Manifolds with aspherical singular Riemannian foliations

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Für meine Eltern
A mis padres

ABSTRACT

In the present work we study A-foliations, i.e. singular Riemannian foliations with regular leaf aspherical. The main result is that, for a simply-connected closed $(n+2)$-manifold M, an A-foliation with regular leaves of codimension 2 in M is homogeneous. In other words it is given by a smooth effective action of the torus $\mathbb{T}^n$ on M by isometries.

We will give some conditions to compare two simply-connected, closed manifolds with A-foliations, up to foliated homeomorphism, via their leaf spaces.

Film is one of the three universal languages, the other two: mathematics and music.

— *Frank Cappra*

Non est regia ad Geometriam via.

— *Euclid.*

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CONTENTS

Everyone knows what a curve is,
until he has studied enough mathematics to become
confused through the countless number
of possible exceptions.

— Felix Klein

1 INTRODUCTION

When studying a Riemannian manifold M, an approach to understand its geometry or its topology is to simplify the problem by "reducing" M to a lower dimensional space B. This can be achieved by considering a partition of the original manifold M into submanifolds which are, roughly speaking, compatible with the Riemannian structure of M. We then study the geometry or topology of B, with the aim of recovering information on M.

As an example of this "reduction", we can consider *Riemannian submersions* from M onto lower dimensional manifolds. We then study the properties of M which remain invariant along the fibers of the submersion. A concrete example of this is present in [GG87] and [Wil01]. The authors prove that a closed, simply-connected, Riemannian manifold M with sectional curvature greater or equal to 1, and diameter equal to $\pi/2$ is either homeomorphic to a sphere, or isometric to a compact symmetric space of rank one (a so called *CROSS*). As a key step in the proof, they show that any Riemannian submersion $\pi\colon \mathbb{S}^n \to B$ onto some Riemannian manifold B is a Hopf fibration.

This "reduction" approach is also present when we consider Riemannian manifolds with an *effective isometric action by a compact Lie group*. In particular, this approach has been applied to the long-standing open problem in Riemannian geometry, of classifying and constructing Riemannian manifolds of positive or nonnegative (sectional) curvature. Namely Grove has proposed in the *symmetry program* to first consider such manifolds with a high degree of symmetry, i.e. with an isometric action of a compact Lie group (see [Gro02]).

The philosophy behind this approach is that by understanding first positively or nonnegatively curved manifolds with symmetry one may gain insight into the general case, either by constructing new examples or by finding possible obstructions. This has proved a successful approach, since many results have come to light by following loosely the symmetry program (see for example [Bre72], [Gro02], [Gro17], [Kob95], [Sea14],[Wil06]). This point of view has even been applied to other lower curvature bounds, such as positive Ricci (see for example [CGG16]), as it provides many tools and much flexibility.

Since, in particular, any compact connected Lie group contains a maximal torus as a Lie subgroup, the study of torus actions is of importance in the study of group actions. The classification up to equivariant diffeomorphism of smooth, closed, simply-connected, manifolds with torus actions is a well studied problem when either the dimension of the manifolds or the cohomogeneity of the action is low (see for example [OR70],[KMP74],[Fin77],[Oh83a],[Oh82]).

Both of these phenomena, Riemannian submersions and compact Lie group actions, are encompassed in the more general concept of *singular Riemannian foliations*. In Riemannian geometry, singular Riemannian foliations have recently attracted the attention of many authors (see, for example, the survey [ABT13]) and led to many interesting results.

Alexandrino has obtained information on the geometry of a manifold admitting certain types of singular Riemannian foliations, called *polar foliations* (see [Ale10, ABT13]). Singular Riemannian foliations have also led to results in differential topology, such as those surveyed in [QG16]. For example, one can obtain a lower bound on the number of distinct smooth structures a manifold with a singular Riemannian foliation can have. Also, as in the case of smooth effective torus actions, Radeschi and Ge obtained in [GR15] an explicit classification up to diffeomorphism of closed simply-connected 4-manifolds admitting a singular Riemannian foliation.

One main difference between group actions and foliations is that foliations may be less rigid (see for example [GR15]), not having several constraints natural to Lie groups. This in turn raises technical challenges, such as the fact that the leaves may carry non-standard smooth structures.

Thus an important problem in the setting of singular Riemannian foliations is to distinguish homogeneous foliations from non-homogeneous ones (see for example [GR15]). This problem does not become more tractable when the topology of the manifold is not complicated. Even in the case of spheres it is not clear how to distinguish *homogeneous foliations* (i.e. those coming from group actions), from non-homogeneous ones.

As a concrete example of this, Radeschi studied in his Ph.D. thesis ([Rad12]) singular Riemannian foliations on round spheres, and showed that when the singular foliation has positive dimension at most 3 they are homogeneous. In contrast when we assume that the foliation has large dimension, for example when the codimension of the singular Riemannian foliation $(\mathbb{S}^n, \mathcal{F})$ is 1, non-homogeneous foliations arise. The leaf space of a codimension one foliation $(\mathbb{S}^n, \mathcal{F})$ is isometric to the closed interval $[0, \pi/g]$ with $g \in \{1, 2, 3, 4, 6\}$ (see for example [Mü80]). For g equal to 1, 2, or 3, Cartan proved that such a foliation is homogeneous (see [Car38, Car39a, Car39b, Car40]), and asked if this was true for all codimension one foliations on $\mathbb{S}^n$. This is answered negatively, when we consider the case $g = 4$, which includes the majority of codimension 1 singular Riemannian foliations. In [FKM81], for $g = 4$, an infinite family of non-homogeneous codimension one foliations on round spheres called of *FKM type* were presented and studied. Following a similar approach, in [Rad14], Radeschi showed the existence of a large family of non-homogeneous singular Riemannian foliations on round spheres with arbitrary codimension.

One of the aims of the present work, is to extend results of the theory of transformation groups to the setting of singular Riemannian foliations. We will focus on the very general problem of comparing two different manifolds, each one endowed with a singular Riemannian foliation, via the leaf space, which is the topological space obtained as a quotient of the foliated manifold by the equivalence relation given by the foliation.

The results presented in this work may in turn be applied to the study of Riemannian manifolds with positive or nonnegative curvature, generalizing the Grove program to the context of singular Riemannian foliations, as first done in [GGR15]. The problem of comparing singular Riemannian foliations encompasses the problem of comparing manifolds with group actions.

To impose some control on this general problem, we impose some control on the topology of the leaves. Namely we focus on compact, simply-connected manifolds with a singular Riemannian foliations with closed aspherical leaves. This means that for any leaf L of such a foliation $\pi_i(L) = 0$ for $i \neq 1$. These type of singular Riemannian foliations are denoted as *A-foliations*, and they where introduced in [GGR15]. The concept of A-foliations are generalizations of smooth effective torus actions on smooth manifolds.

Galaz-García and Radeschi in [GGR15] give a classification up to foliated diffeomorphism of all compact, simply-connected manifolds with a codimension one A-foliations. They show that they are homogeneous, i.e. these foliations arise from torus actions. They also classify up to homeomorphism all compact, simply-connected, Riemannian manifolds of dimensions 4 and 5 with nonnegative sectional curvature that admit an A-foliation of codimension 2. For the 4-dimension case, from [GR15], it follows that the classification given in [GGR15] is up to diffeomorphism.

The main result of the present work is to prove that A-foliations of codimension 2 on compact, simply-connected manifolds are homogeneous up to foliated diffeomorphism.

Theorem G. *For a compact, simply-connected, Riemannian n-manifold M with $n \geqslant 4$, an A-foliation of codimension 2 is homogeneous.*

To be able to prove Theorem G we need to develop a method for comparing two compact, simply-connected manifolds with A-foliations. A technique for classifying up to homeomorphism compact, manifolds admitting a smooth effective Lie group action, is to compare the orbit space (see for example [OR70], [KMP74],[Fin77], [Oh83a]). We would like to apply the same idea to smooth manifolds admitting a singular Riemannian foliation. Let $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ be two compact manifolds admitting singular Riemannian foliations (not necessarily A-foliations). In order to be able to compare them by comparing their leaf spaces M_1^* and M_2^*, the existence of cross-sections $\sigma_i\colon M_i^* \to M_i$ for the quotient map $\pi_i\colon M_i \to M_i^*$, (i.e. $\pi_i \circ \sigma_i = \mathrm{Id}_{M^*}$) is extremely useful.

We show the existence of such cross-sections under certain technical topological conditions. We define the principal stratum of a singular Riemannian foliation $(M, \mathcal{F})$ to be the set of all leaves of maximal dimension with trivial holonomy and denote it by M_{prin}. The holonomy condition means that a small tubular neighborhood in M of a leaf in M_{prin} looks like a product of the leaf and a disk (a tube). We also assume that principal leaves are simple. Loosely this means that for a principal leaf L we have $\pi_k(L) = [\mathbb{S}^k, L]$ for all $k \geqslant 0$. We denote the mapping cylinder of $\pi\colon M \to M^*$ by M_π. We recall that there is an action of $\pi_1(M_{\mathrm{prin}})$ on the groups $\pi_k(M_\pi, M_{\mathrm{prin}})$. When this action is trivial, we call the pair $(M_\pi, M_{\mathrm{prin}})$ simple (this will be precised in section 4.1, or the reader can consult [Hat10, DK01]). With these concepts we find a family of obstructions to the existence of a cross-section over M_{prin}.

Theorem A. *Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation with M simply-connected, quotient map $\pi\colon M \to M^*$, and principal leaf L, which is simple and connected. Furthermore assume M^*_{prin} is simply-connected and (M_π, M_{prin}) is simple. Then there is a family of obstructions $\omega^1_k \in H^{k+1}(M^*_{\text{prin}}; \pi_k(L))$ such that a cross-section $\sigma\colon M^*_{\text{prin}} \to M_{\text{prin}}$ exists if $\omega^1_k = 0$ for all k.*

Next we consider the problem of extending a cross-section $\sigma\colon M^*_{\text{prin}} \to M_{\text{prin}}$ to the whole orbit space M^*. We show the existence of a second family of obstructions to the extension problem.

Theorem B. *Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation with M simply connected, and consider the quotient map $\pi\colon M \to M^*$. Furthermore assume that the homotopy fiber F_π is simple, and setting $A = M^*_{\text{prin}}$ assume there is already a defined cross-section $\sigma\colon A \to M_{\text{prin}}$. Then there is a family of obstructions $\omega^2_k \in H^{k+1}(M^*, A; \pi_k(F_\pi))$ such that a cross-section $\tilde{\sigma}\colon M^* \to M$ extending σ exists if $\omega^2_k = 0$ for all k.*

From these two theorems we get the following corollary, which gives a sufficient condition for the existence of a cross-section.

Corollary C. *Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation on a simply-connected manifold. Suppose that there is a section $\tilde{\sigma}\colon M^*_{\text{prin}} \to M_{\text{prin}}$, and the that hypotheses of Theorem B are satisfied. If M^*_{prin} has the same homotopy type of M^*, then the cross-section $\tilde{\sigma}$ can be extended to a section σ.*

We then proceed to study A-foliations on compact, simply-connected manifolds, namely the homeomorphism type of the leaves. We will prove that, except for possibly dimension 4, they are all homeomorphic to tori, or Bieberbach manifolds. This is due to the positive answer to the Borel conjecture for virtually Abelian groups, except in dimension 4 (see for example [FH83],[KL09]).

Conjecture (Borel conjecture)**.** *Given two aspherical closed topological manifolds X and Y, and $f\colon X\to Y$ a homotopy equivalence. Then f is homotopic to an homeomorphism.*

We also study the infinitesimal foliations of A-foliations, as well as the holonomy of the leaves, and propose a finer stratification of the manifold. Both of these concepts in the particular case of homogeneous foliations are encoded in the isotropy of a leaf. We define *the weights of the foliation*, which encode the information of the infinitesimal foliation and the holonomy. They generalize the weights of smooth effective torus actions (defined in [OR70], [Fin77], [Oh83a]), which encode the isotropy information of torus actions. In the case of existence of a cross-section, the weights characterize up to foliated homeomorphism the manifold.

Theorem D. *If $(M_1,\mathcal{F}_1)$ and $(M_2,\mathcal{F}_2)$ are compact, simply connected manifolds, with A-foliations, such that they have isomorphic weighted leaf spaces and admit cross-sections $\sigma_i\colon M_i^*\to M_i$, then $(M_1,\mathcal{F}_1)$ is foliated homeomorphic to $(M_2,\mathcal{F}_2)$.*

In the general setting of classifying manifolds with singular Riemannian foliations via cross-sections, the best one can obtain is a classification up to foliated homeomorphism. This is because the leaf spaces are only metric spaces (i.e. they may not even be topological manifolds). In the case of A-foliations of codimension 2 on compact, simply-connected manifolds, the authors in [GGR15] proved that the leaf space is homeomorphic to a 2-dimensional disk $\mathbb{D}^2$. For this case the boundary points of the leaf space correspond exactly to the singular leaves of $\mathcal{F}$. Since this leaf space satisfies the conditions of Corollary C, it follows from Theorem D that compact, simply-connected manifolds with A-foliations of codimension 2 are characterized up to foliated homeomorphism by the weights of the foliations.

Theorem E. *Let $(M_1,\mathcal{F}_1)$ and $(M_2,\mathcal{F}_2)$ be two compact, simply connected smooth $(n+2)$-manifolds, admitting singular A-foliations of codimension 2 and $n\geqslant 2$.*

Then M_1 is foliated homeomorphic to M_2 if and only if the weighted leaf spaces M_1^* and M_2^* are isomorphic.

Furthermore Oh in [Oh83a], shows that given a weighted 2-disk, with the weights satisfying some conditions, there is a procedure to construct a smooth manifold with an effective smooth action of cohomogeneity two realizing the weighted disk as an orbit space.

Theorem 2.22 [Oh83a]. *For $n \geqslant 2$ and a family of legal weights $(a_{i1}, \ldots, a_{in}) \in \mathbb{Z}^n$ there exists a closed, simply-connected $(n+2)$-manifold admitting a cohomogeneity two $\mathbb{T}^n$-action, that realizes the family $(a_{i1}, \ldots, a_{in})$ as weights.*

We will show that the weights of an A-foliation of codimension two on a compact, simply-connected manifold M are legal weights in the sense of Oh. Thus there is a torus action on M with the same weights as the foliation. By Theorem D we conclude that an A-foliation of codimension two on a compact, simply-connected manifold is, up to foliated homeomorphism, a homogeneous foliation.

Theorem F. *Let $(M, \mathcal{F}_1)$ be a compact, simply connected $(n+2)$-manifold with an A-foliation of codimension 2 and $n \geqslant 2$. Then there exists a closed, simply-connected $(n+2)$-manifold $(N, \mathcal{F}_2)$ with a homogeneous A-foliation of codimension 2 (i.e. with an effective smooth torus action of cohomogeneity 2), such that $(N, \mathcal{F}_2)$ is foliated homeomorphic to $(M, \mathcal{F}_1)$.*

As mentioned before in the problem of classifying manifolds with singular Riemannian foliations via cross-sections, the best one can obtain is a classification up to foliated homeomorphism. But in the case of A-foliations of codimension two on compact, simply-connected spaces, the leaf space is a 2-disk (see [GGR15]), and thus it is a smooth manifold with boundary in a unique way (it admits a unique smooth structure). So we can expect, in this case, to get a classification up to foliated diffeomorphism. The next obstruction is the existence of exotic smooth structures on torus (see for example [HS69], [HS70]). We study the diffeomorphism type of the leaves of an A-foliation of codimension two on a simply-connected smooth manifold, and prove that they are diffeomorphic to standard tori. With this we conclude the main theorem of this work:

Theorem G. *For $n \geqslant 2$, every A-foliation of codimension 2 on a compact, simply-connected Riemannian $(n+2)$-manifold M is homogeneous.*

We state the presentation order of the present work. In the first part of this work we will give an overview of the theory of Lie group actions and singular Riemannian foliations. In the second part, we will focus on the study of cross-sections and A-foliations on compact, simply-connected manifolds. In section 4.1 we give proofs for Theorem A, Theorem B and Corollary C. In section 4.4 we define the weights of an A-foliation and prove Theorem D. In chapter 5 we focus on the study of A-foliations of codimension 2 on simply-connected manifolds, and prove Theorem E, and Theorem F. Finally on chapter 6 we study the diffeomorphism type of the leaves of an A-foliation of codimension 2 on a simply-connected manifold, and finish the proof of Theorem G.

Part I

BACKGROUND

2 GROUP ACTIONS

In this chapter we review the theory of compact Lie group actions on smooth manifolds, stating some basic results, and then explain some results of Orlik, Raymond and Oh in [OR70, Oh83a]. For a more comprehensive presentation of the subject, the interested reader can consult [AB15, Bre72]

2.1 COMPACT LIE GROUP ACTIONS

Let us begin by stating the basic concepts and results for differentiable group actions.

If M is a smooth manifold and G is a Lie group with identity element e, a smooth *group action* of G on M is a smooth map

$$\mu\colon G \times M \to M,$$

such that for any elements g, h of G and any element x in M the following hold:

(i) $\mu(e, x) = x$,

(ii) $\mu(gh, x) = \mu(g, \mu(h, x))$.

By setting $\mu(g, x) = x$ for any $g \in G$ we see that there always exists a group action. This action is called the *trivial action*. From now on we will use the following more compact and generally used notation for group actions $g(x) := \mu(g, x)$. This actually defines a representation of G into $\mathrm{Diff}(M)$, the diffeomorphism group of M. Thus we may consider the *differential* Dg of an element $g \in G$ as the differential of the map $\mu_g \colon \{g\} \times M \to M$.

The *kernel of the action* μ is the closed normal subgroup

$$\ker \mu = \{g \in G \mid g(x) = x \text{ for all } x \in M\}.$$

The action μ is called *effective* if $\ker \mu$ is trivial. From the following proposition we see that from now on we may consider only effective group actions.

Proposition 2.1 (Proposition 1.1, Chapter I in [Bre72])**.** *Let μ be an action of G on M and set $N = \ker \mu$. Then there is a canonically induced effective action $\mu/\ker \mu$ of G/N on M.*

When working with compact Lie groups G one has, for the map μ, the following nice properties.

Theorem 2.2 (Theorem 1.2, Chapter I in [Bre72])**.** *If $\mu\colon G \times M \to M$ is an action of a compact Lie group G on M, then μ is a closed map.*

Corollary 2.3. *If G is a compact Lie group acting on M, then $G(A)$ is closed (compact) in M for each closed (compact) $A \subset M$.*

An action $\mu\colon G \times M \to M$ is *proper* if the map $\varphi\colon G \times M \to M \times M$ given by

$$\varphi(g, x) = (g(x), x),$$

is proper. We have the following characterization for proper actions.

Proposition 2.4 (Proposition 3.19 in [AB15])**.** *An action $\mu\colon G \times M \to M$ is proper if and only if for any sequence $\{g_n\}$ in G and any convergent sequence $\{x_n\}$ in M, such that $\{g_n(x_n)\}$ converges, the sequence $\{g_n\}$ has a convergent subsequence*

Proof. Suppose that the action is proper and $\{g_n\}$ is a sequence in G, $\{x_n\}$ is a sequence converging in M to x, such that $\{g_n(x_n)\}$ is a convergent sequence in M with limit y. Then the sequence $\{(g_n(x_n), x_n)\}$ is convergent in $M \times M$, with limit (y, x). The set $K = \{(g_n(x_n), x_n)\} \cup \{(y, x)\}$ is a compact subset of $M \times M$, and thus $\varphi^{-1}(K)$ is compact in $G \times M$. Since we have $\{g_n, x_n\} \subset \varphi^{-1}(K)$ there is a convergent subsequence of $\{g_n, x_n\}$ in $G \times M$ and thus there is a convergent subsequence of $\{g_n\}$ in G.

Conversely we assume that for any sequence $\{g_n\}$ in G and any convergent sequence $\{x_n\}$ in M, such that $\{g_n(x_n)\}$ converges, the sequence $\{g_n\}$ has a convergent subsequence. Take $K \subset M \times M$ compact. Now consider a sequence $\{g_n, x_n\}$ in $\varphi^{-1}(K)$. Then the sequence $\{g_n(x_n), x_n\}$ has a convergent subsequence $\{(g_{n_k}(x_{n_k}), x_{n_k})\}$ in K. Thus there is a convergent subsequence $\{g_{n_{k_i}}\}$ of $\{g_{n_k}\}$. Thus we have showed that for any sequence $\{g_n, x_n\}$ in $\varphi^{-1}(K)$ there is a convergent subsequence $\{(g_{n_{k_i}}, x_{n_{k_i}})\}$. $\square$

Corollary 2.5. *For an action $\mu\colon G \times M \to M$, if G is a compact group, then the action is proper.*

Proof. Every sequence $\{g_n\}$ of G has a convergent subsequence in G. $\square$

Now we consider two smooth manifolds M and N that have a group action $\mu\colon G \times M \to M$ and $\theta\colon G \times N \to N$. We say that a smooth function $f\colon M \to N$ is *equivariant* if for all $p \in M$ and all $g \in G$ the following is true:

$$f(g(x)) = g(f(x)).$$

If the equivariant smooth function $f\colon M \to N$ is also a diffeomorphism then we say that the action μ of G on M is *equivalent* to the action θ of G on N, and we say that M is *equivariantly diffeomorphic* to N.

The following two concepts are at the core of the study of group actions. First, for a point $p \in M$, we define the *orbit* of p under G as

$$G(p) = \{g(p) \mid g \in G\}.$$

We also define the *isotropy subgroup* of G at p as

$$G_p = \{g \in G \mid g(p) = p\},$$

i.e. the subgroup of all elements in G which fix p. Since $gG_pg^{-1}(g(p)) = g(p)$ we have that $gG_pg^{-1} \subset G_{g(p)}$, and conversely since $g^{-1}G_{g(p)}g(p) = p$ we have,

$$G_{g(p)} = gG_pg^{-1}. \tag{2.1.1}$$

Thus for a given point $p \in M$ its isotropy subgroup G_p changes by conjugation as the point p moves along its orbit $G(p)$. It is easy to prove that if two orbits $G(p)$ and $G(q)$ have non-empty intersection, then they coincide, and thus the orbits of the action of G on M form a partition of M.

Proposition 2.6. *For a smooth Lie group action $\mu\colon G \times M \to M$, the orbits of μ give a partition of M.*

Proof. Clearly since $p \in G(p)$ for any $p \in M$, no orbit is empty, and $M = \cup_{p\in M} G(p)$. Let $q, p \in M$ be such that $G(p) \cap G(q) \neq \emptyset$. Then there exists $x \in G(p) \cap G(q)$. Since x is an element in $G(p)$ we have by definition that $x = g_1(p)$ for some $g_1 \in G$. Analogously we have that $x = g_2(q)$ for some $g_2 \in G$. Thus $q = g_2^{-1}(x) = g_2^{-1}g_1(p)$, and therefore q is an element of $G(p)$. From this it follows that $G(q) \subset G(p)$. In a similar fashion, interchanging the role of p and q we prove that $G(p) \subset G(q)$ and thus we obtain that $G(p) = G(q)$. Thus we have proven that if we have two non-equal orbits, their intersection must be empty. $\square$

Hence we can consider the quotient given by the equivalence relation induced by the partition, namely,

$$M^* = M/G = \{G(p) \mid p \in M\},$$

which is called the *orbit space* of the action. The natural projection $\pi\colon M \to M/G$, given by $\pi(p) = G(p)$, is called the *quotient map*. We set a topology on M/G by declaring that $U \subset M/G$ is open if and only if its preimage $\pi{-}1(U) \subset M$ is open, i.e. the quotient topology. This implies that π is continuous and open. For compact groups G, the orbit space has reasonable properties.

Theorem 2.7 (Theorem 3.1 in [Bre72])**.** *If a compact group G acts on M, then*

(i) M^* is Hausdorff.

(ii) $\pi\colon M \to M^*$ is closed.

(iii) $\pi\colon M \to M^*$ is proper.

(iv) M is compact if and only if M^* is compact.

