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Chapter 1

Introduction

Understanding the interplay between large scale geometry and the topology of manifolds represents a central challenge in modern mathematics. A basis for the investigation of this relationship is provided by the intriguing fact that certain topological invariants encode information about Riemannian geometric quantities. One such invariant is the so-called *simplicial volume*, which has been central to the study of the relationship between topology and geometry.

The simplicial volume is a real-valued homotopy invariant of oriented manifolds measuring the complexity of real fundamental cycles with respect to the ℓ^1-norm: If M is an oriented d-dimensional manifold, then the simplicial volume of M is defined by

$$\|M\| := \inf\{\|c\|_1 \mid c \text{ is an } \mathbb{R}\text{-fundamental cycle of } M\},$$

where $\left\|\sum_{j=1}^k a_j\sigma_j\right\|_1 = \sum_{j=1}^k |a_j|$. Intuitively, the simplicial volume can be seen as a measure for the complexity of the manifold since it is bounded from above by the number of simplices in a triangulation. Despite being defined in a purely homological way, the simplicial volume has a non-negligible impact on the possible geometric structures on a manifold.

A method to compute the simplicial volume is exploiting its connection with bounded cohomology [5, 18, 23, Sec.1.1; Prop. F.2.2; Sec.7.5], but in most cases even positivity of the simplicial volume is incredibly difficult to establish. There are a number of manifolds with vanishing simplicial volume, for example manifolds with abelian fundamental group or flat manifolds. On the other hand, the simplicial volume is positive for all oriented, closed and connected, negatively curved manifolds [23, 32, 50]. Still, there remain a lot of spaces for which it is not known if their simplicial volume is non-vanishing, let alone exact values of this invariant.

Gromov introduced the simplicial volume in order to re-prove parts of the Mostow rigidity theorem [5, 42]. This was an early spectacular application of this invariant. In his seminal paper "Volume and bounded cohomology" [23], Gromov established various relations between simplicial volume and invariants of geometric nature. The most fundamental theorem in this regard is Gromov's Main Inequality [23, Sec. 0.5], which bounds the simplicial volume of a Riemannian manifold in terms of the Riemannian volume provided the manifold satisfies a Ricci curvature bound. More precisely, let (M,g) be an oriented, closed and connected Riemannian manifold of dimension d with $\mathrm{Ric}(M,g) \geqslant -(d-1)$, then

$$\|M\| \leqslant (d-1)^d d!\, \mathrm{vol}(M).$$

The proof is based on the elementary duality which relates simplicial volume and bounded cohomology. The article [23] contains a proof of the Main Inequality for open manifolds as well, though it is quite difficult and it has never been written up in a detailed way. The Main Inequality is an important tool to prove positivity of another invariant of compact differentiable manifolds, the minimal volume: For a smooth manifold M consider all complete metrics g on M such that the sectional curvature $\sec(g)$ is pinched between -1 and 1. Then define the minimal volume $\mathrm{minvol}(M)$ of the manifold as the infimum of volumes of M over all these metrics. A sufficient condition for the Ricci curvature bound in the Main Inequality is that the sectional curvature satisfies $\sec(M) \geqslant -1$. Consequently, one obtains the volume estimate [23, Sec. 0.5]

$$\|M\| \leqslant \frac{(d-1)^d d!}{d^{d/2}}\, \mathrm{minvol}(M).$$

The improved constant is due to Besson, Courtois and Gallot [6, Théorème D]. In particular, non-vanishing of the simplicial volume implies non-vanishing of the minimal volume. Hence the simplicial volume encodes non-trivial information about the Riemannian volume and related quantities and therefore on the possible geometries of the manifold. It can be seen as a "topological approximation" of the Riemannian volume, which is corroborated by the facts that simplicial volume coincides (up to a factor) with Riemannian volume if the manifold is hyperbolic [23, 50, Sec. 0.4; Cor. 6.1.7] and by the general proportionality principle [23, Sec. 0.4].

With regard to these geometric implications, it is worthwhile to expand the scope and find new results in the spirit of the Main Inequality. First, one can prove results analogously to Gromov's estimate for other topological invariants. Second, one can try to relax the geometric conditions, i.e. replace the Ricci curvature bound by some weaker condition on the manifold. We give an overview of the results regarding both directions in this introduction. The present thesis aims to combine the two approaches. We derive estimates for simplicial volume under weaker geometric conditions. In particular, we derive the following theorem (Theorem 5.7)

Theorem 1.1. *For every real number S_0 and every dimension d, there is a constant $C(S_0, d) > 0$ with the following property: Let (M, g) be an oriented, closed and connected, d-dimensional Riemannian manifold with torsion-free fundamental group, such that the macroscopic scalar curvature of M at scale 1 is at least S_0. Then for the simplicial volume of M we have*

$$\|M\| \leqslant C(S_0, d)\,\mathrm{vol}(M).$$

The notion of macroscopic scalar curvature was introduced by Guth [29]. We will expand on it below. The techniques used in the proof are tailored to show volume estimates for other topological invariants as well: the integral foliated simplicial volume and the L^2-Betti numbers of aspherical manifolds.

Simplicial volume, integral foliated simplicial volume and L^2-Betti numbers

A long standing open question regarding simplicial volume was posed by Gromov in [25, p. 232].

Question 1.2. *Is there a universal upper bound for the L^2-Betti numbers of an aspherical manifold in terms of its simplicial volume?*

This conjecture is based on the observation that both invariants show similar behaviour. For example, both are multiplicative under finite coverings and satisfy a proportionality principle [23, 38, Sec. 0.4; Thm. 3.183]. In combination with the main inequality, a positive answer would yield a deep, general relationship between the L^2-Betti numbers of an aspherical manifold and the Riemannian volume. It is known that the above question is answered positively if the simplicial volume is replaced by a variant, the *integral foliated simplicial volume*. The proof was sketched by Gromov [26] and executed in detail by Schmidt [47]. The integral foliated simplicial volume of M, denoted by $\left| M \right|$, is defined in terms of a measure-preserving action of the fundamental group of the manifold on a probability space (X, μ) and $L^\infty(X, \mathbb{Z})$-valued cycles on the universal cover $\widetilde{M}$ of the manifold. More precisely, $\left| M \right|$ is obtained as the infimum of the ℓ^1-norms of cycles representing the image of the fundamental class under the chain homomorphism

$$C_*(M, \mathbb{Z}) \cong \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M}, \mathbb{Z}) \to L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M}, \mathbb{Z})$$

induced by the inclusion $\mathbb{Z} \hookrightarrow L^\infty(X, \mathbb{Z})$ as constant functions. In particular, one can show that $\|M\| \leqslant |M|$. The result of Schmidt indicates that the integral foliated simplicial volume might be useful to approach Gromov's question. More recent results on this topic can be found in [19, 37]. To obtain an affirmative answer to the question one would have to show equality of the integral foliated simplicial volume and the simplicial volume of aspherical manifolds, or at least universal inequalities between them. It has been proven that the two invariants do not coincide in general for aspherical manifolds [19]. For hyperbolic manifolds of dimension at least 4 the simplicial volume is strictly smaller than the integral foliated simplicial volume [19, Theorem 5.1]. So far, there are no further insights whether the two invariants satisfy some universal inequalities. Gromov's question remains open.

Nevertheless, the analogue of Gromov's Main Inequality for L^2-Betti numbers holds true, as shown by R. Sauer [45, Thm. A]. Provided that an aspherical manifold satisfies a lower Ricci curvature bound, its L^2-Betti numbers are bounded in terms of the volume. Sauer introduced a variant of the integral foliated simplicial volume, the so-called support mass, which satisfies an estimate corresponding to Schmidt's theorem. He showed that this invariant is bounded in terms of the Riemannian volume. The proof is based on techniques motivated by Gaboriau's theory of L^2-Betti numbers of equivalence relations [20].

Curvature conditions

More recently, Guth developed techniques that allow to replace the Ricci curvature bound in the Main Inequality by weaker assumptions [30]. Gromov asked the question if a similar volume estimate for the simplicial volume holds as well if the manifold has a lower scalar curvature bound [24, Conj. 3A]. However, in an incomplete and unedited version of an article available on his website *101 Questions, Problems and Conjectures around Scalar Curvature* he presumes that one has to pose stronger conditions for this estimate to become realistic [27].

A major problem in differential geometry concerns the relationship between scalar curvature and the topology of a manifold. While the topology of a manifold is a global invariant, the scalar curvature at a point is best described by an asymptotic expansion: If (M, g) is a d-dimensional Riemannian manifold and $p \in M$ is a point, then the volume of balls satisfies the asymptotic

$$\mathrm{vol}(B(p,r)) = \omega_d r^d \left(1 - \frac{\mathrm{scal}(p)}{6(d+2)} r^2 + \mathcal{O}(r^3)\right) \quad (r \to 0),$$

where ω_d is the volume of the unit ball in d-dimensional Euclidean space [21, Thm. 3.98]. Thus scalar curvature describes how the volume of very small balls compares to the volume of Euclidean balls of the same radius. If the scalar curvature at a point is positive, the volume of small balls is a bit less than in the Euclidean case. On the other hand, if $\mathrm{scal}(p) < 0$, then the volume of balls is slightly bigger than in the Euclidean case. This provides only local information. It encodes no information about the volume of balls of an arbitrary radius $r > 0$. It is rather difficult to derive insights on the topology from such a local estimate. On the other side, a condition on the Ricci curvature is much stronger. By the Bishop-Gromov inequality [21, Theorem 4.19], the volume of a ball of radius $r > 0$ in a manifold of non-negative Ricci curvature is bounded by the volume of the r-ball in the Euclidean space of the same dimension. This holds at every scale, so it is easier to derive information about the topology from this estimate.

In order to encode the behaviour of the volume of arbitrary balls in comparison with the Euclidean case, Guth introduced the notion of macroscopic scalar curvature [29, Sec. 7].

Definition 1.3. Let (M, g) be a d-dimensional Riemannian manifold and $\widetilde{M}$ be the universal cover with the induced metric. For a radius $r > 0$ and a point $p \in M$, let $B_{\widetilde{M}}(\tilde{p}, r)$ denote the r-ball in the universal cover around a lift $\tilde{p}$ of p. The *macroscopic scalar curvature of M at scale r in p* is defined as the real number S such that

$$\mathrm{vol}(B_{\widetilde{M}}(\tilde{p}, r)) = \mathrm{vol}(B_H(r)),$$

where H is the simply connected d-dimensional space of constant curvature with scalar curvature S. We denote it by $\mathrm{Sc}_r(p)$. The *macroscopic scalar curvature of M at scale r* is defined as

$$\mathrm{Sc}_r(M) \coloneqq \inf_{p \in M} \mathrm{Sc}_r(p).$$

Note that for a fixed $r > 0$, any positive real number can be realized as the volume of an r-ball in some scaling of either hyperbolic space, Euclidean space or the sphere. So the notion of macroscopic scalar curvature at a point is well-defined. Using the asymptotic expansion describing scalar curvature one can easily verify that $\mathrm{scal}(p) = \lim_{r \to 0} \mathrm{Sc}_r(p)$.

By using the universal cover in the definition, one ensures that any flat torus has macroscopic scalar curvature zero at any scale. Let (M,g) be an oriented, closed and connected, d-dimensional Riemannian manifold with $\operatorname{Ric}(M) \geqslant -(d-1)$ as in Gromov's Main Inequality. Denote the supremal volume of a 1-ball in the universal cover $\widetilde{M}$ by $V_{\widetilde{M}}(1)$. Then the Bishop-Gromov inequality implies that $V_{\widetilde{M}}(1)$ is bounded from above by the volume of the 1-ball in d-dimensional hyperbolic space $\mathbb{H}^d$. With the above definition, this translates to a lower bound on the macroscopic scalar curvature at scale 1. More precisely, $\mathrm{Sc}_1(M) \geqslant \mathrm{scal}(\mathbb{H}^d) = -d(d-1)$. This consideration motivates the following result by Guth, which is contained in [30, Lemma 7, Lemma 9]

Theorem 1.4. *For every real number S_0 and every dimension d, there is a constant $C(S_0,d) > 0$ with the following property: Let (M,g) be an oriented, closed and connected, d-dimensional aspherical Riemannian manifold with systole at least 1. Moreover, suppose that the macroscopic scalar curvature of M at scale 1 is at least S_0. Then we have*

$$\|M\| \leqslant C(S_0,d)\operatorname{vol}(M).$$

The systole bound can be replaced by the assumption that the fundamental group of the manifold is residually finite. In this case, one can work on a finite cover of M, for which the systole is at least 1, and use the multiplicativity of the simplicial volume and the Riemannian volume. Based on this theorem, Sauer proved the analogous result for L^2-Betti numbers of aspherical manifolds in [46, Thm. 1.3]. Note that in Gromov's main inequality the constants are explicitly known. In Guth's proof as well as in the estimates derived in the present thesis this is not the case.

Together with the fact due to Gromov and Thurston that the hyperbolic volume is proportional to the simplicial volume [23, 50] the above theorem yields

Theorem 1.5. *[30, Thm. 2] For every dimension d, there is a constant $C(d) > 0$ such that the following holds: Let (M,hyp) be a d-dimensional closed hyperbolic manifold and let g be another metric on M. Suppose $V_{(\widetilde{M},\tilde{g})}(1) \leqslant V_{\mathbb{H}^d}(1)$, i.e. $\mathrm{Sc}_1(M,g) \geqslant \mathrm{scal}(\mathbb{H}^d)$. Then it holds*

$$\operatorname{vol}(M,\mathrm{hyp}) \leqslant C(d)\operatorname{vol}(M,g).$$

This corresponds to a non-sharp macroscopic version of the following Schoen Conjecture for scalar curvature.

Conjecture 1.6. *[48] If (M,hyp) is a d-dimensional closed hyperbolic manifold and g is* *another metric on* M *with* $\mathrm{scal}(M,g) \geqslant \mathrm{scal}(\mathbb{H}^d)$*, then*

$$\mathrm{vol}(M, \mathrm{hyp}) \leqslant \mathrm{vol}(M, g).$$

This conjecture remains open in high dimensions. Balacheff and Karam obtained another theorem, which can be interpreted as a macroscopic version of the Schoen Conjecture [4]. Their proof is based on a *smoothing inequality* due to Gromov.

The proof of Guth's theorem relies on a nerve construction, which is a general method to bound the homology of a manifold M. One realizes M as a homotopy retract of a simplicial complex which comes from the nerve of a covering and for which one can control the number of simplices. This allows to bound the simplicial volume of the manifold by the norm of the image of the fundamental class under the map to the nerve complex. Provided there holds a packing inequality on the manifold, it is a standard trick to construct a Vitali cover of 1-balls whose multiplicity is bounded in terms of a dimensional constant [26, Chapter G_+]. The Ricci curvature in the Main Inequality ensures such a packing inequality. In Guth's setting, by replacing the Ricci curvature bound by a lower bound on the macroscopic scalar curvature at scale 1, there is no way to obtain a universal bound on the multiplicity. Guth managed to carefully construct a so-called *good cover* of the manifold allowing different radii such that the multiplicity is bounded on a subset of large volume. In order to encode the additional information on the radii of the cover sets, he introduced a modification of the well-known simplicial nerve construction, the so-called *rectangular nerve*, which is a subcomplex of a rectangular cuboid. With this new nerve techniques he managed to bound the simplicial volume of the manifold, where he relied on the asphericity and the lower systole bound in order to realize M as a retract of the nerve.

The purpose of this thesis is to expand on Guth's result and prove the estimate of the simplicial volume without these two conditions on the manifold as stated in Theorem 1.1.

Strategy: Randomization methods

In order to work around Guth's assumption that the manifold satisfies a lower bound on the systole or that the fundamental group is residually finite, we rely on a strategy outlined by Gromov in [26, Chapter G_+]. He intended it to be a guideline in order to solve his question on the relationship between simplicial volume and L^2-Betti numbers

(Question 1.2). Sauer implemented this strategy in a strict way for his direct proof of the Main Inequality for L^2-Betti numbers of compact aspherical manifolds [45].

The basic idea is to do a nerve construction on the universal cover $\widetilde{M}$ which is compatible with the action by the fundamental group $\pi_1(M) = \Gamma$, i.e. construct the cover in an equivariant way and the nerve as a free Γ-CW complex. Then in the case of aspherical manifolds, i.e. $\widetilde{M} = E\Gamma$, one automatically gets a retract to the nerve map by the universal property of $E\Gamma$. The problem is that in general one cannot force the cover to be equivariant without losing control on the multiplicity. Gromov's strategy, which he attributes to Connes, is to construct an equivariant measurable cover of a different space instead, namely the product $X \times \widetilde{M}$, where (X, μ) is a probability space with a free measure-preserving Γ-action. We consider this product with the diagonal group action. One can cover this space in an equivariant way by product sets of the form $A \times B$ for measurable sets $A \subset X$ and open balls $B \subset \widetilde{M}$. Then restricted to an arbitrary element $x \in X$, this yields an induced cover of $\{x\} \times \widetilde{M} \cong \widetilde{M}$. Intuitively, one picks a random cover of $\widetilde{M}$, which, after carefully constructing the measurable cover, has the desired multiplicity bounds. For this reason, this approach is called *randomization* [44].

Sauer elaborates on this strategy in [45]. He sets up a framework which involves techniques from measured equivalence relations, Gaboriau's theory of L^2-Betti numbers of $\mathcal{R}$-simplicial complexes and other themes of measured group theory. He works in the category of $\mathcal{R}$-spaces, where $\mathcal{R}$ is the orbit equivalence relation attributed to a probability space (X, μ) equipped with a free measure-preserving action by a countable discrete group Γ. These spaces arise as realizations of $\mathcal{R}$-simplicial complexes, which were introduced by Gaboriau [20]. The easiest examples are product spaces $X \times Z$ fibering over X, where Z is the realization of a Γ-simplicial complex. In this case the action of the orbit equivalence relation corresponds to the diagonal Γ-action on the space. In Sauer's setting [45], the manifold satisfies a packing inequality, thus the author constructs a suitable equivariant cover of $X \times \widetilde{M}$ such that the induced covers on $\{x\} \times \widetilde{M}$ are Vitali covers by 1-balls and have the correct multiplicity. Starting from this $\mathcal{R}$-cover one can do a nerve construction. Moreover, Sauer introduces a homology theory called foliated singular homology and a variant of the integral foliated simplicial volume, for which he manages to derive an upper bound in terms of the Riemannian volume using the nerve techniques.

In the present thesis this randomization strategy is combined with Guth's nerve techniques. So for a given probability space (X, μ) with a measure-preserving $\pi_1(M)$-action, we construct a measurable equivariant cover of $X \times \widetilde{M}$ such that the induced covers on every fibre $\{x\} \times \widetilde{M}$ are good covers in Guth's sense. Starting from this, one can construct a nerve which is fibrewise a rectangular nerve according to Guth's definition. A careful construction of the cover allows us to work around the use of foliated singular homology and work with chain complexes of the form $L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\widetilde{M},\mathbb{Z})$ instead. As a result, our proof yields, in particular, an estimate for the integral foliated simplicial volume. Thus in addition to Theorem 1.1, we derive the following result, where we have to require the manifold to be aspherical (Theorem 5.1).

Theorem 1.7. *For every real number S_0 and every dimension d, there is a constant $C(S_0,d) > 0$ with the following property: Let (M,g) be an oriented, closed and connected, d-dimensional aspherical Riemannian manifold such that the macroscopic scalar curvature of M at scale 1 is at least S_0. Then for the integral foliated simplicial volume of M we have*

$$|M| \leqslant C(S_0,d)\,\mathrm{vol}(M).$$

Using the upper bound on L^2-Betti numbers by Schmidt [47] and the definition of minimal volume we can derive the following estimates (Corollaries 5.3 and 5.4).

Corollary 1.8. *For every real number S_0 and every dimension d, there is a constant $C(S_0,d) > 0$ with the following property. Let (M,g) be a closed and connected, d-dimensional aspherical Riemannian manifold such that the macroscopic scalar curvature of M at scale 1 is at least S_0. Then for the L^2-Betti numbers of M we have*

$$b_k^{(2)}(M) \leqslant C(S_0,d)\,\mathrm{vol}(M) \quad \text{for all } k \geqslant 0.$$

Corollary 1.9. *For every dimension d, there is a constant $C(d) > 0$ with the following property: If M is an oriented, closed and connected, d-dimensional aspherical Riemannian manifold, then*

$$|M| \leqslant C(d)\,\mathrm{minvol}(M).$$

The framework used in [45] is too strong for our purposes. It was set up in order to prove two other results, which are analogues of statements in Gromov's work [23], the vanishing theorem and the isolation theorem for L^2-Betti numbers of aspherical manifolds [45, Theorems B and C].

In our setting, the $\mathcal{R}$-spaces occurring will be of the form $X \times Z$ with Z being a free Γ-CW complex, e.g. $Z = \widetilde{M}$. For spaces like this the action of the orbit equivalence relation corresponds to the diagonal group action. To cover these spaces, we introduce the category of *equivariant simple X-spaces* for a given probability space (X, μ) (Section 2.3). We will see that morphisms between such spaces induce chain maps of chain complexes of the form $L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(Z, \mathbb{Z})$ (Section 2.4). In order to implement Guth's idea of the rectangular nerve, we introduce the notion of *cuboid complexes*, which are metric polyhedral complexes where the cells are rectangular cuboids (Section 2.2). We can adapt Sauer's definition of $\mathcal{R}$-covers and define a rectangular nerve for such covers (Section 2.5).

Organisation of this work

The outline of the remaining chapters is as follows. Chapter 2 details the definitions and facts indicated above. Moreover, we recall the definition of simplicial volume and integral foliated simplicial volume and some of their properties (Section 2.1). Chapter 3 and Chapter 4 form the core of this thesis. In the first of these chapters we construct a good equivariant cover of $X \times \widetilde{M}$ in Guth's sense (Section 3.1). Moreover, we show that the measure of the high-multiplicity set is bounded, which is of fundamental importance for the volume estimate (Section 3.2). Chapter 4 details the construction of the rectangular nerve corresponding to the constructed cover (Section 4.1) as well as establishes the connection between the norms of fundamental cycles and the volume of the manifold (Section 4.2). Chapter 5 concludes with the proofs of the indicated Theorems 1.7 and 1.1 and their corollaries using the results derived in the preceding chapters.

Chapter 2

Basics

In this chapter we introduce the definitions and basic concepts needed for the proof of our main theorems. We start with an overview of simplicial volume and integral foliated simplicial volume (Section 2.1). In Section 2.2 we introduce the notion of metric cuboid complexes which are polyhedral complexes where the cells are rectangular cuboids. In the following two Sections 2.3 and 2.4 we introduce the category of equivariant simple X-spaces and show that the morphisms induce chain maps of twisted chain complexes. Finally, we adapt the notion of $\mathcal{R}$-covers from [45] for our purposes and define the rectangular nerve corresponding to such covers (Section 2.5).

Throughout the thesis we require all topological spaces to be path-connected, locally path-connected and semi-locally simply connected if not otherwise stated, hence allowing a universal covering space.

2.1 Simplicial and integral foliated simplicial volume

In this section we recall the definitions of simplicial volume and one of its variants, the integral foliated simplicial volume.

2.1.1 Simplicial volume

Everything presented in this section can be found in the literature, e.g. in [35]. We restrict ourselves to the definition and a short summary of some results. For a more detailed account we refer to the literature [23, 35, 36]. Simplicial volume is a real-valued homotopy invariant of an oriented manifold defined in terms of the singular chain complex with real coefficients. This chain complex can be equipped with a norm, the ℓ^1-norm, which induces a seminorm in homology. Simplicial volume is then defined as the seminorm of the fundamental homology class.

Definition 2.1.1 (ℓ^1-norm)**.** Let Z be a topological space. We define the ℓ^1-norm on the singular chain complex $C_*(Z,\mathbb{R})$ with real coefficients as follows. Let $n \in \mathbb{N}$. For a chain $\sum_{j=1}^k a_j\sigma_j \in C_n(Z,\mathbb{R})$ we define

$$\Big\|\sum_{j=1}^k a_j\sigma_j\Big\|_1 \coloneqq \sum_{j=1}^k |a_j|.$$

This norm is well-defined since the sum is finite. Thus it turns the singular chain group into a normed $\mathbb{R}$-vector space.

Definition 2.1.2. Suppose Z is a topological space and $\alpha \in H_n(Z,\mathbb{R})$ for some $n \in \mathbb{N}$. Then the ℓ^1-norm $\|.\|_1$ induces a seminorm on $H_n(Z,\mathbb{R})$ via

$$\|\alpha\|_1 \coloneqq \inf\{\Big\|\sum_{j=1}^k a_j\sigma_j\Big\|_1 \mid \sum_{j=1}^k a_j\sigma_j \in C_n(Z,\mathbb{R}) \text{ is a cycle representing } \alpha\}.$$

Remark 2.1.3. This seminorm has a *functoriality property*, which follows immediately from the definition. Let $f\colon Z \to Z'$ be a continuous map, $n \in \mathbb{N}$ and $\alpha \in H_n(Z,\mathbb{R})$. Then we have

$$\|H_n(f)(\alpha)\|_1 \leqslant \|\alpha\|_1.$$

An oriented, closed and connected d-dimensional manifold M comes with a distinguished homology class, the fundamental class. This is the generator of $H_d(M,\mathbb{Z}) \cong \mathbb{Z}$ corresponding to the orientation of M. We denote the fundamental class by $[M]_\mathbb{Z}$. The inclusion of coefficients $\iota\colon C_*(M,\mathbb{Z}) \hookrightarrow C_*(M,\mathbb{R})$ induces a change of coefficient homomorphism

$$H_d(\iota)\colon H_d(M,\mathbb{Z}) \to H_d(M,\mathbb{R}).$$

We call the image $[M] \coloneqq H_d(\iota)([M]_\mathbb{Z})$ the *real fundamental class* of M. The cycles in $C_d(M,\mathbb{R})$ representing $[M]$ are called *real fundamental cycles*.

Definition 2.1.4 (simplicial volume)**.** Let M be an oriented, closed and connected, d-dimensional manifold. The *simplicial volume* of M is defined as

$$\|M\| \coloneqq \|[M]\|_1 = \inf\Big\{\Big\|\sum_{j=1}^k a_j\sigma_j\Big\|_1 \,\Big|\, \sum_{j=1}^k a_j\sigma_j \in C_d(Z,\mathbb{R}) \text{ is a cycle representing } [M]\Big\}.$$

We define the *integral simplicial volume* by

$$\|M\|_{\mathbb{Z}} := \inf\Big\{\Big\|\sum_{j=1}^{k} a_j\sigma_j\Big\|_1 \;\Big|\; \sum_{j=1}^{k} a_j\sigma_j \in C_d(Z,\mathbb{Z}) \text{ is a cycle representing } [M]_{\mathbb{Z}}\Big\}.$$

In particular, we have $\|M\| \leqslant \|M\|_{\mathbb{Z}}$. Intuitively, the simplicial volume measures the complexity of a manifold. If one has a triangulation of the manifold, one obtains a fundamental cycle as sum of the top-dimensional simplices [39, 8.16, p. 138]. Hence the (integral) simplicial volume is bounded by the number of simplices in a triangulation. But the concept is much more flexible, in particular, if one considers fundamental cycles with real coefficient.

Remark 2.1.5. The functoriality property of the seminorm yields the following behaviour of simplicial volume under maps. For a continuous map $f\colon M \to N$ of oriented, closed and connected manifolds of the same dimension we have

$$|\deg(f)| \cdot \|N\| \leqslant \|M\|.$$

If f is a covering map, we obtain equality. One easily verifies the other inequality by using a suitable transfer map on the level of singular chains. Thus simplicial volume behaves multiplicatively under finite coverings.

Example 2.1.6. The functioriality implies that the simplicial volume of spheres and tori of non-zero dimension vanishes, since they allow self-maps of degree at least 2. The fact that $\|S^1\| = 0$ can be seen in an elementary way by observing that for every $k \in \mathbb{N}, k \geqslant 1$ the singular simplex

$$\begin{aligned} \sigma_k\colon [0,1] &\to S^1 \\ s &\mapsto e^{2\pi i k s}, \end{aligned}$$

given by wrapping the unit interval k-times around the circle S^1, defines a representative $\frac{1}{k}\sigma_k$ of the real fundamental class $[S^1]$. Consequently,

$$0 \leqslant \|S^1\| \leqslant \Big\|\tfrac{1}{k}\sigma_k\Big\|_1 = \tfrac{1}{k}$$

and this holds for every $k \in \mathbb{N}, k \geqslant 1$.

These are the easiest examples of manifolds with vanishing simplicial volume. In general, all manifolds with amenable fundamental group have simplicial volume zero [23, Corollary (C), p. 40]. In particular, the simplicial volume of simply connected manifolds vanishes. One the other hand, one source of non-vanishing simplicial volume is negative curvature. In the case of hyperbolic manifolds, the exact value of the simplicial volume is more or less known. By a result of Gromov and Thurston, the volume of a d-dimensional hyperbolic manifold coincides up to a constant with its simplicial volume [23, 50, Sec. 0.4; Cor. 6.1.7]. Hence hyperbolic volume is a topological invariant.

We indicated the geometric implications of simplicial volume in Chapter 1 as well as its connection with other invariants. As mentioned, Gromov's question remains open, whether for aspherical manifolds vanishing of the simplicial volume implies vanishing of the L^2-Betti numbers and therefore of the Euler characteristic (see Question 1.2).

2.1.2 Integral foliated simplicial volume

Describing a strategy on how to answer this question, Gromov suggested to define a variant of simplicial volume [23, p.305 f]. The precise definition of this *integral foliated simplicial volume* was given by Schmidt in his thesis [47]. There the author shows as well that vanishing of this homotopy invariant implies vanishing of the Euler characteristic of a manifold. Integral foliated simplicial volume is defined using homology with twisted coefficients which are induced by actions of the fundamental group on probability spaces. In the following we give the precise definition and some basic properties. Additional information and background on this invariant can be found in the literature [19, 37, 47].

First we recall some definitions. A measurable space is called a *standard Borel space* if it is measurably isomorphic to some Polish space with its Borel σ-algebra. Measurable subsets of standard Borel spaces will be called *Borel sets*. Equipped with a probability measure a standard Borel space becomes a *standard Borel probability space*. Note that a standard Borel probability space (X, μ) with an atom-free probability measure is measurably isomorphic to the interval $[0, 1]$ with the Lebesgue measure. For more information on the well-behaved category of standard Borel spaces we refer to the book of Kechris [33].

Remark 2.1.7. Throughout the whole thesis the abbreviation a.e. means either *almost every* or *almost everywhere*.

Given a measure space (X, μ) one can consider the function space $L^\infty(X)$ of essentially bounded measurable functions (see for example [3, Sec. 1.15, p. 52]). For a measurable function $f\colon X \to \mathbb{R}$ the essential supremum is defined via

$$\operatorname*{ess\,sup}_X f = \inf \Big\{ C \in \mathbb{R} \,\Big|\, \mu(\{x \in X \mid f(x) > C\}) = 0 \Big\}.$$

A function f is *essentially bounded* on X if $\operatorname{ess\,sup}_X|f| < \infty$. Then we define the space of essentially bounded measurable functions on X by

$$L^\infty(X,\mu) \coloneqq \{f\colon X \to \mathbb{R} \mid f \text{ measurable and essentially bounded with respect to } \mu\} /_\sim$$

equipped with the equivalence relation

$$f \sim g \quad \text{in } L^\infty(X,\mu) \quad :\Longleftrightarrow \quad f = g \quad \mu-\text{a.e.}.$$

Hence we deal with equivalence classes of functions. The essential supremum of a function depends only on its μ-a.e. class. If the measure is clear from the context we abbreviate $L^\infty(X,\mu)$ by $L^\infty(X)$. The space $L^\infty(X)$ is a normed vector space with

$$\|f\|_\infty \coloneqq \operatorname*{ess\,sup}_X |f| = \inf_{\substack{N\subseteq X\\ \mu(N)=0}} \sup_{x\in X\setminus N} |f(x)|.$$

Let $L^\infty(X,\mu,\mathbb{Z}) = L^\infty(X,\mathbb{Z})$ be the additive group of measurable essentially bounded functions $f\colon X \to \mathbb{Z}$. It holds the following.

Lemma 2.1.8. *$L^\infty(X,\mathbb{Z})$ is a free abelian group.*

Proof. The norm $\|.\|_\infty$ restricts to a norm on $L^\infty(X,\mathbb{Z})$. Moreover, it is a discrete norm as defined in [49], i.e. we can find an $\varepsilon > 0$ such that $\|f\|_\infty > \varepsilon$ if $f \neq 0$. This holds by the following consideration. If $f \neq 0$ there is a subset $A \subseteq X$ of positive measure where $|f(x)| \geqslant 1$, hence $\sup_{x\in A}|f(x)| \geqslant 1$. This remains true if we replace A by $A\setminus N$ for a μ-null set N. Hence

$$\|f\|_\infty \geqslant \operatorname*{ess\,sup}_A |f| = \inf_{\substack{N\subseteq A\\ \mu(N)=0}} \sup_{x\in A\setminus N} |f(x)| \geqslant 1.$$

According to the theorem proven by Steprāns in [49], an arbitrary abelian group having a discrete norm is free. As a result $L^\infty(X,\mathbb{Z})$ is a free $\mathbb{Z}$-module. □

Remark 2.1.9. Every element in $L^\infty(X,\mathbb{Z})$ can be represented as a finite linear combination of characteristic functions. Let $f \in L^\infty(X,\mathbb{Z})$. Without loss of generality, we may assume that f is represented as bounded (and not only essentially bounded). A bounded function $f\colon X \to \mathbb{Z}$ takes a finite number of distinct values $a_1,\dots,a_m$. Set $A_i = \{x \in X \mid f(x) = a_i\}$. Then $f = \sum_{i=1}^m a_i\chi_{A_i}$ where the sets $A_i \subseteq X$ are measurable since f is.

Definition 2.1.10. Let Γ be a countable group. A *standard* Γ-*space* is a standard Borel probability space (X,μ) endowed with a measurable μ-preserving left Γ-action.

Example 2.1.11. Let Γ be a countable group. An example for a standard Γ-space is given by the *Bernoulli shift*. This is the standard Borel probability space $(\{0,1\}^\Gamma, \otimes_\Gamma(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1))$ endowed with the shift action $\gamma_0(a_\gamma)_{\gamma\in\Gamma} = (a_{\gamma_0\gamma})_{\gamma\in\Gamma}$. The left action is measure-preserving. If the group is infinite, it is essentially free and ergodic [47, Lemma 3.37].

Remark 2.1.12. For a standard Γ-space (X,μ) the group $L^\infty(X,\mathbb{Z})$ of essentially bounded measurable functions with integer values is a right $\mathbb{Z}\Gamma$-module via

$$\begin{aligned} L^\infty(X,\mathbb{Z}) \times \Gamma &\to L^\infty(X,\mathbb{Z}) \\ (f,\gamma) &\mapsto \big(x \mapsto (f\cdot\gamma)(x) = f(\gamma x)\big). \end{aligned}$$

Let Z be a topological space with universal cover $\widetilde{Z}$ and fundamental group $\pi_1(Z) = \Gamma$. The singular chain complex $C_*(\widetilde{Z},\mathbb{Z})$ is a free $\mathbb{Z}\Gamma$-chain complex in a natural way. Then a $\mathbb{Z}\Gamma$-module structure on $L^\infty(X,\mathbb{Z}) \otimes_\mathbb{Z} C_n(\widetilde{Z},\mathbb{Z})$ is given by $\gamma(f\otimes\sigma) = (f\cdot\gamma^{-1})\otimes\gamma\sigma$ for $f\otimes\sigma \in L^\infty(X,\mathbb{Z})\otimes_\mathbb{Z} C_n(\widetilde{Z},\mathbb{Z})$. By passing over to the coinvariants, we obtain a $\mathbb{Z}$-module $L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{Z},\mathbb{Z})$. Tensoring the identity on $L^\infty(X,\mathbb{Z})$ with the differential of $C_*(\widetilde{Z},\mathbb{Z})$ gives a differential of $L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{Z},\mathbb{Z})$, turning it into a chain complex of $\mathbb{Z}$-modules. Note that elements in $L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{Z},\mathbb{Z})$ are equivalence classes. The equivalence relation is generated by the relation given by $f\otimes\sigma \sim f'\otimes\sigma'$ if and only if there is an element $\gamma\in\Gamma$ such that $f\otimes\sigma = \gamma(f'\otimes\sigma') = (f'\cdot\gamma^{-1})\otimes\gamma\sigma'$. We obtain a norm on singular chains with twisted coefficients as follows (see forward [47, Definition 5.20] and [14, Definition 3.5]).

Definition 2.1.13 (parametrised ℓ^1-norm)**.** Let Z be a topological space with universal cover $\widetilde{Z}$ and fundamental group $\pi_1(Z) = \Gamma$. Let (X,μ) be a standard Γ-space. We define the *parametrised ℓ^1-norm* on the $\mathbb{Z}$-chain complex $L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{Z},\mathbb{Z})$ as follows. Let $n\in\mathbb{N}$. For a chain $\sum_{j=1}^k f_j\otimes\sigma_j \in L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{Z},\mathbb{Z})$ we define

$$\left|\sum_{j=1}^k f_j\otimes\sigma_j\right|^X := \inf\left\{\sum_{j=1}^{k'} \int_X |f'_j|\,d\mu \;\middle|\; \sum_{j=1}^{k'} f'_j\otimes\sigma'_j \sim \sum_{j=1}^k f_j\otimes\sigma_j\right\}.$$

Remark 2.1.14. The parametrised ℓ^1-norm is a norm in the sense of [49], where norms on abelian groups are defined.

The infimum in the definition of the parametrised ℓ^1-norm of a chain is not changed if we regard chains in reduced form, which means that all the singular simplices σ_j belong to different Γ-orbits.

Moreover, the norm is well-defined since the group action of Γ on (X,μ) is measure-preserving and therefore

$$\int_X f(x)\,d\mu(x) = \int_X f(\gamma x)\,d\mu(x)$$

holds for every $\gamma \in \Gamma$ and every function $f \in L^\infty(X, \mathbb{Z})$.

Definition 2.1.15. Suppose Z is a topological space with universal cover $\widetilde{Z}$ and fundamental group Γ and (X, μ) is a standard Γ-space. Let $\alpha \in H_n(Z, L^\infty(X, \mathbb{Z})) = H_n(L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{Z}, \mathbb{Z}))$ for some $n \in \mathbb{N}$. Then the parametrised ℓ^1-norm $|.|^X$ induces a seminorm

$$|\alpha|^X := \inf \left\{ \left| \sum_{j=1}^{k} f_j \otimes \sigma_j \right|^X \;\middle|\; \sum_{j=1}^{k} f_j \otimes \sigma_j \in L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{Z}, \mathbb{Z}) \text{ is a cycle representing } \alpha \right\}.$$

Note that by the above remark we could as well take the infimum over all reduced cycles representing the homology class.

Via this parametrised norm we can define the integral foliated simplicial volume of a manifold (see [37, Definitions 4.1 and 4.2]).

Definition 2.1.16 (X-parametrised fundamental class)**.** For an oriented, closed and connected d-dimensional manifold M let $\widetilde{M}$ be its universal cover and $\pi_1(M) = \Gamma$ be its fundamental group. Suppose (X, μ) is a standard Γ-space.

The inclusion $\mathbb{Z} \hookrightarrow L^\infty(X, \mathbb{Z})$ as constant functions induces a change of coefficient homomorphism

$$i_M^X \colon C_*(M, \mathbb{Z}) \cong \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M}, \mathbb{Z}) \to L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M}, \mathbb{Z}) = C_*(M, L^\infty(X, \mathbb{Z}))$$
$$1 \otimes \sigma \mapsto 1 \otimes \sigma$$

The class $[M]^X := H_d(i_M^X)([M]_\mathbb{Z})$ in $H_d(L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M}, \mathbb{Z})) = H_d(M, L^\infty(X, \mathbb{Z}))$ is called *X-parametrised fundamental class of M*. The cycles in $L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_d(\widetilde{M}, \mathbb{Z}) = C_d(M, L^\infty(X, \mathbb{Z}))$ representing $[M]^X$ are called *X-parametrised fundamental cycles.*

Definition 2.1.17 (Integral foliated simplicial volume)**.** Let M be an oriented, closed and connected d-dimensional manifold with $\pi_1(M) = \Gamma$ and let (X, μ) be a standard Γ-space. The *X-parametrised simplicial volume of M* is defined as

$$|M|^X := |[M]^X|^X.$$

It is the infimum of the parametrised ℓ^1-norms of all X-parametrised fundamental cycles of M.

The *integral foliated simplicial volume of M* is defined as

$$|M| := \inf_X |M|^X,$$

where the infimum is over all isomorphism classes of standard Γ-spaces (X, μ).

Remark 2.1.18. The class of isomorphism classes of standard Γ-spaces forms indeed a set if the group Γ is countable [47, Remark 5.26].

In the original definition [47, Definition 5.25], the action of the fundamental group on the parameter space is required to be essentially free. As shown in [37, Corollary 4.14], the infimum is not affected if one also allows actions which are not essentially free. Moreover, the infimum is attained.

The integral foliated simplicial volume fits into the following sandwich of simplicial volume and integral simplicial volume.

Proposition 2.1.19. *[47, Remark 5.23] Suppose M is an oriented, closed and connected d-dimensional manifold with fundamental group Γ and (X,μ) is a standard Γ-space. Then the following holds:*

$$\|M\| \leqslant |M| \leqslant |M|^X \leqslant \|M\|_{\mathbb{Z}}.$$

Proof. We recall the proof of this proposition from [47, Remark 5.23]. For a standard Γ-space (X,μ) let $i_M^X\colon C_*(M,\mathbb{Z}) \cong \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M},\mathbb{Z}) \to L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M},\mathbb{Z})$ be the change of coefficients homomorphism induced by the inclusion $\mathbb{Z} \hookrightarrow L^\infty(X,\mathbb{Z})$ as constant functions. Let $\sum_{j=1}^k a_j\sigma'_j \in C_*(M,\mathbb{Z})$ be a fundamental cycle, i.e. a representative of $[M]_{\mathbb{Z}} \in H_d(M,\mathbb{Z})$. Then $i_M^X(\sum_{j=1}^k a_j\sigma'_j)$ is by definition an X-parametrised fundamental cycle of M. It is given by

$$i_M^X(\sum_{j=1}^k a_j\sigma'_j) = \sum_{j=1}^k a_j\chi_X \otimes \sigma_j$$

where $\sigma_j\colon \Delta_d \to \widetilde{M}$ is a lift of the simplex σ'_j for $j = 1,\dots,k$. The parametrised fundamental cycle is independent of the choice of lifts. We obtain

$$\begin{aligned} |M|^X &\leqslant \big|i_M^X(\sum_{j=1}^k a_j\sigma'_j)\big|^X = \big|\sum_{j=1}^k a_j\chi_X \otimes \sigma_j\big|^X \\ &\leqslant \sum_{j=1}^k \int_X |a_j\chi_X(x)|\, d\mu(x) = \sum_{j=1}^k |a_j| = \|\sum_{j=1}^k a_j\sigma'_j\|_1. \end{aligned}$$

By taking the infimum over all $\mathbb{Z}$-fundamental cycles of M this yields

$$|M| \leqslant |M|^X \leqslant \|M\|_{\mathbb{Z}}.$$

To see that the integral foliated simplicial volume bounds the simplicial volume from above we prove for every standard Γ-space (X,μ) that $\|M\| \leqslant |M|^X$.

Let $\pi\colon \widetilde{M} \to M$ denote the universal cover. Consider the homomorphism

$$\begin{aligned} \rho\colon L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_d(\widetilde{M},\mathbb{Z}) &\longrightarrow C_d(M,\mathbb{R}) \\ f\otimes\sigma &\longmapsto \Big(\int_X f(x)\,d\mu(x)\Big)(\pi\circ\sigma), \end{aligned}$$

which is well-defined since

$$\begin{aligned} \rho(\gamma(f\otimes\sigma)) = \rho(f\cdot\gamma^{-1}\otimes\gamma\sigma) &= \Big(\int_X f(\gamma^{-1}x)\,d\mu(x)\Big)(\pi\circ\gamma\sigma) \\ &= \Big(\int_X f(x)\,d\mu(x)\Big)(\pi\circ\sigma) = \rho(f\otimes\sigma) \end{aligned}$$

This holds since the group action on (X,μ) is measure-preserving. We obtain the following commutative diagram

$$\begin{tikzcd} C_d(M,\mathbb{Z}) \arrow[r, "i_M^X"] \arrow[d, hook, "\iota"'] & L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_d(\widetilde{M},\mathbb{Z}) \arrow[dl, "\rho"] \\ C_d(M,\mathbb{R}) & \end{tikzcd}$$

where $\iota\colon C_d(M,\mathbb{Z}) \hookrightarrow C_d(M,\mathbb{R})$ is induced by the inclusion of coefficients. Let $\sum_{j=1}^k f_j\otimes\sigma_j$ be an X-parametrised fundamental cycle in reduced form. It represents $H_d(i_M^X)([M]_\mathbb{Z})$. Then $\rho(\sum_{j=1}^k f_j\otimes\sigma_j) = \sum_{j=1}^k(\int_X f_j(x)\,d\mu(x))(\pi\circ\sigma_j)$ represents

$$H_d(\rho)H_d(i_M^X)([M]_\mathbb{Z}) = H_d(\iota)([M]_\mathbb{Z}) = [M] \in H_d(M,\mathbb{R}).$$

We obtain

$$\begin{aligned} \|M\| &\leqslant \Big\|\rho\Big(\sum_{j=1}^k f_j\otimes\sigma_j\Big)\Big\|_1 = \Big\|\sum_{j=1}^k \Big(\int_X f_j(x)\,d\mu(x)\Big)(\pi\circ\sigma_j)\Big\|_1 \\ &= \sum_{j=1}^k \Big|\int_X f_j(x)\,d\mu(x)\Big| \leqslant \sum_{j=1}^k \int_X |f_j(x)|\,d\mu(x) = \Big|\sum_{j=1}^k f_j\otimes\sigma_j\Big|^X, \end{aligned}$$

where we used that $\sum_{j=1}^k f_j\otimes\sigma_j$ is in reduced from. By taking the infimum over all X-parametrised fundamental cycles in reduced form we obtain $\|M\| \leqslant |M|^X$. This holds for every standard Γ-space, thus $\|M\| \leqslant |M|$ which concludes the proof. $\square$

We recall the relation between parametrised fundamental cycles and locally finite fundamental cycles of the universal cover $\widetilde{M}$. For this we introduce the following definition from [37, Definition 4.19 and Remark 4.20], which allows us to work on "strict" function spaces and avoid problems with sets of measure zero.

Remark 2.1.20. Let Z be a topological space with universal cover $\widetilde{Z}$ and fundamental group $\pi_1(Z) = \Gamma$ and (X, μ) be a standard Γ-space.

Let $B(X, \mathbb{Z})$ be the abelian group of all bounded measurable functions $X \to \mathbb{Z}$ which are everywhere defined. Denote the subgroup of functions which vanishe μ-a.e. by $N(X, \mu, \mathbb{Z})$. We equip $B(X, \mathbb{Z})$ and $N(X, \mu, \mathbb{Z})$ with the obvious right-action. Then

$$L^\infty(X, \mu, \mathbb{Z}) \cong \frac{B(X, \mathbb{Z})}{N(X, \mu, \mathbb{Z})}.$$

Moreover, we get the following isomorphism of $\mathbb{Z}$-chain complexes

$$L^\infty(X, \mu, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{Z}, \mathbb{Z}) \cong \frac{B(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{Z}, \mathbb{Z})}{N(X, \mu, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{Z}, \mathbb{Z})},$$

since $C_*(\widetilde{Z}, \mathbb{Z})$ is a free hence flat $\mathbb{Z}\Gamma$-chain complex.

Proposition 2.1.21. *Assume Z is a topological space allowing a universal cover $\widetilde{Z}$. Let $\pi_1(Z) = \Gamma$ and (X, μ) be a standard Γ-space.*

For a chain $c = \sum_{j=1}^k f_j \otimes \sigma_j \in L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{Z}, \mathbb{Z})$ define

$$ev_x := \sum_{\gamma \in \Gamma} \sum_{j=1}^k f_j(\gamma^{-1}x)\gamma\sigma_j.$$

For a.e. $x \in X$, this assignment defines a well-defined chain map

$$ev_x \colon L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{Z}, \mathbb{Z}) \to C_*^{\mathrm{lf}}(\widetilde{Z}, \mathbb{Z}).$$

Proof. For a.e. $x \in X$ there is a well-defined evaluation map

$$\begin{aligned} ev_x \colon B(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{Z}, \mathbb{Z}) &\to C_*^{\mathrm{lf}}(\widetilde{Z}, \mathbb{Z}) \\ f \otimes \sigma &\mapsto \sum_{\gamma \in \Gamma} f(\gamma^{-1}x)\gamma\sigma. \end{aligned}$$

The sum on the right hand side is locally finite, since Γ acts on $\widetilde{Z}$ properly discontinuously by deck transformations. The assignment is compatible with the boundary operator, i.e. the square

$$\begin{tikzcd} B(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{Z}, \mathbb{Z}) \arrow[r, "ev_x"] \arrow[d, "\mathrm{id} \otimes \partial_n"'] & C_n^{\mathrm{lf}}(\widetilde{Z}, \mathbb{Z}) \arrow[d, "\partial_n"] \\ B(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_{n-1}(\widetilde{Z}, \mathbb{Z}) \arrow[r, "ev_x"] & C_{n-1}^{\mathrm{lf}}(\widetilde{Z}, \mathbb{Z}) \end{tikzcd}$$

is commutative for a.e. $x \in X$ and every $n \in \mathbb{N}$. For a.e. $x \in X$ we have $ev_x(f \otimes \sigma) = 0$ for $f \otimes \sigma \in N(X, \mu, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{Z}, \mathbb{Z})$. This holds true since $f = 0$ on a subset $N \subseteq X$ of measure 1 and $\mu(\bigcap_{\gamma \in \Gamma} \gamma N) = 1$ since the group is countable. Hence $f(\gamma^{-1}x) = 0$ for a.e. $x \in X$ and every $\gamma \in \Gamma$. As a result, the above map descends to a well-defined chain map

$$ev_x \colon L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{Z}, \mathbb{Z}) \to C_*^{\mathrm{lf}}(\widetilde{Z}, \mathbb{Z}).$$

$\square$

For manifolds, evaluation of X-parametrised fundamental cycles yields locally finite fundamental cycles of the universal cover. We state the following lemma [19, Lemma 2.5] without proof.

Lemma 2.1.22. *Let M be an oriented, closed and connected d-dimensional manifold with $\pi_1(M) = \Gamma$ and let (X, μ) be a standard Γ-space. Let $c = \sum_{j=1}^k f_j \otimes \sigma_j$ be an X-parametrised fundamental cycle of M. Then the chain*

$$c_x := ev_x(c) = \sum_{\gamma \in \Gamma} \sum_{j=1}^{k} f_j(\gamma^{-1}x)\gamma\sigma_j$$

is a well-defined locally finite fundamental cycle of $\widetilde{M}$ for a.e. $x \in X$.

As indicated in Chapter 1, the integral foliated simplicial volume provides an upper bound for the L^2-Betti numbers of the manifold. This relation was conjectured by Gromov and proved by Schmidt. We state the theorem without proof.

Theorem 2.1.23. *[47, Corollary 5.28] For an oriented, closed and connected manifold of dimension d, the following connection holds between the L^2-Betti numbers and the integral foliated simplicial volume:*

$$b_k^{(2)}(M) \leqslant \binom{d+1}{k} \left| M \right| .$$

Consequently,

$$\sum_{k=0}^{d} b_k^{(2)}(M) \leqslant 2^{d+1} \left| M \right| .$$

In the theorem we don't assume that the manifold is aspherical. Note that asphericity is a necessary condition in Gromov's question. While the simplicial volume of the sphere S^2 vanishes, the L^2-Betti numbers do not vanish in all degrees. We have

$b_0^{(d)}(S^2) = b_2^{(2)}(S^2) = 1$. In view of the previous theorem, to answer the question in the affirmative, one would have to establish universal inequalities between the simplicial volume and the integral foliated simplicial volume for aspherical manifolds.

There are only a few manifolds for which the integral foliated simplicial volume is known. We conclude this section with the most basic examples. For more results we refer to the literature [19, 37, 47].

Proposition 2.1.24. *The integral foliated simplicial volume of an oriented, closed, connected and simply connected manifold coincides with the integral simplicial volume and it holds*

$$|M| = \|M\|_{\mathbb{Z}} \geqslant 1.$$

For a proof see [47, Proposition 5.29]. This result stands in contrast to the vanishing of the simplicial volume of simply connected manifolds.

Proposition 2.1.25. $|S^1| = 0$.

Proof. This has been proven in [47, Proposition 5.30] by first showing that for an ergodic standard $\mathbb{Z}$-space (X, μ) the X-parametrised simplicial volume of S^1 vanishes. Consequently, we obtain $|S^1| = 0$. The proof is based on a classical result from ergodic theory, the Rokhlin Lemma [1, Theorem 1.5.9, p. 47].

We illustrate the proof by restricting to a specific standard $\mathbb{Z}$-space (X, μ) (see [47, Remark 5.31]). We have $M = S^1$ with fundamental group $\pi_1(M) = \mathbb{Z} = \langle t \rangle$. Let $X = S^1$ with the Lebesgue measure. Then the rotation by an irrational angle gives rise to an ergodic measure-preserving $\mathbb{Z}$-action on $X = S^1$ (see [52, Example 3.2, p. 245]). The $\mathbb{Z}$-fundamental cycle $\sigma_1 \colon [0,1] \to S^1, s \mapsto e^{2\pi i s}$ defines an X-parametrised fundamental cycle $\chi_X \otimes \sigma$, where

$$\begin{aligned} \sigma \colon [0,1] &\to \mathbb{R} \\ s &\mapsto s \end{aligned}$$

is a lift of σ_1 to the universal cover $\widetilde{S^1} = \mathbb{R}$. This fundamental cycle is indicated in Figure 2.2 part (A).

