Solutions of ordinary differential equations in closed subsets of a Banach space

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Dedicated to Professor Karol Baron on his 70th Birthday

1. Notations and main result. Let $E$ be a real Banach space. For $x \in E$ and $\emptyset \neq B \subseteq E$ we write

$$\text{dist}(x, B) = \inf\{\|x - y\| \mid y \in B\}.$$  

For bounded sets $B \subseteq E$ we define

$$\text{diam}B = \sup\{\|x - y\| \mid x, y \in B\}$$

(diam $\emptyset = 0$) and furthermore

$$\alpha(B) = \inf\{\delta \geq 0 \mid B \subseteq B_1 \cup \ldots \cup B_n, \text{diam}B_\nu \leq \delta (\nu = 1, \ldots, n), \ n \in \mathbb{N}\},$$

where $\mathbb{N} = \{1, 2, 3, \ldots\}$; $\alpha(B)$ is the Kuratowski measure of non-compactness of $B$ (cf. Kuratowski [1]).

Finally we use the notation

$$[x, y]_\tau = \lim_{h \to 0} \frac{1}{h}\{\|x + hy\| - \|x\|\} \quad (x, y \in E).$$

**Theorem 1** Suppose $T > 0$ and $f = g + k$, where $g, k : [0, T] \times E \to E$ are continuous and bounded functions, $g$ satisfying the one-sided Lipschitz condition

$$[x - y, g(t, x) - g(t, y)]_\tau \leq L\|x - y\| \quad (0 \leq t \leq T; \ x, y \in E)$$

and $k$ the $\alpha$-Lipschitz condition

$$\alpha(k([0, T] \times B)) \leq K\alpha(B) \quad (B \subseteq E, \ B \text{ bounded});$$

here $K, L$ are given non-negative numbers.

Moreover let $M$ be a closed subset of the Banach space $E$ such that

$$\liminf_{h \to 0} \frac{1}{h}\text{dist}(x + hf(t, x), M) = 0 \quad (0 \leq t \leq T, \ x \in M).$$

Then for every $(\tau, a) \in [0, T] \times M$ the initial value problem (i.v.p.)

$$u(\tau) = a, \quad u'(t) = f(t, u(t)) \quad (\tau \leq t \leq T)$$

has a solution $u : [\tau, T] \to M$. 

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Proof. Without loss of generality (w.l.o.g.) we assume \( L > 0 \). We choose \( l \in [0, T - \tau] \) according to

\[
\frac{1}{L} (e^{Ll} - 1) \leq \frac{1}{4(2K + 1)}.
\]

The function \( f : [0, T] \times E \to E \) being continuous, bounded and having property (1), there exist \( C^1 \)-functions \( u_n : [\tau, \tau + l] \to E \) \( (n \in \mathbb{N}) \) such that

\[
u_n(\tau) = a, \quad \|u'_n(t) - f(t, u_n(t))\| \leq \frac{1}{n} \quad (\tau \leq t \leq \tau + l),\]

(4)

\[
\text{dist}(u_n(t), M) \leq \frac{1}{n} \quad (\tau \leq t \leq \tau + l);
\]

the proof of this will be given in the next section.

Schmidt’s proof of her Satz 2.3 in [3] shows that a subsequence of \( (u_n)_{n \in \mathbb{N}} \) uniformly converges to a solution \( u : [\tau, \tau + l] \to E \) of the i.v.p. given by (2). As a consequence of (4) we also get \( u : [\tau, \tau + l] \to M \).

If \( \tau + l = T \), then we are done. If \( \tau + l < T \), then it is sufficient to repeat the foregoing reasoning finitely many times in a standard way.

Remark. Schmidt’s form of inequality (3) is a bit different, but in her Satz 2.3 she rather uses the Hausdorff measure of non-compactness instead of \( \alpha \). The here given inequality (3) is appropriate for applying Schmidt’s results to our case.

2. Existence of approximate \( C^1 \)-solutions. The existence of the functions \( u_n \) \( (n \in \mathbb{N}) \) in the proof of Theorem 1 is a consequence of the following Theorem 2.

