor, which proves Proposition 3.3.

3.9.4. Proof of Theorem 3.2

Consider the constant sequence of consumption rates $E^* = \{e^*, e^*, \ldots\}$, where e^* is given by (3.11). Consider a fictitious agent with the discount factor β_{med} and felicity function $u_i(c)$, i.e., the agent with the median discount factor and the felicity function of agent i. It is clear that e^* is the preferred time τ consumption rate for this agent, and E^* is her optimum in terms of consumption rates.

Let τ be an arbitrary point in time. Suppose that expectations of agents are constant and equal to $E_{\tau+1,\infty} = \{e^*, e^*, \ldots\}$. It follows from Proposition 3.2 that the preferences of each agent in voting over the time τ consumption rate are strictly concave. Consider agent i with the discount factor β_i and the felicity function $u_i(c)$. She has the unique preferred time τ consumption rate e_τ^{i*}. It follows from the results of Section 3.9.3 that if $\beta_i > \beta_{med}$ ($\beta_i < \beta_{med}$) then $e_\tau^{i*} < e^*$ ($e_\tau^{i*} > e^*$).

Hence the winner in voting over the time τ consumption rate under constant expectations $E_{\tau+1,\infty} = \{e^*, e^*, \ldots\}$ is precisely e^* given by (3.11). Since the point in time τ is chosen arbitrarily, e^* is the winner in voting over each consumption rate under the expectations $\{e^*, e^*, \ldots\}$. Thus the sequence $\{e^*, e^*, \ldots\}$ is an intertemporal voting equilibrium, and hence e^* is a stationary voting equilibrium.

Let us now prove that e^* is the unique stationary voting equilibrium. Suppose that there is another stationary voting equilibrium $\tilde{e}$. Consider an arbitrary point in time τ. Suppose that the expectations of agents are constant and equal to $E_{\tau+1,\infty} = \tilde{E} = \{\tilde{e}, \tilde{e}, \ldots\}$. It follows from Proposition 3.2 that the preferences of each agent in voting over the time τ consumption rate are strictly concave. By the median voter theorem, $\tilde{e}$ is the most preferred consumption rate for some "median" voter. Clearly, $\tilde{e}$ is the preferred consumption rate for this same agent for all $\tau = 0, 1, \ldots$. Therefore, the sequence $\{\tilde{e}, \tilde{e}, \ldots\}$ is a step-by-step intertemporal optimum for this agent.

Denote the discount factor of this agent by $\tilde{\beta}$. Consider the corresponding $\tilde{k}$, which is the unique positive solution to the equation $k = (1 - \tilde{e})f(k)$. Clearly, $\tilde{k}$ is determined by the "modified golden rule" for this agent: $\tilde{\beta} f'(\tilde{k}) = 1$. It follows that $\tilde{e}$ depends only on $\tilde{\beta}$, and is independent of the felicity function of this agent. In other words, $\tilde{e}$ is the preferred time τ consumption rate for the fictitious agent with the discount factor $\tilde{\beta}$ and any felicity function, in voting over e_τ given $k_\tau = \tilde{k}$ and the expectations $\tilde{E}$.

Now suppose that $\tilde{e} > e^*$ ($\tilde{e} < e^*$), and thus $\tilde{\beta} < \beta_{med}$ ($\tilde{\beta} > \beta_{med}$). Consider agent i with the discount factor $\beta_i \geq \beta_{med}$ ($\beta_i \leq \beta_{med}$) and the felicity function $u_i(c)$. Since $\beta_i > \tilde{\beta}$ ($\beta_i < \tilde{\beta}$), it follows from the results of Section 3.9.3 that $e_\tau^{i*} < \tilde{e}$ ($e_\tau^{i*} > \tilde{e}$). Hence for at least $\frac{N+1}{2}$ agents their preferred time τ consumption rates are are lower (resp. greater) than $\tilde{e}$. It follows that $\tilde{e}$ is not a Condorcet winner in voting over the time τ consumption rate under expectations $\tilde{E}$, and cannot be a stationary voting equilibrium. Thus e^* is the unique stationary voting equilibrium.

3.9.5. Proof of Theorem 3.3

Let τ be an arbitrary point in time. Consider how agents vote over the time τ consumption rate under constant non-degenerate expectations $E_{\tau+1,\infty} = \{e, e, \ldots\}$.

The objective function of agent i in the time τ voting problem under these expectations is given by:

$$V_\tau^i(k_\tau, e_\tau, E_{\tau+1,\infty}) = u_i\left(e_\tau A k_\tau\right) + \beta_i u_i\left(e(1-e_\tau)A^2 k_\tau\right) + \beta_i^2 u_i\left(e(1-e)(1-e_\tau)A^3 k_\tau\right) + \ldots.$$

It follows from Proposition 3.2 that the preferences of agent i are concave, and her preferred time τ consumption rate, e_τ^{i*}, is the unique solution to the following equation:

$$\frac{\partial V_\tau^i(k_\tau, e_\tau, E_{\tau+1,\infty})}{\partial e_\tau} = 0,$$

which can be rewritten as

$$Ak_\tau \left(e_\tau A k_\tau\right)^{-\rho_i} = \beta_i e A^2 k_\tau \left(e(1-e_\tau)A^2 k_\tau\right)^{-\rho_i} + \beta_i^2 e(1-e)A^3 k_\tau \left(e(1-e)(1-e_\tau)A^3 k_\tau\right)^{-\rho_i} + \ldots .$$

Dividing both parts of the above equation by $(Ak_\tau)^{1-\rho_i}$, we get

$$(e_\tau)^{-\rho_i} = (1-e_\tau)^{-\rho_i}\left(A\beta_i e(Ae)^{-\rho_i} + A^2\beta_i^2 e(1-e)(A^2 e(1-e))^{-\rho_i} + \ldots\right),$$

and hence, taking into account that expectations $\{e, e, \ldots\}$ are non-degenerate, we obtain

$$\left(\frac{1-e_\tau}{e_\tau}\right)^{\rho_i} = A\beta_i\left(e\,(Ae)^{-\rho_i} + A\beta_i e(1-e)\left(A^2 e(1-e)\right)^{-\rho_i} + \ldots\right) = \frac{\beta_i (Ae)^{1-\rho_i}}{1-\beta_i\left(A(1-e)\right)^{1-\rho_i}}.$$

Using (3.12) and the above equation, we conclude that the preferred time τ consumption rate for agent i is the solution to the following equation in e_τ:

$$\left(\frac{1-e_\tau}{e_\tau}\right)^{\rho_i} = \frac{\left(\frac{1+\gamma_i}{A}\right)^{\rho_i} e^{1-\rho_i}}{1-\left(\frac{1+\gamma_i}{A}\right)^{\rho_i}(1-e)^{1-\rho_i}}. \tag{3.33}$$

Note that the preferred time τ consumption rate for each agent is independent of the current capital stock k_τ, and depends only on constant expectations e.

Denote the solution to equation (3.33) depending on e by $e_\tau^{i*}(e)$.

Lemma 3.5. *If* $1+\gamma_i \gtreqless A(1-e)$, *then* $e_\tau^{i*}(e) \lesseqgtr e$.

Proof. Suppose that $1+\gamma_i \gtreqless A(1-e)$. Since $e_\tau^{i*}(e)$ is the solution to equation (3.33), we have

$$\left(\frac{1-e_\tau^{i*}(e)}{e_\tau^{i*}(e)}\right)^{\rho_i} \gtreqless \frac{(1-e)^{\rho_i}e^{1-\rho_i}}{1-(1-e)^{\rho_i}(1-e)^{1-\rho_i}} = \frac{(1-e)^{\rho_i}e^{1-\rho_i}}{e} = \left(\frac{1-e}{e}\right)^{\rho_i}.$$

The expression $\frac{1-e}{e}$ is decreasing in e for $0<e<1$. Therefore, $e_\tau^{i*} \lesseqgtr e$. □

Let e^* be given by (3.13). Then $1 + \gamma_{med} = A(1 - e^*)$. If the expectations are given by $\{e^*, e^*, \ldots\}$, then, by Lemma 3.5,

$$\gamma_i \gtreqless \gamma_{med} \Rightarrow e_\tau^{i*} \lesseqgtr e^*.$$

Therefore, for any time τ, e^* is the Condorcet winner in voting over the time τ consumption rate under the expectations $\{e^*, e^*, \ldots\}$. Hence e^* is a balanced-growth voting equilibrium.

Moreover, any other consumption rate $\tilde{e}$ cannot be a balanced-growth voting equilibrium. It follows from Lemma 3.5 that if $\tilde{e} > e^*$ ($\tilde{e} < e^*$), then $A(1 - \tilde{e}) < 1 + \gamma_{med}$ ($A(1 - \tilde{e}) > 1 + \gamma_{med}$), and for at least $\frac{N+1}{2}$ agents their preferred time τ consumption rates in voting over the time τ consumption rate under expectations $\{\tilde{e}, \tilde{e}, \ldots\}$ are lower (resp. greater) than $\tilde{e}$. Hence $\tilde{e}$ is not a Condorcet winner, and is not a balanced-growth voting equilibrium.

4. Economic Growth and Property Rights on Natural Resources

In Chapter 3 we have proposed a simple and natural voting procedure (intertemporal majority voting) in which agents vote step by step not over the absolute values, but over the relative values. Change of variable over which agents vote and make expectations, from level to rate, allowed us to obtain a stable outcome of voting.

We have also noted that in the evident interpretation of the model from Chapter 3 as the common property resource problem, consumption rate becomes harvest or extraction rate. In the field of resource economics, rates are the most natural variables, and this observation suggests that our procedure can be successfully applied to the choice of extraction rate in models where heterogeneous in their time preferences agents collectively manage a stock of exhaustible natural resource. Indeed, in the present chapter we show how this idea may be implemented.

We consider two general equilibrium Ramsey-type models with exhaustible natural resources and agents who are heterogeneous in their time preferences. In the first model, we assume private ownership of natural resources. The privately owned resource stock is an asset to its owner; agents can invest in natural resources as well as in physical capital. Extraction rates are determined in an equilibrium by the market forces of supply and demand. In the private property regime we define a competitive equilibrium and show that it converges to a balanced-growth equilibrium with the long-run growth rate being determined by the discount factor of the most patient agents.

In the second model, natural resources are public property. The resource stock is controlled by a benevolent government that acts in the interest of the agents. Resource income is equally distributed among agents, and extraction rates are determined by majority voting. In the public property regime, our voting procedure is naturally applied, because agents vote precisely over the rates. We show that the sequence of winners in one-dimensional votes over current extraction rate under perfect foresight is determined by the agent with the median discount factor. For this model we define an intertemporal voting equilibrium (which consists of the voting equilibrium sequence of extraction rates along with the corresponding competitive equilibrium) and prove that it also converges to a balanced-growth equilibrium. In the public property regime the long-run rate of growth is determined by the median discount factor.

Our results suggest that if the most patient agents do not constitute a majority of the population, private ownership of natural resources results in a higher rate of growth than public ownership. At the same time, private ownership leads to higher inequality than public ownership, and if inequality impedes growth, then the public property regime is likely to result in a higher long-run rate of growth. However, an appropriate redistributive policy can eliminate the negative impact of inequality on growth.

This chapter is based on the published article "Economic Growth and Property Rights on Natural Resources" (Borissov and Pakhnin, 2018) and is organized as follows. The main body of the chapter focuses on the description of the models and on the general statement of the results. Section 4.1 provides a preliminary discussion of the topic. Section 4.2 presents the basic building blocks of the model and the descriptions of property regimes. In Section 4.3 we study the model with private ownership of natural resources. We define competitive and balanced-growth equilibria, and present the explicit expression for the equilibrium rate of growth. In Section 4.4 the model with public ownership of natural resources is considered. We define a competitive equilibrium under given extraction rates, characterize a temporary voting equilibrium, and study an intertemporal voting equilibrium, deriving the expression for the long-run rate of growth. Section 4.5 compares the long-run consequences of the two different property regimes. In Section 4.6 we modify the two models by taking into account the impact of sociopolitical instability caused by inequality on the growth rates. In Section 4.7 a model with private ownership is modified to include capital taxation. Section 4.8 concludes.

All technical details and proofs are relegated to the additional sections. Section 4.9 contains mathematical details and proofs of the statements related to the private property regime. Section 4.10 provides a thorough formulation of the public property regime. Section 4.11 is devoted to the private property regime with capital taxation.

4.1. Introduction

The question of property rights[1] is one of the most controversial and complicated issues concerning the regulation of natural resources. Who should own natural resources and in what form? Which individual, group or institution will best manage the resource stock in the short and long run? How do different forms and extents of property rights on natural resources affect both present and future generations? These are important and inherently complex problems.

[1] As many other scholars, we do not make any difference between property rights and ownership throughout this chapter. On our level of abstraction these two constructs are essentially the same. Therefore we use the terms "property regime" and "ownership" interchangeably.

The vast amount of literature on property regimes over natural resources (see, e.g., Ostrom, 1990; Barnes, 2009; Cole and Ostrom, 2011) usually places great emphasis on the market failure that occurs when property rights are not properly specified. The relative advantages of private and common property rights in terms of efficiency, equity, and sustainability of natural resource use patterns have been widely discussed and studied.

However, even if property rights are clearly defined and assigned, the optimal choice from a wide array of diverse property regimes is not so obvious. Especially significant in this connection is the choice between private and public property. There has been much debate on the economic and political merits of private versus public ownership in general (see, e.g., a survey by Shleifer, 1998). It is believed that private firms are more efficient, mainly because of strong incentives to invest in improving the ways of using the assets. At the same time, state firms are usually considered inefficient for a number of reasons, e.g., weak incentives to reduce costs (agency problem), and government subsidies to the state-owned sector (soft budget constraints).[2] Moreover, it is widely recognized that public enterprises pursue political rather than economic goals (see Shleifer and Vishny, 1994). However, in this chapter we do not take into account any political considerations and other possible sources of efficiency losses. We assume a government that acts in the interest of the people. Since we assume that extraction costs are zero, it is irrelevant for us, which kinds of enterprises (private or state-owned) extract and sell resources held in public ownership. There is no such an enterprise in our model. Our goal is to compare private and public property regimes over natural resources in terms of economic growth, and not in terms of profitability or efficiency.

Interestingly enough, economists have only recently begun to pay attention to the comparison of private and public ownership in the particular case of natural resources like crude oil or gold. This is even more surprising considering the ambiguity of this issue and its consequences for societies in resource-rich countries. There are many countries in the world that maintain full state ownership of their natural resources. In such countries private firms, especially foreign firms, have little or no operational and managerial control. Examples include Uzbekistan, Turkmenistan, Nigeria, and modern Venezuela. At the same time, there are countries like Kazakhstan or the Russian Federation, where leaders chose to privatize their energy sector (see Jones Luong and Weinthal, 2001). One should also mention the USA and Japan, where private firms own and control much of the countries' subsurface minerals.

There are certain rationales for such cleavage. On the one hand, exhaustible resource stocks (oil and gas fields, coal and ore mines) are universally regarded by the public at

[2] Bajona and Chu (2010) show that reduction in government subsidies leads to an increase in economic efficiency, and Gupta (2005) reports that even partial privatization increase productivity and profitability of state-owned enterprises.

large as public property. It is felt that natural resources should belong to local peoples, who claim sovereign rights on their territorial habitat. It is often argued that "resources found in the territory of a state belong to the population of that state (...) The right to natural resources is a right of peoples or communities to determine how their natural resources should be protected, managed and explored" (Blanco and Razzaque, 2011, p. 76). As Joseph Stiglitz put it, "a country's natural resources should belong to all of its people" (Stiglitz, 2016, p. 354). Moreover, for many countries around the world, especially developing countries, natural resources represent a significant share of income and are too important to be left to the market. It is believed that direct state control over resources is an indispensable feature of national sovereignty and political decision-making (see Mommer, 2002).

On the other hand, most economists are convinced that private ownership of natural resources leads to higher efficiency than public ownership. Empirical evidence shows that private natural resource companies are more efficient and profitable than nationalized firms, though the effects of privatization on employment and income distribution are not as desirable (see, e.g., Chong and de Silanes, 2005; Schmitz and Teixeira, 2008).

One may conjecture that this divergence between the positions of the public at large and economists partly explains the fact that privatization–nationalization cycles tend to occur more often in the natural resource sector (see, e.g., Kobrin, 1984; Chua, 1995; Hogan et al., 2010). This tendency provides an additional incentive to study the impact of property regime over natural resources on macroeconomic performance.

The existing empirical literature on ownership of the primary sectors (e.g., Megginson, 2005; Wolf, 2009) concentrates mostly on the productive efficiency and profitability of firms. The effects of different property regimes on aggregate income are studied by Brunnschweiler and Valente (2013), though they use a slightly different classification of ownership instead of the usual dichotomy between the categories of "private" and "public".

In this chapter, we study private and public property regimes over exhaustible natural resources from the standpoint of economic growth theory. We do not compare private and public ownership in terms of efficiency or optimality. Thus we can abstract from any political considerations and focus only on the following question: which of the two property regimes does lead to a higher long-run rate of economics growth?

Developing the ideas of Borissov and Surkov (2010), we consider two models of economic growth with heterogeneous agents and exhaustible resources. These models are modifications to a well-known Ramsey-type model of economic growth with exhaustible resources (see, e.g., Dasgupta and Heal, 1979). Technical progress is exogenous. Under this assumption the long-run rate of growth is fully determined by the extraction rate.

The two models differ in the property regimes over natural resources. The first model assumes private ownership of natural resources. The resource stock is an asset. Agents can invest their savings in natural resources as well as in physical capital. This implies that resource income belongs to the owners of natural resources. The extraction rate is determined by market forces. In the second model we assume that the resource stock is held in trust by the government for the common benefit. Resource income is equally distributed among all agents, and it is up to the agents to determine the extraction rate.

Following Becker (1980, 2006), we assume that agents are heterogeneous in their time preferences. The rates of time preference, or the degrees of impatience, are represented by agents' discount factors. The discount factors are higher for more patient agents and lower for less patient ones.

In the private property regime, only the most patient agents obtain income from the capital and resource stocks in the long run. We show that the discount factor of the most patient agents determines both the long-run extraction rate and the rate of growth. The extraction rate is decreasing and the growth rate is increasing in the discount factor of the most patient agents.

In the public property regime, the heterogeneity of agents results in different preferences over the resource extraction rate. Relatively impatient agents care less about the future and prefer to extract resources faster than relatively patient agents. Thus there naturally arises a problem of aggregating heterogeneous preferences. We use a conventional collective choice mechanism and suppose that the resource extraction rate is chosen by majority voting.

The problem of social choice in dynamic settings has attracted growing interest and attention in recent years (see, e.g., Rangel, 2003; Zuber, 2011; Le Kama et al., 2014; Asheim and Ekeland, 2016). A number of papers (see, e.g., Krusell et al., 1997; Bernheim and Slavov, 2009) study appropriate dynamic generalizations of standard solution concepts. One of these generalizations is presented by Borissov and Surkov (2010), who consider voting on extraction rates. The same voting mechanism is considered also in Borissov et al. (2014a), where heterogeneous agents vote for a tax aimed at environmental maintenance. In both cases the outcome of voting is the optimal policy for the agent with the median discount factor. However, this voting mechanism is oversimplified; it does not imply perfect rationality of agents, and allows one to analyze voting outcomes only in a balanced-growth equilibrium.

In this chapter, we apply the approach to voting in a dynamic general equilibrium framework described in Chapter 3. We use the intertemporal setting of the model, and ask agents to vote over the extraction rate at each point in time under given expectations about future extraction rates. The sequence of winners in these one-dimensional votes under perfect foresight determines an intertemporal voting equilibrium. We show that in the long run an intertemporal voting equilibrium converges to a balanced-growth voting equilibrium. The long-run extraction rate and the rate of growth are determined by the median discount factor. The extraction rate is decreasing and the growth rate is increasing in the median discount factor.

Our results suggest that the long-run growth rate in the case of private ownership is equal to that of public ownership if the most patient agents constitute the majority of the population, and is higher otherwise. It seems reasonable to conclude that the private property regime is more favorable for promoting long-run economic growth than the public property regime, but this conclusion is somewhat hasty. The private property regime over natural resources, all other things being equal, results in higher income inequality. This can have detrimental effects on economic growth.

High inequality increases sociopolitical instability, the probability of revolutions and mass violence, and the risk of expropriation, thus creating uncertainty in the politico-economic environment. These factors reduce investment incentives and affect the security of property rights. Increased social tension can lead either to a higher extraction rate,[3] or to unproductive costs and losses in output. In both cases the growth rate of the economy is adversely affected.

We consider two different channels through which sociopolitical instability caused by inequality affects economic growth. The first channel assumes that the discount factors of agents are formed endogenously, and the rise in income inequality increases the impatience of agents (see Borissov and Lambrecht, 2009).[4] The second channel assumes that a certain share of output, depending on inequality, is unproductively thrown away. This share might be used to increase military expenditures, to support and expand various social programs to pacify the population, etc. Under both assumptions, if inequality in the society is sufficiently high, then the public property regime over natural resources is likely to result in a higher long-run rate of growth compared with the private property regime.

In the most of the chapter we compare two different institutional frameworks, private and public ownership of natural resources, in a purely positive manner. However, it may be interesting to ask whether differences in growth rate and inequality between private and public ownership could be undone by a social planner implementing certain economic policies. We show that the private property regime with an appropriately chosen capital

[3] The common wisdom (see, e.g., Long, 1975; Long and Sorger, 2006) has been that ownership risk induces a firm to overuse the stock of a resource, though the empirical evidence is ambiguous. For instance, Jacoby et al. (2002) support this point of view by reporting that a higher risk of expropriation reduces private investments and raises the current extraction. At the same time, Bohn and Deacon (2000) show that insecure ownership reduces present extraction for resources with capital-intensive extraction technology.

[4] The reasoning behind this assumption is as follows. All risks emerging from high inequality can be reduced to the threat of total political and economic breakdown. When making their decisions, agents do not take into account a new economic order which will be established after breakdown of the current economic order. This new order will be better for some agents, and worse for others, but agents can behave rationally only within the current economic order, and cannot extend their rationality beyond its end. Therefore, an increase in the probability of breakdown increases the impatience of all agents.

income tax can have less inequality and a higher long-run growth rate than the public property regime.

4.2. The model

We consider a discrete time dynamic general equilibrium model of an economy endowed with exhaustible resources. The economy is populated with L agents who are heterogeneous in their time preferences.

4.2.1. Production and resource extraction

Firms use physical capital, labor and natural resources to produce a homogeneous good, which is a numeraire in the model. Extraction is costless, and all markets are competitive.

Output is given by the Cobb–Douglas production function:

$$Y_t = A_t K_t^{\alpha_1} L^{\alpha_2} E_t^{\alpha_3}, \quad \alpha_i > 0 \ (i = 1, 2, 3), \quad \sum_{i=1}^{3} \alpha_i = 1,$$

where A_t is total factor productivity, K_t is the physical capital stock, E_t is the amount of resources extracted in period t (which is identified with the amount of resources utilized in production), and L is the constant over time labor supply. Capital fully depreciates during one time period. We assume that total factor productivity grows at an exogenously given constant rate $\lambda > 0$: $A_t = (1 + \lambda)\, A_{t-1}$.

The production function in intensive form is given by

$$y_t = \frac{Y_t}{L} = A_t k_t^{\alpha_1} e_t^{\alpha_3},$$

where $k_t = K_t/L$ and $e_t = E_t/L$.

The amount of resources extracted for production decreases the available stock: $R_t = R_{t-1} - E_t$. We denote the resource extraction rate by

$$\varepsilon_t = \frac{E_t}{R_{t-1}},$$

so that the per capita volume of extraction e_t and the dynamics of the resource stock R_t are given by

$$e_t = \frac{\varepsilon_t R_{t-1}}{L}, \qquad R_t = (1 - \varepsilon_t) R_{t-1}.$$

Since all markets are competitive, the interest rate r_t, the wage rate w_t, and the price of natural resources q_t coincide with the respective marginal products:

$$1 + r_t = \alpha_1 A_t (k_t)^{\alpha_1 - 1} (e_t)^{\alpha_3}, \quad w_t = \alpha_2 A_t (k_t)^{\alpha_1} (e_t)^{\alpha_3}, \quad q_t = \alpha_3 A_t (k_t)^{\alpha_1} (e_t)^{\alpha_3 - 1}.$$

4.2.2. Households

There is an odd number L of agents, indexed by $j = 1, \ldots, L$. Each agent is endowed with one unit of labor. Agent j discounts future utilities by the factor β_j. We assume that

$$1 > \beta_1 \geq \beta_2 \geq \ldots \geq \beta_L > 0,$$

i.e., agents are ordered by their patience, from more to less patient. We denote by $J = \{j \mid \beta_j = \beta_1\}$ the set of agents with the highest discount factor. These agents appreciate the future higher than the others, and we refer to them as the most patient agents.

Agents obtain utility from their consumption over an infinite time horizon. Preferences of agent j over consumption stream $\{c_t^j\}_{t=0}^{\infty}$ are given by the log-linear utility function:

$$U^j = \sum_{t=0}^{\infty} \beta_j^t \ln c_t^j.$$

4.2.3. Property regimes

In almost every country of the world the state is the *de jure* owner of domestic natural resources. Thus the primary questions are: who has control over the rights to exploit the resource stock, and who has the right to obtain resource income?

Following Borissov and Surkov (2010), we consider two different property rights regimes over exhaustible resources: private and public. We suppose that the stock of natural resources (e.g., oil or gas fields, coal mines, diamond mine with kimberlite pipes) is divisible, and do not consider common-pool resources (e.g., oil in a common underground reservoir).[5] It is possible to divide the stock into individual parcels and to assign property rights over each parcel.

If proprietary rights over these parcels are established, we refer to this situation as the private property regime. The privately owned resource stock *in situ* is an asset to its owner. Agents can invest in natural resources as well as in physical capital. By the Hotelling (1931) rule, the equilibrium rate of return on the resource stock as an asset is equal to the return on capital. Resource income goes to the resource owner.

[5] For a dynamic model of the common property resource exploitation, see, e.g., Mitra and Sorger (2014).

There can be other property rights regimes over natural resources. It is reasonable to consider the situation in which the exhaustible resource stock is controlled by a government that acts in the interest of the agents. We refer to this situation as the public property regime. In this case resource income is equally distributed among agents, who choose the resource extraction rate by majority voting.

Note that in the public property regime there is no reason to expect that the Hotelling rule holds. There are incentives for arbitrage operations, as the rate of change of the natural resource price can differ from the interest rate. However, we assume that private storages are forbidden. There is no possibility to store resources; they are utilized immediately after extraction.[6] Thus no arbitrage opportunities can be exploited.

4.3. Private property regime

Consider first the case in which the exhaustible resource stock is privately owned. In this section our exposition follows Borissov and Surkov (2010). We introduce the model and specify its main properties. Formal definitions and proofs can be found in Section 4.9.

4.3.1. Competitive equilibrium

Suppose we start at time 0. Agent j is endowed with some amount of physical capital $\hat{k}^j \geq 0$ and natural resources $\hat{R}^j \geq 0$. Given the equilibrium price of natural resources at time $t = -1$, q_{-1}, the initial savings of agent j are determined by $s^j_{-1} = q_{-1}\hat{R}^j + \hat{k}^j \geq 0$.[7]

Agent j chooses her consumption plan by solving the problem of maximizing lifetime utility:

$$\begin{aligned} \max \ & \sum_{t=0}^{\infty} \beta_j^t \ln c_t^j, \\ \text{s. t.} \ & c_t^j + s_t^j \leq (1 + r_t)\, s_{t-1}^j + w_t, \\ & s_t^j \geq 0, \quad t = 0, 1, \ldots, \\ & s_{-1}^j = q_{-1}\hat{R}^j + \hat{k}^j. \end{aligned}$$

Here c_t^j and s_t^j are consumption and savings of agent j, r_t is the interest rate, and w_t is the wage rate at time t.

Agents are prohibited from borrowing against their future earnings. Thus their savings must be non-negative. They can be invested in both physical capital and natural

[6] See, e.g., Bommier et al. (2017) for the discussion of exhaustible resource markets when resource storage is possible.

[7] Note that the price of natural resources at time $t = -1$ is determined endogenously. Therefore, the initial savings of agent j are not given exogenously. To ensure that the initial savings are non-negative, we impose the non-negativity constraints on initial holdings of physical capital and resources.

resources.[8] From the agents' point of view, the two assets are perfect substitutes, so we do not distinguish between physical capital and natural resource in agents' portfolio. The return to investment into physical capital and natural resources must be equal (the Hotelling rule), hence the price of natural resources q_t grows at the rate r_t:

$$q_t = (1 + r_t)\, q_{t-1}.$$

We define a *competitive equilibrium* in the private property regime,

$$\mathcal{E}^* = \left\{(c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^*\right\}_{t=0,1,\dots},$$

in a standard way by the following conditions:

- agents maximize their utilities subject to budget constraints;
- capital, labor and natural resources are paid their marginal products;
- the Hotelling rule holds;
- aggregate savings are equal to investment into physical capital and natural resources.

A competitive equilibrium exists (see Appendix A), and *if initially the stocks of physical capital and natural resources belong to the most patient agents, then the competitive equilibrium starting from this state is unique* (see Proposition 4.1 in Section 4.9). Also, *in each competitive equilibrium from some time onward only the most patient agents can make positive savings, and from this time resources are extracted at a constant rate* $\varepsilon^* = 1 - \beta_1$ (Proposition 4.2).

4.3.2. Balanced-growth equilibrium

A *balanced-growth equilibrium* is a competitive equilibrium in which output, consumption, savings, the capital stock and the wage rate grow at a constant rate γ^*, while the interest rate r^* is constant over time. The price of natural resources grows at a constant rate equal to r^* (by the Hotelling rule), and resources are depleted at a constant extraction rate ε^*.

In our model, *there exists a balanced-growth equilibrium* (see Proposition 4.3). In such an equilibrium only the most patient agents make positive savings, while relatively impatient

[8] Formally speaking, the non-negativity constraint on savings does not rule out the possibility that some agents have positive holdings of physical capital and negative holdings of resources, or vice versa. However, in an equilibrium only agents' savings are of interest for us, and it is irrelevant in which form they are held.

agents make no savings and consume their wages. This important property is known in the literature as the Ramsey (1928) conjecture.[9]

It can be checked that *the interest rate* r^**, the extraction rate* ε^**, and the growth rate* γ^* *are the same for every balanced-growth equilibrium* (Proposition 4.4). Moreover, *every competitive equilibrium converges to a balanced-growth equilibrium* (for the precise meaning of this statement see Proposition 4.5). Thus we can concentrate on the characterization of balanced-growth equilibria when exploring the long-run perspective.

It is well known that unlike Ramsey-type models without natural resources, in which the long-run growth rate is determined by the exogenously given rate of technical progress, in economies with natural resources the growth rate typically depends on the extraction rate, and hence is affected by the time preferences of the agents. For instance, in models with a representative agent, the optimal extraction rate is $\varepsilon = 1 - \beta$, where β is the discount factor of the representative agent (see, e.g., Stiglitz, 1974; Dasgupta and Heal, 1979).

In our model, the equilibrium extraction rate is determined by the discount factor of the most patient agents,

$$\varepsilon^* = 1 - \beta_1,$$

and the equilibrium rate of balanced growth depends on the extraction rate and is given by

$$1 + \gamma^* = (1 + \lambda)^{\frac{1}{1-\alpha_1}} (1 - \varepsilon^*)^{\frac{\alpha_3}{1-\alpha_1}} .$$

Thus the rate of balanced growth is determined by the discount factor of the most patient agents:

$$1 + \gamma^* = (1 + \lambda)^{\frac{1}{1-\alpha_1}} \beta_1^{\frac{\alpha_3}{1-\alpha_1}} .$$

The equilibrium extraction rate is decreasing and the rate of balanced growth is increasing in the discount factor of the most patient agents. An increase in patience of the resource stock owners means that they put more weight on additional future consumption compared to additional present consumption. Thus they prefer to extract less amount of resources today, and the rate of natural resource utilization becomes lower. In turn, a lower extraction rate leads to a higher growth rate in the future.

4.4. Public property regime

Consider now the case in which the stock of exhaustible resources is held in trust by the government. Resources are extracted and sold to the private production sector. Income from the sale of natural resources is equally distributed among agents who choose the

[9] For the history and discussion of this conjecture, see Becker (2006).

resource extraction rate by majority voting.[10] In this section we focus on the description of the model and on the general statement of the results. See Section 4.10 for formal definitions and proofs.

4.4.1. Competitive equilibrium under given extraction rates

Let us first define a competitive equilibrium under given extraction rates starting from an arbitrary point in time. Suppose that instead of time 0, we start at time τ. Each agent j has savings $\hat{s}^j_{\tau-1} \geq 0$ such that the corresponding stock of physical capital is positive, $k_\tau = \frac{1}{L}\sum_{j=1}^{L} \hat{s}^j_{\tau-1} > 0$. The stock of natural resources is also positive, $\hat{R}_{\tau-1} > 0$.

Suppose that at time τ agents have some non-degenerate expectations about future extraction rates, $\{\varepsilon^e_t\}_{t=\tau+1}^{\infty}$.[11] For any $\varepsilon_\tau \in (0,1)$, denote

$$\mathbb{E}_\tau(\varepsilon_\tau) = \{\varepsilon_\tau, \varepsilon^e_{\tau+1}, \varepsilon^e_{\tau+2}, \ldots\},$$

and notice that the sequence of extraction rates $\mathbb{E}_\tau(\varepsilon_\tau)$ is in fact arbitrary.

Clearly, given the sequence of extraction rates, the per capita volumes of extraction e_t and the dynamics of the exhaustible resource stock R_t are predetermined as follows:

$$R_\tau(\varepsilon_\tau) = (1-\varepsilon_\tau)\hat{R}_{\tau-1}, \quad R_t(\varepsilon_\tau) = (1-\varepsilon^e_t)R_{t-1}(\varepsilon_\tau), \; t = \tau+1, \tau+2, \ldots;$$
$$e_\tau(\varepsilon_\tau) = \frac{\varepsilon_\tau \hat{R}_{\tau-1}}{L}, \quad e_t(\varepsilon_\tau) = \frac{\varepsilon^e_t R_{t-1}(\varepsilon_\tau)}{L}, \; t = \tau+1, \tau+2, \ldots.$$

Our notation emphasizes the fact that the future volumes of extraction and the dynamics of the resource stock depend on the time τ extraction rate ε_τ.

Given the future volumes of extraction, agent j solves the following maximization problem:

$$\begin{aligned} &\max \sum_{t=\tau}^{\infty} \beta_j^t \ln c_t^j, \\ &\text{s. t.}\; c_t^j + s_t^j \leq (1+r_t)\, s_{t-1}^j + w_t + v_t, \\ &\qquad s_t^j \geq 0, \quad t = \tau, \tau+1, \ldots, \\ &\qquad s_{\tau-1}^j = \hat{s}_{\tau-1}^j. \end{aligned}$$

Here c_t^j and s_t^j are consumption and savings of agent j, r_t is the interest rate, w_t is the wage rate, and v_t is per capita resource income at time t. The latter is the income from

[10] This regime is also discussed by Borissov and Surkov (2010). However, their voting approach has certain major drawbacks. The voting mechanism in their model allows one to define voting only in a balanced-growth equilibrium, and agents do not take into account the fact that policy changes have general equilibrium effects.

[11] We call a sequence of extraction rates *non-degenerate* if it is bounded away from 0 and 1.

the sale of the extracted resource to the production sector, equally distributed among all agents.

In this model, agents are also prohibited from borrowing against their future income, and their savings must be non-negative. Savings can be invested only in physical capital, but not in natural resources. This is an important difference between the public property regime and the private property regime considered in Section 4.3.

A *competitive equilibrium*

$$\mathcal{E}^{**}_{\tau}(\varepsilon_\tau) = \left\{(c^{j**}_t(\varepsilon_\tau))^L_{j=1}, (s^{j**}_t(\varepsilon_\tau))^L_{j=1}, k^{**}_t(\varepsilon_\tau), r^{**}_t(\varepsilon_\tau), w^{**}_t(\varepsilon_\tau), q^{**}_t(\varepsilon_\tau), v^{**}_t(\varepsilon_\tau)\right\}_{t=\tau,\tau+1\dots}$$

is defined in a standard way by the following conditions:

- agents maximize their utilities subject to the budget constraints, perfectly anticipating the profile of factor returns and resource incomes, $\{r^{**}_t(\varepsilon_\tau), w^{**}_t(\varepsilon_\tau), v^{**}_t(\varepsilon_\tau)\}^{\infty}_{t=\tau}$;[12]
- capital, labor and natural resources are paid their marginal products;
- resource income is determined by the marginal product of natural resources;
- aggregate agents' savings are equal to investment into physical capital.

Let us clarify our notation $\mathcal{E}^{**}_{\tau}(\varepsilon_\tau)$. We use two stars ($**$) to denote the equilibrium values in the public property regime. The equilibrium starts at time τ, hence the subscript. The equilibrium also depends on agents' expectations about future extraction rates, and on the parameters of the model. However, we are interested in the dependence of equilibrium variables on the current extraction rate ε_τ. For instance, the notation $\{c^{j**}_t(\varepsilon_\tau)\}^{\infty}_{t=\tau}$ emphasizes the dependence of the equilibrium consumption stream for agent j (and thus her utility) on ε_τ.

In the above definition we do not suppose that the Hotelling rule holds. Indeed, the Hotelling rule corresponds to the equilibrium on the asset market, i.e., to the private property regime, where the stock of natural resources is an asset in which agents can invest. In the public property regime, the natural resource stock is not an asset, so the Hotelling rule can be violated.[13] Under some circumstances the rate of change of the resource price may not be equal to the interest rate. However, since we assume that resources cannot be stored privately, all arbitrage opportunities that arise from the violation of the Hotelling rule are forbidden.

A competitive equilibrium exists (see Appendix B), and, similarly to the private property regime, *if initially the stock of physical capital is owned by the most patient agents,*

[12] Note that the definition of a competitive equilibrium does not presume that agents perfectly anticipate future extraction rates; here they are considered as given.

[13] See Chermak and Patrick (2002) for a discussion of the Hotelling rule applicability to the observable price dynamics.

then the competitive equilibrium is unique (see Proposition 4.6 in Section 4.10). Also, *in every competitive equilibrium all but the most patient agents run their capital to zero* (Proposition 4.7). Eventually the whole capital stock belongs to the most patient agents. Thus in the public property regime the Ramsey conjecture also holds true.

4.4.2. Balanced-growth equilibrium under given extraction rate

Suppose that a sequence of extraction rates is constant over time. A *balanced-growth equilibrium* under given extraction rate ε is a competitive equilibrium in which output, consumption, savings, the capital stock, the wage rate and resource income grow at a constant rate γ^{**}, while the rate of change of the resource price and the interest rate are constant over time.

We show that *for any ε there exists a balanced-growth equilibrium* (its characterization is given in Proposition 4.8). In any balanced-growth equilibrium only the most patient agents make positive savings and thus own the whole capital stock. It follows that *the rate of balanced growth, the interest rate, and the rate of change of the resource price depend on the parameters of the model and on the given extraction rate ε* (see Proposition 4.9). Moreover, *every competitive equilibrium under a constant sequence of extraction rates converges to a balanced-growth equilibrium* (Proposition 4.10).

Thus, we can give a qualitative description of competitive equilibria under given extraction rates. In every competitive equilibrium from some time onward only the most patient agents own the whole capital stock. They obtain not only wages and resource income, but also capital income. The incomes of all other agents consist only of wages and resource income. If, in addition, from some time onward the sequence of extraction rates is constant, then a competitive equilibrium converges to a balanced-growth equilibrium.

4.4.3. Time τ voting equilibrium

Now we make extraction rates endogenous and introduce a voting procedure into our model. Our approach to voting in a dynamic general equilibrium framework follows Borissov et al. (2014b).[14] In our model agents vote over the current extraction rate at each point in time.

First we define a voting equilibrium under the assumption that a competitive equilibrium under given extraction rates is unique. Recall that this is true, in particular, when the stock of physical capital is initially owned only by the most patient agents. Further, in Section 4.4.6, we generalize our voting procedure to the case in which this assumption may not hold.

[14] Borissov et al. (2014b) study voting on the shares of public goods in the GDP. Here we apply their approach to voting on extraction rates.

Under the uniqueness assumption a voting equilibrium is defined as follows. At each point in time agents choose the today's extraction rate by majority voting under given expectations about future extraction rates. We show that the agents' preferences are single-peaked, and hence the median voter theorem applies: at each point in time there exists an instantaneous Condorcet winner, i.e., an extraction rate which is preferred by a majority of agents to any other extraction rate. A time τ (temporary) equilibrium is determined by this Condorcet winner. We obtain the closed-form solution for the preferred extraction rate for each agent, and show that it depends only on the discount factor of the agent and is independent of expectations.[15] Hence the instantaneous Condorcet winner is the preferred extraction rate for the agent with the median discount factor. Since the outcome of voting at each point in time does not depend on expectations, an intertemporal voting equilibrium is defined in a natural way.

