Scheduling of Measurement Transmission in Networked Control Systems Subject to Communication Constraints

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***Abstract*— In this paper, we consider Networked Control Systems where the transmission of sensor data is restricted in terms of a fixed communication budget due to the limited capacity of the underlying network. Therefore, the remote estimator cannot be supplied with measurements every time step, which impacts the accuracy of the estimates and consequently the achievable control performance. In order to trade off estimation accuracy against the costs of measurement transmission, we formulate the considered task as an optimal control problem, so that it fits into the broader class of sensor and measurement scheduling problems. As the optimal solution of such problems is generally intractable, we derive a suboptimal algorithm based on randomized rounding. In two numerical examples, we show that the proposed approach can outperfom a state-of-the-art sensor selection algorithm.**

I. INTRODUCTION

In a Networked Control System (NCS), a packet-based communication infrastructure based on general-purpose networks like WiFi or Ethernet is typically employed to connect the individual components of the control loop. Compared to traditional, dedicated point-to-point connections, this offers enhanced flexibility and easier maintenance [1], [2]. On the downside, additional challenges emerge from the utilization of such networks, which have to be addressed in conjunction with the control task. Apart from the occurrence of random packet delays and losses, usually the limited availability of communication bandwidth imposes a major restriction to the achievable system performance [2], [3], which can even result in instability of the closed loop system [4].

In order to deal with limited communication capabilities, promising approaches to reduce the amount of data to be transmitted are referred to as *event-based communication* [5]–[7] or *controlled communication* [2], [8] in literature. Event-based communication primarily aims at a more efficient usage of the communication media by only sending data when required, i.e., only when a certain triggering event occurs. A common strategy for event-based state estimation is, for instance, known as send-on-delta [5], while for event-based control criteria based on Lyapunov functions have been devised [7]. On the other hand, controlled communication in the first place aims at trading off system performance

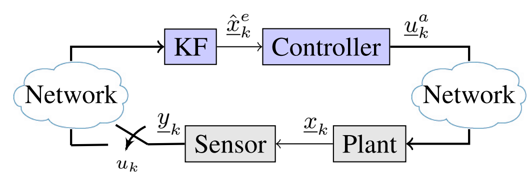

Fig. 1: Considered setup. The control inputs $\underline{u}_k^a$ are computed using state estimates $\underline{\hat{x}}_k^e$ and then transmitted to the plant over a network. The estimates are obtained from measurements $\underline{y}_k$ which are communicated from the sensor according to a transmission schedule, i.e., only at time steps when $u_k = 1$.

against communication requirements by not sending data every time step. This can, for example, be achieved by formulating the trade-off as an optimal control problem, which is then solved to derive (deterministic) communication policies. With respect to sensor data to be communicated via a network, these approaches can be seen as instances of measurement or sensor scheduling and selection problems. In such problems it is typically desired to optimally schedule a set of different sensors with respect to a cost function which attempts to trade off estimation accuracy against the costs of sensor utilization.

However, the number of possible solutions usually increases exponentially with the length of the optimization horizon so that the considered problems are in general NP-hard. Thus, computing the optimal solution is often intractable and one has to resort to suboptimal solutions and heuristics. Due to the relevance of these kind of problems in a variety of fields, plenty of research has been carried out for a long time, resulting in both important theoretical insights into the structure of optimal solutions, and suboptimal algorithms based on, e.g., pruning strategies or relaxation techniques (cf. [9]–[12] or, for more recent results, [13]–[16]).

In this paper, we consider the case where measurements of a single sensor have to be transmitted to a remote estimator attached to a controller over a network as depicted in Fig. 1. The communication is subject to transmission constraints given in terms of limited bandwidth and a fixed budget that is available to the sensor and that can be spent for communication within a fixed horizon. To the best of our knowledge, this particular scenario has not been considered in literature although it belongs to the general problem class described above and resembles problems where only a fixed, limited number of (costly) measurements can be taken, as discussed, for instance, in [14].

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More precisely, the contribution of this paper is threefold. First, we present a formal description of the considered scenario and translate it into an optimal control problem, for which we then, second, derive necessary optimality conditions. Third, we introduce and analyze a suboptimal, probabilistic algorithm for the considered problem based on a suitable relaxation.

Notation: Throughout the paper, vectors will be indicated by underlined letters ($\underline{x}$) while random vectors will be underlined and in bold ($\underline{\boldsymbol{x}}$). To denote matrices, we will employ boldface capital letters, e.g., $\mathbf{A}$. $\mathbf{0}$ and $\mathbf{I}$ are used to denote zero and identity matrix, respectively, and a subscript k indicates the time step. The notation $\mathbf{A} \geq \mathbf{0}$ means that the symmetric matrix $\mathbf{A}$ is positive semidefinite. Finally, transposition of a matrix is indicated by $(.)^{\mathrm{T}}$, $\mathrm{tr}\,[.]$ denotes the trace operator, and $\lfloor x \rfloor$ and $\lceil x \rceil$, respectively, denote floor and ceiling function.