(v) M is locally compact if and only if M^* is locally compact.

The following proposition shows that for a proper action by a smooth Lie group the orbit of a point is an embedded submanifold.

Proposition 2.8 (Proposition 3.41 in [AB15])**.** *Let $\mu\colon G \times M \to M$ be a smooth Lie group action and $x \in M$. Define $\alpha_x\colon G/G_x \to M$ as $\alpha_x(gG_x) = g(x)$. Then α_x is a G-equivariant injective immersion with image $G(x)$. If in addition the action is proper, then α_x is an embedding, and $G(x)$ is an embedded submanifold of M.*

2.2 ORBIT TYPES

In this section we compare the orbits of an action by comparing the isotropy subgroups. We say that an orbit $G(x)$ is *a principal orbit* if there exists a neighborhood V of x in M such that for every $y \in V$ there exists some $g \in G$ such that $G_x \subset G_{g(y)}$. This definition does not depend on the choice of the representative of the orbit $G(x)$ since we have already showed that isotropy groups are conjugate along orbits. The principal orbits are the ones that have the smallest isotropy subgroup among nearby orbits. The existence of principal orbits is guaranteed by [Bre72, IV Theorem 3.1] and [AB15, Theorem 3.82] (see Theorem 2.9).

If $G(x) = G/G_x$ is a principal orbit and $G(y) = G/G_y$ is another orbit then, by definition, we have that G_x is conjugate to a subgroup of G_y, and so we may assume without loss of generality that $G_x \subset G_y$. We consider the map $p\colon G(x) \to G(y)$, given by $p(g(x)) = g(y)$. This map is an equivariant surjection, and in fact it is a fiber bundle projection with fiber G_y/G_x. If $G(x)$ is a principal orbit, we say that $G(y)$ is a *exceptional orbit* if $\dim(G(x)) = \dim(G(y))$, but they are not equivalent, meaning that G_y/G_x is a finite nontrivial group. In case that $\dim(G(y)) < \dim(G(x))$ we say that $G(y)$ is a *singular orbit*. We denote here as P the set of all principal orbits, E the set of all exceptional orbits and Q the set of all the singular orbits.

We can define a relation between different isotropy subgroups. If H is an isotropy group of G we say that an orbit $G(x)$ has *type* (H) if G_x is conjugate to H in G. Since along an orbit the isotropy subgroups are conjugated, the type is well defined. For K, another isotropy subgroup of G, we say that the orbit type (H) is less than or equal to the type of orbit (K) and we denote it by $(H) \prec (K)$, if K is conjugate to a subgroup of H in G. With this relation we see that if $G(x)$ is a principal orbit, then there exists an open neighborhood V of x in M such that for every $y \in V$ we have that $(G_y) \prec (G_x)$. We say that two orbits $G(x)$ and $G(y)$ have *the same orbit type* if $(G_x) \prec (G_y)$ and $(G_y) \prec (G_x)$. With the following theorem we ensure the existence of principal orbits, and that there is only one principal orbit type

Theorem 2.9 (Principal Orbit Theorem 3.82 in [AB15])**.** *Denote by M_{princ} the set of points contained in principal orbits. Then the following hold:*

(i) *M_{princ} is open and dense in M.*

(ii) *The subset M_{princ}/G of M/G is a connected manifold.*

(iii) *If $G(x)$ and $G(y)$ are principal orbits, then there exists $g \in G$ such that $G_x = G_{g(y)}$.*

Now we look at the decomposition of the tangent space of the orbit $G(p)$ at the point p. We say that an embedded submanifold S_p of M containing p is a *slice* at p if it satisfies the following properties:

(i) $T_pM = T_pG(x) \oplus T_pS_p$.

(ii) $T_xM = T_xG(x) + T_xS_p$ for all $x \in S_p$.

(iii) S_p is invariant under the action of G_p, i.e. if $x \in S_p$ and $g \in G_p$, then $g(x) \in S_p$.

(iv) If $x \in S_p$ and $g \in G$ are such that $g(x) \in S_p$, then $g \in G_p$.

Thus we have a way to decompose at each point p the tangent space T_pM. We will refer to T_pS_p as the *normal space* of $G(p)$ at p. The existence of a slice is guaranteed by the following theorem.

Theorem 2.10 (Slice Theorem, Theorem 3.49 in [AB15])**.** *For any compact (proper) group action $\mu\colon G \times M \to M$ there exists a slice S_{x_0} at x_0 for any $x_0 \in M$.*

Let $\mu\colon G \times M \to M$ be a proper smooth action, for $x_0 \in M$ fixed, we define a *tubular neighborhood* of the orbit $G(x_0)$ to be the image of S_{x_0}, the slice through x_0, under the G-action:

$$\mathrm{Tub}(G_(x_0)) = \mu(G, S_{x_0}).$$

By the following theorem the tubular neighborhood $\mathrm{Tub}(G(x_0))$ is the total space of a fiber bundle.

Theorem 2.11 (Tubular Neighborhood Theorem, Theorem 3.57 in [AB15]). *Let $\mu\colon G \times M \to M$ be a smooth proper action. For every point x_0 in M there exist a G-equivariant diffeomorphism between $\mathrm{Tub}(G_(x_0))$ and the total space of the S_{x_0}-fiber bundle with,*

$$S_{x_0} \to G \times_H S_{x_0} \to G/H,$$

associated to the principal bundle $H \to G \to G/H$. Here $H = G_{x_0}$ is the isotropy subgroup at x_0.

If we now consider $x \in M$ we can get an invariant tubular neighborhood around the orbit $G(x)$, considering the slice S_x through x and setting

$$\mathrm{Tub}(G(x)) = G(S_x).$$

Furthermore in the tubular neighborhood we will have a finite number of orbit types.

Theorem 2.12 (Theorem 3.91 in [AB15]). *For a compact group action, for every $x \in M$ there exists a slice S_x at x, such that the tubular neighborhood $G(S_x)$ contains only finitely many different orbit types.*

Thus when the manifold M is compact, since we can cover it by a finite number of such tubular neighborhoods, we conclude that the number of orbit types is finite.

Corollary 2.13. *If M is a compact manifold and G is a compact group acting on M, then the number of orbit types is finite.*

For a compact group action $\mu\colon G \times M \to M$ we have that for each $g \in G$ we can define a diffeomorphism μ_g of M as $\mu_g(x) = g(x)$. Now fix a point $p \in M$ and consider $g \in G_p$, so that μ_g fixes p. Thus for each point $p \in M$ we can define an action $\tilde{\mu}\colon G_p \times T_pM \to T_pM$, by setting for $g \in G_p$ and $v \in T_pM$ the function $\tilde{\mu}$ as follows,

$$\tilde{\mu}(g, v) = D_p(\mu_g)(v),$$

where $D_p(\mu_g)$ is the derivative of μ_g at p. If we consider the slice S_p at p, since it is invariant under the action of G_p, we have that G_p acts on the normal space of $G(p)$ at p a via $\tilde{\mu}$. This action is called the *isotropy representation* of the action at p.

We end this section by reviewing the following concept. For an action of G over an n-dimensional manifold M we define the *cohomogeneity* as the codimension of a principal orbit (which has maximal dimension). If the cohomogeneity is small enough we have the following lemma that helps us to understand the orbit space M^*.

Lemma 2.14 (Chapter IV, Lemma 4.1 in [Bre72])**.** *For a compact group action G over M, if the cohomogeneity is less than or equal to 2 then the orbit space M^* is a manifold with boundary of dimension equal to the cohomogeneity.*

From now on we will be working with cohomogeneity 2 group actions.

2.3 TORUS ACTIONS

Since every compact Lie group G contains a unique maximal torus, torus actions play an important role in the study of compact group actions. In this section we will concentrate on the case where the group G acting effectively is an n-torus and the $(n+2)$-manifold M is simply connected and closed. Following the notation of [BFJ16], we will denote the standard n-torus by $\mathbb{T}^n$, and the circle group by $\mathbb{T}^1$, to distinguish it from the circle $\mathbb{S}^1$ as topological space (i.e. without a group structure).

Proposition 2.15. *An effective action of $\mathbb{T}^n$ on a simply-connected $(n+2)$-manifold has cohomogeneity 2.*

Proof. The trivial subgroup $\{e\} \subset \mathbb{T}^n$ is an isotropy subgroup of the action for some point $p \in M$, and since $\{e\} \subset G_x$ for any point x, we have that the orbit is principal. Thus we have from 2.8 that $\mathbb{T}^n(P)$ is diffeomorphic to $\mathbb{T}^n$ and therefore the action has cohomogeneity 2. □

In general for an action of cohomogeneity 2, the following theorem tells us the orbit space structure.

Theorem 2.16 (Chapter IV, Theorem 8.6 in [Bre72])**.** *If G is a connected Lie group acting on a compact simply-connected manifold M with cohomogeneity 2 and there exists a singular orbit, then the set of exceptional orbits is empty and the orbit space M^* is a 2-disk whit boundary Q^*.*

Furthermore from [KMP74, Theorem 1.3] and [GGK14] the only possible non-trivial isotropy subgroups of an effective action of $\mathbb{T}^n$ on M, are $\mathbb{T}^1$ and $\mathbb{T}^2$. The boundary circle Q^* is a union of $m \geqslant n$ arcs by [KMP74, Corollary 1.7]. The interior points of the arcs corresponds to orbits with isotropy $\mathbb{T}^1$ and the end points correspond to orbits with isotropy $\mathbb{T}^2$, as shown in Figure 2.1.

Now considering the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ as the quotient of $\mathbb{R}^n$ by an integer lattice, we define a *circle subgroup* $G(a_1, \dots, a_n)$ of $\mathbb{T}^n$ as a the projection under $\mathbb{R}^n \to \mathbb{T}^n$ of a line in the direction of $(a_1, \dots, a_n) \in \mathbb{Z}^n$. The vectors $(1, 0, \dots, 0)$ and $(2, 0, \dots, 0)$ represent the same circle subgroup in $\mathbb{T}^n$, so in order to represent uniquely the possible circle subgroups, the integers $a_1, a_2, \dots, a_n$ must be relatively prime.

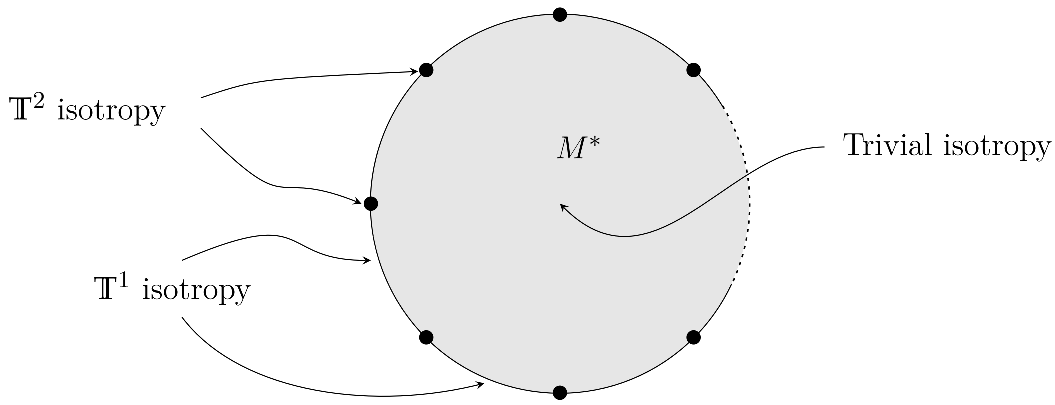

Figure 2.1.: Orbit space structure of a cohomogeneity-two torus action on a closed, simply-connected manifold.

By the *determinant of n-circle subgroups*

$$G(a_{11}, a_{12}, \ldots, a_{1n}), G(a_{21}, a_{22}, \ldots, a_{2n}), \ldots, G(a_{n1}, a_{n2}, \ldots, a_{nn}),$$

of T^n we mean the determinant of the matrix:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

The determinant of n-circles subgroups characterizes the intersection of two circle subgroups, as seen in the following lemma.

Lemma 2.17 (Lemma 1.2 in [Oh83a]). *Two circle subgroups $G(a_1, \ldots, a_n)$ and $G(b_1, \ldots, b_n)$ of $\mathbb{T}^n$ have trivial intersection if and only if there exist $n-3$ vectors $G_i \in \mathbb{Z}^n$ with $i = 3, \ldots, n$, such that the determinant of $(a_1, \ldots, a_n), (b_1, \ldots, b_n)$, $, G_3, \ldots, G_n$ is ± 1.*

Furthermore the determinant of a family of integer vectors

$$\{(a_{11},\ldots,a_{1n}),(a_{21},\ldots,a_{2n}),\ldots,(a_{n1},\ldots,a_{nn})\}\subset\mathbb{Z}^n$$

tells us when do they generate a torus.

Lemma 2.18 (Lemma 1.4 in [Oh83a]). *The n-circles $G(a_{11},\ldots,a_{1n})$, $,G(a_{21},\ldots,a_{2n}),\ldots,G(a_{n1},\ldots,a_{nn})$ generate $\mathbb{T}^n$ if and only if the determinant of the n circles is ± 1.*

We order the edges of Q^* and label them by $\gamma_1,\ldots,\gamma_m$. Next we show that for two orbits $G(x)$ and $G(y)$ that project to the same edge γ_i under $\pi\colon M\to M/G$, they have the same type $G(a_{i1},\ldots,a_{in})$. We also now that for a vertex F_i the circle subgroups of γ_i and γ_{i+1} are subgroups of the isotropy subgroup $\mathbb{T}^2$ at F_i (see [KMP74, Theorem 1.3]).

The following proposition tells us that for a $\mathbb{T}^n$-action of cohomogeneity two, there exist at least n circle subgroups, which are isotropy groups of some orbits, and they generate the n-torus.

Proposition 2.19 (Corollary 1.7 in [KMP74]). *If $\mathbb{T}^n$ acts effectively on a simply-connected closed $(n+2)$-manifold M, then all isotropy subgroups generate the whole group $\mathbb{T}^n$ and there are at least n different circle isotropy subgroups of $\mathbb{T}^n$.*

Combining Lemma 2.17 with Proposition 2.19 we see that actually for the vertex F_i the isotropy subgroup $\mathbb{T}^2$ is generated by the isotropy circle subgroups $G(a_{i1},\ldots,a_{in})$ and $G(a_{i+11},\ldots,a_{i+1n})$ associated to γ_i and γ_{i+1} respectively. Thus the circle subgroups carry all the isotropy information of the action. We define the *weights of orbit space* M^* for the action of $\mathbb{T}^n$ as the isotropy circle subgroups $G(a_{i1},\ldots,a_{in})$. These weights have a crucial role in solving the problem of classifying the smooth simply-connected closed $(n+2)$-manifolds M that admit an effective action by $\mathbb{T}^n$. The following theorems show that the action is classified by the weights previously defined.

Theorem 2.20 ([OR70, KMP74, Oh83a]). *For an effective $\mathbb{T}^n$ action on a simply-connected closed $(n+2)$-manifold M, the orbit map $\pi\colon M \to M^*$ has a cross-section.*

We will give an alternate proof of the previous theorem in Section 4.1. Using the existence of cross-sections we can prove the following theorem, which implies that cohomogeneity-two torus actions on simply-connected, closed, smooth manifolds are classified by the weights. A proof of this theorem will also be given in Section 4.1.

Theorem 2.21 ([OR70, KMP74, Oh83a]). *Let $\mathbb{T}^n$ act effectively on the simply-connected closed $(n+2)$-manifolds M and N. There is an equivariant diffeomorphism f of M onto N if and only if there is a weight preserving diffeomorphism f^* of M^* onto N^*.*

Now consider the disk D^2 and split its circle boundary into m edges, and we order them. If we attach to each edge an n-tuple $(a_{i1}, \dots, a_{in}) \in \mathbb{Z}^n$ with $\gcd(a_{i1}, \dots, a_{in}) = 1$, we say that D^2 is *legally weighted* with weights

$$(a_{11}, \dots, a_{1n}), (a_{21}, \dots, a_{2n}), \dots, (a_{m1}, \dots, a_{mn}),$$

if any two adjacent vectors $(a_{i1}, \dots, a_{in})$ and $(a_{i+11}, \dots, a_{i+1n})$ have trivial intersection in the sense of lemma 2.17. Oh showed in [Oh83a] that given a legally weighted disk, there exists a simply-connected closed $(n+2)$ manifold M and an effective action of $\mathbb{T}^n$ on M that has as weighted orbit space M^* the weighted disk we started with. This means that having legal weights is a sufficient and necessary condition to characterize simply-connected, compact, $(n+2)$-manifolds admitting a $\mathbb{T}^n$-action.

Theorem 2.22 (Remark 4.7 in [Oh83a]). *For $n \geqslant 2$ and a family of legal weights $(a_{i1}, \dots, a_{in}) \in \mathbb{Z}^n$ there exists a closed, simply-connected $(n+2)$-manifold admitting a cohomogeneity two $\mathbb{T}^n$-action that realizes the family $(a_{i1}, \dots, a_{in})$ as weights.*

We end this section and this chapter presenting the topological classification for simply-connected closed $(n+2)$-manifold M that admit an effective $\mathbb{T}^n$ action.

Theorem 2.23. *If $\mathbb{T}^n$ acts effectively on a simply-connected closed $(n+2)$-manifolds M then the following hold:*

(i) [OR70] If $n = 2$, then M is a connected sum of $\mathbb{S}^4$, $\pm\mathbb{C}P^2$ and $\mathbb{S}^2 \times \mathbb{S}^2$.

If there are k orbits with isotropy subgroup $\mathbb{T}^2$, and $w_2(M)$ denotes the second Stiefel-Whitney class of M, we have that

(ii) [Oh83a] If $n = 3$, then $k \geqslant 3$ and M is diffeomorphic to

- $\mathbb{S}^5$ *if* $k = 3$.
- $\#(k-3)(\mathbb{S}^3 \times \mathbb{S}^2)$ *if* $w_2(M) = 0$.
- $(\mathbb{S}^3 \tilde{\times} \mathbb{S}^2)\#(k-4)(\mathbb{S}^3 \times \mathbb{S}^2)$ *if* $w_2(M) \neq 0$,

where $\mathbb{S}^3 \tilde{\times} \mathbb{S}^2$ is the nontrivial $\mathbb{S}^3$ bundle over $\mathbb{S}^2$.

(iii) [Oh82] If $n = 4$, then $k \geqslant 4$ and we have that M is diffeomorphic to

- $\#(k-4)(\mathbb{S}^4 \times \mathbb{S}^2)\#(k-3)(\mathbb{S}^3 \times \mathbb{S}^3)$ *if* $w_2(M) = 0$.
- $(\mathbb{S}^4 \tilde{\times} \mathbb{S}^2)\#(k-5)(\mathbb{S}^5 \times \mathbb{S}^2)\#(k-3)(\mathbb{S}^3 \times \mathbb{S}^3)$ *if* $w_2(M) \neq 0$,

where $\mathbb{S}^4 \tilde{\times} \mathbb{S}^2$ is the nontrivial $\mathbb{S}^4$ bundle over $\mathbb{S}^2$.

In the case of 4-dimensional manifolds the classification of diffeomorphism type is obtained as follows. First, the authors show that there are 7-basic pieces into which any legally weighted disk can be decomposed. This decomposition may not be unique, with each piece corresponding to one of the basic configurations. Next it is shown that for each basic piece there exists a unique diffeomorphism type of a 4-dimensional, closed, simply-connected manifold with a torus action, whose orbit space is the given basic piece. Namely to each basic piece, the smooth manifold with a torus action realizing this piece as an orbit space is one of $\mathbb{S}^4$, $\pm\mathbb{C}P^2$, $\mathbb{S}^3 \times \mathbb{S}^2$ or

$\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$ (see [OR70, Table at pp. 552]). Last it is proved that the decomposition of the orbit space, given by the pieces, corresponds to a connected sum decomposition of the original manifold, by the previous list of manifolds (see [OR70, Theorem 5.7]). It is also observed that the connected sum presentation is not unique, i.e. it is not equivariant (see [OR70, Remark 5.8]).

For the case of 5-dimensional manifolds, the classification of diffeomorphism type is achieved by showing that the number of orbits with circle isotropy is the rank of the second homology group (see [Oh83a, Lemma 5.4]), and explicitly constructing two families of closed, simply-connected 5-manifolds with a $\mathbb{T}^3$-action, realizing all the second homology groups. In the first one, all manifolds have non-trivial second Stiefel-Whitney class, while in the second one they have trivial second Stiefel-Whitney class. The classification follows then from the work of Barden-Smale, in which they show that the second homology group and the second Stiefel-Whitney class classify closed, simply-connected 5-manifolds up to diffeomorphism (see [Bar65, Sma62] and [Oh83a, Theorem 5.5]).

The 6-dimensional case is done as follows. As in the 4-dimensional case, we can show the existence of basic closed, simply-connected, 6-manifolds which determine basic legally weighted orbit spaces. Then we prove that any legally weighted orbit space is obtained via an inductive process from these basic pieces. By computing the homology groups, the first Pontryagin classes, and a trilinear form associated to the manifolds obtained via the inductive process, we can apply classification theorems by Wall and Jupp to obtain the explicit list (see [Jup73, Wal66]).

2.4 ISOMETRIC ACTIONS

For a Riemannian manifold (M, g) we say that a Lie group action $\mu \colon G \times M \to M$ is an *action by isometries*, or an *isometric action*, if for each $h \in G$ the diffeomorphisms μ_h are isometries, i.e. if for any $X, Y \in TM$ the following occurs:

$$g(X, Y) = g(D\mu_h(X), D\mu_h(Y)).$$

In this case we also say that the Riemannian metric g is *G-invariant*.

Recall that for any metric g on a smooth manifold M since $\mathrm{Isom}(M, g)$ is a subset of $C^\infty(M, M)$, the set of all smooth functions from M to M, by endowing $C^\infty(M, M)$ with the *open-compact topology*, we can give a topology to $\mathrm{Isom}(M, g)$, which we also call the open-compact topology. With this topology on $\mathrm{Isom}(M, g)$ Myers and Steenrod showed that the isometry group of a Riemannian manifold is a Lie group.

Theorem 2.24 ([MS39])**.** *Let (M, g) be a Riemannian manifold. Any closed subgroup of $\mathrm{Isom}(M, g)$ with the compact-open topology is a Lie group. In particular, $\mathrm{Isom}(M, g)$ is a Lie group.*

Furthermore they show that in the particular case when M is compact, the group $\mathrm{Isom}(M, g)$ is a compact Lie group. This implies, via the following theorem, that $\mathrm{Isom}(M, g)$ always contains a torus as a subgroup.

Theorem 2.25 (Maximal Torus Theorem, Theorem 4.1 in [AB15])**.** *Let G be a connected, compact Lie group. Then the following hold:*

(i) There exists a maximal torus T in G;

(ii) Any two maximal tori in G are conjugate;

(iii) Every element of G is contained in a maximal torus.

Furthermore given a smooth group action μ of G on M, the following result shows that there exists a metric under which the given action μ is an isometric action.

Theorem 2.26 (Theorem 3.65 in [AB15])**.** *Given a proper smooth action $\mu\colon G \times M \to M$, there exists a G-invariant metric g on M such that G is a closed subgroup of* $\mathrm{Isom}(M, g)$.

Isometric actions have been successfully used to characterize Riemannian manifolds admitting nonnegative or sectional positive curvature as proposed by the so-called *Grove Program* (see [Gro02]). The general idea is the following: assume (M, g) is a Riemannian manifold with nonnegative (positive) curvature, and assume that a given Lie group G is contained in the isometry group $\mathrm{Isom}(M, g)$. In other words we assume that we have a smooth faithful representation $\rho\colon G \to \mathrm{Isom}(M, g)$. From this we can use several results from smooth group actions to characterize the diffeomorphism type of M. For example in dimension 4 Hsiang and Kleiner showed that if M is a simply-connected, 4-dimensional Riemannian manifold of positive sectional curvature, and the circle T^1 acts by isometries, then M is homeomorphic to either $\mathbb{S}^4$ or $\mathbb{C}P^4$. This can in fact be improved to diffeomorphism (see [GW14]).