Formally, suppose at time τ agents are asked to vote over the time τ extraction rate. *Suppose that for given expectations about future extraction rates,* $\{\varepsilon_t^e\}_{t=\tau+1}^{\infty}$*, and for any* $\varepsilon_\tau \in (0,1)$*, a competitive equilibrium* $\mathcal{E}_\tau^{**}(\varepsilon_\tau)$ *is unique.* Then we can unambiguously define agents' preferences over the time τ extraction rate by the indirect utility functions:

$$\mathcal{U}_\tau^j(\varepsilon_\tau) = \ln c_\tau^{j**}(\varepsilon_\tau) + \beta_j \ln c_{\tau+1}^{j**}(\varepsilon_\tau) + \dots, \quad j = 1, \dots, L, \tag{4.1}$$

where $\{c_t^{j**}(\varepsilon_\tau)\}_{t=\tau}^{\infty}$ is the equilibrium consumption stream for agent j. Consumption streams and objective functions depend on expectations and on the parameters of the model as well. However, we use a notation that emphasizes the dependence on the time τ extraction rate ε_τ on which agents vote.

The voting method is majority rule. We define a *time* τ *(temporal) voting equilibrium* as a couple $\{\varepsilon_\tau^{**}, \mathcal{E}_\tau^{**}\}$ such that the equilibrium extraction rate ε_τ^{**} is a Condorcet winner in voting on the time τ extraction rate, and $\mathcal{E}_\tau^{**} = \mathcal{E}_\tau^{**}(\varepsilon_\tau^{**})$ is the corresponding competitive equilibrium.

When voting, agents maximize their indirect utility functions given by (4.1). Therefore, it is crucial to know how the competitive equilibrium $\mathcal{E}_\tau^{**}(\varepsilon_\tau)$ changes when ε_τ changes. Suppose that the time τ extraction rate increases, while the initial resource stock and expectations about future extraction rates remain intact. In other words, let ε_τ be replaced by some other extraction rate, $\tilde{\varepsilon}_\tau > \varepsilon_\tau$.

The time τ volume of extraction increases by the factor $\tilde{\varepsilon}_\tau / \varepsilon_\tau$. Hence the output at time τ increases by the factor $(\tilde{\varepsilon}_\tau / \varepsilon_\tau)^{\alpha_3}$. Some simple but tedious calculations (see Lemma 4.13) show that consumption and savings of all agents, the wage rate, the gross interest rate,

[15] This is due to the log-linear utility functions and the Cobb–Douglas production function. Only in this particular case we can apply our approach to voting in a dynamic general equilibrium framework. A model with general utility and production functions in a dynamic optimization context is proposed by Borissov et al. (2017).

resource income, and the time $\tau + 1$ capital stock also increase by the factor $(\tilde{\varepsilon}_\tau/\varepsilon_\tau)^{\alpha_3}$, proportionally to the changed output.

Further, for all future periods of time the available resource stock decreases: it is multiplied by the factor $(1-\tilde{\varepsilon}_\tau)/(1-\varepsilon_\tau) < 1$. Since expectations about future extraction rates do not change, this leads to the proportional decline in the volumes of extraction. Thus there is a trade-off between the future and today's extraction, which leads to a trade-off between the future and today's consumption. The agent's decision on the time τ extraction rate explicitly affects her future consumption by changing the available resource stock.

Under our assumptions, *agents' preferences in voting on the time τ extraction rate, defined by (4.1), are single-peaked.* We show that *for each agent j there exists a unique preferred time τ extraction rate, i.e., the value ε_τ^j that maximizes her indirect utility function $\mathcal{U}^j(\varepsilon_\tau)$.* It turns out that *the preferred time τ extraction rate for each agent j is given by*

$$\varepsilon_\tau^j = 1 - \beta_j \tag{4.2}$$

(see Proposition 4.11). The preferred time τ extraction rate for each agent is constant over time, depends only on this agent's discount factor, and does not depend on expectations or on the current state of the economy.

By the median voter theorem, there exists a Condorcet winner in majority voting on the time τ extraction rate, which is the preferred extraction rate for the agent with the median discount factor. It follows that *there is a unique time τ voting equilibrium, with the extraction rate ε_τ^{**} given by*

$$\varepsilon_\tau^{**} = 1 - \beta_{med}, \tag{4.3}$$

where β_{med} is the median discount factor (see Theorem 4.3).

Note that the equilibrium time τ extraction rate actually depends neither on the current state of the economy nor on the expectations of agents. The equilibrium extraction rate is constant over time and depends only on the distribution of the discount factors across the population. This result in particular eliminates a strategic motive to influence the outcomes of future votes. This is the reason why they can really be taken as given.

4.4.4. Intertemporal voting equilibrium

Now let us assume perfect foresight about future extraction rates and define an intertemporal voting equilibrium.

Suppose we are given a sequence of extraction rates

$$\mathbb{E}^{**} = \mathbb{E}_0^{**} = \{\varepsilon_t^{**}\}_{t=0}^\infty.$$

Denote by

$$\mathcal{E}^{**} = \mathcal{E}_0^{**} = \left\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\right\}_{t=0,1,\dots}$$

the corresponding competitive equilibrium starting at time $t = 0$.

Let also for $\tau = 1, 2, \dots$

$$\mathcal{E}_\tau^{**} = \left\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\right\}_{t=\tau,\tau+1,\dots}$$

be the corresponding tail of $\mathcal{E}_0^{**}$. Clearly, it is a competitive equilibrium starting at time $t = \tau$ corresponding to the sequence of extraction rates $\mathbb{E}_\tau^{**} = \{\varepsilon_t^{**}\}_{t=\tau}^\infty$.

An *intertemporal voting equilibrium* is a couple which consists of the sequence of extraction rates $\mathbb{E}^{**}$ and the corresponding competitive equilibrium $\mathcal{E}^{**}$ such that for every $\tau = 0, 1, \dots$, the time τ extraction rate ε_τ^{**} and the competitive equilibrium $\mathcal{E}_\tau^{**}$ constitute a time τ voting equilibrium under perfect foresight about future extraction rates ($\varepsilon_t^e = \varepsilon_t^{**},\ t = \tau+1, \tau+2, \dots$).

It is clear that *in every intertemporal voting equilibrium the sequence of extraction rates* $\mathbb{E}^{**}$ *is constant over time*:

$$\mathbb{E}^{**} = \{\varepsilon^{**}, \varepsilon^{**}, \dots\}, \tag{4.4}$$

where $\varepsilon^{**} = 1 - \beta_{med}$ (see Theorem 4.4).

The existence and uniqueness of an intertemporal voting equilibrium are related to the corresponding properties of an underlying competitive equilibria. In particular, *if initially the whole capital stock belongs to the most patient agents, then an intertemporal equilibrium exists and is unique* (Theorem 4.5).

4.4.5. Balanced-growth voting equilibrium

A balanced-growth equilibrium for which the sequence of extraction rates is given by (4.4) is called a *balanced-growth voting equilibrium.* Put differently, a balanced-growth voting equilibrium is an intertemporal voting equilibrium in which output, consumption, savings, the capital stock, the wage rate, and resource income grow at a constant rate γ^{**}, while the rate of change of the resource price and the interest rate are constant over time.

It can be checked that *if initially the whole capital stock is owned by the most patient agents, then the intertemporal voting equilibrium converges to a balanced-growth voting equilibrium* (Theorem 4.6).

The most important conclusion to emerge from the characterization of a balanced-growth voting equilibrium is that the voting equilibrium extraction rate is fully determined by the median discount factor:

$$\varepsilon^{**} = 1 - \beta_{med}.$$

Consequently, the rate of balanced growth depends on the median discount factor:

$$1 + \gamma^{**} = (1+\lambda)^{\frac{1}{1-\alpha_1}} (1-\varepsilon^{**})^{\frac{\alpha_3}{1-\alpha_1}} = (1+\lambda)^{\frac{1}{1-\alpha_1}} \beta_{med}^{\frac{\alpha_3}{1-\alpha_1}}.$$

It is clear that the more patient the population as a whole is (the higher β_{med} is), the lower the voting equilibrium extraction rate is and the higher the long-run growth rate is. These results capture the intuition that the more patient agents value the future more highly and tend to smooth their consumption. They vote for the lower extraction rate today in order to maintain a higher resource stock level in the future, which leads to a higher growth rate.

As we have mentioned, the Hotelling rule can be violated. The rate of change of the resource price π^{**} is not necessarily equal to the interest rate r^{**}. Namely, we have

$$\frac{1+\pi^{**}}{1+r^{**}} = \frac{\beta_1}{\beta_{med}}.$$

It follows that if $\beta_{med} < \beta_1$, then π^{**} is larger than r^{**}. However, this should not be a great surprise. In the public property regime agents cannot invest in natural resources, so there is no reason for the Hotelling rule to hold. Indeed, in this model the return on capital is related to the discount factor of the most patient agents, and the resource extraction rate is determined by the agents with the median discount factor. Unless these types of agents coincide (i.e., unless $\beta_{med} = \beta_1$), the Hotelling rule is violated.

4.4.6. Generalized intertemporal voting equilibria

Our definition of an intertemporal voting equilibrium is given under the assumption that there is a unique competitive equilibrium corresponding to a given sequence of extraction rates. In particular, this assumption ensures the uniqueness of the time τ voting equilibrium and the existence of an intertemporal voting equilibrium.

Let us discuss the general case in which the uniqueness of a competitive equilibrium $\mathcal{E}_\tau^{**}(\varepsilon_\tau)$ is not guaranteed. The difficulty here is that we cannot unambiguously define agents' indirect utility functions and obtain agents' preferred values of extraction rates. However, if we apply the technique proposed by Borissov et al. (2014b), we can get around this difficulty. To do this, we impose a certain additional assumption on the beliefs of agents.

Let us assume that *agents do not take into account the multiplicity of equilibria and believe that a competitive equilibrium after the change of the time τ extraction rate is associated with a competitive equilibrium before the change in the way discussed in Section 4.4.3 (and described in Lemma 4.13).* Our assumption implies that agents simply act as if the competitive equilibrium $\mathcal{E}^{**}_\tau(\varepsilon_\tau)$ is unique for any given extraction rate ε_τ.

We define a *generalized intertemporal voting equilibrium* in essentially the same way as an intertemporal voting equilibrium. The only difference is that we do not assume the uniqueness of the competitive equilibrium; it is replaced with the additional assumption about agents' beliefs.

Clearly, any intertemporal voting equilibrium is a generalized intertemporal voting equilibrium. Moreover, if initially the whole capital stock belongs to the most patient agents, then any generalized intertemporal voting equilibrium is an intertemporal voting equilibrium.

Under our additional assumption all the results concerning voting equilibria remain the same as in the case of a unique competitive equilibrium. Namely, the preferred value of the time τ extraction rate for agent j is given by (4.2), and the equilibrium time τ extraction rate is constant over time and given by (4.3). What is important, *there always exists a generalized intertemporal voting equilibrium, and the sequence of extraction rates in every generalized intertemporal voting equilibrium is constant over time and given by (4.4)* (this is the statement of Theorem 4.7). Furthermore, *every generalized intertemporal voting equilibrium converges to a balanced-growth voting equilibrium* (see Theorem 4.8).

4.5. Comparison of the balanced-growth equilibria

Now we can analyze the long-run consequences of different property regimes in terms of economic growth. In the private property regime, every competitive equilibrium converges to a balanced-growth equilibrium. In the long run only the most patient agents obtain income from the capital and resource stocks. Therefore, the equilibrium extraction rate and the rate of balanced growth are determined by the discount factor of the most patient agents and given by

$$\varepsilon^* = 1 - \beta_1, \qquad 1 + \gamma^* = (1+\lambda)^{\frac{1}{1-\alpha_1}} \, \beta_1^{\frac{\alpha_3}{1-\alpha_1}}.$$

In the public property regime, the sequence of the resource extraction rates is chosen by majority voting. Every generalized intertemporal voting equilibrium converges to a balanced-growth voting equilibrium. In the long run the economy is characterized by the voting equilibrium extraction rate and the rate of balanced growth. They are given by structurally similar equations as in the private property regime, but depend on the median discount factor:

$$\varepsilon^{**} = 1 - \beta_{med}, \qquad 1 + \gamma^{**} = (1+\lambda)^{\frac{1}{1-\alpha_1}} \beta_{med}^{\frac{\alpha_3}{1-\alpha_1}}.$$

It follows that in both regimes the more patient the decision makers are, the lower the extraction rate is and the higher the long-run growth rate is. This is reasonable since patient agents decide to extract less today in order to provide a higher standard of consumption in the future. Therefore it is possible to sustain a higher growth rate in the long run.

The question of which property regime leads to a higher long-run growth rate reduces then to the question of the relationship between the median discount factor and the highest discount factor. In other words, this is the question of whether the most patient agents constitute the majority of the population. If the most patient agents do not constitute the majority of the population ($\beta_{med} < \beta_1$), then the equilibrium extraction rate in the private property regime is lower than in the public property regime. This means that the long-run growth rate is higher in the private property regime. If the most patient agents constitute the majority of the population ($\beta_{med} = \beta_1$), then there is no difference in the growth rates between the two regimes in the long run.

It should be emphasized that we do not argue in favor of private or public ownership, and do not claim that a higher growth rate is better than a lower one. To answer the question of what property regime is better for the society, a social welfare function should be used, and the optimal growth rate is determined by the discount factor of the social planner. However, aggregation of individual preferences and the choice of the discount factor of the social planner is not an easy task when agents are heterogeneous in their time preferences. For instance, Zuber (2011) shows that if agents have different discount factors, then no Paretian social welfare function satisfying natural requirements (history independence, time consistency and stationary) exists.

4.6. Property regimes and income inequality

Recall that in the case of public ownership, resource income is equally distributed among agents, while the capital stock in the long run belongs only to the most patient agents. In the case of private ownership, the most patient agents obtain both capital and resource incomes in the long run. Hence private ownership, all other things being equal, results in higher income inequality than public ownership. Our results suggest that unless we take into account inequality, the private property regime over exhaustible resources is more favorable for promoting long-run economic growth than the public property regime.

However, since inequality can have detrimental effects on economic growth, this conclusion is somewhat hasty.

There is conflicting evidence concerning the relationship between income inequality and economic growth (see, e.g., Henderson et al., 2015). However, there are some recent convincing theoretical and empirical arguments that inequality has a negative and statistically significant long-lasting impact on economic development (see, e.g., Keefer and Knack, 2002; Herzer and Vollmer, 2012; Cingano, 2014).

In particular, Alesina and Perotti (1996) emphasize the role of social conflict as a link between inequality and growth. Inequality increases sociopolitical instability and causes social tension. It creates uncertainty in the politico-economic environment, which in turn reduces investment incentives and affects the security of property rights. Uncertainty and social tension induce fear of losing the sources of income. All these can lead either to a faster depletion of resources, or to unproductive costs and losses in output. Both channels reduce the growth rate of the economy. Thus higher inequality in the private property regime may result in a lower long-run rate of growth compared with the public property regime.

Let us model two channels through which uncertainty and social tension caused by inequality affect economic growth. Suppose that there is a value p which reflects a detrimental effect of inequality on the economy. Let us assume that $p = \psi(G)$, where G is the Gini coefficient, and the function $\psi : [0, 1] \to [0, 1]$ satisfies the following properties:

- $\psi(G)$ is continuous;
- $\psi(G) = 0$ for G smaller than some $\bar{G}$;
- $\psi(G)$ is increasing for $G > \bar{G}$.

An exemplary form of this function is shown in Figure 4.1. Below we will embed the function $\psi(G)$ in our model and clarify its role in affecting the long-run variables.

Clearly, income inequality changes over time. However, for a balanced-growth equilibrium it is constant over time. Let us compare balanced-growth equilibria in the private and public property regimes, taking into account inequality effects.

In a balanced-growth equilibrium the income distribution depends on two characteristics. The first characteristic is the fraction of stock owners, which we denote by σ. Stock owners are the agents who obtain income from both the capital and resource stocks in the private property regime, or the agents who own the capital stock in the public property regime. It is clear that the set of stock owners is a subset of the set of the most patient agents, J. If every agent from the set J makes positive savings in a balanced-growth equilibrium, then $\sigma = |J|/N$. Otherwise, $\sigma < |J|/N$.

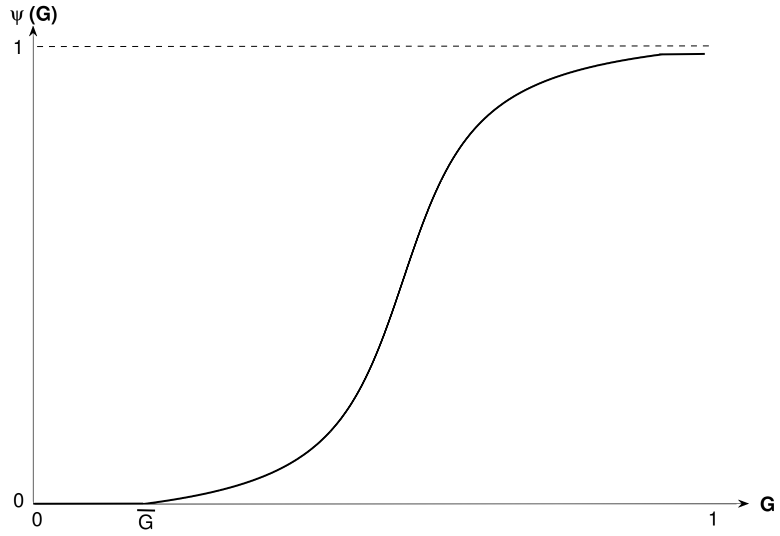

Figure 4.1.: An exemplary form of the function $\psi(G)$

The second characteristic is the distribution of savings across the set of the stock owners. For simplicity we consider the case in which savings are distributed evenly (i.e., initial holdings of capital and resources are equal) across the stock owners. Given σ, any other pattern of the savings distribution in a balanced-growth equilibrium results in a higher Gini coefficient. Therefore, for a given σ, the even distribution of savings across the stock owners provides a lower bound on inequality in the society.[16]

It is not difficult to calculate the Gini coefficient based on the income distribution of agents in a balanced-growth equilibrium for both property regimes. It is given by

$$G = \alpha(1 - \sigma),$$

where $\alpha = \alpha_1$ in the case of the public property regime, and $\alpha = \alpha_1 + \alpha_3$ in the case of the private property regime.

Let us explore two channels through which inequality affects economic growth. Consider first how uncertainty and social tension caused by inequality lead to a faster resource depletion. Following Borissov and Lambrecht (2009), we model this possibility by assuming that the discount factors of agents are formed endogenously. Insecure property rights reduce confidence about the future and decrease the discount factors of agents. Agents

[16] We assume that agents are negligible, and their decisions have no effect on inequality, which they take as given. However, if patient agents can by their actions affect the general equilibrium (e.g., in the case when they own shares in a single firm), then they should be treated as strategic actors who would anticipate that high income inequality causes instability, and thus inequality would be endogenous. The analysis of these issues typically adopts the Markov voting equilibria framework (see, e.g., Acemoglu et al., 2012, 2015), which is beyond the scope of this chapter.

are not sure of the future and are not able to put the high weight on their future utility. Namely, let us assume that the objective function of agent j is given by

$$U^j = \sum_{t=0}^{\infty} ((1-p)\beta_j)^t \ln c_t^j,$$

where $p = \psi(G)$, as discussed above. In this case p reflects the insecurity of property rights that lowers the discount factors of all agents.[17] The value $(1-p)\beta_j$ may be called the effective discount factor of agent j.

The long-run equilibrium extraction rates are now given by

$$\varepsilon^* = 1 - (1 - \psi[(\alpha_1 + \alpha_3)(1-\sigma)])\,\beta_1,$$
$$\varepsilon^{**} = 1 - (1 - \psi[\alpha_1(1-\sigma)])\,\beta_{med}.$$

Thus the rates of balanced growth are given by

$$1 + \gamma^* = (1+\lambda)^{\frac{1}{1-\alpha_1}} \left((1 - \psi[(\alpha_1 + \alpha_3)(1-\sigma)])\beta_1\right)^{\frac{\alpha_3}{1-\alpha_1}},$$
$$1 + \gamma^{**} = (1+\lambda)^{\frac{1}{1-\alpha_1}} \left((1 - \psi[\alpha_1(1-\sigma)])\beta_{med}\right)^{\frac{\alpha_3}{1-\alpha_1}}.$$

It follows that high inequality can lead to an increase in the resource extraction rate and to a decrease in the rate of balanced growth. Indeed, suppose that σ satisfies the following condition:

$$(\alpha_1 + \alpha_3)(1-\sigma) > \bar{G},$$

or, equivalently,

$$\sigma < 1 - \frac{\bar{G}}{\alpha_1 + \alpha_3}. \tag{4.5}$$

Clearly, this may happen when inequality in the society is sufficiently high, i.e., when there are only a few stock owners. Then, the extraction rate ε^* in the private property regime becomes higher, and the corresponding rate of growth $1+\gamma^*$ becomes lower, compared with the case in which we do not take into account the impact of the insecure property rights.

If inequality is so high that

$$\sigma < 1 - \frac{\bar{G}}{\alpha_1},$$

then the equilibrium extraction rates under both property regimes become higher, and both long-run growth rates become lower, compared with the case without the inequality impact.

[17] Similar reasoning is used by Gaddy and Ickes (2005).

Let us compare the growth rates between the two property regimes. It is clear that for $1+\gamma^{**} > 1+\gamma^{*}$, the following condition must hold:

$$\frac{1-\psi[\alpha_1(1-\sigma)]}{1-\psi[(\alpha_1+\alpha_3)(1-\sigma)]} > \frac{\beta_1}{\beta_{med}}. \tag{4.6}$$

Notice that when $\beta_{med} = \beta_1$, condition (4.6) is equivalent to condition (4.5). Thus if the most patient agents constitute the majority of the population, and inequality is just sufficient to increase the extraction rate in the private property regime, then our previous conclusion about the same growth rates in the two regimes is no longer true. Taking into account the impact of insecure property rights, it is the public property regime that results in a higher rate of growth compared with the private property regime.

If the most patient agents do not constitute the majority of the population, then condition (4.6) is more likely to hold when inequality is high, i.e., for the low values of σ. In other words, when there are only a few stock owners, it is more likely that the public property regime leads to a lower equilibrium extraction rate ($\varepsilon^{**} < \varepsilon^{*}$) and a higher rate of growth ($\gamma^{**} > \gamma^{*}$) than the private property regime.

Suppose now that uncertainty and social tension caused by inequality lead to high social costs and losses in output. Namely, assume that in each period a certain share of output which depends on inequality is wasted. If inequality is low, then the wasted fraction is zero. If inequality is high, then instability is also high. In order to pacify the population, a certain share of output will be unproductively thrown away. This wasted share may represent military, police or other special forces expenditure, social spending, etc.

It follows that output per capita is given by

$$y_t = (1-p)A_t k_t^{\alpha_1} e_t^{\alpha_3},$$

where $p = \psi(G)$ now reflects the share of GDP which is drawn away to maintain public order and to prevent possible dissatisfaction of the population about inequality in the society.

Here, the impact of uncertainty and social tension caused by inequality does not influence the equilibrium extraction rates. However, it changes the rates of balanced growth. The growth rate in the case of private property is given by

$$1+\gamma^{*} = \left(1-\psi[(\alpha_1+\alpha_3(1-\sigma)]\right)\left((1+\lambda)\beta_1^{\alpha_3}\right)^{\frac{1}{1-\alpha_1}}.$$

The growth rate in the case of public property is given by

$$1+\gamma^{**} = \left(1-\psi[\alpha_1(1-\sigma)]\right)\left((1+\lambda)\beta_{med}^{\alpha_3}\right)^{\frac{1}{1-\alpha_1}}.$$

It follows that $1 + \gamma^{**} > 1 + \gamma^{*}$ if

$$\frac{1 - \psi[\alpha_1(1-\sigma)]}{1 - \psi[(\alpha_1 + \alpha_3)(1-\sigma)]} > \left(\frac{\beta_1}{\beta_{med}}\right)^{\frac{\alpha_3}{1-\alpha_1}}. \tag{4.7}$$

Again note that when $\beta_{med} = \beta_1$, condition (4.7) is equivalent to condition (4.5). This means that if the most patient agents constitute the majority of the population, then as soon as inequality matters, the public property regime results in a higher long-run rate of growth than the private property regime. If the most patient agents do not constitute the majority of the population, then condition (4.7) is more likely to hold for low values of σ.

Therefore, the two considered channels through which uncertainty and social tension caused by inequality affect economic growth lead to similar results. When inequality in the society is sufficiently high, the public property regime may lead to a higher long-run rate of growth than the private property regime.

4.7. Income inequality and capital taxation

Up to this point our analysis was purely positive, and our concern was in comparing two different institutional frameworks, private and public ownership. As we have noticed above, it is difficult to make normative judgements and discuss economic policy within models where agents are heterogeneous in their time preferences. The very existence of a social welfare function satisfying certain natural requirements is questionable (see Zuber, 2011), and, in particular, it is unclear, what discount factor a social planner should have.

One might conjecture that a social planner can achieve her goals by implementing an appropriate economic policy in the form of taxes (e.g., introducing a capital income or resource income tax). Following the logic of our models, it is natural to assume that the tax rate is also chosen by majority voting. This seems to be a difficult task[18] which may be a fruitful topic for further study, as we have no developed theory of voting for this case.[19]

Having said that, suppose that a social planner seeks to maximize the long-run growth rate of the economy, and simultaneously tries to reach tolerable income inequality. Then it is reasonable to consider the question of whether differences in growth rate and inequality between private and public ownership could be undone by capital income taxation.

[18] It should be noted that the impact of capital taxation on inequality depends on the distribution of capital among the most patient agents, which is indeterminate on balanced-growth paths in our models (cf. Alesina and Rodrik, 1994; Lindner and Strulik, 2004).

[19] See, however, Benhabib and Przeworski (2006).

Our initial model with private ownership of natural resources is easily modified to include capital taxation (see Section 4.11 for details). Suppose that capital income is taxed at some fixed rate θ, and tax revenues are lump-sum distributed to the agents. Then *in the private property regime with capital tax the long-run extraction and growth rates are exactly the same as in the private property regime without capital tax:*

$$\varepsilon^* = 1 - \beta_1, \qquad 1 + \gamma^* = (1+\lambda)^{\frac{1}{1-\alpha_1}} \, \beta_1^{\frac{\alpha_3}{1-\alpha_1}}$$

(see Proposition 4.12 in Section 4.11). The introduction of a capital income tax does not affect the long-run growth rate, and, at the same time, increases the share of wages with transfers in the total income (from α_2 to $\alpha_2 + \theta\alpha_1$), thus decreasing inequality.[20]

Therefore, if a social planner seeks to maximize the long-run growth rate and explicitly takes into account that inequality adversely affects the growth rate, she may effectively implement a tax on capital income. Private ownership with capital tax is equivalent to private ownership without tax in terms of growth rates, while an appropriately chosen capital tax ($\theta = \alpha_3/\alpha_1$) generates the same income inequality as public ownership. Moreover, it is possible to achieve less income inequality and obtain a higher long-run growth rate in the private property regime with capital taxation compared with the public property regime.

4.8. Conclusion

In this chapter, we consider two Ramsey-type models of economic growth with exhaustible natural resources and agents who are heterogeneous in their time preferences. The important difference between the two models lies in the property regime over natural resources. The first model assumes that the resource stock is privately owned. The second model assumes that the resource stock is controlled by a government for the common benefit.

In the private property regime, resource income belongs to the owners of natural resources. The extraction rate is determined by the market forces of supply and demand. Eventually only the most patient agents obtain income from both the capital and resource stocks. We show that every competitive equilibrium in this model converges to a balanced-growth equilibrium. In the long run the discount factor of the most patient agents determines the long-run extraction and growth rates.

In the public property regime, resource income is equally distributed among all agents, who choose the resource extraction rate by majority voting. We define an intertemporal

[20] Recall that in a balanced-growth equilibrium in the private property regime impatient agents consume only their wages. In a balanced-growth equilibrium with capital taxation, each impatient agent receives her wage, $w_t = \alpha_2 y_t$, and a lump-sum transfer payment, $\theta\alpha_1 y_t$. In both cases the capital stock belongs only to the most patient agents.

voting equilibrium and establish its convergence to a balanced-growth voting equilibrium. It turns out that the preferences of the agents with the median discount factor determine all voting decisions. In particular, the long-run extraction rate and the rate of growth are determined by the median discount factor.

When comparing the long-run effects of the two property regimes in terms of economic growth, we have to distinguish between two cases. The first is the case in which the most patient agents constitute the majority of the population. Here the rates of balanced growth under the two regimes are equal. The second is the case in which the most patient agents do not constitute the majority of the population. It follows that the equilibrium extraction rate in the private property regime is lower than that in the public property regime. Correspondingly, the growth rate is higher in the private property regime. The intuition behind this result is as follows. More patient agents prefer to extract less amount of resource today in order to provide a higher standard of consumption in the future. Therefore, the private property regime, in which the extraction rate in the long run is determined by the discount factor of the most patient agents, sustains a higher growth rate.

The conclusion that the private property regime is better for economic growth than the public property regime is not necessarily true if we take into account the detrimental impact of inequality on economic development. In the case of private ownership, only the most patient agents obtain resource income in the long run, while in the case of public ownership, resource income is equally distributed among all agents. Hence the private property regime, all other things being equal, results in higher income inequality than the public property regime.

We explore two different channels through which uncertainty and social tension caused by inequality affect economic growth. The first channel assumes that the discount factors of agents are formed endogenously. When inequality is high, agents' confidence about the future reduces, their effective discount factors decrease, and thus inequality explicitly affects the long-run extraction rates. The second channel assumes that a certain share of output, depending on inequality, is unproductively wasted. This wasted share may be thought of as military expenditures, or social spending to pacify the population. In both cases if inequality is sufficiently high, then the public property regime will likely result in a higher long-run rate of growth.

These results suggest that in the absence of redistributive policy, there is no unambiguous answer to the question, which of the two property regimes does lead to a higher long-run rate of economics growth. In societies with moderate inequality, private ownership of natural resources is likely to provide a higher long-run growth rate than public ownership. In societies with high inequality, the public property regime may result in a higher long-run growth rate compared to the private property regime.

It turns out, however, that if a social planner seeks to maximize the growth rate of the economy taking into account the detrimental effect of inequality on the economy, then she may effectively implement a capital income tax and lump-sum transfers. The private property regime accompanied with such a redistributive policy will lead to less income inequality and a higher long-run growth rate compared with the public property regime.

4.9. Proofs. Private property regime

4.9.1. Competitive equilibrium

Suppose we are given an initial state of the economy, $\mathcal{I}_0 = \{(\hat{k}_0^j)_{j=1}^L, (\hat{R}_{-1}^j)_{j=1}^L\}$, where $(\hat{k}_0^j)_{j=1}^L$ and $(\hat{R}_{-1}^j)_{j=1}^L$ are initial distributions of physical capital and natural resources among agents.

We assume that $\mathcal{I}_0$ is a non-degenerate initial state,[21] i.e.,

$$\hat{k}_0^j \geq 0, \quad \hat{R}_{-1}^j \geq 0, \quad j = 1, \ldots, L; \qquad \frac{1}{L}\sum_{j=1}^{L} \hat{k}_0^j > 0; \qquad \sum_{j=1}^{L} \hat{R}_{-1}^j > 0.$$

Definition. *A competitive equilibrium starting from the initial state $\mathcal{I}_0$ is a sequence*

$$\mathcal{E}^* = \left\{(c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^*\right\}_{t=0,1,\ldots}$$

such that

1. *For each $j = 1, \ldots, L$, the sequence $\{c_t^{j*}, s_t^{j*}\}_{t=0}^\infty$ is a solution to the following utility maximization problem:*

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \beta_j^t \ln c_t^j, \\ &\text{s. t. } \; c_t^j + s_t^j \leq (1 + r_t)\, s_{t-1}^j + w_t, \quad t = 0, 1, \ldots, \\ &\qquad\;\; s_t^j \geq 0, \quad t = 0, 1, \ldots \end{aligned} \tag{4.8}$$

   at $r_t = r_t^*$, $w_t = w_t^*$, and $s_{-1}^j = \frac{q_0^*}{1+r_0^*}\hat{R}_{-1}^j + \hat{k}_0^j$;

2. *Capital is paid its marginal product:*

$$1 + r_t^* = \alpha_1 A_t (k_t^*)^{\alpha_1 - 1} (e_t^*)^{\alpha_3}, \quad t = 0, 1, \ldots,$$

[21] We impose the non-negativity constraints on initial distributions of physical capital and natural resources only for technical convenience. The individual holdings of capital and resources are indeterminate on the equilibrium path, they may be positive as well as negative. However, they do not appear in the definition of equilibrium; it is irrelevant, in which proportion agents invest their savings in different assets. Since the two assets are perfect substitutes, only agents' savings are important in an equilibrium.

where $k_0^* = \frac{1}{L}\sum_{j=1}^{L} \hat{k}_0^j$;

3. *Labor is paid its marginal product:*

$$w_t^* = \alpha_2 A_t (k_t^*)^{\alpha_1} (e_t^*)^{\alpha_3}, \quad t = 0, 1, \ldots;$$

4. *The price of natural resources is equal to the marginal product:*

$$q_t^* = \alpha_3 A_t (k_t^*)^{\alpha_1} (e_t^*)^{\alpha_3 - 1}, \quad t = 0, 1, \ldots;$$

5. *The Hotelling rule holds:*

$$q_{t+1}^* = (1 + r_{t+1}^*) q_t^*, \quad t = 0, 1, \ldots;$$

6. *The natural balance of exhaustible resources is fulfilled:*[22]

$$R_t^* = R_{t-1}^* - L e_t^*, \quad t = 0, 1, \ldots,$$

where $R_{-1}^* = \sum_{j=1}^{L} \hat{R}_{-1}^j$;

7. *Aggregate savings are equal to investment into physical capital and natural resources:*

$$\sum_{j=1}^{L} s_t^{j*} = \frac{q_{t+1}^*}{1 + r_{t+1}^*} R_t^* + L k_{t+1}^*, \quad t = 0, 1, \ldots.$$

Theorem 4.1. *For any initial state* $\mathcal{I}_0$ *there exists a competitive equilibrium starting from* $\mathcal{I}_0$.

Proof. See Appendix A. □

Let us prove two important results about this equilibrium. The following proposition states that if at the initial instant the stocks of physical capital and natural resources are owned by the most patient agents, then the competitive equilibrium starting from this state is unique.

Proposition 4.1. *Suppose that the initial state* $\mathcal{I}_0$ *is such that*

$$\hat{k}_0^j = 0, \qquad \hat{R}_{-1}^j = 0, \qquad j \notin J.$$

[22] Note that extraction rates are determined here in an equilibrium by the market forces of supply and demand.

Then there exists a unique competitive equilibrium starting from $\mathcal{I}_0$,

$$\mathcal{E}^* = \left\{(c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^*\right\}_{t=0,1,\dots},$$

which is given for $t = 0, 1, \dots$ by

$$\begin{gathered} c_t^{j*} = (1-\beta_1)(1+r_t^*)s_{t-1}^{j*} + w_t^*, \qquad s_t^{j*} = \beta_1(1+r_t^*)s_{t-1}^{j*}, \qquad j \in J, \\ c_t^{j*} = w_t^*, \qquad s_t^{j*} = 0, \qquad j \notin J, \\ k_{t+1}^* = \beta_1(1+r_t^*)k_t^*, \qquad 1+r_t^* = \alpha_1 A_t (k_t^*)^{\alpha_1-1}(e_t^*)^{\alpha_3}, \\ w_t^* = \alpha_2 A_t (k_t^*)^{\alpha_1}(e_t^*)^{\alpha_3}, \qquad q_t^* = \alpha_3 A_t (k_t^*)^{\alpha_1}(e_t^*)^{\alpha_3-1}, \\ R_t^* = \beta_1 R_{t-1}^*, \qquad e_t^* = \frac{1-\beta_1}{L} R_{t-1}^*, \end{gathered}$$

where $R_{-1}^* = \sum_{j=1}^L \hat{R}_{-1}^j$, $k_0^* = \frac{1}{L}\sum_{j=1}^L \hat{k}_0^j$, and $s_{-1}^{j*} = \frac{q_0^*}{1+r_0^*}\hat{R}_{-1}^j + \hat{k}_0^j$.

The following proposition verifies that in every competitive equilibrium from some time onward only the most patient agents can make positive savings. From this time relatively less patient agents make no savings, and the extraction rate is constant over time and equals $1 - \beta_1$.

Proposition 4.2. *Suppose that*

$$\mathcal{E}^* = \left\{(c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^*\right\}_{t=0,1,\dots}$$

is a competitive equilibrium starting from an arbitrary initial state $\mathcal{I}_0$. Then there exists a point in time T such that for all $t > T$,

$$\begin{gathered} s_t^{j*} = 0, \quad j \notin J, \\ R_t^* = \beta_1 R_{t-1}^*, \qquad e_{t+1}^* = \beta_1 e_t^*. \end{gathered}$$

Proof of Propositions 4.1 and 4.2.

Consider a competitive equilibrium

$$\mathcal{E}^* = \left\{(c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^*\right\}_{t=0,1,\dots}$$

starting from a non-degenerate state $\mathcal{I}_0 = \{(\hat{k}_0^j)_{j=1}^L, (\hat{R}_{-1}^j)_{j=1}^L\}$. Since for each $j = 1, \dots, L$, the sequence $\{c_t^{j*}, s_t^{j*}\}_{t=0}^\infty$ is a solution to problem (4.8), it satisfies the first-order conditions,

$$c_{t+1}^{j*} \geq \beta_j(1+r_{t+1}^*)c_t^{j*} \ (= \text{ if } s_t^{j*} > 0), \quad t = 0, 1, \dots, \tag{4.9}$$

and the transversality condition,

$$\lim_{t\to\infty} \frac{\beta_j^t s_t^{j*}}{c_t^{j*}} = 0. \tag{4.10}$$

Lemma 4.1. *Let $\beta > 0$ be such that for some T*

$$k_{t+1}^* > \beta(1 + r_t^*)k_t^* = \beta\alpha_1 A_t(k_t^*)^{\alpha_1}(e_t^*)^{\alpha_3}, \quad t > T.$$

If $\beta_j < \beta$, then $s_t^{j*} = 0$ for all sufficiently large t.

Proof. First let us show that if $\beta_j < \beta$, then $s_t^{j*} = 0$ for some $t \geq T$. Assume the converse. By (4.9), for all $t \geq T$,

$$c_{t+1}^{j*} = \beta_j(1 + r_{t+1}^*)c_t^{j*},$$

and hence

$$\frac{c_t^{j*}}{k_{t+1}^*} \leq \frac{\beta_j(1 + r_t^*)c_{t-1}^{j*}}{\beta(1 + r_t^*)k_t^*} \leq \frac{\beta_j}{\beta}\frac{c_{t-1}^{j*}}{k_t^*}.$$

By assumption, $\beta_j/\beta < 1$, and thus $c_t^{j*}/k_{t+1}^* \xrightarrow[t\to\infty]{} 0$. Furthermore, it is clear that $k_{t+1}^* \leq A_t(k_t^*)^{\alpha_1}(e_t^*)^{\alpha_3}$, and therefore

$$\frac{c_t^{j*}}{w_t^*} = \frac{c_t^{j*}}{\alpha_2 A_t(k_t^*)^{\alpha_1}(e_t^*)^{\alpha_3}} \leq \frac{c_t^{j*}}{\alpha_2 k_{t+1}^*} \xrightarrow[t\to\infty]{} 0.$$

Thus for all sufficiently large t, $c_t^{j*} < w_t^*$, which is not optimal for agent j.