Let $k \in \mathbb{N}, k \geqslant 1$ and $\varepsilon > 0$. Choose an irrational $\alpha \in [0, \frac{1}{k}]$ as angle of rotation such that $1 - \alpha k < \varepsilon$. We regard the diagonal $\mathbb{Z}$-action on the product $X \times \widetilde{M} = S^1 \times \mathbb{R}$, which is given by rotation by α in S^1 and translation by 1 in $\mathbb{R}$. Moreover, $\mathbb{Z}$ acts on the chain module $L^\infty(S^1, \mathbb{Z}) \otimes_{\mathbb{Z}[\mathbb{Z}]} C_1(\mathbb{R}, \mathbb{Z})$ by $t(f \otimes \sigma) = f \cdot t^{-1} \otimes t\sigma$, where $t\sigma \colon [0,1] \to \mathbb{R}$ is defined by $t\sigma(s) = s + t$. By setting $A_1 = [0, \alpha) \subset S^1$ and $A_2 = S^1 \setminus \bigcup_{n=0}^{k-1} t^n A_1 = S^1 \setminus [0, k\alpha)$ we can decompose the canonical parametrised fundamental cycle

$$\chi_X \otimes \sigma = \chi_{A_2} \otimes \sigma + \sum_{n=0}^{k-1} \chi_{t^n A_1} \otimes \sigma. \tag{2.1}$$

This decomposition is indicated in Figure 2.2 (B).

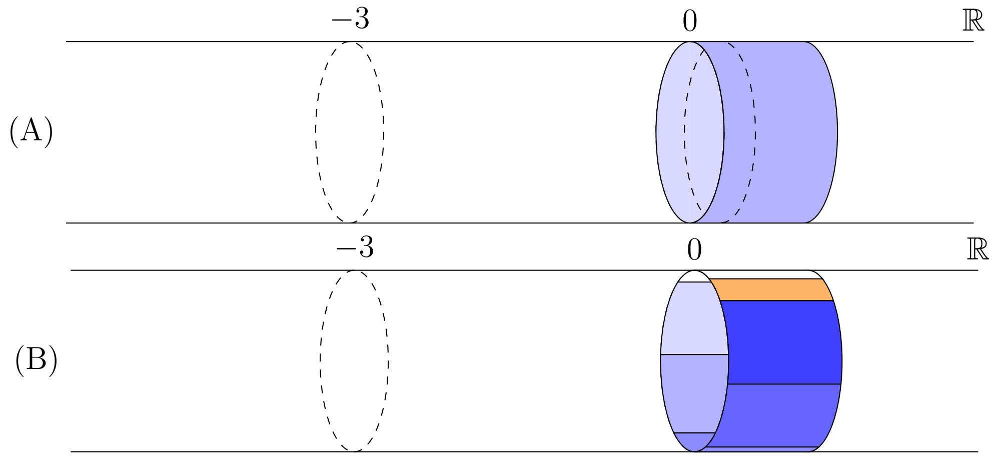

Figure 2.1: Construction of X-parametrised fundamental cycles: (A) indicates the canonical fundamental cycle $\chi_X \otimes \sigma$ which can be decomposed as in (B), see (2.1)*.*

We obtain the following equivalence

$$\chi_{A_2} \otimes \sigma + \sum_{n=0}^{k-1} \chi_{t^n A_1} \otimes \sigma = \chi_{A_2} \otimes \sigma + \sum_{n=0}^{k-1} \chi_{A_1} \otimes t^{-n}\sigma \tag{2.2}$$

(see Figure 2.2 (C)). If we set

$$\begin{aligned} \tilde{\sigma} \colon [0,1] &\to \mathbb{R} \\ s &\mapsto ks - (k-1) \end{aligned}$$

this cycle is homologous to

$$\chi_{A_2} \otimes \sigma + \chi_{A_1} \otimes \tilde{\sigma}. \tag{2.3}$$

As a result, this defines a S^1-parametrised fundamental cycle as well. It is illustrated in Figure 2.2 (D). We obtain

$$\left| \chi_{A_2} \otimes \sigma + \chi_{A_1} \otimes \tilde{\sigma} \right|^{S^1} = \mu(A_2) + \mu(A_1) = (1 - k\alpha) + \alpha < \varepsilon + \alpha < \varepsilon + \frac{1}{k}.$$

Since $\varepsilon > 0$ and $k \geqslant 1$ can be chosen arbitrarily, we can construct S^1-parametrised fundamental cycles of arbitrary small parametrised ℓ^1-norm. $\square$

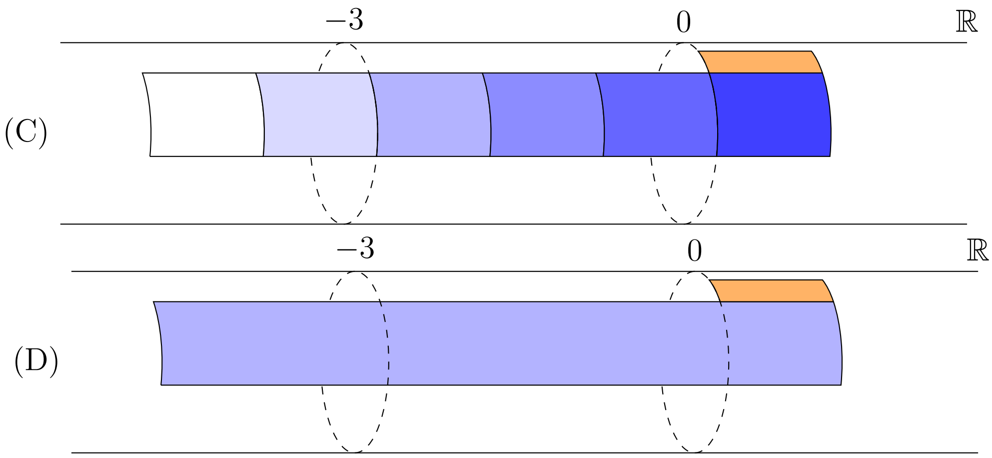

Figure 2.2: Construction of X-parametrised fundamental cycles: (C) indicates the equivalence stated in (2.2) *which yields the parametrised fundamental cycle* (2.3) *as indicated in (D).*

2.2 Cuboid complexes

The metric spaces appearing in this thesis will be either Riemannian manifolds or Euclidean polyhedral complexes. Polyhedral complexes can be seen as a generalization of the geometric realization of a simplicial complex. They are cell complexes, which are constructed by gluing polyhedra isometrically along their sides. The most common examples for such complexes are Euclidean simplicial complexes and cube complexes, which are constructed by gluing geodesic simplices or unit Euclidean cubes, respectively. A precise definition of Euclidean polyhedral complexes and more background on them can be found in the literature [8, Chapter I.7, p. 97-130]. Most of this section is based on what is presented there.

Let $\mathbb{R}^k$ be endowed with the standard Euclidean structure. If we want to emphasize this, we write $\mathbb{E}^k$ for Euclidean k-space.

A cube complex is constructed from a disjoint union of unit cubes, i.e. k-fold products $[0,1]^k$ which are isometric to a cube in Euclidean k-space with side length one, which are glued isometrically along their faces (see [8, Definition I.7.32, p. 112]). We want to mimic this definition but instead of working with unit cubes we want to allow rectangular cuboids, i.e. various side lengths are allowed. The following definitions are analogous to [8, Chapter I.7, p. 111-112].

Definition 2.2.1. A *k-dimensional (rectangular) cuboid with side lengths $a_1, \dots, a_k \in \mathbb{R}_{>0}$* is a metric space which is isometric to a product $\prod_{i=1}^{k}[0, a_i] \subset \mathbb{E}^k$. A 0-*dimensional cuboid* is a point.

For every k-cuboid we can define its set of faces. We consider the cuboid as a face of itself. A 1-dimensional cuboid $[0,a]$ for $a \in \mathbb{R}_{>0}$ has two 0-dimensional faces $\{0\}$ and $\{a\}$ and a 1-dimensional face $[0,a]$. A *face* of the k-cuboid $C = \prod_{i=1}^{k}[0,a_i]$ is given as a product $F_1 \times \ldots \times F_k$ of faces of the $[0,a_i]$. Its dimension is the sum of dimensions of the F_i. Then each k'-dimensional face is a k'-cuboid in its own right. Each face of dimension $k' < k$ is called a *proper face*. The interior of a k-cuboid is given by the points which do not lie in any proper face whereas the union of all proper faces defines the *boundary* of the cuboid. The 0-dimensional faces of a cuboid are called *vertices*. Note that C is the convex hull of its vertices.

Definition 2.2.2. Let $\mathcal{C} = \{C_\lambda\}_{\lambda\in\Lambda}$ be a family of cuboids, i.e. every element C_λ is a k-dimensional cuboid for a $k \in \mathbb{N}$. Let $\sim$ be an equivalence relation on the disjoint union $H = \coprod_{\lambda\in\Lambda} C_\lambda$. We consider $K = H/\sim$ with the natural projection $p\colon H \to K = H/\sim$. Then K is a *(Euclidean) cuboid complex* if the restrictions $p_\lambda := p_{|C_\lambda} : C_\lambda \to K$ satisfy the following:

i) The map p_λ is injective for every $\lambda \in \Lambda$.

ii) For every two elements $\lambda, \lambda' \in \Lambda$ we have: If $p_\lambda(C_\lambda) \cap p_{\lambda'}(C_{\lambda'}) \neq \emptyset$, there is an isometry f from a face $F_\lambda \subseteq C_\lambda$ onto a face $F_{\lambda'} \subseteq C_{\lambda'}$ such that $p_\lambda(q) = p_{\lambda'}(q')$ if and only if $f(q) = q'$.

Hence a cuboid complex is constructed from a disjoint union of cuboids by gluing faces. Condition i) ensures that sides are not folded. It allows us to identify a cuboid with its image in K. A subset $C \subset K$ is called a *k-dimensional cuboid* of K if it is the image under some p_λ of a k-dimensional face of C_λ. A subset of C is called a *face* if its preimage under p_λ is a face of $p_\lambda^{-1}(C)$. The second condition in the above definition ensures that cuboids are glued via isometries along their faces. The intersection of two cuboids in K is empty or a single face.

We fix some more notation. The *interior* of a cuboid is the image of the interior of its preimage. We denote it by $\mathring{C}$. Then the *boundary* is defined as $\partial C := C \backslash \mathring{C}$; it is the union of all proper faces of C. The *(open) carrier* of a point $x \in K$ is the interior of the cuboid containing x in its interior and is denoted by $\operatorname{carr}(x)$. For a cuboid $C \in K$ define the *open star* $\operatorname{Star}(C)$ as the union of the interiors of the cuboids containing C. This means the open star consists of C and the interiors of all higher dimensional cuboids which contain C as a face. It holds that $C \subseteq \overline{\operatorname{carr}(x)}$ for a $x \in K$ if and only if $x \in \operatorname{Star}(C)$.

Note that we can view every cuboid complex as a CW complex where the k-cuboids define the k-cells. Then we have the weak topology on the complex. Hence a subset is closed if and only if it meets each cuboid in a closed set.

Metric on cuboid complexes

Let K be a cuboid complex constructed from a family of cuboids $\{C_\lambda\}_{\lambda\in\Lambda}$. Each C_λ is equipped with a metric d_{C_λ}, which is given by restricting the standard Euclidean metric. Let $C \subset K$ be a k-cuboid, i.e. the image of some k-face of a C_λ under p_λ. Define a metric d_C on C by

$$d_C(p_\lambda(q), p_\lambda(q')) = d_{C_\lambda}(q, q').$$

By the second condition in Definition 2.2.2 this is well-defined. For a face $F \subset C$, d_F is the restriction of d_C to F. The cuboid complex $K = \coprod_{\lambda\in\Lambda} C_\lambda/\sim$ is equipped with the quotient pseudometric d associated to the natural projection $p\colon \coprod_{\lambda\in\Lambda} C_\lambda \to K$ (see [8, Definition I.5.19, p. 65]). As explained in [8, I.7.38, p. 114], one easily verifies that this is equivalent to define $d(q, q')$ via the lengths of piecewise geodesic paths in K joining the points. A *piecewise geodesic path in K* is a map $g\colon [a, b] \to K$ together with a subdivision $a = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n = b$ and geodesic paths $g_i\colon [t_i, t_{i+1}] \to C_{\lambda_i}$ such that for each $t \in [t_i, t_{i+1}]$ we have $g(t) = p_{\lambda_i}(g_i(t))$. The length $l(g)$ is given as the sum of the lengths of the g_i, which is independent of the subdivision. Then define for $q, q' \in K$

$$d(q, q') \coloneqq \inf\{l(g) \mid g \text{ piecewise geodesic path from } q \text{ to } q'\}.$$

One can show that with some minor assumption on a cuboid complex this defines indeed a metric and (K, d) is a length space. For general polyhedral complexes this is the case if there are only finitely many isometry classes of faces. In dealing with cuboid complexes we have a weaker assumption:

Lemma 2.2.3. *Let K be a cuboid complex built from a family of cuboids such that the infimum of all appearing side lengths is positive. Then the above pseudometric is a metric and (K, d) is a length space.*

Proof. This follows from [8, Lemma I.7.9 and Corollary I.7.10, p. 100-101] if we can show that the following number is strictly positive for every $q \in K$:

$$\varepsilon(q) \coloneqq \inf\{\varepsilon(q, C) \mid C \subset K \text{ cuboid containing } q\}$$

with $\varepsilon(q, C) := \inf\{d_C(q, F) \mid F \subset C \text{ face and } q \notin F\}$. In the case that $C = \{q\}$, we set $\varepsilon(q, C) := \infty$. If $C = \overline{\operatorname{carr}(q)}$, then $\varepsilon(q, C) = d_C(q, \partial C)$, which is positive since q is an inner point of C. In all other cases $\varepsilon(q, C) \geqslant \inf\{a \mid a \text{ side length of a cuboid}\}$, which is positive by assumption. This yields $\varepsilon(q) > 0$ for all points in K. $\square$

As a result, provided we have a strictly positive lower bound on the appearing side lengths, a cuboid complex is equipped with the length metric that restricts to the standard Euclidean metric on cuboids. We will also talk about *metric cuboid complexes*. The topology obtained from the metric coincides with the weak topology if the cuboid complex is locally finite.

Barycentric Subdivision

As described in [8, Chapter I.7, p. 115-118] any Euclidean polyhedral complex can be subdivided to obtain an isometric simplicial complex (see [8, Definition I.7.2, p. 98]). This is done by subdividing each cuboid into geodesic simplices. For this we define:

Definition 2.2.4. Let $C = \prod_{i=1}^{k}[0, a_i]$ be a k-cuboid with side lengths $a_1, \ldots, a_k$. Define its *barycentre* by $b_C = (\frac{1}{2}a_1, \ldots, \frac{1}{2}a_k)$. The barycentre of a 0-cuboid is the point itself. Let $F \subset C = \prod_{i=1}^{k}[0, a_i]$ be a face, i.e. a product $F = F_1 \times \ldots \times F_k$ of faces of the $[0, a_i]$. Then the barycentre of F is given as $b_F = b_{F_1} \times \ldots \times b_{F_k}$. Under the identification $F \cong \prod_{j=1}^{k'}[0, a_{i_j}]$ this point corresponds to $(\frac{1}{2}a_{i_1}, \ldots, \frac{1}{2}a_{i_{k'}})$.

A k-cuboid C is the convex hull of its 2^k vertices $v_1, \ldots, v_{2^k} \in \mathbb{R}^k$. Then $b_C = \frac{1}{2k}\sum_{i=1}^{2^k} v_i$. By [8, Lemma I.7.43, p. 116] it lies in the interior of C and is fixed by any isometry.

Let C be a k-cuboid and $F_0, \ldots, F_{k'}$ be faces of C. We say $F_0 \subset F_1 \ldots \subset F_{k'}$ is a *strictly ascending sequence* of faces, if $\dim(F_{i+1}) = \dim(F_i) + 1$.

Definition 2.2.5. [8, I.7.44, p. 116] Let C be a k-cuboid. For every strictly ascending sequence $F_0 \subset F_1 \ldots \subset F_{k'}$ of faces define a geodesic simplex as the convex hull of the barycentres of the F_i. The intersection of two simplices is a simplex as well. Consider the disjoint union of the simplices corresponding to strictly ascending sequences and the natural map from this disjoint union to C. This defines the structure of a Euclidean simplicial complex on C. We call this complex the *(first) barycentric subdivision* of C and denote it by $\operatorname{Sd}(C)$.

Note that the barycentric subdivision of a k-cuboid C consists of $2^k k!$ many simplices of dimension k. These correspond to the strictly ascending sequences $F_0 \subset F_1 \ldots \subset F_k = C$. The barycentric subdivision of a 2-dimensional cuboid is illustrated in Figure 2.3.

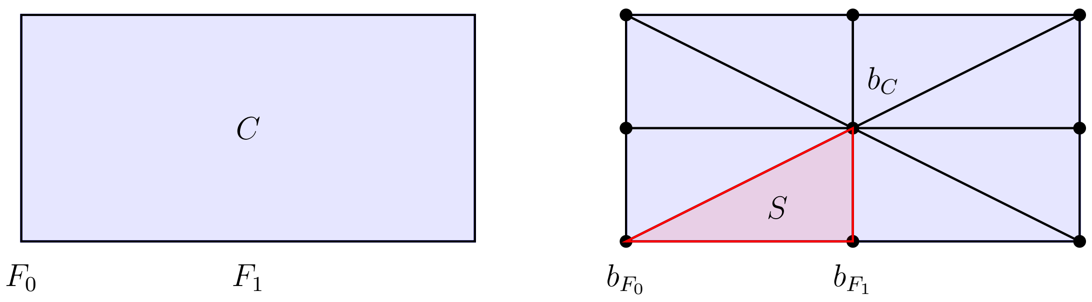

Figure 2.3: Barycentric subdivision of a 2-dimensional cuboid: The red simplex S corresponds to a strictly ascending sequence $F_0 \subset F_1 \subset F_2 = C$.

We obtain the barycentric subdivision of a cuboid complex by subdividing each cuboid (see [8, I.7.47, p. 117]). Let K be a cuboid complex obtained from a family $\{C_\lambda\}_{\lambda\in\Lambda}$ of cuboids. For each cuboid C_λ we consider the set of simplices in the barycentric subdivision $\mathrm{Sd}(C_\lambda)$. Denote this set $\{S_i\}_{i\in I_\lambda}$ for an index set I_λ, so $\mathrm{Sd}(C_\lambda)$ is the simplicial complex resulting as a quotient of the disjoint union $\coprod_{i\in I_\lambda} S_i$ and set $\Lambda' = \bigcup_{\lambda\in\Lambda} I_\lambda$. We consider the natural maps $\coprod_{i\in I_\lambda} S_i \to C_\lambda$ and $p\colon \coprod_{\lambda\in\Lambda} C_\lambda \to K$. Their composition yields a projection $p'\colon \coprod_{i\in\Lambda'} S_i \to K$.

Definition 2.2.6. Define an equivalence relation $\sim$ on the disjoint union $\coprod_{i\in\Lambda'} S_i$ by $q \sim q'$ if and only if $p'(q) = p'(q')$. Then $\coprod_{i\in\Lambda'} S_i/\sim$ is called the *(first) barycentric subdivision of* K and is denoted by $\mathrm{Sd}(K)$.

By [8, Lemma I.7.45,46 and 48, p. 116-117] we obtain the following.

Lemma 2.2.7. *The barycentric subdivision* $\mathrm{Sd}(K)$ *is a simplicial complex obtained from the family of simplices* $\{S_i\}_{i\in\Lambda'}$. *There a natural identification of sets* $K \to \mathrm{Sd}(K)$, *which is an isometry.*

In particular, the restriction of p' to each simplex S_i is injective. In order to see that the first barycentric subdivision of a cuboid complex is indeed a simplicial complex and not just a polyhedral complex, observe that an isometry between cuboids induces a simplicial isometry of their barycentric subdivisions. Hence simplices are glued via isometries. In the cuboid complex K two cuboids either intersect in a single face or not at all, which ensures that the same holds true for the simplices in $\mathrm{Sd}(K)$.

Cuboid complex associated to a set of real numbers

We conclude this section by introducing a specific cuboid complex, which will appear later in the nerve construction (see Section 2.5 and Section 4.1). Given a set of positive real numbers $\{a_j\}_{j\in J}$ indexed over a countable set J we construct a cuboid complex which can be seen as a subcomplex of the 'infinite cuboid' $\coprod_{j\in J}[0,a_j]$. We first fix a set of vertices, i.e. 0-cuboids, and then inductively glue edges and higher dimensional cuboids. To do this in a precise way, recall the definition of the generalized Euclidean space $\mathbb{E}^J$ (see [43, Chapter 1 §2, p. 13]). Let $\mathbb{R}^J$ be the J-fold product of $\mathbb{R}$ with itself. An element is written as a tuple $(y_j)_{j\in J}$. In order to get a Euclidean structure, we restrict to the points which have only finitely many non-vanishing entries. This set is denoted by $\mathbb{E}^J$. It has the structure of a vector space with component-wise addition and scalar multiplication and we can equip it with a Euclidean metric. For $y=(y_j)_{j\in J}$ and $y'=(y'_j)_{j\in J}$ we have

$$d(y,y') := \left(\sum_{j\in J}(y_j-y'_j)^2\right)^{1/2}$$

A basis of $\mathbb{E}^J$ is given by $\mathcal{B}=\{v_i\}_{i\in J}$ where $v_i := (y^i_j)_{j\in J}$ with $y^i_j=1$ if $j=i$ and $y^i_j=0$ otherwise. The space $\mathbb{E}^J$ is a union of the finite-dimensional subspaces spanned by finite subsets of $\mathcal{B}$. These are copies of a $\mathbb{R}^k$ for some k, the restriction of the above metric is the standard Euclidean metric on $\mathbb{R}^k$.

Any finite set of points of $\mathbb{E}^J$ is contained in a subspace as described. Moreover, their convex hull is contained in this subspace. Hence a k-cuboid with vertex set in $\mathbb{E}^J$ lies in a copy of $\mathbb{R}^n$ for some n. In the same way, any finite collection of cuboids lies in a finite dimensional Euclidean space.

Now let $\{a_j\}_{j\in J}$ be a set of positive real numbers. In order to construct a cuboid complex, we first fix a set of vertices

$$V := \{(y_j)_{j\in J}\,|\,(y_j)_{j\in J}\neq 0,\ y_j=0 \text{ or } a_j \text{ with } y_j\neq 0 \text{ only for finitely many } j\in J\}\subset \mathbb{E}^J.$$

Two vertices are *adjacent* if they differ in exactly one entry. Note that any two adjacent vertices v_1 and v_2 lie in a subspace of $\mathbb{E}^J$ spanned by finitely many basis vectors in $\mathcal{B}$. We define an edge between v_1 and v_2 as the Euclidean line segment between these points. If v_1 and v_2 differ in the j-th entry, this edge is isometric to $[0,a_j]$ hence is a 1-cuboid of length a_j. We then glue in higher dimensional cuboids whenever the 1-skeleton is defined. Here 2^k vertices in V span a k-cuboid if each vertex is adjacent to exactly k of these vertices. In this case the vertices are contained in a copy of $\mathbb{R}^n$ and there are k indices

$j_1, \ldots, j_k$ such that the edges between the vertices are of side lenghts $a_{j_1}, \ldots, a_{j_k}$. The k-cuboid spanned by the vertices is contained in the same subspace and is isomorphic to $\prod_{i=1}^k [0, a_{j_i}]$. This procedure yields an infinite cuboid complex Z which is a subspace of $\mathbb{E}^J$.

Definition 2.2.8. The constructed cuboid complex Z is called the *cuboid complex associated to the set* $\{a_j\}_{j \in J}$.

If $\inf_{j\in J} a_j > 0$, Z is equipped with the unique path length metric restricting to the Euclidean standard metric on each cuboid. In general, the topology induced by this metric does not coincide with the weak topology but for locally compact subcomplexes of Z this is the case.

There is a one-to-one correspondence between the cuboids of Z and their barycentres. A k-cuboid $C \subset Z$ is spanned by 2^k vertices $v_1, \ldots, v_k \in \mathbb{E}^J$ and its barycentre is given by $\frac{1}{2^k}\sum_{i=1}^{2^k} v_i$. Under the identification $C \cong \prod_{i=1}^k [0, a_{j_i}]$ this corresponds to the point $(\frac{1}{2}a_{j_1} \ldots, \frac{1}{2}a_{j_k})$. For $k \geqslant 0$ the set of barycentres of k-faces of Z is then given by

$$V_k := \left\{(y_j)_{j\in J} \in Z \,|\, y_j = 0, \tfrac{1}{2}a_j \text{ or } a_j \text{ with exactly } k \text{ entries } \tfrac{1}{2}a_j\right\}$$

and we have $V_0 = V$.

Remark 2.2.9. We can see Z as a subcomplex of an infinite cuboid with side lengths $\{a_j\}_{j\in J}$. With this in mind we call a k-cuboid in Z a *closed k-face* of the cuboid complex. If we talk about an *(open) k-face* in Z we mean the interior of a k-cuboid.

2.3 The category of equivariant simple X-spaces

In Sections 2.3 - 2.5 let (X, μ) be a standard Borel probability space (X, μ) equipped with an atom-free probability measure μ. Suppose Γ is a countable discrete group acting on (X, μ) in a measurable μ-preserving way, i.e. (X, μ) is a standard Γ-space. Further, we require the action to be (essentially) free, i.e. the set of elements in X with non-trivial stabilizer is a μ-null set.

As remarked in Chapter 1, the framework used in [45] is too strong for our purposes. The proof in this article was conducted using the category of $\mathcal{R}$-spaces, where $\mathcal{R}$ is the orbit equivalence relation attributed to a standard Γ-space (X, μ) [45, Section 2]. In our setting, the $\mathcal{R}$-spaces occurring will be of the form $X \times Z$ with Z being a free Γ-CW complex. In this case, the $\mathcal{R}$-action corresponds to the diagonal group actions. Based on this consideration, we introduce a new categroy covering these spaces. Assigned to the standard Borel space (X, μ) we consider a category of spaces fibred over X with a measurable projection to X. Maps between such spaces are measurable maps which preserve the fibre and are fibrewise continuous. More precisely, we have the following definition.

Definition 2.3.1. Given a topological space Z, the product space $X \times Z$ with the product Borel structure together with the measurable projection to the first component $p_Z \colon X \times Z \to X$ is called *simple X-space*. For $x \in X$, $p_Z^{-1}(x) = \{x\} \times Z \cong Z$ is the *fibre* over x. A *map of simple X-spaces* is a Borel map $\Psi \colon X \times Z \to X \times Z'$ such that $p_{Z'} \circ \Psi = p_Z$ and the restrictions to the fibres $\Psi_x \colon Z \to Z'$ are continuous for a.e. $x \in X$. A *subspace* of a simple X-space is a subset $H \subset X \times Z$ such that for a.e. $x \in X$ the fibre $H_x := p_Z^{-1}(x) \cap H$ is a subspace of Z. If Z is a CW-complex, then H is a *subcomplex* if a.e. fibre H_x is a subcomplex of Z.

We need the following notation.

Definition 2.3.2. A *finite Borel partition* of a measure space (X, μ) is a finite family $\{X_l\}_{l \in \{1,\ldots,L\}}$ of Borel subsets of X such that $\mu(X \setminus \bigcup_{l=1}^{L} X_l) = 0$ and $\mu(X_l \cap X_k) = 0$ for $l \neq k$. We write $X = \bigcup_{l=1}^{L} X_l$ (by abuse of notation).

Based on the definition of countable variance in [45, Definition 2.20] we then define the following property of a map between simple X-spaces.

Definition 2.3.3. A map $\Psi \colon X \times Z \to X \times Z'$ between simple X-spaces is of *finite variance*, if for any product set $A \times K$, with a Borel set $A \subseteq X$ and a compact set $K \subseteq Z$, there is a finite Borel partition $A = \bigcup_{l=1}^{N} A_l$ such that the restriction of Ψ to $A_l \times K$ is a product. This means for every $l \in \{1, \ldots, N\}$ we have $\Psi_{|A_l \times K} = \mathrm{id}_{A_l} \times \psi_l$, where ψ_l is a continuous map $\psi_l \colon Z \to Z'$.

In particular, finite variance implies measurability of a map.

As remarked before, we want to consider spaces as described equipped with an action by the group Γ. If Z is a Γ-space, i.e. Γ acts continuously on Z (from the left), we look at the diagonal left Γ-action on the simple X-space $X \times Z$

$$\gamma(x, z) = (\gamma x, \gamma z) \text{ for } \gamma \in \Gamma, (x, z) \in X \times Z.$$

This is a measurable Γ-action on $X \times Z$ such that $\gamma p_Z^{-1}(x) \subseteq p_Z^{-1}(\gamma x)$. Given two simple X-spaces equipped with such a diagonal Γ-action, a map $\Psi \colon X \times Z \to X \times Z'$ is *equivariant* if it is equivariant on a subset of full measure, i.e. for a.e. $x \in X$ and $z \in Z$ we have $\Psi(\gamma(x,z)) = \gamma\Psi(x,z)$ for $\gamma \in \Gamma$. With this in mind we define the category of equivariant simple X-spaces as follows.

Definition 2.3.4. For a standard Γ-space (X,μ) and a topological Γ-space Z the *equivariant simple X-space* $X \times Z$ is the simple X-space $X \times Z$ equipped with the diagonal Γ-action. A Γ-invariant subspace of $X \times Z$ is a *subspace* of the equivariant simple X-space. If Z is a Γ-CW complex, then a *subcomplex* of the equivariant simple X-space $X \times Z$ is a Γ-invariant subspace such that a.e. fibre is a subcomplex of Z.

The morphisms between equivariant simple X-spaces are maps of the underlying simple X-spaces which are Γ-equivariant in the above sense. Furthermore we require them to be of finite variance. The morphisms are called *equivariant X-maps.*

Definition 2.3.5. An equivariant X-map which is in addition proper on a.e. fibre is called an *equivariant geometric map.*

Lemma 2.3.6. *The composition of equivariant X-maps is an equivariant X-map. Moreover the composition of equivariant geometric maps is an equivariant geometric map.*

Proof. We only need to show that the composition of two maps of finite variance has the same property. Let $\Psi\colon X \times Z \to X \times Z'$ and $\Theta\colon X \times Z' \to X \times Z''$ be two equivariant X-maps and $A \times K \subseteq X \times Z$ with K compact. There is a finite Borel partition $A = \bigcup_{l=1}^{N} A_l$ such that $\Psi_{|A_l \times K} = \mathrm{id}_{A_l} \times \psi_l$. Regard the subsets $\mathrm{im}(\Psi_{|A_l \times K}) = A_l \times \psi_l(K) \subseteq X \times Z'$ and note that $\psi_l(K)$, $l = 1, \ldots, N$, is compact as well. Thus there are finite Borel partitions $A_l = \bigcup_{q=1}^{N_l} Y_q^l$ for $l = 1, \ldots, N$ such that $\Theta_{|Y_q^l \times \psi_l(K)} = \mathrm{id}_{Y_q^l} \times \vartheta_q^l$. Then $A = \bigcup_{l=1}^{N} \bigcup_{q=1}^{N_l} Y_q^l$ is a finite Borel partition such that the restriction of the composition $\Theta \circ \Psi$ to $Y_q^l \times K$ is a product. We have $\Theta \circ \Psi_{|Y_q^l \times K} = \mathrm{id}_{Y_q^l} \times (\vartheta_q^l \circ \psi_l)$ and $\Theta \circ \Psi$ is of finite variance as well. □

Remark 2.3.7. Let $\Psi\colon X \times Z \to X \times Z'$ be an equivariant X-map. In particular, Ψ is of finite variance. Let K_1, K_2 be two compact subsets of Z. We get finite Borel partitions $X = \bigcup_{l=1}^{L} X_l$ and $X = \bigcup_{m=1}^{M} Y_m$ such that $\Psi_{|X_l \times K_1} = \mathrm{id}_{X_l} \times \psi_l$ and $\Psi_{|Y_m \times K_2} = \mathrm{id}_{Y_m} \times \psi'_m$. Let l, m be indices such that $\mu(X_l \cap Y_m) > 0$. Restricted to $(X_l \cap Y_m) \times (K_1 \cap K_2)$ the map Ψ is a product. We have $\Psi_{|(X_l \cap Y_m) \times (K_1 \cap K_2)} = \mathrm{id} \times \psi_l = \mathrm{id} \times \psi'_m$, i.e. $\psi_{l|K_1 \cap K_2} = \psi'_{m|K_1 \cap K_2}$. This means, if we restrict to the intersection of two compact subsets, the representation of Ψ as a product map is independent of the partition we choose.

Moreover, we have the following fact. Let $K \subset Z$ be a compact subset and $X = \bigcup_{l=1}^{L} X_l$ be an associated finite Borel partition such that $\Psi_{|X_l \times K} = \mathrm{id}_{X_l} \times \psi_l$. Looking at the compact subset $\gamma K \subset Z$ for a $\gamma \in \Gamma$ we can use the partition $X = \bigcup_{l=1}^{L} X'_l$ with $X'_l = \gamma X_l$.

For this partition we have $\Psi_{|X'_l \times \gamma K} = \mathrm{id}_{X'_l} \times \psi'_l$ with $\psi'_l(\gamma k) = \gamma \psi_l(k)$ for all $k \in K$. This follows easily from the equivariance of the map. For an element $(\gamma x, \gamma k) \in \gamma X_l \times \gamma K$ we have $\Psi(\gamma x, \gamma k) = \gamma \Psi(x, k) = \gamma(x, \psi_l(k)) = (\gamma x, \gamma \psi_l(k)) = (\mathrm{id}_{\gamma X'_l} \times \psi'_l)(\gamma x, \gamma k)$.

Definition 2.3.8. A *(Borel) fundamental domain* $\mathcal{F}$ of an equivariant simple X-space $X \times Z$ is a Borel subset whose intersection with every Γ-orbit consists of exactly one element.

Measure on simple X-spaces

Let Γ be a countable group acting on X in a (ess.) free and measure-preserving way. Assume that the topological space Z comes with a measure μ' on its Borel-σ-algebra and let Γ act on Z continously and measure preserving. Then we can equip the equivariant simple X-space $X \times Z$ with a Γ-invariant measure ν, constructed as the product measure of μ and the measure on Z. For a Borel subset $U \subseteq X \times Z$ we obtain

$$\nu(U) = \int_X \mu'(p_Z^{-1}(x) \cap U)\, d\mu(x). \tag{2.4}$$

Since this measure is Γ-invariant the following holds if we regard Borel fundamental domains of $X \times \widetilde{M}$.

Lemma 2.3.9. *For an equivariant simple X-space $X \times Z$ let ν be the Γ-invariant measure as in* (2.4)*. Suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ are two Borel fundamental domains of $X \times Z$. Then we have*

i) $\nu(\mathcal{F}_1) = \nu(\mathcal{F}_2)$.

ii) Let $U \subseteq X \times Z$ be a subspace, so in particular U is Γ-invariant, and let $\mathcal{F}' \subset U$ be a Borel fundamental domain for it. Then we have $\nu(U \cap \mathcal{F}_1) = \nu(U \cap \mathcal{F}_2) = \nu(\mathcal{F}')$.

Proof. The first statement holds since

$$\begin{aligned} \nu(\mathcal{F}_1) &= \nu\left((\bigcup_{\gamma \in \Gamma} \gamma \mathcal{F}_2) \cap \mathcal{F}_1 \right) = \nu\left(\bigcup_{\gamma \in \Gamma} (\gamma \mathcal{F}_2 \cap \mathcal{F}_1) \right) = \nu\left(\bigcup_{\gamma \in \Gamma} (\mathcal{F}_2 \cap \gamma^{-1} \mathcal{F}_1) \right) \\ &= \nu\left(\bigcup_{\gamma \in \Gamma} (\mathcal{F}_2 \cap \gamma \mathcal{F}_1) \right) = \nu\left(\mathcal{F}_2 \cap (\bigcup_{\gamma \in \Gamma} \gamma \mathcal{F}_1) \right) = \nu(\mathcal{F}_2). \end{aligned}$$

In a similar way we can calculate

$$\begin{aligned}\nu(\mathcal{F}_1 \cap U) &= \nu\left((\bigcup_{\gamma\in\Gamma} \gamma\mathcal{F}_2) \cap \mathcal{F}_1 \cap U \right) = \nu\left(\bigcup_{\gamma\in\Gamma} (\gamma\mathcal{F}_2 \cap \mathcal{F}_1 \cap U) \right) \\ &= \nu\left(\bigcup_{\gamma\in\Gamma} (\mathcal{F}_2 \cap \gamma^{-1}(\mathcal{F}_1 \cap U)) \right) = \nu\left(\bigcup_{\gamma\in\Gamma} (\mathcal{F}_2 \cap \gamma\mathcal{F}_1 \cap \gamma U) \right) \\ &\leqslant \nu\left(\mathcal{F}_2 \cap U \cap (\bigcup_{\gamma\in\Gamma} \gamma\mathcal{F}_1) \right) = \nu(\mathcal{F}_2 \cap U),\end{aligned}$$

where we use in the second to last step that $\gamma U \subseteq U$. By interchanging $\mathcal{F}_1$ and $\mathcal{F}_2$ we obtain the other inequality as well. The last statement follows in a similar way using $\bigcup_{\gamma\in\Gamma} \gamma\mathcal{F}' = U$. □

We conclude this section by giving examples of the equivariant simple X-spaces appearing in this thesis and the Γ-invariant measures on them.

Example 2.3.10. Let (M, g) be a d-dimensional Riemannian manifold with fundamental group Γ. Let $\widetilde{M}$ be the universal Riemannian covering with the induced metric. Then Γ acts on $\widetilde{M}$ by deck transformations and this action is by isometries. The equivariant simple X-space $X \times \widetilde{M}$ is equipped with the product of μ and the Riemannian measure vol on $\widetilde{M}$. Note that for any Γ-fundamental domain $\mathcal{F}$ of $X \times \widetilde{M}$ we get $\nu(\mathcal{F}) = \mathrm{vol}(M)$.

The above example is one kind of equivariant simple X-spaces appearing in the proof of the main theorem. The other kind are equivariant simple X-spaces where the fibres are locally finite metric polyhedral complexes. More precisely we will deal with simplicial or cuboid complexes equipped with the unique length metric that restricts to the standard Euclidean metric on faces. Since the fibres are locally finite the weak topology coincides with the topology obtained from the metric. The appearing complexes will be equipped with a Γ-action by isometries.

Let $X \times Z$ be an equivariant simple X-space of this form. For some positive real number $s > 0$ we can regard the s-dimensional Hausdorff measure of Z, which defines a measure on the Borel σ-algebra of Z. For the definition and properties of this measure see [41, Chapter 2, p. 7-19]. If s is a positive integer, the s-dimensional Hausdorff measure restricted to an s-face coincides with the s-dimensional Lebesgue measure [41, Corollary 2.8, p. 16]. Since the Hausdorff measure is defined solely in terms of the metric it behaves well under isometries. Hence the group action will be measure-preserving. Therefore we can equip $X \times Z$ with the product of μ and the s-dimensional Hausdorff measure on Z. We denote the s-dimensional Hausdorff measure by vol_s.

2.4 Geometric maps induce chain maps

We show that equivariant X-maps between equivariant simple X-spaces induce chain maps of specific chain complexes.

Recall that for a standard Γ-space (X,μ) the group of essentially bounded measurable function with integer values $L(X,\mu,\mathbb{Z}) = L^\infty(X,\mathbb{Z})$ is a right $\mathbb{Z}\Gamma$-module (see Remark 2.1.12). If Z is a Γ-space the singular chain complex $C_*(Z,\mathbb{Z})$ is a $\mathbb{Z}\Gamma$-chain complex in a natural way. Then a $\mathbb{Z}\Gamma$-module structure on $L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_n(Z,\mathbb{Z})$ is given by $\gamma(f\otimes\sigma) = (f\cdot\gamma^{-1})\otimes\gamma\sigma$ for $f\otimes\sigma \in L^\infty(X,\mathbb{Z})\otimes_{\mathbb{Z}} C_n(Z,\mathbb{Z})$ and by passing over to the coinvariants, we get a $\mathbb{Z}$-module $L^\infty(X,\mathbb{Z})\otimes_{\mathbb{Z}\Gamma} C_n(Z,\mathbb{Z})$. Hence associated to any equivariant simple X-space $X\times Z$ there is a chain complex of $\mathbb{Z}$-modules $L^\infty(X,\mathbb{Z})\otimes_{\mathbb{Z}\Gamma} C_*(Z,\mathbb{Z})$ with differential $(\mathrm{id}_{L^\infty(X,\mathbb{Z})}\otimes\partial_*)$.

Theorem 2.4.1. *This assignment extends to a functor from the category of equivariant simple X-spaces with equivariant X-maps to the category of $\mathbb{Z}$-chain complexes with chain maps, i.e. any equivariant X-map $\Psi\colon X\times Z \to X\times Z'$ induces a chain map of $\mathbb{Z}$-chain complexes*

$$C_*^X(\Psi)\colon L^\infty(X,\mathbb{Z})\otimes_{\mathbb{Z}\Gamma} C_*(Z,\mathbb{Z}) \longrightarrow L^\infty(X,\mathbb{Z})\otimes_{\mathbb{Z}\Gamma} C_*(Z',\mathbb{Z})$$

such that $C_*^X(\mathrm{id}\times\varphi) = \mathrm{id}_{L^\infty(X,\mathbb{Z})}\otimes C_*(\varphi)$ for a continuous equivariant map $\varphi\colon Z\to Z'$.

Proof. We first show that an equivariant X-map $\Psi\colon X\times Z\to X\times Z'$ of equivariant simple X-spaces induces a chain map of (left) $\mathbb{Z}\Gamma$-chain complexes

$$L^\infty(X,\mathbb{Z})\otimes_{\mathbb{Z}} C_*(Z,\mathbb{Z}) \longrightarrow L^\infty(X,\mathbb{Z})\otimes_{\mathbb{Z}} C_*(Z',\mathbb{Z}).$$

Let $n\in\mathbb{N}$. Given an arbitrary n-chain $\sum_{j=1}^k f_j\otimes\sigma_j \in L^\infty(X,\mathbb{Z})\otimes_{\mathbb{Z}} C_n(Z,\mathbb{Z})$, we can assume the f_j being represented as bounded and we have $f_j = \sum_{i=1}^m a_{j,i}\chi_{A_i}$. For a compact subset $K\subset Z$ with $\bigcup_{j=1}^k \mathrm{im}(\sigma_j)\subseteq K$ there is a Borel partition $X = \bigcup_{l=1}^L X_l$ such that $\Psi_{|X_l\times K} = \mathrm{id}_{X_l}\times\psi_l$ since Ψ is of finite variance. Note that the functions ψ_l are continuous. We define a map $\Psi_n\colon L^\infty(X,\mathbb{Z})\otimes_{\mathbb{Z}} C_n(Z,\mathbb{Z})\to L^\infty(X,\mathbb{Z})\otimes_{\mathbb{Z}} C_n(Z',\mathbb{Z})$ as follows. For $\sum_{j=1}^k f_j\otimes\sigma_j = \sum_{j=1}^k\sum_{i=1}^m a_{j,i}\chi_{A_i}\otimes\sigma_j = \sum_{j=1}^k\sum_{i=1}^m\sum_{l=1}^L a_{j,i}\chi_{A_i\cap X_l}\otimes\sigma_j \in L^\infty(X,\mathbb{Z})\otimes_{\mathbb{Z}} C_n(Z,\mathbb{Z})$ we set

$$\begin{aligned}\Psi_n\Big(\sum_{j=1}^k f_j\otimes\sigma_j\Big) &= \Psi_n\Big(\sum_{j=1}^k\sum_{i=1}^m\sum_{l=1}^L a_{j,i}\chi_{A_i\cap X_l}\otimes\sigma_j\Big) \\ &\coloneqq \sum_{j=1}^k\sum_{i=1}^m\sum_{l=1}^L a_{j,i}\chi_{A_i\cap X_l}\otimes\psi_l(\sigma_j).\end{aligned} \tag{2.5}$$

By the considerations in Remark 2.3.7 this is independent of the choice of K and the choice of Borel partition. Then the maps Ψ_n for $n \in \mathbb{N}$ define a chain map. We have

$$\begin{aligned}(\mathrm{id} \otimes \partial_n) \circ \Psi_n(\sum_{j=1}^{k} f_j \otimes \sigma_j) &= \sum_{j=1}^{k}\sum_{i=1}^{m}\sum_{l=1}^{L} a_{j,i}\chi_{A_i \cap X_l} \otimes \partial_n(\psi_l(\sigma_j)) \\ &= \sum_{j=1}^{k}\sum_{i=1}^{m}\sum_{l=1}^{L} a_{j,i}\chi_{A_i \cap X_l} \otimes \psi_l(\partial_n(\sigma_j)) \\ &= \Psi_n \circ (\mathrm{id} \otimes \partial_n)(\sum_{j=1}^{k} f_j \otimes \sigma_j).\end{aligned}$$

Note that $\bigcup_{j=1}^{k} \mathrm{im}(\partial_n(\sigma_j)) \subseteq K$, so the same Borel partition works.

It remains to check that the maps Ψ_n are $\mathbb{Z}\Gamma$-module homomorphisms. In order to show this, look at $\chi_A \otimes \sigma \in L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}} C_n(Z, \mathbb{Z})$. Let $\mathrm{im}(\sigma)$ lie in the compact set K, $X = \bigcup_{l=1}^{L} X_l$ be a Borel partition such that $\Psi_{|X_l \times K} = \mathrm{id}_{X_l} \times \psi_l$. Then $\mathrm{im}(\gamma\sigma) \subset \gamma K$. By Remark 2.3.7 we have for $X = \bigcup_{l=1}^{L} X'_l$ with $X'_l = \gamma X_l$ that $\Psi_{|X'_l \times \gamma K} = \mathrm{id}_{X'_l} \times \psi'_l$. Here $\psi'_l(\gamma k) = \gamma\psi_l(k)$ for all $k \in K$. As a result

$$\begin{aligned}\gamma\Psi_n(\chi_A \otimes \sigma) &= \gamma(\sum_{l=1}^{L} \chi_{A\cap X_l} \otimes \psi_l(\sigma)) = \sum_{l=1}^{L} \chi_{\gamma A \cap \gamma X_l} \otimes \gamma\psi_l(\sigma) \\ &= \sum_{q=1}^{L} \chi_{\gamma A \cap X'_l} \otimes \psi'_l(\gamma\sigma) = \Psi_n(\chi_{\gamma A} \otimes \gamma\sigma) = \Psi_n(\gamma(\chi_A \otimes \sigma)).\end{aligned}$$

By passing over to the coinvariants, (2.5) defines a chain map of $\mathbb{Z}$-chain complexes

$$C_*^X(\Psi)\colon L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(Z, \mathbb{Z}) \longrightarrow L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(Z', \mathbb{Z}).$$

□

Remark 2.4.2. As shown, an equivariant X-map $\Psi\colon X \times Z \to X \times Z'$ induces a chain map $C_*^X(\Psi)$ of $\mathbb{Z}$-chain complexes as defined in the proof of Theorem 2.4.1. Therefore it induces a map in homology for every $n \in \mathbb{N}$

$$H_n(\Psi)\colon H_n(L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(Z, \mathbb{Z})) \longrightarrow H_n(L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(Z', \mathbb{Z}))$$

via $H_n(\Psi)([\sum_{j=1}^{k} f_j \otimes \sigma_j]) = [C_n^X(\Psi)(\sum_{j=1}^{k} f_j \otimes \sigma_j)]$. Chain homotopic maps induce the same map in homology.

In view of this induced map in homology, the seminorm defined in Definition 2.1.15 has a certain *functoriality* property:

Lemma 2.4.3. *Suppose Z and Z' are topological spaces with fundamental group Γ and $\widetilde{Z}$ and $\widetilde{Z'}$ are their universal covers. Let $\Psi\colon X\times\widetilde{Z}\to X\times\widetilde{Z'}$ be an equivariant X-map of the corresponding equivariant X-spaces. Further let $n\in\mathbb{N}$, α be a class in $H_n(Z,L^\infty(X,\mathbb{Z}))=H_n(L^\infty(X,\mathbb{Z})\otimes_{\mathbb{Z}\Gamma}C_*(\widetilde{Z},\mathbb{Z}))$ and $H_n(\Psi)$ be the induced map in homology as defined in Remark 2.4.2. Then we get for the seminorm induced by the parametrised ℓ^1-norm*

$$\lvert H_n(\Psi)(\alpha)\rvert^X \leqslant \lvert\alpha\rvert^X.$$

Proof. Let $\sum_{j=1}^k f_j\otimes\sigma_j\in L^\infty(X,\mathbb{Z})\otimes_{\mathbb{Z}\Gamma}C_n(\widetilde{Z},\mathbb{Z})$ be a reduced cycle representing α. Thus

$$\Big|\sum_{j=1}^k f_j\otimes\sigma_j\Big|^X=\sum_{j=1}^k\int_X|f_j|\,d\mu.$$

Then $H_n(\Psi)(\alpha)$ is represented by $C_n^X(\Psi)(\sum_{j=1}^k f_j\otimes\sigma_j)$ (see Remark 2.4.2). We can rewrite $f_j=\sum_{i=1}^m a_{j,i}\chi_{A_i}$ for suitable $a_{j,i}\in\mathbb{Z}$ and disjoint measurable subsets $A_i\subseteq X$. For a compact subset $K\subset Z$ with $\bigcup_{j=1}^k\operatorname{im}(\sigma_j)\subseteq K$ and a corresponding Borel partition $X=\bigcup_{l=1}^L X_l$ we have

$$C_n^X(\Psi)(\sum_{j=1}^k f_j\otimes\sigma_j)=\sum_{j=1}^k\sum_{i=1}^m\sum_{l=1}^L a_{j,i}\chi_{A_i\cap X_l}\otimes\psi_l(\sigma_j).$$

(see (2.5)). Note that this cycle is not necessarily in reduced form. We obtain

$$\begin{aligned}
\lvert H_n(\Psi)(\alpha)\rvert^X &= \Big|[C_n^X(\Psi)(\sum_{j=1}^k f_j\otimes\sigma_j)]\Big|^X \leqslant \Big|C_n^X(\Psi)(\sum_{j=1}^k f_j\otimes\sigma_j)\Big|^X\\
&= \lvert\sum_{j=1}^k\sum_{i=1}^m\sum_{l=1}^L a_{j,i}\chi_{A_i\cap X_l}\otimes\psi_l(\sigma_j)\rvert^X\\
&\leqslant \sum_{j=1}^k\sum_{i=1}^m\sum_{l=1}^L\int_X|a_{j,i}\chi_{A_i\cap X_l}|\,d\mu=\sum_{j=1}^k\sum_{i=1}^m|a_{j,i}|\mu(A_i)\\
&= \sum_{j=1}^k\int_X|\sum_{i=1}^m a_{j,i}\chi_{A_i}|\,d\mu=\sum_{j=1}^k\int_X|f_j|\,d\mu\\
&= \Big|\sum_{j=1}^k f_j\otimes\sigma_j\Big|^X.
\end{aligned}$$

Taking the infimum over all reduced cycles representing α yields the desired inequality $\lvert H_n(\Psi)(\alpha)\rvert^X\leqslant\lvert\alpha\rvert^X$. □

Proposition 2.4.4. *Let Z and Z' be topological spaces with fundamental group Γ and universal covers $\widetilde{Z}$ and $\widetilde{Z'}$, respectively. Suppose $\Psi\colon X\times\widetilde{Z}\to X\times\widetilde{Z'}$ is an equivariant X-map of the corresponding equivariant X-spaces.*

If Ψ is equivariant geometric, i.e. Ψ_x proper for a.e. $x\in X$, then the following diagram is a commutative diagram of chain maps for a.e. $x\in X$.

$$\begin{tikzcd}
L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{Z},\mathbb{Z}) \arrow[r, "C_*^X(\Psi)"] \arrow[d, "ev_x"'] & L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{Z'},\mathbb{Z}) \arrow[d, "ev_x"] \\
C_*^{\mathrm{lf}}(\widetilde{Z}) \arrow[r, "C_*^{\mathrm{lf}}(\Psi_x)"] & C_*^{\mathrm{lf}}(\widetilde{Z}')
\end{tikzcd}$$

Proof. For a.e. $x \in X$ the map Ψ_x is proper hence it induces a well-defined chain map $C_*^{\mathrm{lf}}(\Psi_x)$ of locally finite chain complexes ([22, Proposition 11.1.2, p. 230]). Let $c = \sum_{j=1}^k f_j \otimes \sigma_j$ be an n-chain in $L^\infty(X,\mathbb{Z})\otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{Z},\mathbb{Z})$. We can rewrite $f_j = \sum_{i=1}^m a_{j,i}\chi_{A_i}$ for suitable $a_{j,i} \in \mathbb{Z}$ and measurable $A_i \subseteq X$. For a compact subset $K \subset Z$ with $\bigcup_{j=1}^k \operatorname{im}(\sigma_j) \subseteq K$ and a corresponding Borel partition $X = \bigcup_{l=1}^L X_l$ we then have

$$C_n^X(\Psi)(c) = \sum_{j=1}^k \sum_{i=1}^m \sum_{l=1}^L a_{j,i}\chi_{A_i \cap X_l} \otimes \psi_l(\sigma_j),$$

with continuous functions ψ_l. Hence for a.e. $x \in X$ we calculate

$$(ev_x \circ C_n^X(\Psi))(c) = \sum_{\gamma\in\Gamma} \sum_{j=1}^k \sum_{i=1}^m \sum_{l=1}^L a_{j,i}\chi_{A_i\cap X_l}(\gamma^{-1}x)\gamma\psi_l(\sigma_j).$$

On the other hand we have

$$\begin{aligned}
(C_n^{\mathrm{lf}}(\Psi_x) \circ ev_x)(c) &= \sum_{\gamma\in\Gamma} \sum_{j=1}^k f_j(\gamma^{-1}x)\Psi_x(\gamma\sigma_j) \\
&= \sum_{\gamma\in\Gamma} \sum_{j=1}^k \sum_{i=1}^m \sum_{l=1}^L a_{j,i}\chi_{A_i\cap X_l}(\gamma^{-1}x)\Psi_x(\gamma\sigma_j).
\end{aligned}$$

Note that $\gamma\sigma_j \subset \gamma K$ and, as shown in the proof of Theorem 2.4.1, for the finite Borel partition $X = \bigcup_{l=1}^L (\gamma X_l)$ we have $\Psi_{|\gamma X_l \times \gamma K} = \mathrm{id} \times \psi_l'$ with $\psi_l'(\gamma k) = \gamma\psi_l(k)$ for all $k \in K$. So if $\chi_{A_i \cap X_l}(\gamma^{-1}x) \neq 0$, i.e. $x \in \gamma X_l$, we have $\Psi_x(\gamma\sigma_j) = \gamma\psi_l(\sigma_j)$. This yields

$$(ev_x \circ C_n^X(\Psi))(c) = (C_n^{\mathrm{lf}}(\Psi_x) \circ ev_x)(c)$$

for every $n \in \mathbb{N}$ and a.e. $x \in X$. $\square$

In the next step we restrict to equivariant simple X-spaces where the topological space is a Γ-CW complex. Recall that a model for the classifying space of the action of Γ is a contractible free Γ-CW complex $E\Gamma$. By its universal property, there is a map from any free Γ-CW complex to $E\Gamma$ which is unique up to Γ-homotopy. Therefore $E\Gamma$ is unique up to Γ-homotopy equivalence.

