Weak martingale solutions for the stochastic nonlinear Schrödinger equation driven by pure jump noise

Zdzisław Brzeźniak, Fabian Hornung, Utpal Manna

KARLSRUHE INSTITUTE OF TECHNOLOGY

CRC 1173

WaVe
phenomena

Participating universities

Universität Stuttgart

EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

Funded by

DFG

WEAK MARTINGALE SOLUTIONS FOR THE STOCHASTIC NONLINEAR SCHRÖDINGER EQUATION DRIVEN BY PURE JUMP NOISE

ZDZISŁAW BRZEŹNIAK, FABIAN HORNUNG, AND UTPAL MANNA

ABSTRACT. We construct a martingale solution of the stochastic nonlinear Schrödinger equation with a multiplicative noise of jump type in the Marcus canonical form. The problem is formulated in a general framework that covers the subcritical focusing and defocusing stochastic NLS in H^1 on compact manifolds and on bounded domains with various boundary conditions. The proof is based on a variant of the Faedo-Galerkin method. In the formulation of the approximated equations, finite dimensional operators derived from the Littlewood-Paley decomposition complement the classical orthogonal projections to guarantee uniform estimates. Further ingredients of the construction are tightness criteria in certain spaces of càdlàg functions and Jakubowski's generalization of the Skorohod-Theorem to nonmetric spaces.

CONTENTS

1. INTRODUCTION

In this paper, we study the stochastic nonlinear Schrödinger equation with pure jump noise in the Marcus form

$$\begin{cases} \mathrm{d}u(t) = (-\mathrm{i}Au(t) - \mathrm{i}F(u(t)))\,\mathrm{d}t - \mathrm{i}\sum_{m=1}^{N} B_m u(t) \diamond \mathrm{d}L_m(t) & t > 0, \\ u(0) = u_0. \end{cases} \tag{1.1}$$

Here, A is a selfadjoint nonnegative operator with a compact resolvent in an L^2-space H and the initial value u_0 is chosen from the energy space $E_A := \mathcal{D}(A^{\frac{1}{2}})$. Typical examples for this setting are

- the negative Laplace-Beltrami operator $A = -\Delta_g$ on a compact riemannian manifold (M, g) without boundary, $E_A = H^1(M)$,
- the negative Laplacian $A = -\Delta$ on a bounded domain $M \subset \mathbb{R}^d$ with Neumann boundary condition, i.e. $E_A = H^1(M)$, or Dirichlet boundary conditions, i.e. $E_A = H_0^1(M)$
- and fractional powers of the first two examples.

Moreover, $F : E_A \to E_A^*$ is a nonlinear map generalizing the two most important examples, namely

- the defocusing power nonlinearity $F_\alpha^+(u) := |u|^{\alpha-1}u$ with subcritical exponents in the sense that the embedding $E_A \hookrightarrow L^{\alpha+1}$ is compact
- and the focusing nonlinearity $F_\alpha^-(u) := -|u|^{\alpha-1}u$ with an additional restriction to the power α.

The stochastic noise term is given by selfadjoint linear bounded operators B_m for $m = 1, \ldots, N$ and an $\mathbb{R}^N-$ valued Lévy process $L(t) := (L_1(t), \cdots, L_N(t))$ with pure jump defined as

$$L(t) = \int_0^t \int_B l\, \tilde{\eta}(\mathrm{d}s, \mathrm{d}l) \tag{1.2}$$

where $B := \{|l| \leqslant 1\} \subset \mathbb{R}^N$. Here, η represents a time homogeneous Poisson random measure with σ-finite intensity measure ν such that

$$\int_B |l|^2 \nu(\mathrm{d}l) < \infty.$$

Moreover, $\tilde{\eta} := \eta - \mathrm{Leb}\otimes\nu$ denotes the corresponding time homogeneous compensated Poisson random measure (see Appendix A for details). Note that by the choice of L in (1.2), we restrict ourselves to the case of small jumps. A generalization of the results of the present article to noise with jumps of arbitrary size will be investigated. Using the abbreviation

$$\mathcal{B}(l) = \sum_{m=1}^{N} l_m B_m, \qquad l \in \mathbb{R}^N,$$

the equation (1.1) including the Marcus product $\diamond$ is understood in the sense of the associated integral equation

$$u(t) = u_0 - \mathrm{i} \int_0^t \left(Au(s) + F(u(s)) \right) \mathrm{d}s + \int_0^t \int_B \left[e^{-\mathrm{i}\mathcal{B}(l)} u(s-) - u(s-) \right] \tilde{\eta}(\mathrm{d}s, \mathrm{d}l)$$

$$+ \int_0^t \int_B \left\{ e^{-\mathrm{i}\mathcal{B}(l)} u(s) - u(s) + \mathrm{i}\mathcal{B}(l)u(s) \right\} \nu(\mathrm{d}l)\mathrm{d}s. \tag{1.3}$$

Before we describe our approach and state our result in detail, we would like to give a general overview of the literature on the stochastic NLS. In the two previous decades, existence and uniqueness results for the stochastic NLS with Gaussian noise have been treated in many articles, most notably [dBD99], [dBD03],[BRZ14],[BRZ16],[Hor18b] in the $\mathbb{R}^d$-setting, [BM14] for general $2D$ compact manifolds and [CM18] for the d-dimensional torus $\mathbb{T}^d$. In these articles, the authors applied Strichartz estimates in a fixed point argument based on the mild formulation. Typically, this argument was either combined with a transformation to a random NLS without stochastic integral or with a truncation of the nonlinearities and suitable estimates of stochastic convolutions.

In their joint papers [BHW17] and [BHW18] together with Lutz Weis, the first and second named author developed a different approach to the stochastic NLS with Gaussian noise. By complementing the classical Faedo-Galerkin approximation with methods from spectral theory and particularly, a general version of the Littlewood-Paley decomposition, they were able to prove the existence of a martingale solution. In contrast to the argument based on Strichartz estimates, the construction only employs the Hamiltonian structure of the NLS and certain compact Sobolev embeddings. Therefore, the result could be formulated in a rather general setting including the stochastic NLS and the stochastic fractional NLS on compact manifolds and bounded domains. Subsequently, the authors concentrated on the special case of $2D$ manifolds with bounded geometry and $3D$ compact manifolds and proved pathwise uniqueness using appropriate Strichartz estimates from [BGT04] and [BS14]. For a slight generalization of the existence result from [BHW17] allowing a certain class of non-conservative nonlinear noise, we refer to the PhD thesis [Hor18a] of the second author.

In contrast to their Gaussian counterpart, stochastic nonlinear Schrödinger equations with jump noise as in (1.1) are less well studied in the literature. Models of this type have been proposed in [VM10] and [VM11] to incorporate amplification of a signal in a fiber at random isolated locations caused by material inhomogeneities. In [dBH17], de Bouard and Hausenblas considered a similar problem as (1.1) on the full space $\mathbb{R}^d$ and obtained the existence of a martingale solution. The authors continued their work and in the recent preprint [dBHO18] with Ondrejat, and proved pathwise uniqueness in the $\mathbb{R}^d$-setting. The analysis of the noise in our present work is different compared to [dBH17, dBHO18] and is motivated by the requirement that the noise must preserve the invariance property under coordinate transformation. This issue is important for the norm-preserving condition, see (1.6) below. Thus, one needs to find an analogue of the Stratonovich integral in the case of stochastic integral with respect to compensated Poisson random measure. The work of Marcus [Mar81], developed later by Applebaum and Kunita, see e.g. Section

6.10 of Applebaum [App09] and Kunita [Kun04]; see also Chechkin and Pavlyukevich [CP14]; provides a framework to resolve this technical issue. Surprisingly, the literature on stochastic partial differential equations driven by Lévy noise in the "Marcus" canonical form is very limited and such work has recently been initiated by the first and third named authors in [BM17] for the Landau-Lifshitz-Gilbert equation. The current paper is motivated by similar question and we believe that the theory developed in this work may help in understanding analysis of many other constrained PDEs (e.g. harmonic map flow, nematic liquid crystal model etc.) driven by jump noise or more general Lévy noise. Also, there are some very recent works, see e.g. Chevyrev and Friz [CF17], where rough differential equations are studied in the spirit of Marcus canonical stochastic differential equations by dropping the assumption of continuity prevalent in the rough path literature. Therefore, we hope that Gubinelli's [Gub04] approach of Lyons' theory of integration over rough paths may be integrated with [CF17] and our approach to gain newer insight into the analysis of constrained SPDEs.

The goal of the present study is to construct a martingale solution of the stochastic NLS with pure jump noise in the Marcus canonical form. For that purpose, we transfer the argument developed in [BHW17] for the NLS with Gaussian noise to the present setting. Let us present our reasoning in detail. First, we introduce a strictly positive operator S which commutes with A and also has a compact resolvent. The operator S is used to present a unified proof for each example and will chosen individually in the different concrete settings from Section 3. Typical choices are $S = A$ or $S = \mathrm{Id} + A$. By means of the functional calculus of S which is based on its series representation, we define operators $P_n = p_n(S)$ and $S_n = s_n(S)$ for $n \in \mathbb{N}_0$. The functions p_n and s_n, $n \in \mathbb{N}_0$ are illustrated in Figure 1. For the precise definition, we refer to Section 5 and particularly the proof of Proposition 5.1. To summarize the most important properties of these operators, we remark that both P_n and S_n have a finite dimensional range, P_n is an orthogonal projection and the operators S_n satisfy the uniform estimate $\sup_{n \in \mathbb{N}_0} \|S_n\|_{\mathcal{L}(L^{\alpha+1})} < \infty$ since we assume that S satisfies (generalized) Gaussian bounds. Let us remark that a similar construction has been employed in [Hor18c] to construct a solution of a stochastic nonlinear Maxwell equation with Gaussian noise. This indicates that using operators like S_n, $n \in \mathbb{N}_0$, significantly increases the field of application of the classical Faedo-Galerkin method for both continuous and jump noise.
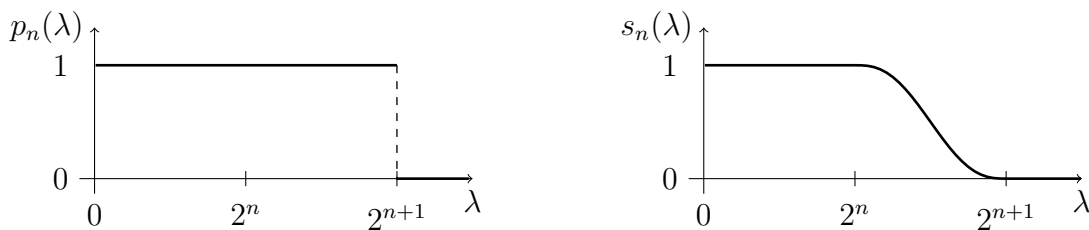

FIGURE 1. Plot of the functions p_n and s_n

Let us denote $\mathcal{B}_n(l) = \sum_{m=1}^{N} l_m S_n B_m S_n$ for $n \in \mathbb{N}$ and $l \in \mathbb{R}^N$ and

$$\widetilde{u_{0,n}} := \begin{cases} S_n u_0 \frac{\|u_0\|_H}{\|S_n u_0\|_H}, & S_n u_0 \neq 0, \\ 0, & S_n u_0 = 0. \end{cases}$$

for $n \in \mathbb{N}$. Then, the finite dimensional approximation

$$u_n(t) = P_n u_0 - \mathrm{i} \int_0^t \left(A u_n(s) + P_n F(u_n(s)) \right) \, \mathrm{d}s + \int_0^t \int_B \left[e^{-\mathrm{i}\mathcal{B}_n(l)} u_n(s-) - u_n(s-) \right] \tilde{\eta}(\mathrm{d}s, \mathrm{d}l)$$

$$+ \int_0^t \int_B \left\{ e^{-\mathrm{i}\mathcal{B}_n(l)} u_n(s) - u_n(s) + \mathrm{i}\mathcal{B}_n(l) u_n(s) \right\} \nu(\mathrm{d}l) \mathrm{d}s \tag{1.4}$$

of problem (1.1) has a unique solution. Due to the properties of P_n and S_n and the Hamiltonian structure of the nonlinear Schrödinger equation combined with the Marcus structure of the noise, we are able to prove the mass identity

$$\|u_n(t)\|_{L^2} = \|P_n u_0\|_{L^2}$$

almost surely for all $t \in [0, T]$ and the uniform estimate

$$\sup_{n \in \mathbb{N}} \mathbb{E}\left[\sup_{t \in [0,T]} \|u_n(t)\|_{E_A}^r \right] < \infty \tag{1.5}$$

for all $r \in [1, \infty)$. Using several compactness Lemmata for spaces of càdlàg functions inspired by [Mot12] and [BM17], (1.5) leads to tightness of the sequence $(u_n)_{n \in \mathbb{N}}$ in

$$Z_T := \mathbb{D}([0,T], E_A^*) \cap L^{\alpha+1}(0,T; L^{\alpha+1}(M)) \cap \mathbb{D}_w\left([0,T], E_A\right).$$

For the precise definition of Z_T, we refer to Section 4. Subsequently, a limit argument based on the Skorohod-Jakubowski Theorem shows the existence of a martingale solution. Altogether, we prove the following result.

Theorem 1.1. *Choose the operator A and the energy space E_A according to Assumption 2.1, the nonlinearity F according to Assumptions 2.4 and 2.6 and the noise according to Assumption 2.7. Then, for any $u_0 \in H$, the problem (1.1) has a martingale solution $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{\eta}, \tilde{\mathbb{F}}, u \right)$ which satisfies*

$$u \in L^q(\tilde{\Omega}, L^\infty(0,T; E_A))$$

for all $q \in [1, \infty)$. Moreover, the equality

$$\|u(t)\|_H = \|u_0\|_H \tag{1.6}$$

holds $\bar{\mathbb{P}}$-almost surely for all $t \in [0, T]$.

The article is organized as follows. In the second section, we fix the setting by stating the general assumptions on the operator A, the nonlinearity F and the noise term. These assumptions are illustrated in the third section by concrete examples. The proof of the main Theorem 1.1 is contained in the sections 4, 5 and 6 that deal with compactness results, the uniform estimates for the Galerkin approximation and the limit procedure. In the appendix, we collect basic material on Poisson random measures and Marcus noise.

2. General Framework and Assumptions

In this section, we formulate the abstract framework for the stochastic nonlinear Schrödinger equation we refer to in Theorem 1.1.

Let $(\tilde{M}, \Sigma, \mu)$ be a σ-finite measure space with metric ρ satisfying the *doubling property*, i.e. $\mu(B(x,r)) < \infty$ for all $x \in \tilde{M}$ and $r > 0$ and

$$\mu(B(x,2r)) \lesssim \mu(B(x,r)). \tag{2.1}$$

Let $M \subset \tilde{M}$ be an open subset with finite measure and $L^q(M)$ for $q \in [1, \infty]$ the space of equivalence classes of $\mathbb{C}$-valued $q-$integrable functions. We further abbreviate $H := L^2(M)$ and equip H with the standard complex L^2-inner product.

Let A be a non-negative self-adjoint operator on H with domain $\mathcal{D}(A)$. We set $E_A := \mathcal{D}((\mathrm{Id} + A)^{\frac{1}{2}})$ and call it *energy space*. Equipped with the inner product

$$\left(x, y\right)_{E_A} = \left((\mathrm{Id} + A)^{\frac{1}{2}} x, (\mathrm{Id} + A)^{\frac{1}{2}} y\right)_{L^2},$$

E_A is a complex Hilbert space. Moreover, we define the extrapolation space $H_{-\frac{1}{2}}$ as the completion of H with respect to the norm

$$\|x\|_{-\frac{1}{2}} := \|(\mathrm{Id} + A)^{-\frac{1}{2}} x\|_{L^2}, \qquad x \in H,$$

and obtain a Hilbert space with the inner product

$$\left(x, y\right)_{-\frac{1}{2}} = \lim_{n,m \to \infty} \left((\mathrm{Id} + A)^{-\frac{1}{2}} x_n, (\mathrm{Id} + A)^{-\frac{1}{2}} y_m\right)_{L^2}, \qquad x, y \in H_{-\frac{1}{2}},$$

for sequences $(x_n)_{n \in \mathbb{N}}, (y_m)_{m \in \mathbb{N}} \subset H$ with $x_n \to x$ and $y_m \to y$ in $H_{-\frac{1}{2}}$ as $n, m \to \infty$. Note that we can identify $H_{-\frac{1}{2}}$ with E_A^* and the duality is given by

$$\langle x, y \rangle_{\frac{1}{2}, -\frac{1}{2}} := \lim_{n \to \infty} \left(x, y_n\right)_{L^2}, \qquad x \in E_A, \quad y \in H_{-\frac{1}{2}},$$

with $(y_n)_{n \in \mathbb{N}} \subset H$ such that $y_n \to y$ in $H_{-\frac{1}{2}}$ as $n \to \infty$. Often, we shortly write $\langle \cdot, \cdot \rangle$ for $\langle \cdot, \cdot \rangle_{\frac{1}{2}, -\frac{1}{2}}$ and write E_A^* instead of $H_{-\frac{1}{2}}$. Note that (E_A, H, E_A^*) is a Gelfand triple, i.e.

$$E_A \hookrightarrow H \cong H^* \hookrightarrow E_A^*.$$

We point out that one can also treat H, E_A and $H_{-\frac{1}{2}}$ as real Hilbert spaces with scalar products $\mathrm{Re} \left(\cdot, \cdot\right)_H$, $\mathrm{Re} \left(\cdot, \cdot\right)_{E_A}$ and $\mathrm{Re} \left(\cdot, \cdot\right)_{-\frac{1}{2}}$, respectively. Then, E_A and $H_{-\frac{1}{2}}$ are dual in the sense that each real-valued continuous linear functional f on E_A has the representation $f = \mathrm{Re} \langle \cdot, y_f \rangle_{\frac{1}{2}, -\frac{1}{2}}$ for some $y_f \in H_{-\frac{1}{2}}$.

We continue with the main Assumption on the functional analytic setting for the stochastic NLS.

Assumption and Notation 2.1. *We assume the following:*

i) *There is a strictly positive self-adjoint operator S on H with compact resolvent commuting with A and $\mathcal{D}(S^k) \hookrightarrow E_A$ for some $k \in \mathbb{N}$. Moreover, we assume that there exists $p_0 \in [1, 2)$, such that S has* generalized Gaussian (p_0, p_0')-bounds, *i.e.*

$$\|\mathbf{1}_{B(x,t^{\frac{1}{m}})} e^{-tS} \mathbf{1}_{B(y,t^{\frac{1}{m}})}\|_{\mathcal{L}(L^{p_0}, L^{p_0'})} \leqslant C\mu(B(x, t^{\frac{1}{m}}))^{\frac{1}{p_0'} - \frac{1}{p_0}} \exp\left\{-c\left(\frac{\rho(x,y)^m}{t}\right)^{\frac{1}{m-1}}\right\}, \quad (2.2)$$

for all $t > 0$ and $(x,y) \in M \times M$ with constants $c, C > 0$ and $m \geqslant 2$.

ii) *Let $\alpha \in (1, p_0' - 1)$ be such that E_A is compactly embedded in $L^{\alpha+1}(M)$. We set*

$$p_{\max} := \sup\{p \in (1, \infty] : E_A \hookrightarrow L^p(M) \quad \text{is continuous}\}$$

and note that $p_{\max} \in [\alpha + 1, \infty]$. In the case $p_{\max} < \infty$, we assume that $E_A \hookrightarrow L^{p_{\max}}(M)$ is continuous, but not necessarily compact.

Remark 2.2.   a) If $p_0 = 1$, then it is proved in [BK03] that (2.2) is equivalent to the usual upper Gaussian estimate, i.e. for all $t > 0$ there is a measurable function $p(t, \cdot, \cdot) : M \times M \to \mathbb{R}$ with

$$(e^{-tS}f)(x) = \int_M p(t, x, y) f(y) \mu(dy), \quad t > 0, \quad \text{a.e. } x \in M$$

for all $f \in H$ and

$$|p(t, x, y)| \leqslant \frac{C}{\mu(B(x, t^{\frac{1}{m}}))} \exp\left\{-c\left(\frac{\rho(x,y)^m}{t}\right)^{\frac{1}{m-1}}\right\}, \quad (2.3)$$

for all $t > 0$ and almost all $(x, y) \in M \times M$ with constants $c, C > 0$ and $m \geqslant 2$. In particular, e^{-tS} can be extended to a C_0-semigroup on $L^p(M)$ for all $p \in [1, \infty)$.

b) In fact, in all our examples in the third section, the upper Gaussian estimate (2.3) holds and therefore, the previous assumption is fulfilled with $p_0 = 1$.

The following Lemma contains some straightforward consequences of 2.1.

Lemma 2.3.   a) *There is a positive self-adjoint operator $\hat{A}$ on E_A^* with $\mathcal{D}(\hat{A}) = E_A$ such that the restriction of $\hat{A}$ to $D(A)$ is equal to A. For simplicity of notation, we will denote the operator $\hat{A}$ by A.*

b) *The embedding $E_A \hookrightarrow H$ is compact.*

c) *There is an orthonormal basis $(h_n)_{n \in \mathbb{N}}$ and a nondecreasing sequence $(\lambda_n)_{n \in \mathbb{N}}$ with $\lambda_n > 0$ and $\lambda_n \to \infty$ as $n \to \infty$ and*

$$Sx = \sum_{n=1}^{\infty} \lambda_n (x, h_n)_H h_n, \quad x \in \mathcal{D}(S) = \left\{x \in H : \sum_{n=1}^{\infty} \lambda_n^2 |(x, h_n)_H|^2 < \infty\right\}.$$

Assumption 2.4. *Let $\alpha \in (1, p_0' - 1)$ be chosen as in Assumption 2.1. Then, we assume the following:*

i) *Let* $F : L^{\alpha+1}(M) \to L^{\frac{\alpha+1}{\alpha}}(M)$ *be a function satisfying the following estimate*

$$\|F(u)\|_{L^{\frac{\alpha+1}{\alpha}}(M)} \leqslant C_{F,1} \|u\|_{L^{\alpha+1}(M)}^{\alpha}, \quad u \in L^{\alpha+1}(M). \tag{2.4}$$

Note that this leads to $F : E_A \to E_A^*$ *by Assumption* 2.1, *because* $E_A \hookrightarrow L^{\alpha+1}(M)$ *implies* $(L^{\alpha+1}(M))^* = L^{\frac{\alpha+1}{\alpha}}(M) \hookrightarrow E_A^*$. *We further assume* $F(0) = 0$ *and*

$$\mathrm{Re}\langle iu, F(u)\rangle = 0, \quad u \in L^{\alpha+1}(M). \tag{2.5}$$

ii) *The map* $F : L^{\alpha+1}(M) \to L^{\frac{\alpha+1}{\alpha}}(M)$ *is continuously real Fréchet differentiable with*

$$\|F'[u]\|_{L^{\alpha+1} \to L^{\frac{\alpha+1}{\alpha}}} \leqslant C_{F,2} \|u\|_{L^{\alpha+1}(M)}^{\alpha-1}, \quad u \in L^{\alpha+1}(M). \tag{2.6}$$

iii) *The map* F *has a real antiderivative* $\hat{F}$, *i.e. there exists a Fréchet-differentiable map* $\hat{F} : L^{\alpha+1}(M) \to \mathbb{R}$ *with*

$$\hat{F}'[u]h = \mathrm{Re}\langle F(u), h\rangle, \quad u, h \in L^{\alpha+1}(M). \tag{2.7}$$

By Assumption 2.4 ii) and the mean value theorem, we get

$$\|F(x) - F(y)\|_{L^{\frac{\alpha+1}{\alpha}}(M)} \leqslant \sup_{t \in [0,1]} \|F'[tx + (1-t)y]\| \|x - y\|_{L^{\alpha+1}(M)}$$

$$\leqslant C_{F,2} \left(\|x\|_{L^{\alpha+1}(M)} + \|y\|_{L^{\alpha+1}(M)}\right)^{\alpha-1} \|x - y\|_{L^{\alpha+1}(M)} \tag{2.8}$$

for $x, y \in L^{\alpha+1}(M)$ which means that the nonlinearity is Lipschitz on bounded sets of $L^{\alpha+1}(M)$. We will cover the following two standard types of nonlinearities.

