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Johannes Ernesti, Christian Wieners

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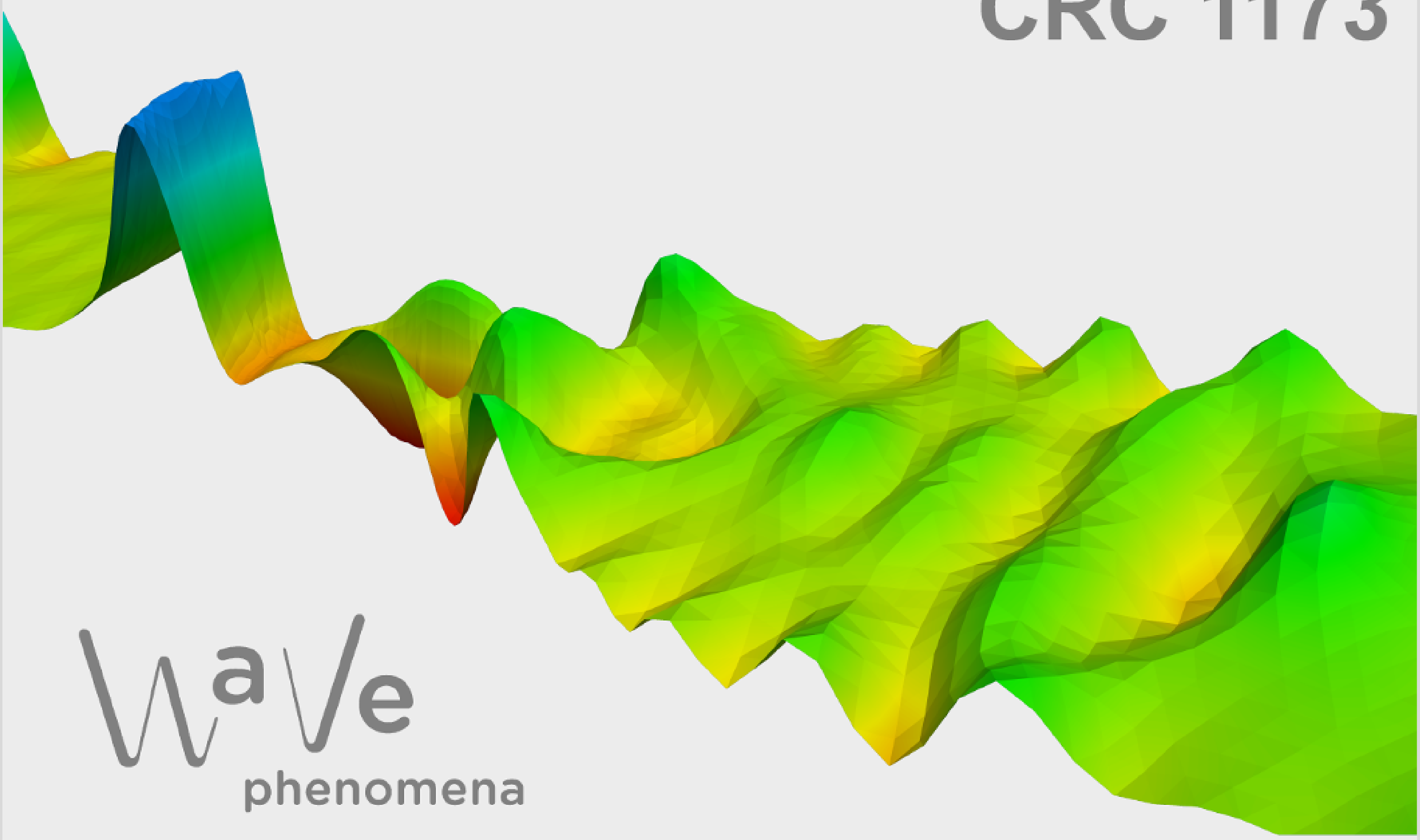

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A space-time discontinuous Petrov-Galerkin method for acoustic waves

Johannes Ernesti and Christian Wieners

Abstract. We apply the discontinuous Petrov-Galerkin (DPG) method to linear acoustic waves in space and time using the framework of first-order Friedrichs systems. Based on results for operators and semigroups of hyperbolic systems, we show that the ideal DPG method is well-posed. The main task is to avoid the explicit use of traces, which are difficult to define in Hilbert spaces with respect to the graph norm of the space-time differential operator. Then, the practical DPG method is analyzed by constructing a Fortin operator numerically.
For our numerical experiments we introduce a simplified DPG method with discontinuous ansatz functions on the faces of the space-time skeleton, where the error is bounded by an equivalent conforming DPG method. Examples for a plane wave configuration confirms the numerical analysis, and the computation of a diffraction pattern illustrates a first step to applications.

Keywords. Discontinuous Petrov-Galerkin method, space-time discretizations, semigroups, variational space-time Hilbert spaces.

AMS classification. 65N30.

1 Introduction

The discontinuous Petrov-Galerkin method (DPG), introduced by Demkowicz et al., provides a very flexible framework to construct and to analyze stable finite element discretizations for general linear first-order systems, see [5] for an overview and many references.

The main idea of the DPG method is to introduce a substructuring, and to use discontinuous approximations in the subdomains and traces on the skeleton. This is combined with discontinuous test functions, so that the discrete solution can be obtained by a symmetric linear system for the skeleton values.

The DPG method can be introduced as a minimal residual method, which allows for an equivalent saddle point formulation. So the main objective for the numerical analysis of the DPG method is to provide the corresponding inf-sup stability. This involves two steps. Firstly, in the ideal DPG method, stability has to be provided with respect to the dual norm of the residual, i.e., by testing with a full Hilbert space.

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Secondly, the explicit construction of a Fortin operator allows to analyze the practical DPG method with discrete test functions.

Here, we discuss the application of the DPG method to linear first-order systems in space and time, where we consider as reference example linear acoustic waves. The required stability for the space-time operator is obtained within a semigroup approach, which also provides an estimate of Poincaré type. To establish a suitable Hilbert space setting for the closure of the operator in space and time, we use the results in [19, Sect. 12] for semigroups, in [20, Sect. 4.5] for polar sets, and the framework for symmetric Friedrich systems in [10].

Then, following the DPG analysis in [6, 14], we show that the ideal and the practical DPG method only rely on the boundary operator using integration by parts without explicit reference to traces, see also [10]. Then, the construction of a Fortin operator follows the approach in [17].

In our realization of the DPG method, we use a simplified approach. Since the traces of space-time cells are different for faces in time and in space, conforming ansatz spaces on the skeleton may require nodal points on faces, edges, and vertices. It turns out that discontinuous ansatz functions on the faces of the space-time skeleton are easier to construct and yield optimal convergence rates in simple test scenarios. This variational crime is analyzed with respect to a discrete norm by comparing the simplified method with an equivalent conforming DPG method.

The method is implemented within the parallel finite element software system [21]. We test the full space-time approach by computing the diffraction pattern of a double slit experiment which demonstrates the advantages of a method which is simultaneously parallel in space *and* time.

2 Linear acoustic waves

We consider the first-order system for linear acoustics

$$\kappa^{-1}\,\partial_t p + \nabla\cdot\mathbf{v} \;=\; 0\,, \tag{2.1a}$$

$$\rho\,\partial_t\mathbf{v} + \nabla p \;=\; \mathbf{0} \tag{2.1b}$$

in the space-time cylinder

$$Q = \boldsymbol{\Omega}\times(0,T)\subset\mathbb{R}^d\times\mathbb{R}$$

depending on a density distribution $\rho > 0$ and bulk modulus $\kappa > 0$ (see [8, Sect. 2] for more details on this model and the relation to elastic waves). For simplicity of the presentation, we set $\rho = 1$ and $\kappa = 1$.

The corresponding the first-order differential operator is given by

$$L(p,\mathbf{v}) = \big(\partial_t p + \nabla\cdot\mathbf{v}, \partial_t\mathbf{v} + \nabla p\big)\,.$$

Now we want to establish an analytic setting for a unique solution of

$$L(p,\mathbf{v}) = (f,\mathbf{g}) \tag{2.2}$$

(subject to initial and boundary conditions) which depends continuously on the data.

2.1 The semigroup setting

We consider the ODE

$$\partial_t(p,\mathbf{v}) = A(p,\mathbf{v}) + (f,\mathbf{g})\,, \qquad A(p,\mathbf{v}) = \big(-\nabla\cdot\mathbf{v}, -\nabla p\big)\,,$$

where the operator A is associated with a dense domain

$$\mathcal{D}(A) \subset \mathrm{L}_2(\Omega;\mathbb{R}\times\mathbb{R}^d)\,.$$

Here, we choose $\mathcal{D}(A) = \mathrm{H}^1_0(\Omega)\times\mathrm{H}(\mathrm{div},\Omega)$ including homogeneous Dirichlet boundary conditions for the pressure on $\partial\Omega$.

We show that the operator A with domain $\mathcal{D}(A)$ generates a semigroup. Therefore, we check the requirements of the Lumer-Phillips theorem. In the first step, we show that $\mathrm{id} - A$ is surjective. For $(f,\mathbf{g}) \in \mathrm{L}_2(\Omega;\mathbb{R}\times\mathbb{R}^d)$, we define $p \in \mathrm{H}^1_0(\Omega)$ solving

$$(\nabla p, \nabla q)_\Omega + (p,q)_\Omega = (f,q)_\Omega + (\mathbf{g},\nabla q)_\Omega\,, \qquad q \in \mathrm{H}^1_0(\Omega)\,,$$

and then we define $\mathbf{v} = \mathbf{g} - \nabla p$. We observe

$$(\mathbf{v}, -\nabla q)_\Omega = (f,q)_\Omega - (p,q)_\Omega\,, \qquad q \in \mathrm{C}^1_\mathrm{c}(\Omega)\,,$$

i.e., $\mathbf{v} \in \mathrm{H}(\mathrm{div},\Omega)$ and $\nabla\cdot\mathbf{v} = f - p$, so that together $(p,\mathbf{v}) - A(p,\mathbf{v}) = (f,\mathbf{g})$. This gives surjectivity. Moreover, we have

$$\big(A(p,\mathbf{v}),(p,\mathbf{v})\big)_\Omega = 0\,, \qquad (p,\mathbf{v}) \in \mathcal{D}(A)\,. \tag{2.3}$$

Thus, the operator A generates a semigroup [19, Thm. 12.22] (see also [15, Sect. 2.2] and [16] for the application to general linear wave equations).

2.2 Duality, adjoint operators and the Hilbert adjoint

In the next section, many arguments will rely on duality. For this purpose, we introduce the Hilbert adjoint A^{ad} of the operator A with domain $\mathcal{D}(A^{\mathrm{ad}})$, cf. [19, Sect. 8.4.2].

The adjoint operator is defined in the domain

$$\mathcal{D}(A^{\mathrm{ad}}) = \Big\{(q,\mathbf{w}) \in \mathrm{L}_2(\Omega;\mathbb{R}\times\mathbb{R}^d)\colon (f,\mathbf{g}) \in \mathrm{L}_2(\Omega;\mathbb{R}\times\mathbb{R}^d) \text{ exists}$$
$$\text{such that } \big((f,\mathbf{g}),(p,\mathbf{v})\big)_\Omega = \big((q,\mathbf{w}),A(p,\mathbf{v})\big)_\Omega \text{ for } (p,\mathbf{v}) \in \mathcal{D}(A)\Big\}\,.$$

For the acoustic wave equation we have $\mathcal{D}(A^{\mathrm{ad}}) = \mathrm{H}^1_0(\Omega)\times\mathrm{H}(\mathrm{div},\Omega) = \mathcal{D}(A)$.

Then, for $(q,\mathbf{w}) \in \mathcal{D}(A^{\mathrm{ad}})$ we define $A^{\mathrm{ad}}(q,\mathbf{w}) \in \mathrm{L}_2(\Omega;\mathbb{R}\times\mathbb{R}^d)$ by

$$\big(A^{\mathrm{ad}}(q,\mathbf{w}),(p,\mathbf{v})\big)_\Omega = \big((q,\mathbf{w}),A(p,\mathbf{v})\big)_\Omega\,, \qquad (p,\mathbf{v}) \in \mathcal{D}(A)\,.$$

Since $\mathcal{D}(A) \subset \mathrm{L}_2(\Omega;\mathbb{R}\times\mathbb{R}^d)$ is dense, the operator A^{ad} is well-defined.

Correspondingly, for the space-time operator $L = \partial_t - A$ the formal adjoint of the differential operator is given by $L^{\mathrm{ad}} = -\partial_t - A^{\mathrm{ad}}$, and we obtain in $Q = \Omega \times (0,T)$

$$\big(L^{\mathrm{ad}}(q,\mathbf{w}),(p,\mathbf{v})\big)_Q = \big((q,\mathbf{w}),L(p,\mathbf{v})\big)_Q\,, \qquad (p,\mathbf{v})\,,\ (q,\mathbf{w}) \in \mathrm{C}^1_{\mathrm{c}}(Q;\mathbb{R}\times\mathbb{R}^d)\,.$$

In our application the adjoint problem describes a wave equation backward in time.

In the next section we will define suitable domains for the operators L and L^{ad} extending the domains $\mathcal{D}(A)$ and $\mathcal{D}(A^{\mathrm{ad}})$ in $\mathrm{L}_2(\Omega;\mathbb{R}\times\mathbb{R}^d)$ to domains of the space-time operators in $\mathrm{L}_2(Q;\mathbb{R}\times\mathbb{R}^d)$, so that L^{ad} is the Hilbert adjoint of L in this setting.

2.3 Polar sets

Below we use polar sets in Hilbert spaces X. Let X' be the topological dual of X. For $Y \subset X$ and $Z \subset X'$ the corresponding annihilator or polar sets are given by

$$\begin{aligned} Y^\perp &= \Big\{\ell \in X' \colon \langle \ell,\mathbf{y}\rangle = 0\,,\ \mathbf{y}\in Y\Big\}\,,\\ {}^\perp Z &= \Big\{\mathbf{z} \in X \colon \langle \ell,\mathbf{z}\rangle = 0\,,\ \ell\in Z\Big\}\,, \end{aligned}$$

see [20, Sect. 4.5]. In particular, ${}^\perp\big(Y^\perp\big)$ is the closure of Y in X, cf. [20, Thm. 4.7].