II. Problem Formulation

Consider the NCS illustrated in Fig. 1, where the discrete-time dynamics of the linear plant is given by

$$\underline{\boldsymbol{x}}_{k+1} = \mathbf{A}_k \underline{\boldsymbol{x}}_k + \mathbf{B}_k \underline{u}_k^a + \underline{\boldsymbol{w}}_k \,,$$

with $\underline{\boldsymbol{x}}_k \in \mathbb{R}^N$ the plant state, $\underline{u}_k^a \in \mathbb{R}^L$ the control input applied to the plant, $\mathbf{A}_k \in \mathbb{R}^{N \times N}$ the system matrix and $\mathbf{B}_k \in \mathbb{R}^{N \times L}$ the input matrix. The noise process $\underline{\boldsymbol{w}}_k$ is white, Gaussian, and zero mean with covariance matrix $\mathbf{W}_k$. At each time step, a sensor takes a noisy measurement $\underline{y}_k \in \mathbb{R}^M$ of the state according to

$$\underline{\boldsymbol{y}}_k = \mathbf{H}_k \underline{\boldsymbol{x}}_k + \underline{\boldsymbol{v}}_k \,,$$

where $\mathbf{H}_k \in \mathbb{R}^{M \times N}$ is the measurement matrix. The measurement noise $\underline{\boldsymbol{v}}_k$ is also white, zero mean, and Gaussian with covariance matrix $\mathbf{V}_k$, and independent of $\underline{\boldsymbol{w}}_j$. The initial plant state $\underline{\boldsymbol{x}}_0$ is assumed to be Gaussian with given mean $\hat{\underline{x}}_0$ and covariance $\mathbf{X}_0$, and independent of $\underline{\boldsymbol{w}}_i$ and $\underline{\boldsymbol{v}}_j$.

At each time step, the control input $\underline{u}_k^a$ is computed by a controller based on an estimate of the plant state provided by an estimator, and then transmitted to the plant over a network. We do not make any particular assumption on the nature of the applied controller and the network between it and the plant, because this is not the focus of this work. Thus, in the most general case, the network might be lossy and subject to random delays, and the employed controller can be tailored to NCS scenarios based on model predictive [17], [18] or sequence-based [19] control approaches.

The estimator is attached to the controller and supplied with measurements transmitted by the sensor over another network. Under the conditions of perfect communication, that is, measurements arrive without delay at the estimator and do not get lost, it is well known that the Kalman filter (KF) is the *minimum mean squared error (MMSE)* estimator [20]. The evolution of the covariance of the estimation error is then governed by the discrete Riccati equation

$$\begin{aligned} \mathbf{C}_{k+1} &= \mathbf{A}_k \mathbf{C}_k \mathbf{A}_k^{\mathrm{T}} + \mathbf{W}_k \\ &\quad - \mathbf{A}_k \mathbf{C}_k \mathbf{H}_k^{\mathrm{T}} \left(\mathbf{H}_k \mathbf{C}_k \mathbf{H}_k^{\mathrm{T}} + \mathbf{V}_k \right)^{-1} \mathbf{H}_k \mathbf{C}_k \mathbf{A}_k^{\mathrm{T}} \,. \end{aligned} \quad (1)$$

However, in the considered setup, the network is subject to a limited capacity. Hence, the measurements cannot be transmitted to the estimator at each time instant k. Instead, at every K-th time step, the sensor is assigned a communication budget that can be spent for sending measurements in a time window of length $K \in \mathbb{N}$. More precisely, at time nK, $n = 0, 1, 2, \ldots$, a budget $B_n > 0$ is available for communication at time steps $nK, nK+1, \ldots, (n+1)K-1$. Within this horizon, the nonnegative costs β_k for transmitting the measurements $\underline{y}_k$ are known but not necessarily equal. The overall goal is to find a schedule of measurement transmissions which i) does not exceed the prescribed budget, ii) avoids sending measurements at costly time steps (that is, when β_k is large) and iii) maintains small error covariances $\mathbf{C}_k$, expressed in terms of the MSE given by $\mathrm{tr}\,[\mathbf{C}_k]$.

The cost function to be minimized at time nK with respect to the binary decision variables $u_{nK}, u_{nK+1}, \ldots, u_{(n+1)K-1}$ is

$$\mathcal{J}_n = \mathrm{tr}\left[\mathbf{C}_{(n+1)K}\right] + \sum\nolimits_{k=nK}^{(n+1)K-1} \mathrm{tr}\,[\mathbf{C}_k] + \beta_k^2 u_k \,, \quad (2)$$

where $u_k = 1$ if the measurement shall be communicated to the estimator, and $u_k = 0$ otherwise. Note that we use the squared transmission costs in (2) to enforce that communication at costly time steps is avoided. The budget constraint can be expressed by the inequality

$$\sum\nolimits_{k=nK}^{(n+1)K-1} u_k \beta_k \leq B_n \,,$$

and the evolution of the error covariance (1) becomes

$$\begin{aligned} \mathbf{C}_{k+1} &= \mathbf{A}_k \mathbf{C}_k \mathbf{A}_k^{\mathrm{T}} + \mathbf{W}_k \\ &\quad - u_k \mathbf{A}_k \mathbf{C}_k \mathbf{H}_k^{\mathrm{T}} \left(\mathbf{H}_k \mathbf{C}_k \mathbf{H}_k^{\mathrm{T}} + \mathbf{V}_k \right)^{-1} \mathbf{H}_k \mathbf{C}_k \mathbf{A}_k^{\mathrm{T}} \,. \end{aligned} \quad (3)$$

This is a modified Riccati equation whose properties have, for instance, been investigated in [21]. For convenience, we will denote the right side of (3) by $\mathbf{G}_k(\mathbf{C}_k, u_k)$ and write $\mathbf{C}_{k+1} = \mathbf{G}_k(\mathbf{C}_k, u_k)$.