Definition 2.5. *Let* F *satisfy Assumption* 2.4. *Then,* F *is called*

defocusing, *if* $\hat{F}(u) \geqslant 0$ *for all* $u \in L^{\alpha+1}(M)$
and
focusing, *if* $\hat{F}(u) \leqslant 0$ *for all* $u \in L^{\alpha+1}(M)$.

Assumption 2.6. *We assume that either condition i) or condition i') holds, where*

i) *The function* F *is defocusing and satisfies*

$$\frac{1}{C_{F,3}} \|u\|_{L^{\alpha+1}(M)}^{\alpha+1} \leqslant \hat{F}(u) \leqslant C_{F,3} \|u\|_{L^{\alpha+1}(M)}^{\alpha+1}, \quad u \in L^{\alpha+1}(M). \tag{2.9}$$

i') *The function* F *is focusing and satisfies*

$$-\hat{F}(u) \leqslant C_{F,4} \|u\|_{L^{\alpha+1}(M)}^{\alpha+1}, \quad u \in L^{\alpha+1}(M), \tag{2.10}$$

and there exists $\theta \in (0, \frac{2}{\alpha+1})$ *such that* [1]

$$(H, E_A)_{\theta,1} \hookrightarrow L^{\alpha+1}(M). \tag{2.11}$$

The model nonlinearities are the defocusing power nonlinearity $F_\alpha^+(u) := |u|^{\alpha-1}u$ with subcritical exponents in the sense that the embedding $E_A \hookrightarrow L^{\alpha+1}$ is compact and the focusing nonlinearity $F_\alpha^-(u) := -|u|^{\alpha-1}u$ with an additional restriction to the power α.

---

[1] In below, the symbol $(\cdot, \cdot)_{\theta,1}$ stands for the real interpolation functor with parameters 1 and ∞, see for instance [Tri92].

Assumption 2.7. (a) *Assume that* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *is a filtered probability space, where* $\mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}$ *is the filtration, and this probability space satisfies the so called usual conditions, i.e.*

(i) $\mathbb{P}$ *is complete on* $(\Omega, \mathcal{F})$,

(ii) *for each* $t \geqslant 0$, $\mathcal{F}_t$ *contains all* $(\mathcal{F}, \mathbb{P})$*-null sets,*

(iii) *the filtration* $\mathbb{F}$ *is right-continuous.*

(b) *Assume that* $(L(t))_{t \geqslant 0}$ *is an* $\mathbb{R}^N$*-valued,* $(\mathcal{F}_t)$*-adapted Lévy process of pure jump type defined on the above probability space with drift* 0 *and the corresponding time homogenous Poisson random measure* η.

(c) *Assume that the intensity measure* $\mathrm{Leb} \otimes \nu$ *is such that* $\mathrm{supp}\, \nu \subset B$, *where* B *is the closed unit ball in* $\mathbb{R}^N$.

d) *Let* $B_1, \ldots, B_M \in \mathcal{L}(H)$ *be self-adjoint operators on* H *with* $B_m|_{E_A} \in \mathcal{L}(E_A)$ *and* $B_m|_{L^{\alpha+1}(M)} \in \mathcal{L}(L^{\alpha+1}(M))$.

We abbreviate

$$b_{E_A} := \sum_{m=1}^{N} \|B_m\|_{\mathcal{L}(E_A)}^2, \qquad b_{L^{\alpha+1}} := \sum_{m=1}^{N} \|B_m\|_{\mathcal{L}(L^{\alpha+1})}^2, \qquad b_H := \sum_{m=1}^{N} \|B_m\|_{\mathcal{L}(H)}^2 \quad (2.12)$$

and for $l \in \mathbb{R}^N$, we introduce the notation

$$\mathcal{B}(l) =: \sum_{m=1}^{N} l_m B_m.$$

Remark 2.8. Note that by the Lévy-Khinchine formula, see [PZ07], Theorem 4.23, the previous assumption yields that the intensity measure ν is a Lévy-measure on $\mathbb{R}^N$, i.e.

$$\int_B |l|^2 \nu(\mathrm{d}l) < \infty. \tag{2.13}$$

Moreover, we have the representation

$$L(t) = \int_0^t \int_B l \, \tilde{\eta}(\mathrm{d}s, \mathrm{d}l).$$

2.1. **The Marcus Mapping.** Let us define a generalized Marcus mapping

$$\Phi : \mathbb{R}_+ \times \mathbb{R}^N \times H \to H, \qquad \Phi(t, l, x) := e^{-it\mathcal{B}(l)} x,$$

i.e. for each fixed $l \in \mathbb{R}^N$, $x \in H$, the function $t \mapsto \Phi(t, l, x)$ is the continuously differentiable solution of

$$\frac{du}{dt}(t) = -\mathrm{i} \sum_{m=1}^{N} l_m B_m u(t), \qquad t \geqslant 0, \tag{2.14}$$

with $u(0) = x \in H$, and $l = (l_1, l_2, \ldots, l_N) \in \mathbb{R}^N$. Equation (1.1) with notation $\diamond$ is defined in the integral form as following

$$u(t) = u_0 - \mathrm{i} \int_0^t \left(Au(s) + F(u(s)) \right) \mathrm{d}s + \int_0^t \int_B \left[e^{-\mathrm{i}\mathcal{B}(l)} u(s-) - u(s-) \right] \tilde{\eta}(\mathrm{d}s, \mathrm{d}l)$$

$$+ \int_0^t \int_B \left\{ e^{-i\mathcal{B}(l)}u(s) - u(s) + i\sum_{m=1}^N l_m B_m u(s) \right\} \nu(\mathrm{d}l)\mathrm{d}s, \tag{2.15}$$

where $\tilde{\eta} := \eta - \mathrm{Leb} \otimes \nu$ denotes the compensated Poisson random measure induced by η. In the next definition, we define the notion of a solution used in the present article.

Definition 2.9. *Let $T > 0$ and $u_0 \in E_A$. A* martingale solution *of the equation* (1.1) *is a system $\left(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \bar{\eta}, \bar{\mathbb{F}}, u\right)$ with*

- *a probability space $\left(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}\right)$;*
- *a time homogeneous Poisson random measure $\bar{\eta}$ on $\mathbb{R}^N$ over $\bar{\Omega}$ with intensity measure ν,*
- *a filtration $\bar{\mathbb{F}} = \left(\bar{\mathcal{F}}_t\right)_{t \in [0,T]}$ with the usual conditions;*
- *an $\bar{\mathbb{F}}$-adapted, E_A^*-valued càdlàg process such that $u \in L^2(\Omega \times [0,T], E_A^*)$ and almost all paths are in $\mathbb{D}_w([0,T], E_A)$,*

such that the equation (2.15) *holds $\bar{\mathbb{P}}$-almost surely in E_A^* for all $t \in [0,T]$ with $\bar{\tilde{\eta}}$ instead of $\tilde{\eta}$.*

3. Examples

In this section, we collect concrete settings which are covered by the general framework of Assumptions 2.1, 2.4 and 2.6. We skip the proofs since they already appeared in [BHW17], where the NLS with Gaussian noise was considered in the same framework.

Corollary 3.1. *Suppose that a) or b) or c) is true.*

- a) *Let M be a d-dimensional compact manifold, $A = -\Delta_g$, $E_A = H^1(M)$.*
- b) *Let $M \subset \mathbb{R}^d$ be a bounded domain and $A = -\Delta_D$ be the Dirichlet-Laplacian, $E_A = H_0^1(M)$.*
- c) *Let $M \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $A = -\Delta_N$ be the Neumann-Laplacian, $E_A = H^1(M)$.*

Choose the nonlinearity from i) or ii).

- i) $F(u) = |u|^{\alpha-1}u$ *with $\alpha \in \left(1, 1 + \frac{4}{(d-2)_+}\right)$, i.e. F is defocusing,*
- ii) $F(u) = -|u|^{\alpha-1}u$ *with $\alpha \in \left(1, 1 + \frac{4}{d}\right)$, i.e. F is focusing,*

Set $B_m x = e_m x$ for $x \in H$ and $m = 1, \ldots, M$ with real-valued functions

$$e_m \in F := \begin{cases} H^{1,d}(M) \cap L^\infty(M), & d \geqslant 3, \\ H^{1,q}(M), & d = 2, \\ H^1(M), & d = 1, \end{cases} \tag{3.1}$$

for some $q > 2$ in the case $d = 2$. Then, the problem

$$\begin{cases} \mathrm{d}u(t) = \left(-iAu(t) - iF(u(t))\right)\mathrm{d}t - i\sum_{m=1}^N B_m u(t) \diamond dL_m(t), \\ u(0) = u_0 \in E_A, \end{cases} \tag{3.2}$$

has a martingale solution which satisfies $\|u(t)\|_H = \|u_0\|_H$ *almost surely for all* $t \in [0, T]$
and

$$u \in L^q(\tilde{\Omega}, L^\infty(0, T; E_A))$$

for all $q \in [1, \infty)$.

Proof. We refer to [BHW17], Section 3, for the verification of the Assumptions in Theorem 1.1. □

Additionally to the stochastic NLS, we can also cover the fractional NLS with the Laplacians replaced by their fractional powers.

Corollary 3.2. *Choose one of the settings* a)*,* b) *or* c) *in Corollary. Let* $\beta > 0$ *and suppose that we have either* i) *or* ii) *below.*

i) $F(u) = |u|^{\alpha-1}u$ *with* $\alpha \in \left(1, 1 + \frac{4\beta}{(d-2\beta)_+}\right)$,

ii) $F(u) = -|u|^{\alpha-1}u$ *with* $\alpha \in \left(1, 1 + \frac{4\beta}{d}\right)$,

Let B_m *for* $m = 1, \ldots, M$ *as in Assumption 2.7. Then, the problem*

$$\begin{cases} du(t) = \left(-iA^\beta u(t) - iF(u(t))\right) dt - i\sum_{m=1}^N B_m u(t) \diamond dL_m(t), \\ u(0) = u_0 \in \mathcal{D}(A^{\frac{\beta}{2}}), \end{cases} \tag{3.3}$$

has a martingale solution which satisfies $\|u(t)\|_H = \|u_0\|_H$ *almost surely for all* $t \in [0, T]$
and

$$u \in L^q(\tilde{\Omega}, L^\infty(0, T; \mathcal{D}(A^{\frac{\beta}{2}})))$$

for all $q \in [1, \infty)$.

4. COMPACTNESS AND TIGHTNESS CRITERIA

This section is devoted to the compactness results which will be used to get a martingale solution of (1.1) by the Faedo-Galerkin method. We begin with a definition of the càdlàg functions and a generalization of the modulus of continuity to this class. Throughout the section, $(\mathbb{S}, d)$ denotes a complete, separable metric space.

Definition 4.1. a) *The space of all* càdlàg *functions* $f : [0, T] \to \mathbb{S}$, *i.e.* f *is right-continuous with left limit in every* $t \in [0, T]$, *is called* $\mathbb{D}([0, T], \mathbb{S})$.

b) *For* $u \in \mathbb{D}([0, T], \mathbb{S})$ *and* $\delta > 0$, *we define the* modulus

$$w_\mathbb{S}(u, \delta) := \inf_{\Pi_\delta} \max_{t_j \in Q} \sup_{t,s \in [t_{j-1}, t_j)} d(u(t), u(s)),$$

where Π_δ *is the set of all partitions* $Q = \{0 = t_0 < t_1 < \cdots < t_N = T\}$ *of* $[0, T]$ *with*

$$t_{j+1} - t_j \geqslant \delta, \qquad j = 0, \ldots, N - 1.$$

c) *We denote the set of increasing homeomorphisms of $[0,T]$ by Λ and we equip $\mathbb{D}([0,T],\mathbb{S})$ with the metric defined by*

$$\rho(u,v) := \inf_{\lambda \in \Lambda} \left[\sup_{t \in [0,T]} \mathrm{d}(u(t),v(\lambda(t))) + \sup_{t \in [0,T]} |t - \lambda(t)| + \sup_{s \neq t} \left| \log \frac{\lambda(t) - \lambda(s)}{t-s} \right| \right]$$

for $u,v \in \mathbb{D}([0,T],\mathbb{S})$.

The following Proposition is about the so-called *Skohorod-topology* on $\mathbb{D}([0,T],\mathbb{S})$.

Proposition 4.2.   a) *The pair $\left(\mathbb{D}([0,T],\mathbb{S}), \rho \right)$ is a complete, separable metric space.*

b) *A sequence $(u_n)_{n \in \mathbb{N}} \in \mathbb{D}([0,T],\mathbb{S})^{\mathbb{N}}$ is convergent to $u \in \mathbb{D}([0,T],\mathbb{S})$ in the metric ρ if and only if there exists $(\lambda_n)_{n \in \mathbb{N}} \in \Lambda^{\mathbb{N}}$ with*

$$\sup_{t \in [0,T]} |\lambda_n(t) - t| \to 0, \qquad \sup_{t \in [0,T]} \mathrm{d}(u_n(\lambda_n(t)), u(t)) \to 0, \qquad n \to \infty.$$

Proof. See [Bil99], page 123 and following for a proof. $\qquad\qquad\qquad\qquad\square$

Definition 4.3. *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let X be a reflexive, separable $\mathbb{K}$-Banach space and X^* its dual.*

a) *Then, we define $\mathbb{D}_w([0,T],X)$ as the space of all $u : [0,T] \to X$ such that*

$$[0,T] \ni t \to \langle u(t), x^* \rangle \in \mathbb{K} \text{ is càdlàg for all } x^* \in X^*.$$

We equip $\mathbb{D}_w([0,T],X)$ with the weakest topology such that the map

$$\mathbb{D}_w([0,T],X) \ni u \mapsto \langle u(\cdot), x^* \rangle \in \mathbb{D}([0,T],\mathbb{K})$$

is continuous for all $x^* \in X^*$.

b) *For $r > 0$, we consider the ball $\mathbb{B}_X^r := \{u \in X : \|u\|_X \leqslant r\}$ and define*

$$\mathbb{D}([0,T],\mathbb{B}_X^r) := \left\{ u \in \mathbb{D}_w([0,T],X) : \sup_{t \in [0,T]} \|u(t)\|_X \leqslant r \right\}.$$

Remark 4.4. By the separability of X, the weak topology on $\mathbb{B}_X^r$ is metrizable and we choose a corresponding metric q. The notation in Definition 4.3 is justified, i.e.

$$\mathbb{D}([0,T],\mathbb{B}_X^r) \text{ coincides with } \mathbb{D}([0,T],\mathbb{S}) \text{ for } (\mathbb{S},d) = (\mathbb{B}_X^r, q). \qquad (4.1)$$

In particular, $\mathbb{D}([0,T],\mathbb{B}_X^r)$ is a complete, separable metric space by Proposition 4.2. To show (4.1), we note that the right-continuity of $\langle u(\cdot), x^* \rangle$ for all $x^* \in X^*$ is equivalent to the right-continuity of u in $(\mathbb{B}_X^r, q)$ by the definition of q. It is also easy to see that the existence of left limits transfers from $(\mathbb{B}_X^r, q)$ to $\langle \cdot, x^* \rangle$ for all $x^* \in X^*$.

For the converse direction, let $t_n \to t-$. Then, for each $x^* \in X^*$, there is $\gamma_{x^*} \in \mathbb{K}$ with $\langle u(t_n), x^* \rangle \to \gamma_{x^*}$. Since X is reflexive, $x^* \mapsto \gamma_{x^*}$ is linear and $|\gamma_{x^*}| \leqslant r\|x^*\|_{X^*}$, there is $v \in X$ such that $\gamma_{x^*} = \langle v, x^* \rangle$. Hence, $q(u(t_n), v) \to 0$.

Lemma 4.5. *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let X be a reflexive, separable $\mathbb{K}$-Banach space and let $u_n, u \in \mathbb{D}_w([0,T],X)$ with $u_n \to u$ in $\mathbb{D}_w([0,T],X)$ as $n \to \infty$. Then, we have*

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} \|u_n(t)\|_X < \infty.$$

Proof. From $u_n \to u$ in $\mathbb{D}_w([0,T],X)$ as $n \to \infty$, we infer that for every $x^* \in X^*$, we have $\langle u_n, x^* \rangle \to \langle u, x^* \rangle$ in $\mathbb{D}([0,T],\mathbb{K})$ as $n \to \infty$. Proposition 4.2 therefore implies that for every $x^* \in X^*$, there exists $(\lambda_n) \in \Lambda^{\mathbb{N}}$ with

$$\sup_{t \in [0,T]} |\lambda_n(t) - t| \to 0, \qquad \sup_{t \in [0,T]} |\langle u_n(\lambda_n(t)), x^* \rangle - \langle u(t), x^* \rangle| \to 0, \qquad n \to \infty.$$

In particular, we obtain

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} |\langle u_n(t), x^* \rangle| = \sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} |\langle u_n(\lambda_n(t)), x^* \rangle| < \infty$$

for every $x^* \in X^*$. The uniform boundedness principle yields

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} \|u_n(t)\|_X = \sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} \sup_{\|x^*\|_{X^*} \leqslant 1} |\langle u_n(t), x^* \rangle| < \infty.$$

$\square$

We recall that the energy space E_A is defined by $E_A := \mathcal{D}((\mathrm{Id} + A)^{\frac{1}{2}})$. We continue with a criterion for convergence of a sequence in $\mathbb{D}([0,T], \mathbb{B}_{E_A}^r)$.

Lemma 4.6. *Let $r > 0$ and $u_n : [0,T] \to E_A$ functions such that*
  a) $\sup_{n \in \mathbb{N}} \sup_{s \in [0,T]} \|u_n(s)\|_{E_A} \leqslant r$,
  b) $u_n \to u$ in $\mathbb{D}([0,T], E_A^*)$ for $n \to \infty$.
Then $u_n, u \in \mathbb{D}([0,T], \mathbb{B}_{E_A}^r)$ for all $n \in \mathbb{N}$ and $u_n \to u$ in $\mathbb{D}([0,T], \mathbb{B}_{E_A}^r)$ for $n \to \infty$.

Proof. See [Mot12], Lemma 3.3. $\square$

We continue with a Lemma stated in Lions [Lio69], p. 58.

Lemma 4.7 (Lions). *Let X, X_0, X_1 be Banach spaces with $X_0 \hookrightarrow X \hookrightarrow X_1$ where the first embedding is compact. Assume furthermore that X_0, X_1 are reflexive and $p \in [1, \infty)$. Then, for each $\varepsilon > 0$ there is $C_\varepsilon > 0$ with*

$$\|x\|_X^p \leqslant \varepsilon \|x\|_{X_0}^p + C_\varepsilon \|x\|_{X_1}^p, \quad x \in X_0.$$

Proof. See [Hor18a], Lemma 2.34. $\square$

We define a space Z_T by

$$Z_T := \mathbb{D}([0,T], E_A^*) \cap L^{\alpha+1}(0,T; L^{\alpha+1}(M)) \cap \mathbb{D}_w([0,T], E_A) =: Z_1 \cap Z_2 \cap Z_3. \qquad (4.2)$$

We equip Z_T with the supremum-topology, i.e. the smallest topology that contains $\bigcup_{j=1}^3 \mathcal{O}_j$, where $\mathcal{O}_j$ is the trace of the Z_j-topology in Z_T.

In the next Proposition, we give a criterion for compactness in Z_T. This result generalises Theorem 2 of Section 3 from [Mot12]. For a continuous version of this result see Proposition 4.2 of [BHW17]. Our proof is along the similar lines to Proposition 5.7 of the first and third named authours [BM17].

Proposition 4.8. *Let K be a subset of Z_T and $r > 0$ such that*
  a) $\sup_{z \in K} \sup_{t \in [0,T]} \|z(t)\|_{E_A} \leqslant r$;
  b) $\lim_{\delta \to 0} \sup_{z \in K} w_{E_A^*}(z, \delta) = 0$.

Then, K is relatively compact in Z_T.

Proof. Let K be a subset of Z_T such that the assumptions $a)$ and $b)$ are fullfilled and $(z_n)_{n\in\mathbb{N}} \subset K$.

Step 1: The relative compactness of K in $\mathbb{D}([0,T], E_A^*)$ is an immediate consequence of Theorem 3.2 in [Mot12]. Hence, we can take a subsequence again denoted by $(z_n)_{n\in\mathbb{N}}$ and $z \in \mathbb{D}([0,T], E_A^*)$ with $z_n \to z$ in $\mathbb{D}([0,T], E_A^*)$. By Lemma 4.6, we infer that $z_n \to z$ in $\mathbb{D}_w([0,T], E_A)$ and $\sup_{t\in[0,T]} \|z(t)\|_{E_A} \leqslant r$.

