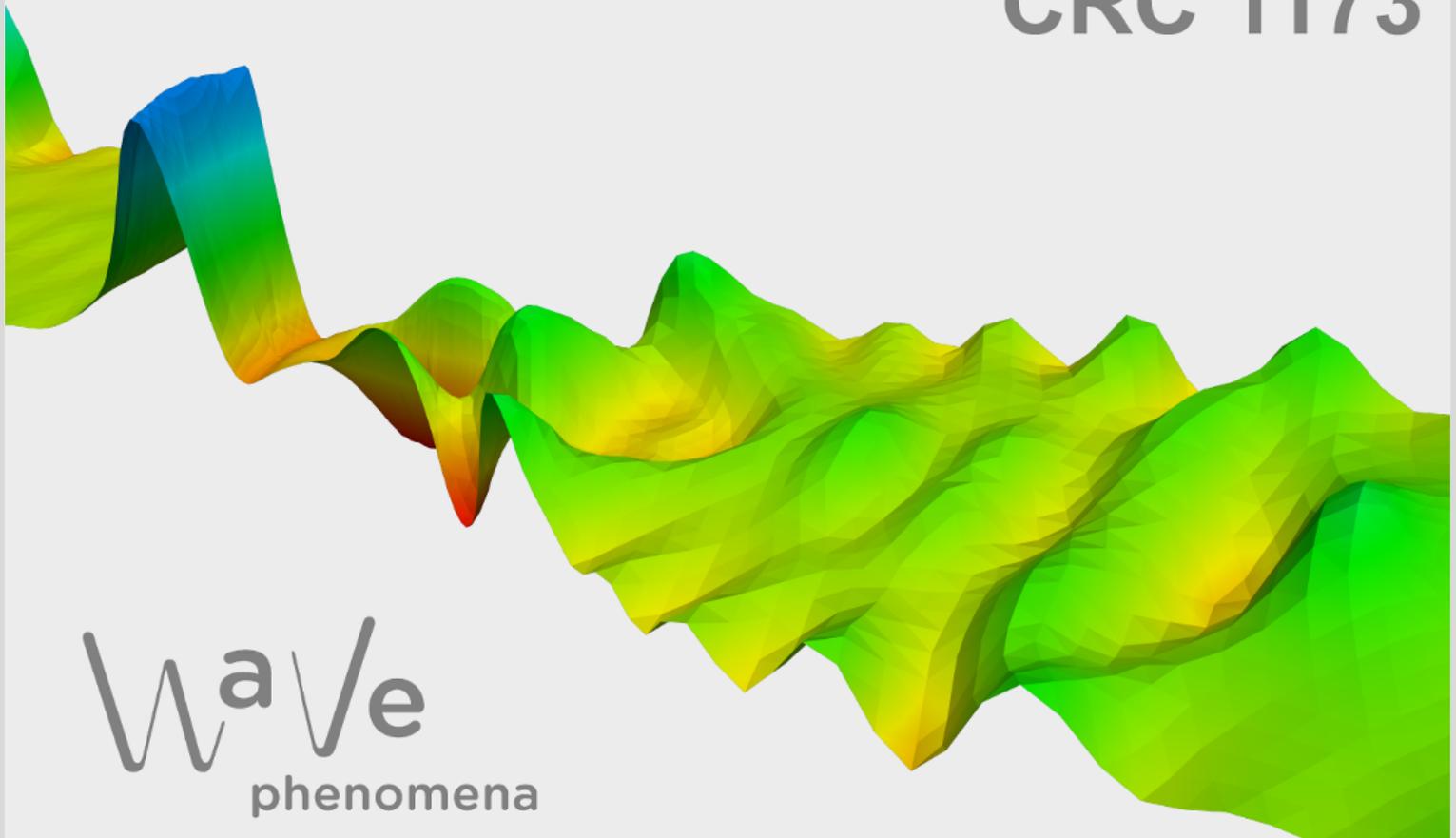# Knocking out teeth in one-dimensional periodic NLS

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KNOCKING OUT TEETH IN ONE-DIMENSIONAL PERIODIC NLS.

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ABSTRACT. We show the existence of weak solutions in the extended sense of the Cauchy problem for the cubic nonlinear Schrödinger equation in one dimension with initial data u_0 in $H^{s_1}(\mathbb{R}) + H^{s_2}(\mathbb{T}), 0 \leq s_1 \leq s_2$. In addition, we show that if $u_0 \in H^s(\mathbb{R}) + H^{\frac{1}{2}+\epsilon}(\mathbb{T})$ where $\epsilon > 0$ and $\frac{1}{6} \leq s \leq \frac{1}{2}$ the solution is unique in $H^s(\mathbb{R}) + H^{\frac{1}{2}+\epsilon}(\mathbb{T})$. Our main tool is a normal form type reduction via the use of the differentiation by parts technique.

1. INTRODUCTION AND MAIN RESULTS

We are interested in the equation

$$
(1) \qquad
\begin{cases}
iu_t - u_{xx} \pm |u|^2 u = 0 & , (t,x) \in \mathbb{R}^2 \\
u(0,x) = u_0(x) & , x \in \mathbb{R}
\end{cases}
$$

with initial data $u_0 \in H^s(\mathbb{R}) + H^s(\mathbb{T})$ for $s \geq 0$, where $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ is the one-dimensional torus, that is, the circle. The Sobolev H^s spaces are defined as

$$
(2) \qquad H^s(\mathbb{R}) := \{f \in L^2(\mathbb{R}) \ / \ \|f\|_{H^s(\mathbb{R})} := \Big(\int_{\mathbb{R}} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \Big)^{\frac{1}{2}} < \infty\}
$$

and

$$
(3) \qquad H^s(\mathbb{T}) := \{f \in L^2(\mathbb{T}) \ / \ \|f\|_{H^s(\mathbb{T})} := \Big(\sum_{n \in \mathbb{Z}} (1+|n|^2)^s |f_n|^2 \Big)^{\frac{1}{2}} < \infty\},
$$

and we will use $\langle k \rangle := (1+|k|^2)^{\frac{1}{2}}$ for the so–called Japanese bracket. $S(\mathbb{R})$ is the Schwartz class, $D(\mathbb{T}) = C^\infty(\mathbb{T})$, $S'(\mathbb{R})$ the tempered distributions, and $D'(\mathbb{T})$ the distributions on the torus $\mathbb{T}$. The Fourier transform of a function $f : \mathbb{R} \to \mathbb{C}$ is given by

$$
(4) \qquad \hat{f}(\xi)(= \mathcal{F}(f)(\xi)) := \int_{\mathbb{R}} e^{-2\pi i \xi x} f(x) dx, \ \xi \in \mathbb{R},
$$

and the Fourier coefficients of a periodic function $f : \mathbb{T} \to \mathbb{C}$ are

$$
(5) \qquad f_n := \int_0^1 e^{-2\pi i n x} f(x) dx, \ n \in \mathbb{Z}.
$$

In [16] it was proved that NLS (1) is locally wellposed in $L^2(\mathbb{R})$ with guaranteed time of existence depending only on the $L^2(\mathbb{R})$ norm of the initial data and since this is a conserved quantity, $\|u(t,\cdot)\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})}$ for all $t \in \mathbb{R}$, it follows that the NLS (1) is globally

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wellposed in $L^2(\mathbb{R})$. In [2] it was proved that the NLS (1) is locally wellposed in $L^2(\mathbb{T})$ and again by the $L^2(\mathbb{T})$ conservation law it follows that is globally wellposed in $L^2(\mathbb{T})$. In [7] the NLS (1) was studied for initial conditions $u_0 \in H^s(\mathbb{T})$ and in [18] for $u_0 \in H^s(\mathbb{R}), s \geq 0$. In both papers unconditional well–posedness was proved for $s \geq \frac{1}{6}$, that is uniqueness of solutions in $C([0, T], H^s(\mathbb{T}))$ (and $C([0, T], H^s(\mathbb{R}))$ respectively) without intersecting with any auxiliary function space (see [9] where this notion first appeared). They used a normal form reduction via the differentiation by parts technique which was originally introduced in [1] in the study of the KdV equation for periodic initial data. We also refer to [13] where the last author introduced a different approach to normal form reduction for the NLS (1) on $\mathbb{R}$ which follows closely what is done in the periodic case and is well suited for modulation spaces defined in equation (13). See also [3] for the case of general modulation spaces. For textbook accounts on these type of results we refer to [12, 15], to [14] for a slightly more applied point of view, and, in particular, [5] for a nice discussion of the differentiation by parts technique.