Remark 1 *A formula for the evolution of the state estimate $\hat{\underline{x}}_k^e$ which complements* (3) *has been derived in [9], [21]. Therein it has also been shown that the resulting KF-like estimator is optimal in the MSE sense.*

Remark 2 *It is worth to mention that digital networks are typically packet-based, so that sending only portions or single entries of the measurements $\underline{y}_k$ requires the same amount of network resources. Hence, the transmission costs β_k stay the same.*

In summary, the considered problem can be stated in terms of the *deterministic* optimal control problem[1]

$$\begin{aligned} \min_{u_0, \ldots, u_{K-1}} \quad & \mathcal{J} = \mathrm{tr}\,[\mathbf{C}_K] + \sum\nolimits_{k=0}^{K-1} \mathrm{tr}\,[\mathbf{C}_k] + \beta_k^2 u_k \\ \text{s.t.} \quad & (3)\,, \quad u_k \in \{0, 1\} \quad \forall k = 0, \ldots, K-1\,, \\ & \sum\nolimits_{k=0}^{K-1} u_k \beta_k \leq B\,. \end{aligned} \quad (4)$$

[1]In the remainder of this paper, we restrict ourselves to $\mathcal{J}_0$, i.e., $n = 0$, without loss of generality and write $\mathcal{J}$ and B instead of $\mathcal{J}_0$ and B_0 for convenience.

Before we derive necessary conditions for an optimal solution of (4) in the next section, it is worth to discuss the semantics of the transmission costs. We regard the individual β_k as a measure of the actual or estimated state of the network at the respective time step in such a way that "large" values reflect a "bad" network. Here, "bad" can be perceived as a quantification of the quality of the communication channel in terms of, e.g., high utilization or little available bandwidth. The corresponding values could result from the application of network management such as bandwidth estimation [22]. On the other hand, we envision that the concepts for event-based communication introduced recently in [6], [23] can be utilized. There, the authors propose to use triggering criteria to predict the next network access in the near future, rather than to decide whether or not to send data at the current time step. When the network is shared among several control loops, these predictions can be used to gather estimates of the expected future utilization and subsequently to derive the communication costs β_k.

Finally, we want to stress that the considered problem is not limited to scenarios where communication is costly. For instance, (4) can easily be adapted to scenarios where several sensors are available but associated with different cost β_k when utilized [14]. Additionally, problems where only a limited number, say q, measurements can be taken in a certain horizon can be handled by setting $\beta_k = 1$, $B = q$ and employing an equality constraint in (4).

III. Necessary Conditions For Optimality

In this section, we present necessary conditions for an optimal solution of the optimization problem (4), which are derived by first formulating the Hamiltonian associated with the problem and then employing the matrix minimum principle [24].

Theorem 1 *For a feasible sequence* $u_0^*, \dots, u_{K-1}^*$ *and corresponding covariance trajectory* $\mathbf{C}_0^*, \dots, \mathbf{C}_K^*$ *to be a solution of the optimization problem* (4) *with initial condition* $\mathbf{C}_0$*, it must hold that*

(i) $\mathbf{C}_0^* = \mathbf{C}_0$,

(ii) if $u_k^* = 0$*, then*

$$\mathrm{tr}\left[\mathbf{S}_k \mathbf{C}_k^* \mathbf{A}_k^{\mathrm{T}} \mathbf{P}_{k+1}^* \mathbf{A}_k\right] \leq \beta_k(\beta_k + \mu) , \tag{5}$$

with

$$\mathbf{S}_k = \mathbf{C}_k^* \mathbf{H}_k^{\mathrm{T}} \left(\mathbf{H}_k \mathbf{C}_k^* \mathbf{H}_k^{\mathrm{T}} + \mathbf{V}_k\right)^{-1} \mathbf{H}_k , \tag{6}$$

and $\mathbf{P}_0^*, \dots, \mathbf{P}_K^* \geq \mathbf{0}$ *given by*

$$\begin{aligned} \mathbf{P}_K^* &= \mathbf{I} , \\ \mathbf{P}_k^* &= \mathbf{I} + \left(\mathbf{I} - u_k^* \mathbf{S}_k\right)^{\mathrm{T}} \mathbf{A}_k^{\mathrm{T}} \mathbf{P}_{k+1}^* \mathbf{A}_k \left(\mathbf{I} - u_k^* \mathbf{S}_k\right) , \end{aligned} \tag{7}$$

and where $\mu \geq 0$ *is a constant which fulfills*

$$\mu \left(B - \textstyle\sum_{k=0}^{K-1} u_k^* \beta_k\right) = 0 . \tag{8}$$

Proof: The proof is given in Appendix A. ■

The conditions given in the theorem bear close resemblance to those derived in [9] where sensor selection optimization was investigated for a continuous-time scenario. Additionally, an iterative scheme for obtaining a (suboptimal) solution which directly exploits the structure of the necessary conditions was devised therein, which was later on adapted to an akin, discrete-time scenario in [25], [26].