Step 2: We fix again $\varepsilon > 0$. By the Lions Lemma 4.7 with $X_0 = E_A$, $X = L^{\alpha+1}(M)$, $X_1 = E_A^*$, $p = \alpha + 1$ and $\varepsilon_0 = \frac{\varepsilon}{2T(2r)^{\alpha+1}}$ we get

$$\|v\|_{L^{\alpha+1}(M)}^{\alpha+1} \leqslant \varepsilon_0 \|v\|_{E_A}^{\alpha+1} + C_{\varepsilon_0} \|v\|_{E_A^*}^{\alpha+1} \tag{4.3}$$

for all $v \in E_A$. Integration with respect to time yields

$$\|z_n - z\|_{L^{\alpha+1}(0,T;L^{\alpha+1}(M))}^{\alpha+1} \leqslant \varepsilon_0 \|z_n - z\|_{L^{\alpha+1}(0,T;E_A)}^{\alpha+1} + C_{\varepsilon_0} \|z_n - z\|_{L^{\alpha+1}(0,T;E_A^*)}^{\alpha+1};$$

$$\varepsilon_0 \|z_n - z\|_{L^{\alpha+1}(0,T;E_A)}^{\alpha+1} \leqslant \varepsilon_0 T \|z_n - z\|_{L^\infty(0,T;E_A)}^{\alpha+1} \leqslant \varepsilon_0 T (2r)^{\alpha+1} \leqslant \frac{\varepsilon}{2}.$$

By [Bil99], p.124, equation (12.14), convergence in $\mathbb{D}([0,T], E_A^*)$ implies $z_n(t) \to u(t)$ in E_A^* for almost all $t \in [0,T]$. By Assumption $a)$, Lebesgue's Theorem yields $z_n \to z$ in $L^{\alpha+1}(0,T;E_A^*)$. Hence,

$$\limsup_{n\to\infty} \|z_n - z\|_{L^{\alpha+1}(0,T;L^{\alpha+1}(M))}^{\alpha+1} \leqslant \frac{\varepsilon}{2}$$

for all $\varepsilon > 0$ and thus, the sequence $(z_n)_{n\in\mathbb{N}}$ is also converges to u in $L^{\alpha+1}(0,T;L^{\alpha+1}(M))$. $\square$

In the following, we want to obtain a criterion for tightness in Z_T. Therefore, we introduce the Aldous condition.

Definition 4.9. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of stochastic processes in a Banach space E. Assume that for every $\varepsilon > 0$ and $\eta > 0$ there is $\delta > 0$ such that for every sequence $(\tau_n)_{n\in\mathbb{N}}$ of $[0,T]$-valued stopping times one has*

$$\sup_{n\in\mathbb{N}} \sup_{0<\theta\leqslant\delta} \mathbb{P}\left\{\|X_n((\tau_n + \theta) \wedge T) - X_n(\tau_n)\|_E \geqslant \eta\right\} \leqslant \varepsilon.$$

In this case, we say that $(X_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition $[A]$.

The following Lemma (see [Mot12], Lemma A.7) gives us a useful consequence of the Aldous condition $[A]$.

Lemma 4.10. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of stochastic processes in a Banach space E, which satisfies the Aldous condition $[A]$. Then, for every $\varepsilon > 0$ there exists a measurable subset $A_\varepsilon \subset \mathbb{D}([0,T], E)$ such that*

$$\mathbb{P}^{X_n}(A_\varepsilon) \geqslant 1 - \varepsilon, \qquad \lim_{\delta\to 0} \sup_{u\in A_\varepsilon} w_E(u,\delta) = 0.$$

The deterministic compactness result in Proposition 4.8 and the last Lemma can be used to get the following tightness criterion in Z_T.

Proposition 4.11. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of adapted E_A^*-valued processes satisfying the Aldous condition $[A]$ in E_A^* and*

$$\sup_{n\in\mathbb{N}}\mathbb{E}\left[\sup_{t\in[0,T]}\|X_n(t)\|_{E_A}^2\right]<\infty.$$

Then, the sequence $\left(\mathbb{P}^{X_n}\right)_{n\in\mathbb{N}}$ is tight in Z_T.

Proof. Let $\varepsilon>0$. With $R_1:=\left(\frac{2}{\varepsilon}\sup_{n\in\mathbb{N}}\mathbb{E}\left[\sup_{t\in[0,T]}\|X_n(t)\|_{E_A}^2\right]\right)^{\frac{1}{2}}$, we obtain

$$\mathbb{P}\left\{\sup_{t\in[0,T]}\|X_n(t)\|_{E_A}>R_1\right\}\leqslant\frac{1}{R_1^2}\mathbb{E}\left[\sup_{t\in[0,T]}\|X_n(t)\|_{E_A}^2\right]\leqslant\frac{\varepsilon}{2}.$$

By Lemma 4.10, one can use the Aldous condition $[A]$ to find a Borel subset A of $\mathbb{D}([0,T],E_A^*)$ such that

$$\inf_{n\in\mathbb{N}}\mathbb{P}^{X_n}(A)\geqslant 1-\frac{\varepsilon}{2},\qquad\text{and }\limsup_{\delta\to 0}\sup_{u\in A}w_{E_A^*}(u,\delta)=0.$$

We define $K:=\overline{A\cap B}$ where $B:=\left\{u\in Z_T:\sup_{t\in[0,T]}\|X_n(t)\|_{E_A}\leqslant R_1\right\}$. This set K is compact in Z_T by Proposition 4.8 and we can estimate

$$\mathbb{P}^{X_n}(K)\geqslant\mathbb{P}^{X_n}(A\cap B)\geqslant\mathbb{P}^{X_n}(A)-\mathbb{P}^{X_n}(B^c)\geqslant 1-\frac{\varepsilon}{2}-\frac{\varepsilon}{2}=1-\varepsilon$$

for all $n\in\mathbb{N}$. $\qquad\square$

In metric spaces, one can apply Prokhorov Theorem (see [Par67], Theorem II.6.7) and Skohorod Theorem (see [Bil99], Theorem 6.7.) to obtain a.s.-convergence from tightness. Since Z_T is not a metric space, we use the following generalization due to Jakubowski [Jak] and Brzeźniak et al [BHR17] in the variant of Motyl, [Mot12], Corollary 7.3.

Proposition 4.12. *Let $\mathcal{X}_1$ be a complete separable metric space and $\mathcal{X}_2$ a topological space such that there is a sequence of continuous functions $f_m:\mathcal{X}_2\to\mathbb{R}$ that separates points of $\mathcal{X}_2$. Define $\mathcal{X}:=\mathcal{X}_1\times\mathcal{X}_2$ and equip $\mathcal{X}$ with the topology induced by the canonical projections $\pi_j:\mathcal{X}_1\times\mathcal{X}_2\to\mathcal{X}_j$. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $(\chi_n)_{n\in\mathbb{N}}$ be a tight sequence of random variables in $(\mathcal{X},\mathcal{B}(\mathcal{X}_1)\otimes\mathcal{A})$, where $\mathcal{A}$ is the σ-algebra generated by f_m, $m\in\mathbb{N}$. Assume that there is a random variable η in $\mathcal{X}_1$ such that $\mathbb{P}^{\pi_1\circ\chi_n}=\mathbb{P}^\eta$.*

Then, there are a subsequence $(\chi_{n_k})_{k\in\mathbb{N}}$ and random variables $\tilde{\chi}_k,\tilde{\chi}$ in $\mathcal{X}$ for $k\in\mathbb{N}$ on a common probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\bar{\mathbb{P}})$ with

   i) $\bar{\mathbb{P}}^{\tilde{\chi}_k}=\mathbb{P}^{\chi_{n_k}}$ *for $k\in\mathbb{N}$,*
   ii) $\tilde{\chi}_k\to\tilde{\chi}$ *in $\mathcal{X}$ almost surely for $k\to\infty$,*
   iii) $\pi_1\circ\tilde{\chi}_k=\pi_1\circ\tilde{\chi}$ *almost surely.*

5. Energy Estimates for the solutions of the Galerkin approximation

In the following section, we formulate an approximation of (1.1) and prove existence and uniqueness, conservation of the L^2-norm as well as uniform bounds of the energy of the solutions to the approximated equation.

Recall from Lemma 2.3, that S has the representation

$$Sx = \sum_{m=1}^{\infty} \lambda_m (x, h_m)_H h_m, \quad x \in \mathcal{D}(S) \text{ and } \mathcal{D}(S) = \left\{ x \in H : \sum_{m=1}^{\infty} \lambda_m^2 |(x, h_m)_H|^2 < \infty \right\},$$

with an orthonormal basis $(h_m)_{m \in \mathbb{N}}$ of the complex Hilbert space $\left(H, (\cdot, \cdot)_H \right)$, eigenvalues $\lambda_m > 0$ such that $\lambda_m \to \infty$ as $m \to \infty$. For $n \in \mathbb{N}_0$, we set

$$H_n := \operatorname{span} \left\{ h_m : m \in \mathbb{N}, \lambda_m < 2^{n+1} \right\}$$

and denote the orthogonal projection from H to H_n by P_n, i.e.

$$P_n x = \sum_{\lambda_m < 2^{n+1}} (x, h_m)_H h_m, \qquad x \in H.$$

Since S and A commute by Assumption 2.1, we deduce that $\|P_n\|_{\mathcal{L}(E_A)} \leqslant 1$ and by density of H in E_A^*, we can extend P_n to an operator $P_n : E_A^* \to H_n$ with $\|P_n\|_{E_A^* \to E_A^*} \leqslant 1$ and

$$\langle v, P_n v \rangle \in \mathbb{R}, \qquad \langle v, P_n w \rangle = (P_n v, w)_H, \qquad v \in E_A^*, \quad w \in E_A. \tag{5.1}$$

Unfortunately, the operators P_n, $n \in \mathbb{N}_0$, are, in general, not uniformly bounded from $L^{\alpha+1}(M)$ to $L^{\alpha+1}(M)$. Therefore, we have to use another sequence operators introduced in [BHW17] to cut off the noise terms.

Proposition 5.1. *There exists a sequence $(S_n)_{n \in \mathbb{N}_0}$ of self-adjoint operators $S_n : H \to H_n$ for $n \in \mathbb{N}_0$ with $S_n \psi \to \psi$ in E_A for $n \to \infty$ and $\psi \in E_A$ and the uniform norm estimates*

$$\sup_{n \in \mathbb{N}_0} \|S_n\|_{\mathcal{L}(H)} \leqslant 1, \quad \sup_{n \in \mathbb{N}_0} \|S_n\|_{\mathcal{L}(E_A)} \leqslant 1, \quad \sup_{n \in \mathbb{N}_0} \|S_n\|_{\mathcal{L}(L^{\alpha+1})} < \infty. \tag{5.2}$$

A proof of this result can be in [BHW17], Proposition 5.2. For convenience of the reader, we present an alternative proof.

Proof. Step 1. We take a function $\rho \in C_c^{\infty}(0, \infty)$ with $\operatorname{supp} \rho \subset [\frac{1}{2}, 2]$ and $\sum_{m \in \mathbb{Z}} \rho(2^{-m}t) = 1$ for all $t > 0$. For the existence of ρ with these properties, we refer to [BL76], Lemma 6.1.7. Then, we fix $n \in \mathbb{N}_0$ and define

$$s_n : (0, \infty) \to \mathbb{C}, \qquad s_n(\lambda) := \sum_{m=-\infty}^{n} \rho(2^{-m}\lambda).$$

Let $k \in \mathbb{Z}$ and $\lambda \in [2^{k-1}, 2^k)$. From $\operatorname{supp} \rho \subset [\frac{1}{2}, 2]$, we infer

$$1 = \sum_{m=-\infty}^{\infty} \rho(2^{-m}\lambda) = \rho(2^{-(k-1)}\lambda) + \rho(2^{-k}\lambda) = \sum_{m=-\infty}^{k} \rho(2^{-m}\lambda).$$

In particular

$$s_n(\lambda) = \begin{cases} 1, & \lambda \in (0, 2^n), \\ \rho(2^{-n}\lambda), & \lambda \in [2^n, 2^{n+1}), \\ 0, & \lambda \geqslant 2^{n+1}. \end{cases} \tag{5.3}$$

We define $S_n := s_n(S)$ for $n \in \mathbb{N}_0$. Since s_n is real-valued and bounded by 1, the operator S_n is selfadjoint with $\|S_n\|_{\mathcal{L}(H)} \leqslant 1$. Furthermore, S_n and A commute due to the assumption that S and A commute. In particular, this implies $\|S_n\|_{\mathcal{L}(E_A)} \leqslant 1$ and $S_n\psi \to \psi$ for all $\psi \in E_A$ by the convergence property of the Borel functional calculus. Moreover, the range of S_n is contained in H_n since we have the representation

$$S_n x = \sum_{\lambda_m < 2^n} (x, h_m)_H h_m + \sum_{\lambda_m \in [2^n, 2^{n+1})} \rho(2^{-n}\lambda_m)(x, h_m)_H h_m, \quad x \in H,$$

as a consequence of (5.3).

Step 2. Next, we show the uniform estimate in $L^{\alpha+1}(M)$ based on a spectral multiplier theorem by Kunstmann and Uhl, [KU15], for operators with generalized Gaussian bounds. In view of Theorem 5.3 in [KU15], Lemma 2.19 and Fact 2.20 in [Uhl11], it is sufficient to show that s_n satisfies the Mihlin condition

$$\sup_{\lambda > 0} |\lambda^k s_n^{(k)}(\lambda)| \leqslant C_k, \qquad k = 0, \dots, \gamma, \tag{5.4}$$

for some $\gamma \in \mathbb{N}$ uniformly in $n \in \mathbb{N}_0$. This can be verified by the calculation

$$\sup_{\lambda > 0} |\lambda^k s_n^{(k)}(\lambda)| = \sup_{\lambda \in [2^n, 2^{n+1})} |\lambda^k s_n^{(k)}(\lambda)| = \sup_{\lambda \in [2^n, 2^{n+1})} \left| \lambda^k \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} \rho(2^{-n}\lambda) \right| \leqslant 2^k \|\rho^{(k)}\|_\infty$$

for all $k \in \mathbb{N}_0$. $\qquad \square$

We set

$$\mathcal{B}_n(l) = \sum_{m=1}^N l_m S_n B_m S_n, \qquad n \in \mathbb{N}, \quad l \in \mathbb{R}^N$$

and

$$\widetilde{u_{0,n}} := \begin{cases} S_n u_0 \frac{\|u_0\|_H}{\|S_n u_0\|_H}, & S_n u_0 \neq 0, \\ 0, & S_n u_0 = 0. \end{cases} \tag{5.5}$$

From $S_n u_0 \to u_0$ in H, we infer

$$\widetilde{u_{0,n}} \to u_0, \qquad n \to \infty. \tag{5.6}$$

Moreover, there is $C_0 > 0$ such that we have

$$1 \leqslant \frac{\|u_0\|_H}{\|S_n u_0\|_H} \leqslant C_0. \tag{5.7}$$

for $n \geqslant n_0(u_0) := \min\{n \in \mathbb{N} : S_n u_0 \neq 0\} \in \mathbb{N} \cup \{\infty\}$. For $n \in \mathbb{N}$, we consider the Galerkin equation

$$u_n(t) = \widetilde{u_{0,n}} - \mathrm{i} \int_0^t \left(Au_n(s) + P_n F(u_n(s))\right) \mathrm{d}s + \int_0^t \int_{\{|l| \leqslant 1\}} \left[e^{-\mathrm{i}\mathcal{B}_n(l)} u_n(s-) - u_n(s-)\right] \tilde{\eta}(\mathrm{d}s, \mathrm{d}l)$$

$$+ \int_0^t \int_{\{|l| \leqslant 1\}} \left\{e^{-\mathrm{i}\mathcal{B}_n(l)} u_n(s) - u_n(s) + \mathrm{i}\mathcal{B}_n(l) u_n(s)\right\} \nu(\mathrm{d}l)\mathrm{d}s, \quad t \in [0, T]. \tag{5.8}$$

In order to prove the global wellposedness of (5.8) and estimates for the solution u_n uniformly in $n \in \mathbb{N}$, we need some auxiliary Lemmata. We start with properties of the operators $\mathcal{B}_n(l)$.

Lemma 5.2. *Let $n \in \mathbb{N}$ and $l \in \mathbb{R}^N$. Then, we have*

$$\|\mathcal{B}_n(l)\|_{\mathcal{L}(H)} \leqslant |l| b_H^{\frac{1}{2}}, \qquad \|\mathcal{B}_n(l)\|_{\mathcal{L}(E_A)} \leqslant |l| b_{E_A}^{\frac{1}{2}}, \qquad \|\mathcal{B}_n(l)\|_{\mathcal{L}(L^{\alpha+1})} \leqslant |l| b_{\alpha+1}^{\frac{1}{2}} \sup_{n \in \mathbb{N}} \|S_n\|_{\mathcal{L}(L^{\alpha+1})}^2.$$

Moreover, $\left(e^{-\mathrm{i}t\mathcal{B}_n(l)}\right)_{t \in \mathbb{R}}$ is a group of unitary operators on H with

$$\|e^{-\mathrm{i}t\mathcal{B}_n(l)}\|_{\mathcal{L}(E_A)} \leqslant e^{|t||l|b_{E_A}^{\frac{1}{2}}}, \qquad \|e^{-\mathrm{i}t\mathcal{B}_n(l)}\|_{\mathcal{L}(L^{\alpha+1})} \leqslant e^{|t||l|b_{\alpha+1}^{\frac{1}{2}} \sup_{n \in \mathbb{N}} \|S_n\|_{\mathcal{L}(L^{\alpha+1})}^2}, \qquad t \in \mathbb{R}.$$

Proof. By the boundedness of $(S_n)_{n \in \mathbb{N}} \in \mathcal{L}(L^{\alpha+1})^{\mathbb{N}}$, we deduce that

$$\|\mathcal{B}_n(l)\|_{\mathcal{L}(L^{\alpha+1})} \leqslant \sum_{m=1}^N |l_m| \|S_n B_m S_n\|_{\mathcal{L}(L^{\alpha+1})} \leqslant |l| \left(\sum_{m=1}^N \|B_m\|_{\mathcal{L}(L^{\alpha+1})}^2\right)^{\frac{1}{2}} \sup_{n \in \mathbb{N}} \|S_n\|_{\mathcal{L}(L^{\alpha+1})}^2$$

$$= |l| b_{\alpha+1}^{\frac{1}{2}} \sup_{n \in \mathbb{N}} \|S_n\|_{\mathcal{L}(L^{\alpha+1})}^2. \tag{5.9}$$

The estimates of $\mathcal{B}_n(l)$ in spaces H and E_A can be shown analogously using $\|S_n\|_{\mathcal{L}(H)} = 1$ and $\|S_n\|_{\mathcal{L}(E_A)} = 1$. Since S_n and B_m are self-adjoint on H for $n \in \mathbb{N}$ and $m \in \{1, \ldots, M\}$, the Stone Theorem yields that $\left(e^{-\mathrm{i}t\mathcal{B}_n(l)}\right)_{t \in \mathbb{R}}$ is a unitary group on H. Moreover,

$$\|e^{-\mathrm{i}t\mathcal{B}_n(l)} x\|_{E_A} \leqslant e^{|t| \|\mathcal{B}_n(l)\|_{\mathcal{L}(E_A)}} \|x\|_{E_A} \leqslant e^{|t||l|b_{E_A}^{\frac{1}{2}}} \|x\|_{E_A}, \qquad x \in E_A, \quad t \in \mathbb{R},$$

$$\|e^{-\mathrm{i}t\mathcal{B}_n(l)} x\|_{L^{\alpha+1}} \leqslant e^{|t| \|\mathcal{B}_n(l)\|_{\mathcal{L}(L^{\alpha+1})}} \|x\|_{L^{\alpha+1}}$$

$$\leqslant e^{|t||l|b_{\alpha+1}^{\frac{1}{2}} \sup_{n \in \mathbb{N}} \|S_n\|_{\mathcal{L}(L^{\alpha+1})}^2} \|x\|_{L^{\alpha+1}}, \qquad x \in L^{\alpha+1}(M), \quad t \in \mathbb{R}.$$

$\square$

In the next Lemma inspired by Lemma 2.2 in [BM17] , we show how to control the differences in (5.8) in the H-norm.

Lemma 5.3. *For every $n \in \mathbb{N}$, $l \in B$ and $x \in H$, the following inequalities hold:*

$$\|e^{-\mathrm{i}\mathcal{B}_n(l)} x - x\|_H \leqslant b_H^{\frac{1}{2}} |l| \|x\|_H,$$

$$\|e^{-i\mathcal{B}_n(l)}x - x + i\mathcal{B}_n(l)x\|_H \leqslant \frac{1}{2}b_H|l|^2\|x\|_H.$$

Proof. The identities

$$e^{-i\mathcal{B}_n(l)}x - x = \int_0^1 \frac{d}{dt}e^{-it\mathcal{B}_n(l)}x dt = -i\mathcal{B}_n(l)\int_0^1 e^{-it\mathcal{B}_n(l)}x dt$$

and

$$e^{-i\mathcal{B}_n(l)}x - x + i\mathcal{B}_n(l)x = \int_0^1 \int_0^s \frac{d^2}{dt^2}e^{-it\mathcal{B}_n(l)}x dt ds = -\mathcal{B}_n(l)^2\int_0^1 \int_0^s e^{-it\mathcal{B}_n(l)}x dt ds$$

and Lemma 5.2 lead to

$$\|e^{-i\mathcal{B}_n(l)}x - x\|_H \leqslant \|\mathcal{B}_n(l)\|_{\mathcal{L}(H)}\int_0^1 \|e^{-it\mathcal{B}_n(l)}x\|_H dt \leqslant b_H^{\frac{1}{2}}|l|\|x\|_H,$$

$$\|e^{-i\mathcal{B}_n(l)}x - x + i\mathcal{B}_n(l)x\|_H \leqslant \|\mathcal{B}_n(l)\|_{\mathcal{L}(H)}^2\int_0^1 \int_0^s \|e^{-it\mathcal{B}_n(l)}x\|_H dt ds \leqslant \frac{1}{2}b_H|l|^2\|x\|_H.$$

$\square$

Next, we prove the well-posedness of the Galerkin equation. Moreover, we show that the Marcus noise and the approximation do not destroy the mass conservation which is well-known for the deterministic nonlinear Schrödinger equation.

Proposition 5.4. *For each $n \in \mathbb{N}$, there is a unique global strong solution $u_n \in \mathbb{D}([0,T], H_n)$ of* (5.8) *and we have the equality*

$$\|u_n(t)\|_H = \|\widetilde{u_{0,n}}\|_H = \|u_0\|_H \tag{5.10}$$

almost surely for all $t \in [0,T]$.