Since in particular for a compact Riemannian manifold (M, g), as observed above, the group $\mathrm{Isom}(M, g)$ contains a torus as a closed subgroup, we may consider this torus as the subgroup G and try to deduce some properties about M. As an example in this direction, Galaz-García and Searle showed in [GGS14] the following.

Theorem 2.27 (Theorem A in [GGS14])**.** *Let M be a closed, simply connected, non-negatively curved 5-manifold. If T^2 acts isometrically and effectively on M, then M is diffeomorphic to one of $\mathbb{S}^5$, $\mathbb{S}^3 \times \mathbb{S}^2$, $\mathbb{S}^3 \tilde{\times} \mathbb{S}^2$, or the Wu manifold $SU(3)/SO(3)$.*

We observe that this approach may be somewhat limited since, by the work of Ebin, for a fixed smooth manifold M most Riemannian metrics g on M have trivial isometry group (see [Ebi70]). To understand how constrained this approach may be, any type of answer (either positive or negative) to the following problem is required.

Problem 2.28 (Problem 5.5 in [Gro02])**.** *Do simply-connected manifolds of non-negative or more generally almost nonnegative curvature have positive symmetry degree?*

In another direction, there is the following question: Given a smooth group action G on a smooth manifold M, does a G-invariant Riemannian metric g exists on M, which admits given lower curvature bounds?

For actions with low cohomogeneity several positive answers have been given, in particular for positive Ricci curvature. For example in [GZ02] the following theorem was proved for cohomogeneity one actions.

Theorem 2.29. *A compact cohomogeneity one manifold admits an invariant metric with positive Ricci curvature if and only if its fundamental group is finite.*

For a torus action of cohomogeneity two, on a simply-connected, closed, smooth manifold the following was proved in [CGG16].

Theorem 2.30. *If M is a closed, simply-connected smooth $(n+2)$-manifold with a smooth, effective action of a torus T^n, then there exists a T^n-invariant Riemannian metric on M with positive Ricci curvature.*

In particular from the previous theorem it follows that all spaces in Theorem 2.23 admit an invariant metric with positive Ricci curvature. Since in dimensions 5 and 6 we have an explicit list we get the following corollary (see [CGG16]).

Corollary 2.31. *For every integer $k \geqslant 4$, every connected sum of the form*

$$\#(k-3)(\mathbb{S}^2 \times \mathbb{S}^3), \tag{2.4.1}$$

$$(\mathbb{S}^2 \tilde{\times} \mathbb{S}^3)\#(k-4)(\mathbb{S}^2 \times \mathbb{S}^3), \tag{2.4.2}$$

$$\#(k-4)(\mathbb{S}^2 \times \mathbb{S}^4)\#(k-3)(\mathbb{S}^3 \times \mathbb{S}^3), \tag{2.4.3}$$

$$(\mathbb{S}^2 \tilde{\times} \mathbb{S}^4)\#(k-5)(\mathbb{S}^2 \times \mathbb{S}^4)\#(k-3)(\mathbb{S}^3 \times \mathbb{S}^3), \tag{2.4.4}$$

has a metric with positive Ricci curvature invariant under a cohomogeneity-two torus action.

Remark 2.32. In dimension 4, this result was known by the work of Bazaĭkin and Matvienko in [BkM07].

3 RIEMANNIAN FOLIATIONS

In this chapter we review concepts and results of singular Riemannian foliations and discuss their relation to group actions. Our main references are [ABT13], [GGR15], [MM03], [Mol88], and [MR18].

3.1 SINGULAR RIEMANNIAN FOLIATIONS.

A *Singular Riemannian Foliation* on a Riemannian manifold M, which we denote by $(M, \mathcal{F})$, is the decomposition of M into a collection $\mathcal{F} = \{L_p \mid p \in M\}$ of disjoint connected, complete, immersed submanifolds L_p, called *leaves*, which may not be of the same dimension, such that (see [ABT13]):

(i) Every geodesic meeting one leaf perpendicularly, stays perpendicular to all the leaves it meets.

(ii) For each point $p \in M$ there exist local smooth vector fields spanning the tangent space of the leaves.

If $(M, \mathcal{F})$ satisfies the first condition, then we say $(M, \mathcal{F})$ is a *transnormal system*. If it satisfies the second one, we say $(M, \mathcal{F})$ is a *singular foliation*. When the dimension of the leaves is constant, we say the foliation is a *regular Riemannian foliation* or just a *Riemannian foliation*. In the remarks at the end of [Wil07] it is stated that for a Riemannian manifold M, if $\mathcal{F}$ is a partition which is a transnormal system, then there are Lipschitz continuous vector fields spanning the tangent spaces of the leaves, i.e. $\mathcal{F}$ is a Lipschitz foliation. It is a question of interest to know if this can be improved in the following sense:

Question. *Is any transnormal system a smooth foliation, that is a foliation where the vector fields spanning it are smooth?*

These are some examples of singular Riemannian foliations:

1. Given (M, g) a Riemannian manifold, define a singular foliation by letting each point be a leaf. This will be a singular Riemannian foliation, which is one of the two *trivial foliations*. The other one is taking the foliation given by one single leaf, namely M.

2. If $(M, \mathcal{F})$ is a regular Riemannian foliation, then the foliation we obtain by taking the closure of the leaves, denoted by $\overline{\mathcal{F}}$, is a singular Riemannian foliation (see [ABT13, Mol88]).

3. If G is a compact Lie group acting on M by isometries, then the orbits of the action give a singular Riemannian foliation. These foliations are called *homogeneous foliations* (see section 3.4).

Another family of interesting examples is given by *isoparametric submanifolds*. Given an immersed isoparametric submanifold $L \subset M$, i.e. a codimension one submanifold with constant mean curvature, we can partition the ambient manifold into the submanifolds parallel to L, which are all isoparametric unless they lie on the focal set of L. In this case they have lower dimension. This partition gives a singular Riemannian foliation of M (see [AR16])

We say that a singular Riemannian foliation is *closed* if all the leaves are closed. The *dimension* of a foliation $\mathcal{F}$, denoted by $\dim \mathcal{F}$, is the maximal dimension of the leaves. The *codimension* of a foliation is the codimension of the maximal dimensional leaves, that is,

$$\mathrm{codim}(M, \mathcal{F}) = \dim M - \dim \mathcal{F}.$$

The leaves of maximal dimension are called *regular leaves* and the leaves that do not have maximal dimension are called *singular leaves*. Points on regular leaves are called *regular points* and points on singular leaves are called *singular points*. Since $\mathcal{F}$ gives a partition of M, for each point $p \in M$ there is a unique leaf, which we denote by L_p, that contains p and we say L_p is the *leaf through* p. Furthermore as with group actions, from the partition, we can consider the quotient space $M/\mathcal{F}$ which we call the *leaf space*. The quotient map $\pi\colon M \to M/\mathcal{F}$ associated to it, is the *leaf projection map*. As with group actions we will denote the image of a subset N of M under the projection map π by N^*. For a point $p \in M$, observe that $p^* = L_p^*$, by the definition of π. Since M carries by definition a Riemannian metric g, in the case where the singular Riemannian foliation $\mathcal{F}$ is closed, the quotient map π induces a metric d^* on M^*. With this metric the quotient π becomes a *submetry*. Recall that a submetry is a map that for any point p and $\varepsilon > 0$ small enough, it sends $B_\varepsilon(p)$, the ball of radius $\varepsilon > 0$ around p, to $B_\varepsilon(\pi(p))$. Furthermore, for a closed singular Riemannian foliation $(M, \mathcal{F})$, if the Riemannian manifold (M, g) is complete and has sectional curvature bounded below by $\lambda \in \mathbb{R}$, then (M^*, d^*) is an *Alexandrov space* of curvature also bounded below by λ (see for example [LT10, BBI01]). This means there M^* is a complete length space, and the curvature is defined via comparison triangles.

For a singular Riemannian foliation $(M, \mathcal{F})$, the foliation gives a stratification of M. For $k \leqslant \dim \mathcal{F}$ we define the k-dimensional stratum as:

$$\Sigma_{(k)} = \{p \in M \mid \dim L_p = k\}.$$

The *regular stratum* $\Sigma_{\text{reg}} = \Sigma_{(\dim \mathcal{F})}$ is an open, dense and connected submanifold of M (see [Rad12, Lemma 2.2.2]). The foliation restricted to the regular stratum yields a Riemannian foliation $(\Sigma_{\text{reg}}, \mathcal{F})$, and Σ^*_{reg} is open and dense in the leaf space M^*. Furthermore by Proposition 3.7 in [Mol88], if $(M, \mathcal{F})$ is a singular Riemannian foliation with compact closed regular leaves, then Σ^*_{reg} is an orbifold. Note that the foliation is regular if and only if $\Sigma_{\text{reg}} = M$. A leaf $L \subset M$ is called *regular* if $\dim L = \dim \mathcal{F}$, and *singular* otherwise.

To close this section we mention an interesting type of singular Riemannian foliations, called *polar foliations* (some authors refer to them as *singular Riemannian foliations with sections*), which has recently attracted the attention of some authors (see for example [Ale04, AG07, ABT13]). A singular Riemannian foliation $(M, \mathcal{F})$ is *polar* if, for each regular point p in M, there is an immersed submanifold Σ_p containing p, called a *section* through p, such that its dimension is equal to the codimension of the foliation, it intersects all the leaves, and it is orthogonal to all the leaves. When the polar foliation is given by a group action (i.e. it is homogeneous), we say it is a *polar group action*. When we consider the distribution $\mathcal{D}$ normal to the leaves on M, then by the following theorem the condition of being polar is equivalent to $\mathcal{D}$ being an integrable distribution.

Theorem 3.1 (Theorem 1.4 in [Ale06])**.** *Let $\mathcal{F}$ be a singular Riemannian foliation on a complete Riemannian manifold M. If the normal distribution $\mathcal{D}$ is integrable, then $\mathcal{F}$ is polar and the set of regular points is open and dense in each section.*

3.2 INFINITESIMAL FOLIATION.

An important tool in the study of singular Riemannian foliations is the *infinitesimal foliation*. Let M be a complete Riemannian manifold with a closed singular Riemannian foliation $\mathcal{F}$. Given a point $p \in M$ we will construct a singular Riemannian foliation on $\nu_p L_p$, the normal space of the leaf through p. We start by fixing $\varepsilon > 0$, and considering $S_p = \exp_p(\nu_p L_p) \cap B_\varepsilon(p)$, where $B_\varepsilon(p) \subset M$ is the ball of radius ε centered at p. The foliation $\mathcal{F}$ induces a foliation $\mathcal{F}|_{S_p}$ on S_p by setting the leaves of $\mathcal{F}|_{S_p}$ to be the connected component of the intersection between the leaves of $\mathcal{F}$ and S_p. This foliation may not be a singular Riemannian foliation with respect to the induced metric of M on S_p, i.e the leaves of $\mathcal{F}|_{S_p}$ may not be equidistant with respect to the induced metric. Nevertheless from, [Mol88, Proposition 6.5], the pull-back foliation $\exp_p^*(\mathcal{F}|_{S_p})$ is a singular Riemannian foliation on $\nu_p L_p \cap B_\varepsilon(0)$ equipped with the euclidean metric.

Theorem 3.2. *Let $(M, \mathcal{F})$ be a singular Riemannian foliation, on a compact manifold, and fix $p \in M$ such that the leaf passing through p is just the point p. Then $(S_p, \mathcal{F}|_{S_p})$ is a singular Riemannian foliation, with the flat metric on $\nu_p L_p$ pulled back to S_p via the exponential map $\exp_p$.*

In an equivalent way, writing $\nu_p^\varepsilon L_p = \nu_p L_p \cap B_\varepsilon(0)$ and considering the pull-back foliation $\mathcal{F}^p = \exp_p^*(\mathcal{F}|_{S_p})$, the space $(\nu_p^\varepsilon L_p, \mathcal{F}^p)$ is a singular Riemannian foliation with respect to the Euclidean metric of $\nu_p L_p$, which we denote as the *infinitesimal foliation at p*. With this description we define for small λ a homothetic transformation of $h_\lambda\colon \nu_p^\varepsilon \to \nu_p^{\lambda\varepsilon}$, by simply sending a vector in ν_p^ε to $\nu_p^{\lambda\varepsilon}$. By the following lemma, the foliation $(\nu_p^\varepsilon L_p, \mathcal{F}^p)$ is invariant under homotheties that fix the origin.

Lemma 3.3 (Lemma 6.2 in [Mol88])**.** *The homothetic transformations h_λ preserves the foliation $\mathcal{F}^p$.*

We note that the origin $\{0\} \subset \nu_p L_p$ is always a leaf of the infinitesimal foliation $\mathcal{F}^p$. Since by definition the leaves of $\mathcal{F}^p$ stay at a constant distance from each other, the fact that the origin is a leaf implies that any leaf of $\mathcal{F}^p$ is at a constant distance from the origin, and thus it is contained in a sphere around the origin. From this last fact it follows that we may consider the infinitesimal foliation restricted to the unit normal sphere, which we denote by $\mathbb{S}_p^{\perp}$, yielding a foliated round sphere $(\mathbb{S}_p^{\perp}, \mathcal{F}^p)$ with respect to the standard round metric of $\mathbb{S}_p^{\perp}$ which is also called the infinitesimal foliation. From here on when we say "infinitesimal foliation" we refer to $(\mathbb{S}_p^{\perp}, \mathcal{F}^p)$. We note that we may refer to $(\nu_p L_p, \mathcal{F}^p)$ and $(\mathbb{S}_p^{\perp}, \mathcal{F}^p)$ as the infinitesimal foliation indistinctly since $(\nu_p L_p, \mathcal{F}^p)$ is invariant under homothetic transformations and thus it can be recovered from $(\mathbb{S}_p^{\perp}, \mathcal{F}^p)$.

A singular Riemannian foliation $(M, \mathcal{F})$ such that for any point $p \in M$ the infinitesimal foliation $(\mathbb{S}_p^{\perp}, \mathcal{F}^p)$ is polar is called an *infinitesimally polar foliation.* By Theorem 4.10 (d) in [ABT13] it follows that polar foliations are infinitesimally polar foliations. Infinitesimally polar foliations have a leaf space with more regularity.

Theorem 3.4 (Proposition 6.7 in [ABT13])**.** *Let $\mathcal{F}$ be a closed singular Riemannian foliation on a complete Riemannian manifold M. The leaf space $M/\mathcal{F}$ is a Riemannian orbifold if and only if $\mathcal{F}$ is infinitesimally polar.*

Infinitesimally polar foliations characterize singular Riemannian foliations which can be covered by a regular Riemannian foliation, or, n other words, by singular foliations where we can resolve the singularities, without losing the transverse geometry. Formally we say that a regular Riemannian foliation $(\hat{M}, \hat{\mathcal{F}})$ is a *geometric resolution* of a singular Riemannian foliation $(M, \mathcal{F})$, if there is a smooth surjective map $F \colon \hat{M} \to M$ mapping leaves of $\hat{\mathcal{F}}$ to leaves of $\mathcal{F}$, and preserving the transverse lengths (see [Lyt10]). By the following theorem, infinitesimally polar foliations are the only foliations which admit a geometric resolution.

Theorem 3.5 (Theorem 1.1 in [Lyt10])**.** *A singular Riemannian foliation $(M, \mathcal{F})$ has a geometric resolution if and only if $\mathcal{F}$ is infinitesimally polar. Furthermore, when $\mathcal{F}$ is infinitesimally polar there is a canonical resolution.*

Remark 3.6. We will see in the next section that for a singular Riemannian foliation $(M, \mathcal{F})$, we can compare two leaves L_q and L_p, which are close, via a fibration whose connected components are the leaves of the infinitesimal foliation.

3.3 HOLONOMY AND TYPES OF LEAVES

In this section we define the *holonomy* group Γ_L of a closed leaf L of a singular Riemannian foliation $(M, \mathcal{F})$. The interpretation of the holonomy is the following: In Section 3.2 we defined the infinitesimal foliation which gives a description of the foliation around a point p in the fixed leaf L. But since the infinitesimal foliation is given by taking connected components of intersections of leaves of $\mathcal{F}$ with a ball around p, it might happen that two of these connected components are contained in a common leaf of $\mathcal{F}$. This means that to recover a small neighborhood of $p^* \in M/\mathcal{F}$, we can consider first the quotient space $\mathbb{S}_p^\perp/\mathcal{F}^p$, and then identify the leaves $\mathcal{L}^* \in \mathbb{S}_p^\perp/\mathcal{F}^p$ which where contained in a leaf of $\mathcal{F}$. This identification is done via an action of $\pi_1(L, p)$ on $\mathbb{S}_p^\perp/\mathcal{F}^p$, which we call the *holonomy action*. All of this discussion is encoded in the following theorem from [MR18]. It is the analog to Theorem 2.11 for singular Riemannian foliations, describing a tubular neighborhood of a leaf in the foliation.

Theorem 3.7 (Slice theorem for singular Riemannian foliations, Theorem A in [MR18])**.** *Let $(M, \mathcal{F})$ be a singular Riemannian foliation, and let L be a closed leaf with infinitesimal foliation $(\mathbb{D}_p^\perp, \mathcal{F}^p)$ at a point $p \in L$. Then there is a group K of foliated isometries of $(\mathbb{D}_p^\perp, \mathcal{F}_p)$ and a principal K-bundle P over L, such that for small enough $\varepsilon > 0$, the ε-tube U around L is foliated diffeomorphic to $(P \times_K \mathbb{D}^\perp, P \times_K \mathcal{F}^p)$.*

Let $(M, \mathcal{F})$ be a singular Riemannian foliation and let p be a point contained in L, a closed leaf. Recall from Section 3.2 that the infinitesimal foliation $(\mathbb{S}_p^\perp, \mathcal{F}^p)$ is invariant under homotheties (see Lemma 3.3). Using this property we can extend the infinitesimal foliation to the whole normal space $\nu_p L \subset T_p M$ of L at p. Given a path $\gamma \colon [0, 1] \to L$ starting at p, the following theorem gives us a foliated transformation from $\nu_p L$ to the normal bundle νL.

Theorem 3.8 (Corollary 1.5 in [MR18]). *Let L be a closed leaf of a singular Riemannian foliation $(M, \mathcal{F})$, and $\gamma\colon [0,1] \to L$ a piecewise smooth curve with $\gamma(0) = p$. Then there is a map $G\colon [0,1] \times \nu_p L \to \nu L$ such that:*

(i) $G(t,v) \in \nu_{\gamma(t)} L$ for every $(t,v) \in [0,1] \times \nu_p L$.

(ii) For every $t \in [0,1]$, the restriction $G\colon \{t\} \times \nu_p L \to \nu_{\gamma(t)} L$ is a linear isometry preserving the leaves of ν_L.

(iii) For every $s \in \mathbb{R}$ the map $\exp_{\gamma(t)}(sG(t,v))$ belongs to the same leaf as $\exp_p(sv)$.

Proof. See Corollary 1.5 in [MR18], or Appendix A. $\square$

Thus if we have a loop γ at p, from Theorem 3.8 we have a foliated linear isometry $G\colon \{1\} \times \nu_p L \to \nu_p L$, which we will denote by G_γ. We denote by $O(\mathbb{S}_p^\perp, \mathcal{F}^p)$ the group of *foliated isometries* of the infinitesimal foliation, i.e. all the isometries which preserve the foliation. We note that such an isometry may map a leaf to a different leaf. By $O(\mathcal{F}^p)$ we denote the foliated isometries which leave the foliation invariant, i.e. the isometries $f \in O(\mathbb{S}_p^\perp, \mathcal{F}^p)$ such that for any leaf $\mathcal{L}$ of $(\mathbb{S}_p^\perp, \mathcal{F}^p)$s we have $f(\mathcal{L}) \subset \mathcal{L}$. There is a natural action of $O(\mathbb{S}_p^\perp, \mathcal{F}^p)$ on the quotient $\mathbb{S}_p^\perp / \mathcal{F}^p$. The kernel of this action is $O(\mathcal{F}^p)$. In Appendix A we will show that if two loops, γ_1 and γ_2, are homotopic then $G_{\gamma_1}^{-1} \circ G_{\gamma_2}$ are in the kernel of the action of $O(\mathbb{S}_p^\perp, \mathcal{F}^p)$ on $\mathbb{S}_p^\perp / \mathcal{F}^p$. Therefore we obtain a group morphism from $\pi_1(L,p)$ to $O(\mathbb{S}_p^\perp, \mathcal{F}^p)$.

Proposition 3.9. *Let $(M, \mathcal{F})$ be a singular Riemannian foliation, L a closed leaf of the foliation and $p \in L$. There is a well defined group morphism,*

$$\rho\colon \pi_1(L,p) \to O(\mathbb{S}_p^\perp, \mathcal{F}^p)/O(\mathcal{F}^p),$$

given by $\rho[\gamma] = [G_\gamma]$.

Proof. See Corollary A.5 in Appendix A. $\square$

For a closed leaf L of a singular Riemannian foliation $(M, \mathcal{F})$ we define the *holonomy of the leaf* L as the image $\Gamma_L < O(\mathbb{S}_p^\perp, \mathcal{F}^p)/O(\mathcal{F}^p)$ of $\pi_1(L, p)$ under the morphism ρ. When we consider the holonomy of a leaf L_p trough a point $p \in M$, we will denote it by Γ_p. We say that a regular leaf L is called a *principal leaf* if the holonomy is trivial, and *exceptional* otherwise.

The holonomy action can be interpreted in the case of regular leaves, as in the work of Molino in [Mol88, Section 1.7], as follows. Suppose that $(M, \mathcal{F})$ is a Riemannian foliation of dimension m on an n-manifold (for the general case consider $(\Sigma_{reg}, \mathcal{F})$). Let L be a regular closed leaf and take $\gamma\colon [0,1] \to L$ a path in L, with $p \in L$ as start point. When we consider for $v \in \mathbb{D}_p^\perp$ the germ of the map $\exp_{\gamma(1)}(G_\gamma(v))$, we obtain an element of the holonomy as defined in [Mol88, Section 1.7], or [MM03, Chapter 2]. Furthermore we will see that for any path γ, the holonomy transformation associated to γ in [Mol88, MM03] is given by the germ of $\exp_{\gamma(1)}(G_\gamma(v))$. In order to do this recall that for a Riemannian foliation $(M, \mathcal{F})$, there exist a neighborhood U of p in M and a diffeomorphism $\phi\colon U \to \mathbb{R}^n$, such that the image of the foliation $(U, \mathcal{F})$ is given by the preimages of the projection map $\mathbb{R}^n \to \mathbb{R}^{n-m}$ (see [Mol88, MM03]). For the sake of clarity we assume first that $\gamma([0,1]) \subset U$. Since $(U, \mathcal{F})$ is foliated diffeomorphic to $(\mathbb{R}^{n-m} \times \mathbb{R}^m, \mathbb{R}^m)$, then we can extend the vector field γ' to a vector field X on U, in such a way that it is invariant under rescalings. This implies that X is a *linearized vector field* (see Appendix A for the definition of linearized vector field). The holonomy element of γ is the germ of the map given by the flow of X (see Figure 3.1). As one can see in Appendix A, this is exactly how the map G_γ is defined. In the general case when there is a foliated atlas of $(M, \mathcal{F})$, we can do the previous analysis on each chart. Thus this shows that the notion of holonomy for a singular Riemannian foliation is an extension of the notion of holonomy for Riemannian foliations. For more details on the theory of Riemannian foliations we invite the reader to consult [Mol88], and [MM03].