Now we know that $s_t^{j*} = 0$ at least for some t. Let us show that $s_t^{j*} = 0$ for all $t \geq T$. Indeed, assume the converse. Then there are only two possibilities. The first is that there exists $T' > T$ such that $s_t^{j*} > 0$ for all $t \geq T'$. However, applying the same argument as above, we obtain that $s_t^{j*} = 0$ for some $t \geq T'$. The second possibility is that there are t_1 and t_2 such that $T \leq t_1 < t_2 - 1$, and

$$s_{t_1}^{j*} = 0, \qquad s_{t_2}^{j*} = 0, \qquad s_t^{j*} > 0, \quad t_1 < t < t_2.$$

It follows from the budget constraints of agent j that

$$c_{t_1+1}^{j*} < w_{t_1+1}^*, \qquad c_{t_2}^{j*} > w_{t_2}^*. \tag{4.11}$$

However, for $t \geq T$,

$$\frac{\alpha_1}{\alpha_2}\frac{w_{t+1}^*}{1 + r_{t+1}^*} = k_{t+1}^* > \beta(1 + r_t^*)k_t^* = \frac{\alpha_1}{\alpha_2}\beta w_t^*.$$

Thus $w^*_{t+1} > \beta(1 + r^*_{t+1})w^*_t$. Using (4.9), we get

$$c^{j*}_{t_1+2} = \beta_j(1 + r^*_{t_1+2})c^{j*}_{t_1+1} < \beta_j(1 + r^*_{t_1+2})w^*_{t_1+1} < \beta(1 + r^*_{t_1+2})w^*_{t_1+1} < w^*_{t_1+2}.$$

Repeating this argument, we obtain

$$c^{j*}_{t+1} < w^*_{t+1}, \quad t_1 < t < t_2,$$

which implies $c^{j*}_{t_2} < w^*_{t_2}$, a contradiction of (4.11). □

Lemma 4.2.

$$k^*_{t+1} \leq \beta_1(1 + r^*_t)k^*_t, \quad t = 0, 1, \ldots.$$

Proof. Assume the converse. Then there are T and $\zeta > 1$ such that

$$k^*_{T+1} \geq \zeta\beta_1(1 + r^*_T)k^*_T. \tag{4.12}$$

Let us show that (4.12) implies

$$k^*_{t+1} \geq \zeta\beta_1(1 + r^*_t)k^*_t, \quad t \geq T. \tag{4.13}$$

Denote

$$J(T) = \left\{j \in \{1, 2, \ldots, L\} \ \mid \ s^{j*}_T > 0\right\},$$

and recall that

$$\frac{(1 + r^*_t)k^*_t}{\alpha_1} = \frac{w^*_t}{\alpha_2} = \frac{q^*_t e^*_t}{\alpha_3}. \tag{4.14}$$

We have

$$\begin{aligned}
&\sum_{j\in J(T)}\left(c^{j*}_{T+1}+s^{j*}_{T+1}\right)-\frac{q^*_{T+2}}{1+r^*_{T+2}}R^*_{T+1}\\
&\qquad=\sum_{j\in J(T)}\left((1+r^*_{T+1})s^{j*}_T+w^*_{T+1}\right)-\frac{q^*_{T+2}}{1+r^*_{T+2}}R^*_{T+1}\\
&\qquad=q^*_{T+1}R^*_T+(1+r^*_{T+1})Lk^*_{T+1}+|J(T)|w^*_{T+1}-q^*_{T+1}R^*_{T+1}\\
&\qquad=q^*_{T+1}Le^*_{T+1}+(1+r^*_{T+1})Lk^*_{T+1}+|J(T)|w^*_{T+1}\\
&\qquad=(1+r^*_{T+1})k^*_{T+1}\left(L+L\frac{\alpha_3}{\alpha_1}+|J(T)|\frac{\alpha_2}{\alpha_1}\right)\\
&\qquad\geq\zeta\beta_1(1+r^*_{T+1})(1+r^*_T)k^*_T\left(L+L\frac{\alpha_3}{\alpha_1}+|J(T)|\frac{\alpha_2}{\alpha_1}\right)\\
&\qquad=\zeta\beta_1(1+r^*_{T+1})\left(L(1+r^*_T)k^*_T+Lq^*_Te^*_T+|J(T)|w^*_T\right)\\
&\qquad=\zeta\beta_1(1+r^*_{T+1})\left((1+r^*_T)\left(\sum_{j=1}^{L}s^{j*}_{T-1}-\frac{q^*_T}{1+r^*_T}R_{T-1}\right)+Lq^*_Te^*_T+\sum_{j\in J(T)}w^*_T\right)\\
&\qquad\geq\zeta\beta_1(1+r^*_{T+1})\left((1+r^*_T)\sum_{j\in J(T)}s^{j*}_{T-1}+\sum_{j\in J(T)}w^*_T-q^*_TR^*_{T-1}+Lq^*_Te^*_T\right)\\
&\qquad=\zeta\beta_1(1+r^*_{T+1})\left(\sum_{j\in J(T)}(c^{j*}_T+s^{j*}_T)-\frac{q^*_{T+1}}{1+r^*_{T+1}}R^*_T\right).
\end{aligned}$$

Thus

$$\sum_{j\in J(T)}\left(c^{j*}_{T+1}+s^{j*}_{T+1}\right)-\frac{q^*_{T+2}}{1+r^*_{T+2}}R^*_{T+1}\geq\zeta\beta_1(1+r^*_{T+1})\left(\sum_{j\in J(T)}(c^{j*}_T+s^{j*}_T)-\frac{q^*_{T+1}}{1+r^*_{T+1}}R^*_T\right).$$

For $j\in J(T)$, (4.9) holds with equality, so

$$\sum_{j\in J(T)}c^{j*}_{T+1}=\sum_{j\in J(T)}\beta_j(1+r^*_{T+1})c^{j*}_T\leq\beta_1(1+r^*_{T+1})\sum_{j\in J(T)}c^{j*}_T<\zeta\beta_1(1+r^*_{T+1})\sum_{j\in J(T)}c^{j*}_T.$$

Clearly, two above inequalities are consistent only if

$$\sum_{j\in J(T)}s^{j*}_{T+1}-\frac{q^*_{T+2}}{1+r^*_{T+2}}R^*_{T+1}\geq\zeta\beta_1(1+r^*_{T+1})\left(\sum_{j\in J(T)}s^{j*}_T-\frac{q^*_{T+1}}{1+r^*_{T+1}}R^*_T\right).$$

Therefore,

$$\begin{aligned} Lk^*_{T+2} &= \sum_{j=1}^{L} s^{j*}_{T+1} - \frac{q^*_{T+2}}{1+r^*_{T+2}} R^*_{T+1} \geq \sum_{j \in J(T)} s^{j*}_{T+1} - \frac{q^*_{T+2}}{1+r^*_{T+2}} R^*_{T+1} \\ &\geq \zeta\beta_1(1+r^*_{T+1}) \left(\sum_{j \in J(T)} s^{j*}_{T} - \frac{q^*_{T+1}}{1+r^*_{T+1}} R^*_{T} \right) \\ &= \zeta\beta_1(1+r^*_{T+1}) \left(\sum_{j=1}^{L} s^{j*}_{T} - \frac{q^*_{T+1}}{1+r^*_{T+1}} R^*_{T} \right) = \zeta\beta_1(1+r^*_{T+1}) Lk^*_{T+1}. \end{aligned}$$

Repeating the argument, we infer that (4.13) holds for all $t \geq T$.

However, from Lemma 4.1 it follows that $s^{j*}_t = 0$ for all j and for all sufficiently large t. This contradicts the evident positivity of k^*_t for all $t = 0, 1, \ldots$. □

Lemma 4.3.

$$w^*_{t+1} \leq \beta_1(1+r^*_{t+1}) w^*_t, \qquad e^*_{t+1} \leq \beta_1 e^*_t, \qquad t = 0, 1, \ldots.$$

Proof. Both inequalities follow from (4.14) and Lemma 4.2. Indeed, for all t

$$\frac{w^*_{t+1}}{1+r^*_{t+1}} = \frac{\alpha_2(1+r^*_{t+1})k^*_{t+1}}{\alpha_1(1+r^*_{t+1})} \leq \frac{\beta_1\alpha_2(1+r^*_t)k^*_t}{\alpha_1} = \beta_1 w^*_t.$$

Moreover, for all t

$$\frac{e^*_{t+1}}{e^*_t} = \frac{q^*_t}{q^*_{t+1}} \frac{(1+r^*_{t+1})k^*_{t+1}}{(1+r^*_t)k^*_t} = \frac{(1+r^*_{t+1})q^*_t}{q^*_{t+1}} \frac{k^*_{t+1}}{(1+r^*_t)k^*_t} \leq \beta_1,$$

since $q^*_{t+1} = (1+r^*_{t+1})q^*_t$ by the Hotelling rule. □

Lemma 4.4.

$$s^{j*}_{t+1} \geq \beta_1(1+r^*_{t+1}) s^{j*}_t, \quad j \in J, \quad t = -1, 0, \ldots. \tag{4.15}$$

Proof. Consider $j \in J$. Then by (4.9),

$$\beta_1^t c^{j*}_0 \leq \frac{c^{j*}_t}{(1+r^*_1)\cdots(1+r^*_t)}, \quad t = 1, 2, \ldots,$$

and hence

$$c^{j*}_0(1+\beta_1+\beta_1^2+\ldots) \leq c^{j*}_0 + \frac{c^{j*}_1}{(1+r^*_1)} + \frac{c^{j*}_2}{(1+r^*_1)(1+r^*_2)} + \ldots. \tag{4.16}$$

Adding together all budget constraints of agent j, we obtain

$$
\begin{aligned}
c_0^{j*} + \frac{c_1^{j*}}{(1+r_1^*)} + \frac{c_2^{j*}}{(1+r_1^*)(1+r_2^*)} + \dots \\
\le (1+r_0^*)s_{-1}^j + w_0^* + \frac{w_1^*}{(1+r_1^*)} + \frac{w_2^*}{(1+r_1^*)(1+r_2^*)} + \dots.
\end{aligned} \tag{4.17}
$$

Moreover, by Lemma 4.3 for $t = 0, 1, \dots$,

$$\frac{w_{t+1}^*}{(1+r_1^*)\cdots(1+r_{t+1}^*)} \le \frac{\beta_1 w_t^*}{(1+r_1^*)\cdots(1+r_t^*)} \le \dots \le \beta_1^{t+1} w_0^*,$$

which implies

$$(1+r_0^*)s_{-1}^{j*}+w_0^*+\frac{w_1^*}{(1+r_1^*)}+\frac{w_2^*}{(1+r_1^*)(1+r_2^*)}+\dots \le (1+r_0^*)s_{-1}^{j*}+w_0^*(1+\beta_1+\beta_1^2+\dots). \tag{4.18}$$

Combining (4.16)–(4.18), we finally get

$$c_0^{j*}(1+\beta_1+\beta_1^2+\dots) \le (1+r_0^*)s_{-1}^{j*} + w_0^*(1+\beta_1+\beta_1^2+\dots),$$

and therefore

$$c_0^{j*} \le (1+r_0^*)(1-\beta_1)s_{-1}^{j*} + w_0^*.$$

Thus,

$$s_0^{j*} = (1+r_0^*)s_{-1}^{j*} + w_0^* - c_0^{j*} \ge (1+r_0^*)s_{-1}^{j*} + w_0^* - (1+r_0^*)(1-\beta_1)s_{-1}^{j*} - w_0^* = \beta_1(1+r_0^*)s_{-1}^{j*}.$$

This proves (4.15) for $t = -1$. To prove it for $t = 0, 1, \dots$, it is sufficient to repeat the argument. □

Lemma 4.5. *For any $\delta > 0$ there exists a point in time T such that for all $t > T$,*

$$k_{t+1}^* > \beta_1(1-\delta)(1+r_t^*)k_t^*.$$

Proof. From (4.9) and Lemma 4.3 it is clear that for $j \in J$

$$\frac{c_{t+1}^{j*}}{w_{t+1}^*} \ge \frac{\beta_1(1+r_{t+1}^*)c_t^{j*}}{\beta_1(1+r_{t+1}^*)w_t^*} = \frac{c_t^{j*}}{w_t^*}, \quad t = 0, 1, \dots.$$

This means that the sequence $\left\{\frac{c_t^{j*}}{w_t^*}\right\}_{t=0}^{\infty}$ is non-decreasing. It is also bounded from above, as consumption cannot exceed total output:

$$c_t^{j*} \le L\frac{w_t^*}{\alpha_2}, \quad t = 0, 1, \dots.$$

Therefore, the sequence $\left\{\frac{c_t^{j*}}{w_t^*}\right\}_{t=0}^{\infty}$ converges, so the sequence $\left\{\frac{c_t^{j*}}{w_t^*}\frac{w_{t+1}^*}{c_{t+1}^{j*}}\right\}_{t=0}^{\infty}$ converges to 1.

It follows from Lemma 4.4 that if $s_{-1}^j > 0$, then $s_t^{j*} > 0$ for all $t \geq 0$ and $j \in J$. Thus,

$$\frac{c_t^{j*}}{w_t^*}\frac{w_{t+1}^*}{c_{t+1}^{j*}} = \frac{w_{t+1}^*}{\beta_1(1+r_{t+1}^*)w_t^*}, \quad t = 0, 1, \dots,$$

and the sequence $\left\{\frac{w_{t+1}^*}{\beta_1(1+r_{t+1}^*)w_t^*}\right\}_{t=0}^{\infty}$ converges to 1 as well. Hence for any $\delta > 0$ there exists T such that for $t > T$,

$$\frac{w_{t+1}^*}{\beta_1(1+r_{t+1}^*)w_t^*} > (1-\delta),$$

which implies

$$k_{t+1}^* = \frac{\alpha_1}{\alpha_2}\frac{w_{t+1}^*}{1+r_{t+1}^*} > \frac{\alpha_1}{\alpha_2}\beta_1(1-\delta)w_t^* = \beta_1(1-\delta)(1+r_t^*)k_t^*.$$

This proves the lemma. □

Consider δ that satisfies $\beta_1(1-\delta) > \max_{j\notin J}\beta_j$. Applying Lemma 4.1 with $\beta = \beta_1(1-\delta)$, we obtain that for any competitive equilibrium starting from a non-degenerate initial state there exists a point in time T such that for all $t > T$, $s_t^{j*} = 0$ for $j \notin J$.

Lemma 4.6. *For all $t > T$,*

$$k_{t+1}^* = \beta_1(1+r_t^*)k_t^*,$$
$$c_t^{j*} = (1-\beta_1)(1+r_t^*)s_{t-1}^{j*} + w_t^*, \qquad s_t^{j*} = \beta_1(1+r_t^*)s_{t-1}^{j*}, \qquad j \in J,$$
$$c_t^{j*} = w_t^*, \qquad s_t^{j*} = 0, \qquad j \notin J.$$

Moreover,

$$e_{t+1}^* = \beta_1 e_t^*, \qquad R_t^* = \beta_1 R_{t-1}^*.$$

Proof. First let us show that

$$\lim_{t\to\infty} R_t^* = 0. \tag{4.19}$$

To prove this, note that for all j,

$$\lim_{t\to\infty}\frac{s_t^{j*}}{(1+r_1^*)\cdots(1+r_t^*)} = 0.$$

Indeed, it follows from (4.17) that this limit exists. Since savings are non-negative, this limit is also non-negative. Suppose that $\lim_{t\to\infty} \frac{s_t^{j*}}{(1+r_1^*)\cdots(1+r_t^*)} > 0$. Then $s_t^{j*} > 0$ for all t, and thus by (4.9),

$$c_t^{j*} = \beta_j(1+r_t^*)c_{t-1}^{j*} = \ldots = \beta_j^t(1+r_t^*)\cdots(1+r_1^*)c_0^{j*}.$$

Hence

$$\frac{\beta_j^t s_t^{j*}}{c_t^{j*}} = \frac{s_t^{j*}}{(1+r_t^*)\cdots(1+r_1^*)c_0^{j*}} = \frac{1}{c_0^{j*}}\frac{s_t^{j*}}{(1+r_1^*)\cdots(1+r_t^*)}.$$

It follows from (4.10) that $\lim_{t\to\infty} \frac{s_t^{j*}}{(1+r_1^*)\cdots(1+r_t^*)} = 0$, which is a contradiction. Therefore,

$$\lim_{t\to\infty} \frac{\sum_{j=1}^L s_t^{j*}}{(1+r_1^*)\cdots(1+r_t^*)} = 0.$$

Since $\sum_{j=1}^L s_t^{j*} = q_t^* R_t^* + Lk_{t+1}^*$, and both terms are non-negative, we get

$$\lim_{t\to\infty} \frac{q_t^* R_t^*}{(1+r_1^*)\cdots(1+r_t^*)} = \lim_{t\to\infty} q_0^* R_t^* = 0.$$

As $q_0^* > 0$, (4.19) indeed holds.

Now suppose that $t > T$. By Lemma 4.4,

$$\begin{aligned} \beta_1(1+r_t^*)\left(Lk_t^* + \frac{q_t^*}{1+r_t^*}R_{t-1}^*\right) &= \beta_1(1+r_t^*)\sum_{j\in J} s_{t-1}^{j*} \\ &\leq \sum_{j\in J} s_t^{j*} = Lk_{t+1}^* + \frac{q_{t+1}^*}{1+r_{t+1}^*}R_t^*. \end{aligned} \tag{4.20}$$

At the same time, by Lemma 4.2,

$$k_{t+1}^* \leq \beta_1(1+r_t^*)k_t^*, \quad t = 0, 1, \ldots.$$

Therefore for $t > T$,

$$\frac{q_{t+1}^*}{1+r_{t+1}^*}R_t^* \geq \beta_1(1+r_t^*)\frac{q_t^*}{1+r_t^*}R_{t-1}^*,$$

or, equivalently,

$$R_t^* \geq \beta_1 R_{t-1}^*. \tag{4.21}$$

It follows from the natural balance of exhaustible resources and (4.19) that

$$R_T^* = R_{T+1}^* + Le_{T+1}^* = R_{T+2}^* + Le_{T+2}^* + Le_{T+1}^*$$
$$= \ldots = Le_{T+1}^* \left(1 + \frac{e_{T+2}^*}{e_{T+1}^*} + \frac{e_{T+3}^*}{e_{T+2}^*} \frac{e_{T+2}^*}{e_{T+1}^*} + \ldots \right).$$

Hence, using Lemma 4.3, we conclude that

$$R_T^* \leq Le_{T+1}^* \left(1 + \beta_1 + \beta_1^2 + \ldots\right) = Le_{T+1}^* \frac{1}{1-\beta_1}.$$

It follows that $(1-\beta_1)\left(R_{T+1}^* + Le_{T+1}^*\right) \leq Le_{T+1}^*$, or

$$R_{T+1}^* \leq \beta_1 \left(R_{T+1}^* + Le_{T+1}^*\right) = \beta_1 R_T^*.$$

Thus, using (4.21) we get

$$R_{T+1}^* = \beta_1 R_T^*.$$

Repeating the argument, we obtain that

$$R_t^* = \beta_1 R_{t-1}^*, \quad t > T.$$

Therefore, for all $t > T$, $Le_t^* = R_{t-1}^* - R_t^* = (1-\beta_1)R_{t-1}^*$, and

$$\frac{e_{t+1}^*}{e_t^*} = \frac{R_t^*}{R_{t-1}^*} = \beta_1, \quad t > T.$$

We have proved that eventually the extraction rate becomes constant over time and equal to $1-\beta_1$.

Since for $t > T$,

$$\frac{q_{t+1}^*}{1+r_{t+1}^*} R_t^* = q_t^* R_t^* = \beta_1 q_t^* R_{t-1}^*,$$

it follows from (4.20) that

$$\beta_1(1+r_t^*)k_t^* \leq k_{t+1}^*, \quad t > T.$$

Using Lemma 4.2, we obtain that

$$k_{t+1}^* = \beta_1(1+r_t^*)k_t^*, \quad t > T,$$

and hence for $t > T$, $s_t^{j*} = \beta_1(1+r_t^*)s_{t-1}^{j*}$ for $j \in J$, while $s_t^{j*} = 0$ for $j \notin J$. □

Proposition 4.2 is a corollary of Lemma 4.6. Proposition 4.1 also easily follows from Lemma 4.6. If the initial state $\mathcal{I}_0$ is such that

$$\hat{k}_0^j = 0, \qquad \hat{R}_{-1}^j = 0, \qquad j \notin J,$$

then $s_{-1}^j = 0$ for $j \notin J$, and we can take $T = -1$. The sequences $\{r_t^*\}$, $\{w_t^*\}$, and $\{q_t^*\}$ are derived from the known sequences $\{k_t^*\}$ and $\{e_t^*\}$, described in Lemma 4.6.

Thus, in every competitive equilibrium from some time onward only the most patient agents make positive savings, and from this time resources are extracted at the constant rate $\varepsilon^* = 1 - \beta_1$.

4.9.2. Balanced-growth equilibrium

Definition. *A competitive equilibrium*

$$\mathcal{E}^* = \left\{ (c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^* \right\}_{t=0,1,\ldots}$$

starting from a non-degenerate initial state $\mathcal{I}_0$ is called a balanced-growth equilibrium if there exist an equilibrium rate of balanced growth γ^* and an equilibrium extraction rate ε^* such that for $t = 0, 1, \ldots,$

$$c_{t+1}^{j*} = (1+\gamma^*)c_t^{j*}, \qquad s_t^{j*} = (1+\gamma^*)s_{t-1}^{j*}, \qquad j = 1, \ldots, L, \tag{4.22}$$

$$k_{t+1}^* = (1+\gamma^*)k_t^*, \qquad w_{t+1}^* = (1+\gamma^*)w_t^*, \tag{4.23}$$

$$1 + r_t^* = 1 + r^*, \qquad q_{t+1}^* = (1+r^*)\, q_t^*, \tag{4.24}$$

$$e_{t+1}^* = (1-\varepsilon^*)\, e_t^*, \qquad R_t^* = (1-\varepsilon^*)\, R_{t-1}^*. \tag{4.25}$$

The following proposition proves the existence of a balanced-growth equilibrium, and provides its characterization. In particular, it maintains that in every balanced-growth equilibrium less patient agents make no savings.

Proposition 4.3. *A balanced-growth equilibrium*

$$\mathcal{E}^* = \left\{ (c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^* \right\}_{t=0,1,\ldots}$$

starting from a non-degenerate initial state $\mathcal{I}_0 = \{(\hat{k}_0^j)_{j=1}^L, (\hat{R}_{-1}^j)_{j=1}^L\}$ exists if and only if

$$\hat{k}_0^j = 0, \qquad \hat{R}_{-1}^j = 0, \qquad j \notin J, \tag{4.26}$$

$$\alpha_1 A_0 \left(\frac{1}{L} \sum_{j=1}^L \hat{k}_0^j \right)^{\alpha_1 - 1} \left(\frac{1-\beta_1}{L} \sum_{j=1}^L \hat{R}_{-1}^j \right)^{\alpha_3} = (1+\lambda)^{\frac{1}{1-\alpha_1}} \beta_1^{\frac{\alpha_1+\alpha_3-1}{1-\alpha_1}}, \tag{4.27}$$

and (4.22)–(4.25) hold.

Proof. **Necessity**. Suppose that there exists a balanced-growth equilibrium

$$\mathcal{E}^* = \left\{(c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^*\right\}_{t=0,1,\dots}$$

starting from a non-degenerate state $\mathcal{I}_0$. It is a competitive equilibrium which satisfies (4.22)–(4.25) for some r^*, ε^* and γ^*.

Repeating a well-known argument by Becker (1980, 2006), we infer that every balanced-growth equilibrium is characterized by the following properties:

$$s_{t-1}^{j*} = 0, \quad j \notin J, \quad t = 0, 1, \dots, \tag{4.28}$$

$$1 + \gamma^* = \beta_1 (1 + r^*). \tag{4.29}$$

Moreover, comparing the definitions of competitive and balanced-growth equilibria, we obtain that for every balanced-growth equilibrium the following relationships hold:

$$(1+\gamma^*)^{1-\alpha_1} = (1+\lambda)\,(1-\varepsilon^*)^{\alpha_3}, \tag{4.30}$$

$$1 + r^* = \frac{1+\gamma^*}{1-\varepsilon^*}. \tag{4.31}$$

Indeed, (4.30) follows from the fact that

$$1 = \frac{1+r_{t+1}^*}{1+r_t^*} = \frac{A_{t+1}}{A_t}\left(\frac{k_{t+1}^*}{k_t^*}\right)^{\alpha_1-1}\left(\frac{e_{t+1}^*}{e_t^*}\right)^{\alpha_3} = (1+\lambda)\,(1+\gamma^*)^{\alpha_1-1}\,(1-\varepsilon^*)^{\alpha_3}.$$

We also have

$$1 + r^* = \frac{q_{t+1}^*}{q_t^*} = \frac{A_{t+1}}{A_t}\left(\frac{k_{t+1}^*}{k_t^*}\right)^{\alpha_1}\left(\frac{e_{t+1}^*}{e_t^*}\right)^{\alpha_3-1} = (1+\lambda)\,(1+\gamma^*)^{\alpha_1}\,(1-\varepsilon^*)^{\alpha_3-1},$$

which is equivalent to (4.31).

Using (4.29)–(4.31), it is easily checked that

$$1 + r^* = (1+\lambda)^{\frac{1}{1-\alpha_1}}\beta_1^{\frac{\alpha_1+\alpha_3-1}{1-\alpha_1}}. \tag{4.32}$$

It follows from (4.28) that $s_{-1}^{j*} = 0$ for $j \notin J$. Since $\mathcal{I}_0$ is non-degenerate, $\hat{k}_0^j \geq 0$ and $\hat{R}_{-1}^j \geq 0$ for all j, and thus (4.26) holds.[23] Furthermore, a constant over time interest rate is consistent with the definition of a competitive equilibrium if and only if

$$1 + r^* = 1 + r_0^* = \alpha_1 A_0 (k_0^*)^{\alpha_1 - 1} (e_0^*)^{\alpha_3} = \alpha_1 A_0 \left(\frac{1}{L} \sum_{j=1}^{L} \hat{k}_0^j \right)^{\alpha_1 - 1} \left(\frac{1 - \beta_1}{L} \sum_{j=1}^{L} \hat{R}_{-1}^j \right)^{\alpha_3}.$$

Taking into account (4.32), we obtain (4.27).

Sufficiency. Suppose that the initial state $\mathcal{I}_0$ is such that (4.26)–(4.27) hold. Consider the sequence

$$\mathcal{E}^* = \left\{ (c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^* \right\}_{t=0,1,\ldots}$$

starting from $\mathcal{I}_0$ and determined by (4.22)–(4.25).

It is easily checked that this sequence is a competitive equilibrium which is described in Proposition 4.1, with the constant interest rate

$$1 + r_t^* = 1 + r_0^* = \alpha_1 A_0 \left(\frac{1}{L} \sum_{j=1}^{L} \hat{k}_0^j \right)^{\alpha_1 - 1} \left(\frac{1 - \beta_1}{L} \sum_{j=1}^{L} \hat{R}_{-1}^j \right)^{\alpha_3}.$$

Therefore, $\mathcal{E}^*$ is a competitive equilibrium which satisfies (4.22)–(4.25), i.e., a balanced-growth equilibrium. □

It follows that the interest rate r^*, the equilibrium extraction rate ε^*, and the equilibrium rate of balanced growth γ^* are uniquely determined by the parameters of the model and are the same for every balanced-growth equilibrium.

Proposition 4.4. *For every balanced-growth equilibrium,*

$$1 + \gamma^* = (1 + \lambda)^{\frac{1}{1-\alpha_1}} \beta_1^{\frac{\alpha_3}{1-\alpha_1}}, \tag{4.33}$$

$$1 + r^* = \frac{1 + \gamma^*}{\beta_1}, \tag{4.34}$$

$$\varepsilon^* = 1 - \beta_1. \tag{4.35}$$

Proof. It is sufficient to repeat the argument used in the proof of Proposition 4.3. Combining (4.29)–(4.31), we obtain (4.33)–(4.35). □

[23] Recall that a balanced-growth equilibrium is defined up to the distribution of physical capital and natural resources in the structure of each agent's savings. Individual holdings of capital and resources are indeterminate in an equilibrium. However, since we assumed for convenience that the initial state is non-degenerate, and initial savings of less patient agents must be zero, it follows that a balanced-growth equilibrium can start only from the state where individual holdings of capital and resources of less patient agents are zero.

The following proposition maintains that every competitive equilibrium converges in some sense to a balanced-growth equilibrium.

Proposition 4.5. *Every competitive equilibrium starting from an arbitrary non-degenerate initial state satisfies the following asymptotic properties:*

$$\lim_{t\to\infty} 1 + r_t^* = 1 + r^* = \frac{1+\gamma^*}{\beta_1}, \tag{4.36}$$

$$\lim_{t\to\infty} \frac{k_{t+1}^*}{k_t^*} = \lim_{t\to\infty} \frac{w_{t+1}^*}{w_t^*} = 1 + \gamma^*, \tag{4.37}$$

$$\lim_{t\to\infty} \frac{s_{t+1}^{j*}}{s_t^{j*}} = 1 + \gamma^*, \quad j \in J, \tag{4.38}$$

$$\lim_{t\to\infty} \frac{c_{t+1}^{j*}}{c_t^{j*}} = 1 + \gamma^*, \quad j = 1, \ldots, L, \tag{4.39}$$

$$\lim_{t\to\infty} \frac{q_{t+1}^*}{q_t^*} = 1 + r^*, \tag{4.40}$$

where $1 + \gamma^* = (1+\lambda)^{\frac{1}{1-\alpha_1}} \beta_1^{\frac{\alpha_3}{1-\alpha_1}}$.

Proof. It follows from Proposition 4.2 that for $t > T$,

$$\frac{1 + r_{t+1}^*}{1 + r_t^*} = \frac{A_{t+1}}{A_t} \left(\frac{k_{t+1}^*}{k_t^*}\right)^{\alpha_1 - 1} \left(\frac{e_{t+1}^*}{e_t^*}\right)^{\alpha_3} = (1+\lambda)\left(\beta_1(1+r_t^*)\right)^{\alpha_1-1} (\beta_1)^{\alpha_3},$$

and thus

$$1 + r_{t+1}^* = (1+\lambda)(\beta_1)^{\alpha_1+\alpha_3-1} (1+r_t^*)^{\alpha_1}.$$

Iterating, we get

$$1 + r_{t+1+n}^* = (1+\lambda)^{1+\alpha_1+\ldots+\alpha_1^n} (\beta_1)^{(\alpha_1+\alpha_3-1)(1+\alpha_1+\ldots+\alpha_1^n)} (1+r_t^*)^{\alpha_1^{n+1}},$$

and

$$\lim_{n\to\infty} 1 + r_{t+n+1}^* = (1+\lambda)^{\frac{1}{1-\alpha_1}} \beta_1^{\frac{\alpha_1+\alpha_3-1}{1-\alpha_1}}.$$

From Lemma 4.6 we know that $\frac{k_{t+1}^*}{k_t^*} = \beta_1(1+r_t^*)$. Moreover, $\frac{w_t^*}{k_t^*} = \frac{\alpha_2}{\alpha_1}(1+r_t^*)$. Now (4.37) is straightforward. It also follows from Lemma 4.6 that $\frac{s_{t+1}^{j*}}{s_t^{j*}} = \beta_1(1+r_t^*)$ for $j \in J$, which proves (4.38).

Clearly, for $j \in J$,

$$\frac{c_t^{j*}}{k_t^*} = (1-\beta_1)(1+r_t^*)\frac{s_{t-1}^{j*}}{k_t^*} + \frac{w_t^*}{k_t^*},$$

and thus the sequence c_t^{j*}/k_t^* converges to a positive constant as $t \to \infty$. For $j \notin J$, we have $c_t^{j*} = w_t^*$. Thus consumption of all agents asymptotically grows at a constant rate. This proves (4.39).

It remains to note that (4.40) follows from the Hotelling rule. □

4.10. Proofs. Public property regime

4.10.1. Competitive equilibrium under given extraction rates

Suppose that the economy at time τ is in a state $\mathcal{I}_{\tau-1} = \{(\hat{s}^j_{\tau-1})_{j=1}^L, \hat{R}_{\tau-1}\}$, where $(\hat{s}^j_{\tau-1})_{j=1}^L$ are agents' savings and $\hat{R}_{\tau-1}$ is the stock of natural resources. We assume that $\mathcal{I}_{\tau-1}$ is a non-degenerate state, i.e.,

$$\hat{s}^j_{\tau-1} \geq 0, \quad j = 1, \dots, L; \qquad \frac{1}{L}\sum_{j=1}^{L} \hat{s}^j_{\tau-1} > 0; \qquad \hat{R}_{\tau-1} > 0.$$

Suppose we are also given a sequence of extraction rates $\mathbb{E}_\tau = \{\varepsilon_t\}_{t=\tau}^\infty$. We call $\mathbb{E}_\tau$ *non-degenerate* if $0 < \varepsilon_t < 1$ for all $t \geq \tau$, and

$$0 < \liminf_{t\to\infty} \varepsilon_t \leq \limsup_{t\to\infty} \varepsilon_t < 1.$$

In other words, the sequence of extraction rates is non-degenerate if there exists $\delta > 0$ such that for all $t \geq \tau$ the following property holds:

$$\delta \leq \varepsilon_t \leq 1 - \delta.$$

Given a sequence of extraction rates $\mathbb{E}_\tau$ and the resource stock $R_{\tau-1} = \hat{R}_{\tau-1}$, the volume of extraction e_t and the dynamics of the exhaustible resource stock R_t are recursively determined for $t \geq \tau$:

$$e_t = e_t(\mathbb{E}_\tau) = \frac{\varepsilon_t R_{t-1}}{L}, \qquad R_t = R_t(\mathbb{E}_\tau) = (1-\varepsilon_t)R_{t-1}, \qquad t = \tau, \tau+1, \dots. \tag{4.41}$$

We use this notation to emphasize that the sequence of extraction rates determines the volume of extraction and the dynamics of the resource stock.

Definition. *Let $\mathbb{E}_\tau$ be a non-degenerate sequence of extraction rates. A sequence*

$$\mathcal{E}^{**}_\tau = \left\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\right\}_{t=\tau,\tau+1,\dots}$$

is a competitive $\mathbb{E}_\tau$-equilibrium starting from $\mathcal{I}_{\tau-1}$ if

1. *For each* $j = 1, \ldots, L$*, the sequence* $\{c_t^{j**}, s_t^{j**}\}_{t=\tau}^{\infty}$ *is a solution to the following utility maximization problem:*

$$\begin{aligned} &\max \sum_{t=\tau}^{\infty} \beta_j^t \ln c_t^j, \\ &\text{s. t. } c_t^j + s_t^j \leq (1 + r_t)\, s_{t-1}^j + w_t + v_t, \quad t = \tau, \tau + 1, \ldots, \\ &\qquad s_t^j \geq 0, \quad t = \tau, \tau + 1, \ldots \end{aligned} \tag{4.42}$$

at $r_t = r_t^{**}$, $w_t = w_t^{**}$, $v_t = v_t^{**}$, $s_{\tau-1}^j = \hat{s}_{\tau-1}^j$*;*

2. *Aggregate savings are equal to the capital stock:*

$$\sum_{j=1}^{L} s_{t-1}^{j**} = Lk_t^{**}, \quad t = \tau, \tau + 1, \ldots;$$

3. *Capital is paid its marginal product:*

$$1 + r_t^{**} = \alpha_1 A_t (k_t^{**})^{\alpha_1 - 1} (e_t)^{\alpha_3}, \quad t = \tau, \tau + 1, \ldots;$$

4. *Labor is paid its marginal product:*

$$w_t^{**} = \alpha_2 A_t (k_t^{**})^{\alpha_1} (e_t)^{\alpha_3}, \quad t = \tau, \tau + 1, \ldots;$$

5. *The price of natural resources is equal to the marginal product:*

$$q_t^{**} = \alpha_3 A_t (k_t^{**})^{\alpha_1} (e_t)^{\alpha_3 - 1}, \quad t = \tau, \tau + 1, \ldots;$$

6. *Resource income is given by:*

$$v_t^{**} = q_t^{**} e_t, \quad t = \tau, \tau + 1, \ldots.$$

Here we do not suppose that the Hotelling rule holds. The Hotelling rule is an equilibrium condition for the asset market. This is the reason why the Hotelling rule holds in the private property regime, where the stock of natural resources is an asset in which agents can invest. In the public property regime, the resource stock is not an asset, so there is no particular reason for the Hotelling rule to hold. Under some circumstances the rate of change of the resource price is not equal to the interest rate.

It is clear that if

$$\mathcal{E}_0^{**} = \{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\}_{t=0,1,\ldots}$$

is a competitive $\mathbb{E}_0$-equilibrium starting from $\{(\hat{s}_{-1}^j)_{j=1}^L, \hat{R}_{-1}\}$, then for each $\tau = 1, 2, \ldots$, the sequence

$$\mathcal{E}_\tau^{**} = \{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\}_{t=\tau,\tau+1,\ldots}$$

is a competitive $\mathbb{E}_\tau$-equilibrium starting from $\{(s_{\tau-1}^{j**})_{j=1}^L, R_{\tau-1}(\mathbb{E}_0)\}$. In other words, competitive equilibria are time consistent.

There always exists a competitive $\mathbb{E}_\tau$-equilibrium.

Theorem 4.2. *For any non-degenerate state $\mathcal{I}_{\tau-1}$ there exists a competitive $\mathbb{E}_\tau$-equilibrium starting from $\mathcal{I}_{\tau-1}$.*

Proof. See Appendix B. □

The issue with uniqueness is more subtle. We can only conjecture that the competitive equilibrium is unique, but we have no proof of this fact. At the same time, the following proposition maintains that the competitive equilibrium starting from the state where the whole stock of physical capital is owned by the most patient agents is unique.

Proposition 4.6. *Suppose that $\mathcal{I}_{\tau-1}$ is such that $\hat{s}_{\tau-1}^j = 0$ for $j \notin J$. Then there exists a unique competitive $\mathbb{E}_\tau$-equilibrium starting from $\mathcal{I}_{\tau-1}$,*

$$\mathcal{E}_\tau^{**} = \{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\}_{t=\tau,\tau+1,\ldots},$$

which is given for $t = \tau, \tau+1, \ldots$ by

$$c_t^{j**} = (1-\beta_1)(1+r_t^{**})s_{t-1}^{j**} + w_t^{**} + v_t^{**}, \qquad s_t^{j**} = \beta_1(1+r_t^{**})s_{t-1}^{j**}, \qquad j \in J,$$
$$c_t^{j**} = w_t^{**} + v_t^{**}, \qquad s_t^{j**} = 0, \qquad j \notin J,$$
$$k_{t+1}^{**} = \beta_1(1+r_t^{**})k_t^{**}, \qquad 1 + r_t^{**} = \alpha_1 A_t (k_t^{**})^{\alpha_1 - 1}(e_t)^{\alpha_3},$$
$$w_t^{**} = \alpha_2 A_t (k_t^{**})^{\alpha_1}(e_t)^{\alpha_3}, \qquad q_t^{**} = \alpha_3 A_t (k_t^{**})^{\alpha_1}(e_t)^{\alpha_3 - 1}, \qquad v_t^{**} = q_t^{**} e_t,$$

where $s_{\tau-1}^{j**} = \hat{s}_{\tau-1}^j$, and $e_t = e_t(\mathbb{E}_\tau)$.

This case is important because in every competitive $\mathbb{E}_\tau$-equilibrium less patient agents inevitably lose their capital with time. The following proposition verifies that the whole capital stock eventually belongs to the most patient agents.

Proposition 4.7. *Suppose that*

$$\mathcal{E}_\tau^{**} = \left\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\right\}_{t=\tau,\tau+1,\dots}$$

is a competitive $\mathbb{E}_\tau$-equilibrium starting from an arbitrary state $\mathcal{I}_{\tau-1}$. Then there exists a point in time T such that for all $t \geq T$,

$$s_t^{j**} = 0, \quad j \notin J.$$

Proof of Propositions 4.6 and 4.7 is very similar to that of Propositions 4.1 and 4.2. Without loss of generality, let us consider a competitive $\mathbb{E}_0$-equilibrium

$$\mathcal{E}_0^{**} = \left\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\right\}_{t=0,1,\dots}$$

starting from $\mathcal{I}_{-1} = \{(\hat{s}_{-1}^j)_{j=1}^L, \hat{R}_{-1}\}$, and give a sketch of the proof.

Lemma 4.7. *Let $\beta > 0$ be such that for some T*

$$k_{t+1}^{**} > \beta(1 + r_t^{**})k_t^{**} = \beta\alpha_1 A_t (k_t^{**})^{\alpha_1}(e_t)^{\alpha_3}, \quad t > T.$$

If $\beta_j < \beta$, then $s_t^{j**} = 0$ for all sufficiently large t.

Proof. This lemma can be proved in the same way as Lemma 4.1. □

Lemma 4.8.

$$k_{t+1}^{**} \leq \beta_1(1 + r_t^{**})k_t^{**}, \quad t = 0, 1, \dots.$$

Proof. It is sufficient to repeat the argument used in the proof of Lemma 4.2. □

Lemma 4.9.

$$w_{t+1}^{**} \leq \beta_1(1 + r_{t+1}^{**})w_t^{**}, \qquad v_{t+1}^{**} \leq \beta_1(1 + r_{t+1}^{**})v_t^{**}, \qquad t = 0, 1, \dots.$$

Proof. This statement follows from Lemma 4.8. □

Lemma 4.10.

$$s_{t+1}^{j**} \geq \beta_1(1 + r_{t+1}^{**})s_t^{j**}, \quad j \in J, \quad t = -1, 0, \dots.$$

Proof. This lemma can be proved in the same way as Lemma 4.4. □

Lemma 4.11. *For any $\delta > 0$ there exists a point in time T such that for all $t > T$,*

$$k_{t+1}^{**} > \beta_1(1 - \delta)(1 + r_t^{**})k_t^{**}.$$

Proof. This lemma can be proved in the same way as Lemma 4.5. □

Proposition 4.7 follows from Lemmas 4.7 and 4.11. Proposition 4.6 follows directly from Lemma 4.12 which explicitly constructs a competitive $\mathbb{E}_0$-equilibrium starting from the state $\mathcal{I}_{-1}$ such that $\hat{s}^j_{-1} = 0$ for $j \notin J$.

Lemma 4.12. *Suppose that*

$$k_0^{**} = \frac{1}{L}\sum_{j\in J} \hat{s}^j_{-1}, \ \ i.e.,\ \ \hat{s}^j_{-1} = 0,\ \ j \notin J.$$

Then for all $t = 0, 1, \ldots,$

$$k_{t+1}^{**} = \beta_1(1 + r_t^{**})k_t^{**},$$
$$c_t^{j**} = (1-\beta_1)(1 + r_t^{**})s_{t-1}^{j**} + w_t^{**} + v_t^{**}, \qquad s_t^{j**} = \beta_1(1 + r_t^{**})s_{t-1}^{j**}, \qquad j \in J,$$
$$c_t^{j**} = w_t^{**} + v_t^{**}, \qquad s_t^{j**} = 0, \qquad j \notin J.$$

Proof. By Lemma 4.10,

$$\beta_1(1 + r_0^{**})k_0^{**} = \beta_1(1 + r_0^{**})\frac{1}{L}\sum_{j\in J} \hat{s}^j_{-1} \le \frac{1}{L}\sum_{j\in J} s_0^{j**} \le k_1^{**}.$$

At the same time, by Lemma 4.8,

$$k_1^{**} \le \beta_1(1 + r_0^{**})k_0^{**}.$$

Therefore, $k_1^{**} = \beta_1(1 + r_0^{**})k_0^{**}$, and hence $s_0^{j**} = \beta_1(1 + r_0^{**})\hat{s}^j_{-1}$ for $j \in J$, while $s_0^{j**} = 0$ for $j \notin J$. We have proved the lemma for $t = 0$. To prove it for $t = 1, 2, \ldots$, it is sufficient to repeat the argument. □

This completes the proof of Propositions 4.6 and 4.7.