So if $X \times Z$ is an equivariant simple X-space with a connected free Γ-CW complex Z we have a unique map $\eta\colon Z \to E\Gamma$ and an equivariant X-map $\mathrm{id}_X \times \eta\colon X \times Z \to X \times E\Gamma$. In particular this map induces a chain map

$$C_*^X(\mathrm{id}_X \times \eta)\colon L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(Z,\mathbb{Z}) \to L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(E\Gamma,\mathbb{Z}).$$

Considering the definition of the induced chain map in the proof of Theorem 2.4.1 one sees that this map equals the chain map $\mathrm{id}_{L^\infty(X,\mathbb{Z})} \otimes C_*(\eta)$ where $C_*(\eta)\colon C_*(Z,\mathbb{Z}) \to C_*(E\Gamma,\mathbb{Z})$ is the induced chain map of the singular chain complexes. The map $C_*^X(\mathrm{id} \times \eta)$ is unique up to chain homotopy.

Lemma 2.4.5. *Let $X \times Z$ be an equivariant simple X-space where Z is a connected free Γ-CW complex. Then any two equivariant X-maps $X \times Z \to X \times E\Gamma$ induce chain homotopic maps*

$$L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(Z,\mathbb{Z}) \to L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(E\Gamma,\mathbb{Z}).$$

Proof. The result will be a consequence of the fundamental lemma of homological algebra [10, Theorem 2.22, p. 36]. Let Ψ, Θ be equivariant X-maps $X \times Z \to X \times E\Gamma$. and denote the induced maps of chain complexes by $C_*^X(\Psi)$ and $C_*^X(\Theta)$. We first show that the induced maps of $\mathbb{Z}\Gamma$-chain complexes $L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_*(Z,\mathbb{Z}) \to L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_*(E\Gamma,\mathbb{Z})$ are chain homotopic. We denote these maps by Ψ_* and Θ_*.

Note that both chain complexes $L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_*(Z,\mathbb{Z})$ and $L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_*(E\Gamma,\mathbb{Z})$ are free chain complexes of $\mathbb{Z}\Gamma$-modules. The singular chain complex $C_*(Z,\mathbb{Z})$ is a free $\mathbb{Z}\Gamma$-chain complex. We have $C_n(Z,\mathbb{Z}) \cong \bigoplus_{I_n} \mathbb{Z}\Gamma$ for some index set I_n and therefore we obtain $L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_n(Z,\mathbb{Z}) \cong L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} (\bigoplus_{I_n} \mathbb{Z}\Gamma) \cong \bigoplus_{I_n}(L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}\Gamma)$. The $\mathbb{Z}\Gamma$-module $L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}\Gamma$ is equipped with the diagonal Γ-action $\gamma_0(f \otimes \gamma) = (f \cdot \gamma_0^{-1}) \otimes \gamma_0\gamma$. There is an isomorphism of $\mathbb{Z}\Gamma$-modules

$$\begin{aligned} L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}\Gamma &\xrightarrow{\cong} L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}\Gamma \\ f \otimes \gamma &\longmapsto (f \cdot \gamma) \otimes \gamma, \end{aligned}$$

where Γ acts on the second module via $\gamma_0(f \otimes \gamma) = f \otimes \gamma_0\gamma$. Equipped with this module structure we have $L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}\Gamma \cong L^\infty(X,\mathbb{Z})[\Gamma]$ which is a free $\mathbb{Z}\Gamma$-module since $L^\infty(X,\mathbb{Z})$ is a free $\mathbb{Z}$-module according to Lemma 2.1.8. As a result the chain modules $L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_n(Z,\mathbb{Z})$ are indeed free $\mathbb{Z}\Gamma$-modules. The same holds true for $L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_*(E\Gamma,\mathbb{Z})$. Moreover, we can show that the latter chain complex is a free resolution of the left $\mathbb{Z}\Gamma$-module $L^\infty(X,\mathbb{Z})$ with the Γ-action given by $\gamma f \coloneqq f \cdot \gamma^{-1}$.

Since $E\Gamma$ is contractible we have $H_n(E\Gamma,\mathbb{Z}) \cong 0$ if $n \geqslant 1$ and $H_0(E\Gamma,\mathbb{Z}) \cong \mathbb{Z}$ with the trivial Γ-action. Therefore, $C_*(E\Gamma,\mathbb{Z})$ is a free resolution of the trivial $\mathbb{Z}\Gamma$-module $\mathbb{Z}$ [9, I. proposition 4.2, p. 15]. Since $L^\infty(X,\mathbb{Z})$ is in particular a flat $\mathbb{Z}$-module, the tensored chain complex $L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_*(E\Gamma,\mathbb{Z})$ gives a free resolution of the left $\mathbb{Z}\Gamma$-module $L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z} \cong L^\infty(X,\mathbb{Z})$.

We look at the augmented chain complexes. Let the augmentation homomorphisms of $C_0(Z,\mathbb{Z})$ and $C_0(E\Gamma,\mathbb{Z})$ be denoted by ε_1 and ε_2. They map a singular 0-simplex to 1. Then we have the augmentation homomorphism $\mathrm{id} \otimes \varepsilon_1 \colon L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_0(Z,\mathbb{Z}) \to L^\infty(X,\mathbb{Z})$ and $\mathrm{id} \otimes \varepsilon_2 \colon L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_0(E\Gamma,\mathbb{Z}) \to L^\infty(X,\mathbb{Z})$. Recall that for an element $\sum_{j=1}^k \sum_{i=1}^m a_{j,i}\chi_{A_i} \otimes \sigma_j \in L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_0(\widetilde{Z},\mathbb{Z})$ we have (for a suitable Borel decomposition $X = \bigcup_{l=1}^L X_l$ and continuous functions ψ_l and ϑ_l)

$$\begin{aligned}
\mathrm{id} \otimes \varepsilon_2 \left(\Psi_0 (\sum_{j=1}^k \sum_{i=1}^m a_{j,i}\chi_{A_i} \otimes \sigma_j) \right) &= \mathrm{id} \otimes \varepsilon_2 \left(\sum_{j=1}^k \sum_{i=1}^m \sum_{l=1}^L a_{j,i}\chi_{A_i \cap X_l} \otimes \psi_l(\sigma_j) \right) \\
= \sum_{j=1}^k \sum_{i=1}^m \sum_{l=1}^L a_{j,i}\chi_{A_i \cap X_l} &= \sum_{j=1}^k \sum_{i=1}^m a_{j,i}\chi_{A_i} = \mathrm{id} \otimes \varepsilon_1 \left(\sum_{j=1}^k \sum_{i=1}^m a_{j,i}\chi_{A_i} \otimes \sigma_j \right), \\
\mathrm{id} \otimes \varepsilon_2 \left(\Theta_0 (\sum_{j=1}^k \sum_{i=1}^m a_{j,i}\chi_{A_i} \otimes \sigma_j) \right) &= \mathrm{id} \otimes \varepsilon_2 \left(\sum_{j=1}^k \sum_{i=1}^m \sum_{l=1}^L a_{j,i}\chi_{A_i \cap X_l} \otimes \vartheta_l(\sigma_j) \right) \\
= \sum_{j=1}^k \sum_{i=1}^m \sum_{l=1}^L a_{j,i}\chi_{A_i \cap X_l} &= \sum_{j=1}^k \sum_{i=1}^m a_{j,i}\chi_{A_i} = \mathrm{id} \otimes \varepsilon_1 \left(\sum_{j=1}^k \sum_{i=1}^m a_{j,i}\chi_{A_i} \otimes \sigma_j \right).
\end{aligned}$$

So we get chain maps of the augmented chain complexes

$$\begin{tikzcd}
{} \arrow[r] & L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_1(\widetilde{Z},\mathbb{Z}) \arrow[r] \arrow[d, "\Psi_1"] & L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_0(\widetilde{Z},\mathbb{Z}) \arrow[r, "\mathrm{id}\otimes\varepsilon_1"'] \arrow[d, "\Psi_0"] & L^\infty(X,\mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
{} \arrow[r] & L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_1(E\Gamma,\mathbb{Z}) \arrow[r] & L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_0(E\Gamma,\mathbb{Z}) \arrow[r, "\mathrm{id}\otimes\varepsilon_2"'] & L^\infty(X,\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
{} \arrow[r] & L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_1(\widetilde{Z},\mathbb{Z}) \arrow[r] \arrow[d, "\Theta_1"] & L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_0(\widetilde{Z},\mathbb{Z}) \arrow[r, "\mathrm{id}\otimes\varepsilon_1"'] \arrow[d, "\Theta_0"] & L^\infty(X,\mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
{} \arrow[r] & L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_1(E\Gamma,\mathbb{Z}) \arrow[r] & L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_0(E\Gamma,\mathbb{Z}) \arrow[r, "\mathrm{id}\otimes\varepsilon_2"'] & L^\infty(X,\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

Since Z and $E\Gamma$ are path-connected we have identifications in 0-th homology

$$H_0(L^\infty(X,\mathbb{Z}) \otimes_\mathbb{Z} C_*(Z,\mathbb{Z})) \cong L^\infty(X,\mathbb{Z}) \otimes_\mathbb{Z} H_0(Z,\mathbb{Z}) \cong L^\infty(X,\mathbb{Z}) \otimes_\mathbb{Z} \mathbb{Z} \cong L^\infty(X,\mathbb{Z})$$

and $H_0(L^\infty(X,\mathbb{Z}) \otimes_\mathbb{Z} C_*(E\Gamma,\mathbb{Z})) \cong L^\infty(X,\mathbb{Z})$. Therefore both chain maps, Ψ_* and Θ_*, induce the same isomorphism of left $\mathbb{Z}\Gamma$-modules

$$H_0(L^\infty(X,\mathbb{Z}) \otimes_\mathbb{Z} C_*(Z,\mathbb{Z})) \to H_0(L^\infty(X,\mathbb{Z}) \otimes_\mathbb{Z} C_*(E\Gamma,\mathbb{Z})).$$

By the fundamental lemma of homological algebra they are Γ-chain homotopic. Passing over to the coinvariants, we obtain chain maps of $\mathbb{Z}$-modules

$$C_*^X(\Psi), C_*^X(\Theta)\colon L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(Z,\mathbb{Z}) \longrightarrow L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(E\Gamma,\mathbb{Z}).$$

which are chain homotopic. □

2.5 Equivariant covers of $X \times \widetilde{M}$

Suppose M is a connected, finite simplicial complex with fundamental group Γ. Hence $\widetilde{M}$ is a free simplicial Γ-complex and the group Γ acts properly discontinuously on it. All metric notions about M refer to the unique length metric that restricts to the standard Euclidean metric on simplices.

We consider the equivariant simple X-space $X \times \widetilde{M}$. Recall that the Γ-action is diagonal. In this section we introduce a nerve construction for certain equivariant covers of $X \times \widetilde{M}$. We can adapt the notion of $\mathcal{R}$-cover from [45, Section 2]. Afterwards we implement a rectangular nerve based on Guth's ideas [30, Section 3].

2.5.1 Equivariant covers

Look at the equivariant simple X-space $X \times \widetilde{M}$. Note that the following definition is the definition of an $\mathcal{R}$-cover and $\mathcal{R}$-packing in [45, Definition 2.27] with some adjustments. We require that an equivariant cover has only finitely many orbits of cover sets and the appearing subsets of $\widetilde{M}$ are relatively compact. In particular, these sets are bounded and their diameters are bounded. Recall that the diameter of an open set $U \subseteq \widetilde{M}$ is defined as

$$\operatorname{diam}(U) \coloneqq \sup\{d_{\widetilde{M}}(p,q) \mid p,q \in \widetilde{M}\}.$$

Definition 2.5.1. Let J be a free Γ-set with finitely many orbits. For every element $j \in J$, let $A_j \subseteq X$ be a Borel subset and $U_j \subseteq \widetilde{M}$ be an open relatively compact subset. The family of products of those sets $\mathcal{U} = \{A_j \times U_j\}_{j\in J}$ is a Γ-*cover of* $X \times \widetilde{M}$ if the following holds true:

i) For all $\gamma \in \Gamma$ and $j \in J$ we have $A_{\gamma j} = \gamma A_j$ and $U_{\gamma j} = \gamma U_j$.

ii) $\mathcal{U}_x := \{U_j \,|\, x \in A_j\}_{j\in J}$ is locally finite in $\widetilde{M}$ for a.e. $x \in X$.

iii) Let $p \in \widetilde{M}$. For a.e. $x \in X$ we have $(x,p) \in \bigcup_{j\in J} A_j \times U_j \subseteq X \times \widetilde{M}$.

The family $\mathcal{U} = \{A_j \times U_j\}_{j\in J}$ is a Γ-*packing* if it satisfies:

i) For all $\gamma \in \Gamma$ and $j \in J$ we have $A_{\gamma j} = \gamma A_j$ and $U_{\gamma j} = \gamma U_j$.

ii) If $U_j \cap U_k \neq \emptyset$ for $j,k \in J$ with $j \neq k$, then it holds $\mu(A_j \cap A_k) = 0$.

Note that by the assumption on the number of orbits of the index set, we immediately obtain that J is countable since the group is countable. The following slightly modified lemmas from [45, Lemma 2.28 and 2.29] hold true.

Lemma 2.5.2. *A Γ-cover $\mathcal{U} = \{A_j \times U_j\}_{j\in J}$ of $X \times \widetilde{M}$ where all Borel sets A_j have positive measure has the following properties:*

i) For every compact set $K \subset \widetilde{M}$ there is a finite Borel partition $X = \bigcup_{l=1}^{L} X_l$ such that for almost every $x, y \in X_l$ and every $k \in K$ we have:

$$(x,k) \in A_j \times U_j \quad \Longleftrightarrow \quad (y,k) \in A_j \times U_j.$$

ii) $\mathcal{U}_x$ is a cover of $\widetilde{M} \cong \{x\} \times \widetilde{M}$ for a.e. $x \in X$.

Proof. In order to prove the first statement, set $J_K = \{j \in J \,|\, K \cap U_j \neq \emptyset\} \subset J$. Since Γ acts properly discontinuously on $\widetilde{M}$, this set is finite. Moreover, for $x \in X$ the set $J_K(x) = \{j \in J \,|\, x \in A_j,\, K \cap U_j \neq \emptyset\}$ is finite. We get a measurable function from X to the finite set of subsets of J_K. Now let $X = \bigcup_{l=1}^{L} X_l$ be a finite Borel partition such that $J_K(x)$ is a constant set on each X_l. This means for a.e. $x, y \in X_l$ we have $J_K(x) = J_K(y)$. Then if $(x,k) \in A_j \times U_j$ we have $j \in J_K(x) = J_K(y)$. Hence $y \in A_j$ and $(y,k) \in A_j \times U_j$. The second assertion is proved in [45, Lemma 2.28]. □

Lemma 2.5.3. *Let $\mathcal{U} = \{A_j \times U_j\}_{j\in J}$ be a Γ-packing where all Borel sets A_j have positive measure. Then $\mathcal{U}_x := \{U_j \,|\, x \in A_j\}_{j\in J}$ is a packing of $\widetilde{M}$ for a.e. $x \in X$. This means, the elements of $\{U_j \,|\, x \in A_j\}_{j\in J}$ are pairwise disjoint.*

For the proof see [45, Lemma 2.29].

2.5.2 Nerves of covers

Corresponding to an open cover of a topological space, there is the construction of an abstract simplicial complex, this nerve of the cover. In [45, Definition 2.30] the nerve construction is adapted for Γ-covers of $X \times \widetilde{M}$. We introduce another nerve construction for such a cover. The goal is to adapt Guth's construction of the *rectangular nerve* [30, Section 3] which takes into account the possible different sizes of the open sets $U_j \subseteq \widetilde{M}$. Guth introduced his notion of rectangular nerve for a specific cover of a manifold by so-called good balls. We generalize this notion to general covers of metric space by open bounded sets.

The rectangular nerve

Let Y be a metric space and $\mathcal{U} = \{U_j\}_{j\in J}$ be a locally finite open cover of Y by relatively compact sets U_j. Set $a_j \coloneqq \frac{1}{2}\operatorname{diam}(U_j)$. Assume further that $\inf_{j\in J} a_j > 0$.

Recall the construction of the cuboid complex associated to the set $\{a_j\}_{j\in J}$ (see Definition 2.2.8). We denote this complex by Z. An (open) k-face in Z is the interior of a k-cuboid in the complex and has side lenghts in $\{a_j\}_{j\in J}$. Z is equipped with the Euclidean path length metric, the unique path length metric restricting to the Euclidean standard metric on each face. Every point in Z can be written as a tuple $(y_j)_{j\in J}$. The vertex set of the complex is given by

$$V \coloneqq \{(y_j)_{j\in J} \,|\, (y_j)_{j\in J} \neq 0,\ y_j = 0 \text{ or } a_j \text{ with } y_j \neq 0 \text{ only for finitely many } j \in J\}.$$

So by construction, for every point $(y_j)_{j\in J} \in Z$ there is at least one index $j \in J$ such that $y_j = a_j$. Recall that there is a one-to-one correspondence between the faces of Z and their barycentres. The barycentres of k-faces for $k \geqslant 0$ can be described by

$$V_k \coloneqq \left\{(y_j)_{j\in J} \in Z \,|\, y_j = 0, \tfrac{1}{2}a_j \text{ or } a_j \text{ with exactly } k \text{ entries } \tfrac{1}{2}a_j\right\}.$$

In particular $V_0 = V$.

We want to define the *rectangular nerve* $\mathcal{N}(\mathcal{U})$ as a subcomplex of Z (see [30, Section 3]). For every open face F of Z consider its barycentre $b_F = (y_j)_{j\in J}$. Divide the index set J into three subsets

$$\begin{aligned} J_0(F) &\coloneqq \{j \in J \,|\, y_j = 0\} \\ J_{1/2}(F) &\coloneqq \{j \in J \,|\, y_j = \tfrac{1}{2}a_j\} \\ J_1(F) &\coloneqq \{j \in J \,|\, y_j = a_j\} \end{aligned}$$

and denote $J_+(F) = J_1(F) \cup J_{1/2}(F)$. Note that, by definition of Z, $J_1(F) \neq \emptyset$. We denote the dimension of an open face F by $d(F)$. It holds $d(F) = |J_{1/2}(F)|$. Each face is the interior of a cuboid with side lengths a_j with $j \in J_{1/2}(F)$. We denote these side lenghts by $a_1(F), \ldots, a_{d(F)}(F)$ corresponding to an order $a_1(F) \leqslant \ldots \leqslant a_{d(F)}(F)$. For each face, let j_F be the element in $J_{1/2}(F)$ such that $a_{j_F} = a_{d(F)}(F)$.

With the above decomposition of the index set J we can define the rectangular nerve.

Definition 2.5.4. Let $\mathcal{U} = \{U_j\}_{j\in J}$ be a locally finite open cover with $a_j \coloneqq \frac{1}{2}\operatorname{diam}(U_j)$ bounded and $\inf_{j\in J} a_j > 0$. The *rectangular nerve* $\mathcal{N}(\mathcal{U})$ is defined as a subcomplex of the cuboid complex Z associated to $\{a_j\}_{j\in J}$. An open face $F \in Z$ belongs to the nerve if and only if $\bigcap_{j\in J_+(F)} U_j \neq \emptyset$.

Note that the nerve is indeed a subcomplex. If an open face F belongs to the nerve, then any face F' in its boundary belongs to the nerve as well since it holds $J_+(F') \subseteq J_+(F)$.

Proposition 2.5.5. *The nerve $\mathcal{N}(\mathcal{U})$ is a locally finite, finite-dimensional complex.*

Proof. This statements holds since the cover is locally finite. Assume the nerve is not locally finite. Then there is a vertex such that a neighbourhood intersects infinitely many cuboids. In particular there are infinitely many edges in $\mathcal{N}(\mathcal{U})$ having this vertex as an endpoint. This implies that there is a set U_i in the cover intersecting infinitely many other elements of $\mathcal{U}$. The set U_i is relatively compact, i.e. its closure is compact. Since $\mathcal{U}$ is locally finite, a compact set can only intersect finitely many cover sets ,which yields a contradiction.

Moreover, the multiplicity of $\mathcal{U}$ is bounded hence the nerve is a finite-dimensional complex. □

Lemma 2.5.6. *Let $\mathcal{U} = \{U_j\}_{j\in J}$ be a locally finite open cover of a metric space Y with bounded sets U_j such that $\inf_{j\in J}(\operatorname{diam}(U_j)) > 0$ and $\mathcal{N}(\mathcal{U})$ be the corresponding rectangular nerve. Then there is a continuous proper map*

$$\tau\colon Y \longrightarrow \mathcal{N}(\mathcal{U}).$$

The preimage of the open star of an open face F is contained in U_{j_F}, the cover set with $\frac{1}{2}\operatorname{diam}(U_{j_F}) = a_{j_F} = a_{d(F)}(F)$.

Proof. Let Z be the cuboid complex associated to $\{a_j = \frac{1}{2}\operatorname{diam}(U_j)\}_{j\in J}$, i.e. the nerve is a subcomplex of Z.

A metric space is paracompact and Hausdorff. Thus there exists a locally finite open refinement $\mathcal{V} = \{W_j\}_{j\in J}$ of $\mathcal{U}$ such that $\bar{W}_j \subseteq U_j$ for all $j \in J$ (see [34, Lemma 4.84, p. 114]). By Urysohn's lemma, for each $j \in J$ there exists a continuous function $\tau_j\colon Y \to [0,1]$ such that $\tau_{|W_j} \equiv a_j$ and $\operatorname{supp}(\tau_j) \subseteq U_j$ [34, Theorem 4.82 and Corollary 4.83, p. 112-114]. Now define

$$\begin{aligned} \tau\colon Y &\longrightarrow Z \\ p &\longmapsto (\tau_j(p))_{j\in J}. \end{aligned}$$

This is indeed a well-defined map, since every point $p \in Y$ is contained in at least one set U_j, i.e. $(\tau_j(p))_{j\in J} \neq 0$, but since $\mathcal{U}$ is locally finite, it is contained in only finitely many cover sets. Moreover, there is an index $j \in J$ such that $\tau_j(p) = a_j$ since $\mathcal{V}$ is a cover as well. Hence τ maps into Z. By definition of the nerve, the image is actually contained in $\mathcal{N}(\mathcal{U})$.

Since the τ_i are continuous, τ is a continuous map. Let K be a compact subset in $\mathcal{N}(\mathcal{U})$. The cuboid complex is Hausdorff thus K is closed. It is contained in a finite subcomplex of the cuboid complex. Hence its preimage is contained in a finite union of cover sets. The cover sets are relatively compact, so is the union of finitely many of them, thus $\tau^{-1}(K)$ is a closed subset of a compact set and therefore itself compact. As a result, the nerve map τ is proper.

The map clearly satisfies the last assertion. Let F' be a face in $\operatorname{Star}(F)$, i.e. it contains F in its boundary. Then $J_+(F) \subseteq J_+(F')$. Hence the index j_F corresponding to the side length $a_{d(F)}(F)$ is contained in $J_+(F')$ for all elements of the open star and the preimage of the open star is contained in U_{j_F}. □

Remark 2.5.7. The preimage of the open star F of an open face is contained in $\bigcap_{j\in J_+(F)} U_j$. Let $j_1, \ldots, j_k$ be the elements of $J_{1/2}$, i.e. F has side lengths $a_{j_1}, \ldots, a_{j_k}$. Assume that $a_{j_1} \leqslant \ldots \leqslant a_{j_{k-1}} = a_{j_k}$, so the largest side length is not unique. But the preimage of the open star of F is contained in $U_{j_{k-1}}$ as well as U_{j_k}. Therefore the statement of the lemma is independent of the chosen order of side lengths and the index j_F such that $a_{d(F)}(F)$ is the length of a largest side.

The rectangular nerve of a Γ-cover

We want to adapt this nerve construction for Γ-covers of $X \times \widetilde{M}$. Let

$$\mathcal{U} = \{A_j \times U_j \mid j \in J\}$$

be a Γ-cover of $X \times \widetilde{M}$. The diameters of $U_j \subseteq \widetilde{M}$ are bounded. We set $a_j \coloneqq \frac{1}{2}\operatorname{diam}(U_j)$. It holds $a_{\gamma j} = a_j$ for all $\gamma \in \Gamma$ and $\inf_{j\in J} a_j > 0$, since there are only finitely many Γ-orbits of cover sets.

We define the nerve as a subcomplex in the product space of X with the metric cuboid complex Z associated to $\{a_j\}_{j\in J}$.

The Γ-action on J induces a Γ-action on this cuboid complex via $\gamma(y_j)_{j\in J} = (y_{\gamma^{-1}j})_{j\in J}$ for a point $(y_j)_{j\in J} \in Z$. This descends to a Γ-action on each V_k for $k \geqslant 0$, the set of barycentres of k-faces in Z. Therefore the action on the cuboid complex permutes k-faces. Moreover, this action is isometric with respect to the path length metric on Z. With this action, Z is a Γ-space and $X \times Z$ with the diagonal Γ-action is an equivariant simple X-space. We want to define the nerve as a subcomplex of this space, i.e. as a Γ-invariant subspace which is fibrewise a subcomplex of Z. As before we regard for every open face F in Z its barycentre b_F and divide the index set into the subsets $J_0(F), J_{1/2}(F)$ and $J_1(F)$, depending on whether the j-th component of b_F is $0, \frac{1}{2}a_j$ or a_j, respectively. Let $J_+(F) = J_{1/2}(F) \cup J_1(F)$.

Definition 2.5.8. For a Γ-cover $\mathcal{U} = \{U_j\}_{j\in J}$ of $X \times \widetilde{M}$ with $a_j \coloneqq \frac{1}{2}\operatorname{diam}(U_j)$ the *rectangular nerve* $\mathcal{N}(\mathcal{U})$ is defined as a subcomplex of the equivariant simple X-space $X \times Z$, where Z is the metric cuboid complex associated to $\{a_j\}_{j\in J}$. For $x \in X$ and an open face $F \in Z$, (x, F) belongs to the nerve if and only if $\bigcap_{j\in J_+(F)} U_j \neq \emptyset$ and $x \in \bigcap_{j\in J_+(F)} A_j$.

Remark 2.5.9. The nerve inherits the Γ-action on $X \times Z$. For $(x, F) \in \mathcal{N}(\mathcal{U})$ we have $\gamma(x, F) = (\gamma x, \gamma F)$. If $b_F = (y_j)_{j\in J}$ is the barycentre of F,

$$\gamma b_F = \gamma(y_j)_{j\in J} = (y_{\gamma^{-1}j})_{j\in J}$$

is the barycentre $b_{\gamma F}$ of γF. It holds $J_+(\gamma F) = \gamma J_+(F)$ and therefore

$$\begin{aligned}\emptyset \neq \gamma\Big(\bigcap_{j\in J_+(F)} U_j\Big) &= \bigcap_{j\in J_+(F)} U_{\gamma j} = \bigcap_{j\in J_+(\gamma F)} U_j\\ \gamma x \in \gamma\Big(\bigcap_{j\in J_+(F)} A_j\Big) &= \bigcap_{j\in J_+(F)} A_{\gamma j} = \bigcap_{j\in J_+(\gamma F)} A_j.\end{aligned}$$

Thus $\mathcal{N}(\mathcal{U})$ is a Γ-invariant subspace of $X \times Z$. Every fibre $\mathcal{N}(\mathcal{U})_x$ is a subcomplex of $\{x\} \times Z \cong Z$. If $(x, F) \in \mathcal{N}(\mathcal{U})_x$ and $F' \in Z$ is a face in the boundary of F, then $J_+(F') \subseteq J_+(F)$ hence (x, F') belongs to $\mathcal{N}(\mathcal{U})_x$ as well. Hence $\mathcal{N}(\mathcal{U})$ is indeed a subcomplex of the equivariant simple X-space $X \times Z$.

Definition 2.5.10. For a Γ-cover $\mathcal{U} = \{A_j \times U_j \mid j \in J\}$ of $X \times \widetilde{M}$, the family of open sets $\mathcal{W} = \{U_j \mid j \in J\}$ is an equivariant cover of $\widetilde{M}$ with finitely many orbits of cover sets. We call $\mathcal{W}$ the *underlying cover of* $\mathcal{U}$.

Proposition 2.5.11. *$\mathcal{W}$ is a locally finite cover.*

Proof. The cover $\mathcal{W}$ consists of finitely many orbits of cover sets. Therefore we can write $\mathcal{W} = \{\gamma U_i \mid \gamma \in \Gamma, i \in \{1, \dots, n\}\}$ for some $n \in \mathbb{N}$. Since the cover sets are relatively compact and the group Γ acts properly discontinuously on $\widetilde{M}$, any set $\gamma U_i \in \mathcal{W}$ intersects only finitely many other cover sets. Therefore $\mathcal{W}$ is locally finite. □

Remark 2.5.12. By the above proposition and Proposition 2.5.5, $\mathcal{N}(\mathcal{W})$ is a locally finite, finite-dimensional subcomplex of the cuboid complex Z associated to $\{\frac{1}{2}\operatorname{diam}(U_j)\}_{j \in J}$. Then the weak topology coincides with the topology induced by the path lenght metric restricting to the Euclidean standard metric on every face. As remarked in the end of Section 2.3, we equip $\mathcal{N}(\mathcal{W})$ with the s-dimensional Hausdorff measure for a $s > 0$. This measure is preserved under the Γ-action inherited from Z.

Look at the collection of product sets $\mathcal{W}' = \{X \times \gamma U_i \mid \gamma \in \Gamma, i \in \{1, \dots, n\}\}$. This defines a Γ-cover of $X \times \widetilde{M}$ as well. By definition, $\mathcal{N}(\mathcal{W}') = X \times \mathcal{N}(\mathcal{W})$ which is a subcomplex of $X \times Z$.

Remark 2.5.13. Recall that, by Lemma 2.5.2, $\mathcal{U}_x = \{U_j \mid x \in A_j\}_{j \in J}$ is a locally finite cover of $\widetilde{M}$ for a.e. $x \in X$. So the corresponding nerve $\mathcal{N}(\mathcal{U}_x)$ is a locally finite complex by Proposition 2.5.5. We describe the relation between $\mathcal{N}(\mathcal{U}_x)$ and the fibre of the nerve $\mathcal{N}(\mathcal{U})_x$. Recall the definition of the former. We introduce a new index set $I_x \subseteq J$ to rewrite $\mathcal{U}_x = \{U_j\}_{j \in I_x}$. Then the rectangular nerve corresponding to this cover $\mathcal{N}(\mathcal{U}_x)$ is a subcomplex of the cuboid complex associated to $\{a_j\}_{j \in I_x}$, which we denote by Z^x. The inclusion of I_x in the original index set J induces an inclusion of Z^x as a subcomplex in the cuboid complex Z. Then the image of $\mathcal{N}(\mathcal{U}_x)$ under this inclusion is $\mathcal{N}(\mathcal{U})_x$.

Shortly speaking we say that for a.e. $x \in X$, $\mathcal{N}(\mathcal{U})_x$ is the rectangular nerve of the cover $\mathcal{U}_x$, which is locally finite, since $\mathcal{U}_x$ is so. Further the $\mathcal{U}_x$ are subcovers of the underlying cover $\mathcal{W}$ of $\mathcal{U}$ and $\mathcal{N}(\mathcal{U})_x$ is a subcomplex of $\mathcal{N}(\mathcal{W})$. Since $\mathcal{N}(\mathcal{U})$ is Γ-invariant, it is a subcomplex of $\mathcal{N}(\mathcal{W}') = X \times \mathcal{N}(\mathcal{W})$.

As before we denote the dimension of an open face in $\mathcal{N}(\mathcal{W})$ with $d(F)$. We fix an order of its side lengths $a_1(F) \leqslant \ldots \leqslant a_{d(F)}(F)$ where each $a_i(F) = a_{j_i}$ for an index $j_i \in J_{1/2}(F)$. The index corresponding to $a_{d(F)}(F)$ is denoted by j_F. Note that the face γF has the same side lengths as F and we can fix an order of the side length such that $j_{\gamma F} = \gamma j_F$.

We define the open star of a face F, $\mathrm{Star}(F)$, to be the open star of F with respect to $\mathcal{N}(\mathcal{W})$. If we want to emphasise the fibre, we write $\mathrm{Star}(x, F)$. The open star of a face with respect to $\mathcal{N}(\mathcal{U})_x$ is then given as $\mathrm{Star}_{\mathcal{N}(\mathcal{U})_x}(F) = \mathrm{Star}(F) \cap \mathcal{N}(\mathcal{U})_x$.

Lemma 2.5.14. *Let $\mathcal{U} = \{A_j \times U_j \mid j \in J\}$ be a Γ-cover of $X \times \widetilde{M}$ and $\mathcal{N}(\mathcal{U})$ the corresponding rectangular nerve. Then there is an equivariant geometric map*

$$\tau \colon X \times \widetilde{M} \longrightarrow X \times \mathcal{N}(\mathcal{W})$$

such that the following holds:

i) The image of τ is contained in $\mathcal{N}(\mathcal{U})$.

ii) For a.e. $x \in X$ the preimage under τ_x of the open star of a face F is contained in U_{j_F}, the cover set with $\frac{1}{2}\,\mathrm{diam}(U_{j_F}) = a_{j_F} = a_{d(F)}(F)$.

Proof. For the proof see also [45, proof of Lemma 2.35]. Let $K \subset \widetilde{M}$ be a compact set which contains the open 1-neighbourhood of a Γ-fundamental domain of $\widetilde{M}$. The set $\{\gamma K \mid \gamma \in \Gamma\}$ covers $\widetilde{M}$ and has Lebesgue number 1, i.e. any set of diameter smaller than 1 is contained in a γK. For a.e. $x \in X$, $\mathcal{U}_x$ is a cover of $\widetilde{M}$. Look at the restriction of these covers to K. There are only finitely many covers of K which can appear as such a restriction, since there can only be finitely many cover sets U_i intersecting K. This is due to the fact that the index set J has finitely many orbits and the group Γ acts properly discontinuously on $\widetilde{M}$. Any cover of the compact set has positive Lebesgue number with respect to the restricted metric on K. Let ε' be the minimal such number. Then the Lebesque number of $\mathcal{U}_x$ is $\varepsilon \coloneqq \min\{\varepsilon', 1\}$ for a.e. $x \in X$.

Then we find a refinement for the Γ-cover $\mathcal{U}$ as follows. For each cover set U_j we regard the closed $\varepsilon/4$-neighbourhood $\overline{N}_{\varepsilon/4}(\partial U_j)$ of its boundary ∂U_j and set $W_j \coloneqq U_j \backslash \overline{N}_{\varepsilon/4}(\partial U_j)$. The W_j are relatively compact, $W_{\gamma j} = \gamma W_j$ for $\gamma \in \Gamma$ and $\bar{W}_j \subset U_j$. Then $\mathcal{V} \coloneqq \{A_j \times W_j\}_{j \in J}$ is a Γ-cover of $X \times \widetilde{M}$, since for a.e. $x \in X$, $\mathcal{V}_x = \{W_j \mid x \in A_j\}$ is a locally finite cover of $\widetilde{M}$. For every element j in a system of representatives $J' \subseteq J$ there exists a continuous function $\tau_j \colon \widetilde{M} \to [0, a_j]$ such that $\tau_{j|W_j} \equiv a_j$ and $\mathrm{supp}(\tau_j) \subseteq U_j$. This holds true by Urysohn's lemma [34, Theorem 4.82 and Corollary 4.83, p. 112-114]. Extend this definition to all $j \in J$ by $\tau_{\gamma j}(p) = \tau_j(\gamma^{-1}p)$. We define a map $\tau\colon X \times \widetilde{M} \to X \times Z$ by

$$\tau(x,p) = \left(x, \left(\chi_{A_j}(x)\tau_j(p)\right)_{j\in J}\right).$$

Here χ_{A_j} denotes the characteristic function of A_j.

This is indeed a well-defined map, since for a.e. $x \in X$ we have $(x,p) \in \bigcup_{j\in J} A_j \times U_j$. Furthermore $\mathcal{V}$ is a Γ-cover as well, so for $p \in \widetilde{M}$ there is an index $j \in J$ such that $\chi_{A_j}(x)\tau_j(p) = a_j$ and, since $\mathcal{U}_x$ is locally finite for a.e. $x \in X$, for a point $p \in \widetilde{M}$ we have $\tau_j(p) \neq 0$ only for finitely many $j \in J$. As a result, the image is contained in $X \times Z$, more precisely in $X \times \mathcal{N}(\mathcal{W})$, since a tuple $(\tau_j(p))_{j\in J}$ belongs to $\mathcal{N}(\mathcal{W})$. Moreover, by definition of the rectangular nerve, the image of τ is contained in $\mathcal{N}(\mathcal{U})$.

The defined map is equivariant. We have

$$\begin{aligned}\tau(\gamma(x,p)) = \tau(\gamma x, \gamma p) &= \left(\gamma x, \left(\chi_{A_j}(\gamma x)\tau_j(\gamma p)\right)_{j\in J}\right) = \left(\gamma x, \left(\chi_{A_{\gamma^{-1}j}}(x)\tau_{\gamma^{-1}j}(p)\right)_{j\in J}\right)\\ &= \left(\gamma x, \gamma\left(\chi_{A_j}(x)\tau_j(p)\right)_{j\in J}\right) = \gamma\left(x, \left(\chi_{A_j}(x)\tau_j(p)\right)_{j\in J}\right) = \gamma\tau(x,p).\end{aligned}$$

Further, τ is fibrewise continuous and proper. By Lemma 2.5.2 i), given a compact subset $K \subseteq \widetilde{M}$, there is a finite Borel partition $X = \bigcup_{l=1}^{L} X_l$ such that for a.e. $x, y \in X_l$ and every $k \in K$ we have $\tau(x,k) = \tau(y,k)$. It follows that restricted to every $X_l \times K$ the nerve map is a product $\mathrm{id}_{X_l} \times \tau_l'$ and hence of finite variance. So the nerve map is an equivariant X-map which is fibrewise proper, hence equivariant geometric. The second assertion follows as in the proof of Lemma 2.5.6. □

Fundamental domains

Let $\mathcal{U} = \{A_j \times U_j \mid j \in J\}$ be a Γ-cover of $X \times \widetilde{M}$ and $\mathcal{W}$ be its underlying cover. Then the rectangular nerve $\mathcal{N}(\mathcal{U})$ is a subcomplex of $X \times \mathcal{N}(\mathcal{W})$. The group Γ acts on both spaces by permuting faces. This action is fibrewise continuous. We want to specify Borel fundamental domains for this group action (see Definition 2.3.8). First we define skeletons of an equivariant simple X-space.

Definition 2.5.15. Let $X \times Z$ be an equivariant simple X-space with Z being a CW complex. Then the *k-skeleton* of $X \times Z$ is given by $(X \times Z)^{(k)} := X \times Z^{(k)}$. The k-skeleton of a subcomplex $H \subseteq X \times Z$ is given as union of the k-skeletons of its fibres, $H^{(k)} = \coprod_{x\in X} H_x^{(k)}$.

Recall that $\mathcal{N}(\mathcal{W})$ is a locally finite, finite dimensional cuboid complex, i.e. there is a $D \in \mathbb{N}$ such that $\mathcal{N}(\mathcal{W}) = \mathcal{N}(\mathcal{W})^{(D)}$. A Borel fundamental domain for the equivariant simple X-space $X \times \mathcal{N}(\mathcal{W})$ is given by taking the union of fundamental domains of $(X \times \mathcal{N}(\mathcal{W}))^{(k)}$ for all $0 \leqslant k \leqslant D$. The intersection of such a fundamental domain with $\mathcal{N}(\mathcal{U})$ defines a Borel fundamental domain of $\mathcal{N}(\mathcal{U})$. We find fundamental domains for the skeleta of $X \times \mathcal{N}(\mathcal{W})$ by fixing a set of representatives for the open k-faces of $\mathcal{N}(\mathcal{W})^{(k)}$ for $k \geqslant 0$. Every k-face F comes with an index $j_F \in J_{1/2}(F)$ corresponding to a largest side length $a_{d(F)}(F)$. Pick a complete set of representatives J' of the free Γ-set J.

Lemma 2.5.16. *Let $\mathcal{F}$ be a Γ-fundamental domain of $\mathcal{N}(\mathcal{W})$ such that for every face $F \in \mathcal{F}$ we have $j_F \in J'$. Then $\mathcal{F}$ consists of finitely many disjoint open faces. Moreover, a fundamental domain of $X \times \mathcal{N}(\mathcal{W})$ is given by finitely many disjoint Borel sets $X \times F$ where $F \in \mathcal{F}$.*

Proof. We proof that there are only finitely many faces in $\mathcal{N}(\mathcal{W}) = \mathcal{N}(\mathcal{W})^{(D)}$ with $j_F \in J'$. Fix $j' \in J'$. Let $F \in \mathcal{N}(\mathcal{W})$ with $j_F = j'$. Then it holds $\emptyset \neq \bigcap_{j \in J_+(F)} U_j \subseteq U_{j'}$. Hence $J_+(F)$ is a finite subset of

$$J_{U_{j'}} \coloneqq \{j \in J \mid U_{j'} \cap U_j \neq \emptyset\},$$

which is a finite set, since $U_{j'}$ is relatively compact and the group Γ acts properly discontinuously on $\widetilde{M}$. The set of subsets of $J_{U_{j'}}$ is finite as well, hence there can only be finitely many faces of the above form. The assertion in the lemma then follows from the fact that the set of representatives J' is finite, since J has finitely many Γ-orbits. $\square$

A fundamental domain for $\mathcal{N}(\mathcal{U})$ results from a fundamental domain of $X \times \mathcal{N}(\mathcal{W})$ by restricting it to $\mathcal{N}(\mathcal{U})$.

Lemma 2.5.17. *Let $\mathcal{F}$ be a Γ-fundamental domain of $\mathcal{N}(\mathcal{W})$ such that for every face $F \in \mathcal{F}$ we have $j_F \in J'$. Then the following holds:*

i) There is a finite Borel partition $X = \bigcup_{l=1}^{L} X_l$ such that for a.e. $x, y \in X_l$ and every face $F \in \mathcal{F}$ we have

$$(x, F) \in \mathcal{N}(\mathcal{U}) \iff (y, F) \in \mathcal{N}(\mathcal{U}).$$

ii) A fundamental domain of $\mathcal{N}(\mathcal{U})$ is given by finitely many disjoint Borel sets of the form $X_l \times F$ for a face $F \in \mathcal{F}$ and some $l \in \{1, \dots, L\}$.

Proof. As in the proof of Lemma 2.5.16, fix $j' \in J'$. Let $(x, F) \in \mathcal{N}(\mathcal{U})$ with $j_F = j'$. Then $J_+(F)$ is a finite subset of the set

$$J_{U_{j'}} \coloneqq \{j \in J \mid U_{j'} \cap U_j \neq \emptyset\}.$$

By assumption, it holds $x \in \bigcap_{j \in J_+(F)} A_j$. We get a measurable function from X to the finite set of subsets of $J_{U_{j'}}$ mapping x to the set

$$\{\bar{J} \subseteq J_{U_{j'}} \,|\, x \in \bigcap_{j \in \bar{J}} A_j\}.$$

Now let $X = \bigcup_{l=1}^{L} X_l$ be a finite Borel partition such that this function is a constant set on each X_l. This means if $x, y \in X_l$ and $\bar{J} \subseteq J_{U_{j'}}$ with $x \in \bigcap_{j \in \bar{J}} A_j$, then $y \in \bigcap_{j \in \bar{J}} A_j$ as well. In particular this holds for $J_+(F)$, which proves the first assertion. The second assertion follows immediately from this and Lemma 2.5.16. □

Chapter 3

The good equivariant cover

In order to prove our main theorems, we follow the randomization strategy of [45] outlined in Chapter 1.

In this chapter, as well as in Chapter 4, let (M, g) be an oriented, closed and connected, d-dimensional Riemannian manifold with fundamental group $\pi_1(M) = \Gamma$. We denote by $\pi\colon \widetilde{M} \to M$ the universal covering with the induced metric $\tilde{g}$. Let $V_{\widetilde{M}}(1)$ denote the largest volume of any metric ball of radius 1 in $(\widetilde{M}, \tilde{g})$.

Let (X, μ) be a standard Borel probability space with an atom-free probability measure μ. Suppose Γ acts on X in a μ-preserving way, i.e. (X, μ) is a standard Γ-space. Further we require the action to be (essentially) free. The fundamental group acts by deck transformations on $\widetilde{M}$. Thus $X \times \widetilde{M}$ with the diagonal Γ-action is an equivariant simple X-space where the topological space has some further structure, namely the structure of a Riemannian manifold. The equivariant simple X-space is equipped with the product measure of μ and the Riemannian measure vol on $\widetilde{M}$. We denote this measure by ν.

In this chapter we construct a suitable Γ-cover of $X \times \widetilde{M}$ implementing Guth's construction of a good cover [30, Section 1] and derive a couple of properties. In particular, we show in Section 3.2 that the induced covers satisfy certain multiplicity bounds. Moreover, we can show that the measure of the subset with high-multiplicity in $X \times \widetilde{M}$ is under control.

By convention if $B = B(p, r) \subset \widetilde{M}$ denotes the concentric open ball of radius r around p, aB is the concentric ball of radius $a \cdot r$ around p.

3.1 Construction of a good equivariant cover

In contrast to the situation in [45], there is no packing type inequality on the manifold (M, g). In order to produce a suitable Γ-cover of $X \times \widetilde{M}$ we adapt ideas of L. Guth. In [30] he introduced a so-called *good cover* of the manifold, which allows to still find certain bounds on the multiplicity of points. The cover sets are *good balls*, open balls which satisfy certain geometric conditions. Recall the exact definition [30, Section 1]:

Definition 3.1.1. Let (N, g) be a connected d-dimensional Riemannian manifold and $V_N(1)$ be the supremal volume of a 1-ball in N. The ball $B(p, r) \subseteq N$ of radius r around a point $p \in M$ is called a *good ball* if the following conditions are satisfied.

i) Reasonable growth: $\mathrm{vol}(B(p, 100r)) \leqslant 10^{4(d+3)} \,\mathrm{vol}(B(p, \frac{1}{100}r))$.

ii) Volume bound: $\mathrm{vol}(B(p, r)) \leqslant 10^{2(d+3)} V_N(1) r^{d+3}$.

iii) Small radius: $r \leqslant \frac{1}{100}$.

Remark 3.1.2. As Guth remarked in [30, Section 1], the exact constants are not important. The given choice of constants ensures that in comparison with the equalities satisfied in Euclidean space, the reasonable growth condition is relaxed whereas the bound on the volume is much stronger than the Euclidean bound if one regards small radii. A good ball with small radius has a very small volume. Furthermore, the choice of the constants ensures that for any point in a complete manifold there exists a concentric good ball [30, Lemma 1].

Definition 3.1.3. A *good cover* of a connected d-dimensional Riemannian manifold is an open cover by good balls where the concentric $\frac{1}{6}$-balls are disjoint and the $\frac{1}{2}$-balls provide a cover of the manifold as well.

Guth showed that any closed Riemannian manifold has a good cover [30, Lemma 2]. Having this in mind we show that there exists a good Γ-cover of $X \times \widetilde{M}$.

Theorem 3.1.4. *Let M be a Riemannian manifold and (X, μ) be a standard Γ-space as in the assumptions stated in the beginning of this chapter.*

Then there are countable families $\{A_j\}_{j \in J}$ of Borel subsets of X and $\{B_j\}_{j \in J}$ of open balls in $\widetilde{M}$ such that:

i) Each B_j is good ball.

ii) $\mathcal{U} := \{A_j \times B_j\}_{j \in J}$ is a Γ-cover of $X \times \widetilde{M}$.

iii) $\mathcal{U}(\frac{1}{2}) \coloneqq \{A_j \times \frac{1}{2}B_j\}_{j\in J}$ *is a* Γ*-cover of* $X \times \widetilde{M}$.

iv) $\mathcal{U}(\frac{1}{6}) \coloneqq \{A_j \times \frac{1}{6}B_j\}_{j\in J}$ *is a* Γ*-packing of* $X \times \widetilde{M}$.

We call the cover $\mathcal{U} \coloneqq \{A_j \times B_j\}_{j\in J}$ *a* good Γ-cover of $X \times \widetilde{M}$.

The idea to prove this theorem is to first cover $\widetilde{M}$ equivariantly by good balls. By compactness of M we can ensure that we only need finitely many orbits of balls. Then we choose a convenient subcover by ideas of the Vitali covering lemma (see for example [13, Theorem 1.24, p. 36]). For every ball we need to fix a suitable Borel set A_j. In order to do this, we need the following lemma, which appears in parts in [45, Proof of Theorem 4.1].

Lemma 3.1.5. *Let* $A \subseteq X$ *be a Borel set with* $\mu(A) > 0$. *For every finite* $F \subset \Gamma$ *with* $1 \notin F$ *there exists a Borel subset* $A' \subseteq A$ *of positive measure such that* $\mu(A' \cap \gamma A') = 0$ *for all* $\gamma \in F$. *In particular, we can choose* A' *in a maximal way, i.e. if there is another Borel subset* $\bar{A} \subseteq A$ *containing almost every element* $x \in A'$ *and* $\mu(\bar{A} \triangle A') > 0$, *then* $\mu(\bar{A} \cap \gamma\bar{A}) > 0$ *for an element* γ *in* F.

Proof. The first part follows analogously to [45, proof of Theorem 4.1]. We use [45, Lemma 4.2] which states that for every element $\gamma \in \Gamma\backslash\{1\}$ there is a Borel subset $A'' \subseteq A$ with $\mu(A'') > 0$ such that $\mu(A'' \cap \gamma A'') = 0$. Applying this repeatedly for $F = \{\gamma_1, \dots, \gamma_k\}$ we obtain Borel sets $A_1, \dots, A_k$, with $A_{m+1} \subseteq A_m$ and $\mu(A_m \cap \gamma A_m) = 0$. Then $A' \coloneqq A_k$ works.

For the second part we look at the set

$$\mathcal{M} \coloneqq \{A' \subseteq A \,|\, \mu(A') > 0,\ \mu(A' \cap \gamma A') = 0 \text{ for all } \gamma \in F\},$$

and consider the equivalence relation on it given by

$$A' \sim A'' \quad :\Longleftrightarrow \quad \mu(A' \triangle A'') = 0.$$

We set $\mathcal{M}' = \mathcal{M}/\sim$. For elements $[A']$ and $[A'']$ in $\mathcal{M}'$ we say $[A'] \leqslant [A'']$ if almost every $x \in A'$ lies in A''. This defines a well-defined partial order on $\mathcal{M}'$. Let $K \subset \mathcal{M}'$ be a totally ordered subset. We consider the injective well-defined function

$$\begin{aligned} \varphi\colon K &\longrightarrow [0,1], \\ [A'] &\longmapsto \mu(A'). \end{aligned}$$

In particular, if $[A'] \leqslant [A'']$, then $\varphi([A']) \leqslant \varphi([A''])$.
Let $m \coloneqq \sup_{[A']\in K} \varphi([A']) \in [0,1]$. If m is attained by an element $[A'] \in K$, this element is

an upper bound in K for the totally ordered subset. Otherwise we can choose a sequence $[A_k]$ from K such that

$$\lim_{k\to\infty} \mu(A_k) = \lim_{k\to\infty} \varphi([A_k]) = m.$$

Then $[\bigcup_{k\in\mathbb{N}} A_k] = \bigcup_{k\in\mathbb{N}}[A_k]$ is an upper bound for K by the following consideration: Let $[A'] \in K$. Then $\mu(A') < m$ and there is a $k \in \mathbb{N}$ such that $\mu(A') \leqslant \mu(A_k)$. Hence $[A'] \leqslant [A_k] \leqslant [\bigcup_{k\in\mathbb{N}} A_k]$. The set $\bigcup_{k\in\mathbb{N}} A_k$ has positive measure. Further, for all $\gamma \in F$ we obtain

$$\begin{aligned}
\mu\left((\bigcup_{k\in\mathbb{N}} A_k) \cap \gamma(\bigcup_{k\in\mathbb{N}} A_k) \right) &= \mu\left((\bigcup_{k\in\mathbb{N}} A_k) \cap (\bigcup_{k\in\mathbb{N}} \gamma A_k) \right) \\
&= \mu\left(\bigcup_{k\in\mathbb{N}} \bigcup_{l\in\mathbb{N}} A_k \cap \gamma A_l \right) \leqslant \mu\left(\bigcup_{k\in\mathbb{N}} A_k \cap \gamma A_k \right) = 0,
\end{aligned}$$

where the inequality holds since the $[A_k]$ are totally ordered.