Proposition 3.10 (Theorem 2.6 in [MM03]). *For a Riemannian foliation $(M, \mathcal{F})$, if L is a compact leaf, then its holonomy group is finite.*

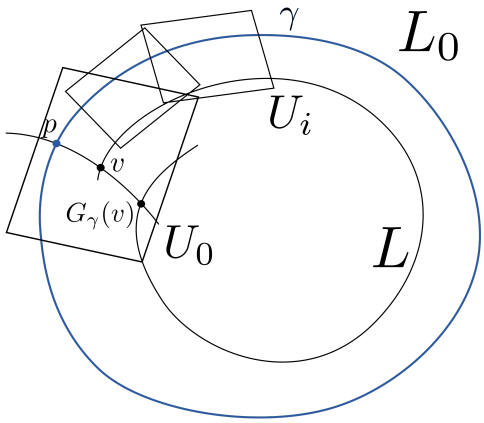

Figure 3.1.: Construction of holonomy action for a regular leaf.

From Proposition 3.10 we have that the holonomy of a compact regular leaf is finite. The following theorem shows that for a singular Riemannian foliation $(M, \mathcal{F})$ such that the Riemannian foliation $(\Sigma_{\text{reg}}, \mathcal{F})$ has compact closed leaves, then the leaf space $\Sigma_{\text{reg}}/\mathcal{F}$ is an orbifold.

Theorem 3.11 (Theorem 2.15 in [MM03]). *Let $(M, \mathcal{F})$ be a Riemannian foliation such that any leaf of $\mathcal{F}$ is closed compact. Then the space of leaves $M/\mathcal{F}$ has a canonical orbifold structure of dimension q. The isotropy group of a leaf in $M/\mathcal{F}$ is its holonomy group.*

If L is a principal leaf of a singular Riemannian foliation $(M, \mathcal{F})$ on an n-dimensional manifold with all regular leaves compact closed, then Theorem 3.11 and the fact that Γ_p is trivial show that L^* is a regular point of the orbifold Σ^*_{reg}. With this we can easily see that when we consider the *stratum of principal leaves*, which we denote by M_{prin}, then M^*_{prin} corresponds to the manifold part of the orbifold Σ^*_{reg}. Thus M_{prin} is open and dense in Σ_{reg}. Since in general for a singular Riemannian foliation Σ_{reg} is open and dense in M, then M^*_{prin} is open and dense in M^*. Furthermore, from the fact that the manifold part of an orbifold is connected (see for example

[Lan18, Lemma 2.3], [Yer14], [Fae11]) it follows that the set M^*_{prin} is connected in M^*. Since it is locally euclidean, it is path connected.

We collect these observations in the following theorem.

Theorem 3.12. *For a singular Riemannian foliation $(M, \mathcal{F})$ with compact closed regular leaves, the principal stratum M_{prin} is open dense in M, and M^*_{prin} is connected and path connected in the leaf space M^*.*

As stated at the end of Section 3.2, for a singular Riemannian foliation $(M, \mathcal{F})$ the infinitesimal foliation $(\mathbb{S}_p^\perp, \mathcal{F}^p)$ at $p \in M$, gives rise to a way of comparing leaves of different types, which will be exploited in the present work. Given a fixed point $p \in M$ and a vector $v \in \mathbb{S}_p^\perp$, set $q = \exp_p(\varepsilon v)$. If ε is small enough, then L_q is contained a tubular neighborhood of L_p, and thus there is a well defined smooth closest-point projection $\operatorname{proj}\colon L_q \to L_p$ which is by Lemma 6.1 in [Mol88] is a submersion. The connected component of the fiber of proj through q can be identified with the leaf $\mathcal{L}_v \in \mathcal{F}^p$, through v. Taking $\overline{L}_p = \widetilde{L}_p / \operatorname{proj}_*(\pi_1(L_q))$, the quotient of the universal cover $\widetilde{L}_p$ of L_p, we have a finite cover $\overline{L}_p \to L_p$ such that $\operatorname{proj}\colon L_q \to L_p$ lifts to a fibration:

$$\mathcal{L}_v \to L_q \xrightarrow{\xi} \overline{L}_p. \tag{3.3.1}$$

Clearly fibration (3.3.1) is a surjective map by construction. The following proposition gives another way of obtaining the covering $\overline{L}$ of L, via a subgroup H of the holonomy group Γ_p.

Proposition 3.13. *For $v \in \mathbb{S}_p^\perp$ with image $v^* \in \mathbb{S}_{p^*}$, set H to be the subgroup of Γ_p fixing v^*. Then, taking $q = \exp_p(v)$, the finite cover $\overline{L}_p$ of L_p in the fibration $\xi\colon L_q \to \overline{L}_p$ is $\widetilde{L}_p / H$.*

Proof. Let $F = \mathrm{proj}^{-1}(p)$ be the fiber of the metric projection $\mathrm{proj}\colon L_q \to L_p$, which may consist of several connected components. The long exact sequence of the fibration looks like

$$\cdots \to \pi_1(F,q) \to \pi_1(L_q,q) \xrightarrow{\mathrm{proj}_*} \pi_1(L_p,p) \xrightarrow{\partial} \pi_0(F,q) \to 0.$$

From exactness, we conclude that $(\mathrm{proj}_*)(\pi_1(L_q,q)) = \ker(\partial)$. We recall how the map $\partial\colon \pi_1(L_q,q) \to \pi_0(F,q)$ is defined, following a modification of the definitions presented in Hatcher (see Sections 4.1 and 4.2 in [Hat10]). Let $\lambda_1\colon \mathbb{D}^1 \to \mathbb{S}^1$ be the map collapsing $\partial\mathbb{D}^1$ to a point. Let $\delta_0\colon \mathbb{S}^0 \to \mathbb{D}^1$ be the inclusion as the boundary. Consider a loop $\varphi\colon \mathbb{S}^1 \to L_p$ with base point p. By the homotopy lifting property there is a lift $\lambda\colon \mathbb{D}^1 \to L_q$ for the map $\varphi\circ\lambda_1\colon \mathbb{D}^1 \to L_p$, with $\lambda(0) = q$. Furthermore, by definition, we have that $\varphi\circ\lambda_1\circ\delta_0 = \mathrm{proj}\circ\lambda\circ\delta_0$ is constant. Therefore the image of the map $\psi = \lambda\circ\delta_0$ is contained in F. Thus we have a map $\psi\colon \mathbb{S}^0 \to F$. We define $\partial[\varphi] = [\psi]$. Let $\alpha\colon \mathbb{S}^1 \to L_p$ be a loop in $\mathrm{proj}_*(\pi_1(L_q,q)) = \ker\partial$. Consider $G_t = G\colon \{t\}\times\nu_pL_p \to \nu L_p$, the transformation given by Theorem 3.8 corresponding to α. Then $\tilde{\alpha}(t) = \exp_{\alpha(t)}(G_t(v))$ is a lift of α in L_q. Since ∂ does not depend on the choice of a lift we have that $0 = \partial[\alpha] = [\tilde{\alpha}]$. It follows that the end point of $\tilde{\alpha}\colon [0,1] \to F$ is in the same connected component of F as q. Therefore we have that $G_1\colon \nu_pL_p \to \nu_pL_p$ fixes the infinitesimal leaf $\mathcal{L}_v$ in $\mathbb{S}^\perp_p/\mathcal{F}_p$. Thus $\mathrm{proj}_*(\pi_1(L_q,q)) \subset H$. Conversely, if we start with $[\alpha]\in H$, then for the map $G\colon [0,1]\times\nu_pL_p \to \nu L_p$ given by Theorem 3.8, we have G_1 maps the infinitesimal leaf $\mathcal{L}_v$ to itself. By definition this means that $\exp_p(G_1(v))$ is in the same connected component of F as $q = \exp_p(v)$. Theorem 3.8 (iii), implies that the path $\exp_p(G_t(v))$ is a path between q and $\exp_p(G_1(v))$ in F. Thus we have that $\partial[\alpha] = 0$. Therefore we conclude that $\mathrm{proj}_*(\pi_1(L_q,q)) = H$. $\square$

In particular Proposition 3.13 gives a way to detect if there is holonomy for a closed leaf of $(M,\mathcal{F})$.

If the map $\xi\colon L_q \to \overline{L}_p$ is proper then the following theorem of Ehresmann says that ξ is a locally trivial fibration.

Theorem 3.14 (Ehresmann's fibration lemma, in [Ehr51]). *If W and N are smooth manifolds, and $f\colon W \to N$ is a smooth surjective submersion which is also proper, then f is a locally trivial fibration. This means that for each point $p \in N$ there exists an open neighborhood $U \subset N$ of p, and a diffeomorphism $\phi\colon f^{-1}(U) \to U \times F$, where $F = f^{-1}(p)$, such that the following diagram commutes:*

$$\begin{tikzcd} f^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "f"'] & & U \times F \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

As a particular case, if in the previous theorem W is a compact manifold, we have the following proposition:

Corollary 3.15. *Let W and N be smooth manifolds, with W compact. If $f\colon W \to N$ is a smooth surjective submersion, then f is a locally trivial fibration.*

Remark 3.16. We note that in Ehresmann's lemma the fiber bundle given by the projection map f, may not have as structure group a Lie group, but rather a very large topological group, namely the diffeomorphism group of the fiber, $\mathrm{Diff}(F)$. Although $\mathrm{Diff}(F)$ is in general not a Lie group, it is a Frobenious group, i.e. the group operations are smooth with respect to a Frobenious atlas (see [GW07]).

3.4 HOMOGENEOUS FOLIATIONS

A classical set of examples of singular Riemannian foliations comes from smooth actions of compact Lie groups on smooth manifolds. Let G be a compact Lie group acting smoothly on a smooth manifold M, and let us assume it acts effectively. Given any Riemannian metric g on M, we can always construct a Riemannian metric g_G which is invariant under the action of G, i.e. $G < \mathrm{Isom}(M, g_G)$ (see [AB15, Proof of Slice Theorem 3.49]). The Lie algebra $\mathfrak{g}$ of the group G provides the vector fields spanning the tangent spaces of the leaves, which in this case are the orbits of the action, so the partition induced by the orbits of G is a foliation. Furthermore, by the following proposition the foliation (M, G) is a transnormal system, and thus a singular Riemannian foliation.

Proposition 3.17 (Proposition 3.78-(i) in [AB15]). *If γ is a geodesic which starts normal to the orbit $G(\gamma(0))$, then $\gamma(t)$ is normal to the orbit $G(\gamma(t))$ for all times.*

Proof. Since the action of G is by isometries, any vector field X which is tangent to orbits is a Killing vector field. So it is sufficient to show that for a geodesic $\gamma\colon I \to M$, if a Killing vector field X is orthogonal to $\gamma'(0)$, then X is orthogonal to $\gamma'(t)$ for all t. Since a vector field X is Killing if and only if $g(\nabla_Y X, Z) = -g(\nabla_Z X, Y)$ for all vector fields Y and Z, then in particular $g(\nabla_{\gamma'(t)} X, \gamma'(t)) = 0$ and so $\frac{d}{dt} g(X, \gamma'(t)) = 0$. Thus $g(X, \gamma'(t))$ is constant, and since at $t = 0$ it is zero, we conclude that X and $\gamma'(t)$ are orthogonal. □

A singular Riemannian foliation $(M; \mathcal{F})$ is called *homogeneous* if it is induced by a group G acting by isometries in M. By the work of Radeschi in [Rad14], there are several examples of non-homogeneous singular Riemannian foliations. In general, given a singular Riemannian foliation it is a difficult problem to show it is either homogeneous or non-homogeneous.

In the case of homogeneous foliations the infinitesimal foliation $(\mathbb{S}_p^\perp, \mathcal{F}^p)$ on the normal sphere at p is given by connected components of the orbits of the action of G_p in $\mathbb{S}_p^\perp$ via the isotropy representation. Therefore, denoting by G_p^0 the connected component of G_p containing the identity element, the infinitesimal foliation is given by considering only the action of G_p^0 on $\mathbb{S}_p^\perp$ given by the isotropy representation. The holonomy of $G(p)$ is given by G_p/G_p^0 (see [MR18, Section 3.1]). For $q \in M$

close to p with G_q a subgroup of G_p, the fiber bundles given by (3.3.1) are of the form:

$$G_p^0/G_q \to G/G_q \to G/G_p^0,$$

where G_p^0 is the connected component of the identity of the isotropy group G_p, and G/G_p^0 is a cover of the orbit G/G_p (see [GGR15, Example 2.4]).

Part II

ASPHERICAL FOLIATIONS

4 CROSS-SECTIONS AND A-FOLIATIONS

In this chapter we will consider *A-foliations*, i.e. singular Riemannian foliation with aspherical leaves. We will focus on the case of A-foliations of codimension 2 on an $(n+2)$-dimensional, compact, simply-connected manifold M. It has been proven in [GGR15, Corollary B] that the leaves of such foliation must be homeomorphic to tori. We exploit this fact to show that several results of torus actions extend to A-foliations. We begin by giving conditions to be able to compare in a foliated sense two foliated manifolds via their leaf spaces.

4.1 CROSS-SECTION FOR THE LEAF SPACE

For homogeneous foliations given by a fixed group G, the existence of cross-sections for the projection map $\pi\colon M \to M/\mathcal{F}$ has been exploited in works such as [Oh83a], [OR70], [OR74], to classify up to homeomorphism the manifolds M admitting a G-action, via the orbit space M^*. In order to get the classification up to equivariant homeomorphism, more structure is needed on M^* which will be discussed in following sections.

In this section we state sufficient topological conditions on the leaf space of a foliated manifold $(M, \mathcal{F})$ for the existence of a cross-section of the projection map $\pi\colon M \to M/^*$.

We start by recalling the *relative lifting problem*. Given a relative CW-complex (W, A), continuous maps $p\colon X \to Y$, $g\colon W \to Y$ and $f\colon A \to X$, we must find sufficient conditions to guarantee the existence of an extension $\tilde{f}\colon W \to Y$ of f lifting g, i.e. a map $\tilde{f}$ which makes the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & X \arrow[d, "p"] \\ W \arrow[r, "g"] \arrow[ur, dashed, "\tilde{f}"] & Y \end{tikzcd} \tag{4.1.1}$$

This problem is well understood when the map p is a fibration (e.g. [Hat10, DK01]). In a more general setting, when p is not a fibration, one can ask that the spaces be nicely behaved to get a family of obstructions.

We recall that a topological space F is called *n-simple* if it is path connected, with Abelian fundamental group $\pi_1(F)$, and the action of the fundamental group on all the higher homotopy groups $\pi_k(F)$ is trivial. This last condition is equivalent to $\pi_k(F) = [\mathbb{S}^k, F]$. A space which is simple for all n is called *simple*.

For a pair of spaces (W, A), with A path connected, we can define the *relative homotopy groups* $\pi_k(W, A)$ by considering $I^k = I^{k-1} \times I$, and considering I^{k-1} as the face $I^{k-1} \times \{0\}$. Set J^{k-1} to be the closure of $\partial I^k \setminus I^{k-1}$, and for a base point $x_0 \in A$, define $\pi_k(W, A, x_0)$ as the set of homotopy classes of maps $(I^k, \partial I^k, J^{k-1}) \to (W, A, x_0)$. Since W and A are path connected spaces, the last definition does not depend on the choice of the base point, just as in the non-relative case. Furthermore, there is a natural action of $\pi_1(A)$ on $\pi_k(W, A)$. Consider a loop $\gamma\colon I \to A$ with $\gamma(0) = x_0$, and a map $f\colon (I^k, \partial I^k, J^{k-1}) \to (W, A, x_0)$. Consider the map γf defined as in Figure 4.1, where for a point on a radial line γf is just γ (see [Hat10, Section 4.1]). If this action of $\pi_1(A)$ on $\pi_n(W, A)$ is trivial, we say that (W, A) is an *n-simple pair*. A pair that is n-simple for all n is called a *simple pair* (see [Hat10, Section 4.1][DK01, Section 6.16]).

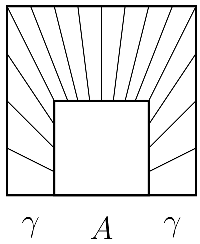

Figure 4.1.: Map γf

We recall some basic topological constructions which will be used frequently in the remainder of the section. We denote by Y^I the space of all continuous paths $\gamma\colon I \to Y$ with the compact-open topology. There is a natural fibration $q\colon Y^I \to Y$ called the *path space fibration*, with q defined as $q(\gamma) = \gamma(0)$. For a map $p\colon X \to Y$, the *mapping path fibration* $\pi_p\colon E_{\pi_p} \to Y$ is the fibration with total space the total space $p^*(Y^I) \subset X \times Y^I$ of the pullback via p. For a path $\gamma\colon I \to Y$, with $\gamma(0) = p(w)$ the projection map π_p is defined as $\pi_p(w, \gamma) = \gamma(1)$. The following theorem gives some properties of the construction $\pi_p\colon E_{\pi_p} \to Y$.

Theorem 4.1 (Theorem 6.18 in [DK01])**.** *Suppose that $p\colon X \to Y$ is a continuous map.*

1. *There exists a homotopy equivalence $h : X \to E_{\pi_p}$ so that the diagram,*

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "p"'] & & E_{\pi_p} \arrow[dl, "\pi_p"] \\ & Y & \end{tikzcd}$$

   commutes.

2. *The map $\pi_p\colon E_{\pi_p} \to Y$ is a fibration.*

3. *If $p\colon X \to Y$ is a fibration, then h is a fiber homotopy equivalence.*

The fiber F_p of π_p is called the *homotopy fiber of* p, and there is a homotopy equivalence between X and E_{π_p}. If p is already a fibration with fibers F, then F_p is homotopy equivalent to F. Next we define the *mapping cylinder* M_p. This is the space obtained by considering the disjoint union of $X \times I$ with Y, and identifying $(x, 1)$ with $p(x)$. There is an inclusion $i\colon X \to M_p$ given by $i(x) = [x, 0]$. The properties of the mapping cylinder are given by the following theorem.

Theorem 4.2 (Theorem 6.27 in [DK01]). *Let $p\colon X \to Y$ be a continuous map, and let $i\colon X \to M_p$ be the inclusion defined above.*

1. *There exists a homotopy equivalence $h\colon M_p \to Y$ such that the following diagram commutes:*

$$\begin{tikzcd} & X \arrow[dl, "p"'] \arrow[dr, "i"] & \\ Y & & M_p \arrow[ll, "h"] \end{tikzcd}$$

2. *The inclusion $i\colon X \to M_p$ is a cofibration.*

Recall that a map $i\colon X \to Y$ is a *cofibration* if the following diagram has a solution for any space Z:

$$\begin{tikzcd} X \times \{0\} \arrow[rr] \arrow[dd, "i"'] & & X \times I \arrow[dl] \arrow[dd, "i \times \mathrm{Id}"] \\ & Z & \\ Y \times \{0\} \arrow[rr] \arrow[ur] & & Y \times I \arrow[ul, dotted] \end{tikzcd}$$

Given a map $f\colon X \to Y$ between path-connected spaces, a *Moore-Postnikov tower* for f is a collection of spaces,

$$\cdots \to Z_{n+1} \overset{\alpha_n}{\to} Z_n \to \cdots \to Z_1,$$

and continuous maps $\alpha_n\colon Z_{n+1} \to Z_n$, $\lambda_n\colon X \to Z_n$, $\mu_n\colon Z_n \to Y$ such that:

(i) $\alpha_n \circ \lambda_{n+1} = \lambda_n$;

(ii) $\mu_n \circ \alpha_n = \mu_{n+1}$;

(iii) for $i < n$ the map λ_n induces an isomorphism between $\pi_i(X)$ and $\pi_i(Z_n)$ and a surjection for $i = n$;

(iv) for $i > n$ the map μ_n induces an isomorphism between $\pi_i(Z_n)$ and $\pi_i(Y)$ and an injection for $i = n$;

(v) the map α_n is a fibration with fiber an Eilenberg-MacLane space $K(\pi_n(F), n)$, where F is the homotopy fiber of f.

The points (i) to (v) can be summarized in the following commutative diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d, "\alpha_3"] & \\
 & Z_3 \arrow[d] \arrow[ddr, "\mu_3"] & \\
 & Z_2 \arrow[d] \arrow[dr] & \\
X \arrow[r] \arrow[ur] \arrow[uur, "\lambda_3"] & Z_1 \arrow[r] & Y
\end{tikzcd}$$

The idea behind a Moore-Postnikov tower is to have a series of fibrations with spaces which start approximating the homotopy type of Y and gradually they approximate the homotopy type of X. In general a fibration $F \to E \to B$ is called *principal* if there is a commutative diagram of the form:

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & B \arrow[d] & \\
\Omega B' \arrow[r] & F' \arrow[r] & E' \arrow[r] & B'
\end{tikzcd}$$

Here the second row is a fibration sequence, and all vertical maps are weak homotopy equivalences, i.e. they induced isomorphisms between homotopy groups of all degrees. When the fibrations $\alpha_n \colon Z_{n+1} \to Z_n$ are principal fibrations we say that we have a *Moore-Postnikov tower of principal fibrations.* In general any map between CW-spaces admits a Moore-Postnikov tower, but the following theorem explicitly states when does a Moore-Postnikov tower of principal fibrations exist.

Theorem 4.3 (Theorem 4.71 in [Hat10], Existence of Moore-Postnikov tower of principal fibrations)**.** *For a given map $f\colon X \to Y$ between connected CW-spaces, a Moore-Postnikov tower of principal fibrations exists if and only if $\pi_1(X)$ acts trivially on $\pi_n(M_f, X)$ for all $n > 1$, where M_f is the mapping cylinder of f.*

With all these concepts at hand we can now give the general solution to the relative lifting problem. The presence of a Moore-Postnikov towers is useful for solving the relative lifting problem associated to Diagram (4.1.1).

The idea is to inductively construct a lift $W \to Z_n$ for each n, and from these lifts obtain a lift $W \to X$. We start with the case where the map $p\colon X \to Y$ is a fibration.

Theorem 4.4 (Obstruction Theory in [Hat10])**.** *Let $p\colon X \to Y$ be a fibration with fiber F, (W, A) a CW-pair with W simply connected. Assume the fibration has a Moore-Postnikov tower of principal fibrations and consider the relative lifting problem:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & X \arrow[d, "p"] \\ W \arrow[r, "g"'] \arrow[ur, dashed, "\tilde{f}"] & Y \end{tikzcd}$$

There exists an obstruction $\omega_n \in H^{n+1}(W, A; \pi_n(F))$, such that a lift $\tilde{f}\colon W \to X$ extending $f\colon A \to X$ exists, if $\omega_n = 0$ for all n.

Proof. First we note that since we have a fibration $p\colon X \to Y$, we may take Z_1 to be the covering space of Y corresponding to the subgroup $p_*(\pi_1(X))$ of $\pi_1(Y)$. Since W is simply-connected we can lift g to $W \to Z_1$, which agrees with $g \circ \lambda_1\colon A \to Z_1$. Since the Moore-Postnikov tower is by principal fibrations, for the inductive step we have a commutative diagram as follows:

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & Z_n \arrow[r] \arrow[d] & PK \arrow[d] \\ W \arrow[r] & Z_{n-1} \arrow[r] & K = K(\pi_n(F), n+1). \end{tikzcd}$$

Here $PK \to K$ is the *path fibration*, defined by fixing a point $b_0 \in K$, and letting PK be the space of all curves in K starting at b_0, and the letting $PK \to K$ be the map that sends each path to its end point. Since Z_n is the pullback, the elements in Z_n are pairs consisting of a point in Z_{n-1} and a path from its image in K to the base point in K. A lift $W \to Z_n$ therefore amounts to a nullhomotopy of the composition $W \to Z_{n-1} \to K$. Since we have already defined such a lift on A, we have a nullhomotopy of $A \to K$, and the desired nullhomotopy of $W \to K$ must extend this nullhomotopy on A. The map $W \to K$ together with the nullhomotopy on A gives a map $W \cup C(A) \to K$, where $C(A)$ is the cone of A. Since K is an Eilenberg-MacLane space $K(\pi_n(F), n+1)$, the map $W \cup C(A) \to K$ determines the desired obstruction

$$\omega_n \in H^{n+1}(W \cup C(A); \pi_n(F)) = H^{n+1}(W, A; \pi_n(F)).$$

If $\omega_n = 0$, by construction we have that there is a nullhomotopy of $W \to K$ extending the given nullhomotopy $A \to K$.