4.10.2. Balanced-growth equilibrium under given extraction rate

Suppose that the sequence of extraction rates is constant,

$$\mathbb{E}_\tau^\varepsilon = \mathbb{E}^\varepsilon = \{\varepsilon, \varepsilon, \varepsilon, \ldots\}.$$

Then, clearly,

$$R_t = (1-\varepsilon)^{t+1-\tau}\hat{R}_{\tau-1}, \qquad e_t = (1-\varepsilon)^{t-\tau}\frac{\varepsilon\hat{R}_{\tau-1}}{L}, \qquad t = \tau, \tau+1, \ldots.$$

Definition. *A competitive* $\mathbb{E}^{\varepsilon}_{\tau}$*-equilibrium*

$$\mathcal{E}^{**}_{\tau} = \left\{(c^{j**}_t)^L_{j=1}, (s^{j**}_t)^L_{j=1}, k^{**}_t, r^{**}_t, w^{**}_t, q^{**}_t, v^{**}_t\right\}_{t=\tau,\tau+1,\ldots}$$

starting from $\mathcal{I}_{\tau-1}$ *is called a balanced-growth* $\mathbb{E}^{\varepsilon}$*-equilibrium if there exist an equilibrium rate of balanced growth* γ^{**} *and the rate of change of the resource price,* π^{**}*, such that for* $t = \tau, \tau+1, \ldots,$

$$c^{j**}_{t+1} = (1+\gamma^{**})c^{j**}_t, \qquad s^{j**}_t = (1+\gamma^{**})s^{j**}_{t-1}, \qquad j = 1, \ldots, L, \tag{4.43}$$

$$k^{**}_{t+1} = (1+\gamma^{**})k^{**}_t, \qquad w^{**}_{t+1} = (1+\gamma^{**})w^{**}_t, \qquad v^{**}_{t+1} = (1+\gamma^{**})v^{**}_t, \tag{4.44}$$

$$q^{**}_{t+1} = (1+\pi^{**})\, q^{**}_t, \qquad 1 + r^{**}_t = 1 + r^{**}. \tag{4.45}$$

The following proposition provides necessary and sufficient conditions for the existence of a balanced-growth $\mathbb{E}^{\varepsilon}$-equilibrium. In particular, this proposition maintains that in a balanced-growth equilibrium only the most patient agents make positive savings and own the whole capital stock.

Proposition 4.8. *Suppose that a constant sequence of extraction rates* $\mathbb{E}^{\varepsilon}$ *is given. A balanced-growth* $\mathbb{E}^{\varepsilon}$*-equilibrium*

$$\mathcal{E}^{**}_{\tau} = \left\{(c^{j**}_t)^L_{j=1}, (s^{j**}_t)^L_{j=1}, k^{**}_t, r^{**}_t, w^{**}_t, q^{**}_t, v^{**}_t\right\}_{t=\tau,\tau+1,\ldots}$$

starting from a non-degenerate state $\mathcal{I}_{\tau-1} = \{(\hat{s}^j_{\tau-1})^L_{j=1}, \hat{R}_{\tau-1}\}$ *exists if and only if*

$$\hat{s}^j_{\tau-1} = 0, \quad j \notin J,$$

$$\alpha_1 A_\tau \left(\frac{1}{L}\sum_{j=1}^{L} \hat{s}^j_{\tau-1}\right)^{\alpha_1-1} \left(\frac{\varepsilon \hat{R}_{\tau-1}}{L}\right)^{\alpha_3} = (1+\lambda)^{\frac{1}{1-\alpha_1}} (1-\varepsilon)^{\frac{\alpha_3}{1-\alpha_1}} \frac{1}{\beta_1},$$

and (4.43)–(4.45) hold.

Proof. It can be proved exactly in the same way as Proposition 4.3. □

The following proposition maintains that the interest rate, the equilibrium rate of balanced growth, and the rate of change of the resource price are determined by the parameters of the model and by the constant over time extraction rate ε.

Proposition 4.9. *Suppose that a constant sequence of extraction rates* $\mathbb{E}^{\varepsilon}$ *is given. In a balanced-growth* $\mathbb{E}^{\varepsilon}$*-equilibrium, the interest rate, the equilibrium rate of balanced growth, and the rate of change of the resource price are determined as follows:*

$$1+\gamma^{**} = (1+\lambda)^{\frac{1}{1-\alpha_1}} (1-\varepsilon)^{\frac{\alpha_3}{1-\alpha_1}}, \qquad 1+\pi^{**} = \frac{1+\gamma^{**}}{1-\varepsilon}, \qquad 1+r^{**} = \frac{1+\gamma^{**}}{\beta_1}.$$

Proof. The proof is similar to that of Proposition 4.4. □

The following proposition maintains that every competitive $\mathbb{E}^{\varepsilon}_{\tau}$-equilibrium under given constant sequence of extraction rates converges in some sense to a balanced-growth $\mathbb{E}^{\varepsilon}$-equilibrium.

Proposition 4.10. *Every competitive $\mathbb{E}^{\varepsilon}_{\tau}$-equilibrium starting from an arbitrary state $\mathcal{I}_{\tau-1}$ satisfies the following asymptotic properties:*

$$\lim_{t\to\infty} 1 + r^{**}_t = 1 + r^{**} = \frac{1+\gamma^{**}}{\beta_1},$$
$$\lim_{t\to\infty} \frac{k^{**}_{t+1}}{k^{**}_t} = \lim_{t\to\infty} \frac{w^{**}_{t+1}}{w^{**}_t} = \lim_{t\to\infty} \frac{v^{**}_{t+1}}{v^{**}_t} = 1 + \gamma^{**},$$
$$\lim_{t\to\infty} \frac{s^{j**}_{t+1}}{s^{j**}_t} = 1 + \gamma^{**}, \quad j \in J,$$
$$\lim_{t\to\infty} \frac{c^{j**}_{t+1}}{c^{j**}_t} = 1 + \gamma^{**}, \quad j = 1, \dots, L,$$
$$\lim_{t\to\infty} \frac{q^{**}_{t+1}}{q^{**}_t} = 1 + \pi^{**},$$

where $1 + \gamma^{**} = (1+\lambda)^{\frac{1}{1-\alpha_1}} (1-\varepsilon)^{\frac{\alpha_3}{1-\alpha_1}}$*, and* $1 + \pi^{**} = \frac{1+\gamma^{**}}{1-\varepsilon}$.

Proof. The proof is similar to that of Proposition 4.5. □

4.10.3. Time τ extraction rate

Before making extraction rates endogenous, let us explore the dependence of a competitive $\mathbb{E}_{\tau}$-equilibrium on the time τ extraction rate.

Suppose we are given a non-degenerate sequence of extraction rates $\mathbb{E}^0_{\tau} = \{\varepsilon^0_t\}^{\infty}_{t=\tau}$. Assume that ε^0_{τ} is replaced by some other extraction rate ε_{τ}, while all future extraction rates remain intact. Clearly, the volumes of extraction and the dynamics of the resource stock before and after this replacement are linked in the following way:

$$\tilde{R}_t = \frac{1-\varepsilon_{\tau}}{1-\varepsilon^0_{\tau}} R_t, \qquad t = \tau, \tau+1, \dots,$$
$$\tilde{e}_{\tau} = \frac{\varepsilon_{\tau}}{\varepsilon^0_{\tau}} e_{\tau}, \quad \tilde{e}_t = \frac{1-\varepsilon_{\tau}}{1-\varepsilon^0_{\tau}} e_t, \qquad t = \tau+1, \tau+2, \dots.$$

A competitive $\mathbb{E}^0_{\tau}$-equilibrium should also change. The change of the competitive $\mathbb{E}^0_{\tau}$-equilibrium and the dependence of a new equilibrium on ε_{τ} is characterized in the following lemma.

Lemma 4.13. *Suppose that for a non-degenerate sequence of extraction rates,* $\mathbb{E}^0_\tau = \{\varepsilon^0_t\}_{t=\tau}^\infty$*, the sequence*

$$\mathcal{E}^{**}_\tau = \left\{(c^{j**}_t)_{j=1}^L, (s^{j**}_t)_{j=1}^L, k^{**}_t, r^{**}_t, w^{**}_t, q^{**}_t, v^{**}_t\right\}_{t=\tau,\tau+1,\dots}$$

is a competitive $\mathbb{E}^0_\tau$*-equilibrium starting from* $\mathcal{I}_{\tau-1} = \{(\hat{s}^j_{\tau-1})_{j=1}^L, \hat{R}_{\tau-1}\}$.

Let

$$\mathbb{E}_\tau = \{\varepsilon_\tau, \varepsilon^0_{\tau+1}, \varepsilon^0_{\tau+2}, \dots\},$$

and

$$\nu_\tau = \frac{1-\varepsilon_\tau}{1-\varepsilon^0_\tau}.$$

Consider the sequence

$$\tilde{\mathcal{E}}_\tau(\varepsilon_\tau) = \left\{(\tilde{c}^j_t(\varepsilon_\tau))_{j=1}^L, (\tilde{s}^j_t(\varepsilon_\tau))_{j=1}^L, \tilde{k}_t(\varepsilon_\tau), \tilde{r}_t(\varepsilon_\tau), \tilde{w}_t(\varepsilon_\tau), \tilde{q}_t(\varepsilon_\tau), \tilde{v}_t(\varepsilon_\tau)\right\}_{t=\tau,\tau+1,\dots},$$

given by

$$\tilde{k}_{\tau+1}(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3} k^{**}_{\tau+1}, \tag{4.46}$$

$$\tilde{k}_{t+1}(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3\alpha_1^{t-\tau}} \nu_\tau^{\alpha_3(1+\alpha_1+\dots+\alpha_1^{t-\tau-1})} k^{**}_{t+1}, \quad t=\tau+1,\tau+2,\dots, \tag{4.47}$$

$$\tilde{w}_\tau(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3} w^{**}_\tau, \tag{4.48}$$

$$\tilde{w}_t(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3\alpha_1^{t-\tau}} \nu_\tau^{\alpha_3(1+\alpha_1+\dots+\alpha_1^{t-\tau-1})} w^{**}_t, \quad t=\tau+1,\tau+2,\dots, \tag{4.49}$$

$$\tilde{v}_\tau(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3} v^{**}_\tau, \tag{4.50}$$

$$\tilde{v}_t(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3\alpha_1^{t-\tau}} \nu_\tau^{\alpha_3(1+\alpha_1+\dots+\alpha_1^{t-\tau-1})} v^{**}_t, \quad t=\tau+1,\tau+2,\dots, \tag{4.51}$$

$$\tilde{c}^j_\tau(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3} c^{j**}_\tau, \quad j=1,\dots,L, \tag{4.52}$$

$$\tilde{c}^j_t(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3\alpha_1^{t-\tau}} \nu_\tau^{\alpha_3(1+\alpha_1+\dots+\alpha_1^{t-\tau-1})} c^{j**}_t, \quad t=\tau+1,\tau+2,\dots, \quad j=1,\dots,L, \tag{4.53}$$

$$\tilde{s}^j_\tau(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3} s^{j**}_\tau, \quad j=1,\dots,L, \tag{4.54}$$

$$\tilde{s}^j_t(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon^0_\tau}\right)^{\alpha_3\alpha_1^{t-\tau}} \nu_\tau^{\alpha_3(1+\alpha_1+\dots+\alpha_1^{t-\tau-1})} s^{j**}_t, \quad t=\tau+1,\tau+2,\dots, \quad j=1,\dots,L, \tag{4.55}$$

$$1+\tilde{r}_\tau(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3}(1+r_\tau^{**}), \tag{4.56}$$

$$1+\tilde{r}_t(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3(\alpha_1-1)\alpha_1^{t-\tau-1}} \nu_\tau^{\alpha_3\alpha_1^{t-\tau-1}}(1+r_t^{**}), \quad t=\tau+1,\tau+2,\ldots, \tag{4.57}$$

$$\tilde{q}_\tau(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3-1} q_\tau^{**}, \tag{4.58}$$

$$\tilde{q}_t(\varepsilon_\tau) = \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1^{t-\tau}} \nu_\tau^{\alpha_3(1+\alpha_1+\ldots+\alpha_1^{t-\tau-1})-1} q_t^{**}, \quad t=\tau+1,\tau+2,\ldots. \tag{4.59}$$

The sequence $\tilde{\mathcal{E}}_\tau(\varepsilon_\tau)$ *is a competitive* $\mathbb{E}_\tau$*-equilibrium starting from the same state* $\mathcal{I}_{\tau-1} = \{(\hat{s}_{\tau-1}^j)_{j=1}^L, \hat{R}_{\tau-1}\}$.

This lemma plays a very important role in our further considerations. If both the competitive $\mathbb{E}_\tau^0$-equilibrium and the competitive $\mathbb{E}_\tau$-equilibrium are unique, then (4.46)–(4.59) provides formulas of transition from the equilibrium before the change of the time τ extraction rate to the equilibrium after the change. If we cannot guarantee the uniqueness of a competitive $\mathbb{E}_\tau$-equilibrium, then the interpretation of this lemma is slightly different. It maintains that after the change of the time τ extraction rate, there exists a competitive equilibrium which is given by (4.46)–(4.59).

Proof. For the simplicity of exposition let us slightly abuse the notation and write simply $\tilde{k}_t$, $\tilde{w}_t$, etc., instead of $\tilde{k}_t(\varepsilon_\tau)$, $\tilde{w}_t(\varepsilon_\tau)$, etc.

Obviously, $\tilde{k}_\tau = k_\tau^{**}$, as the initial state is the same. Directly from (4.46)–(4.47) and (4.54)–(4.55), we get

$$\tilde{k}_{t+1} = \sum_{j=1}^L \tilde{s}_t^j, \quad t=\tau,\tau+1,\ldots.$$

Let us show that capital is paid its marginal product:

$$1+\tilde{r}_\tau = \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3}(1+r_\tau^{**}) = \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3}\alpha_1 A_\tau (e_\tau^{**})^{\alpha_3}(k_\tau^{**})^{\alpha_1-1} = \alpha_1 A_\tau(\tilde{e}_\tau)^{\alpha_3}(\tilde{k}_\tau)^{\alpha_1-1},$$

$$\begin{aligned}
1+\tilde{r}_{\tau+1} &= \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3(\alpha_1-1)} \nu_\tau^{\alpha_3}(1+r_{\tau+1}^{**}) \\
&= \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3(\alpha_1-1)} \nu_\tau^{\alpha_3}\alpha_1 A_{\tau+1}(e_{\tau+1}^{**})^{\alpha_3}(k_{\tau+1}^{**})^{\alpha_1-1} \\
&= \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3(\alpha_1-1)} \nu_\tau^{\alpha_3}\alpha_1 A_{\tau+1}(\tilde{e}_{\tau+1})^{\alpha_3}(\tilde{k}_{\tau+1})^{\alpha_1-1}\nu_\tau^{-\alpha_3}\left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3(1-\alpha_1)} \\
&= \alpha_1 A_{\tau+1}(\tilde{e}_{\tau+1})^{\alpha_3}(\tilde{k}_{\tau+1})^{\alpha_1-1},
\end{aligned}$$

$$
\begin{aligned}
1+\tilde{r}_t &= \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3(\alpha_1-1)\alpha_1^{t-\tau-1}} \nu_\tau^{\alpha_3\alpha_1^{t-\tau-1}}(1+r_t^{**}) \\
&= \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3(\alpha_1-1)\alpha_1^{t-\tau-1}} \nu_\tau^{\alpha_3\alpha_1^{t-\tau-1}}\alpha_1 A_t(e_t^{**})^{\alpha_3}(k_t^{**})^{\alpha_1-1} \\
&= \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3(\alpha_1-1)\alpha_1^{t-\tau-1}} \nu_\tau^{\alpha_3\alpha_1^{t-\tau-1}}\alpha_1 A_t(\tilde{e}_t)^{\alpha_3}(\tilde{k}_t)^{\alpha_1-1}\nu_\tau^{-\alpha_3}\left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{(\alpha_3\alpha_1^{t-\tau-1})(1-\alpha_1)} \times \\
&\times \nu_\tau^{\alpha_3(1+\alpha_1+\ldots+\alpha_1^{t-\tau-2})(1-\alpha_1)} = \alpha_1 A_t(\tilde{e}_t)^{\alpha_3}(\tilde{k}_t)^{\alpha_1-1}, \quad t=\tau+2,\tau+3,\ldots.
\end{aligned}
$$

Similar calculations show that

$$
\tilde{w}_t = \alpha_2 A_t(\tilde{k}_t)^{\alpha_1}(\tilde{e}_t)^{\alpha_3}, \qquad \tilde{q}_t = \alpha_3 A_t(\tilde{k}_t)^{\alpha_1}(\tilde{e}_t)^{\alpha_3-1}, \qquad t=\tau,\tau+1,\ldots,
$$

and it is easy to check that

$$
\tilde{v}_t = \tilde{q}_t\tilde{e}_t, \quad t=\tau,\tau+1,\ldots.
$$

It remains to show that the sequence $\{(\tilde{c}_t^j)_{j=1}^L, (\tilde{s}_t^j)_{j=1}^L\}_{t=\tau}^\infty$ is a solution to the problem

$$
\begin{aligned}
\max \ & \sum_{t=\tau}^{\infty} \beta_j^t \ln c_t^j, \\
\text{s. t.} \ & c_t^j + s_t^j = (1+r_t)s_{t-1}^j + w_t + v_t, \quad t=\tau,\tau+1,\ldots, \\
& s_t^j \geq 0, \quad t=\tau,\tau+1,\ldots,
\end{aligned}
$$

at $r_t = \tilde{r}_t$, $w_t = \tilde{w}_t$, and $v_t = \tilde{v}_t$, or, equivalently, that the following conditions hold ($j=1,\ldots,L$):

$$
\tilde{c}_t^j + \tilde{s}_t^j = (1+\tilde{r}_t)\tilde{s}_{t-1}^j + \tilde{w}_t + \tilde{v}_t, \quad t=\tau,\tau+1,\ldots, \tag{4.60}
$$

$$
\tilde{c}_{t+1}^j \geq \beta_j(1+\tilde{r}_{t+1})\tilde{c}_t^j \ (= \text{if } \tilde{s}_t^j > 0), \quad t=\tau,\tau+1,\ldots, \tag{4.61}
$$

$$
\frac{\beta_j^t \tilde{s}_t^j}{\tilde{c}_t^j} \xrightarrow[t\to\infty]{} 0. \tag{4.62}
$$

To do this, note that the sequence $\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L\}_{t=\tau}^\infty$ is a solution to maximization problem (4.42) and hence satisfies the following conditions:

$$\begin{gathered} c_t^{j**} + s_t^{j**} = (1 + r_t^{**}) s_{t-1}^{j**} + w_t^{**} + v_t^{**}, \quad t = \tau, \tau+1, \dots, \\ c_{t+1}^{j**} \geq \beta_j (1 + r_{t+1}^{**}) c_t^{j**} \ (= \text{if } s_t^{j**} > 0), \quad t = \tau, \tau+1, \dots, \\ \frac{\beta_j^t s_t^{j**}}{c_t^{j**}} \xrightarrow[t\to\infty]{} 0. \end{gathered}$$

Consider (4.60) for $t = \tau$. We have

$$\begin{gathered} \tilde{c}_\tau^j + \tilde{s}_\tau^j = \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3} (c_\tau^{j**} + s_\tau^{j**}) = \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3} \left((1 + r_\tau^{**}) \hat{s}_{\tau-1}^j + w_\tau^{**} + v_\tau^{**}\right) \\ = \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3} (1 + r_\tau^{**}) \hat{s}_{\tau-1}^j + \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3} w_\tau^{**} + \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3} v_\tau^{**} = (1 + \tilde{r}_\tau) \hat{s}_{\tau-1}^j + \tilde{w}_\tau + \tilde{v}_\tau. \end{gathered}$$

Consider (4.60) for $t = \tau + 1$:

$$\begin{gathered} \tilde{c}_{\tau+1}^j + \tilde{s}_{\tau+1}^j = \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1} \nu_\tau^{\alpha_3} (c_{\tau+1}^{j**} + s_{\tau+1}^{j**}) = \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1} \nu_\tau^{\alpha_3} \left((1 + r_{\tau+1}^{**}) s_\tau^{j**} + w_{\tau+1}^{**} + v_{\tau+1}^{**}\right) \\ = \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3(\alpha_1-1)} \nu_\tau^{\alpha_3} (1 + r_{\tau+1}^{**}) \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3} s_\tau^{j**} + \left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1} \nu_\tau^{\alpha_3} (w_{\tau+1}^{**} + v_{\tau+1}^{**}) \\ = (1 + \tilde{r}_{\tau+1}) \tilde{s}_\tau^j + \tilde{w}_{\tau+1} + \tilde{v}_{\tau+1}. \end{gathered}$$

The validity of conditions (4.60) for $t \geq \tau+2$ can be proved similarly. The same arguments prove (4.61) for $t \geq \tau$. Notice also that

$$\lim_{t\to\infty} \frac{\beta_j^t \tilde{s}_t^j}{\tilde{c}_t^j} = \lim_{t\to\infty} \beta_j^t \frac{\left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1^{t-\tau}}}{\left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1^{t-\tau}}} \frac{\nu_\tau^{\alpha_3(1+\alpha_1+\ldots+\alpha_1^{t-\tau-1})}}{\nu_\tau^{\alpha_3(1+\alpha_1+\ldots+\alpha_1^{t-\tau-1})}} \frac{s_t^{j**}}{c_t^{j**}} = \lim_{t\to\infty} \frac{\beta_j^t s_t^{j**}}{c_t^{j**}} = 0.$$

This completes the proof of the lemma. □

4.10.4. Time τ voting equilibrium

We have characterized a competitive equilibrium and a balanced-growth equilibrium under given sequence of extraction rates. Now we make extraction rates endogenous and introduce voting into our model.

Suppose that we start at time τ. The economy is in a non-degenerate state $\mathcal{I}_{\tau-1} = \{(\hat{s}_{\tau-1}^j)_{j=1}^L, \hat{R}_{\tau-1}\}$. Suppose further that agents have some expectations about future extraction rates, represented by a non-degenerate sequence $\{\varepsilon_t^e\}_{t=\tau+1}^\infty$, and they vote over the time τ extraction rate.

For any $\varepsilon_\tau \in (0,1)$, consider the non-degenerate sequence of extraction rates

$$\mathbb{E}_\tau(\varepsilon_\tau) = \{\varepsilon_\tau, \varepsilon^e_{\tau+1}, \varepsilon^e_{\tau+2}, \ldots\}.$$

Let us assume that for any $\varepsilon_\tau \in (0,1)$ there is a unique competitive $\mathbb{E}_\tau(\varepsilon_\tau)$-equilibrium starting from $\mathcal{I}_{\tau-1}$,

$$\mathcal{E}^{**}_\tau(\varepsilon_\tau) = \left\{(c^{j**}_t(\varepsilon_\tau))^L_{j=1}, (s^{j**}_t(\varepsilon_\tau))^L_{j=1}, k^{**}_t(\varepsilon_\tau), r^{**}_t(\varepsilon_\tau), w^{**}_t(\varepsilon_\tau), q^{**}_t(\varepsilon_\tau), v^{**}_t(\varepsilon_\tau)\right\}_{t=\tau,\ldots}.$$

It is clear that $\mathcal{E}^{**}_\tau(\varepsilon_\tau)$ depends on the expectations and on the parameters of the model as well. However, here we underline its dependence only on ε_τ, as it is the value on which agents vote.

Under the uniqueness assumption, agents' preferences over the time τ extraction rate are represented by the following indirect utility functions:

$$\mathcal{U}^j_\tau(\varepsilon_\tau) = \ln c^{j**}_\tau(\varepsilon_\tau) + \beta_j \ln c^{j**}_{\tau+1}(\varepsilon_\tau) + \ldots, \quad j = 1, \ldots, L.$$

Definition. *Suppose that for any $\varepsilon_\tau \in (0,1)$ there is a unique competitive $\mathbb{E}_\tau(\varepsilon_\tau)$-equilibrium starting from $\mathcal{I}_{\tau-1}$, $\mathcal{E}^{**}_\tau(\varepsilon_\tau)$. We call a couple $\{\varepsilon^{**}_\tau, \mathcal{E}^{**}_\tau\}$ a time τ voting equilibrium if ε^{**}_τ is a Condorcet winner in voting on the time τ extraction rate, and $\mathcal{E}^{**}_\tau = \mathcal{E}^{**}_\tau(\varepsilon^{**}_\tau)$.*

Since the functions $\mathcal{U}^j(\varepsilon_\tau)$, $j = 1, \ldots, L$, are strictly concave, the agents' preferences are single-peaked. Hence the median voter theorem applies, and at each point in time there exists a Condorcet winner. Note that the time τ voting equilibrium consists of the time τ voting equilibrium extraction rate ε^{**}_τ and the corresponding competitive equilibrium.

In order to determine a Condorcet winner, let us consider *the preferred time τ extraction rate for agent j*. This is the value ε^j_τ such that

$$\mathcal{U}^j_\tau(\varepsilon^j_\tau) > \mathcal{U}^j_\tau(\varepsilon_\tau) \quad \forall\ \varepsilon_\tau \neq \varepsilon^j_\tau.$$

From Lemma 4.13 we know how the consumption stream of every agent depends on the time τ extraction rate, which allows us to obtain agents' preferred values of time τ extraction rate.

Proposition 4.11. *Suppose that for any $\varepsilon_\tau \in (0,1)$ there is a unique competitive $\mathbb{E}_\tau(\varepsilon_\tau)$-equilibrium starting from $\mathcal{I}_{\tau-1}$. The preferred time τ extraction rate for each agent j is given by*

$$\varepsilon^j_\tau = 1 - \beta_j. \tag{4.63}$$

Proof. Let us take some $\varepsilon_\tau^0 \in (0,1)$, and consider the non-degenerate sequence

$$\mathbb{E}_\tau(\varepsilon_\tau^0) = \{\varepsilon_\tau^0, \varepsilon_{\tau+1}^e, \varepsilon_{\tau+2}^e, \ldots\}.$$

By assumption, there is a unique competitive $\mathbb{E}_\tau(\varepsilon_\tau^0)$-equilibrium

$$\mathcal{E}_\tau^{**}(\varepsilon_\tau^0) = \big\{(c_t^{j**}(\varepsilon_\tau^0))_{j=1}^L, (s_t^{j**}(\varepsilon_\tau^0))_{j=1}^L, k_t^{**}(\varepsilon_\tau^0), r_t^{**}(\varepsilon_\tau^0), w_t^{**}(\varepsilon_\tau^0), q_t^{**}(\varepsilon_\tau^0), v_t^{**}(\varepsilon_\tau^0)\big\}_{t=\tau,\ldots}$$

starting from $\mathcal{I}_{\tau-1}$. We use this equilibrium as a benchmark.

Further, for any $\varepsilon_\tau \in (0,1)$, there is a unique competitive $\mathbb{E}_\tau(\varepsilon_\tau)$-equilibrium

$$\mathcal{E}_\tau^{**}(\varepsilon_\tau) = \big\{(c_t^{j**}(\varepsilon_\tau))_{j=1}^L, (s_t^{j**}(\varepsilon_\tau))_{j=1}^L, k_t^{**}(\varepsilon_\tau), r_t^{**}(\varepsilon_\tau), w_t^{**}(\varepsilon_\tau), q_t^{**}(\varepsilon_\tau), v_t^{**}(\varepsilon_\tau)\big\}_{t=\tau,\ldots}$$

starting from $\mathcal{I}_{\tau-1}$.

From Lemma 4.13 we know that if the time τ extraction rate changes from ε_τ^0 to ε_τ, the benchmark equilibrium $\mathcal{E}_\tau^{**}(\varepsilon_\tau^0)$ transforms to the "new" equilibrium $\mathcal{E}_\tau^{**}(\varepsilon_\tau)$ according to the formulas (4.46)–(4.59). In particular, the consumption stream of agent j is given by (4.52)–(4.53). Therefore, the indirect utility function of agent j in this equilibrium is given by:

$$\begin{aligned}
&\mathcal{U}_\tau^j(\varepsilon_\tau) = \ln c_\tau^{j**}(\varepsilon_\tau) + \beta_j \ln c_{\tau+1}^{j**}(\varepsilon_\tau) + \ldots \\
&= \alpha_3 \ln\left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right) + \ln c_\tau^{j**}(\varepsilon_\tau^0) + \beta_j\alpha_3\alpha_1 \ln\left(\frac{\varepsilon_\tau}{\varepsilon_\tau^0}\right) + \beta_j\alpha_3 \ln\left(\frac{1-\varepsilon_\tau}{1-\varepsilon_\tau^0}\right) + \beta_j \ln c_{\tau+1}^{j**}(\varepsilon_\tau^0) + \ldots \\
&= \Gamma^j + \alpha_3 \ln\varepsilon_\tau(1 + \beta_j\alpha_1 + \beta_j^2\alpha_1^2 + \ldots) \\
&+ \beta_j\alpha_3 \ln(1-\varepsilon_\tau)\big(1 + \beta_j(1+\alpha_1) + \beta_j^2(1+\alpha_1+\alpha_1^2) + \ldots\big) \\
&= \Gamma^j + \frac{\alpha_3}{1-\beta_j\alpha_1}\ln\varepsilon_\tau + \frac{\beta_j\alpha_3}{1-\beta_j}(1 + \beta_j\alpha_1 + \beta_j^2\alpha_1^2 + \ldots)\ln(1-\varepsilon_\tau) \\
&= \Gamma^j + \frac{\alpha_3}{1-\beta_j\alpha_1}\ln\varepsilon_\tau + \frac{\alpha_3}{1-\beta_j\alpha_1}\frac{\beta_j}{1-\beta_j}\ln(1-\varepsilon_\tau),
\end{aligned}$$

where

$$\begin{aligned}
\Gamma^j = \ln\left[\left(\frac{1}{\varepsilon_\tau^0}\right)^{\alpha_3} c_\tau^{j**}(\varepsilon_\tau^0)\right] + \beta_j \ln\left[\left(\frac{1}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1}\left(\frac{1}{1-\varepsilon_\tau^0}\right)^{\alpha_3} c_{\tau+1}^{j**}(\varepsilon_\tau^0)\right] + \ldots \\
+ \beta_j^t \ln\left[\left(\frac{1}{\varepsilon_\tau^0}\right)^{\alpha_3\alpha_1^t}\left(\frac{1}{1-\varepsilon_\tau^0}\right)^{\alpha_3(1+\alpha_1+\ldots+\alpha_1^{t-1})} c_{\tau+t}^{j**}(\varepsilon_\tau^0)\right] + \ldots.
\end{aligned}$$

is the term that depends on the parameters of the model and on the characteristics of the benchmark equilibrium $\mathcal{E}_\tau^{**}(\varepsilon_\tau^0)$ (the extraction rate ε_τ^0 and the consumption stream), but does not depend on ε_τ, on which agents vote. Since the benchmark equilibrium exists, $-\infty < \Gamma^j < +\infty$, and hence the indirect utility function of each agent is well-defined.

When voting on ε_τ, agent j maximizes her indirect utility $\mathcal{U}^j_\tau(\varepsilon_\tau)$, i.e., solves

$$\frac{\partial \mathcal{U}^j_\tau(\varepsilon_\tau)}{\partial \varepsilon_\tau} = 0.$$

This equation can be rewritten as

$$\frac{1}{\varepsilon_\tau} - \frac{\beta_j}{1-\beta_j}\frac{1}{1-\varepsilon_\tau} = 0.$$

The solution to this equation is $\varepsilon^j_\tau = 1 - \beta_j$. □

Proposition 4.11 maintains that the preferred time τ extraction rate for every agent is constant over time and depends only on this agent's discount factor. In particular, the preferred time τ extraction rate for agent j is time- and expectations-independent.

Now it is straightforward to see that the Condorcet winner in voting on the time τ extraction rate is

$$\varepsilon^{**}_\tau = 1 - \beta_{med},$$

where β_{med} is the median discount factor. Thus the following theorem takes place.

Theorem 4.3. *Suppose that for any $\varepsilon_\tau \in (0,1)$ there is a unique competitive $\mathbb{E}_\tau(\varepsilon_\tau)$-equilibrium starting from $\mathcal{I}_{\tau-1}$. Then there exists a unique time τ voting equilibrium $\{\varepsilon^{**}_\tau, \mathcal{E}^{**}_\tau\}$. The equilibrium extraction rate is constant over time and given by*

$$\varepsilon^{**}_\tau = \varepsilon^{**} = 1 - \beta_{med}. \tag{4.64}$$

4.10.5. Intertemporal voting equilibrium

Suppose we are given an initial state $\mathcal{I}_{-1} = \{(\hat{s}^j_{-1})^L_{j=1}, \hat{R}_{-1}\}$ and a non-degenerate sequence of extraction rates $\mathbb{E}^{**} = \mathbb{E}^{**}_0 = \{\varepsilon^{**}_t\}^\infty_{t=0}$. Therefore, the volumes of extraction and the dynamics of the resource stock are also known:

$$e^{**}_t = e_t(\mathbb{E}^{**}), \qquad R^{**}_t = R_t(\mathbb{E}^{**}), \qquad t = 0, 1, \ldots.$$

Let

$$\mathcal{E}^{**}_0 = \left\{(c^{j**}_t)^L_{j=1}, (s^{j**}_t)^L_{j=1}, k^{**}_t, r^{**}_t, w^{**}_t, q^{**}_t, v^{**}_t\right\}_{t=0,1,\ldots}$$

be a competitive $\mathbb{E}^{**}$-equilibrium starting from $\mathcal{I}_{-1}$. Let for $\tau = 1, 2, \ldots$

$$\mathcal{E}^{**}_\tau = \left\{(c^{j**}_t)^L_{j=1}, (s^{j**}_t)^L_{j=1}, k^{**}_t, r^{**}_t, w^{**}_t, q^{**}_t, v^{**}_t\right\}_{t=\tau,\tau+1,\ldots}$$

be the corresponding tail of $\mathcal{E}_0^{**}$, which is a competitive $\mathbb{E}_\tau^{**}$-equilibrium starting from $\mathcal{I}_{\tau-1}^{**} = \{(s_{\tau-1}^{j**})_{j=1}^L, R_{\tau-1}^{**}\}$.

Definition. *We call a couple $\{\mathbb{E}^{**}, \mathcal{E}_0^{**}\}$ an intertemporal voting equilibrium starting from $\mathcal{I}_{-1}$ if for each time $\tau = 0, 1, \ldots$, a couple $\{\varepsilon_\tau^{**}, \mathcal{E}_\tau^{**}\}$ is a time τ voting equilibrium starting from $\mathcal{I}_{\tau-1}^{**}$ under perfect foresight about future extraction rates ($\varepsilon_t^e = \varepsilon_t^{**},\ t = \tau+1, \tau+2, \ldots$).*

The following theorem provides the characterization of the sequence of extraction rates in every intertemporal voting equilibrium.

Theorem 4.4. *In every intertemporal voting equilibrium $\{\mathbb{E}^{**}, \mathcal{E}_0^{**}\}$ the sequence of extraction rates $\mathbb{E}^{**}$ is constant over time and given by*

$$\mathbb{E}^{**} = \mathbb{E}^{\varepsilon^{**}} = \{\varepsilon^{**}, \varepsilon^{**}, \ldots\}, \tag{4.65}$$

where ε^{**} is defined by (4.64).

Proof. The sequence of extraction rates in every intertemporal voting equilibrium is a sequence of time τ equilibrium extraction rates. It follows from Theorem 4.3 that every equilibrium extraction rate is constant and given by (4.64). □

The answer to the question about the existence and uniqueness of an intertemporal voting equilibrium is provided by the following theorem. It states that if the initial state is such that the whole capital stock belongs to the most patient agents, then an intertemporal voting equilibrium exists and is unique.

Theorem 4.5. *Suppose that the initial state $\mathcal{I}_{-1}$ is such that $\hat{s}_{-1}^j = 0$ for $j \notin J$. Then there exists a unique intertemporal voting equilibrium $\{\mathbb{E}^{**}, \mathcal{E}_0^{**}\}$ starting from $\mathcal{I}_{-1}$. The equilibrium sequence of extraction rates $\mathbb{E}^{**}$ is constant over time and given by (4.65), and $\mathcal{E}_0^{**}$ is a unique competitive $\mathbb{E}^{**}$-equilibrium starting from $\mathcal{I}_{-1}$, as described in Proposition 4.6.*

Proof. It follows from Proposition 4.6 and Theorem 4.4. □

4.10.6. Balanced-growth voting equilibrium

Definition. *An intertemporal voting equilibrium $\{\mathbb{E}^{**}, \mathcal{E}_0^{**}\}$ starting from $\mathcal{I}_{-1}$ is called a balanced-growth voting equilibrium if $\mathcal{E}_0^{**}$ is a balanced-growth $\mathbb{E}^{\varepsilon^{**}}$-equilibrium starting from $\mathcal{I}_{-1}$, where ε^{**} is given by (4.64).*

The following theorem maintains that if at the initial instant the whole capital stock belongs to the most patient agents, then the intertemporal voting equilibrium converges to a balanced-growth voting equilibrium.

Theorem 4.6. *Suppose that the initial state $\mathcal{I}_{-1}$ is such that $\hat{s}^j_{-1} = 0$ for $j \notin J$. A unique intertemporal voting equilibrium starting from $\mathcal{I}_{-1}$ satisfies the following asymptotic properties:*

$$\lim_{t\to\infty} 1 + r_t^{**} = 1 + r^{**} = \frac{1+\gamma^{**}}{\beta_1}, \tag{4.66}$$

$$\lim_{t\to\infty} \frac{k_{t+1}^{**}}{k_t^{**}} = \lim_{t\to\infty} \frac{w_{t+1}^{**}}{w_t^{**}} = \lim_{t\to\infty} \frac{v_{t+1}^{**}}{v_t^{**}} = 1 + \gamma^{**}, \tag{4.67}$$

$$\lim_{t\to\infty} \frac{s_{t+1}^{j**}}{s_t^{j**}} = 1 + \gamma^{**}, \quad j \in J, \tag{4.68}$$

$$\lim_{t\to\infty} \frac{c_{t+1}^{j**}}{c_t^{j**}} = 1 + \gamma^{**}, \quad j = 1, \ldots, L, \tag{4.69}$$

$$\lim_{t\to\infty} \frac{q_{t+1}^{**}}{q_t^{**}} = 1 + \pi^{**}, \tag{4.70}$$

where

$$1 + \gamma^{**} = (1+\lambda)^{\frac{1}{1-\alpha_1}} (\beta_{med})^{\frac{\alpha_3}{1-\alpha_1}}, \tag{4.71}$$

and

$$1 + \pi^{**} = \frac{1+\gamma^{**}}{\beta_{med}}. \tag{4.72}$$

Proof. It follows from Proposition 4.10 and Theorem 4.5. □

4.10.7. Generalized intertemporal voting equilibria

Our definition of an intertemporal voting equilibrium is given under the assumption of uniqueness of a competitive $\mathbb{E}_\tau(\varepsilon_\tau)$-equilibrium for any $\varepsilon_\tau \in (0,1)$. This assumption is crucial in the statement of Theorem 4.3 about the constant equilibrium extraction rate. Moreover, we obtained the existence and uniqueness of an intertemporal voting equilibrium (Theorem 4.5) only for the case in which the underlying competitive equilibria are unique. Thus to guarantee the mere existence of an intertemporal voting equilibrium, we have to prove the uniqueness of a competitive $\mathbb{E}_\tau(\varepsilon_\tau)$-equilibrium starting from an arbitrary state $\mathcal{I}_{\tau-1}$ for any $\varepsilon_\tau \in (0,1)$, which is not an easy task.

Let us discuss the general case in which the competitive $\mathbb{E}_\tau(\varepsilon_\tau)$-equilibrium starting from an arbitrary state $\mathcal{I}_{\tau-1}$ is not necessarily unique. The difficulty here is that we cannot unambiguously define agents' indirect utility functions and obtain from them agents' preferred values of extraction rates. However, if we apply the technique proposed by Borissov et al. (2014b), we can get around this difficulty. Namely, let us impose an additional assumption on the beliefs of agents. Assume that agents simply act as if a competitive $\mathbb{E}_\tau(\varepsilon_\tau)$-equilibrium is unique, and do not take into account the possible multiplicity of equilibria.

Formally, let

$$\mathbb{E}^{**} = \mathbb{E}_0^{**} = \{\varepsilon_t^{**}\}_{t=0}^{\infty},$$

and

$$e_t^{**} = e_t(\mathbb{E}^{**}), \qquad R_t^{**} = R_t(\mathbb{E}^{**}), \quad t = 0, 1, \ldots.$$

Consider a competitive $\mathbb{E}_0^{**}$-equilibrium

$$\mathcal{E}_0^{**} = \left\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\right\}_{t=0,1,\ldots}$$

starting from $\mathcal{I}_{-1} = \left\{(\hat{s}_{-1}^j)_{j=1}^L, \hat{R}_{-1}\right\}$. Let also for $\tau = 1, 2, \ldots$,

$$\mathcal{E}_\tau^{**} = \left\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\right\}_{t=\tau,\tau+1,\ldots}$$

be the corresponding tail of $\mathcal{E}_0^{**}$.