Proof. Step 1. We fix $n \in \mathbb{N}$. To obtain a global solution, we regard H_n as a finite dimensional real Hilbert space equipped with the scalar product $(u,v)_{H_n} := \mathrm{Re}\,(u,v)_H$ and check the assumptions of [ABW10], Theorem 3.1 for the coefficients defined by

$$\xi = \widetilde{u_{0,n}}, \qquad \sigma(u) = 0,$$

$$b(u) = -iAu - iP_nF(u) + \int_{\{|l|\leqslant 1\}} \left\{e^{-i\mathcal{B}_n(l)}u - u + i\mathcal{B}_n(l)u\right\}\nu(dl),$$

$$g(u,l) = \left[e^{-i\mathcal{B}_n(l)}u - u\right]$$

for $u \in H_n$ and $l \in B$. Let $R > 0$. We take $u, v \in H_n$ such that $\|u\|_H, \|v\|_H \leqslant R$ and estimate

$$\|b(u) - b(v)\|_H \leqslant \|A|_{H_n}\|_{\mathcal{L}(H)}\|u - v\|_H + \|F(u) - F(v)\|_H$$

$$+ \int_{\{|l|\leqslant 1\}} \|e^{-i\mathcal{B}_n(l)}(u-v) - (u-v) + i\mathcal{B}_n(l)(u-v)\|_H\,\nu(dl). \tag{5.11}$$

By Lemma 5.3 and (2.13)

$$\int_{\{|l|\leqslant 1\}} \|e^{-i\mathcal{B}_n(l)}(u-v)-(u-v)+i\mathcal{B}_n(l)(u-v)\|_H \, \nu(\mathrm{d}l) \leqslant \frac{1}{2}b_H \int_{\{|l|\leqslant 1\}} |l|^2 \nu(\mathrm{d}l)\|u-v\|_H$$

$$\lesssim \|u-v\|_H. \tag{5.12}$$

To estimate the nonlinearity, we use the equivalence of all norms in H_n and (2.8) to get

$$\|P_nF(u)-P_nF(v)\|_H \lesssim_n \|P_nF(u)-P_nF(v)\|_{E_A^*} \lesssim \|F(u)-F(v)\|_{L^{\frac{\alpha+1}{\alpha}}}$$

$$\lesssim \left(\|u\|_{L^{\alpha+1}(M)}+\|v\|_{L^{\alpha+1}(M)}\right)^{\alpha-1}\|u-v\|_{L^{\alpha+1}(M)}$$

$$\lesssim \left(\|u\|_H+\|v\|_H\right)^{\alpha-1}\|u-v\|_H \lesssim_R \|u-v\|_H. \tag{5.13}$$

We insert (5.13) and (5.12) in (5.11) to get a constant $C=C(R)$ such that

$$\|b(u)-b(v)\|_H \leqslant C\|u-v\|_H. \tag{5.14}$$

Moreover, we have

$$\int_{\{|l|\leqslant 1\}} \|g(u,l)-g(v,l)\|_H^2 \nu(\mathrm{d}l) \leqslant b_H \int_{\{|l|\leqslant 1\}} |l|^2\nu(\mathrm{d}l)\|u-v\|_H^2 \lesssim \|u-v\|_H^2 \tag{5.15}$$

where we used Lemma 5.3 and (2.13). To check the one-sided linear growth condition, we use (2.5) and (5.12) for $v=0$ and obtain a constant $K_1>0$ with

$$2\big(u,b(u)\big)_{H_n} + \int_{\{|l|\leqslant 1\}} \|g(u,l)\|_H^2 \nu(\mathrm{d}l) \leqslant 2\|A|_{H_n}\|_{\mathcal{L}(H)}\|u\|_H^2 + 2\,\mathrm{Re}\,\big(u,-iF(u)\big)_H$$

$$+2\|u\|_H \int_{\{|l|\leqslant 1\}} \|e^{-i\mathcal{B}_n(l)}u-u+i\mathcal{B}_n(l)u\|_H \, \nu(\mathrm{d}l)$$

$$\leqslant K_1\|u\|_H^2. \tag{5.16}$$

In view of (5.14), (5.15) and (5.16), we can apply Theorem 3.1 of [ABW10] and get a unique global strong solution of (5.8) for each $n \in \mathbb{N}$.

Step 2. It remains to show (5.10). The function $\mathcal{M}: H_n \to \mathbb{R}$ defined by $\mathcal{M}(v):=\|v\|_H^2$ for $v \in H_n$ is continuously Fréchet-differentiable with

$$\mathcal{M}'[v]h_1 = 2\,\mathrm{Re}\,\big(v,h_1\big)_{L^2},$$

for $v,h_1,h_2 \in H_n$. By the Itô formula and (2.15), we get almost surely, for all $t \in [0,T]$,

$$\|u_n(t)\|_H^2 = \|\widetilde{u_{0,n}}\|_H^2 + 2\int_0^t \mathrm{Re}\,\big(u_n(s),-iAu_n(s)-iP_nF\left(u_n(s)\right)\big)_{L^2}\mathrm{d}s$$

$$+\int_0^t \int_{\{|l|\leqslant 1\}} \left[\|e^{-i\mathcal{B}_n(l)}u_n(s-)\|_H^2 - \|u_n(s-)\|_H^2\right]\tilde{\eta}(\mathrm{d}l,\mathrm{d}s)$$

$$+\int_0^t \int_{\{|l|\leqslant 1\}} \left[\|e^{-i\mathcal{B}_n(l)}u_n(s)\|_H^2 - \|u_n(s)\|_H^2\right]\nu(\mathrm{d}l)\mathrm{d}s$$

$$- 2 \int_0^t \int_{\{|l| \leqslant 1\}} \operatorname{Re} \left(u_n(s), -\mathrm{i} \sum_{m=1}^N l_m S_n B_m S_n u_n(s) \right)_{L^2} \nu(\mathrm{d}l) \mathrm{d}s.$$

By

$$\operatorname{Re} \left(v, -\mathrm{i} A v \right)_{L^2} = \operatorname{Re} \left[\mathrm{i} \| A^{\frac{1}{2}} v \|_H^2 \right] = 0, \qquad \operatorname{Re} \left(v, -\mathrm{i} P_n F(v) \right)_{L^2} = 0, \qquad \operatorname{Re} \left(v, \mathrm{i} B_m v \right)_{L^2} = 0$$

for $v \in H_n$ and the fact that $S_n \mathcal{B}(l) S_n$ is self-adjoint and hence, $e^{-\mathrm{i}\mathcal{B}_n(l)}$ unitary, this simplifies to

$$\| u_n(t) \|_H^2 = \| \widetilde{u_{0,n}} \|_H^2 = \| u_0 \|_H^2$$

almost surely for all $t \in [0, T]$. □

Recall that by Assumption 2.4, the nonlinearity F has a real antiderivative denoted by $\hat{F}$. The second ingredient for uniform estimates in E_A is to control the energy associated to the NLS.

Definition 5.5. *We define the energy $\mathcal{E}$ function by*

$$\mathcal{E}(u) := \frac{1}{2} \| A^{\frac{1}{2}} u \|_H^2 + \hat{F}(u), \qquad u \in E_A.$$

Note that $\mathcal{E}(u)$ is well defined for every $u \in E_A$ by the continuity of the embedding $E_A \hookrightarrow L^{\alpha+1}(M)$. The compactness of this embedding formulated in Assumption 2.1 is not needed here. Before we estimate the energy of the solutions u_n of (5.8), we need some preparations.

Lemma 5.6. a) *There is a constant $C = C(b_{E_A}, b_{\alpha+1}, \alpha, F) > 0$ such that for every $n \in \mathbb{N}$, we have*

$$|\mathcal{E}(e^{-\mathrm{i}\mathcal{B}_n(l)} x) - \mathcal{E}(x)| \leqslant C|l| \left(\| x \|_{E_A}^2 + \| x \|_{L^{\alpha+1}}^{\alpha+1} \right)$$

for all $x \in H_n$, and $l \in \mathbb{R}^N$ with $|l| \leqslant 1$.
b) *There is a constant $C = C(b_{E_A}, b_{\alpha+1}, q, \alpha, F) > 0$ such that for every $n \in \mathbb{N}$, we have*

$$|\mathcal{E}(e^{-\mathrm{i}\mathcal{B}_n(l)} x) - \mathcal{E}(x) + \mathcal{E}'[x](\mathrm{i}\mathcal{B}_n(l) x)| \leqslant C|l|^2 \left(\| x \|_{E_A}^2 + \| x \|_{L^{\alpha+1}}^{\alpha+1} \right)$$

for all $x \in H_n$, and $l \in \mathbb{R}^N$ with $|l| \leqslant 1$.

Proof. ad a): The map $\mathcal{E}$ is twice continuously Fréchet-differentiable with

$$\mathcal{E}'[v] h = \operatorname{Re} \langle A v + F(v), h \rangle,$$

$$\mathcal{E}''[v](h_1, h_2) = \operatorname{Re} \left(A^{\frac{1}{2}} h_1, A^{\frac{1}{2}} h_2 \right)_{L^2} + \operatorname{Re} \langle F'[v] h_1, h_2 \rangle$$

for $v, h_1, h_2 \in H_n$. Let us fix $x \in H_n$ and $l \in B$. Then, we get

$$\mathcal{E}(e^{-\mathrm{i}\mathcal{B}_n(l)} x) - \mathcal{E}(x) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(e^{-\mathrm{i}t\mathcal{B}_n(l)} x) \mathrm{d}t = \int_0^1 \mathcal{E}'[e^{-\mathrm{i}t\mathcal{B}_n(l)} x] \left(-\mathrm{i}\mathcal{B}_n(l) e^{-\mathrm{i}t\mathcal{B}_n(l)} x \right) \mathrm{d}t$$

$$= \int_0^1 \operatorname{Re} \left\langle A e^{-\mathrm{i}t\mathcal{B}_n(l)} x + F(e^{-\mathrm{i}t\mathcal{B}_n(l)} x), -\mathrm{i}\mathcal{B}_n(l) e^{-\mathrm{i}t\mathcal{B}_n(l)} x \right\rangle \mathrm{d}t. \quad (5.17)$$

We define $f : [0,1] \times \mathbb{R}^N \to [0,\infty)$ by

$$f(t,l) := \max\left\{1, e^{2t|l|b_{E_A}^{\frac{1}{2}}} + e^{(\alpha+1)t|l|b_{\alpha+1}^{\frac{1}{2}}\sup_{n\in\mathbb{N}}\|S_n\|^2_{\mathcal{L}(L^{\alpha+1})}}\right\}, \qquad t \in [0,1], \quad l \in \mathbb{R}^N,$$

and by the properties of $\mathcal{B}_n(l)$ from Lemma 5.2, we estimate the integrand of (5.17):

$$|\left(Ae^{-it\mathcal{B}_n(l)}x, -i\mathcal{B}_n(l)e^{-it\mathcal{B}_n(l)}x\right)_{L^2}| \leqslant \|A^{\frac{1}{2}}e^{-it\mathcal{B}_n(l)}x\|_{L^2}\|A^{\frac{1}{2}}\mathcal{B}_n(l)e^{-it\mathcal{B}_n(l)}x\|_{L^2}$$

$$\leqslant e^{t|l|b_{E_A}^{\frac{1}{2}}}\|x\|_{E_A}|l|b_{E_A}^{\frac{1}{2}}\|e^{-it\mathcal{B}_n(l)}x\|_{E_A}$$

$$\leqslant e^{2t|l|b_{E_A}^{\frac{1}{2}}}|l|b_{E_A}^{\frac{1}{2}}\|x\|^2_{E_A} \qquad (5.18)$$

and

$$\left|\left\langle F(e^{-it\mathcal{B}_n(l)}x), -i\mathcal{B}_n(l)e^{-it\mathcal{B}_n(l)}x\right\rangle\right| \leqslant \|F(e^{-it\mathcal{B}_n(l)}x)\|_{L^{\frac{\alpha+1}{\alpha}}}\|\mathcal{B}_n(l)e^{-it\mathcal{B}_n(l)}x\|_{L^{\alpha+1}}$$

$$\leqslant C_{F,1}\|\mathcal{B}_n(l)\|_{\mathcal{L}(L^{\alpha+1})}\|e^{-it\mathcal{B}_n(l)}x\|^{\alpha+1}_{L^{\alpha+1}}$$

$$\leqslant C_{F,1}|l|b_{\alpha+1}^{\frac{1}{2}}\sup_{n\in\mathbb{N}}\|S_n\|^2_{\mathcal{L}(L^{\alpha+1})}\|x\|^{\alpha+1}_{L^{\alpha+1}}$$

$$e^{(\alpha+1)t|l|b_{\alpha+1}^{\frac{1}{2}}\sup_{n\in\mathbb{N}}\|S_n\|^2_{\mathcal{L}(L^{\alpha+1})}}. \qquad (5.19)$$

We obtain

$$|\mathcal{E}(e^{-i\mathcal{B}_n(l)}x) - \mathcal{E}(x)|$$

$$\leqslant |l|\max\left\{b_{E_A}^{\frac{1}{2}}, C_{F,1}b_{\alpha+1}^{\frac{1}{2}}\sup_{n\in\mathbb{N}}\|S_n\|^2_{\mathcal{L}(L^{\alpha+1})}\right\}\left(\|x\|^2_{E_A} + \|x\|^{\alpha+1}_{L^{\alpha+1}}\right)\int_0^1 f(t,l)\mathrm{d}t$$

and the assertion follows from

$$\int_0^1 f(t,l)\mathrm{d}t = \int_0^1 \max\left\{1, e^{2t|l|b_{E_A}^{\frac{1}{2}}} + e^{(\alpha+1)t|l|b_{\alpha+1}^{\frac{1}{2}}\sup_{n\in\mathbb{N}}\|S_n\|^2_{\mathcal{L}(L^{\alpha+1})}}\right\}\mathrm{d}t$$

$$\leqslant \max\left\{1, e^{2b_{E_A}^{\frac{1}{2}}} + e^{(\alpha+1)b_{\alpha+1}^{\frac{1}{2}}\sup_{n\in\mathbb{N}}\|S_n\|^2_{\mathcal{L}(L^{\alpha+1})}}\right\} < \infty, \qquad |l| \leqslant 1. \qquad (5.20)$$

ad b): Let us fix $x \in H_n$ and $l \in B$. We start with the identity

$$\mathcal{E}(e^{-i\mathcal{B}_n(l)}x) - \mathcal{E}(x) + \mathcal{E}'[x](i\mathcal{B}_n(l))x = \int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(e^{-is\mathcal{B}_n(l)}x) - \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(e^{-is\mathcal{B}_n(l)}x)\bigg|_{s=0}\right)\mathrm{d}s$$

$$= \int_0^1\int_0^s \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{E}(e^{-it\mathcal{B}_n(l)}x)\mathrm{d}t\mathrm{d}s$$

$$= \int_0^1\int_0^s \mathcal{E}'[e^{-it\mathcal{B}_n(l)}x]\left(-\mathcal{B}_n(l)^2e^{-it\mathcal{B}_n(l)}x\right)\mathrm{d}t\mathrm{d}s$$

$$+ \int_0^1\int_0^s \mathcal{E}''[e^{-it\mathcal{B}_n(l)}x]\left(-i\mathcal{B}_n(l)e^{-it\mathcal{B}_n(l)}x, -i\mathcal{B}_n(l)e^{-it\mathcal{B}_n(l)}x\right)\mathrm{d}t\mathrm{d}s$$

$$=: I_1 + I_2.$$

As above

$$|I_1| \leqslant |l|^2 \max\left\{ b_{E_A}, C_{F,1} b_{\alpha+1} \sup_{n\in\mathbb{N}} \|S_n\|_{\mathcal{L}(L^{\alpha+1})}^4 \right\} \left(\|x\|_{E_A}^2 + \|x\|_{L^{\alpha+1}}^{\alpha+1} \right) \int_0^1 f(t,l) \mathrm{d}t.$$

We further decompose $I_2 = I_{2,1} + I_{2,2}$ with

$$I_{2,1} = \int_0^1 \int_0^s \|A^{\frac{1}{2}} \mathcal{B}_n(l) e^{-it\mathcal{B}_n(l)} x\|_{L^2}^2 \mathrm{d}t\mathrm{d}s,$$

$$I_{2,2} = \int_0^1 \int_0^s \mathrm{Re}\left\langle F'[e^{-it\mathcal{B}_n(l)} x] \mathcal{B}_n(l) e^{-it\mathcal{B}_n(l)} x, \mathcal{B}_n(l) e^{-it\mathcal{B}_n(l)} x \right\rangle \mathrm{d}t\mathrm{d}s.$$

By Lemma 5.2,

$$|I_{2,1}| \leqslant \int_0^1 \int_0^s |l|^2 b_{E_A} \|e^{-it\mathcal{B}_n(l)} x\|_{E_A}^2 \mathrm{d}t\mathrm{d}s \leqslant \|x\|_{E_A}^2 |l|^2 b_{E_A} \int_0^1 f(t,l) \mathrm{d}t.$$

Moreover, the estimate

$$\left| \left\langle F'[e^{-it\mathcal{B}_n(l)} x] \mathcal{B}_n(l) e^{-it\mathcal{B}_n(l)} x, \mathcal{B}_n(l) e^{-it\mathcal{B}_n(l)} x \right\rangle \right|$$

$$\leqslant \|F'[e^{-it\mathcal{B}_n(l)} x] \mathcal{B}_n(l) e^{-it\mathcal{B}_n(l)} x\|_{L^{\frac{\alpha+1}{\alpha}}} \|\mathcal{B}_n(l) e^{-it\mathcal{B}_n(l)} x\|_{L^{\alpha+1}}$$

$$\leqslant C_{F,2} \|\mathcal{B}_n(l)\|_{\mathcal{L}(L^{\alpha+1})}^2 \|e^{-it\mathcal{B}_n(l)} x\|_{L^{\alpha+1}}^{\alpha+1}$$

$$\leqslant C_{F,2} |l|^2 b_{\alpha+1} \sup_{n\in\mathbb{N}} \|S_n\|_{\mathcal{L}(L^{\alpha+1})}^4 f(t,l) \|x\|_{L^{\alpha+1}}^{\alpha+1}$$

yields

$$|I_{2,2}| \leqslant \int_0^1 \int_0^s \left| \left\langle F'[e^{-it\mathcal{B}_n(l)} x] \mathcal{B}_n(l) e^{-it\mathcal{B}_n(l)} x, \mathcal{B}_n(l) e^{-it\mathcal{B}_n(l)} x \right\rangle \right| \mathrm{d}t\mathrm{d}s$$

$$\leqslant C_{F,2} |l|^2 b_{\alpha+1} \sup_{n\in\mathbb{N}} \|S_n\|_{\mathcal{L}(L^{\alpha+1})}^4 \|x\|_{L^{\alpha+1}}^{\alpha+1} \int_0^1 f(t,l) \mathrm{d}t$$

and finally, we find a constant $C = C(b_{\alpha+1}, b_{E_A}, \sup_{n\in\mathbb{N}} \|S_n\|_{\mathcal{L}(L^{\alpha+1})}, F)$ such that

$$\left| \mathcal{E}(e^{-i\mathcal{B}_n(l)} x) - \mathcal{E}(x) + \mathcal{E}'[x](i\mathcal{B}_n(l) x) \right| \leqslant C|l|^2 \left(\|x\|_{E_A}^2 + \|x\|_{L^{\alpha+1}}^{\alpha+1} \right) \int_0^1 f(t,l) \mathrm{d}t$$

and the second assertion also follows from (5.20). □

The next observation will be useful to simplify the following arguments based on the Gronwall Lemma to estimate of the energy. It has already appeared in [BHW17], Lemma 5.6, but we need it in a slightly more general form.

Lemma 5.7. *Let $r \in [1,\infty)$, $q \in (1,\infty)$, $\varepsilon > 0$, $T > 0$ and $X \in L^r(\Omega, L^\infty(0,T))$. Then,*

$$\|X\|_{L^r(\Omega, L^q(0,t))} \leqslant \varepsilon \|X\|_{L^r(\Omega, L^\infty(0,t))} + \varepsilon^{1-q} \frac{1}{q} \left(1 - \frac{1}{q}\right)^{q-1} \int_0^t \|X\|_{L^r(\Omega, L^\infty(0,s))} \mathrm{d}s, \qquad t \in [0,T].$$

Proof. As a consequence of Young's inequality, we obtain

$$a^{1-\frac{1}{q}}b^{\frac{1}{q}} \leqslant \varepsilon a + \varepsilon^{1-q}\frac{1}{q}\left(1-\frac{1}{q}\right)^{q-1}b, \qquad a,b \geqslant 0, \quad \varepsilon > 0. \tag{5.21}$$

Then, interpolation of $L^q(0,t)$ between $L^\infty(0,t)$ and $L^1(0,t)$ and (5.21) yield

$$\|X\|_{L^q(0,t)} \leqslant \|X\|_{L^\infty(0,t)}^{1-\frac{1}{q}}\|X\|_{L^1(0,t)}^{\frac{1}{q}} \leqslant \varepsilon\|X\|_{L^\infty(0,t)} + \varepsilon^{1-q}\frac{1}{q}\left(1-\frac{1}{q}\right)^{q-1}\|X\|_{L^1(0,t)}.$$

Now, we take the $L^r(\Omega)$-norm and apply Minkowski's inequality to get

$$\|X\|_{L^r(\Omega,L^q(0,t))} \leqslant \varepsilon\|X\|_{L^r(\Omega,L^\infty(0,t))} + \varepsilon^{1-q}\frac{1}{q}\left(1-\frac{1}{q}\right)^{q-1}\int_0^t\|X(s)\|_{L^r(\Omega)}\mathrm{d}s$$

$$\leqslant \varepsilon\|X\|_{L^r(\Omega,L^\infty(0,t))} + \varepsilon^{1-q}\frac{1}{q}\left(1-\frac{1}{q}\right)^{q-1}\int_0^t\|X\|_{L^r(\Omega,L^\infty(0,s))}\mathrm{d}s.$$

$\square$

Now, we are ready prove that the solutions of (5.8) have uniform energy estimates and satisfy the Aldous condition.

Proposition 5.8. *Let us assume Assumption* 2.6 *i). Then, the following assertions hold:*

a) *For all* $q \in [1,\infty)$ *there exists* $C = C(\mathcal{E}(u_0),T,b_{E_A},b_{\alpha+1},q,\alpha,F) > 0$ *such that*

$$\sup_{n\in\mathbb{N}}\mathbb{E}\left[\sup_{t\in[0,T]}\left[\|u_n(t)\|_H^2 + \mathcal{E}(u_n(t))\right]^q\right] \leqslant C.$$

b) *The sequence* $(u_n)_{n\in\mathbb{N}}$ *satisfies the Aldous condition* [A] *in* E_A^*.

c) *The sequence* $(\mathbb{P}^{u_n})_{n\in\mathbb{N}}$ *is tight in* Z_T.