In principle this algorithm from [9], being simple and easy to implement, could be adopted to the problem considered in this paper. In essence, it would consist of the repetitive improvement of a given initial solution by i) solving the forward recursion (3), ii) evaluating the costate equations (7) backwards in time and then iii) determining an improved solution according to the *switching function* (5). However, the incorporation of the budget constraint, that is, the additional condition (8), which was not present in [9], [25], [26], is not straightforward. Moreover, oscillatory behavior of the algorithm was reported in [9] and dependence on appropriate initial guesses was pointed out in [25].

IV. Proposed Algorithm

Instead of employing an optimization algorithm to directly (suboptimally) solve the original problem (4), we solve a relaxed variant where the binary constraints are replaced by convex ones. This is similar to what was proposed in [14] for a sensor selection problem. Then, to convert the relaxed solution into a feasible solution of the original problem, we apply a strategy based on randomized rounding [27], [28].

A. Problem Relaxation

Substituting the binary constraints in (4) with the convex constraints $u_k \in [0, 1]$ yields a relaxation of the original problem which can be summarized as

$$\begin{aligned} &\min_{\bar{u}_0, \dots, \bar{u}_{K-1}} \mathcal{J} \\ \text{s.t.} \quad &(3) , \quad 0 \leq \bar{u}_k \leq 1 \quad \forall k = 0, \dots, K-1 , \\ &\textstyle\sum_{k=0}^{K-1} \bar{u}_k \beta_k \leq B . \end{aligned} \tag{9}$$

Efficient numerical algorithms for solving this problem are, e.g., interior point methods or sequential quadratic programming [29]. However, the optimal solution of (9) will in general be fractional and must thus be converted into a feasible integral solution, which, commonly, will be suboptimal. On the other hand, since the feasible set of the original problem (4) is included in the feasible set of (9), we can deduce that the optimal value, i.e., the minimal costs, of the latter constitutes a lower bound for the optimal value of the former.

B. Randomized Rounding

In case of uniform transmission costs, that is, $\beta_k = \beta$, a simple approach to obtain a feasible solution of the original problem is to set the $\left\lfloor \frac{B}{\beta} \right\rfloor$ largest elements of the solution of the relaxation to 1 and the remaining ones to 0. Yet, as stressed in [14], very little can be said on how close to the optimal solution of (4) the thus obtained schedule will be. Moreover, for the general case of non-uniform transmission costs, the described procedure is likely to produce solutions that violate the imposed budget constraint.

We propose to instead employ a strategy based on randomized rounding [27], [28] to recover a solution $u_0, \ldots, u_{K-1}$ of the original problem from a given feasible solution $\bar{u}_0, \ldots, \bar{u}_{K-1}$ of the relaxation. Although being probabilistic, this scheme is superior to the one discussed above as it allows us in the following i) to explicitly derive an upper bound for the expected objective value of the obtained solution, that is, the expected costs $\mathrm{E}\{\mathcal{J}\}$ and ii) to prove that this solution is feasible with non-zero probability.

The basic idea of randomized rounding is straightforward and can be summarized as follows. For each element $\bar{u}_k$, the corresponding element u_k is set to 1 with probability $\bar{u}_k$ and to 0 with probability $1 - \bar{u}_k$. More formally, each u_k is the result of a Bernoulli trial $\boldsymbol{\psi}_k$ with success probability $\bar{u}_k$, i.e.,

$$\mathrm{P}\left[\boldsymbol{\psi}_k = 1\right] = \bar{u}_k \, .$$

By exploiting that the values of the u_k are determined independently of each other, we can find an upper bound for the expected costs $\mathrm{E}\{\mathcal{J}\}$ of the resulting solution.

Theorem 2 *Suppose a not necessarily feasible solution* $u_0, \ldots, u_{K-1}$ *of the original problem* (4) *is given where each* u_k *is the outcome of a Bernoulli trial* $\boldsymbol{\psi}_k$ *with success probability* $\bar{u}_k$*. Then, the expected objective value* $\mathrm{E}\{\mathcal{J}\}$ *is bounded from above according to*

$$\mathrm{E}\{\mathcal{J}\} \leq \mathrm{tr}\left[\mathbf{\Lambda}_K\right] + \textstyle\sum_{k=0}^{K-1} \mathrm{tr}\left[\mathbf{\Lambda}_k\right] + \beta_k^2 \bar{u}_k \, , \qquad (10)$$

with $\mathbf{\Lambda}_k$ *computed by the recursion*

$$\mathbf{\Lambda}_0 = \mathbf{C}_0 \, , \quad \mathbf{\Lambda}_{k+1} = \mathbf{G}_k(\mathbf{\Lambda}_k, \bar{u}_k) \, .$$

Proof: The proof is given in Appendix B. ■

Despite the derived bound, using this simple rounding scheme as is, however, is of similarly little avail as the procedure discussed above because violations of the budget constraint are likely. In fact, it is typically not possible to determine in advance, that is, before a feasible solution of the relaxation is at hand, whether the probability that the rounding procedure yields a feasible solution is non-zero [27]. Main reason for this is that the right side of the inequality constraint in (4) is independent of the cost function $\mathcal{J}$. As a consequence, we adopt a technique named *scaling*, introduced in [27], to bound the probability that the budget constraint is violated by a value less than 1. This in turn implies that the corresponding solution is feasible with non-zero probability. The basic idea here is to decrease the number of variables which are set to 1 by scaling the individual rounding probabilities. This is achieved by multiplying them with a constant $1 - \delta$, where $\delta \in (0, 1)$. The following result shows that for every such δ the obtained solution $u_0, \ldots, u_{K-1}$ is indeed feasible with non-zero probability.