3 A variational space-time setting

We consider the ODE

$$\partial_t\mathbf{y} = A\mathbf{y} + \mathbf{b} \quad \text{in } [0,T]\,, \quad \mathbf{y}(0) = \mathbf{0}\,, \tag{3.1}$$

where A is an operator with a dense domain $\mathcal{D}(A)$ in $Y = \mathrm{L}_2(\Omega;\mathbb{R}^m)$. We assume that the operator A generates a semigroup. Then, for all

$$\mathbf{b} \in \mathrm{C}^0\big([0,T];\mathcal{D}(A)\big)$$

a solution $\mathbf{y} \in \mathrm{C}^1\big([0,T];Y\big) \cap \mathrm{C}^0\big([0,T];\mathcal{D}(A)\big)$ of (3.1) exists and is of the form

$$\mathbf{y}(t) = \int_0^t \exp\big((t-s)A\big)\mathbf{b}(s)\,\mathrm{d}s\,. \tag{3.2}$$

This extends to right-hand sides in

$$\mathrm{W}^{1,1}\big((0,T);Y\big) = \Big\{v \in \mathrm{L}_1\big((0,T);Y\big) \colon \exists f_v \in \mathrm{L}_1\big((0,T);Y\big) \colon \int_0^T \varphi f_v\,\mathrm{d}t = -\int_0^T \varphi' v\,\mathrm{d}t \in Y\,,\ \varphi \in \mathrm{C}^1_{\mathrm{c}}(0,T)\Big\}$$

[19, Thm. 12.16], [1, Def. II.5.7]. Equation (3.2) directly implies

$$\|\mathbf{y}(t)\|_\Omega \le \int_0^t \big\|\exp\big((t-s)A\big)\big\|_\Omega \big\|\mathbf{b}(s)\big\|_\Omega \,\mathrm{d}s\,.$$

In case of hyperbolic operators satisfying (2.3) we have $\|\exp(A)\|_\Omega = 1$, see, e.g., [19, Thm. 12.22]. Then, $\|\mathbf{y}(t)\|_\Omega \le \int_0^t \|\mathbf{b}(s)\|_\Omega\,\mathrm{d}s$, and integration in time yields

$$\begin{aligned} \|\mathbf{y}\|_{\Omega\times(0,T)} &\le \left(\int_0^T \left(\int_0^t \|\mathbf{b}(s)\|_\Omega\,\mathrm{d}s\right)^2 \mathrm{d}t\right)^{1/2} \le \left(\int_0^T t\,\|\mathbf{b}\|^2_{\Omega\times(0,t)}\mathrm{d}t\right)^{1/2} \\ &\le \left(\int_0^T t\,\mathrm{d}t\,\|\mathbf{b}\|^2_{\Omega\times(0,T)}\right)^{1/2} = \frac{T}{\sqrt{2}}\,\|\mathbf{b}\|_{\Omega\times(0,T)}\,. \end{aligned} \tag{3.3}$$

The ODE solution (3.1) belongs to the Banach space

$$\mathcal{V} = \big\{\mathbf{y}\in \mathrm{C}^1\big([0,T];Y\big)\cap \mathrm{C}^0\big([0,T];\mathcal{D}(A)\big)\colon \mathbf{y}(0)=\mathbf{0}\big\}\,,$$

and inserting the operator $L = \partial_t - A$ we obtain for all $\mathbf{b}\in \mathrm{W}^{1,1}\big((0,T);Y\big)$ a solution $\mathbf{y}\in\mathcal{V}$ with $L\mathbf{y} = \mathbf{b}$, see [19, Thm. 12.16]. Note that L is not a closed operator in $\mathcal{V}$.

Since $\mathrm{W}^{1,1}\big((0,T);Y\big)$ is dense in $\mathrm{L}_2\big((0,T);Y\big)$, we obtain the following result.

Lemma 3.1. *$L(\mathcal{V})$ is dense in $\mathrm{L}_2\big((0,T);Y\big)$.*

In [6] a corresponding density result is obtained for the linear Schrödinger equation (see also [14] for acoustic waves).

In our application also the adjoint operator A^{ad} generates a semigroup, so that this result transfers to the adjoint problem, given by the ODE backward in time

$$-\partial_t\mathbf{z} = A^{\mathrm{ad}}\mathbf{z} + \mathbf{c} \quad \text{in } [0,T]\,, \quad \mathbf{z}(T) = \mathbf{0}\,. \tag{3.4}$$

Thus, for $\mathbf{c}\in\mathrm{W}^{1,1}\big((0,T);Y\big)$ the solution of $L^{\mathrm{ad}}\mathbf{z} = \mathbf{c}$ is given by

$$\mathbf{z}(t) = -\int_t^T \exp\big((s-t)A^{\mathrm{ad}}\big)\mathbf{c}(s)\,\mathrm{d}s\,.$$

Defining

$$\mathcal{V}^{\mathrm{ad}} = \big\{\mathbf{z}\in \mathrm{C}^1\big([0,T];Y\big)\cap \mathrm{C}^0\big([0,T];\mathcal{D}(A^{\mathrm{ad}})\big)\colon \mathbf{z}(T)=\mathbf{0}\big\}$$

this shows that $L^{\mathrm{ad}}(\mathcal{V}^{\mathrm{ad}})$ is dense in $\mathrm{L}_2\big((0,T);Y\big)$, and we have

$$\big(L^{\mathrm{ad}}(q,\mathbf{w}),(p,\mathbf{v})\big)_Q = \big((q,\mathbf{w}),L(p,\mathbf{v})\big)_Q\,, \qquad (p,\mathbf{v})\in\mathcal{V}\,,\ (q,\mathbf{w})\in\mathcal{V}^{\mathrm{ad}}\,.$$

3.1 A Hilbert space setting for the space-time operator

In $W = \mathrm{L}_2\big((0,T);Y\big) = \mathrm{L}_2(Q;\mathbb{R}^m)$ we define the Hilbert space

$$\begin{aligned} \mathrm{H}(L,Q) &= \big\{\mathbf{y} \in W\colon L\mathbf{y} \in W\big\} \\ &= \big\{\mathbf{y} \in W\colon \mathbf{b} \in W \text{ exists such that} \\ &\qquad (\mathbf{b},\mathbf{z})_Q = (\mathbf{y}, L^{\mathrm{ad}}\mathbf{z})_Q \text{ for } \mathbf{z} \in \mathrm{C}^1_{\mathrm{c}}(Q;\mathbb{R}^m)\big\} \end{aligned} \tag{3.5}$$

with respect to the graph norm

$$\|\mathbf{y}\|_{Q,L} = \sqrt{\|\mathbf{y}\|_Q^2 + \|L\mathbf{y}\|_Q^2}\,, \qquad \mathbf{y} \in \mathrm{H}(L,Q)\,.$$

Analogously, we define $\mathrm{H}(L^{\mathrm{ad}},Q) = \big\{\mathbf{y} \in W\colon L^{\mathrm{ad}}\mathbf{y} \in W\big\}$, and $\mathrm{H}(L^{\mathrm{ad}},Q)'$ denotes its dual space. We define the operator $D \in \mathcal{L}\big(\mathrm{H}(L,Q),\mathrm{H}(L^{\mathrm{ad}},Q)'\big)$ by

$$\big\langle D\mathbf{y},\mathbf{z}\big\rangle = (L\mathbf{y},\mathbf{z})_Q - (\mathbf{y}, L^{\mathrm{ad}}\mathbf{z})_Q\,, \qquad \mathbf{y} \in \mathrm{H}(L,Q)\,,\ \mathbf{z} \in \mathrm{H}(L^{\mathrm{ad}},Q)\,,$$

and the kernel of D is denoted by

$$\mathcal{N}(D) \;=\; \Big\{\mathbf{y} \in \mathrm{H}(L,Q)\colon D\mathbf{y} = \mathbf{0}\Big\}\,.$$

By definition of the adjoint operator L^{ad}, we have $C^1_{\mathrm{c}}\big(Q;\mathbb{R}^m\big) \subset \mathcal{N}(D)$. Thus, the operator D describes traces obtained using integration by parts in abstract form.

Let $\mathrm{H}_0(L,Q) \subset \mathrm{H}(L,Q)$ be the closure of $C^1_{\mathrm{c}}\big(Q;\mathbb{R}^m\big) \subset \mathcal{N}(D)$. Then, also $\mathrm{H}_0(L,Q) \subset \mathcal{N}(D)$.

In fact, we can establish equality. The proof is based on duality using the operator $D^{\mathrm{ad}} \in \mathcal{L}(\mathrm{H}(L^{\mathrm{ad}},Q),\mathrm{H}(L,Q)')$ with $\big\langle D^{\mathrm{ad}}\mathbf{z},\mathbf{y}\big\rangle = (L^{\mathrm{ad}}\mathbf{z},\mathbf{y})_Q - (\mathbf{z},L\mathbf{y})_Q$, i.e., $-D^{\mathrm{ad}}$ is the transposed operator of D.

Theorem 3.2. *We have*

$$\mathrm{H}_0(L,Q) = \mathcal{N}(D)\,.$$

Proof. We only have to show $\mathcal{N}(D) \subset \mathrm{H}_0(L,Q)$. Provided we have established $\mathrm{C}^1_{\mathrm{c}}\big(Q;\mathbb{R}^m\big)^\perp \subset D^{\mathrm{ad}}\big(\mathrm{H}(L^{\mathrm{ad}},Q)\big)$, the assertion follows from

$$\begin{aligned} \mathcal{N}(D) &= \Big\{\mathbf{y} \in \mathrm{H}(L,Q)\colon \big\langle D\mathbf{y},\mathbf{z}\big\rangle = \mathbf{0} = \big\langle D^{\mathrm{ad}}\mathbf{z},\mathbf{y}\big\rangle \text{ for } \mathbf{z} \in \mathrm{H}(L^{\mathrm{ad}},Q)\Big\} \\ &= {}^\perp D^{\mathrm{ad}}\big(\mathrm{H}(L^{\mathrm{ad}},Q)\big) \subset {}^\perp\big(\mathrm{C}^1_{\mathrm{c}}\big(Q;\mathbb{R}^m\big)^\perp\big) = \mathrm{H}_0(L,Q)\,. \end{aligned}$$

The proof uses the technique in [10, Lem. 2.4], see also [4, Lem. 2.2] and [22, Lem. 1]. For a given functional $\ell \in \mathrm{C}^1_{\mathrm{c}}\big(Q;\mathbb{R}^m\big)^\perp \subset \mathrm{H}(L,Q)'$ we construct $\mathbf{z} \in \mathrm{H}(L^{\mathrm{ad}},Q)$ with $D^{\mathrm{ad}}\mathbf{z} = \ell$. Therefore, we define $\mathbf{y} \in \mathrm{H}(L,Q)$ solving

$$\big(L\mathbf{y}, L\boldsymbol{\phi}\big)_Q + \big(\mathbf{y},\boldsymbol{\phi}\big)_Q = -\langle \ell,\boldsymbol{\phi}\rangle\,, \quad \boldsymbol{\phi} \in \mathrm{H}(L,Q)\,. \tag{3.6}$$

Then, since $\langle \ell, \mathbf{w}\rangle = 0$ for test functions $\mathbf{w} \in \mathrm{C}^1_{\mathrm{c}}(Q;\mathbb{R}^m)$, we observe

$$\big(\mathbf{y}, \boldsymbol{\phi}\big)_Q = -\big(L\mathbf{y}, L\boldsymbol{\phi}\big)_Q\,, \qquad \boldsymbol{\phi} \in \mathrm{C}^1_{\mathrm{c}}\big(Q;\mathbb{R}^m\big)\,.$$

Inserting $\mathbf{z} = L\mathbf{y}$ and using the definition of $\mathrm{H}(L^{\mathrm{ad}}, Q)$, we observe $\mathbf{z} \in \mathrm{H}(L^{\mathrm{ad}}, Q)$ and $L^{\mathrm{ad}}\mathbf{z} = -\mathbf{y}$. From (3.6) we now obtain

$$\begin{aligned}\big\langle D^{\mathrm{ad}}\mathbf{z}, \boldsymbol{\phi}\big\rangle &= \big(L^{\mathrm{ad}}\mathbf{z}, \boldsymbol{\phi}\big)_Q - \big(\mathbf{z}, L\boldsymbol{\phi}\big)_Q \\ &= -\big(\mathbf{y}, \boldsymbol{\phi}\big)_Q - \big(L\mathbf{y}, L\boldsymbol{\phi}\big)_Q = \big\langle \ell, \boldsymbol{\phi}\big\rangle\,, \quad \boldsymbol{\phi} \in \mathrm{H}(L, Q)\,,\end{aligned}$$

i.e., $D^{\mathrm{ad}}\mathbf{z} = \ell$. □

3.2 The closure of the space-time operator $(L, \mathcal{V})$

We assume that $C_L > 0$ exists such that

$$\|\mathbf{y}\|_Q \le C_L\,\|L\mathbf{y}\|_Q\,, \qquad \mathbf{y} \in \mathcal{V}\,. \tag{3.7}$$

In case of linear hyperbolic operators this is obtained from (3.3) with $C_L = \frac{1}{\sqrt{2}}T$, see also [18, Thm. 3.1], [7, Lem. 1], and [23, Lem. 6].

In particular, L is injective on $\mathcal{V}$. Now we define

$$V = {}^{\perp}(\mathcal{V}^{\perp}) \subset \mathrm{H}(L, Q)\,,$$

i.e., V is the closure of $\mathcal{V}$ in $\mathrm{H}(L, Q)$ with respect to the graph norm [20, Thm. 4.7]. The estimate (3.7) also holds for the closure, i.e.,

$$\|\mathbf{y}\|_Q \le C_L\,\|L\mathbf{y}\|_Q\,, \qquad \mathbf{y} \in V\,. \tag{3.8}$$

Theorem 3.3. $L \in \mathcal{L}(V, W)$ *is a bijection.*

This is a general result for operators: if L satisfies (3.7) and $L(\mathcal{V}) \subset W$ is dense, then L extends to a bijection in the closure $V = {}^{\perp}(\mathcal{V})^{\perp}$.

