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CRC Preprint 2018/13, August 2018

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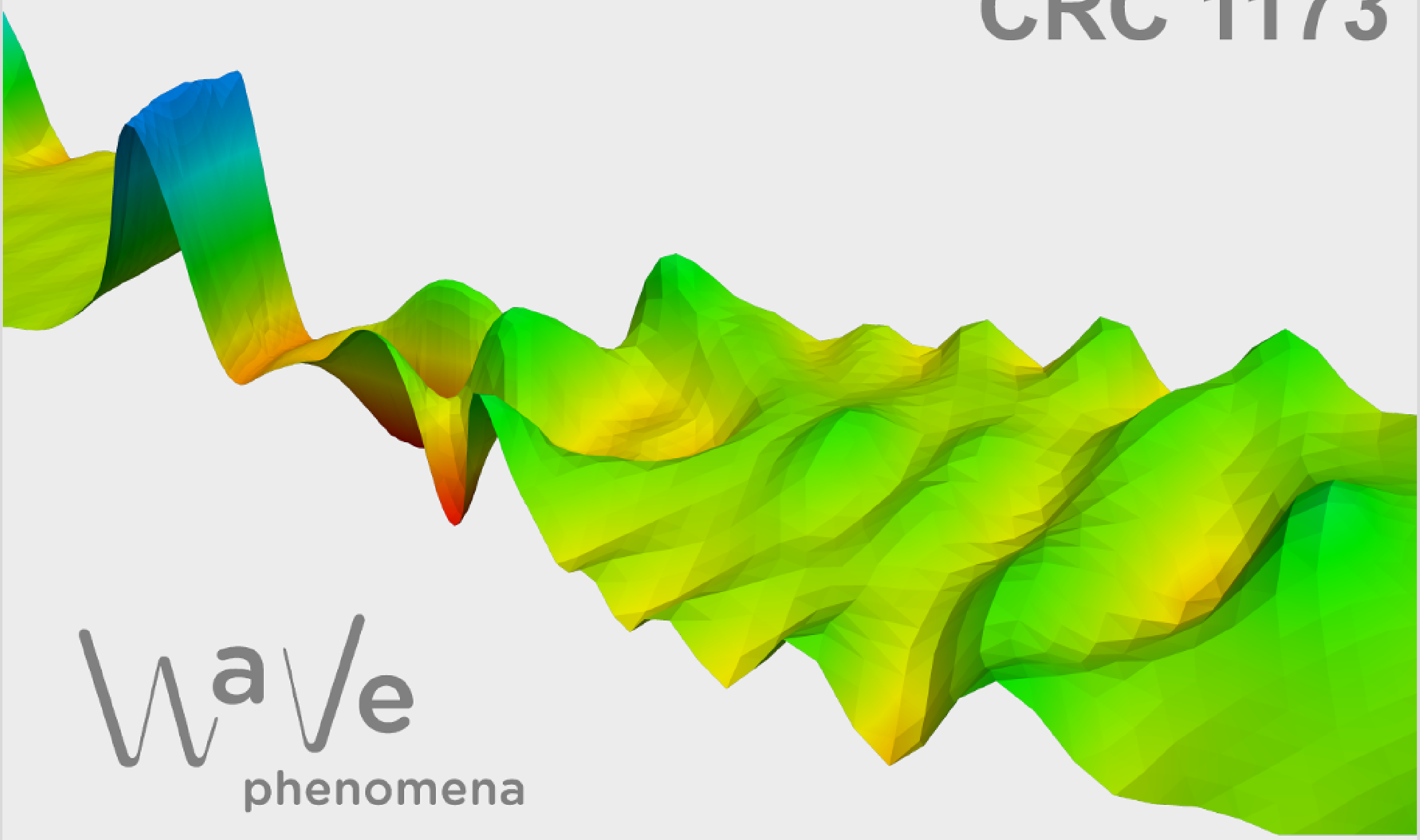
CRC 1173



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ISSN 2365-662X

SPACE-TIME DISCONTINUOUS PETROV-GALERKIN METHODS FOR LINEAR WAVE EQUATIONS IN HETEROGENEOUS MEDIA

Johannes Ernesti and Christian Wieners[1]

Abstract. We establish an abstract space-time DPG framework for the approximation of linear waves in heterogeneous media. The estimates are based on a suitable variational setting in the energy space. The analysis combines the approaches for acoustic waves in Gopalakrishnan / Sepulveda (A space-time DPG method for acoustic waves, arXiv 2017) and in Ernesti / Wieners (RICCAM proceedings, submitted 2017) and is based on the abstract definition of traces on the skeleton of the time-space substructuring. The method is evaluated by large-scale parallel computations motivated from applications in seismic imaging, where the computational domain can be restricted substantially to a subset of the full space-time cylinder.

1991 Mathematics Subject Classification. 65N30.

August 14, 2018.

1. Introduction

Space-time finite elements aim for a unified analysis of discretization and solution methods in space and time. In particular they allow for an efficient combined error control and for scaling of the solution scheme to the next generation of massively parallel computers.

The discontinuous Petrov-Galerkin method (DPG) is a well-suited finite element class for space-time applications which provides robust a-priori estimates, reliable error control, and the efficient hybridization to a symmetric positive definite Schur complement system. This is attractive for hyperbolic systems and allows to transfer features of discretizations for elliptic problems to wave-type equations. The long-term goal is, as it is discussed in [3] for the transport equation, to establish optimality of the solution process and of adaptive schemes. For a general discussion on the DPG technology we refer to [7].

First results of space-time DPG methods are established in [8] for the Schrödinger equations and in [13, 15] for acoustic waves. Here, we show that the analysis transfers to general wave equations in heterogeneous media and provides robust estimated in the energy norm. Therefore, we recall in Lem. 3 and Lem. 4 the abstract DPG analysis based on the technique introduced in [15] which avoids explicit traces. Then, following the arguments in [3] we show that a test space exists which guarantees discrete inf-sup stability for general wave equations, and we extend the analysis for the simplified DPG method with nonconforming traces as in [13] to this more general setting. Finally, we apply a Strang-type argument to estimate the consistency error of the DPG method due to inexact quadrature in heterogeneous media.

Keywords and phrases: space-time methods, discontinuous Petrov-Galerkin finite elements, linear hyperbolic systems, heterogenous media

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The analysis is complemented by numerical results for wave propagation in heterogeneous media. Here we discuss an application scenario motivated from seismic measurements, were the wave signal is initiated by a point source and the results are only measured at selected points. In this application class the finite propagation speed of wave solutions results into an a priori information about the region of interest within the space-time cylinder and which allows to truncate the computational domain substantially.

2. Linear hyperbolic systems

A semigroup framework. We consider the evolution equation

$$M\partial_t u + Au = f \quad \text{in} \quad (0,T)\times\Omega_0 \subset \mathbb{R}\times\mathbb{R}^d$$

subject to homogeneous initial conditions $u(0)=0$ in Ω_0, where Ω_0 is a Lipschitz domain, $f(t)\in \mathrm{L}_2(\Omega_0;\mathbb{R}^m)$ is a source function, and with

a) a symmetric positive definite operator M in $\mathrm{L}_2(\Omega_0;\mathbb{R}^m)$ represented by $M\in \mathrm{L}_\infty(\Omega_0;\mathbb{R}^{m\times m}_{\mathrm{sym}})$;
b) a hyperbolic operator A with domain $\mathcal{D}(A)\subset \mathrm{L}_2(\Omega_0;\mathbb{R}^m)$ such that

$$(Av,z)_{0,\Omega_0} = -(v,Az)_{0,\Omega_0}\,, \qquad v,z\in\mathcal{D}(A) \tag{1}$$

and such that $M+A$ is surjective, i.e.,

$$(M+A)(\mathcal{D}(A)) = \mathrm{L}_2(\Omega_0;\mathbb{R}^m)\,. \tag{2}$$

Then, $M^{-1}A$ generates a semigroup in $\mathrm{L}_2(\Omega_0;\mathbb{R}^m)$ and for $f\in \mathrm{C}^0\big((0,T);\mathcal{D}(A)\big)$ the solution is given by

$$u(t) = \int_0^t \exp\big((t-s)M^{-1}A\big)M^{-1}f(s)\,\mathrm{d}s\,. \tag{3}$$

The solution belongs to the Banach space

$$\mathcal{V} = \big\{v\in \mathrm{C}^1\big([0,T];\mathrm{L}_2(\Omega_0;\mathbb{R}^m)\big)\cap \mathrm{C}^0\big([0,T];\mathcal{D}(A)\big)\colon v(0)=0\big\}\,,$$

see, e.g., [19, Thm. 12.22]. In particular, we obtain for the range of the space-time operator $L=M\partial_t+A$

$$\mathrm{C}^1_{\mathrm{c}}\big((0,T)\times\Omega_0;\mathbb{R}^m\big)\subset \mathrm{C}^0\big((0,T);\mathcal{D}(A)\big)\subset L(\mathcal{V})\subset \mathrm{L}_2(0,T)\times\Omega_0;\mathbb{R}^m) \tag{4}$$

so that $L(\mathcal{V})$ is dense in $\mathrm{L}_2\big((0,T)\times\Omega_0;\mathbb{R}^m\big)$, cf. [19, Thm. 12.16].

Linear wave equations. Our basic example is the acoustic wave equation for velocity and pressure with $(v,p)\in\mathcal{D}(A)=\mathrm{H}(\mathrm{div},\Omega_0)\times \mathrm{H}^1_0(\Omega_0)$, $M(v,p)=(\rho v,\kappa^{-1}p)$, and $A(v,p)=(\nabla p,\nabla\cdot v)$.

Note that in this example the definition of the domain $\mathcal{D}(A)$ includes homogeneous Dirichlet boundary conditions for the pressure p; more general boundary conditions can be included into the domain of the operator A, provided the conditions (1) and (2) are satisfied (see also [16, Sect. 2.2] for various examples).

This framework also applies to linear elastic waves described by $(v,\sigma)\in\mathcal{D}(A)=\mathrm{H}^1_0(\Omega_0;\mathbb{R}^d)\times \mathrm{H}(\mathrm{div},\Omega_0;\mathbb{R}^{d\times d}_{\mathrm{sym}})$, $M(v,\sigma)=(\rho v,\mathbb{C}^{-1}\sigma)$, and $A(v,\sigma)=(\mathrm{div}\,\sigma,\varepsilon(v))$, and to electro-magnetic waves described by Maxwell's equations with $(E,H)\in \mathrm{H}_0(\mathrm{curl},\Omega_0)\times \mathrm{H}^1(\mathrm{curl},\Omega_0)$, $M(E,H)=(\varepsilon E,\mu H)$, and $A(E,H)=(-\nabla\times H,\nabla\times E)$.

Note that in all cases $\frac{1}{2}\big(Mu(t),u(t)\big)_{\Omega_0}$ is the free energy, i.e., for elastic waves the kinetic and potential energy, and for the Maxwell case the electro-magnetic energy.

A main property of the linear wave equation is the finite speed of propagation $c_{\max} > 0$, which allows – in case of local support of the source function f – to restrict the computation of $u \in \mathcal{V}$ solving $Lu = f$ to the cone

$$\mathcal{C}_+(\operatorname{supp} f) = \Big\{(t,x) \in (0,T) \times \Omega_0 \colon |x - x_0| \le c_{\max}(t - t_0) \text{ for all } (t_0,x_0) \in \operatorname{supp} f\Big\}\,, \tag{5}$$

i.e., $\operatorname{supp} u \in \mathcal{C}_+(\operatorname{supp} f)$, cf. [14, Chap. 7.2.4]. The maximal wave speed can be determined by the equivalent formulation as symmetric Friedrichs system, i.e., by the representation of the linear operator in the form $Av = \sum A_j \partial_j v$ with symmetric matrices $A_j \in \mathbb{R}^{m\times m}_{\mathrm{sym}}$. Then, the maximal speed of propagation in heterogeneous media is given by

$$c_{\max} = \|c\|_{\infty,\Omega_0}\,, \qquad c(x) = \max_{|n|=1} \max_{|w|=1} \frac{w^\top A_n w}{w^\top M(x) w}\,, \qquad A_n = \sum_{j=1}^{d} n_j A_j\,. \tag{6}$$

E.g., in the acoustic case we have $c(x) = \sqrt{\kappa(x)/\rho(x)}$.