Hence every totally ordered subset of $\mathcal{M}'$ has an upper bound and by the Lemma of Zorn there exists a maximal element $[A'] \in \mathcal{M}'$. A representative A' of this equivalence class is an element of $\mathcal{M}$ which is maximal in the sense described in the lemma. □

Proof of Theorem 3.1.4. By [30, Lemma 1] there exists a good ball $B(p, r)$ for every $p \in \widetilde{M}$. In particular, it holds $r \leqslant \frac{1}{100}$. The fundamental group Γ acts freely on $\widetilde{M}$, thus $\widetilde{M} = \bigcup_{l\in L} \Gamma x_l$ for some set of representatives $\{x_l\}_{l\in L}$.

By fixing good balls $B_l := B(x_l, r_l)$ and setting $B_{\gamma l} := \gamma B_l$ (good ball around γx_l) we get an equivariant cover of $\widetilde{M}$ by good balls. In particular, the family $\{\frac{1}{6}B_{\gamma l} \,|\, \gamma \in \Gamma, l \in L\}$ covers $\widetilde{M}$ as well. The projections of the cover sets form a cover of M given by $\{\pi(\frac{1}{6}B_{\gamma l}) \,|\, \gamma \in \Gamma, l \in L\} = \{\pi(\frac{1}{6}B_l) \,|\, l \in L\}$. Since the covering map is open, this is an open cover of M and by compactness of the manifold we find a finite subcover $\{\pi(\frac{1}{6}B_i) \,|\, i \in \{1, \ldots n\}\}$. This yields equivariant covers of $\widetilde{M}$ with finitely many orbits of balls

$$\begin{aligned}
\mathcal{V}\left(\tfrac{1}{6}\right) &= \{\tfrac{1}{6}B_{\gamma i} \,|\, i \in \{1, \ldots, n\}, \gamma \in \Gamma\} \\
\mathcal{V} &= \{B_{\gamma i} \,|\, i \in \{1, \ldots, n\}, \gamma \in \Gamma\}.
\end{aligned}$$

We denote the centres of the balls $B_1, \ldots, B_n$ by $p_1, \ldots, p_n$ and the radii by $r_1, \ldots, r_n$ where we arrange the indices such that $r_1 \geqslant r_2 \geqslant \ldots \geqslant r_n$. In particular, $B_{\gamma i} = \gamma B_i$ is the ball with centre $p_{\gamma i} = \gamma p_i$ and radius $r_{\gamma i} = r_i$. We aim to construct a Γ-cover of $X \times \widetilde{M}$ by elements in $\mathcal{B}(X) \times \mathcal{V}$, where $\mathcal{B}(X)$ denotes the set of Borel sets in X. We achieve this by constructing a Γ-packing by elements in $\mathcal{B}(X) \times \mathcal{V}(\frac{1}{6})$ first. So we have to choose a suitable Borel subset $A_i \subseteq X$ for each ball B_i for $i = 1, \ldots, n$. Lemma 3.1.5 allows us to adjust the measurable component. We start with the first ball $\frac{1}{6}B_1$ and set

$$F_1 \coloneqq \{\gamma \in \Gamma \mid \gamma \neq 1,\ \tfrac{1}{6}B_1 \cap \gamma\tfrac{1}{6}B_1 \neq \emptyset\} \subset \Gamma.$$

This is a finite subset since Γ acts properly discontinuously on $\widetilde{M}$. Then by the above Lemma 3.1.5, there is a maximal Borel subset $A_1 \subseteq X$ of positive measure such that $\mu(A_1 \cap \gamma A_1) = 0$ for all $\gamma \in F_1$. We set

$$\mathcal{U}_1(\tfrac{1}{6}) \coloneqq \{\gamma(A_1 \times \tfrac{1}{6}B_1) \mid \gamma \in \Gamma\},$$

which is a Γ-packing by construction.
Set

$$F_k \coloneqq \{\gamma \in \Gamma \mid \gamma \neq 1,\ \tfrac{1}{6}B_k \cap \gamma\tfrac{1}{6}B_k \neq \emptyset\}$$

for $k = 2, \ldots, n$, which are finite sets since Γ acts properly discontinuously on $\widetilde{M}$. Assume that for some $k \in \{2, \ldots, n\}$ we have chosen Borel sets $A_1, \ldots, A_{k-1}$ from $\mathcal{B}(X)$ and sets

$$\mathcal{U}_m(\tfrac{1}{6}) \coloneqq \{\gamma(A_m \times \tfrac{1}{6}B_m) \mid \gamma \in \Gamma\}$$

for $m = 1, \ldots, k-1$ such that $\mathcal{U}_1(\frac{1}{6}) \cup \ldots \cup \mathcal{U}_{k-1}(\frac{1}{6})$ is a Γ-packing. For $m = 1, \ldots, k-1$ set

$$G_k^m \coloneqq \{\gamma \in \Gamma \mid \tfrac{1}{6}B_k \cap \gamma\tfrac{1}{6}B_m \neq \emptyset\}$$

and $G_k = \bigcup_{m=1}^{k-1} G_k^m$, which are finite subsets of Γ. Then consider the two sets

$$\begin{aligned}
\mathcal{S}_k &\coloneqq \{A \subseteq X \mid \mu(A) > 0,\ \mu(A \cap \gamma A) = 0 \text{ for all } \gamma \in F_k\},\\
\mathcal{T}_k &\coloneqq \{A \in \mathcal{S}_k \mid \forall\, m = 1, \ldots, k-1\colon \mu(A \cap \gamma A_m) = 0 \text{ for all } \gamma \in G_k^m\}.
\end{aligned}$$

If $\mathcal{T}_k = \emptyset$, we set $A_k = \emptyset$, which is the same as omitting the ball B_k in the packing or the cover we want to construct. Otherwise we can show, by using the same arguments as in the proof of Lemma 3.1.5, that we can choose a maximal set $A_k \in \mathcal{T}_k$. Here, maximal means that there is no other element A' in $\mathcal{T}_k$ containing almost every point of A_k such that $\mu(A' \triangle A_k) > 0$. We set

$$\mathcal{U}_k(\tfrac{1}{6}) \coloneqq \{\gamma(A_k \times \tfrac{1}{6}B_k) \mid \gamma \in \Gamma\}$$

and get a Γ-packing $\mathcal{U}_1(\frac{1}{6}) \cup \ldots \cup \mathcal{U}_k(\frac{1}{6})$.
In the end this construction yields a Γ-packing of $X \times \widetilde{M}$ by

$$\mathcal{U}(\tfrac{1}{6}) \coloneqq \bigcup_{i=1}^{n} \mathcal{U}_i(\tfrac{1}{6}) = \{\gamma(A_i \times \tfrac{1}{6}B_i) \mid \gamma \in \Gamma, i \in \{1, \ldots, n\}\}.$$

Note that at this point $A_i = \emptyset$ is possible. Otherwise we have $\mu(A_i) > 0$. We get a candidate for the good Γ-cover by

$$\mathcal{U} = \{\gamma(A_i \times B_i) \mid \gamma \in \Gamma, i \in \{1, \ldots, n\}\}.$$

The balls γB_i are good balls hence it only remains to check that $\mathcal{U}(\frac{1}{2})$ and $\mathcal{U}$ are Γ-covers. The first property in the definition of a Γ-cover (Definition 2.5.1) is satisfied by construction. It remains to check the other two conditions. We show that for $p \in \widetilde{M}$ and a.e. $x \in X$ we have $(x, p) \in \bigcup_{i=1}^{n} \gamma(A_i \times \frac{1}{2}B_i)$. Then the analogous statement holds for $\mathcal{U}$ as well.

Assume there is a $p \in \widetilde{M}$ and Borel subset $A \subseteq X$ of positive measure such that

$$(x, p) \notin \bigcup_{\gamma \in \Gamma} \bigcup_{i=1}^{n} \gamma(A_i \times \tfrac{1}{2}B_i) \quad \text{for all } x \in A.$$

Since $\mathcal{V}(\frac{1}{6}) = \{\frac{1}{6}\gamma B_i \mid \gamma \in \Gamma, i \in \{1, \ldots, n\}\}$ covers $\widetilde{M}$, p belongs to a ball $\frac{1}{6}\gamma B_i$. We can assume $p \in \frac{1}{6}B_k$ for a $k \in \{1, \ldots, n\}$. Otherwise, if $p \in \frac{1}{6}\gamma B_k$, we consider $p' = \gamma^{-1}p$ and $A' = \gamma^{-1}A$ instead. In particular, we have for all $x \in A$

$$(x, p) \notin \bigcup_{\gamma \in \Gamma} \bigcup_{i=1}^{k-1} \gamma(A_i \times \tfrac{1}{2}B_i).$$

Hence for all $i \in \{1, \ldots, k-1\}$ and every element $\gamma \in \Gamma$ one of the following cases has to be given:
Case 1: $p \notin \frac{1}{2}\gamma B_i$, i.e. $d_{\widetilde{M}}(p, \gamma p_i) \geqslant \frac{1}{2}r_i$ where p_i is the centre of B_i. Since $p \in \frac{1}{6}B_k$ we have $\frac{1}{6}B_k \subseteq B(p, \frac{1}{3}r_k)$ and, using $r_i \geqslant r_k$,

$$d(p, \gamma p_i) \geqslant \tfrac{1}{2}r_i \geqslant \tfrac{1}{6}r_i + \tfrac{1}{3}r_k.$$

This implies $\frac{1}{6}B_k \cap \gamma(\frac{1}{6}B_i) = \emptyset$ hence $\gamma \notin G_k$.
Case 2: $x \notin \gamma A_i$ for all $x \in A$ and thus

$$\mu(A \cap \gamma A_i) = 0.$$

In particular this holds for all elements $\gamma \in G_k$.

As a result, the assumption that $(x,p) \notin \bigcup_{\gamma\in\Gamma}\bigcup_{i=1}^{n}\gamma(A_i \times \frac{1}{2}B_i)$ for all $x \in A$ implies that if $\gamma \in G_k$, i.e. $\frac{1}{6}B_k \cap \gamma\frac{1}{6}B_i \neq \emptyset$ for some element $\gamma \in \Gamma$ and some $i \in \{1,\dots,k-1\}$, then $\mu(A \cap \gamma A_i) = 0$. Further we have $\mu(A \cap A_k) = 0$, since $(x,p) \notin A_k \times \frac{1}{2}B_k$. By Lemma 3.1.5 we can choose a subset of positive measure $A' \subseteq A$ lying in $\mathcal{S}_k$ satisfying the same property. Thus it belongs to $\mathcal{T}_k$. If $A_k = \emptyset$ this contradicts $\mathcal{T}_k = \emptyset$. Otherwise $A' \cup A_k$ belongs to $\mathcal{T}_k$ as well contradicting the maximality of A_k.

As a last step we show that $\mathcal{U}$ (and therefore $\mathcal{U}(\frac{1}{2})$) satisfies the following property of a Γ-cover: The induced cover $\mathcal{U}_x = \{B_{\gamma i} \mid x \in A_{\gamma i} \mid \gamma \in \Gamma, i \in \{1,\dots,n\}\}$ of $\{x\} \times \widetilde{M} \cong \widetilde{M}$ is locally finite for a.e. $x \in X$. This holds true since the induced covers are subcovers of the locally finite cover $\{B_{\gamma i} \mid \gamma \in \Gamma, i \in \{1,\dots,n\}\}$. Any ball $B_{\gamma i}$ intersects only finitely many other sets, since the balls are relatively compact and the group Γ acts properly discontinuously on $\widetilde{M}$.

Therefore, $\mathcal{U}(\frac{1}{2})$ and $\mathcal{U}$ are Γ-covers and $\mathcal{U}$ is the desired good Γ-cover of $X \times \widetilde{M}$. This concludes the proof. □

We have constructed a Γ-packing of $X \times \widetilde{M}$

$$\mathcal{U}(\tfrac{1}{6}) = \{\gamma(A_i \times \tfrac{1}{6}B_i) \mid \gamma \in \Gamma, i \in \{1,\dots,n\}\}$$

leading to a good Γ-cover. Without loss of generality, we simply omit the balls with $A_i = \emptyset$ and assume that $\mu(A_i) > 0$ for all $i = 1,\dots,n$. Introducing a free Γ-set $J \coloneqq \{\gamma i \mid \gamma \in \Gamma,\ i \in \{1,\dots,n\}\}$ we can simplify the above expression to

$$\mathcal{U}(\tfrac{1}{6}) = \{A_j \times \tfrac{1}{6}B_j\}_{j\in J}.$$

Then a good Γ-cover of $X \times \widetilde{M}$ is given by

$$\mathcal{U} = \{\gamma(A_i \times B_i) \mid \gamma \in \Gamma, i \in \{1,\dots,n\}\} = \{A_j \times B_j\}_{j\in J}.$$

Note that the index set J is countable by construction. We already stated properties of a Γ-cover in Lemma 2.5.2. In particular, $\mathcal{U}_x = \{B_j \mid x \in A_j\}$ is a cover of $\widetilde{M}$ for a.e. $x \in X$. For the constructed good Γ-cover we have the following additional property.

Lemma 3.1.6. *Let $\mathcal{U}$ be a good Γ-cover of $X \times \widetilde{M}$. Then $\mathcal{U}_x$ is a good cover for $\widetilde{M} \cong \{x\} \times \widetilde{M}$ for a.e. $x \in X$.*

Proof. For a.e. $x \in X$ the set $\mathcal{U}_x = \{B_j \mid x \in A_j\}$ is a cover of $\widetilde{M} = \{x\} \times \widetilde{M}$ where the open sets B_j are good balls. By the same reason, $\mathcal{U}(\frac{1}{2})_x$ covers $\{x\} \times \widetilde{M}$ whereas $\mathcal{U}(\frac{1}{6})_x$ is a packing, i.e. the balls $\frac{1}{6}B_j$ with $x \in A_j$ are disjoint (see Lemma 2.5.3). Thus the induced covers $\mathcal{U}_x$ yield good covers as introduced by Guth for a.e. $x \in X$. □

3.1.1 Some remarks on multiplicity

Assume that V_0 is a positive real number such that the supremal volume of a 1-ball in the universal cover $\widetilde{M}$, $V_{\widetilde{M}}(1)$, is bounded by this number.

Remark 3.1.7. Note that the volume of the appearing good balls is bounded from below. Since M is compact, the injectivity radius of M, $\mathrm{inj}(M)$, is positive and the sectional curvature is bounded from above by some constant κ [7, Corollary to Proposition 1, p. 167]. Take $0 < r' \leqslant \inf\{\mathrm{inj}(M), \frac{1}{6}r_1, \ldots, \frac{1}{6}r_n\}$ and consider the balls $\pi(B(p_i, r')) = B(\pi(p_i), r') \subseteq M$. By the Bishop-Gunther inequality [21, Theorem 3.101, p.140] the volume of these balls is bounded from below by the volume $\mathrm{vol}(B^\kappa(r'))$ of the r'-ball in the simply connected Riemannian d-manifold H with constant curvature κ. In particular, we obtain

$$\mathrm{vol}(B_i) \geqslant \mathrm{vol}(\tfrac{1}{6}B_i) \geqslant \mathrm{vol}(B(p_i, r')) \geqslant \mathrm{vol}(B(\pi(p_i), r')) \geqslant \mathrm{vol}(B_H(r')) =: C_0$$

for $i = 1, \ldots, n$.

Remark 3.1.8. The multiplicity of the induced covers $\mathcal{U}_x$ is bounded from above by some constant N_0. Let $p \in \widetilde{M}$ be an arbitrary point. By construction $\mathcal{U}(\frac{1}{6})$ is a Γ-packing of $X \times \widetilde{M}$ hence for a.e. $x \in X$, $\mathcal{U}(\frac{1}{6})_x$ is a packing of $\widetilde{M}$ by Lemma 2.5.3. Thus the elements $\frac{1}{6}B_{\gamma i} \in \mathcal{U}(\frac{1}{6})_x$ are disjoint. By Remark 3.1.7 the volume of these elements is bounded from below by some constant C_0, hence the number of elements of $\mathcal{U}(\frac{1}{6})_x$ contained in the concentric 1-ball $B(p, 1)$ is bounded from above by $\mathrm{vol}(B(p, 1))/C_0$. By assumption this number is at most $V_0/C_0 =: N_0$. Since the radius of a good ball is at most $\frac{1}{100}$, a ball of radius $R < 1 - 2\frac{1}{100}$ can intersect at most N_0 elements of $\mathcal{U}_x$. Therefore, the multiplicity of $\mathcal{U}_x$ at the point p is bounded by N_0.

Note that this upper bound on the multiplicity is not a dimensional constant. There is no way to find such a universal bound on the multiplicity. Following Guth [30] we will prove a weaker estimate in the next section, bounding the volume of the set where the multiplicity with respect to $\mathcal{U}_x$ is high. Moreover, we show that such an estimate even holds for the whole good Γ-cover.

3.2 Properties of the good equivariant cover

We have constructed a good Γ-cover $\mathcal{U} = \{A_j \times B_j \mid j \in J\}$ of $X \times \widetilde{M}$ such that $\mathcal{U}_x = \{B_j \mid x \in A_j\}$ is a good cover of $\{x\} \times \widetilde{M} \cong \widetilde{M}$ for a.e. $x \in X$ (Lemma 3.1.6).

Hence the properties of a good cover proven in [30] hold also in the given setting with slight modifications. In particular, we can show that for the constructed good Γ-cover $\mathcal{U}$ the measure of the set of high-multiplicity is bounded. First, we regard the properties of the induced covers $\mathcal{U}_x$.

The following lemma by Guth is local hence applies on the complete manifold $\widetilde{M}$. It allows to estimate the number of balls of comparably large radius intersecting a given ball.

Lemma 3.2.1. *[30, Lemma 3] There is a dimensional constant $C(d)$ such that for a.e. $x \in X$ the following holds true. Let $s < 1$ and $B(s) \subseteq \widetilde{M}$ be a ball of radius s, which does not necessarily belong to $\mathcal{U}$. Then the number of balls $B_j \in \mathcal{U}_x$ intersecting $B(s)$, whose radius is in the range $\frac{1}{2}s \leqslant r_j \leqslant 2s$, is bounded by $C(d)$. In particular, the constant $C(d)$ is independent of $x \in X$.*

Proof. For a given $x \in X$ we look at all balls $\{B_j\}$ in $\mathcal{U}_x$ that intersect $B(s)$ and satisfy the above condition on the radius. Take the ball of smallest volume and denote it by B_{j_0}. By the conditions on the radii all the balls B_j are contained in the ball $B(5s)$, which is itself contained in $20B_{j_0}$. By construction of the good cover the balls $\frac{1}{6}B_j$ are disjoint, hence $\sum_j \mathrm{vol}(\frac{1}{6}B_j) \leqslant \mathrm{vol}(20B_{j_0})$. By the reasonable growth condition of a good ball (see Definition 3.1.1) we have the estimates

$$\begin{aligned} \mathrm{vol}(B_j) &\leqslant \mathrm{vol}(100B_j) \leqslant 10^{4(d+3)} \cdot \mathrm{vol}(\tfrac{1}{100}B_j) \leqslant 10^{4(d+3)} \cdot \mathrm{vol}(\tfrac{1}{6}B_j) \\ \mathrm{vol}(20B_{j_0}) &\leqslant \mathrm{vol}(100B_{j_0}) \leqslant 10^{4(d+3)} \cdot \mathrm{vol}(\tfrac{1}{100}B_{j_0}) \leqslant 10^{4(d+3)} \cdot \mathrm{vol}(B_{j_0}), \end{aligned}$$

hence $\sum_j \mathrm{vol}(B_j) \leqslant (10^{4(d+3)})^2 \cdot \mathrm{vol}(B_{j_0})$. Having chosen B_{j_0} as the ball of smallest volume we can estimate the number of balls intersecting $B(s)$ and having roughly equal radius by $C(d) = (10^{4(d+3)})^2$. □

3.2.1 Estimates for the parts of high-multiplicity

As in Guth's paper [30, Section 2] we want to estimate the volume of the high-multiplicity set. For the induced cover $\mathcal{U}_x$ we denote the multiplicity function on $\{x\} \times \widetilde{M} \cong \widetilde{M}$ by m_x, i.e.

$$m_x : \widetilde{M} \longrightarrow \mathbb{N},\ p \longmapsto m_x(p) := |\{B_j \in \mathcal{U}_x \,|\, p \in B_j\}|.$$

The set of points which are contained in at least λ elements of the cover $\mathcal{U}_x$ is given by $\widetilde{M}^x(\lambda) := \{p \in \widetilde{M} \,|\, m_x(p) \geqslant \lambda\}$. In the same way as Guth we can bound the size of this set $\widetilde{M}^x(\lambda)$ for large λ. We define the ω-neighbourhood of an open set $U \subseteq \widetilde{M}$ by $N_\omega(U) := \{p \in \widetilde{M} \mid d_{\widetilde{M}}(p, U) < \omega\}$. For the set of points in U with multiplicity at least λ with respect to $\mathcal{U}_x$ we write $\widetilde{M}^x_U(\lambda) := U \cap \widetilde{M}^x(\lambda)$. Then the proof of [30, Lemma 4] applies as long as U is bounded and gives

Lemma 3.2.2. *There are dimensional constants $\alpha = \alpha(d), \beta = \beta(d)$ such that for a.e. $x \in X$ the following statement holds. For any bounded open set $U \subseteq \widetilde{M}$, any $\lambda \geqslant 0$ and $\omega < \frac{1}{100}$ we can estimate*

$$\operatorname{vol}\left(\widetilde{M}^x_U(\beta \log(\tfrac{1}{\omega}) + \lambda)\right) \leqslant e^{-\alpha\lambda} \operatorname{vol}(N_\omega(U)).$$

Moreover, for a good ball B of radius r in $\mathcal{U}_x$ this implies

$$\operatorname{vol}\left(\widetilde{M}^x_B(\beta \log(\tfrac{1}{r}) + \lambda)\right) \leqslant c(d) \cdot e^{-\alpha\lambda} \operatorname{vol}(B)$$

for some dimensional constant $c = c(d)$.

Remark 3.2.3. Following Guth's proof of [30, Lemma 4], we see that for the occurring dimensional constants we have $\alpha(d) = -\log(1 - (10^{4(d+3)})^{-4})$ and $\beta(d) \leqslant 2(10^{4(d+3)})^2$. The same constants will appear in the modified proof of Lemma 3.2.4 below.
For the estimate on good balls in the above lemma, note that $N_{2r}(B) = 3B$ and by the reasonable growth property of a good ball we have $\operatorname{vol}(3B) \leqslant 10^{4(d+3)} \operatorname{vol}(B)$, hence $c(d) = 10^{4(d+3)}$.

However, we require a more general version of the above lemma. Recall from Section 2.3 that the equivariant simple X-space $X \times \widetilde{M}$ is equipped with the product measure ν of the probability measure μ and the Riemannian measure vol. In particular, for a Borel fundamental domain $\mathcal{F}$ of $X \times \widetilde{M}$ we have $\nu(\mathcal{F}) = \operatorname{vol}(M)$.

Define the multiplicity of a point $(x, p) \in X \times \widetilde{M}$ by $m(x, p) := m_x(p)$. By Lemma 2.5.2 the following holds: For every compact set $K \subset \widetilde{M}$ there is a finite Borel partition $X = \bigcup_{l=1}^L X_l$ such that $m(x, k)$ is constant on X_l for every $k \in K$. This implies that the set of points of multiplicity at least λ, given by

$$(X \times \widetilde{M})(\lambda) := \{(x, p) \in X \times \widetilde{M} \mid m(x, p) = m_x(p) \geqslant \lambda\},$$

is measurable. For a Borel set $\widetilde{V} \subset X \times \widetilde{M}$ let $\widetilde{V}_x$ be the points in the fibre over x, i.e. $\widetilde{V}_x = \widetilde{V} \cap p_{\widetilde{M}}^{-1}(x)$. We can define the ω-neighbourhood of $\widetilde{V}$ in $X \times \widetilde{M}$ by $N_\omega(\widetilde{V}) := \bigcup_{x \in X} N_\omega(\widetilde{V}_x)$ and the points of multiplicity at least λ by $(X \times \widetilde{M})_{\widetilde{V}}(\lambda) = \widetilde{V} \cap (X \times \widetilde{M})(\lambda)$. Then we can prove the following generalized version of the above lemma.

Lemma 3.2.4. *There are constants $\alpha(d)$, $\beta(d)$ only depending on the dimension d such that the following estimate holds: For any open bounded subset $V \subseteq \mathcal{F}_{\widetilde{M}}$ of a Γ-fundamental domain $\mathcal{F}_{\widetilde{M}}$ of $\widetilde{M}$, any $\lambda \geqslant 0$ and $\omega < 1/100$ we have for $\widetilde{U} := \cup_{\gamma \in \Gamma} \gamma(X \times V)$ and any Borel fundamental domain $\mathcal{F}$ of $X \times \widetilde{M}$ the estimate*

$$\nu\left((X \times \widetilde{M})_{\widetilde{U}}(\beta \log(\tfrac{1}{\omega}) + \lambda) \cap \mathcal{F}\right) \leqslant e^{-\alpha\lambda} \nu(N_\omega(\widetilde{U}) \cap \mathcal{F}).$$

Thus if $V = \mathcal{F}_{\widetilde{M}}$, we have $\widetilde{U} = X \times \widetilde{M}$ and it follows

$$\nu\left((X \times \widetilde{M})(\beta \log(\tfrac{1}{\omega}) + \lambda) \cap \mathcal{F}\right) \leqslant e^{-\alpha\lambda} \operatorname{vol}(M).$$

Proof. The proof is a careful adaptation of the ideas of the proof of [30, lemma 4]. We mainly follow the notation introduced there.

The proof consists of several steps. First, we sort all elements of the cover intersecting $\widetilde{U} = \cup_{\gamma \in \Gamma} \gamma(X \times V)$ into layers such that the orbit of each set belongs to exactly one layer and the layers consist of disjoint sets. Then, for each layer we define its *core*, a subset which contains a large part of the measure of the layer but its intersection with lower layers is under control. Based on these estimates on the core, it is then possible to show the exponential decay of the measure of the set of high-multiplicity.

Let

$$\mathcal{U} = \{\gamma(A_i \times B_i) \,|\, \gamma \in \Gamma, i \in \{1, \ldots, n\}\} = \{A_j \times B_j \,|\, j \in J\}$$

be the good Γ-cover constructed before. We want to define $\mathcal{W}$ as the set of subsets of the cover sets intersecting $\widetilde{U} = \cup_{\gamma \in \Gamma} \gamma(X \times V)$. This means there needs to be a subset $\{i_1 < i_2 < \ldots < i_{n'}\}$ of $\{1, \ldots, n\}$ and subsets $A'_{i_k} \subseteq A_{i_k}$ of positive measure such that

$$\mathcal{W} = \{\gamma(A'_{i_k} \times B_{i_k}) \,|\, \gamma \in \Gamma, k \in \{1, \ldots, n'\}\}$$

satisfies the following: For every $k = 1, \ldots, n'$ and every $\gamma \in \Gamma$ we have

$$\gamma B_{i_k} \cap \widetilde{U}_x \neq \emptyset \quad \text{for a.e. } x \in \gamma A_{i_k}. \tag{3.1}$$

Note that the bounded set $V \subset \mathcal{F}_{\widetilde{M}}$ is relatively compact, since $\widetilde{M}$ is complete. Then the set $J_V = \{j \in J \,|\, V \cap B_j \neq \emptyset\}$ is finite since Γ acts properly discontinuously on $\widetilde{M}$. Moreover, the sets $J_V(x) = \{j \in J_V \,|\, x \in A_j\}$ are finite and the assignment $x \mapsto J_V(x)$ defines a measurable function from X to the finite set of subsets of J_V. Let $X = \bigcup_{l=1}^{L} X_l$

be a finite Borel partition such that $J_V(x)$ is constant on each X_l. Then we have for every $k \in V$ and a.e. $x, y \in X_l$ that

$$(x,k) \in A_j \times B_j \quad \Longleftrightarrow \quad (y,k) \in A_j \times B_j.$$

This implies that for a.e. $x, y \in X_l$ we have $B_j \cap \{x\} \times V \neq \emptyset$ if and only if $B_j \cap \{y\} \times V \neq \emptyset$. By this considerations we obtain that

$$\{x \in A_j \mid B_j \cap \{x\} \times V \neq \emptyset\}$$

is a measurable subset of A_j. In the same way, using Borel partitions corresponding to γV, we can show for every $\gamma \in \Gamma$ that $\{x \in A_j \mid B_j \cap \{x\} \times \gamma V \neq \emptyset\}$ is measurable. Then

$$\begin{aligned} A'_j &= \bigcup_{\gamma \in \Gamma} \{x \in A_j \mid B_j \cap \{x\} \times \gamma V \neq \emptyset\} \\ &= \{x \in A_j \mid B_j \cap \widetilde{U}_x \neq \emptyset\} \end{aligned} \tag{3.2}$$

is a measurable subset of A_j for every $j \in J$. The definition (3.2) is compatible with the group action on X. One easily verifies, that for every $\gamma \in \Gamma$ we have $\gamma A'_j = A'_{\gamma j}$. We restrict to the indices where $\mu(A'_j) > 0$. This yields a subset $\{i_1, i_2, \ldots, i_{n'}\} \subset \{1, \ldots, n\}$ such that

$$\mathcal{W} = \{\gamma(A'_{i_k} \times B_{i_k}) \mid \gamma \in \Gamma, k \in \{1, \ldots, n'\}\}$$

is the suitable subset of the cover. Note that for the radii of the B_{i_k} it holds $r_{i_1} \geqslant \ldots \geqslant r_{i_{n'}}$.

To simplify the notation in the rest of the proof we use the subscripts $\{1, \ldots, n'\}$ instead of $\{i_1, i_2, \ldots, i_{n'}\}$. Further we write A_i instead of A'_i. Since we only focus on $\mathcal{W}$, not on the whole cover $\mathcal{U}$, this should not lead to any confusion.
With this notation we have

$$\mathcal{W} = \{\gamma(A_i \times B_i) \mid \gamma \in \Gamma, i \in \{1, \ldots, n'\}\} = \{A_j \times B_j \mid j \in J'\}$$

for the Γ-set $J' := \{\gamma i \mid \gamma \in \Gamma,\, i \in \{1, \ldots, n'\}\}$. The radius of $B_j = B_{\gamma i}$ is $r_j = r_{\gamma i} = r_i$ and it holds $r_1 \geqslant \ldots \geqslant r_{n'}$. Note that for every $i = 1, \ldots, n'$ we have a subset

$$F_i := \{\gamma \in \Gamma \mid \gamma \neq 1, \gamma B_i \cap B_i \neq \emptyset\} \subset \Gamma,$$

which is finite since the group Γ acts properly discontinuously on $\widetilde{M}$.

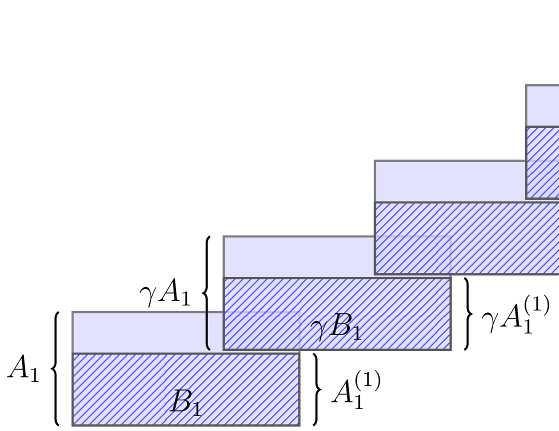

Figure 3.1: Adjusting the measurable component: Choose a maximal Borel subset $A_1^{(1)} \subset A_1$ such that the product sets $\gamma(A_1^{(1)} \times B_1)$ do not intersect up to null sets.

Construction of the layers

We divide $\mathcal{W}$ into layers by possibly subdividing the measurable sets A_j. The procedure is similar to the construction of $\mathcal{U}$ in Section 3.1. For the first layer, Layer(1), we look at $A_1 \times B_1$, the set where the ball in $\widetilde{M}$ has the largest radius. We want to adjust A_1 such that the orbit of the product set has no self-intersections (see Figure 3.1). Set

$$\mathcal{S}_1^{(1)} := \{A \subseteq A_1 \mid \mu(A) > 0,\ \mu(A \cap \gamma A) = 0\ \ \forall \gamma \in F_1\}.$$

By Lemma 3.1.5 we have $\mathcal{S}_1^{(1)} \neq \emptyset$ and there is a maximal set in $\mathcal{S}_1^{(1)}$. We denote it by $A_1^{(1)}$, where the upper index indicates the layer we are working on, and the subscript specifies the initial ball $A_1 \times B_1$. The sets $\{\gamma(A_1^{(1)} \times B_1) \mid \gamma \in \Gamma\}$ will be part of Layer(1).

In the next step we look at the set $A_2 \times B_2$ and adjust A_2 such that the Γ-translates of the resulting product set are disjoint to one another and to the previous constructed set. For this we consider the finite subset

$$G_2 = G_2^1 := \{\gamma \in \Gamma \mid\ B_2 \cap \gamma B_1 \neq \emptyset\}$$

and define

$$\begin{aligned} \mathcal{S}_2^{(1)} &:= \{A \subseteq A_2 \mid \mu(A) > 0,\ \mu(A \cap \gamma A) = 0 \text{ for all } \gamma \in F_2\} \\ \mathcal{T}_2^{(1)} &:= \{A \in \mathcal{S}_2^{(1)} \mid \mu(A \cap \gamma A_1^{(1)}) = 0 \text{ for all } \gamma \in G_2\}. \end{aligned}$$

As in the construction of the cover in Section 3.1, Lemma 3.1.5 allows us to choose a maximal element in $\mathcal{T}_2^{(1)}$, which we denote by $A_2^{(1)}$. If $\mathcal{T}_2^{(1)} = \emptyset$, we set $A_2^{(1)} = \emptyset$, which is

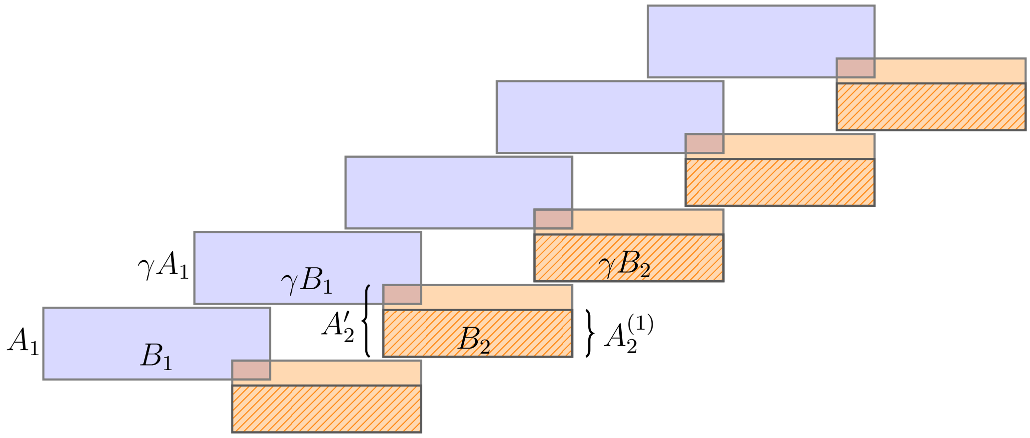

Figure 3.2: We adjust the elements $\gamma(A_2 \times B_2)$. The blue rectangles indicate the sets $\gamma(A_1^{(1)} \times B_1)$. The orange rectangles result from $\gamma(A_2 \times B_2)$ by adjusting the measurable component to $A'_2 \subset A_2$ as in the first step. After passing to a suitable subset $A_2^{(1)} \in \mathcal{T}_2^{(1)}$ the sets $\gamma(A_2^{(1)} \times B_2)$ have no intersection with the previously constructed elements in the same layer.

the same as omitting the ball B_2 in Layer(1). This is in particular the case if $\mathcal{S}_2^{(1)} = \emptyset$. The sets $\{\gamma(A_2^{(1)} \times B_2) \,|\, \gamma \in \Gamma\}$ will be part of Layer(1).

Basically, the step of constructing these sets consists of two parts. We first adjust A_2 to an element $A'_2 \in \mathcal{S}_2^{(1)}$ as in the first step (see Figure 3.1) to avoid self-intersections and then adjust the measurable component further to avoid intersections with the previously constructed product sets $\{\gamma(A_1^{(1)} \cap B_1)\}$ (see Figure 3.2). We construct Layer(1) by applying these two steps on all remaining sets $\gamma(A_k \times B_k)$ for $k = 3, \dots, n'$.

Assume that for some $k \geqslant 3$ we have already chosen subset $A_1^{(1)}, \dots, A_{k-1}^{(1)}$ of the sets $A_1, \dots, A_{k-1}$ and elements $\gamma(A_i^{(1)} \times B_i)$ $(\gamma \in \Gamma, i = 1, \dots, k-1)$ in Layer(1). For $m = 1, \dots, k-1$ set

$$G_k^m := \{\gamma \in \Gamma \,|\, B_k \cap \gamma B_m \neq \emptyset\} \tag{3.3}$$

and $G_k = \bigcup_{m=1}^{k-1} G_k^m$, which are finite subsets of Γ. We look at the sets

$$\begin{aligned} \mathcal{S}_k^{(1)} &:= \{A \subseteq A_k \,|\, \mu(A) > 0,\ \mu(A \cap \gamma A) = 0 \text{ for all } \gamma \in F_k\} \\ \mathcal{T}_k^{(1)} &:= \{A \in \mathcal{S}_k^{(1)} \,|\, \forall\, m = 1, \dots, k-1 \colon \mu(A \cap \gamma A_m^{(1)}) = 0 \text{ for all } \gamma \in G_k^m\}. \end{aligned}$$

Then we can choose $A_k^{(1)}$ as a maximal element of $\mathcal{T}_k^{(1)}$ or $A_k^{(1)} = \emptyset$ in case $\mathcal{T}_k^{(1)}$ is empty. In the end we obtain the first layer

$$\mathrm{Layer}(1) := \{\gamma(A_i^{(1)} \times B_i) \,|\, \gamma \in \Gamma, i \in \{1, \dots, n'\}\} = \{A_j^{(1)} \times B_j \,|\, j \in J'\}$$

where $A^{(1)}_{\gamma i} = \gamma A^{(1)}_i$.

After adjusting the A_i we have to take into account the remaining parts. Thus for Layer(2) we consider the set

$$\mathcal{W}^{(2)} := \Big\{\gamma\Big((A_i \backslash A^{(1)}_i) \times B_i\Big) \,\Big|\, \gamma \in \Gamma, i \in \{1, \dots, n'\}\Big\}.$$

Note that if $A^{(1)}_i = \emptyset$, i.e. the ball has been omitted in Layer(1), we regard the initial set $A_i \times B_i$. Repeat the same process as before to find $A^{(2)}_i \subseteq A_i \backslash A^{(1)}_i$. Here, $A^{(2)}_1$ is a maximal element in

$$\mathcal{S}^{(2)}_1 := \{A \subseteq A_1 \backslash A^{(1)}_1 \,|\, \mu(A) > 0,\ \mu(A \cap \gamma A) = 0 \text{ for all } \gamma \in F_1\}.$$

If $\mathcal{S}^{(2)}_1 = \emptyset$, which is the case if $A_1 = A^{(1)}_1$ up to null sets, we set $A^{(2)}_1 = \emptyset$. This means, that the ball is omitted in Layer(2). Inductively we set

$$\begin{aligned}
\mathcal{S}^{(2)}_k &:= \{A \subseteq A_k \backslash A^{(1)}_k \,|\, \mu(A) > 0,\ \mu(A \cap \gamma A) = 0 \text{ for all } \gamma \in F_k\} \\
\mathcal{T}^{(2)}_k &:= \{A \in \mathcal{S}^{(2)}_k \,|\, \forall\, m = 1, \dots, k-1\colon \mu(A \cap \gamma A^{(2)}_m) = 0 \text{ for all } \gamma \in G^m_k\}
\end{aligned}$$

for $k \geqslant 2$, where G^m_k is given in (3.3). We set $A^{(2)}_k$ as a maximal element of $\mathcal{T}^{(2)}_k$. This yields

$$\text{Layer}(2) := \{\gamma(A^{(2)}_i \times B_i) \,|\, \gamma \in \Gamma, i \in \{1, \dots, n'\}\} = \{A^{(2)}_j \times B_j \,|\, j \in J'\},$$

where $A^{(2)}_{\gamma i} = \gamma A^{(2)}_i$.

Assume we have constructed Layer(1), …, Layer($l-1$) in this way for some $l \geqslant 3$. Apply the above procedure on

$$\mathcal{W}^{(l)} := \left\{\gamma\Big(\Big(A_i \backslash \bigcup_{h=1}^{l-1} A^{(h)}_i\Big) \times B_i\Big) \,\middle|\, \gamma \in \Gamma, i \in \{1, \dots, n'\}\right\}.$$

That means we first regard

$$\mathcal{S}^{(l)}_1 := \left\{A \subseteq \Big(A_1 \backslash \bigcup_{h=1}^{l-1} A^{(h)}_1\Big) \,\middle|\, \mu(A) > 0,\ \mu(A \cap \gamma A) = 0 \text{ for all } \gamma \in F_1\right\}.$$

and then step by step for $k \geqslant 2$

$$\begin{aligned}
\mathcal{S}^{(l)}_k &:= \left\{A \subseteq \Big(A_k \backslash \bigcup_{h=1}^{l-1} A^{(h)}_k\Big) \,\middle|\, \mu(A) > 0,\ \mu(A \cap \gamma A) = 0 \text{ for all } \gamma \in F_k\right\} \\
\mathcal{T}^{(l)}_k &:= \{A \in \mathcal{S}^{(l)}_k \,|\, \forall\, m = 1, \dots, k-1\colon \mu(A \cap \gamma A^{(l)}_m) = 0 \text{ for all } \gamma \in G^m_k\}
\end{aligned}$$

with G_k^m given in (3.3) and we choose maximal elements $A_1^{(l)}$ in $\mathcal{S}_1^{(l)}$ and $A_k^{(l)}$ in $\mathcal{T}_k^{(l)}$, respectively. If $\mathcal{S}_1^{(l)} = \emptyset$, which is the case if $A_1 = \bigcup_{h=1}^{l-1} A_1^{(h)}$ up to null sets, we set $A_1^{(l)} = \emptyset$ and in the same way $A_k^{(l)} = \emptyset$ if $\mathcal{T}_k^{(l)} = \emptyset$. This yields

$$\mathrm{Layer}(l) := \{\gamma(A_i^{(l)} \times B_i) \mid \gamma \in \Gamma, i \in \{1, \ldots, n'\}\} = \{A_j^{(l)} \times B_j \mid j \in J'\}. \tag{3.4}$$

We say Layer(l') is *lower* than Layer(l) if $l' > l$. Layer(1) is called the *top layer*. The constructed layers have the following properties: After subdividing the measurable component of a set $A_i \times B_i$ the resulting sets belong to exactly one layer and its Γ-orbit is part of the same layer. Furthermore, every layer consists of disjoint sets. This means, if $A_k^{(l)} \times B_k$ and $A_m^{(l)} \times B_m$ belong to a Layer(l) $(k, m \in J')$ and $B_k \cap B_m \neq \emptyset$ then $\mu(A_k \cap A_m) = 0$. Analogously to the proof of Lemma 2.5.2 we can show that for every compact set $K \subset \widetilde{M}$ there is a finite Borel partition $X = \bigcup_{q=1}^{R_l} X_q^{(l)}$ such that for almost every $x, y \in X_q^{(l)}$ and every $k \in K$ we have

$$(x, k) \in A_j^{(l)} \times B_j \quad \Longleftrightarrow \quad (y, k) \in A_j^{(l)} \times B_j. \tag{3.5}$$

Moreover,

$$B_j \subseteq K,\ B_j \in \mathrm{Layer}(l)_x \quad \Longleftrightarrow \quad B_j \subseteq K,\ B_j \in \mathrm{Layer}(l)_y.$$

For each Layer$(l) = \{A_j^{(l)} \times B_j \mid j \in J'\}$ we define

$$L(l) := \bigcup_{j \in J'} A_j^{(l)} \times B_j,$$

the union of all the elements in the layer. If we restrict to a $x \in X$ we obtain what we call *induced layer*

$$\mathrm{Layer}(l)_x = \{B_j \mid x \in A_j^{(l)}, j \in J'\} \subseteq \mathcal{U}_x,$$
$$L(l)_x = \bigcup_{B_j \in \mathrm{Layer}(l)_x} B_j.$$

Note that the balls in Layer$(l)_x$ are disjoint for a.e. $x \in X$. If $B_j \in \mathrm{Layer}(l)_x$, then by construction it holds that $\gamma B_j = B_{\gamma j} \in \mathrm{Layer}(l)_{\gamma x}$ for all $\gamma \in \Gamma$. Moreover, $\gamma L(l)_x = L(l)_{\gamma x}$.

By the above consideration (see (3.5)), for every compact set $K \subset \widetilde{M}$ there is a finite Borel partition $X = \bigcup_{q=1}^{R_l} X_q^{(l)}$ such that $L(l)_x \cap K$ is constant on $X_q^{(l)}$. This implies that $L(l)$ is a measurable subset of $X \times \widetilde{M}$.

Note that a priori the described proceeding for the construction of the layers yields countably many layers. As we will show subsequently to the proof of the lemma every ball appears in some layer after finitely many steps (see Remark 3.2.7). Moreover, we show in Proposition 3.2.6 that for each $i = 1, \dots, n'$ the choice of the $A_i^{(l)}$ as maximal subsets of positive measure which are disjoint to the sets constructed before ensures that

$$\lim_{l\to\infty} \mu\Big(A_i \setminus \bigcup_{h=1}^{l-1} A_i^{(h)}\Big) = 0.$$

Hence for every $\eta > 0$ there is a $l_0 \in \mathbb{N}$ such that the remaining parts not sorted into a layer are given by

$$\mathcal{W}^{(l_0+1)} := \left\{\gamma\left(\tilde{A}_i \times B_i\right) \mid \gamma \in \Gamma, i \in \{1, \dots, n'\}\right\} = \left\{\tilde{A}_j \times B_j \,|\, j \in J'\right\},$$

where $\tilde{A}_i = (A_i \setminus \bigcup_{h=1}^{l_0} A_i^{(h)})$ and $\mu(\tilde{A}_i) \leqslant \eta$. We set

$$\mathcal{W}(l_0+1) = \bigcup_{j\in J'} \tilde{A}_j \times B_j. \tag{3.6}$$

In the same way as for $L(l)$ we can show that this set is measurable. Moreover, for every relatively compact set $K \subset \widetilde{M}$ there is a finite Borel partition $X = \bigcup_{q=1}^{R} X_q$ such that $\mathcal{W}(l_0+1)_x \cap K$ is constant on X_q. Since the manifold M is compact, a Borel fundamental domain of $X \times \widetilde{M}$ is given by $X \times K$ for a fundamental domain K of $\widetilde{M}$, which is a relatively compact set. By Lemma 2.3.9 we obtain for any fundamental domain $\mathcal{F}$ of $X \times \widetilde{M}$

$$\begin{aligned}
\nu\left(\mathcal{W}(l_0+1) \cap \mathcal{F}\right) &= \nu\left(\mathcal{W}(l_0+1) \cap (X \times K)\right) = \nu\Big(\bigcup_{i=1}^{n'} \bigcup_{\gamma\in\Gamma} \bigcup_{q=1}^{R} (\gamma\tilde{A}_i \cap X_q) \times (\gamma B_i \cap K)\Big) \\
&= \sum_{i=1}^{n'} \sum_{\gamma\in\Gamma} \sum_{q=1}^{R} \nu((\gamma\tilde{A}_i \cap X_q) \times (\gamma B_i \cap K)) = \sum_{i=1}^{n'} \sum_{\gamma\in\Gamma} \sum_{q=1}^{R} \mu(\gamma\tilde{A}_i \cap X_q)\operatorname{vol}(\gamma B_i \cap K) \\
&= \sum_{i=1}^{n'} \sum_{\gamma\in\Gamma} \mu(\gamma\tilde{A}_i)\operatorname{vol}(\gamma B_i \cap K) = \sum_{i=1}^{n'} \mu(\tilde{A}_i) \sum_{\gamma\in\Gamma} \operatorname{vol}(B_i \cap \gamma^{-1}K) = \sum_{i=1}^{n'} \mu(\tilde{A}_i)\operatorname{vol}(B_i),
\end{aligned}$$

where we use the Γ-invariance of the measures and the fact that K is a Γ-fundamental domain of $\widetilde{M}$. Hence by the properties of a good ball - the volume bound and the small radius - and $\mu(\tilde{A}_i) < \eta$ for $i = 1, \dots, n'$, we obtain

$$\nu\left(\mathcal{W}(l_0+1) \cap \mathcal{F}\right) = \sum_{i=1}^{n'} \mu(\tilde{A}_i)\operatorname{vol}(B_i) \leqslant n'\eta \cdot 10^{2(d+3)} V_{\widetilde{M}}(1) r_i^{d+3} \leqslant n'\eta V_{\widetilde{M}}(1).$$

As a result, for every $\varepsilon > 0$ there exists a $Q_0 \in \mathbb{N}$ such that

$$\nu\left(\mathcal{W}(Q_0+1) \cap \mathcal{F}\right) \leqslant \varepsilon. \tag{3.7}$$

In the following we will assume that the layers up to some very large number $Q \in \mathbb{N}$ are constructed. In all statements on some layer $\mathrm{Layer}(l)$, we imply that $l \leqslant Q$. We assume that $Q \gg (10^{4(d+3)})^2 \cdot \log(\frac{1}{\omega})$.

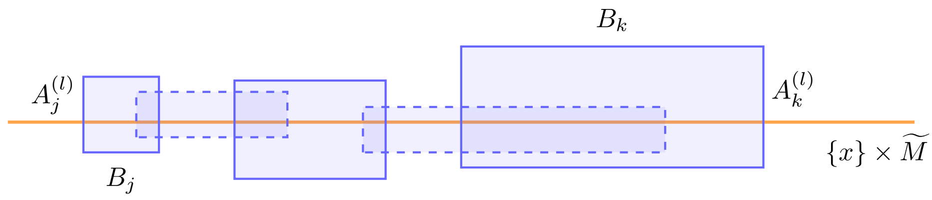

Figure 3.3: Partial order on Layer$(l)_x$: The image indicates a situation in which $B_j < B_k$ in Layer$(l)_x$. The rectangles drawn with solid lines indicate elements in Layer(l) whereas the rectangles drawn with dashed lines indicate sets in lower layers.

The core of a layer

As a next step we define a subset of each layer up to Layer(Q), the *core*, Core$(l) \subseteq L(l)$. We will see that it contains a large fraction of the measure of this layer but intersects only a bounded number of sets from lower layers. For the definition of the core Guth introduced a partial order on the layers. We regard this partial order on Layer$(l)_x = \{B_j \mid x \in A_j^{(l)}, j \in J'\}$ for a.e. $x \in X$. We recall the definition from [30, Proof of Lemma 4].