Here we make the differentiation by parts approach work in a *hybrid case*, namely the case where the initial data u_0 is the sum of a periodic function w_0 on $\mathbb{R}$ and an $L^2(\mathbb{R})$ function v_0. A tooth, as referred to in the title of this paper, is, for example, w_0 restricted to one period. We think of the addition of v_0 to w_0 as eliminating, or knocking out, finitely many of these teeth in the underlying periodic signal.

Our work is motivated by high–speed optical fiber communications, where in a certain approximation the behavior of pulses in glass–fiber cables is described by a NLS. A periodic signal is the simplest type of a non-decaying signal, encoding, for example, an infinite string of ones if there is exactly one tooth per period. However, such a purely periodic signal carries no information. One would like to be able to change it, at least locally. This leads necessarily to a hybrid formulation of the NLS where the signal is the sum of a periodic and a localized part. The localized part being able to knock out, i.e., remove, one or more of the teeth in the underlying periodic signal. This way one can model, for example, a signal consisting of two infinite blocks of ones which are separated by a single zero, or even far more complicated patterns. An interesting question then naturally arises: Can the missing teeth regrow, which means that the original signal gets distorted. Is there an optimal choice of a periodic signal, which makes this distortion very weak or even impossible?

From a mathematics point of view, in order to be able to address these type of questions, one should have first solved the corresponding local existence and uniqueness problems, which is the main purpose of this work: We solve the local existence problem and provide an unconditional uniqueness result. Since the underlying periodic signal can also be the constant function, we also cover the case of so–called dark solitons, that is, NLS with a non-zero boundary conditions at infinity, where the signals are of the form $u = c + v$ with c a constant, see [10] and [11] for a review on dark solitons from a point of view of applied mathematics and physics.

Our solution of NLS (1) with initial data $u_0 = v_0 + w_0 \in H^{s_1}(\mathbb{R}) + H^{s_2}(\mathbb{T})$ will be constructed as the sum of the solutions of the following partial differential equations

(6)
$$\begin{cases} iw_t - w_{xx} \pm |w|^2 w = 0 & , \ (t,x) \in \mathbb{R} \times \mathbb{T} \\ w(0,x) = w_0(x) \in H^{s_2}(\mathbb{T}) & , \ x \in \mathbb{T}, \end{cases}$$

which is the periodic cubic NLS on the real line, and the modified cubic NLS

(7)
$$\begin{cases} iv_t - v_{xx} \pm G(w,v) = 0 & , \ (t,x) \in \mathbb{R} \times \mathbb{R} \\ v(0,x) = v_0(x) \in H^{s_1}(\mathbb{R}) & , \ x \in \mathbb{R}, \end{cases}$$

where $G(w,v)$ is the nonlinearity

(8) $\qquad G(w,v) = |w + v|^2(w + v) - |w|^2 w = |v|^2 v + v^2 \bar{w} + w^2 \bar{v} + 2w|v|^2 + 2v|w|^2.$

In order to give a meaning to solutions of NLS (6) in $C([0,T], H^s(\mathbb{T}))$ and NLS (7) in $C([0,T], H^{\tilde{s}}(\mathbb{R})), s, \tilde{s} \in \mathbb{R}$ and to the nonlinearities $\mathcal{N}(w) := w|w|^2$ and $G(w,v)$ we need the following definitions, which first appeared in [4] for the periodic NLS.

Definition 1. *A sequence of Fourier cutoff operators is a sequence of Fourier multiplier operators $\{T_N\}_{N \in \mathbb{N}}$ with multipliers $m_N : \mathbb{R} \to \mathbb{C}$ such that*
- *m_N has compact support on $\mathbb{R}$ for every $N \in \mathbb{N}$,*
- *m_N is uniformly bounded,*
- *$\lim_{N \to \infty} m_N(x) = 1$, for any $x \in \mathbb{R}$.*

Definition 2 (Periodic case)**.** *Let $w \in C([0,T], H^s(\mathbb{T}))$. We say that $\mathcal{N}(w)$ exists and is equal to a distribution $\tilde{w} \in [C^\infty((0,T), D(\mathbb{T}))]'$ if, for every sequence $\{T_N\}_{N \in \mathbb{N}}$ of Fourier cutoff operators, we have*

(9)
$$\lim_{N \to \infty} \mathcal{N}(T_N w) = \tilde{w},$$

in the sense of distributions on $(0,T) \times \mathbb{T}$.

Definition 3 (Periodic case)**.** *Let $r \geq 0$. We say that $w \in C([0,T], H^r(\mathbb{T}))$ is a weak solution in the extended sense of the NLS (6) if*
- *$w(0,x) = w_0(x)$,*
- *the nonlinearity $\mathcal{N}(w)$ exists in the sense of Definition 2,*
- *w satisfies (6) in the sense of distributions on $(0,T) \times \mathbb{T}$, where the nonlinearity $\mathcal{N}(w) = w|w|^2$ is interpreted as above.*

For a fixed such solution w of equation (6), in the sense of Definition 3, we define a solution v of equation (7) as

Definition 4 (Continuous case)**.** *Let $s \geq 0$ and $v \in C([0,T], H^s(\mathbb{R}))$. We say that $G(w,v)$ exists and is equal to a distribution $\tilde{v} \in [C^\infty((0,T), S(\mathbb{R})]'$ if, for every sequence $\{T_N\}_{N \in \mathbb{N}}$ of Fourier cutoff operators, we have*

(10)
$$\lim_{N \to \infty} G(T_N w, T_N v) = \tilde{v},$$

in the sense of distributions on $(0,T) \times \mathbb{R}$.

Similarly to the periodic case, we also introduce

Definition 5 (Continuous case)**.** *We say that $v \in C([0,T], H^s(\mathbb{R}))$ is a weak solution in the extended sense of NLS* (7) *if*

- $v(0,x) = v_0(x)$,
- *the nonlinearity $G(w,v)$ exists in the sense of Definition 4,*
- *v satisfies* (7) *in the sense of distributions on $(0,T) \times \mathbb{R}$, where the nonlinearity $G(w,v)$ is interpreted as above.*

The main results of the paper are the following

Theorem 6 (Local existence and well–posedness)**.** *Let $0 \le s_1 \le s_2$ and $u_0 = v_0 + w_0 \in H^{s_1}(\mathbb{R}) + H^{s_2}(\mathbb{T})$. There exists a weak solution in the extended sense $u = v + w \in C([0,T], H^{s_1}(\mathbb{R})) + C([0,T], H^{s_2}(\mathbb{T}))$ of NLS* (1) *with initial condition u_0 where w solves NLS* (6) *in the sense of Definition 3, v solves NLS* (7) *in the sense of Definition 5 and the time T of existence depends only on $\|v_0\|_{H^{s_1}(\mathbb{R})} + \|w_0\|_{H^{s_2}(\mathbb{T})}$.*

Moreover, the solution map is locally Lipschitz continuous.