Proof. Ad c): Follows from the two other parts by applying Proposition 4.11.
Ad a): Since $\widetilde{u_{0,n}} = 0$ already implies $u_n \equiv 0$, we may assume $\widetilde{u_{0,n}} \neq 0$ without loss of generality. Furthermore, we only prove the assertion for $q > 2$. The case $q \in [1,2]$ is a simple consequence of the Hölder inequality. Recall that the energy $\mathcal{E}$ is twice Frechet differentiable. In particular, the function $\mathcal{E}'$ is Hölder continuous. Hence, we can use Proposition 5.4 and the Itô formula B.2 to deduce

$$\frac{1}{2}\|u_n(s)\|_H^2 + \mathcal{E}(u_n(s)) = \frac{1}{2}\|\widetilde{u_{0,n}}\|_H^2 + \mathcal{E}(\widetilde{u_{0,n}})$$

$$+ \int_0^s \mathrm{Re}\langle Au_n(r) + F(u_n(r)), -\mathrm{i}Au_n(r) - \mathrm{i}P_nF(u_n(r))\rangle\mathrm{d}r$$

$$+ \int_0^s \int_{\{|l|\leqslant 1\}} \left[\mathcal{E}(e^{-\mathrm{i}\mathcal{B}_n(l)}u_n(r-)) - \mathcal{E}(u_n(r-))\right]\tilde{\eta}(\mathrm{d}l,\mathrm{d}r)$$

$$+ \int_0^s \int_{\{|l|\leqslant 1\}} \left[\mathcal{E}(e^{-\mathrm{i}\mathcal{B}_n(l)}u_n(r)) - \mathcal{E}(u_n(r)) + \mathcal{E}'[u_n(r)]\left(\mathrm{i}\mathcal{B}_n(l)u_n(r)\right)\right]\nu(\mathrm{d}l)\mathrm{d}r$$

$$=: \frac{1}{2}\|\widetilde{u_{0,n}}\|_H^2 + \mathcal{E}(\widetilde{u_{0,n}}) + I_1(s) + I_2(s) + I_3(s) \tag{5.22}$$

almost surely for all $s \in [0, T]$. The first integral $I_1(s)$ cancels due to the following three identities which hold for all for all $v \in H_n$:

$$\text{Re}\langle F(v), -\mathrm{i}P_n F(v)\rangle = \text{Re}\left[\mathrm{i}\langle F(v), P_n F(v)\rangle\right] = 0;$$

$$\text{Re}\left[\langle Av, -\mathrm{i}P_n F(v)\rangle + \langle F(v), -\mathrm{i}Av\rangle\right] = \text{Re}\left[-\langle Av, \mathrm{i}F(v)\rangle + \overline{\langle Av, \mathrm{i}F(v)\rangle}\right] = 0;$$

$$\text{Re}\left(Av, -\mathrm{i}Av\right)_{L^2} = \text{Re}\left[\mathrm{i}\|Av\|_H^2\right] = 0$$

By the maximal inequality for the Poisson stochastic integral, see Theorem 4.5 in [DMN13], and Lemma 5.6, we obtain

$$\left(\mathbb{E}\left[\sup_{s\in[0,t]} |I_2(s)|^q\right]\right)^{\frac{1}{q}} \lesssim \left(\mathbb{E}\left(\int_0^t \int_{\{|l|\leqslant 1\}} \left|\mathcal{E}(e^{-\mathrm{i}\mathcal{B}_n(l)}u_n(s)) - \mathcal{E}(u_n(s))\right|^2 \nu(\mathrm{d}l)\mathrm{d}s\right)^{\frac{q}{2}}\right)^{\frac{1}{q}}$$

$$+ \left(\mathbb{E}\int_0^t \int_{\{|l|\leqslant 1\}} \left|\mathcal{E}(e^{-\mathrm{i}\mathcal{B}_n(l)}u_n(s)) - \mathcal{E}(u_n(s))\right|^q \nu(\mathrm{d}l)\mathrm{d}s\right)^{\frac{1}{q}}$$

$$\lesssim \left(\mathbb{E}\left(\int_0^t \int_{\{|l|\leqslant 1\}} |l|^2 \left(\|u_n(s)\|_{E_A}^2 + \|u_n(s)\|_{L^{\alpha+1}}^{\alpha+1}\right)^2 \nu(\mathrm{d}l)\mathrm{d}s\right)^{\frac{q}{2}}\right)^{\frac{1}{q}}$$

$$+ \left(\mathbb{E}\int_0^t \int_{\{|l|\leqslant 1\}} |l|^q \left(\|u_n(s)\|_{E_A}^2 + \|u_n(s)\|_{L^{\alpha+1}}^{\alpha+1}\right)^q \nu(\mathrm{d}l)\mathrm{d}s\right)^{\frac{1}{q}}.$$

$$(5.23)$$

We introduce the abbreviation

$$X_n := \frac{1}{2}\|u_n\|_{L^2}^2 + \mathcal{E}(u_n)$$

and observe

$$\|u_n\|_{E_A}^2 + \|u_n\|_{L^{\alpha+1}}^{\alpha+1} \lesssim X_n. \tag{5.24}$$

Moreover, we have

$$\int_{\{|l|\leqslant 1\}} |l|^q \, \nu(\mathrm{d}l) \leqslant \int_{\{|l|\leqslant 1\}} |l|^2 \, \nu(\mathrm{d}l) < \infty, \qquad q \geqslant 2. \tag{5.25}$$

Thus, we can conclude

$$\left(\mathbb{E}\left[\sup_{s\in[0,t]} |I_2(s)|^q\right]\right)^{\frac{1}{q}} \lesssim \left(\mathbb{E}\left(\int_0^t X_n(s)^2 \mathrm{d}s\right)^{\frac{q}{2}}\right)^{\frac{1}{q}} + \left(\mathbb{E}\int_0^t X_n(s)^q \mathrm{d}s\right)^{\frac{1}{q}}$$

$$= \|X_n\|_{L^q(\Omega, L^2(0,t))} + \|X_n\|_{L^q(\Omega, L^q(0,t))}, \quad t \in [0, T]. \tag{5.26}$$

By Lemma 5.6 b), (5.24) and the Minkowski inequality

$$
\left(\mathbb{E}\Big[\sup_{s\in[0,t]} |I_3(s)|^q \Big] \right)^{\frac{1}{q}} \lesssim \int_{\{|l|\leqslant 1\}} |l|^2 \nu(\mathrm{d}l) \left(\mathbb{E}\left(\int_0^t \big(\|u_n(r)\|_{E_A}^2 + \|u_n(r)\|_{L^{\alpha+1}}^{\alpha+1} \big)\, \mathrm{d}r \right)^q \right)^{\frac{1}{q}}
$$

$$
\lesssim \int_{\{|l|\leqslant 1\}} |l|^2 \nu(\mathrm{d}l) \int_0^t \|X_n(r)\|_{L^q(\Omega)}\, \mathrm{d}r \lesssim \int_0^t \|X_n\|_{L^q(\Omega, L^\infty(0,r))}\, \mathrm{d}r, \quad t\in[0,T].
$$

Therefore, from (5.22) and the previous estimates we get

$$
\|X_n\|_{L^q(\Omega, L^\infty(0,t))} \leqslant \frac{1}{2}\|\widetilde{u_{0,n}}\|_H^2 + \mathcal{E}(\widetilde{u_{0,n}}) + \left(\mathbb{E}\Big[\sup_{s\in[0,t]} |I_2(s)|^q \Big] \right)^{\frac{1}{q}} + \left(\mathbb{E}\Big[\sup_{s\in[0,t]} |I_3(s)|^q \Big] \right)^{\frac{1}{q}}
$$

$$
\lesssim \frac{1}{2}\|u_0\|_H^2 + \mathcal{E}(\widetilde{u_{0,n}}) + \|X_n\|_{L^q(\Omega, L^2(0,t))} + \|X_n\|_{L^q(\Omega, L^q(0,t))}
$$

$$
+ \int_0^t \|X_n\|_{L^q(\Omega, L^\infty(0,s))}\mathrm{d}s, \quad t\in[0,T]. \tag{5.27}
$$

Using Lemma 5.7 with $\varepsilon > 0$ to estimate $\|X_n\|_{L^q(\Omega, L^2(0,t))}$ and $\|X_n\|_{L^q(\Omega, L^q(0,t))}$, we get for $t\in[0,T]$,

$$
\|X_n\|_{L^q(\Omega, L^\infty(0,t))} \lesssim \frac{1}{2}\|u_0\|_H^2 + \mathcal{E}(\widetilde{u_{0,n}}) + \varepsilon\|X_n\|_{L^q(\Omega, L^\infty(0,t))} + \int_0^t \|X_n\|_{L^q(\Omega, L^\infty(0,s))}\mathrm{d}s.
$$

Taking ε sufficiently small we end up with

$$
\|X_n\|_{L^q(\Omega, L^\infty(0,t))} \lesssim \frac{1}{2}\|u_0\|_H^2 + \mathcal{E}(\widetilde{u_{0,n}}) + \int_0^t \|X_n\|_{L^q(\Omega, L^\infty(0,s))}\mathrm{d}s, \quad t\in[0,T].
$$

Finally, the Gronwall Lemma yields

$$
\|X_n\|_{L^q(\Omega, L^\infty(0,t))} \leqslant C \left(\frac{1}{2}\|u_0\|_H^2 + \mathcal{E}(\widetilde{u_{0,n}}) \right) e^{Ct}, \qquad t\in[0,T],
$$

where the constant $C = C(b_{E_A}, b_{\alpha+1}, q, \alpha, F) > 0$ is uniform in $n\in\mathbb{N}$. As a consequence of (5.7) and Proposition 5.1, we obtain

$$
\mathcal{E}(\widetilde{u_{0,n}}) \lesssim \frac{\|u_0\|_H^2}{\|S_n u_0\|_H^2}\|A^{\frac{1}{2}} S_n u_0\|_H^2 + \frac{\|u_0\|_H^{\alpha+1}}{\|S_n u_0\|_H^{\alpha+1}}\|S_n u_0\|_{L^{\alpha+1}}^{\alpha+1}
$$

$$
\lesssim \|A^{\frac{1}{2}} u_0\|_H^2 + \|u_0\|_{L^{\alpha+1}}^{\alpha+1} \lesssim \mathcal{E}(u_0) \tag{5.28}
$$

for $n \geqslant n_0$ and $\mathcal{E}(\widetilde{u_{0,n}}) = 0$ for $n < n_0$. This completes the proof of Proposition 5.8 a).

Ad b): Now, we continue with the proof of the Aldous condition. Let us fix $n\in\mathbb{N}$. We have for all $t\in[0,T]$, almost surely

$$
u_n(t) - \widetilde{u_{0,n}} = -\mathrm{i}\int_0^t A u_n(s)\mathrm{d}s - \mathrm{i}\int_0^t P_n F(u_n(s))\mathrm{d}s
$$

$$+ \int_0^t \int_{\{|l| \leqslant 1\}} \left[e^{-i\mathcal{B}_n(l)} u_n(s-) - u_n(s-) \right] \tilde{\eta}(ds, dl)$$

$$+ \int_0^t \int_{\{|l| \leqslant 1\}} \left\{ e^{-i\mathcal{B}_n(l)} u(s) - u(s) + i\mathcal{B}_n(l)u(s) \right\} \nu(dl)ds$$

$$=: J_1(t) + J_2(t) + J_3(t) + J_4(t)$$

in H_n. Let us next fix a sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times and $\theta > 0$. By the above we infer that

$$\|u_n((\tau_n + \theta) \wedge T) - u_n(\tau_n)\|_{E_A^*} \leqslant \sum_{k=1}^4 \|J_k((\tau_n + \theta) \wedge T) - J_k(\tau_n)\|_{E_A^*}.$$

Hence, for a fixed $\eta > 0$, we get

$$\mathbb{P}\left\{ \|u_n((\tau_n + \theta) \wedge T) - u_n(\tau_n)\|_{E_A^*} \geqslant \eta \right\} \leqslant \sum_{k=1}^4 \mathbb{P}\left\{ \|J_k((\tau_n + \theta) \wedge .T) - J_k(\tau_n)\|_{E_A^*} \geqslant \frac{\eta}{4} \right\}$$

$$(5.29)$$

We aim to apply the Chebyshev inequality and estimate the expected value of each term in the sum on the RHS of (5.29). We use part a) for

$$\mathbb{E}\|J_1((\tau_n + \theta) \wedge T) - J_1(\tau_n)\|_{E_A^*} \leqslant \mathbb{E} \int_{\tau_n}^{(\tau_n+\theta)\wedge T} \|Au_n(s)\|_{E_A^*} ds \leqslant \mathbb{E} \int_{\tau_n}^{(\tau_n+\theta)\wedge T} \|A^{\frac{1}{2}} u_n(s)\|_H ds$$

$$\lesssim \theta \mathbb{E}\left[\sup_{s \in [0,T]} \|u_n(s)\|_{E_A} \right] \leqslant \theta \mathbb{E}\left[\sup_{s \in [0,T]} \|u_n(s)\|_{E_A}^2 \right]^{\frac{1}{2}} \leqslant \theta C_1;$$

the embedding $L^{\frac{\alpha+1}{\alpha}}(M) \hookrightarrow E_A^*$ and the nonlinear estimates (2.4) and (2.9) for

$$\mathbb{E}\|J_2((\tau_n + \theta) \wedge T) - J_2(\tau_n)\|_{E_A^*} \leqslant \mathbb{E} \int_{\tau_n}^{(\tau_n+\theta)\wedge T} \|P_n F(u_n(s))\|_{E_A^*} ds$$

$$\leqslant \mathbb{E} \int_{\tau_n}^{(\tau_n+\theta)\wedge T} \|F(u_n(s))\|_{E_A^*} ds \lesssim \mathbb{E} \int_{\tau_n}^{(\tau_n+\theta)\wedge T} \|F(u_n(s))\|_{L^{\frac{\alpha+1}{\alpha}}(M)} ds$$

$$\lesssim \mathbb{E} \int_{\tau_n}^{(\tau_n+\theta)\wedge T} \|u_n(s)\|_{L^{\alpha+1}(M)}^\alpha ds \lesssim \theta \mathbb{E}\left[\sup_{s \in [0,T]} \|u_n(s)\|_{E_A}^\alpha \right] \leqslant \theta C_2$$

By the Levy-Itô-isometry, Lemma 5.3, (2.13) and Proposition 5.4 we get

$$\mathbb{E}\|J_3((\tau_n + \theta) \wedge T) - J_3(\tau_n)\|_{E_A^*}^2 \lesssim \mathbb{E} \left\| \int_{\tau_n}^{(\tau_n+\theta)\wedge T} \int_{\{|l| \leqslant 1\}} \left[e^{-i\mathcal{B}_n(l)} u_n(s-) - u_n(s-) \right] \tilde{\eta}(ds, dl) \right\|_H^2$$

$$= \mathbb{E} \int_{\tau_n}^{(\tau_n+\theta)\wedge T} \int_{\{|l| \leqslant 1\}} \|e^{-i\mathcal{B}_n(l)} u_n(s) - u_n(s)\|_H^2 \nu(dl)ds$$

$$\leqslant b_H \int_{\{|l| \leqslant 1\}} |l|^2 \nu(dl) \mathbb{E} \int_{\tau_n}^{(\tau_n+\theta)\wedge T} \|u_n(s)\|_H^2 ds \lesssim \theta \|u_0\|_H^2$$

and

$$\mathbb{E}\|J_4((\tau_n + \theta) \wedge T) - J_4(\tau_n)\|_{E_A^*}$$

$$= \mathbb{E}\left\|\int_{\tau_n}^{(\tau_n+\theta)\wedge T} \int_{\{|l|\leqslant 1\}} \left\{e^{-i\mathcal{B}_n(l)}u_n(s) - u_n(s) + i\mathcal{B}_n(l)u_n(s)\right\} \nu(\mathrm{d}l)\mathrm{d}s\right\|_{E_A^*}$$

$$\lesssim \mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T} \int_{\{|l|\leqslant 1\}} \left\|e^{-i\mathcal{B}_n(l)}u_n(s) - u_n(s) + i\mathcal{B}_n(l)u_n(s)\right\|_H \nu(\mathrm{d}l)\mathrm{d}s$$

$$\leqslant \frac{1}{2}b_H \int_{\{|l|\leqslant 1\}} |l|^2\nu(\mathrm{d}l)\mathbb{E}\int_{\tau_n}^{(\tau_n+\theta)\wedge T} \|u_n(s)\|_H\mathrm{d}s \lesssim \theta\|u_0\|_H.$$

By the Chebyshev inequality, we obtain for a given $\eta > 0$

$$\mathbb{P}\left\{\|J_k((\tau_n + \theta) \wedge T) - J_k(\tau_n)\|_{E_A^*} \geqslant \frac{\eta}{4}\right\} \leqslant \frac{4}{\eta}\mathbb{E}\|J_k((\tau_n + \theta) \wedge T) - J_k(\tau_n)\|_{E_A^*} \leqslant \frac{4C_k\theta}{\eta} \tag{5.30}$$

for $k \in \{1, 2, 4\}$ and

$$\mathbb{P}\left\{\|J_3((\tau_n + \theta) \wedge T) - J_3(\tau_n)\|_{E_A^*} \geqslant \frac{\eta}{4}\right\} \leqslant \frac{16}{\eta^2}\mathbb{E}\|J_3((\tau_n + \theta) \wedge T) - J_3(\tau_n)\|_{E_A^*}^2 \leqslant \frac{16C_4\theta}{\eta^2}. \tag{5.31}$$

Let us fix $\varepsilon > 0$. Due to estimates (5.30) and (5.31) we can choose $\delta_1, \ldots, \delta_4 > 0$ such that

$$\mathbb{P}\left\{\|J_k((\tau_n + \theta) \wedge T) - J_k(\tau_n)\|_{E_A^*} \geqslant \frac{\eta}{4}\right\} \leqslant \frac{\varepsilon}{4}$$

for $0 < \theta \leqslant \delta_k$ and $k = 1, \ldots, 4$. With $\delta := \min\{\delta_1, \ldots, \delta_4\}$, using (5.29) we get

$$\mathbb{P}\left\{\|J_k((\tau_n + \theta) \wedge T) - J_k(\tau_n)\|_{E_A^*} \geqslant \eta\right\} \leqslant \varepsilon$$

for all $n \in \mathbb{N}$ and $0 < \theta \leqslant \delta$ and therefore, the Aldous condition $[A]$ holds in E_A^*. $\qquad\square$

We continue with the a priori estimate for solutions of (5.8) with a focusing nonlinearity. Note that this case is harder since the expression

$$\frac{1}{2}\|v\|_H^2 + \mathcal{E}(v) = \frac{1}{2}\|v\|_{E_A}^2 + \hat{F}(v), \qquad v \in H_n,$$

does not dominate $\|v\|_{E_A}^2$, because $\hat{F}$ is negative. Nevertheless, we will see that the E_A-norm is still the dominating part under the additional Assumption 2.6 i'), which leads to a restriction to the maximal degree of the nonlinearity F. In particular, uniform estimates in E_A are still possible.

Proposition 5.9. *Under Assumption 2.6 i'), the following assertions hold:*

a) *For all $r \in [1, \infty)$, there is a constant*

$$C = C(\|u_0\|_H, \|A^{\frac{1}{2}}u_0\|_H, \|u_0\|_{L^{\alpha+1}}, \gamma, \alpha, T, F, b_{E_A}, b_{\alpha+1}, r) > 0$$

with

$$\sup_{n\in\mathbb{N}}\mathbb{E}\left[\sup_{t\in[0,T]}\|u_n(t)\|_{E_A}^r\right]\leqslant C;$$

b) *The sequence $(u_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition [A] in E_A^*.*

In particular, the sequence $(\mathbb{P}^{u_n})_{n\in\mathbb{N}}$ is tight in Z_T by Proposition 4.11.

Proof. ad a): Let $\varepsilon > 0$. Assumption 2.6 i') and Young's inequality imply that there are $\gamma > 0$ and $C_\varepsilon > 0$ such that

$$\|u\|_{L^{\alpha+1}(M)}^{\alpha+1}\lesssim\varepsilon\|u\|_{E_A}^2+C_\varepsilon\|u\|_H^\gamma,\qquad u\in E_A,\tag{5.32}$$

and therefore by Proposition 5.4, we infer that

$$-\hat{F}(u_n(s))\lesssim\|u_n(s)\|_{L^{\alpha+1}(M)}^{\alpha+1}\lesssim\varepsilon\|u_n(s)\|_{E_A}^2+C_\varepsilon\|u_n(s)\|_H^\gamma$$
$$\lesssim\varepsilon\|A^{\frac{1}{2}}u_n(s)\|_H^2+\varepsilon\|u_0\|_H^2+C_\varepsilon\|u_0\|_H^\gamma,\qquad s\in[0,T].\tag{5.33}$$

By analogous calculations as in the proof of Proposition 5.8 we get

$$\frac{1}{2}\|A^{\frac{1}{2}}u_n(s)\|_H^2=-\hat{F}(u_n(s))+\mathcal{E}\left(u_n(s)\right)$$
$$=-\hat{F}(u_n(s))+\mathcal{E}\left(\widetilde{u_{0,n}}\right)$$
$$+\int_0^s\int_{\{|l|\leqslant1\}}\left[\mathcal{E}(e^{-\mathrm{i}\mathcal{B}_n(l)}u_n(r-))-\mathcal{E}(u_n(r-))\right]\tilde{\eta}(\mathrm{d}l,\mathrm{d}r)$$
$$+\int_0^s\int_{\{|l|\leqslant1\}}\left[\mathcal{E}(e^{-\mathrm{i}\mathcal{B}_n(l)}u_n(r))-\mathcal{E}(u_n(r))+\mathcal{E}'[u_n(r)]\left(\mathrm{i}\mathcal{B}_n(l)u_n(s)\right)\right]\nu(\mathrm{d}l)\mathrm{d}r$$
$$=:-\hat{F}(u_n(s))+\mathcal{E}\left(\widetilde{u_{0,n}}\right)+I_1(s)+I_2(s)\tag{5.34}$$

almost surely for all $t\in[0,T]$. We abbreviate

$$Y_n(s):=\|u_0\|_{L^2}^2+\|A^{\frac{1}{2}}u_n(s)\|_{L^2}^2+\|u_n(s)\|_{L^{\alpha+1}}^{\alpha+1},\qquad s\in[0,T].$$