Proof. From (3.8) we observe that L is injective, and since $V \subset \mathrm{H}(L, Q)$ is closed, $L(V) \subset W$ has closed range. This is shown as follows: for any sequence $(\mathbf{y}_n)_n \subset V$ with $\lim L\mathbf{y}_n = \mathbf{b} \in W$ we have

$$\|\mathbf{y}_n - \mathbf{y}_k\|_Q + \|L\mathbf{y}_n - L\mathbf{y}_k\|_Q \le (C_L + 1)\|L\mathbf{y}_n - L\mathbf{y}_k\|_Q \longrightarrow 0\,,$$

so that $(\mathbf{y}_n)_n$ is a Cauchy sequence in V; since $V \subset \mathrm{H}(L, Q)$ is closed, $\mathbf{y} = \lim \mathbf{y}_n \in V$ with $L\mathbf{y} = \mathbf{b}$ exists. Since $L(\mathcal{V}) \subset W$ is dense (Lem. 3.1), we obtain $L(V) = W$. □

The estimate (3.7) transfers to the adjoint operator, i.e., we have for $\mathbf{z} \in \mathcal{V}^{\mathrm{ad}}$

$$\|\mathbf{z}\|_Q = \sup_{\mathbf{b}\in W} \frac{(\mathbf{z},\mathbf{b})_Q}{\|\mathbf{b}\|_Q} = \sup_{\mathbf{y}\in\mathcal{V}} \frac{(\mathbf{z},L\mathbf{y})_Q}{\|L\mathbf{y}\|_Q} = \sup_{\mathbf{y}\in\mathcal{V}} \frac{(L^{\mathrm{ad}}\mathbf{z},\mathbf{y})_Q}{\|L\mathbf{y}\|_Q} \le C_L\,\|L^{\mathrm{ad}}\mathbf{z}\|_Q$$

again using the density of $L(\mathcal{V})$ in W, and using

$$\big\langle D^{\mathrm{ad}}\mathbf{z},\mathbf{y}\big\rangle = (L^{\mathrm{ad}}\mathbf{z},\mathbf{y})_Q - (\mathbf{z},L\mathbf{y})_Q = 0\,, \qquad \mathbf{y}\in\mathcal{V}\,,\ \mathbf{z}\in\mathcal{V}^{\mathrm{ad}}\,, \tag{3.9}$$

which holds by construction of $\mathcal{V}$ and $\mathcal{V}^{\mathrm{ad}}$. Defining $V^{\mathrm{ad}} = {}^\perp(\mathcal{V}^{\mathrm{ad}})^\perp \subset \mathrm{H}(L^{\mathrm{ad}},Q)$, the estimate corresponding to (3.8) holds also for the closure of the adjoint, i.e.,

$$\|\mathbf{z}\|_Q \le C_L\,\|L^{\mathrm{ad}}\mathbf{z}\|_Q\,, \qquad \mathbf{z}\in V^{\mathrm{ad}}\,. \tag{3.10}$$

Theorem 3.4. *We have*

$$\begin{aligned} V &= {}^\perp D^{\mathrm{ad}}\big(\mathcal{V}^{\mathrm{ad}}\big) \\ &= \big\{\mathbf{y}\in\mathrm{H}(L,Q)\colon (L\mathbf{y},\mathbf{z})_Q = (\mathbf{y},L^{\mathrm{ad}}\mathbf{z})_Q \ \textit{for all}\ \mathbf{z}\in\mathcal{V}^{\mathrm{ad}}\big\}\,. \end{aligned}$$

In particular this shows that the operator L with domain V is the Hilbert adjoint of the operator L^{ad} with domain V^{ad}.

Proof. We have $\mathcal{V}\subset {}^\perp D^{\mathrm{ad}}\big(\mathcal{V}^{\mathrm{ad}}\big)$ by (3.9), so that $V\subset {}^\perp D^{\mathrm{ad}}\big(\mathcal{V}^{\mathrm{ad}}\big)$, since ${}^\perp D^{\mathrm{ad}}\big(\mathcal{V}^{\mathrm{ad}}\big)$ is closed in $\mathrm{H}(L,Q)$.
Now, for $\mathbf{w}\in {}^\perp D^{\mathrm{ad}}\big(\mathcal{V}^{\mathrm{ad}}\big)\subset \mathrm{H}(L,Q)$ set $\mathbf{b}=L\mathbf{w}$ and let $\mathbf{y}\in V$ be the unique solution of $L\mathbf{y}=\mathbf{b}$, cf. Thm. 3.3, yielding $L(\mathbf{y}-\mathbf{w})=\mathbf{0}$. Since $\mathbf{y}\in V\subset {}^\perp D^{\mathrm{ad}}(\mathcal{V}^{\mathrm{ad}})$, we have $\mathbf{y}-\mathbf{w}\in {}^\perp D^{\mathrm{ad}}\big(\mathcal{V}^{\mathrm{ad}}\big)$, and we obtain for all $\mathbf{z}\in\mathcal{V}^{\mathrm{ad}}$

$$0 = \big\langle D^{\mathrm{ad}}\mathbf{z},\mathbf{y}-\mathbf{w}\big\rangle = \big(L^{\mathrm{ad}}\mathbf{z},\mathbf{y}-\mathbf{w}\big)_Q - \big(\mathbf{z},L(\mathbf{y}-\mathbf{w})\big)_Q = \big(L^{\mathrm{ad}}\mathbf{z},\mathbf{y}-\mathbf{w}\big)_Q\,.$$

Since $L^{\mathrm{ad}}(\mathcal{V}^{\mathrm{ad}})\subset W$ is dense, we obtain $\mathbf{w}=\mathbf{y}\in V$. □

Remark. By the definition of V, Thm. 3.4 directly extends to $V = {}^\perp D^{\mathrm{ad}}\big(V^{\mathrm{ad}}\big)$.

4 Space-time substructuring

For a decomposition $Q_h = \bigcup_{R\in\mathcal{R}_h} R$ into open disjoint space-time cells R, we consider the corresponding discontinuous space $\mathrm{H}(L,Q_h) = \prod_R \mathrm{H}(L,R)$.

Introducing local operators $D_R \in \mathcal{L}\big(\mathrm{H}(L,R),\mathrm{H}(L^{\mathrm{ad}},R)'\big)$ defined by

$$\big\langle D_R\mathbf{y}_R,\mathbf{z}_R\big\rangle = (L\mathbf{y}_R,\mathbf{z}_R)_R-(\mathbf{y}_R,L^{\mathrm{ad}}\mathbf{z}_R)_R\,,\qquad \mathbf{y}_R\in\mathrm{H}(L,R)\,,\ \mathbf{z}_R\in\mathrm{H}(L^{\mathrm{ad}},R)\,,$$

we extend the operator D to $D_h \in \mathcal{L}\big(\mathrm{H}(L,Q_h),\mathrm{H}(L^{\mathrm{ad}},Q_h)'\big)$ by

$$\big\langle D_h\mathbf{y},\mathbf{z}\big\rangle = \sum_R \big\langle D_R\mathbf{y}_R,\mathbf{z}_R\big\rangle$$

with $\mathbf{y}_R = \mathbf{y}|_R$ and $\mathbf{z}_R = \mathbf{z}|_R$. In particular, we obtain

$$\big\langle D\mathbf{y},\mathbf{z}\big\rangle = \sum_R \big((L\mathbf{y})|_R,\mathbf{z}|_R\big)_R - \big(\mathbf{y}|_R,(L^{\mathrm{ad}}\mathbf{z})|_R\big)_R = \big\langle D_h\mathbf{y},\mathbf{z}\big\rangle \tag{4.1}$$

for conforming functions $\mathbf{y}\in\mathrm{H}(L,Q)$ and $\mathbf{z}\in\mathrm{H}(L^{\mathrm{ad}},Q)$.

Analogously, we define $D_h^{\mathrm{ad}} \in \mathcal{L}\big(\mathrm{H}(L^{\mathrm{ad}},Q_h),\mathrm{H}(L,Q_h)'\big)$.

Lemma 4.1. *We have*

$$\begin{aligned} V &= {}^\perp D_h^{\mathrm{ad}}\big(V^{\mathrm{ad}}\big)\\ &= \big\{\mathbf{y}\in\mathrm{H}(L,Q_h)\colon \big\langle D_h\mathbf{y},\mathbf{z}\big\rangle = 0 \textit{ for all } \mathbf{z}\in V^{\mathrm{ad}}\big\}\,. \end{aligned}$$

Proof. It is sufficient to show ${}^\perp D_h^{\mathrm{ad}}(V^{\mathrm{ad}}) \subset \mathrm{H}(L,Q)$. Then, (4.1) yields the assertion by ${}^\perp D_h^{\mathrm{ad}}(V^{\mathrm{ad}})\cap\mathrm{H}(L,Q) = {}^\perp D^{\mathrm{ad}}(V^{\mathrm{ad}}) = V$, cf. Thm. 3.4.

For $\mathbf{y}\in {}^\perp D_h^{\mathrm{ad}}(V^{\mathrm{ad}}) \subset \mathrm{H}(L,Q_h)$ and $\mathbf{b} = L\mathbf{y}\in W$, we have $\big\langle D_h\mathbf{y},\mathbf{z}\big\rangle = 0$ for $z\in\mathrm{C}_\mathrm{c}^1(Q,\mathbb{R}^m)\subset V^{\mathrm{ad}}$. Thus, we obtain

$$(\mathbf{b},\mathbf{z})_Q = (L\mathbf{y},\mathbf{z})_{Q_h} = (\mathbf{y},L^{\mathrm{ad}}\mathbf{z})_{Q_h} = (\mathbf{y},L^{\mathrm{ad}}\mathbf{z})_Q\,,\qquad \mathbf{z}\in\mathrm{C}_\mathrm{c}^1(Q,\mathbb{R}^m)\,,$$

so that indeed $\mathbf{y}\in\mathrm{H}(L,Q)$ by definition (3.5). □

Lemma 4.2. *We have*

$$\mathrm{H}_0(L,Q_h) = \mathcal{N}(D_h)\,.$$

Proof. We have $\mathrm{H}_0(L,R) = \mathcal{N}(D_R)$, cf. Thm. 3.2. Thus, the assertion follows from

$$\mathrm{H}_0(L,Q_h) = \textstyle\prod_R \mathrm{H}_0(L,R) = \prod_R \mathcal{N}(D_R) = \mathcal{N}(D_h)\,.$$ □

This shows that the operator D_h is well-defined on the quotient space (associated with the quotient norm) denoted by

$$\hat{\mathrm{H}}(L,Q_h) = \mathrm{H}(L,Q_h)/\mathrm{H}_0(L,Q_h)\,, \qquad \|\hat{\mathbf{y}}\|_{L,\partial Q_h} = \inf_{\hat{\mathbf{y}}=\mathbf{y}+\mathrm{H}_0(L,Q_h)} \|\mathbf{y}\|_{L,Q_h}\,,$$

i.e., $\hat{D}_h \in \mathcal{L}\big(\hat{\mathrm{H}}(L,Q_h),\mathrm{H}(L^{\mathrm{ad}},Q_h)'\big)$ is well-defined with

$$\big\langle \hat{D}_h\hat{\mathbf{y}},\mathbf{z}\big\rangle = \big\langle D_h\mathbf{y},\mathbf{z}\big\rangle\,, \qquad \hat{\mathbf{y}} = \mathbf{y} + \mathrm{H}_0(L,Q_h)\,.$$

By construction, $\hat{D}_h$ is injective, i.e., $\mathcal{N}(\hat{D}_h) = \{\mathbf{0}\}$.

With respect to the substructuring, we represent the solution in $W \times \hat{\mathrm{H}}(L,Q_h)$ as follows. For given $\mathbf{b} \in W$ let $\mathbf{y} \in V$ be the unique solution of $L\mathbf{y} = \mathbf{b}$, and define $\hat{\mathbf{y}} = \mathbf{y} + \mathrm{H}_0(L,Q_h) \in \hat{\mathrm{H}}(L,Q_h)$. Then, inserting $\hat{D}_h$ yields

$$\big(\mathbf{b},\mathbf{z}\big)_Q = \big(L\mathbf{y},\mathbf{z}\big)_Q = \big(\mathbf{y},L^{\mathrm{ad}}\mathbf{z}\big)_{Q_h} + \big\langle \hat{D}_h\hat{\mathbf{y}},\mathbf{z}\big\rangle\,, \qquad \mathbf{z} \in \mathrm{H}(L^{\mathrm{ad}},Q_h)\,.$$

For the corresponding Petrov-Galerkin method in $W \times \hat{\mathrm{H}}(L,Q_h)$, we define the operator

$$B_h \in \mathcal{L}\big(W\times\hat{\mathrm{H}}(L,Q_h),\mathrm{H}(L^{\mathrm{ad}},Q_h)'\big)\,, \qquad \big\langle B_h(\mathbf{y},\hat{\mathbf{y}}),\mathbf{z}\big\rangle = \big(\mathbf{y},L^{\mathrm{ad}}\mathbf{z}\big)_Q + \big\langle \hat{D}_h\hat{\mathbf{y}},\mathbf{z}\big\rangle$$

As a result, the pair $(\mathbf{y},\hat{\mathbf{y}})$ also fulfills the equation

$$\big\langle B_h(\mathbf{y},\hat{\mathbf{y}}),\mathbf{z}\big\rangle = (\mathbf{b},\mathbf{z})_Q\,, \qquad \mathbf{z} \in \mathrm{H}(L^{\mathrm{ad}},Q_h)\,,$$

for trial functions $(\mathbf{y},\hat{\mathbf{y}}) \in W \times \hat{\mathrm{H}}(L,Q_h)$ and test functions $\mathbf{z} \in \mathrm{H}(L^{\mathrm{ad}},Q_h)$. The norm in $W \times \hat{\mathrm{H}}(L,Q_h)$ is denoted by $\|(\mathbf{y},\hat{\mathbf{y}})\|_{Q;L,\partial Q_h} = \sqrt{\|\mathbf{y}\|_Q^2 + \|\hat{\mathbf{y}}\|_{L,\partial Q_h}^2}$.

Now we show that B_h is invertible in $W \times \hat{V}$ with $\hat{V} = V/\mathrm{H}_0(L,Q_h) \subset \hat{\mathrm{H}}(L,Q_h)$.