Suppose that the economy has settled on $\mathcal{E}_0^{**}$. At each time τ, when the economy is in the state $\mathcal{I}_{\tau-1}^{**} = \{(s_{\tau-1}^{j**})_{j=1}^L, R_{\tau-1}^{**}\}$, agents are asked to vote over the time τ extraction rate. To do this, agents' indirect utility functions should be unambiguously specified. Originally this was done under the assumption of uniqueness of the competitive $\mathbb{E}_\tau^{**}$-equilibrium starting from $\mathcal{I}_{\tau-1}^{**}$. Now let us instead assume that *when voting on the time τ extraction rate, all agents believe that if ε_τ^{**} is replaced by the other extraction rate ε_τ, then the economy will settle on the path $\tilde{\mathcal{E}}_\tau(\varepsilon_\tau)$, which is linked with the "initial" equilibrium $\mathcal{E}_\tau^{**}$ in the way described in Lemma 4.13.*

Recall that under the uniqueness assumption, the interpretation of Lemma 4.13 is simple. After changing the time τ extraction rate from ε_τ^{**} to ε_τ, a unique competitive $\mathbb{E}^{**}$-equilibrium also changes, and becomes a unique competitive $\mathbb{E}_\tau$-equilibrium, described in Lemma 4.13. Here the interpretation is slightly different. After changing the time τ extraction rate, the competitive $\mathbb{E}^{**}$-equilibrium can change unpredictably, and the economy can settle on one of multiple $\mathbb{E}_\tau$-equilibria. Under our assumption about agents' beliefs, agents ignore the possible multiplicity of equilibria and believe that after the change of the time τ extraction rate, the economy settles on the path $\tilde{\mathcal{E}}_\tau(\varepsilon_\tau)$, which is described in Lemma 4.13.

Under this additional assumption, agents' indirect utility functions, which represent their preferences over the time τ extraction rate, can be defined unambiguously as follows:

$$\mathcal{U}_\tau^j(\varepsilon_\tau) = \ln \tilde{c}_\tau^j(\varepsilon_\tau) + \beta_j \ln \tilde{c}_{\tau+1}^j(\varepsilon_\tau) + \ldots, \quad j = 1, \ldots, L,$$

where the sequence $\{\tilde{c}_\tau^j(\varepsilon_\tau), \tilde{c}_{\tau+1}^j(\varepsilon_\tau), \ldots\}$ is constructed according to (4.52)–(4.53).

Definition. *If for each $t = 0, 1, \ldots$ there is a Condorcet winner in voting on ε_t described above, and it coincides with ε_t^{**}, then we call a couple $\{\mathbb{E}^{**}, \mathcal{E}_0^{**}\}$ a generalized intertemporal voting equilibrium starting from $\mathcal{I}_{-1}$.*

Clearly, any intertemporal voting equilibrium is a generalized intertemporal voting equilibrium. Moreover, any generalized intertemporal voting equilibrium starting from an initial state where the whole capital stock belongs to the most patient agents is an intertemporal voting equilibrium. Under the additional assumption about agents' beliefs, there always exists a generalized intertemporal voting equilibrium starting from an arbitrary initial state.

Theorem 4.7. *For any non-degenerate initial state there exists a generalized intertemporal voting equilibrium $\{\mathbb{E}^{**}, \mathcal{E}_0^{**}\}$ starting from this state. The equilibrium sequence of extraction rates is constant over time and given by (4.65).*

Proof. It is sufficient to repeat the argument used in the proof of Theorem 4.4, and refer to Theorem 4.2. □

Furthermore, every generalized intertemporal voting equilibrium converges to a balanced-growth voting equilibrium.

Theorem 4.8. *Every generalized intertemporal voting equilibrium starting from an arbitrary initial state satisfies asymptotic properties (4.66)–(4.70), where γ^{**} and π^{**} are given by (4.71) and (4.72) respectively.*

Proof. It follows from Proposition 4.10 and Theorem 4.7. □

4.11. Proofs. Private property regime with capital taxation

Consider a competitive equilibrium in the private property regime, and assume in addition that capital income paid by competitive firms to the capital holders is taxed at some constant rate θ and the revenue is lump-sum redistributed among all agents.

The definition of competitive equilibrium with a capital income tax,

$$\mathcal{E}^* = \left\{(c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^*\right\}_{t=0,1,\ldots},$$

repeats the definition of competitive equilibrium in Subsection 4.9.1, except for the following changes:

1′. *For each* $j = 1, \ldots, L$, *the sequence* $\{c_t^{j*}, s_t^{j*}\}_{t=0}^{\infty}$ *is a solution to the following utility maximization problem:*

$$\begin{aligned} \max\ & \sum_{t=0}^{\infty} \beta_j^t \ln c_t^j, \\ \text{s. t.}\ & c_t^j + s_t^j \leq (1+r_t)(1-\theta)\, s_{t-1}^j + w_t + \theta\,(1+r_t)\, k_t \quad t = 0, 1, \ldots, \\ & s_t^j \geq 0, \quad t = 0, 1, \ldots \end{aligned}$$

at $r_t = r_t^*$, $w_t = w_t^*$, $k_t = k_t^*$, *and* $s_{-1}^j = \frac{q_0^*}{(1+r_0^*)(1-\theta)} \hat{R}_{-1}^j + \hat{k}_0^j$;

5′. *The Hotelling rule takes the form*

$$q_{t+1}^* = (1 + r_{t+1}^*)(1-\theta) q_t^*, \quad t = 0, 1, \ldots;$$

7′. *Aggregate savings are equal to investment into physical capital and natural resources:*

$$\sum_{j=1}^{L} s_t^{j*} = \frac{q_{t+1}^*}{(1+r_{t+1}^*)(1-\theta)} R_t^* + L k_{t+1}^*, \quad t = 0, 1, \ldots.$$

Taking into account the new form of the Hotelling rule, we can define a balanced-growth equilibrium with capital taxation along the lines of the definition of a balanced-growth equilibrium in Subsection 4.9.2. Slightly modifying the arguments from the proof of Propositions 4.3 and 4.4, we can provide a characterization of a balanced-growth equilibrium with capital taxation.

Proposition 4.12. *For every balanced-growth equilibrium with capital taxation,*

$$1 + \gamma^* = (1+\lambda)^{\frac{1}{1-\alpha_1}} \beta_1^{\frac{\alpha_3}{1-\alpha_1}}, \tag{4.73}$$

$$(1 + r^*)(1-\theta) = \frac{1+\gamma^*}{\beta_1}, \tag{4.74}$$

$$\varepsilon^* = 1 - \beta_1. \tag{4.75}$$

Proof. A balanced-growth equilibrium with capital taxation

$$\mathcal{E}^* = \left\{ (c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^* \right\}_{t=0,1,\ldots}$$

is a competitive equilibrium with capital taxation in which real variables grow at a constant rate γ^*, while the interest rate r^* and extraction rate ε^* are constant over time. In a competitive equilibria with capital taxation the post-tax interest rate received by agents is equal to the pre-tax gross interest rate $(1 + r^*)$ multiplied by $(1 - \theta)$. Repeating the argument by Becker (1980, 2006), we obtain that a balanced-growth equilibrium with capital taxation is characterized as follows:

$$s_{t-1}^{j*} = 0, \quad j \notin J, \quad t = 0, 1, \ldots,$$
$$1 + \gamma^* = \beta_1 (1 + r^*)(1 - \theta). \tag{4.76}$$

Moreover, since in a balanced-growth equilibrium with capital taxation the extraction rate is constant,

$$1 = \frac{1 + r_{t+1}^*}{1 + r_t^*} = \frac{A_{t+1}}{A_t} \left(\frac{k_{t+1}^*}{k_t^*} \right)^{\alpha_1 - 1} \left(\frac{e_{t+1}^*}{e_t^*} \right)^{\alpha_3} = (1 + \lambda) \left(1 + \gamma^*\right)^{\alpha_1 - 1} \left(1 - \varepsilon^*\right)^{\alpha_3},$$

and we get

$$\left(1 + \gamma^*\right)^{1 - \alpha_1} = (1 + \lambda) \left(1 - \varepsilon^*\right)^{\alpha_3}. \tag{4.77}$$

We also have

$$(1 + r^*)(1 - \theta) = \frac{q_{t+1}^*}{q_t^*} = \frac{A_{t+1}}{A_t} \left(\frac{k_{t+1}^*}{k_t^*} \right)^{\alpha_1} \left(\frac{e_{t+1}^*}{e_t^*} \right)^{\alpha_3 - 1} = (1 + \lambda) \left(1 + \gamma^*\right)^{\alpha_1} \left(1 - \varepsilon^*\right)^{\alpha_3 - 1},$$

and it follows that

$$(1 + r^*)(1 - \theta) = \frac{1 + \gamma^*}{1 - \varepsilon^*}. \tag{4.78}$$

Comparing (4.76) and (4.78), we immediately obtain (4.75). Now (4.74) follows from (4.75) and (4.78), while (4.73) follows from (4.75) and (4.77). □

It follows that the long-run growth rate, the post-tax interest rate and the extraction rate in the model with a capital income tax are the same as in the model without capital tax.

A. Existence of a competitive equilibrium in the private property regime

The existence of a competitive equilibrium in the general equilibrium Ramsey-type model with private property over exhaustible natural resources is established in the following theorem.

Theorem A.1. *For any initial state $\mathcal{I}_0$ there exists a competitive equilibrium starting from $\mathcal{I}_0$.*

Theorem A.1 is proved below following the ideas presented in Borissov and Dubey (2015). The existence of equilibrium in the considered Ramsey-type model can be also proved along the lines of Becker et al. (1991). See also Becker et al. (2015a) and Le Van and Pham (2016) for similar proofs of the existence of equilibria in Ramsey-type models with heterogeneous agents and borrowing constraints.

Proof. We divide the proof of the theorem into two steps. First we show the existence of a competitive equilibrium in the finite horizon model. We prove that for any $T > 0$ there exists a finite T-period competitive equilibrium. Second, we construct a candidate for a competitive equilibrium in the infinite horizon model by applying some kind of diagonalization procedure to the sequence of finite T-period equilibrium paths, and then prove that this candidate is indeed a competitive equilibrium in the infinite horizon model.

Step I. Competitive equilibrium in the finite horizon model.

Let us define a finite T-period competitive equilibrium along the lines of the above definition.

Definition A.1. *A finite T-period competitive equilibrium starting from the initial state $\mathcal{I}_0$ is a sequence*

$$\mathcal{E}_T^* = \left\{(c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^*\right\}_{t=0,1,\ldots,T}$$

such that

1. *For each* $j = 1, \ldots, L$, *the sequence* $\{c_t^{j*}, s_t^{j*}\}_{t=0}^{T}$ *is a solution to the following utility maximization problem:*

$$\begin{aligned} &\max \sum_{t=0}^{T} \beta_j^t \ln c_t^j, \\ &\text{s. t.} \;\; c_t^j + s_t^j \leq (1 + r_t)\, s_{t-1}^j + w_t, \quad t = 0, 1, \ldots, T, \\ &\qquad s_t^j \geq 0, \quad t = 0, 1, \ldots, T, \end{aligned} \tag{A.1}$$

at $r_t = r_t^*$, $w_t = w_t^*$, *and*

$$s_{-1}^j = \frac{q_0^*}{1 + r_0^*} \hat{R}_{-1}^j + \hat{k}_0^j;$$

2. *Capital is paid its marginal product:*

$$1 + r_t^* = \alpha_1 A_t (k_t^*)^{\alpha_1 - 1} (e_t^*)^{\alpha_3}, \quad t = 0, 1, \ldots, T,$$

where $k_0^* = \frac{1}{L} \sum_{j=1}^{L} \hat{k}_0^j$;

3. *Labor is paid its marginal product:*

$$w_t^* = \alpha_2 A_t (k_t^*)^{\alpha_1} (e_t^*)^{\alpha_3}, \quad t = 0, 1, \ldots, T;$$

4. *The price of natural resources is equal to the marginal product:*

$$q_t^* = \alpha_3 A_t (k_t^*)^{\alpha_1} (e_t^*)^{\alpha_3 - 1}, \quad t = 0, 1, \ldots, T;$$

5. *The Hotelling rule holds:*

$$q_{t+1}^* = (1 + r_{t+1}^*) q_t^*, \quad t = 0, 1, \ldots, T - 1;$$

6. *The natural balance of exhaustible resources is fulfilled:*

$$R_t^* = R_{t-1}^* - L e_t^*, \quad t = 0, 1, \ldots T,$$

where $R_{-1}^* = \sum_{j=1}^{L} \hat{R}_{-1}^j$, *and* $R_t \geq 0$, $e_t \geq 0$, *for all* $t = 0, 1, \ldots, T$;

7. *Total agents' savings are equal to the investment into physical capital and natural resources:*

$$\sum_{j=1}^{L} s_t^{j*} = \frac{q_{t+1}^*}{1 + r_{t+1}^*} R_t^* + L k_{t+1}^*, \quad t = 0, 1, \ldots, T - 1.$$

It is clear that $\{c_t^{j*}, s_t^{j*}\}_{t=0}^T$ is a solution to (A.1) if and only if it satisfies the following conditions:

$$c_t^{j*} + s_t^{j*} = (1 + r_t)s_{t-1}^{j*} + w_t^*, \quad t = 0, 1, \ldots, T, \tag{A.2}$$

$$c_{t+1}^{j*} \geq \beta_j(1 + r_{t+1}^*)c_t^{j*} \ (= \text{if } s_t^{j*} > 0), \quad t = 0, 1, \ldots, T-1, \tag{A.3}$$

$$s_T^{j*} = 0.$$

Let ε_t^* be the extraction rate at time t:

$$\varepsilon_t^* = \frac{Le_t^*}{R_{t-1}^*}.$$

The existence of a competitive equilibrium in the finite horizon model is shown via the following steps. First we present some preliminary definitions and results that will be useful in what follows. Second, we reduce our finite horizon model to a game, and show that there exists a Nash equilibrium in this game. Third, we prove that a Nash equilibrium in the game that represents our model determines a competitive equilibrium in the finite horizon model.

Step I.1. Preliminaries

We use the notation

$$e = e(\varepsilon, R) := \frac{\varepsilon R}{L},$$

for the volume of extraction as depending on the extraction rate ε and resource stock R, and the notation

$$f(k, e, A) := Ak^{\alpha_1}e^{\alpha_3},$$
$$1 + r(k, e, A) := \alpha_1 Ak^{\alpha_1 - 1}e^{\alpha_3},$$
$$w(k, e, A) := \alpha_2 Ak^{\alpha_1}e^{\alpha_3},$$
$$q(k, e, A) := \alpha_3 Ak^{\alpha_1}e^{\alpha_3 - 1},$$

for the output (production function), interest rate, wage rate, and resource price as depending on the capital stock k, the volume of extraction e and total factor productivity A. It is clear that for all k, e, A,

$$(1 + r(k, e, A))k + w(k, e, A) + q(k, e, A)e = f(k, e, A), \tag{A.4}$$

$$f(k, e, A) = \frac{(1 + r(k, e, A))k}{\alpha_1} = \frac{w(k, e, A)}{\alpha_2} = \frac{q(k, e, A)e}{\alpha_3}. \tag{A.5}$$

In particular,

$$\frac{q(k,e,A)}{1+r(k,e,A)} = \frac{\alpha_3}{\alpha_1}\frac{k}{e} = \frac{\alpha_3}{\alpha_1}\frac{k}{\varepsilon}\frac{L}{R}. \tag{A.6}$$

Denote

$$\tilde{\varepsilon} := \frac{\alpha_3(1-\beta_1)}{1-(\alpha_1+\alpha_2)(1-\beta_1)},$$
$$\bar{\varepsilon} := \frac{1}{1+\beta_L(1-\beta_1)^2}.$$

Let

$$\begin{aligned} \tilde{R}_{-1} &:= \hat{R}_{-1}, \\ \tilde{R}_t &:= (1-\bar{\varepsilon})\tilde{R}_{t-1}, \quad t=0,1,\dots, \\ \tilde{e}_t &:= \frac{\tilde{\varepsilon}\tilde{R}_{t-1}}{L}, \quad t=0,1,\dots, \\ \bar{e} &:= \frac{\hat{R}_{-1}}{L}. \end{aligned} \tag{A.7}$$

Denote

$$1+\bar{g} = \max\left\{(1+\lambda)^{\frac{1}{1-\alpha_1}}, (1+\lambda)^{\frac{1}{1-\alpha_1}}\left(\frac{\bar{\varepsilon}(1-\tilde{\varepsilon})}{\tilde{\varepsilon}}\right)^{\frac{\alpha_3}{1-\alpha_1}}\right\},$$

where λ is the growth rate of the total factor productivity:

$$A_t = (1+\lambda)A_{t-1} = (1+\lambda)^t A_0. \tag{A.8}$$

Let also

$$1+\tilde{g} = \min\left\{(1+\lambda)^{\frac{1}{1-\alpha_1}}\left(\frac{\tilde{\varepsilon}(1-\bar{\varepsilon})}{\bar{\varepsilon}}\right)^{\frac{\alpha_3}{1-\alpha_1}}, \frac{A_0(\tilde{e}_0)^{\alpha_3}}{(\hat{k}_0)^{1-\alpha_1}}\right\},$$

and

$$1+\underline{g} = \beta_L\alpha_1(1+\tilde{g}).$$

It is clear that

$$\begin{aligned} 1+\bar{g} &\geq (1+\lambda)^{\frac{1}{1-\alpha_1}}, \\ 1+\bar{g} &\geq (1+\lambda)^{\frac{1}{1-\alpha_1}}\left(\tfrac{\bar{\varepsilon}(1-\tilde{\varepsilon})}{\tilde{\varepsilon}}\right)^{\frac{\alpha_3}{1-\alpha_1}}, \end{aligned} \tag{A.9}$$

$$\begin{aligned} 1+\tilde{g} &\leq \frac{A_0(\tilde{e}_0)^{\alpha_3}}{(\hat{k}_0)^{1-\alpha_1}}, \\ 1+\tilde{g} &\leq (1+\lambda)^{\frac{1}{1-\alpha_1}}\left(\tfrac{\tilde{\varepsilon}(1-\bar{\varepsilon})}{\bar{\varepsilon}}\right)^{\frac{\alpha_3}{1-\alpha_1}}, \end{aligned} \tag{A.10}$$

and

$$1+\bar{g} > 1+\tilde{g} > 1+\underline{g}. \tag{A.11}$$

Suppose that $\bar{\kappa} > 0$ is given by

$$(1+\tilde{g})\bar{\kappa} = (\bar{\kappa})^{\alpha_1}, \tag{A.12}$$

Let the sequence $\{\bar{k}_t\}$ be given by

$$\bar{k}_{t+1} = (1+\bar{g})\bar{k}_t,$$

where

$$\bar{k}_0 = \bar{\kappa}(A_0 \bar{e}^{\alpha_3})^{\frac{1}{1-\alpha_1}}.$$

We show that

$$\bar{\kappa}(A_t(\bar{e})^{\alpha_3})^{\frac{1}{1-\alpha_1}} \leq \bar{k}_t. \tag{A.13}$$

Due to (A.8), (A.9), and the choice of $\bar{k}_0$, we get

$$\bar{\kappa}(A_t(\bar{e})^{\alpha_3})^{\frac{1}{1-\alpha_1}} = \bar{\kappa}\left((1+\lambda)^t A_0(\bar{e})^{\alpha_3}\right)^{\frac{1}{1-\alpha_1}} \leq \bar{\kappa}\left(A_0(\bar{e})^{\alpha_3}\right)^{\frac{1}{1-\alpha_1}}(1+\bar{g})^t = (1+\bar{g})^t \bar{k}_0 = \bar{k}_t.$$

Moreover,

$$f(\bar{k}_t, \bar{e}, A_t) < \bar{k}_{t+1}, \quad t = 0, 1, \ldots. \tag{A.14}$$

Indeed, using (A.13), (A.12) and (A.11), we get

$$\begin{aligned} f(\bar{k}_t, \bar{e}, A_t) - \bar{k}_{t+1} &= (\bar{k}_t)^{\alpha_1} A_t(\bar{e})^{\alpha_3} - (1+\bar{g})\bar{k}_t \\ &= \bar{k}_t\left(\frac{A_t(\bar{e})^{\alpha_3}}{(\bar{k}_t)^{1-\alpha_1}} - (1+\bar{g})\right) \leq \bar{k}_t\left(\frac{A_t(\bar{e})^{\alpha_3}}{(\bar{\kappa})^{1-\alpha_1}A_t(\bar{e})^{\alpha_3}} - (1+\bar{g})\right) \\ &= \bar{k}_t\left(\frac{\bar{\kappa}^{\alpha_1}}{\bar{\kappa}} - (1+\bar{g})\right) = \bar{k}_t\left((1+\tilde{g}) - (1+\bar{g})\right) < 0. \end{aligned}$$

Denote

$$\bar{c}_t := L\bar{k}_{t+1}. \tag{A.15}$$

Clearly,

$$\bar{c}_{t+1} = (1+\bar{g})\bar{c}_t. \tag{A.16}$$

Let the sequence $\{\tilde{k}_t\}_{t=0}^{\infty}$ be defined recursively as follows. We take $\tilde{k}_0$ such that $0 < \tilde{k}_0 < \hat{k}_0$. Suppose we are given $\tilde{k}_t > 0$. Consider the following equation in k:

$$\left(1 + \frac{\alpha_3}{\alpha_1}\frac{1}{\tilde{\varepsilon}}\right)k + \frac{\bar{c}_{t+1}}{\beta_L(1 + r(k, \tilde{e}_{t+1}, A_{t+1}))} = f(\tilde{k}_t, \tilde{e}_t, A_t).$$

The left-hand side of the above equation is increasing in k, and equals to 0 when $k = 0$. Thus there is a unique positive solution to this equation. We take $\tilde{k}_{t+1} > 0$ as this solution. Clearly, the sequence $\{\tilde{k}_t\}_{t=0}^{\infty}$ satisfies the following equation:

$$\left(1 + \frac{\alpha_3}{\alpha_1}\frac{1}{\tilde{\varepsilon}}\right)\tilde{k}_{t+1} + \frac{\bar{c}_{t+1}}{\beta_L(1 + r(\tilde{k}_{t+1}, \tilde{e}_{t+1}, A_{t+1}))} = f(\tilde{k}_t, \tilde{e}_t, A_t), \quad t = 0, 1, \ldots . \tag{A.17}$$

Step I.2. A game.

We reduce our finite horizon model to a game $\Gamma = (X_k, G_k)_{k\in I}$. Recall that to specify a game, we need to describe a set of players, I, and for each player $k \in I$ define the strategy set X_k and the loss function

$$G_k \; : \; \prod_{i\in I} X_i \to \mathbb{R}.$$

Elements of $\prod_{i\in I} X_i$ are called multistrategies. The equilibrium of the game Γ is defined as follows.

Definition. *A multistrategy* $\left(x_1^*, \ldots, x_{|I|}^*\right)$ *is called a Nash equilibrium of the game* Γ *if for each* $k \in I$, x_k^* *is a solution to*

$$\min_{x_k} G_k\left(x_1^*, \ldots, x_{k-1}^*, x_k, x_{k+1}^*, \ldots, x_{|I|}^*\right), \quad \text{s. t.} \quad x_k \in X_k.$$

The sufficient conditions for the existence of a Nash equilibrium of this game are well-known (see, e.g., Ichiishi, 2014): for each $k \in I$ the set X_k is a convex and compact subset of a finite dimensional space, and the function $G_k(x_1, \ldots, x_k, \ldots, x_{|I|})$ is continuous in all variables and quasi-convex in x_k.

Consider the following game Γ_T with $3T + (2T+1)L$ players where

1. for each agent $j = 1, \ldots, L$,

   a) T players determine s_t^j, $t = 0, 1, \ldots, T-1$, by solving

$$\min_s s\left(c_{t+1}^j - \beta_j(1 + r(k_{t+1}, e(\varepsilon_{t+1}, R_t), A_{t+1}))c_t^j\right), \quad \text{s. t.} \quad 0 \le s \le \tfrac{L\bar{k}_{t+1}}{\tilde{\varepsilon}}. \tag{A.18}$$

   b) $T+1$ players determine c_t^j, $t = 0, 1, \ldots, T$, by solving

$$\min_c \left|c - \left((1 + r(k_t, e(\varepsilon_t, R_{t-1}), A_t))s_{t-1}^j + w(k_t, e(\varepsilon_t, R_{t-1}), A_t) - s_t^j\right)\right|, \quad \text{s. t.} \quad 0 \le c \le \tfrac{\bar{c}_t}{\tilde{\varepsilon}}, \tag{A.19}$$

   where $s_T^j = 0$ and $R_{-1} = \hat{R}_{-1}$.

2. T players determine k_t, $t = 1, 2, \ldots, T$, by solving

$$\begin{aligned} &\min_k \left| k - \frac{1}{L} \sum_{j=1}^{L} s_{t-1}^j \frac{\alpha_1 \varepsilon_t}{\alpha_3 + \alpha_1 \varepsilon_t} \right|, \\ &\text{s. t.} \quad \tilde{k}_t \le k \le \tfrac{\bar{k}_{t+1}}{\tilde{\varepsilon}} \end{aligned} \tag{A.20}$$

3. T players determine R_t, $t = 0, 1, \ldots, T-1$, by solving

$$\begin{aligned} &\min_R \left| R - (1 - \varepsilon_t) R_{t-1} \right|, \\ &\text{s. t.} \quad \tilde{R}_t \le R \le \hat{R}_{-1}, \end{aligned} \tag{A.21}$$

   where $R_{-1} = \hat{R}_{-1}$.

4. T players determine ε_t, $t = 0, 1, \ldots, T-1$, by solving

$$\begin{aligned} &\min_\varepsilon \left| \frac{\left(e(\varepsilon, R_{t-1})\right)^{1-\alpha_3}}{A_t k_t^{\alpha_1}} - \alpha_1 \frac{e(\varepsilon_{t+1}, R_t)}{k_{t+1}} \right|, \\ &\text{s. t.} \quad \tilde{\varepsilon} \le \varepsilon \le \bar{\varepsilon}, \end{aligned} \tag{A.22}$$

   where $R_{-1} = \hat{R}_{-1}$, and $\varepsilon_T = 1$.

Lemma. *There exists a Nash equilibrium in the game Γ_T with $3T + (2T+1)L$ players having the strategy sets and loss functions described by (A.18)–(A.22).*

Proof. All the strategy sets are closed intervals, and for each player the loss function is continuous in all variables and quasi-convex in the player's own strategy variable. □

Step I.3. Nash equilibrium vs. competitive equilibrium.

The following lemma maintains that the Nash equilibrium of the game Γ_T determines a finite T-period competitive $\mathbb{E}_0$-equilibrium.

Lemma A.1. *Let*

$$\left\{ (c_t^{j*})_{t=0,1,\ldots,T}^{j=1,\ldots,L}, (s_t^{j*})_{t=0,1,\ldots,T-1}^{j=1,\ldots,L}, (k_t^*)_{t=1,2,\ldots,T}, (R_t^*)_{t=0,1,\ldots,T-1}, (\varepsilon_t^*)_{t=0,1,\ldots,T-1} \right\}$$

be a Nash equilibrium of the game Γ_T. *Let* $k_0^* = \hat{k}_0$, $R_{-1}^* = \hat{R}_{-1}$, $R_T^* = 0$, $\varepsilon_T^* = 1$, *and* $s_T^{j*} = 0$ *for all* j. *Let also*

$$\begin{aligned} e_t^* &= e(\varepsilon_t^*, R_{t-1}^*), \quad t = 0, 1, \dots, T, \\ 1 + r_t^* &= 1 + r(k_t^*, e_t^*, A_t), \quad t = 0, 1, \dots, T, \\ w_t^* &= w(k_t^*, e_t^*, A_t), \quad t = 0, 1, \dots, T, \\ q_t^* &= q(k_t^*, e_t^*, A_t), \quad t = 0, 1, \dots, T. \end{aligned}$$

Then

$$\left\{(c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^*\right\}_{t=0,1,\dots,T}$$

is a finite T*-period competitive equilibrium starting from the initial state* $\mathcal{I}_0$.

Proof. First, observe that

- if $c_{t+1}^j > \beta_j(1 + r(k_{t+1}, e(\varepsilon_{t+1}, R_t), A_{t+1}))c_t^j$, then the only solution to the problem (A.18) is $s = 0$;
- if $c_{t+1}^j = \beta_j(1 + r(k_{t+1}, e_{t+1}, A_{t+1}))c_t^j$, then any s from the interval $[0, \frac{L\bar{k}_{t+1}}{\tilde{\varepsilon}}]$ is a solution to the problem (A.18);
- if $c_{t+1}^j < \beta_j(1 + r(k_{t+1}, e_{t+1}, A_{t+1}))c_t^j$, then the only solution to the problem (A.18) is $s = \frac{L\bar{k}_{t+1}}{\tilde{\varepsilon}}$.

Second, notice that minimization problems (A.19)–(A.21) are of the form

$$\begin{aligned} &\min_x |x - \hat{x}|, \\ &\text{s. t.} \quad a_1 \le x \le a_2. \end{aligned}$$

The unique solution to this problem, x^*, is given by

$$x^* = \begin{cases} a_1, & \text{if} \quad \hat{x} < a_1; \\ a_2, & \text{if} \quad \hat{x} > a_2; \\ \hat{x}, & \text{if} \quad a_1 \le \hat{x} \le a_2. \end{cases}$$

Remark A.1. *When* $\hat{x} \ge a_1$, *we have* $\hat{x} \ge x^*$.

Remark A.2. *When* $\hat{x} \le a_2$, *we have* $\hat{x} \le x^*$.

Third, note that minimization problems (A.22) are of the form

$$\begin{aligned} &\min_x |g(x) - \hat{x}|, \\ &\text{s. t.} \quad a_1 \le x \le a_2, \end{aligned}$$

where the function $g(x)$ is increasing in x. The unique solution to this problem, x^*, is given by

$$x^* = \begin{cases} a_1, & \text{if} \quad \hat{x} < g(a_1); \\ a_2, & \text{if} \quad \hat{x} > g(a_2); \\ g^{-1}(\hat{x}), & \text{if} \quad g(a_2) \leq \hat{x} \leq g(a_1). \end{cases}$$

Let

$$\left\{ (c_t^{j*})_{t=0,1,\ldots,T}^{j=1,\ldots,L}, (s_t^{j*})_{t=-1,0,\ldots,T-1}^{j=1,\ldots,L}, (k_t^*)_{t=1,2,\ldots,T}, (R_t^*)_{t=0,1,\ldots,T-1}, (\varepsilon_t^*)_{t=0,1,\ldots,T-1} \right\}$$

be a Nash equilibrium of the game Γ_T. Denote $s_{-1}^{j*} = \frac{q_0^*}{1+r_0^*}\hat{R}_{-1}^j + \hat{k}_0^j$. Notice that that for all $t = 0, 1, \ldots, T$, $k_t^* \geq \tilde{k}_t > 0$, and

$$0 < \tilde{\varepsilon} \leq \varepsilon_t^* \leq \bar{\varepsilon} < 1. \tag{A.23}$$

Therefore, for all $t = 0, 1, \ldots, T$, $e_t^* > 0$, $w_t^* > 0$, $0 < 1 + r_t^* < \infty$, and $0 < q_t^* < \infty$.

The proof of Lemma A.1 is divided into several claims.

Claim A.1. *For each $j = 1, \ldots, L$,*

$$0 < c_t^{j*} \leq (1 + r_t^*)s_{t-1}^{j*} + w_t^* - s_t^{j*}, \quad t = 0, 1, \ldots, T, \tag{A.24}$$

and hence

$$c_t^{j*} + s_t^{j*} \leq (1 + r_t^*)s_{t-1}^{j*} + w_t^*, \quad t = 0, 1, \ldots, T, \tag{A.25}$$

Proof. Assume the converse. Then, by the structure of the problem (A.19), there are j and $0 \leq \tau \leq T$ such that

$$0 < c_t^{j*} \leq (1 + r_t^*)s_{t-1}^{j*} + w_t^* - s_t^{j*}, \quad t = 0, 1, \ldots, \tau - 1,$$

and

$$0 = c_\tau^{j*} \geq (1 + r_\tau^*)s_{\tau-1}^{j*} + w_\tau^* - s_\tau^{j*}. \tag{A.26}$$

Consider two cases. First, let $\tau \leq T - 1$. By (A.26),

$$s_\tau^{j*} \geq (1 + r_\tau^*)s_{\tau-1}^{j*} + w_\tau^* \geq w_\tau^* > 0.$$

Then, by the structure of the problem (A.18),

$$c_{\tau+1}^{j*} \leq \beta_j(1 + r_{\tau+1}^*)c_\tau^{j*} = 0,$$

because otherwise we would have $s_\tau^{j*} = 0$. Therefore, using Remark A.1, we conclude that

$$0 = c_{\tau+1}^{j*} \geq (1 + r_{\tau+1}^*) s_\tau^{j*} + w_{\tau+1}^* - s_{\tau+1}^{j*}.$$

Repeating the argument, and using the structure of the problem (A.18), we obtain for $t = \tau, \tau+1, \ldots, T-1$,

$$s_t^{j*} > 0,$$
$$c_{t+1}^{j*} = 0.$$

However, $c_T^{j*} = 0$ is impossible, because $s_T^{j*} = 0$, and by the structure of the problem (A.19) we have

$$0 = c_T^{j*} = c_T^{j*} + s_T^{j*} \geq (1 + r_T^*) s_{T-1}^{j*} + w_T^* > 0,$$

a contradiction.

Second, let $\tau = T$. Since $c_{T-1}^{j*} > 0$, and $c_T^{j*} = 0$, we have

$$c_T^{j*} - \beta_j (1 + r_T^*) c_{T-1}^{j*} = -\beta_j (1 + r_T^*) c_{T-1}^{j*} < 0,$$

and, by the structure of the problem (A.18), $s_{T-1}^{j*} = \frac{L\bar{k}_T}{\tilde{\varepsilon}}$. Using the fact that $s_T^{j*} = 0$, by the structure of the problem (A.19) we have

$$0 = c_T^{j*} + s_T^{j*} \geq (1 + r_T^*) s_{T-1}^{j*} + w_T^* > 0,$$

a contradiction. □

Claim A.2. *For each $j = 1, \ldots, L$,*

$$(1 + r_t^*) s_{t-1}^{j*} + w_t^* < \frac{L\bar{k}_{t+1}}{\tilde{\varepsilon}}, \quad t = 0, 1, \ldots, T, \tag{A.27}$$

$$\frac{1}{L} \sum_{j=1}^{L} s_{t-1}^{j*} \frac{\alpha_1 \varepsilon_t^*}{\alpha_3 + \alpha_1 \varepsilon_t^*} < \frac{\bar{k}_t}{\tilde{\varepsilon}}, \quad t = 0, 1, \ldots, T, \tag{A.28}$$

and

$$\frac{1}{L} \sum_{j=1}^{L} s_{t-1}^{j*} \leq k_t^* + \frac{q_t^*}{1 + r_t^*} \frac{R_{t-1}^*}{L}, \quad t = 0, 1, \ldots, T. \tag{A.29}$$

Proof. Using the definition of s_{-1}^{j*}, we obtain

$$\frac{1}{L} \sum_{j=1}^{L} s_{-1}^{j*} = \frac{q_0^*}{1 + r_0^*} \frac{1}{L} \sum_{j=1}^{L} \hat{R}_{-1}^j + \frac{1}{L} \sum_{j=1}^{L} \hat{k}_0^j = \frac{q_0^*}{1 + r_0^*} \frac{R_{-1}^*}{L} + k_0^*. \tag{A.30}$$

Therefore, by (A.30), (A.23), and (A.14)

$$
\begin{aligned}
(1+r_0^*)s_{-1}^{j*} + w_0^* &\le \sum_{j=1}^{L} \left((1+r_0^*)s_{-1}^{j*} + w_0^*\right) \\
&= (1+r_0^*)\frac{q_0^*}{1+r_0^*}R_{-1}^* + L(1+r_0^*)k_0^* + Lw_0^* = L(1+r_0^*)k_0^* + Lw_0^* + q_0^* R_{-1}^* \\
&= \alpha_1 L f(k_0^*, e_0^*, A_0) + \alpha_2 L f(k_0^*, e_0^*, A_0) + \alpha_3 f(k_0^*, e_0^*, A_0)\frac{R_{-1}^*}{e_0^*} \\
&= (\alpha_1+\alpha_2) L f(k_0^*, e_0^*, A_0) + \alpha_3 L f(k_0^*, e_0^*, A_0)\frac{1}{\varepsilon_0^*} \\
&= L f(k_0^*, e_0^*, A_0) + \alpha_3 L f(k_0^*, e_0^*, A_0)\frac{1-\varepsilon_0^*}{\varepsilon_0^*} \\
&\le L f(\bar{k}_0, \bar{e}, A_0) + \alpha_3 L f(\bar{k}_0, \bar{e}, A_0)\frac{1-\tilde{\varepsilon}}{\tilde{\varepsilon}} < L\bar{k}_1 + L\bar{k}_1\frac{1-\tilde{\varepsilon}}{\tilde{\varepsilon}} = \frac{L\bar{k}_1}{\tilde{\varepsilon}},
\end{aligned}
$$

which proves (A.27) for $t = 0$.

Moreover, by (A.25),

$$
\frac{1}{L}\sum_{j=1}^{L} s_0^{j*} \le \frac{1}{L}\sum_{j=1}^{L}\left(c_0^{j*} + s_0^{j*}\right) < \frac{1}{L}\sum_{j=1}^{L}\left((1+r_0^*)s_{-1}^{j*} + w_0^*\right) \le \frac{\bar{k}_1}{\tilde{\varepsilon}}.
$$

Since $\varepsilon_1^* > 0$,

$$
\frac{\alpha_1 \varepsilon_1^*}{\alpha_3 + \alpha_1 \varepsilon_1^*} < 1,
$$

and therefore

$$
\frac{1}{L}\sum_{j=1}^{L} s_0^{j*}\frac{\alpha_1 \varepsilon_1^*}{\alpha_3 + \alpha_1 \varepsilon_1^*} < \frac{\bar{k}_1}{\tilde{\varepsilon}},
$$

which proves (A.28) for $t = 0$.

It follows from Remark A.2 that

$$
\frac{1}{L}\sum_{j=1}^{L} s_0^{j*}\frac{\alpha_1 \varepsilon_1^*}{\alpha_3 + \alpha_1 \varepsilon_1^*} \le k_1^*.
$$

Using (A.6),

$$
\frac{1}{L}\sum_{j=1}^{L} s_0^{j*} \le k_1^* + \frac{\alpha_3}{\alpha_1}\frac{k_1^*}{\varepsilon_1^*} = k_1^* + \frac{q_1^*}{1+r_1^*}\frac{R_0^*}{L}.
$$

Thus, (A.29) holds for $t = 0$.

To obtain inequalities (A.27)–(A.29) for all $t \le T$, it is sufficient to repeat the argument.

□

Claim A.3. *For each agent $j = 1, \dots, L$,*

$$c_t^{j*} + s_t^{j*} = (1 + r_t^*)s_{t-1}^{j*} + w_t^*, \quad t = 0, 1, \dots, T. \tag{A.31}$$

Proof. Using (A.27), (A.15) and the fact that $s_t^{j*} \geq 0$ for all $t = 0, 1, \dots, T$, we get

$$(1 + r_t^*)s_{t-1}^{j*} + w_t^* - s_t^{j*} < \frac{L\bar{k}_{t+1}}{\tilde{\varepsilon}} = \frac{\bar{c}_t}{\tilde{\varepsilon}}.$$

Therefore, by the structure of the problem (A.19), for each $j = 1, \dots, L$,

$$c_t^{j*} \geq (1 + r_t^*)s_{t-1}^{j*} + w_t^* - s_t^{j*}, \quad t = 0, 1, \dots, T.$$

Combining this inequality with (A.24), we obtain (A.31). □

Claim A.4. *For each $j = 1, \dots, L$,*

$$c_{t+1}^{j*} \geq \beta_j(1 + r_{t+1}^*)c_t^{j*} \ (= \textit{if } s_t^{j*} > 0), \quad t = 0, 1, \dots, T. \tag{A.32}$$

Proof. Assume that for some j and $t < T$,

$$c_{t+1}^{j*} < \beta_j(1 + r_{t+1}^*)c_t^{j*}.$$

Then, by the structure of the problem (A.18), $s_t^{j*} = \frac{L\bar{k}_{t+1}}{\tilde{\varepsilon}}$. By (A.27),

$$(1 + r_t^*)s_{t-1}^{j*} + w_t^* < \frac{L\bar{k}_{t+1}}{\tilde{\varepsilon}} = s_t^{j*},$$

and hence

$$(1 + r_t^*)s_{t-1}^{j*} + w_t^* - s_t^{j*} < 0,$$

which contradicts (A.24). Thus we have proved that

$$c_{t+1}^{j*} \geq \beta_j(1 + r_{t+1}^*)c_t^{j*}.$$

Moreover, if

$$c_{t+1}^{j*} > \beta_j(1 + r_{t+1}^*)c_t^{j*},$$

then by the structure of the problem (A.18), $s_t^{j*} = 0$. □

Claim A.5. *For $t = 0, 1, \dots, T - 1$,*

$$R_t^* = (1 - \varepsilon_t^*)R_{t-1}^*. \tag{A.33}$$

Proof. Due to the (A.7), (A.23) and the bounds for R in (A.21), for all $t = 0, 1, \dots, T-1$, we have

$$\tilde{R}_t = (1-\bar{\varepsilon})\tilde{R}_{t-1} \le (1-\varepsilon_t^*)R_{t-1}^* \le (1-\varepsilon_t^*)\hat{R}_{-1} < \hat{R}_{-1}.$$

Now (A.33) follows from the structure of the problem (A.21). □

It follows from (A.7), (A.23), the bounds for R in (A.21) and the definition of e_t^* that

$$\tilde{e}_t \le e_t^* \le \bar{e}, \quad t = 0, 1, \dots, T. \tag{A.34}$$

Claim A.6. *For all $t = 0, 1, \dots, T$,*

$$k_t^* > \tilde{k}_t, \tag{A.35}$$

and

$$\frac{1}{L}\sum_{j=1}^{L} s_{t-1}^{j*} = \frac{q_t^*}{1+r_t^*}\frac{R_{t-1}^*}{L} + k_t^*. \tag{A.36}$$

Proof. Note that by the choice of $\tilde{k}_0$,

$$k_0^* > \tilde{k}_0,$$

and it follows from (A.30) that (A.36) holds for $t = 0$.