Let B_j, B_k be two balls in Layer$(l)_x$. If there is a ball $B_m \in$ Layer$(l')_x$ with $Q \geqslant l' > l$ intersecting both B_j and B_k and if the radii satisfy

$$2r_j \leqslant r_m \leqslant r_k, \tag{3.8}$$

then we say that $B_j < B_k$. The desired partial order on Layer$(l)_x$ is the minimal partial order generated by these relations. This means for two balls B_j, B_k in Layer$(l)_x$ the relation $B_j < B_k$ holds if there is a chain of balls $B_j = B_{h_0}, B_{h_1}, \ldots, B_{h_s}, B_{h_{s+1}} = B_k$ in the induced layer Layer$(l)_x$ and balls from a lower layers $B_{m_t} \in$ Layer$(l_t)_x$ for $t = 1, \ldots, s+1$ with $Q \geqslant l_t > l$ intersecting $B_{h_{t-1}}$ and B_{h_t} so that the radii obey the condition in (3.8), i.e. we have

$$2^{s+1} r_j \leqslant 2^s r_{m_1} \leqslant 2^s r_{h_1} \leqslant \ldots \leqslant 2r_{h_s} \leqslant r_{m_{s+1}} \leqslant r_k. \tag{3.9}$$

This is illustrated in Figure 3.3 for $s = 1$. This order is Γ-equivariant. For $\gamma B_j, \gamma B_k \in$ Layer$(l)_{\gamma x}$ the balls $\gamma B_{h_r} \in$ Layer$(l)_{\gamma x}$ and $\gamma B_{m_t} \in$ Layer$(l_t)_{\gamma x}$ fulfil the conditions described above, i.e. $\gamma B_j < \gamma B_k$. We get a partial order on every Layer$(l)_{\gamma x}$ for $\gamma \in \Gamma$. Moreover, this partial order has the following properties. If $B_j < B_k$ holds in some Layer$(l)_x$, the distance of their centers p_j and p_k is bounded. In particular, we have

$$d_{\widetilde{M}}(p_k, p_j) \leqslant 7r_k \leqslant 7r_1. \tag{3.10}$$

This holds by the following consideration. If $B_j < B_k$ in $\mathrm{Layer}(l)_x$ there exists a chain of overlapping balls $B_j = B_{h_0}, B_{h_1}, \dots, B_{h_s}, B_{h_{s+1}} = B_k \in \mathrm{Layer}(l)_x$ and balls of lower layers $B_{m_t} \in \mathrm{Layer}(l_t)_x$ for $t = 1, \dots, s+1$ with $Q \geqslant l_t > l$ such that the radii satisfy (3.9). Then $d_{\widetilde{M}}(p_k, p_j) \leqslant r_k + 2(r_{m_{s+1}} + r_{h_s} + \ldots + r_{h_1} + r_{m_1}) + r_j \leqslant 7r_k$.
Based on this fact we prove the following
Claim: Let $K \subset \widetilde{M}$ be a compact set. Then there is a finite Borel partition $X = \bigcup_{q=1}^{R} X_q$ such that for a.e. $x, y \in X_q$ we have

$$\begin{array}{lcl} B_j \in \mathrm{Layer}(l)_x & & B_j \in \mathrm{Layer}(l)_y \\ B_j \cap K \neq \emptyset & \Longleftrightarrow & B_j \cap K \neq \emptyset \\ B_j < B_k \text{ for a } B_k \in \mathrm{Layer}(l)_x & & B_j < B_k \text{ for a } B_k \in \mathrm{Layer}(l)_y \end{array} \tag{3.11}$$

and the same statement holds if we replace $B_j < B_k$ by $B_k < B_j$.

This is due to the following consideration. If $B_j, B_k \in \mathrm{Layer}(l)_x$ for a $x \in X$ where $B_j \cap K \neq \emptyset$ and $B_j < B_k$, we obtain by (3.10) that B_k is contained in the closed $8r_1$-neighbourhood of K, $\overline{N}_{8r_1}(K) =: K'$. This is a compact set as well. For every $l = 1, \dots, Q$ we obtain a finite Borel partition $X = \bigcup_{q=1}^{R_l} X_q^{(l)}$ such that for a ball B_m we have

$$B_m \subseteq K',\ B_m \in \mathrm{Layer}(l)_x \quad \Longleftrightarrow \quad B_m \subseteq K',\ B_m \in \mathrm{Layer}(l)_y$$

for a.e. $x, y \in X_q^{(l)}$. Let $X = \bigcup_{q=1}^{R} X_q$ be a refinement of these partitions. Hence for a.e. $x, y \in X_q$ we have

$$\begin{array}{lcl} B_m \subseteq K' & & B_m \subseteq K' \\ & \Longleftrightarrow & \\ B_m \in \mathrm{Layer}(l)_x \text{ for some } l \in \{1, \dots, Q\} & & B_m \in \mathrm{Layer}(l)_y \text{ for some } l \in \{1, \dots, Q\} \end{array}$$

Then if $B_j < B_k$ in $\mathrm{Layer}(l)_x$ for $x \in X_q$ and both balls are contained in K', we have a chain of balls in $\mathrm{Layer}(l)_x$ or lower layers $\mathrm{Layer}(l_t)_x$ such that the consecutive balls intersect and fulfil (3.8). All these balls are itself contained in K'. Thus for a.e. $y \in X_q$ we get the same chain of balls with respect to the layers restricted to y as well. This implies that $B_j < B_k$ in $\mathrm{Layer}(l)_y$ and concludes the proof of the above claim.

An element $B_j \in \mathrm{Layer}(l)_x$ is a maximal element of the partial order if there is no other ball B_k in this layer with $B_j < B_k$. Maximal elements exist by the Lemma of Zorn. A totally ordered chain is bounded above, since the radii of balls in $\mathrm{Layer}(l)_x$ are bounded and by (3.8) the radii in such a chain increase. Then we can define

$$\mathrm{Max}(l)_x = \{B_j \in \mathrm{Layer}(l)_x \mid B_j \text{ maximal}\}.$$

It holds $B_j \in \mathrm{Max}(l)_x$ if and only if $\gamma B_j \in \mathrm{Max}(l)_{\gamma x}$. Set $\mathrm{Max}(l) \coloneqq \bigcup_{x\in X} \mathrm{Max}(l)_x \subseteq L(l)$.

For a maximal ball $B_j \in \mathrm{Max}(l)_x$ the concentric ball $\frac{1}{10}B_j$ is called the core, and the union of all those define the core of the induced layer $\mathrm{Layer}(l)_x$ given by

$$\mathrm{Core}(l)_x = \bigcup_{B_j\in \mathrm{Max}(l)_x} \tfrac{1}{10}B_j \subseteq L(l)_x.$$

Note that this is a disjoint union of balls and $\gamma\mathrm{Core}(l)_x = \mathrm{Core}(l)_{\gamma x}$. The core of $\mathrm{Layer}(l)$ is then given by $\mathrm{Core}(l) = \bigcup_{x\in X} \mathrm{Core}(l)_x \subseteq L(l)$.

By the considerations above (see (3.11)), given a compact set $K \subset \widetilde{M}$, there is a finite Borel partition $X = \bigcup_{q=1}^{R} X_q$ such that $\mathrm{Max}(l)_x \cap K$ and $\mathrm{Core}(l)_x \cap K$ is constant on X_q. For this reason, $\mathrm{Core}(l)$ is a measurable subset of $X \times \widetilde{M}$.

Properties of the core

As a next step we show that a point in $\mathrm{Core}(l)_x$ lies only in a controlled number of balls in lower layers up to $\mathrm{Layer}(Q)$. Thus in particular the multiplicity with respect to $\bigcup_{m=1}^{Q} \mathrm{Layer}(m)_x$ of such a point is bounded by a certain constant.

Claim 1: For a.e. $x \in X$ the following holds true: The number of balls in lower layers (up to $\mathrm{Layer}(Q)_x$) containing a point in $\mathrm{Core}(l)_x$ is bounded by a dimensional constant.

To see this, let $p \in \frac{1}{10}B_j \subset \mathrm{Core}(l)_x$ be a point in the core with $B_j \in \mathrm{Max}(l)_x$. If p lies in a ball B_k of some lower layer, i.e. $p \in B_k \in \mathrm{Layer}(l')_x$ for $Q \geqslant l' > l$, then we show that its radius is in the range

$$\tfrac{1}{15}r_j \leqslant r_k \leqslant 2r_j. \tag{3.12}$$

If $k = \gamma i$ for a $i \in \{1,\dots,n'\}$ we can consider $\gamma^{-1}p \in \mathrm{Core}(l)_{\gamma^{-1}x}$ instead. Thus $\gamma^{-1}p \in \gamma^{-1}B_i = B_i$ with $i \in \{1,\dots,n'\}$. Hence we can assume that k is in $\{1,\dots,n'\}$. Let $A_j^{(l)} \times B_j$ and $A_k^{(l')} \times B_k$ be the specific sets of $\mathrm{Layer}(l)$ and $\mathrm{Layer}(l')$, respectively, i.e. we have $p \in \frac{1}{10}B_j \cap B_k$ and $x \in A_j^{(l)} \cap A_k^{(l')}$ and we can assume that we deal with sets of positive measure.

To show the upper bound in (3.12) assume $r_k > 2r_j$. The ball B_k does not belong to Layer$(l)_x$. In the construction of Layer(l) we considered the set of balls

$$\mathcal{W}^{(l)} := \left\{ \gamma\Big(\big(A_i \setminus \bigcup_{h=1}^{l-1} A_i^{(h)}\big) \times B_i\Big) \,\Big|\, \gamma \in \Gamma, i \in \{1, \ldots, n'\} \right\}.$$

and the sets

$$\mathcal{S}_k^{(l)} := \left\{ A \subseteq \big(A_k \setminus \bigcup_{h=1}^{l-1} A_k^{(h)}\big) \,|\, \mu(A) > 0,\ \mu(A \cap \gamma A) = 0 \text{ for all } \gamma \in F_k \right\}$$
$$\mathcal{T}_k^{(l)} := \{A \in \mathcal{S}_k^{(l)} \,|\, \mu(A \cap \gamma A_m^{(l)}) = 0 \text{ for all } \gamma, m \in \{1, \ldots, k-1\} \text{ s.th. } B_k \cap \gamma B_m \neq \emptyset\}.$$

In case $k = 1$ we only look at $\mathcal{S}_1^{(l)}$ but this is just a special case of the following considerations. Note that $A_k^{(l')} \in \mathcal{S}_k^{(l')} \subset \mathcal{S}_k^{(l)}$ since $l' > l$ and hence $A_k \setminus \bigcup_{h=1}^{l'-1} A_k^{(h)} \subseteq A_k \setminus \bigcup_{h=1}^{l-1} A_k^{(h)}$. As a result $\mathcal{S}_k^{(l)}$ cannot be empty. There are two cases which can occur: either $\mathcal{T}_k^{(l)}$ is the empty set or not.

Case 1: Suppose $\mathcal{T}_k^{(l)} = \emptyset$. Then we have in particular that $A_k^{(l')}$ is not an element in $\mathcal{T}_k^{(l)}$. Set

$$S := \{(\gamma, m) \,|\, \gamma \in \Gamma, m \in \{1, \ldots, k-1\} \text{ s.th. } B_k \cap \gamma B_m \neq \emptyset \text{ and } \mu(A_k^{(l')} \cap \gamma A_m^{(l)}) > 0\}.$$

Let $(\gamma, m) \in S$ be a pair satisfying the above conditions. If $x \in A_k^{(l')} \cap \gamma A_m^{(l)}$, we have $\gamma B_m = B_{\gamma m} \in$ Layer$(l)_x$ and $r_{\gamma m} = r_m \geqslant r_k$. By assumption we have $r_m \geqslant r_k > 2r_j$. Thus $B_m > B_j$ (see (3.8)) contradicting the maximality of B_j.
On the other hand, if $x \notin A_k^{(l')} \cap \gamma A_m^{(l)}$ but $\mu(A_k^{(l')} \cap \gamma A_m^{(l)}) > 0$ there might be another pair $(\gamma, m) \in S$ such that the above argument works. Otherwise, there has to be a subset of positive measure $A' \subseteq A_k^{(l')}$ containing x such that

$$A' \subseteq A_k^{(l')} \setminus \bigcup_{(\gamma,m) \in S} (A_k^{(l')} \cap \gamma A_m^{(l)}).$$

Thus for every $\gamma \in \Gamma$ and $m \in \{1, \ldots, k-1\}$ with $B_k \cap \gamma B_m \neq \emptyset$ we have $\mu(A' \cap \gamma A_m^{(l)}) = 0$. This implies that A' belongs to $\mathcal{T}_k^{(l)}$, which contradicts the assumption.

Case 2: Assume $\mathcal{T}_k^{(l)} \neq \emptyset$. By Lemma 3.1.5 there is a maximal element $A_k^{(l)} \in \mathcal{T}_k^{(l)}$. If $A_k^{(l')} \notin \mathcal{T}_k^{(l)}$, the arguments of the first case apply. Either there is a ball in Layer$(l)_x$ contradicting the maximality of B_j or we find a subset of positive measure containing x which lies in $\mathcal{T}_k^{(l)}$. In the latter case we can argue in the same way as for $A_k^{(l')} \in \mathcal{T}_k^{(l)}$, which is described in the following.

Thus we restrict ourselves to the case that $A_k^{(l')} \in \mathcal{T}_k^{(l)}$. Since $A_k^{(l')} \subseteq A_k \backslash \bigcup_{h=1}^{l'-1} A_k^{(h)} \subseteq A_k \backslash \bigcup_{h=1}^{l} A_k^{(h)}$ we have $\mu(A_k^{(l')} \cap A_k^{(l)}) = 0$. Then the union of these two sets, $A_k^{(l')} \cup A_k^{(l)}$, has positive measure and for all $\gamma \in \Gamma, m \in \{1, \dots, k-1\}$ such that $B_k \cap \gamma B_m \neq \emptyset$ we have

$$\begin{aligned}&\mu\left((A_k^{(l')} \cup A_k^{(l)}) \cap \gamma A_m^{(l)}\right)\\&\leqslant \mu\left(A_k^{(l')} \cap \gamma A_m^{(l)}\right) + \mu\left(A_k^{(l)} \cap \gamma A_m^{(l)}\right) - \mu\left(A_k^{(l')} \cap A_k^{(l)} \cap \gamma A_m^{(l)}\right) = 0.\end{aligned}$$

The first two terms vanishe since $A_k^{(l)}$ and $A_k^{(l')}$ are in $\mathcal{T}_k^{(l)}$, the last term vanishes since $\mu(A_k^{(l')} \cap A_k^{(l)}) = 0$. As a result, the union $A_k^{(l')} \cup A_k^{(l)}$ does not belong to $\mathcal{S}_k^{(l)}$. Otherwise it would lie in $\mathcal{T}_k^{(l)}$ as well, contradicting the maximality of $A_k^{(l)}$. Hence $A_k^{(l')} \cup A_k^{(l)} \notin \mathcal{S}_k^{(l)}$ and there is an element $\gamma \in F_k = \{\gamma \in \Gamma \mid \gamma \neq 1, \gamma B_k \cap B_k \neq \emptyset\}$ such that

$$\mu\left((A_k^{(l')} \cup A_k^{(l)}) \cap \gamma(A_k^{(l')} \cup A_k^{(l)})\right) > 0.$$

We resolve the term using $A_k^{(l')}, A_k^{(l)} \in \mathcal{S}_k^{(l)}$, i.e. $\mu(A_k^{(l')} \cap \gamma A_k^{(l')}) = 0$ and $\mu(A_k^{(l)} \cap \gamma A_k^{(l)}) = 0$ for all $\gamma \in F_k$. Then the above inequality implies that for an element $\gamma \in F_k$ we have $\mu(A_k^{(l')} \cap \gamma A_k^{(l)}) > 0$ or $\mu(A_k^{(l')} \cap \gamma^{-1} A_k^{(l)}) = \mu(A_k^{(l)} \cap \gamma A_k^{(l')}) > 0$. Note that the set $F_k \subset \Gamma$ is symmetric, i.e. $\gamma{-}1 \in F_k$ if $\gamma \in F_k$.

If $x \in A_k^{(l')} \cap \gamma A_k^{(l)}$ ($x \in A_k^{(l')} \cap \gamma^{-1} A_k^{(l)}$) we have $\gamma B_k \in \mathrm{Layer}(l)_x$ ($\gamma^{-1} B_k \in \mathrm{Layer}(l)_x$) intersecting B_k. Since by assumption $2r_j < r_k = r_{\gamma k} = r_{\gamma^{-1} k}$, we have $B_j < B_{\gamma k}$ ($B_j < B_{\gamma^{-1} k}$) by (3.8). This is a contradiction since B_j is maximal. On the other hand if there is a set of positive measure $A' \subseteq A_k^{(l')}$ containing x which is not a subset of one of these two intersections, then $A' \cup A_k^{(l)}$ would be in $\mathcal{S}_k^{(l)}$ and $\mathcal{T}_k^{(l)}$ contradicting the maximality of $A_k^{(l)}$.

In the end, both cases lead to contradictions, which proves the upper bound of (3.12), i.e. $r_k \leqslant 2r_j$.

The lower bound is independent of the maximality of B_j. Note that B_j and B_k are in particular elements of $\mathcal{U}_x$ and we constructed the cover such that $\mathcal{U}(\frac{1}{6})_x$ is a packing of $\widetilde{M}$. Hence the balls $\frac{1}{6}B_j$ and $\frac{1}{6}B_k$ are disjoint. With p_j, p_k being the centres of these balls and $p \in B_k \cap \frac{1}{10}B_j$ we obtain

$$r_k + \tfrac{1}{10} r_j > d_{\widetilde{M}}(p_k, p) + d_{\widetilde{M}}(p, p_j) \geqslant d_{\widetilde{M}}(p_k, p_j) > \tfrac{1}{6}(r_j + r_k) > \tfrac{1}{6} r_j.$$

Hence $r_k > \frac{1}{15} r_j$ and (3.12) is proven.

By Lemma 3.2.1, the number of balls $B_k \in \mathcal{U}_x$ intersecting the given B_j with radius satisfying $\frac{1}{15} r_j \leqslant r_k \leqslant 2r_j$ is bounded by a dimensional constant $C(d) = (10^{4(d+3)})^2$.

Therefore, the number of balls B_k in lower layers up to $\text{Layer}(Q)_x$ containing a point in $\text{Core}(l)_x$ is bounded by this constant as claimed. Thus the multiplicity with respect to $\bigcup_{m=1}^{Q} \text{Layer}(m)$ of points in the core has a specific upper bound.

Remark 3.2.5. The proof of the upper bound of (3.12) shows the following, which we state for later use: If $B_k \in \text{Layer}(l)_x$ for some $1 < l \leqslant Q$, then for every $1 \leqslant l' < l$ there is a ball $B_m \in \text{Layer}(l')_x$ intersecting B_k which has at least radius r_k.

As a next step we show that the core of a layer contains a large fraction of its measure. Recall that on $X \times \widetilde{M}$ we regard the measure ν given as the product of the probability measure μ on X and the Riemannian measure vol on $\widetilde{M}$. We have the following connection between $\nu(\text{Core}(l) \cap \mathcal{F})$ and $\nu(L(l) \cap \mathcal{F})$.

Claim 2: It holds $\nu(L(l) \cap \mathcal{F}) \leqslant c(d)\nu(\text{Core}(l) \cap \mathcal{F})$ for some dimensional constant $c(d) = 10^{4(d+3)}$.

To see this we first show

$$L(l)_x = \bigcup_{B_j \in \text{Layer}(l)_x} B_j \subseteq \bigcup_{B_k \in \text{Max}(l)_x} 10B_k. \tag{3.13}$$

Let $B_j \in \text{Layer}(l)_x$, i.e. $A_j^{(l)} \times B_j \in \text{Layer}(l)$ with $x \in A_j^{(l)}$. If B_j is maximal, then (3.13) holds. Otherwise there is a maximal ball $B_k \in \text{Max}(l)_x$ of radius r_k with $B_j < B_k$. This implies that $d_{\widetilde{M}}(p_k, p_j) \leqslant 7r_k$ by (3.10), where p_k and p_j are the centres of B_k and B_j, respectively. Therefore the concentric $8r_k$-ball around p_k, $8B_k = B(p_k, 8r_k)$, contains B_j and we obtain $B_j \subseteq 10B_k \subseteq \bigcup_{B_k \in \text{Max}(l)_x} 10B_k$.

In order to prove Claim 2 recall Lemma 2.3.9 and start with two Borel fundamental domains of the Γ-invariant set $L(l)$. For a Borel fundamental domain $\mathcal{F}$ of $X \times \widetilde{M}$, $L(l) \cap \mathcal{F} = \bigcup_{x \in X} \{B_j \cap \mathcal{F}_x \mid B_j \in \text{Layer}(l)_x\}$ is a fundamental domain of $L(l)$. On the other hand we get another fundamental domain if we look at balls with centre lying in the fundamental domain. Taking any point $(y, q) \in L(l) \subset X \times \widetilde{M}$ there is a unique ball $B_k \in \text{Layer}(l)_y$ with centre p_k which contains q. This is due to the fact that every layer consists of disjoint sets. There exists exactly one element $\gamma \in \Gamma$ such that γq is contained in a ball $\gamma B_k = B_{\gamma k}$ whose centre $\gamma p_k = p_{\gamma k}$ lies in $\mathcal{F}_{\gamma y}$. Thus $\bigcup_{x \in X} \{B_j \in \text{Layer}(l)_x \mid p_j \in \mathcal{F}_x\}$

is a fundamental domain of $L(l)$ as well. By Lemma 2.3.9 we have

$$\nu\left(L(l)\cap\mathcal{F}\right)=\nu\Big(\bigcup_{x\in X}\{B_j\in\mathrm{Layer}(l)_x\mid p_j\in\mathcal{F}_x\}\Big).$$

Using Fubini's theorem and the fact that the balls in the induced layers $\mathrm{Layer}(l)_x$ are disjoint we obtain

$$\nu\left(L(l)\cap\mathcal{F}\right)=\int_X \mathrm{vol}\Big(\bigcup_{\substack{B_j\in\mathrm{Layer}(l)_x\\ p_j\in\mathcal{F}_x}} B_j\Big)\,d\mu(x)=\int_X\sum_{\substack{B_j\in\mathrm{Layer}(l)_x\\ p_j\in\mathcal{F}_x}}\mathrm{vol}(B_j)\,d\mu(x).$$

By (3.13), for every ball $B_j\in\mathrm{Layer}(l)_x$ with $p_j\in\mathcal{F}$ we get a maximal ball $B_k\in\mathrm{Max}(l)_x$ such that $B_j\subseteq 10B_k$. For B_k there is exactly one Γ-translate γB_k with centre γp_k in $\mathcal{F}$. The translate is a maximal ball as well. Therefore, $\mathrm{vol}(B_j)\leqslant\mathrm{vol}(10B_k)=\mathrm{vol}(10B_{\gamma B_k})$ yields

$$\begin{aligned}\int_X\sum_{\substack{B_j\in\mathrm{Layer}(l)_x\\ p_j\in\mathcal{F}_x}}\mathrm{vol}(B_j)\,d\mu(x)&\leqslant\int_X\sum_{\substack{B_k\in\mathrm{Max}(l)_x\\ p_k\in\mathcal{F}_x}}\mathrm{vol}(10B_k)\,d\mu(x)\\&\leqslant 10^{4(d+3)}\int_X\sum_{\substack{B_k\in\mathrm{Max}(l)_x\\ p_k\in\mathcal{F}_x}}\mathrm{vol}(\tfrac{1}{10}B_k)\,d\mu(x)\\&=10^{4(d+3)}\int_X\mathrm{vol}\Big(\bigcup_{\substack{B_k\in\mathrm{Max}(l)_x\\ p_k\in\mathcal{F}_x}}\tfrac{1}{10}B_k\Big)\,d\mu(x)\\&=10^{4(d+3)}\nu\left(\mathrm{Core}(l)\cap\mathcal{F}\right).\end{aligned}$$

Here we use that good balls have reasonable growth, i.e. $\mathrm{vol}(10B_k)\leqslant 10^{4(d+3)}\,\mathrm{vol}(\frac{1}{10}B_k)$, and that the $\frac{1}{10}$-balls are disjoint, since $\mathcal{U}(\frac{1}{6})_x$ is a packing. The last equality holds by Lemma 2.3.9, since $\bigcup_{x\in X}\{B_k\in\mathrm{Max}(l)_x\mid p_k\in\mathcal{F}_x\}$ is a Borel fundamental domain of the $\mathrm{Core}(l)$. Finally, we obtain

$$\nu(L(l)\cap\mathcal{F})\leqslant 10^{4(d+3)}\nu(\mathrm{Core}(l)\cap\mathcal{F})=c(d)\nu(\mathrm{Core}(l)\cap\mathcal{F})\tag{3.14}$$

and therefore Claim 2.

Establishing the exponential decay

We prove the exponential decay of the set of high-multiplicity. Following Guth we define

$$\begin{aligned}L^\vartheta(\lambda)_x&:=\{p\in\widetilde{M}\mid p\in L(l)_x\text{ for at least }\vartheta\text{ different values of }l\text{ with }Q\geqslant l\geqslant\lambda\}\\ L^\vartheta(\lambda)&:=\bigcup_{x\in X}L^\vartheta(\lambda)_x.\end{aligned}$$

One can easily deduce from the measurability of $L(\lambda)$ that $L^{\vartheta}(\lambda)$ is measurable. We have a nested sequence $L^1(\lambda) \supseteq L^2(\lambda) \supseteq \dots$ and $L^{\vartheta}(\lambda+1) \subseteq L^{\vartheta}(\lambda)$. Note that $L^{\vartheta}(\lambda) \subseteq L^1(\lambda) = \bigcup_{l=\lambda}^{Q} L(l) = \bigcup_{l=\lambda}^{Q} \bigcup_{x\in X} \bigcup_{B_j \in \mathrm{Layer}(l)_x} B_j$. Due to the construction of the layers we have that every ball from a lower layer, $B_j \in \mathrm{Layer}(l)_x, Q \geqslant l \geqslant \lambda$ is contained in a concentric $3r_k$-ball for a ball $B_k \in \mathrm{Layer}(\lambda)_x$. This is clear for $l = \lambda$. If $Q \geqslant l > \lambda$, for every ball $B_j \in \mathrm{Layer}(l)_x$ there is some ball $B_k \in \mathrm{Layer}(\lambda)_x$ of the same or larger radius intersecting B_j (see Remark 3.2.5). Hence $B_j \subseteq 3B_k$ and $L^1(\lambda) \subseteq \bigcup_{x\in X} \bigcup_{B_k\in \mathrm{Layer}(\lambda)_x} 3B_k$. We obtain

$$\nu\left(L^1(\lambda)\cap\mathcal{F}\right) \leqslant \nu\Big(\big(\bigcup_{x\in X}\ \bigcup_{B_k\in\mathrm{Layer}(\lambda)_x} 3B_k\big)\cap\mathcal{F}\Big).$$

As before we get another Borel fundamental domain for $\bigcup_{x\in X}\bigcup_{B_k\in\mathrm{Layer}(\lambda)_x} 3B_k$ if we regard all balls with centre in $\mathcal{F}$, i.e. $\bigcup_{x\in X}\{3B_k \in \mathrm{Layer}(\lambda)_x \,|\, p_k \in \mathcal{F}_x\}$. By Fubini's theorem and the fact that the balls in one layer are disjoint and have reasonable growth we obtain

$$\begin{aligned}
&\nu\Big(\bigcup_{x\in X}\{3B_k\in\mathrm{Layer}(\lambda)_x \,|\, p_k\in\mathcal{F}\}\Big) = \int_X \mathrm{vol}\Big(\bigcup_{\substack{B_k\in\mathrm{Layer}(\lambda)_x\\ p_k\in\mathcal{F}_x}} 3B_k\Big)\, d\mu(x)\\
&\leqslant \int_X \sum_{\substack{B_k\in\mathrm{Layer}(\lambda)_x\\ p_k\in\mathcal{F}_x}} \mathrm{vol}(3B_k)\, d\mu(x) \leqslant 10^{4(d+3)} \int_X \sum_{\substack{B_k\in\mathrm{Layer}(\lambda)_x\\ p_k\in\mathcal{F}_x}} \mathrm{vol}(B_k)\, d\mu(x)\\
&= 10^{4(d+3)}\nu\left(L(\lambda)\cap\mathcal{F}\right).
\end{aligned}$$

The second to last step is due to the fact that $\bigcup_{x\in X}\{B_k\in\mathrm{Layer}(\lambda)_x \,|\, p_k\in\mathcal{F}_x\}$ is a Borel fundamental domain for $L(\lambda)$. As a result

$$\nu(L^{\vartheta}(\lambda)\cap\mathcal{F}) \leqslant \nu(L^1(\lambda)\cap\mathcal{F}) \leqslant 10^{4(d+3)}\nu\left(L(\lambda)\cap\mathcal{F}\right) = c'(d)\nu\left(L(\lambda)\cap\mathcal{F}\right). \tag{3.15}$$

In Claim 1 we showed that a point in $\mathrm{Core}(\lambda)_x \subseteq L(\lambda)_x$ is contained in at most $C(d) = (10^{4(d+3)})^2$ lower layers up to $\mathrm{Layer}(Q)$. We average over the measures of $L^{\vartheta}(\lambda)$ for values of ϑ being smaller than this constant. Define the function

$$T(\lambda) \coloneqq \tfrac{1}{C(d)} \sum_{\vartheta=1}^{C(d)} \nu\left(L^{\vartheta}(\lambda)\cap\mathcal{F}\right).$$

In particular, we have

$$\nu(L^{C(d)}(\lambda)\cap\mathcal{F}) \leqslant T(\lambda) \leqslant \nu(L^1(\lambda)\cap\mathcal{F}) \leqslant 10^{4(d+3)}\nu(L(\lambda)\cap\mathcal{F}). \tag{3.16}$$

Since the multiplicity of points in $\mathrm{Core}(\lambda)$ is under control we obtain

$$\mathrm{Core}(\lambda)_x = \bigcup_{B_j \in \mathrm{Max}(\lambda)_x} \tfrac{1}{10} B_j \subseteq \bigcup_{\vartheta=1}^{C(d)} \Big(L^{\vartheta}(\lambda)_x \backslash L^{\vartheta}(\lambda+1)_x \Big)$$

$$\mathrm{Core}(\lambda) \subset \bigcup_{\vartheta=1}^{C(d)} L^{\vartheta}(\lambda) \backslash L^{\vartheta}(\lambda+1).$$

To see this, note that $L^{\vartheta}(\lambda)\backslash L^{\vartheta}(\lambda+1)$ contains all points (x,p) from $\mathrm{Layer}(\lambda)$ which are in exactly $(\vartheta-1)$ different lower layers up to $\mathrm{Layer}(Q)$. Every point in $\mathrm{Core}(\lambda)$ is in at most $C(d)$ lower layers up to $\mathrm{Layer}(Q)$ and the above relation follows. Note that we deal with a disjoint union. We can estimate the measure of the core of $\mathrm{Layer}(\lambda)$ as follows

$$\begin{aligned}
\nu(\mathrm{Core}(\lambda) \cap \mathcal{F}) &\leqslant \nu\Big(\big(\bigcup_{\vartheta=1}^{C(d)} (L^{\vartheta}(\lambda)\backslash L^{\vartheta}(\lambda+1)) \big) \cap \mathcal{F} \Big) \\
&= \nu\Big(\bigcup_{x\in X} \bigcup_{\vartheta=1}^{C(d)} (L^{\vartheta}(\lambda)_x \backslash L^{\vartheta}(\lambda+1)_x \cap \mathcal{F}_x) \Big) \\
&= \int_X \mathrm{vol}\Big(\bigcup_{\vartheta=1}^{C(d)} L^{\vartheta}(\lambda)_x \backslash L^{\vartheta}(\lambda+1)_x \cap \mathcal{F}_x \Big)\, d\mu(x) \\
&\leqslant \sum_{\vartheta=1}^{C(d)} \int_X \mathrm{vol}\Big((L^{\vartheta}(\lambda)_x \cap \mathcal{F}_x) \backslash (L^{\vartheta}(\lambda+1)_x \cap \mathcal{F}_x) \Big)\, d\mu(x) \\
&= \sum_{\vartheta=1}^{C(d)} \int_X \Big(\mathrm{vol}(L^{\vartheta}(\lambda)_x \cap \mathcal{F}_x) - \mathrm{vol}(L^{\vartheta}(\lambda+1)_x \cap \mathcal{F}_x) \Big)\, d\mu(x) \\
&= \sum_{\vartheta=1}^{C(d)} \Big(\int_X \mathrm{vol}(L^{\vartheta}(\lambda)_x \cap \mathcal{F}_x)\, d\mu(x) - \int_X \mathrm{vol}(L^{\vartheta}(\lambda+1)_x \cap \mathcal{F}_x)\, d\mu(x) \Big) \\
&= \sum_{\vartheta=1}^{C(d)} \Big(\nu(L^{\vartheta}(\lambda) \cap \mathcal{F}) - \nu(L^{\vartheta}(\lambda+1) \cap \mathcal{F}) \Big) = C(d) \cdot (T(\lambda) - T(\lambda+1)).
\end{aligned}$$

By (3.14) and (3.15) and the definition of $T(\lambda)$ we have

$$T(\lambda) - T(\lambda+1) \geqslant \tfrac{1}{C(d)} \nu(\mathrm{Core}(\lambda) \cap \mathcal{F}) \geqslant C'(d) \nu(L^1(\lambda) \cap \mathcal{F}) \geqslant C'(d) T(\lambda)$$

where $0 < C'(d) = (10^{4(d+3)})^{-4}$, i.e. $C'(d)$ is a small dimensional constant. This yields

$$T(\lambda+1) \leqslant (1 - C'(d)) T(\lambda).$$

Then we have for every $\xi > 0$

$$T(\xi + \lambda) \leqslant e^{-\alpha(d)\lambda} \cdot T(\xi) \tag{3.17}$$

for $\alpha(d) = -\log(1 - (10^{4(d+3)})^{-4})$, so in particular

$$\alpha(d) = -\log(1 - C'(d)) \quad \Longleftrightarrow \quad e^{-\alpha(d)} = (1 - C'(d)).$$

Estimating the measure of the high-multiplicity set

We aim to control the measure of the set of high-multiplicity, thus we need to control $T(\lambda)$ for large values of λ. By (3.17) we obtain this control by first estimating $T(\lambda)$ for small λ. Basically, we show that large balls are put into the top layers.

Claim 3: For a.e. $x \in X$ the following holds true: Let B_j be a ball in $\mathrm{Layer}(l)_x \subseteq \mathcal{U}_x$ with radius r_j. Then there is a dimensional constant $\beta = \beta(d)$ such that $l \leqslant \beta \log(\frac{1}{r_j})$.

To see this, first note that B_j does not belong to $\mathrm{Layer}(l')_x$ for all $l' = 1, \ldots, l-1$. By Remark 3.2.5 there is a ball of radius at least r_j in every such induced layer intersecting B_j. Let $B_{k_{l'}} \in \mathrm{Layer}(l')_x$ denote such balls ($l' = 1, \ldots, l-1$). The radii satisfy $r_j \leqslant r_{k_{l'}} \leqslant \frac{1}{100}$. Thus l is bounded by the number of larger balls in $\mathcal{U}_x$ which intersect B_j. By Lemma 3.2.1 for $m \geqslant 1$ the number of balls with radius in $[2^{m-1}r_j, 2^{m+1}r_j]$ intersecting the concentric $2^m r_j$-ball is bounded by $C(d) = (10^{4(d+3)})^2$. Therefore, the number of balls which intersect B_j and have radius in this interval is bounded by $C(d)$. The radius of good balls is at most $\frac{1}{100}$ so the values of m which are relevant are smaller than $\log(\frac{1}{r_j})$. Hence the number of larger balls in $\mathcal{U}_x$ which possibly intersect B_j is at most $(10^{4(d+3)})^2 \log(\frac{1}{r_j}) = \beta(d) \log(\frac{1}{r_j})$. This yields an upper bound for l as claimed.

Let $\omega < \frac{1}{100}$. For a.e. $x \in X$, the radius r_j of a ball $B_j \in \mathrm{Layer}(l)_x$ with $l \geqslant \beta \log(\frac{1}{\omega})$ is at most ω. Hence all balls in the induced layers $\mathrm{Layer}(l)_x$ lower than $\mathrm{Layer}(\beta \log(\frac{1}{\omega}))_x$ are contained in the 2ω-neighbourhood $N_{2\omega}(\widetilde{U}_x)$ since they intersect $\widetilde{U}_x$ by the choice of $\mathcal{W}$ in the beginning of the proof (see (3.1)). In view of this fact, the assumption $Q \gg (10^{4(d+3)})^2 \log(\frac{1}{\omega}) = \beta(d) \log(\frac{1}{\omega})$ makes sense. All elements of higher layers are contained in the 2ω-neighbourhood of $\widetilde{U}$.

We have

$$\begin{aligned} L^1(\beta \log(\tfrac{1}{\omega})) = \bigcup_{l=\lfloor \beta \log(\frac{1}{\omega})+1 \rfloor}^{Q} L(l) &= \bigcup_{x \in X} \bigcup_{l=\lfloor \beta \log(\frac{1}{\omega})+1 \rfloor}^{Q} \bigcup_{B_j \in \mathrm{Layer}(l)_x} B_j \\ &\subseteq \bigcup_{x \in X} N_{2\omega}(\widetilde{U}_x) = N_{2\omega}(\widetilde{U}). \end{aligned}$$

Hence we obtain

$$T(\beta \log(\tfrac{1}{\omega})) \leqslant \nu(L^1(\beta \log(\tfrac{1}{\omega})) \cap \mathcal{F}) \leqslant \nu(N_{2\omega}(\widetilde{U}) \cap \mathcal{F}).$$

Together with (3.17) this yields

$$T(\beta \log(\tfrac{1}{\omega}) + \lambda) \leqslant e^{-\alpha\lambda} \nu(N_{2\omega}(\widetilde{U}) \cap \mathcal{F}). \tag{3.18}$$

It remains to establish the link between $T(\cdot)$ and the set $(X\times\widetilde{M})_{\widetilde{U}}(\lambda)$. The above statements can be shown for every Q. In the remaining part of the proof we specify the chosen Q by adding the subscript Q to all defined objects, e.g. $T_Q(\cdot)$. For a given Q we define a certain subset of $(X\times\widetilde{M})_{\widetilde{U}}(\lambda)$ whose measure can be controlled. In order to do this, we define the multiplicity of a point in $X\times\widetilde{M}$ with respect to the first Q layers. Set

$$m_x^Q:\ \widetilde{M}\to\mathbb{N},\ p\mapsto m_x^Q(p):=|\{B_j\in\bigcup_{m=1}^{Q}\mathrm{Layer}(m)_x\,|\,p\in B_j\}|.$$

and $m^Q(x,p):=m_x^Q(p)$. Define the subset

$$(X\times\widetilde{M})_{\widetilde{U}}^Q(\lambda):=\{(x,p)\in\widetilde{U}\,|\,m^Q(x,p)\geqslant\lambda\},$$

which is measurable since $(X\times\widetilde{M})_{\widetilde{U}}(\lambda)$ and the sets $L(m), m=1,\dots,Q$, are measurable. Note that for all Q, we have $(X\times\widetilde{M})_{\widetilde{U}}^Q(\lambda)\subseteq(X\times\widetilde{M})_{\widetilde{U}}^{Q+1}(\lambda)$ and $(X\times\widetilde{M})_{\widetilde{U}}^Q(\lambda)\subseteq(X\times\widetilde{M})_{\widetilde{U}}(\lambda)$. If a point $(x,p)\in(X\times\widetilde{M})_{\widetilde{U}}(\lambda)$ does not lie in $(X\times\widetilde{M})_{\widetilde{U}}^Q(\lambda)$ for a Q, then it is contained in the parts which remain after constructing Q layers, i.e. in $\mathcal{W}(Q+1)\cap\widetilde{U}$ (see (3.6)). As shown before, for every $\varepsilon>0$ there is a Q_0 such that the measure of $\mathcal{W}(Q_0+1)\cap\mathcal{F}\cap\widetilde{U}$ is at most ε (see (3.7)) and therefore

$$\nu\left((X\times\widetilde{M})_{\widetilde{U}}(\lambda)\cap\mathcal{F}\right)\leqslant\nu\left((X\times\widetilde{M})_{\widetilde{U}}^{Q_0}(\lambda)\cap\mathcal{F}\right)+\varepsilon. \tag{3.19}$$

For every Q, look at $(X\times\widetilde{M})_{\widetilde{U}}^Q(\lambda+C(d))=\{(x,p)\in\widetilde{U}\,|\,m_x^Q(p)\geqslant\lambda+C(d)\}$ for $C(d)=(10^{4(d+3)})^2$. A point (x,p) from this set is in at least $\lambda+C(d)$ different layers up to $L(Q)$. But it belongs to at most λ of the first layers $L(1),\dots,L(\lambda)$ since the layers consist of disjoint balls. With the condition on the multiplicity this implies that (x,p) is in at least $C(d)$ different layers lower than $L(\lambda)$. We get $(X\times\widetilde{M})_{\widetilde{U}}^Q(\lambda+C(d))\subseteq L_Q^{C(d)}(\lambda)$ and

$$\begin{aligned}\nu\left((X\times\widetilde{M})_{\widetilde{U}}^Q(\lambda+\beta\log(\tfrac{1}{\omega})+C(d))\cap\mathcal{F}\right)&\leqslant\nu\left(L_Q^{C(d)}(\lambda+\beta\log(\tfrac{1}{\omega}))\cap\mathcal{F}\right)\\ \overset{(3.16)}{\leqslant}T_Q(\lambda+\beta\log(\tfrac{1}{\omega}))&\overset{(3.18)}{\leqslant}e^{-\alpha\lambda}\nu(N_{2\omega}(\widetilde{U})\cap\mathcal{F}).\end{aligned}$$

By setting $\tilde{\omega}=2\omega$ and $\widetilde{\beta}(d)=((\beta(d)\log(2)+C(d))/\log(1/\tilde{\omega})+\beta(d))$ we obtain

$$\nu\left((X\times\widetilde{M})_{\widetilde{U}}^Q(\widetilde{\beta}\log(\tfrac{1}{\tilde{\omega}})+\lambda)\cap\mathcal{F}\right)\leqslant e^{-\alpha\lambda}\nu(N_{\tilde{\omega}}(\widetilde{U})\cap\mathcal{F}).$$

Note that $\widetilde{\beta}(d)\leqslant 2\cdot(10^{4(d+3)})^2$. These inequalities hold for every Q. Thus by (3.19), for every $\varepsilon>0$ there is a Q_0 such that

$$\begin{aligned}\nu\left((X\times\widetilde{M})_{\widetilde{U}}(\widetilde{\beta}\log(\tfrac{1}{\tilde{\omega}})+\lambda)\cap\mathcal{F}\right)&\leqslant\nu\left((X\times\widetilde{M})_{\widetilde{U}}^{Q_0}(\widetilde{\beta}\log(\tfrac{1}{\tilde{\omega}})+\lambda)\cap\mathcal{F}\right)+\varepsilon\\&\leqslant e^{-\alpha\lambda}\nu(N_{\tilde{\omega}}(\widetilde{U})\cap\mathcal{F})+\varepsilon.\end{aligned} \tag{3.20}$$

Note that this holds in particular if $\widetilde{\omega} < \frac{1}{100}$. If $V = \mathcal{F}_{\widetilde{M}}$, we have $\widetilde{U} = X \times \widetilde{M}$. This means, we divide the whole cover $\mathcal{U}$ into layers. It follows

$$\nu\Big((X \times \widetilde{M})(\widetilde{\beta}\log(\tfrac{1}{\widetilde{\omega}}) + \lambda) \cap \mathcal{F}\Big) \leqslant e^{-\alpha\lambda}\nu(X \times \widetilde{M} \cap \mathcal{F}) + \varepsilon = e^{-\alpha\lambda}\operatorname{vol}(M) + \varepsilon. \quad (3.21)$$

Letting ε go to 0 in (3.20) and (3.21) concludes the proof of the lemma. □

3.2.2 Remark on the layering

The proof of Lemma 3.2.4 is based on the construction of the layers. A priori, the procedure yields countably many layers. Let the notation be as in the proof. For $k = 1, \ldots, n'$ we have finite subsets of the group Γ

$$F_k := \{\gamma \in \Gamma \mid \gamma \neq 1, \gamma B_k \cap B_k \neq \emptyset\},$$
$$G_k^m := \{\gamma \in \Gamma \mid B_k \cap \gamma B_m \neq \emptyset\}$$

for $m = 1, \ldots, k-1$ and $G_k = \bigcup_{m=1}^{k-1} G_k^m$.

Recall the inductive construction of the layers (see Figure 3.4). Assume that for a $l \in \mathbb{N}$ the layers $\operatorname{Layer}(1), \ldots, \operatorname{Layer}(l-1)$ are constructed in the form of (3.4). Then $\operatorname{Layer}(l)$ is constructed by choosing a maximal element in

$$\mathcal{S}_1^{(l)} := \left\{A \subseteq \Big(A_1 \backslash \bigcup_{h=1}^{l-1} A_1^{(h)}\Big) \,\middle|\, \mu(A) > 0,\ \mu(A \cap \gamma A) = 0 \ \ \forall \gamma \in F_1\right\}$$

and then inductively maximal elements in

$$\mathcal{T}_k^{(l)} := \{A \in \mathcal{S}_k^{(l)} \mid \forall\, m = 1, \ldots, k-1\colon \mu(A \cap \gamma A_m^{(l)}) = 0 \text{ for all } \gamma \in G_k^m\}$$

with

$$\mathcal{S}_k^{(l)} := \left\{A \subseteq \Big(A_k \backslash \bigcup_{h=1}^{l-1} A_k^{(h)}\Big) \,\middle|\, \mu(A) > 0,\ \mu(A \cap \gamma A) = 0 \ \ \forall \gamma \in F_k\right\}.$$

We set $A_1^{(l)} = \emptyset$ and $A_k^{(l)} = \emptyset$ if $\mathcal{S}_1^{(l)} = \emptyset$ or $\mathcal{T}_k^{(l)} = \emptyset$, respectively. As long as we have $\mu(A_k \backslash \bigcup_{h=1}^{l-1} A_k^{(h)}) > 0$, Lemma 3.1.5 ensures that $\mathcal{S}_k^{(l)} \neq \emptyset$. We prove the following

Proposition 3.2.6. *For every $k = 1, \ldots, n'$ we have*

$$\lim_{l \to \infty} \mu\Big(A_k \backslash \bigcup_{h=1}^{l-1} A_k^{(h)}\Big) = 0.$$

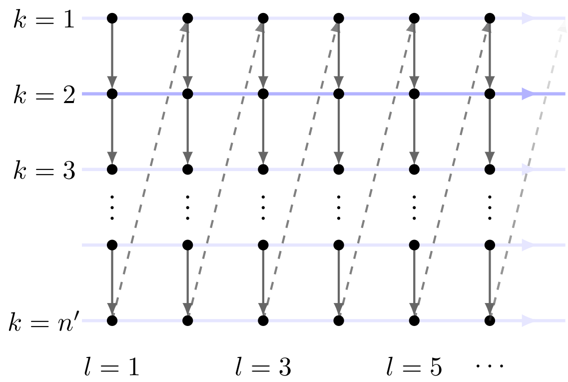

Figure 3.4: Construction of the layers: Each point (k, l) indicates one step in the algorithm, i.e. choosing a measurable set $A_k^{(l)} \subseteq A_k \setminus \bigcup_{h=1}^{l-1} A_k^{(h)}$. Each column l describes one layer, Layer(l), and points on the same horizontal line k correspond to one initial set $A_k \times B_k$. The choice of $A_k^{(l)}$ depends on the choices of $A_m^{(l)}$ for $m = 1, \dots, k-1$, i.e. in the choices taken in the same layer, as well as on the choices of $A_k^{(h)}$ for $h = 1, \dots, l-1$, i.e. choices for the same ball in the previous layers. If we assign the value $\mu(A_k \setminus \bigcup_{h=1}^{l} A_k^{(h)})$ to each point (k, l) in the grid, the limit along the horizontal line is zero.

Proof. We first show that for arbitrary $l \in \mathbb{N}$ and every $k = 1, \dots, n'$ the measure of a maximal element A in $\mathcal{S}_k^{(l)}$ is bounded from below. More precisely, we have

$$\mu(A) \geqslant \frac{1}{|F_k|}\, \mu\Big(A_k \setminus \bigcup_{h=1}^{l-1} A_k^{(h)}\Big). \tag{3.22}$$

This follows if $A_k \setminus \bigcup_{h=1}^{l-1} A_k^{(h)} \subseteq \bigcup_{\gamma \in F_k} \gamma A$. Assume this is not the case. That means there is a subset A' of positive measure in $A_k \setminus \bigcup_{h=1}^{l-1} A_k^{(h)}$ such that $\mu(A' \cap \gamma A) = 0$ for all $\gamma \in F_k$. By Lemma 3.1.5 we can assume that this set belongs to $\mathcal{S}_k^{(l)}$. Then we obtain

$$\mu((A \cup A') \cap \gamma(A \cup A')) = \mu((A' \cap \gamma A) \cup (\gamma A' \cap A)) = 0$$

since F_1 is symmetric. This implies $A \cup A' \in \mathcal{S}_k^{(l)}$, which contradicts the maximality of A in $\mathcal{S}_k^{(l)}$.

We want to show a similar statement for maximal elements in $\mathcal{T}_k^{(l)}$. Let $k \geqslant 2$ and C be a maximal element in $\mathcal{T}_k^{(l)}$. Then C belongs to $\mathcal{S}_k^{(l)}$ as well. If it is already maximal in this set, (3.22) applies. Otherwise C is contained in a maximal element $\widetilde{A} \in \mathcal{S}_k^{(l)}$. Then $\widetilde{A} = C \cup C'$ with $C' \notin \mathcal{T}_k^{(l)}$. For $m = 1, \dots, k-1$ let $\{\gamma_1^m, \dots, \gamma_{k_m}^m\}$ be the subset of G_k^m such that $\mu(C' \cap \gamma A_m^{(l)}) > 0$ for $\gamma \in \{\gamma_1^m, \dots, \gamma_{k_m}^m\}$. Then

$$C' \subseteq \bigcup_{m=1}^{k-1} \bigcup_{s=1}^{k_m} \gamma_s^m A_m^{(l)}.$$

If this is not the case, there is a subset $C'' \subset C'$ of positive measure which belongs to $\mathcal{S}_k^{(l)}$ and satisfies for every $m = 1, \dots, k-1$

$$\mu(C'' \cap \gamma A_m^{(l)}) = 0 \quad \text{for all } \gamma \in G_m^k.$$

But then one easily verifies that $C \cup C''$ belongs to $\mathcal{T}_k^{(l)}$, which contradicts the maximality of C. Therefore it holds, since $A_m^{(l)} \subseteq A_m \backslash \bigcup_{h=1}^{l-1} A_m^{(h)}$ for $m = 1, \dots, k-1$,

$$\begin{aligned}\mu(C') &\leqslant \sum_{m=1}^{k-1} |G_k^m| \mu\Big(A_m \backslash \bigcup_{h=1}^{l-1} A_m^{(h)}\Big) \\ &\leqslant |G_k| \sum_{m=1}^{k-1} \mu\Big(A_m \backslash \bigcup_{h=1}^{l-1} A_m^{(h)}\Big).\end{aligned}$$

As a result, we obtain the following estimate

$$\begin{aligned}\mu(C) &= \mu(\widetilde{A}) - \mu(C') \\ &\geqslant \frac{1}{|F_k|} \mu\Big(A_k \backslash \bigcup_{h=1}^{l-1} A_k^{(h)}\Big) - |G_k| \sum_{m=1}^{k-1} \mu\Big(A_m \backslash \bigcup_{h=1}^{l-1} A_m^{(h)}\Big).\end{aligned} \tag{3.23}$$

The same holds if C is already maximal in $\mathcal{S}_k^{(l)}$. Hence (3.23) holds for maximal elements $C \in \mathcal{T}_k^{(l)}$.

For $k = 1, \dots, n'$ we define sequences $(a_l^{(k)})_{l \in \mathbb{N}}$ by

$$a_l^{(k)} = \mu\Big(A_k \backslash \bigcup_{h=1}^{l-1} A_k^{(h)}\Big) = \mu(A_k) - \sum_{h=1}^{l-1} \mu\Big(A_k^{(h)}\Big).$$

Here we use that the sets $A_k^{(h)}$ are disjoint by construction. Note that

$$\begin{aligned}0 \leqslant a_l^{(k)} \leqslant a_1^{(k)} &= \mu(A_k) \leqslant 1 \\ a_l^{(k)} - a_{l+1}^{(k)} &= \mu(A_k^{(l)}) \geqslant 0\end{aligned}$$

hence the sequences are bounded and monton decreasing. Thus each sequence converges. It remains to prove that

$$\lim_{l \to \infty} a_l^{(k)} = 0 \tag{3.24}$$

for each $k = 1, \dots, n'$. We show this inductively. Let $k = 1$. In the construction of Layer(l) we choose $A_k^{(l)} = A_1^{(l)}$ as maximal element in $\mathcal{S}_1^{(l)}$. By (3.22) we obtain

$$\mu(A_1^{(l)}) \geqslant \frac{1}{|F_1|} \mu\Big(A_1 \backslash \bigcup_{h=1}^{l-1} A_1^{(h)}\Big)$$

which translates to

$$a_l^{(1)} - a_{l+1}^{(1)} \geqslant \frac{1}{|F_1|} a_l^{(1)} \quad \Longleftrightarrow \quad a_{l+1}^{(1)} \leqslant (1 - \frac{1}{|F_1|}) a_l^{(1)}.$$

We obtain

$$\lim_{l\to\infty} a_{l+1}^{(1)} \leqslant (1 - \frac{1}{|F_1|}) \lim_{l\to\infty} a_l^{(1)}$$

which holds only if $\lim_{l\to\infty} a_l^{(1)} = 0$.

Assume we have shown (3.24) up to $k-1$ for some $k \in \{2, \dots, n'-1\}$. Recall that in Layer(l) we choose $A_k^{(l)}$ maximal in $\mathcal{T}_k^{(l)}$. Then (3.23) yields

$$\mu(A_k^{(l)}) \geqslant \frac{1}{|F_k|} \mu\Big(A_k \setminus \bigcup_{h=1}^{l-1} A_k^{(h)}\Big) - |G_k| \sum_{m=1}^{k-1} \mu\Big(A_m \setminus \bigcup_{h=1}^{l-1} A_m^{(h)}\Big)$$

which translates to

$$a_l^{(k)} - a_{l+1}^{(k)} \geqslant \frac{1}{|F_k|} a_l^{(k)} - |G_k| \sum_{m=1}^{k-1} a_l^{(m)} \quad \Longleftrightarrow \quad a_{l+1}^{(k)} \leqslant (1 - \frac{1}{|F_1|}) a_l^{(k)} + |G_k| \sum_{m=1}^{k-1} a_l^{(m)}.$$

In the end this yields

$$\lim_{l\to\infty} a_{l+1}^{(k)} = \lim_{l\to\infty} a_l^{(k)} = (1 - \frac{1}{|F_k|}) \lim_{l\to\infty} a_l^{(k)} + |G_k| \sum_{m=1}^{k-1} \lim_{l\to\infty} a_l^{(m)}.$$

The terms in the sum are 0 by hypothesis, hence $\lim_{l\to\infty} a_l^{(k)} = 0$ has to be satisfied as well. This shows (3.24) and concludes the proof of the proposition. $\square$

Remark 3.2.7. Note that the above proposition implies in particular, that after finitely many steps every ball appears in some layer.