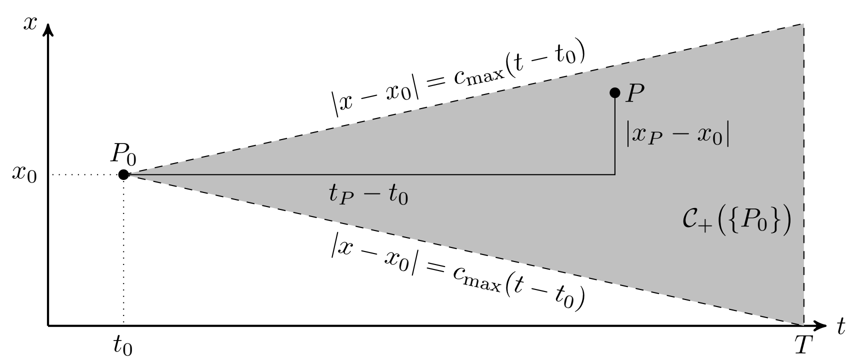

FIGURE 1. The grey area depicts the cone of dependence $C_+(\{P_0\}) \subset (0,T) \times \Omega_0$ in 1D for a single point source P_0. Due to the limit wave speed, information originating from P_0 only affects this space-time region, resulting in $C_+(\operatorname{supp} f) = \bigcup_{P_0 \in \operatorname{supp} f} C_+(\{P_0\})$ for a right-hand side f.

The adjoint equation. Let A^* be the adjoint operator of A with domain $\mathcal{D}(A^*)$, and let L^* be the adjoint operator for the wave equation backward in time defined in

$$\mathcal{V}^* = \Big\{z \in \mathrm{C}^1\big([0,T]; \mathrm{L}_2(\Omega_0;\mathbb{R}^m)\big) \cap \mathrm{C}^0\big([0,T]; \mathcal{D}(A^*)\big) \colon z(T) = 0\Big\}\,.$$

Then, $L^*(\mathcal{V}^*) \subset \mathrm{L}_2(\Omega;\mathbb{R}^m)$ is dense, we have $L^*v = -Lv$ for $v \in C^1_{\mathrm{c}}((0,T)\times\Omega_0;\mathbb{R}^m)$, and

$$(Lv, z)_{0,(0,T)\times\Omega_0} = (v, L^* z)_{0,(0,T)\times\Omega_0}\,, \qquad v \in \mathcal{V}\,,\ z \in \mathcal{V}^*\,. \tag{7}$$

The corresponding backward cone for a domain of interest $\overline{\omega} \subset [0,T] \times \Omega_0$ is given by

$$\mathcal{C}_-(\overline{\omega}) = \Big\{(t,x) \in (0,T) \times \Omega_0 \colon |x - x_0| \le c_{\max}(t_0 - t) \text{ for all } (t_0,x_0) \in \overline{\omega}\Big\}\,. \tag{8}$$

In the following, we consider applications where the source f has local support, and where the solution is evaluated only in the domain of interest $\overline{\omega}$, so that the solution process can be restricted to $C_+(\operatorname{supp} f) \cap C_-(\overline{\omega})$.

Subsets of the space-time cylinder. We consider $\Omega \subset (0,T)\times\Omega_0 \subset \mathbb{R}\times\mathbb{R}^d$ combining time slices in the form $\overline{\Omega} = \bigcup [t_{n-1},t_n]\times\overline{\Omega}_n$ with open subsets $\Omega_n \subset \Omega_0$ for $n = 1,\dots,N$ and a decomposition of the time interval $0 = t_0 < t_1 < \cdots < t_N = T$. In every time slice we select $\mathcal{D}(A;\Omega_n) \subset \mathcal{D}(A)$ such that the conditions (1) and (2) are satisfied in $\mathrm{L}_2(\Omega_n;\mathbb{R}^m)$. This defines $\mathcal{V}_n = \mathrm{C}^1\big([t_{n-1},t_n];\mathrm{L}_2(\Omega_n;\mathbb{R}^m)\big)\cap\mathrm{C}^0\big([t_{n-1},t_n];\mathcal{D}(A;\Omega_n)\big)$. Using $v_n \in \mathrm{C}^0\big([t_{n-1},t_n];\mathrm{L}_2(\Omega_n;\mathbb{R}^m)\big)$ for $v_n\in\mathcal{V}_n$ we define

$$\mathcal{V}(\Omega) = \big\{v\in \mathrm{L}_2(\Omega;\mathbb{R}^m)\colon v_n\in\mathcal{V}_n\,, v(0)=0\,, v_n(t_n) = v_{n+1}(t_n) \text{ on } \Omega_n\cap\Omega_{n+1} \text{ and } v_{n+1}(t_n)=0 \text{ on } \Omega_{n+1}\setminus\Omega_n\big\},$$

$$\mathcal{V}^*(\Omega) = \big\{z\in \mathrm{L}_2(\Omega;\mathbb{R}^m)\colon z_n\in\mathcal{V}_n\,, z(T)=0\,, z_n(t_n) = z_{n+1}(t_n) \text{ on } \Omega_n\cap\Omega_{n+1} \text{ and } z_n(t_n)=0 \text{ on } \Omega_n\setminus\Omega_{n+1}\big\}$$

with $v_n = v|_{(t_{n-1},t_n)}$. By construction, we have

$$(Lv,z)_{0,\Omega} = (v,L^*z)_{0,\Omega}\,, \qquad v\in\mathcal{V}(\Omega)\,,\ z\in\mathcal{V}^*(\Omega)\,, \tag{9}$$

and the ranges $L\big(\mathcal{V}(\Omega)\big)$, $L^*\big(\mathcal{V}^*(\Omega)\big)$ are dense in the energy space $W = \mathrm{L}_2(\Omega;\mathbb{R}^m)$. The corresponding energy norm in space and time is given by $\|w\|_W = \sqrt{(Mw,w)_{0,\Omega}}$ for $w\in W$.

Lemma 1. *We have for $u\in\mathcal{V}(\Omega)$ and $f = Lu$*

$$\|u\|_W \le \frac{T}{\sqrt{2}}\|M^{-1}f\|_W\,.$$

This is a Poincaré type estimate since it relies on the initial condition $u(0)=0$.

Proof. The estimate relies on the representation (3) of the solution in every slice $[t_{n-1},t_n]\times\Omega_n$. In every slice define $W_n = \mathrm{L}_2(\Omega_n;\mathbb{R}^m)$ and we use the energy inner product $\big(v_n,w_n\big)_{W_n} = (Mv_n,w_n)_{0,\Omega_n}$ for $v_n,w_n\in W_n$. Since the operator $M^{-1}A$ is skew-adjoint in W_n, i.e., $(M^{-1}Av_n,v_n)_{W_n} = (Av_n,v_n)_{0,\Omega_n} = 0$ for $v_n\in\mathcal{D}(A;\Omega_n)$, the spectrum is contained in $\mathrm{i}\mathbb{R}$ which yields $\|\exp(sM^{-1}A)v_n\|_{W_n}\le\|v_n\|_{W_n}$ for all $s\in\mathbb{R}$. Now, inserting

$$u(t) = \exp\big((t-t_{n-1})M^{-1}A\big)u(t_{n-1}) + \int_{t_{n-1}}^t \exp\big((t-s)M^{-1}A\big)M^{-1}f(s)\,\mathrm{d}s\,, \qquad t\in(t_{n-1},t_n]\,,$$

yields recursively for $t\in(t_{n-1},t_n]$

$$\begin{aligned}
\|u(t)\|_{W_n} &\le \|u(t_{n-1})\|_{W_n} + \int_{t_{n-1}}^t \|M^{-1}f(s)\|_{W_n}\,\mathrm{d}s\\
&\le \|u(t_{n-1})\|_{W_n} + \sqrt{t-t_{n-1}}\left(\int_{t_{n-1}}^t \|M^{-1}f(s)\|_{W_k}^2\,\mathrm{d}s\right)^{1/2}\\
&\le \sum_{k=1}^{n-1}\sqrt{t_k-t_{k-1}}\left(\int_{t_{k-1}}^{t_k}\|M^{-1}f(s)\|_{W_k}^2\,\mathrm{d}s\right)^{1/2} + \sqrt{t-t_{n-1}}\left(\int_{t_{n-1}}^t\|M^{-1}f(s)\|_{W_n}^2\,\mathrm{d}s\right)^{1/2}\\
&\le \sqrt{t}\left(\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\|M^{-1}f(s)\|_{W_k}^2\,\mathrm{d}s + \int_{t_{n-1}}^t\|M^{-1}f(s)\|_{W_n}^2\,\mathrm{d}s\right)^{1/2}.
\end{aligned}$$

Together, this yields

$$\|u\|_W^2 = \sum_{n=1}^N\int_{t_{n-1}}^{t_n}\|u(t)\|_{W_n}^2\,\mathrm{d}t \le \int_0^T t\left(\sum_{k=1}^N\int_{t_{k-1}}^{t_k}\|M^{-1}f(s)\|_{W_k}^2\,\mathrm{d}s\right)\mathrm{d}t = \frac{1}{2}T^2\|M^{-1}f\|_W^2\,.$$

□

A variational setting. We extend the operator L in $\mathcal{V}(\Omega) \subset W$ to a suitable Hilbert space defining

$$\begin{aligned}\mathrm{H}(L,\Omega) &= \{v \in W : Lv \in W\} \\ &= \big\{v \in W : w \in W \text{ exists such that } (w,z)_{0,\Omega} = (v, L^* z)_{0,\Omega} \text{ for all } z \in \mathrm{C}^1_{\mathrm{c}}(\Omega;\mathbb{R}^m)\big\}\,.\end{aligned}$$

We use the weighted graph norm $\|v\|_{\mathrm{H}(L,\Omega)} = \sqrt{\|v\|^2_W + \|M^{-1}Lv\|^2_W} = \sqrt{(Mv,v)_{0,\Omega} + (M^{-1}Lv, Lv)_{0,\Omega}}$. The closure of $\mathrm{C}^1_{\mathrm{c}}(\Omega,\mathbb{R}^m)$ with respect to $\|\cdot\|_{\mathrm{H}(L,\Omega)}$ is denoted by $\mathrm{H}_0(L,\Omega)$.
In the hyperbolic case we have $\mathrm{H}(L,\Omega) = \mathrm{H}(L^*,\Omega)$. Nevertheless, since L is associated to the forward problem and L^* to the backward problem, we will need different subspaces in the following arguments.
Since $\mathrm{C}^1(\Omega,\mathbb{R}^m)$ is dense in $\mathrm{H}(L,\Omega)$, unique extensions $L \in \mathcal{L}(\mathrm{H}(L,\Omega),W)$ and $L^* \in \mathcal{L}(\mathrm{H}(L^*,\Omega),W)$ exist.

Let V be the closure of $\mathcal{V}(\Omega)$ in $\mathrm{H}(L,\Omega)$, and let V^* be the closure of $\mathcal{V}^*(\Omega)$ in $\mathrm{H}(L^*,\Omega)$. Then, Lem. 1 yields

$$\|v\|_W \leq C_L \,\|M^{-1}Lv\|_W\,, \qquad v \in V \tag{10}$$

with $C_L = T/\sqrt{2}$, and, correspondingly, $\|z\|_W \leq C_L\,\|M^{-1}L^*z\|_W$ for $z \in V^*$ for the adjoint problem backwards in time. Moreover, (10) implies that $L(V) \subset W$ is closed, and since $L(\mathcal{V}) \subset W$ is dense, we obtain $L(V) = W$, so that $L \in \mathcal{L}(V,W)$ is a bijection. Furthermore, (9) extends to

$$(Lv,z)_{0,\Omega} = (v, L^*z)_{0,\Omega}\,, \qquad v \in V\,,\ z \in V^*\,. \tag{11}$$

The solution in the restricted space-time domain $\Omega \subset (0,T)\times\Omega_0$ is now compared with the solution in the full space-time cylinder $(0,T)\times\Omega_0$. Therefore, let V_0 and V_0^* be the closures of $\mathcal{V} = \mathcal{V}((0,T)\times\Omega_0)$ and $\mathcal{V}^* = \mathcal{V}^*((0,T)\times\Omega_0)$. Furthermore, set $W_0 = \mathrm{L}_2\big((0,T)\times\Omega_0;\mathbb{R}^m\big)$.