Theorem 4.3. *We have for* $(\mathbf{y},\hat{\mathbf{y}}) \in W \times \hat{V}$

$$\sup_{\mathbf{z}\in\mathrm{H}(L^{\mathrm{ad}},Q_h)} \frac{\big\langle B_h(\mathbf{y},\hat{\mathbf{y}}),\mathbf{z}\big\rangle}{\|\mathbf{z}\|_{L^{\mathrm{ad}},Q_h}} \ge \frac{1}{\sqrt{4C_L^2+2}}\|(\mathbf{y},\hat{\mathbf{y}})\|_{Q;L,\partial Q_h}\,. \tag{4.2}$$

Proof. In the first step, we establish

$$\sup_{(\mathbf{y},\hat{\mathbf{y}})\in W\times\hat{V}} \frac{\big\langle B_h(\mathbf{y},\hat{\mathbf{y}}),\mathbf{z}\big\rangle}{\|(\mathbf{y},\hat{\mathbf{y}})\|_{Q;L,\partial Q_h}} \ge \frac{1}{\sqrt{4C_L^2+2}}\|\mathbf{z}\|_{L^{\mathrm{ad}},Q}\,, \qquad \mathbf{z} \in \mathrm{H}(L^{\mathrm{ad}},Q_h)\,. \tag{4.3}$$

For given $\mathbf{z} \in \mathrm{H}(L^{\mathrm{ad}},Q_h) \subset W$, we find a unique function $\mathbf{y}_0 \in V$ solving $L\mathbf{y}_0 = \mathbf{z}$, and we set $\hat{\mathbf{y}}_0 = \mathbf{y}_0 + \mathrm{H}_0(L,Q_h) \in \hat{V}$. Then,

$$\big\langle B_h(\mathbf{y}_0,\hat{\mathbf{y}}_0),\mathbf{z}\big\rangle = \big(\mathbf{y}_0,L^{\mathrm{ad}}\mathbf{z}\big)_{Q_h} + \big\langle \hat{D}_h\hat{\mathbf{y}}_0,\mathbf{z}\big\rangle = \big(L\mathbf{y}_0,\mathbf{z}\big)_Q = \|\mathbf{z}\|_Q^2\,,$$

and inserting (3.8) yields

$$\begin{aligned}
\|(\mathbf{y}_0,\hat{\mathbf{y}}_0)\|^2_{Q;L,\partial Q_h} &= \|\mathbf{y}_0\|^2_Q + \|\hat{\mathbf{y}}_0\|^2_{L,\partial Q}\\
&\le \|\mathbf{y}_0\|^2_Q + \|\mathbf{y}_0\|^2_{L,Q} = 2\|\mathbf{y}_0\|^2_Q + \|L\mathbf{y}_0\|^2_Q\\
&\le (2C_L^2+1)\|L\mathbf{y}_0\|^2_Q = (2C_L^2+1)\|\mathbf{z}\|^2_Q\,,
\end{aligned}$$

so that we obtain

$$\begin{aligned}
\sup_{(\mathbf{y},\hat{\mathbf{y}})\in W\times\hat V} \frac{\big\langle B_h(\mathbf{y},\hat{\mathbf{y}}),\mathbf{z}\big\rangle}{\|(\mathbf{y},\hat{\mathbf{y}})\|_{Q;L,\partial Q_h}} &\ge \frac{\big\langle B_h(\mathbf{y}_0,\hat{\mathbf{y}}_0),\mathbf{z}\big\rangle}{\|(\mathbf{y}_0,\hat{\mathbf{y}}_0)\|_{Q;L,\partial Q_h}} = \frac{\|\mathbf{z}\|^2_Q}{\|(\mathbf{y}_0,\hat{\mathbf{y}}_0)\|_{Q;L,\partial Q_h}}\\
&\ge \frac{1}{\sqrt{2C_L^2+1}}\|\mathbf{z}\|_Q\,.
\end{aligned}$$

Then, choosing $(\mathbf{y},\hat{\mathbf{y}}) = (L^{\mathrm{ad}}\mathbf{z},\mathbf{0})$ yields

$$\sup_{(\mathbf{y},\hat{\mathbf{y}})\in W\times\hat V} \frac{\big\langle B_h(\mathbf{y},\hat{\mathbf{y}}),\mathbf{z}\big\rangle}{\|(\mathbf{y},\hat{\mathbf{y}})\|_{Q;L,\partial Q_h}} \ge \frac{\big\langle B_h(L^{\mathrm{ad}}\mathbf{z},\mathbf{0}),\mathbf{z}\big\rangle}{\|(L^{\mathrm{ad}}\mathbf{z},\mathbf{0})\|_{Q;L,\partial Q_h}} = \|L^{\mathrm{ad}}\mathbf{z}\|_Q\,.$$

Now, (4.3) follows from $\|\mathbf{z}\|^2_{L^{\mathrm{ad}},Q} \le 2\max\big\{\|\mathbf{z}\|^2_Q, \|L^{\mathrm{ad}}\mathbf{z}\|^2_Q\big\}$, i.e.,

$$\max\big\{\|\mathbf{z}\|_Q, \|L^{\mathrm{ad}}\mathbf{z}\|_Q\big\} \ge \frac{1}{\sqrt{2}}\|\mathbf{z}\|_{L^{\mathrm{ad}},Q}\,.$$

In the second step, we show that the operator B_h is injective in $W\times\hat V$; then, (4.2) is obtained by duality [2, Lem. 4.4.2].
Therefore, we consider $(\mathbf{y},\hat{\mathbf{y}})\in W\times\hat V$ with

$$\big\langle B_h(\mathbf{y},\hat{\mathbf{y}}),\mathbf{z}\big\rangle = 0\,,\qquad \mathbf{z}\in \mathrm{H}(L^{\mathrm{ad}},Q_h)\,.$$

This yields

$$0 = \big\langle B_h(\mathbf{y},\hat{\mathbf{y}}),\mathbf{z}\big\rangle = \big(\mathbf{y},L^{\mathrm{ad}}\mathbf{z}\big)_{Q_h}\,,\qquad \mathbf{z}\in \mathrm{C}^1_{\mathrm{c}}(Q_h,\mathbb{R}^m)\,,$$

i.e., $\mathbf{y}\in\mathrm{H}(L,Q_h)$ and $L\mathbf{y}=0$. Thus, from $\hat{\mathbf{y}}\in\hat V$ and $\big\langle \hat D_h\hat{\mathbf{y}},\mathbf{z}\big\rangle = 0$ for $\mathbf{z}\in V^{\mathrm{ad}}$, we conclude for all $\mathbf{z}\in V^{\mathrm{ad}}$

$$\begin{aligned}
0 &= \big\langle B_h(\mathbf{y},\hat{\mathbf{y}}),\mathbf{z}\big\rangle - \big(L\mathbf{y},\mathbf{z}\big)_{Q_h} = \big(\mathbf{y},L^{\mathrm{ad}}\mathbf{z}\big)_{Q_h} + \big\langle \hat D_h\hat{\mathbf{y}},\mathbf{z}\big\rangle - \big(L\mathbf{y},\mathbf{z}\big)_{Q_h}\\
&= \big\langle \hat D_h\hat{\mathbf{y}},\mathbf{z}\big\rangle - \big\langle D_h\mathbf{y},\mathbf{z}\big\rangle = -\big\langle D_h\mathbf{y},\mathbf{z}\big\rangle\,,
\end{aligned}$$

which shows $\mathbf{y}\in V$, cf. Lem. 4.1. Together with $L\mathbf{y}=\mathbf{0}$ and (3.8) this implies $\mathbf{y}=\mathbf{0}$. Thus,

$$0 = \big\langle B_h(\mathbf{y},\hat{\mathbf{y}}),\mathbf{z}\big\rangle = \big\langle \hat D_h\hat{\mathbf{y}},\mathbf{z}\big\rangle\,,\qquad \mathbf{z}\in\mathrm{H}(L^{\mathrm{ad}},Q_h)\,,$$

which implies $\hat{\mathbf{y}}=\mathbf{0}$, see Lem. 4.2. □

Thm. 4.3 provides stability for the *ideal DPG method* with discrete approximations in $W\times\hat V$, and for test functions the continuous space $\mathrm{H}(L^{\mathrm{ad}},Q_h)$. Using a discrete test space yields the *practical DPG method*.

5 The DPG method

Now we select a global conforming discrete ansatz space $\hat V_h \subset \hat V$ on the skeleton and local ansatz and test spaces $W_{R,h} \subset \mathrm{L}_2(R,\mathbb{R}^m)$ and $Z_{R,h} \subset \mathrm{H}(L^{\mathrm{ad}},R)$. We set $\hat V_{R,h} = \hat V_h|_{\partial R}$, $W_h = \prod W_{R,h}$ and $Z_h = \prod Z_{R,h}$.

In order to verify discrete inf-sup stability, we construct a suitable local Fortin operator $\Pi_{R,h} \in \mathcal{L}\big(\mathrm{H}(L^{\mathrm{ad}},R), Z_{R,h}\big)$ in every space-time cell R following the approach presented in [17, Sect. 3.1.4], see also the construction in [9, Thm. 1]. This yields a mesh-dependent estimate. Then we show by a scaling argument, that it is sufficient to construct a local Fortin operator on a reference cell, so that finally a mesh-independent a-priori bound for the DPG approximation is obtained.

5.1 The local construction of the Fortin operator

We define $B_R \in \mathcal{L}\big(\mathrm{L}_2(R,\mathbb{R}^m) \times \hat{\mathrm{H}}(L,R), \mathrm{H}(L^{\mathrm{ad}},R)'\big)$ by

$$\big\langle B_R(\mathbf{y}_R,\hat{\mathbf{y}}_R),\mathbf{z}_R\big\rangle = \big(\mathbf{y}_R, L^{\mathrm{ad}}\mathbf{z}_R\big)_R + \big\langle \hat D_R\hat{\mathbf{y}}_R,\mathbf{z}_R\big\rangle\,.$$

We assume that for given $\hat V_{R,h}$ and $W_{R,h}$ the local test spaces $Z_{R,h}$ are selected so that for all $\mathbf{z}_R \in \mathrm{H}(L^{\mathrm{ad}},R)$ the affine space

$$\mathcal{N}(\mathbf{z}_R) \;=\; \Big\{\mathbf{z}_{R,h} \in Z_{R,h}\colon \big\langle B_R(\mathbf{y}_{R,h},\hat{\mathbf{y}}_{R,h}),\mathbf{z}_{R,h}\big\rangle = \big\langle B_R(\mathbf{y}_{R,h},\hat{\mathbf{y}}_{R,h}),\mathbf{z}_R\big\rangle\,, \quad (\mathbf{y}_{R,h},\hat{\mathbf{y}}_{R,h}) \in W_{R,h}\times \hat V_{R,h}\Big\}$$

is not empty for all $R \in \mathcal{R}$. Then, a Fortin operator with $\Pi_{R,h}\mathbf{z}_R \in \mathcal{N}(\mathbf{z}_R)$ exists. For the scaling argument below we require the additionally $|\Pi_{R,h}\mathbf{z}_R|_{L^{\mathrm{ad}},R} \le |\mathbf{z}_R|_{L^{\mathrm{ad}},R}$ with respect to the semi-norm $|\mathbf{z}_R|_{L^{\mathrm{ad}},R} = \|L^{\mathrm{ad}}\mathbf{z}_R\|_R$. This can easily be achieved by extending $W_{R,h}$ to $W_{R,h}^{\mathrm{ext}} \supset W_{R,h} + L^{\mathrm{ad}} Z_{R,h}$, since the orthogonality

$$0 = \big\langle B_R(\mathbf{y}_{R,h},\mathbf{0}),\mathbf{z}_{R,h}-\mathbf{z}_R\big\rangle = \big(\mathbf{y}_{R,h}, L^{\mathrm{ad}}(\mathbf{z}_{R,h}-\mathbf{z}_R)\big)_R\,, \qquad \mathbf{y}_{R,h} \in W_{R,h}^{\mathrm{ext}}$$

implies $|\mathbf{z}_{R,h}|_{L^{\mathrm{ad}},R} \le |\mathbf{z}_R|_{L^{\mathrm{ad}},R}$. We assume that also $\mathcal{N}^{\mathrm{ext}}(\mathbf{z}_R) \subset \mathcal{N}(\mathbf{z}_R)$ obtained by testing with the larger space $W_{R,h}^{\mathrm{ext}} \supset W_{R,h}$ is not empty.

In order to compute a bound for the norm of $\Pi_{R,h}$ numerically, we assume that extensions $V_{R,h} \subset \mathrm{H}(L,R)$ of $\hat V_{R,h}$ exists with $\dim V_{R,h} = \dim \hat V_{R,h}$, so that for every trace function $\hat{\mathbf{y}}_{R,h} \in \hat V_{R,h}$ a unique extension $\bar{\mathbf{y}}_{R,h} \in V_{R,h}$ exists which can be locally evaluated in R and which satisfies

$$\big\langle D_R\bar{\mathbf{y}}_{R,h},\mathbf{z}_R\big\rangle \;=\; \big\langle \hat D_R\hat{\mathbf{y}}_{R,h},\mathbf{z}_R\big\rangle\,, \qquad \mathbf{z}_R \in \mathrm{H}(L^{\mathrm{ad}},R)\,, \tag{5.1}$$

i.e., $\hat{\mathbf{y}}_{R,h} = \bar{\mathbf{y}}_{R,h} + \mathrm{H}_0(L,Q_h)$. This defines a well-defined bijection

$$\hat I_{R,h}\colon V_{R,h} \to \hat V_{R,h}$$

such that $\hat{\mathbf{y}}_{R,h} = \hat I_{R,h}\bar{\mathbf{y}}_{R,h}$ satisfies (5.1).