Theorem 7 (Unconditional uniqueness)**.** *Let $\epsilon > 0$ and $\frac{1}{6} \le s \le \frac{1}{2}$. For any initial condition $u_0 \in H^s(\mathbb{R}) + H^{\frac{1}{2}+\epsilon}(\mathbb{T})$ the solution $u = v + w$ constructed in Theorem 6 is unique in $C([0,T], H^s(\mathbb{R}) + H^{\frac{1}{2}+\epsilon}(\mathbb{T}))$.*

Remark 8. The result of Theorem 7 is also true for $s > \frac{1}{2}$, but in this case the spaces $H^s(\mathbb{R})$ and $H^{\frac{1}{2}+\epsilon}(\mathbb{T})$ embed continuously into $L^\infty(\mathbb{R})$, thus also their sum. Hence $H^s(\mathbb{R}) + H^{\frac{1}{2}+\epsilon}(\mathbb{T})$ is a Banach algebra and existence and uniqueness results become much easier with the help of straightforward direct Banach contraction mapping arguments. The condition $s \ge \frac{1}{6}$ guarantees that $v \in H^s(\mathbb{R}) \hookrightarrow L^3(\mathbb{R})$ which means that $|v|^2 v \in L^1(\mathbb{R})$ and together with $H^{\frac{1}{2}+\epsilon}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$, allows us to control non–linear interaction terms which pair v and w together. For example, integrals of the form $\int w^2 \bar{v}$ and $\int v^2 \bar{w}$ which appear naturally due to the nonlinearity $G(w,v)$.

For the proof of Theorem 6 we will need to localise our functions on the Fourier side and this is achieved through the box operators that are defined as follows: Let $Q_0 = [-\frac{1}{2}, \frac{1}{2})$ and its translations $Q_k = Q_0 + k$ for all $k \in \mathbb{Z}$. Consider a partition of unity $\{\sigma_k = \sigma_0(\cdot - k)\}_{k \in \mathbb{Z}} \subset C^\infty(\mathbb{R})$ satisfying

- $\exists c > 0 : \forall \eta \in Q_0 : |\sigma_0(\eta)| \ge c$,
- $\mathrm{supp}(\sigma_0) \subseteq \{\xi \in \mathbb{R} : |\xi| < 1\}$.

Note that this implies $1 = \sigma_0(0) = \sigma_k(k)$ for all $k \in \mathbb{Z}$. Given a partition of unity as above, we define the isometric decomposition operators (box operators)

$$(11) \qquad \qquad \Box_k := \mathcal{F}^{(-1)} \sigma_k \mathcal{F}, \quad (\forall k \in \mathbb{Z}).$$

It is not difficult to see that for $1 \le p_1 \le p_2 \le \infty$ the following holds

$$(12) \qquad \qquad \|\Box_k f\|_{p_2} \lesssim \|\Box_k f\|_{p_1},$$

where the implicit constant is independent of k and the function f. Having the box operators we may define the modulation spaces $M_{p,q}^s(\mathbb{R}), s \in \mathbb{R}, 1 \le p, q \le \infty$ as

$$(13) \qquad M_{p,q}^s(\mathbb{R}) := \{f \in S'(\mathbb{R}) \ / \ \|f\|_{M_{p,q}^s} := \Big(\sum_{k \in \mathbb{Z}} \langle k \rangle^{sq} \|\Box_k f\|_p^q\Big)^{\frac{1}{q}} < \infty\},$$

with the usual interpretation when the index q is equal to infinity. It can be proved that different choices of the function σ_0 lead to equivalent norms in $M_{p,q}^s(\mathbb{R})$. When $s = 0$ we denote the space $M_{p,q}^0(\mathbb{R})$ by $M_{p,q}(\mathbb{R})$. In the special case where $p = q = 2$ we have $M_{2,2}^s(\mathbb{R}) = H^s(\mathbb{R})$. The usual Sobolev spaces as in (2). Modulation spaces were introduced by Feichtinger in [6]. In [3] and [13] the NLS (1) was studied with initial data $u_0 \in M_{p,q}^s(\mathbb{R})$ and under the restrictions $s \in [0, \infty), q \in [1, 2]$ and $p \in [2, \frac{10q'}{q'+6})$, existence of weak solutions in the extended sense was proved. Moreover, under the extra assumption that $M_{p,q}^s(\mathbb{R}) \hookrightarrow L^3(\mathbb{R})$ unconditional well–posedness of the Cauchy problem was shown to be true. Unfortunately, the space $M_{\infty,2}(\mathbb{R})$ is not included in the previously mentioned family of modulation spaces. Nevertheless, we are able to obtain an existence result (and uniqueness of solutions under some extra assumptions) for initial data u_0 in its subspace $H^s(\mathbb{R}) + H^s(\mathbb{T}) \subset M_{\infty,2}(\mathbb{R})$ for $s \ge 0$.

1.1. **Preliminaries.** The following lemma will be needed in the proof of Theorem 6. It is a straightforward consequence of Young's inequality.

Lemma 9. *Let $1 \le p \le \infty$ and $\sigma \in C_c^\infty(\mathbb{R})$. Then the multiplier operator $T_\sigma : S'(\mathbb{R}) \to S'(\mathbb{R})$ defined by*

$$(T_\sigma f) = \mathcal{F}^{-1}(\sigma \cdot \hat{f}), \quad \forall f \in S'(\mathbb{R})$$

is bounded on $L^p(\mathbb{R})$ and

$$\|T_\sigma\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \lesssim \|\check{\sigma}\|_{L^1(\mathbb{R})}.$$

We also need for $S(t) = e^{it\Delta}$, the Schrödinger semigroup, the 'conservation of mass'

$$(14) \qquad \|S(t)f\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}.$$

Lastly, let us recall the following number theoretic fact (see [8], Theorem 315) which is going to be used throughout the proof of Theorem 6: Given an integer m, let $d(m)$ denote the number of divisors of m. Then

$$(15) \qquad d(m) \lesssim e^{c\frac{\log m}{\log \log m}} = o(m^\epsilon)$$

for all $\epsilon > 0$.

The paper is organised as follows: In Section 2 we consider initial data $u_0 = v_0 + w_0$ with v_0, w_0 sufficiently smooth and we show that NLS (1) is locally wellposed. In Section 3 we describe the first steps of the differentiation by parts technique and in Section 4 we define the trees which allow us to continue with the infinite iteration procedure. Finally, in Section 5 we show that the solution u described in Theorem 6 exists through a smooth approximation procedure and in Section 6 we prove Theorem 7.

2. SMOOTH INITIAL DATA

Let us assume that the initial data is smooth, that is, $u_0 = v_0 + w_0$ where $v_0 \in H^{s_1}(\mathbb{R}), w_0 \in H^{s_2}(\mathbb{T})$ for sufficiently large $s_1, s_2 \in \mathbb{R}$. We choose $s_1 > 1, s_2 = s_1 + 1$. Then the spaces $H^{s_1}(\mathbb{R})$ and $H^{s_2}(\mathbb{T})$ are Banach algebras and an easy Banach contraction argument for the operator

$$(16) \qquad Tw = e^{it\partial_x^2}w_0 \pm \int_0^t e^{i(t-\tau)\partial_x^2}|w|^2 w \, d\tau$$

shows that the NLS (6) is locally wellposed in $X_2 := C([0,T], H^{s_2}(\mathbb{T}))$ for some $T = T(\|w_0\|_{H^{s_2}})$. Let w be that solution of NLS (6) in the ball $\{w \in X_2 : \|w\|_{X_2} \leq 2\|w_0\|_{H^{s_2}}\}$ and consider the operator

$$(17) \qquad Tv = e^{it\partial_x^2}v_0 \pm \int_0^t e^{i(t-\tau)\partial_x^2}G(w,v) \, d\tau.$$

Our goal is to show that T is a contraction in a suitable ball in $X_1 := C([0,T], H^{s_1}(\mathbb{R}))$.