If we succeed in extending the lifts $A \to Z_n$ to lifts $W \to Z_n$ for all n, then we obtain a map $W \to \varprojlim Z_n$, to the inverse limit $\varprojlim Z_n$, extending the given $A \to X \to \varprojlim Z_n$. Let M be the mapping cylinder of $X \to \varprojlim Z_n$. From the hypothesis that the restriction of $W \to \varprojlim Z_n \subset M$ to A factors through X, this gives a homotopy of this restriction to the map $A \to X \subset M$. We extend this homotopy to a homotopy of $W \to M$ producing a map $(W, A) \to (M, X)$. Since the map $X \to \varprojlim Z_n$ is a weak homotopy equivalence, then $\pi_i(M, X) = 0$ for all i, and from the so-called *Compression Lemma* (see Lemma 4.6 in [Hat10]), we conclude that the map $(W, A) \to (M, X)$ is homotopic relative to A to a map $W \to X$. Hence the map $W \to X$ extends the given map $A \to X$. $\square$

Since in the previous theorem we started with a fibration $p \colon X \to Y$, we state the relevant existence of obstructions but for a general continuous map.

Theorem 4.5. *(Obstruction to extension) Let (W, A) be a relative CW-complex, with W simply-connected, and assume we have continuous maps $p\colon X \to Y$, $f\colon A \to X$ and $g\colon W \to Y$. Furthermore, suppose that the homotopy fiber F_p of p is simple, and that (M_p, X) is a simple pair. Then the following are true:*

(i) There is a family of obstructions $\omega_k \in H^{k+1}(W, A; \pi_k(F_p))$ such that there exists a lift $\tilde{f}$ of f solving diagram (4.1.1) if $\omega_k = 0$ for all k.

(ii) If F_p is an Eilenberg-McLane space $K(\pi, \ell)$, then there is a unique obstruction $\omega_\ell \in H^{\ell+1}(W, A; \pi)$, and the lift $\tilde{f}$ of f solving diagram (4.1.1) exists if and only if $\omega_\ell = 0$.

Proof. We sketch here the proof. For further details we invite the interested reader to see, for example, [Hat10, Chapter 4] for a more detailed discussion. First we show that the pair (M_p, X) is simple only when the pair (M_{π_p}, E_{π_p}) is simple. To prove this we start by noting that there is a homotopy equivalence between X and the total space E_{π_p} of the path space fibration for $p\colon X \to Y$ given by Theorem 4.1. For the maps $p\colon X \to Y$ and $\pi_p\colon E_{\pi_p} \to Y$, from Theorem 4.2 we have cofibrations $j\colon X \to M_p$ and $i\colon E_{\pi_p} \to M_{\pi_p}$. Furthermore Theorem 4.2 also yields a homotopy equivalence between the mapping cylinder M_p and M_{π_p}. Combining the diagrams obtained by applying Theorems 4.1 and 4.2 to the map $p\colon X \to Y$, and the diagram obtained by applying Theorem 4.2 to $\pi_p\colon E_{\pi_p} \to Y$ we obtain the following diagram, which commutes up to homotopy:

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook, "j"'] & E_{\pi_p} \arrow[d, hook, "i"] \\ M_p \arrow[r] & M_{\pi_p} \end{tikzcd}$$

Observe that the arrows going down are cofibrations. Then from the previous commutative diagram and [Bro06, 7.4.2], the homotopy groups $\pi_k(M_p, X)$ are (equivariantly under the action of $\pi_1(X)$) isomorphic to $\pi_k(M_{\pi_q}, E_{\pi_p})$ (with the action of

$\pi_1(E_{\pi_p})$). Thus (M_p, X) is simple if and only if (M_{π_p}, E_{π_p}) is simple. Therefore, for the fibration $\pi_p \colon E_{\pi_p} \to Y$, there exists a Moore-Postnikov tower by principal fibrations which yield the desired family of obstructions. Thus we may apply Theorem 4.4 to the fibration $\pi_p \colon E_{\pi_p} \to Y$.

We also note that we may apply the last argument in the proof of Theorem 4.4, and use the fact that the restriction of $W \to \varprojlim Z_n \subset M$ to A factors through X, to construct the lift $W \to X$ which extends $A \to X$. $\square$

Remark 4.6. The reason why in Theorem 4.5(i) we have an "if... then..." statement and on Theorem 4.5(ii) we have an "if and only if" statement lies in the fact that for the proofs of these theorems we use a Moore-Postnikov tower of principal fibrations $\cdots \to Z_2 \to Z_1 \to Y$ for p. In the case of Theorem 4.5(i) the lifts may be not unique, and in some examples this may yield non trivial ω_k even when an extension exists. An exception to this, is the case when F_p is an Eilenberg-McLane space (see [Hat10, Section 4.3]).

Remark 4.7. The condition of W being simply connected is used to ensure a unique lift from W to Z_1 in the Moore-Postnikov chain.

Remark 4.8. When we consider a principal $\mathbb{S}^1$-bundle $p \colon X \to Y$, the only obstruction to a cross-section of p is the Euler class of the bundle (see [Mor01]). This class coincides with the obstruction $\omega_2 \in H^2(Y; \pi_1(\mathbb{S}^1))$ given by part (ii) in Theorem 4.5. Thus for maps $p \colon X \to Y$ with homotopy fiber F_p aspherical, the obstruction obtained in Theorem 4.5 (ii) is a generalized Euler class.

Given a singular Riemannian foliation $(M, \mathcal{F})$, we consider the subset M_{prin} of M, consisting of principal leafs. The projection map $M \to M^*$ restricted to M_{prin} yields a fibration:

$$L \to M_{\text{prin}} \to M^*_{\text{prin}}.$$

We apply Theorem 4.5 to get a family of obstructions, which we will call *first obstructions*.

Theorem A. *Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation with M simply-connected, quotient map $\pi\colon M \to M^*$, and principal leaf L, which is simple and connected. Furthermore assume M^*_{prin} is simply-connected and (M_π, M_{prin}) is simple. Then there is a family of obstructions $\omega^1_k \in H^{k+1}(M^*_{\text{prin}}; \pi_k(L))$ such that a cross-section $\sigma\colon M^*_{\text{prin}} \to M_{\text{prin}}$ exists if $\omega^1_k = 0$ for all k.*

Proof. By applying Theorem 4.5 with $X = M_{\text{prin}}$, $W = Y = M^*_{\text{prin}}$, and $A = \emptyset$ we get the result. □

Even if a section exists on the principal part of the foliation, it may happen that it cannot be extended to the whole leaf space (as an example see [Fin76] or [Fin77]). To solve this new extension problem we need another family of obstructions which we call *second obstructions*.

Theorem B. *Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation with M simply connected, and consider the quotient map $\pi\colon M \to M^*$. Furthermore assume that the homotopy fiber F_π is simple, and setting $A = M^*_{\text{prin}}$, assume we have already defined a cross-section $\sigma\colon A \to M_{\text{prin}}$. Then there is a family of obstructions $\omega^2_k \in H^{k+1}(M^*, A; \pi_k(F_\pi))$ such that a cross-section $\tilde{\sigma}\colon M^* \to M$ extending σ exists if $\omega^2_k = 0$ for all k.*

Proof. Since M is simply-connected, then M^* is also simply-connected. We apply Theorem 4.5 to obtain the desired result. □

In particular, when we cannot distinguish M^* from M^*_{prin} from a homotopical view point, we get the following corollary from Theorem B.

Corollary C. *Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation on a simply-connected manifold. Suppose that there is a section $\tilde{\sigma}\colon M^*_{\text{prin}} \to M_{\text{prin}}$, and the that hypothesis of Theorem B are satisfied. If M^*_{prin} has the same homotopy type of M^*, then the cross-section $\tilde{\sigma}$ can be extended to a section σ.*

Remark 4.9. Since the holonomy is only defined for closed leaves (see Section 3.3), we ask that the foliation is closed in order to ensure the existence of a principal stratum in Theorem A, Theorem B, and Corollary C.

The following are particular applications of Corollary C in the setting of group actions.

We begin by considering M a closed, simply-connected, smooth $(n+2)$-manifold, with an effective and smooth $\mathbb{T}^n$-action. From [Bre72], it follows that, the orbit space $M^* = M/\mathbb{T}^n$ is a 2-disk with all the isotropy information contained in the boundary of M^*. Thus via Corollary C, we recover Theorem 2.20, which was proved first by Orlik and Raymond in [OR70] for 4-manifolds, and extended to arbitrary dimensions by Oh in [Oh83a].

Theorem 4.10 ([Oh83a],[KMP74], [OR70],Theorem 2.20)**.** *Let M be a closed simply-connected smooth $(n+2)$-manifold, with an effective and smooth $\mathbb{T}^n$-action. Then there exists a cross-section $\sigma\colon M^* \to M$.*

Proof. First we point out that the fibers of the fibration $M_{\mathrm{prin}} \to M^*_{\mathrm{prin}}$ are the group $\mathbb{T}^n$, a $K(\mathbb{Z}^n, 1)$ Eilenberg-McLane space (since they are connected and aspherical). Furthermore since the principal orbits all lie in the interior of M^*, which is contractible, we may apply Theorem A (ii) to show that the only obstruction vanishes. Thus we have a section from $\sigma\colon M^*_{\mathrm{prin}} \to M_{\mathrm{prin}}$. Second, we point out that M^* and M^*_{prin} have the same homotopy type since they both are contractible. Thus we can extend the section σ to the whole orbit space M^* by virtue of corollary C. $\square$

We see that the previous argument works in general when the group G is an Eilenberg-McLane space, and the orbit space is a k-disk, with all the isotropy contained in the boundary. In particular we obtain the following result, which generalizes [ES17, Theorem 1.3]. The original result for smooth, effective, cohomogeneity three torus actions on closed, simply-connected 6-manifolds was proven in [McG76].

Theorem 4.11. *Let T^k act smoothly and effectively on a smooth, closed n-dimensional manifold M, such that the orbit space M^* is an $(n-k)$-disk. Furthermore, suppose that all interior points of the orbit space correspond to principal orbits and that points on the boundary of M^* correspond to non-principal orbits. Then the orbit map $\pi\colon M \to M^*$ admits a cross-section.*

Remark 4.12. In Theorem 4.11, since effective torus actions have trivial principal isotropy, we could have said that the boundary of $M/\mathbb{T}^k$ consist of all orbits with non-trivial isotropy.

Last we present a general example which shows, in the setting of homogeneous foliations, why we are interested in the existence of cross-sections. Namely we expose a simple case where the presence of cross-section allows to lift a homeomorphism between orbit spaces to foliated homeomorphisms between the foliated manifolds.

Theorem 4.13. *Suppose that M and N are compact manifolds, with a homogeneous foliation, by a proper, effective action by a fixed Lie group G. Furthermore, suppose there exist cross-sections $s_1\colon M^* \to M$, $s_2\colon N^* \to N$. If there exists a homeomorphism*

$$f^*\colon M^* \to N^*,$$

that preserves the isotropy type of the orbits, then M and N are foliated homeomorphic.

Proof. Take $p \in M$ and set $q = s_1[p]$ and $\tilde{q} = s_2(f^*[p])$. Since f^* preserves the isotropy type we have that $G_q = G_{\tilde{q}}$. Because $G(q) = G/G_q$ we then have that $p = gG_q$ for some $g \in G$. We now define $f\colon M \to N$ by $f(p) = gG_{\tilde{q}} \in G(\tilde{q})$. The action of G is continuous, therefore f is an equivariant homeomorphism. $\square$

Remark 4.14. Theorem 2.21 is a corollary of Theorem 4.13.

4.2 A-FOLIATIONS

Since in the proof of Proposition 4.13 the isotropy information was used to give auxiliary points in the leaves that helped construct the homeomorphism, and in general, for foliations we do not have a natural choice of something that plays the same role as the isotropy information, it is not clear that a result similar to Proposition 4.13 exists for an arbitrary Riemannian foliation.

By assuming more conditions on the topology of the leaves, namely on the homotopy type of the leaves, we can define extra information on the leaf space that will be analogous to the weights defined for torus actions.

In this section we discuss a particular type of foliations, for which the construction of these weights is possible. These foliations have been already studied by Galaz-García and Radeschi in [GGR15]. In particular the authors give a complete description of the leaf space for the codimension 2 case in [GGR15].

An *A-foliation* is a foliation where all the leaves are closed, connected, and *aspherical*, i.e. for $n > 1$ the n-th homotopy group of the leaves is trivial. The following corollary in [GGR15] shows that the principal leaves of an A-foliation on a compact, simply-connected, Riemannian manifold are homeomorphic to tori.

Theorem 4.15 (Corollary B in [GGR15])**.** *Let $(M, \mathcal{F})$ be an A-foliation on a compact Riemannian manifold M. If M is simply-connected, then the regular leaves are homeomorphic to tori.*

We recall that for q close to p in M with respect to the metric of $(M, \mathcal{F})$, if L_q is a principal leaf and L_p is any leaf in M, then there is a fibration:

$$\mathcal{L} \to L_q \to \overline{L}_p, \tag{3.3.1}$$

where $\overline{L}_p = \tilde{L}_p/H$ is a finite cover of L_p, and $\mathcal{L}$ is a leaf in the infinitesimal foliation $\mathcal{F}^p$ (see Section 3.2). Using this description we will describe the topology of the other leaves types in an A-foliation.

First we consider the case when the leaves of the infinitesimal foliation $(\mathbb{S}_p^{\perp}, \mathcal{F}^p)$ are connected. In this case the finite covering $\overline{L}_p$ is trivial, i.e. $\overline{L}_p = L_p$. Thus following the proof of Theorem 3.7 in [GGR15], we prove the following result:

Proposition 4.16. *Let F, M and N be topological manifolds, with F connected, and let $F \to M \to N$ be a fibration. If M is homeomorphic to a torus, then F and N are tori.*

Proof. Since M is aspherical we have from Theorem 3.7 in [GGR15] that F and N are also aspherical. From the long exact sequence of the fibration we get:

$$0 \to \pi_1(F) \to \pi_1(M) \to \pi_1(N) \to 0.$$

Since $\pi_1(M)$ is an Abelian, torsion-free, finitely-generated group, and $\pi_1(F)$ is a subgroup of $\pi_1(M)$, then $\pi_1(F)$ is an Abelian, torsion free, finitely generated group. Thus by classification of finitely generated Abelian groups and the the Borel conjecture F is homeomorphic to a torus. Now assume that $\pi_1(N)$ has torsion. Then for some $k \in \mathbb{Z}$, the cyclic group $\mathbb{Z}_k$ acts freely on the contractible manifold $\tilde{N}$. Therefore it follows that $\tilde{N}/\mathbb{Z}_k$ is an Eilenberg-MacLane space $K(\mathbb{Z}_k, 1)$. This contradicts the fact that $K(\mathbb{Z}_k, 1)$ has infinite cohomological dimension. Thus $\pi_1(N)$ is an Abelian, torsion-free, finitely generated group. Again by the classification of finitely generated Abelian groups and the Borel conjecture, N is homeomorphic to a torus. $\square$

Corollary 4.17. *In an A-foliation all leaves with trivial holonomy are homeomorphic to tori.*

In the case when the leaf L_p has non-trivial holonomy, applying Proposition 4.16 to fibration (3.3.1) we have that the covering $\overline{L}_p$ is homeomorphic to a torus. Thus, applying the long exact sequence of homotopy groups to the fibration $\overline{L}_p \to L_p$ with finite fiber F, we get,

$$0 \to \pi_1(\overline{L}_p) \to \pi_1(L_p) \to \pi_0(F) \to 0.$$

Therefore $\pi_1(L_p)$ is a finite extension of $\pi_0(F)$ by $\pi_1(\overline{L}_p)$. Assume that $\pi_1(L_p)$ is not torsion-free, and recall that since $\overline{L}_p$ is a torus, we have $\tilde{L}_p = \mathbb{R}^n$. Then there exists a finite cyclic subgroup $\mathbb{Z}_k$ acting on the contractible manifold $\tilde{L}_p = \mathbb{R}^n$. As in the proof of Proposition 4.16 this contradicts the fact that the Eilenberg-MacLane space $K(\mathbb{Z}_k, 1)$ has infinite cohomological dimension. Since $\pi_1(\overline{L}_p)$ is $\mathbb{Z}^n$ and F is finite, we have that $\pi_1(L_p) = G$ is a crystallographic group (see [FH83, Section 6],[AK57],[Zas48]). Thus $\pi_1(L_p)$ is a Bieberbach group, since it is a torsion free crystallographic group. By theorem 6.1 in [FH83], for $n \neq 3, 4$, L_p is homeomorphic to a Bieberbach manifold. In Theorem 0.7 in [KL09], it is proved that the Borel conjecture is true in dimension 3.

From the previous discussion it follows that we have proved the following proposition:

Proposition 4.18. *The leaves (of* $\dim \neq 4$*) with non-trivial holonomy of an A-foliation are homeomorphic to Bieberbach manifolds.*

Remark 4.19. In [GGR15] the authors define a B-foliation as an A-foliation with all leaves homeomorphic to Bieberbach manifolds. Since a torus is a Bieberbach space, it follows from Propositions 4.17 and 4.18 that any A-foliation is a B-foliation. Because of this fact, we will not distinguish them in this work.

Corollary 4.20. *In an A-foliation all leaves (of* $\dim \neq 4$*) are homeomorphic to Bieberbach manifolds.*

Remark 4.21. The diffeomorphism type of the leaves of an A-foliation may not be unique. If the leaves have trivial holonomy, i.e. are homeomorphic to tori, then for dimensions $k \geqslant 5$, there exist different smooth structures $\{U_{\alpha_1}, \varphi_{\alpha_1}\}$, $\{U_{\alpha_2}, \varphi_{\alpha_2}\}$ on the k-torus T^k, such that $\tau_1^k = (T^k, U_{\alpha_1}, \varphi_{\alpha_1})$ is homeomorphic (as a topological manifold) to $\tau_2^k = (T^k, U_{\alpha_2}, \varphi_{\alpha_2})$, but τ_1^k is not diffeomorphic to τ_2^k (see for example [HS70]) .

As a concrete example of this exotic phenomena, we may consider Σ^k an exotic sphere and the standard torus:

$$\mathbb{T}^k = \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{k\text{-times}}.$$

The manifold $\mathbb{T}^k \# \Sigma^k$ is homeomorphic to $\mathbb{T}^k$ but not diffeomorphic to $\mathbb{T}^k$ (see Remark pp.18 in [FJ90] and Theorem 3 in [FJO07]).

We end this section by stating for $q \in M$, with the leaf L_q through q singular, which is the homeomorphism type of the leaves of the infinitesimal foliation $(\mathbb{S}_p^{\perp}, \mathcal{F})$. Since they are connected it follows from Theorem 4.15 and Proposition 4.16 that the infinitesimal foliation is an A-foliation by tori. We collect this fact in the following corollary.

Corollary 4.22. *The infinitesimal foliations of an A-foliation are A-foliations on round spheres whose leaves are all homeomorphic to tori.*

4.3 MOLINO BUNDLE

Next we introduce the so called *Molino bundle* of a Riemannian foliation. Let $(S, \mathcal{F})$ be a Riemannian foliation of codimension q (recall that this means that the dimension of the leaves is constant). We consider the subbundle $N(S, \mathcal{F})$ of the tangent bundle TS, consisting of vector fields on S which are orthogonal to the leaves of the foliation $\mathcal{F}$. This is called the *normal bundle* of the Riemannian foliation $(S, \mathcal{F})$. The normal bundle $N(S, \mathcal{F})$ is the orthogonal complement to the smooth distribution (and thus a bundle over S) given by the tangent spaces of the leaves of the foliation $\mathcal{F}$. We denote by $\widehat{S}$ the bundle of orthonormal frames of $N(S, \mathcal{F})$, and call it *the transverse orthonormal frame bundle* or the *Molino bundle* of the Riemannian foliation $(S, \mathcal{F})$ (some authors denote $\widehat{S}$ by $OF(S, \mathcal{F})$, see for example [MM03]). Note that the projection map $p\colon \widehat{S} \to S$ has the structure of a principal $O(q)$-bundle. Therefore we can lift the foliation $\mathcal{F}$ of S to obtain a regular foliation $\widehat{\mathcal{F}}$ on $\widehat{S}$, called the *lifted foliation on the orthonormal frame bundle* of the Riemannian foliation.

We describe some properties of the foliated Molino bundle $(\widehat{S}, \widehat{\mathcal{F}})$. The projection map $p\colon (\widehat{S}, \widehat{\mathcal{F}}) \to (S, \mathcal{F})$ is a foliated map by construction of $\widehat{\mathcal{F}}$. The foliation $\mathcal{F}$ is closed if and only if $\widehat{\mathcal{F}}$ is a closed foliation. The restriction of the map p to a leaf $\widehat{L}$ of $\widehat{\mathcal{F}}$ is a covering map of the leaf $L = p(\widehat{L})$. The group of deck transformations of the covering $p\colon \widehat{L} \to L$ is the holonomy group Γ_L. Thus if L is a principal leaf of $(S, \mathcal{F})$, the leaf of $\widehat{\mathcal{F}}$ corresponding to L is diffeomorphic to L. For details on the construction of this foliation $\widehat{\mathcal{F}}$, and proofs of all the previous statements we invite the reader to check for instance Example 4.19 in [MM03].

We will use other properties of the Molino bundle, which are stated in the following theorem:

Theorem 4.23 (Molino's structure theorem, Theorem 4.26 in [MM03], Theorem 10.1 in [Ton97], [Mol82])**.** *Let $(S, \mathcal{F})$ be a Riemannian foliation of codimension q on a compact, connected, Riemannian manifold. The following hold:*

(i) There exists a manifold $\widehat{W}$ with an $O(q)$-action and a fiber bundle $\widehat{\pi}\colon \widehat{S} \to \widehat{W}$ such that $\widehat{\pi}$ is $O(q)$-equivariant.

(ii) The fibers of $\widehat{\pi}$ are the closure of the leaves of $\widehat{\mathcal{F}}$.

(iii) *Let* $\overline{\mathcal{F}}$ *denote the singular Riemannian foliation on* S *given by the closure of the leaves of* $\mathcal{F}$ *and consider* $W = S/\overline{\mathcal{F}}$*, the space of leaves. Then* $W = \widehat{W}/O(q)$ *and, for the quotient maps* $\widehat{\pi}\colon \widehat{S} \to W$ *and* $\pi\colon S \to W$*, the following diagram commutes,*

$$\begin{tikzcd} \widehat{S} \arrow[r, "p"] \arrow[d, "\widehat{\pi}"'] & S \arrow[d, "\pi"] \\ \widehat{W} \arrow[r, "\beta"'] & W \end{tikzcd}$$

Next we consider the principal universal bundle $EO(q) \to BO(q)$ associated to $O(q)$. For a Riemannian foliation $(S, \mathcal{F})$ on a compact manifold, we define the *Borel constructions* of the spaces $\widehat{S}$ and $\widehat{W}$ given by Theorem 4.23, as the quotient manifolds $\widehat{S}_O = (\widehat{S} \times EO(q))/O(q)$ and $\widehat{W}_O = (\widehat{W} \times EO(q))/O(q)$. The action of $O(q)$ on both products, $\widehat{S} \times EO(q)$ and $\widehat{W} \times EO(q)$, is given by the diagonal action. By functoriality we obtain a fiber bundle $\widehat{\pi}_O\colon \widehat{S}_O \to \widehat{W}_O$. The fibers of this bundle are homeomorphic to the fibers of the bundle $\widehat{\pi}\colon \widehat{S} \to \widehat{W}$. If $(S, \mathcal{F})$ is closed, then this induces a regular foliation $\widehat{\mathcal{F}}_O$ on $\widehat{S}_O$. The projection map $p_O\colon (\widehat{S}_O, \widehat{\mathcal{F}}_O) \to (S, \mathcal{F})$ is a foliated map. Again the restriction of p_O to $\widehat{L}$, a leaf of $\widehat{\mathcal{F}}$, is a covering map $p_O\colon \widehat{L} \to L$, where L is the corresponding leaf of $\mathcal{F}$. Furthermore from Theorem 4.23 (iii), we get the following commutative diagram:

$$\begin{tikzcd} \widehat{S}_O \arrow[r, "\widehat{\pi}_O"] \arrow[d, "\widehat{p_O}"'] & \widehat{W}_O \arrow[d, "\widehat{\beta}"] \\ S \arrow[r, "\pi"'] & W \end{tikzcd}$$

Since by construction $O(q)$ acts freely on $\widehat{S}$, we have a fiber bundle $p_O\colon \widehat{S}_O \to S$ with fibers $EO(q)$. Since $EO(q)$ is contractible, p_O is a homotopy equivalence between $\widehat{S}_O$ and S. Namely we have proved the following proposition.