Lemma 2. *Let $f_0 \in W_0$ be a source function and let $u_0 \in V_0$ be the unique solution of $Lu_0 = f_0$ in $(0,T)\times\Omega_0$. Assume that $\Omega \supset \mathcal{C}_+(\operatorname{supp} f_0) \cap \mathcal{C}_-(\overline{\omega})$ for a domain of interest $\overline{\omega} \subset [0,T]\times\overline{\Omega}_0$. Then, we have for the unique solution $u \in V$ of $Lu = f$ with restricted source function $f = f_0|_\Omega \in W$ the identical values in ω, i.e.,*

$$u|_\omega = u_0|_\omega\,.$$

Proof. Let $z_0 \in V_0^*$ be the dual solution with respect to the goal functional $(u-u_0)|_\omega$ defined by

$$(L^*z_0, w_0)_{0,(0,T)\times\Omega} = (u-u_0, w_0)_{0,\omega}\,, \qquad w_0 \in W_0\,.$$

Then we observe for the dual solution $\operatorname{supp} z_0 \subset \mathcal{C}_-(\overline{\omega})$.

Let $\chi \in \mathrm{C}^1_{\mathrm{c}}\big((0,T)\times\Omega_0\big)$ be a function with $\operatorname{supp}\chi \subset \Omega$ and $\chi \equiv 1$ in $\mathcal{C} = \mathcal{C}_+(\operatorname{supp} f_0) \cap \mathcal{C}_-(\overline{\omega})$. This extends $u \in V$ to $\chi u \in V_0$ with $u(t,x) = 0$ for $(t,x) \in (0,T)\times\Omega_0 \setminus \Omega$. Using $\operatorname{supp} u \in \mathcal{C}_+(\operatorname{supp} f_0)$ and (11) we obtain

$$(L^*z_0, \chi u)_{0,\Omega} = (L^*z_0,\chi u)_{0,(0,T)\times\Omega_0} = (z_0, L\chi u)_{0,(0,T)\times\Omega_0} = (z_0, Lu)_{0,\mathcal{C}} = (z_0, Lu)_{0,\Omega}$$

and $(L^*z_0, \chi u_0)_{0,(0,T)\times\Omega_0} = (z_0, Lu_0)_{0,(0,T)\times\Omega_0}$. This yields the assertion by

$$\begin{aligned}\|u-u_0\|^2_{0,\omega} &= \big(u-u_0, \chi(u-u_0)\big)_{0,\omega} \\ &= (L^*z_0,\chi u)_{0,\Omega} - (L^*z_0,\chi u_0)_{0,(0,T)\times\Omega_0} \\ &= (z_0, Lu)_{0,\Omega} - (z_0, Lu_0)_{0,(0,T)\times\Omega_0} \\ &= (z_0, f)_{0,\Omega} - (z_0, f_0)_{0,(0,T)\times\Omega_0} = (z_0,f)_{0,\Omega} - (z_0,f_0)_{0,\mathcal{C}} = 0\,.\end{aligned}$$

□

Integration by parts defines the operators $D_\Omega \in \mathcal{L}\big(\mathrm{H}(L,\Omega),\mathrm{H}(L^*,\Omega)'\big)$ and $D'_\Omega \in \mathcal{L}\big(\mathrm{H}(L^*,\Omega),\mathrm{H}(L,\Omega)'\big)$ with

$$\langle D_\Omega v, z\rangle \;=\; (Lv,z)_{0,\Omega} - (v,L^*z)_{0,\Omega} \;=\; \langle D'_\Omega z, v\rangle\,, \qquad v\in \mathrm{H}(L,\Omega)\,,\; z\in\mathrm{H}(L^*,\Omega)\,.$$

The kernel of D_Ω is denoted by $\mathcal{N}(D_\Omega) = \big\{y \in \mathrm{H}(L,\Omega)\colon \langle D_\Omega y, z\rangle = 0 \text{ for all } z\in \mathrm{H}(L^*,\Omega)\big\} \supset \mathrm{H}_0(L,\Omega)$. In fact, the boundary conditions $\mathrm{H}_0(L,\Omega)$ and in V are characterized by duality.

Lemma 3. *We have* $\mathrm{H}_0(L,\Omega) = \mathcal{N}(D_\Omega)$ *and* $V = \big\{v\in \mathrm{H}(L,\Omega)\colon \langle D_\Omega v, z\rangle = 0$ *for all* $z\in\mathcal{V}^*\big\} = {}^\perp D'_\Omega(\mathcal{V}^*)$.

The proof is based on [11, Lem. 2.4] and [5] and uses properties of polar sets, cf. [20, Sect. 4.5].

Proof. For a given functional $\ell \in \mathrm{C}^1_{\mathrm{c}}(\Omega;\mathbb{R}^m)^\perp = \big\{\eta\in \mathrm{H}(L,\Omega)'\colon \langle \eta, v\rangle = 0 \text{ for all } v\in \mathrm{C}^1_{\mathrm{c}}(\Omega;\mathbb{R}^m)\big\}$ we define $u_\ell \in \mathrm{H}(L,Q)$ solving

$$\big(M^{-1}Lu_\ell, Lv\big)_{0,\Omega} + \big(Mu_\ell, v\big)_{0,\Omega} = \langle \ell, v\rangle\,, \quad v\in\mathrm{H}(L,Q)\,. \tag{12}$$

Then, $\big(Mu_\ell,v\big)_{0,\Omega} = -\big(M^{-1}Lu_\ell, Lv\big)_{0,\Omega}$ for all $v\in \mathrm{C}^1_{\mathrm{c}}(\Omega;\mathbb{R}^m)$, which yields $z_\ell = M^{-1}Lu_\ell \in \mathrm{H}(L^*,Q)$ and $L^*z_\ell = -Mu_\ell$. This is inserted in (12), and we obtain for the adjoint operator D'_Ω

$$\langle D'_\Omega z_\ell, v\rangle = \langle D_\Omega v, z_\ell\rangle = (Lv,z_\ell)_{0,\Omega} - (v,L^*z_\ell)_{0,\Omega} = \big(Lv, M^{-1}Lu_\ell\big)_{0,\Omega} + \big(v, Mu_\ell\big)_{0,\Omega} = \langle \ell, v\rangle\,, \quad v\in\mathrm{H}(L,Q)\,,$$

i.e., $D'_\Omega z_\ell = \ell$. This proves $\mathrm{C}^1_{\mathrm{c}}(\Omega;\mathbb{R}^m)^\perp \subset D'_\Omega\big(\mathrm{H}(L^*,\Omega)\big)$, and by duality we conclude the first assertion

$$\mathcal{N}(D_\Omega) = {}^\perp D'_\Omega\big(\mathrm{H}(L^*,\Omega)\big) \subset {}^\perp\big(\mathrm{C}^1_{\mathrm{c}}(\Omega;\mathbb{R}^m)^\perp\big) = \mathrm{H}_0(L,Q)\,.$$

Since ${}^\perp D'_\Omega\big(\mathcal{V}^*\big)$ is closed in $\mathrm{H}(L,Q)$ and $\mathcal{V}\subset {}^\perp D'_\Omega\big(\mathcal{V}^*\big)$ by (9), we have $V\subset {}^\perp D'_\Omega(\mathcal{V}^*)$. On the other hand, for $v\in {}^\perp D'_\Omega\big(\mathcal{V}^*\big)\subset \mathrm{H}(L,Q)$ set $f = Lv$ and let $u\in V$ be the unique solution of $Lu = f$. Together, we have by construction $u - v\in {}^\perp D'_\Omega(\mathcal{V}^*)$ and $L(u-v) = 0$, so that

$$0 = \big\langle D'_\Omega z, u-v\big\rangle = \big(L^*z, u-v\big)_Q - \big(z, L(u-v)\big)_Q = \big(L^*z, u-v\big)_Q\,, \qquad z\in\mathcal{V}^*\,.$$

Since $L^*(\mathcal{V}^*)\subset W$ is dense, we obtain $u = v\in V$. □

3. Space-time substructuring

Let $\Omega_h = \bigcup K \subset \Omega$ be a decomposition into open convex subsets K with skeleton $\partial\Omega_h$ satisfying $\overline{\Omega} = \Omega_h\cup\partial\Omega_h$. We consider the broken space $Z = \mathrm{H}(L^*,\Omega_h) = \prod_K \mathrm{H}(L^*,K)$ and its dual space $Z' = \mathrm{H}(L^*,\Omega_h)'$, again using the weighted norm $\|z\|_Z = \sqrt{(Mz,z)_{0,\Omega} + (M^{-1}L^*z, L^*z)_{0,\Omega_h}}$

Integration by parts defines the operators $D_K \in \mathcal{L}\big(\mathrm{H}(L,K),\mathrm{H}(L^*,K)\big)$ and $D_{\Omega_h}\in\mathcal{L}\big(\mathrm{H}(L,\Omega_h),Z'\big)$ by

$$\langle D_{\Omega_h} v, z\rangle = (Lv,z)_{0,\Omega_h} - (v,L^*z)_{0,\Omega_h} = \sum_K (Lv,z)_{0,K} - (v,L^*z)_{0,K} = \sum_K \langle D_K v|_K, z|_K\rangle\,,\; v\in\mathrm{H}(L,\Omega_h)\,,\; z\in Z\,.$$

Lem. 3 yields $\mathrm{H}_0(L,K) = \mathcal{N}(D_K)$ and thus $\mathrm{H}_0(L,\Omega_h) = \prod \mathrm{H}_0(L,K) = \mathcal{N}(D_{\Omega_h})$. This allows to identify traces on the skeleton $\partial\Omega_h$ with functionals $D_{\Omega_h}\big(\mathrm{H}(L,\Omega_h)\big)\subset Z'$. By construction, we have

$$|\langle D_{\Omega_h}v,z\rangle| \;=\; \big|(M^{-1}Lv, Mz)_{0,\Omega_h} - (Mv, M^{-1}L^*z)_{0,\Omega_h}\big| \le \|v\|_{\mathrm{H}(L,\Omega_h)}\|z\|_Z\,, \qquad v\in\mathrm{H}(L,\Omega_h)\,,\; z\in \mathrm{Z}\,,$$

i.e., $\|D_{\Omega_h}v\|_{Z'}\le \|v\|_{\mathrm{H}(L,\Omega_h)}$.

For given $f \in W$ let $u \in V$ be the unique solution of $Lu = f$. Then, we observe

$$(Lu, z)_{0,\Omega_h} = (u, L^* z)_{0,\Omega_h} + \langle D_{\Omega_h} u, z\rangle = (f, z)_{0,\Omega_h}\,, \qquad z \in Z\,.$$

This allows for a weak characterization in the energy space W and the abstract trace space $\hat V = D_{\Omega_h}(V) \subset Z'$. Therefore, we define the bilinear form in $(W \times Z') \times Z$ by

$$b(w, \hat v; z) = (v, L^* z)_{0,\Omega_h} + \langle \hat v, z\rangle\,, \qquad (w, \hat v) \in W \times Z'\,,\ z \in Z\,.$$

We now show that a unique weak solution $(u, \hat u) \in W \times \hat V$ exists with of $b(u, \hat u; z) = (f, z)_{0,\Omega_h}$ for all $z \in Z$.

In order to obtain improved optimal estimates, we use in $W \times Z'$ the norm $\|(w, \hat v)\|_{W\times Z'} = \max\{\|w\|_W, \|\hat v\|_{Z'}\}$. Then, the bilinear form $b(\cdot;\cdot)$ is continuous with $|b(w, \hat v; z)| \le \|(w, \hat v)\|_{W\times Z'}\|z\|_Z$, and we obtain the following estimate for the inf-sup stability for the ideal DPG method, cf. [7].