Before we prove this, let us estimate the norm of $\|wv\|_{H^{s_1}(\mathbb{R})}$ for $w \in H^{s_2}(\mathbb{T})$ and $v \in H^{s_1}(\mathbb{R})$. First we need to calculate $\mathcal{F}(wv)(\xi)$ which equals

$$\hat{w} * \hat{v}(\xi) = \Big(\sum_{n\in\mathbb{Z}} w_n \delta_n\Big) * \hat{v}(\xi) = \sum_{n\in\mathbb{Z}} w_n \hat{v}(\xi - n).$$

Thus,

$$|\mathcal{F}(wv)(\xi)|^2 = \sum_{n,m\in\mathbb{Z}} w_n \bar{w}_m \hat{v}(\xi - n)\overline{\hat{v}(\xi - m)}$$

and, therefore,

$$\begin{aligned}
\|wv\|_{H^{s_1}}^2 &= \int_{\mathbb{R}} (1 + |\xi|^2)^{s_1} \sum_{n,m\in\mathbb{Z}} w_n \bar{w}_m \hat{v}(\xi - n)\overline{\hat{v}(\xi - m)}d\xi \\
&= \Big| \int_{\mathbb{R}} (1 + |\xi|^2)^{s_1} \sum_{n,m\in\mathbb{Z}} w_n \bar{w}_m \hat{v}(\xi - n)\overline{\hat{v}(\xi - m)}d\xi \Big| \\
&\leq \sum_{n,m\in\mathbb{Z}} |w_n||\bar{w}_m| \int_{\mathbb{R}} (1 + |\xi|^2)^{s_1} |\hat{v}(\xi - n)||\overline{\hat{v}(\xi - m)}|d\xi.
\end{aligned}$$

For the integral we apply Hölder's inequality

$$\int_{\mathbb{R}} (1 + |\xi|^2)^{s_1} |\hat{v}(\xi - n)||\overline{\hat{v}(\xi - m)}|d\xi$$
$$\leq \Big(\int_{\mathbb{R}} (1 + |\xi|^2)^{s_1} |\hat{v}(\xi - n)|^2 d\xi \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}} (1 + |\xi|^2)^{s_1} |\hat{v}(\xi - m)|^2 d\xi \Big)^{\frac{1}{2}},$$

and this can be estimated from above by the product

$$\lesssim_{s_1} (1 + |n|^{s_1})(1 + |m|^{s_1})\|v\|_{H^{s_1}}^2,$$

which implies

$$\|wv\|_{H^{s_1}}^2 \lesssim_{s_1} \|v\|_{H^{s_1}}^2 \Big(\sum_{n\in\mathbb{Z}} (1+|n|)^{s_1}|w_n| \Big)^2.$$

Since $s_1 > 1$, the last sum is again easily estimated using Hölder's inequality as follows

$$\Big(\sum_{n\in\mathbb{Z}} (1+|n|)^{s_1}|w_n| \Big)^2 = \Big(\sum_{n\in\mathbb{Z}} \frac{(1+|n|)^{s_1+1}}{(1+|n|)}|w_n| \Big)^2$$

$$\leq \Big(\sum_{n\in\mathbb{Z}} (1+|n|)^{2s_1+2}|w_n|^2 \Big) \Big(\sum_{n\in\mathbb{Z}} \frac{1}{(1+|n|)^2} \Big) \lesssim \|w\|_{H^{s_1+1}}^2.$$

Thus

(18) $$\|wv\|_{H^{s_1}(\mathbb{R})} \lesssim_{s_1} \|w\|_{H^{s_1+1}(\mathbb{T})} \|v\|_{H^{s_1}(\mathbb{R})}.$$

From (18) and (8) we also obtain

(19) $$\|Tv\|_{X_1} \lesssim \|v_0\|_{H^{s_1}(\mathbb{R})} + T\|G(w,v)\|_{X_1} \lesssim \|v_0\|_{X_1} + T(\|v\|_{X_1} + \|w\|_{X_2})^3,$$

which implies

(20) $$\|Tv\|_{X_1} \lesssim \|v_0\|_{H^{s_1}(\mathbb{R})} + T(\|v\|_{X_1} + 2\|w_0\|_{H^{s_2}(\mathbb{T})})^3.$$

If we assume $v \in B := \{v \in X_1 : \|v\|_{X_1} \leq 2\|v_0\|_{H^{s_1}} := R\}$, then T maps B into itself for sufficiently small $T = T(\|v_0\|_{H^{s_1}(\mathbb{R})} + \|w_0\|_{H^{s_2}(\mathbb{T})})$. Indeed, for $T > 0$ such that $16T(\|v_0\|_{H^{s_1}} + \|w_0\|_{H^{s_2}})^3 \leq R$, we see from (20) that $Tv \in B$.

Also, for $v_1, v_2 \in B$, it is easy to see

(21) $$Tv_1 - Tv_2 = \pm \int_0^t (G(w,v_1) - G(w,v_2))\, d\tau$$

where the difference inside the integral equals

$$|v_1|^2 v_1 - |v_2|^2 v_2 + v_1^2 w - v_2^2 w + w^2 \bar{v}_1 - w^2 \bar{v}_2 + 2w|v_1|^2 - 2w|v_2|^2 + 2v_1|w|^2 - 2v_2|w|^2 =$$

$$(v_1 - v_2)(|v_1|^2 + \bar{v}_1 v_2) + (\bar{v}_1 - \bar{v}_2)v_2^2 + w(v_1 + v_2)(v_1 - v_2) + w^2(\bar{v}_1 - \bar{v}_2) +$$

$$2w(v_1(\bar{v}_1 - \bar{v}_2) + \bar{v}_2(v_1 - v_2)) + 2|w|^2(v_1 - v_2).$$

Thus,

(22) $$\|Tv_1 - Tv_2\|_{X_1} \leq T(\|v_1\|_{X_1} + \|v_2\|_{X_1} + \|w\|_{X_2})^2 \|v_1 - v_2\|_{X_1},$$

which implies, for sufficiently small $T = T(\|v_0\|_{H^{s_1}} + \|w_0\|_{H^{s_2}}) > 0$, that the operator $T : B \to B$ is a contraction. Therefore, we have proved

Lemma 10. *Let $s > 1$ and $u_0 = v_0 + w_0 \in H^s(\mathbb{R}) + H^{s+1}(\mathbb{T})$. Then NLS (1) is locally wellposed with a solution $u = v + w \in C([0,T], H^s(\mathbb{R})) + C([0,T], H^{s+1}(\mathbb{T}))$ where w solves (6) in the sense that satisfies (16) and v solves (7) in the sense that it satisfies (17) for a sufficiently small $T = T(\|v_0\|_{H^s} + \|w_0\|_{H^{s+1}}) > 0$.*

3. FIRST STEPS OF THE ITERATION PROCESS

From here on, we consider only the case $s_1 = s_2 = 0$ in Theorem 6 since for the other cases similar considerations apply. See Remark 24 at the end of the Section 4 for a more detailed argument. We also assume in the following calculations that the functions v and w are sufficiently smooth.

Let us define the function $\Phi : \mathbb{R}^4 \to \mathbb{R}, \Phi(\xi, \xi_1, \xi_2, \xi_3) := \xi^2 - \xi_1^2 + \xi_2^2 - \xi_3^2$ and observe that, under the hypothesis $\xi = \xi_1 - \xi_2 + \xi_3$, it factorizes into $\Phi(\xi, \xi_1, \xi_2, \xi_3) = 2(\xi - \xi_1)(\xi - \xi_3)$.