The minimizer $\mathbf{z}_{R,h} = \Pi_{R,h}\mathbf{z}_R \in \mathcal{N}^{\text{ext}}(\mathbf{z}_R)$ with respect to the norm in $\mathrm{H}(L^{\text{ad}}, R)$ can be computed by a discrete linear saddle point problem as follows. We define the discrete operators

$$\begin{aligned} B_{R,h} &\in \mathcal{L}\big(W_{R,h}^{\text{ext}} \times V_{R,h}, Z'_{R,h}\big)\,, \\ C_{R,h} &\in \mathcal{L}\big(Z_{R,h}, Z'_{R,h}\big)\,, \\ G_{R,h} &\in \mathcal{L}\big(W_{R,h}^{\text{ext}} \times V_{R,h}, (W_{R,h}^{\text{ext}} \times V_{R,h})'\big) \end{aligned}$$

by

$$\begin{aligned} \big\langle B_{R,h}(\mathbf{y}_{R,h}, \bar{\mathbf{y}}_{R,h}), \mathbf{z}_{R,h}\big\rangle &= \big\langle B_R(\mathbf{y}_{R,h}, \hat{I}_{R,h}\bar{\mathbf{y}}_{R,h}), \mathbf{z}_{R,h}\big\rangle\,, \\ \big\langle C_{R,h}\mathbf{z}_{R,h}, \boldsymbol{\psi}_{R,h}\big\rangle &= \big(\mathbf{z}_{R,h}, \boldsymbol{\psi}_{R,h}\big)_{L^{\text{ad}},R}\,, \\ \big\langle G_{R,h}(\mathbf{y}_{R,h}, \bar{\mathbf{y}}_{R,h}), (\boldsymbol{\phi}_{R,h}, \bar{\boldsymbol{\phi}}_{R,h})\big\rangle &= \big(\mathbf{y}_{R,h}, \boldsymbol{\phi}_{R,h}\big)_R + \big(\bar{\mathbf{y}}_{R,h}, \bar{\boldsymbol{\phi}}_{R,h}\big)_{L,R}\,, \end{aligned}$$

and the embedding $E_{R,h} \in \mathcal{L}\big(W_{R,h}^{\text{ext}} \times V_{R,h}, W \times \hat{\mathrm{H}}(L,R)\big)$. Then, $\mathbf{z}_{R,h} = \Pi_{R,h}\mathbf{z}_R$ solves the discrete saddle point problem

$$C_{R,h}\mathbf{z}_{R,h} + B_{R,h}(\mathbf{y}_{R,h}, \bar{\mathbf{y}}_{R,h}) = \mathbf{0}\,, \tag{5.2a}$$

$$B'_{R,h}\mathbf{z}_{R,h} = E'_{R,h}B'_R\mathbf{z}_R\,, \tag{5.2b}$$

where $(\mathbf{y}_{R,h}, \bar{\mathbf{y}}_{R,h}) \in W_{R,h}^{\text{ext}} \times V_{R,h}$ is the Lagrange multiplier.

Remark 5.1. Inf-sup stability requires that B_h is injective in $W_h \times \hat{V}_h$, but locally we cannot expect that $B_{R,h}$ is injective, since $B_{R,h}(\mathbf{y}_{R,h}, \mathbf{y}_{R,h}) = \mathbf{0}$ for all functions $\mathbf{y}_{R,h} \in V_{R,h} \cap W_{R,h} \cap \mathcal{N}(L)$.
On the other hand, since we assume that $\mathcal{N}^{\text{ext}}(\mathbf{z}_R)$ is not empty for all $\mathbf{z}_R$, the equation (5.2b) has always a solution, and since $C_{R,h}$ is positive definite, $\mathbf{z}_{R,h} = \Pi_{R,h}\mathbf{z}_R$ is the unique solution of the optimization problem. The Lagrange parameter $(\mathbf{y}_{R,h}, \bar{\mathbf{y}}_{R,h})$ is only unique up to $\mathcal{N}(B_{R,h})$.

Inserting $\mathbf{z}_{R,h} = -C_{R,h}^{-1}B_{R,h}(\mathbf{y}_{R,h}, \bar{\mathbf{y}}_{R,h})$ yields

$$S_{R,h}(\mathbf{y}_{R,h}, \bar{\mathbf{y}}_{R,h}) = -E'_{R,h}B'_R\mathbf{z}_R$$

with the Schur complement operator

$$S_{R,h} = B'_{R,h}C_{R,h}^{-1}B_{R,h} \in \mathcal{L}\big(W_{R,h}^{\text{ext}} \times V_{R,h}, (W_{R,h}^{\text{ext}} \times V_{R,h})'\big)\,.$$

Inserting the pseudo-inverse (with respect to the inner product in $W_{R,h}^{\text{ext}} \times V_{R,h}$)

$$S_{R,h}^{+} = \lim_{\delta \longrightarrow 0} \big(S_{R,h}G_{R,h}^{-1}S_{R,h} + \delta G_{R,h}\big)^{-1} S_{R,h}G_{R,h}^{-1}$$

satisfying $S_{R,h}^{+}S_{R,h}S_{R,h}^{+} = S_{R,h}^{+}$ defines

$$\Pi_{R,h} = C_{R,h}^{-1}B_{R,h}S_{R,h}^{+}E'_{R,h}B'_R\,. \tag{5.3}$$

We compute $\alpha_{R,h} > 0$ such that

$$\big\langle \ell_{R,h}, S_{R,h}^{+}\ell_{R,h}\big\rangle \le \alpha_{R,h}\big\langle \ell_{R,h}, G_{R,h}^{-1}\ell_{R,h}\big\rangle\,, \qquad \ell_{R,h} \in (W_{R,h}^{\mathrm{ext}} \times V_{R,h})' \tag{5.4}$$

i.e., we determine the largest eigenvalue of a finite dimensional symmetric generalized eigenvalue problem. For given $\mathbf{z}_R \in \mathrm{H}(L^{\mathrm{ad}}, R)$ we select the discrete functional $\ell_{R,h} = E'_{R,h}B'_R\mathbf{z}_R$, and the norm of the Fortin operator is estimated by

$$\begin{aligned}
\|\Pi_{R,h}\mathbf{z}_R\|_{L^{\mathrm{ad}},R}^2 &= \big\langle B_{R,h}S_{R,h}^{+}\ell_{R,h}, C_{R,h}^{-1}B_{R,h}S_{R,h}^{+}\ell_{R,h}\big\rangle\\
&= \big\langle \ell_{R,h}, S_{R,h}^{+}S_{R,h}S_{R,h}^{+}\ell_{R,h}\big\rangle\\
&= \big\langle \ell_{R,h}, S_{R,h}^{+}\ell_{R,h}\big\rangle\\
&\le \alpha_{R,h}\big\langle \ell_{R,h}, G_{R,h}^{-1}\ell_{R,h}\big\rangle\\
&\le 2\alpha_{R,h}\|\mathbf{z}_R\|_{L^{\mathrm{ad}},R}^2
\end{aligned}$$

using

$$\begin{aligned}
\sqrt{\big\langle \ell_{R,h}, G_{R,h}^{-1}\ell_{R,h}\big\rangle} &= \sup_{(\mathbf{y}_{R,h},\bar{\mathbf{y}}_{R,h})\in W_{R,h}^{\mathrm{ext}}\times V_{R,h}} \frac{\big\langle \ell_{R,h}, (\mathbf{y}_{R,h}, \bar{\mathbf{y}}_{R,h})\big\rangle}{\sqrt{\big\langle G_{R,h}(\mathbf{y}_{R,h}, \bar{\mathbf{y}}_{R,h}), (\mathbf{y}_{R,h}, \bar{\mathbf{y}}_{R,h})\big\rangle}}\\
&= \sup_{(\mathbf{y}_{R,h},\bar{\mathbf{y}}_{R,h})\in W_{R,h}^{\mathrm{ext}}\times V_{R,h}} \frac{\big\langle E'_{R,h}B'_R\mathbf{z}_R, (\mathbf{y}_{R,h}, \bar{\mathbf{y}}_{R,h})\big\rangle}{\|(\mathbf{y}_{R,h}, \bar{\mathbf{y}}_{R,h})\|_{R;L,R}}\\
&= \sup_{(\mathbf{y}_{R,h},\bar{\mathbf{y}}_{R,h})\in W_{R,h}^{\mathrm{ext}}\times V_{R,h}} \frac{\big\langle B_R E_{R,h}(\mathbf{y}_{R,h}, \bar{\mathbf{y}}_{R,h}), \mathbf{z}_R\big\rangle}{\|(\mathbf{y}_{R,h}, \bar{\mathbf{y}}_{R,h})\|_{R;L,R}} \le \sqrt{2}\|\mathbf{z}_R\|_{L^{\mathrm{ad}},R}
\end{aligned}$$

with $\big\langle G_{R,h}(\mathbf{y}_{R,h}, \bar{\mathbf{y}}_{R,h}), (\mathbf{y}_{R,h}, \bar{\mathbf{y}}_{R,h})\big\rangle = \|(\mathbf{y}_{R,h}, \bar{\mathbf{y}}_{R,h})\|_{R;L,R}^2$ and

$$\begin{aligned}
\big\langle B_R(\mathbf{y}_{R,h}, \hat{I}_{R,h}\bar{\mathbf{y}}_{R,h}), \mathbf{z}_R\big\rangle &= \big(\mathbf{y}_{R,h}, L^{\mathrm{ad}}\mathbf{z}_R\big)_R + \big(L\bar{\mathbf{y}}_{R,h}, \mathbf{z}_R\big)_R - \big(\bar{\mathbf{y}}_{R,h}, L^{\mathrm{ad}}\mathbf{z}_R\big)_R\\
&\le \|\mathbf{y}_{R,h}\|_R\|L^{\mathrm{ad}}\mathbf{z}_R\|_R + \|L\bar{\mathbf{y}}_{R,h}\|_R\|\mathbf{z}_R\|_R + \|\bar{\mathbf{y}}_{R,h}\|_R\|L^{\mathrm{ad}}\mathbf{z}_R\|_R\\
&\le \sqrt{\|\mathbf{y}_{R,h}\|_R^2 + \|L\bar{\mathbf{y}}_{R,h}\|_R^2 + \|\bar{\mathbf{y}}_{R,h}\|_R^2}\sqrt{2\|L^{\mathrm{ad}}\mathbf{z}_R\|_R^2 + \|\mathbf{z}_R\|_R^2}\\
&\le \sqrt{2}\|(\mathbf{y}_{R,h}, \bar{\mathbf{y}}_{R,h})\|_{R;L,R}\|\mathbf{z}_R\|_{L^{\mathrm{ad}},R}\,.
\end{aligned}$$

The construction is completely local, so that it extends to

$$\|\Pi_h\mathbf{z}\|_{L^{\mathrm{ad}},Q_h} \le \sqrt{2\alpha_h}\,\|\mathbf{z}\|_{L^{\mathrm{ad}},Q_h}\,, \qquad \mathbf{z} \in \mathrm{H}(L^{\mathrm{ad}}, Q_h) \tag{5.5}$$

with $\alpha_h = \max \alpha_{R,h}$. This gives discrete inf-sup stability

$$\sup_{\mathbf{z}_h\in Z_h} \frac{\big\langle B_h(\mathbf{y}_h, \hat{\mathbf{y}}_h), \mathbf{z}_h\big\rangle}{\|\mathbf{z}_h\|_{L^{\mathrm{ad}},Q_h}} \ge \beta_h\|(\mathbf{y}_h, \hat{\mathbf{y}}_h)\|_{Q;L,\partial Q_h} \tag{5.6}$$

for $(\mathbf{y}_h, \hat{\mathbf{y}}_h) \in W_h \times \hat{V}_h$ with $\beta_h = \frac{1}{2\sqrt{\alpha_{R,h}}\sqrt{2C_L^2+1}}$ [3, Prop. II.2.8].

5.2 A scaling argument

Numerically we observe that the eigenvalue estimate (5.4) is mesh-dependent. Thus we compute $\alpha_0 = \alpha_{R_0,h_0}$ on a reference element $R_0 = (0,h_0)^d \times (0,h_0/c)$ for the speed of sound $c > 0$, and we analyze the transformation $\varphi_R \colon R_0 \longrightarrow R$. For simplicity, we only discuss a scaling of the form $\varphi_R(\mathbf{x},t) = (\mathbf{x}_R,t_R) + (h/h_0)(\mathbf{x},t)$ with $R = (\mathbf{x}_R,t_R) + (0,h)^d \times (0,h/c)$.

Let Π_{R_0,h_0} be a local Fortin operator on the reference cell R_0. For the semi-norm $|\mathbf{z}_R|_{L^{\mathrm{ad}},R} = \|L^{\mathrm{ad}}\mathbf{z}_R\|_R$ and the operator B_R we assume the scaling properties

$$\begin{aligned} h^{-d+1}\,|\mathbf{z}_R|^2_{L^{\mathrm{ad}},R} &= h_0^{-d+1}\,|\mathbf{z}_R\circ\varphi_R|^2_{L^{\mathrm{ad}},R_0}\,,\\ h^{-d}\,\big\langle B_R(\mathbf{y}_{R,h},\hat{\mathbf{y}}_{R,h}),\mathbf{z}_R\big\rangle &= h_0^{-d}\,\big\langle B_{R_0}(\mathbf{y}_{R,h}\circ\varphi_R,\hat{\mathbf{y}}_{R,h}\circ\varphi_R),\mathbf{z}_R\circ\varphi_R\big\rangle\,. \end{aligned}$$

This holds for acoustic waves with constant coefficients. Then, the transformation

$$\Pi_{R,h}\mathbf{z}_R = \Big(\Pi_{R_0,h_0}(\mathbf{z}_R\circ\varphi_R)\Big)\circ\varphi_R^{-1}$$

defines a local Fortin operator in R. By scaling we obtain for $h \le h_0$

$$\begin{aligned} h^{-d-1}\,\|\Pi_{R,h}\mathbf{z}_R\|^2_R &= h_0^{-d-1}\,\|(\Pi_{R,h}\mathbf{z}_R)\circ\varphi_R\|^2_{R_0}\\ &= h_0^{-d-1}\,\|\Pi_{R_0,h_0}(\mathbf{z}_R\circ\varphi_R)\|^2_{R_0}\\ &\le h_0^{-d-1}\,\|\Pi_{R_0,h_0}(\mathbf{z}_R\circ\varphi_R)\|^2_{L^{\mathrm{ad}},R_0}\\ &\le h_0^{-d-1}\,\|\Pi_{R_0,h_0}\|^2_{L^{\mathrm{ad}},R_0}\|\mathbf{z}_R\circ\varphi_R\|^2_{L^{\mathrm{ad}},R_0}\,,\\ h_0^{-d-1}\,\|\mathbf{z}_R\circ\varphi_R\|^2_{L^{\mathrm{ad}},R_0} &= h^{-d-1}\,\|\mathbf{z}_R\|^2_R + h_0^{-2}\,h^{-d+1}\,|\mathbf{z}_R|^2_{L^{\mathrm{ad}},R}\\ &\le h^{-d-1}\,\|\mathbf{z}_R\|^2_{L^{\mathrm{ad}},R}\,,\\ h^{-d+1}\,|\Pi_{R,h}\mathbf{z}_R|^2_{L^{\mathrm{ad}},R} &= h_0^{-d+1}\,|(\Pi_{R,h}\mathbf{z}_R)\circ\varphi_R|^2_{L^{\mathrm{ad}},R_0}\\ &= h_0^{-d+1}\,|\Pi_{R_0,h_0}(\mathbf{z}_R\circ\varphi_R)|^2_{L^{\mathrm{ad}},R_0}\\ &\le h_0^{-d+1}\,|\mathbf{z}_R\circ\varphi_R|^2_{L^{\mathrm{ad}},R_0} = h^{-d+1}\,|\mathbf{z}_R|^2_{L^{\mathrm{ad}},R}\,, \end{aligned}$$

which yields together

$$\|\Pi_{R,h}\mathbf{z}_R\|_{L^{\mathrm{ad}},R_0} \le \sqrt{1+\|\Pi_{R_0,h_0}\|^2_{L^{\mathrm{ad}},R_0}}\,\|\mathbf{z}_R\|_{L^{\mathrm{ad}},R}\,.$$