Let $k \in \{2, \dots, n'\}$. For every $\epsilon > 0$ there is a $l_0 \in \mathbb{N}$ such that $a_{l_0}^{(k)} < \varepsilon$. This means $\mu\Big(A_k \setminus \bigcup_{h=1}^{l_0-1} A_k^{(h)}\Big) < \varepsilon$. In particular, there is a $l_0 \in \mathbb{N}$ such that

$$\mu\Big(A_k \setminus \bigcup_{h=1}^{l_0-1} A_k^{(h)}\Big) < \mu(A_k).$$

Therefore at some finite step $h \in \{1 \dots, l_0 - 1\}$, a set $A_k^{(h)}$ of positive measure is chosen and $A_k^{(h)} \times B_k$ belongs to Layer(h).

Chapter 4

The rectangular nerve

Based on the good Γ-cover of $X \times \widetilde{M}$ constructed in the previous chapter, we introduce the corresponding nerve construction.

Retain the assumptions from the beginning of Chapter 3. Further we assume throughout this chapter that the fundamental group $\pi_1(M) = \Gamma$ of the Riemannian manifold M is torsion-free and that the supremal volume of a 1-ball in the universal cover $\widetilde{M}$, $V_{\widetilde{M}}(1)$, is bounded by some constant V_0.

Let $\mathcal{U}$ be the good Γ-cover constructed in Theorem 3.1.4, a family of product sets

$$\begin{aligned} \mathcal{U} &= \{\gamma(A_i \times B_i) \mid \gamma \in \Gamma, i \in \{1, \dots, n\}\} \\ &= \{A_j \times B_j \mid j \in J\} \end{aligned}$$

where $J \coloneqq \{\gamma i \mid \gamma \in \Gamma,\ i \in \{1, \dots, n\}\}$ is a free Γ-set. Here $A_j \subset X$ are Borel subsets and $B_j \subset \widetilde{M}$ are good balls of radii r_j in the universal cover, such that $A_{\gamma i} = \gamma A_i$, $B_{\gamma i} = \gamma B_i$ and $r_{\gamma i} = r_i$. As proven in Section 3.2 the measure of the set of high-multiplicity of this cover is bounded (Lemma 3.2.4).

We consider the corresponding rectangular nerve $\mathcal{N}(\mathcal{U})$ (Definition 2.5.8). Furthermore, we look at the underlying cover of $\mathcal{U}$, given by $\mathcal{W} = \{B_j \mid j \in J\}$, which is an equivariant locally finite cover of $\widetilde{M}$ as remarked in Proposition 2.5.11. The corresponding nerve $\mathcal{N}(\mathcal{W})$ as defined in Definition 2.5.4 is a locally finite, finite-dimensional cuboid complex equipped with the unique length metric which restricts to the standard Euclidean metric on faces (Remark 2.5.12). The group Γ acts on $\mathcal{N}(\mathcal{W})$ by permuting faces, which is an isometric action. Regard the equivariant simple X-space $X \times \mathcal{N}(\mathcal{W})$ equipped with the diagonal Γ-action. Then the rectangular nerve $\mathcal{N}(\mathcal{U})$ of the good Γ-cover $\mathcal{U}$ is a subcomplex of $X \times \mathcal{N}(\mathcal{W})$. Hence the fibres of $\mathcal{N}(\mathcal{U})$ are subcomplexes of the cuboid complex $\mathcal{N}(\mathcal{W})$ and are equipped with the restricted metric.

As explained in the end of Section 2.3 we have a measure on $X \times \mathcal{N}(\mathcal{W})$ as the product measure of μ and a measure on the cuboid complex. In the following we will always regard the d-dimensional Hausdorff measure on $\mathcal{N}(\mathcal{W})$ and denote it by vol_d. It behaves well under isometries and coincides with the d-dimensional Lebesgue measure on d-faces. We denote the product measure of μ and vol_d on $X \times \mathcal{N}(\mathcal{W})$ by $\nu_{\mathcal{N}}$. On the other hand the equivariant simple X-space $X \times \widetilde{M}$ is equipped with the product measure of μ and the Riemannian measure vol. This measure is denoted by ν as in the previous chapter. Note that for a d-dimensional manifold the Riemannian measure vol coincides with the d-dimensional Hausdorff measure.

Throughout the section all appearing metric cuboid complexes are equipped with the d-dimensional Hausdorff measure if not stated otherwise. Moreover, we assume that all topological spaces are equipped with a free action by Γ, the fundamental group of M.

4.1 The nerve map

Since we assume that the group Γ is torsion-free, the fibres of the rectangular nerve $\mathcal{N}(\mathcal{U})$ are subcomplexes of a free Γ-CW complex.

Lemma 4.1.1. *Let Γ be a torsion-free countable group and $\{a_j\}_{j\in J}$ be a set of positive real numbers indexed over a free Γ-set J. Then the cuboid complex associated to $\{a_j\}_{j\in J}$ is a free Γ-CW complex.*

Proof. The cuboid complex Z associated to a set $\{a_j\}_{j\in J}$ was defined in Definition 2.2.8. As seen in Section 2.5.2 the Γ-action on the index set J induces a Γ-action on Z via $\gamma(y_j)_{j\in J} = (y_{\gamma^{-1}j})_{j\in J}$ for a point $(y_j)_{j\in J} \in Z$. For the barycentres V_k of k-faces of Z we obtain that $\Gamma.V_k \subseteq V_k$, i.e the Γ-action permutes k-faces. Suppose this action has a fixed point $(y_j)_{j\in J} \in V_k$, i.e. there is a $\gamma \in \Gamma$ such that $\gamma(y_j)_{j\in J} = (y_{\gamma^{-1}j})_{j\in J} = (y_j)_{j\in J}$. There is an index j such that $y_j = a_j$ and therefore $y_{\gamma^l j} = a_j$ for all $l \in \mathbb{N}$. A point in Z has by definition only finitely many non-vanishing entries. For this reason, γ has to be a torsion element. This contradicts the fact that Γ is a torsion-free group. Thus the action on V_k is free for every $k \geqslant 0$. Since the barycentres V_k correspond to the k-faces of Z, the action of Γ on Z permutes the k-faces freely, i.e. is a free and cellular action on a CW complex. As a result, Z is a free Γ-CW complex [51, II. Proposition 1.15, p. 101]. □

In the given setting, $X \times \mathcal{N}(\mathcal{W})$ with the diagonal group action is a subcomplex of $X \times Z$, where Z is the cuboid complex associated to the set of radii $\{r_j\}_{j\in J}$ of the good balls B_j (Remark 2.5.12). Hence $\mathcal{N}(\mathcal{W})$ is a Γ-invariant subcomplex of the free Γ-CW complex Z and therefore a free Γ-CW complex in his own right. As a result, the fibres of $\mathcal{N}(\mathcal{U})$ are free Γ-CW complexes as well.

Having in mind Lemma 2.5.14, we define an equivariant geometric nerve map

$$\Phi\colon X \times \widetilde{M} \longrightarrow X \times \mathcal{N}(\mathcal{W})$$

whose image lies in $\mathcal{N}(\mathcal{U})$. Let J' be a complete set of representatives of the free Γ-set J. For each of the appearing good balls B_j, $j \in J'$, we define a map $\varphi_j\colon \widetilde{M} \longrightarrow [0, r_j]$ which is supported on B_j such that $\varphi_j(p) = r_j$ if $p \in \frac{1}{2}B_j$ and φ_j decreasing to zero as p approaches the boundary $\partial B_j = \bar{B}_j \backslash B_j$ (see [31, Section 1]). It is possible to choose φ_j piecewise-smooth and with Lipschitz constant at most 3. We can extend this definition equivariantly to all indices $j \in J$, i.e. $\varphi_j(p) = \varphi_{\gamma i}(p) = \varphi_i(\gamma^{-1}p)$ where we have that $r_{\gamma i} = r_i$ for all $\gamma \in \Gamma$ and $i \in \{1, \ldots, n\}$. Set

$$\begin{aligned} \Phi\colon X \times \widetilde{M} &\longrightarrow X \times \mathcal{N}(\mathcal{W}) \\ (x, p) &\longmapsto \left(x, \left(\chi_{A_j}(x)\varphi_j(p)\right)_{j\in J}\right), \end{aligned} \tag{4.1}$$

where χ_{A_j} denotes the characteristic function of A_j. This is indeed a well defined map, since for a fixed $p \in \widetilde{M}$ and a.e. $x \in X$ we have $(x, p) \in \bigcup_{j\in J} A_j \times B_j$ by the property of the Γ-cover $\mathcal{U}$. Furthermore, by Theorem 3.1.4, $\mathcal{U}(\frac{1}{2})$ is a Γ-cover as well, so for $(x, p) \in X \times \widetilde{M}$ there is an index $j \in J$ such that $\chi_{A_j}(x)\varphi_j(p) = r_j$. Since $\mathcal{U}_x$ is locally finite for every $p \in \widetilde{M}$ we have $\varphi_j(p) \neq 0$ only for finitely many $j \in J$. By definition of the rectangular nerve, the image is contained in $\mathcal{N}(\mathcal{U})$. The fact that Φ is equivariant geometric follows as in the proof of Lemma 2.5.14.

Remark 4.1.2. For a.e. $x \in X$ we have the induced locally finite good cover $\mathcal{U}_x$ of $\widetilde{M}$. By Remark 2.5.13, its associated rectangular nerve can be seen as the fibre of $\mathcal{N}(\mathcal{U})$ over x. With this identification the above map restricted to x gives a continuous proper nerve map $\Phi_x\colon \widetilde{M} \to \mathcal{N}(\mathcal{U})_x \subset \mathcal{N}(\mathcal{W})_x$. This is the nerve map as defined by Guth (see [30, Section 3] and [31, Section 1]). It is piecewise-smooth and its Lipschitz constant at a point $p \in \widetilde{M}$ is bounded in terms of the multiplicity $m_x(p)$ of p with respect to $\mathcal{U}_x$ by $3m_x(p)^{1/2}$.

Remark 4.1.3. Recall from Section 2.5.2 that we denote the dimension of an open face in $\mathcal{N}(\mathcal{W})$ with $d(F)$. For each face we divided the index set J into three subsets, $J_0(F)$, $J_1(F)$ and $J_{1/2}(F)$, depending on the coordinates of its barycenre. A face F has side lenghts r_j corresponding to the indices $J_{1/2}(F)$ and for each face we fix an order of its side lengths and write $r_1(F) \leqslant \ldots \leqslant r_{d(F)}(F)$. The index corresponding to $r_{d(F)}(F)$ is denoted by j_F. We fix an order for every face consistent with the Γ-action, i.e. $r_i(F) = r_i(\gamma F)$ for $i = 1, \ldots, d(F)$. The open star of a face F, $\mathrm{Star}(F)$, is the star of F with respect to $\mathcal{N}(\mathcal{W})$, i.e. the union of F and all open faces $F' \in \mathcal{N}(\mathcal{W})$ containing F in their closures. The open star of a face with respect to $\mathcal{N}(\mathcal{U})_x$ is then given as $\mathrm{Star}_{\mathcal{N}(\mathcal{U})_x}(F) = \mathrm{Star}(F) \cap \mathcal{N}(\mathcal{U})_x$. As shown in Lemma 2.5.14 we have $\Phi_x^{-1}(\mathrm{Star}(F)) = \Phi_x^{-1}(\mathrm{Star}_{\mathcal{N}(\mathcal{U})_x}(F)) \subseteq B_{j_F}$.

Using the bounds on the high-multiplicity set of $\widetilde{M}$ in Lemma 3.2.2 we can bound the volume in the image of the nerve map and the volume contained in certain regions of the nerve. We obtain the following version of [30, Lemma 5].

Lemma 4.1.4. *There are dimensional constants $c = c(d), \eta = \eta(d) > 0$ such that for a.e. $x \in X$ the following estimate holds for a face $F \in \mathcal{N}(\mathcal{W})$:*

$$\mathrm{vol}_d(\Phi_x(\widetilde{M}) \cap \mathrm{Star}(F)) < cV_{\widetilde{M}}(1) r_1(F)^{d+1} e^{-\eta d(F)}.$$

Proof. Note that if F does not lie in the image of Φ_x, in particular if $F \notin \mathcal{N}(\mathcal{U})_x$, the measure on the left-hand side vanishes. Let B_1 denote the ball in $\mathcal{U}_x$ with radius $r_1 = r_1(F)$, i.e. the ball corresponding to the index $j \in J_{1/2}(F)$ with $r_j = r_1(F)$. Let F' be a face in $\mathrm{Star}(F)$, i.e. it contains F in its boundary. Then $J_+(F) \subseteq J_+(F')$. Hence the index corresponding to the smallest side length $r_1(F)$ is contained in $J_+(F')$ for all elements of the open star and the preimage $\Phi_x^{-1}(\mathrm{Star}(F))$ is contained in B_1. Since B_1 is a good ball, we have $\mathrm{vol}(B_1) \leqslant 10^{2(d+3)} V_{\widetilde{M}}(1) r_1^{d+3}$. The Lipschitz constant of the nerve map Φ_x is bounded by $3m_x(p)^{1/2}$. In order to estimate $\mathrm{vol}_d(\Phi_x(B_1))$, we divide the good ball B_1 into regions of different multiplicity and add up their contributions. Here we use Lemma 3.2.2, stating that the volume of the set of points in B_1 with multiplicity with respect to $\mathcal{U}_x$ at least $(\beta \log(\frac{1}{r_1}) + \lambda)$ has volume bounded by $C \cdot e^{-\alpha\lambda} \mathrm{vol}(B_1)$ for dimensional constants $C > 1$ and α.
Define the following subsets of B_1 for $i \in \mathbb{N}$:

$$\begin{aligned} S = S^{(1)} :&= \widetilde{M}^x_{B_1}(\beta \log(\tfrac{1}{r_1})) = \{p \in B_1 \,|\, m_x(p) \geqslant \beta \log(\tfrac{1}{r_1})\} \\ &= \{p \in B_1 \,|\, m_x(p) \geqslant \lfloor \beta \log(\tfrac{1}{r_1}) + 1 \rfloor\}, \\ S^{(i)} :&= \widetilde{M}^x_{B_1}(\beta \log(\tfrac{1}{r_1}) + (i-1)) = \{p \in B_1 \,|\, m_x(p) \geqslant \lfloor \beta \log(\tfrac{1}{r_1}) + i \rfloor\}, \\ S_{(i)} :&= \{p \in B_1 \,|\, m_x(p) = \lfloor \beta \log(\tfrac{1}{r_1}) + i \rfloor\} \subseteq S^{(i)}. \end{aligned}$$

By Lemma 3.2.2, we have the following estimates for every i

$$\mathrm{vol}(S_{(i)}) \leqslant \mathrm{vol}(S^{(i)}) \leqslant C \cdot e^{-\alpha(i-1)} \mathrm{vol}(B_1) \leqslant C' e^{-\alpha i} \mathrm{vol}(B_1). \tag{4.2}$$

Note that $C' = Ce^{\alpha} > C > 1$ (see Remark 3.2.3). Using the Lipschitz property we obtain

$$\begin{aligned}\operatorname{vol}_d(\Phi_x(B_1\backslash S)) = \operatorname{vol}_d(\Phi_x(B_1\backslash S^{(1)})) &\leqslant 3^d(\beta\log(\tfrac{1}{r_1}))^{d/2}\operatorname{vol}(B_1\backslash S^{(1)})\\ &\leqslant 3^d(\beta\log(\tfrac{1}{r_1}))^{d/2}\operatorname{vol}(B_1)\\ &< 3^d(\beta\log(\tfrac{1}{r_1}))^{d/2}C'\operatorname{vol}(B_1)\end{aligned}$$

and

$$\operatorname{vol}_d(\Phi_x(S_{(i)})) \leqslant 3^d(\lfloor\beta\log(\tfrac{1}{r_1})+i\rfloor)^{d/2}\operatorname{vol}(S_{(i)}) \overset{(4.2)}{\leqslant} 3^d(\beta\log(\tfrac{1}{r_1})+i)^{d/2}C'e^{-\alpha i}\operatorname{vol}(B_1).$$

We add up the volume of the different regions. Note that the multiplicity of the locally finite cover $\mathcal{U}_x$ is bounded, so the appearing sum is in fact finite. We get

$$\begin{aligned}\operatorname{vol}_d(\Phi_x(B_1)) &= \operatorname{vol}_d(\Phi_x(B_1\backslash S)) + \sum_{i=1}^{\infty}\operatorname{vol}_d(\Phi_x(S_{(i)}))\\ &< C'\cdot 3^d\left(\sum_{i=0}^{\infty}(\beta\log(\tfrac{1}{r_1})+i)^{d/2}e^{-\alpha i}\right)\operatorname{vol}(B_1)\\ &\leqslant C''(d)\tfrac{1}{r_1}\operatorname{vol}(B_1) \leqslant C''(d)\cdot 10^{2(d+3)}V_{\widetilde{M}}(1)r_1^{d+2}.\end{aligned}$$

The second to last inequality arises by the estimate

$$\sum_{i=0}^{\infty}(\beta\log(\tfrac{1}{r_1})+i)^{d/2}e^{-\alpha i} < c'(d)\cdot\tfrac{1}{r_1}. \tag{4.3}$$

To see this look at the following calculation. With $k = \beta\log(1/r_1)$ we obtain

$$e^{\alpha k}\sum_{i=0}^{\infty}(k+i)^{d/2}e^{-\alpha(k+i)} \leqslant e^{\alpha k}\int_0^{\infty}x^{d/2}e^{-\alpha x}\,dx = e^{\alpha k}\alpha^{-((d/2)+1)}\int_0^{\infty}t^{((d/2)+1)-1}e^{-t}\,dt, \tag{4.4}$$

where α and d are positive numbers. The integral is given by the gamma function $\Gamma((d/2)+1)$, which is a constant only depending on d [2, Sec. 6.1, p. 255]. To get the above claim (4.3) observe that $\alpha\beta < 1$ (see Remark 3.2.3) hence

$$e^{\alpha k} = e^{\alpha\beta\log(\frac{1}{r_1})} = (\tfrac{1}{r_1})^{\alpha\beta} < \tfrac{1}{r_1}$$

using that $r_1 < 1$ (see Definition 3.1.1). The constant in (4.3) is then given by $c'(d) = \alpha^{-((d/2)+1)}\Gamma((d/2)+1)$.

To obtain the volume estimate stated in the lemma, it remains to prove the following claim.

Claim: There is some constant $\eta = \eta(d)$ such that $r_1 = r_1(F) \leqslant e^{-\eta d(F)}$.

The dimension of the face F, $d(F)$, can be estimated from above by the number of balls in $\mathcal{U}_x$ intersecting B_1. This number can be estimated using arguments as in the proof of Claim 3 in the proof of Lemma 3.2.4. Lemma 3.2.1 allows us to bound the number of such balls with radii in the interval $[2^{l-1}r_1, 2^{l+1}r_1]$, with $l = 0, 1, \dots$ by a constant $C(d) = 10^{4(d+3)}$. Since $r_1(F) \leqslant 1/100$, l is bounded by approximately $\log(\frac{1}{r_1})$, we get

$$d(F) \leqslant \log(\tfrac{1}{r_1}) \cdot C(d) \tag{4.5}$$

and therefore the claimed estimate on $r_1(F)$ where $\eta(d) = C(d)^{-1} = 10^{-4(d+3)}$.

As a result, using $\Phi_x(\widetilde{M}) \cap \operatorname{Star}(F) \subseteq \Phi_x(B_1)$, we obtain

$$\operatorname{vol}_d(\Phi_x(\widetilde{M}) \cap \operatorname{Star}(F)) \leqslant \operatorname{vol}_d(\Phi_x(B_1)) < c(d) V_{\widetilde{M}}(1) r_1(F)^{d+1} e^{-\eta(d)d(F)}$$

for a dimensional constant $c(d)$. □

4.2 Estimates for simplicial norms

By the results in Section 2.4, the nerve map $\Phi\colon X \times \widetilde{M} \to X \times \mathcal{N}(\mathcal{W})$ constructed as an equivariant geometric map induces a chain map of $\mathbb{Z}$-chain complexes

$$C_*^X(\Phi)\colon L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M},\mathbb{Z}) \longrightarrow L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\mathcal{N}(\mathcal{W}),\mathbb{Z})$$

and moreover a map in homology for every $n \in \mathbb{N}$

$$H_n(\Phi)\colon H_n(L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M},\mathbb{Z})) \longrightarrow H_n(L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\mathcal{N}(\mathcal{W}),\mathbb{Z})).$$

As stated in the beginning of this chapter, we assume that the supremal volume of a 1-ball in $\widetilde{M}$, $V_{\widetilde{M}}(1)$, is bounded by some constant V_0. The goal of this section is to establish an upper bound for the parametrised ℓ^1-norm of the image of the X-parametrised fundamental class $[M]^X$ of M under the induced map in homology. In order to do this, we construct a deformation of the nerve map, which lands in the d-skeleton of the nerve, and consider the corresponding induced map in homology. We will see that it is enough to bound the parametrised ℓ^1-norm of the image of $[M]^X$ under the resulting map. The goal of this section is to prove the following theorem.

Theorem 4.2.1. *Let $\Phi\colon X \times \widetilde{M} \to X \times \mathcal{N}(W)$ be the nerve map defined in* (4.1)*. Its image is contained in $\mathcal{N}(\mathcal{U})$. Let $V_0 > 0$ be a constant such that $V_{\widetilde{M}}(1) \leqslant V_0$. There exists an equivariant geometric map*

$$\Phi'\colon X \times \widetilde{M} \longrightarrow X \times \mathcal{N}(\mathcal{W})^{(d)}$$

with image contained in the d-skeleton of the rectangular nerve $\mathcal{N}(\mathcal{U})$ such that the following is satisfied. There is a constant $C(V_0, d)$ only depending on V_0 and the dimension d of the manifold such that

$$\| H_d(\Phi')([M]^X) \|^X \leqslant C(V_0, d)\, \mathrm{vol}(M).$$

To prove this we adapt the proof of [30, Lemma 9] in our setting, which will be done in the remainder of this section. This is building on ideas from [45, Section 4]. First we divide the elements (x, F) of $X \times \mathcal{N}(\mathcal{W})$ into two classes depending on the faces $F \in \mathcal{N}(\mathcal{W})$. Afterwards we will define deformations on both classes, which will allow us to deform the nerve map down to the d-skeleton. This means we construct a map

$$\Phi' \colon X \times \widetilde{M} \longrightarrow X \times \mathcal{N}(\mathcal{W})^{(d)}$$

via deformations from Φ. These steps are based on a sequence of lemmas, which are contained in Section 4.2.1. In the subsequent Section 4.2.2 we show that the image of the X-parametrised fundamental class under $H_d(\Phi')$ can be bound in terms of the volume of the manifold.

4.2.1 Deformation into the d-skeleton

We introduce deformations of maps landing in a finite skeleton of an equivariant simple X-space $X \times Z$ where Z is a metric cuboid complex equipped with a Γ-action. We establish different kinds of deformations such that by going over to the new map the increase of the volume is under control. In combining the deformations one can push the image of the map down to a lower skeleton. We first state the deformations in a general way and then apply them to the nerve map Φ in order to construct the map Φ' addressed in Theorem 4.2.1.

Remark 4.2.2. Let $X \times Z$ be an equivariant simple X-space with Z being a metric cuboid complex. In view of Remark 2.2.9 we call a k-cuboid in Z a *closed k-face*. An *(open) k-face* is the interior of a k-cuboid. If we talk about k-faces in $X \times Z$, we refer to elements $(x, F) \in X \times Z$ with F being a k-face in Z. The k-skeleton of $X \times Z$ is denoted by $X \times Z^{(k)}$ (see Definition 2.5.15). If $F \in Z$ is an open k-face, we denote the closed face by $\bar{F}$ and its boundary by ∂F. For $x \in X$ we have $(x, F) \in X \times Z^{(k)}$. We write $(x, \partial F) \subset X \times Z^{(k-1)}$ for the union of faces $(x, F') \in X \times Z^{(k-1)}$ such that $F' \in \partial F$. Then $(x, \bar{F}) = (x, F) \cup (x, \partial F)$.

Federer-Fleming deformations

In [45, Section 4] the nerve map is pushed down to the d-skeleton of the nerve by a sequence of so-called *ε-projections* in the simplices, i.e. radial projections from an interior point to the boundary with a fixed Lipschitz constant. In the framework used there, the equivariant cover has a universal multiplicity bound, which is a dimensional constant, which causes a similar bound on the Lipschitz constant of the nerve map. The volume estimate depends on this Lipschitz constant [45, Theorem 4.3, Lemma 4.10]. The ε-projections are tailored to keep the Lipschitz constant and therefore the increase of the volume under control.

Since the good Γ-cover $\mathcal{U}$ does not have a suitable universal bound on the multiplicity the ε-projections ar not suited for our purpose. We introduce projections based on the Federer-Fleming Deformation Theorem as suggested by Guth. Nevertheless, the present section is building on the ideas presented in [45].

We start with introducing Federer-Fleming projections on cuboids. For this we need radial projections on cuboids (see [16, Definition 4.3.14]).

Definition 4.2.3. Let $\bar{F}$ be a k-dimensional cuboid and p be a point in the interior F. Then the *radial projection (within F) from p* to the boundary ∂F is denoted by $\rho_{p,F}$ and defined as follows

$$\begin{aligned} \rho_{p,F}\colon \bar{F}\backslash\{p\} &\longrightarrow \partial F \\ q &\longmapsto q' \in [p,q) \cap \partial F, \end{aligned}$$

where $[p,q)$ is the infinite geodesic ray starting from p through q which is just a euclidean ray.

This radial projection is the identity on the boundary. Moreover, its restriction to $\bar{F}\backslash U$ is Lipschitz for all open sets U containing p. If we look at a subset of the face, which has finite d-dimensional Hausdorff measure, and its image under a radial projection from a point not lying in this subset, the d-volume might become very large, since pieces which are near to the projection point will be stretched by the projection. The following lemma from [16, Lemma 4.3.15] (see also [17, Lemma 5]) shows that there exist projection points such that the increase of the volume is under control. Figure 4.1 illustrates the situation of the lemma for a 2-dimensional cuboid.

Lemma 4.2.4. *Let $1 \leqslant d < k < N$ be integers. There exists a constant $C(k,d) \geqslant 1$ only depending on k and d such that the following holds: For any k-dimensional cuboid $F \subset \mathbb{R}^N$ and any closed vol_d-measurable set $E \subset \bar{F}$ with $\mathrm{vol}_d(E \cap \bar{F}) < \infty$ we find a*

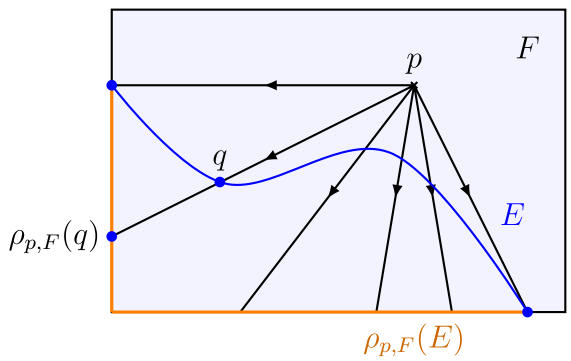

Figure 4.1: Radial projection from p within F to the boundary.

subset Q of F with $\mathrm{vol}_k(Q) \neq 0$ such that for all $p \in Q$ the following holds for the radial projection from p

$$\mathrm{vol}_d(\rho_{p,F}(E \cap \bar{F})) \leqslant C(k,d) \cdot \mathrm{vol}_d(E \cap \bar{F}).$$

We say a point in Q is suitable for a Federer-Fleming projection *within F with respect to E.*

Here vol_d and vol_k denotes the d-dimensional and k-dimensional Hausdorff measure, respectively. We stated the lemma in the context of cuboids instead of general polyhedra as in [16]. Note that the constant in the original lemma contains the so-called roundness of a polygon, a number in $[0,1]$ measuring how far a polyhedron is from a circle (see [16, Definition 1.2.25]). In the case of a cuboid of side lengths $r_1(F) \leqslant \ldots \leqslant r_k(F)$ this is given by the fraction of $r_1(F)$ and the length of the diagonal of the cuboid $\sqrt{\sum_{m=1}^{k} r_m(F)^2}$, hence it is bounded by $1/\sqrt{k}$. Basically, the lemma is a simple case of the Federer-Fleming Deformation Theorem proved in [15, Section 5].

The proof presented in [16] is based on a mean value argument and Fubini's Theorem. It works as well if we regard the relatively closed set $E \cap F \subseteq F$. This set is vol_d-measurable with $\mathrm{vol}_d(E \cap F) < \infty$. As a result, we obtain stricter estimates.

Corollary 4.2.5. *In the situation of Lemma 4.2.4 we have the following inequality for the relatively closed set $E \cap F \subseteq F$: For all $p \in Q$ the radial projection from p within F satisfies*

$$\mathrm{vol}_d(\rho_{p,F}(E \cap F)) \leqslant C(k,d) \cdot \mathrm{vol}_d(E \cap F)$$

and

$$\mathrm{vol}_d(\rho_{p,F}(E \cap \bar{F})) \leqslant \mathrm{vol}_d(E \cap \partial F) + C(k,d) \cdot \mathrm{vol}_d(E \cap F).$$

Remark 4.2.6. Let $X \times Z$ be an equivariant simple X-space with Z being a finite-dimensional metric cuboid complex. Let $(x, F) \in X \times Z$ with a k-face F for $k > d$. We say a point $(x, p) \in (x, F)$ is suitable for a Federer-Fleming projection within (x, F) with respect to a subset $\{x\} \times E \subset \{x\} \times \bar{F}$, if p is suitable for a Federer-Fleming projection within F with respect to E. We call the map given by $id_{\{x\}} \times \rho_{p,F}$ the radial projection (within (x, F)) from (x, p) to the boundary $(x, \partial F)$.

The following definition and lemmas are based on [45, Def.4.7 and Lemmas 4.8 and 4.9].

Definition 4.2.7. Let $\tau\colon X \times Z' \to X \times Z$ be an equivariant geometric map between equivariant simple X-spaces. Here Z is a finite dimensional, locally finite metric cuboid complex. We denote by $pr\colon X \times Z \to Z$ the projection. Let $\Sigma \subseteq X \times Z^{(k)}$ be a Γ-invariant Borel subset of k-faces and $\{(x, p_F)\}_{(x,F)\in\Sigma}$ be a family of points $(x, p_F) \in (x, F)$ such that the following holds:

i) $\{(x, p_F)\}_{(x,F)\in\Sigma} \subset X \times Z$ is a Γ-invariant subset with finitely many Γ-orbits.

ii) The set $\{pr(x, p_F)\}_{(x,F)\in\Sigma}$ of points in Z is countable and to every face of Z belong only finitely many of these points.

iii) Every point $(x, p_F) \in (x, F)$ is chosen suitable for a Federer-Fleming projection within (x, F) with respect to $\tau_x(Z') \cap \bar{F}$.

iv) For a.e. $x \in X$ and every face F in the fibre Σ_x there is an $\varepsilon_{(x,F)} > 0$ such that
$$d_Z(p_F, \tau_x(Z') \cap \bar{F}) > \varepsilon_{(x,F)}.$$
In particular we require $\varepsilon_{\gamma(x,F)} = \varepsilon_{(x,F)}$.

Then we call $\{(x, p_F)\}_{(x,F)\in\Sigma}$ a *family of Federer-Fleming projectors for* τ.

Lemma 4.2.8. *Let Z be a finite-dimensional, locally finite metric cuboid complex which is equipped with the d-dimensional Hausdorff measure. For $d < k$, let $\tau\colon X \times Z' \to X \times Z$ be an equivariant geometric map with image contained in the k-skeleton of $X \times Z$. Suppose $\Sigma \subseteq X \times Z^{(k)}$ is a Γ-invariant subset of k-faces such that $\mathrm{vol}_d(\tau_x(Z') \cap \bar{F}) < \infty$ for a.e. $x \in X$ and every $F \in \Sigma_x$. Let $\{(x, p_F)\}_{(x,F)\in\Sigma}$ be a family of Federer-Fleming projectors for τ.*

We regard the map $\tau'\colon X \times Z' \to X \times Z$ obtained from τ by post-composition with the radial projections within (x, F) from (x, p_F) to the boundary $(x, \partial F)$ for every $(x, F) \in \Sigma$. Then the following holds or holds a.e., respectively:

i) τ' is an equivariant geometric map.

ii) If τ_x is Lipschitz with Lipschitz constant Λ, then τ'_x is Lipschitz with Lipschitz constant Λ/ε for $\varepsilon := \min_{F\in\Sigma_x} \varepsilon_{(x,F)}$.

iii) There is a constant $C(k,d) \geqslant 1$ such that for every face $(x,F) \in \Sigma$ we have

$$\operatorname{vol}_d(\tau'_x(Z') \cap \bar{F}) \leqslant \operatorname{vol}_d(\tau_x(Z') \cap \partial F) + C(k,d)\operatorname{vol}_d(\tau_x(Z') \cap F).$$

For all other faces $(x,F) \in X \times Z$ it holds that

$$\operatorname{vol}_d(\tau'_x(Z') \cap F) = \operatorname{vol}_d(\tau_x(Z') \cap F).$$

iv) For every face $F \in Z$ the preimage of $\operatorname{Star}(F)$ under τ'_x is contained in the τ_x-preimage.

We say that the map τ' is obtained from τ via the family of Federer-Fleming projectors $\{(x,p_F)\}_{(x,F)\in\Sigma}$.

Proof. First we show that τ' is an equivariant X-map. It is clearly equivariant and fibrewise continuous for a.e. fibre. Due to the first property of $\{(x,p_F)\}_{(x,F)\in\Sigma}$ in Definition 4.2.7, τ' is of finite variance. To see this, regard a compact subset $K \subseteq Z'$. The map τ is of finite variance hence there is a finite Borel partition $X = \bigcup_{l=1}^{L} X_l$ such that $\tau_{|X_l\times K} = \operatorname{id}\times\varphi_l$ for continuous $\varphi_l \colon Z' \to Z^{(k)}$. For every $l \in \{1,\dots,L\}$ the set $\varphi_l(K)$ is compact and therefore is contained in a finite subcomplex of $Z^{(k)}$. Let $pr \colon X \times Z \to Z$ be the projection on the cuboid complex. The image of $(X_l \times \varphi_l(K)) \cap \{(x,p_F)\}_{(x,F)\in\Sigma}$ under this projection is contained in $\varphi_l(K) \cap \{pr(x,p_F)\}_{(x,F)\in\Sigma}$, which is a finite number of points, since every face contains only finitely many points. Then for every $l = 1,\dots,L$ we can find a finite Borel partition $X_l = \bigcup_{q=1}^{Q_l} X_q^l$ such that $pr((X_q^l \times \varphi_l(K)) \cap \{(x,p_F)\}_{(x,F)\in\Sigma})$ is a finite set of points in $\varphi_l(K)$ such that every face contains at most one point of this set. On $X_q^l \times K$ the post-composition of τ with the radial projections within faces $(x,F) \in \Sigma \cap (X_q^l \times K)$ from (x,p_F) is independent of x. As a result, $\tau'_{|X_q^l\times K} = \operatorname{id}\times\varphi_q^l$ for continuous φ_q^l. Hence τ' is of finite variance and therefore an equivariant X-map.

We show that τ'_x is proper for a.e. $x \in X$. Then τ' is an equivariant geometric map. Let $K' \subseteq Z^{(k)}$ be a compact subcomplex. K' is closed and it is contained in a subcomplex which is the union of finitely many open faces $\{F_1,\dots,F_m\}$. For every point $p \in (\tau'_x)^{-1}(K')$ we have $\tau'_x(p) \in \overline{\operatorname{carr}(\tau_x(p))} \cap K' \neq \emptyset$. There is a face F_i for some $i \in \{1,\dots,m\}$ which is a subset of $\overline{\operatorname{carr}(\tau_x(p))}$. This implies that $\tau_x(p) \in \operatorname{Star}(F_i)$ and

$$(\tau'_x)^{-1}(K') \subseteq \bigcup_{i=1}^{m} \tau_x^{-1}(\operatorname{Star}(F_i)).$$

The latter is a relatively compact set, since Z is a finite-dimensional locally finite cuboid complex and τ_x is proper. Therefore, the closed set $(\tau_x')^{-1}(K')$ is compact and τ_x' proper.

To see the last statement iv), note that the preimage of a face under the radial projections lies in its open star. So $(\tau')_x^{-1}(\mathrm{Star}(F)) \subseteq \tau_x^{-1}(\bigcup_{F'\in\mathrm{Star}(F)} \mathrm{Star}(F')) \subseteq \tau_x^{-1}(\mathrm{Star}(F))$, since $\mathrm{Star}(F') \subseteq \mathrm{Star}(F)$ for every face $F' \in \mathrm{Star}(F)$.

The third assertion is due to the fact that we deal with a family of Federer-Fleming projectors. By Corollary 4.2.5, given a face $(x,F) \in \Sigma$, the radial projection from (x, p_F) within (x,F) with respect to $(\tau_x(Z') \cap \bar{F})$ has the property that

$$\mathrm{vol}_d(\rho_{p_F,F}(\tau_x(Z') \cap \bar{F})) \leqslant \mathrm{vol}_d(\tau_x(Z') \cap \partial F) + C(k,d)\,\mathrm{vol}_d(\tau_x(Z') \cap F).$$

and we have that $\rho_{p_F,F}(\tau_x(Z')) \cap \bar{F} \subseteq \rho_{p_F,F}(\tau_x(Z') \cap \bar{F})$. For all other faces in $X \times \mathcal{N}(\mathcal{W})$ it holds $\tau_x'(Z') \cap F = \tau_x(Z') \cap F$. This is in particular valid if a face is not contained in the image of τ.

Let τ_x be Λ-Lipschitz. The radial projections $\rho_{p_F,F}$ are Lipschitz with Lipschitz constant less than or equal to $\varepsilon_{(x,F)}^{-1}$. Using that Z is equipped with a length metric one obtains that τ_x' has Lipschitz constant Λ/ε where $\varepsilon = \min_{(x,F)\in\Sigma} \varepsilon_{(x,F)}$. Here we use that Σ has only finitely many orbits, hence we deal with finitely many numbers $\varepsilon_{(x,F)}$. This concludes the proof of the lemma. $\square$

Remark 4.2.9. Retain the setting of Lemma 4.2.8 and assume that Z' is a Riemannian manifold. Then if τ is a piecewise-smooth map into $X \times Z^{(k)}$, the map τ' obtained from τ via a family of Federer-Fleming projectors is piecewise-smooth as well. This is due to the fact that the radial projection from a point p defines a smooth map on closed subsets of $\mathbb{R}^N$

Lemma 4.2.10. *Suppose $d < k$. Let $\tau\colon X \times \widetilde{M} \to X \times \mathcal{N}(\mathcal{W})^{(k)}$ be an equivariant geometric map such that for a.e. $x \in X$ and every face $F \in \mathcal{N}(\mathcal{W})$ the preimage $\tau_x^{-1}(\mathrm{Star}(F))$ is contained in B_{j_F}, the ball in $\mathcal{U}_x$ corresponding to $r_{d(F)}(F)$. Let $\Sigma \subseteq X \times \mathcal{N}(\mathcal{W})^{(k)}$ be a Γ-invariant subset of k-faces such that*

$$\mathrm{vol}_d(\tau_x(\widetilde{M}) \cap \bar{F}) < \infty \quad \textit{for } (x,F) \in \Sigma.$$

Then there exists a family of Federer-Fleming projectors for τ.

Proof. Recall the construction of a fundamental domain of $X \times \mathcal{N}(\mathcal{W})^{(k)}$ in Lemma 2.5.16. By choosing a complete set $J' \subset J$ of Γ-representatives, a Borel fundamental domain $\mathcal{F}$ of $X \times \mathcal{N}(\mathcal{W})^{(k)}$ is given by a finite disjoint union of Borel sets $X \times F_l$, where F_l is contained in a Γ-fundamental domain of $\mathcal{N}(\mathcal{W})$ and we have that $j_{F_l} \in J'$ for all F_l. Since $|J'| = n$ and the balls B_j are relatively compact, the union $\bigcup_{j \in J'} B_j$ is relatively compact. Let K be the compact closure of $\bigcup_{j \in J'} 3B_j$. Given the above fundamental domain we have that $\tau_x^{-1}(\bar{F}_l) \subseteq K$. This holds true, since $\tau_x^{-1}(F_l) \subset B_{j_{F_l}}$ and for every face $F' \subset \partial F_l$ its preimage is contained in the ball corresponding to its largest side length. Since $J_{1/2}(F') \subset J_{1/2}(F_l)$, we have

$$\tau_x^{-1}(\bar{F}) \subseteq \bigcup_{j \in J_{1/2}(F_l)} B_j \subseteq 3B_{j_{F_l}}$$

by using that $\bigcap_{j \in J_{1/2}(F_l)} B_j \neq 0$ and $B_{j_{F_l}}$ is the ball of largest radius. Hence we obtain

$$\tau_x(\widetilde{M}) \cap \bar{F}_l = \tau_x(K) \cap \bar{F}_l \tag{4.6}$$

for a.e. $x \in X$ and every $F_l \in \mathcal{F}_x$. This holds in particular if a face $(x, F_l) \in \mathcal{F}$ is not in the image of τ. Given that τ is of finite variance there is a finite Borel partition $X = \bigcup_{q=1}^{Q} X_q$ such that every restriction $\tau_{|X_q \times K} = \mathrm{id} \times \varphi_q$ is a product map with φ_q continuous. Then by the above (4.6), the set

$$\tau_x(\widetilde{M}) \cap \bar{F}_l = \tau_x(K) \cap \bar{F}_l = \varphi_q(K) \cap \bar{F}_l$$

is constant for a.e. $x \in X_q$. Since $\tau_x(K)$ is compact, $\tau_x(\widetilde{M}) \cap \bar{F}_l$ is closed and by assumption it is of finite d-dimensional Hausdorff measure if $(x, F_l) \in \Sigma$. By Lemma 4.2.4, for every q and every l we can pick a point (x, p_q^l) which is suitable for a Federer-Fleming projection within $(x, F_l) \in \Sigma$ with respect to $\tau_x(\widetilde{M}) \cap \bar{F}_l$. Moreover, since $\tau_x(\widetilde{M}) \cap \bar{F}_l$ is closed and $p_q^l \notin \tau_x(\widetilde{M}) \cap \bar{F}_l$, there is an $\varepsilon_q^l > 0$ such that $d_{\widetilde{M}}(p_q^l, \tau_x(\widetilde{M}) \cap \bar{F}_l) > \varepsilon_q^l$.

For $(x, F_l) \in \Sigma \cap \mathcal{F}$ with $x \in X_q$ define $(x, p_{F_l}) := (x, p_q^l)$ and set $\varepsilon_{(x,F)} := \varepsilon_q^l$.We extend this definition equivariantly to all of Σ. Then $\{(x, p_F)\}_{(x,F) \in \Sigma}$ is a family of Federer-Fleming projectors. By construction it satisfies the second property in Definition 4.2.7. Let $pr\colon X \times \mathcal{N}(\mathcal{W})^{(k)} \to \mathcal{N}(\mathcal{W})^{(k)}$ be the projection on the cuboid complex. We picked one point (x, p_q^l) for every q and l. Hence the set $\{pr(x, p_{F_l})\}_{(x,F_l) \in \Sigma \cap \mathcal{F}} = \{p_q^l\}_{(x,F_l) \in \Sigma \cap \mathcal{F}}$ is finite and there are finitely many points in every face. So $\{pr(x, p_{F_l})\}_{(x,F_l) \in \Sigma}$ is a countable set of points in $\mathcal{N}(\mathcal{W})^{(k)}$ where every face contains only finitely many points. It is clear that Σ satisfies the other properties of a family of Federer-Fleming projectors. This concludes the proof of the lemma. $\square$

The pushout lemma for small surfaces

Another kind of deformation of maps such that the increase of volume is under control is due to the following *pushout lemma for small surfaces* by Guth [31, Lemma 0.6].

Lemma 4.2.11. *For each dimension $d \geqslant 2$ there is a constant σ_d such that the following is satisfied.*

Let N be a compact piecewise-smooth d-dimensional manifold with boundary and $K \subset \mathbb{R}^N$ be a convex set. Let $\tau\colon (N, \partial N) \to (K, \partial K)$ be a piecewise-smooth map. Then there is a map $\tau'\colon (N, \partial N) \to (K, \partial K)$ such that the following holds:

i) *The maps τ and τ' are homotopic.*

ii) *τ and τ' agree on the boundary ∂N.*

iii) $\mathrm{vol}_d(\tau'(N)) \leqslant \mathrm{vol}_d(\tau(N))$.

iv) *The image $\tau'(N)$ is contained in the W-neighbourhood of the boundary ∂K where $W = \sigma_d \cdot \mathrm{vol}_d(\tau(N))^{1/d}$.*

Based on this lemma,we can prove the following lemma.

Lemma 4.2.12. *For each dimension $d \geqslant 2$ there is a constant σ_d such that the following holds.*

For $d < k$, let $\tau\colon X \times \widetilde{M} \to X \times \mathcal{N}(\mathcal{W})^{(k)}$ be an equivariant geometric map such that for every face $F \in \mathcal{N}(\mathcal{W})$ the preimage $\tau_x^{-1}(\mathrm{Star}(F))$ is contained in B_{j_F}, the ball in $\mathcal{U}_x$ corresponding to $r_{d(F)}(F)$. Moreover, let τ_x be piecewise-smooth for a.e. $x \in X$. Suppose $\Sigma \subseteq X \times \mathcal{N}(\mathcal{W})^{(k)}$ is a Γ-invariant subset of k-faces. Then τ can be modified to a map $\tau'\colon X \times \widetilde{M} \to X \times \mathcal{N}(\mathcal{W})^{(k)}$ such that the following holds or holds a.e., respectively:

i) *The map τ' is an equivariant geometric map. Moreover, it is piecewise-smooth on a.e. fibre.*

ii) *For every face $F \in \mathcal{N}(\mathcal{W})$ we have*

$$\mathrm{vol}_d(\tau'_x(\widetilde{M}) \cap F) \leqslant \mathrm{vol}_d(\tau_x(\widetilde{M}) \cap F).$$

iii) *For every face $(x, F) \in \Sigma$ the image $\tau'_x(\widetilde{M}) \cap F$ is contained in the W'-neighbourhood of ∂F, where $W' = s\sigma_d \cdot \mathrm{vol}_d(\tau_x(\widetilde{M}) \cap F)^{1/d}$ for a real number $s > 1$. This number s can be chosen arbitrary close to 1.*

iv) *For every face $F \in \mathcal{N}(\mathcal{W})$ the preimage of $\mathrm{Star}(F)$ under τ'_x is contained in the τ_x-preimage.*

Proof. We proof this lemma by adapting ideas of the proof of Theorem 0.1 in [31, Section 1]. First we fix a Borel fundamental domain of $X \times \mathcal{N}(\mathcal{W})^{(k)}$ as in the proof of Lemma 4.2.10. It consists of finitely many disjoint Borel sets $X \times F_l$, where F_l belongs to a Γ-fundamental domain of $\mathcal{N}(\mathcal{W})$ and j_F is contained in a complete set of representatives $J' \subset J$. It holds $|J'| = n$. Let K be the compact closure of $\bigcup_{j \in J'} B_j$. Then $\tau_x^{-1}(F_l) \subseteq K$ and therefore

$$\tau_x(\widetilde{M}) \cap F_l = \tau_x(K) \cap F_l$$

for a.e. $x \in X$ and every $F_l \in \mathcal{F}_x$. By finite variance of τ, there is a finite Borel partition $X = \bigcup_{q=1}^{Q} X_q$ such the restricted maps are products $\tau_{|X_q \times K} = \mathrm{id} \times \varphi_q$ with continuous maps φ_q. Hence

$$\tau_x(\widetilde{M}) \cap F_l = \tau_x(K) \cap F_l = \varphi_q(K) \cap F_l$$

is constant for a.e. $x \in X_q$. For every q and l such that $(x, F_l) \in \Sigma \cap \mathcal{F}$ with $x \in X_q$, let $K_q^l \subseteq F_l$ be a closed and convex subset of F_l which contains a large fraction of F_l, i.e. it lies arbitrarily close to ∂F_l. Following Guth ([31, Section 1, p. 206]), by a general position argument we can choose K_q^l such that τ_x is transversal to its boundary ∂K_q^l while $x \in X_q$ (see forward Remark 4.2.13). We set $Y_q^l := \tau_x^{-1}(K_q^l)$ for $x \in X_q$, which is a compact set. Since $\tau_x(\widetilde{M}) \cap F_l$ is constant on X_q, this preimage is independent of x for fixed q. By the transversality of τ_x and ∂X_q^l, Y_q^l is a compact, piecewise-smooth d-dimensional manifold with boundary $\partial Y_q^l = \tau_x^{-1}(\partial K_q^l)$. Thus the restriction of τ_x yields a piecewise-smooth map $\tau_x \colon (Y_q^l, \partial Y_q^l) \to (K_q^l, \partial K_q^l)$. By Lemma 4.2.12, there is a homotopic map $\tau_x' \colon (Y_q^l, \partial Y_q^l) \to (K_q^l, \partial K_q^l)$ such that τ_x and τ_x' coincide on ∂Y_q^l. It holds that $\mathrm{vol}_d(\tau_x'(Y_q^l)) \leqslant \mathrm{vol}_d(\tau_x(Y_q^l))$ and the image $\tau_x'(Y_q^l)$ is contained in the W_q^l-neighbourhood of ∂K_q^l for $W_q^l = \sigma_d \,\mathrm{vol}_d(\tau_x(Y_q^l))^{1/d}$. Since we can choose K_q^l as close as we like to ∂F_l, we can ensure that $\tau_x'(Y_q^l)$ is contained in the sW_q^l-neighbourhood of ∂F_l for a fixed real number s which is slightly bigger than 1.

These properties will also be fulfilled if we extend the definition of τ' to all elements in Σ by equivariance. For $(\gamma x, \gamma F_l) \in \Sigma$ such that $(x, F_l) \in \mathcal{F}$ with $x \in X_q$, look at $\gamma K_q^l \subset \gamma F_l$ and its preimage $\tau_{\gamma x}^{-1}(\gamma K_q^l) = \gamma Y_q^l$ and set $\tau_{\gamma x}' = \gamma \tau_x'$. So for a.e. $x \in X$ we defined τ_x on the subset

$$S := \bigcup_{\gamma \in \Gamma} \bigcup_{\substack{(x, F_l) \in \Sigma \cap \mathcal{F} \\ x \in X_q}} \gamma Y_q^l.$$

By setting $\tau'_x = \tau_x$ on $\widetilde{M}\backslash S$ we can extend it to all of $\{x\} \times \widetilde{M} \cong \widetilde{M}$. This defines a piecewise-smooth map $\tau'_x\colon \widetilde{M} \to \mathcal{N}(\mathcal{W})^{(k)}$, since τ'_x agrees with τ_x on each $\gamma\partial(Y^l_q)$ and the boundary of $\widetilde{M}\backslash S$ is given by $\bigcup_{\gamma\in\Gamma} \bigcup_{\substack{(x,F_l)\in\Sigma\cap\mathcal{F}\\ x\in X_q}} (\gamma\partial Y^l_q)$.

By the properties of τ'_x, resulting from Lemma 4.2.12, we get the second assertion. Moreover, let $(\gamma x, \gamma F_l)$ be a face in Σ such that $(x, F_l) \in \mathcal{F}$ with $x \in X_q$. Then $\tau'_{\gamma x}(\gamma Y^l_q)$ is contained in the sW^l_q-neighbourhood of $\partial\gamma F_l$. This implies that $\tau'_x(\widetilde{M}) \cap \gamma F_l$ is contained in the sW^l_q-neighbourhood of $\partial\gamma F_l$. Here we have

$$sW^l_q = s\sigma_d \operatorname{vol}_d(\tau_x(Y^l_q))^{1/d} \leqslant s\sigma_d \operatorname{vol}_d(\tau_x(\widetilde{M}) \cap F)^{(1/d)} =: W'.$$

This proves the third statement.