By making the change of variables $w \mapsto e^{-it\partial_x^2}w$, we can rewrite the periodic NLS (6) in terms of its Fourier coefficients as

$$
(23) \qquad \partial_t w_n = \sum_{n = n_1 - n_2 + n_3} e^{-2i(n-n_1)(n-n_3)t} w_{n_1} \bar{w}_{n_2} w_{n_3} - |w_n|^2 w_n + 2\Big(\int_{\mathbb{T}} |w|^2 dx \Big) w_n
$$
$$
= \mathscr{N}_1^t(w)(n) - \mathscr{R}_1^t(w)(n) + \mathscr{R}_2^t(w)(n).
$$

In a similar fashion, we would like to rewrite the modified NLS (7), which contains both periodic and non–periodic functions. For this we again make the change of variables $v \mapsto e^{-it\partial_x^2}v$ and introduce, with the help of the isometric decomposition operators, $v_n := \Box_n v$ for $n \in \mathbb{Z}$. Note that its Fourier transform, $\hat{v}_n$, is a function supported within the interval $(n-1, n+1)$, so, in general, products of the form $\hat{v}_n \hat{v}_m$ can be non-zero only if $|n - m| \le 1$, that is, only neighbouring $\hat{v}_n$ can overlap. Thus it is convenient to define

$$
(24) \qquad\qquad n \approx m \text{ iff } n = m \text{ or } n = m + 1 \text{ or } n = m - 1
$$

for $n, m \in \mathbb{Z}$. Recall that for a 1–periodic function w its Fourier transform is given by $\widehat{w} = \sum_{n \in \mathbb{Z}} w_n \delta_n$, where δ_n is Dirac delta centered at n. Thus $\Box_n w = w_n \delta_n$, since the partition of unity we use in the definition of $\Box_n$ obeys $1 = \sigma_n(n)$. With this we may rewrite the modified NLS (7) on the Fourier side, up to constants, as

$$
(25) \quad
\begin{aligned}
\partial_t \hat{v}_n = {} & E_{I,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}) + E_{II,n}^{1,t}(w_{n_1}, \bar{w}_{n_2}, v_{n_3}) + E_{III,n}^{1,t}(w_{n_1}, \bar{v}_{n_2}, w_{n_3}) \\
& + E_{IV,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, w_{n_3}) + E_{V,n}^{1,t}(v_{n_1}, \bar{w}_{n_2}, v_{n_3}) \,.
\end{aligned}
$$

where we also introduced
(26)

$$
E_{I,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3})(\xi) := \sum_{n \approx n_1 - n_2 + n_3} \sigma_n(\xi) \overbrace{\iint_{\mathbb{R}^2} e^{-2i(\xi-\xi_1)(\xi-\xi_3)t} \hat{v}_{n_1}(\xi_1) \hat{\bar{v}}_{n_2}(\xi - \xi_1 - \xi_3) \hat{v}_{n_3}(\xi_3) d\xi_1 d\xi_3}^{=:\mathcal{F}(Q_{I,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}))}
$$

$$
(27) \quad E_{II,n}^{1,t}(w_{n_1}, \bar{w}_{n_2}, v_{n_3})(\xi) := \sum_{n \approx n_1 - n_2 + n_3} \overbrace{\sigma_n(\xi) e^{-2i(\xi-n_1)(n_1-n_2)t} w_{n_1} \bar{w}_{n_2} \hat{v}_{n_3}(\xi - n_1 + n_2)}^{=:\mathcal{F}(Q_{II,n}^{1,t}(w_{n_1}, \bar{w}_{n_2}, v_{n_3}))}
$$

$$(28) \quad E_{III,n}^{1,t}(w_{n_1}, \bar{v}_{n_2}, w_{n_3})(\xi) := \sum_{n \approx n_1 - n_2 + n_3} \overbrace{\sigma_n(\xi) e^{-2i(\xi - n_1)(\xi - n_3)t} w_{n_1} \hat{\bar{v}}_{n_2}(\xi - n_1 - n_3) w_{n_3}}^{=: \mathcal{F}(Q_{III,n}^{1,t}(w_{n_1}, \bar{v}_{n_2}, w_{n_3}))}$$

(29)

$$E_{IV,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, w_{n_3})(\xi) := \sum_{n \approx n_1 - n_2 + n_3} \overbrace{\sigma_n(\xi) w_{n_3} \int_{\mathbb{R}} e^{-2i(\xi - n_3)(\xi - \xi_1)t} \hat{v}_{n_1}(\xi_1) \hat{\bar{v}}_{n_2}(\xi - n_3 - \xi_1) d\xi_1}^{=: \mathcal{F}(Q_{IV,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, w_{n_3}))}$$

(30)

$$E_{V.n}^{1,t}(v_{n_1}, \bar{w}_{n_2}, v_{n_3})(\xi) := \sum_{n \approx n_1 - n_2 + n_3} \overbrace{\sigma_n(\xi) \bar{w}_{n_2} \int_{\mathbb{R}} e^{-2i(\xi - \xi_1)(\xi_1 - n_2)t} \hat{v}_{n_1}(\xi_1) \hat{v}_{n_3}(\xi + n_2 - \xi_1) d\xi_1}^{=: \mathcal{F}(Q_{V,n}^{1,t}(v_{n_1}, \bar{w}_{n_2}, v_{n_3}))} .$$

Remark 11. A short note on our notation is necessary here: The expression $E_{I,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3})$ above depends not only on the single v_{n_1}, $\bar{v}_{n_2}$, or v_{n_3}, but on the sequences $(v_{n_1})_{n_1 \in \mathbb{Z}}$, $(\bar{v}_{n_2})_{n_2 \in \mathbb{Z}}$, and $(v_{n_3})_{n_3 \in \mathbb{Z}}$. So one should instead write $E_{I,n}^{1,t}((v_{n_1})_{n_1 \in \mathbb{Z}}, (\bar{v}_{n_2})_{n_2 \in \mathbb{Z}}, (v_{n_3})_{n_3 \in \mathbb{Z}})$, or simply, $E_{I,n}^{1,t}(v, \bar{v}, v)$. However, when we construct a tree–type expansion later, it will be very important to know in which order the v_n and w_m appear in considerably more involved expressions. Thus it will be convenient to write $E_{I,n}^{1,t}(v, \bar{v}, v)$ as $E_{I,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3})$, keeping in mind, that one sums over n_1, n_2, and n_3. The same applies to the other terms on the right–hand side of equation (25).

Remark 12. The operator $Q_{I,n}^{1,t}$ in the definition of $E_{I,n}^{1,t}$ in equation (26) is the same as the operator $Q_n^{1,t}$ studied in [3] and [13]. Here let us notice that if we choose functions such that $\hat{v}_{n_1} = w_{n_1} \delta_{n_1}$ and $\hat{v}_{n_2} = w_{n_2} \delta_{n_2}$ then we obtain the relation $Q_{I,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}) = Q_{II,n}^{1,t}(w_{n_1}, \bar{w}_{n_2}, v_{n_3})$. Similar relations hold between $Q_{I,n}^{1,t}$ and the remaining operators $Q_{III,n}^{1,t}, Q_{IV,n}^{1,t}$ and $Q_{V,n}^{1,t}$.