Let $q>2$ and recall (5.25) as well as the mass conservation from Proposition 5.4. As in the proof of Proposition 5.8, we estimate

$$|\mathcal{E}(\widetilde{u_{0,n}})|\lesssim\|A^{\frac{1}{2}}u_0\|_H^2+\|u_0\|_{L^{\alpha+1}}^{\alpha+1},\tag{5.35}$$

$$\left(\mathbb{E}\left[\sup_{s\in[0,t]}|I_1(s)|^q\right]\right)^{\frac{1}{q}}\lesssim\left(\int_{\{|l|\leqslant1\}}|l|^2\nu(\mathrm{d}l)\right)^{\frac{1}{2}}\left(\mathbb{E}\left(\int_0^t\left(\|u_n(s)\|_{E_A}^2+\|u_n(s)\|_{L^{\alpha+1}}^{\alpha+1}\right)^2\mathrm{d}s\right)^{\frac{q}{2}}\right)^{\frac{1}{q}}$$
$$+\left(\int_{\{|l|\leqslant1\}}|l|^q\nu(\mathrm{d}l)\right)^{\frac{1}{q}}\left(\mathbb{E}\int_0^t\left(\|u_n(s)\|_{E_A}^2+\|u_n(s)\|_{L^{\alpha+1}}^{\alpha+1}\right)^q\mathrm{d}s\right)^{\frac{1}{q}}$$
$$\lesssim\|Y_n\|_{L^q(\Omega,L^2(0,t))}+\|Y_n\|_{L^q(\Omega,L^q(0,t))};\tag{5.36}$$

$$\left(\mathbb{E}\Big[\sup_{s \in [0,t]} |I_2(s)|^q \Big] \right)^{\frac{1}{q}} \lesssim \int_{\{|l| \leqslant 1\}} |l|^2 \nu(\mathrm{d}l) \int_0^t \big\| \|u_n\|_{E_A}^2 + \|u_n\|_{L^{\alpha+1}}^{\alpha+1} \big\|_{L^q(\Omega, L^\infty(0,r))} \mathrm{d}r$$

$$\lesssim \int_0^t \|Y_n\|_{L^q(\Omega, L^\infty(0,r))} \mathrm{d}r. \tag{5.37}$$

Using (5.33), (5.35), (5.36) and (5.37) in (5.34), we obtain

$$\left\| \|A^{\frac{1}{2}} u_n\|_{L^2}^2 \right\|_{L^q(\Omega, L^\infty(0,t))} \lesssim \left\| \|A^{\frac{1}{2}} u_n\|_{L^2}^2 \right\|_{L^q(\Omega, L^\infty(0,t))} \varepsilon + \varepsilon \|u_0\|_{L^2}^2 + C_\varepsilon \|u_0\|_{L^2}^\gamma$$

$$+ \|A^{\frac{1}{2}} u_0\|_{L^2}^2 + \|u_0\|_{L^{\alpha+1}}^{\alpha+1} + \|Y_n\|_{L^q(\Omega, L^2(0,t))}$$

$$+ \|Y_n\|_{L^q(\Omega, L^q(0,t))} + \int_0^t \|Y_n\|_{L^q(\Omega, L^\infty(0,r))} \mathrm{d}r.$$

If we employ Lemma 5.7 to estimate $\|Y_n\|_{L^q(\Omega, L^2(0,t))}$ and $\|Y_n\|_{L^q(\Omega, L^q(0,t))}$, we get

$$\left\| \|A^{\frac{1}{2}} u_n\|_{L^2}^2 \right\|_{L^q(\Omega, L^\infty(0,t))} \lesssim \left\| \|A^{\frac{1}{2}} u_n\|_{L^2}^2 \right\|_{L^q(\Omega, L^\infty(0,t))} \varepsilon + \varepsilon \|u_0\|_{L^2}^2 + C_\varepsilon \|u_0\|_{L^2}^\gamma$$

$$+ \|A^{\frac{1}{2}} u_0\|_{L^2}^2 + \|u_0\|_{L^{\alpha+1}}^{\alpha+1} + \varepsilon \|Y_n\|_{L^q(\Omega, L^\infty(0,t))}$$

$$+ \int_0^t \|Y_n\|_{L^q(\Omega, L^\infty(0,r))} \mathrm{d}r. \tag{5.38}$$

In order to estimate the terms with Y_n by the LHS of (5.38), we exploit (5.33) to get

$$\|Y_n\|_{L^q(\Omega, L^\infty(0,t))} \leqslant \|u_0\|_H^2 + \left\| \|A^{\frac{1}{2}} u_n\|_H^2 \right\|_{L^q(\Omega, L^\infty(0,t))} + \left\| \|u_n\|_{L^{\alpha+1}}^{\alpha+1} \right\|_{L^q(\Omega, L^\infty(0,t))}$$

$$\leqslant (1 + \varepsilon) \left\| \|A^{\frac{1}{2}} u_n\|_H^2 \right\|_{L^q(\Omega, L^\infty(0,t))} + C(\varepsilon, \|u_0\|_H).$$

Now, we choose $\varepsilon > 0$ sufficiently small and end up with

$$\left\| \|A^{\frac{1}{2}} u_n\|_{L^2}^2 \right\|_{L^q(\Omega, L^\infty(0,t))} \leqslant C \left(1 + \int_0^t \left\| \|A^{\frac{1}{2}} u_n\|_{L^2}^2 \right\|_{L^q(\Omega, L^\infty(0,r))} \mathrm{d}r \right)$$

for some $C = C(\|u_0\|_{L^2}, \|A^{\frac{1}{2}} u_0\|_H, \|u_0\|_{L^{\alpha+1}}, \gamma, \alpha, T, F, b_{E_A}, b_{\alpha+1}, q)$ independent of n. From the Gronwall Lemma, we infer

$$\left\| \|A^{\frac{1}{2}} u_n\|_{L^2}^2 \right\|_{L^q(\Omega, L^\infty(0,t))} \leqslant C e^{Ct}, \qquad t \in [0, T]. \tag{5.39}$$

In view of Proposition 5.4, we have proved the assertion for $r = 2q > 4$. The case $r \in [1, 4]$ is an easy consequence of the Hölder inequality.

ad b). The proof of the Aldous condition is similar to the defocusing case, see Proposition 5.8 b). $\qquad \square$

Corollary 5.10. *Under Assumption 2.6, the sequence $(u_n)_{n \in \mathbb{N}}$ of Galerkin solutions is tight on Z_T.*

Proof. Immediate consequence of Propositions 4.11, 5.8 and 5.9. $\qquad \square$

6. Construction of a martingale solution

In this section, we will use the compactness results and the uniform estimates from the previous sections to complete the proof of Theorem 1.1. Let us recall

$$Z_T := \mathbb{D}([0,T], E_A^*) \cap L^{\alpha+1}(0,T; L^{\alpha+1}(M)) \cap \mathbb{D}_w([0,T], E_A).$$

The first step is to prove that Proposition 4.12 can be applied with

$$\mathcal{X}_1 := M_{\bar{\mathbb{N}}}^\nu([0,T] \times \mathbb{R}^M), \qquad \mathcal{X}_2 := Z_T.$$

Here, $M_{\bar{\mathbb{N}}}^\nu([0,T] \times \mathbb{R}^M)$ denotes the set of all $\bar{\mathbb{N}}$-valued Borel measures ξ on $[0,T] \times \mathbb{R}^N$ with $\xi(\mathcal{S}_n) < \infty$ for all $n \in \mathbb{N}$, for some sequence $\mathcal{S}_n \subset [0,T] \times \mathbb{R}^N$ of Borel sets with $\mathcal{S}_n \uparrow [0,T] \times \mathbb{R}^N$ and $\mathrm{Leb} \otimes \nu(\mathcal{S}_n) < \infty$ for all $n \in \mathbb{N}$. It is well known, see e.g. Lemma 2.53 in the second authors dissertation [Hor18a] or Section 1 in [BDM+11], that $M_{\bar{\mathbb{N}}}^\nu([0,T] \times \mathbb{R}^M)$ is a complete separable metric space.

Moreover, we determine the σ-algebra $\mathcal{A}$. Of course, it would be natural to equip Z_T with the Borel σ-algebra $\mathcal{B}(Z_T)$, but it turns out that $\mathcal{A}$ is strictly contained in $\mathcal{B}(Z_T)$. Given real-valued functions f_m on a topological space Z, we will frequently use the notation $f = (f_1, f_2, \dots)$ and the fact that $\sigma(f_m : m \in \mathbb{N}) = f^{-1}(\mathcal{B}(\mathbb{R}^\infty))$, where $\mathbb{R}^\infty$ is equipped with the locally convex topology induced by the seminorms $p_k(x) := |x_k|$.

Lemma 6.1. *Let X be a set and $f_m : X \to \mathbb{R}$, $m \in \mathbb{N}$. Let $\mathcal{O}_X$ be the coarsest topology such f_m is continuous for all $m \in \mathbb{N}$. Then, we have*

$$\mathcal{B}(X) := \sigma(\mathcal{O}_X) = \sigma(f_m : m \in \mathbb{N}).$$

Proof. The direction " $\supset$ " is obvious by the continuity of f_m for $m \in \mathbb{N}$. In view of the good set principle, it is sufficient for the other inclusion to show that each $O \in \mathcal{O}_X$ is contained in the $f^{-1}(\mathcal{B}(\mathbb{R}^\infty))$. Since each $O \in \mathcal{O}_X$ is of the form

$$O = \bigcup_{i \in I} \bigcap_{k=1}^K f_3^{-1}(O_{i,k}), \qquad O_{i,k} \text{ open in } \mathbb{R}^\infty,$$

see [Fol99], Proposition 4.4, we can write represent O as the inverse image of the open set $\bigcup_{i \in I} \bigcap_{k=1}^K O_{i,k}$ under the continuous function f, which verifies the assertion. $\square$

Lemma 6.2. *There is a countable family F of real-valued continuous functions on Z_T that separates points of Z_T and generates the σ-algebra*

$$\mathcal{A} = \sigma\left(\mathcal{B}(Z_1 \cap Z_2)|_{Z_T} \cup \sigma(F_3)\right), \tag{6.1}$$

where F_3 consists of real-valued continuous functions on Z_3 separating points of Z_3.

Proof. Step 1. For each Z_i, we give a sequence $(f_{m,i})_{m \in \mathbb{N}}$ of continuous functions $f_{m,i} : Z_i \to \mathbb{R}$ separating points and determine the generated σ-algebras.

Let $\{\varphi_k : k \in \mathbb{N}\}$ be a sequence with $\|\varphi_k\|_{E_A} \leqslant 1$ and $\|x\|_{E_A^*} = \sup_{k \in \mathbb{N}} |\mathrm{Re}\langle x, \varphi_k \rangle|$ for all $x \in E_A^*$ and $\{t_l : l \in \mathbb{N}\}$ be dense in $[0,T]$. We set

$$f_{k,l,1}(u) := \mathrm{Re}\langle u(t_l), \varphi_k \rangle, \qquad u \in Z_1, \quad k, l \in \mathbb{N}$$

and for $n \in \mathbb{N}$, we denote

$$\pi_{t_1,\ldots,t_n} : Z_1 \to (E_A^*)^n, \qquad u \mapsto (u(t_1),\ldots,u(t_n)).$$

From [Jak], Corollary 2.4, we know that

$$\mathcal{B}(Z_1) = \sigma(\pi_{t_1,\ldots,t_n} : n \in \mathbb{N}).$$

But since $\pi_{t_1,\ldots,t_n}$ is strongly measurable in $(E_A^*)^n$ if and only if

$$Z_1 \ni u \mapsto \operatorname{Re}\langle \pi_{t_1,\ldots,t_n}(u), (\varphi_{k_1},\ldots,\varphi_{k_n})\rangle_{(E_A^*)^n,(E_A)^n} = \sum_{j=1}^n f_{k_j,j,1}(u)$$

for all $k_1,\ldots,k_n \in \mathbb{N}$, we obtain $\mathcal{B}(Z_1) = \sigma(f_{k,l} : k \in \mathbb{N}, l \in \mathbb{N})$. By right-continuity and the choice of φ_k, $k \in \mathbb{N}$, the $f_{k,l}$ separate points in Z_1 and they are continuous since convergence in Z_1 implies pointwise convergence.

The existence of $(f_{m,2})_{m\in\mathbb{N}}$ is a consequence of the Hahn-Banach-Theorem in Z_2. For the details, we refer to [Hor18a], Lemma 2.28. Let $\{h_k : k \in \mathbb{N}\}$ and $\{t_l : l \in \mathbb{N}\}$ be dense subsets of E_A^* and $[0,T]$, respectively. We set

$$f_{k,l,3}(u) := \operatorname{Re}\langle u(t_l), h_k\rangle, \qquad u \in Z_3, \quad k,l \in \mathbb{N}.$$

and denote the enumeration of $(f_{k,l,3})_{k,l\in\mathbb{N}}$ by $(f_{m,3})_{m\in\mathbb{N}}$. By the definition of the topology in Z_3 and the fact that convergence in $\mathbb{D}([0,T])$ implies pointwise convergence, we obtain that $f_{m,3}$ is continuous. Suppose that $f_{m,3}(u_1) = f_{m,3}(u_2)$ for all $u_1,u_2 \in Z_3$. From the right-continuity of $[0,T] \ni t \mapsto \operatorname{Re}\langle u_j(t), h_k\rangle$ and the density of $(t_l)_l$ $(h_k)_k$, we infer $u_1(t) = u_2(t)$ for all $t \in \mathbb{N}$, i.e. $(f_{m,3})_{m\in\mathbb{N}}$ separates points in Z_3.

Step 2. We define $F_j := \{f_{m,j}|_{Z_T} : m \in \mathbb{N}\}$ and set $\mathcal{A} := \sigma(F)$, where $F := F_1 \cup F_2 \cup F_3$. We would like to prove (6.1). Above, we obtained $\sigma(f_{m,j} : m \in \mathbb{N}) = \mathcal{B}(Z_j)$ for $j = 1,2$. Since we have

$$\sigma\left(f_{m,j}|_{Z_1\cap Z_2} : m \in \mathbb{N}\right) = \sigma(f_{m,j} : m \in \mathbb{N})|_{Z_1\cap Z_2}$$

and

$$\mathcal{B}(Z_1 \cap Z_2) = \sigma\left(\bigcup_{j=1,2} \mathcal{B}(Z_j)|_{Z_1\cap Z_2}\right),$$

we conclude

$$\mathcal{B}(Z_1 \cap Z_2) = \sigma\left(\bigcup_{j=1,2} \sigma\left(f_{m,j}|_{Z_1\cap Z_2} : m \in \mathbb{N}\right)\right)$$

and thus,

$$\mathcal{B}(Z_1 \cap Z_2)|_{Z_T} = \sigma\left(f_{m,1}|_{Z_T}, f_{m,2}|_{Z_T} : m \in \mathbb{N}\right) = \sigma(F_1 \cup F_2).$$

Similarly, we obtain $\mathcal{A} = \sigma\left(\mathcal{B}(Z_1 \cap Z_2)|_{Z_T} \cup \sigma(F_3)\right).$

$\square$

Remark 6.3. By Lemma 6.1, we have $\sigma\left(f_{m,3} : m \in \mathbb{N}\right) = \sigma(\tilde{\mathcal{O}}_{Z_3})$, where $\tilde{\mathcal{O}}_{Z_3}$ is the coarsest topology such that $f_{m,3}$ is continuous for all $m \in \mathbb{N}$. In particular, we have

$$\sigma\left(f_{m,3} : m \in \mathbb{N}\right) \subsetneq \mathcal{B}(Z_3),$$

since convergence in $\mathbb{D}([0,T])$ implies pointwise convergence, but not vice versa. In particular, we would get $\mathcal{A} = \mathcal{B}(\tilde{Z}_T)$ where $\tilde{Z}_T$ is the topological space arising when we replace the topology on Z_3 by $\tilde{\mathcal{O}}_{Z_3}$.

By the previous Lemma and the uniform estimates from Propositions 5.8 and 5.9, we can apply Proposition 4.12 to the sequence $(u_n)_{n\in\mathbb{N}}$ of Galerkin solutions. As a result, we obtain a candidate v for the martingale solution.

Corollary 6.4. *Let $(u_n)_{n\in\mathbb{N}}$ be the sequence of solutions to the Galerkin equation* (5.8) *on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{A}$ be the σ-algebra on Z_T defined in* (6.1).

  a) *There are a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and random variables $v, v_k : \bar{\Omega} \to Z_T$ and $\bar{\eta}_k, \bar{\eta} : \bar{\Omega} \to M_{\bar{\mathbb{N}}}^\nu([0,T] \times \mathbb{R}^M)$ with*
       i) *$\bar{\mathbb{P}}^{(\bar{\eta}_k, v_k)} = \mathbb{P}^{(\eta, u_{n_k})}$ for $k \in \mathbb{N}$,*
      ii) *$(\bar{\eta}_k, v_k) \to (\bar{\eta}, v)$ in $M_{\bar{\mathbb{N}}}^\nu([0,T] \times \mathbb{R}^M) \times Z_T$ almost surely for $k \to \infty$,*
     iii) *$\bar{\eta}_k = \bar{\eta}$ almost surely.*
  Moreover, $\bar{\eta}_k, \bar{\eta}$ are time-homogeneous Poisson random measures on $[0,T] \times \mathbb{R}^N$ with intensity measure $\mathrm{Leb} \otimes \nu$. w.r.t to the filtration $\bar{\mathbb{F}}$ defined by the augmentation of

$$\bar{\mathcal{F}}_t := \sigma\left(\bar{\eta}_k(s), v_m(s), v(s) : k \in \mathbb{N}, m \in \mathbb{N}, s \in [0,t]\right),$$

  where by the notation $\bar{\eta}_k(s)$ we mean all random variables of the form $\bar{\eta}_k((0,s] \times B_1)$, where B_1 is a measurable set in B.
  b) *We have $v_k \in \mathbb{D}\left([0,T], H_k\right)$ $\bar{\mathbb{P}}$-a.s. and for all $r \in [1, \infty)$, there is $C = C(T, \|u_0\|_{E_A}, r) > 0$ with*

$$\sup_{k\in\mathbb{N}} \bar{\mathbb{E}}\left[\|v_k\|_{L^\infty(0,T;E_A)}^r\right] \leqslant C.$$

  c) *For all $r \in [1, \infty)$, we have*

$$\bar{\mathbb{E}}\left[\|v\|_{L^\infty(0,T;E_A)}^r\right] \leqslant C$$

  with the same constant $C > 0$ as in b).

Remark 6.5. The fact that for each $n \in \mathbb{N}$, u_n is an $(Z_T, \mathcal{A})$-valued random variable is true since $\mathbb{D}([0,T], H_n) \subset Z_j$ for each $n \in \mathbb{N}$ and each $j = 1, 2, 3$, see (4.2) for the definition of the spaces Z_j, with continuity of the canonical embedding. In particular

$$\{B \cap \mathbb{D}([0,T], H_n) : B \in \mathcal{B}(Z_T)\} = \sigma\left(\{B \cap \mathbb{D}([0,T], H_n) : B \text{ closed in } Z_T\}\right)$$

$$\subset \sigma(\{\tilde{B} : \tilde{B} \text{ closed in } \mathbb{D}([0,T], H_n)\}) = \mathcal{B}(\mathbb{D}([0,T], H_n)).$$

Since u_n is random variable in $\mathbb{D}([0,T], H_n)$, we infer that

$$\{u_n \in B\} = \{u_n \in B \cap \mathbb{D}([0,T], H_n)\} \in \mathcal{F}$$

for all $B \in \mathcal{A}$.

Proof. *ad a).* We apply Proposition 4.12 with

$$\mathcal{X}_1 := M_{\bar{\mathbb{N}}}^{\nu}([0,T] \times \mathbb{R}^M), \qquad \mathcal{X}_2 := Z_T$$

and $\chi_n = (\eta, u_n)$, $n \in \mathbb{N}$. The tightness of χ_n is guaranteed by Corollary 5.10 and the fact that random variables on metric spaces are tight, see [Par67], Theorem 3.2. In Lemma 6.2, we have checked that Z_T fulfills the assumptions of Proposition 4.12 with the σ-algebra $\mathcal{A}$ from above. For the proof of the last assertion, we refer to [BHR17], Section 8, Step III.

ad b). Since $\mathbb{D}([0,T], H_k)$ is contained in Z_j for $j = 1, \ldots, 4$, the definition of $\mathcal{A}$ yields that $\mathbb{D}([0,T], H_k) \in \mathcal{A}$. Hence, we obtain $v_k \in \mathbb{D}([0,T], H_k)$ $\bar{\mathbb{P}}$-a.s. as an immediate consequence of the identity of the laws of v_k and u_{n_k}.

The uniform estimate follows from the respective estimates for $(u_{n_k})_{k \in \mathbb{N}}$, see Propositions 5.8 and 5.9, via the identity of laws, since $\mathbb{D}([0,T], H_k) \ni w \mapsto \sup_{t \in [0,T]} \|w(t)\|_{E_A}$ is a measurable function.

ad c). We can follow the lines of the proof of Proposition 6.1 c) in [BHW17]. $\qquad\square$

Corollary 6.6. *In the framework of Corollary 6.4, we have $\bar{\mathbb{P}}$-almost surely, for each $k \in \mathbb{N}$,*

$$\|v_k(t)\|_H = \|u_0\|_H \text{ for all } t \in [0,T].$$

Proof. Let us fix $k \in \mathbb{N}$. Then, the set

$$\mathcal{S} = \left\{ u \in \mathbb{D}([0,T], H_{n_k}) : \|u(t)\|_H = \|u_0\|_H \text{ for all } t \in [0,T] \right\}$$

is closed in $\mathbb{D}([0,T], H_{n_k})$ by Corollary C.2. Therefore, $\mathcal{S}$ is a Borel set in Z_T. By Corollary 6.4, the laws of v_k and u_{n_k} are equal. Since by Proposition 5.4 the law of u_{n_k} is concentrated on $\mathcal{S}$, so is the law of v_k. The proof is thus complete. $\qquad\square$

It remains to show that $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \bar{\eta}, \bar{\mathbb{F}}, u)$ is indeed martingale solution. The compensated Poisson random measure induced by $\bar{\eta}$ is denoted by $\tilde{\bar{\eta}} := \bar{\eta} - \text{Leb} \otimes \nu$. We need the following convergence results.