The map $\tau'\colon X \times \widetilde{M} \to X \times \mathcal{N}(\mathcal{W})^{(k)}$ is equivariant by construction and fibrewise continuous. Note that $(\tau'_x)^{-1}(F) \subseteq \tau_x^{-1}(F)$ for all faces $F \in \mathcal{N}(\mathcal{W})$. In particular, the preimage of $\operatorname{Star}(F)$ under τ'_x is contained in the τ_x-preimage as stated in the last assertion. Moreover, τ'_x is proper for a.e. $x \in X$. Let $K \subseteq \mathcal{N}(\mathcal{W})^{(k)}$ be a compact subset. It is contained in a finite subcomplex of $\mathcal{N}(\mathcal{W})$ which is the union of open faces $F_1, \ldots, F_m$. Then

$$(\tau'_x)^{-1}(K) \subseteq \bigcup_{i=1}^m (\tau'_x)^{-1}(F_i) \subseteq \bigcup_{i=1}^m (\tau_x)^{-1}(F_i) \subseteq \bigcup_{i=1}^m (\tau_x)^{-1}(\operatorname{Star}(F_i)).$$

The last set is by assumption contained in the relatively compact set $\bigcup_{i=1}^m B_{j_{F_i}}$. Since $(\tau'_x)^{-1}(K)$ is closed, it is compact and τ'_x is proper. For τ' being an equivariant geometric map it remains to prove that the map is of finite variance. Let $K' \subset \widetilde{M}$ be a compact subset and $X = \bigcup_{h=1}^H X_h$ be a finite Borel decomposition such that $\tau_{|X_h \times K} = \mathrm{id} \times \varphi_h$ with continuous φ_h. Then $\varphi_h(K')$ is contained in a finite subcomplex of $\mathcal{N}(\mathcal{W})$, since it is a compact subset. A finite subcomplex consists of finitely many open faces, which are some Γ-translates of faces in the Γ-fundamental domain of $\mathcal{N}(\mathcal{W})$ used in the beginning. So for fixed h, we have $\gamma^h_1, \ldots, \gamma^h_m$ such that this finite complex consists of faces $\gamma^h_1 F_1, \ldots, \gamma^h_m F_m$ with $F_1, \ldots, F_m$ being in the Γ-fundamental domain. Recall that in the beginning of this proof we fixed a finite Borel decomposition $X = \bigcup_{q=1}^Q X_q$ such that $\tau_x(\tilde{M}) \cap F_l$ is constant for a.e. $x \in X_q$ and every face $F_l \in \mathcal{F}_x$.

Claim: For $h \in \{1, \ldots, H\}$ the following holds: For a.e. $x, y \in X_h \cap (\bigcap_{i=1}^m \gamma^h_i X_q)$ and $p \in K'$ we have $\tau'_x(p) = \tau'_y(p)$.

This implies that τ' is of finite variance. Since $x, y \in X_h$ and $p \in K'$ we have $\tau_x(p) = \tau_y(p)$. There exists a $i \in \{1, \ldots, m\}$ such that $\tau_x(p) \in \gamma^h_i F_i$. Then by construction, since $x, y \in \gamma^h_i X_q$, we get $\tau'_x(p) = \tau'_y(p)$ as well. This concludes the proof of the first assertion. $\square$

Remark 4.2.13. We elaborate on the general position argument in the case that τ_x is smooth. Retain the setting as in the proof of Lemma 4.2.12. Let $F_l \in \mathcal{F}_x$ and $x \in X_q$. Consider the smooth map

$$\begin{aligned} T\colon \tau_x^{-1}(F_l) \times F_l &\longrightarrow \mathbb{R}^k \\ (p, r) &\longmapsto \tau_x(p) + r, \end{aligned}$$

where $\mathbb{R}^k$ is the hyperplane containing F_l. We can assume that $0 \in F_l$. For a fixed point $p \in \tau_x^{-1}(F_l)$, T is a translation of F_l. Therefore, its differential is surjective at every point. The same holds for the map T on $\tau_x^{-1}(F_l) \times F_l$. As a result, T is transversal to every submanifold of $\mathbb{R}^k$.

Let $\widetilde{K} \subseteq F_l$ be a submanifold with boundary which lies close to ∂F and contains a large fraction of F_l. In particular the boundary $\partial\widetilde{K}$ is a submanifold of codim 1 and τ_x is transversal to it. Then by the Transversality Theorem [28, Chapter 2 §3, p. 68], the map

$$\begin{aligned} T_r\colon \tau_x^{-1}(F_l) &\longrightarrow \mathbb{R}^k \\ p &\longmapsto \tau_x(p) + r \end{aligned}$$

is transversal to $\partial\widetilde{K}$ for almost every $r \in F_l$. We can arrange that for $r \in F_l$ small enough, the set $K := \widetilde{K} - r$ lies in F_l and τ_x is transversal to $\partial(\widetilde{K}) - r = \partial(\widetilde{K} - r) = \partial K$. So $K_q^l = K$ is a suitable subset. By transversality (see [28, Chapter 1 §5, Theorem on p. 28]), $\tau_x^{-1}(\partial K)$ is a submanifold of $\widetilde{M}$ of codim 1, i.e. a $(d-1)$-dimensional submanifold and $\tau_x^{-1}(K)$ is a d-dimensional smooth manifold with boundary $\tau_x^{-1}(\partial K)$.

Pushing a map down to a lower skeleton

The above lemmas describe deformations of maps with image in the k-skeleton such that the increase of the volume is under control. The resulting maps are still landing in the k-skeleton. In the end, our goal is to deform maps into lower skeleta. Following Guth (see [30, proof of Lemma 8]), we describe a Lipschitz map, which pulls a small neighbourhood of the $(k-1)$-skeleton of the nerve complex into the $(k-1)$-skeleton.

Lemma 4.2.14. *Let $\tau\colon X \times \widetilde{M} \to X \times \mathcal{N}(\mathcal{W})^{(k)}$ be an equivariant geometric map with image in $\mathcal{N}(\mathcal{U})^{(k)}$. Assume that there is a constant $\frac{1}{2} > \delta > 0$ such that for every face $(x, F) \in X \times \mathcal{N}(\mathcal{W})$ the image $\tau_x(\widetilde{M}) \cap F$ is contained in the $\delta \cdot r_1(F)$-neighbourhood of ∂F. Here $r_1(F) \leqslant \ldots \leqslant r_{d(F)}(F)$ are the side lengths of F according to the order we chose in the beginning (see Remark 4.1.3). Then there is a map $R_\delta : X \times \mathcal{N}(\mathcal{W}) \to X \times \mathcal{N}(\mathcal{W})$ satisfying the following (for a.e. $x \in X$):*

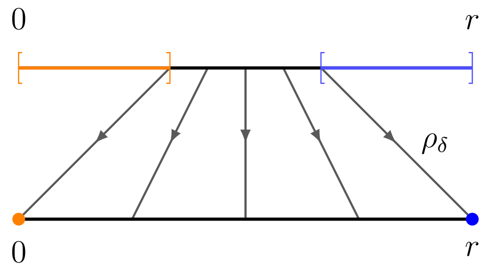

Figure 4.2: The map ρ_δ for $\delta = \frac{1}{3}$.

i) $R_\delta \circ \tau \colon X \times \widetilde{M} \to X \times \mathcal{N}(\mathcal{W})^{(k-1)}$ *is an equivariant geometric map with image in* $\mathcal{N}(\mathcal{U})^{(k-1)}$.

ii) $(R_\delta)_x$ *is Lipschitz with Lipschitz constant* $(1-2\delta(k))^{-1}$ *and piecewise-smooth.*

iii) For any face $F \in \mathcal{N}(\mathcal{W})$ *the preimage of* $\mathrm{Star}(F)$ *under* $(R_\delta)_x$ *is contained in* $\mathrm{Star}(F)$.

Proof. As in the proof of [30, Lemma 8] the map R_δ is defined via a basic map on an interval $[0,r]$. We set

$$\rho_\delta \colon [0,r] \to [0,r],\; y \mapsto \begin{cases} 0, & y \in [0,\delta r] \\ \frac{1}{1-2\delta}y - \frac{\delta r}{1-2\delta}, & y \in [\delta r, r-\delta r] \\ r, & y \in [r-\delta r, r] \end{cases}.$$

Thus the map ρ_δ takes the boundary regions of the interval into its end points and linearly stretches the inner set $[\delta r, r-\delta r]$ to cover $[0,r]$ (see Figure 4.2). The Lipschitz constant of this map is $(1-2\delta)^{-1}$. Then we define

$$\begin{aligned} R_\delta \colon X \times \mathcal{N}(\mathcal{W}) &\to X \times \mathcal{N}(\mathcal{W}) \\ (x,(y_j)_{j\in J}) &\mapsto (x,(\rho_\delta(y_j))_{j\in J}), \end{aligned}$$

where ρ_δ in the j-th component is the above map applied on $[0,r_j]$ (see Figure 4.3). Then the assertions in the lemma follow easily. The map is obviously equivariant and continuous on a.e. fibre. For $x \in X$, $(R_\delta)_x$ is Lipschitz with Lipschitz constant $(1-2\delta)^{-1}$ arising from the Lipschitz constant of ρ_δ. $(R_\delta)_x$ maps an open face F to its closure $\bar{F}$. We have $(R_\delta)_x^{-1}(F) \subseteq F$ and therefore $(R_\delta)_x^{-1}(\mathrm{Star}(F)) \subseteq \mathrm{Star}(F)$ for each face $F \in \mathcal{N}(\mathcal{W})$. Moreover, it follows easily that the map $(R_\delta)_x$ is proper for a.e. $x \in X$. Since $(R_\delta)_x$ is independent of x, the map R_δ is of finite variance and thus an equivariant geometric map. By assumption the image of τ is contained in $\mathcal{N}(\mathcal{U})^{(k)}$. Moreover, for every face

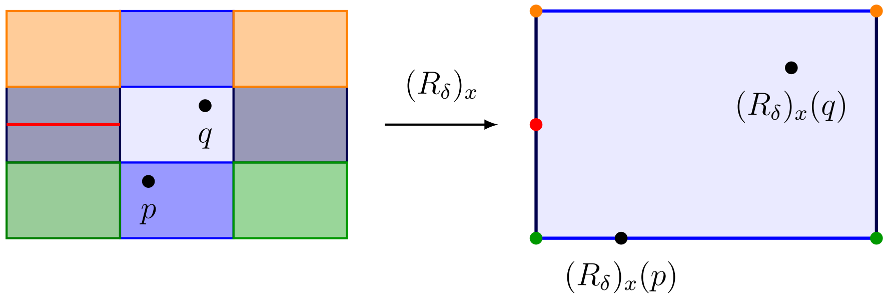

Figure 4.3: The map R_δ on a 2-face of $\mathcal{N}(\mathcal{W})$ with $\delta = \frac{1}{3}$. The different colors indicate on which part of the boundary the different regions are projected. The inner rectangle on the left-hand side is stretched to cover the whole face.

$(x, F) \in \mathcal{N}(\mathcal{U})^{(k)}$, the set $\tau_x(\widetilde{M}) \cap F$ is contained in the $\delta \cdot r_1(F)$-neighbourhood of the boundary ∂F. Since no side length of F is smaller than $r_1(F)$, the set $\tau_x(\widetilde{M}) \cap F$ is mapped completely into ∂F if we apply $(R_\delta)_x$. Note that every face in the boundary of F belongs to $\mathcal{N}(\mathcal{U})$ if F does. As a result, $R_\delta \circ \tau$ lands in the $(k-1)$-skeleton of $\mathcal{N}(\mathcal{U})$. This concludes the proof. $\square$

Thin and thick faces

Before we apply the defined deformations on the nerve map $\Phi\colon X \times \widetilde{M} \to X \times \mathcal{N}(\mathcal{W})$, we specify two categories of faces (x, F) in $X \times \mathcal{N}(\mathcal{W})$ (see [30, Proof of Lemma 9]).

Recall that we can bound the volume of $\Phi_x(\widetilde{M})$ in certain regions of $\mathcal{N}(\mathcal{W})$. We showed in Lemma 4.1.4 that for a.e. $x \in X$ and every face $F \in \mathcal{N}(\mathcal{W})$ we have

$$\mathrm{vol}_d(\Phi_x(\widetilde{M}) \cap \mathrm{Star}(F)) < cV_{\widetilde{M}}(1) r_1(F)^{d+1} e^{-\eta d(F)},$$

where $c = c(d)$ and $\eta = \eta(d)$ are dimensional constants, which are independent of x. Let $\varepsilon = \varepsilon(\eta, d) < \frac{1}{2^{d+1} d!}$ be a small constant such that the following inequality holds:

$$\prod_{k=d+1}^{\infty} \left(1 - 2(3\varepsilon\sigma_d^d e^{-\eta k})^{1/d}\right)^{-d} < 2, \tag{4.7}$$

where σ_d is the constant given in Lemma 4.2.12. By the exponential decay in the term $e^{-\eta k}$, the product converges and we can make the inequality valid by choosing ε sufficiently small. Its value depends on d and η hence it is a dimensional constant.

Definition 4.2.15. Assume that $V_{\widetilde{M}}(1) \leqslant V_0$ holds for the manifold.

Let F be an open k-face in the cuboid complex $\mathcal{N}(\mathcal{W})$ with side lengths $r_1(F) \leqslant \ldots \leqslant r_k(F)$, c be the constant from Lemma 4.1.4 and ε chosen as in (4.7). F is called a *thin face* if $cV_0 r_1(F) < \varepsilon$. Otherwise it is called *thick face*.

We prove some simple remarks on thin and thick faces.

Proposition 4.2.16. *The following properties hold for thin and/or thick faces of* $\mathcal{N}(\mathcal{W})$*:*

i) The property of a face being thin or thick is invariant under the Γ*-action on* $\mathcal{N}(\mathcal{W})$*.*

ii) If F *is thin, then any higher-dimensional face containing* F *in its boundary is thin as well.*

iii) If F *is a thick face, its dimension is bounded by a constant* $D(V_0, d)$ *depending only on* d *and* V_0*.*

Proof. The first statement holds, since for a face $F \in \mathcal{N}(\mathcal{W})$ and an element $\gamma \in \Gamma$ we have $r_1(\gamma F) = r_1(F)$ and the constants are independent of the face. The second assertion is a statement on the elements in $\mathrm{Star}(F)$. For $F' \in \mathrm{Star}(F)$ we have $J_{1/2}(F) \subseteq J_{1/2}(F')$. Thus $r_1(F') \leqslant r_1(F)$, which implies that F' is thin if F is.

The last assertion follows from (4.5) in the proof of Lemma 4.1.4. We showed that the dimension of a face F is bounded by $d(F) \leqslant \log(\frac{1}{r_1}) \cdot C(d)$ with $C(d) = 10^{4(d+3)}$ being the constant from Lemma 3.2.1. Since for a thick face we have $r_1(F) \geqslant \frac{\varepsilon}{cV_0}$, its dimension satisfies

$$d(F) \leqslant C(d) \log(\tfrac{cV_0}{\varepsilon}) =: D(V_0, d).$$

□

Note that if we regard elements $(x, F) \in X \times \mathcal{N}(\mathcal{W})$, the property of F being thin or thick is independent of $x \in X$, since the constants are independent of x. We sometimes talk about thick or thin faces in $X \times \mathcal{N}(\mathcal{W})$ meaning an element of the form (x, F) with F being thick or thin respectively. This property is invariant under the diagonal Γ-action on $X \times \mathcal{N}(\mathcal{W})$.

Deformation of the nerve map down to the d-skeleton

Lemma 4.2.17. *Let* $V_0 > 0$ *be a constant such that* $V_{\widetilde{M}}(1) \leqslant V_0$*. Let* $\Phi\colon X \times \widetilde{M} \to X \times \mathcal{N}(\mathcal{W})$ *be the nerve map as defined in* (4.1)*. Its image is contained in* $\mathcal{N}(\mathcal{W})^{(N_0)}$ *for* N_0 *given in Remark 3.1.8.*

Then Φ *can be deformed to an equivariant geometric map* $\Phi'\colon X \times \widetilde{M} \to X \times \mathcal{N}(\mathcal{W})^{(d)}$ *into the d-skeleton such that the following holds or holds a.e.,respectively:*

i) The image of Φ' *is contained in* $\mathcal{N}(\mathcal{U})^{(d)}$*.*

ii) For every face $F \in \mathcal{N}(\mathcal{W})$ *the preimage of* $\mathrm{Star}(F)$ *under* Φ'_x *is contained in* B_{j_F}.

iii) For every thin face $(x, F) \in X \times \mathcal{N}(\mathcal{W})$ *the map* Φ' *obeys the estimate*

$$\mathrm{vol}_d(\Phi'_x(\widetilde{M}) \cap \mathrm{Star}(F)) < 2\varepsilon r_1(F)^d.$$

iv) Starting with $\mathcal{F}^{(d)}$*, a Borel fundamental domain of* $X \times \mathcal{N}(\mathcal{W})^{(d)}$*, there is a Borel fundamental domain* $\mathcal{F}^{(N_0)}$ *of* $X \times \mathcal{N}(\mathcal{W})^{(N_0)}$ *such that*

$$\nu_{\mathcal{N}}\left(\Phi'(X \times \widetilde{M}) \cap \mathcal{F}^{(d)}\right) \leqslant 2G(V_0, d)^{D(V_0,d)} \nu_{\mathcal{N}}\left(\Phi(X \times \widetilde{M}) \cap \mathcal{F}^{(N_0)}\right).$$

for constants $G(V_0, d) \geqslant 1$ *and* $D(V_0, d)$ *only depending on* d *and* V_0*, where the latter is the constant given in Proposition 4.2.16 iii).*

Proof. Let $\Phi \colon X \times \widetilde{M} \to X \times \mathcal{N}(\mathcal{W})$ be the nerve map as defined in (4.1) with image contained in $\mathcal{N}(\mathcal{U})$. Since the maximal volume of a 1-ball in $\widetilde{M}$ is at most V_0, the multiplicity of the induced covers $\mathcal{U}_x$ is bounded by some constant N_0 by Remark 3.1.8. Thus by Remark 4.1.2, for a.e. $x \in X$ the maps Φ_x are Lipschitz with Lipschitz constant $3N_0^{1/2}$. Moreover, the map Φ lands in the N_0-skeleton $X \times \mathcal{N}(\mathcal{W})^{(N_0)}$. If $N_0 \leqslant d$, we can set $\Phi' = \Phi$. So we consider the case $d < N_0$.

Strategy: Constructing a sequence of maps

Note that for a.e. $x \in X$ and every face $F \in \mathcal{N}(\mathcal{W})$ the preimage of $\mathrm{Star}(F)$ under Φ_x is contained in the ball B_{j_F}. Moreover, it holds the following estimate

$$\mathrm{vol}_d(\Phi_x(\widetilde{M}) \cap \mathrm{Star}(F)) < cV_{\widetilde{M}}(1) r_1(F)^{d+1} e^{-\eta d(F)}, \tag{4.8}$$

with dimensional constants c and η (see Lemma 4.1.4). Recall that we fixed a small constant ε (see (4.7)) and divided the faces of $\mathcal{N}(\mathcal{W})$ into thin and thick faces (see Definition 4.2.15). For thin faces the estimate (4.8) yields

$$\mathrm{vol}_d(\Phi_x(\widetilde{M}) \cap \mathrm{Star}(F)) < \varepsilon r_1(F)^d e^{-\eta d(F)}. \tag{4.9}$$

For the proof of the lemma we construct, starting with $\Phi = \Phi^{(N_0)}$, a sequence of equivariant geometric maps $\Phi = \Phi^{(N_0)}, \Phi^{(N_0-1)}, \ldots, \Phi^{(d)}$ where

$$\Phi^{(k-1)} \colon X \times \widetilde{M} \to X \times \mathcal{N}(\mathcal{W})^{(k-1)}$$

lands in the $(k-1)$-skeleton and results from a deformation of $\Phi^{(k)}$. The maps should satisfy the following:

i) The image of $\Phi^{(k-1)}$ is contained in $\mathcal{N}(\mathcal{U})^{(k-1)}$.

ii) $\Phi_x^{(k-1)}$ is Lipschitz and piecewise-smooth for a.e. $x \in X$.

iii) For a.e. $x \in X$ and every face $F \in \mathcal{N}(\mathcal{W})$ the preimage of $\operatorname{Star}(F)$ under $\Phi_x^{(k-1)}$ is contained in the $\Phi_x^{(k)}$-preimage.

iv) For a.e. $x \in X$ and every thin face $F \in \mathcal{N}(\mathcal{W})$ the map $\Phi^{(k-1)}$ satisfies the estimate

$$\operatorname{vol}_d(\Phi_x^{(k-1)}(\widetilde{M}) \cap \operatorname{Star}(F)) < 2\varepsilon r_1(F)^d e^{-\eta d(F)}. \tag{4.10}$$

v) Given a Borel fundamental domain $\mathcal{F}^{(k-1)}$ of $X \times \mathcal{N}(\mathcal{W})^{(k-1)}$ there is a Borel fundamental domain $\mathcal{F}^{(k)}$ of $X \times \mathcal{N}(\mathcal{W})^{(k)}$, such that

$$\nu_{\mathcal{N}}\left(\Phi^{(k-1)}(X \times \widetilde{M}) \cap \mathcal{F}^{(k-1)}\right) \leqslant (1-2\delta(k))^{-d} G(V_0, d) \cdot \nu_{\mathcal{N}}\left(\Phi^{(k)}(X \times \widetilde{M}) \cap \mathcal{F}^{(k)}\right). \tag{4.11}$$

for a constant $G(V_0, d) \geqslant 1$ only depending on V_0 and d. Here $\delta(k) = (3\varepsilon\sigma_d^d e^{-\eta d(F)})^{1/d}$ where σ_d is the constant in Lemma 4.2.12.

Moreover, if $k > D(V_0, d)$, the constant given in Proposition 4.2.16 iii), it actually holds:

$$\nu_{\mathcal{N}}\left(\Phi^{(k-1)}(X \times \widetilde{M}) \cap \mathcal{F}^{(k-1)}\right) \leqslant (1-2\delta(k))^{-d} \cdot \nu_{\mathcal{N}}\left(\Phi^{(k)}(X \times \widetilde{M}) \cap \mathcal{F}^{(k)}\right). \tag{4.12}$$

Note that the estimate in iv) is slightly weaker than the estimate Φ obeys.

If we can arrange this for all maps $\Phi = \Phi^{(N_0)}, \Phi^{(N_0-1)}, \ldots, \Phi^{(d)}$, then $\Phi^{(d)}$ satisfies the assertions in the lemma. This is clear for the first three statements. To see the last estimate, we run through the sequence of maps $\Phi^{(d)}, \Phi^{(d+1)}, \ldots, \Phi^{(N_0)} = \Phi$ and use the estimates given in (4.11) and (4.12). As a result, starting with $\mathcal{F}^{(d)}$, there is a Borel fundamental domain $\mathcal{F}^{(N_0)}$ of $X \times \mathcal{N}(\mathcal{W})^{(N_0)}$ such that

$$\begin{aligned}
\nu_{\mathcal{N}}\left(\Phi^{(d)}(X \times \widetilde{M}) \cap \mathcal{F}^{(d)}\right) &\leqslant G(V_0, d)^{D(V_0,d)-d} \prod_{k=d+1}^{N_0} (1-2\delta(k))^{-d} \nu_{\mathcal{N}}\left(\Phi(X \times \widetilde{M}) \cap \mathcal{F}^{(N_0)}\right) \\
&\leqslant G(V_0, d)^{D(V_0,d)} \prod_{k=d+1}^{\infty} (1-2\delta(k))^{-d} \nu_{\mathcal{N}}\left(\Phi(X \times \widetilde{M}) \cap \mathcal{F}^{(N_0)}\right) \\
&= G(V_0, d)^{D(V_0,d)} \prod_{k=d+1}^{\infty} (1-2(3\varepsilon\sigma_d^d e^{-\eta d(F)})^{1/d})^{-d} \nu_{\mathcal{N}}\left(\Phi(X \times \widetilde{M}) \cap \mathcal{F}^{(N_0)}\right) \\
&\leqslant 2G(V_0, d)^{D(V_0,d)} \nu_{\mathcal{N}}\left(\Phi(X \times \widetilde{M}) \cap \mathcal{F}^{(N_0)}\right),
\end{aligned}$$

where the last inequality arises by (4.7). So $\Phi' := \Phi^{(d)}$ will do.

Construction of $\Phi^{(k-1)}$ from $\Phi^{(k)}$

Assume that we have already constructed $\Phi^{(m)}$ for $m = N_0, N_0-1, \ldots, k$ for a $N_0 > k > d$. Then $\Phi^{(k-1)}$ results from $\Phi^{(k)}$ by applying the deformations we described before. Here we will apply Federer-Fleming deformations to thick faces (Lemma 4.2.8), use the pushout lemma for small surfaces on thin faces (Lemma 4.2.12) and push everything into the $(k-1)$-skeleton by a map $R_{\delta(k)}$ as described in Lemma 4.2.14.

By Proposition 4.2.16 iii), the dimension of a thick face is bounded by some constant $D(V_0, d)$. Thus if $k > D(V_0, d)$ we do not need any Federer-Fleming construction and it is enough to run through the second and the third of the following steps.

Step 1: Federer-Fleming deformation in thick faces: Note that the deformed nerve map $\Phi^{(k)}: X \times \widetilde{M} \to X \times \mathcal{N}(\mathcal{W})^{(k)}$ is an equivariant geometric map such that for every face $F \in \mathcal{N}(\mathcal{W})^{(k)}$ the preimage $(\Phi_x^{(k)})^{-1}(\mathrm{Star}(F))$ is contained in B_{j_F}, the ball in $\mathcal{U}_x$ corresponding to the largest side length in the fixed order of the side lengths. Let $\Sigma_1 = \{(x, F) \in X \times \mathcal{N}(\mathcal{W}) \mid F \text{ is a thick } k\text{-face}\}$. By Proposition 4.2.16 i) this is a Γ-invariant subset of $X \times \mathcal{N}(\mathcal{W})$. Moreover, we have that $\mathrm{vol}_d(\Phi_x^{(k)}(\widetilde{M}) \cap \bar{F}) < \infty$ for all faces $(x, F) \in \Sigma_1$. This holds by the area formula for Lipschitz maps [41, Chapter 3]. Then Lemma 4.2.10 implies that there exists a family of Federer-Fleming projectors for the map $\Phi^{(k)}$. We denote the map obtained from $\Phi^{(k)}$ via this family by $\widehat{\Phi}^{(k)}$. By Lemma 4.2.8 this is an equivariant geometric map such that for a.e. $x \in X$ and every thick k-face $F \in \mathcal{N}(\mathcal{W})$ we have

$$\mathrm{vol}_d(\widehat{\Phi}_x^{(k)}(\widetilde{M}) \cap \bar{F}) \leqslant \mathrm{vol}_d(\Phi_x^{(k)}(\widetilde{M}) \cap \partial F) + C(k, d)\, \mathrm{vol}_d(\Phi_x^{(k)}(\widetilde{M}) \cap F)$$

for a constant $C(k, d) \geqslant 1$ only depending on k and d. The dimension k of the thick faces is bounded in terms of $D(V_0, d)$ hence there is a constant $G(V_0, d)$ only depending on V_0 and the dimension d, such that we obtain

$$\mathrm{vol}_d(\widehat{\Phi}_x^{(k)}(\widetilde{M}) \cap \bar{F}) \leqslant \mathrm{vol}_d(\Phi_x^{(k)}(\widetilde{M}) \cap \partial F) + G(V_0, d)\, \mathrm{vol}_d(\Phi_x^{(k)}(\widetilde{M}) \cap F). \tag{4.13}$$

For all other faces $(x, F) \in X \times \mathcal{N}(\mathcal{W})$ we have the inequality

$$\mathrm{vol}_d(\widehat{\Phi}_x^{(k)}(\widetilde{M}) \cap F) \leqslant \mathrm{vol}_d(\Phi_x^{(k)}(\widetilde{M}) \cap F).$$

Further, for a.e. $x \in X$, $\widehat{\Phi}_x^{(k)}$ is Lipschitz and piecewise-smooth and for every face $F \in \mathcal{N}(\mathcal{W})^{(k)}$ the preimage $(\widehat{\Phi}_x^{(k)})^{-1}(\mathrm{Star}(F))$ is contained in the $\Phi_x^{(k)}$-preimage, hence in B_{j_F}.

If $k > D(V_0, d)$ we skip this step and set $\widehat{\Phi}^{(k)} = \Phi^{(k)}$.

Step 2: Pushout lemma on small surfaces applied to thin faces: Consider the subset $\Sigma_2 = \{(x, F) \in X \times \mathcal{N}(\mathcal{W}) \mid F \text{ is a thin } k\text{-face}\} \subset X \times \mathcal{N}(\mathcal{W})$. It is Γ-invariant by Proposition 4.2.16 i). We apply Lemma 4.2.12 to this subset and the map $\widehat{\Phi}^{(k)}$. This yields an equivariant geometric map $\widetilde{\Phi}^{(k)} \colon X \times \widetilde{M} \to X \times \mathcal{N}(\mathcal{W})^{(k)}$ which is piecewise-smooth and therefore Lipschitz on a.e. fibres such that for every face the preimage of $\mathrm{Star}(F)$ under $\widetilde{\Phi}^{(k)}_x$ is contained in B_{j_F}.

Moreover, we have for a.e. $x \in X$ and every face $F \in \mathcal{N}(\mathcal{W})$ that

$$\mathrm{vol}_d(\widetilde{\Phi}^{(k)}_x(\widetilde{M}) \cap F) \leqslant \mathrm{vol}_d(\widehat{\Phi}^{(k)}_x(\widetilde{M}) \cap F). \tag{4.14}$$

For a.e. element (x, F) in Σ_2 the image $\widetilde{\Phi}^{(k)}_x(\widetilde{M}) \cap F$ is contained in the W-neighbourhood of the boundary for

$$W = s\sigma_d \,\mathrm{vol}_d(\widehat{\Phi}^{(k)}_x(\widetilde{M}) \cap F)^{1/d} = s\sigma_d \,\mathrm{vol}_d(\Phi^{(k)}_x(\widetilde{M}) \cap F)^{1/d}.$$

The map $\Phi^{(k)}$ satisfies a volume estimate for thin faces as in (4.10). This yields

$$s\sigma_d \,\mathrm{vol}_d(\Phi^{(k)}_x(\widetilde{M}) \cap F)^{1/d} < s\sigma_d \,\mathrm{vol}_d(\Phi^{(k)}_x(\widetilde{M}) \cap \mathrm{Star}(F))^{1/d} < s\sigma_d(2\varepsilon r_1(F)^d e^{-\eta d(F)})^{1/d}.$$

We can arrange such that $2s^d < 3$. Hence the image $\widetilde{\Phi}^{(k)}_x(\widetilde{M}) \cap F$ is contained in the $\delta(k) r_1(F)$-neighbourhood of the boundary ∂F with

$$\delta(k) \coloneqq (3\varepsilon\sigma_d^d e^{-\eta d(F)})^{1/d}.$$

Step 3: Pushing everything into the $(k-1)$-skeleton: The map $\widehat{\Phi}^{(k)}$ has the property that for every open face $(x, F) \in X \times \mathcal{N}(\mathcal{W})$ the set $\widehat{\Phi}^{(k)}_x(\widetilde{M}) \cap F$ is empty or is contained in the $\delta(k) r_1(F)$-neighbourhood of the boundary ∂F. Set $\Phi^{(k-1)} \coloneqq R_{\delta(k)} \circ \widetilde{\Phi}^{(k)}$ with $R_{\delta(k)}$ being the map defined in Lemma 4.2.14. Then $\Phi^{(k-1)}$ is an equivariant geometric map with image in $\mathcal{N}(\mathcal{U})^{(k-1)}$. If (x, F) is a k-face in $\mathrm{im}(\Phi^{(k)})$, so in particular in $\mathcal{N}(\mathcal{U})^{(k)}$, then by construction the image of $\Phi^{(k-1)}$ lies in the boundary $(x, \partial F)$. But every face $(x, F') \in (x, \partial F)$ belongs to $\mathcal{N}(\mathcal{U})$ as well. The preimage of the open star of any face F under $(R_{\delta(k)})_x$ is contained in $\mathrm{Star}(F)$ hence the preimage under $\Phi^{(k-1)}_x$ is contained in the $\Phi^{(k)}_x$-preimage.

The maps $(R_{\delta(k)})_x$ are Lipschitz with Lipschitz constant $(1 - 2\delta(k))^{-1}$ and the map $\Phi^{(k-1)}_x$ is Lipschitz and piecewise-smooth for a.e. $x \in X$.

Volume estimates for $\Phi^{(k-1)}$

We have to check that $\Phi^{(k-1)}$ obeys certain volume estimates (see (4.10),(4.11) and (4.12)). At first, check that for a thin face $(x, F) \in X \times \mathcal{N}(\mathcal{W})$ we have the estimate

$$\mathrm{vol}_d(\Phi^{(k-1)}_x(\widetilde{M}) \cap \mathrm{Star}(F)) < 2\varepsilon r_1(F)^d e^{-\eta d(F)}.$$

To see this, note that $\mathrm{Star}(F)$ only contains thin faces if F is thin (Proposition 4.2.16 ii)). We obtain by the Lipschitz property of $(R_{\delta(k)})_x$ and the estimate (4.14)

$$\begin{aligned}
\mathrm{vol}_d(\Phi_x^{(k-1)}(\widetilde{M}) \cap \mathrm{Star}(F)) &= \mathrm{vol}_d((R_{\delta(k)})_x \circ \widetilde{\Phi}_x^{(k)}(\widetilde{M}) \cap \mathrm{Star}(F)) \\
&\leqslant (1-2\delta(k))^{-d}\,\mathrm{vol}_d(\widetilde{\Phi}_x^{(k)}(\widetilde{M}) \cap \mathrm{Star}(F)) \\
&\leqslant (1-2\delta(k))^{-d}\,\mathrm{vol}_d(\widehat{\Phi}_x^{(k)}(\widetilde{M}) \cap \mathrm{Star}(F)) \\
&= (1-2\delta(k))^{-d}\,\mathrm{vol}_d(\Phi_x^{(k)}(\widetilde{M}) \cap \mathrm{Star}(F)).
\end{aligned}$$

This estimate holds for all preceding maps $\Phi^{(k)}, \dots, \Phi^{(N_0-1)}$. Therefore,

$$\mathrm{vol}_d(\Phi_x^{(k-1)}(\widetilde{M}) \cap \mathrm{Star}(F)) \leqslant \prod_{l=k}^{N_0}(1-2\delta(l))^{-d}\,\mathrm{vol}_d(\Phi_x(\widetilde{M}) \cap \mathrm{Star}(F)) < 2\varepsilon r_1(F)^d e^{-\eta d(F)}.$$

The last inequality holds by the estimate Φ obeys for thin faces, see (4.9), and by the definition of $\delta(l)$ and the choice of ε in (4.7).

Let $\mathcal{F}^{(k-1)}$ be a Borel fundamental domain of $X \times \mathcal{N}(\mathcal{W})^{(k-1)}$. We have $\Phi^{(k-1)} = R_{\delta(k)} \circ \widetilde{\Phi}^{(k)}$ and $R_{\delta(k)}^{-1}(\mathcal{F}^{(k-1)})$ is contained in a Borel fundamental domain $\mathcal{F}^{(k)}$ of $X \times \mathcal{N}(\mathcal{W})^{(k)}$. Using that $(R_{\delta(k)})_x$ is $(1-2\delta(k))^{-1}$-Lipschitz we obtain

$$\begin{aligned}
\nu_{\mathcal{N}}\left(\Phi^{(k-1)}(X \times \widetilde{M}) \cap \mathcal{F}^{(k-1)}\right) &\leqslant \nu_{\mathcal{N}}\left(R_{\delta(k)}\left(\widetilde{\Phi}^{(k)}(X \times \widetilde{M}) \cap \mathcal{F}^{(k)}\right)\right) \\
&= \int_X \mathrm{vol}_d\left((R_{\delta(k)})_x\left(\widetilde{\Phi}_x^{(k)}(\widetilde{M}) \cap \mathcal{F}^{(k)}\right)\right) d\mu(x) \\
&\leqslant (1-2\delta(k))^{-d}\int_X \mathrm{vol}_d\left(\widetilde{\Phi}_x^{(k)}(\widetilde{M}) \cap \mathcal{F}^{(k)}\right) d\mu(x) \\
&= (1-2\delta(k))^{-d}\nu_{\mathcal{N}}\left(\widetilde{\Phi}^{(k)}(X \times \widetilde{M}) \cap \mathcal{F}^{(k)}\right).
\end{aligned} \tag{4.15}$$

For $x \in X$, $\widetilde{\Phi}_x^{(k)}$ is formed from $\widehat{\Phi}_x^{(k)}$ by a surgery in each thin k-face. By (4.14) the volume is not increasing under these surgeries. We obtain

$$\begin{aligned}
\nu_{\mathcal{N}}\left(\widetilde{\Phi}^{(k)}(X \times \widetilde{M}) \cap \mathcal{F}^{(k)}\right) &= \int_X \mathrm{vol}_d\left(\widetilde{\Phi}_x^{(k)}(\widetilde{M}) \cap \mathcal{F}^{(k)}\right) d\mu(x) \\
&\leqslant \int_X \mathrm{vol}_d\left(\widehat{\Phi}_x^{(k)}(\widetilde{M}) \cap \mathcal{F}^{(k)}\right) d\mu(x) \\
&= \nu_{\mathcal{N}}\left(\widehat{\Phi}^{(k)}(X \times \widetilde{M}) \cap \mathcal{F}^{(k)}\right).
\end{aligned} \tag{4.16}$$

If $k > D(V_0, d)$, we have $\widehat{\Phi}^{(k)} = \Phi^{(k)}$, since there are no thick k-faces (see Proposition 4.2.16 iii)). Together with (4.15) we obtain (4.12). Otherwise $\widehat{\Phi}^{(k)}$ results from $\Phi^{(k)}$ by a Federer-Fleming deformation in the thick faces. For every thick k-face $(x, F) \in X \times \mathcal{N}(\mathcal{W})$ the estimate (4.13) holds. For all other faces we have $\mathrm{vol}_d(\widehat{\Phi}_x^{(k)}(\widetilde{M}) \cap F) \leqslant \mathrm{vol}_d(\Phi_x^{(k)}(\widetilde{M}) \cap F)$.

In order to estimate $\widehat{\Phi}_x^{(k)}(\widetilde{M}) \cap \mathcal{F}^{(k)}$, we divide the Borel fundamental domain $\mathcal{F}^{(k)}$ into two subsets $\mathcal{F}_1$ and $\mathcal{F}_2$. The first subset contains all faces which are not involved in the Federer-Fleming projections in the thick k-faces, i.e. all faces of dimension less than $(k-1)$, $(k-1)$-faces which are not in the boundary of a thick face and the thin k-faces. $\mathcal{F}_2$ contains all thick k-faces in $\mathcal{F}^{(k)}$ and the $(k-1)$-faces in $\mathcal{F}^{(k)}$ which are in the boundary of some thick k-face. We have

$$\begin{aligned}\nu_{\mathcal{N}}\left(\widehat{\Phi}^{(k)}(X \times \widetilde{M}) \cap \mathcal{F}_1\right) &= \int_X \mathrm{vol}_d\left(\widehat{\Phi}_x^{(k)}(\widetilde{M}) \cap \mathcal{F}_1\right)\, d\mu(x)\\ &= \int_X \mathrm{vol}_d\left(\Phi_x^{(k)}(\widetilde{M}) \cap \mathcal{F}_1\right)\, d\mu(x) = \nu_{\mathcal{N}}\left(\Phi^{(k)}(X \times \widetilde{M}) \cap \mathcal{F}_1\right).\end{aligned}$$

Let $\mathcal{F}_2'$ be the union of all closed k-faces $(x, \bar{F})$ with $(x, F) \in \mathcal{F}_2$ being a thick k-face. Note that $\mathcal{F}_2'$ arises from $\mathcal{F}_2$ by replacing some $(k-1)$-faces by a Γ-translate. There is only one face of every orbit in $\mathcal{F}_2'$. Then by Lemma 2.3.9 we have

$$\begin{aligned}\nu_{\mathcal{N}}\left(\widehat{\Phi}^{(k)}(X \times \widetilde{M}) \cap \mathcal{F}_2\right) &= \int_X \mathrm{vol}_d\left(\widehat{\Phi}_x^{(k)}(\widetilde{M}) \cap \mathcal{F}_2\right)\, d\mu(x)\\ &= \int_X \mathrm{vol}_d\left(\widehat{\Phi}_x^{(k)}(\widetilde{M}) \cap \mathcal{F}_2'\right)\, d\mu(x) = \nu_{\mathcal{N}}\left(\widehat{\Phi}^{(k)}(X \times \widetilde{M}) \cap \mathcal{F}_2'\right)\end{aligned} \tag{4.17}$$

and

$$\nu_{\mathcal{N}}\left(\Phi^{(k)}(X \times \widetilde{M}) \cap \mathcal{F}_2\right) = \nu_{\mathcal{N}}\left(\Phi^{(k)}(X \times \widetilde{M}) \cap \mathcal{F}_2'\right). \tag{4.18}$$

The estimate on thick faces in (4.13) yields

$$\sum_{\bar{F} \in (\mathcal{F}_2')_x} \mathrm{vol}_d(\widehat{\Phi}_x^{(k)}(\widetilde{M}) \cap \bar{F}) \leqslant \sum_{\bar{F} \in (\mathcal{F}_2')_x} \left(\mathrm{vol}_d(\Phi_x^{(k)}(\widetilde{M}) \cap \partial F) + G(V_0, d)\, \mathrm{vol}_d(\Phi_x^{(k)}(\widetilde{M}) \cap F)\right)$$

for a constant $G(V_0, d) \geqslant 1$. Note that in the sum, some of the $(k-1)$-faces might appear more than once, if they are in the boundary of more than one thick k-face. For such faces F' we have

$$\mathrm{vol}_d(\Phi_x^{(k)}(\widetilde{M}) \cap F') \leqslant \mathrm{vol}_d(\widehat{\Phi}_x^{(k)}(\widetilde{M}) \cap F').$$

We can subtract their contributions of the form $\mathrm{vol}_d(\widehat{\Phi}_x^{(k)}(\widetilde{M}) \cap F')$ and $\mathrm{vol}_d(\Phi_x^{(k)}(\widetilde{M}) \cap F')$, respectively, on the two sides of the inequality and obtain

$$\begin{aligned}\mathrm{vol}_d(\widehat{\Phi}_x^{(k)}(\widetilde{M}) \cap \mathcal{F}_2') &\leqslant \sum_{\substack{(k-1)\text{-faces}\\ F' \in (\mathcal{F}_2')_x}} \mathrm{vol}_d(\Phi_x^{(k)}(\widetilde{M}) \cap F') + G(V_0, d) \sum_{\substack{k\text{-faces}\\ F \in (\mathcal{F}_2')_x}} \mathrm{vol}_d(\Phi_x^{(k)}(\widetilde{M}) \cap F)\\ &\leqslant G(V_0, d)\Big(\sum_{\substack{(k-1)\text{-faces}\\ F' \in (\mathcal{F}_2')_x}} \mathrm{vol}_d(\Phi_x^{(k)}(\widetilde{M}) \cap F') + \sum_{\substack{k\text{-faces}\\ F \in (\mathcal{F}_2')_x}} \mathrm{vol}_d(\Phi_x^{(k)}(\widetilde{M}) \cap F)\Big)\\ &= G(V_0, d)\, \mathrm{vol}_d(\Phi_x^{(k)}(\widetilde{M}) \cap \mathcal{F}_2').\end{aligned}$$

Together with (4.17) and (4.18) this yields

$$\nu_{\mathcal{N}}\left(\widehat{\Phi}^{(k)}(X\times\widetilde{M})\cap\mathcal{F}_2\right)\leqslant G(V_0,d)\cdot\nu_{\mathcal{N}}\left(\Phi^{(k)}(X\times\widetilde{M})\cap\mathcal{F}_2\right)$$

and

$$\begin{aligned}\nu_{\mathcal{N}}\left(\widehat{\Phi}^{(k)}(X\times\widetilde{M})\cap\mathcal{F}^{(k)}\right)&=\nu_{\mathcal{N}}\left(\widehat{\Phi}^{(k)}(X\times\widetilde{M})\cap\mathcal{F}_1\right)+\nu_{\mathcal{N}}\left(\widehat{\Phi}^{(k)}(X\times\widetilde{M})\cap\mathcal{F}_2\right)\\&\leqslant G(V_0,d)\left(\nu_{\mathcal{N}}\left(\Phi^{(k)}(X\times\widetilde{M})\cap\mathcal{F}_1\right)+\nu_{\mathcal{N}}\left(\Phi^{(k)}(X\times\widetilde{M})\cap\mathcal{F}_2\right)\right)\\&=G(V_0,d)\nu_{\mathcal{N}}\left(\Phi^{(k)}(X\times\widetilde{M})\cap\mathcal{F}^{(k)}\right).\end{aligned}$$

As a result, for $k\leqslant D(V_0,d)$, we obtain

$$\nu_{\mathcal{N}}\left(\Phi^{(k-1)}(X\times\widetilde{M})\cap\mathcal{F}^{(k-1)}\right)\leqslant(1-2\delta(k))^{-d}G(V_0,d)\cdot\nu_{\mathcal{N}}\left(\Phi^{(k)}(X\times\widetilde{M})\cap\mathcal{F}^{(k)}\right),$$

where we used (4.15) and (4.16). Therefore the constructed map $\Phi^{(k-1)}$ satisfies all the stated properties. This concludes the proof of the lemma. □

Remark 4.2.18. One can show that the map Φ' constructed in Lemma 4.2.17 is equivariant geometric homotopic to the original nerve map Φ, i.e. there exists a map

$$H\colon (X\times\widetilde{M})\times[0,1]\longrightarrow X\times\mathcal{N}(\mathcal{W}),$$

such that for every $t\in[0,1]$ the map $H_t\colon X\times\widetilde{M}\to X\times\mathcal{N}(\mathcal{W})$ is an equivariant geometric map with image in $\mathcal{N}(\mathcal{U})$ with $H_0=\Phi$ and $H_1=\Phi'$. This implies that the corresponding chain maps $C_*^X(\Phi)$ and $C_*^X(\Phi')$ are chain homotopy equivalent and the induced maps in homology coincide. As we will see, this is not necessary for the proof of Theorem 5.2.

4.2.2 Bounds on the simplicial norm

So far we constructed a map Φ' landing in the d-skeleton of the nerve by deformation of the nerve map Φ. In this subsection we show how to establish an upper bound on $|H_d(\Phi')([M]^X)|^X$ in terms of the volume as claimed in Theorem 4.2.1. Apart from the properties of Φ' stated in Lemma 4.2.17 we will need some more lemmas. We start by showing how to connect the measure of the image of the nerve map with the volume of the manifold.

Lemma 4.2.19. *Assume that $V_{\widetilde{M}}(1)\leqslant V_0$.*

Let $\Phi\colon X\times\widetilde{M}\to X\times\mathcal{N}(\mathcal{W})$ be the nerve map defined in (4.1). *Its image is contained in $\mathcal{N}(\mathcal{U})^{(N_0)}$ for some constant N_0 given in Remark 3.1.8. Let $\mathcal{F}^{(N_0)}$ be a Borel* *fundamental domain of the N_0-skeleton $X \times \mathcal{N}(\mathcal{W})^{(N_0)}$. Then the following holds: There is a constant $C''(d)$ only depending on the dimension d of the manifold M such that we have*

$$\nu_{\mathcal{N}}\left(\Phi(X \times \widetilde{M}) \cap \mathcal{F}^{(N_0)}\right) \leqslant C''(d)\operatorname{vol}(M).$$

Proof. The proof is based on the same ideas as the proof of Lemma 4.1.4. We need the adjusted version of [30, Lemma 4], which we stated in Lemma 3.2.4. Note that the preimage of $\mathcal{F}^{(N_0)}$ in $X \times \widetilde{M}$ lies in a fundamental domain $\mathcal{F}$ of $X \times \widetilde{M}$. We decompose $\mathcal{F}$ hence $X \times \widetilde{M}$ into sets of different multiplicity. Let $\omega < \frac{1}{100}$. We define the following subsets for $i \in \mathbb{N}$:

$$\begin{aligned}
S = S^{(1)} :&= (X \times \widetilde{M})(\beta \log(\tfrac{1}{\omega})) = \{(x,p) \in X \times \widetilde{M} \mid m(x,p) = m_x(p) \geqslant \beta \log(\tfrac{1}{\omega})\} \\
&= \{(x,p) \in X \times \widetilde{M} \mid m_x(p) \geqslant \lfloor \beta \log(\tfrac{1}{\omega}) + 1 \rfloor\} \\
S^{(i)} :&= (X \times \widetilde{M})(\beta \log(\tfrac{1}{\omega}) + (i-1)) = \{(x,p) \in X \times \widetilde{M} \mid m_x(p) \geqslant \lfloor \beta \log(\tfrac{1}{\omega}) + i \rfloor\} \\
S_{(i)} :&= \{(x,p) \in X \times \widetilde{M} \mid m_x(p) = \lfloor \beta \log(\tfrac{1}{\omega}) + i \rfloor\} \subseteq S^{(i)}.
\end{aligned}$$

Lemma 3.2.4 yields the following inequalities for every $i \in \mathbb{N}$:

$$\nu\left(S_{(i)} \cap \mathcal{F}\right) \leqslant \nu\left(S^{(i)} \cap \mathcal{F}\right) \leqslant e^{-\alpha(i-1)}\operatorname{vol}(M) < Ce^{-\alpha i}\operatorname{vol}(M),$$

where $C = e^{\alpha} > 1$ is a dimensional constant. Using that Φ_x is locally Lipschitz with Lipschitz constant $3m_x(p)^{(1/2)}$ only depending on the multiplicity (see Remark 4.1.2), we obtain

$$\begin{aligned}
\nu_{\mathcal{N}}\left(\Phi(X \times \widetilde{M} \backslash S) \cap \mathcal{F}^{(N_0)}\right) &\leqslant \nu_{\mathcal{N}}\left(\Phi(X \times \widetilde{M} \backslash S \cap \mathcal{F})\right) \\
&= \int_X \operatorname{vol}_d\left(\Phi_x(\widetilde{M} \backslash S_x \cap \mathcal{F})\right) d\mu(x) \\
&\leqslant 3^d(\beta \log(\tfrac{1}{\omega}))^{d/2} \int_X \operatorname{vol}_d\left((\widetilde{M} \backslash S_x) \cap \mathcal{F}\right) d\mu(x) \\
&= 3^d(\beta \log(\tfrac{1}{\omega}))^{d/2} \nu\left(X \times \widetilde{M} \backslash S \cap \mathcal{F}\right) \\
&\leqslant 3^d(\beta \log(\tfrac{1}{\omega}))^{d/2} \nu\left(X \times \widetilde{M} \cap \mathcal{F}\right) \\
&< 3^d(\beta \log(\tfrac{1}{\omega}))^{d/2} C \operatorname{vol}(M).
\end{aligned}$$

The same way we get

$$\begin{aligned}
\nu_{\mathcal{N}}\left(\Phi(S_{(i)}) \cap \mathcal{F}^{(N_0)}\right) &\leqslant 3^d(\lfloor \beta \log(\tfrac{1}{\omega}) + i \rfloor)^{d/2} \nu\left(S_{(i)} \cap \mathcal{F}\right) \\
&\leqslant 3^d(\beta \log(\tfrac{1}{\omega}) + i)^{d/2}(Ce^{-\alpha i}\operatorname{vol}(M)).
\end{aligned}$$

We add up the measures of the different regions. Note that the multiplicity of the induced covers $\mathcal{U}_x$ is bounded by some constant N_0 (see Remark 3.1.8), hence we obtain

$$\begin{aligned}
\nu_{\mathcal{N}}\left(\Phi(X\times\widetilde{M})\cap\mathcal{F}^{(N_0)}\right) &= \nu_{\mathcal{N}}\left(\Phi(X\times\widetilde{M}\backslash S)\cap\mathcal{F}^{(N_0)}\right)+\sum_{i=1}^{N_0}\nu_{\mathcal{N}}\left(\Phi(S_{(i)})\cap\mathcal{F}^{(N_0)}\right)\\
&< 3^d\left(\sum_{i=0}^{N_0}(\beta\log(\tfrac{1}{\omega})+i)^{d/2}\left(C\cdot e^{-\alpha i}\operatorname{vol}(M)\right)\right)\\
&\leqslant C\cdot 3^d\left(\sum_{i=0}^{\infty}(\beta\log(\tfrac{1}{\omega})+i)^{d/2}e^{-\alpha i}\right)\operatorname{vol}(M)\\
&< C'(d)\tfrac{1}{\omega}\operatorname{vol}(M).
\end{aligned}$$

The last inequality is due to (4.4). The lowest upper bound is achieved for $\omega=\frac{1}{100}$ and we obtain

$$\nu_{\mathcal{N}}\left(\Phi(X\times\widetilde{M})\cap\mathcal{F}^{(N_0)}\right)\leqslant C''(d)\operatorname{vol}(M).$$

□

We can find an upper bound for the parametrised ℓ^1-norm of the image of the X-parametrised fundamental class of M by looking at a suitable representative of this homology class. We obtain the canonical parametrised fundamental cycle by considering a triangulation of the manifold. It gives rise to a representative $\sum_{j=1}^k\tau'_j\in C_d(M,\mathbb{Z})$ of the fundamental class $[M]_{\mathbb{Z}}$ of M. Let $\tau_j\colon\Delta_d\to\widetilde{M}$ be a lift of τ'_j for $j=1,\dots,k$. Then $\kappa=\sum_{j=1}^k\chi_X\otimes\tau_j\in L^\infty(X,\mathbb{Z})\otimes_{\mathbb{Z}\Gamma}C_d(\widetilde{M},\mathbb{Z})$ is a reduced X-parametrised fundamental cycle of M by definition. $H_d(\Phi')([M]^X)$ is then represented by $C_d^X(\Phi')(\kappa)$ where $C_*^X(\Phi')$ is the induced chain map

$$C_*^X(\Phi')\colon L^\infty(X,\mathbb{Z})\otimes_{\mathbb{Z}\Gamma}C_*(\widetilde{M},\mathbb{Z})\longrightarrow L^\infty(X,\mathbb{Z})\otimes_{\mathbb{Z}\Gamma}C_*(\mathcal{N}(\mathcal{W}),\mathbb{Z}).$$

as defined in Theorem 2.4.1. Let $K\subset\widetilde{M}$ be a compact subset containing $\operatorname{im}(\tau_j)$ for $j=1,\dots,k$. Since Φ' is of finite variance, there is a Borel partition $X=\bigcup_{q=1}^Q X_q$ such that $\Phi'_{|X_q\times K}=\operatorname{id}\times\varphi_q$ with continuous maps φ_q. Then by the definition of $C_*^X(\Phi')$ we have

$$C_d^X(\Phi')(\kappa)=C_d^X(\Phi')(\sum_{j=1}^k\chi_X\otimes\tau_j)=\sum_{j=1}^k\sum_{q=1}^Q\chi_{X_q}\otimes\varphi_q(\tau_j).$$

For a.e. $x\in X$ the evaluated cycle $(C_d^X(\Phi')(\kappa))_x$ is a locally finite cycle in $C_d^{\mathrm{lf}}(\mathcal{N}(\mathcal{W}),\mathbb{Z})$ and by Proposition 2.4.4 it holds $(C_d^X(\Phi')(\kappa))_x=C_d^{\mathrm{lf}}(\Phi'_x)(\kappa_x)$. This is due to the fact that Φ' is equivariant geometric, so fibrewise proper. Note that κ_x is a locally finite fundamental cycle of $\widetilde{M}$ by Lemma 2.1.22.