Proposition 4.24 (Proposition 2.4 in [FGLT15])**.** *For a singular Riemannian foliation* $(S, \mathcal{F})$ *on a compact manifold, the Borel construction* $\widehat{S}_O$ *of the Molino bundle* $\widehat{S}$ *is homotopy equivalent to* S*.*

Remark 4.25. By Theorem 3.11, if $(S, \mathcal{F})$ is a closed Riemannian foliation on a compact manifold, the space of leaves $S^* = S/\mathcal{F}$ is an orbifold. Furthermore, the foliation $\widehat{\mathcal{F}}$ on $\widehat{S}$ is closed. Therefore the fibers of $\widehat{\pi}\colon \widehat{S} \to \widehat{W}$ are the leaves of $\mathcal{F}$. Since the fibers of $\widehat{\pi}\colon \widehat{S} \to \widehat{W}$ are covering spaces of the fibers of $\mathcal{F}$, they are diffeomorphic via the map $p\colon \widehat{S} \to S$ to the principal leaves of $\mathcal{F}$. Moreover for a closed Riemannian foliation $(S, \mathcal{F})$ on a compact manifold S, the Borel construction $\widehat{W}_O$ coincides with Haefliger's classifying space $B(S^*)$ of the orbifold S^* (see [ALR07, Corollary 5.2] [Hae84, Section 4], [FGLT15]). Therefore for $n \geqslant 1$ we have that $\pi_n^{\mathrm{orb}}(S^*) = \pi_n(\widehat{W}_O)$.

Consider L_0 and L principal leaves in a closed singular Riemannian foliation $(M, \mathcal{F})$ on a compact, simply-connected manifold M. Take any path $\gamma\colon I \to \Sigma^*_{\mathrm{reg}}$, with $\gamma(0) = L_0^*$ and $\gamma(1) = L^*$. For a fixed point $x \in L_0$, from Proposition 1.3.1 in [GW09] there exists a unique curve $\gamma^x\colon I \to \Sigma_{\mathrm{reg}}$, such that $\gamma^x(0) = x$ and it is perpendicular to all the leaves of $\mathcal{F}$ it intersects. Such a curve γ^x is called the *horizontal lift of* γ *through* x (see [GW09, Chapter 1] for more details). With this we are able to define a homeomorphism $h_\gamma\colon L_0 \to L$, by setting $h_\gamma(x) = \gamma^x(1)$. We will show that if we consider two such curves γ_0 and γ_1 connecting L_0^* to L^*, then the homeomorphisms h_{γ_0} and h_{γ_1} are homotopic.

In order to proof this, we need to define a *full singular Riemannian foliation*. We say that a singular Riemannian foliation $\mathcal{F}$ on a Riemannian manifold M is *full*, if for each leaf L there is some $\epsilon > 0$, such that, the map $v \mapsto \exp(\epsilon v)$ is defined for any unit vector v in the normal bundle νL of L. If M is complete this is the case. If $\mathcal{F}$ is a full singular Riemannian foliation on a Riemannian manifold M with all leaves closed, then M^* is a metric space, with a natural inner metric that has curvature locally bounded below in the sense of Alexandrov (see [LT10]). Finally, a full regular Riemannian foliation is simple, i.e. has closed leaves with trivial holonomy, if and only if the quotient M^* is a Riemannian manifold. (see [Lyt10]). In particular for full foliations on simply-connected manifolds we have the following result.

Lemma 4.26 (Corollary 5.3 in [Lyt10])**.** *Let $(M, \mathcal{F})$ be a full singular Riemannian foliation on a simply-connected Riemannian manifold M, with all the leaves closed. Then the quotient $B = \Sigma_{\text{reg}}/\mathcal{F}$, of the restriction of $\mathcal{F}$ to the regular part Σ_{reg} is a Riemannian orbifold with $\pi_1^{\text{orb}}(B) = 1$.*

With the previous lemma we can then show that the quotient M^*_{prin} of the principal stratum M_{prin} is simply connected.

Lemma 4.27. *Let $(M, \mathcal{F})$ be a singular Riemannian foliation with closed leaves on a compact, simply-connected, Riemannian manifold. Then M^*_{prin} is simply-connected in M^**

Proof. Recall from Proposition 3.7 in [Mol88], that $\Sigma^*_{\text{reg}} = \Sigma_{\text{reg}}/\mathcal{F}$ is a Riemannian orbifold since the leaves of $(\Sigma_{\text{reg}}, \mathcal{F})$ are closed. Furthermore, from the fact that M is compact it follows that M is a complete Riemannian foliation. Therefore $(M, \mathcal{F})$ is a full foliation. Since M is simply connected, by applying Lemma 4.26 it follows that the orbifold fundamental group $\pi_1^{\text{orb}}(\Sigma^*_{\text{reg}})$ of Σ^*_{reg} is trivial. Therefore there are no codimension one strata in Σ^*_{reg} (see for example [Lan18]). Following the notation of Section 1.3 in [Dav11] if $\Sigma^*_{\text{reg}(1)}$ denotes the complement of the strata of codimension at least 2, then $\Sigma^*_{\text{reg}(1)}$ consists only of strata of codimension one and zero. Since the codimension one stratum is empty, then $\Sigma^*_{\text{reg}(1)}$ corresponds exactly to the codimension zero stratum. From the fact that the codimension zero strata is the regular part of the orbifold (i.e. the manifold part), we conclude that $\Sigma^*_{\text{reg}(1)} = M^*_{\text{prin}}$. Taking x_0 in the interior of M^*_{prin}, from Section 1.3 in [Dav11] the orbifold fundamental group $\pi_1^{\text{orb}}(\Sigma^*_{\text{reg}}, x_0)$ is generated by taking $\pi_1(M_{\text{prin}}, x_0)$ and adding the following generators:

(i) For each component T of a codimension 2 stratum in the interior of Σ^*_{reg}, choose a loop α_T, starting at x_0, which makes a small loop around T.

And the following relations:

(ii) $[\alpha_T]^{n(T)}$, for some positive integer $n(t)$.

With this the group $\pi_1(M^*_{\text{prin}}, x_0)$ is a subgroup of $\pi_1^{\text{orb}}(\Sigma^*_{\text{reg}}, x_0) = 1$. Thus we conclude that $\pi_1(M^*_{\text{prin}}, x_0) = 1$ □

Corollary 4.28. *Consider a singular Riemannian foliation $(M, \mathcal{F})$ with closed leaves on a compact simply-connected Riemannian manifold. Fix L_0 and L principal leaves of $\mathcal{F}$ and consider two paths $\gamma_0 \colon I \to M^*_{\text{prin}}$ and $\gamma_1 \colon I \to M^*_{\text{prin}}$, connecting L_0^* and L^*. Then the homeomorphism h_{γ_0} is homotopic to h_{γ_1}.*

Proof. From Lemma 4.27 we have that M^*_{prin} is simply connected. Therefore there is a homotopy from $H \colon I \to I \to M^*_{\text{prin}}$ from γ_0 to $\gamma(1)$ fixing the end points L_0^* and L^*. This defines a continuous family of curves $\gamma_s \colon I \to M^*_{\text{prin}}$, by setting $\gamma_s(t) = H(t, s)$. We define a homotopy $\widetilde{H} \colon L_0 \times I \to L$ by setting $\widetilde{H}(x, s) = \gamma_s^x(1)$. □

4.4 WEIGHTS OF AN A-FOLIATION

For a homogeneous A-foliation $(M, \mathcal{F})$ of low codimension (i.e. one induced by an effective torus action), Orlik and Raymond in [OR70], encoded the isotropy information of the orbits into weights of the orbit space M^*. This approach was followed by Oh in [Oh83a], and Fintushel in [Fin77], to give equivariant classifications of homogeneous A-foliations by encoding the isotropy information as weights. In this section we extend the notion of weights to an arbitrary A-foliation $(M, \mathcal{F})$ on a compact, simply-connected manifold M.

We start by fixing a principal leaf L_0. We consider any arbitrary point $p \in M$ and fix it. Next we take $v \in \mathbb{S}_p^\perp$, a normal vector to T_pL_p, such that $q = \exp_p(v)$ is contained in a principal leaf. From Theorem 3.12 there exists a path $\gamma \colon I \to \Sigma^*_{\text{reg}}$ connecting q^* and L_0^*. We consider the horizontal lift γ^q of γ, through q, and we set $q_0 = \gamma^q(1) \in L_0$. Recall from Section 3.3 that, in this setting, for some cover $\overline{L}_p \to L_p$, we have a fibration

$$\mathcal{L}_v \to L_q \to \overline{L}_p. \tag{3.3.1}$$

From Theorem 4.15 (cf. [GGR15, Corollary B]) and Proposition 4.16, the principal leaf $L_q = T^n$, $\overline{L}_p = T^{n-k}$, and the leaf of the infinitesimal foliation $\mathcal{L}_v = T^k$, for some $k \leqslant n$. From the homotopy long exact sequences of the fibration we get a short exact sequence

$$0 \to \pi_1(\mathcal{L}_v, q) \to \pi_1(L_q, q) \to \pi_1(\overline{L}_p, p) \to 1.$$

The path $\gamma\colon I \to M^*_{\text{prin}}$ connecting L_0^* to L_q^* induces a homeomorphism $h_\gamma\colon L_0 \to L_q$. Via this homeomorphism, from the previous short sequence of homotopy groups of the fibration, we obtain the following short exact sequence

$$0 \to \pi_1(\mathcal{L}_v, q) \to \pi_1(L_0, q_0) \to \pi_1(\overline{L}_p, p) \to 1.$$

Since all spaces in this short exact sequence are tori, the short exact sequence becomes

$$0 \to \mathbb{Z}^k \to \mathbb{Z}^n \to \mathbb{Z}^{n-k} \to 0.$$

Consider $e_1, \ldots, e_k$, generators of $\pi_1(\mathcal{L}_v, q) = \mathbb{Z}^k$. They are mapped to elements $a_{p1}, \ldots, a_{pk}$ in $\pi_1(L_0, q_0) = \mathbb{Z}^n$.

The definition of the integers $a_{p1}, \ldots, a_{pk}$ depends a priori on the choice of path γ joining L_0^* to L_q^*. The following lemma shows that in fact, they are independent of the choice of γ.

Lemma 4.29. *The elements $a_{p1}, \ldots, a_{pk} \in \pi_1(L_0, q_0)$ do not depend on the path $\gamma\colon I \to M^*$.*

Proof. If we choose any other path γ_1 from L_0^* to L_q^*, then Corollary 4.28, shows that the group isomorphisms induced by $(h_\gamma)_*\colon \pi_1(L_0, q_0) \to \pi_1(L_q, q)$ and $(h_{\gamma_1})_*\colon \pi_1(L_0, q_0) \to \pi_1(L_q, q)$ are equal. Therefore the set of integer vectors $a_{p1}, \dots, a_{pk}$ do not depend of the curve γ. $\square$

Next we prove that if we choose another vector $w \in \mathbb{S}_p^\perp$ such that $\exp_p(w)$ lies in a principal leaf, then we recover the same integers $a_{p1}, \dots, a_{pk}$.

Lemma 4.30. *The integers $a_{p1}, \dots, a_{pk}$ do not depend on the choice of $v \in \mathbb{S}_p^\perp$.*

Proof. Take $w \in \mathbb{S}_p^\perp$ another vector with $w \neq v$, such that $q_1 = \exp_p(w)$ lies on a principal leaf L_{q_1}. Since $(\mathbb{S}_p^\perp, \mathcal{F}_p)$ is a singular Riemannian foliation with closed, compact leaves, by Theorem 3.12, the space $(\mathbb{S}_p^\perp/\mathcal{F}^p)_{\text{prin}}$ is path-connected. Therefore there exists a path $\beta\colon I \to (\mathbb{S}_p^\perp/\mathcal{F}^p)_{\text{prin}}$ from $\mathcal{L}_v^* \in \mathbb{S}_p^\perp/\mathcal{F}^p$ to $\mathcal{L}_w^* \in \mathbb{S}_p^\perp/\mathcal{F}^p$. By taking horizontal lifts of β in $(\mathbb{S}_p^\perp, \mathcal{F}^p)_{\text{reg}}$ we obtain a homeomorphism $h_\beta\colon \mathcal{L}_v \to \mathcal{L}_w$. By setting $q_1' = \exp_p(h_\beta(v))$, the homeomorphism h_β induces an isomorphism $(h_\beta)_*\colon \pi_1(\mathcal{L}_v, q) \to \pi_1(\mathcal{L}_w, q_1')$. From Corollary 4.28, this isomorphism is independent of the choice of β.

Let σ be a path in $\mathcal{L}_w$ from q_1 to q_1'. This gives an isomorphism from $\pi_1(\mathcal{L}_w, q_1')$ onto $\pi_1(\mathcal{L}_w, q_1)$, given by mapping an element $[\delta] \in \pi_1(\mathcal{L}_w, q_1')$ to $[\sigma^{-1}\delta\sigma]$. Let α be another path in $\mathcal{L}_w$ from q_1 to q_1'. Consider the concatenation of paths $\sigma\alpha^{-1}\delta\alpha\sigma^{-1}$. The path $\alpha\sigma^{-1}$ is a loop based at q_1'. Thus we have a conjugation $[\sigma\alpha^{-1}][\delta][\alpha\sigma^{-1}]$ in $\pi_1(\mathcal{L}_v, q_1')$. Since we have an A-foliation, $\mathcal{L}_w$ is homeomorphic to a torus. Thus $\pi_1(\mathcal{L}_v, q_1')$ is an Abelian group. Therefore the path $\sigma\alpha^{-1}\delta\alpha\sigma^{-1}$ is homotopic to σ, relative to the end points. Thus $\alpha^{-1}\delta\alpha$ is homotopic to $\sigma^{-1}\delta\sigma$. Therefore the isomorphism from $\pi_1(\mathcal{L}_w, q_1')$ onto $\pi_1(\mathcal{L}_w, q_1)$, does not depend on the path σ. It follows that we have a well defined isomorphism from $\pi_1(\mathcal{L}_v, q)$ to $\pi_1(\mathcal{L}_w, q_1)$.

Let $h_\gamma\colon L_q \to L_0$ and $h_\lambda\colon L_{q_1} \to L_0$, be homeomorphisms given by paths $\gamma\colon I \to M_{\text{reg}}^*$ and $\lambda\colon I \to M_{\text{reg}}^*$. Set $x_0 = h_\lambda(q_1')$, $y_0 = h_\lambda(q_1)$ and $q_0 = h_\gamma(q)$ (see Figure 4.2). Denote by $i_1\colon \mathcal{L}_v \to L_q$ and $i_2\colon \mathcal{L}_w \to L_{q_1}$ the inclusions, given by the bundles (3.3.1), of the infinitesimal leaves into the leaves L_q and L_{q_1}, respectively. The homeomorphism h_β induces an isomorphism from $(h_\gamma \circ i_1)_*(\pi_1(\mathcal{L}_v, q)$ onto $(h_\lambda \circ i_2)_*(\pi_1(\mathcal{L}_w, q_1')$. The path $\sigma\colon I \to \mathcal{L}_w$ gives a well defined isomorphism from $(h_\gamma \circ i_2)_*(\pi_1(\mathcal{L}_w, q_1'))$ onto $(h_\lambda \circ i_2)_*(\pi_1(\mathcal{L}_w, q_1))$. Thus a generator of $\pi_1(\mathcal{L}_v, q)$ in $\pi_1(L_0, q_0)$ is mapped to a generator of $\pi_1(\mathcal{L}_w, q_1)$. From this we see that the integer vectors $a_{p1}, \ldots, a_{pk}$ do not depend on v. □

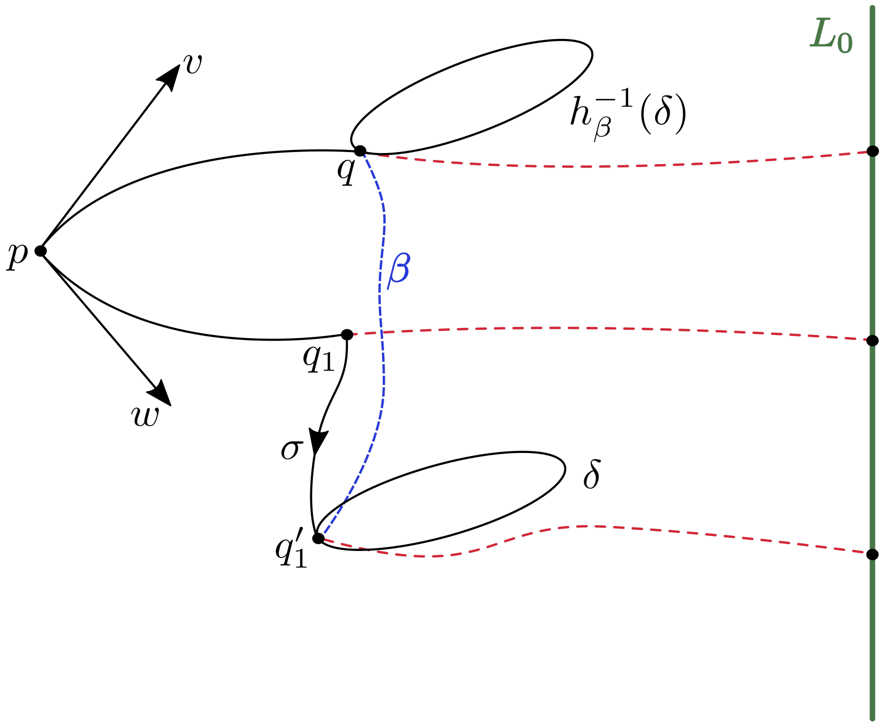

Figure 4.2.: Well defined weights.

From the proof of the previous lemma, by using the fact that the fundamental groups of $\overline{L}_p$ and L_0 are Abelian, it follows that the definition of the integer vectors $a_{p1}, \ldots, a_{pk}$ does not depend on the choice of basepoint p in L_p.

Lemma 4.31. *The weights $a_{p1}, \ldots, a_{pk}$ of L_p do not depend on the choice of $p \in L_p$.*

We recall from Proposition 4.18 that for a leaf L_p with non-trivial holonomy, L_p is a Bieberbach space. The finite covering $\overline{L}_p \to L_p$ implies the existence of the following short exact sequence of groups, where the group H is the subgroup of the holonomy group Γ_L fixing v:

$$0 \to \pi_1(\overline{L}_p, p) \to \pi_1(L_p, p) \to H \to 0.$$

We define the *weights* of the leaves of an A-foliation on a compact, simply-connected, manifold as follows. A principal leaf has no weight associated to it. To an exceptional leaf L_p, we associate the collection $\{\pi_1(L_p), H\}$. For a singular leaf L_p without holonomy we associate $\{a_{p1}, \ldots, a_{pk}\}$. Finally, the weight of a singular leaf with holonomy L_p is the collection $\{a_{p1}, \ldots, a_{pk}; \pi_1(L_p, p), H\}$. With this information we can recover the homeomorphism type of a leaf, as well as its leaf type. Therefore we have encoded the leaf type information in the weights.

We say two weighted leaf spaces, M_1^* and M_2^*, are *isomorphic* if there is a homeomorphism $\varphi\colon M_1^* \to M_2^*$ sending the weights of M_1^* to the weights of M_2^*. The map φ is called an *isomorphism between the weighted leaf spaces*, or just simply an *isomorphism between the leaf spaces.* The following theorem, analogous to Theorem 4.13, shows the weighted space classifies the topology of M as well as the foliation $\mathcal{F}$.

Theorem D. *If $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ are compact simply connected manifolds, with A-foliations, such that they have isomorphic weighted leaf spaces and admit cross-sections $\sigma_i\colon M_i^* \to M_i$, then $(M_1, \mathcal{F}_1)$ is foliated homeomorphic to $(M_2, \mathcal{F}_2)$.*

Proof. Given a weighted isomorphism $\phi^*\colon M_1^* \to M_2^*$, between the leaf spaces we will define a foliated homeomorphism $\phi\colon (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$. Fix $x \in M_1$ and for the cross-section $\sigma_1\colon M_1^* \to M$, set $y = \sigma_1(x^*)$. The leaf $L_x = L_y$ is homeomorphic to $\mathbb{R}^k/\Gamma$, where Γ is a Bieberbach group and $0 \leqslant k \leq \dim(\mathcal{F})$. The Dirichlet domain $D \subset R^k$, of the action of Γ on $\mathbb{R}^k$, is a convex fundamental domain (see for example [Rö10, Theorem 2]). We may assume that a preimage of y corresponds to the center of the Dirichlet domain. Furthermore we may assume (via a translation) that in turn the center of the Dirichlet domain is the origin $\overline{0} \in \mathbb{R}^k$. Then there is a unique vector $v_x \in \mathbb{R}^k$ connecting the origin to a preimage of x in the Dirichlet domain D.

We set $\phi(y) = \sigma_2(\phi^*(y^*))$. Since ϕ^* preserves the weights, then it preserves the leaf type, and thus we have that $L_{\phi(y)}$ is homeomorphic to $L_y = \mathbb{R}^k/\Gamma$. Last we set $\phi(x)$ as the point in the Dirichlet domain of $\phi(y)$ which corresponds to the vector v_x. In the same fashion we can construct a continuous foliated inverse map. Thus we have that ϕ is a foliated homeomorphism. □

Lets assume that the leaf spaces M_i^* admit a smooth structure (for example when they are homeomorphic to disks). In this case it may happen that the cross-sections $\sigma_i \colon M_i^* \to M_i$ are smooth. Furthermore if the leaves of $(M_i, \mathcal{F}_i)$ have a standard smooth structure then they are diffeomorphic to $\mathbb{R}^n/\Gamma$. If this three hypothesis are met, i.e. the orbit space are smooth manifolds, the cross-sections are smooth, and the leaves have a standard smooth structure the map $\phi\colon (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ is a foliated diffeomorphism.

Lemma 4.32. *Let $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F})$ be compact, simply-connected manifolds, with A-foliations with standard diffeomorphism type, and isometric leaf spaces. If the leaf spaces M_1^* and M_2^* are homeomorphic to smooth manifolds, there is a weighted diffeomorphism $f^*\colon M_1 \to M_2$, and the cross-sections $\sigma_i \colon M_i^* \to M_i$ are smooth with respect to these smooth structure, then the foliated homeomorphism of Theorem D is a foliated diffeomorphism.*

Proof. This follows from the fact that the foliated homeomorphism defined in the proof of Theorem D, is defined by composition of the map f^* and the cross-sections, which by hypothesis are smooth. The fact that the leaves are diffeomorphic to $\mathbb{R}^n/\Gamma$ is used to show that once we have chosen our center of the Dirichlet domain y, the dependency of $x \in L$ with respect to this center is smooth. □

Let us consider $(M, \mathcal{F})$ a compact, manifold with a singular Riemannian foliation admitting a cross-section $\sigma\colon M^* \to M$. When M^* is homeomorphic to a smooth manifold (i.e. it admits a smooth structure), then the following lemma gives sufficient conditions to the existence of a smooth cross-section.