Lemma 4. *The bilinear form $b(\cdot;\cdot)$ is injective on $W \times \hat V$ and inf-sup stable with $\beta = (2C_L^2 + 2)^{-1/2}$, i.e.,*

$$\sup_{z\in Z} \frac{b(w, \hat v; z)}{\|z\|_Z} \ge \beta\, \|(w, \hat v)\|_{W\times Z'}\,, \qquad (w, \hat v) \in W \times \hat Y\,. \tag{13}$$

Proof. Let $(w, \hat v) \in W \times \hat V$ be in the kernel of b, i.e, $b(w, \hat v; z) = 0$ for $z \in Z$. Then we have

$$0 = (w, L^* z)_{0,\Omega_h} + \langle \hat v, z\rangle = \big(w, L^* z\big)_{0,\Omega_h}\,, \qquad z \in \mathrm{C}^1_{\mathrm{c}}(\Omega_h, \mathbb{R}^m)\,,$$

i.e., $w \in \mathrm{H}(L, \Omega_h)$ and $Lw = 0$ in Ω_h. This yields for all $v \in V$ with $D_{\Omega_h} v = \hat v$

$$\langle D_{\Omega_h}(w - v), z\rangle = \big(Lw, z\big)_{0,\Omega_h} - \big(w, L^* z\big)_{0,\Omega_h} - \langle \hat v, z\rangle = 0 - b(w, \hat v; z) = 0\,, \qquad z \in \mathrm{H}(L^*, \Omega_h)\,,$$

i.e., $w - v \in \mathcal{N}(D_{\Omega_h}) = \mathrm{H}_0(L, \Omega_h)$. Thus we have $w \in v + \mathrm{H}_0(L, \Omega_h) \subset V$, and together with $Lw = 0$ and using (10) we obtain $w = 0$. Thus, b is injective on $W \times \hat V$.
In the second step we show inf-sup stability. For given $z \in Z \subset W$ we select $u_z \in V$ with $Lu_z = Mz$, which yields

$$b(u_z, D_{\Omega_h} u_z; z) = \big(u_z, L^* z\big)_{0,\Omega_h} + \big(Lu_z, z\big)_{0,\Omega_h} - \big(u_z, L^* z\big)_{0,\Omega_h} = \big(Lu_z, z\big)_{0,\Omega_h} = \|z\|_W^2\,.$$

Inserting (10) we obtain

$$\begin{aligned} \|(u_z, D_{\Omega_h} u_z)\|_{W\times Z'} &= \max\big\{\|u_z\|_W, \|D_{\Omega_h} u_z\|_{Z'}\big\} \le \max\big\{\|u_z\|_W, \|u_z\|_{\mathrm{H}(L,\Omega_h)}\big\} \\ &\le \|u_z\|_{\mathrm{H}(L,\Omega_h)} \le \sqrt{C_L^2 + 1}\|M^{-1} L u_z\|_W = \sqrt{C_L^2 + 1}\|z\|_W\,. \end{aligned}$$

This establishes for all $z \in Z$ by selecting $(u_z, D_{\Omega_h} u_z)$ and $(M^{-1} L^* z, 0)$ in $W \times Z'$

$$\begin{aligned} \sup_{(v,\hat v)\in W\times\hat V} \frac{b(v, \hat v; z)}{\|(v, \hat v)\|_{W\times Z'}} &\ge \max\left\{\frac{b(u_z, D_{\Omega_h} u_z; z)}{\|(u_z, D_{\Omega_h} u_z)\|_{W\times Z'}}, \frac{b(M^{-1} L^* z, 0; z)}{\|(M^{-1} L^* z, 0)\|_{W\times Z'}}\right\} \\ &\ge \max\left\{\frac{\|z\|_W^2}{\sqrt{C_L^2 + 1}\|z\|_W}, \frac{\|M^{-1} L^* z\|_W^2}{\|M^{-1} L^* z\|_W}\right\} \\ &\ge \frac{\max\big\{\|z\|_W, \|M^{-1} L^* z\|_W\big\}}{\sqrt{C_L^2 + 1}} \ge \frac{\|z\|_Z}{\sqrt{2}\sqrt{C_L^2 + 1}}\,. \end{aligned} \tag{14}$$

Since $b(\cdot;\cdot)$ is injective in $W \times \hat V$, we obtain (13) by duality [1, Lem. 4.4.2]. □

4. **Petrov-Galerkin estimates**

In Lem. 4 we established a variational space-time setting for weak solutions in the product space $Y = W \times \hat{V}$ with a broken test space Z. Introducing the trial-to-test operator T_Z, see Tab. 1, we observe

$$\|T_Z y\|_Z = \sup_{z \in Z} \frac{b(y;z)}{\|z\|_Z} \geq \beta \|y\|_Y \,, \qquad y \in Y \,,$$

i.e., inf-sup stability of the bilinear form $b(\cdot;\cdot)$ in $Y \times Z$ implies ellipticity $\|T_Z y\|_Z^2 = \langle Sy, y\rangle \geq \beta^2 \|y\|_Y^2$ of the corresponding symmetric Schur complement problem in Y.

Table 1. Operators for the Petrov-Galerkin analysis.

	$B \in \mathcal{L}(Y, Z')$	$\langle By, z\rangle = b(y;z)$	$y \in Y\,,\ z \in Z$
Riesz operator	$A_Z \in \mathcal{L}(Z, Z')$	$\langle A_Z z, \psi\rangle = (z, \psi)_Z$	$z, \psi \in Z$
trial-to-test operator	$T_Z = A_Z^{-1} B \in \mathcal{L}(Y, Z)$	$(Ty, z)_Z = b(y;z)$	$y \in Y\,,\ z \in Z$
Schur complement	$S = B' A_Z^{-1} B \in \mathcal{L}(Y, Y')$		
natural embedding	$E_{Y_h} \in \mathcal{L}(Y_h, Y)$	$E_{Y_h} y_h = y_h$	$y_h \in Y_h$
	$E_{Z_h} \in \mathcal{L}(Z_h, Z)$	$E_{Z_h} z_h = z_h$	$z_h \in Z_h$
	$A_{Z_h} = E'_{Z_h} A_Z E_{Z_h} \in \mathcal{L}(Z_h, Z'_h)$	$\langle A_{Z_h} z_h, \psi_h\rangle = (z_h, \psi_h)_Z$	$z_h, \psi_h \in Z_h$
Galerkin projection	$P_{Z_h} = A_{Z_h}^{-1} E'_{Z_h} A_Z \in \mathcal{L}(Z, Z_h)$	$(P_{Z_h} z, \psi_h)_Z = (z, \psi_h)_Z$	$z \in Z\,,\ \psi_h \in Z_h$
	$B_h = E'_{Z_h} B E_{Y_h} \in \mathcal{L}(Y_h, Z'_h)$	$\langle B_h y_h, z_h\rangle = b(y_h; z_h)$	$y_h \in Y_h\,, z_h \in Z_h$
	$T_{Z_h} = A_{Z_h}^{-1} B_h = P_{Z_h} T_Z E_{Y_h} \in \mathcal{L}(Y_h, Z_h)$	$(T_{Z_h} y_h, z_h)_Z = b(y_h; z_h)$	$y_h \in Y_h,\ z_h \in Z_h$
	$S_h = B'_h A_{Z_h}^{-1} B_h \in \mathcal{L}(Y_h, Y'_h)$		

Now we select discrete spaces $W_h \subset W$ and $V_h \subset V$, and we set $Y_h = W_h \times D_{\Omega_h}(V_h) \subset Y$. An appropriate discrete test space always exists (see, e.g., [3, Thm. 4.8] for the transport equation and [24, Sect. 6]).

Lemma 5. *For given $\beta_h \in (0, \beta)$ a discrete test space $Z_h \subset Z$ exists such that*

$$\|T_{Z_h} y_h\|_Z = \sup_{z_h \in Z_h} \frac{b(y_h; z_h)}{\|z_h\|_Z} \geq \beta_h \|y_h\|_Y \,, \qquad y_h \in Y_h \,. \tag{15}$$

Again this implies ellipticity $\langle S_h y_h, y_h\rangle \geq \beta_h^2 \|y_h\|_Y^2$ of the corresponding discrete Schur complement problem.

Proof. Let $Z_h^{\mathrm{opt}} = T_Z E_{Y_h}(Y_h)$ be the optimal test space, and let $Z_{h,k}$, $k \in \mathbb{N}$ be a dense family of discrete spaces so that $z = \lim_{k \longrightarrow \infty} P_{Z_{h,k}} z$ for all $z \in Z$. Since Z_h^{opt} is discrete, for all $\varepsilon \in (0,1)$ some $k = k(\varepsilon) \in \mathbb{N}$ exists such that $\|P_{Z_{h,k}} z_h - z_h\|_Z \leq \varepsilon \|z_h\|_Z$ for all $z_h \in Z_h^{\mathrm{opt}}$. This yields

$$\|P_{Z_{h,k}} z_h\|_Z^2 = \|z_h\|_Z^2 - \|P_{Z_{h,k}} z_h - z_h\|_Z^2 \geq \|z_h\|_Z^2 - \varepsilon^2 \|z_h\|_Z^2 = (1 - \varepsilon^2)\, \|z_h\|_Z^2 \,, \qquad z_h \in Z_h^{\mathrm{opt}} \,.$$

Now, for given $\beta_h \in (0, \beta)$, select $\varepsilon = \sqrt{1 - \beta_h^2/\beta^2} > 0$ and set $Z_h = Z_{h,k(\varepsilon)}$. This yields for all $y_h \in Y_h$

$$\sup_{z_h \in Z_h} \frac{b(y_h; z_h)}{\|z_h\|_Z} = \|T_{Z_h} y_h\|_Z = \|P_{Z_h} T_Z E_{Y_h} y_h\|_Z \geq \sqrt{1 - \varepsilon^2} \|T_Z E_{Y_h} y_h\|_Z \geq \sqrt{1 - \varepsilon^2} \beta \|y_h\|_Y = \beta_h \|y_h\|_Y \,.$$

□

Table 2. Operators and bilinear forms for the local Petrov-Galerkin analysis with restricted spaces $Y_K = \mathrm{L}_2(K;\mathbb{R}^m) \times \mathrm{H}(K,L)$ and $Z_K = \mathrm{H}(K,L^*)$ for $K \subset \Omega_h$, and local finite element spaces $W_{K,h} = W_h|_K$, $V_{K,h} = V_h|_K$, $Y_{K,h} = W_{K,h} \times V_{K,h}$, and $Z_{K,h} = Z_h|_K$.

$B_K \in \mathcal{L}(Y_K, Z_K')$	$\langle B_K y_K, z_K\rangle = b_K(y_K; z_K)$	$y_K \in Y_K\,,\ z_K \in Z_K$
$B_{K,h} \in \mathcal{L}(Y_{K,h}, Z_{K,h}')$	$\langle B_{K,h} y_{K,h}, z_{K,h}\rangle = b_K(y_{K,h}; z_{K,h})$	$y_{K,h} \in Y_{K,h}\,,\ z_{K,h} \in Z_{K,h}$
$A_{Z_{K,h}} \in \mathcal{L}(Z_{K,h}, Z_{K,h}')$	$\langle A_{Z_{K,h}} z_{K,h}, \psi_{K,h}\rangle = (z_{K,h}, \psi_{K,h})_{Z_K}$	$z_{K,h}, \psi_{K,h} \in Z_{K,h}$
$A_{Y_{K,h}} \in \mathcal{L}(Y_{K,h}, Y_{K,h}')$	$\langle A_{Y_{K,h}} y_{K,h}, \phi_{K,h}\rangle = (y_{K,h}, \phi_{K,h})_{Y_K}$	$y_{K,h}, \phi_{K,h} \in Y_{K,h}$
$E_{Y_{K,h}} \in \mathcal{L}(Y_{K,h}, Y_K)$	$E_{Y_{K,h}} y_{K,h} = y_{K,h}$	$y_{K,h} \in Y_{K,h}$
$S_{K,h} = B_{K,h}' A_{Z_{K,h}}^{-1} B_{K,h} \in \mathcal{L}(Y_{K,h}, Y_{K,h}')$		

Since the optimal test space Z_h^{opt} is not accessible, the proof in Lem. 5 is not constructive, and the norm in $Y_h = W_h \times \hat{V}_h \subset W \times Z'$ cannot be evaluated. Nevertheless, for broken test spaces $Z_h = \prod Z_{K,h}$ the well-posedness of the discrete problem can be tested by a local criterion, and norm estimates can be evaluated in $W_{K,h} \times V_{K,h} = (W_h \times V_h)|_K$ using the norm in $V_{K,h} \subset \mathrm{H}(L,K)$, so that for the computable estimates it is not required to evaluate the norm in the trace space $\hat{V}_{K,h} = D_K(V_{K,h}) \subset Z_{K,h}'$.