**Theorem 2** Let \( f : [\tau, T] \times E \to E \) be a bounded continuous function, where \( E \) is a Banach space and \( \tau, T \) are reals, \( \tau < T \). Let \( M \) be a closed subset of \( E \) and suppose

\[
\lim_{h \searrow 0} \frac{1}{h} \text{dist}(x + hf(t, x), M) = 0 \quad (\tau \leq t \leq T, \ x \in M).
\]

Finally let \( a \in M \) and \( \varepsilon > 0 \) be given. Then there exists a \( C^1 \)-function \( u : [\tau, T] \to E \) such that

\[
u(\tau) = a, \quad \|u'(t) - f(t, u(t))\| \leq \varepsilon, \quad \text{dist}(u(t), M) \leq \varepsilon \quad (\tau \leq t \leq T).
\]

Proof. 1. W.l.o.g. we suppose \( \varepsilon \leq 1 \). According to Martin [2] there is a polygonal line \( p : [\tau, T] \to E \) satisfying

\[
p(\tau) = a, \quad \|p'_\pm(t) - f(t, p(t))\| \leq \frac{\varepsilon}{4}, \quad \text{dist}(p(t), M) \leq \frac{\varepsilon}{4} \quad (\tau \leq t \leq T),
\]

(6)
where \( p'_+ \), \( p'_- \) mean right- and left-hand derivatives (with the natural exceptions of \( p'_-(\tau), p'_+(T) \)).

2. Let \((s, c)\) be a corner of \( p \), where \( \tau < s < T \), \( c = p(s) \). Then in a small interval \( [s - \beta, s + \beta] \) (where \( \beta > 0 \)) there is no second corner of \( p \). We shall show that for sufficiently small positive \( \eta < \frac{\beta}{2} \) there is a \( C^1 \)-function \( u : [s - \beta, s + \beta] \to E \) which coincides with \( p \) on \( [s - \beta, s - \eta] \cup [s + \eta, s + \beta] \) and fulfills the inequalities

\[
\|u'(t) - f(t, u(t))\| \leq \varepsilon, \quad \text{dist}(u(t), M) \leq \varepsilon (s - \eta \leq t \leq s + \eta).
\]

When changing \( p \) in a neighborhood of every corner \((s, c)\) according to the just given description into a \( C^1 \)-function \( u \), then we get a \( C^1 \)-function \( u : [\tau, T] \to E \) satisfying (5).

3. Now we like to prove the statements of the preceding paragraph. So let \((s, c)\) be a corner of \( p \), w.l.o.g. we assume \( s = 0 \). Then in a small interval \([-\beta, \beta]\) the function \( p \) has the form

\[
p(t) = \begin{cases} 
  c + tb_1 & (0 \leq t \leq \beta) \\
  c + tb_2 & (-\beta \leq t \leq 0),
\end{cases}
\]

where \( \beta > 0 \) and \( b_1, b_2 \in E \). With \( v = \frac{1}{2}(b_1 + b_2) \), \( w = \frac{1}{2}(b_1 - b_2) \) we get

\[
p(t) = \begin{cases} 
  c + tv + tw & (0 \leq t \leq \beta) \\
  c + tv - tw & (-\beta \leq t \leq 0),
\end{cases}
\]

\[
p'(t) = \begin{cases} 
  v + w & (0 < t < \beta) \\
  v - w & (-\beta < t < 0).
\end{cases}
\]

Therefore (6) leads to

\[
\|v + w - f(t, c + tv + tw)\| \leq \varepsilon \quad (0 \leq t \leq \beta),
\]

\[
\|v - w - f(t, c + tv - tw)\| \leq \varepsilon \quad (-\beta \leq t \leq 0).
\]

Using these inequalities for \( t = 0 \) implies

\[
2\|w\| = \|v + w - f(0, c) - (v - w - f(0, c))\| \leq \varepsilon / 2,
\]

hence

\[
\|w\| \leq \varepsilon / 4, \quad \|v - f(0, c)\| \leq \varepsilon / 2.
\]

The continuity of \( f \) at \((0, c)\) shows the existence of an \( \eta \in]0, \min\{1, \frac{\beta}{2}\} [\) such that

\[
(t, x) \in \mathbb{R} \times E, \ |t| \leq \eta, \ |x| \leq \eta \Rightarrow \|f(t, c + tv + x) - f(0, c)\| \leq \varepsilon / 4.
\]
Hence we have

\[ |t| \leq \eta, \quad \|x\| \leq \eta \Rightarrow \|f(t, c + tv + x) - v\| \leq \frac{3\varepsilon}{4}. \tag{8} \]

We define \( u : ] - \beta, \beta[ \to E \) by

\[
u(t) = \begin{cases} 
  c + tv + tw & (\eta \leq t < \beta) \\
  c + \left(\frac{1}{2\eta} t^2 + \frac{\eta}{2}\right) w + tv & (|t| \leq \eta) \\
  c + tv - tw & (-\beta < t \leq -\eta).
\end{cases}
\]