Theorem 3 *Let* $\bar{u}_0, \ldots, \bar{u}_{K-1}$ *be a feasible solution of the relaxation* (9)*, fix a* $\delta \in (0, 1)$ *and obtain* $u_0, \ldots, u_{K-1}$ *by randomized rounding with scaled success probabilities* $(1 - \delta)\bar{u}_k$*. Then*

$$\mathrm{P}\left[\textstyle\sum_{k=0}^{K-1} \boldsymbol{\psi}_k \beta_k > B\right] \leq \frac{\sigma^2}{\sigma^2 + \gamma^2} < 1 \, , \qquad (11)$$

with

$$\sigma^2 = (1 - \delta) \textstyle\sum_{k=0}^{K-1} \beta_k^2 \bar{u}_k \left(1 - (1 - \delta)\bar{u}_k\right) \, ,$$
$$\gamma = B - (1 - \delta) \textstyle\sum_{k=0}^{K-1} \beta_k \bar{u}_k \, ,$$

i.e., the probability that $u_0, \ldots, u_{K-1}$ *exceeds the prescribed transmission budget is strictly less than 1.*

Proof: The proof is given in Appendix C. ■

An immediate consequence of Theorem 3 is that the expected number of trials required until the rounding procedure yields a feasible solution of the original problem (4) can be expressed as the mean of a geometric distribution with success probability

$$p \geq 1 - \frac{\sigma^2}{\sigma^2 + \gamma^2} = \frac{\gamma^2}{\sigma^2 + \gamma^2} \, . \qquad (12)$$

The mean of a geometric distribution with parameter p is $\frac{1}{p}$ [28], we thus have the following corollary.

Corollary 1 *An upper bound for the expected number of trials required until* $\sum_{k=0}^{K-1} u_k \beta_k \leq B$ *holds is* $\left\lceil \frac{\sigma^2 + \gamma^2}{\gamma^2} \right\rceil$.

It is apparent from the above results that the choice of the scaling parameter δ impacts both the quality of the resulting solutions and the runtime of the algorithm. More precisely, by Theorem 3, choosing δ close to 1 decreases the probability of infeasible solutions and hence, by Corollary 1, the expected number of trials required. On the downside, it becomes more likely that only few u_k will be 1 and thus, by (3), the quality of the obtained solutions will be poor. On the contrary, if δ is near to 0, it can be expected that the resulting solutions will be closer to the optimal one, however at the cost of an increased risk of infeasibility and, in turn, increased runtime.

To summarize the results of this section, the proposed algorithm consists of solving the relaxation (9) and randomized rounding with suitably scaled success probabilities where the latter step is repeated until a feasible solution is obtained.

V. Evaluation

In this section, we compare the proposed approach to the suboptimal algorithm for sensor selection presented in [13] which we adapted to our purposes and implemented using yalmip [30]. The comparison will be conducted by means of simulations. To that end, we consider an uncontrolled double integrator plant with parameters

$$\mathbf{A}_k = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \, , \quad \mathbf{W}_k = \begin{bmatrix} 0.4^2 & 0 \\ 0 & 0.4^2 \end{bmatrix} \, ,$$
$$\mathbf{H}_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \, , \quad \mathbf{V}_k = 0.8^2 \, ,$$

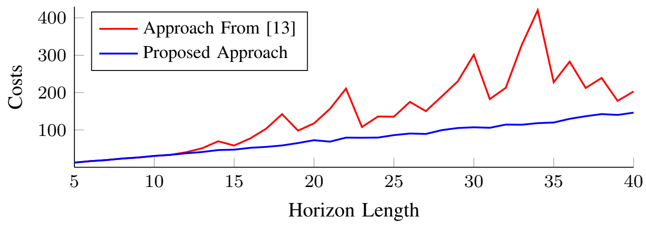

Fig. 2: Average costs of the the proposed approach compared to the costs of the approach from [13] for $\beta_k = 1$.

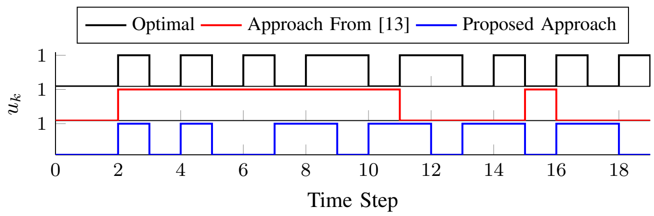

Fig. 3: Best found solution of the proposed approach, the solution found by the pruning approach from [13] and the optimal solution for $K = 20$ and $\beta_k = 1$.

and initial condition

$$\mathbf{X}_0 = \mathbf{C}_0 = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix} .$$

The suboptimal algorithm proposed in [13] is deterministic and utilizes a concept named $\epsilon-$redundancy to prune the search tree, where ϵ is a parameter to be supplied by the user. We choose $\epsilon = 0.1$, since this value exhibited a fairly good trade-off between computational demand and solution quality in the numerical examples in [13]. For the proposed randomized rounding approach, we use the scaling parameter $\delta = 10^{-3}$.