Therefore we introduce local operators (see Tab. 2), the local bilinear form $b_K(\cdot;\cdot)$ defined by

$$b_K(w_K, v_K; z_K) = (w_K, L^* z_K)_{0,K} + \langle D_K v_K, z_K\rangle\,, \qquad (w_K, v_K) \in Y_K\,,\ z_K \in Z_K\,,$$

and for all $z_K \in Z_K$ the local affine spaces

$$Z_{K,h}(z_K) = \big\{ z_{K,h} \in Z_{K,h} \colon b_K(y_{K,h}, z_K - z_{K,h}) = 0 \text{ for all } y_{K,h} \in Y_{K,h} \big\}\,.$$

Lemma 6. *If $Z_{K,h}(z_K)$ is not empty for all $z_K \in Z_K$ and $K \subset \Omega_h$, the operator B_h is injective in Y_h and a Fortin operator exists, i.e., a projection $\Pi_h \in \mathcal{L}(Z, Z_h)$ with $b(y_h; z - \Pi_h z) = 0$ for all $y_h \in Y_h$ and $z \in Z$.*

This provides discrete stability and the estimate $\beta_h \geq \beta/\|\Pi_h\|_Z$ for the inf-sup constant [2, Prop. II.2.8].

Proof. For $z_K \in Z_K$ and $K \subset \Omega_h$ we define $\Pi_{K,h} z \in Z_{K,h}(z_K)$ as the element with minimal norm. Therefore, we compute a critical point $(z_{K,h}, y_{K,h}) \in Z_{K,h} \times Y_{K,h}$ of the corresponding local Lagrange functional

$$\begin{aligned} F_{K,h}(z_{K,h}, y_{K,h}) &= \frac{1}{2}\|z_{K,h}\|_{Z_K}^2 + b_K(y_{K,h}; z_K - z_{K,h}) \\ &= \frac{1}{2}\langle A_{Z_{K,h}} z_{K,h}, z_{K,h}\rangle + \langle B_K E_{Y_{K,h}} y_{K,h}, z_K\rangle - \langle B_{K,h} y_{K,h}, z_{K,h}\rangle\,, \end{aligned}$$

i.e., $A_{Z_{K,h}} z_{K,h} = B_{K,h} y_{K,h}$ and $B_{K,h}' z_{K,h} = E_{Y_{K,h}}' B_K' z_K$. This yields

$$S_{K,h} y_{K,h} = B_{K,h}' A_{Z_{K,h}}^{-1} B_{K,h} y_{K,h} = E_{Y_{K,h}}' B_K' z_K\,. \tag{16}$$

We have

$$\langle E_{Y_{K,h}}' B_K' z_K, \psi_{K,h}\rangle = b_K(\psi_{K,h}; z_K) = \langle B_{K,h}\psi_{K,h}, E_{Z_{K,h}} z_K\rangle = 0\,, \qquad \psi_{K,h} \in \mathcal{N}(B_{K,h})\,,$$

i.e., $E_{Y_{K,h}}' B_K' z_K \in \mathcal{N}(B_{K,h})^\perp$. Since $S_{K,h}$ is self-adjoint and $\mathcal{N}(S_{K,h}) = \mathcal{N}(B_{K,h})$, this shows that a Lagrange multiplier $y_{K,h} \in Y_{K,h}$ solving the local Schur complement problem (16) exists, but in general, the solution is not unique. Thus we select the solution with minimal norm in $Y_{K,h}$. This is determined by the pseuso-inverse with respect to the topology in Y_K

$$S_{K,h}^+ \;=\; \lim_{\delta \longrightarrow 0} \Big(S_{K,h} A_{Y_{K,h}}^{-1} S_{K,h} + \delta A_{Y_{K,h}} \Big)^{-1} S_{K,h} A_{Y_{K,h}}^{-1} \in \mathcal{L}(Y_{K,h}', Y_{K,h})\,.$$

Then, $y_{K,h} = S^+_{K,h} E'_{Y_{K,h}} B'_K z_K$ is a Lagrange multiplier and the minimizer is given by $z_{K,h} = \Pi_{K,h} z_K$ with

$$\Pi_{K,h} = A^{-1}_{Z_{K,h}} B_{K,h} S^+_{K,h} E'_{Y_{K,h}} B'_K \in \mathcal{L}(Z, Z_{K,h})\,.$$

This defines $\Pi_h z = \Big(\sum_{K,h} E_{Z_{K,h}} \Pi_{K,h} z|_K\Big)_K$, and thus $b(y_h, z - \Pi_h z) = \sum_K b_K(y_h|_K, z|_K - \Pi_{K,h} z|_K) = 0$ for all $y_h \in Y_h$, i.e., Π_h is a Fortin operator. Then, for all $y_h \in \mathcal{N}(B_h)$

$$0 = b(y_h; \Pi_h z) = b(y_h; z) = \langle B E_{Y_h} y_h, z \rangle\,, \qquad z \in Z\,.$$

Since B is injective in Y, this implies $y_h = 0$ and thus the assertion. □

Remark 7. If B_h is injective, a Fortin operator exists (see, e.g., [10] for a general Banach space case and [6] for the application to DPG). An optimal Fortin operator $\Pi_h^{\rm opt}$ can be determined as follows: for given $z \in Z$ find $z_h \in Z_h$ with minimal norm $\|z_h\|_Z$ subject to the constraint $b(y_h; z - z_h) = 0$ for all $y_h \in Y_h$. Again, this can be computed from a critical point $(z_h, y_h) \in Z_h \times Y_h$ of the corresponding Lagrange functional

$$F(z_h, y_h) = \frac{1}{2}\|z_h\|_Z^2 + b(y_h; z - z_h)\,,$$

i.e., $A_{Z_h} z_h = B_h y_h$, $B'_h z_h = E'_{Y_h} B' z$, and thus $z_h = A^{-1}_{Z_h} B_h y_h$ and $S_h y_h = B'_h A^{-1}_{Z_h} B_h y_h = B'_h z_h = E'_{Y_h} B' z$. This yields $z_h = \Pi_h^{\rm opt} z$ with

$$\Pi_h^{\rm opt} = A^{-1}_{Z_h} B_h S_h^{-1} E'_{Y_h} B'\,.$$

Bounding $\Pi_h^{\rm opt}$ involves an estimate for S_h^{-1} which requires a bound for the inf-sup constant. The construction in Lem. 6 requires only local estimates for $S^+_{K,h}$ which can be computed by local discrete symmetric eigenvalue problems

$$A_{Y_{K,h}} S^+_{K,h} A_{Y_{K,h}} y_{K,h} = \lambda_{K,h} A_{Y_{K,h}} y_{K,h}\,, \qquad (\lambda_{K,h}, y_{K,h}) \in [0,\infty) \times Y_{K,h}\,. \tag{17}$$

Let $\|(w_K, v_K)\|_{Y_K} = \max\big\{\|w_K\|_{W_K}, \|v_K\|_{V_K}\big\}$ be the norm in $Y_K = W_K \times V_K$.

Lemma 8. *If the constants $\alpha_{K,h} > 0$ satisfy*

$$\|S^+_{K,h} \ell_{K,h}\|_{Y_K} \le \alpha_{K,h} \|A^{-1}_{Y_{K,h}} \ell_{K,h}\|_{Y_K}\,, \qquad \ell_{K,h} \in Y'_{K,h}\,,\ K \subset \Omega_h\,, \tag{18}$$

the Fortin operator constructed in Lem. 6 is bounded by $\|\Pi_h\|_Z \le \max \alpha_{K,h}$.

Solving the eigenvalue problem (17) yields the estimate $\alpha_{K,h} = \sqrt{2} \max \lambda_{K,h}$.

Proof. For $z_K \in Z_K$ we define $\ell_{K,h} = E'_{Y_{K,h}} B'_K z_K$ and $y_{K,h} = S^+_{K,h} \ell_{K,h}$, so that

$$\begin{aligned} \Pi_{K,h} z_K &= A^{-1}_{Z_{K,h}} B_{K,h} S^+_{K,h} E'_{Y_{K,h}} B'_K z_K \\ &= A^{-1}_{Z_{K,h}} B_{K,h} S^+_{K,h} \ell_{K,h} \\ &= A^{-1}_{Z_{K,h}} B_{K,h} y_{K,h}\,. \end{aligned}$$

The definition of the norm in Y_K yields $|b_K(y_{K,h}; z_{K,h})| = |\langle B_{K,h} y_{K,h}, z_{K,h} \rangle| \le \|y_{K,h}\|_{Y_K} \|z_{K,h}\|_{Z_K}$, $z_{K,h} \in Z_{K.h}$, which gives

$$\|\Pi_{K,h} z_K\|_{Z_K} = \|A^{-1}_{Z_{K,h}} B_{K,h} y_{K,h}\|_{Z_K} \le \|y_{K,h}\|_{Y_K}\,.$$

From (18) we we obtain $\|y_{K,h}\|_{Y_K} = \|S^+_{K,h}\ell_{K,h}\|_{Y_K} \le \alpha_{K,h}\|A^{-1}_{Y_{K,h}}\ell_{K,h}\|_{Y_K}$, and finally

$$\|A^{-1}_{Y_{K,h}}\ell_{K,h}\|_{Y_K} \quad = \quad \|A^{-1}_{Y_{K,h}}E'_{Y_{K,h}}B'_K z_K\|_{Y_K} \le \|z_K\|_{Z_K} \,.$$

Together this yields $\|\Pi_{K,h}z\|_{Z_{K,h}} \le \alpha_{K,h}\|z\|_{Z_K}$. Then, $\|\Pi_h z\|_Z^2 \le \sum \alpha_{K,h}^2 \|z|_K\|_{Z_K}^2 \le \max \alpha_{K,h}^2\|z\|_Z^2$ yields the assertion. □

Remark 9. Scaling arguments on uniformly shape regular meshes in [18] and [13, Sect. 5.2] show that a bound for $\alpha_{K,h}$ can be estimated on the reference cell. Results for the acoustic wave equation on tensor-product space-time cells are presented in [13, Sect. 7.1]

5. The realization of the DPG method

In heterogeneous materials the finite element error also depends on the approximation error of the PDE, and the realization of the DPG method uses an approximation $M_h \in \mathrm{L}_\infty(\Omega_0;\mathbb{R}^{m\times m}_{\mathrm{sym}})$ of M; then we set $L_h = M_h\partial_t + A$ and $L_h^* z = -L_h z$ for $z \in \mathrm{C}^1_{\mathrm{c}}(\Omega_h;\mathbb{R}^m)$.

The estimates for the DPG analysis use functionals in $D_{\Omega_h}(V) \subset Z'$. In the implementation, we use representations of these functionals in $\widetilde{W} = \mathrm{L}_2(\partial\Omega_h;\mathbb{R}^m)$. For this purpose we introduce trace mappings for sufficiently smooth functions.