Assume that for some $t = 1, 2, \dots T$,

$$\frac{1}{L}\sum_{j=1}^{L} s_{t-2}^{j*}\frac{\alpha_1\varepsilon_{t-1}^*}{\alpha_3+\alpha_1\varepsilon_{t-1}^*} = \left(\frac{q_{t-1}^*}{1+r_{t-1}^*}R_{t-2}^* + k_{t-1}^*\right)\frac{\alpha_1\varepsilon_{t-1}^*}{\alpha_3+\alpha_1\varepsilon_{t-1}^*} = k_{t-1}^* > \tilde{k}_{t-1},$$

and

$$\frac{1}{L}\sum_{j=1}^{L} s_{t-1}^{j*}\frac{\alpha_1\varepsilon_t^*}{\alpha_3+\alpha_1\varepsilon_t^*} \le \left(\frac{q_t^*}{1+r_t^*}R_{t-1}^* + k_t^*\right)\frac{\alpha_1\varepsilon_t^*}{\alpha_3+\alpha_1\varepsilon_t^*} = k_t^* = \tilde{k}_t.$$

By (A.31), (A.23), (A.5) and (A.34),

$$\begin{aligned}\frac{1}{L}\sum_{j=1}^{L}\left(c_{t-1}^{j*}+s_{t-1}^{j*}\right) &= \frac{1}{L}\sum_{j=1}^{L}\left((1+r_{t-1}^*)s_{t-2}^{j*}+w_{t-1}^*\right)\\ &= (1+r_{t-1}^*)k_{t-1}^* + q_{t-1}^*\frac{R_{t-2}^*}{L} + w_{t-1}^* = (1+r_{t-1}^*)k_{t-1}^* + w_{t-1}^* + q_{t-1}^*e_{t-1}^*\frac{1}{\varepsilon_{t-1}^*}\\ &> (1+r_{t-1}^*)k_{t-1}^* + w_{t-1}^* + q_{t-1}^*e_{t-1}^* = f(k_{t-1}^*, e_{t-1}^*, A_{t-1}) > f(\tilde{k}_{t-1}, \tilde{e}_{t-1}, A_{t-1}).\end{aligned}$$

Hence, due to (A.23),

$$\frac{1}{L}\sum_{j=1}^{L} c_{t-1}^{j*} > f(\tilde{k}_{t-1}, \tilde{e}_{t-1}, A_{t-1}) - \frac{1}{L}\sum_{j=1}^{L} s_{t-1}^{j*}$$
$$\geq f(\tilde{k}_{t-1}, \tilde{e}_{t-1}, A_{t-1}) - \tilde{k}_t - \frac{\alpha_3}{\alpha_1}\frac{\tilde{k}_t}{\varepsilon_t^*} \geq f(\tilde{k}_{t-1}, \tilde{e}_{t-1}, A_{t-1}) - \tilde{k}_t\left(1 + \frac{\alpha_3}{\alpha_1}\frac{1}{\tilde{\varepsilon}}\right).$$

Therefore, there is j such that

$$c_{t-1}^{j*} > f(\tilde{k}_{t-1}, \tilde{e}_{t-1}, A_{t-1}) - \tilde{k}_t\left(1 + \frac{\alpha_3}{\alpha_1}\frac{1}{\tilde{\varepsilon}}\right) > 0. \tag{A.37}$$

Using (A.17), the bounds for c in (A.19), and taking into account that $\tilde{k}_t = k_t^*$, we get

$$c_{t-1}^{j*} > f(\tilde{k}_{t-1}, \tilde{e}_{t-1}, A_{t-1}) - \tilde{k}_t\left(1 + \frac{\alpha_3}{\alpha_1}\frac{1}{\tilde{\varepsilon}}\right)$$
$$= \frac{\bar{c}_t}{\beta_L(1 + r(\tilde{k}_t, \tilde{e}_t, A_t))} \geq \frac{\bar{c}_t}{\beta_j(1 + r(\tilde{k}_t, e_t^*, A_t))} \geq \frac{c_t^{j*}}{\beta_j(1 + r_t^*)},$$

and hence

$$c_t^{j*} < \beta_j(1 + r_t^*)c_{t-1}^{j*}.$$

It follows from the structure of the problem (A.18) that

$$s_{t-1}^{j*} = \frac{L\bar{k}_t}{\tilde{\varepsilon}}.$$

By (A.31) and (A.27), we have

$$c_{t-1}^{j*} = (1 + r_{t-1}^*)s_{t-2}^{j*} + w_{t-1}^* - s_{t-1}^{j*} < \frac{L\bar{k}_t}{\tilde{\varepsilon}} - \frac{L\bar{k}_t}{\tilde{\varepsilon}} = 0,$$

a contradiction of (A.37). This proves (A.35).

Now it follows from (A.28) and (A.35) that

$$\tilde{k}_t < \frac{1}{L}\sum_{j=1}^{L} s_{t-1}^{j*}\frac{\alpha_1\varepsilon_t^*}{\alpha_3 + \alpha_1\varepsilon_t^*} < \frac{\bar{k}_t}{\tilde{\varepsilon}}, \quad t = 0, 1, \ldots, T.$$

By the structure of the problem (A.20),

$$\frac{1}{L}\sum_{j=1}^{L} s_{t-1}^{j*}\frac{\alpha_1\varepsilon_t^*}{\alpha_3 + \alpha_1\varepsilon_t^*} = k_t^*, \quad t = 0, 1, \ldots, T.$$

Therefore, using (A.6), for all $t = 0, 1, \ldots, T$, we obtain

$$\frac{1}{L}\sum_{j=1}^{L} s_{t-1}^{j*} = k_t^* + \frac{\alpha_3}{\alpha_1}\frac{k_t^*}{\varepsilon_t^*} = k_t^* + \frac{q_t^*}{1+r_t^*}\frac{R_{t-1}^*}{L}.$$

□

Claim A.7. *For all* $t = 0, 1, \ldots, T$ *and for all* $j = 1, 2, \ldots, L$,

$$c_t^{j*} \geq (1-\beta_1)\left((1+r_t^*)s_{t-1}^{j*} + w_t^*\right). \tag{A.38}$$

Proof. Let us prove (A.38) for $t = 0$. It is clear that there is $0 \leq \tau \leq T$ such that $s_t > 0$ for all $t < \tau$ and $s_\tau = 0$. If $\tau = 0$, then it is sufficient to note that

$$c_0^{j*} = (1+r_0^*)s_{-1}^{j*} + w_0^* \geq (1-\beta_1)\left((1+r_0^*)s_{-1}^{j*} + w_0^*\right).$$

If $\tau > 0$, then, by (A.32),

$$c_1^{j*} = \beta_j(1+r_1^*)c_0^{j*}, \quad \ldots, \quad c_\tau^{j*} = \beta_j^\tau(1+r_1^*)\cdots(1+r_\tau^*)c_0^{j*},$$

and, by (A.31),

$$\begin{aligned} c_0^{j*} + \frac{1}{1+r_1^*}c_1^{j*} + \ldots + \frac{1}{(1+r_1^*)\cdots(1+r_\tau^*)}c_\tau^{j*} \\ = (1+r_0^*)s_{-1}^{j*} + w_0^* + \frac{1}{1+r_1^*}w_1^* + \ldots + \frac{1}{(1+r_1^*)\cdots(1+r_\tau^*)}w_\tau^*. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{1-\beta_1}c_0^{j*} &> c_0^{j*} + \beta_j c_0^{j*} + \ldots + \beta_j^\tau c_0^{j*} \\ &= c_0^{j*} + \frac{1}{1+r_1^*}c_1^{j*} + \ldots + \frac{1}{(1+r_1^*)\cdots(1+r_\tau^*)}c_\tau^{j*} \\ &= (1+r_0^*)s_{-1}^{j*} + w_0^* + \frac{1}{1+r_1^*}w_1^* + \ldots + \frac{1}{(1+r_1^*)\cdots(1+r_\tau^*)}w_\tau^* \geq (1+r_0^*)s_{-1}^{j*} + w_0^*, \end{aligned}$$

which proves (A.38) for $t = 0$. To prove it for $t > 0$ it is sufficient to repeat the argument.

□

Claim A.8.

$$k_{t+1}^* > \alpha_1\beta_L(1-\beta_1)^2 f(k_t^*, e_t^*, A_t), \quad t = 0, 1, \ldots, T-1. \tag{A.39}$$

Proof. Note that

$$\alpha_1 + \alpha_2 + \frac{\alpha_3}{\tilde{\varepsilon}} = \alpha_1 + \alpha_2 + \frac{1 - (\alpha_1 + \alpha_2)(1 - \beta_1)}{1 - \beta_1} = \frac{1}{1 - \beta_1}. \tag{A.40}$$

Due to (A.31), (A.30), (A.4), (A.23), and (A.40), we get for $t = 0, 1, \ldots, T - 1$,

$$\begin{aligned}\frac{1}{L}\sum_{j=1}^{L} c_t^{j*} &\le \left((1 + r_t^*)\frac{1}{L}\sum_{j=1}^{L} s_{t-1}^{j*} + w_t^*\right) \\ &= \left(q_t^* \frac{R_{t-1}^*}{L} + (1 + r_t^*)k_t^* + w_t^*\right) = \left(\frac{\alpha_3}{\varepsilon_t^*} + \alpha_1 + \alpha_2\right) f(k_t^*, e_t^*, A_t) \\ &\le \left(\frac{\alpha_3}{\tilde{\varepsilon}} + \alpha_1 + \alpha_2\right) f(k_t^*, e_t^*, A_t) = \frac{f(k_t^*, e_t^*, A_t)}{1 - \beta_1},\end{aligned}$$

or

$$f(k_t^*, e_t^*, A_t) \ge (1 - \beta_1)\frac{1}{L}\sum_{j=1}^{L} c_t^{j*}. \tag{A.41}$$

By (A.38), (A.30), (A.4), (A.5), and (A.23), we get

$$\begin{aligned}\frac{1}{L}\sum_{j=1}^{L} c_0^{j*} &\ge (1-\beta_1)\left((1 + r_0^*)\frac{1}{L}\sum_{j=1}^{L} s_{-1}^{j*} + w_0^*\right) = (1-\beta_1)\left(q_0^*\frac{R_{-1}^*}{L} + (1 + r_0^*)k_0^* + w_0^*\right) = \\ &(1 - \beta_1)\left(\frac{\alpha_3}{\varepsilon_0^*} + \alpha_1 + \alpha_2\right) f(k_0^*, e_0^*, A_0) > (1 - \beta_1) f(k_0^*, e_0^*, A_0),\end{aligned}$$

and therefore, taking into account (A.41) and (A.32),

$$\begin{aligned}f(k_1^*, e_1^*, A_1) &\ge (1 - \beta_1)\frac{1}{L}\sum_{j=1}^{L} c_1^{j*} \ge (1 - \beta_1)\frac{1}{L}\sum_{j=1}^{L} \beta_j (1 + r_1^*) c_0^{j*} \\ &\ge (1 - \beta_1)\beta_L(1 + r_1^*)\frac{1}{L}\sum_{j=1}^{L} c_0^{j*} > \beta_L(1 - \beta_1)^2(1 + r_1^*) f(k_0^*, e_0^*, A_0).\end{aligned}$$

Repeating the argument, we obtain for $t = 0, 1, \ldots, T - 1$,

$$f(k_{t+1}^*, e_{t+1}^*, A_{t+1}) > \beta_L(1 - \beta_1)^2(1 + r_{t+1}^*) f(k_t^*, e_t^*, A_t). \tag{A.42}$$

It follows from (A.5) and (A.42) that

$$\frac{(1 + r_{t+1}^*)k_{t+1}^*}{\alpha_1} = f(k_{t+1}^*, e_{t+1}^*, A_{t+1}) > \beta_L(1 - \beta_1)^2(1 + r_{t+1}^*) f(k_t^*, e_t^*, A_t),$$

and hence (A.39) holds. □

Claim A.9. *For* $t = 0, 1, \dots, T-1$,

$$\frac{\big(e(\varepsilon_t^*, R_{t-1}^*)\big)^{1-\alpha_3}}{A_t (k_t^*)^{\alpha_1}} = \alpha_1 \frac{e(\varepsilon_{t+1}^*, R_t^*)}{k_{t+1}^*}, \tag{A.43}$$

or, equivalently,

$$\frac{1}{q(k_t^*, e_t^*, A_t)} = \frac{1 + r(k_{t+1}^*, e_{t+1}^*, A_{t+1})}{q(k_{t+1}^*, e_{t+1}^*, A_{t+1})}. \tag{A.44}$$

Proof. Using (A.6) and the definition of q_t^*, it is easily seen that (A.43) is equivalent to (A.44).

Suppose that for some $t = 0, 1, \dots, T-1$ equality (A.43) does not hold. Then either

$$\frac{\big(e(\varepsilon_t^*, R_{t-1}^*)\big)^{1-\alpha_3}}{A_t (k_t^*)^{\alpha_1}} > \alpha_1 \frac{e(\varepsilon_{t+1}^*, R_t^*)}{k_{t+1}^*}, \tag{A.45}$$

or

$$\frac{\big(e(\varepsilon_t^*, R_{t-1}^*)\big)^{1-\alpha_3}}{A_t (k_t^*)^{\alpha_1}} < \alpha_1 \frac{e(\varepsilon_{t+1}^*, R_t^*)}{k_{t+1}^*}. \tag{A.46}$$

Consider the first case. It follows from the structure of the problem (A.22) that

$$\varepsilon_t^* = \tilde{\varepsilon} = \frac{\alpha_3(1-\beta_1)}{1 - (\alpha_1 + \alpha_2)(1-\beta_1)}.$$

By (A.31),

$$\frac{1}{L}\sum_{j=1}^{L} \left(c_t^{j*} + s_t^{j*}\right) = (1 + r_t^*)\frac{1}{L}\sum_{j=1}^{L} s_{t-1}^{j*} + w_t^*, \quad t = 0, 1, \dots, T-1,$$

and, using (A.36) and (A.33), we get for $t = 0, 1, \dots, T-1$,

$$\begin{aligned} \frac{1}{L}\sum_{j=1}^{L} c_t^{j*} + k_{t+1}^* &= q_t^* \frac{R_{t-1}^*}{L} - \frac{q_{t+1}^*}{1 + r_{t+1}^*}\frac{R_t^*}{L} + (1 + r_t^*)k_t^* + w_t^* \\ &= (1 + r_t^*)k_t^* + w_t^* + q_t^* e_t^* + q_t^* \frac{R_t^*}{L} - \frac{q_{t+1}^*}{1 + r_{t+1}^*}\frac{R_t^*}{L} \\ &= f(k_t^*, e_t^*, A_t) + \frac{R_t^*}{L}\left(q_t^* - \frac{q_{t+1}^*}{1 + r_{t+1}^*}\right). \end{aligned} \tag{A.47}$$

Using (A.44), it is easily seen that (A.45) is equivalent to

$$\frac{q_{t+1}^*}{1 + r_{t+1}^*} > q_t^*,$$

and thus from (A.47) we get

$$f(k_t^*, e_t^*, A_t) > \frac{1}{L}\sum_{j=1}^{L} c_t^{j*}, \quad t = 0, 1, \ldots, T-1. \tag{A.48}$$

By (A.48), (A.38), (A.30), (A.4), (A.5), and (A.40), we get

$$\begin{aligned} f(k_t^*, e_t^*, A_t) &> \frac{1}{L}\sum_{j=1}^{L} c_t^{j*} \geq (1-\beta_1)\left((1+r_t^*)\frac{1}{L}\sum_{j=1}^{L} s_{t-1}^{j*} + w_t^*\right) \\ &= (1-\beta_1)\left(q_t^*\frac{R_{t-1}^*}{L} + (1+r_t^*)k_t^* + w_t^*\right) = (1-\beta_1)\left(\frac{\alpha_3}{\varepsilon_t^*} + \alpha_1 + \alpha_2\right) f(k_t^*, e_t^*, A_t) \\ &= (1-\beta_1)\left(\frac{\alpha_3}{\tilde{\varepsilon}} + \alpha_1 + \alpha_2\right) f(k_t^*, e_t^*, A_t) = (1-\beta_1)\frac{1}{1-\beta_1} f(k_t^*, e_t^*, A_t) = f(k_t^*, e_t^*, A_t), \end{aligned}$$

a contradiction.

Consider the second case. It follows from the structure of the problem (A.22) that $\varepsilon_t^* = \bar{\varepsilon}$. By (A.23),

$$\varepsilon_t^* = \bar{\varepsilon} = \frac{1}{1+\beta_L(1-\beta_1)^2} \geq \frac{\varepsilon_{t+1}^*}{\varepsilon_{t+1}^* + \beta_L(1-\beta_1)^2},$$

or

$$\varepsilon_{t+1}^*(1-\varepsilon_t^*) \leq \beta_L(1-\beta_1)^2\varepsilon_t^*.$$

By (A.33) and the definition of e_t^*,

$$e_{t+1}^* \leq \beta_L(1-\beta_1)^2 e_t^*.$$

At the same time, it follows from (A.46) that

$$\frac{e_{t+1}^*}{e_t^*} > \frac{k_{t+1}^*}{\alpha_1 f(k_t^*, e_t^*, A_t)}.$$

Hence, by (A.39),

$$e_{t+1}^* > \beta_L(1-\beta_1)^2 e_t^*,$$

a contradiction.

Thus equality (A.43) holds for all $t = 0, 1, \ldots, T-1$. □

Claims A.3 and A.4 show that condition 1 of Definition A.1 holds. Due to the choice of e_t^*, r_t^*, q_t^* and w_t^*, conditions 2–4 of Definition A.1 are satisfied. Claims A.9, A.5 and A.6 show that conditions 5, 6 and 7 of Definition A.1 are valid. Thus the proof of Lemma A.1 is complete. □

Step II. Competitive equilibrium in the infinite horizon model.

Step II.1. A candidate equilibrium path.

Let for $T = 1, 2, \ldots,$

$$\mathcal{E}_T^* = \left\{(c_t^{j*}(T))_{j=1}^L, (s_t^{j*}(T))_{j=1}^L, k_t^*(T), r_t^*(T), w_t^*(T), q_t^*(T), e_t^*(T), R_t^*(T)\right\}_{t=0,1,\ldots,T}$$

be a finite T-period equilibrium path. Let us apply the following procedure to the sequence $\{\mathcal{E}_T^*\}_{T=1,2,\ldots}$.

At the first step of the process we take a cluster point of the sequence

$$\left\{(c_0^{j*}(T))_{j=1}^L, (s_0^{j*}(T))_{j=1}^L, k_0^*(T), r_0^*(T), w_0^*(T), q_0^*(T), e_0^*(T), R_0^*(T)\right\}_{T=1,2,\ldots},$$

denote it as

$$\left\{(c_0^{j*})_{j=1}^L, (s_0^{j*})_{j=1}^L, k_0^*, r_0^*, w_0^*, q_0^*, e_0^*, R_0^*\right\},$$

and extract a subsequence $\{T_{0n}\}_{n=1}^\infty$ from $\{T\}_{T=1,2,\ldots}$ such that

$$\left\{(c_0^{j*}(T_{0n}))_{j=1}^L, (s_0^{j*}(T_{0n}))_{j=1}^L, k_0^*(T_{0n}), r_0^*(T_{0n}), w_0^*(T_{0n}), q_0^*(T_{0n}), e_0^*(T_{0n}), R_0^*(T_{0n})\right\}_{n=1}^\infty$$

converges to $\left\{(c_0^{j*})_{j=1}^L, (s_0^{j*})_{j=1}^L, k_0^*, r_0^*, w_0^*, q_0^*, e_0^*, R_0^*\right\}$.

At the second step we take a cluster point of the sequence

$$\left\{(c_1^{j*}(T_{0n}))_{j=1}^L, (s_1^{j*}(T_{0n}))_{j=1}^L, k_1^*(T_{0n}), r_1^*(T_{0n}), w_1^*(T_{0n}), q_1^*(T_{0n}), e_1^*(T_{0n}), R_1^*(T_{0n})\right\}_{n=1}^\infty,$$

denote it as

$$\left\{(c_1^{j*})_{j=1}^L, (s_1^{j*})_{j=1}^L, k_1^*, r_1^*, w_1^*, q_1^*, e_1^*, R_1^*\right\},$$

and extract a subsequence $\{T_{1n}\}_{n=1}^\infty$ from the sequence $\{T_{0n}\}_{n=1}^\infty$ such that $T_{11} > 1$ and

$$\left\{(c_1^{j*}(T_{1n}))_{j=1}^L, (s_1^{j*}(T_{1n}))_{j=1}^L, k_1^*(T_{1n}), r_1^*(T_{1n}), w_1^*(T_{1n}), q_1^*(T_{1n}), e_1^*(T_{1n}), R_1^*(T_{1n})\right\}_{n=1}^\infty$$

converges to $\left\{(c_1^{j*})_{j=1}^L, (s_1^{j*})_{j=1}^L, k_1^*, r_1^*, w_1^*, q_1^*, e_1^*, R_1^*\right\}$. This procedure continues ad infinitum.

As a result, we obtain an infinite sequence

$$\mathcal{E}_\infty^* = \left\{(c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L, k_t^*, r_t^*, w_t^*, q_t^*, e_t^*, R_t^*\right\}_{t=0,1,\ldots}. \tag{A.49}$$

This sequence is a natural candidate to be a competitive equilibrium in our model.

Step II.2. Bounds of the T-period equilibrium capital sequence.

It follows from Lemma A.1 that in any T-period finite equilibrium $\varepsilon_t^* = \frac{Le_t^*}{R_{t-1}^*}$ satisfies (A.23), and e_t^* satisfies (A.34). Moreover, by condition 6 of Definition A.1,

$$\frac{e_{t+1}^*}{e_t^*} = \frac{\varepsilon_{t+1}^* R_t^*}{L} \frac{L}{\varepsilon_t^* R_{t-1}^*} = \frac{\varepsilon_{t+1}^*(1-\varepsilon_t^*)}{\varepsilon_t^*},$$

and hence, for $t = 0, 1, \ldots$,

$$\frac{\tilde{\varepsilon}(1-\bar{\varepsilon})}{\bar{\varepsilon}} < \frac{e_{t+1}^*}{e_t^*} < \frac{\bar{\varepsilon}(1-\tilde{\varepsilon})}{\tilde{\varepsilon}}. \tag{A.50}$$

We also know that k_t^*, is bounded from below by $\tilde{k}_t$. However, we need to establish a more precise lower bound of the capital sequence in a T-period finite equilibrium.

Let the value $1 + r'$ be such that

$$\beta_L(1+r') > 2(1+\bar{g}), \tag{A.51}$$

and k' be given by

$$\alpha_1 A_0 (\tilde{e}_0)^{\alpha_3} (k')^{\alpha_1 - 1} = 1 + r'. \tag{A.52}$$

Let further the sequence $\{k_t'\}$ be given by

$$k_{t+1}' = (1+\underline{g})k_t', \tag{A.53}$$

where

$$0 < k_0' < \min\{\hat{k}_0, k'\}.$$

Claim A.10. *For all t,*

$$(1+\underline{g}) < \frac{f(k_{t+1}', e_{t+1}^*, A_{t+1})}{f(k_t', e_t^*, A_t)} < (1+\bar{g}). \tag{A.54}$$

Proof. By (A.53), (A.8), (A.50), (A.10), and (A.11),

$$\begin{aligned}\frac{f(k_{t+1}', e_{t+1}^*, A_{t+1})}{f(k_t', e_t^*, A_t)} &= \frac{(k_{t+1}')^{\alpha_1}}{(k_t')^{\alpha_1}} \frac{A_{t+1}}{A_t} \frac{(e_{t+1}^*)^{\alpha_3}}{(e_t^*)^{\alpha_3}} \\ &> \left(1+\underline{g}\right)^{\alpha_1} (1+\lambda) \left(\frac{\tilde{\varepsilon}(1-\bar{\varepsilon})}{\bar{\varepsilon}}\right)^{\alpha_3} \geq \left(1+\underline{g}\right)^{\alpha_1} (1+\tilde{g})^{1-\alpha_1} > (1+\underline{g}).\end{aligned}$$

Analogously, using (A.53), (A.8), (A.50), (A.9), and (A.11), we have

$$\begin{aligned}\frac{f(k'_{t+1}, e^*_{t+1}, A_{t+1})}{f(k'_t, e^*_t, A_t)} &= \frac{(k'_{t+1})^{\alpha_1}}{(k'_t)^{\alpha_1}} \frac{A_{t+1}}{A_t} \frac{(e^*_{t+1})^{\alpha_3}}{(e^*_t)^{\alpha_3}} \\ &< \left(1+\underline{g}\right)^{\alpha_1} (1+\lambda) \left(\frac{\bar{\varepsilon}(1-\tilde{\varepsilon})}{\tilde{\varepsilon}}\right)^{\alpha_3} \leq (1+\bar{g})^{\alpha_1} (1+\bar{g})^{1-\alpha_1} = (1+\bar{g}).\end{aligned}$$

□

Claim A.11. *For all* $t = 0, 1, \dots, T$,

$$1 + r(k'_t, e^*_t, A_t) > 1 + r'. \tag{A.55}$$

Proof. It follows from (A.5), (A.53), and (A.54) that

$$1 + r(k'_t, e^*_t, A_t) = \alpha_1 \frac{f(k'_t, e^*_t, A_t)}{k'_t} > \alpha_1 \frac{f(k'_{t-1}, e^*_{t-1}, A_{t-1})}{k'_{t-1}} = 1 + r(k'_{t-1}, e^*_{t-1}, A_{t-1}).$$

Repeating the argument, and using (A.34) along with (A.52), we get

$$\begin{aligned}1 + r(k'_t, e^*_t, A_t) &> 1 + r(k'_t, e^*_0, A_t) \geq 1 + r(k'_0, \tilde{e}_0, A_0) \\ &> 1 + r(k', \tilde{e}_0, A_0) = \alpha_1 A_0 (\tilde{e}_0)^{\alpha_3} (k')^{\alpha_1 - 1} = 1 + r'.\end{aligned}$$

□

Let

$$w'_{t+1} = (1+\underline{g}) w'_t, \quad t = 0, 1, \dots,$$

where

$$w'_0 = \alpha_2 A_0 (k'_0)^{\alpha_1} (\tilde{e}_0)^{\alpha_3} > 0.$$

Claim A.12. *In any finite* T*-period competitive equilibrium*

$$\left\{(c^{j*}_t)^L_{j=1}, (s^{j*}_t)^L_{j=1}, k^*_t, r^*_t, w^*_t, q^*_t, e^*_t, R^*_t\right\}_{t=0,1,\dots,T},$$

for $t \leq T-1$,

$$k^*_t > k'_t > 0, \tag{A.56}$$

and

$$w^*_t > w'_t > 0.$$

Proof. First let us prove (A.56). Assume the converse. Then there is $\tau > 0$ such that $k^*_\tau > k'_\tau$, and $k^*_{\tau+1} \le k'_{\tau+1}$. It follows from (A.3), (A.34), and (A.55), that for all j,

$$\begin{aligned} c^{j*}_{\tau+1} \ge \beta_j(1+r^*_{\tau+1})c^{j*}_\tau \ge \beta_L(1+r^*_{\tau+1})c^{j*}_\tau &= \beta_L(1+r(k^*_{\tau+1}, e^*_{\tau+1}, A_{\tau+1}))c^{j*}_\tau \\ &\ge \beta_L(1+r(k'_{\tau+1}, e^*_{\tau+1}, A_{\tau+1}))c^{j*}_\tau > \beta_L(1+r')c^{j*}_\tau. \end{aligned}$$

By (A.51),

$$c^{j*}_{\tau+1} > 2(1+\bar{g})c^{j*}_\tau. \tag{A.57}$$

Adding together the budget constraints of all agents at time t in (A.2), and using conditions 5–7 of Definition A.1, we get

$$\frac{1}{L}\sum_{j=1}^{L} c^{j*}_t + k^*_{t+1} = (1+r^*_t)\, k^*_t + w^*_t = f(k^*_t, e^*_t, A_t), \quad t = 0, 1, \ldots, T. \tag{A.58}$$

Applying (A.58) for $t = \tau + 1$ and $t = \tau$, and using (A.57), we have

$$\begin{aligned} f(k^*_{\tau+1}, e^*_{\tau+1}, A_{\tau+1}) - k^*_{\tau+2} &= \frac{1}{L}\sum_{j=1}^{L} c^{j*}_{\tau+1} \\ &> 2(1+\bar{g})\frac{1}{L}\sum_{j=1}^{L} c^{j*}_\tau = 2(1+\bar{g})\left(f(k^*_\tau, e^*_\tau, A_\tau) - k^*_{\tau+1}\right). \end{aligned}$$

Hence, by the choice of τ and (A.34),

$$\begin{aligned} k^*_{\tau+2} &< 2(1+\bar{g})k^*_{\tau+1} + f(k^*_{\tau+1}, e^*_{\tau+1}, A_{\tau+1}) - 2(1+\bar{g})f(k^*_\tau, e^*_\tau, A_\tau) \\ &\le (1+\bar{g})\left(2k'_{\tau+1} - f(k'_\tau, e^*_\tau, A_\tau)\right) + f(k'_{\tau+1}, e^*_{\tau+1}, A_{\tau+1}) - (1+\bar{g})f(k'_\tau, e^*_\tau, A_\tau). \end{aligned} \tag{A.59}$$

It follows from (A.54) that

$$f(k'_{\tau+1}, e^*_{\tau+1}, A_{\tau+1}) < (1+\bar{g})f(k'_\tau, e^*_\tau, A_\tau). \tag{A.60}$$

Moreover, using (A.55) and (A.51), we get

$$\frac{f(k'_\tau, e^*_\tau, A_\tau)}{k'_\tau} = \frac{1 + r(k'_\tau, e^*_\tau, A_\tau)}{\alpha_1} > \frac{1+r'}{\alpha_1} > \frac{2(1+\bar{g})}{\alpha_1\beta_L} > 2(1+\bar{g}),$$

and hence, by (A.53) and (A.11), we get

$$2k'_{\tau+1} = 2(1+\underline{g})k'_\tau < 2(1+\bar{g})k'_\tau < f(k'_\tau, e^*_\tau, A_\tau). \tag{A.61}$$

Combining (A.60) and (A.61), we have

$$(1+\bar{g})\left(2k'_{\tau+1} - f(k'_\tau, e^*_\tau, A_\tau)\right) + f(k'_{\tau+1}, e^*_{\tau+1}, A_{\tau+1}) - (1+\bar{g})f(k'_\tau, e^*_\tau, A_\tau) < 0.$$

Now it follows from (A.59) that $k^*_{\tau+2} < 0$, which is impossible. Hence $\tau + 2 > T$, and therefore the inequality (A.56) holds for $t \le T-1$.

Using (A.56), (A.34), (A.53), (A.8), (A.50), (A.10) and (A.11), we obtain for all $t = 0, 1, \ldots, T-1$,

$$\begin{aligned} w^*_t = \alpha_2 A_t (k^*_t)^{\alpha_1} (e^*_t)^{\alpha_3} &> \alpha_2 A_t (k'_t)^{\alpha_1} (e^*_t)^{\alpha_3} \\ &= \alpha_2 (1+\lambda)^t A_0 (1+\underline{g})^{t\alpha_1} (k'_0)^{\alpha_1} \left(\frac{e^*_t}{e^*_0}\right)^{\alpha_3} (e^*_0)^{\alpha_3} \\ &> (1+\underline{g})^{t\alpha_1} (1+\lambda)^t \left(\frac{\tilde{\varepsilon}(1-\bar{\varepsilon})}{\bar{\varepsilon}}\right)^{t\alpha_3} \alpha_2 A_0 (k'_0)^{\alpha_1} (\tilde{e}_0)^{\alpha_3} \\ &\ge (1+\underline{g})^{t\alpha_1} (1+\tilde{g})^{t(1-\alpha_1)} w'_0 > (1+\underline{g})^t w'_0 = w'_t. \end{aligned}$$

□

Step II.3. Existence of an equilibrium.

Now we are ready to prove the following lemma which maintains that the sequence $\mathcal{E}^*_\infty$ defined by (A.49) is a competitive equilibrium under given extraction rates in our model.

Lemma A.2. *The sequence $\mathcal{E}^*_\infty$ defined by (A.49) is a competitive equilibrium starting from $\mathcal{I}_0$.*

Proof. It is clear that by construction $\mathcal{E}^*_\infty$ satisfies conditions 2–7 for the competitive equilibrium in the private property regime. Thus to prove that $\mathcal{E}^*_\infty$ is a competitive equilibrium it is sufficient to show that $\{(c^{j*}_t)^L_{j=1}, (s^{j*}_t)^L_{j=1}\}^\infty_{t=0}$ is a solution to the problem (4.8) at $r_t = r^*_t$, $w_t = w^*_t$, and $s^j_{-1} = \frac{q^*_0}{1+r^*_0}\hat{R}^j_{-1} + \hat{k}^j_0$.

Let c'_t be such that

$$c'_t = \frac{w'_t}{2}, \quad t = 0, 1, \ldots. \tag{A.62}$$

It is clear that

$$c'_{t+1} = (1+\underline{g})c'_t,$$

and hence

$$\sum_{t=0}^{\infty} \beta^t \ln c'_t = \frac{\ln c'_0}{1-\beta} + \ln(1+\underline{g}) \sum_{t=0}^{\infty} t\beta^t = \frac{\ln c'_0}{1-\beta} + \frac{\beta}{(1-\beta)^2} \ln(1+\underline{g}).$$

Consider the instantaneous utility function

$$u_t(c) = \ln c - \ln c'_t.$$

Clearly, the solution to the problem (4.8) will not change if we replace the instantaneous utility function $\ln c$ with the function $u_t(c)$. It is also clear that $u_t(c'_t) = 0$.

Note that for all t,

$$\bar{c}_t > Lf(\bar{k}_t, \bar{e}, A_t) > Lf(k'_t, \tilde{e}_t, A_t) > c'_t,$$

and hence $u_t(\bar{c}_t) > 0$. Moreover, it follows from (A.16) that

$$\begin{aligned}\sum_{t=0}^{\infty} \beta^t u_t(\bar{c}_t) = \frac{\ln \bar{c}_0}{1-\beta} + \frac{\beta}{(1-\beta)^2}\ln(1+\bar{g}) - \frac{\ln c'_0}{1-\beta} + \frac{\beta}{(1-\beta)^2}\ln(1+\underline{g}) \\ = \frac{1}{1-\beta}\ln\left(\frac{\bar{c}_0}{c'_0}\right) + \frac{\beta}{(1-\beta)^2}\ln\left(\frac{1+\bar{g}}{1+\underline{g}}\right).\end{aligned}$$

Now assume that $\{(c_t^{j*})_{j=1}^L, (s_t^{j*})_{j=1}^L\}_{t=0}^{\infty}$ is not a solution to the problem (4.8). Then for some j (we fix this j and omit it in the remaining part of the proof for the simplicity of notation) there is a feasible sequence $\{\widehat{c}_t, \widehat{s}_t\}_{t=0}^{\infty}$ such that

$$\widehat{U} > U^*, \text{ where } \widehat{U} = \sum_{t=0}^{\infty} \beta^t u_t(\widehat{c}_t), \text{ and } U^* = \sum_{t=0}^{\infty} \beta^t u_t(c_t^*).$$

Let $0 < \Delta < \widehat{U} - U^*$, and let Θ be such that

$$\sum_{t=\Theta+1}^{\infty} \beta^t u_t(\bar{c}_t) < \min\left\{\frac{\Delta}{2}, \ln 2\right\}.$$

Further, let

$$U^{*\Theta} = \sum_{t=0}^{\Theta} \beta^t u_t(c_t^*), \quad \widehat{U}^{\Theta} = \sum_{t=0}^{\Theta} \beta^t u_t(\widehat{c}_t),$$

and

$$U^*(T) = \sum_{t=0}^{T} \beta^t u_t(c_t^*(T)), \quad U^{*\Theta}(T) = \sum_{t=0}^{\Theta} \beta^t u_t(c_t^*(T)),$$

for $T = \Theta+1, \Theta+2, \ldots$.

Claim A.13. *There is a sequence* $\{T_{\Theta n}\}_{n=1}^{\infty}$ *such that*

$$U^{*\Theta}(T_{\Theta n}) \underset{n\to\infty}{\longrightarrow} U^{*\Theta}.$$

Proof. It is sufficient to note that since $\mathcal{E}^*_\infty$ is obtained as a result of the application of the process described at Step II.1 to the sequence $\{\mathcal{E}^*_T\}_{T=1,2,\dots}$, there is a sequence $\{T_{\Theta n}\}_{n=1}^\infty$ such that for $t = 0, 1, \dots, \Theta$:

$$\lim_{n\to\infty} k^*_t(T_{\Theta n}) = k^*_t, \quad \lim_{n\to\infty} r^*_t(T_{\Theta n}) = r^*_t, \quad \lim_{n\to\infty} w^*_t(T_{\Theta n}) = w^*_t,$$
$$\lim_{n\to\infty} q^*_t(T_{\Theta n}) = q^*_t, \quad \lim_{n\to\infty} e^*_t(T_{\Theta n}) = e^*_t, \quad \lim_{n\to\infty} R^*_t(T_{\Theta n}) = R^*_t,$$
$$\lim_{n\to\infty} c^*_t(T_{\Theta n}) = c^*_t, \quad \lim_{n\to\infty} s^*_t(T_{\Theta n}) = s^*_t.$$

□

Let us formulate a claim that will be useful in what follows.

Statement 1. *Suppose that $F_r(x, y)$, $r = 1, \dots, R$, are continuous and concave in y functions defined on $X \times Y$, where X and Y are convex compact subsets of finite dimensional spaces. If there exists $\hat{y} \in Y$ such that $F_r(x, \hat{y}) > 0$ for all $x \in X$, $r = 1, \dots, R$, then the correspondence*

$$x \to \bigcap_{r=1}^{R} \{y \in Y \mid F_r(x, y) \geq 0\}$$

is upper and lower semi-continuous, and all sets

$$\bigcap_{r=1}^{R} \{y \in Y \mid F_r(x, y) \geq 0\}$$

are non-empty, convex and closed.

Proof. It is trivial. □

Let $W^{*\Theta}$ be the maximum value of utility in the problem

$$\begin{aligned} \max \; & \sum_{t=0}^{\Theta} \beta^t u_t(c_t), \\ \text{s. t.} \; & c_t + s_t \leq (1 + r^*_t)\, s_{t-1} + w^*_t, \\ & s_t \geq 0, \quad t = 0, 1, \dots, \Theta, \end{aligned}$$

and $W^{*\Theta}(T)$ be the maximum value of utility in the problem

$$\begin{aligned} \max \; & \sum_{t=0}^{\Theta} \beta^t u_t(c_t), \\ \text{s. t.} \; & c_t + s_t \leq (1 + r^*_t(T))\, s_{t-1} + w^*_t(T), \\ & s_t \geq 0, \quad t = 0, 1, \dots, \Theta, \end{aligned} \tag{A.63}$$

for $T = \Theta + 1, \Theta + 2, \ldots$.

Claim A.14.

$$W^{*\Theta}(T_{\Theta n}) \xrightarrow[n\to\infty]{} W^{*\Theta}.$$

Proof. Consider the correspondence that takes to each

$$\{(1 + r_0, w_0), \ldots, (1 + r_\Theta, w_\Theta)\} \\ \in \prod_{t=0}^{\Theta} \Big([1 + r(\bar{k}_t, \tilde{e}_t, A_t), 1 + r(\tilde{k}_t, \bar{e}_t, A_t)] \times [w(\tilde{k}_t, \tilde{e}_t, A_t), w(\bar{k}_t, \bar{e}_t, A_t)] \Big)$$

the set

$$\{(c_0, s_0), \ldots, (c_\Theta, s_\Theta\} \in \mathbb{R}^{2(\Theta+1)}$$

which is such that, with $s_{-1} = \hat{s}_{-1}$ being given,

$$c_t + s_t \le (1 + r_t^*(T))\, s_{t-1} + w_t^*(T), \text{ and } s_t \ge 0,$$

hold for all $t = 0, 1, \ldots, \Theta$.