We split the sums in (26)), (27)), (28)), (29)) and (30)) into

$$\sum_{\substack{n_1 \approx n \\ or \\ n_3 \approx n}} \cdots + \sum_{n_1, n_3 \not\approx n} \cdots$$

and define the resonant operators

$$(31) \qquad R_2^t(v)(n) := \Big(\sum_{n_1 \approx n} + \sum_{n_3 \approx n} \Big) \Big(Q_{I,n}^{1,t} + Q_{II,n}^{1,t} + Q_{III,n}^{1,t} + Q_{IV,n}^{1,t} + Q_{V,n}^{1,t} \Big)$$

$$R_1^t(v)(n) := \sum_{\substack{n_1 \approx n \\ and \\ n_3 \approx n}} \Big(Q_{I,n}^{1,t} + Q_{II,n}^{1,t} + Q_{III,n}^{1,t} + Q_{IV,n}^{1,t} + Q_{V,n}^{1,t} \Big)$$

and the non-resonant operator

$$(32) \qquad N_1^t(v)(n) := \sum_{n_1,n_3 \napprox n} \left(Q_{I,n}^{1,t} + Q_{II,n}^{1,t} + Q_{III,n}^{1,t} + Q_{IV,n}^{1,t} + Q_{V,n}^{1,t} \right).$$

With this notation, equation (25)) can be written in the form

$$(33) \qquad \partial_t v_n = R_2^t(v)(n) - R_1^t(v)(n) + N_1^t(v)(n),$$

keeping in mind that the operators appearing in the RHS above depend also on the periodic function w, which we suppress in our notation, for simplicity. For the resonant part we have the estimate

Lemma 13. *For $j = 1, 2$*

$$\|R_j^t(v)\|_{l^2(\mathbb{Z})L^2(\mathbb{R})} \lesssim \|v\|_{L^2(\mathbb{R})}^3 + \|w\|_{L^2(\mathbb{T})}\|v\|_{L^2(\mathbb{R})}^2 + \|w\|_{L^2(\mathbb{T})}^2\|v\|_{L^2(\mathbb{R})}$$

and

$$\|R_j^t(v) - R_j^t(u)\|_{l^2(\mathbb{Z})L^2(\mathbb{R})} \lesssim$$
$$(\|v\|_{L^2(\mathbb{R})}^2 + \|u\|_{L^2(\mathbb{R})}^2 + \|w\|_{L^2(\mathbb{T})}(\|v\|_{L^2(\mathbb{R})} + \|u\|_{L^2(\mathbb{R})}) + \|w\|_{L^2(\mathbb{T})}^2)\|v - u\|_{L^2(\mathbb{R})}.$$

Proof. Both resonant operators contain a sum that only involves the v function, that is

$$\left(\sum_{n_1 \approx n} + \sum_{n_3 \approx n} - \sum_{\substack{n_1 \approx n \\ and \\ n_3 \approx n}} \right) Q_{I,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}).$$

As mentioned in Remark 12 this operator was estimated in [13], it gives the upper bound of $\|v\|_2^3$ and we refer the interested reader to Lemma 10 of that paper.

For the sum that contains $Q_{II,n}^{1,t}$ and $Q_{III,n}^{1,t}$ it suffices to estimate only $Q_{II,n}^{1,t}$, the bound for the sum involving $Q_{III,n}^{1,t}$ is very similar to one for $Q_{II,n}^{1,t}$ Moreover, since, for fixed $n \in \mathbb{Z}$, the sum

$$\sum_{\substack{n_1 \approx n \\ and \\ n_3 \approx n}} Q_{II,n}^{1,t}(w_{n_1}, \bar{w}_{n_2}, v_{n_3})$$

is only over the neighbours of n, we only look at the part where $n_1 = n$ and $n_3 = n$, the other summands are bounded in the same way. Then we have the estimate

$$\left\| \sigma_n(\xi) w_n \bar{w}_n \hat{v}_n(\xi) \right\|_{l^2(\mathbb{Z})L_\xi^2(\mathbb{R})} \lesssim \left(\sum_{n \in \mathbb{Z}} |w_n|^4 \|v_n\|_2^2 \right)^{\frac{1}{2}} \lesssim \|w_n\|_{l^\infty(\mathbb{Z})}^2 \|v\|_2 \leq \|w\|_{L^2(\mathbb{T})}^2 \|v\|_{L^2(\mathbb{R})},$$

by the embedding $l^2(\mathbb{Z}) \hookrightarrow l^\infty(\mathbb{Z})$. To continue it suffices to look at the sum

$$\sum_{n_1 \approx n} Q_{II,n}^{1,t}(w_{n_1}, \bar{w}_{n_2}, v_{n_3}).$$

Again, since it consists of finitely many summands, depending on whether $n_1 = n - 1$ or $n_1 = n$ or $n_1 = n + 1$, it is enough to estimate the part where $n_1 = n$. In this case, we have

$$\left\| \sigma_n(\xi) w_n \sum_{n_2 \in \mathbb{Z}} e^{-2it(\xi-n)(n-n_2)} \bar{w}_{n_2} \hat{v}_{n_2}(\xi - n + n_2) \right\|_{L^2(\mathbb{R})} \lesssim |w_n| \sum_{n_2 \in \mathbb{Z}} |w_{n_2}| \|v_{n_2}\|_2,$$

so with Hölder's inequality we get the upper bound

$$|w_n| \Big(\sum_{n_2 \in \mathbb{Z}} |w_{n_2}|^2 \Big)^{\frac{1}{2}} \Big(\sum_{n_2 \in \mathbb{Z}} \|v_{n_2}\|_2^2 \Big)^{\frac{1}{2}} = |w_n| \|w\|_{L^2(\mathbb{T})} \|v\|_{L^2(\mathbb{R})}.$$

Taking the $l^2(\mathbb{Z})$ norm we obtain

$$\|w\|_{L^2(\mathbb{T})}^2 \|v\|_{L^2(\mathbb{R})}.$$

For the sum that contains $Q_{IV,n}^{1,t}$ and $Q_{V,n}^{1,t}$ it suffices to estimate only $Q_{V,n}^{1,t}$. As before, from the sum

$$\sum_{\substack{n_1 \approx n \\ and \\ n_3 \approx n}} Q_{V,n}^{1,t}(v_{n_1}, \bar{w}_{n_2}, v_{n_3})$$

we may look only at the part where $n_1 = n$ and $n_3 = n$. Thus, we have

$$\Big\| \sigma_n(\xi) \bar{w}_n \int_{\mathbb{R}} e^{-2it(\xi - \xi_1)(\xi_1 - n)} \hat{v}_n(\xi_1) \hat{v}_n(\xi - \xi_1 + n) d\xi_1 \Big\|_{L^2(\mathbb{R})}$$

which, by setting $\hat{V}_n = e^{it\xi^2} \hat{v}_n$ and using (14)), we may rewrite as

$$\Big\| \sigma_n(\xi) \bar{w}_n e^{-itn^2} \int_{\mathbb{R}} \hat{V}_n(\xi_1) \hat{V}_n(\xi - \xi_1 + n) d\xi_1 \Big\|_{L^2(\mathbb{R})} \lesssim |w_n| \Big\| \hat{V}_n * \hat{V}_n(\cdot + n) \Big\|_{L^2(\mathbb{R})}.$$