Since $\mathcal{N}(D_K) = \mathrm{H}_0(L,K) \supset \mathrm{C}^1_{\mathrm{c}}(K;\mathbb{R}^m)$, we can define trace mappings $\mathrm{tr}_K, \mathrm{tr}^*_K \in \mathcal{L}\big(\mathrm{C}^1(K;\mathbb{R}^m), \mathrm{L}_2(\partial K;\mathbb{R}^m)\big)$ such that

$$(L_h v_K, z_K)_{0,K} - (L_h^* v_K, z_K)_{0,K} = (\mathrm{tr}_K\, v_K, \mathrm{tr}^*_K\, z_K)_{0,\partial K}\,, \qquad v_K, z_K \in \mathrm{C}^1(K;\mathbb{R}^m)\,.$$

This extends to $\mathrm{tr}_h \in \mathcal{L}\big(\mathrm{C}^1(\Omega_h;\mathbb{R}^m)\cap V, \widetilde{W}\big)$ and $\mathrm{tr}^*_h \in \mathcal{L}\big(\mathrm{C}^1(\Omega_h;\mathbb{R}^m), \widetilde{W}\big)$ such that

$$(L_h v, z)_{0,\Omega_h} - (L_h^* v, z)_{0,\Omega_h} = (\mathrm{tr}_h\, v, \mathrm{tr}^*_h\, z)_{0,\partial\Omega_h}\,, \qquad v \in \mathrm{C}^1(\Omega_h;\mathbb{R}^m)\cap V\,,\ z \in \mathrm{C}^1(\Omega_h;\mathbb{R}^m)$$

by the selection of an orientation on inner faces $\overline{F} = \partial K \cap \partial K_F$ and the restriction $\mathrm{tr}_h\, v|_F = \mathrm{tr}_K\, v|_F$ for conforming functions, and the jump term $\mathrm{tr}^*_h\, z|_F = \mathrm{tr}_K\, z|_F - \mathrm{tr}_{K_F}\, z|_F$ for functions in the broken space (see [13, Sect. 6.1] for the acoustic case).

The trace defines the approximation of the bilinear form $b(\cdot;\cdot)$

$$b_h(w_h, \tilde v_h; z_h) = (w_h, L_h^* z_h)_{0,\Omega_h} + (\tilde v_h, \mathrm{tr}^*_h\, z_h)_{0,\partial\Omega_h}\,, \qquad w_h \in W\,,\ \tilde v_h \in \widetilde{W}\,,\ z_h \in Z_h\,.$$

Now we select $W_h \subset W$, $\widetilde{V}_h \subset \widetilde{W}$, and $Z_h \subset \mathrm{C}^1(\Omega_h;\mathbb{R}^m)$ such that

$$\|\tilde v_h\|_{Z'_h} = \sup_{z_h\in Z_h} \frac{(\tilde v_h, \mathrm{tr}^*_h\, z_h)_{0,\partial\Omega_h}}{\|z_h\|_Z}\,, \qquad v_h \in \widetilde{V}_h$$

is a norm in $\widetilde{V}_h$ and such that the bilinear form $b_h(\cdot;\cdot)$ is inf-sup stable, i.e., $\beta_0 > 0$ exists such that

$$\sup_{z_h\in Z_h} \frac{b_h(w_h,\tilde v_h;z_h)}{\|z_h\|_{Z_h}} \ge \beta_0 \|(w_h,\tilde v_h)\|_{W_h\times Z'_h}\,, \qquad (w_h,\tilde v_h) \in W_h \times \widetilde{V}_h \tag{19}$$

with respect to the norms for $w \in W$, $z \in Z$, and $(w_h,\tilde v_h) \in W_h \times \widetilde{V}_h$

$$\|w\|_{W_h} = \sqrt{(M_h w, w)_{0,\Omega_0}}\,, \quad \|z\|_{Z_h} = \sqrt{\|z\|^2_{W_h} + \|M_h^{-1}L^* z\|^2_{W_h}}\,, \quad \|(w_h,\tilde v_h)\|_{W\times Z'_h} = \max\{\|w_h\|_W, \|\tilde v_h\|_{Z'_h}\}\,.$$

For the error analysis we construct a conforming extension $V_h \subset V$ of the trace space $\widetilde{V}_h$ so that for all $\tilde{v}_h \in \widetilde{V}_h$ a reconstruction $v_h \in V_h$ exist satisfying

$$(\mathrm{tr}_h\, v_h, \mathrm{tr}_h^*\, z_h)_{0,\partial\Omega_h} = (\tilde{v}_h, \mathrm{tr}_h^*\, z_h)_{0,\partial\Omega_h}\,, \qquad z_h \in Z_h\,. \tag{20}$$

Then, we denote by $\widehat{\mathrm{tr}}_h v_h \in Z_h'$ the corresponding functional defined by

$$\langle \widehat{\mathrm{tr}}_h v_h, z_h \rangle = (\mathrm{tr}_h\, v_h, \mathrm{tr}_h^*\, z_h)_{0,\partial\Omega_h}\,, \qquad z_h \in Z_h\,, \tag{21}$$

and we set $\hat{V}_h = \widehat{\mathrm{tr}}_h(V_h) \subset Z_h'$.

The DPG approximation $(u_h, \tilde{u}_h) \in W_h \times \widetilde{V}_h$ for right-hand side $f \in W$ minimizes the residual

$$(u_h, \tilde{u}_h) = \arg \min_{(w_h,\tilde{v}_h)\in W_h\times\widetilde{V}_h} \sup_{z_h\in Z_h} \frac{b_h(w_h, \tilde{v}_h; z_h) - (f, z_h)_{0,\Omega}}{\|z_h\|_{Z_h}}\,.$$

It is computed by

$$b_h(u_h, \tilde{u}_h; z_h) = (f, z_h)_{0,\Omega}\,, \qquad z_h \in \tilde{T}_h(W_h \times \widetilde{V}_h)\,, \tag{22}$$

where the discrete trial-to-test operator $\tilde{T}_h \in \mathcal{L}(W_h \times \widetilde{V}_h, Z_h)$ is defined by

$$\big(\tilde{T}_h(w_h, \tilde{v}_h), z_h\big)_Z = b_h(w_h, \tilde{v}_h; z_h)\,, \qquad (w_h, \tilde{v}_h) \in W_h \times \widetilde{V}_h\,,\ z_h \in Z_h\,.$$

Now, the first Strang Lemma [9, Lem. 2.27] takes the following form.

Theorem 10. *For $f \in W$ let $u \in V$ be the solution of $Lu = f$, and let $(u_h, \tilde{u}_h) \in W_h \times \widetilde{V}_h$ be the DPG approximation solving (22). We assume that a conforming reconstruction space $V_h \subset V$ satisfying (20) and $\hat{u}_h \in \hat{V}_h$ exist with*

$$\langle \hat{u}_h, z_h \rangle = (\tilde{u}_h, \mathrm{tr}_h^*\, z_h)_{0,\partial\Omega_h}\,, \qquad z_h \in Z_h\,. \tag{23}$$

Let $\widehat{\mathrm{tr}}_h u \in Z_h'$ be defined by

$$\langle \widehat{\mathrm{tr}}_h u, z_h \rangle = (L_h u, z_h)_{0,\Omega_h} - (u, L_h^* z_h)_{0,\Omega_h}\,, \qquad z_h \in Z_h\,, \tag{24}$$

and we set $\hat{u} = D_{\Omega_h} u \in Z'$. Then, the error is bounded by

$$\|(u - u_h, \widehat{\mathrm{tr}}_h u - \hat{u}_h)\|_{W_h\times Z_h'} \le \inf_{(w_h,v_h)\in W_h\times V_h} \Bigg(\|(u - w_h, \widehat{\mathrm{tr}}_h u - \widehat{\mathrm{tr}}_h v_h)\|_{W_h\times Z_h'} + \beta_0^{-1} \sup_{z_h\in Z_h} \frac{b_h(w_h, \mathrm{tr}_h\, v_h; z_h) - b(u, \hat{u}; z_h)}{\|z_h\|_{Z_h}} \Bigg)\,.$$

Furthermore, if $u \in \mathcal{V}$, the error is bounded by

$$\|(u - u_h, \widehat{\mathrm{tr}}_h u - \hat{u}_h)\|_{W_h\times Z_h'} \le \big(1 + \beta_0^{-1}\big) \inf_{(w_h,v_h)\in W_h\times V_h} \|(u - w_h, \widehat{\mathrm{tr}}_h u - \widehat{\mathrm{tr}}_h v_h)\|_{W_h\times Z_h'} + \beta_0^{-1} \|\,\mathrm{id} - M^{-1} M_h\|_\infty \|\partial_t u\|_W\,.$$

Proof. For all $(w_h, v_h) \in W_h \times V_h$ we have

$$\|(u-u_h, \widehat{\mathrm{tr}}_h u - \hat{u}_h)\|_{W_h\times Z_h'} \leq \|(u-w_h, \widehat{\mathrm{tr}}_h u - \widehat{\mathrm{tr}}_h v_h)\|_{W_h\times Z_h'} + \|(w_h-u_h, \widehat{\mathrm{tr}}_h v_h - \hat{u}_h)\|_{W_h\times Z_h'} ,$$

and inserting (21), (23), and (19) gives

$$\begin{aligned}\|(w_h-u_h, \widehat{\mathrm{tr}}_h v_h - \hat{u}_h)\|_{W_h\times Z_h'} &= \|(w_h-u_h, \mathrm{tr}_h\, v_h - \tilde{u}_h)\|_{W_h\times Z_h'} \\ &\leq \beta_0^{-1} \sup_{z_h\in Z_h} \frac{b_h(w_h-u_h, \mathrm{tr}_h\, v_h - \tilde{u}_h; z_h)}{\|z_h\|_{Z_h}} .\end{aligned}$$

Using $b_h(u_h, \tilde{u}_h; z_h) = (f, z_h)_{0,\Omega} = (Lu, z_h)_{0,\Omega} = b(u, \hat{u}; z_h)$ for $z_h \in \tilde{T}_h(W_h \times \widetilde{V}_h)$ and

$$\sup_{z_h\in Z_h} \frac{b_h(w_h, \tilde{v}_h; z_h)}{\|z_h\|_{Z_h}} = \sup_{z_h\in \tilde{T}_h(W_h\times\widetilde{V}_h)} \frac{b_h(w_h, \tilde{v}_h; z_h)}{\|z_h\|_{Z_h}} , \qquad (w_h, \tilde{v}_h) \in W_h \times \widetilde{V}_h$$

yields first estimate. The second estimate follows from $b_h(w_h, \mathrm{tr}_h\, v_h; z_h) = (w_h, L_h^* z_h)_{0,\Omega_h} + \langle \widehat{\mathrm{tr}}_h v_h, z_h\rangle$ and

$$\begin{aligned} b_h(w_h-u_h, \mathrm{tr}_h\, v_h - \tilde{u}_h; z_h) &= b_h(w_h, \mathrm{tr}_h\, v_h; z_h) - (Lu, z_h)_{0,\Omega} \\ &= (w_h, L_h^* z_h)_{0,\Omega_h} + \langle \widehat{\mathrm{tr}}_h v_h, z_h\rangle + \big((M_h - M)\partial_t u, z_h\big)_{0,\Omega} - (L_h u, z_h)_{0,\Omega} \\ &= (w_h - u, L_h^* z_h)_{0,\Omega_h} + \langle \widehat{\mathrm{tr}}_h v_h - \widehat{\mathrm{tr}}_h u, z_h\rangle + \big((M_h - M)\partial_t u, z_h\big)_{0,\Omega} . \end{aligned}$$