Lemma 6.7. *Let $\psi \in E_A$. Then, we have the following convergences in $L^2(\bar{\Omega} \times [0,T])$ as $n \to \infty$:*

$$\text{Re}\left(v_n - \widetilde{u_{0,n}}, \psi\right)_H \to \text{Re}\left(v - u_0, \psi\right)_H \tag{6.2}$$

$$\int_0^{\cdot} \text{Re}\left(Av_n(s) + P_n F(v_n(s)), \psi\right)_H \mathrm{d}s \to \int_0^{\cdot} \text{Re}\langle Av(s) + F(v(s)), \psi\rangle \mathrm{d}s; \tag{6.3}$$

$$\int_0^{\cdot} \int_{\{|l| \leqslant 1\}} \text{Re}\left(e^{-i\mathcal{B}_n(l)} v_n(s-) - v_n(s-), \psi\right)_H \tilde{\bar{\eta}}(\mathrm{d}s, \mathrm{d}l)$$

$$\to \int_0^{\cdot} \int_{\{|l| \leqslant 1\}} \text{Re}\left(e^{-i\mathcal{B}(l)} v(s-) - v(s-), \psi\right)_H \tilde{\bar{\eta}}(\mathrm{d}s, \mathrm{d}l); \tag{6.4}$$

$$\int_0^\cdot \int_{\{|l|\leqslant 1\}} \mathrm{Re}\left(e^{-\mathrm{i}\mathcal{B}_n(l)}v_n(s) - v_n(s) + \mathrm{i}\mathcal{B}_n(l)v_n(s), \psi\right)_H \nu(\mathrm{d}l)\mathrm{d}s$$

$$\rightarrow \int_0^\cdot \int_{\{|l|\leqslant 1\}} \mathrm{Re}\left(e^{-\mathrm{i}\mathcal{B}(l)}v(s) - v(s) + \mathrm{i}\mathcal{B}(l)v(s), \psi\right)_H \nu(\mathrm{d}l)\mathrm{d}s. \tag{6.5}$$

Proof. ad (6.2). We get (6.2) pointwise in $\bar\Omega \times [0, T]$ from (5.6) and $v_n \rightarrow v$ in $L^2(0, T; H)$. In view of

$$\bar{\mathbb{E}} \int_0^T |\mathrm{Re}\left(v_n(t) - \widetilde{u_{0,n}}, \psi\right)_H|^r \mathrm{d}t \leqslant \|\psi\|_H^r \bar{\mathbb{E}} \int_0^T \left(\|v_n(t)\|_H + \|u_0\|_H\right)^r \mathrm{d}t \leqslant \|\psi\|_H^r T 2^r \|u_0\|_H^r < \infty$$

for $r > 2$, Vitali's convergence Theorem yields the assertion.

ad (6.3). Let us fix $\omega \in \bar\Omega$ and $t \in [0, T]$. Then,

$$\int_0^t \mathrm{Re}\left(P_n F(v_n(s)), \psi\right)_H \mathrm{d}s \rightarrow \int_0^t \mathrm{Re}\langle F(v(s)), \psi\rangle \mathrm{d}s$$

follows from $v_n \rightarrow v$ in $L^{\alpha+1}(0, T; L^{\alpha+1}(M))$, see [BHW17], Lemma 6.2, step 3. Moreover,

$$\mathrm{Re}\langle A(v_n(s) - v(s)), \psi\rangle = \mathrm{Re}\langle v_n(s) - v(s), A\psi\rangle \rightarrow 0$$

for all $s \in [0, T]$ by $v_n \rightarrow v$ in $\mathbb{D}_w([0, T], E_A)$. Via

$$\bar{\mathbb{E}} \int_0^T \int_0^t |\mathrm{Re}\langle Av_n(s), \psi\rangle|^r \mathrm{d}s\mathrm{d}t \leqslant \|\psi\|_{E_A}^r T^2 \bar{\mathbb{E}}\left[\sup_{s\in[0,T]} \|v_n(s)\|_{E_A}^r\right] < \infty,$$

$$\bar{\mathbb{E}} \int_0^T \left|\int_0^t \mathrm{Re}\left(P_n F(v_n(s)), \psi\right)_H \mathrm{d}s\right|^r \mathrm{d}t \leqslant T^{1+r} \|\psi\|_{E_A}^r \bar{\mathbb{E}}\left[\sup_{s\in[0,T]} \|F(v_n(s))\|_{E_A^*}^r\right]$$

$$\lesssim T^{1+r} \|\psi\|_{E_A}^r \bar{\mathbb{E}}\left[\sup_{s\in[0,T]} \|v_n(s)\|_{E_A}^{r\alpha}\right] < \infty$$

for $r > 2$, Vitali yields (6.3) in $L^2(\bar\Omega \times [0, T])$.

ad (6.4). In view of the Itô isometry, it is equivalent to prove

$$\int_0^\cdot \int_{\{|l|\leqslant 1\}} |\mathrm{Re}\left(e^{-\mathrm{i}\mathcal{B}_n(l)}v_n(s) - v_n(s) - \left[e^{-\mathrm{i}\mathcal{B}(l)}v(s) - v(s)\right], \psi\right)_H|^2 \nu(\mathrm{d}l)\mathrm{d}s \rightarrow 0, \qquad n \rightarrow \infty, \tag{6.6}$$

in $L^1(\bar\Omega \times [0, T])$. For $x \in H$, Lebesgue yields

$$\|e^{-\mathrm{i}\mathcal{B}_n(l)}x - e^{-\mathrm{i}\mathcal{B}(l)}x\|_H = \left\|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\left[e^{-\mathrm{i}s\mathcal{B}_n(l)}e^{-\mathrm{i}(1-s)\mathcal{B}(l)}x\right]\mathrm{d}s\right\|_H$$

$$\leqslant \int_0^1 \|\left(\mathcal{B}_n(l) - \mathcal{B}(l)\right)e^{-\mathrm{i}s\mathcal{B}_n(l)}e^{-\mathrm{i}(1-s)\mathcal{B}(l)}x\|_H \mathrm{d}s \rightarrow 0, \qquad n \rightarrow \infty.$$

From $v_n \to v$ almost surely in $L^2(0,T;H)$ and again Lebesgue, we infer

$$\int_0^t |\operatorname{Re}\left(e^{-i\mathcal{B}_n(l)}v_n - v_n - \left[e^{-i\mathcal{B}(l)}v - v\right], \psi\right)_H|^2 \mathrm{d}s \tag{6.7}$$

$$\leqslant 2\int_0^t \left(\|e^{-i\mathcal{B}_n(l)}(v-v_n)\|_H^2 + \|v_n - v\|_H^2 + \|\left[e^{-i\mathcal{B}_n(l)} - e^{-i\mathcal{B}(l)}\right]v\|_H\right)\|\psi\|_H^2 \mathrm{d}s \to 0 \tag{6.8}$$

as $n \to \infty$ almost surely for all $t \in [0,T]$ and $l \in B(0,1)$. Since we have

$$\int_0^t |\operatorname{Re}\left(e^{-i\mathcal{B}_n(l)}v_n - v_n - \left[e^{-i\mathcal{B}(l)}v - v\right], \psi\right)_H|^2 \mathrm{d}s$$

$$\leqslant 2\|\psi\|_H^2 b_H|l|^2 \left(\|v_n\|_{L^2(0,t;H)}^2 + \|v\|_{L^2(0,t;H)}^2\right) \lesssim |l|^2 \in L^1(B(0,1);\nu), \tag{6.9}$$

by Lemma 5.3 and Remark 2.8, we get

$$\int_{\{|l|\leqslant 1\}} \int_0^t |\operatorname{Re}\left(e^{-i\mathcal{B}_n(l)}v_n - v_n - \left[e^{-i\mathcal{B}(l)}v - v\right], \psi\right)_H|^2 \mathrm{d}s\nu(\mathrm{d}l) \to 0$$

as $n \to \infty$ almost surely for all $t \in [0,T]$. For $r > 1$, we employ similar estimates as in (6.9) for

$$\bar{\mathbb{E}}\int_0^T \left(\int_{\{|l|\leqslant 1\}} \int_0^t |\operatorname{Re}\left(e^{-i\mathcal{B}_n(l)}v_n - v_n - \left[e^{-i\mathcal{B}(l)}v - v\right], \psi\right)_H|^2 \mathrm{d}s\nu(\mathrm{d}l)\right)^r \mathrm{d}r$$

$$\lesssim \|\psi\|_H^{2r}\bar{\mathbb{E}}\int_0^T \left(\|v_n\|_{L^2(0,t;H)}^2 + \|v\|_{L^2(0,t;H)}^2\right)^r \mathrm{d}r$$

$$\lesssim \|\psi\|_H^{2r}T^{1+r}\bar{\mathbb{E}}\left[\sup_{s\in[0,T]}\left(\|v_n\|_H^2 + \|v\|_H^2\right)^r\right] < \infty,$$

and thus, we get (6.4) by Vitali's Theorem.

ad (6.5). From (6.7),

$$\int_0^t |\operatorname{Re}\left(i\mathcal{B}_n(l)v_n - i\mathcal{B}(l)v, \psi\right)_H|\mathrm{d}s$$

$$\leqslant \|\psi\|_H \left(\|\mathcal{B}_n(l)(v_n - v)\|_{L^1(0,t;H)} + \|\left[\mathcal{B}_n(l) - \mathcal{B}(l)\right]v\|_{L^1(0,t;H)}\right)$$

$$\leqslant \|\psi\|_H t^{\frac{1}{2}} \left(\|\mathcal{B}(l)\|_{\mathcal{L}(H)}\|v_n - v\|_{L^2(0,t;H)} + \|\left[\mathcal{B}_n(l) - \mathcal{B}(l)\right]v\|_{L^2(0,t;H)}\right) \to 0$$

and the bound

$$\int_0^t |\operatorname{Re}\left(e^{-i\mathcal{B}_n(l)}v_n(s) - v_n(s) + i\mathcal{B}_n(l)v_n(s), \psi\right)_H|\mathrm{d}s \leqslant \frac{1}{2}b_H\|\psi\|_H|l|^2\|v_n\|_{L^2(0,t;H)}^2$$

$$\lesssim_{\omega,t} |l|^2 \in L^1(B(0,1);\nu)$$

by Lemma 5.3, we infer (6.5) pointwise in $\bar{\Omega} \times [0, T]$. The $L^2(\bar{\Omega} \times [0, T])$-convergence follows similarly as in the previous step by the Vitali type argument based on the uniform bounds on v_n, $n \in \mathbb{N}$. $\qquad \square$

Finally, we are ready to summarize our results and obtain the existence of a martingale solution.

Proof of Theorem 1.1. Step 1. Let us define the maps

$$M_{n,\psi}(w, t) = \widetilde{u_{0,n}} - \mathrm{i} \int_0^t \mathrm{Re}\langle Aw(s) + P_n F(w(s)), \psi \rangle \mathrm{d}s$$
$$+ \int_0^t \int_{\{|l| \leqslant 1\}} \mathrm{Re}\left(e^{-\mathrm{i}\mathcal{B}_n(l)} w(s-) - w(s-), \psi\right)_H \tilde{\tilde{\eta}}(\mathrm{d}s, \mathrm{d}l)$$
$$+ \int_0^t \int_{\{|l| \leqslant 1\}} \mathrm{Re}\left(e^{-\mathrm{i}\mathcal{B}_n(l)} w(s) - w(s) + \mathrm{i}\mathcal{B}_n(l) w(s), \psi\right)_H \nu(\mathrm{d}l) \mathrm{d}s;$$

$$M_\psi(w, t) = u_0 - \mathrm{i} \int_0^t \mathrm{Re}\langle Aw(s) + F(w(s)), \psi \rangle \mathrm{d}s$$
$$+ \int_0^t \int_{\{|l| \leqslant 1\}} \mathrm{Re}\left(e^{-\mathrm{i}\mathcal{B}(l)} w(s-) - w(s-), \psi\right)_H \tilde{\tilde{\eta}}(\mathrm{d}s, \mathrm{d}l)$$
$$+ \int_0^t \int_{\{|l| \leqslant 1\}} \mathrm{Re}\left(e^{-\mathrm{i}\mathcal{B}(l)} w(s) - w(s) + \mathrm{i}\mathcal{B}(l) w(s), \psi\right)_H \nu(\mathrm{d}l) \mathrm{d}s.$$

The results of Lemma 6.7 can be summarized as

$$\mathrm{Re}\left(v_n, \psi\right)_H - M_{n,\psi}(v_n, \cdot) \to \mathrm{Re}\left(v, \psi\right)_H - M_\psi(v, \cdot), \qquad n \to \infty,$$

in $L^2(\bar{\Omega} \times [0, T])$ for all $\psi \in E_A$ and from the definition of u_n via the Galerkin equation, we infer $\mathrm{Re}\left(u_n(t), \psi\right)_H = M_{n,\psi}(u_n, t)$ almost surely for all $t \in [0, T]$. Due to the identity $\mathrm{Leb}_{[0,T]} \otimes \mathbb{P}^{u_n} = \mathrm{Leb}_{[0,T]} \otimes \bar{\mathbb{P}}^{v_n}$, we obtain

$$\bar{\mathbb{E}} \int_0^T |\mathrm{Re}\left(v(t), \psi\right)_H - M_\psi(v, t)|^2 \mathrm{d}t = \lim_{n \to \infty} \bar{\mathbb{E}} \int_0^T |\mathrm{Re}\left(v_n(t), \psi\right)_H - M_{n,\psi}(v_n, t)|^2 \mathrm{d}t$$
$$= \lim_{n \to \infty} \mathbb{E} \int_0^T |\mathrm{Re}\left(u_n(t), \psi\right)_H - M_{n,\psi}(u_n, t)|^2 \mathrm{d}t = 0$$

and thus,

$$\bar{\mathbb{P}}\left\{\mathrm{Re}\left(v(t), \psi\right)_H = M_\psi(v, t) \quad \text{f.a.a. } t \in [0, T]\right\} = 1.$$

Since both $\mathrm{Re}\left(v, \psi\right)_H$ and $M_\psi(v, \cdot)$ are almost surely in $\mathbb{D}([0, T])$, we obtain

$$\bar{\mathbb{P}}\left\{\mathrm{Re}\left(v(t), \psi\right)_H = M_\psi(v, t) \quad \forall t \in [0, T]\right\} = 1,$$

which means that $\left(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \bar{\eta}, \bar{\mathbb{F}}, u\right)$ is a martingale solution to (2.15).

Step 2. In order to conclude the proof, we need to show that the process v satisfies the mass preservation condition (1.6). Let us first fix $\omega \in \bar{\Omega}$ such that

$$v_k(\cdot, \omega) \to v(\cdot, \omega) \text{ in } Z_T, \tag{6.10}$$

as $k \to \infty$. By part (a)(ii) of Corollary 6.4, the set of such elements is a full set in $\bar{\Omega}$. Together with Lemma 4.5, (6.10) implies that there exists $r = r(\omega) > 0$ such that $\sup_{t \in [0,T]} \|v_k(t, \omega)\| \leqslant r$ for every $k \in \mathbb{N}$. From (6.10) and Proposition 4.2, we infer that there is a sequence $(\lambda_k)_{n \in \mathbb{N}} = (\lambda_k(\omega))_{n \in \mathbb{N}} \in \Lambda^{\mathbb{N}}$, such that

$$\sup_{t \in [0,T]} \|v_k(\lambda_k(t), \omega) - v(t, \omega)\|_{E_A^*} \to 0, \quad k \to \infty.$$

Hence, we get

$$\sup_{t \in [0,T]} \|v_k(\lambda_k(t), \omega) - v(t, \omega)\|_H$$

$$\lesssim \sup_{t \in [0,T]} \left[\|v_k(\lambda_k(t), \omega) - v(t, \omega)\|_{E_A^*}^{\frac{1}{2}} \|v_k(\lambda_k(t), \omega) - v(t, \omega)\|_{E_A}^{\frac{1}{2}} \right]$$

$$\leqslant (2r)^{\frac{1}{2}} \sup_{t \in [0,T]} \|v_k(\lambda_k(t), \omega) - v(t, \omega)\|_{E_A^*}^{\frac{1}{2}} \to 0, \qquad k \to \infty.$$

In view of Proposition 4.2, this implies $v_k(\cdot, \omega) \to v(\cdot, \omega)$ in $\mathbb{D}([0,T], H)$ as $k \to \infty$. Since the norm function $\| \cdot \|_H : H \to \mathbb{R}$ is Lipschitz continuous we deduce that

$$\|v_k(\cdot, \omega)\|_H \to \|v(\cdot, \omega)\|_H \text{ in } \mathbb{D}([0,T], \mathbb{R}).$$

On the other hand, by Corollary 6.6, we infer that

$$\|v_k(t, \omega)\|_H = \|u_0\|_H \text{ for all } t \in [0, T].$$

Applying finally Lemma C.1 we infer that

$$\|v(t, \omega)\|_H = \|u_0\|_H \text{ for all } t \in [0, T].$$

$\square$

Appendix A. Time Homogeneous Poisson Random Measure

Let $\bar{\mathbb{N}}$ denote the set of extended natural numbers, i.e., $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ and $\mathbb{R}^+ := [0, \infty)$. Let $(S, \mathscr{S})$ be a measurable space and $M_{\bar{\mathbb{N}}}(S)$ be the set of all $\bar{\mathbb{N}}$-valued measures on the measurable space $(S, \mathscr{S})$. On the set $M_{\bar{\mathbb{N}}}(S)$ we consider the σ-field $\mathscr{M}_{\bar{\mathbb{N}}}(S)$ defined as the smallest σ-field such that for all $C \in \mathscr{S}$: the map

$$i_C : M_{\bar{\mathbb{N}}}(S) \ni \mu \to \mu(C) \in \bar{\mathbb{N}}$$

is measurable.

Definition A.1. *Let $(Y, \mathscr{B}(Y))$ be a measurable space. A time homogeneous Poisson random measure η on $(Y, \mathscr{B}(Y))$ over $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a measurable function*

$$\eta : (\Omega, \mathcal{F}) \to (M_{\bar{\mathbb{N}}}(\mathbb{R}^+ \times Y), \mathscr{M}_{\bar{\mathbb{N}}}(\mathbb{R}^+ \times Y))$$

such that

(a) *for each $C \in \mathscr{B}(\mathbb{R}^+) \otimes \mathscr{B}(Y), \eta(C) := i_C \circ \eta : \Omega \to \bar{\mathbb{N}}$ is a Poisson random variable with parameter $\mathbb{E}[\eta(C)]$;*

(b) *η is independently scattered, i.e., if the sets $C_1, C_2, \ldots, C_n \in \mathscr{B}(\mathbb{R}^+) \otimes \mathscr{B}(Y)$ are disjoint, then the random variables $\eta(C_1), \eta(C_2), \ldots, \eta(C_n)$ are mutually independent;*

(c) *for all $U \in \mathscr{B}(Y)$ the $\bar{\mathbb{N}}-$valued process $(N(t, U))_{t \geqslant 0}$ defined by*

$$N(t, U) := \eta((0, t] \times U), \quad t \geqslant 0$$

is $\mathcal{F}_t$-adapted and its increments are independent of the past, i.e., if $t > s \geqslant 0$, then $N(t, U) - N(s, U) = \eta((s, t] \times U)$ is independent of $\mathcal{F}_s$.

If η is a time homogeneous Poisson random measure then the formula

$$\nu(A) := \mathbb{E}[\eta((0, 1] \times A)], \quad A \in \mathscr{B}(Y)$$

defines a measure on $(Y, \mathscr{B}(Y))$ called the intensity measure of η. We assume that ν is σ-finite. Moreover, for all $T < \infty$ and all $A \in \mathscr{B}(Y)$ such that $\mathbb{E}[\eta((0, T] \times A)] < \infty$, the $\mathbb{R}-$valued process $\{\tilde{N}(t, A)\}_{t \in [0, T]}$ defined by

$$\tilde{N}(t, A) := \eta((0, t] \times A) - t \, \nu(A), \quad t \in (0, T],$$

is an integrable martingale on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. The random measure $m \otimes \nu$ on $\mathscr{B}(\mathbb{R}^+) \otimes \mathscr{B}(Y)$, where m stands for the Lebesgue measure (often denoted also as Leb), is called a compensator of η and the difference between a time homogeneous Poisson random measure η and its compensator, i.e.,

$$\tilde{\eta} := \eta - m \otimes \nu,$$

is called a compensated time homogeneous Poisson random measure.

We follow the notion of Ikeda and Watanabe [IW81], Peszat and Zabczyk [PZ07], to list some of the basic properties of the stochastic integral with respect to $\tilde{\eta}$. Let E be

a separable Hilbert space and let $\mathcal{P}$ be a predictable σ-field on $[0,T] \times \Omega$. Let $\mathfrak{L}^2_{\nu,T}(\mathcal{P} \otimes \mathscr{B}(Y), m \otimes \mathbb{P} \otimes \nu; E)$ be a space of all E-valued, $\mathcal{P} \otimes \mathscr{B}(Y)$-measurable processes such that

$$\mathbb{E}\left[\int_0^T \int_Y \|\xi(s,\cdot,y)\|^2_E \, d\nu(y) \, ds\right] < \infty.$$

If $\xi \in \mathfrak{L}^2_{\nu,T}(\mathcal{P} \otimes \mathscr{B}(Y), m \otimes \mathbb{P} \otimes \nu; E)$ then the integral process $\int_0^T \int_Y \xi(s,\cdot,y)\tilde{\eta}(ds,dy)$, $t \in [0,T]$, is a càdlàg square-integrable E-valued martingale. Moreover, we have the following isometry formula

$$\mathbb{E}\left[\left\|\int_0^T \int_Y \xi(s,\cdot,y)\,\tilde{\eta}(ds,dy)\right\|^2_E\right] = \mathbb{E}\left[\int_0^T \int_Y \|\xi(s,\cdot,y)\|^2_E \, d\nu(y) \, ds\right], \; t \in [0,T]. \quad \text{(A.1)}$$

Appendix B. Marcus Canonical SDEs

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a probability space equipped with a filtration $\mathbb{F} := \{\mathcal{F}_t, t \geqslant 0\}$ that satisfies the usual hypothesis (i.e., $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets and $\mathcal{F}$ is right continuous). Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_N : \mathbb{R}^d \to \mathbb{R}^d$ be complete C^1-vector fields. Define $\mathbf{v} : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^N, \mathbb{R}^d)$ such that $\mathbf{v}(y)(h) := \sum_{j=1}^N \mathbf{v}_j(y)h_j, h \in \mathbb{R}^N, y \in \mathbb{R}^d$.

Let $L(t) := (L_1(t), \cdots, L_N(t))$ be a $\mathbb{R}^N-$ valued Lévy process with pure jump,

$$L(t) = \int_0^t \int_B l \, \tilde{\eta}(ds, dl) + \int_0^t \int_{B^c} l \, \eta(ds, dl)$$

where $B := \mathbb{B}(0,1) \subset \mathbb{R}^N$, $l = (l_1, \ldots, l_N) \in \mathbb{R}^N$; η, $\tilde{\eta}$ represent homogeneous Poisson random measure and the compensated one with the compensator $m \otimes \nu$ respectively. We always assume that η is independent of $\mathcal{F}_0$.