The last expression equals

$$|w_n| \Big\| V_n e^{in(\cdot)} V_n \Big\|_{L^2(\mathbb{R})} = |w_n| \|V_n\|_{L^4(\mathbb{R})}^2 \lesssim |w_n| \|V_n\|_{L^2(\mathbb{R})}^2,$$

where we used (12)) and, since $\|V_n\|_2 = \|v_n\|_2$ (by (14))), we can take the $l^2(\mathbb{Z})$ norm in n and obtain the upper bound

$$\Big(\sum_n |w_n|^2 \|v_n\|_2^4 \Big)^{\frac{1}{2}} \le \|w\|_{L^2(\mathbb{T})} \|\{\|v_n\|_2\}_{n\in\mathbb{Z}}\|_{l^\infty(\mathbb{Z})}^2 \le \|w\|_{L^2(\mathbb{T})} \|v\|_{L^2(\mathbb{R})}^2,$$

by the embedding $l^2(\mathbb{Z}) \hookrightarrow l^\infty(\mathbb{Z})$. Finally, we look at the sum

$$\sum_{n_1 \approx n} Q_{V,n}^{1,t}(v_{n_1}, \bar{w}_{n_2}, v_{n_3}).$$

As before, it suffices to look at the term where $n_1 = n$. In this case we have

$$\Big\| \sigma_n(\xi) \sum_{n_2 \in \mathbb{Z}} \bar{w}_{n_2} \int_{\mathbb{R}} e^{-2it(\xi - \xi_1)(\xi_1 - n_2)} \hat{v}_{n_1}(\xi_1) \hat{v}_{n_2}(\xi - \xi_1 + n_2) d\xi_1 \Big\|_{L^2(\mathbb{R})},$$

and setting again $\hat{V}_n = e^{it\xi^2} \hat{v}_n$, we arrive at the upper bound

$$\sum_{n_2 \in \mathbb{Z}} |w_{n_2}| \|\hat{V}_n * \hat{V}_{n_2}(\cdot + n_2)\|_2 = \sum_{n_2 \in \mathbb{Z}} |w_{n_2}| \|V_n e^{in_2(\cdot)} V_{n_2}\|_2 = \sum_{n_2 \in \mathbb{Z}} |w_{n_2}| \|V_n V_{n_2}\|_2.$$

Applying Hölder's inequality, (12)) and (14)) we continue the estimate as follows

$$\sum_{n_2 \in \mathbb{Z}} |w_{n_2}| \|V_n\|_4 \|V_{n_2}\|_4 \lesssim \|V_n\|_2 \sum_{n_2 \in \mathbb{Z}} |w_{n_2}| \|V_{n_2}\|_2 = \|v_n\|_2 \sum_{n_2 \in \mathbb{Z}} |w_{n_2}| \|v_{n_2}\|_2$$

$$\leq \|v_n\|_2 \Big(\sum_{n_2 \in \mathbb{Z}} |w_{n_2}|^2 \Big)^{\frac{1}{2}} \Big(\sum_{n_2 \in \mathbb{Z}} \|v_{n_2}\|^2 \Big)^{\frac{1}{2}} = \|v_n\|_2 \|w\|_{L^2(\mathbb{T})} \|v\|_{L^2(\mathbb{R})}.$$

Taking the $l^2(\mathbb{Z})$ norm in n finishes the proof. □

Remark 14. In [7] it was proved that the resonant part of the periodic solution w satisfies

$$\|\mathscr{R}_j^t(w)\|_{L^2(\mathbb{T})} \lesssim \|w\|_{L^2(\mathbb{T})}^3$$

for $j = 1, 2$. This will be used later in Lemma 22 for the estimate of the $N_r^{(J)}$ operator.

In order to continue the iteration process we define the sets

(34) $A_N(n) = \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n_1 - n_2 + n_3 \approx n, n_1 \not\approx n \not\approx n_3, |\Phi(n, n_1, n_2, n_3)| \leq N\}$

and

(35) $A_N(n)^c = \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n_1 - n_2 + n_3 \approx n, n_1 \not\approx n \not\approx n_3, |\Phi(n, n_1, n_2, n_3)| > N\}.$

The number $N > 0$ is considered to be large and will be fixed later in the proof. The non-resonant operator N_1^t we split as

(36) $$N_1^t(v)(n) = N_{11}^t(v)(n) + N_{12}^t(v)(n),$$

where

$$N_{11}^t(v)(n) = \sum_{A_N(n)} \Big(Q_{I,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}) + Q_{II,n}^{1,t}(w_{n_1}, \bar{w}_{n_2}, v_{n_3}) + Q_{III,n}^{1,t}(w_{n_1}, \bar{v}_{n_2}, w_{n_3})$$

$$+ Q_{IV,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, w_{n_3}) + Q_{V,n}^{1,t}(v_{n_1}, \bar{w}_{n_2}, v_{n_3}) \Big),$$

and the following yields a convenient bound on N_{11}^t.

Lemma 15.

$$\|N_{11}^t(v)\|_{l^2(\mathbb{Z})L^2(\mathbb{R})} \lesssim N^{\frac{1}{2}+}(\|v\|_{L^2(\mathbb{R})}^3 + \|w\|_{L^2(\mathbb{T})}\|v\|_{L^2(\mathbb{R})}^2 + \|w\|_{L^2(\mathbb{T})}^2\|v\|_{L^2(\mathbb{R})})$$

and

$$\|N_{11}^t(v) - N_{11}^t(u)\|_{l^2(\mathbb{Z})L^2(\mathbb{R})} \lesssim$$

$$N^{\frac{1}{2}+}(\|v\|_{L^2(\mathbb{R})}^2 + \|u\|_{L^2(\mathbb{R})}^2 + \|w\|_{L^2(\mathbb{T})}(\|v\|_{L^2(\mathbb{R})} + \|u\|_{L^2(\mathbb{R})}) + \|w\|_{L^2(\mathbb{T})}^2)\|v - u\|_{L^2(\mathbb{R})}.$$

Proof. The part

$$\sum_{A_N(n)} Q_{I,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3})$$

has been estimated in [13], Lemma 11 giving an upper bound of the form $N^{\frac{1}{2}+}\|v\|_2^3$. For the sum that contains $Q_{II,n}^{1,t}$ and $Q_{III,n}^{1,t}$ it suffices to estimate only $Q_{II,n}^{1,t}$, the other one being similar. We have

$$\sum_{A_N(n)} \left\| \sigma_n(\xi) e^{-2it(\xi-n_1)(n_1-n_2)} w_{n_1} \bar{w}_{n_2} \hat{v}_{n_3}(\xi - n_1 + n_2) \right\|_{L^2(\mathbb{R})} \lesssim \sum_{A_N(n)} |w_{n_1}||w_{n_2}|\|v_{n_3}\|_2,$$

which by Hölder's inequality implies the estimate

$$(37) \qquad \Big(\sum_{A_N(n)} 1 \Big)^{\frac{1}{2}} \Big(\sum_{A_N(n)} |w_{n_1}|^2 |w_{n_2}|^2 \|v_{n_3}\|_2^2 \Big)^{\frac{1}{2}}.$$

The first factor is estimated by $N^{\frac{1}{2}+}$ with the use of (15)) and then, by taking the $l^2(\mathbb{Z})$ norm of the second sum and applying Young's inequality in $l^1(\mathbb{Z})$, we obtain the upper bound

$$N^{\frac{1}{2}+}\Big(\sum_{n\in\mathbb{Z}} \sum_{A_N(n)} |w_{n_1}|^2 |w_{n_2}|^2 \|v_{n_3}\|_2^2 \Big)^{\frac{1}{2}} \le N^{\frac{1}{2}+}\|w\|_{L^2(\mathbb{T})}^2 \|v\|_{L^2(\mathbb{R})}.$$

For the sum that contains $Q_{IV.n}^{1,t}$ and $Q_{V,n}^{1,t}$ it again suffices to estimate only $Q_{V,n}^{1,t}$. In this case, letting $\hat{V}_n = e^{it\xi^2}\hat{v}_n$, we have

$$\sum_{A_N(n)} \left\| \sigma_n(\xi)\bar{w}_{n_2} \int_{\mathbb{R}} e^{-2it(\xi-\xi_1)(\xi_1-n_2)} \hat{v}_{n_1}(\xi_1)\hat{v}_{n_3}(\xi-\xi_1+n_2)d\xi_1 \right\|_{L^2(\mathbb{R})} \lesssim$$

$$\sum_{A_N(n)} |w_{n_2}| \left\| \hat{V}_{n_1} * \hat{V}_{n_3}(\cdot + n_2) \right\|_{L^2(\mathbb{R})} = \sum_{A_N(n)} |w_{n_2}| \left\| V_{n_1} V_{n_3} \right\|_{L^2(\mathbb{R})} \le$$

$$\sum_{A_N(n)} |w_{n_2}| \|V_{n_1}\|_4 \|V_{n_3}\|_4 \lesssim \sum_{A_N(n)} |w_{n_2}| \|V_{n_1}\|_2 \|V_{n_3}\|_2 = \sum_{A_N(n)} |w_{n_2}| \|v_{n_1}\|_2 \|v_{n_3}\|_2,$$

where we used (12)) and (14)). Then the estimate continues as in (37)) giving the upper bound $N^{\frac{1}{2}+}\|w\|_2\|v\|_2^2$. $\qquad\qquad\square$