For simple meshes this results into the computable inf-sup constant

$$\beta_h = \frac{1}{\sqrt{1+2\alpha_{R_0,h_0}}\sqrt{4C_L^2+2}}\,.$$

5.3 An a priori error estimate for the practical DPG method

The discrete DPG solution $(\mathbf{y}_h, \hat{\mathbf{y}}_h) \in W_h \times \hat{V}_h$ is obtained by minimizing the residual $B_h(\mathbf{y}_h, \hat{\mathbf{y}}_h) - \mathbf{b}$ in Z_h', i.e., by minimizing the functional

$$\Psi_h(\mathbf{y}_h, \hat{\mathbf{y}}_h) = \sup_{\mathbf{z}_h \in Z_h} \frac{\big\langle B_h(\mathbf{y}_h, \hat{\mathbf{y}}_h), \mathbf{z}_h \big\rangle - \big(\mathbf{b}, \mathbf{z}_h\big)_Q}{\|\mathbf{z}_h\|_{L^{\mathrm{ad}},Q}} .$$

The unique minimizer is the Petrov-Galerkin solution obtained by testing with the optimal test space $Z_h^{\mathrm{opt}} = C_h^{-1} B_h(W_h \times \hat{V}_h)$, i.e.,

$$\big\langle B_h(\mathbf{y}_h, \hat{\mathbf{y}}_h), \mathbf{z}_h \big\rangle \quad = \quad \big(\mathbf{b}, \mathbf{z}_h\big)_Q , \qquad \mathbf{z}_h \in Z_h^{\mathrm{opt}} . \tag{5.7}$$

Since B_h is continuous and since we assume that $\mathcal{N}(\mathbf{z}_R) \neq \{\mathbf{0}\}$ for all $\mathbf{z}_R \in \mathrm{H}(L^{\mathrm{ad}}, R)$ (so that a computable but in general mesh dependent inf-sup constant exists as discussed in 5.1), Petrov-Galerkin estimates apply [24, Thm. 2]. In simple cases where the scaling argument applies, this yields a mesh-independent estimate for α_h and thus for the inf-sup constant β_h.

In our experiments, the assumption $\mathcal{N}(\mathbf{z}_R) \neq \{\mathbf{0}\}$ can be achieved easily by choosing polynomials of higher order in the local test spaces $Z_{R,h}$ than in the local ansatz spaces $W_{R,h}, V_{R,h}$.

Theorem 5.2. *Let $\mathbf{y} \in V$ be the solution of $L\mathbf{y} = \mathbf{b}$, and set $\hat{\mathbf{y}} = \mathbf{y} + \mathrm{H}_0(L, Q_h) \in \hat{V}$. If a Fortin operator can be constructed and bounded by* (5.5)*, a unique Petrov-Galerkin approximation $(\mathbf{y}_h, \hat{\mathbf{y}}_h) \in W_h \times \hat{V}_h$ of* (5.7) *exists and satisfies the a priori error estimate*

$$\|(\mathbf{y} - \mathbf{y}_h, \hat{\mathbf{y}} - \hat{\mathbf{y}}_h)\|_{Q;L,\partial Q_h} \leq \frac{\sqrt{2}}{\beta_h} \inf_{(\phi_h, \hat{\phi}_h) \in W_h \times \hat{V}_h} \|(\mathbf{y} - \phi_h, \hat{\mathbf{y}} - \hat{\phi}_h)\|_{Q;L,\partial Q_h} .$$

6 The simplified DPG method

For the realization of the practical DPG method it is advantageous to use traces on the skeleton ∂Q_h. This process depends on the application and is now described for linear acoustic waves. For space-time tensor-product decompositions with space-time cells $R = K \times (a,b) \subset \Omega \times (0,T)$, we define a trace mapping $I_{\partial R}$ to $\mathrm{L}_2(\partial R; \mathbb{R} \times \mathbb{R}^d)$ by

$$I_{\partial R}(p_R, \mathbf{v}_R) = \begin{cases} (p_R, \mathbf{v}_R)|_{K\times\{t\}} & \text{for traces at time } t \in \{a,b\}, \\ \big(p_R, (\mathbf{v}_R \cdot \mathbf{n}_F)\mathbf{n}_F\big)|_{F\times(a,b)} & \text{in space with } F \subset \partial K \end{cases}$$

for all sufficiently smooth functions $(p_R, \mathbf{v}_R)$. We define local trace bilinear forms

$$\begin{aligned} &\gamma_R\big((\tilde p_R, \tilde{\mathbf{v}}_R), (q_R, \mathbf{w}_R)\big) \\ &\quad = \big((\tilde p_R, \tilde{\mathbf{v}}_R), (q_R, \mathbf{w}_R)\big)_{K\times\{b\}} - \big((\tilde p_R, \tilde{\mathbf{v}}_R), (q_R, \mathbf{w}_R)\big)_{K\times\{a\}} \\ &\qquad + \sum_{F\subset\partial K} \big(\tilde p_R, \mathbf{w}_R \cdot \mathbf{n}_K\big)_{F\times(a,b)} + \big(\tilde{\mathbf{v}}_R \cdot \mathbf{n}_K, q_R\big)_{F\times(a,b)} \end{aligned}$$

for $(\tilde p_R, \tilde{\mathbf{v}}_R) \in \mathrm{L}_2(\partial R; \mathbb{R} \times \mathbb{R}^d)$ and $(q_R, \mathbf{w}_R) \in \mathrm{H}(L^{\mathrm{ad}}, R)$ sufficiently smooth with $I_{\partial R}(q_R, \mathbf{w}_R) \in \mathrm{L}_2(\partial R; \mathbb{R} \times \mathbb{R}^d)$, and we define

$$\begin{aligned} b_h\big(\big((p,\mathbf{v}), (\tilde p, \tilde{\mathbf{v}})\big), (q, \mathbf{w})\big) &= -\big(p, \partial_t q + \nabla\cdot\mathbf{w}\big)_{Q_h} - \big(\mathbf{v}, \partial_t \mathbf{w} + \nabla q\big)_{Q_h} \\ &\quad + \gamma_h\big((\tilde p, \tilde{\mathbf{v}}), (q, \mathbf{w})\big) \end{aligned}$$

for $(p, \mathbf{v}) \in \mathrm{L}_2(Q; \mathbb{R} \times \mathbb{R}^d)$, $(\tilde p, \tilde{\mathbf{v}}) \in \mathrm{L}_2(\partial Q_h; \mathbb{R} \times \mathbb{R}^d)$ and for $(q, \mathbf{w}) \in \mathrm{H}(L^{\mathrm{ad}}, Q_h)$ with traces in L_2, where $\gamma_h\big((\tilde p, \tilde{\mathbf{v}}), (q, \mathbf{w})\big) = \sum_R \gamma_R\big((\tilde p_R, \tilde{\mathbf{v}}_R), (q_R, \mathbf{w}_R)\big)$.

By construction, we observe

$$\gamma_R\big(I_{\partial R}(p_R, \mathbf{v}_R), (q_R, \mathbf{w}_R)\big) = \big\langle D_R(p_R, \mathbf{v}_R), (q_R, \mathbf{w}_R)\big\rangle$$

for $(p_R, \mathbf{v}_R) \in \mathrm{H}(L, R)$ and $(q_R, \mathbf{w}_R) \in \mathrm{H}(L^{\mathrm{ad}}, R)$ with traces in L_2, and

$$b_h\big(\big((p,\mathbf{v}), I_{\partial Q_h}(\tilde p, \tilde{\mathbf{v}})\big), (q, \mathbf{w})\big) = \big\langle B_h\big((p,\mathbf{v}), \hat I_h(\tilde p, \tilde{\mathbf{v}})\big), (q, \mathbf{w})\big\rangle$$

for $(p, \mathbf{v}) \in \mathrm{L}_2(Q; \mathbb{R} \times \mathbb{R}^d)$, and for $(\tilde p, \tilde{\mathbf{v}}) \in \mathrm{H}(L, Q_h)$, and $(q, \mathbf{w}) \in \mathrm{H}(L^{\mathrm{ad}}, Q_h)$ with traces in L_2.

Thus, in the realization of the DPG method we can replace the operator B_h by the bilinear form $b_h(\cdot,\cdot)$, so that it is sufficient to represent $\hat V_h$ by its trace values on ∂Q_h.

In the simplified DPG method, we select independently polynomial ansatz spaces for the traces on every space-time face of the skeleton ∂Q_h, i.e., we choose a discontinuous space

$$\tilde V_h = \prod_{K\times\{a\}\subset\partial Q_h} V_{K\times\{a\},h} \times \prod_{F\times(a,b)\subset\partial Q_h} V_{F\times(a,b),h} \subset \mathrm{L}_2(\partial Q_h;\mathbb{R}\times\mathbb{R}^d)\,.$$

The representation of Neumann traces for $(\tilde p_h,\tilde{\mathbf{v}}_h)\in\tilde V_h$ requires to select an orientation $\mathbf{n}_F\in\{\pm\mathbf{n}_K\}$. Then, $\tilde{\mathbf{v}}_h|_{F\times(a,b)} = \tilde v_h\mathbf{n}_F$ with $\tilde v_h\in\mathrm{L}_2(F\times(a,b))$.

In case that $\tilde V_h$ is the trace of a conforming subspace $V_h\subset V$, i.e., $\tilde V_h = I_{\partial Q_h}V_h$, the simplified method coincides with a conforming DPG method. In general, the skeleton space $\tilde V_h$ may be nonconforming. Then, we assume a weaker condition which is described in the following.

In order to obtain a well-defined method and to provide an a priori error analysis, we assume that a conforming reconstruction $V_h\subset V$ of $\tilde V_h$ exists such that for given $(\tilde p_h,\tilde{\mathbf{v}}_h)\in\tilde V_h$ the reconstruction $(\bar p_h,\bar{\mathbf{v}}_h)\in V_h$ is uniquely defined by

$$\gamma_R\big((\tilde p_{R,h},\tilde{\mathbf{v}}_{R,h}),(q_{R,h},\mathbf{w}_{R,h})\big) = \gamma_R\big(I_{\partial R,h}(\bar p_h,\bar{\mathbf{v}}_h),(q_{R,h},\mathbf{w}_{R,h})\big) \tag{6.1}$$

for all $(q_{R,h},\mathbf{w}_{R,h})\in Z_{R,h}$ and all space-time cells R. In particular, this implies $\dim V_h = \dim\tilde V_h$. Note that the traces in V_h only coincide with functions $\tilde V_h$ when tested with the finite dimensional space $Z_{R,h}$.

Then, by construction, the simplified method with ansatz space $W_h\times\tilde V_h$ and test space Z_h yields the same discrete linear system as the practical method with $\tilde V_h$ replaced by $\hat V_h = V_h/\mathrm{H}_0(L,Q_h)$. For the error analysis we introduce the discrete norm

$$\|(\tilde p_h,\tilde{\mathbf{v}}_h)\|_{Z_h'} = \sup_{(q_h,\mathbf{w}_h)\in Z_h}\frac{\gamma_h\big((\tilde p_h,\tilde{\mathbf{v}}_h),(q_h,\mathbf{w}_h)\big)}{\|(q_h,\mathbf{w}_h)\|_{L^{\mathrm{ad}},Q_h}}\,,\quad (\tilde p_h,\tilde{\mathbf{v}}_h)\in\tilde V_h\,. \tag{6.2}$$

This extends to a (mesh-dependent) semi-norm in $\mathrm{L}_2(\partial Q_h;\mathbb{R}\times\mathbb{R}^d)$, and we observe for $(p,\mathbf{v})\in V$ with trace $(\tilde p,\tilde{\mathbf{v}}) = I_{\partial Q_h}(p,\mathbf{v})\in\mathrm{L}_2(\partial Q_h;\mathbb{R}\times\mathbb{R}^d)$

$$\begin{aligned}
\|(\tilde p,\tilde{\mathbf{v}})\|_{Z_h'} &= \sup_{(q_h,\mathbf{w}_h)\in Z_h}\frac{\gamma_h\big((\tilde p,\tilde{\mathbf{v}}),(q_h,\mathbf{w}_h)\big)}{\|(q_h,\mathbf{w}_h)\|_{L^{\mathrm{ad}},Q_h}}\\
&= \sup_{(q_h,\mathbf{w}_h)\in Z_h}\ \inf_{(p_0,\mathbf{v}_0)\in\mathrm{H}_0(L,Q_h)}\frac{\big\langle D_h(p+p_0,\mathbf{v}+\mathbf{v}_0),(q_h,\mathbf{w}_h)\big\rangle}{\|(q_h,\mathbf{w}_h)\|_{L^{\mathrm{ad}},Q_h}}\\
&\le \sup_{(q,\mathbf{w})\in\mathrm{H}(L^{\mathrm{ad}},Q_h)}\ \inf_{(p_0,\mathbf{v}_0)\in\mathrm{H}_0(L,Q_h)}\frac{\big\langle D_h(p+p_0,\mathbf{v}+\mathbf{v}_0),(q,\mathbf{w})\big\rangle}{\|(q,\mathbf{w})\|_{L^{\mathrm{ad}},Q_h}}\\
&\le \inf_{(p_0,\mathbf{v}_0)\in\mathrm{H}_0(L,Q_h)}\|(p+p_0,\mathbf{v}+\mathbf{v}_0)\|_{L,Q_h} = \|(\hat p,\hat{\mathbf{v}})\|_{L,\partial Q_h}
\end{aligned} \tag{6.3}$$

with $(\hat p,\hat{\mathbf{v}}) = (p,\mathbf{v})+\mathrm{H}_0(L,Q_h)\in\hat V$.