The general procedure of the simulations is as follows. First, we compute a solution of the original problem (4) using the deterministic method from [13]. Then, we compute a solution of the relaxation (9) and perform the proposed randomized rounding to obtain a solution of the original problem. Instead of repeating the rounding step until a feasible solution is obtained, we conduct 50 000 single trials and discard all infeasible solutions.

A. Uniform Transmission Costs

First, we consider uniform transmission costs, $\beta_k = 1$, and the possibility to send measurements at least 50 % of the time, which corresponds to $B = \lceil \frac{K}{2} \rceil$.

In Fig. 2 the costs of both approaches for different horizons $K = 5, 6, \ldots, 40$ are depicted, where for the proposed approach the costs are averaged over all feasible solutions. The results show that the proposed approach produces significantly better results for optimization horizons $K \geq 12$. Moreover, in contrast to the approach from [13], where we can observe sharp increases, the costs increase only slightly with the horizon length. For $K = 20$, the best found solution of the proposed algorithm is shown in Fig. 3, in comparison to the solution of the pruning approach and the optimal solution, which has been obtained by enumerating all feasible solutions. From the figure, we can conclude that the proposed

	Uniform β_k	Non-Uniform β_k
Optimal Solution	65.307	499.055
Approach From [13]	117.627	499.055
Proposed Approach	68.226	516.944

TABLE I: Costs of the solutions for $K = 20$ shown in Figs. 3 and 5.

	Uniform β_k	Non-Uniform β_k
Percentage of Feasible Solutions	0.948	1
Computed Lower Bound (12)	$3.815 \cdot 10^{-3}$	0.999

TABLE II: Percentage of feasible solutions in comparison to the derived lower bound (12).

approach is only slightly suboptimal in this scenario, because it coincides with the optimal solution most of the time. This can also be seen from the corresponding values of the cost function, which are given in the middle column of Table I. The periodic nature of the optimal solution and the solution computed by the proposed approach is another interesting observation, as it is in line with previous results [6], [15]. There, however, in contrast to this work, only the estimation error was considered as decision criterion.

In Table II the percentage of all feasible solutions and the (averaged) lower bound of the probability that the proposed rounding procedure yields a feasible solution, computed according to (12), are given. The numbers suggest that the derived bound, and consequently the bounds given in Theorem 3 and Corollary 1, are too conservative in this scenario. On the other hand, they indicate that a feasible solution is obtained with at most two repetitions of the rounding procedure.

B. Non-Uniform Transmission Costs

Now we consider non-uniform transmission costs which increase linearly within the horizon according to $\beta_k = 1, 2, \ldots, K$. This corresponds to a network whose quality gradually decreases over time. The available transmission budget is $B = \left\lceil \frac{\sum_{k=0}^{K-1} \beta_k}{2} \right\rceil$.

The resulting costs for different horizon lengths are plotted in Fig. 4. The costs for the proposed approach are again averaged over all feasible solutions. Note that, in contrast to the previous scenario, we used $\epsilon = 0.5$ for the algorithm from [13]. This became necessary since for smaller values the algorithm was not able to prune the search tree efficiently, and hence, computation became intractable already for small K. The results indicate that both approaches perform similarly, which is remarkable because the pruning approach from [13] was able to compute the optimal solution in this setup. This is illustrated for $K = 20$ in Fig. 5, where the best found solution of the proposed algorithm, the solution of the pruning approach and the optimal solution are shown. The corresponding costs are given in the right column of Table I. Due to the relatively large transmission costs, the number of feasible solutions and hence the resulting search tree might be fairly small. This could explain the good performance of the algorithm from [13] in this scenario. However, we did not verify this supposition. From Fig. 5, it also becomes obvious that a large portion of the available

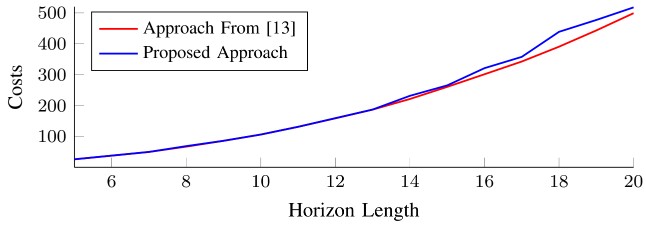

Fig. 4: Average costs of the the proposed approach compared to the costs of the approach from [13] for non-uniform β_k.

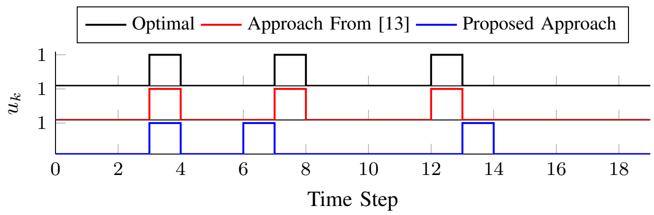

Fig. 5: Best found solution of the proposed approach, the solution found by the pruning approach from [13] and the optimal solution for $K = 20$ and non-uniform β_k.

transmission budget, which is $B = 105$ for $K = 20$, remains unused. This is an expected outcome, since the transmission costs enter the cost function (4) quadratically. Nonetheless, it is necessary to transmit a measurement at a costlier time step in the second half of the horizon in order to minimize the estimation error and hence the value of the cost function.