□

Remark 11. In the case that the solution $u \in V$ has the additional regularity $u \in \mathrm{H}^{1+s}(\Omega; \mathbb{R}^m)$ with $s > k \geq 0$, a trace function $\tilde{u} \in \mathrm{L}_2(\partial\Omega_h; \mathbb{R}^m)$ with

$$(\tilde{u}, \mathrm{tr}_h^*\, z)_{0,\partial\Omega_h} = (Lu, z)_{0,\Omega_h} - (u, L^* z)_{0,\Omega_h} , \qquad z \in \mathrm{C}^1(\Omega_h; \mathbb{R}^m)$$

exists. In this case, the proof of Thm. 10 only relies on the traces in L_2, and no conforming reconstruction space V_h is required for the estimates. For $W_h \supset \mathbb{P}_k(\Omega_h)$ we obtain

$$\inf_{w_h\in W_h} \|u - w_h\|_{W_h} \leq \|M_h\|_\infty^{1/2} \inf_{w_h\in W_h} \|u - w_h\|_{0,\Omega} \leq C_1 \|M_h\|_\infty^{1/2} h^s \|u\|_{s,\Omega} ,$$

and for $\widetilde{V}_h \supset \mathbb{P}_k(\Gamma_h)$ with $\Gamma_h = \bigcup F \subset \partial\Omega_h$, we obtain, provided that $M, M_h \in \mathrm{L}_\infty(\partial\Omega_h; \mathbb{R}^{m\times m}_{\mathrm{sym}})$, the estimate

$$\inf_{v_h\in V_h} \|\widehat{\mathrm{tr}}_h u - \widehat{\mathrm{tr}}_h v_h\|_{Z_h'} = \inf_{\tilde{v}_h\in \widetilde{V}_h} \|\tilde{u} - \tilde{v}_h\|_{Z_h'} \leq C_2 h^{-1/2} \inf_{\tilde{v}_h\in \widetilde{V}_h} \|\tilde{u} - \tilde{v}_h\|_{0,\partial\Omega_h} \leq C_3 h^{s-1} \|u\|_{s,\Omega}$$

with constants $C_1, C_2, C_3 > 0$ depending on the shape regularity of Ω_h and on M_h. Together, this results in the convergence estimate

$$\|(u-u_h, \widehat{\mathrm{tr}}_h u - \hat{u}_h)\|_{W_h\times Z_h'} \leq (1+\beta_0^{-1}) \max\{C_1, C_3\} h^{s-1} \|u\|_{s,\Omega} + \beta_0^{-1} \|\,\mathrm{id} - M^{-1} M_h\|_\infty \|\partial_t u\|_W .$$

Remark 12. Choosing an extended space $Z_h^{\mathrm{ext}} \supset Z_h$ allows for a conforming reconstruction $V_h \subset Z_h^{\mathrm{ext}} \cap V$ satisfying (20), see [13, Sect. 6] for the local construction and for examples in case of acoustic waves.
In the nonconforming case $\widetilde{V} \not\subset \mathrm{tr}_h(V_h)$ the construction of the reconstruction space V_h depends on Z_h, and the asymptotic arguments in Lem. 5 do not apply. In particular, the reconstruction space may have strong oscillations at the corners and edges of ∂K. The numerical experiments in [12] indicate that the optimal choice of Z_h has to be well balanced to ensure discrete inf-sup stability on the one hand and to limit the nonconformity on the other hand. Nevertheless, for a given choice of W_h, $\widetilde{V}_h$, and Z_h the well-posedness of the discrete system can be guaranteed by Lem. 6, and explicitly computing a reconstruction V_h a lower bound for the inf-sup constant can be provided by Lem. 8.

6. A numerical example for an application in geophysics

Many applications rely on accurate numerical simulations of waves through complex material structures. For instance, geophysical structures like the earth's crust below the sea bed feature complex varying material properties. A typical example is the problem of full waveform inversion (FWI), where the material distribution is reconstructed from measurements of the wave field close to the surface [21]. Here, in a field survey a wave is excited at some point $S_0 \in \Omega_0$, and the scattered wave field is measured by receiver devices located at $R_0, \dots, R_N \in \Omega_0$. During the experiment, each receiver records a time series of approximate point measurements $u(t, R_n)$, $t \in [0, T]$. The collection of all these measurements is called a seismogram, see Fig. 5 for examples.

The recorded seismogram contains information about the material structure the wave has traveled through. Full waveform inversion techniques attempt to reconstruct from this information by applying iterative schemes of Newton-type (see, e.g., [17]). During the iteration, a large number of wave equations has to be solved numerically for different right-hand sides and varying material parameters.

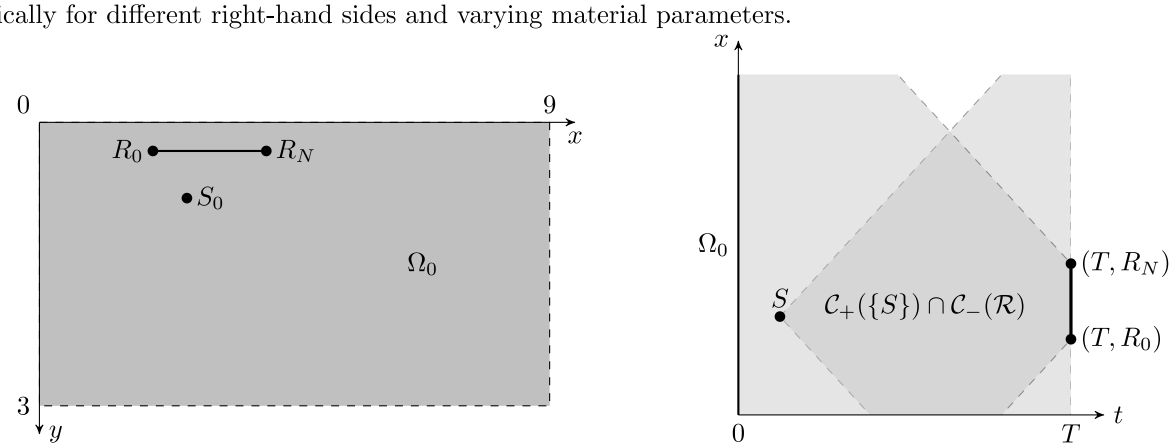

FIGURE 2. The left figure shows the spatial domain $\Omega_0 = (0,9) \times (0,3)$, the source position $S_0 = (2.602, 0.802)$, and the first and last receiver $R_0 = (2.002, 0.303)$ and $R_N = (4.002, 0.303)$. In the space-time cylinder, the source is located at $S = (0.4, S_0)$ and the signal is measured in $\mathcal{R} = \operatorname{conv}\big(\{(t, R_0), \dots, (t, R_N) \colon t \in (0,T)\}\big)$, which results in the domain of interest intersecting of the forward and backward cone $\mathcal{C}_+(\{S\}) \cap \mathcal{C}_-(\mathcal{R})$. This is illustrated on the right for $\{y = 0\}$.

To demonstrate the flexibility and the accuracy of the space-time DPG method for heterogeneous media, we consider a numerical example corresponding to the forward problem within FWI. We use the acoustic wave equation for the Marmousi benchmark, a synthetic model problem for geophysical structures in two space dimensions featuring a material distribution that is similar to what is located inside the earth crust (see, e.g., [4]). Fig. 3 shows rescaled variants of the spatial varying mass density and bulk modulus that have similar structure to the materials in this benchmark. We select the spatial domain $\Omega_0 \subset \mathbb{R}^2$, the receiver positions $R_0, \dots, R_N$, and the source position $S = (t_0, S_0) \in (0,T) \times \Omega_0$ as shown in Fig. 2.

For the inversion, we only require the wave field in the space-time region that contributes to the transport of information from the source to the receiver array. Thus, using Lem. 2 for the maximal wave speed $c_{\max} = 1.37$, we find a superset of the relevant region of the full space-time domain $(0,T) \times \Omega_0$, see Fig. 2. At the source position S, the wave is excited by the right-hand side $f(x,t) = 100\, \phi_t(t_0 - t)\phi_x(|S_0 - x|)$, $(t,x) \in (0,T) \times \Omega_0$ with a Ricker wavelet for $f_0 = 100$ and $r_0 = 0.05$ in time

$$\phi_t(\tau) = (1 - 2a\tau^2)\exp(-a\tau^2), \quad a = \pi^2 f_0^2 \tau^2,$$

and in space $\phi_x(r) = \chi_{[0,r_0)} \cos\left(\frac{\pi}{2}\frac{r}{r_0}\right)^6$.

The numerical simulations were performed using the DPG approximations with local finite element spaces

$$W_{K,h} = \mathbb{Q}_3(K)^3\,, \quad \tilde V_h|_{(t_{n-1},t_n)\times\partial K} = \mathbb{Q}_4(F)^3\,, \quad \tilde V_h|_{\{t_n\}\times K_0} = \mathbb{Q}_4(F)^2\,, \qquad Z_{K,h} = \mathbb{Q}_6(K)^3$$

on a quadrilateral mesh, yielding a scheme that converges with order 4 in $\mathrm{L}_2\big((0,T)\times\Omega_0\big)$ for smooth solutions, see [12, Sec. 4.7]. The material parameters κ and ρ are cell-wise constant as shown in Fig. 3.

The method is realized in the parallel Finite Element system M++ [22], the linear systems are solved with a preconditioned cg method using the reduction to the symmetric positive definite Schur complement system for the skeleton approximation $\widetilde V_h$, cf. [23]. The implementation of the space-time DPG method is been evaluated systematically in [12] for configurations where the analytic solution is known. Moreover, basic applications of space-time DPG to FWI are discussed in [12].

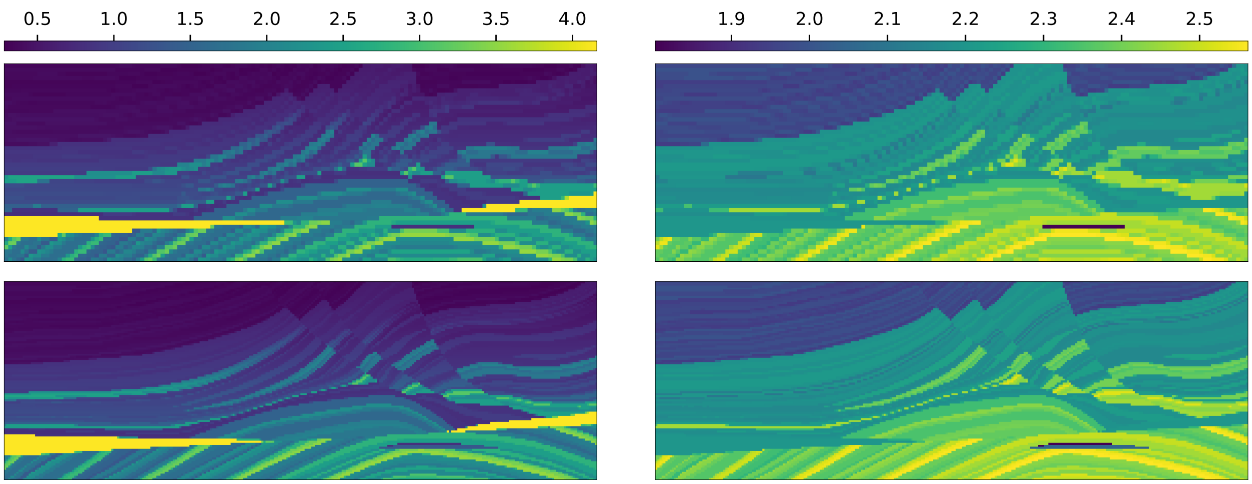

FIGURE 3. Approximations of the spatial material distributions for Marmousi, κ on the left and ρ on the right. The first row corresponds to the discrete material parameters on a coarse quadrilateral mesh with $N_x \cdot N_y = 144 \cdot 48 = 6\,912$ cells, the second row shows the material on a refined mesh with $N_x \cdot N_y = 288 \cdot 96 = 27\,648$ cells.

The pressure component of the full numerical space-time solution is shown in Fig. 4, also see Fig. 6. This is compared with the solution on a mesh that has been truncated to the space-time region of interest. The seismograms corresponding to the materials are shown in Fig. 5.

The numerical experiments with the truncated space-time cylinder correspond to our results in Lem. 2. Up to a small relative $\mathrm{L}_2\big(0,T;\mathbb{R}^3\big)$ difference of 0.00318, the seismograms on the full mesh and the truncated mesh coincide. The linear system of all space-time unknowns is solved approximately using an iterative solver, which can explain the small difference. On the other hand, only explicit time stepping methods guarantee finite speed of the discrete wave propagation. Since DPG is an implicit scheme, this also may explain a small difference of the results for the full space-time cylinder and the truncated domain.

The comparison of the solutions on different refinement levels of the mesh shows that both seismograms have a similar structure at the beginning but differ strongly at later times. Here, we have two possible reasons for this difference: on the one hand, the approximation quality for the wave field increases on mesh refinement, and on the other hand, on the finer mesh we have a better resolution of the material parameters M_h, i.e., a different wave equation is solved, see Thm. 10.