Consider the following "Marcus" stochastic differential equation:

$$dY(t) = \mathbf{v}_0(Y(t)) \, dt + \mathbf{v}(Y(t-)) \diamond dL(t)$$

$$= \mathbf{v}_0(Y(t)) \, dt + \sum_{j=1}^N \mathbf{v}_j(Y(t-)) \diamond dL_j(t), \quad \text{(B.1)}$$

which is defined in the integral form as follows

$$Y(t) = Y_0 + \int_0^t \mathbf{v}_0(Y(s)) \, ds + \int_0^t \int_B \left[\Phi\big(1,l,Y(s-)\big) - Y(s-)\right]\tilde{\eta}(ds,dl)$$

$$+ \int_0^t \int_{B^c} \left[\Phi\big(1,l,Y(s-)\big) - Y(s-)\right]\eta(ds,dl)$$

$$+ \int_0^t \int_B \left[\Phi\big(1,l,Y(s)\big) - Y(s) - \sum_{j=1}^N l_j\mathbf{v}_j(Y(s))\right]\nu(dl)ds, \quad \text{(B.2)}$$

where $y(t) := \Phi(t,l,y_0)$ solves

$$\frac{dy}{dt} = \sum_{j=1}^N l_j\mathbf{v}_j(y), \quad \text{with initial condition } y(0) = y_0. \quad \text{(B.3)}$$

Theorem B.1 (Itô's formula 1). *Let $\varphi : \mathbb{R}^d \to \mathbb{R}^k$ is a C^1-class function. If Y is an $\mathbb{R}^d$-valued process a solution to* (B.1)*, then*

$$
\varphi(Y(t)) - \varphi(Y_0)
$$
$$
= \int_0^t \varphi'(Y(s))(\mathbf{v}_0(Y(s)))\,ds + \int_0^t \int_{B^c} \Big[\varphi\big(\Phi(1,l,Y(s-))\big) - \varphi(Y(s-))\Big]\eta(ds,dl)
$$
$$
+ \int_0^t \int_B \Big[\varphi\big(\Phi(1,l,Y(s-))\big) - \varphi(Y(s-))\Big]\tilde{\eta}(ds,dl)
$$
$$
+ \int_0^t \int_B \Big[\varphi\big(\Phi(1,l,Y(s))\big) - \varphi(Y(s)) - \sum_{j=1}^N l_j\varphi'(Y(s))(\mathbf{v}_j(Y(s)))\Big]\nu(dl)ds. \quad \text{(B.4)}
$$

Moreover, when $k = d$ and $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ is a C^1-diffeomorphism, we define for each $j = 0, 1, \ldots, N$, the "Push-forward" of the vector fields $\mathbf{v}_j$ by φ' as $\hat{\mathbf{v}}_j : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$
z \mapsto (d_{\varphi^{-1}(z)}\varphi)\big(\mathbf{v}_j(\varphi^{-1}(z))\big) := \varphi'(\varphi^{-1}(z))\big(\mathbf{v}_j(\varphi^{-1}(z))\big).
$$

Let $\hat{\mathbf{v}} : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^N, \mathbb{R}^d)$ be as before.
Then Y is a solution to (B.1) *iff*

$$
Z(t) := \varphi(Y(t))
$$

is a solution to

$$
dZ = \hat{\mathbf{v}}_0(Z(t))\,dt + \hat{\mathbf{v}}(Z(t)) \diamond dL(t), \quad Z_0 = \varphi(Y_0). \quad \text{(B.5)}
$$

We will now present an infinite dimensional version of the above result, which has been used in this work.

As before let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability space. Let E be a separable Hilbert space. Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_N : E \to E$ be complete C^1-vector fields. Define $\mathbf{v} : E \to \mathcal{L}(\mathbb{R}^N, E)$ such that $\mathbf{v}(y)(h) := \sum_{j=1}^N \mathbf{v}_j(y)h_j, h \in \mathbb{R}^N, y \in E$. Define the Lévy process $L(t)$ as before. Define the Marcus mapping

$$
\Phi : \mathbb{R}_+ \times \mathbb{R}^N \times E \to E
$$

such that for each fixed $l \in \mathbb{R}^N$, $y_0 \in E$, the function

$$
t \mapsto \Phi(t, l, u_0)
$$

is the continuously differentiable solution of the ordinary differential equation

$$
\frac{dy}{dt} = \sum_{j=1}^N l_j v_j(y), \ t \geqslant 0,
$$

with $y(0) = y_0 \in E$, and $l = (l_1, l_2, \ldots, l_N) \in B$, i.e.,

$$
\Phi(t, l, y_0) = \Phi(0, l, y_0) + \int_0^t \sum_{j=1}^N l_j v_j(\Phi(s, l, y_0))\,ds, \ t \geqslant 0.
$$

With the above setting, let us consider the E-valued process Y given by (B.2). Then we have the following result.

Theorem B.2 (Itô's formula 2)**.** *Let G be a separable Hilbert space and $\varphi : E \to G$ be a C^1-class function such that the first derivative $\varphi' : E \to \mathcal{L}(E, G)$ is $(p-1)$-Hölder continuous. If Y is an E-valued process given by* (B.2), *then for every $t > 0$, we have ¶-a.s.*

$$\varphi(Y(t)) - \varphi(Y_0)$$
$$= \int_0^t \varphi'(Y(s))(\mathbf{v}_0(Y(s)))\, ds + \int_0^t \int_{B^c} \Big[\varphi\big(\Phi\big(1, l, Y(s-)\big)\big) - \varphi(Y(s-))\Big]\eta(ds, dl)$$
$$+ \int_0^t \int_B \Big[\varphi\big(\Phi\big(1, l, Y(s-)\big)\big) - \varphi(Y(s-))\Big]\tilde{\eta}(ds, dl)$$
$$+ \int_0^t \int_B \Big[\varphi\big(\Phi\big(1, l, Y(s)\big)\big) - \varphi(Y(s)) - \sum_{j=1}^N l_j \varphi'(Y(s))(\mathbf{v}_j(Y(s)))\Big]\nu(dl)ds. \quad \text{(B.6)}$$

Moreover, when $\varphi : E \to E$ is a C^1-diffeomorphism, we define for each $j = 0, 1, \dots, N$, the "Push-forward" of the vector fields $\mathbf{v}_j$ by φ' as $\hat{\mathbf{v}}_j : E \to E$ such that

$$z \mapsto (d_{\varphi^{-1}(z)}\varphi)\big(\mathbf{v}_j(\varphi^{-1}(z))\big) := \varphi'(\varphi^{-1}(z))\big(\mathbf{v}_j(\varphi^{-1}(z))\big).$$

Let $\hat{\mathbf{v}} : E \to \mathcal{L}(\mathbb{R}^N, E)$ be as before.
Then Y is a solution to (B.1) *iff*

$$Z(t) := \varphi(Y(t))$$

is a solution to

$$dZ = \hat{\mathbf{v}}_0(Z(t))\, dt + \hat{\mathbf{v}}(Z(t)) \diamond dL(t), \quad Z_0 = \varphi(Y_0). \quad \text{(B.7)}$$

Proof. Let us assume, for the sake of simplicity, $\eta = 0$ on B^c. For $y \in E$, define

$$f(y, l) := \Phi\big(1, l, y\big) - y \quad \text{for all } l \in B \quad \text{(B.8)}$$

$$a(y) := \mathbf{v}_0(y) + \int_B \Big[\Phi\big(1, l, y\big) - y - \sum_{j=1}^N l_j \mathbf{v}_j(y)\Big]\nu(dl)$$

$$= \mathbf{v}_0(y) + \int_B \Big[f(y, l) - \sum_{j=1}^N l_j \mathbf{v}_j(y)\Big]\nu(dl). \quad \text{(B.9)}$$

Then the E-valued process Y given in (B.2) takes the form

$$Y(t) = Y_0 + \int_0^t a(Y(s))\, ds + \int_0^t \int_B f(Y(s-), l)\tilde{\eta}(ds, dl). \quad \text{(B.10)}$$

Then by the Itô's formula (see Theorem B.1 in Brzeźniak *et al.* [BHZ13]), for every $t > 0$, we have $\mathbb{P}$-a.s.

$$\varphi(Y(t)) = \varphi(Y_0) + \int_0^t \varphi'(Y(s))(a(Y(s)))\, ds + \int_0^t \int_B \varphi'(Y(s-))(f(Y(s-), l))\tilde{\eta}(ds, dl)$$

$$+ \int_0^t \int_B \Big[\varphi\big(Y(s-) + f(Y(s-), l)\big) - \varphi(Y(s-)) - \varphi'(Y(s-))(f(Y(s-), l))\Big]\eta(ds, dl)$$

$$= \varphi(Y_0) + \sum_{i=1}^{3} I_i. \tag{B.11}$$

Note that by the definition of a in (B.9)

$$I_1 := \int_0^t \varphi'(Y(s))(a(Y(s)))\, ds$$

$$= \int_0^t \varphi'(Y(s))(\mathbf{v}_0(Y(s)))\, ds + \int_0^t \int_B \varphi'(Y(s))(f(Y(s), l))\, \nu(dl)\, ds$$

$$- \int_0^t \int_B \Big[\sum_{j=1}^{N} l_j \varphi'(Y(s))(\mathbf{v}_j(Y(s))) \Big] \nu(dl)\, ds. \tag{B.12}$$

Using the definitions of f in (B.8) and that of compensated Poisson random measure $\tilde{\eta} := \eta - m \otimes \nu$, we have

$$I_3 := \int_0^t \int_B \Big[\varphi\big(Y(s-) + f(Y(s-), l)\big) - \varphi(Y(s-)) - \varphi'(Y(s-))(f(Y(s-), l)) \Big] \eta(ds, dl)$$

$$= \int_0^t \int_B \Big[\varphi\big(\Phi(1, l, Y(s-))\big) - \varphi(Y(s-)) - \varphi'(Y(s-))(f(Y(s-), l)) \Big] \eta(ds, dl)$$

$$= \int_0^t \int_B \Big[\varphi\big(\Phi(1, l, Y(s-))\big) - \varphi(Y(s-)) \Big] \tilde{\eta}(ds, dl) - \int_0^t \int_B \varphi'(Y(s-))(f(Y(s-), l)) \tilde{\eta}(ds, dl)$$

$$+ \int_0^t \int_B \Big[\varphi\big(\Phi(1, l, Y(s))\big) - \varphi(Y(s)) \Big] \nu(dl)\, ds - \int_0^t \int_B \varphi'(Y(s))(f(Y(s), l))\, \nu(dl)\, ds. \tag{B.13}$$

Note that, while adding up I_1, I_2 and I_3, the second term of (B.12) and the last term of (B.13) cancel each other. Also note the 2nd term on the right hand of (B.13) is $-I_2$, and thus it gets cancelled with I_2. Hence using (B.12) and (B.13) in (B.11), and grouping the similar integrals we have the desired result (B.6).

To prove the second part of the Theorem, let us define a map

$$\hat{\Phi} : \mathbb{R}_+ \times \mathbb{R}^N \times E \to E$$

such that for all $l \in \mathbb{R}^N, z \in E$, the function $t \mapsto \hat{\Phi}(t, l, z)$ solves

$$\frac{dz}{dt} = \sum_{j=1}^{N} l_j \hat{\mathbf{v}}_j(z), \quad t \geqslant 0, \quad z(0) = z.$$

Let us assume that $\varphi : E \to E$ is a C^1-diffeomorphism. Then one can show that for all $l \in \mathbb{R}^N$ and $t \geqslant 0$

$$\hat{\Phi}(t, l, z) = \varphi\big(\Phi(t, l, y)\big) \text{ where } z = \varphi(y), \ y \in E. \tag{B.14}$$

Then from the Itô's formula (B.6), we deduce

$$
\begin{aligned}
Z(t) = Z_0 &+ \int_0^t \hat{\mathbf{v}}_0(Z(s))\, ds + \int_0^t \int_{B^c} \Big[\hat{\Phi}\big(1, l, Z(s-)\big) - Z(s-) \Big] \eta(ds, dl) \\
&+ \int_0^t \int_B \Big[\hat{\Phi}\big(1, l, Z(s-)\big) - Z(s-) \Big] \tilde{\eta}(ds, dl) \\
&+ \int_0^t \int_B \Big[\hat{\Phi}\big(1, l, Z(s)\big) - Z(s) - \sum_{j=1}^N l_j \hat{\mathbf{v}}_j(Z(s)) \Big] \nu(dl) ds.
\end{aligned}
$$

This proves $Z(t) = \varphi(Y(t))$ is an E-valued process satisfying

$$
dZ = \hat{\mathbf{v}}_0(Z(t))\, dt + \hat{\mathbf{v}}(Z(t)) \diamond dL(t), \quad Z_0 = \varphi(Y_0).
$$

Converse part can similarly be proven. $\qquad\square$

Appendix C. A simple convergence result

Lemma C.1. *Suppose that for each $n \in \mathbb{N}$, a function $f_n \in \mathbb{D}([0,T], \mathbb{R})$ is constant and that for some $f \in \mathbb{D}([0,T], \mathbb{R})$, $f_n \to f$ in $\mathbb{D}([0,T], \mathbb{R})$ as $n \to \infty$. Then f is also a constant function and $f_n \to f$ in $C([0,T], \mathbb{R})$ as $n \to \infty$.*

Proof. Let us denote, for each $n \in \mathbb{N}$, the value of the function f_n by c_n, for some $c_n \in \mathbb{R}$. By part (b) of Proposition 4.2 there exists a sequence $(\lambda_n) \in \Lambda^{\mathbb{N}}$ such that

$$
\sup_{t \in [0,T]} |\lambda_n(t) - t| \to 0 \tag{C.1}
$$

and

$$
\sup_{t \in [0,T]} |f_n(\lambda_n(t)) - f(t)| \to 0, \qquad n \to \infty. \tag{C.2}
$$

This yields

$$
\sup_{t \in [0,T]} |f_n(t) - f(t)| = \sup_{t \in [0,T]} |c_n - f(t)| = \sup_{t \in [0,T]} |f_n(\lambda_n(t)) - f(t)| \to 0, \qquad n \to \infty.
$$

Moreover, (C.2) implies

$$
|f(t) - f(s)| \leqslant |f(t) - c_n| + |c_n - f(s)| = |f(t) - f_n(\lambda_n(t))| + |f_n(\lambda_n(s)) - f(s)| \to 0
$$

as $n \to \infty$ for $s, t \in [0,T]$. Hence, f is a constant function as claimed. $\qquad\square$

We conclude this section with the following result.

Corollary C.2. *Let $n \in \mathbb{N}_0$ and $c \geqslant 0$. Then, the set*

$$
\mathcal{S} = \big\{ u \in \mathbb{D}([0,T], H_n) : \|u(t)\|_H = c \text{ for all } t \in [0,T] \big\}
$$

is closed in $\mathbb{D}([0,T], H_n)$.

Proof. Take an $\mathcal{S}$-valued sequence (u_k) such that $u_k \to u$ in $\mathbb{D}([0, T], H_n)$ for some $u \in \mathbb{D}([0, T], H_n)$. For $t \in [0, T]$ and $k \in \mathbb{N}$, we define $f_k(t) = \|u_k(t)\|_H$ and $f(t) = \|u(t)\|_H$. Since the H-norm function is Lipschitz on H_n, we infer that $f_k \to f$ in $\mathbb{D}([0, T], \mathbb{R})$. In view of Lemma C.1 we obtain $f(t) = c$ for all $t \in [0, T]$ which implies $u \in \mathcal{S}$. $\qquad\square$

References

[ABW10] S. Albeverio, Z. Brzeźniak, and J.(iang)-L.(un) Wu. Existence of global solutions and invariant measures for stochastic differential equations driven by Poisson type noise with non-Lipschitz coefficients. *Journal of Mathematical Analysis and Applications*, 371(1):309 – 322, 2010.

[App09] David Applebaum. *Lévy processes and stochastic calculus*. Cambridge university press, 2009.

[BDM⁺11] Amarjit Budhiraja, Paul Dupuis, Vasileios Maroulas, et al. Variational representations for continuous time processes. In *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, volume 47, pages 725–747. Institut Henri Poincaré, 2011.

[BGT04] N. Burq, P. Gérard, and N. Tzvetkov. Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds. *American Journal of Mathematics*, 126 (3):569–605, 2004.

[BHR17] Z. Brzeźniak, E. Hausenblas, and P. Razafimandimby. Martingale solutions for stochastic equation of reaction diffusion type driven by Lévy noise or Poisson random measure. *Potential Analysis*, 2017. to appear.

[BHW17] Z. Brzeźniak, F. Hornung, and L. Weis. Martingale solutions for the stochastic nonlinear Schrödinger equation in the energy space. *arXiv preprint arXiv:1707.05610*, 2017.

[BHW18] Z. Brzeźniak, F. Hornung, and L. Weis. Uniqueness of martingale solutions for the stochastic nonlinear Schrödinger equation on 3d compact manifolds. *Preprint in preparation*, 2018.

[BHZ13] Z. Brzeźniak, E. Hausenblas, and J. Zhu. 2d stochastic Navier-Stokes equations driven by jump noise. *Nonlinear Analysis: Theory, Methods & Applications*, 79:122–139, 2013.

[Bil99] P. Billingsley. *Convergence of probability measures*. Wiley series in probability and statistics: Probability and statistics sectionA Wiley-Interscience publication. Wiley, New York, 2. ed. edition, 1999.

[BK03] S. Blunck and P.C. Kunstmann. Calderon-Zygmund theory for non-integral operators and the h-infinity functional calculus. *Revista Matematica Iberoamericana*, 19(3):919–942, 2003.

[BL76] J. Bergh and J. Löfström. *Interpolation spaces: an introduction*. Springer, 1976.

[BM14] Z. Brzeźniak and A. Millet. On the stochastic Strichartz estimates and the stochastic nonlinear Schrödinger equation on a compact riemannian manifold. *Potential Analysis*, 41(2):269–315, 2014.

[BM17] Z. Brzeźniak and U. Manna. Weak solutions of a stochastic Landau-Lifshitz-Gilbert equation driven by pure jump noise. *preprint*, 2017. submitted.

[BRZ14] V. Barbu, M. Röckner, and D. Zhang. Stochastic nonlinear Schrödinger equations with linear multiplicative noise: Rescaling approach. *Journal of Nonlinear Science*, 24(3):383–409, 2014.

[BRZ16] V. Barbu, M. Röckner, and D. Zhang. Stochastic nonlinear Schrödinger equations. *Nonlinear Analysis: Theory, Methods & Applications*, 136:168–194, 2016.

[BS14] F. Bernicot and V. Samoyeau. Dispersive estimates with loss of derivatives via the heat semigroup and the wave operator. *arXiv preprint arXiv:1407.4086*, 2014.

[CF17] I. Chevyrev and P. K. Friz. Canonical RDEs and general semimartingales as rough paths. *arXiv preprint arXiv:1704.08053*, 2017.

[CM18] K. Cheung and R. Mosincat. Stochastic nonlinear Schrödinger equations on tori. *arXiv preprint arXiv:1803.02817*, 2018.

[CP14]   A. Chechkin and I. Pavlyukevich. Marcus versus Stratonovich for systems with jump noise. *Journal of Physics A: Mathematical and Theoretical*, 47(34):342001, 2014.

[dBD99]   A. de Bouard and A. Debussche. A stochastic nonlinear Schrödinger equation with multiplicative noise. *Communications in Mathematical Physics*, 205(1):161–181, 1999.

[dBD03]   A. de Bouard and A. Debussche. The stochastic nonlinear Schrödinger equation in h 1. *Stochastic Analysis and Applications*, 21(1):97–126, 2003.

[dBH17]   A. de Bouard and E. Hausenblas. The nonlinear Schrödinger equation driven by jump processes. *arXiv preprint arXiv:1702.02523*, 2017.

[dBHO18]   A. de Bouard, E. Hausenblas, and M. Ondrejat. Uniqueness of the nonlinear Schrödinger equation driven by jump processes. *arXiv preprint arXiv:1805.10797*, 2018.

[DMN13]   S. Dirksen, J. Maas, and J.M.A.M Neerven. Poisson stochastic integration in Banach spaces. *Electronic Journal of Probability*, 18, 2013.

[Fol99]   G.B. Folland. *Real Analysis: Modern Techniques and Their Applications*. Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts. Wiley, 1999.

[Gub04]   M. Gubinelli. Controlling rough paths. *Journal of Functional Analysis*, 216(1):86–140, 2004.

[Hor18a]   F. Hornung. *Global solutions of the nonlinear Schrödinger equation with multiplicative noise*. PhD thesis, Karlsruhe Institute of Technology, 2018.

[Hor18b]   F. Hornung. The nonlinear stochastic Schrödinger equation via stochastic Strichartz estimates. *Journal of Evolution Equations*, Feb 2018.

[Hor18c]   L. Hornung. Strong solutions to a nonlinear stochastic Maxwell equation with a retarded material law. *Journal of Evolution Equations*, 2018.

[IW81]   N. Ikeda and S. Watanabe. *Stochastic Differential Equations and Diffusion Processes*. North-Holland mathematical library. North-Holland Publishing Company, 1981.

[Jak]   A. Jakubowski. On the Skorokhod topology. In *Annales de l'IHP Probabilités et statistiques*, volume 22.

[KU15]   P. C. Kunstmann and M. Uhl. Spectral multiplier theorems of Hörmander type on Hardy and Lebesgue spaces. *Journal of Operator Theory*, 73(1):27–69, 2015.

[Kun04]   H. Kunita. Stochastic differential equations based on Lévy processes and stochastic flows of diffeomorphisms. In *Real and stochastic analysis*, pages 305–373. Springer, 2004.

[Lio69]   J. L. Lions. *Quelques methodes de resolution des problemes aux limites non lineaires*. Dunod Paris, 1969.

[Mar81]   S.I. Marcus. Modeling and approximation of stochastic differential equations driven by semi-martingales. *Stochastics: An International Journal of Probability and Stochastic Processes*, 4(3):223–245, 1981.

[Mot12]   E. Motyl. Stochastic Navier-Stokes equations driven by Lévy noise in unbounded 3d domains. *Potential Analysis*, 38(3):863–912, 2012.

[Par67]   K. R. Parthasarathy. *Probability measures on metric spaces*. Probability and mathematical statistics ; 3. Acad. Press, New York [u.a.], 1967.

[PZ07]   S. Peszat and J. Zabczyk. *Stochastic Partial Differential Equations with Lévy Noise: An Evolution Equation Approach*. Encyclopedia of Mathematics an. Cambridge University Press, 2007.

[Tri92]   H. Triebel. *Theory of function spaces II*. Monographs in mathematics ; 84. Birkhäuser, Basel [u.a.], 1992.

[Uhl11]   M: Uhl. *Spectral multiplier theorems of Hörmander type via generalized Gaussian estimates*. PhD thesis, Karlsruhe Institute of Technology, 2011.

[VM10]   Javier Villarroel and Miquel Montero. On the effect of random inhomogeneities in Kerr media modelled by a nonlinear Schrödinger equation. *Journal of Physics B: Atomic, Molecular and Optical Physics*, 43(13):135404, 2010.

[VM11]   Javier Villarroel and Miquel Montero. On the integrability of the Poisson driven stochastic nonlinear Schrödinger equations. *Studies in Applied Mathematics*, 127(4):372–393, 2011.

Department of Mathematics, The University of York, Heslington, York YO10 5DD, UK
Email address: `zdzislaw.brzezniak@york.ac.uk`

Institute for Analysis, Karlsruhe Institute for Technology (KIT), 76128 Karlsruhe, Germany
Email address: `fabian.hornung@kit.edu`

School of Mathematics, Indian Institute of Science Education and Research Thiru-vananthapuram, Trivandrum 695016, INDIA