Lemma 4.33. *Let $(M, \mathcal{F})$ be compact manifold with a singular Riemannian foliation. Assume that there is a cross-section $\sigma\colon M^* \to M$, the leaf space M^* admits a smooth structure and the quotient map $\pi\colon M \to M^*$ is smooth with respect to this smooth structure. Then there exists a smooth cross-section $\overline{\sigma}\colon M^* \to M$.*

Proof. We observe that by Theorem 3.3. in [Hir94], it follows that the space of smooth functions $C^\infty(M^*, M)$ is dense in the space of continuous functions $C^0(M^*, M)$ with respect to the strong topology. Therefore there exists a smooth map $h\colon M^* \to M$ close to σ in $C^0(M^*, M)$. Since the quotient map $\pi\colon M \to M^*$ is smooth, the map $\overline{\sigma}\colon M^* \to M$ defined as $\overline{\sigma} = h \circ (\pi \circ h)^{-1}$ is smooth. By construction the map $\overline{\sigma}$ is a cross-section for the map $\pi\colon M \to M^*$. $\square$

5 A-FOLIATIONS OF CODIMENSION 2

Using the frame work we developed in Chapter 4, we will concentrate in this chapter on the study of A-foliations of codimension 2 on compact, simply-connected, Riemannian manifolds. In particular we will compare such foliations to homogeneous ones, and show that we can apply Theorem D. With this we will prove that any A-foliation of codimension 2 on a compact, simply-connected, Riemannian manifold is, up to foliated homeomorphism, a homogeneous foliation.

5.1 LEAF SPACE OF A-FOLIATIONS OF CODIMENSION 2

We give a short review of how to prove that, for a compact, simply-connected manifold M with a singular A-foliation of codimension 2, the leaf space is a 2-disk. We begin by recalling that A-foliations of codimension 1 are homogeneous, and likewise regular A-foliations of codimension 2 are homogeneous, provided the manifold is closed and simply connected.

Theorem 5.1 (Theorem E in [GGR15]). *Let $(M, \mathcal{F})$ be a simply-connected manifold equipped with regular a A-foliation of codimension 2. Then $M = \mathbb{S}^3$ and the foliation is given by a weighted Hopf action, or the following hold.*

(i) *The leaf space $B = M/\mathcal{F}$ is homeomorphic to a 2-disk, the interior of B is smooth, and the boundary ∂B consists of at least n totally geodesic segments meeting in an angle of $\pi/2$.*

(ii) *Let L_0 be a generic leaf and L_1 be a singular leaf. Then there is a submersion $L_0 \to L_1$, with fiber $\mathbb{S}^1$ if L_1 belongs to a geodesic in ∂B, or with fiber T^2, if L_1 belongs to a vertex of ∂B.*

Thus we will concentrate only on singular (i.e. where the dimensions of the leaves is not constant) A-foliations of codimension 2. Let $(M, \mathcal{F})$ be a compact, simply-connected manifold with such a foliation. For $p \in M$ we define the *quotient codimension* of the stratum Σ^p as:

$$\mathrm{codim}(M, \mathcal{F}) - \mathrm{codim}(\Sigma^p, \mathcal{F}).$$

Clearly, if $(M, \mathcal{F})$ is a singular Riemannian foliation of codimension 2, then, for any $p \in M$, the quotient codimension of Σ^p is less than or equal to 2. Thus the following proposition establishes that codimension 2 singular Riemannian foliations are infinitesimally polar.

Proposition 5.2 (Proposition 3.1 in [LT10])**.** *Let $(M, \mathcal{F})$ be a singular Riemannian foliation. Let $x \in M$ be a point with stratum Σ^x of quotient codimension at most 2. Then $\mathcal{F}$ is infinitesimally polar at x.*

Corollary 5.3. *Singular A-foliations of codimension 2 are infinitesimally polar.*

With this information we see that the orbit space of a codimension 2 singular Riemannian foliation is an orbifold. For M simply-connected, applying the following theorems by Lytchak, we see that there are no exceptional leaves, and that the leaf space M^* has non-empty boundary. Furthermore the boundary corresponds to singular strata.

Theorem 5.4 (Theorem 1.6 in [Lyt10])**.** *Let $(M, \mathcal{F})$ be a closed infinitesimally polar singular Riemannian foliation on a complete manifold with quotient orbifold*

M^*. Then all a singular leaves are contained in the boundary of ∂M^*. If M is simply connected, then the quotient M^* has no boundary if and only if $\mathcal{F}$ is regular.

Theorem 5.5 (Corollary 1.7 in [Lyt10])**.** *Let $(M, \mathcal{F})$ be a singular Riemannian foliation on a complete simply connected manifold, with quotient M^* of dimension 2. Then either the foliation is regular or there are no exceptional leaves.*

As in the case of compact Lie group actions, the fundamental group of M surjects onto the fundamental group of the leaf space via $\pi_*\colon \pi_1(M) \to \pi_1(M^*)$ (see [Bre72, Chp. II, Thm. 6.2], [Bre72, Chp. II, Cor.6.3]). Therefore M^* is a simply-connected 2-orbifold with boundary. Thus it is homeomorphic to a 2-disk. The boundary ∂M^* is divided into k edges γ_i, and k vertexes F_i, labeled as pictured in Figure 5.1.

The leaves that project under $\pi\colon M \to M^*$ to interior points of the arc γ_i, contained in ∂M^*, which we call *least singular leaves*, are singular leaves of codimension 3 in M and thus are homeomorphic to T^{n-1}. The leaves that project to the vertexes of ∂M^*, called *most singular leaves*, are singular leaves homeomorphic to T^{n-2}. For a point q in a singular leaf, we have by corollary 4.22 and [GGR15, Theorem D], that the infinitesimal foliation $(\mathbb{S}_q^{\perp}, \mathcal{F}_q)$ at q is one of the homogeneous foliations $(\mathbb{S}^2, \mathbb{S}^1)$ or $(\mathbb{S}^3, T^2)$, induced by orthogonal actions.

The first case occurs when the singular leaf projects to an interior point of an edge in M^*, and the second case occurs when the singular leaf projects to a vertex. Since there are no exceptional leaves, the holonomy action is trivial. Thus for any point $q \in M$, we have $\overline{L}_q = L_q$. Therefore, from Proposition 4.16 all the leaves of an A-foliation of codimension 2 are homeomorphic to tori. Furthermore for each edge in M^* we have the following type of fibration:

$$\mathbb{S}^1 \to T^n \to T^{n-1}, \tag{5.1.1}$$

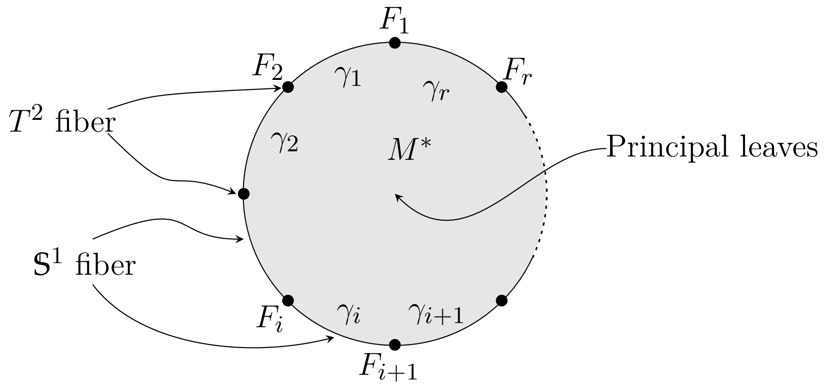

Figure 5.1.: Leaf space of A-foliation of codimension 2.

For each vertex in M^* we have the following two type of fibration:

$$T^2 \to T^n \to T^{n-2}. \tag{5.1.2}$$

In both cases the maps are smooth submersions.

5.2 WEIGHTS OF A-FOLIATION OF CODIMENSION 2

We know introduce the weights we developed in the preceding chapter for the special case of an A-foliation of codimension 2 on a compact, simply-connected manifold. We recall from the previous section that we have three types of leaves: the principal ones, the least singular ones, and the most singular ones.

The weights of the least singular leaves, $(a_{i1}, \ldots, a_{in}) \in \mathbb{Z}^n$ correspond to the image of the generator $\alpha_i \in \pi_1(\mathbb{S}^1)$ under the inclusion $\pi_1(\mathbb{S}^1) \to \pi_1(T^n)$.

Recall from Section 2.3 that for a homogeneous foliation given by a cohomogeneity 2 torus action, there are no exceptional orbits, and furthermore the singular orbits are tori $\mathbb{T}^{n-1}$ and $\mathbb{T}^{n-2}$. Furthermore the possible non-trivial isotropy groups are $\mathbb{T}^1$ and $\mathbb{T}^2$. Thus the weights for the leaves of codimension 3 in a homogeneous foliation given by a cohomogeneity two torus action, correspond to the following fibrations:

$$\mathbb{T}^1 = G_p \to G = \mathbb{T}^n \to G/G_p = \mathbb{T}^{n-1}.$$

Therefore the weights show how the isotropy subgroup $\mathbb{T}^1$ is immersed in the group $\mathbb{T}^n$. From these observations it follows that the weights defined for A-foliations coincide with the weights defined by Oh for torus actions in [Oh83a].

5.3 TOPOLOGICAL CLASSIFICATION OF A-FOLIATIONS OF CODIMENSION 2

In this section we prove the following equivalence theorem for A-foliations of codimension 2 on compact, simply-connected $(n+2)$-manifolds, which is one of our main results.

Theorem E. *Let $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ be two compact, simply-connected smooth $(n+2)$-manifolds, admitting singular A-foliations of codimension 2 and $n \geqslant 2$. Then M_1 is foliated homeomorphic to M_2 if and only if the weighted leaf spaces $M_1/\mathcal{F}_1$ and $M_2/\mathcal{F}_2$ are isomorphic.*

Proof. We begin by observing that, for any A-foliation of codimension 2 on a compact, simply-connected, smooth, Riemannian manifold M the leaf space is a disk, which is a 2-dimensional CW-complex, with all interior points corresponding to principal leaves. Since the interior of the disk is contractible, by Theorem A, there is a cross-section defined on the interior of the disk. Furthermore we note that the disk and its interior are homotopic equivalent, and thus we apply Corollary C to get the existence of a cross-section $\sigma\colon M^* \to M$. Then applying Theorem D we get the desired conclusion. □

Remark 5.6. For an A-foliation $(M, \mathcal{F})$ of codimension 2 on a compact simply-connected $(n+2)$-manifold with $n \geqslant 2$, an other approach to obtaining a cross-section $\sigma\colon M^* \to M$ can be done using obstruction theory, and the procedure of Orlik and Raymond in [OR70]. Namely we can split the leaf space M^* into an open interior disk Y^*, and quadrilaterals $D_1^*, \ldots, D_r^*$, as in Figure 5.2. Then we apply obstruction theory to show the existence of cross-section over Y^*. We apply again obstruction theory to show that we can extend the given cross-section to D_1^*. We then continue applying obstruction theory to extend the cross-section to each D_i^*.

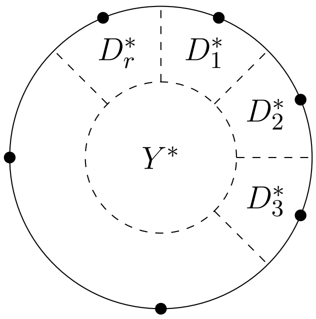

Figure 5.2.: Decomposition of M^* in [OR70].

Moreover we remark that, for $(M, \mathcal{F})$ a compact, simply-connected manifold with an A-foliation of codimension 2, the leaf space M^* is homeomorphic to a 2-disk. Thus the leaf space M^* admits a unique smooth structure. We will show that the hypothesis of Lemma 4.33 are satisfied in this case, and thus we may consider smooth cross-sections for A-foliations of codimension 2 .

Lemma 5.7. *Consider an A-foliation $(M, \mathcal{F})$ of codimension 2 on a compact, simply-connected manifold. Let $p \in M$ be such that L_p is a least singular leaf (i.e. L_p has codimension 3 in M). Then the following hold for the infinitesimal foliation $(\mathbb{S}_p^\perp, \mathcal{F}^p)$ at p.*

(i) The quotient space $\mathbb{S}_p^\perp / \mathcal{F}^p$ is homeomorphic to the closed interval $[0, \pi]$, and thus it admits a unique smooth structure.

(ii) *The quotient map* $\mathbb{S}^\perp_p \to \mathbb{S}^\perp_p/\mathcal{F}^p$ *is smooth.*

Proof. We note that $(\mathbb{S}^\perp_p, \mathcal{F}^p)$ is an A-foliation of codimension 1, with principal leaf homeomorphic to $\mathbb{S}^1$. It follows from Theorem D in [GGR15] that $(\mathbb{S}^\perp_p, \mathcal{F}^p)$ is the homogeneous foliation $(\mathbb{S}^2, \mathbb{S}^1)$. Furthermore, from [GGZ39] and [Mos57] it follows that, any smooth action of $\mathbb{S}^1$ on $\mathbb{S}^2$ is equivalent (i.e. there exists an equivariant diffeomorphism) to the linear $\mathbb{S}^1$ action on $\mathbb{S}^2$. We describe this linear action. We consider, for $\mathbb{S}^2$, the following spherical coordinates:

$$(\theta, \varphi) \mapsto (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),$$

with $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$, and parametrizing $\mathbb{S}^1$ by the angle ψ, the action of $\mathbb{S}^1$ on $\mathbb{S}^2$ is given by

$$\psi(\theta, \varphi) = (\theta, \varphi + \psi).$$

Thus the quotient map $\mathbb{S}^2 \to \mathbb{S}^2/\mathbb{S}^1$ is given by $(\theta, \varphi) \mapsto \theta$ (see Figure 5.3). This proves both claims. $\square$

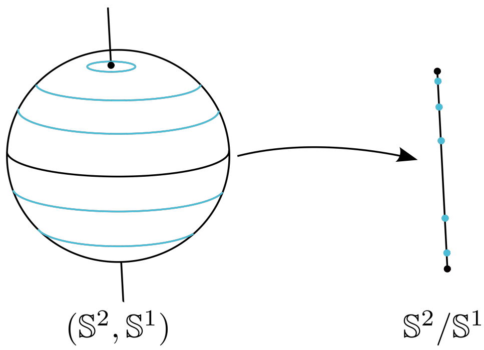

Figure 5.3.: Quotient map of the homogeneous foliation $(\mathbb{S}^2, \mathbb{S}^1)$

Lemma 5.8. *Consider an A-foliation $(M, \mathcal{F})$ of codimension 2 on a compact, simply-connected manifold. Let $p \in M$ be such that L_p is a most singular leaf (i.e. L_p has codimension 4 in M). Then the following hold for the infinitesimal foliation $(\mathbb{S}^\perp_p, \mathcal{F}^p)$ at p.*

(i) *The quotient space* $\mathbb{S}_p^\perp/\mathcal{F}^p$ *is homeomorphic to the closed interval* $[0,\pi/2]$, *and thus it admits a unique smooth structure.*

(ii) *The quotient map* $\mathbb{S}_p^\perp \to \mathbb{S}_p^\perp/\mathcal{F}^p$ *is smooth.*

Proof. We note that $(\mathbb{S}_p^\perp, \mathcal{F}^p)$ is an A-foliation of codimension 1, with principal leaf homeomorphic to $\mathbb{T}^2$. It follows from Theorem D in [GGR15] that $(\mathbb{S}_p^\perp, \mathcal{F}^p)$ is the homogeneous foliation $(\mathbb{S}^3, \mathbb{T}^2)$, given by the standard linear action.

We consider $\mathbb{S}^3$ as the unit sphere in $\mathbb{C}^2$ and we use the so-called *Hopf coordinates* for $\mathbb{S}^3$, given by

$$(\theta_1, \theta_2, \eta) \mapsto (\sin\eta e^{i\theta_1}, \sin\eta e^{i\theta_2}, \cos\eta),$$

with $\theta_1 \in [0,2\pi]$, $\theta_2 \in [0,2\pi]$, and $\eta \in [0,\pi/2]$. We parametrize the 2-torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ by the angles (α,β). With these coordinates the action of $\mathbb{T}^2$ on $\mathbb{S}^3$ is given by:

$$(\alpha,\beta)(\theta_1,\theta_2,\eta) = (\theta_1+\alpha, \theta_2+\beta, \eta).$$

Thus the quotient map $\mathbb{S}^3 \to \mathbb{S}^3/\mathbb{T}^2$ is given by $(\theta_1,\theta_1,\eta) \mapsto \eta$. This proves both claims. □

Proposition 5.9. *For* $(M,\mathcal{F})$ *a compact, simply-connected manifold with an A-foliation of codimension* 2*, the leaf space* M^* *admits a unique smooth structure. Furthermore there is a smooth cross-section* $\overline{\sigma}\colon M^* \to M$ *with respect to this smooth structure.*

Proof. Since, for a (singular) A-foliation $(M,\mathcal{F})$ of codimension 2 on a simply-connected closed manifold the leaf space M^* is a 2-disk, it carries a unique smooth structure proving the first claim of the proposition. In this case in we get a smooth cross-section $\overline{\sigma}\colon M^* \to M$ as follows. Let $\sigma\colon M^* \to M$ be a cross-section obtained from Corollary C. Also we note that, for $p \in M$, the infinitesimal foliation $(\mathbb{S}_p^\perp, \mathcal{F}^p)$ is homogeneous (see Section 5.1). By Lemma 5.7 and Lemma 5.8, each infinitesimal foliation, the quotient map $\mathbb{S}_p^\perp \to \mathbb{S}_p^\perp/\mathcal{F}^p$ is smooth. Since for any point $p \in M$, it has trivial holonomy group, a local neighborhood of p^* is given by a cone over $\mathbb{S}_p^\perp/\mathcal{F}^p$. This implies that the quotient map $\pi\colon M \to M^*$ is smooth. Thus by applying Lemma 4.33 we obtain a smooth cross-section $\overline{\sigma}\colon M^* \to M$. □

Before showing that A-foliations of codimension 2 on simply-connected manifolds are homogeneous we will state some facts about the weights.

From the proof of Theorem A in [GGR15], we are able to determine the number of different bundles of the form (5.1.1) for an A-foliation of codimension 2 on a compact, simply-connected manifold.

Theorem 5.10. *Let $(M, \mathcal{F})$ be a compact, simply-connected $(n+2)$-manifold with an A-foliation of codimension 2, and L_0 a regular leaf of dimension n. If the leaf space M^* has r-edges in the boundary, then $r \geqslant n$.*

Proof. We first note that for A-foliations of codimension 2, a regular leaf is a principal leaf, and fix $p_0 \in L_0$. We consider $M_0 = M_{\mathrm{reg}}$, and $B = B(M_0^*)$ the Haefliger classifying space of M_0^*. Then from Theorem 4.23, Proposition 4.24, and Remark 4.25 we obtain the following long exact sequence:

$$\cdots \to \pi_2(B, b_0) \to \pi_1(L_0, p_0) \to \pi_1(M_0, p_0) \to \pi_1(B, b_0) \to 1.$$

By taking H to be the image of $\pi_2(B, b_0)$ under the group morphism $\pi_2(B, b_0) \to \pi_1(L_0, p_0)$, we obtain the following short exact sequence:

$$0 \to H \to \pi_1(L_0, p_0) \to \pi_1(M_0, p_0) \to \pi_1(B, b_0) \to 1.$$

Using the fact that for an A-foliation of codimension 2 on a compact, simply-connected manifold, the leaf space is a 2-disk, we conclude that $H = 0$. Consider the fibers of the fibrations given by the codimension 3 leafs. I.e. we consider the fibers of the fibrations of the from (5.1.1). Observe that by hypothesis, there are a number r of these fibrations. We consider their homotopy class in L_0 and denote by K the subgroup they generate in $\pi_1(L_0, p_0)$. It follows from the proof of Theorem A in [GGR15] that $\pi_1(L_0, p_0)$ is generated by K and H. Furthermore K splits as an Abelian group and a finite 2 step nilpotent 2-group. Since by Theorem 4.15 L_0 is a torus, then we conclude that the finite 2 step nilpotent 2-group is trivial. Thus from this discussion it follows that there are at least n fibrations of the form (5.1.1). $\square$

Recalling Lemma 2.18, since the fibers of the fibrations of the form (5.1.1) generate a the fundamental group of a principal leaf, we deduce the following property of the weights $(a_{i1}, \ldots, a_{in}) \in \mathbb{Z}^n$, associated to the least singular leaves.

Lemma 5.11. *For an A foliation of codimension* 2*, the determinant of the weights* $(a_{11}, \ldots, a_{1n})$*,*$(a_{21}, \ldots, a_{2n}), \ldots, (a_{k1}, \ldots, a_{kn})$ *is* ± 1*.*

Now we are able to prove the main result for this chapter.

Theorem F. *Let* $(M, \mathcal{F}_1)$ *be closed, simply-connected* $(n+2)$*-manifold with an A-foliation of codimension* 2 *and* $n \geqslant 2$*. Then there exist a closed, simply-connected* $(n+2)$*-manifold* $(N, \mathcal{F}_2)$ *with a homogeneous A-foliation of codimension* 2 *(i.e. with an effective smooth torus action of cohomogeneity* 2*), such that* $(N, \mathcal{F}_2)$ *is foliated homeomorphic to* $(M, \mathcal{F}_1)$*.*

Proof. By Lemma 5.11, for an A-foliation of codimension 2 on a closed, simply-connected $(n+2)$-manifold M, the weights $(a_{i1}, \ldots, a_{in})$ are legal weights in the sense of Oh (see [Oh83a]). Thus by Theorem 2.22 there is a closed, simply-connected, $(n+2)$-manifold N together with a $\mathbb{T}^n$-action realizing the weights. By Theorem E, the manifolds M and N are foliated homeomorphic. $\square$

Remark 5.12. In the case of $n = 2$ or $n = 3$ the differentiable type of the torus is unique, so by Proposition 5.9 and Lemma 4.32, the homeomorphism in Theorem F will be a diffeomorphism.

6 SMOOTH STRUCTURE OF LEAVES OF AN A-FOLIATION

As we have seen in Remark 5.12, for $n = 2, 3$, any A-foliation $(M, \mathcal{F})$ of codimension 2 on a compact, simply-connected $(n+2)$-manifold is, up to foliated diffeomorphism, homogeneous.

In this chapter, we prove that the same statement is true for $n \geqslant 4$. Namely we will show that for $n \geqslant 4$ an A-foliation $(M, \mathcal{F})$ on a compact, simply-connected $(n+2)$-manifold is, up to foliated diffeomorphism, homogeneous. In other words, we can strengthen the conclusions of Theorem F, to obtain the same result up to foliated diffeomorphism.

To achieve this, we need to study the differentiable structure of a principal leaf of $(M, \mathcal{F})$, which, as proved in Section 4.2, is homeomorphic to a torus T^k. Recall, as mentioned in the same section, for $k \geqslant 5$ there are examples of tori with exotic *differentiable structure* (see for example [HS70]). In the particular case of $k = 4$, for the torus T^4, to the best knowledge of the author, there is very few facts in the literature about its smooth structure (as it is with many other cases of four dimensional manifolds). In the present chapter we focus on studying the smooth structure of leaves of an A-foliation of codimension 2 on a compact, simply-connected manifolds.

6.1 FIBRATIONS BETWEEN LEAVES

We consider an A-foliation $(M, \mathcal{F})$ on a compact, simply-connected manifold M. Let $p \in M$ be a fixed point, and consider a nearby point q, such that the leaf L_q has trivial holonomy. We recall that by Corollary 4.17 the leaf L_q is homeomorphic to a torus T^n. We also observe that there is a finite covering $\overline{L}_p$ of the leaf L_p, with $\overline{L}_p$ homeomorphic to a torus T^k. Now we consider the fibration given by

$$\mathcal{L}_v \to L_q \xrightarrow{\xi} \overline{L}_p. \tag{3.3.1}$$

In particular from Proposition 4.16, fibration (3.3.1) takes the form:

$$T^k \to T^m \xrightarrow{\xi} T^{m-k}. \tag{6.1.1}$$

Remark 6.1. Furthermore if the leaf L_p has also trivial holonomy, recall that $\overline{L}_p$ is homeomorphic to T^{m-k}.

Furthermore, we note that such fibrations come from an orthogonal metric projection (see Section 3.3), and thus this fibration is actually a smooth submersion. Since the total space of the fibration is a torus T^n, it is compact. We restate Corollary 3.15 for completeness.