By Statement 1, this correspondence is lower- and upper-semicontinuous, and it is sufficient to apply the Maximum Theorem. □

Claim A.15.

$$U^*(T) \ge W^{*\Theta}(T).$$

Proof. Let for some $T > \Theta + 1$, the sequence $\{(\breve{c}_0, \breve{s}_0), \ldots, (\breve{c}_\Theta, \breve{s}_\Theta)\}$ be a solution to (A.63). Let further for $t = \Theta + 1, \ldots, T$, $\{(\breve{c}_t, \breve{s}_t)\}$ be defined recursively by

$$\breve{c}_t = c_t', \quad \breve{s}_t = (1 + r_t^*(T))\, \breve{s}_{t-1} + w_t^*(T) - \breve{c}_t. \tag{A.64}$$

We show that given $s_{-1} = \hat{s}_{-1}$, the sequence

$$\{(\breve{c}_0, \breve{s}_0), \ldots, (\breve{c}_\Theta, \breve{s}_\Theta), (\breve{c}_{\Theta+1}, \breve{s}_{\Theta+1}), \ldots, (\breve{c}_T, \breve{s}_T)\} \tag{A.65}$$

is feasible for the problem

$$\begin{aligned} &\max \sum_{t=0}^{T} \beta^t u_t(c_t), \\ &\text{s. t. } c_t + s_t \le (1 + r_t^*(T))\, s_{t-1} + w_t^*(T), \\ &\qquad s_t \ge 0, \quad t = 0, 1, \ldots, T. \end{aligned} \tag{A.66}$$

It is sufficient to check that $\breve{s}_t \geq 0$ for $t = \Theta + 1, \dots, T$. By Claim A.12, we have for $\Theta + 1 \leq t \leq T - 1$,

$$\breve{c}_t = c'_t = \frac{w'_t}{2} < w^*_t(T).$$

We prove that $\breve{s}_t > 0$ for $t = \Theta + 1, \dots, T - 1$ recursively. Clearly, $\breve{s}_\Theta = 0$. Suppose that $\breve{s}_{t-1} \geq 0$ for $\Theta + 1 \leq t < T - 2$. Then

$$\breve{s}_t = (1 + r^*_t(T))\, \breve{s}_{t-1} + w^*_t(T) - \breve{c}_t \geq w^*_t(T) - c'_t > 0.$$

In particular, $\breve{s}_{T-2} > 0$. For $t = T - 1$ we have

$$\begin{aligned} \breve{s}_{T-1} = \big(1 + r^*_{T-1}(T)\big)\, \breve{s}_{T-2} + w^*_{T-1}(T) - \breve{c}_{T-1} \\ \geq w^*_{T-1}(T) - c'_{T-1} > 2c'_{T-1} - c'_{T-1} = c'_{T-1}. \end{aligned}$$

For $t = T$ we know from Claim A.12 that either

$$w^*_T(T) > c'_T,$$

or

$$1 + r^*_T \geq 1 + r'.$$

In the first case, we can apply the same reasoning as before:

$$\breve{s}_T = (1 + r^*_T(T))\, \breve{s}_{T-1} + w^*_T(T) - \breve{c}_T \geq w^*_T(T) - c'_T > 0.$$

In the second case, using (A.51) and (A.11), we have

$$\begin{aligned} \breve{s}_T &= (1 + r^*_T(T))\, \breve{s}_{T-1} + w^*_T(T) - \breve{c}_T > (1 + r')c'_{T-1} - c'_T \\ &> \frac{2}{\beta_L}(1 + \bar{g})c'_{T-1} - c'_T > (1 + \bar{g})c'_{T-1} - c'_T > (1 + \underline{g})c'_{T-1} - c'_T = 0. \end{aligned}$$

Thus we have proved that the sequence (A.65) is feasible for the problem (A.66). Since the sequence

$$\{(c^*_0(T), s^*_0(T)), \dots, (c^*_T(T), s^*_T(T))\}$$

is the solution to this problem, we have

$$U^*(T) = \sum_{t=0}^{T} \beta^t u_t(c^*_t(T)) \geq \sum_{t=0}^{\Theta} \beta^t u_t(\breve{c}_t) + \sum_{t=\Theta+1}^{T} \beta^t u_t(c'_t) = \sum_{t=0}^{\Theta} \beta^t u_t(\breve{c}_t) = W^{*\Theta}(T).$$

□

Let us prove another useful claim.

Claim A.16. *For all* $t = 0, 1, \ldots, T$,

$$k_t^*(T) \leq \bar{\kappa}(A_t(e_t^*(T))^{\alpha_3})^{\frac{1}{1-\alpha_1}}. \tag{A.67}$$

Proof. It is sufficient to show that

$$\kappa_t^* \leq \bar{\kappa}, \quad t = 0, 1, \ldots, T, \tag{A.68}$$

where

$$\kappa_t^* := \frac{k_t^*(T)}{(A_t(e_t^*(T))^{\alpha_3})^{\frac{1}{1-\alpha_1}}}.$$

By (A.12), (A.10), and (A.34),

$$\bar{\kappa} = \frac{1}{(1+\tilde{g})^{\frac{1}{1-\alpha_1}}} \geq \frac{\hat{k}_0}{(A_0\tilde{e}_0)^{\alpha_3})^{\frac{1}{1-\alpha_1}}} \geq \frac{\hat{k}_0}{(A_0(e_0^*(T))^{\alpha_3})^{\frac{1}{1-\alpha_1}}} = \kappa_0^*,$$

which proves (A.68) for $t = 0$. We prove it for $t = 1, \ldots, T$ recursively. Suppose that $\kappa_t^* \leq \bar{\kappa}$. If follows from (A.58) that for all t,

$$k_{t+1}^*(T) \leq f(k_t^*(T), e_t^*(T), A_t) = (k_t^*(T))^{\alpha_1}\left((A_t(e_t^*(T))^{\alpha_3})^{\frac{1}{1-\alpha_1}}\right)^{1-\alpha_1},$$

and hence, due to (A.8), (A.50), and (A.10),

$$(\kappa_t^*)^{\alpha_1} \geq \frac{k_{t+1}^*(T)}{(A_t(e_t^*(T))^{\alpha_3})^{\frac{1}{1-\alpha_1}}} = \kappa_{t+1}^* \frac{(A_{t+1}(e_{t+1}^*(T))^{\alpha_3})^{\frac{1}{1-\alpha_1}}}{(A_t(e_t^*(T))^{\alpha_3})^{\frac{1}{1-\alpha_1}}}$$
$$= \kappa_{t+1}^*\left((1+\lambda)\left(\frac{e_{t+1}^*(T)}{e_t^*(T)}\right)^{\alpha_3}\right)^{\frac{1}{1-\alpha_1}} > \kappa_{t+1}^*(1+\lambda)^{\frac{1}{1-\alpha_1}}\left(\frac{\tilde{\varepsilon}(1-\bar{\varepsilon})}{\bar{\varepsilon}}\right)^{\frac{\alpha_3}{1-\alpha_1}} \geq (1+\tilde{g})\kappa_{t+1}^*.$$

Therefore, by (A.12),

$$\kappa_{t+1}^* \leq \frac{(\kappa_t^*)^{\alpha_1}}{1+\tilde{g}} \leq \frac{(\bar{\kappa})^{\alpha_1}}{1+\tilde{g}} = \bar{\kappa}.$$

Thus (A.68) holds for all $t = 0, 1, \ldots, T$. □

Denote

$$1 + \bar{r} = \alpha_1 \frac{1}{(\bar{\kappa})^{1-\alpha_1}}.$$

By (A.12) and the choice of $1 + \underline{g}$,

$$\beta_L(1+\bar{r}) = \beta_L\alpha_1\frac{(\bar{\kappa})^{\alpha_1}}{\bar{\kappa}} = \beta_L\alpha_1(1+\tilde{g}) = (1+\underline{g}), \tag{A.69}$$

and hence, by (A.67), for all $t = 0, 1, \dots, T$, we have

$$1 + r_t^*(T) = \alpha_1 \frac{A_t(e_t^*(T))^{\alpha_3}}{(k_t^*(T))^{1-\alpha_1}} \geq \alpha_1 \frac{A_t(e_t^*(T))^{\alpha_3}}{(\bar{\kappa})^{1-\alpha_1} A_t(e_t^*(T))^{\alpha_3}} = \alpha_1 \frac{1}{(\bar{\kappa})^{1-\alpha_1}} = 1 + \bar{r}. \tag{A.70}$$

Claim A.17.

$$U^* \geq U^{*\Theta}.$$

Proof. Let us prove that for any $T > \Theta + 1$,

$$c_\Theta^*(T) \geq c'_\Theta. \tag{A.71}$$

Assume that $c_\Theta^*(T) < c'_\Theta$. We show that this inequality implies $c_t^*(T) < c'_t$ for all $t \leq \Theta$. Indeed, if $c_t^*(T) < c'_t$ for some $t < \Theta$, then it follows from (A.3), (A.70) and (A.69) that

$$c_t^*(T) \geq \beta_j(1 + r_t^*(T))c_{t-1}^*(T) \geq \beta_L(1 + \bar{r})c_{t-1}^*(T) = (1 + \underline{g})c_{t-1}^*(T), \tag{A.72}$$

and thus

$$c_{t-1}^*(T) \leq \frac{c_t^*(T)}{1 + \underline{g}} < \frac{c'_t}{1 + \underline{g}} = c'_{t-1}.$$

Hence

$$\sum_{t=0}^{\Theta} \beta^t u_t(c_t^*(T)) = \sum_{t=0}^{\Theta} \beta^t \left(\ln c_t^*(T) - \ln c'_t\right) < 0.$$

At the same time, by the choice of Θ, we have

$$\sum_{t=\Theta+1}^{T} \beta^t u_t(c_t^*(T)) \leq \sum_{t=\Theta+1}^{T} \beta^t u_t(\bar{c}_t) \leq \sum_{t=\Theta+1}^{\infty} \beta^t u_t(\bar{c}_t) < \ln 2.$$

Therefore

$$\sum_{t=0}^{T} \beta^t u_t(c_t^*(T)) = \sum_{t=0}^{\Theta} \beta^t u_t(c_t^*(T)) + \sum_{t=\Theta+1}^{T} \beta^t u_t(c_t^*(T)) < \ln 2. \tag{A.73}$$

Consider the sequence

$$\{(\check{c}_0, \check{s}_0), \dots, (\check{c}_T, \check{s}_T)\}, \tag{A.74}$$

defined as follows: for $t \leq \Theta$

$$\check{c}_t = w_t^*(T), \quad \check{s}_t = 0,$$

and for $t = \Theta + 1, \dots, T$, $\{(\check{c}_t, \check{s}_t)\}$ is given by (A.64). It follows from Claim A.12 that for $t \leq \Theta$,

$$\check{c}_t = w_t^*(T) > w'_t > c'_t. \tag{A.75}$$

Repeating the argument from the proof of Claim A.15, we obtain that the sequence (A.74) is feasible for the problem (A.66). At the same time, the sequence

$$\{(c_0^*(T), s_0^*(T)), \ldots, (c_T^*(T), s_T^*(T))\}$$

is the solution to this problem. Hence, using (A.75), we get

$$\begin{aligned}\sum_{t=0}^{T} \beta^t u_t(c_t^*(T)) \geq \sum_{t=0}^{T} \beta^t u_t(\breve{c}_t) = \sum_{t=0}^{\Theta} \beta^t u_t(w_t^*(T)) + \sum_{t=\Theta+1}^{T} \beta^t u_t(c_t') \\ = u_0(w_0^*(T)) + \sum_{t=1}^{\Theta} \beta^t u_t(\breve{c}_t) > u_0(w_0^*(T)) = \ln w_0^*(T) - \ln c_0' \\ > \ln w_0' - \ln c_0' = \ln\left(\frac{w_0'}{c_0'}\right) = \ln 2,\end{aligned}$$

a contradiction of (A.73).

Thus (A.71) holds, and using the fact that c_Θ^* is a limit of the sequence $\{c_\Theta^*(T_{\Theta n})\}_{n=1}^{\infty}$, we have

$$c_\Theta^* \geq c_\Theta'.$$

It immediately follows from (A.72) that for all $\Theta + 1 \leq t \leq T$,

$$c_t^*(T) \geq c_t'.$$

Since every c_t^* is a cluster point of the sequence $\{c_t^*(T)\}_{T=1,2,\ldots}$, we get

$$c_t^* \geq c_t', \quad t = \Theta + 1, \Theta + 2, \ldots.$$

It follows that

$$U^* - U^{*\Theta} = \sum_{t=\Theta+1}^{\infty} \beta^t u_t(c_t^*) = \sum_{t=\Theta+1}^{\infty} \beta^t \left(\ln c_t^* - \ln c_t'\right) \geq 0,$$

which completes the proof. □

Claim A.18.

$$U^{*\Theta}(T) > U^*(T) - \frac{\Delta}{2}, \quad T = \Theta + 1, \Theta + 2, \ldots; \tag{A.76}$$

$$W^{*\Theta} > \widehat{U} - \frac{\Delta}{2}. \tag{A.77}$$

Proof. Clearly, $\bar{c}_t > c_t^*(T)$ and $\bar{c}_t > \widehat{c}_t$ for all t. It follows from the choice of Θ that

$$U^*(T) - U^{*\Theta}(T) = \sum_{t=0}^{T} \beta^t u_t(c_t^*(T)) - \sum_{t=0}^{\Theta} \beta^t u_t(c_t^*(T)) = \sum_{t=\Theta+1}^{T} \beta^t u_t(c_t^*(T))$$
$$< \sum_{t=\Theta+1}^{T} \beta^t u_t(\bar{c}_t) \le \sum_{t=\Theta+1}^{\infty} \beta^t u_t(\bar{c}_t) < \frac{\Delta}{2},$$

which proves (A.76).

Due to the definition of $W^{*\Theta}$, we have $W^{*\Theta} \ge \widehat{U}^{\Theta}$. Now it is easily seen that

$$W^{*\Theta} \ge \widehat{U}^{\Theta} = \sum_{t=0}^{\Theta} \beta^t u_t(\widehat{c}_t) = \widehat{U} - \sum_{t=\Theta+1}^{\infty} \beta^t u_t(\widehat{c}_t) \ge \widehat{U} - \sum_{t=\Theta+1}^{\infty} \beta^t u_t(\bar{c}_t) > \widehat{U} - \frac{\Delta}{2},$$

which proves (A.77). □

Now, combining Claims A.13–A.15 and A.17–A.18, we obtain

$$U^* \ge U^{*\Theta} = \lim_{n\to\infty} U^{*\Theta}(T_{\Theta n}) \ge \lim_{n\to\infty} U^*(T_{\Theta n}) - \frac{\Delta}{2}$$
$$\ge \lim_{n\to\infty} W^{*\Theta}(T_{\Theta n}) - \frac{\Delta}{2} = W^{*\Theta} - \frac{\Delta}{2} > \widehat{U} - \Delta,$$

which contradicts the choice of Δ. This contradiction completes the proof of the lemma. □

Thus the proof of Theorem A.1 is finally complete, and there exists a competitive equilibrium. □

B. Existence of a competitive equilibrium in the public property regime

The existence of a competitive equilibrium in the general equilibrium Ramsey-type model with public property over exhaustible natural resources under given non-degenerate extraction rates is established in the following theorem.

Theorem B.1. *For any non-degenerate state $\mathcal{I}_{\tau-1}$ there exists a competitive $\mathbb{E}_\tau$-equilibrium starting from $\mathcal{I}_{\tau-1}$.*

The proof of Theorem B.1 also follows the ideas presented in Borissov and Dubey (2015) and is in many respects similar to the proof of Theorem A.1 in Appendix 4.9 However, for the sake of completeness, we provide below a full proof for Theorem B.1.

Proof. Without loss of generality, let us consider the case $\tau = 0$ and prove the existence of a competitive $\mathbb{E}_0$-equilibrium starting from $\mathcal{I}_{-1} = \{(\hat{s}^j_{-1})^L_{j=1}, \hat{R}_{-1}\}$.

The proof is divided into two steps. First we show the existence of a competitive equilibrium in the finite horizon model. We prove that for any $T > 0$ there exists a finite T-period competitive equilibrium under given extraction rates. Second, we construct a candidate for a competitive equilibrium in the infinite horizon model by applying some kind of diagonalization procedure to the sequence of finite T-period equilibrium paths, and then prove that this candidate is indeed a competitive equilibrium in the infinite horizon model.

Step I. Competitive equilibrium under given extraction rates in the finite horizon model.

Let us define a finite T-period competitive equilibrium under given extraction rates along the lines of the above definition. Suppose that $\mathcal{I}_{-1} = \{(\hat{s}^j_{-1})^L_{j=1}, \hat{R}_{-1}\}$ is a non-degenerate initial state, $\mathbb{E}_0 = \{\varepsilon_t\}^\infty_{t=0}$ is a non-degenerate sequence of extraction rates, and recall that $e_t = e_t(\mathbb{E}_0)$, $t = 0, 1, \ldots, T$.

Definition B.1. *A sequence*

$$\left\{(c^{j**}_t)^L_{j=1}, (s^{j**}_t)^L_{j=1}, k^{**}_t, r^{**}_t, w^{**}_t, q^{**}_t, v^{**}_t\right\}_{t=0,1,\ldots,T}$$

is a finite T-period competitive $\mathbb{E}_0$-equilibrium starting from $\mathcal{I}_{-1}$ if

1. *For each* $j = 1, \ldots, L$, *the sequence* $\{c_t^{j**}, s_t^{j**}\}_{t=0}^T$ *is a solution to the following utility maximization problem:*

$$\begin{aligned} \max \ & \sum_{t=0}^{T} \beta_j^t \ln c_t^j, \\ \text{s. t.} \ & c_t^j + s_t^j \leq (1 + r_t)\, s_{t-1}^j + w_t + v_t, \quad t = 0, 1, \ldots, T, \\ & s_t^j \geq 0, \quad t = 0, 1, \ldots, T, \end{aligned} \tag{B.1}$$

at $r_t = r_t^{**}$, $w_t = w_t^{**}$, $v_t = v_t^{**}$, *and* $s_{-1}^j = \hat{s}_{-1}^j$;

2. *Aggregate savings are equal to the capital stock:*

$$\sum_{j=1}^{L} s_{t-1}^{j**} = Lk_t^{**}, \quad t = 0, 1, \ldots, T;$$

3. *Capital is paid its marginal product:*

$$1 + r_t^{**} = \alpha_1 A_t (k_t^{**})^{\alpha_1 - 1} (e_t)^{\alpha_3}, \quad t = 0, 1, \ldots, T;$$

4. *Labor is paid its marginal product:*

$$w_t^{**} = \alpha_2 A_t (k_t^{**})^{\alpha_1} (e_t)^{\alpha_3}, \quad t = 0, 1, \ldots, T;$$

5. *The price of natural resources is equal to the marginal product:*

$$q_t^{**} = \alpha_3 A_t (k_t^{**})^{\alpha_1} (e_t)^{\alpha_3 - 1}, \quad t = 0, 1, \ldots, T;$$

6. *The resource income is given by:*

$$v_t^{**} = q_t^{**} e_t, \quad t = 0, 1, \ldots, T.$$

Clearly, the solution to the problem (B.1), $\{c_t^{j**}, s_t^{j**}\}_{t=0}^T$, satisfies the following conditions:

$$c_t^{j**} + s_t^{j**} = (1 + r_t^{**}) s_{t-1}^{j**} + w_t^{**} + v_t^{**}, \quad t = 0, 1, \ldots, T, \tag{B.2}$$

$$c_{t+1}^{j**} \geq \beta_j (1 + r_{t+1}^{**}) c_t^{j**} \ (= \text{if } s_t^{j**} > 0), \quad t = 0, 1, \ldots, T-1, \tag{B.3}$$

$$s_T^{j**} = 0,$$

where $s_{-1}^{j**} = \hat{s}_{-1}^j$.

The existence of a competitive equilibrium under given extraction rates in the finite horizon model is shown via the following steps. First we present some preliminary definitions and results that will be useful in what follows. Second, we reduce our finite horizon model to a game, and show that there exists a Nash equilibrium in this game. Third, we prove that a Nash equilibrium in the game that represents our model determines a competitive equilibrium under given extraction rates in the finite horizon model.

Step I.1. Preliminaries.

We use the notation

$$\begin{aligned} f(k,e,A) &:= Ak^{\alpha_1}e^{\alpha_3}, \\ 1+r(k,e,A) &:= \alpha_1 Ak^{\alpha_1-1}e^{\alpha_3}, \\ w(k,e,A) &:= \alpha_2 Ak^{\alpha_1}e^{\alpha_3}, \\ q(k,e,A) &:= \alpha_3 Ak^{\alpha_1}e^{\alpha_3-1}, \\ v(k,e,A) &:= \alpha_3 Ak^{\alpha_1}e^{\alpha_3}, \end{aligned}$$

for the output (production function), interest rate, wage rate, resource price and the resource income as depending on the capital stock k, the volume of extraction e and total factor productivity A. Clearly,

$$(1+r(k,e,A))k + w(k,e,A) + v(k,e,A) = f(k,e,A). \tag{B.4}$$

Denote

$$\bar{e} = \frac{\hat{R}_{-1}}{L}.$$

Claim B.1. *For all* t,

$$\bar{e}\delta^{t+1} \le e_t \le \bar{e}, \tag{B.5}$$

and

$$\frac{\delta^2}{1-\delta} < \frac{e_{t+1}}{e_t} < \frac{(1-\delta)^2}{\delta}. \tag{B.6}$$

Proof. It follows from (4.41) that for all $t > 0$,

$$e_t = \frac{\varepsilon_t R_{t-1}}{L} = \frac{\varepsilon_t(1-\varepsilon_{t-1})R_{t-2}}{L} = \ldots = \frac{\hat{R}_{-1}}{L}\varepsilon_t(1-\varepsilon_{t-1})\cdots(1-\varepsilon_0).$$

Since we require a sequence of extraction rates to be non-degenerate, i.e., $\delta \le \varepsilon_t \le 1-\delta$, it follows that for all $t \ge 0$,

$$\varepsilon_t \ge \delta, \qquad 1-\varepsilon_t \ge \delta. \tag{B.7}$$

Hence

$$e_t = \bar{e}\varepsilon_t(1-\varepsilon_{t-1})\cdots(1-\varepsilon_0) \geq \bar{e}\delta^{t+1} > 0.$$

It is also clear that

$$e_t \leq \bar{e},$$

which proves (B.5).

Moreover, for all t,

$$\frac{e_{t+1}}{e_t} = \frac{\bar{e}\varepsilon_{t+1}(1-\varepsilon_t)(1-\varepsilon_{t-1})\cdots(1-\varepsilon_0)}{\bar{e}\varepsilon_t(1-\varepsilon_{t-1})\cdots(1-\varepsilon_0)} = \frac{\varepsilon_{t+1}(1-\varepsilon_t)}{\varepsilon_t}.$$

Using (B.7), we obtain (B.6). □

Denote

$$1+\bar{g} = (1+\lambda)^{\frac{1}{1-\alpha_1}}\left(\frac{(1-\delta)^2}{\delta}\right)^{\frac{\alpha_3}{1-\alpha_1}},$$

where λ is the growth rate of the total factor productivity:

$$A_t = (1+\lambda)A_{t-1} = (1+\lambda)^t A_0. \tag{B.8}$$

Let also

$$1+\tilde{g} = \min\left\{(1+\lambda)^{\frac{1}{1-\alpha_1}}\left(\frac{\delta^2}{1-\delta}\right)^{\frac{\alpha_3}{1-\alpha_1}}, \frac{A_0(\delta\bar{e})^{\alpha_3}}{(\hat{k}_0)^{1-\alpha_1}}\right\},$$

and

$$1+\underline{g} = \beta_L\alpha_1(1+\tilde{g}).$$

It is clear that

$$1+\tilde{g} \leq (1+\lambda)^{\frac{1}{1-\alpha_1}}\left(\frac{\delta^2}{1-\delta}\right)^{\frac{\alpha_3}{1-\alpha_1}}, \tag{B.9}$$

and

$$1+\bar{g} > 1+\tilde{g} > 1+\underline{g}. \tag{B.10}$$

Suppose that $\bar{\kappa} > 0$ is given by

$$(1+\tilde{g})\bar{\kappa} = (\bar{\kappa})^{\alpha_1}, \tag{B.11}$$

Let the sequence $\{\bar{k}_t\}$ be given by

$$\bar{k}_{t+1} = (1+\bar{g})\bar{k}_t,$$

where

$$\bar{k}_0 = \bar{\kappa}(A_0\bar{e}^{\alpha_3})^{\frac{1}{1-\alpha_1}}.$$

We show that

$$\bar{\kappa}(A_t e_t^{\alpha_3})^{\frac{1}{1-\alpha_1}} \le \bar{k}_t. \tag{B.12}$$

It follows from (B.8), (B.6), and the choice of $\bar{k}_0$ that

$$\begin{aligned}
\bar{\kappa}(A_t e_t^{\alpha_3})^{\frac{1}{1-\alpha_1}} &= \bar{\kappa}\left((1+\lambda)^t A_0 \left(\frac{e_t}{e_0}\right)^{\alpha_3} e_0^{\alpha_3}\right)^{\frac{1}{1-\alpha_1}} \\
&\le \bar{\kappa}\left((1+\lambda)^t \left(\frac{(1-\delta)^2}{\delta}\right)^{t\alpha_3} A_0 \bar{e}^{\alpha_3}\right)^{\frac{1}{1-\alpha_1}} \\
&= \bar{\kappa}\left(A_0 \bar{e}^{\alpha_3}\right)^{\frac{1}{1-\alpha_1}} (1+\lambda)^{\frac{t}{1-\alpha_1}} \left(\frac{(1-\delta)^2}{\delta}\right)^{\frac{t\alpha_3}{1-\alpha_1}} = (1+\bar{g})^t \bar{k}_0 = \bar{k}_t.
\end{aligned}$$

Furthermore,

$$f(\bar{k}_t, e_t, A_t) < \bar{k}_{t+1}. \tag{B.13}$$

Indeed, by (B.12), (B.11) and (B.10),

$$\begin{aligned}
f(\bar{k}_t, e_t, A_t) - \bar{k}_{t+1} &= (\bar{k}_t)^{\alpha_1} A_t e_t^{\alpha_3} - (1+\bar{g})\bar{k}_t \\
&= \bar{k}_t\left(\frac{A_t e_t^{\alpha_3}}{(\bar{k}_t)^{1-\alpha_1}} - (1+\bar{g})\right) \le \bar{k}_t\left(\frac{A_t e_t^{\alpha_3}}{(\bar{\kappa})^{1-\alpha_1} A_t e_t^{\alpha_3}} - (1+\bar{g})\right) \\
&= \bar{k}_t\left(\frac{\bar{\kappa}^{\alpha_1}}{\bar{\kappa}} - (1+\bar{g})\right) = \bar{k}_t\left((1+\tilde{g}) - (1+\bar{g})\right) < 0.
\end{aligned}$$

Denote

$$\bar{c}_t := L\bar{k}_{t+1}. \tag{B.14}$$

Clearly,

$$\bar{c}_{t+1} = (1+\bar{g})\bar{c}_t. \tag{B.15}$$

Let the sequence $\{\tilde{k}_t\}_{t=0}^{\infty}$ be defined recursively as follows. We take $\tilde{k}_0$ such that $0 < \tilde{k}_0 < \hat{k}_0$. Suppose we are given $\tilde{k}_t > 0$. Consider the following equation in k:

$$k + \frac{\bar{c}_{t+1}}{\beta_L(1 + r(k, e_{t+1}, A_{t+1}))} = f(\tilde{k}_t, e_t, A_t).$$

The left-hand side of the above equation is increasing in k, and equals to 0 when $k = 0$. Thus there is a unique positive solution to this equation. We take $\tilde{k}_{t+1} > 0$ as this solution. Clearly, the sequence $\{\tilde{k}_t\}_{t=0}^{\infty}$ satisfies the following equation:

$$\tilde{k}_{t+1} + \frac{\bar{c}_{t+1}}{\beta_L(1 + r(\tilde{k}_{t+1}, e_{t+1}, A_{t+1}))} = f(\tilde{k}_t, e_t, A_t), \quad t = 0, 1, \ldots. \tag{B.16}$$

Step I.2. A game.

We reduce our finite horizon model to a game $\Gamma = (X_k, G_k)_{k\in I}$. To specify a game, we need to describe a set of players, I, and for each player $k \in I$, define the strategy set X_k and the loss function

$$G_k \ : \ \prod_{i\in I} X_i \to \mathbb{R}.$$

Elements of $\prod_{i\in I} X_i$ are called multistrategies. The equilibrium of the game Γ is defined as follows.

Definition. *A multistrategy* $\left(x_1^*, \ldots, x_{|I|}^*\right)$ *is called a Nash equilibrium of the game* Γ *if for each* $k \in I$*,* x_k^* *is a solution to*

$$\min_{x_k} G_k\left(x_1^*, \ldots, x_{k-1}^*, x_k, x_{k+1}^*, \ldots, x_{|I|}^*\right), \quad \text{s. t.} \quad x_k \in X_k.$$

The sufficient conditions for the existence of a Nash equilibrium of this game are well-known (see, e.g., Ichiishi, 2014): for each $k \in I$ the set X_k is a convex and compact subset of a finite dimensional space, and the function $G_k(x_1, \ldots, x_k, \ldots, x_{|I|})$ is continuous in all variables and quasi-convex in x_k.

Let us specify the game Γ_T that represents our model. There are $T + (2T + 1)L$ players, and

1. for each $j = 1, \ldots, L$,
   a) T players determine s_t^j, $t = 0, 1, \ldots, T-1$, by solving

$$\min_s s\left(c_{t+1}^j - \beta_j(1 + r(k_{t+1}, e_{t+1}, A_{t+1}))c_t^j\right), \quad \text{s. t.} \quad 0 \le s \le L\bar{k}_{t+1}. \tag{B.17}$$

   b) $T+1$ players determine c_t^j, $t = 0, 1, \ldots, T$, by solving

$$\min_c \left|c - \left((1 + r(k_t, e_t, A_t))s_{t-1}^j + w(k_t, e_t, A_t) + v(k_t, e_t, A_t) - s_t^j\right)\right|, \quad \text{s. t.} \quad 0 \le c \le \bar{c}_t, \tag{B.18}$$

   where $s_{-1}^j = \hat{s}_{-1}^j$, and $s_T^j = 0$.

2. T players determine k_t, $t = 1, 2, \ldots, T$, by solving

$$\min_k \left| k - \frac{1}{L}\sum_{j=1}^{L} s_{t-1}^j \right|, \quad \text{s. t.} \quad \tilde{k}_t \le k \le \bar{k}_t. \tag{B.19}$$

Lemma. *There exists a Nash equilibrium in the game Γ_T with $T + (2T + 1)L$ players having the strategy sets and loss functions described by (B.17)–(B.19).*

Proof. All strategy sets are closed intervals, and for each player the loss function is continuous in all variables and quasi-convex in the player's own strategy variable. Hence the sufficient conditions for the existence of a Nash equilibrium in the game Γ_T are satisfied. □

Step I.3. Nash equilibrium and competitive equilibrium.

The following lemma maintains that a Nash equilibrium of the game Γ_T determines a finite T-period competitive $\mathbb{E}_0$-equilibrium.

Lemma B.1. *Let*

$$\left\{(c_t^{j**})_{j=1,\ldots,L;t=0,1,\ldots,T}, (s_t^{j**})_{j=1,\ldots,L;t=0,1,\ldots,T-1}, (k_t^{**})_{t=1,2,\ldots,T}\right\}$$

be a Nash equilibrium of the game Γ_T. Let $k_0^{**} = \hat{k}_0$, and $s_{-1}^{j**} = \hat{s}_{-1}^j$, $s_T^{j**} = 0$ for all j. Let also

$$\begin{aligned} 1 + r_t^{**} &= 1 + r(k_t^{**}, e_t, A_t), \quad t = 0, 1, \ldots, T, \\ w_t^{**} &= w(k_t^{**}, e_t, A_t), \quad t = 0, 1, \ldots, T, \\ q_t^{**} &= q(k_t^{**}, e_t, A_t), \quad t = 0, 1, \ldots, T, \\ v_t^{**} &= v(k_t^{**}, e_t, A_t), \quad t = 0, 1, \ldots, T. \end{aligned}$$

Then

$$\left\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\right\}_{t=0,1,\ldots,T}$$

is a finite T-period competitive $\mathbb{E}_0$-equilibrium starting from $\mathcal{I}_{-1}$.

Proof. First, observe that

- if $c_{t+1}^j > \beta_j(1 + r(k_{t+1}, e_{t+1}, A_{t+1}))c_t^j$, then the only solution to the problem (B.17) is $s = 0$;
- if $c_{t+1}^j = \beta_j(1 + r(k_{t+1}, e_{t+1}, A_{t+1}))c_t^j$, then any s from the interval $[0, L\bar{k}_{t+1}]$ is a solution to the problem (B.17);
- if $c_{t+1}^j < \beta_j(1 + r(k_{t+1}, e_{t+1}, A_{t+1}))c_t^j$, then the only solution to the problem (B.17) is $s = L\bar{k}_{t+1}$.

Second, notice that minimization problems (B.18) and (B.19) are of the form

$$\min_x |x-\hat{x}|,$$
$$\text{s. t.} \quad a_1 \le x \le a_2.$$

The unique solution to this problem, x^*, is given by

$$x^* = \begin{cases} a_1, & \text{if} \quad \hat{x} < a_1; \\ a_2, & \text{if} \quad \hat{x} > a_2; \\ \hat{x}, & \text{if} \quad a_1 \le \hat{x} \le a_2. \end{cases}$$

Remark B.1. *When $\hat{x} \ge a_1$, we have $\hat{x} \ge x^*$.*

Remark B.2. *When $\hat{x} \le a_2$, we have $\hat{x} \le x^*$.*

Let $\left\{(c_t^{j**})_{j=1,\dots,L;t=0,1,\dots,T}, (s_t^{j**})_{j=1,\dots,L;t=0,1,\dots,T-1}, (k_t^{**})_{t=1,2,\dots,T}\right\}$ be a Nash equilibrium of the game Γ_T. Note that for all $t=0,1,\dots,T$, $k_t^{**} \ge \tilde{k}_t > 0$. It follows that for all $t=0,1,\dots,T$, $w_t^{**}>0$, $v_t^{**}>0$, and $0<1+r_t^{**}<\infty$.

We divide the proof of Lemma B.1 into several claims.

Claim B.2. *For each $j=1,\dots,L$,*

$$0 < c_t^{j**} \le (1+r_t^{**})s_{t-1}^{j**} + w_t^{**} + v_t^{**} - s_t^{j**}, \quad t=0,1,\dots,T, \tag{B.20}$$

and hence

$$0 < c_t^{j**} + s_t^{j**} \le (1+r_t^{**})s_{t-1}^{j**} + w_t^{**} + v_t^{**}, \quad t=0,1,\dots,T, \tag{B.21}$$

Proof. Assume the converse. Then, by the structure of the problem (B.18), there are j and $0 \le \tau \le T$ such that (B.20) holds for $t<\tau$, and

$$0 = c_\tau^{j**} \ge (1+r_\tau^{**})s_{\tau-1}^{j**} + w_\tau^{**} + v_\tau^{**} - s_\tau^{j**}. \tag{B.22}$$

Consider two cases. First, let $\tau \le T-1$. By (B.22),

$$s_\tau^{j**} \ge (1+r_\tau^{**})s_{\tau-1}^{j**} + w_\tau^{**} + v_\tau^{**} > 0.$$

Hence, by the structure of the problem (B.17),

$$c_{\tau+1}^{j**} \le \beta_j(1+r_{\tau+1}^{**})c_\tau^{j**} = 0,$$

because otherwise we would have $s_\tau^{j**} = 0$. Therefore, using Remark B.1, we conclude that

$$0 = c_{\tau+1}^{j**} \geq (1 + r_{\tau+1}^{**})s_\tau^{j**} + w_{\tau+1}^{**} + v_{\tau+1}^{**} - s_{\tau+1}^{j**}.$$

Repeating the argument, and using the structure of the problem (B.17), we have

$$\begin{aligned} s_t^{j**} &> 0, \quad t = \tau, \tau+1, \dots, T-1, \\ c_{t+1}^{j**} &= 0, \quad t = \tau, \tau+1, \dots, T-1. \end{aligned}$$

However, $c_T^{j**} = 0$ is impossible, because $s_T^{j**} = 0$, and by the structure of the problem (B.18) we have

$$0 = c_T^{j**} = c_T^{j**} + s_T^{j**} \geq (1 + r_T^{**})s_{T-1}^{j**} + w_T^{**} + v_T^{**} > 0,$$

a contradiction.

Second, let $\tau = T$. Since $c_{T-1}^{j**} > 0$, and $c_T^{j**} = 0$, we have

$$c_T^{j**} - \beta_j(1 + r_T^{**})c_{T-1}^{j**} = -\beta_j(1 + r_T^{**})c_{T-1}^{j**} < 0,$$

and, by the structure of the problem (B.17), $s_{T-1}^{j**} = L\bar{k}_T$. Using the fact that $s_T^{j**} = 0$, by the structure of the problem (B.18) we obtain

$$0 = c_T^{j**} + s_T^{j**} \geq (1 + r_T^{**})s_{T-1}^{j**} + w_T^{**} + v_T^{**} > 0,$$

a contradiction. □

Claim B.3. *For each* $j = 1, \dots, L$,

$$(1 + r_t^{**})s_{t-1}^{j**} + w_t^{**} + v_t^{**} \leq Lf(k_t^{**}, e_t, A_t), \quad t = 0, 1, \dots, T, \tag{B.23}$$

and

$$\frac{1}{L}\sum_{j=1}^{L} s_{t-1}^{j**} \leq k_t^{**}, \quad t = 0, 1, \dots, T. \tag{B.24}$$

Proof. Using (B.21), (B.4), the bounds for k in (B.19), and (B.13), for each $j = 1, \dots, L$, we obtain

$$\begin{aligned} c_0^{j**} + s_0^{j**} &\leq (1 + r_0^{**})s_{-1}^{j**} + w_0^{**} + v_0^{**} \\ &\leq \sum_{j=1}^{L}\left((1 + r_0^{**})s_{-1}^{j**} + w_0^{**} + v_0^{**}\right) \leq L(1 + r_0^{**})k_0^{**} + Lw_0^{**} + Lv_0^{**} \\ &= Lf(k_0^{**}, e_0, A_0) \leq Lf(\bar{k}_0, e_0, A_0) < L\bar{k}_1. \end{aligned}$$

Hence

$$\frac{1}{L}\sum_{j=1}^{L} s_0^{j**} \leq \frac{1}{L}\sum_{j=1}^{L}\left(c_0^{j**} + s_0^{j**}\right) < \bar{k}_1,$$

and, by Remark B.2,

$$\frac{1}{L}\sum_{j=1}^{L} s_0^{j**} \leq k_1^{**}.$$

Thus, inequalities (B.23) and (B.24) hold for $t = 0$. To obtain these inequalities for all $t \leq T$, it is sufficient to repeat the argument. □

Claim B.4. *For each* $j = 1, \dots, L$,

$$c_t^{j**} + s_t^{j**} = (1 + r_t^{**})s_{t-1}^{j**} + w_t^{**} + v_t^{**}, \quad t = 0, 1, \dots, T. \tag{B.25}$$

Proof. It follows from the constraints in (B.17) that $s_t^{j**} \geq 0$ for all $t = 0, 1, \dots, T$. By (B.23) and (B.14),

$$(1 + r_t^{**})s_{t-1}^{j**} + w_t^{**} + v_t^{**} - s_t^{j**} < L\bar{k}_{t+1} = \bar{c}_t.$$

Therefore, by the structure of the problem (B.18), for each $j = 1, \dots, L$,

$$c_t^{j**} \geq (1 + r_t^{**})s_{t-1}^{j**} + w_t^{**} + v_t^{**} - s_t^{j**}, \quad t = 0, 1, \dots, T.$$

Combining this inequality with (B.20), we obtain (B.25). □

Claim B.5. *For each* $j = 1, \dots, L$,

$$c_{t+1}^{j**} \geq \beta_j(1 + r_{t+1}^{**})c_t^{j**} \ (= \textit{if } s_t^{j**} > 0), \quad t = 0, 1, \dots, T.$$

Proof. Assume that for some j and $t < T$,

$$c_{t+1}^{j**} < \beta_j(1 + r_{t+1}^{**})c_t^{j**}.$$

Then, by the structure of the problem (B.17), $s_t^{j**} = L\bar{k}_{t+1}$. It follows from (B.23), the bounds for k in (B.19), and (B.13) that

$$(1 + r_t^{**})s_{t-1}^{j**} + w_t^{**} + v_t^{**} \leq Lf(k_t^{**}, e_t, A_t) \leq Lf(\bar{k}_t, e_t, A_t) < L\bar{k}_{t+1} = s_t^{j**},$$

and thus

$$(1 + r_t^{**})s_{t-1}^{j**} + w_t^{**} + v_t^{**} - s_t^{j**} \leq 0,$$

which contradicts (B.20). Thus we have proved that

$$c_{t+1}^{j**} \geq \beta_j (1 + r_{t+1}^{**}) c_t^{j**}.$$

It remains to note that if

$$c_{t+1}^{j**} > \beta_j (1 + r_{t+1}^{**}) c_t^{j**},$$

then by the structure of the problem (B.17), $s_t^{j**} = 0$. □

Claim B.6. *For all* $t = 0, 1, \ldots, T$,

$$k_t^{**} > \tilde{k}_t, \tag{B.26}$$

and

$$\frac{1}{L} \sum_{j=1}^{L} s_{t-1}^{j**} = k_t^{**}. \tag{B.27}$$

Proof. By the choice of $\tilde{k}_0$,

$$\frac{1}{L} \sum_{j=1}^{L} s_{-1}^{j**} = k_0^{**} > \tilde{k}_0.$$

Assume that for some $t = 1, 2, \ldots T$,

$$\frac{1}{L} \sum_{j=1}^{L} s_{t-2}^{j**} = k_{t-1}^{**} > \tilde{k}_{t-1}, \text{ and } \frac{1}{L} \sum_{j=1}^{L} s_{t-1}^{j**} \leq k_t^{**} = \tilde{k}_t.$$

By (B.25),

$$\begin{aligned} \frac{1}{L} \sum_{j=1}^{L} \left(c_{t-1}^{j**} + s_{t-1}^{j**} \right) &= \frac{1}{L} \sum_{j=1}^{L} \left((1 + r_{t-1}^{**}) s_{t-2}^{j**} + w_{t-1}^{**} + v_{t-1}^{**} \right) \\ &= (1 + r_{t-1}^{**}) k_{t-1}^{**} + w_{t-1}^{**} + v_{t-1}^{**} = f(k_{t-1}^{**}, e_{t-1}, A_{t-1}) > f(\tilde{k}_{t-1}, e_{t-1}, A_{t-1}). \end{aligned}$$

Hence

$$\frac{1}{L} \sum_{j=1}^{L} c_{t-1}^{j**} > f(\tilde{k}_{t-1}, e_{t-1}, A_{t-1}) - \frac{1}{L} \sum_{j=1}^{L} s_{t-1}^{j**} \geq f(\tilde{k}_{t-1}, e_{t-1}, A_{t-1}) - \tilde{k}_t.$$

Therefore, there is j such that

$$c_{t-1}^{j**} > f(\tilde{k}_{t-1}, e_{t-1}, A_{t-1}) - \tilde{k}_t > 0. \tag{B.28}$$

Using (B.16), the bounds for c in (B.18), and taking into account that $\tilde{k}_t = k_t^{**}$, we get

$$c_{t-1}^{j**} > f(\tilde{k}_{t-1}, e_{t-1}, A_{t-1}) - \tilde{k}_t = \frac{\bar{c}_t}{\beta_L(1 + r(\tilde{k}_t, e_t, A_t))} \geq \frac{\bar{c}_t}{\beta_j(1 + r(\tilde{k}_t, e_t, A_t))} \geq \frac{c_t^{j**}}{\beta_j(1 + r_t^{**})},$$

and hence

$$c_t^{j**} \leq \beta_j(1 + r_t^{**})c_{t-1}^{j**}.$$

It follows from the structure of the problem (B.17) that for this j we have $s_{t-1}^{j**} = L\bar{k}_t$. By (B.25), (B.23), the bounds for k in (B.19), and (B.13), we have

$$c_{t-1}^{j**} = (1 + r_{t-1}^{**})s_{t-2}^{j**} + w_{t-1}^{**} + v_{t-1}^{**} - s_{t-1}^{j**} \leq Lf(k_{t-1}^{**}, e_{t-1}, A_{t-1}) - s_{t-1}^{j**} \leq Lf(\bar{k}_{t-1}, e_{t-1}, A_{t-1}) - L\bar{k}_t < 0,$$

a contradiction of (B.28). This proves (B.26).