With respect to the semi-norm (6.2), we can transfer the result in Thm. 5.2 to the simplified DPG method.

Theorem 6.1. *Assume that a conforming reconstruction $V_h \subset V$ of $\tilde V_h$ exists satisfying* (6.1) *and* $\dim V_h = \dim \tilde V_h$.

a) If a Fortin operator can be constructed and bounded by (5.5)*, a unique Petrov-Galerkin approximation $\big((p_h,\mathbf{v}_h),(\tilde p_h,\tilde{\mathbf{v}}_h)\big) \in W_h \times \tilde V_h$ exists solving*

$$b_h\big(\big((p_h,\mathbf{v}_h),(\tilde p_h,\tilde{\mathbf{v}}_h)\big),(q_h,\mathbf{w}_h)\big) = \big((f,\mathbf{g}),(q_h,\mathbf{w}_h)\big)_Q\,, \qquad (q_h,\mathbf{w}_h) \in Z_h^{\mathrm{opt}}\,.$$

b) Let $(p,\mathbf{v}) \in V$ be the solution of (2.2)*, and assume that $(p,\mathbf{v})$ is sufficiently regular with traces $(\tilde p,\tilde{\mathbf{v}}) = I_{\partial Q_h}(p,\mathbf{v}) \in \mathrm{L}_2(\partial Q_h;\mathbb{R}\times\mathbb{R}^d)$.*
Then, the error can be bounded by

$$\begin{aligned}&\big\|\big((p,\mathbf{v})-(p_h,\mathbf{v}_h)\big),\big((\tilde p,\tilde{\mathbf{v}})-(\tilde p_h,\tilde{\mathbf{v}}_h)\big)\big\|_{W\times Z_h'}\\ &\quad\le (1+\sqrt{2}\beta_h^{-1})\\ &\qquad\inf_{((\phi_h,\boldsymbol{\psi}_h),(\tilde\phi_h,\tilde{\boldsymbol{\psi}}_h))\in W_h\times\tilde V_h} \big\|\big((p,\mathbf{v})-(\phi_h,\boldsymbol{\psi}_h)\big),\big((\tilde p,\tilde{\mathbf{v}})-(\tilde\phi_h,\tilde{\boldsymbol{\psi}}_h)\big)\big\|_{W\times Z_h'}\,.\end{aligned}$$

Proof. For the conforming reconstruction $V_h \subset V$ of $\tilde V_h$ and $\hat V_h = V_h/\mathrm{H}_0(L,Q_h)$, the practical DPG method applies, providing an error estimate in $W_h \times \hat V_h$ by Thm. 5.2.

Let $(p,\mathbf{v}) \in V$ be the solution of (2.2), let $(\tilde p,\tilde{\mathbf{v}}) = I_{\partial Q_h}(p,\mathbf{v}) \in \mathrm{L}_2(\partial Q_h;\mathbb{R}\times\mathbb{R}^d)$ its trace, and set $(\hat p,\hat{\mathbf{v}}) = (p,\mathbf{v}) + \mathrm{H}_0(L,Q_h) \in \hat V$.

For the discrete solution $\big((p_h,\mathbf{v}_h),(\tilde p_h,\tilde{\mathbf{v}}_h)\big) \in W_h \times \tilde V_h$ let $(\bar p_h,\bar{\mathbf{v}}_h) \in V_h$ be the conforming reconstruction of $(\tilde p_h,\tilde{\mathbf{v}}_h)$, and set $(\hat p_h,\hat{\mathbf{v}}_h) = (\bar p_h,\bar{\mathbf{v}}_h)+\mathrm{H}_0(L,Q_h) \in \hat V_h$. Then, (6.1) and (6.3) yield $\|(\tilde p_h,\tilde{\mathbf{v}}_h)\|_{Z_h'} = \|I_{\partial R,h}(\bar p_h,\bar{\mathbf{v}}_h)\|_{Z_h'} \le \|(\hat p_h,\hat{\mathbf{v}}_h)\|_{L,\partial Q_h}$.

Now, for some $\big((\phi_h,\boldsymbol{\psi}_h),(\tilde\phi_h,\tilde{\boldsymbol{\psi}}_h)\big) \in W_h\times\tilde V_h$ let $(\bar\phi_h,\bar{\boldsymbol{\psi}}_h) \in V_h$ be the conforming reconstruction of $(\tilde\phi_h,\tilde\psi_h)$, and set $(\hat\phi_h,\hat{\boldsymbol{\psi}}_h) = (\bar\phi_h,\bar{\boldsymbol{\psi}}_h)+\mathrm{H}_0(L,Q_h)$. Then

$$\begin{aligned}&\beta_h\big\|\big((p_h,\mathbf{v}_h)-(\phi_h,\boldsymbol{\psi}_h),(\hat p_h,\hat{\mathbf{v}}_h)-(\hat\phi_h,\hat{\boldsymbol{\psi}}_h)\big)\big\|_{Q;L,\partial Q_h}\\ &\quad\le \sup_{(q_h,\mathbf{w}_h)\in Z_h}\frac{\big\langle B_h\big((p_h,\mathbf{v}_h)-(\phi_h,\boldsymbol{\psi}_h),(\hat p_h,\hat{\mathbf{v}}_h)-(\hat\phi_h,\hat{\boldsymbol{\psi}}_h)\big),(q_h,\mathbf{w}_h)\big\rangle}{\|(q_h,\mathbf{w}_h)\|_{L^{\mathrm{ad}},Q_h}}\\ &\quad\le \big\|(p,\mathbf{v})-(\phi_h,\boldsymbol{\psi}_h)\big\|_W + \sup_{(q_h,\mathbf{w}_h)\in Z_h}\frac{\big\langle \hat D_h\big((\hat p,\hat{\mathbf{v}})-(\hat\phi_h,\hat{\boldsymbol{\psi}}_h)\big),(q_h,\mathbf{w}_h)\big\rangle}{\|(q_h,\mathbf{w}_h)\|_{L^{\mathrm{ad}},Q_h}}\\ &\quad= \big\|(p,\mathbf{v})-(\phi_h,\boldsymbol{\psi}_h)\big\|_W + \sup_{(q_h,\mathbf{w}_h)\in Z_h}\frac{\gamma_h\big((\tilde p,\tilde{\mathbf{v}})-(\tilde\phi_h,\tilde{\boldsymbol{\psi}}_h),(q_h,\mathbf{w}_h)\big)}{\|(q_h,\mathbf{w}_h)\|_{L^{\mathrm{ad}},Q_h}}\\ &\quad= \big\|(p,\mathbf{v})-(\phi_h,\boldsymbol{\psi}_h)\big\|_W + \big\|(\tilde p,\tilde{\mathbf{v}})-(\tilde\phi_h,\tilde{\boldsymbol{\psi}}_h)\big\|_{Z_h'}\\ &\quad\le \sqrt{2}\,\big\|\big((p,\mathbf{v})-(\phi_h,\boldsymbol{\psi}_h),(\tilde p,\tilde{\mathbf{v}})-(\tilde\phi_h,\tilde{\boldsymbol{\psi}}_h)\big)\big\|_{W\times Z_h'}\end{aligned}$$

using discrete inf-sup stability (5.6). Together with

$$\begin{aligned}\big\|(\tilde\phi_h,\tilde{\boldsymbol{\psi}}_h)-(\tilde p_h,\tilde{\mathbf{v}}_h)\big\|_{Z_h'} &= \big\|I_{\partial Q_h}(\bar\phi_h,\bar{\boldsymbol{\psi}}_h)-I_{\partial Q_h}(\bar p_h,\bar{\mathbf{v}}_h)\big\|_{Z_h'}\\ &\le \big\|(\hat\phi_h,\hat{\boldsymbol{\psi}}_h)-(\hat p_h,\hat{\mathbf{v}}_h)\big\|_{L,\partial Q_h}\end{aligned}$$

we finally obtain

$$
\begin{aligned}
&\big\|\big((p,\mathbf{v})-(p_h,\mathbf{v}_h),(\tilde p,\tilde{\mathbf{v}})-(\tilde p_h,\tilde{\mathbf{v}}_h)\big)\big\|_{W\times Z_h'}\\
&\quad\le \big\|\big((p,\mathbf{v})-(\phi_h,\boldsymbol{\psi}_h),(\tilde p,\tilde{\mathbf{v}})-(\tilde\phi_h,\tilde{\boldsymbol{\psi}}_h)\big)\big\|_{W\times Z_h'}\\
&\qquad\quad+\big\|\big((\phi_h,\boldsymbol{\psi}_h)-(p_h,\mathbf{v}_h),(\tilde\phi_h,\tilde{\boldsymbol{\psi}}_h)-(\tilde p_h,\tilde{\mathbf{v}}_h)\big)\big\|_{W\times Z_h'}\\
&\quad\le \big\|\big((p,\mathbf{v})-(\phi_h,\boldsymbol{\psi}_h),(\tilde p,\tilde{\mathbf{v}})-(\tilde\phi_h,\tilde{\boldsymbol{\psi}}_h)\big)\big\|_{W\times Z_h'}\\
&\qquad\quad+\big\|\big((\phi_h,\boldsymbol{\psi}_h)-(p_h,\mathbf{v}_h),(\hat\phi_h,\hat{\boldsymbol{\psi}}_h)-(\hat p_h,\hat{\mathbf{v}}_h)\big)\big\|_{Q;L,\partial Q_h}\\
&\quad\le \Big(1+\frac{\sqrt{2}}{\beta_h}\Big)\big\|\big((p,\mathbf{v})-(\phi_h,\boldsymbol{\psi}_h),(\tilde p,\tilde{\mathbf{v}})-(\tilde\phi_h,\tilde{\boldsymbol{\psi}}_h)\big)\big\|_{W\times Z_h'}\,. \qquad \square
\end{aligned}
$$

The reconstruction space V_h is completely virtual, it is not required for the realization of the simplified DPG solution. On the other hand, one needs an explicit representation of V_h for the estimate of the discrete inf-sup constant as it described in the previous section.

For the numerical solution, the discrete Petrov-Galerkin equation is reduced to a positive definite Schur complement problem for $(\tilde p_h,\tilde{\mathbf{v}}_h)$; see [22, Lem. 9] for explicit estimates for the Schur complement depending on β_h and C_L.

7 Numerical experiments

To evaluate the efficiency of the simplified space-time DPG-method, we consider two examples in two space dimensions, i.e., $d = 2$. In the first numerical test, we use a configuration where the exact solution is known, so that we can compare the approximation results with the a priori estimate in Thm. 6.1. The second test illustrates the application to a double-slit experiment. Also see [11, Sect. 5] for a more detailed evaluation of this space-time DPG method.

For both examples we use the discrete spaces

$$W_{R,h} = \mathbb{Q}_2(R) \times \mathbb{Q}_2(R)^2, \quad Z_{R,h} = \mathbb{Q}_4(R) \times \mathbb{Q}_4(R)^2$$

with tensor product polynomial spaces $\mathbb{Q}_2(R) = \mathbb{P}_2 \otimes \mathbb{P}_2 \otimes \mathbb{P}_2$ in the space-time cells $R = K \times (a,b)$, and on the skeleton ∂Q_h

$$\begin{aligned} V_{K\times\{t\},h} &= \mathbb{Q}_2(K) \times \mathbb{Q}_2(K)^2 \text{ for faces in time, and} \\ V_{F\times(a,b),h} &= \mathbb{Q}_2\big(F \times (a,b)\big) \times \mathbb{Q}_2\big(F \times (a,b)\big)\mathbf{n}_F \text{ for faces in space.} \end{aligned}$$

7.1 The construction of the Fortin Operator

In case of conforming trace approximations $\tilde{V}_h$ and simple meshes it is sufficient to construct the Fortin operator in a reference element R_0, and then the estimates for the Fortin operator in $R \subset Q_h$ follows from the scaling argument in Sect. 5.2.

In the nonconforming case, a conforming reconstruction $V_h \subset V$ with (6.1) has to be computed. Therefore, we compute a minimum energy extension of trace functions in $\tilde{V}_{R,h}$. On each cell R we select a basis $\{(\tilde{p}_1, \tilde{\mathbf{v}}_1), \dots, (\tilde{p}_N, \tilde{\mathbf{v}}_N)\}$ of $\tilde{V}_{R,h}$ and an extension space $V_{R,h} \subset \mathrm{H}(L,R)$. Then we compute $(\bar{p}_1, \bar{\mathbf{v}}_1), \dots, (\bar{p}_N, \bar{\mathbf{v}}_N) \in V_{R,h}$ by solving the minimization problem

$$\min_{(\bar{p}_n,\bar{\mathbf{v}}_n)\in V_{R,h}(\tilde{p}_n,\tilde{\mathbf{v}}_n)} \|(\bar{p}_n, \bar{\mathbf{v}}_n)\|_{L,R}$$

in the affine space

$$V_{R,h}(\tilde{p}_n, \tilde{\mathbf{v}}_n) = \Big\{(\bar{p}_n, \bar{\mathbf{v}}_n) \in V_{R,h} \colon \gamma_R\big((\bar{p}_n, \bar{\mathbf{v}}_n) - (\tilde{p}_n, \tilde{\mathbf{v}}_n), (q_{R,h}, \mathbf{w}_{R,h})\big) = 0 \\ \text{for } (q_{R,h}, w_{R,h}) \in Z_{R,h}\Big\},$$

see Fig. 1 for an illustration. The resulting estimates for the Fortin operator for different polynomial degrees are listed in Tab. 1.

Computational setup The numerical calculations were performed on a parallel computer consisting of two nodes each of which features AMD Opteron(TM) 6274 Processors with 64 cores and 512 GB memory. We used a GMRES method preconditioned by a local symmetric Gauß-Seidel iteration on every parallel subdomain to solve the linear systems. This performs reasonably well for our examples; it remains an open task to realize an efficient multigrid preconditioner for this application.