This \( u \) is a \( C^1 \)-function and it coincides with \( p \) on \( ] - \beta, -\eta[ \cup [\eta, \beta[. \) We finally have to verify (7) (with \( s = 0 \)). For \( |t| \leq \eta \) we get

\[
u'(t) = \frac{t}{\eta} w + v,
\]

hence

\[ \|\nu'(t) - v\| \leq \|w\| \leq \frac{\varepsilon}{4}. \tag{9} \]

Furthermore

\[
\|\nu(t) - c - tv\| = \left(\frac{t^2}{2\eta} + \frac{\eta}{2}\right) \|w\| \leq \eta \frac{\varepsilon}{4} \leq \eta.
\]

Using (8), we get

\[ \|f(t, \nu(t)) - v\| \leq \frac{3\varepsilon}{4}. \]

From this and (9) we derive

\[ \|\nu'(t) - f(t, \nu(t))\| \leq \varepsilon \quad \text{(for } |t| \leq \eta), \]

which is the first inequality in (7). Concerning the second one, we observe for \( |t| \leq \eta \) that

\[
p(t) - \nu(t) = |t|w - \left(\frac{1}{2\eta} t^2 + \frac{\eta}{2}\right) w,
\]

hence

\[ \|p(t) - \nu(t)\| \leq 2\eta \|w\| \leq 2\eta \frac{\varepsilon}{4}, \]

and thus

\[ \text{dist}(\nu(t), M) \leq \text{dist}(p(t), M) + \|p(t) - \nu(t)\| \leq \frac{\varepsilon}{4} + 2\eta \frac{\varepsilon}{4} = \frac{\varepsilon}{4}(1 + 2\eta) \leq \varepsilon. \]

3. A local result. Let us consider Theorem 1 for \( M = E \). Then (1) holds for every function \( f : [0, T] \times E \to E \). In this case Theorem 1 gives a solution \( u : [\tau, T] \to E \) of the i.v.p. (2), which means that we find back Satz 2.3 of Schmidt [3].

As a consequence of Satz 2.3, Schmidt proves a local version of it. Now we are doing the same with our Theorem 1.
**Theorem 3** Suppose $T > 0$ and $M \subseteq D \subseteq E$, where $M$ is a closed and $D$ an open subset of the Banach space $E$. Consider $f = g + k$, where $g, k : [0, T] \times D \to E$ are continuous and such that

$$[x - y, g(t, x) - g(t, y)]_\leq \leq L\|x - y\| \quad (0 \leq t \leq T; \ x, y \in D),$$

$$\alpha(k([0, T] \times B)) \leq K\alpha(B) \quad (B \subseteq D, B \text{ bounded}).$$

Let $f$ satisfy condition (1). Then for every $a \in M$ there exists a $\tilde{T} \in [0, T]$ such that the i.v.p.

$$(10) \quad u(0) = a, \ u'(t) = f(t, u(t)) \quad (0 \leq t \leq \tilde{T})$$

has a solution $u : [0, \tilde{T}] \to M$.

**Proof.** We simply follow the proof of Satz 2.4 in [3]: We fix $a \in M$ and we choose $T_0, r > 0$ such that $g, k$ are defined and bounded on

$$[0, T_0] \times \{x \mid x \in E, \|x - a\| \leq r\};$$

let $\mu$ be a positive bound for their norms. We take

$$\tilde{T} = \min\{T_0, \frac{r}{4\mu}\}.$$

We define $p : [0, \infty[ \to [0, 1]$ by

$$p(s) = \begin{cases} 1 & (0 \leq s \leq \frac{r}{2}) \\ 2 - \frac{2}{r} s & (\frac{r}{2} \leq s \leq r) \\ 0 & (s \geq r). \end{cases}$$

Then we define $\tilde{f}, \tilde{g}, \tilde{k} : [0, \tilde{T}] \times E \to E$ by

$$(11) \quad \tilde{f}(t, x) = \begin{cases} p(\|x - a\|)f(t, x) & (0 \leq t \leq \tilde{T}, \|x - a\| \leq r) \\ 0 & (\|x - a\| \geq r), \end{cases}$$

and $\tilde{g}, \tilde{k}$ in an analogous way. We can apply Theorem 1 with $T, f, g, k$ replaced by $\tilde{T}, \tilde{f}, \tilde{g}, \tilde{k}$. Especially (1) holds after this replacement also for $\tilde{f}$, because in (11) we have $p(\|x - a\|) \geq 0$.

We thus get a solution $u : [0, \tilde{T}] \to M$ of the i.v.p.

$$u(0) = a, \ u'(t) = \tilde{f}(t, u(t)) \quad (0 \leq t \leq \tilde{T}).$$

Because of $\|u(t) - a\| \leq 2\mu\tilde{T} \leq \frac{r}{2}$, the function $u$ also solves (10).
References


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