Now (B.27) follows from (B.24), (B.26), and the structure of the problem (B.19). □

Claims B.2–B.6 complete the proof of Lemma B.1. □

Step II. Competitive equilibrium under given extraction rates in the infinite horizon model.

Step II.1. A candidate for an equilibrium path.

Let for $T = 1, 2, \ldots$,

$$\mathcal{E}_{0,T}^{**} = \left\{(c_t^{j**}(T))_{j=1}^L, (s_t^{j**}(T))_{j=1}^L, k_t^{**}(T), r_t^{**}(T), w_t^{**}(T), q_t^{**}(T), v_t^{**}(T)\right\}_{t=0,1,\ldots,T}$$

be a finite T-period equilibrium path. Let us apply the following procedure to the sequence $\{\mathcal{E}_{0,T}^{**}\}_{T=1,2,\ldots}$.

At the first step of the process we take a cluster point of the sequence

$$\left\{(c_0^{j**}(T))_{j=1}^L, (s_0^{j**}(T))_{j=1}^L, k_0^{**}(T), r_0^{**}(T), w_0^{**}(T), q_0^{**}(T), v_0^{**}(T)\right\}_{T=1,2,\ldots},$$

denote it as

$$\left\{(c_0^{j**})_{j=1}^L, (s_0^{j**})_{j=1}^L, k_0^{**}, r_0^{**}, w_0^{**}, q_0^{**}, v_0^{**}\right\},$$

and extract a subsequence $\{T_{0n}\}_{n=1}^{\infty}$ from $\{T\}_{T=1,2,\ldots}$ such that

$$\left\{(c_0^{j**}(T_{0n}))_{j=1}^L, (s_0^{j**}(T_{0n}))_{j=1}^L, k_0^{**}(T_{0n}), r_0^{**}(T_{0n}), w_0^{**}(T_{0n}), q_0^{**}(T_{0n}), v_0^{**}(T_{0n})\right\}_{n=1}^{\infty}$$

converges to $\left\{(c_0^{j**})_{j=1}^L, (s_0^{j**})_{j=1}^L, k_0^{**}, r_0^{**}, w_0^{**}, q_0^{**}, v_0^{**}\right\}$.

At the second step we take a cluster point of the sequence

$$\left\{(c_1^{j**}(T_{0n}))_{j=1}^L, (s_1^{j**}(T_{0n}))_{j=1}^L, k_1^{**}(T_{0n}), r_1^{**}(T_{0n}), w_1^{**}(T_{0n}), q_1^{**}(T_{0n}), v_1^{**}(T_{0n})\right\}_{n=1}^{\infty},$$

denote it as

$$\left\{(c_1^{j**})_{j=1}^L, (s_1^{j**})_{j=1}^L, k_1^{**}, r_1^{**}, w_1^{**}, q_1^{**}, v_1^{**}\right\},$$

and extract a subsequence $\{T_{1n}\}_{n=1}^{\infty}$ from the sequence $\{T_{0n}\}_{n=1}^{\infty}$ such that $T_{11} > 1$, and

$$\left\{(c_1^{j**}(T_{1n}))_{j=1}^L, (s_1^{j**}(T_{1n}))_{j=1}^L, k_1^{**}(T_{0n}), r_1^{**}(T_{1n}), w_1^{**}(T_{1n}), q_1^{**}(T_{1n}), v_1^{**}(T_{1n})\right\}_{n=1}^{\infty}$$

converges to $\left\{(c_1^{j**})_{j=1}^L, (s_1^{j**})_{j=1}^L, k_1^{**}, r_1^{**}, w_1^{**}, q_1^{**}, v_1^{**}\right\}$. This procedure continues ad infinitum.

As a result, we obtain an infinite sequence

$$\mathcal{E}_{0,\infty}^{**} = \left\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\right\}_{t=0,1,\dots}. \tag{B.29}$$

This sequence is a natural candidate for a competitive equilibrium under given extraction rates in our model.

Step II.2. Bounds of the T-period equilibrium capital sequence.

We already know that every element of the capital sequence for any T-period finite equilibrium, k_t^{**}, is bounded from below by $\tilde{k}_t$. However, we need to establish a more precise estimate for the lower bound of the capital sequence of a T-period finite equilibrium. Again, we begin with some preliminary definitions.

Let the value $1 + r'$ be such that

$$\beta_L(1 + r') > 2(1 + \bar{g}), \tag{B.30}$$

and k' be given by

$$\alpha_1(\delta\bar{e})^{\alpha_3} A_0 (k')^{\alpha_1 - 1} = 1 + r'. \tag{B.31}$$

Let further the sequence $\{k_t'\}$ be given by

$$k_{t+1}' = (1 + \underline{g})k_t', \tag{B.32}$$

where

$$0 < k_0' < \min\{\hat{k}_0, k'\}.$$

Claim B.7. *For all t,*

$$(1+\underline{g}) < \frac{f(k'_{t+1}, e_{t+1}, A_{t+1})}{f(k'_t, e_t, A_t)} < (1+\bar{g}). \tag{B.33}$$

Proof. By (B.32), (B.8), (B.6), (B.9) and (B.10),

$$\begin{aligned}\frac{f(k'_{t+1}, e_{t+1}, A_{t+1})}{f(k'_t, e_t, A_t)} &= \frac{(k'_{t+1})^{\alpha_1}}{(k'_t)^{\alpha_1}} \frac{A_{t+1}}{A_t} \frac{e_{t+1}^{\alpha_3}}{e_t^{\alpha_3}} = \left(1+\underline{g}\right)^{\alpha_1} (1+\lambda) \frac{e_{t+1}^{\alpha_3}}{e_t^{\alpha_3}} \\ &> \left(1+\underline{g}\right)^{\alpha_1} (1+\lambda) \left(\frac{\delta^2}{1-\delta}\right)^{\alpha_3} \geq \left(1+\underline{g}\right)^{\alpha_1} (1+\tilde{g})^{1-\alpha_1} > (1+\underline{g}).\end{aligned}$$

Analogously, using (B.32), (B.8), (B.10) and (B.6), we have

$$\begin{aligned}\frac{f(k'_{t+1}, e_{t+1}, A_{t+1})}{f(k'_t, e_t, A_t)} &= \frac{(k'_{t+1})^{\alpha_1}}{(k'_t)^{\alpha_1}} \frac{A_{t+1}}{A_t} \frac{e_{t+1}^{\alpha_3}}{e_t^{\alpha_3}} = \left(1+\underline{g}\right)^{\alpha_1} (1+\lambda) \frac{e_{t+1}^{\alpha_3}}{e_t^{\alpha_3}} \\ &< (1+\bar{g})^{\alpha_1} (1+\lambda) \frac{e_{t+1}^{\alpha_3}}{e_t^{\alpha_3}} < (1+\bar{g})^{\alpha_1} (1+\lambda) \left(\frac{(1-\delta)^2}{\delta}\right)^{\alpha_3} = (1+\bar{g}).\end{aligned}$$

□

Claim B.8. *For all $t = 0, 1, \ldots, T$,*

$$1 + r(k'_t, e_t, A_t) > 1 + r'. \tag{B.34}$$

Proof. It follows from (B.32) and (B.33) that

$$\begin{aligned}1 + r(k'_t, e_t, A_t) &= \alpha_1 A_t e_t^{\alpha_3} (k'_t)^{\alpha_1 - 1} \\ &= \alpha_1 \frac{f(k'_t, e_t, A_t)}{k'_t} > \alpha_1 \frac{f(k'_{t-1}, e_{t-1}, A_{t-1})}{k'_{t-1}} = 1 + r(k'_{t-1}, e_{t-1}, A_{t-1}).\end{aligned}$$

Repeating the argument, and using (B.31), we get

$$\begin{aligned}1 + r(k'_t, e_t, A_t) &> 1 + r(k'_0, e_0, A_0) > 1 + r(k', e_0, A_0) \\ &= \alpha_1 A_0 e_0^{\alpha_3} (k')^{\alpha_1 - 1} > \alpha_1 A_0 (\delta \bar{e})^{\alpha_3} (k')^{\alpha_1 - 1} = 1 + r'.\end{aligned}$$

□

Let

$$\begin{aligned}w'_{t+1} &= (1+\underline{g}) w'_t, \quad t = 0, 1, \ldots, \\ v'_{t+1} &= (1+\underline{g}) v'_t, \quad t = 0, 1, \ldots,\end{aligned}$$

where

$$w_0' = \alpha_2 A_0 (k_0')^{\alpha_1} (\delta \bar{e})^{\alpha_3} > 0,$$
$$v_0' = \alpha_3 A_0 (k_0')^{\alpha_1} (\delta \bar{e})^{\alpha_3} > 0.$$

Claim B.9. *In any finite T-period competitive $\mathbb{E}_0$-equilibrium*

$$\left\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L, k_t^{**}, r_t^{**}, w_t^{**}, q_t^{**}, v_t^{**}\right\}_{t=0,1,\ldots,T},$$

for $t \leq T-1$,

$$k_t^{**} > k_t' > 0, \tag{B.35}$$

and

$$w_t^{**} > w_t' > 0,$$
$$v_t^{**} > v_t' > 0.$$

Proof. First let us prove (B.35). Assume the converse. Then there is $\tau > 0$ such that $k_\tau^{**} > k_\tau'$, and $k_{\tau+1}^{**} \leq k_{\tau+1}'$. It follows from (B.3) and (B.34), that for all j,

$$\begin{aligned} c_{\tau+1}^{j**} &\geq \beta_j (1 + r_{\tau+1}^{**}) c_\tau^{j**} \geq \beta_L (1 + r_{\tau+1}^{**}) c_\tau^{j**} \\ &= \beta_L (1 + r(k_{\tau+1}^{**}, e_{\tau+1}, A_{\tau+1})) c_\tau^{j**} \geq \beta_L (1 + r(k_{\tau+1}', e_{\tau+1}, A_{\tau+1})) c_\tau^{j**} \\ &> \beta_L (1 + r') c_\tau^{j**}. \end{aligned}$$

By (B.30),

$$c_{\tau+1}^{j**} > 2(1+\bar{g}) c_\tau^{j**}. \tag{B.36}$$

Adding together the budget constraints of all agents at time t in (B.2), and using condition 2 in Definition B.1, we get

$$\frac{1}{L}\sum_{j=1}^{L} c_t^{j**} + k_{t+1}^{**} = (1 + r_t^{**})\, k_t^{**} + w_t^{**} + v_t^{**} = f(k_t^{**}, e_t, A_t), \quad t = 0, 1, \ldots, T. \tag{B.37}$$

Applying (B.37) for $t = \tau + 1$ and $t = \tau$, and using (B.36), we have

$$\begin{aligned} f(k_{\tau+1}^{**}, e_{\tau+1}, A_{\tau+1}) - k_{\tau+2}^{**} &= \frac{1}{L}\sum_{j=1}^{L} c_{\tau+1}^{j**} \\ &> 2(1+\bar{g}) \frac{1}{L}\sum_{j=1}^{L} c_\tau^{j**} = 2(1+\bar{g}) \left(f(k_\tau^{**}, e_\tau, A_\tau) - k_{\tau+1}^{**} \right). \end{aligned}$$

Hence, by the choice of τ,

$$\begin{aligned} k^{**}_{\tau+2} &< 2(1+\bar{g})k^{**}_{\tau+1} + f(k^{**}_{\tau+1}, e_{\tau+1}, A_{\tau+1}) - 2(1+\bar{g})f(k^{**}_{\tau}, e_\tau, A_\tau) \\ &\leq (1+\bar{g})\left(2k'_{\tau+1} - f(k'_\tau, e_\tau, A_\tau)\right) + f(k'_{\tau+1}, e_{\tau+1}, A_{\tau+1}) - (1+\bar{g})f(k'_\tau, e_\tau, A_\tau). \end{aligned} \tag{B.38}$$

It follows from (B.33) that

$$f(k'_{\tau+1}, e_{\tau+1}, A_{\tau+1}) < (1+\bar{g})f(k'_\tau, e_\tau, A_\tau). \tag{B.39}$$

Moreover, using (B.34) and (B.30), we get

$$\frac{f(k'_\tau, e_\tau, A_\tau)}{k'_\tau} = \frac{1 + r(k'_\tau, e_\tau, A_\tau)}{\alpha_1} > \frac{1+r'}{\alpha_1} > \frac{2(1+\bar{g})}{\alpha_1 \beta_L} > 2(1+\bar{g}),$$

and hence, by (B.32) and (B.10), we get

$$2k'_{\tau+1} = 2(1+\underline{g})k'_\tau < 2(1+\bar{g})k'_\tau < f(k'_\tau, e_\tau, A_\tau). \tag{B.40}$$

Combining (B.39) and (B.40), we have

$$(1+\bar{g})\left(2k'_{\tau+1} - f(k'_\tau, e_\tau, A_\tau)\right) + f(k'_{\tau+1}, e_{\tau+1}, A_{\tau+1}) - (1+\bar{g})f(k'_\tau, e_\tau, A_\tau) < 0.$$

Now it follows from (B.38) that $k^{**}_{\tau+2} < 0$, which is impossible. Hence $\tau + 2 > T$, and therefore the inequality (B.35) holds for $t \leq T-1$.

Using (B.35), (B.8), (B.6), (B.9) and (B.10), we obtain that for all $t = 0, 1, \ldots, T-1$,

$$\begin{aligned} w^{**}_t = \alpha_2 A_t (k^{**}_t)^{\alpha_1} (e_t)^{\alpha_3} &> \alpha_2 A_t (k'_t)^{\alpha_1} \left(\frac{e_t}{e_0}\right)^{\alpha_3} e_0^{\alpha_3} \\ &> \alpha_2 (1+\lambda)^t A_0 (1+\underline{g})^{t\alpha_1} (k'_0)^{\alpha_1} \left(\frac{\delta^2}{1-\delta}\right)^{t\alpha_3} (\delta\bar{e})^{\alpha_3} \\ &= (1+\underline{g})^{t\alpha_1} (1+\lambda)^t \left(\frac{\delta^2}{1-\delta}\right)^{t\alpha_3} \alpha_2 A_0 (k'_0)^{\alpha_1} (\delta\bar{e})^{\alpha_3} \\ &\geq (1+\underline{g})^{t\alpha_1} (1+\tilde{g})^{t(1-\alpha_1)} w'_0 > (1+\underline{g})^t w'_0 = w'_t. \end{aligned}$$

Applying the same argument, it can be easily seen that for $t \leq T-1$,

$$v^{**}_t > v'_t > 0.$$

□

Step II.3. Existence of an equilibrium.

Now we are ready to prove the following lemma which maintains that the sequence $\mathcal{E}^{**}_{0,\infty}$ defined by (B.29) is a competitive equilibrium under given extraction rates in our model.

Lemma B.2. *The sequence $\mathcal{E}^{**}_{0,\infty}$ defined by (B.29) is a competitive $\mathbb{E}_0$-equilibrium starting from $\mathcal{I}_{-1}$.*

Proof. It is clear that by construction $\mathcal{E}^{**}_{0,\infty}$ satisfies conditions 2–6 for the competitive equilibrium under given extraction rates in the public property regime. Thus to prove that $\mathcal{E}^{**}_{0,\infty}$ is a competitive $\mathbb{E}_0$-equilibrium it is sufficient to show that $\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L\}_{t=0}^{\infty}$ is a solution to the problem

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \beta_j^t \ln c_t^j, \\ &\text{s. t. } c_t^j + s_t^j \leq (1 + r_t^{**})\, s_{t-1}^j + w_t^{**} + v_t^{**}, \quad t = 0, 1, \dots, \\ &\qquad s_t^j \geq 0, \quad t = 0, 1, \dots. \end{aligned} \tag{B.41}$$

Let c_t' be such that

$$c_t' = \frac{1}{2}(w_t' + v_t'), \quad t = 0, 1, \dots. \tag{B.42}$$

It is clear that

$$c_{t+1}' = (1 + \underline{g})c_t',$$

and hence

$$\sum_{t=0}^{\infty} \beta^t \ln c_t' = \frac{\ln c_0'}{1-\beta} + \ln(1+\underline{g}) \sum_{t=0}^{\infty} t\beta^t = \frac{\ln c_0'}{1-\beta} + \frac{\beta}{(1-\beta)^2} \ln(1+\underline{g}).$$

Consider the instantaneous utility function

$$u_t(c) = \ln c - \ln c_t'.$$

Clearly, the solution to the problem (B.41) will not change if we replace the instantaneous utility function $\ln c$ with the function $u_t(c)$. It is also clear that $u_t(c_t') = 0$.

Note that for all t,

$$\bar{c}_t > Lf(\bar{k}_t, e_t, A_t) > Lf(k_t', e_t, A_t) > c_t',$$

and hence $u_t(\bar{c}_t) > 0$. Moreover, it follows from (B.15) that

$$\sum_{t=0}^{\infty} \beta^t u_t(\bar{c}_t) = \frac{\ln \bar{c}_0}{1-\beta} + \frac{\beta}{(1-\beta)^2} \ln(1+\bar{g}) - \frac{\ln c_0'}{1-\beta} + \frac{\beta}{(1-\beta)^2} \ln(1+\underline{g})$$
$$= \frac{1}{1-\beta} \ln\left(\frac{\bar{c}_0}{c_0'}\right) + \frac{\beta}{(1-\beta)^2} \ln\left(\frac{1+\bar{g}}{1+\underline{g}}\right).$$

Now assume that $\{(c_t^{j**})_{j=1}^L, (s_t^{j**})_{j=1}^L\}_{t=0}^{\infty}$ is not a solution to the problem (B.41). Then for some j (we fix this j and omit it in the remaining part of the proof for the simplicity of notation) there is a feasible sequence $\{\widehat{c}_t, \widehat{s}_t\}_{t=0}^{\infty}$ such that

$$\widehat{U} > U^*, \text{ where } \widehat{U} = \sum_{t=0}^{\infty} \beta^t u_t(\widehat{c}_t), \text{ and } U^* = \sum_{t=0}^{\infty} \beta^t u_t(c_t^{**}).$$

Let $0 < \Delta < \widehat{U} - U^*$, and let Θ be such that

$$\sum_{t=\Theta+1}^{\infty} \beta^t u_t(\bar{c}_t) < \min\left\{\frac{\Delta}{2}, \ln 2\right\}.$$

Further, let

$$U^{*\Theta} = \sum_{t=0}^{\Theta} \beta^t u_t(c_t^{**}), \quad \widehat{U}^{\Theta} = \sum_{t=0}^{\Theta} \beta^t u_t(\widehat{c}_t),$$

and

$$U^*(T) = \sum_{t=0}^{T} \beta^t u_t(c_t^{**}(T)), \quad U^{*\Theta}(T) = \sum_{t=0}^{\Theta} \beta^t u_t(c_t^{**}(T)),$$

for $T = \Theta+1, \Theta+2, \ldots$.

Claim B.10. *There is a sequence $\{T_{\Theta n}\}_{n=1}^{\infty}$ such that*

$$U^{*\Theta}(T_{\Theta n}) \xrightarrow[n\to\infty]{} U^{*\Theta}.$$

Proof. It is sufficient to note that since $\mathcal{E}_{0,\infty}^{**}$ is obtained as a result of the application of the process described at Step II.1 to the sequence $\{\mathcal{E}_{0,T}^{**}\}_{T=1,2,\ldots}$, there is a sequence $\{T_{\Theta n}\}_{n=1}^{\infty}$ such that for $t = 0, 1, \ldots, \Theta$:

$$\lim_{n\to\infty} k_t^{**}(T_{\Theta n}) = k_t^{**}, \quad \lim_{n\to\infty} w_t^{**}(T_{\Theta n}) = w_t^{**}, \quad \lim_{n\to\infty} v_t^{**}(T_{\Theta n}) = v_t^{**},$$
$$\lim_{n\to\infty} r_t^{**}(T_{\Theta n}) = r_t^{**}, \quad \lim_{n\to\infty} c_t^{**}(T_{\Theta n}) = c_t^{**}, \quad \lim_{n\to\infty} s_t^{**}(T_{\Theta n}) = s_t^{**}.$$

□

Let $W^{*\Theta}$ be the maximum value of utility in the problem

$$\begin{aligned}
&\max \sum_{t=0}^{\Theta} \beta^t u_t(c_t),\\
&\text{s. t. } \; c_t + s_t \le (1 + r_t^{**})\, s_{t-1} + w_t^{**} + v_t^{**}, \quad t = 0, 1, \ldots, \Theta,\\
&\qquad s_t \ge 0, \quad t = 0, 1, \ldots, \Theta,
\end{aligned}$$

and $W^{*\Theta}(T)$ be the maximum value of utility in the problem

$$\begin{aligned}
&\max \sum_{t=0}^{\Theta} \beta^t u_t(c_t),\\
&\text{s. t. } \; c_t + s_t \le (1 + r_t^{**}(T))\, s_{t-1} + w_t^{**}(T) + v_t^{**}(T),\\
&\qquad s_t \ge 0, \quad t = 0, 1, \ldots, \Theta,
\end{aligned} \tag{B.43}$$

for $T = \Theta + 1, \Theta + 2, \ldots$.

Claim B.11.

$$W^{*\Theta}(T_{\Theta n}) \xrightarrow[n\to\infty]{} W^{*\Theta}.$$

Proof. Consider the correspondence that takes to each

$$\{(1 + r_0, w_0, v_0), \ldots, (1 + r_\Theta, w_\Theta, v_\Theta)\} \in \prod_{t=0}^{\Theta} \Big([1 + r(\bar{k}_t, e_t, A_t), 1 + r(\tilde{k}_t, e_t, A_t)] \times [w(\tilde{k}_t, e_t, A_t), w(\bar{k}_t, e_t, A_t)] \times [v(\tilde{k}_t, e_t, A_t), v(\bar{k}_t, e_t, A_t)] \Big)$$

the set

$$\{(c_0, s_0), \ldots, (c_\Theta, s_\Theta\} \in \mathbb{R}^{2(\Theta+1)}$$

which is such that, with $s_{-1} = \hat{s}_{-1}$ being given,

$$c_t + s_t \le (1 + r_t^{**}(T))\, s_{t-1} + w_t^{**}(T) + v_t^{**}(T), \text{ and } s_t \ge 0,$$

hold for all $t = 0, 1, \ldots, \Theta$.

By Statement 1, this correspondence is lower- and upper-semicontinuous, and it is sufficient to apply the Maximum Theorem. □

Claim B.12.

$$U^*(T) \ge W^{*\Theta}(T).$$

Proof. Let for some $T > \Theta + 1$, the sequence $\{(\breve{c}_0, \breve{s}_0), \dots, (\breve{c}_\Theta, \breve{s}_\Theta)\}$ be a solution to (B.43). Let further for $t = \Theta + 1, \dots, T$, $\{(\breve{c}_t, \breve{s}_t)\}$ be defined recursively by

$$\breve{c}_t = c'_t, \quad \breve{s}_t = (1 + r_t^{**}(T))\, \breve{s}_{t-1} + w_t^{**}(T) + v_t^{**}(T) - \breve{c}_t. \tag{B.44}$$

We show that given $s_{-1} = \hat{s}_{-1}$, the sequence

$$\{(\breve{c}_0, \breve{s}_0), \dots, (\breve{c}_\Theta, \breve{s}_\Theta), (\breve{c}_{\Theta+1}, \breve{s}_{\Theta+1}), \dots, (\breve{c}_T, \breve{s}_T)\} \tag{B.45}$$

is feasible for the problem

$$\begin{aligned} &\max \sum_{t=0}^{T} \beta^t u_t(c_t), \\ &\text{s. t. } \; c_t + s_t \le (1 + r_t^{**}(T))\, s_{t-1} + w_t^{**}(T) + v_t^{**}(T), \\ &\qquad s_t \ge 0, \quad t = 0, 1, \dots, T. \end{aligned} \tag{B.46}$$

It is sufficient to check that $\breve{s}_t \ge 0$ for $t = \Theta + 1, \dots, T$. By Claim B.9, we have for $\Theta + 1 \le t \le T - 1$,

$$\breve{c}_t = c'_t = \frac{1}{2}(w'_t + v'_t) < w_t^{**}(T) + v_t^{**}(T).$$

We prove that $\breve{s}_t > 0$ for $t = \Theta + 1, \dots, T - 1$ recursively. Clearly, $\breve{s}_\Theta = 0$. Suppose that $\breve{s}_{t-1} \ge 0$ for $\Theta + 1 \le t < T - 2$. Then

$$\breve{s}_t = (1 + r_t^{**}(T))\, \breve{s}_{t-1} + w_t^{**}(T) + v_t^{**}(T) - \breve{c}_t \ge w_t^{**}(T) + v_t^{**}(T) - c'_t > 0.$$

In particular, $\breve{s}_{T-2} > 0$. For $t = T - 1$ we have

$$\begin{aligned} \breve{s}_{T-1} &= \big(1 + r_{T-1}^{**}(T)\big)\, \breve{s}_{T-2} + w_{T-1}^{**}(T) + v_{T-1}^{**}(T) - \breve{c}_{T-1} \\ &\ge w_{T-1}^{**}(T) + v_{T-1}^{**}(T) - c'_{T-1} > 2c'_{T-1} - c'_{T-1} = c'_{T-1}. \end{aligned}$$

For $t = T$ we know from Claim B.9 that either

$$w_T^{**}(T) + v_T^{**}(T) > c'_T,$$

or

$$1 + r_T^{**} \ge 1 + r'.$$

In the first case, we can apply the same reasoning as before:

$$\breve{s}_T = (1 + r_T^{**}(T))\, \breve{s}_{T-1} + w_T^{**}(T) + v_T^{**}(T) - \breve{c}_T \ge w_T^{**}(T) + v_T^{**}(T) - c'_T > 0.$$

In the second case, using (B.30) and (B.10), we have

$$\begin{aligned}\check{s}_T &= (1 + r_T^{**}(T))\,\check{s}_{T-1} + w_T^{**}(T) + v_T^{**}(T) - \check{c}_T > (1+r')c'_{T-1} - c'_T \\ &> \frac{2}{\beta_L}(1+\bar{g})c'_{T-1} - c'_T > (1+\bar{g})c'_{T-1} - c'_T > (1+\underline{g})c'_{T-1} - c'_T = 0.\end{aligned}$$

Thus we have proved that the sequence (B.45) is feasible for the problem (B.46). Since the sequence

$$\{(c_0^{**}(T), s_0^{**}(T)), \ldots, (c_T^{**}(T), s_T^{**}(T))\}$$

is the solution to this problem, we have

$$U^*(T) = \sum_{t=0}^{T} \beta^t u_t(c_t^{**}(T)) \geq \sum_{t=0}^{\Theta} \beta^t u_t(\check{c}_t) + \sum_{t=\Theta+1}^{T} \beta^t u_t(c'_t) = \sum_{t=0}^{\Theta} \beta^t u_t(\check{c}_t) = W^{*\Theta}(T).$$

□

Let us prove another useful claim.

Claim B.13. *For all $t = 0, 1, \ldots, T$,*

$$k_t^{**}(T) \leq \bar{\kappa}(A_t e_t^{\alpha_3})^{\frac{1}{1-\alpha_1}}. \tag{B.47}$$

Proof. It is sufficient to show that

$$\kappa_t^{**} \leq \bar{\kappa}, \quad t = 0, 1, \ldots, T, \tag{B.48}$$

where

$$\kappa_t^{**} := \frac{k_t^{**}(T)}{(A_t e_t^{\alpha_3})^{\frac{1}{1-\alpha_1}}}.$$

By (B.11), the choice of $1 + \tilde{g}$, and (B.5),

$$\bar{\kappa} = \frac{1}{(1+\tilde{g})^{\frac{1}{1-\alpha_1}}} \geq \frac{\hat{k}_0}{(A_0(\delta\bar{e})^{\alpha_3})^{\frac{1}{1-\alpha_1}}} \geq \frac{\hat{k}_0}{(A_0 e_0^{\alpha_3})^{\frac{1}{1-\alpha_1}}} = \kappa_0^{**},$$

which proves (B.48) for $t = 0$. We prove it for $t = 1, \ldots, T$ recursively. Suppose that $\kappa_t^{**} \leq \bar{\kappa}$. If follows from (B.37) that for all t,

$$k_{t+1}^{**}(T) \leq f(k_t^{**}(T), e_t, A_t) = (k_t^{**}(T))^{\alpha_1}\left((A_t e_t^{\alpha_3})^{\frac{1}{1-\alpha_1}}\right)^{1-\alpha_1},$$

and hence, due to (B.8), (B.6) and (B.9),

$$(\kappa_t^{**})^{\alpha_1} \geq \frac{k_{t+1}^{**}(T)}{(A_t e_t^{\alpha_3})^{\frac{1}{1-\alpha_1}}} = \kappa_{t+1}^{**} \frac{(A_{t+1} e_{t+1}^{\alpha_3})^{\frac{1}{1-\alpha_1}}}{(A_t e_t^{\alpha_3})^{\frac{1}{1-\alpha_1}}} = \kappa_{t+1}^{**} \left((1+\lambda) \left(\frac{e_{t+1}}{e_t} \right)^{\alpha_3} \right)^{\frac{1}{1-\alpha_1}}$$

$$> \kappa_{t+1}^{**} (1+\lambda)^{\frac{1}{1-\alpha_1}} \left(\frac{\delta^2}{1-\delta} \right)^{\frac{\alpha_3}{1-\alpha_1}} \geq (1+\tilde{g}) \kappa_{t+1}^{**}.$$

Therefore, by (B.11),

$$\kappa_{t+1}^{**} \leq \frac{(\kappa_t^{**})^{\alpha_1}}{1+\tilde{g}} \leq \frac{(\bar{\kappa})^{\alpha_1}}{1+\tilde{g}} = \bar{\kappa}.$$

Thus (B.48) holds for all $t = 0, 1, \ldots, T$. □

Denote

$$1 + \bar{r} = \alpha_1 \frac{1}{(\bar{\kappa})^{1-\alpha_1}}.$$

By (B.11) and the choice of $1 + \underline{g}$,

$$\beta_L (1 + \bar{r}) = \beta_L \alpha_1 \frac{(\bar{\kappa})^{\alpha_1}}{\bar{\kappa}} = \beta_L \alpha_1 (1 + \tilde{g}) = (1 + \underline{g}), \tag{B.49}$$

and hence, by (B.47), for all $t = 0, 1, \ldots, T$, we have

$$1 + r_t^{**}(T) = \alpha_1 \frac{A_t e_t^{\alpha_3}}{(k_t^{**}(T))^{1-\alpha_1}} \geq \alpha_1 \frac{A_t e_t^{\alpha_3}}{(\bar{\kappa})^{1-\alpha_1} A_t e_t^{\alpha_3}} = \alpha_1 \frac{1}{(\bar{\kappa})^{1-\alpha_1}} = 1 + \bar{r}. \tag{B.50}$$

Claim B.14.

$$U^* \geq U^{*\Theta}.$$

Proof. Let us prove that for any $T > \Theta + 1$,

$$c_\Theta^{**}(T) \geq c'_\Theta. \tag{B.51}$$

Assume that $c_\Theta^{**}(T) < c'_\Theta$. We show that this inequality implies $c_t^{**}(T) < c'_t$ for all $t \leq \Theta$. Indeed, if $c_t^{**}(T) < c'_t$ for some $t < \Theta$, then it follows from (B.3), (B.50) and (B.49) that

$$c_t^{**}(T) \geq \beta_j (1 + r_t^{**}(T)) c_{t-1}^{**}(T) \geq \beta_L (1 + \bar{r}) c_{t-1}^{**}(T) = (1 + \underline{g}) c_{t-1}^{**}(T), \tag{B.52}$$

and thus

$$c_{t-1}^{**}(T) \leq \frac{c_t^{**}(T)}{1+\underline{g}} < \frac{c'_t}{1+\underline{g}} = c'_{t-1}.$$

Hence
$$\sum_{t=0}^{\Theta} \beta^t u_t(c_t^{**}(T)) = \sum_{t=0}^{\Theta} \beta^t \left(\ln c_t^{**}(T) - \ln c_t'\right) < 0.$$
At the same time, by the choice of Θ, we have
$$\sum_{t=\Theta+1}^{T} \beta^t u_t(c_t^{**}(T)) \leq \sum_{t=\Theta+1}^{T} \beta^t u_t(\bar{c}_t) \leq \sum_{t=\Theta+1}^{\infty} \beta^t u_t(\bar{c}_t) < \ln 2.$$
Therefore
$$\sum_{t=0}^{T} \beta^t u_t(c_t^{**}(T)) = \sum_{t=0}^{\Theta} \beta^t u_t(c_t^{**}(T)) + \sum_{t=\Theta+1}^{T} \beta^t u_t(c_t^{**}(T)) < \ln 2. \tag{B.53}$$

Consider the sequence
$$\{(\breve{c}_0, \breve{s}_0), \ldots, (\breve{c}_T, \breve{s}_T)\}, \tag{B.54}$$
defined as follows: for $t \leq \Theta$
$$\breve{c}_t = w_t^{**}(T) + v_t^{**}(T), \quad \breve{s}_t = 0,$$
and for $t = \Theta + 1, \ldots, T$, $\{(\breve{c}_t, \breve{s}_t)\}$ is given by (B.44). It follows from Claim B.9 that for $t \leq \Theta$,
$$w_t^{**}(T) + v_t^{**}(T) > w_t' + v_t' > c_t'. \tag{B.55}$$

Repeating the argument from the proof of Claim B.12, we obtain that the sequence (B.54) is feasible for the problem (B.46). At the same time, the sequence
$$\{(c_0^{**}(T), s_0^{**}(T)), \ldots, (c_T^{**}(T), s_T^{**}(T))\}$$
is the solution to this problem. Hence, using (B.55), we get
$$\begin{aligned}\sum_{t=0}^{T} \beta^t u_t(c_t^{**}(T)) &\geq \sum_{t=0}^{T} \beta^t u_t(\breve{c}_t) = \sum_{t=0}^{\Theta} \beta^t u_t(w_t^{**}(T) + v_t^{**}(T)) + \sum_{t=\Theta+1}^{T} \beta^t u_t(c_t') \\ &= u_0(w_0^{**}(T) + v_0^{**}(T)) + \sum_{t=1}^{\Theta} \beta^t u_t(\breve{c}_t) > u_0(w_0^{**}(T) + v_0^{**}(T)) \\ &= \ln(w_0^{**}(T) + v_0^{**}(T)) - \ln c_0' > \ln(w_0' + v_0') - \ln c_0' = \ln\left(\frac{w_0' + v_0'}{c_0'}\right) = \ln 2,\end{aligned}$$
a contradiction of (B.53).

Thus (B.51) holds, and using the fact that c_Θ^{**} is a limit of the sequence $\{c_\Theta^{**}(T_{\Theta n})\}_{n=1}^\infty$, we have

$$c_\Theta^{**} \geq c'_\Theta.$$

It immediately follows from (B.52) that for all $\Theta + 1 \leq t \leq T$,

$$c_t^{**}(T) \geq c'_t.$$

Since every c_t^{**} is a cluster point of the sequence $\{c_t^{**}(T)\}_{T=1,2,\ldots}$, we get

$$c_t^{**} \geq c'_t, \quad t = \Theta+1, \Theta+2, \ldots.$$

It follows that

$$U^* - U^{*\Theta} = \sum_{t=\Theta+1}^{\infty} \beta^t u_t(c_t^{**}) = \sum_{t=\Theta+1}^{\infty} \beta^t \left(\ln(c_t^{**}) - \ln c'_t\right) \geq 0,$$

which completes the proof. □

Claim B.15.

$$U^{*\Theta}(T) > U^*(T) - \frac{\Delta}{2}, \quad T = \Theta+1, \Theta+2, \ldots; \tag{B.56}$$

$$W^{*\Theta} > \widehat{U} - \frac{\Delta}{2}. \tag{B.57}$$

Proof. Clearly, $\bar{c}_t > c_t^{**}(T)$ and $\bar{c}_t > \widehat{c}_t$ for all t. It follows from the choice of Θ that

$$\begin{aligned} U^*(T) - U^{*\Theta}(T) &= \sum_{t=0}^{T} \beta^t u_t(c_t^{**}(T)) - \sum_{t=0}^{\Theta} \beta^t u_t(c_t^{**}(T)) = \sum_{t=\Theta+1}^{T} \beta^t u_t(c_t^{**}(T)) \\ &\leq \sum_{t=\Theta+1}^{T} \beta^t u_t(\bar{c}_t) \leq \sum_{t=\Theta+1}^{\infty} \beta^t u_t(\bar{c}_t) < \frac{\Delta}{2}, \end{aligned}$$

which proves (B.56).

Due to the definition of $W^{*\Theta}$, we have $W^{*\Theta} \geq \widehat{U}^\Theta$. Now it is easily seen that

$$W^{*\Theta} \geq \widehat{U}^\Theta = \sum_{t=0}^{\Theta} \beta^t u_t(\widehat{c}_t) = \widehat{U} - \sum_{t=\Theta+1}^{\infty} \beta^t u_t(\widehat{c}_t) \geq \widehat{U} - \sum_{t=\Theta+1}^{\infty} \beta^t u_t(\bar{c}_t) > \widehat{U} - \frac{\Delta}{2},$$

which proves (B.57). □

Now, combining Claims B.10–B.12 and B.14–B.15, we obtain

$$U^* \geq U^{*\Theta} = \lim_{n\to\infty} U^{*\Theta}(T_{\Theta n}) \geq \lim_{n\to\infty} U^*(T_{\Theta n}) - \frac{\Delta}{2}$$
$$\geq \lim_{n\to\infty} W^{*\Theta}(T_{\Theta n}) - \frac{\Delta}{2} = W^{*\Theta} - \frac{\Delta}{2} > \widehat{U} - \Delta,$$

which contradicts the choice of Δ. This contradiction completes the proof of the lemma. □

Thus the proof of Theorem B.1 is finally complete, and there exists a competitive equilibrium under given extraction rates. □

List of Figures

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