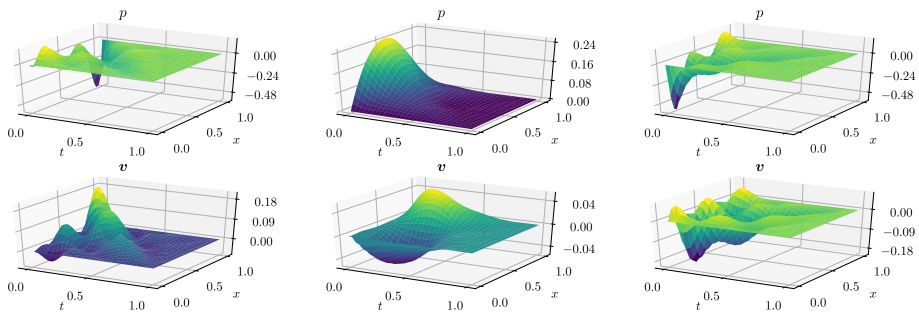

Figure 1. Conforming reconstructions in $V_{R,h} = \mathbb{Q}_6(R) \times \mathbb{Q}_6(R)$ for $d = 1$ of the trace space $\tilde V_K = \mathbb{P}_2 \times \mathbb{P}_2$ on a face $K \subset \partial R$, and test space $Z_{R,h} = \mathbb{Q}_4(R)^2$. We show the extensions $\bar p_n$ and $\bar v_n$ for the three nodal basis functions in $\mathbb{P}_2$.

$\|\Pi_{R,h}\|_{L^{\mathrm{ad}},R}$	h_0	h_1	h_2	h_3	h_0
$p = 0$	2.067	2.161	2.182	2.19	2.91
$p = 1$	12.039	18.817	32.87	123.71	34.85
$p = 2$	35.861	64.140	116.78	239.71	144.78

Table 1. We present two upper bounds for $\|\Pi_{R,h}\|_{L^{\mathrm{ad}},R}$ in two space-dimension with $R = (0, a_1h_k) \times (0, a_2h_k) \times (0, ch_k)$ and $a_1 \approx a_2 \approx c \approx 1$.
Left: Numerical norm estimates with ansatz space $W_{R,h} = \mathbb{Q}_p(R)^3$, test space $Z_{R,h} = \mathbb{Q}_{p+2}(R)^3$, and extension space $\mathbb{Q}_{p+4}(R)^3 \supset \tilde V_{R,h}$. The estimates depend on the mesh size $h_k = 2^k$ and the polynomial degree p.
Right: Numerical estimate on the reference cell R_0 with $W^{\mathrm{ext}}_{R,h} = \mathbb{Q}_{p+1}(R)^3$. This yields an inf-sup constant independent of h by the scaling argument in Sect. 5.2.

7.2 A plane wave solution

We consider a rectangular domain $\Omega = (-2, 2) \times (0, 1)$ with a non-homogeneous material distribution for density ρ and bulk modulus κ

$$\big(\rho(x_1, x_2), \kappa(x_1, x_2)\big) = \begin{cases} (1, 1) & x_1 < 0\,, \\ (2, 0.5) & x_1 \in (0, 1)\,, \\ (0.5, 2) & x_1 > 1\,. \end{cases}$$

so that the system (2.1) has a plane wave solution with amplitude $a(\cdot)$

$$\big(p(x_1, x_2, t), \mathbf{v}(x_1, x_2, t)\big) = \begin{cases} a(x_1 - t)(1, (1, 0)) & x_1 < 0\,, \\ a(2x_1 - t)(1, (1, 0)) & x_1 \in (0, 1)\,, \\ a(1.5 + 0.5x_1 - t)(1, (1, 0)) & x_1 > 1\,, \end{cases}$$

cf. [13, Sec. 3.5]. For our test we use $a(s) = \cos(\pi s)^2$ for $|s| < 1$ and $a(s) = 0$ else. We use homogenous Neumann boundary conditions $\mathbf{v} \cdot \mathbf{n} = 0$ for $x_2 = 0$ and $x_2 = 1$, and homogeneous Dirichlet boundary conditions $p = 0$ for $x_1 = \pm 2$.

Note that due to the special choice of material parameters, the analytic solution does not feature reflections on the interfaces. Fig. 2 illustrates the evolution of this solution in the space-time cylinder.

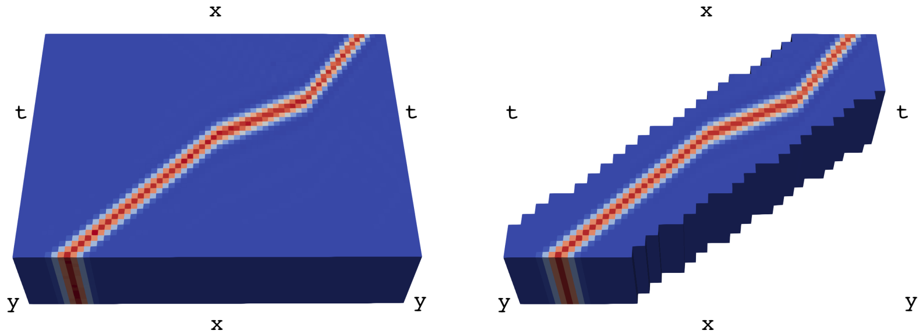

Figure 2. A wave front traveling from left to the right through three different materials. Here, the the full space-time mesh with 3 193 344 DoFs can be truncated resulting in 1 284 984 DoFs while the approximation quality remains unchanged.

The convergence of the DPG-method for this example is evaluated for a sequence of regular meshes, where each cuboid is subdivided in 8 congruent parts on refinement, see Tab. 2.

DoFs	7 560	54 432	411 264	3 193 344
e_h	0.493784	0.244424	0.081376	0.028764
e_{2h}/e_h		2.020197	3.003630	2.829075
$\log_2 e_{2h}/e_h$		1.014495	1.586707	1.500330

Table 2. The convergence of $(p_h, \mathbf{v}_h)$ in $\mathrm{L}_2(Q; \mathbb{R} \times \mathbb{R}^2)$ on a sequence of uniformly refined meshes with $h_k = 2^{-k} h_0$, $h_0 = 1$.

Now we exploit that the support of the solution is contained in a small fraction of the space-time cylinder Q. To this end, we use the simulation on a coarse mesh to identify a superset of this support. In a second step we truncate the space-time mesh by dropping cells where the solution vanishes. The resulting new space-time boundaries are enhanced with zero boundary or initial conditions. As a result, we have reduced

DoFs	168 534	1 284 984	10 026 720	79 201 152
e_h	0.08127	0.02902	0.00519	0.00114
e_{2h}/e_h		2.79989	5.59012	4.51952
$\log_2 e_{2h}/e_h$		1.485373	2.482881	2.176172

Table 3. The convergence of $(p, \mathbf{v})$ in the $\mathrm{L}_2(Q)$ norm on a sequence of truncated meshes for $h_k = 2^{-k} h_0$, $h_0 = 2^{-3}$, is shown.

the amount of DoFs to approximate the solution while conserving the approximation quality. See Fig. 2 for an example of this procedure and Tab. 3 for a convergence study.

7.3 The diffraction pattern from a double slit

The second example considers a double-slit experiment, where two coherent wave fronts enter the domain through a pair of small slits. By Huygens principle, a circular wave is propagated from each of the slits yielding a characteristic inference pattern, cf. Fig. 3 for a description of the setup and Fig. 4 for visualizations of the solution. The boundary $\partial\Omega = \Gamma_{\mathrm{N}} \cup \Gamma_{\mathrm{D}}$ is partitioned in a Neumann part Γ_{N} and a Dirichlet part Γ_{D}, where we use $\mathbf{v} \cdot \mathbf{n} = 0$ on $\Gamma_{\mathrm{N}} \times (0, T)$ and $p(x, t) = \sin(2\pi\omega(x - t))$ for $(x, t) \in \Gamma_{\mathrm{D}} \times (0, T)$ with $\omega = 2$.

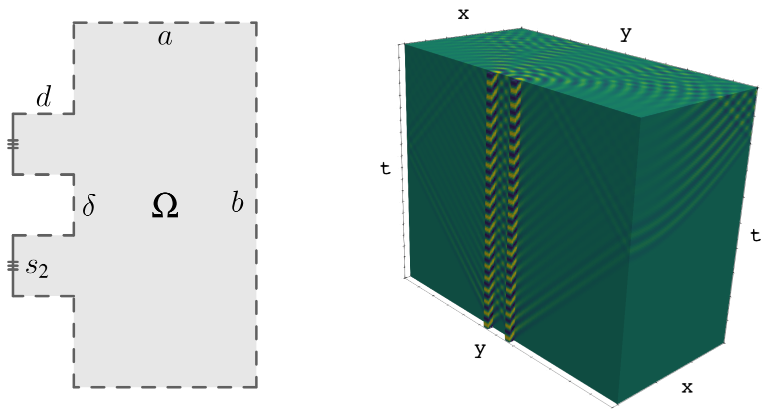

Figure 3. The spatial domain Ω is described on the left, where the slit dimensions are $d = s_1 = s_2 = 0.25$, their distance is $\delta = 1$, and the dimensions of the large rectangle are $a = 6$, $b = 12$. Ω is substructured using a regular mesh Ω_h of squares with side lengths 0.25. The corresponding space-time cylinder $Q = \Omega \times (0, T)$ is discretized using tensor-product elements $R = K \times (t_{n-1}, t_n)$ for each cell $K \in \Omega_h$ and $t_n = nT/N$, $n = 0, \ldots, N$, with $T = 10$ and $N = 50$. The dashed portion of $\partial\Omega$ indicates Γ_{N} and the remaining faces, marked by three lines, represent Γ_{D}. On the right, a space-time plot of the solution is given on a two times refined version of this mesh featuring 3 692 800 space-time cells and 234 210 528 face DoFs.

This example compares the results in [12, Sect. 8.3.2] for a space-time discontinuous Galerkin method for 2D Maxwell's equations and a similar double-slit setup. The experiment demonstrates that the space-time DPG method is able to approximate a complex wave pattern in a physically motivated example. An in-depth comparison of the space-time DPG method to other methods like discontinuous Galerkin remains as a challenge for the future.

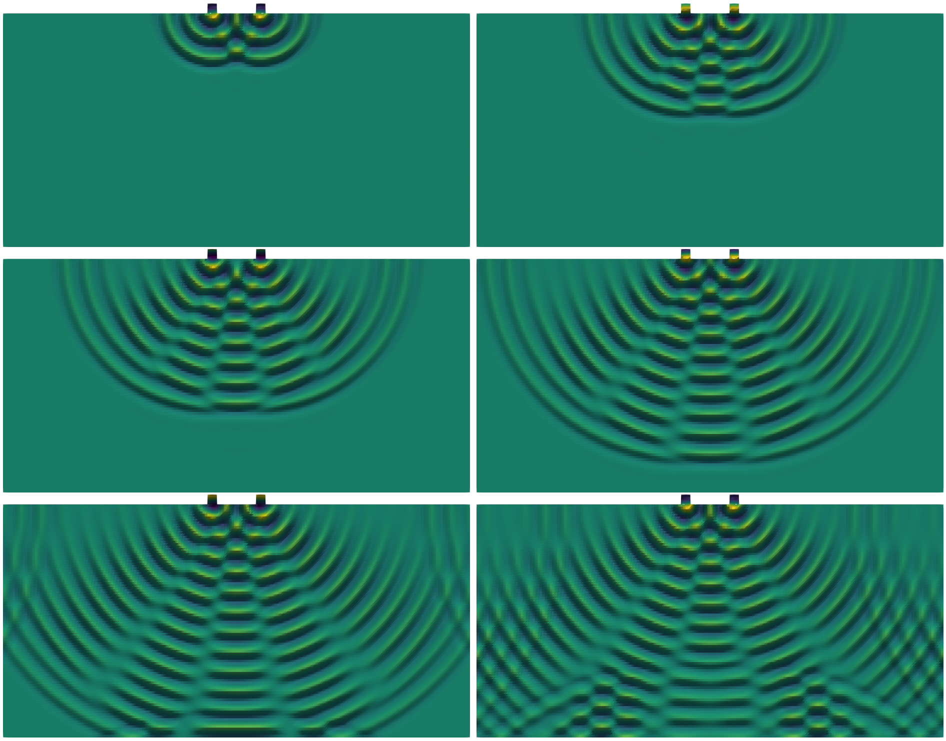

Figure 4. Snapshots of the pressure component at times $t = 0.6, 2.08, 3.56, 5.04, 6.52, 8$. These were obtained by slicing the space-time solution from Fig. 3 along planes that are orthogonal to the time direction.

8 Conclusion and Outlook

We presented a space-time framework for acoustic waves including an appropriate abstract trace space and we constructed and analyzed a space-time DPG method within this setting. Moreover, we considered a non-conforming variant with appealing properties from an implementation point of view and we provided a numerically accessible criterion to compute a bound for the norm of the Fortin operator. We demonstrated the flexibility of the method by providing a numerical example with a truncated space-time mesh which reduces the size of the linear system by a factor of three. Furthermore, we presented an example for a double-slit experiment as a starting point for upcoming in-depth comparisons of space-time DPG to established discretization methods.

For the presented theory, we used tools from semi-groups and functional analysis which are also applicable to other first-order systems such as electro-magnetic or elastic waves. Moreover, we assumed a constant material distribution to keep our notation simple. A consideration of more general wave equations taking into account spatially varying material properties remains as a future challenge. To render this space-time discretization competitive to classical schemes, a preconditioner is necessary that scales well with respect to the mesh size and also with the number of processes used. In the future, we would like to consider multigrid-algorithms which are promising candidates in this respect, see e.g. [8, 12] for a multigrid preconditioned space-time discontinuous Galerkin method.

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Author information

Johannes Ernesti, Institut für Angewandte und Numerische Mathematik, KIT,
D-76128 Karlsruhe, Germany.
E-mail: `Johannes.Ernesti@kit.edu`

Christian Wieners, Institut für Angewandte und Numerische Mathematik, KIT,
D-76128 Karlsruhe, Germany.
E-mail: `Christian.Wieners@kit.edu`