Finally, the percentage of all feasible solutions computed by the proposed approach and the (averaged) lower bound (12) are listed in Table II. In contrast to the previous scenario, the bound is relatively sharp this time. It is striking that in this scenario all of the 50 000 computed solutions are feasible, which again illustrates the applicability of the approach. It is important to emphasize that this result also suggests that the algorithm has only polynomial computational complexity in practice. As opposed to this, the algorithm from [13] exhibits exponentially increasing computational demand, as already mentioned above, due to the employed pruning strategy.

VI. Conclusions

In this work, we investigated the scheduling of measurements in Networked Control Systems, where communication between sensor and remote estimator was constrained in terms of a fixed communication budget due to limited network capacity. We derived necessary optimality conditions for the resulting optimal control problem and presented a suboptimal algorithm based on a suitable problem relaxation and subsequent randomized rounding. Also, we derived upper bounds for the expected costs of a solution computed by the algorithm and for the probability that this solution is infeasible. The evaluation results indicated that the proposed approach is superior to a state-of-the-art sensor selection algorithm in both quality and computational demand.

Future work might be concerned with deriving a lower bound for the expected costs, similar to what has been done in [16]. Likewise, it may be interesting to elaborate on the impact of the scaling parameter δ on the presented bounds and, more importantly, to further investigate the trade-off between solution quality and the required number of rounding steps. Also, future research should aim at providing clues on how to determine appropriate transmission costs β_k.

Appendix

A. Proof of Theorem 1

With the scalar state s_k with initial condition $s_0 = B$ and dynamics $s_{k+1} = s_k - u_k\beta_k$ the global inequality constraint in (4) becomes $s_K \geq 0$, which is equivalent but only local [11]. Then, the Hamiltonian $\mathcal{H}_k$ of (4) is [24]

$$\begin{aligned}\mathcal{H}_k &= \mathrm{tr}\left[\mathbf{C}_k\right] + \beta_k^2 u_k + \mathrm{tr}\left[\mathbf{G}_k(\mathbf{C}_k, u_k)\mathbf{P}_{k+1}^{\mathrm{T}}\right] \\ &\quad + \lambda_{k+1}(s_k - u_k\beta_k)\,,\end{aligned}$$

where $\mathbf{P}_{k+1}$ and λ_{k+1} are the costates associated with $\mathbf{C}_k$ and s_k, respectively. For the former, the dynamics for an optimal solution are given by

$$\mathbf{P}_K^* = \frac{\partial \mathrm{tr}\left[\mathbf{C}_K^*\right]}{\partial \mathbf{C}_K^*}\,, \quad \mathbf{P}_k^* = \frac{\partial \mathcal{H}_k}{\partial \mathbf{C}_k^*}\,.$$

Performing the differentiation using identities from, e.g., [31], yields

$$\begin{aligned}\mathbf{P}_K^* &= \mathbf{I}\,,\\ \mathbf{P}_k^* &= \mathbf{I} + \mathbf{A}_k^{\mathrm{T}}\mathbf{P}_{k+1}^*\mathbf{A}_k + u_k^*\mathbf{H}_k^{\mathrm{T}}\left(\mathbf{H}_k\mathbf{C}_k^*\mathbf{H}_k^{\mathrm{T}} + \mathbf{V}_k\right)^{-1}\\ &\quad \cdot \mathbf{C}_k^*\mathbf{A}_k^{\mathrm{T}}\mathbf{P}_{k+1}^*\mathbf{A}_k\mathbf{C}_k^*\mathbf{H}_k^{\mathrm{T}}\left(\mathbf{H}_k\mathbf{C}_k^*\mathbf{H}_k^{\mathrm{T}} + \mathbf{V}_k\right)^{-1}\mathbf{H}_k\\ &\quad - u_k^*\mathbf{A}_k^{\mathrm{T}}\mathbf{P}_{k+1}^*\mathbf{A}_k\mathbf{C}_k^*\mathbf{H}_k^{\mathrm{T}}\left(\mathbf{H}_k\mathbf{C}_k^*\mathbf{H}_k^{\mathrm{T}} + \mathbf{V}_k\right)^{-1}\mathbf{H}_k\end{aligned}$$

which, after the substitution (6) and completing the square, gives (7). Symmetry of $\mathbf{P}_k^*$ then follows from the symmetry of $\mathbf{P}_K^*$ by inductive reasoning. Likewise, positive semidefiniteness of $\mathbf{P}_K^*$ is apparent. Now suppose that this holds for some $k+1 < K$. Then $\mathbf{A}_k^{\mathrm{T}}\mathbf{P}_{k+1}^*\mathbf{A}_k \geq \mathbf{0}$ and hence $\mathbf{P}_k^* \geq (\mathbf{I} - u_k^*\mathbf{S}_k)^{\mathrm{T}}\mathbf{A}_k^{\mathrm{T}}\mathbf{P}_{k+1}^*\mathbf{A}_k(\mathbf{I} - u_k^*\mathbf{S}_k) \geq \mathbf{0}$, which concludes the induction.