Corollary 3.15. *Let W and N be smooth manifolds, with W compact. If $f\colon W \to N$ is a smooth surjective submersion, then f is a locally trivial fibration.*

Thus the fibration (6.1.1) is a locally trivial fibration, i.e. a fiber bundle. We collect this fact in the following result:

Corollary 6.2. *The submersions $\xi\colon T^m \to T^{m-k}$ are fiber bundles with fiber T^k.*

From now on we study the tori fiber bundles (6.1.1).

6.2 FOUR DIMENSIONAL TORUS.

We begin by discussing the 4-dimensional case, i.e. when the total space of (6.1.1) is T^4. Let $(M, \mathcal{F})$ be an A-foliation of codimension 2 on a compact-simply connected 6 manifold. In this case the least singular leaves are homeomorphic to T^3, and thus they admit a unique smooth structure. The least singular leaves are homeomorphic to T^2, which also admit a unique smooth structure. The only leaf type that may admit an exotic smooth structure is a principal leaf, which is homeomorphic to T^4.

Consider x a point in M, such that L_x^* lies on a vertex of ∂M^* (i.e. L_x is a most singular leaf). Consider L_q a principal leaf with q close to p in M. For this particular setting. the fibration (6.1.1) becomes:

$$T^2 \to T^4 \to T^2.$$

Even though, as mentioned before it is unstated in the literature, if the four torus admits a non-standard smooth structure, the total space of T^2 bundles over T^2 have been classified by Fukuhara and Sakamoto in [SF83]. Using this classification, Ue showed in [Ue90] that a bundle of the form

$$T^2 \to E \to T^2,$$

admits a geometric structure in the sense of Thurston (see for example [Sco83]). First, in [Ue90], the author proves that the total space E is classified among the geometric 4-manifolds, up to diffeomorphism, by $\pi_1(E)$. From this, it follows that it is enough to find a discrete faithful representation of $\pi_1(E)$ on a suitable geometry

X. An explicit list of orientable T^2-bundles over T^2 can be found in [Gei92, Table 1]. From this table, it follows that for the case of the bundle:

$$T^2 \to T^4 \to T^2,$$

induced by the infinitesimal foliation (3.3.1), that the torus T^4 has an Euclidean geometry. This means that it is the quotient of *Euclidean space*, denoted by $\mathbb{E}^4$, by some group of finite isometries. This implies that a principal leaf of $(M, \mathcal{F})$ is diffeomorphic to the standard torus. Since the other possible singular leaves (the least singular ones) are homeomorphic to T^3, and the 3-torus admits a unique smooth structure, it follows that for an A-foliation of codimension 2 on a compact, simply-connected 6-manifold all leaves are diffeomorphic to standard tori. Thus by Proposition 5.9, Lemma 4.32, Lemma 4.33, and Theorem F we prove that for a compact, simply-connected 6-manifold M, any A-foliation $(M, \mathcal{F})$ of codimension 2 is homogeneous.

Theorem 6.3. *If $(M, \mathcal{F})$ is a 6-dimensional, simply-connected, compact Riemannian manifold with an A-foliation of codimension 2, then the foliation is homogeneous.*

6.3 HIGHER DIMENSIONAL TORUS.

In this section we show that, for $n \geqslant 4$, the smooth structure of the leaves of an A-foliation $(M, \mathcal{F})$ of codimension 2 on a compact, simply-connected $(n+2)$-manifold is the standard one. Recall that for $(M, \mathcal{F})$ the least singular leaves are singular leaves of codimension 3 in M. The most singular leaves of $(M, \mathcal{F})$ are singular leaves of codimension 4 in M.

We begin by recalling that, for the foliated manifold $(M, \mathcal{F})$, the leaf space M^* is a disk. The boundary of M^* consists of a number $r \geqslant n$ of edges, each edge denoted by γ, and a number r of vertexes. We label the edges and vertexes as in Figure 6.1.

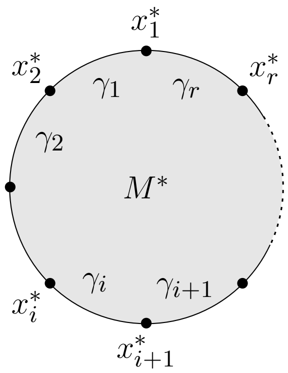

Figure 6.1.: Labels in the leaf space of A-foliation of codimension 2.

We fix $p_i \in M$, such that $L_{p_i}^*$ lies on the i-th edge, γ_i, in ∂M^* (i.e. L_{p_i} is a least singular leaf). This implies that L_{p_i} is homeomorphic to T^{n-1}. Take $q_i \in M$ close enough to p_i in M, such that the leaf L_{q_i} is principal. Then L_{q_i} is homeomorphic to T^n. In this case the fiber bundle (6.1.1) takes the form:

$$\mathbb{S}_i^1 \hookrightarrow T^n \to T^{n-1}. \tag{6.3.1}$$

Remark 6.4. We note that we have exactly r of these bundles. One for each edge in ∂M^*. The index i on the fiber is added to be able to distinguish the edge we are referring to.

With these bundles we will first show a principal leaf (and thus all principal leaves) of an A-foliation of codimension 2 on a compact, simply-connected $(n+2)$-manifold are the standard n-torus. We begin by observing that such bundles are orientable.

Proposition 6.5. *The bundle (6.3.1) is orientable.*

Proof. We can choose an arbitrary orientation for the fiber $\mathbb{S}^1_i$ in local charts, to obtain a vector field, tangent to the circles in the total space. Since the n-torus is orientable, we can extend this vector field to a base, such that the transition maps have positive determinant in this base.

Indeed if we choose on a local chart an orientation of the fiber $\mathbb{S}^1_i \subset T^n$, we can extend it to a basis of the tangent spaces of T^n. Since T^n is orientable we can do this construction in such a way that for two open trivial neighborhoods, the orientations of the fibers are positive. □

We combine the previous proposition with the following result, which tells us that the bundle (6.3.1) is principal.

Theorem 6.6 (Proposition 6.15 in [Mor01])**.** *Every oriented* $\mathbb{S}^1$*-bundle admits the structure of a principal* $\mathbb{S}^1$*-bundle.*

Corollary 6.7. *The fiber bundle* (6.3.1) *is a principal* $\mathbb{S}^1$*-bundle.*

We recall from Theorem F that an A-foliation $(M, \mathcal{F})$ of codimension 2 on a compact, simply-connected $(n+2)$-manifold is homogeneous, up to foliated homeomorphism. Furthermore, from Theorem 5.10 we have $r \geqslant n$ fiber bundles of the form (6.3.1). The circles, which are the fibers of these bundles, play the role of the isotropy circle subgroups for smooth (continuous) effective actions of T^n on M. We denote by $(a_{i1}, a_{i2}, \dots, a_{in}) \in \mathbb{Z}^n$ the weight associated to the leaf L_{p_i}. We recall that $(a_{i1}, a_{i2}, \dots, a_{in})$ defines how $\mathbb{S}_i$ is embedded into the principal leaf T^n.

Since the homotopy classes of the circles $\mathbb{S}^1_1, \mathbb{S}^1_2, \cdots, \mathbb{S}^1_r \subset T^n$ generate the fundamental group $\pi_1(T^n)$ of the principal leaf of $(M, \mathcal{F})$, then by Lemma 5.11 there exists a subcollection of labels $\{i_1, i_2, \dots, i_n\}$ such that the collection of weights

$$\Big\{(a_{i_1 1}, \dots, a_{i_1 n}), \dots, (a_{i_1 1}, \dots, a_{i_1 n})\Big\},$$

have determinant ± 1. We conclude from [Oh83a, Lemma 1.4], that the principal leaf T^n of $(M, \mathcal{F})$ is homeomorphic to the torus

$$\mathbb{S}^1_{i_1} \times \ldots \times \mathbb{S}^1_{i_n},$$

for the collection $\{i_1, i_2, \ldots, i_n\}$ of distinct edge labels of ∂M^*.

From this observations, we can prove the following proposition:

Proposition 6.8. *There exists a free smooth $\mathbb{T}^n$-action on the principal leaf T^n of the foliation.*

Proof. Consider the bundles $\mathbb{S}^1_{i_j} \hookrightarrow T^n \to T^{n-1}$, associated to the weights generating the fundamental group of the principal leaf T^n. From Corollary 6.7 these bundles are principal. Thus for each i_j there is a free smooth action $\mu_{i_j} \colon \mathbb{S}^1_{i_j} \times T^n \to T^n$. The image of this action is exactly the fiber $\mathbb{S}^1_{i_j}$ of the bundle (6.3.1). We now define the $\mathbb{T}^n$-action $\mu \colon \mathbb{T}^n \times T^n \to T^n$ on the principal leaf T^n as

$$\mu((\xi_1, \ldots, \xi_n), p) = \mu_{i_1}(\xi_1, \mu_{i_2}(\xi_2, \cdots, \mu_{i_n}(\xi_n(p)) \cdots)).$$

The actions μ_{i_j} commute, since the principal leaf T^n is homeomorphic to the product $\mathbb{S}^1_{i_1} \times \cdots \times \mathbb{S}^1_{i_n}$. Therefore μ gives a continuous action of the standard n-torus, $\mathbb{T}^n$, on the principal leaf T^n. Furthermore, the action μ is free and smooth since each of the transformations μ_{i_j} are free and smooth. $\square$

Corollary 6.9. *For $n \geq 5$, the principal leaf of an A-foliation $(M, \mathcal{F})$ of codimension 2 on a compact, simply-connected manifold is diffeomorphic to the standard torus $\mathbb{T}^n$.*

Proof. Since $T^n = \mathbb{S}^1_{i_1} \times \ldots \times \mathbb{S}^1_{i_n}$ the action μ is transitive, and therefore T^n is the standard torus $\mathbb{T}^n$. $\square$

We end this section by proving that the singular leaves of an A-foliation of codimension 2 on a compact, simply-connected manifold are also diffeomorphic to standard tori.

Corollary 6.10. *The least singular leaf of an A-foliation $(M, \mathcal{F})$ of codimension 2 on a compact, simply-connected $(n+2)$-manifold M is diffeomorphic to the standard torus.*

Proof. For the least singular leaf L_{p_i} the claim follows from the fact that the fiber bundle (6.3.1) is an $\mathbb{S}^1_i$-principal bundle, combined with the fact that the total space is the standard torus $\mathbb{T}^n$. Thus the least singular leaf L_{p_i} is diffeomorphic to $\mathbb{T}^n/\mathbb{S}^1_i = \mathbb{T}^{n-1}$, i.e. the standard $(n-1)$-dimensional torus. □

We recall that, if x_i is a point in $(M, \mathcal{F})$, such that $L^*_{x_i}$ is a vertex in M^*, then L_{x_i} is a most singular leaf of $\mathcal{F}$, and it is homeomorphic to T^{n-1}. Furthermore we can choose p_i close enough to x_i in M, such that L_{p_i} is a least singular leaf. We point out that the leaf L_{p_i} has trivial holonomy. Thus for the leaves L_{p_i} and L_{x_i} fibration (3.3.1) is a fiber bundle of the form:

$$\mathbb{S}^1 \hookrightarrow T^{n-1} \to T^{n-2}. \tag{6.3.2}$$

By following the same arguments as in the proof of Proposition 6.5 we prove that this bundle is orientable. Applying Theorem 6.6 we conclude that the fibration (6.3.2) is a principal $\mathbb{S}^1$-bundle. With theses remarks we can prove the following proposition:

Proposition 6.11. *The most singular leaf of an A-foliation $(M, \mathcal{F})$ of codimension 2 on a compact, simply-connected $(n+2)$-manifold M is diffeomorphic to the standard torus.*

Proof. Since we have a smooth principal $\mathbb{S}^1$-bundle:

$$\mathbb{S}^1 \hookrightarrow L_{p_i} \to L_{x_i}, \tag{6.3.2}$$

we conclude that L_{x_i} is diffeomorphic to $L_{p_i}/\mathbb{S}^1$. From Corollary 6.10, we have that the least singular leaf L_{p_i} is diffeomorphic to $\mathbb{T}^{n-1}$. Thus the most singular leaf is diffeomorphic to $\mathbb{T}^{n-1}/\mathbb{S}^1 = \mathbb{T}^{n-2}$. □

Thus all leaves in an A-foliation $(M, \mathcal{F})$ of codimension 2 on a compact simply-connected $(n+2)$-manifold M, are diffeomorphic to standard tori. Following the proof of Theorem F, together with Lemma 4.32, and Remark 5.9 we get a proof of the main theorem of the present work:

Theorem G. *Every A-foliation of codimension 2 on a compact, simply-connected Riemannian manifold is homogeneous.*

APPENDIX

A LINEARIZED FLOWS

In this appendix we prove that for a singular Riemannian foliation $(M, \mathcal{F})$, given a closed leaf L and a point $p \in L$, the map $\rho\colon \pi_1(L,p) \to O(\mathbb{S}_p^\perp, \mathcal{F}^p)$ defined in Section 3.3 is well defined. To do this, we need to understand the correspondence given in Theorem 3.8 between a path $\gamma\colon [0,1] \to L$, starting at p, and a foliated map $G\colon \nu_p L \to \nu L$. We recall the needed concepts and results from Section 3.2 in [MR18], and the notes [Rad17].

1 LINEARIZED VECTOR FIELDS

We consider a complete Riemannian manifold M, and L a closed submanifold of M. Let X be a vector field on M, which is tangent to L when restricted to L. Recall that there exists an open neighborhood $W \subset \nu L$ of the zero-section of the normal bundle $\nu L \to L$, and an open neighborhood $U \subset M$ of L, such that normal exponential map $\exp^\perp\colon W \to U$ is a diffeomorphism. We will consider the preimage $(\exp^\perp)_*^{-1}(X)$ of X, which we will also denote by X. Observe that this preimage is also a smooth vector field on W. Given $\lambda > 0$ we consider the transformation $r_\lambda\colon \nu L \to \nu L$ given by taking a normal vector field V to λV. If λ is small enough, the image of the restriction of r_λ to W lies in W again, i.e. $r_\lambda(W) \subset W$.

Given a smooth vector field X on $W \subset \nu_L$, we define the *linearization of* X *around* L, denoted by X^ℓ, as the vector field obtained as the limit

$$X^\ell = \lim_{\lambda \to 0} (r_\lambda)_*^{-1}(X \circ r_\lambda).$$

If $X^\ell = X$, we say that X is a *linearized vector field.* By the following proposition shows that X^ℓ is well defined and it is invariant under rescalings, i.e. $(r_\lambda)_*(X^\ell) = X^\ell$.

Proposition A.1 (Linearization of vector fields, Proposition 13 in [MR18])**.** *Let X be a smooth vector field on W. Then its linearization X^ℓ is a well-defined, smooth vector field defined on the whole of νL, which is invariant under rescalings.*

Now we consider the case where L is a closed leaf of a singular Riemannian foliation $(M, \mathcal{F})$. Recall, from Section 3.2, that there exists a singular foliation, which we denote in this appendix by $\nu\mathcal{F}$, on νL, which is scaling invariant. Its leaves are the preimages of $\mathcal{F}|_U$ given by the map $\exp^\perp$. The next proposition shows that flows of linearized vectors preserve this foliation.

Proposition A.2 (Linear flows, Proposition 14 in [MR18])**.** *Let $(M, \mathcal{F})$ be a singular Riemannian foliation, L a closed leaf, and X a vector field tangent to the leaves of $\mathcal{F}$. Then for any $t \in \mathbb{R}$, the linearization X^ℓ around L and its flow $\Phi^t \colon \nu L \to \nu L$ satisfy:*

(i) *X^ℓ is a tangent to the leaves of the singular foliation $(\nu L, \nu\mathcal{F})$, and Φ^t preserves the leaves of $(\nu L, \nu\mathcal{F})$.*

(ii) *For any $p \in L$, the restriction of Φ^t to $\nu_p L$ is a linear orthogonal transformation from $\nu_p L$ to $\nu_{\Phi^t(p)} L$.*

We recall the following property of singular Riemannian foliations, the so-called *equifocallity*, as stated in [MR18].

Theorem A.3 (Equifocality, Proposition 5 in [MR18])**.** *Let $\mathcal{F}$ be a singular Riemannian foliation of a Riemannian manifold M, and let L be a leaf. If $v, w \in \nu L$ are two normal vectors (at possibly different points) such that $\exp^{\perp}(tv)$ and $\exp^{\perp}(tw)$ belong to the same leaf for all small $t > 0$, then they belong to the same leaf for all $t \in \mathbb{R}$ for which $\exp^{\perp}(tv)$ and $\exp^{\perp}(tw)$ are defined.*

Proof. See Proposition 4.3 in [LT10], and Theorem 2.9 in [Ale10]. □

With Proposition A.2 and Theorem A.3 we are able to give a proof of Theorem 3.8, as done in [MR18].

Theorem 3.8. *Let L be a closed leaf of a singular Riemannian foliation $(M, \mathcal{F})$, and*
$\gamma\colon [0,1] \to L$ a piece-wise smooth curve with $\gamma(0) = p$. Then there is a map $G\colon [0,1] \times \nu_p L \to \nu L$ such that:

(i) $G(t, v) \in \nu_{\gamma(t)} L$ *for every* $(t, v) \in [0,1] \times \nu_p L$.

(ii) *For every $t \in [0,1]$, the restriction $G\colon \{t\} \times \nu_p L \to \nu_{\gamma(t)} L$ is a linear isometry preserving the leaves of ν_L.*

(iii) *For every $s \in \mathbb{R}$ the map $\exp_{\gamma(t)}(sG(t,v))$ belongs to the same leaf as the point $\exp_p(sv)$.*

Proof. First we consider a partition $0 = t_0 < t_1 < \cdots < t_N = 1$ of $[0,1]$, such that the restriction $\gamma_i = \gamma\colon [t_{i-1}, t_i] \to L$ is an embedding for $1 \leqslant i \leqslant N$. Thus for all i, the curves γ_i are integral curves of some smooth vector field X_i on L. We extend each vector field X_i to M, and obtain a vector field $\hat{X}_i$ on M. We then use the Riemannian metric to consider the component of $\hat{X}_i$ tangent to the leaves of $\mathcal{F}$. In this way we obtain a vector field, which we denote by X_i, which is an extension of the original X_i. Observe that by construction X_i is tangent to the leaves. Next we consider a neighborhood W of the zero section in νL, and $U \subset M$ a neighborhood of L such that $\exp^{\perp}\colon W \to U$ is a diffeomorphism. We identify each vector field X_i with a vector field X_i on W via the inverse of $(\exp^{\perp})_*$. We now consider the linearizations X_i^{ℓ} around L of the vector fields X_i, and we denote by Φ_i^t the flow of each X_i^{ℓ}. Given a normal vector $v \in \nu_p L$, by Proposition A.2, on $[0, t_1]$ we have a linear orthogonal transformation $\Phi_1^{t_1}(v) \in \nu_{\gamma(t_1)}$. We may apply the same construction on $[t_1, t_2]$ for $v_1 \in \nu_{\gamma(t_1)} L$, and so on for the rest of the partition of $[0, 1]$. In this way we can define a map $G\colon [0, 1] \times \nu_p L \to \nu L$ as follows: for $t \in [t_{j-1}, j]$ and $v \in \nu_p L$ we define G as

$$G(t, v) = \Phi_j^{t-t_{j-1}} \circ \Phi_{j-1}^{t_{j-1}-t_{j-2}} \circ \cdots \circ \Phi_2^{t_2-t_1} \circ \Phi_1^{t_1}(v).$$

Parts (i) and (ii) follow from Proposition A.2. Part (iii) follows from Theorem A.3.

□

Consider two curves, γ_0 and γ_1, in a closed leaf L, with $\gamma_0(0) = p = \gamma_1(0)$ and $\gamma_0(1) = q = \gamma_1(1)$. Let $G_i\colon \nu_p \to \nu_q$ be the linear transformation given by Theorem 3.8, associated to γ_i. We end this appendix by showing that, if the two paths γ_0 and γ_1 are homotopic relative to the endpoints, then $(G_1)^{-1} \circ G_0$ is homotopic to the identity map of $\nu_p L$. We follow the proof of Lemma 2.36 in [Rad17].

Proposition A.4. *Let γ_0 and γ_1 be two curves in a closed leaf L which are homotopic relative to the end points, with $\gamma_0(0) = p = \gamma_1(0)$, and $\gamma_0(1) = q = \gamma_1(1)$. Then $(G_1)^{-1} \circ G_0\colon \nu_p L \to \nu_p$ is homotopic to the identity map. Furthermore it takes every leaf of the infinitesimal foliation $\mathcal{F}^p$ to itself.*

Proof. Let $H\colon [0, 1] \times I \to L$ be the homotopy between γ_0 and γ_1. By applying Whitney's Approximation Theorem (see for example Theorem 9.27 in [Lee13]), we can assume that H is a smooth map. For $s \in I$ fixed we consider the smooth curve $\gamma_s(t) = H(t, s)$. From the compactness of $[0, 1] \times I$ we can find a partition $0 = t_0 < t_1 < \cdots t_N = 1$ of $[0, 1]$ such that for any $s \in I$ the curves γ_s restricted to

$[t_{i-1}, t_i]$ is an embedding. By extending the vector field $\gamma_s'(t)$ for $t \in [t_{i-1}, t_i]$ to L, we obtain smooth vector fields V_{si} on L. Since the family of curves γ_s varies continuously with respect to s by construction, for each $1 \leqslant i \leqslant N$ the family of vector fields V_{si} varies smoothly with respect to s. This implies that when we consider for each γ_s the map $G_s\colon \nu_p L \to \nu_q L$ given by Theorem 3.8, then G_s varies continuously with respect to s (see Proof of Theorem 3.8). Defining $K(v,s) = (G_s)^{-1}(G_0(v))$ we obtain a homotopy $K\colon \nu_p L \times I \to \nu_p L$, between the identity $Id\colon \nu_p L \to \nu_p L$ and $(G_1)^{-1} \circ G_0\colon \nu_p L \to \nu_p L$. For $v \in \nu_p L$ fixed, we have, from Theorem 3.8 (iii), that $\exp_p((G_s)^{-1}(G_0(v)))$ lies in the same leaf of $\mathcal{F}$ as $\exp_p(v)$. Since $K(v,s)$ defines a path between v and $(G_1)^{-1}(G_0(v))$, we have that $(G_1)^{-1}(G_0(v))$ lies in the same leaf $\mathcal{L}_v$ of $\mathcal{F}^p$ as v. Thus $(G_1)^{-1}(G_0(\mathcal{L}_v)) \subset \mathcal{L}_v$. □

Recall from Section 3.3, that the group $O(\mathbb{S}_p^\perp, \mathcal{F}^p)$ consists of all the foliated isometries of the infinitesimal foliation at p, and the subgroup $O(\mathcal{F}^p)$ consists of all the foliated isometries which leave invariant the leaves of $\mathcal{F}^p$. The last part of Proposition A.4 states that $(G_1)^{-1} \circ G_0$ is an element of $O(\mathcal{F}^p)$.

Corollary A.5. *For a point p in a closed leaf L of a singular Riemannian foliation $(M, \mathcal{F})$, the map $\rho\colon \pi_1(L_p, p) \to O(\mathbb{S}_p^\perp, \mathcal{F}^p)/O(\mathcal{F}^p)$,defined in Section 3.3 is well defined.*

Proof. We recall how the map ρ is defined. Given a loop γ_0, we consider $G_0\colon \nu_p L \to \nu_p L$ the linear foliated transformation given by Theorem 3.8, and set $\rho[\gamma_0] = [G_0] \in O(\mathbb{S}_p^\perp, \mathcal{F}^p)/O(\mathcal{F}^p)$. From Proposition A.4 if γ_1 is a loop homotopic to γ_0, then we have $(G_1)^{-1} \circ G_0 \in O(\mathcal{F}^p)$. Therefore $[G_0] = [G_1]$ in $O(\mathbb{S}_p^\perp, \mathcal{F}^p)/O(\mathcal{F}^p)$. □

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