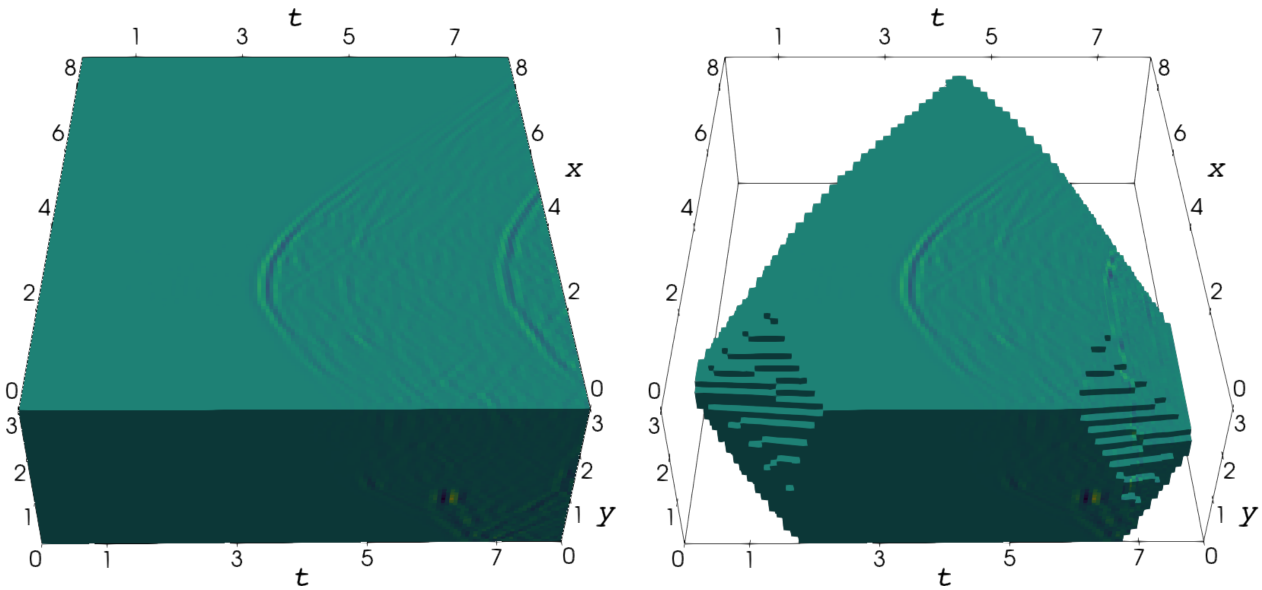

FIGURE 4. The left picture shows the pressure component of the numerical solution on a regular grid with $N_t \cdot N_x \cdot N_y = 128 \cdot 144 \cdot 48 = 884\,736$ space-time cells resulting in $156\,576\,000$ space-time DoFs for all components in the skeleton space $\tilde{V}_h$. On the right, a truncated version of the space-time cylinder is depicted that respects the position of the source and the receivers. The truncation reduced the amount of cells by approx. 43% to $503\,024$ cells and $89\,394\,900$ DoFs.

Conclusion and Outlook. In this example we show that the method also yields accurate results for application driven simulations. We demonstrate that for wave-type equations the finite wave speed can be exploited to reduce the size of the linear system considerably while yielding the same results. Further numerical experiments are necessary to compare the roles of the discretization error for fixed material M and $h \to 0$ to the model error resulting from the approximate material M_h. For that reason, we intend to evaluate the performance of an established discretization like discontinuous Galerkin with time stepping in comparison to the space-time DPG method.

Acknowledgment. We gratefully acknowledge financial support by the Deutsche Forschungsgemeinschaft (DFG) through CRC 1173.

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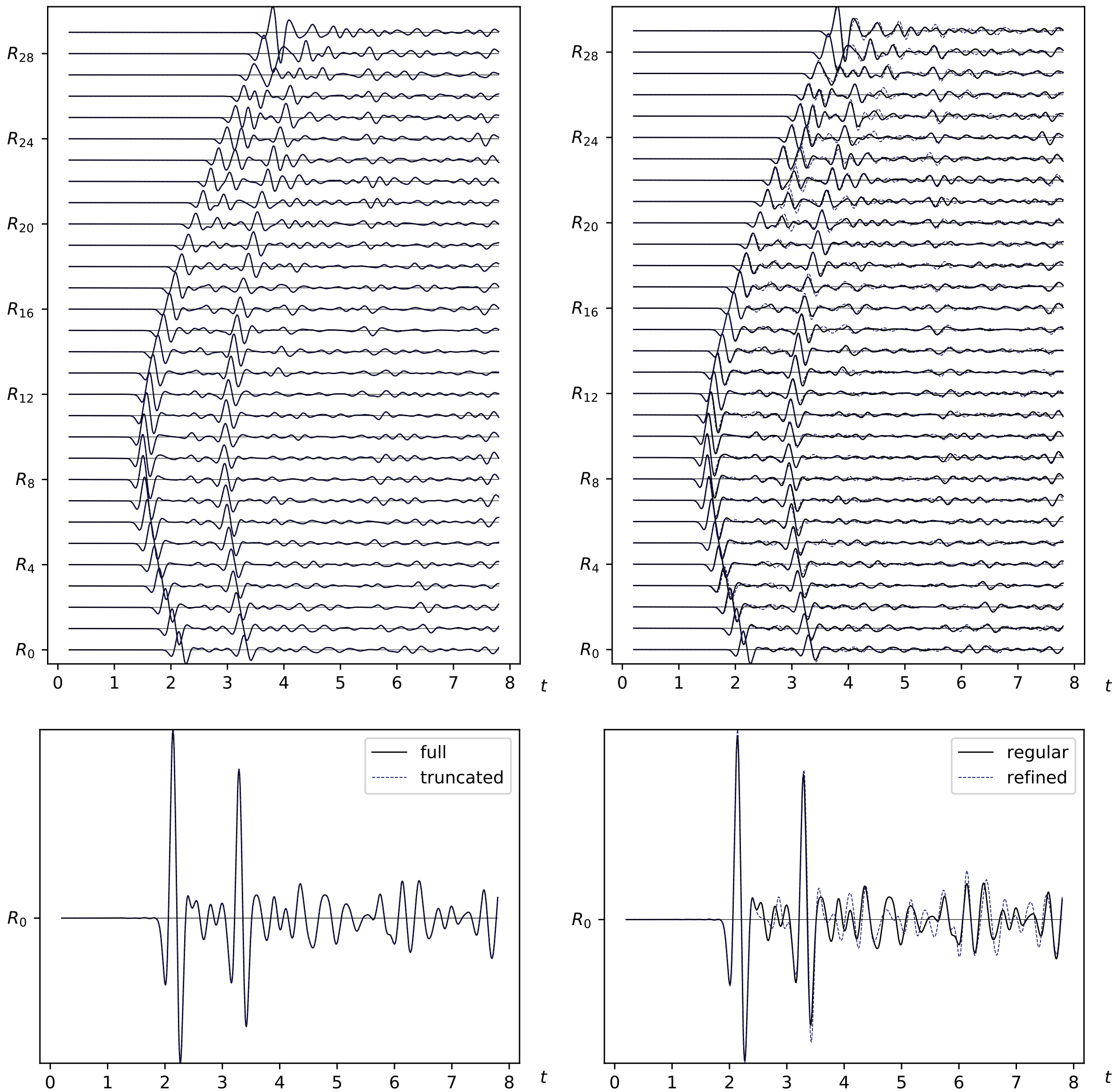

FIGURE 5. All plots show measurements of the form $t \longmapsto \int \phi(s-t, x-R_n) \cdot u(s,x)\, \mathrm{d}(s,x)$ of the pressure component at the corresponding receivers. Here ϕ is a measurement kernel with small support in space and time. On the left-hand side, a comparison of the seismograms on the full mesh (solid) and the truncated mesh (dotted) is depicted. At the bottom, the recording at R_0 is shown for illustration purposes. Since the difference is very small (relative $\mathrm{L}_2\big(0,T;\mathbb{R}^3\big)$ difference is 0.00318), both seismograms look very similar. On the right, we compare the seismogram corresponding to the truncated mesh (solid) to the seismogram obtained on a mesh that has been uniformly refined (dotted). While being similar to the measurements on the coarser grid, we see a different fine structure due to the different material approximation M_h.

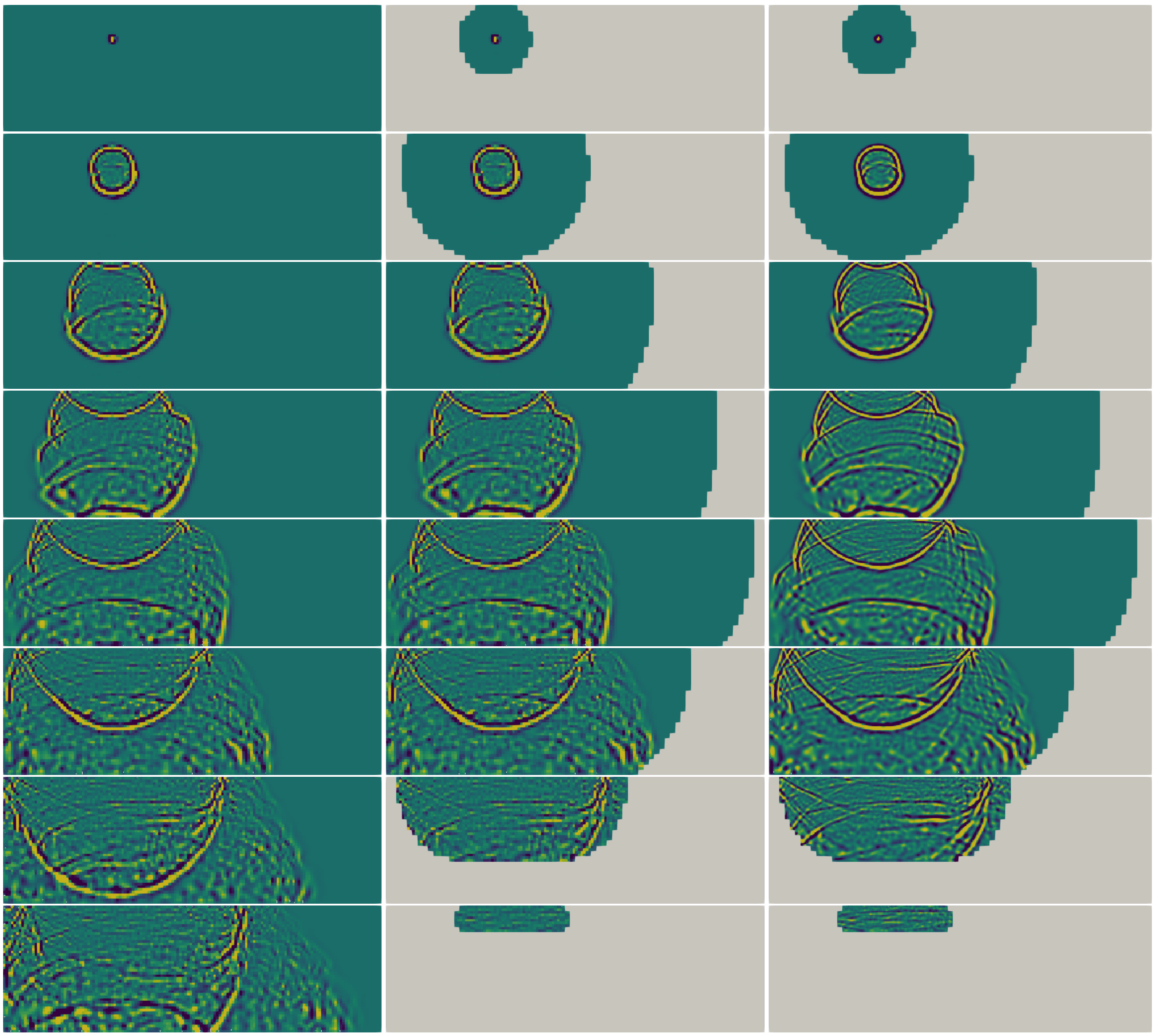

FIGURE 6. Pressure distribution for three different numerical simulations at selected time steps $t = 0.4$, 1.49, 2.57, 3.66, 4.74, 5.83, 6.91, 8. The first column shows the results for the space-time DPG method using the full space-time cylinder on level 4. The second and third column correspond to space-time DPG in the truncated space-time cylinder on level 4 and level 5 respectively.

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