To show the remaining parts, we first observe that (8) is an immediate consequence of the complementary condition [29] $s_K^*\mu = 0$ imposed on the nonnegative multiplier μ corresponding to the constraint $-s_K^* \leq 0$. Finally, for (5), we use that the optimal input must minimize $\mathcal{H}_k$, i.e.,

$$\begin{aligned}&\beta_k^2 u_k^* - u_k^*\,\mathrm{tr}\left[\mathbf{S}_k\mathbf{C}_k^*\mathbf{A}_k^{\mathrm{T}}\mathbf{P}_{k+1}^*\mathbf{A}_k\right] - \lambda_{k+1}^* u_k^*\beta_k\\ &\leq \beta_k^2 u_k - u_k\,\mathrm{tr}\left[\mathbf{S}_k\mathbf{C}_k^*\mathbf{A}_k^{\mathrm{T}}\mathbf{P}_{k+1}^*\mathbf{A}_k\right] - \lambda_{k+1}^* u_k\beta_k\,,\end{aligned} \tag{13}$$

must hold, with the optimal costate λ_k^* determined by the recursion

$$\lambda_K^* = \mu\frac{\partial(-s_K^*)}{\partial s_K^*} = -\mu\,, \quad \lambda_k^* = \frac{\partial \mathcal{H}_k}{\partial s_k^*} = \lambda_{k+1}^*\,.$$

The claim then follows by setting $u_k^* = 0$ in (13).

B. Proof of Theorem 2

The proof works similar to the one of Theorem 3 in [16]. We have $\mathrm{E}\{\boldsymbol{\psi}_k\} = \bar{u}_k$ so that the expected costs are given by

$$\begin{aligned}\mathrm{E}\{\mathcal{J}\} &= \mathrm{E}\{\mathrm{tr}\left[\mathbf{C}_K\right]\} + \textstyle\sum_{k=0}^{K-1}\mathrm{E}\{\mathrm{tr}\left[\mathbf{C}_k\right]\} + \beta_k^2\mathrm{E}\{\boldsymbol{\psi}_k\}\\ &= \mathrm{tr}\left[\mathrm{E}\{\mathbf{C}_K\}\right] + \textstyle\sum_{k=0}^{K-1}\mathrm{tr}\left[\mathrm{E}\{\mathbf{C}_k\}\right] + \beta_k^2\bar{u}_k\,,\end{aligned}$$

where the expectations are with respect to $\psi_0, \ldots, \psi_{K-1}$. Since ψ_k and $\mathbf{C}_k$ are independent, we get

$$\mathrm{E}\{\mathbf{C}_1\} = \mathrm{E}\{\mathbf{G}_0(\mathbf{\Lambda}_0, \psi_0)\} = \mathbf{G}_0(\mathbf{\Lambda}_0, \bar{u}_0) = \mathbf{\Lambda}_1 \,,$$
$$\mathrm{E}\{\mathbf{C}_2\} = \mathrm{E}\{\mathbf{G}_1(\mathbf{C}_1, \bar{u}_1)\} \,.$$

Finally, as $\mathbf{G}_k$ is both increasing and concave with respect to $\mathbf{C}_k$ [21, Lemma 1c),e)], due to Jensen's inequality we obtain

$$\mathrm{E}\{\mathbf{C}_2\} \leq \mathbf{G}_1(\mathrm{E}\{\mathbf{C}_1\}, \bar{u}_1) = \mathbf{G}_1(\mathbf{\Lambda}_1, \bar{u}_1) = \mathbf{\Lambda}_2 \,,$$

from which the indicated recursion follows inductively and hence the bound (10).

C. Proof of Theorem 3

Note that $\mathrm{E}\{\psi_k\} = (1-\delta)\bar{u}_k$ and consider the random variables

$$\psi = \textstyle\sum_{k=0}^{K-1} \psi_k \beta_k \,, \;\; \xi = \psi - \mathrm{E}\{\psi\} \,.$$

Then, observe that mean and variance of ψ are given by

$$\mathrm{E}\{\psi\} = (1-\delta)\textstyle\sum_{k=0}^{K-1} \beta_k \bar{u}_k \,, \;\; \mathrm{Var}\{\psi\} = \sigma^2 \,,$$

and that ξ is zero mean with same variance as ψ. Since $\bar{u}_0, \ldots, \bar{u}_{K-1}$ is feasible, we have $\mathrm{E}\{\psi\} \leq (1-\delta)B$ and thus $\gamma > 0$. Hence, for any $\epsilon > -\gamma$

$$\begin{aligned} \mathrm{P}\left[\psi > B\right] &= \mathrm{P}\left[\xi > \gamma\right] = \mathrm{P}\left[\xi + \epsilon > \gamma + \epsilon\right] \\ &= \mathrm{P}\left[\frac{\xi+\epsilon}{\gamma+\epsilon} > 1\right] \leq \mathrm{P}\left[\left(\frac{\xi+\epsilon}{\gamma+\epsilon}\right)^2 > 1\right] . \end{aligned}$$

Applying Markov's inequality then yields

$$\mathrm{P}\left[\psi > B\right] \leq \mathrm{E}\left\{\left(\frac{\xi+\epsilon}{\gamma+\epsilon}\right)^2\right\} = \frac{\sigma^2+\epsilon^2}{(\gamma+\epsilon)^2} \,. \qquad (14)$$

The right side of (14) attains its minimum at $\epsilon = \frac{\sigma^2}{\gamma}$, from which (11) follows.

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