Note that $C_d^X(\Phi')(\kappa)$ is a cycle in $L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_d(\mathcal{N}(\mathcal{W})^{(d)},\mathbb{Z})$, since $\operatorname{im}(\Phi')$ belongs to the d-skeleton $X \times \mathcal{N}(\mathcal{W})^{(d)}$. Let $\mathrm{Sd}(\mathcal{N}(\mathcal{W})^{(d)})$ be the first barycentric subdivision of the cubical complex $\mathcal{N}(\mathcal{W})^{(d)}$ which is a locally finite metric simplicial complex by Lemma 2.2.7. Note that $\mathrm{Sd}(\mathcal{N}(\mathcal{W})^{(d)}) = \mathrm{Sd}(\mathcal{N}(\mathcal{W}))^{(d)}$. By Lemma 2.2.7, we have an isometry $a\colon \mathcal{N}(\mathcal{W})^{(d)} \to \mathrm{Sd}(\mathcal{N}(\mathcal{W})^{(d)})$, which is in particular Γ-equivariant. The group Γ acts on $\mathcal{N}(\mathcal{W})$ by permuting faces, which is an isometric action. Isometries of the cuboids induce isometries of the corresponding barycentric subdivisions, so the map a is compatible with the group action. It induces an isomorphism of chain complexes $C_*(a)\colon C_*(\mathcal{N}(\mathcal{W})^{(d)},\mathbb{Z}) \to C_*(\mathrm{Sd}(\mathcal{N}(\mathcal{W})^{(d)}),\mathbb{Z})$ and (after passing over to the coinvariants)

$$\mathrm{id} \otimes C_*(a)\colon L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\mathcal{N}(\mathcal{W}),\mathbb{Z}) \longrightarrow L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\mathrm{Sd}(\mathcal{N}(\mathcal{W})^{(d)}),\mathbb{Z}). \tag{4.19}$$

The inverse is given by $\mathrm{id} \otimes C_*(a^{-1})$.
Associated to the metric simplicial complex $\mathrm{Sd}(\mathcal{N}(\mathcal{W})^{(d)})$ we get an locally finite abstract simplicial complex Δ [8, Appendix Chapter I.7, p. 126]. Its set of vertices V is given by the vertices of $\mathrm{Sd}(\mathcal{N}(\mathcal{W})^{(d)})$. The vertex set of any simplex in the metric simplicial complex defines a simplex in Δ. Δ inherits the Γ-action. Its geometric realization, $|\Delta|$, is homeomorphic to $\mathrm{Sd}(\mathcal{N}(\mathcal{W})^{(d)})$, and this homeomorphism is Γ-equivariant. As a result

$$L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\mathrm{Sd}(\mathcal{N}(\mathcal{W})^{(d)}),\mathbb{Z}) \cong L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(|\Delta|,\mathbb{Z}). \tag{4.20}$$

We want to use the chain homotopy equivalence between the simplicial and singular chain complexes for a simplicial complex respectively its realization. Recall the definition of the ordered simplicial chain complex for an abstract simplicial complex Θ [43, §13, p. 76]. We assume that Θ is locally finite. An ordered n-simplex is an $(n+1)$-tuple of vertices which belong to one simplex of Θ but are not necessarily distinct. We denote the group of ordered n-chains, i.e. the free abelian group generated by the ordered n-simplices, by $C_n^{simp}(\Theta)$. We can identify an ordered n-simplex with the n-chain which takes the value 1 on this n-simplex and vanishes for all others. The boundary homomorphism $\partial_n\colon C_n^{simp}(\Theta) \to C_{n-1}^{simp}(\Theta)$ is given by $\partial_n(v_0,\dots,v_n) = \sum_{i=0}^n (-1)^i (v_0,\dots,\hat{v}_i,\dots,v_n)$, where $(v_0,\dots,\hat{v}_i,\dots,v_n)$ is the n-tuple we get by omitting the i-th entry in $(v_0,\dots,v_n)$.

We get the ordered simplicial chain complex of Θ, $C_*^{simp}(\Theta) = (C_*^{simp}(\Theta), \partial_*)$. Furthermore any simplicial map $f : \Theta \to \Theta'$ between abstract simplicial complexes yields a chain map via $f_*(v_0, \ldots, v_n) = (f(v_0), \ldots, f(v_n))$. Look at the map

$$\theta_n \colon C_n^{simp}(\Theta) \longrightarrow C_n(|\Theta|, \mathbb{Z}),$$

assigning to every ordered n-simplex $(v_0, \ldots, v_n)$ the linear singular simplex $\Theta_n \to |\Theta|$ mapping the vertices of the standard simplex e_i to v_i. This map defines a chain map $C_*^{simp}(\Theta) \to C_*(|\Theta|, \mathbb{Z})$ which is natural with respect to simplicial maps [43, §34, p. 190 ff.]. Eilenberg showed that this map is a chain homotopy equivalence by constructing a chain homotopy inverse [11]. Note that this inverse is not natural. Nevertheless, Eilenberg showed in [12], that we get an equivariant chain homotopy equivalence if we consider a simplicial group action on a locally finite simplicial complex. If a group Λ acts simplicially on Θ, the ordered simplicial chain complex has the structure of a $\mathbb{Z}\Lambda$-chain complex. The group acts on ordered simplices, so $C_n^{simp}(\Theta)$ is a $\mathbb{Z}\Lambda$-module and the boundary homomorphism commutes with the group action. In the same way $C_*(|\Theta|, \mathbb{Z})$ is a chain complex of $\mathbb{Z}\Lambda$-modules. The above map θ is equivariant hence a chain map of $\mathbb{Z}\Lambda$-chain complex. By [12, Thm. 10.1] it is a Λ-chain homotopy equivalence.

Now consider the free Γ-simplicial complex Δ. The group Γ acts freely on ordered simplices, so $C_*^{simp}(\Delta)$ is a chain complex of free $\mathbb{Z}\Gamma$-modules just the same as the singular chain complex $C_*(|\Delta|, \mathbb{Z})$. We get a chain map of $\mathbb{Z}\Gamma$-chain complexes

$$\theta_* \colon C_*^{simp}(\Delta) \longrightarrow C_*(|\Delta|, \mathbb{Z}),$$

which is a Γ-chain homotopy equivalence. Then

$$\mathrm{id}_{L^\infty(X,\mathbb{Z})} \otimes_{\mathbb{Z}} \theta_* \colon L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}} C_*^{simp}(\Delta) \longrightarrow L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}} C_*(|\Delta|, \mathbb{Z})$$

is a chain homotopy equivalence of $\mathbb{Z}\Gamma$-chain complexes as well. Passing over to the coinvariants, we obtain a chain map of $\mathbb{Z}$-modules

$$\alpha_* := \mathrm{id} \otimes \theta_* \colon L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*^{simp}(\Delta) \longrightarrow L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(|\Delta|, \mathbb{Z})$$

which is a chain homotopy equivalence. As a result, together with (4.19) and (4.20), we get a sequence of chain maps, which are either isomorphisms or chain homotopy equivalences

$$\begin{aligned} &L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*^{simp}(\Delta) \xrightarrow{\alpha_*} L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(|\Delta|, \mathbb{Z}) \cong \\ &\cong L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\mathrm{Sd}(\mathcal{N}(\mathcal{W})^{(d)}), \mathbb{Z}) \xrightarrow{\mathrm{id} \otimes C_*(a^{-1})} L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\mathcal{N}(\mathcal{W}), \mathbb{Z}). \end{aligned}$$

Thus the fundamental cycle $C_d^X(\Phi')(\kappa)$ is homologous to the image of a cycle

$$\sum_{l=1}^{m} g_l \otimes (v_0^l, \dots, v_d^l) \in L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_d^{simp}(\Delta)$$

under this composition. For $l = 1, \dots, m$, $(v_0^l, \dots, v_d^l)$ is an ordered d-simplices of Δ. We denote the linear singular simplex mapping the i-th vertex of Δ_d to v_i^l by $\tilde{\sigma}_l \colon \Delta_d \to \mathrm{Sd}(\mathcal{N}(\mathcal{W})^{(d)})$ where we use the homeomorphism $|\Delta| \cong \mathrm{Sd}(\mathcal{N}(\mathcal{W})^{(d)})$. Note that the image of $\tilde{\sigma}_l$ is a d-simplex of $\mathrm{Sd}(\mathcal{N}(\mathcal{W})^{(d)})$ Then by the definition of α_* we have that $C_d^X(\Phi')(\kappa)$ is homologous to a cycle

$$(\mathrm{id} \otimes C_d(a^{-1}))(\sum_{l=1}^{m} g_l \otimes \tilde{\sigma}_l) = \sum_{l=1}^{m} g_l \otimes (a^{-1} \circ \tilde{\sigma}_l) =: \sum_{l=1}^{m} g_l \otimes \sigma_l$$

in $L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_d(\mathcal{N}(\mathcal{W})^{(d)}, \mathbb{Z})$. Note that the images of σ_l and $\tilde{\sigma}_l$ coincide if we forget the structure of the simplicial complex and they have the same volume, since a^{-1} is an isometry. We set $S_l = \mathrm{im}(\sigma_l)$, which is a d-simplex in the barycentric subdivision of some closed d-face in $\mathcal{N}(\mathcal{W})^{(d)}$. With this preparatory work we can show the following estimate.

Lemma 4.2.20. *$H_d(\Phi')([M]^X)$ is represented by a d-cycle $\sum_{l=1}^m g_l \otimes \sigma_l$ where σ_l is an affine linear simplex. We set $S_l = \mathrm{im}(\sigma_l)$. Then it holds*

$$|g_l(x)| \operatorname{vol}_d(S_l) \leqslant \operatorname{vol}_d \left(\Phi'_x(\widetilde{M}) \cap S_l \right).$$

Proof. Let κ be the reduced X-parametrised fundamental cycle of M arising from a triangulation of the manifold as we described it preceding this lemma. Hence it is given by $\kappa = \sum_{j=1}^k \chi_X \otimes \tau_j \in L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_d(\widetilde{M}, \mathbb{Z})$ and $C_d^X(\Phi')(\kappa)$ represents $H_d(\Phi')([M]^X)$. We showed above that $C_d^X(\Phi')(\kappa)$ is homologous to a cycle $\sum_{l=1}^m g_l \otimes \sigma_l$ where σ_l is the linear singular simplex mapping the standard simplex to S_l, a d-simplex in the barycentric subdivision of $\mathcal{N}(\mathcal{W})^{(d)}$. We can assume that this cycle is in reduced form. For a.e. $x \in X$ the evaluations

$$(C_d^X(\Phi')(\kappa))_x = C_d^{\mathrm{lf}}(\Phi'_x)(\kappa_x)$$

and

$$(\sum_{l=1}^{m} g_l \otimes \sigma_l)_x = \sum_{\gamma \in \Gamma} \sum_{l=1}^{m} g_l(\gamma^{-1} x) \gamma \sigma_l$$

are homologous locally finite cycles on $\mathcal{N}(\mathcal{W})^{(d)}$. We get the above estimates on the coefficient functions g_l by evaluating these locally finite cycles by suitable cocycles of compact support. Note that for a.e. $x \in X$ the map $\Phi'_x\colon \widetilde{M} \to \mathcal{N}(\mathcal{W})^{(d)}$ is continuous and proper. For $l_0 = 1, \dots, m$ we regard the following projection maps

$$\pi_{l_0}\colon \mathcal{N}(\mathcal{W})^{(d)} \longrightarrow \mathcal{N}(\mathcal{W})^{(d)}/(\mathcal{N}(\mathcal{W})^{(d)} \backslash \mathring{S}_{l_0}) \cong S_{l_0}/\partial S_{l_0}.$$

The latter space is homeomorphic to a d-sphere thus we can define a volume form on this space. Note that S_{l_0} is a d-simplex in the barycentric subdivision of a closed d-face $\overline{F(l_0)}$ with side lengths $r_1 \leqslant \ldots \leqslant r_d$, i.e. $\overline{F(l_0)} \cong \prod_{i=1}^{d}[0, r_i] \subset \mathbb{R}^d$. The subdivision $\mathrm{Sd}(\overline{F(l_0)})$ has $2^d d!$ simplices of dimension d and it holds

$$\mathrm{vol}_d(\overline{F(l_0)}) = \sum_{\substack{S \subset \mathrm{Sd}(\overline{F(l_0)}) \\ d\text{-simplex}}} \mathrm{vol}_d(S)$$

hence $\mathrm{vol}_d(\overline{F(l_0)}) = 2^d d! \,\mathrm{vol}_d(S_0)$. Let U be a small neighbourhood of ∂S_{l_0} under this identification and define a cutoff-function for the simplex, $g : \mathbb{R}^d \to \mathbb{R}$. This means g vanishes outside of S_{l_0}, is identical 1 on $S_{l_0} \backslash U$ and decreases rapidly on U as approaching ∂S_0. Set $\omega_{l_0} := g \cdot d\,\mathrm{vol}\,\mathbb{R}^d$ which defines a d-form on $S_{l_0}/\partial S_{l_0}$. We get a cocycle $d\,\mathrm{vol}_{l_0} \in C^d(S_{l_0}/\partial S_{l_0})$, which is given on smooth chains by

$$\begin{aligned} d\,\mathrm{vol}_{l_0}\colon C_d^{C^\infty}(S_{l_0}/\partial S_{l_0}) &\longrightarrow \mathbb{R} \\ \sigma &\longmapsto \int_{\Delta_d} \sigma^* \omega_{l_0}. \end{aligned}$$

We have $[d\,\mathrm{vol}_{l_0}] \in H^d(S_{l_0}/\partial S_{l_0}) \cong H^d(S_{l_0}, \partial S_{l_0})$. The fundamental class, i.e. the generator of $H_d(S_{l_0}, \partial S_{l_0})$, is represented by the relative cycle $\pi_{l_0} \circ \sigma_{l_0} \in C_d(S_{l_0}, \partial S_{l_0})$, where $\mathrm{im}(\sigma_{l_0}) = S_{l_0}$. We get for the volume of the d-simplex S_{l_0}

$$\mathrm{vol}_d(S_{l_0}) = |\langle [d\,\mathrm{vol}_{l_0}], [S_{l_0}, \partial S_{l_0}] \rangle| = |\langle d\,\mathrm{vol}_{l_0}, \pi_{l_0} \circ \sigma_{l_0} \rangle|.$$

More precisely we have

$$\begin{aligned} \mathrm{vol}_d(S_{l_0}) = |\langle d\,\mathrm{vol}_{l_0}, \pi_{l_0} \circ \sigma_{l_0} \rangle| = \Big| \int_{\Delta_d} (\pi_{l_0} \circ \sigma_{l_0})^* \omega_{l_0} \Big| &= \int_{\pi_{l_0}(S_{l_0})} g \cdot d\,\mathrm{vol}\,\mathbb{R}^d \\ &= \tfrac{1}{2^d d!} \prod_{i=1}^{d} r_i = \tfrac{1}{2^d d!} \,\mathrm{vol}_d(\overline{F(l_0)}), \end{aligned}$$

with σ_{l_0} being an affine linear simplex. Note that the Lebesgue measure and the d-dimensional Hausdorff measure coincide on $\mathbb{R}^d$.

Now we look at the pullback $\pi_{l_0}^*[d\,\mathrm{vol}_{l_0}] \in H^d(\mathcal{N}(\mathcal{W})^{(d)})$, which has compact support, since for the d-cocycle $\pi_{l_0}^*(d\,\mathrm{vol}_{l_0})$ we have

$$\pi_{l_0}^*(d\,\mathrm{vol}_{l_0})(\sigma) = d\,\mathrm{vol}_{l_0}(\pi_{l_0} \circ \sigma) = 0$$

if $\operatorname{im}(\sigma) \cap S_{l_0} \neq \emptyset$. We evaluate this cocycle on the locally finite cycles $(C_d^X(\Phi')(\kappa))_x$ and $(\sum_{l=1}^m g_l \otimes \sigma_l)_x$. For the latter we obtain

$$
\begin{aligned}
|\langle \pi_{l_0}^*(d\operatorname{vol}_{l_0}), (\sum_{l=1}^m g_l \otimes \sigma_l)_x\rangle| &= |\langle \pi_{l_0}^*(d\operatorname{vol}_{l_0}), \sum_{\gamma\in\Gamma}\sum_{l=1}^m g_l(\gamma^{-1}x)\gamma\sigma_l\rangle| \qquad (4.21)\\
&= |\langle d\operatorname{vol}_{l_0}, \sum_{\gamma\in\Gamma}\sum_{l=1}^m g_l(\gamma^{-1}x)\pi_{l_0}\circ(\gamma\sigma_l)\rangle|\\
&= |g_{l_0}(x)\langle d\operatorname{vol}_{l_0}, \pi_{l_0}\circ\sigma_{l_0}\rangle|\\
&= |g_{l_0}(x)|\operatorname{vol}_d(S_{l_0}),
\end{aligned}
$$

since $\pi_{l_0}\circ(\gamma\sigma_l) \neq \{*\}$ only for $l = l_0$ and $\gamma = 1$. This holds true because the original cycle is in reduced form, hence $\sigma_l \neq \gamma\sigma_n$ for $l \neq n$ and every $\gamma \in \Gamma$. Note that $(\pi_{l_0})_*(\sum_{l=1}^m g_l \otimes \sigma_l)$ is a locally finite cycle even if π_{l_0} is not proper.

Note that it holds $(C_d^X(\Phi')(\kappa))_x = C_d^{\mathrm{lf}}(\Phi'_x)(\kappa_x)$ and κ_x represents the locally finite fundamental class of $\widetilde{M}$. Since Φ'_x is proper, $(\pi_{l_0}\circ\Phi'_x)^*(d\operatorname{vol}_{l_0})$ is a cocycle with compact support $(\Phi'_x)^{-1}(S_{l_0})$ on $\widetilde{M}$. Let $\mathcal{C} = \pi_0((\Phi'_x)^{-1}(S_{l_0}))$ be the connected components of this preimage. We obtain

$$
\begin{aligned}
\langle \pi_{l_0}^*(d\operatorname{vol}_{l_0}), C_d^{\mathrm{lf}}(\Phi'_x)(\kappa_x)\rangle &= \langle(\pi_{l_0}\circ\Phi'_x)^*(d\operatorname{vol}_{l_0}), \kappa_x\rangle = \int_{\widetilde{M}}(\pi_{l_0}\circ\Phi'_x)^*(d\operatorname{vol}_{l_0})\\
&= \int_{(\Phi'_x)^{-1}(S_l)}(\pi_{l_0}\circ\Phi'_x)^*(d\operatorname{vol}_{l_0}) = \sum_{C\in\mathcal{C}}\int_C(\pi_{l_0}\circ\Phi'_x)^*(d\operatorname{vol}_{l_0}).
\end{aligned}
$$

For every $C \in \mathcal{C}$ we have by the coarea formula (see [13, Theorem 3.10, p. 134])

$$
\Big|\int_C(\pi_{l_0}\circ\Phi'_x)^*(d\operatorname{vol}_{l_0})\Big| = \Big|\int_{\Phi'_x(C)\cap S_{l_0}} \#\left((\Phi'_x)^{-1}(q)\right)\pi_{l_0}^*(d\operatorname{vol}_{l_0})(q)\Big| = \operatorname{vol}_d(\Phi'_x(C)\cap S_{l_0})
$$

where we take into consideration that Φ'_x can be orientation preserving or reversing on each connected component. As a result

$$
\begin{aligned}
|\langle \pi_{l_0}^*(d\operatorname{vol}_{l_0}), C_d^{\mathrm{lf}}(\Phi'_x)(\kappa_x)\rangle| &= |\sum_{C\in\mathcal{C}}\int_C(\pi_{l_0}\circ\Phi'_x)^*(d\operatorname{vol}_{l_0})| \leqslant \sum_{C\in\mathcal{C}}\Big|\int_C(\pi_{l_0}\circ\Phi'_x)^*(d\operatorname{vol}_{l_0})\Big|\\
&= \sum_{C\in\mathcal{C}}\operatorname{vol}_d(\Phi'_x(C)\cap S_{l_0}) = \operatorname{vol}_d(\Phi'_x(\widetilde{M})\cap S_{l_0}).
\end{aligned}
$$

Together with (4.21) we obtain for $l_0 \in \{1,\dots,m\}$

$$
|g_{l_0}(x)|\operatorname{vol}_d(S_{l_0}) \leqslant \operatorname{vol}_d(\Phi'_x(\widetilde{M})\cap S_{l_0}).
$$

□

Finally we can proof the theorem stated in the beginning of this section.

Proof of Theorem 4.2.1. By Lemma 4.2.20 the class $H_d(\Phi')([M]^X)$ is represented by the cycle $\sum_{l=1}^m g_l \otimes \sigma_l \in L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_d(\mathcal{N}(\mathcal{W})^{(d)},\mathbb{Z})$ with $\mathrm{im}(\sigma_l) = S_l$ and $|g_l(x)| \operatorname{vol}_d(S_l) \leqslant \operatorname{vol}_d\left(\Phi'_x(\widetilde{M}) \cap S_l\right)$. Here we can assume that $\sum_{l=1}^m g_l \otimes \sigma_l$ is in reduced form. Let $\overline{F(l)}$ be the closed d-face containing $\mathrm{im}(\sigma_l) = S_l$. This means S_l is identified with a d-simplex in the barycentric subdivision of $\overline{F(l)}$, $\mathrm{Sd}(\overline{F(l)})$. We can arrange that in the set of open faces $\mathcal{S} := \{F \mid F = F(l) \text{ for an } l \in \{1,\dots,m\}\}$ there is only one d-face of every Γ-orbit. So if for $l \neq l'$ with $\mathring{S}_l \subset F(l)$ and $\mathring{S}_{l'} \subset F(l')$, $F(l)$ and $F(l')$ belong to the same Γ-orbit, we can assume that they already coincide and S_l and $S_{l'}$ are different d-simplices in the same face. The number of d-simplices in the subdivision $\mathrm{Sd}(\overline{F(l)})$ is $2^d d!$ and it holds

$$\operatorname{vol}_d(\overline{F(l)}) = 2^d d! \operatorname{vol}_d(S_l).$$

Moreover, since $\operatorname{vol}_d(\partial S_l) = \operatorname{vol}_d(\partial F(l)) = 0$, for the open d-face we have

$$\operatorname{vol}_d(F(l)) = 2^d d! \operatorname{vol}_d(\mathring{S}_l). \tag{4.22}$$

By the same reason the estimate in Lemma 4.2.20 gives

$$|g_l(x)| \operatorname{vol}_d(\mathring{S}_l) \leqslant \operatorname{vol}_d\left(\Phi'_x(\widetilde{M}) \cap \mathring{S}_l\right). \tag{4.23}$$

Since $\operatorname{vol}_d(\Phi'_x(\widetilde{M}) \cap \mathring{S}_l) \leqslant \operatorname{vol}_d(\Phi'_x(\widetilde{M}) \cap F(l))$, the above inequalities imply for a.e. $x \in X$

$$|g_l(x)| \leqslant 2^d d! \frac{\operatorname{vol}_d\left(\Phi'_x(\widetilde{M}) \cap F(l)\right)}{\operatorname{vol}_d(F(l))}. \tag{4.24}$$

If $F(l)$ is a thin face, by Lemma 4.2.17 iii) for a.e. $x \in X$ it holds $\operatorname{vol}_d(\Phi'_x(\widetilde{M}) \cap F(l)) \leqslant 2\varepsilon r_1(F(l))^d$. Here $r_1(F(l))$ denotes the smallest side length of the face hence the fraction in (4.24) is bounded by 2ε. By the choice of ε in (4.7) this yields $|g_l(x)| < 1$. Since $g_l \in L^\infty(X,\mathbb{Z})$, the function has to vanish for a.e. $x \in X$. In the same way the functions vanish a.e. if the corresponding face does not intersect the image of $\Phi'_x(\widetilde{M})$ for a.e. $x \in X$. So we can assume that all faces in $\mathcal{S} := \{F \mid F = F(l) \text{ for an } l \in \{1,\dots,m\}\}$ are thick.

Now, let $\mathcal{F}^{(d)}$ be a Borel fundamental domain of $X \times \mathcal{N}(\mathcal{W})^{(d)}$, i.e. a finite disjoint union of Borel sets of the form $X \times F$ where F belongs to a Γ-fundamental domain of $\mathcal{N}(\mathcal{W})^{(d)}$ (see Lemma 2.5.16). We can arrange that the sets $\mathrm{supp}(g_l) \times F(l)$ are subsets of elements in $\mathcal{F}^{(d)}$.

By Lemma 4.2.20 we obtain

$$\left| H_d(\Phi')([M]^X) \right|^X \leqslant \left| \sum_{l=1}^m g_l \otimes \sigma_l \right|^X = \sum_{l=1}^m \int_X |g_l(x)|\, d\mu(x). \tag{4.25}$$

Recall that for thick faces it holds $r_1(F) \geqslant \frac{\varepsilon}{cV_0}$ (see Definition 4.2.15) hence the volume $\mathrm{vol}_d(F)$ of a thick d-face is bounded below by some constant depending on V_0 and d. Then we can estimate

$$\begin{aligned}
\sum_{l=1}^{m} \int_X |g_l(x)|\, d\mu(x) &\overset{(4.23)}{\leqslant} \sum_{l=1}^{m} \int_X \frac{1}{\mathrm{vol}_d(\mathring{S}_l)} \mathrm{vol}_d(\Phi'_x(\widetilde{M}) \cap \mathring{S}_l)\, d\mu(x) \\
&\overset{(4.22)}{\leqslant} \sum_{l=1}^{m} \int_X \frac{2^d d!}{\mathrm{vol}_d(F(l))} \mathrm{vol}_d(\Phi'_x(\widetilde{M}) \cap \mathring{S}_l)\, d\mu(x) \\
&\leqslant c(V_0,d) \int_X \sum_{l=1}^{m} \mathrm{vol}_d(\Phi'_x(\widetilde{M}) \cap \mathring{S}_l)\, d\mu(x) \\
&\leqslant c(V_0,d) \int_X \sum_{\substack{S \subset \mathrm{Sd}(\overline{F}) \\ d\text{-simplex} \\ F \in \mathcal{S}}} \mathrm{vol}_d(\Phi'_x(\widetilde{M}) \cap \mathring{S})\, d\mu(x) \\
&\leqslant c(V_0,d) \int_X \sum_{F \in \mathcal{S}} \mathrm{vol}_d\left(\Phi'_x(\widetilde{M}) \cap F\right)\, d\mu(x) \\
&\leqslant c(V_0,d) \int_X \sum_{F \in \mathcal{F}_x^{(d)}} \mathrm{vol}_d\left(\Phi'_x(\widetilde{M}) \cap F\right) d\mu(x) \\
&= c(V_0,d) \int_X \mathrm{vol}_d\left(\Phi'_x(\widetilde{M}) \cap \mathcal{F}^{(d)}\right) d\mu(x) \\
&= c(V_0,d) \nu_{\mathcal{N}}\left(\Phi'(X \times \widetilde{M}) \cap \mathcal{F}^{(d)}\right).
\end{aligned}$$

Here $c(V_0,d)$ is a constant only depending on V_0 and the dimension d. The last inequality is valid, since we can assume that the sets $\mathrm{supp}(g_l) \times F(l)$ are contained in elements of $\mathcal{F}^{(d)}$. Hence for $F \in \mathcal{S}$ we have $F \in \mathcal{F}_x^{(d)}$ for certain $x \in X$. The appearing sum is finite by the definition of $\mathcal{F}^{(d)}$.

Finally we can combine this result with (4.25) and the results from Lemma 4.2.17 iv) and Lemma 4.2.19. Thus there is a Borel fundamental domain $\mathcal{F}^{(N_0)}$ for $X \times \mathcal{N}(\mathcal{W})^{(N_0)}$ such that we obtain

$$\begin{aligned}
\left| H_d(\Phi')([M]^X) \right|^X &\leqslant c(V_0,d) \nu_{\mathcal{N}}\left(\Phi'(X \times \widetilde{M}) \cap \mathcal{F}^{(d)}\right) \\
&\overset{4.2.17}{\leqslant} c(V_0,d) 2G(V_0,d)^{D(V_0,d)} \nu_{\mathcal{N}}\left(\Phi(X \times \widetilde{M}) \cap \mathcal{F}^{(N_0)}\right) \\
&\overset{4.2.19}{\leqslant} c(V_0,d) 2G(V_0,d)^{D(V_0,d)} C''(d)\, \mathrm{vol}(M) \\
&=: C(V_0,d)\, \mathrm{vol}(M),
\end{aligned}$$

where all constants only depend on V_0 and d. This is precisely the statement of Theorem 4.2.1 which concludes the proof. □

Chapter 5

Proof of the main theorems

Using the cover and nerve construction and the properties we derived in Chapter 3 and 4 we can finally give the proof of the main theorems as indicated in the introductory Chapter 1. Moreover, we state a number of corollaries arising from these theorems.

We first focus on estimates for the integral foliated simplicial volume of a manifold. We show the following theorem.

Theorem 5.1. *For every real number S_0 and every dimension d, there is a constant $C(S_0, d) > 0$ with the following property: Let (M, g) be an oriented, closed and connected, d-dimensional aspherical Riemannian manifold such that the macroscopic scalar curvature of M at scale 1 is at least S_0. Then for the integral foliated simplicial volume of M we have*

$$|M| \leqslant C(S_0, d) \operatorname{vol}(M).$$

In Chapter 1 we introduced the notion of macroscopic scalar curvature as Guth defined it [29]. By definition, a lower bound on the macroscopic scalar curvature of M at scale 1 is equivalent to an upper bound on the supremal volume of 1-balls in the universal cover $\widetilde{M}$ (see Definition 1.3). Thus it is sufficient to prove the following theorem.

Theorem 5.2. *For each number $V_0 > 0$ and every dimension d, there is a constant $C(V_0, d) > 0$ with the following property: Let (M, g) be an oriented, closed and connected, d-dimensional aspherical Riemannian manifold and $(\widetilde{M}, \tilde{g})$ be the universal cover with the metric induced by g. If $V_{\widetilde{M}}(1) \leqslant V_0$, then for the integral foliated simplicial volume of M we have*

$$|M| \leqslant C(V_0, d) \operatorname{vol}(M).$$

Proof. Fix a standard Borel probability space (X, μ) with an atom-free probability measure, such that the fundamental group $\pi_1(M) = \Gamma$ acts on it (ess.) free and μ-preservingly. Then $|M| \leqslant |M|^X$.

By Theorem 3.1.4 and Theorem 4.2.1 there is a good Γ-cover $\mathcal{U}$ of the equivariant simple X-space $X \times \widetilde{M}$ with underlying cover $\mathcal{W}$ and an equivariant geometric nerve map

$$\Phi'\colon X \times \widetilde{M} \to X \times \mathcal{N}(\mathcal{W})^{(d)},$$

with image contained in the d-skeleton $\mathcal{N}(\mathcal{U})^{(d)}$ of the rectangular nerve of $\mathcal{U}$. The image of the X-parametrised fundamental class of M, $[M]^X \in H_d(L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M},\mathbb{Z}))$, satisfies

$$|H_d(\Phi')([M]^X)|^X \leqslant C(V_0, d)\,\mathrm{vol}(M)$$

by Theorem 4.2.1. Here the constant only depends on the dimension d and V_0. The nerve of the underlying cover $\mathcal{N}(\mathcal{W})$ is a free Γ-CW complex by Lemma 4.1.1, since Γ is torsion-free as fundamental group of an aspherical manifold [40, Lemma 3.1]. Moreover, M being aspherical implies that $\widetilde{M}$ is a model of the classifying space $E\Gamma$. There is a (up to Γ-homotopy) unique equivariant map $\eta\colon \mathcal{N}(\mathcal{W}) \to \widetilde{M}$ and an equivariant X-map $\mathrm{id} \times \eta\colon X \times \mathcal{N}(\mathcal{W}) \to X \times \widetilde{M}$. By Theorem 2.4.1, the maps Φ' and $\mathrm{id} \times \eta$ induce chain maps of $\mathbb{Z}$-modules

$$L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M},\mathbb{Z}) \xrightarrow{C_*^X(\Phi')} L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\mathcal{N}(\mathcal{W}),\mathbb{Z})$$
$$L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\mathcal{N}(\mathcal{W}),\mathbb{Z}) \xrightarrow{C_*^X(\mathrm{id} \times \eta)} L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M},\mathbb{Z})$$

and their composition is homotopic to the identity on $L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M},\mathbb{Z})$ as shown in Lemma 2.4.5. Using the functoriality of the parametrised ℓ^1-norm in Lemma 2.4.3 we have

$$|M|^X = |H_d(\mathrm{id} \times \eta) \circ H_d(\Phi')([M]^X)|^X \leqslant |H_d(\Phi')([M]^X)|^X \overset{4.2.1}{\leqslant} C(V_0, d)\,\mathrm{vol}(M),$$

which concludes the proof. □

As stated in Theorem 2.1.23, integral foliated simplicial volume provides an upper bound on the L^2-Betti numbers. This is due to Schmidt [47, Corollary 5.28, p.78]. Then our result implies the following upper bound for L^2-Betti numbers.

Corollary 5.3. *For every real number S_0 and every dimension d, there is a constant $C(S_0, d) > 0$ such that the following property holds: Let (M, g) be a closed and connected,* *d-dimensional aspherical Riemannian manifold such that the macroscopic scalar curvature of M at scale* 1 *is at least S_0. Then for the L^2-Betti numbers of M we have*

$$b_k^{(2)}(M) \leqslant C(S_0, d)\,\mathrm{vol}(M) \quad \textit{for all } k \geqslant 0.$$

Proof. We can assume that the manifold is oriented. Then the result is a direct consequence of Theorem 5.1 and Schmidt's bound on the L^2-Betti numbers.

If M is non-orientable there is a two-fold orientation cover $\pi'\colon \widehat{M} \to M$ equipped with the induced Riemannian metric, which satisfies the assumption on the macroscopic scalar curvature if M does. Since L^2-Betti numbers ([38, Example 1.37, p. 40]) and the Riemannian volume behave multiplicatively under finite coverings, we have

$$b_k^{(2)}(\widehat{M}) = 2b_k^{(2)}(M)$$

and it is sufficient to proof the statement for $\widehat{M}$. □

Moreover, the above theorem yields a connection between integral foliated simplicial volume and the minimal volume. As indicated before, the minimal volume of a smooth manifold M is defined as

$$\mathrm{minvol}(M) := \inf\{\mathrm{vol}(M, g) \mid g \text{ a complete Riemannian metric on } M \text{ with } |\sec(g)| \leqslant 1)\}.$$

The analogue of the following estimate for simplicial volume is a corollary of Gromov's Main Inequality [23, Section 0.5] and holds also in the non-aspherical case, whereas the analogue statement for the L^2-Betti numbers of aspherical manifolds has been shown in [45].

Corollary 5.4. *For every dimension d, there is a constant $C(d) > 0$ with the following property: If M is an oriented, closed and connected, d-dimensional smooth, aspherical manifold, then*

$$| M | \leqslant C(d)\,\mathrm{minvol}(M).$$

Proof. A manifold (M, g) with sectional curvature $\sec(g) \geqslant -1$, has Ricci curvature $\mathrm{Ric}(M) \geqslant -(d-1)$. By the Bishop-Gromov inequality [21, Theorem 4.19, p. 214] this implies that $V_{\widetilde{M}}(1) \leqslant \mathrm{vol}(B_{hyp}(1))$ and $\mathrm{Sc}_1(M) \geqslant -d(d-1)$, where $-d(d-1)$ is the scalar curvature of $\mathbb{H}^d$. □

Remark 5.5. As we have seen before, a manifold (M, g) with a lower Ricci curvature bound of the form $\mathrm{Ric}(M) \geqslant -d(d-1)$ has a lower bound on the macroscopic scalar curvature at scale 1 by the Bishop-Gromov inequality. Therefore, the Main Inequality for L^2-Betti numbers of aspherical manifols, which has been proven in [45, Corollary to Theorem A], can be deduced also from our Theorem 5.1. Note that in this particular case, we could simplify the proof. The construction of the $\mathcal{R}$-cover used in Sauer's proof ([45, Theorem 4.1]) can be easily modified to give a Γ-cover $\mathcal{U}$. The construction works as described in the proof of Theorem 3.1.4 though we start with balls of a fixed radius instead of good balls. This yields an equivariant cover $\mathcal{U}$ of $X \times \widetilde{M}$ with only finitely many orbits of balls, such that the induced covers $\mathcal{U}_x$ have a uniform multiplicity bound as in [45, Theorem 4.1]. Starting from this cover, one can derive the Main Inequality for L^2-Betti numbers, following Sauer's presentation, i.e. one uses the simplicial nerve construction instead of the rectangular nerve. Since the cover $\mathcal{U}$ has only finitely many orbits of balls, the nerve map and its deformations to lower skeleta will be of finite variance as opposed to countable variance as in [45]. This allows to estimate the integral foliated simplicial volume without bringing the singular foliated homology into the game.

We conclude our treatment of the integral foliated simplicial volume with the following version of Yano's theorem. It has been proven in [14, Corollary 1.2] using Yano's construction. As remarked there it follows by Corollary 5.4 as well.

Corollary 5.6. *Let M be an oriented, closed and connected, aspherical smooth manifold which admits a non-trivial smooth S^1-action. Then the integral foliated simplicial volume of M vanishes.*

Proof. By [38, Cor. 1.43, p. 48] a non-trivial S^1-action on a closed aspherical manifold has no fixed points and the inclusion of any orbit into M is π_1-injective. Given this, the minimal volume of the manifold is zero (see [23, Appendix 2]). Then the above Corollary 5.4 implies that the integral foliated simplicial volume is vanishing. □

To proof the estimates for integral foliated simplicial volume and L^2-Betti numbers we need that the manifolds in question are aspherical. Restricting to simplicial volume we can omit this assumption exploiting the connection of simplicial volume and bounded cohomology. More precisely, we make use of Gromov's mapping theorem [23, Section 3.1, p. 40]. We obtain the following estimate.

Theorem 5.7. *For every real number S_0 and every dimension d, there is a constant $C(S_0, d) > 0$ with the following property: Let (M, g) be an oriented, closed and connected d-dimensional Riemannian manifold with torsion-free fundamental group, such that the* *macroscopic scalar curvature of M at scale 1 is at least S_0. Then for the simplicial volume of M we have*

$$\|M\| \leqslant C(S_0, d)\,\mathrm{vol}(M).$$

Proof. If the manifold is in addition aspherical, this follows from Theorem 5.1 since $\|M\| \leqslant |M|$ (see Proposition 2.1.19).

The lower bound on the macroscopic scalar curvature is by definition equivalent to an upper bound $V_{\widetilde{M}}(1) \leqslant V_0$, where $\widetilde{M}$ denotes the universal cover with the induced metric. The covering map is denoted by $\pi\colon \widetilde{M} \to M$. Let $\pi_1(M) = \Gamma$ be the fundamental group of the manifold and $B\Gamma$ be a model of its classifying space. Further we fix a classifying map $c\colon M \to B\Gamma$, i.e. a map inducing an isomorphism on fundamental groups. Let $E\Gamma$ be the universal cover of $B\Gamma$ with covering map $p\colon E\Gamma \to B\Gamma$ and $\tilde{c}$ be a lift of c such that the following diagram commutes

$$\begin{tikzcd}
\widetilde{M} \arrow[r, "\tilde{c}"] \arrow[d, "\pi"'] & E\Gamma \arrow[d, "p"] \\
M \arrow[r, "c"] & B\Gamma
\end{tikzcd}$$

As before, we fix a standard Borel probability space (X, μ) with an atom-free probability measure, such that the fundamental group Γ acts on it (ess.) free and μ-preservingly. Then $|M| \leqslant |M|^X$.

By Theorem 3.1.4 we can fix a good Γ-cover $\mathcal{U}$ of the equivariant simple X-space $X \times \widetilde{M}$ with underlying cover $\mathcal{W}$. Theorem 4.2.1 provides an equivariant geometric nerve map $\Phi'\colon X \times \widetilde{M} \to X \times \mathcal{N}(\mathcal{W})^{(d)}$ with image contained in the d-skeleton $\mathcal{N}(\mathcal{U})^{(d)}$ of the rectangular nerve of $\mathcal{U}$. For the X-parametrised fundamental class of M, $[M]^X$, we can estimate the parametrised ℓ^1-norm of the image by

$$|H_d(\Phi')([M]^X)|^X \leqslant C(V_0, d)\,\mathrm{vol}(M),$$

where the constant only depends on the dimension d and V_0 (see Theorem 4.2.1). By Lemma 4.1.1, the nerve of the underlying cover $\mathcal{N}(\mathcal{W})$ is a free Γ-CW complex. There is a unique (up to Γ-homotopy) equivariant map $\eta\colon \mathcal{N}(\mathcal{W}) \to E\Gamma$ and an equivariant X-map $\mathrm{id} \times \eta\colon X \times \mathcal{N}(\mathcal{W}) \to X \times E\Gamma$. The maps Φ' and $\mathrm{id} \times \eta$ induce chain maps of $\mathbb{Z}$-modules (see Theorem 2.4.1)

$$L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M},\mathbb{Z}) \xrightarrow{C_*^X(\Phi')} L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\mathcal{N}(\mathcal{W}),\mathbb{Z})$$
$$L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\mathcal{N}(\mathcal{W}),\mathbb{Z}) \xrightarrow{C_*^X(\mathrm{id}\times\eta)} L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{E\Gamma},\mathbb{Z})$$

and their composition is homotopic to the map $C_*^X(\mathrm{id}\times\tilde{c})$ induced by $\mathrm{id}\times\tilde{c}\colon X\times\widetilde{M}\to X\times E\Gamma$. This is due to Lemma 2.4.5. Using the functoriality of the parametrised ℓ^1-norm in Lemma 2.4.3 we obtain

$$\begin{aligned} \left| H_d(\mathrm{id}\times\tilde{c})([M]^X)\right|^X &= \left|H_d(\mathrm{id}\times\eta)\circ H_d(\Phi')([M]^X)\right|^X \\ &\leqslant \left|H_d(\Phi')([M]^X)\right|^X \overset{4.2.1}{\leqslant} C(V_0,d)\operatorname{vol}(M). \end{aligned} \tag{5.1}$$

We get the following commutative diagram

$$\begin{tikzcd} C_d(M,\mathbb{Z}) \arrow[r, "i_M^X"] \arrow[d, hook, "\iota"'] & L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_d(\widetilde{M},\mathbb{Z}) \arrow[r, "C_d^X(\mathrm{id}\times\tilde{c})"] & L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_d(E\Gamma,\mathbb{Z}) \arrow[d, "\rho"] \\ C_d(M,\mathbb{R}) \arrow[rr, "C_d(c)"] & & C_d(B\Gamma,\mathbb{R}) \end{tikzcd}$$

Here $i_M^X\colon C_d(M,\mathbb{Z})\to L^\infty(X,\mathbb{Z})\otimes_{\mathbb{Z}\Gamma} C_d(\widetilde{M},\mathbb{Z})$ is the change of coefficient homomorphism as in Definition 2.1.16 and $\rho\colon L^\infty(X,\mathbb{Z})\otimes_{\mathbb{Z}\Gamma} C_d(E\Gamma,\mathbb{Z})\to C_d\,(B\Gamma,\mathbb{R})$ is the map defined by integration as in the proof of Proposition 2.1.19. Note that $C_*^X(\mathrm{id}\times\tilde{c}) = \mathrm{id}\otimes C_*(\tilde{c})$ with $C_*(\tilde{c})\colon C_*(\widetilde{M},\mathbb{Z})\to C_*(E\Gamma,\mathbb{Z})$ is the induced map of singular chain complexes.

Recall the definitions of the real fundamental class and the X-parametrised fundamental class from Section 2.1, i.e. $[M]=H_d(\iota)[M]_{\mathbb{Z}}$ and $[M]^X = H_d(i_M^X)[M]_{\mathbb{Z}}$. By the mapping theorem [23, Section 3.1, p. 40] the classifying map c induces an isometric isomorphism, thus $\|[M]\|_1 = \|H_d(c)[M]\|_1$. Using Proposition 2.1.19 and (5.1), we obtain

$$\begin{aligned} \|M\| = \|[M]\|_1 &= \|H_d(c)[M]\|_1 = \|H_d(c)H_d(\iota)[M]_{\mathbb{Z}}\|_1 \\ &= \left\|H_d(\rho)H_d(\mathrm{id}\times\tilde{c})H_d(i_M^X)[M]_{\mathbb{Z}}\right\|_1 = \left\|H_d(\rho)H_d(\mathrm{id}\times\tilde{c})\,[M]^X\right\|_1 \\ &\overset{2.1.19}{\leqslant} \left|H_d(\mathrm{id}\times\tilde{c})([M]^X)\right|^X \overset{(5.1)}{\leqslant} C(V_0,d)\operatorname{vol}(M), \end{aligned}$$

which concludes the proof. □

Remark 5.8. Since the simplicial volume is multiplicative under finite coverings, the assumption that the fundamental group is virtually torsion-free is sufficient. It seems possible to omit this assumption on the fundamental group. This would yield a complete proof of Gromov's Main Inequality in the compact case by other means than in [23].

References

[1] J. Aaronson. *An Introduction to Infinite Ergodic Theory.* Mathematical surveys and monographs. American Mathematical Society, 1997.

[2] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*, volume 55. Courier Corporation, 1964.

[3] H. W. Alt. *Lineare Funktionalanalysis: Eine anwendungsorientierte Einführung.* Masterclass. Springer Berlin Heidelberg, 2012.

[4] F. Balacheff and S. Karam. Macroscopic schoen conjecture for manifolds with non-zero simplicial volume. *arXiv preprint arXiv:1709.10049*, 2017.

[5] R. Benedetti and C. Petronio. *Lectures on hyperbolic geometry.* Universitext. Springer-Verlag, Berlin, 1992.

[6] G. Besson, G. Courtois, and S. Gallot. Volume et entropie minimale des espaces localement symétriques. *Invent. Math.*, 103(2):417–445, 1991.

[7] R. L. Bishop and R. J. Crittenden. *Geometry of manifolds.* Pure and Applied Mathematics, Vol. XV. Academic Press, New York-London, 1964.

[8] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1999.

[9] K. S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics.* Springer-Verlag, New York-Berlin, 1982.

[10] J. F. Davis and P. Kirk. *Lecture notes in algebraic topology.* Graduate studies in mathematics ; 35. American Mathematical Society, Providence, RI, 2001.

[11] S. Eilenberg. Singular homology theory. *Ann. of Math. (2)*, 45:407–447, 1944.

[12] S. Eilenberg. Homology of spaces with operators. I. *Trans. Amer. Math. Soc.*, 61:378–417; errata, 62, 548 (1947), 1947.

[13] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions.* Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.

[14] D. Fauser. Integral foliated simplicial volume and S^1-actions. *arXiv preprint arXiv:1704.08538*, 2017.

[15] H. Federer and W. H. Fleming. Normal and integral currents. *Ann. of Math. (2)*, 72:458–520, 1960.

[16] V. Feuvrier. *A result about existence for minimal sets using polyhedral optimization.* Theses, Université Paris Sud - Paris XI, Sept. 2008. URL `https://tel.archives-ouvertes.fr/tel-00348735`.

[17] V. Feuvrier. Condensation of polyhedric structures onto soap films. *arXiv preprint arXiv:0906.3505*, 2009.

[18] R. Frigerio. *Bounded cohomology of discrete groups*, volume 227 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017.

[19] R. Frigerio, C. Löh, C. Pagliantini, and R. Sauer. Integral foliated simplicial volume of aspherical manifolds. *Israel J. Math.*, 216(2):707–751, 2016.

[20] D. Gaboriau. Invariants l^2 de relations d'équivalence et de groupes. *Publ. Math. Inst. Hautes Études Sci.*, (95):93–150, 2002.

[21] S. Gallot, D. Hulin, and J. Lafontaine. *Riemannian geometry*. Universitext. Springer-Verlag, Berlin, 3rd ed. edition, 2004.

[22] R. Geoghegan. *Topological methods in group theory*, volume 243 of *Graduate Texts in Mathematics*. Springer, New York, 2008.

[23] M. Gromov. Volume and bounded cohomology. *Inst. Hautes Études Sci. Publ. Math.*, (56):5–99 (1983), 1982.

[24] M. Gromov. Large Riemannian manifolds. In *Curvature and topology of Riemannian manifolds (Katata, 1985)*, volume 1201 of *Lecture Notes in Math.*, pages 108–121. Springer, Berlin, 1986.

[25] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[26] M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston MA, english edition, 2007. Based on the 1981 French original, With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.

[27] M. Gromov. 101 questions, problems and conjectures around scalar curvature.(incomplete and unedited version). 2017. URL `http://www.ihes.fr/~/gromov/PDF/101-problemsOct1-2017.pdf`.

[28] V. Guillemin and A. Pollack. *Differential topology*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1974.

[29] L. Guth. Metaphors in systolic geometry. In *Proceedings of the International Congress of Mathematicians. Volume II*, pages 745–768. Hindustan Book Agency, New Delhi, 2010.

[30] L. Guth. Volumes of balls in large Riemannian manifolds. *Ann. of Math. (2)*, 173(1):51–76, 2011.

[31] L. Guth. Volumes of balls in Riemannian manifolds and Uryson width. *J. Topol. Anal.*, 9(2):195–219, 2017.

[32] H. Inoue and K. Yano. The Gromov invariant of negatively curved manifolds. *Topology*, 21(1):83–89, 1982.

[33] A. S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[34] J. M. Lee. *Introduction to topological manifolds*, volume 202 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2011.

[35] C. Löh. The proportionality principle of simplicial volume. diploma thesis, Westfälische Wilhelms-Universität Münster, 2004. arXiv preprint math/0504106 (2005).

[36] C. Löh. *ℓ^1-Homology and Simplicial Volume.* PhD thesis, Westfälische Wilhelms-Universität Münster, 2007. URL `http://nbn-resolving.de/urn:nbn:de:hbz:6-37549578216`.

[37] C. Löh and C. Pagliantini. Integral foliated simplicial volume of hyperbolic 3-manifolds. *Groups Geom. Dyn.*, 10(3):825–865, 2016.

[38] W. Lück. *L^2-invariants: theory and applications to geometry and K-theory*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics].* Springer-Verlag, Berlin, 2002.

[39] W. Lück. *Algebraische Topologie: Homologie und Mannigfaltigkeiten.* Springer-Verlag, 2005.

[40] W. Lück. Survey on aspherical manifolds. In *European Congress of Mathematics*, pages 53–82. Eur. Math. Soc., Zürich, 2010.

[41] F. Morgan. *Geometric measure theory: a beginner's guide.* Academic Press, Inc., San Diego, CA, third edition, 2000.

[42] H. J. Munkholm. Simplices of maximal volume in hyperbolic space, Gromov's norm, and Gromov's proof of Mostow's rigidity theorem (following Thurston). In *Topology Symposium, Siegen 1979 (Proc. Sympos., Univ. Siegen, Siegen, 1979)*, volume 788 of *Lecture Notes in Math.*, pages 109–124. Springer, Berlin, 1980.

[43] J. R. Munkres. *Elements of algebraic topology.* Addison-Wesley Publishing Company, Menlo Park, CA, 1984.

[44] R. Sauer. Volume and L^2-Betti numbers of aspherical manifolds. In *Trends in mathematics*, pages 97–105. Universitätsdrucke Göttingen, Göttingen, 2008.

[45] R. Sauer. Amenable covers, volume and L^2-Betti numbers of aspherical manifolds. *J. Reine Angew. Math.*, 636:47–92, 2009.

[46] R. Sauer. Volume and homology growth of aspherical manifolds. *Geom. Topol.*, 20(2):1035–1059, 2016.

[47] M. Schmidt. *L^2-Betti numbers of R-spaces and the integral foliated simplicial volume.* PhD thesis, Westfälische Wilhelms-Universität Münster, 2005. URL `http://nbn-resolving.de/urn:nbn:de:hbz:6-05699458563`.

[48] R. M. Schoen. Variational theory for the total scalar curvature functional for Riemannian metrics and related topics. In *Topics in calculus of variations (Montecatini Terme, 1987)*, volume 1365 of *Lecture Notes in Math.*, pages 120–154. Springer, Berlin, 1989.

[49] J. Steprāns. A characterization of free abelian groups. *Proceedings of the American Mathematical Society*, 93(2):347–349, 1985.

[50] W. P. Thurston. Geometry and topology of three-manifolds. princeton lecture notes, 1979. *Revised version*, 1991.

[51] T. tom Dieck. *Transformation groups*, volume 8. Walter de Gruyter, 1987.

[52] A. Valette. Amenability and Margulis super-rigidity. In *Representation theory and complex analysis*, volume 1931 of *Lecture Notes in Math.*, pages 235–258. Springer, Berlin, 2008.