For the N_{12}^t operator we only look at frequencies where $|\Phi(n, n_1, n_2, n_3)| > N$, which means that we can apply the differentiation by parts techniques, in order to take advantage of possible cancellations, due to the fact that the exponential terms contain the phase factor $\Phi(n, n_1, n_2, n_3)$, having a large magnitude. By doing this separately to the $Q_{I,n}^{1,t} \ldots, Q_{V,n}^{1,t}$ operators we obtain the following expressions

$$(38) \qquad \partial_t \Bigg(\overbrace{ \sigma_n(\xi) \int_{\mathbb{R}^2} \frac{e^{-2it(\xi-\xi_1)(\xi-\xi_3)}}{-2i(\xi-\xi_1)(\xi-\xi_3)} \hat{v}_{n_1}(\xi_1)\hat{\bar{v}}_{n_2}(\xi-\xi_1-\xi_3)\hat{v}_{n_3}(\xi_3) \, d\xi_1 d\xi_3 }^{=:\mathcal{F}(\tilde{Q}_{I,n}^{1,t})} \Bigg)$$

$$- \underbrace{ \sigma_n(\xi) \int_{\mathbb{R}^2} \frac{e^{-2it(\xi-\xi_1)(\xi-\xi_3)}}{-2i(\xi-\xi_1)(\xi-\xi_3)} \partial_t \Big(\hat{v}_{n_1}(\xi_1)\hat{\bar{v}}_{n_2}(\xi-\xi_1-\xi_3)\hat{v}_{n_3}(\xi_3) \Big) \, d\xi_1 d\xi_3 }_{=:\mathcal{F}(T_{I,n}^{1,t})},$$

$$\overbrace{\partial_t\Big(\sigma_n(\xi)\,\frac{e^{-2it(\xi-n_1)(n_1-n_2)}}{-2i(\xi-n_1)(n_1-n_2)}\,w_{n_1}\bar{w}_{n_2}\hat{v}_{n_3}(\xi-n_1+n_2)\Big)}^{=:\mathcal{F}(\tilde{Q}_{II,n}^{1,t})}$$

$$(39)$$

$$-\,\overbrace{\sigma_n(\xi)\,\frac{e^{-2it(\xi-n_1)(n_1-n_2)}}{-2i(\xi-n_1)(n_1-n_2)}\,\partial_t\Big(w_{n_1}\bar{w}_{n_2}\hat{v}_{n_3}(\xi-n_1+n_2)\Big)}^{=:\mathcal{F}(T_{II,n}^{1,t})},$$

$$\overbrace{\partial_t\Big(\sigma_n(\xi)\,\frac{e^{-2it(\xi-n_1)(\xi-n_3)}}{-2i(\xi-n_1)(\xi-n_3)}\,w_{n_1}\hat{\bar{v}}_{n_2}(\xi-n_1-n_3)w_{n_3}\Big)}^{=:\mathcal{F}(\tilde{Q}_{III,n}^{1,t})}$$

$$(40)$$

$$-\,\overbrace{\sigma_n(\xi)\,\frac{e^{-2it(\xi-n_1)(\xi-n_3)}}{-2i(\xi-n_1)(\xi-n_3)}\,\partial_t\Big(w_{n_1}\hat{\bar{v}}_{n_2}(\xi-n_1-n_3)w_{n_3}\Big)}^{=:\mathcal{F}(T_{III,n}^{1,t})}$$

$$\overbrace{\partial_t\Big(\sigma_n(\xi)w_{n_3}\,\int_{\mathbb{R}}\frac{e^{-2it(\xi-n_3)(\xi-\xi_1)}}{-2i(\xi-n_3)(\xi-\xi_1)}\,\hat{v}_{n_1}(\xi_1)\hat{\bar{v}}_{n_2}(\xi-\xi_1-n_3)d\xi_1\Big)}^{=:\mathcal{F}(\tilde{Q}_{IV.n}^{1,t})}$$

$$(41)$$

$$-\,\overbrace{\sigma_n(\xi)\,\int_{\mathbb{R}}\frac{e^{-2it(\xi-n_3)(\xi-\xi_1)}}{-2i(\xi-n_3)(\xi-\xi_1)}\,\partial_t\Big(\hat{v}_{n_1}(\xi_1)\hat{\bar{v}}_{n_2}(\xi-\xi_1-n_3)w_{n_3}\Big)d\xi_1}^{=:\mathcal{F}(T_{IV,n}^{1,t})}$$

and

$$\overbrace{\partial_t\Big(\sigma_n(\xi)\bar{w}_{n_2}\,\int_{\mathbb{R}}\frac{e^{-2it(\xi-\xi_1)(\xi_1-n_2)}}{-2i(\xi-\xi_1)(\xi_1-n_2)}\,\hat{v}_{n_1}(\xi_1)\hat{v}_{n_3}(\xi-\xi_1+n_2)d\xi_1\Big)}^{=:\mathcal{F}(\tilde{Q}_{V,n}^{1,t})}-$$

$$(42)$$

$$\overbrace{\sigma_n(\xi)\,\int_{\mathbb{R}}\frac{e^{-2it(\xi-\xi_1)(\xi_1-n_2)}}{-2i(\xi-\xi_1)(\xi_1-n_2)}\,\partial_t\Big(\hat{v}_{n_1}(\xi_1)\bar{w}_{n_2}\hat{v}_{n_3}(\xi-\xi_1+n_2)\Big)d\xi_1}^{\mathcal{F}(T_{V,n}^{1,t})}\,.$$

This allows us to express

$$N_{12}^t(v) = \sum_{A_N(n)} \left(Q_{I,n}^{1,t} + Q_{II,n}^{1,t} + Q_{III,n}^{1,t} + Q_{IV,n}^{1,t} + Q_{V,n}^{1,t} \right)$$

$$\overbrace{= \partial_t \Big(\sum_{A_N(n)} \left(\tilde{Q}_{I,n}^{1,t} + \tilde{Q}_{II,n}^{1,t} + \tilde{Q}_{III,n}^{1,t} + \tilde{Q}_{IV,n}^{1,t} + \tilde{Q}_{V,n}^{1,t} \right) \Big)}^{=:N_{21}^t(v)}$$

(43)

$$+ \overbrace{\sum_{A_N(n)} \left(T_{I,n}^{1,t} + T_{II,n}^{1,t} + T_{III,n}^{1,t} + T_{IV,n}^{1,t} + T_{V,n}^{1,t} \right)}^{=:N_{22}^t(v)}.$$

At this point let us also define the operators

$$(44) \qquad \mathcal{F}(R_{I,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}))(\xi) = \sigma_n(\xi) \int_{\mathbb{R}^2} \frac{\hat{v}_{n_1}(\xi_1) \hat{\bar{v}}_{n_2}(\xi - \xi_1 - \xi_3) \hat{v}_{n_3}(\xi_3)}{(\xi - \xi_1)(\xi - \xi_3)} \, d\xi_1 d\xi_3 \,,$$

$$(45) \qquad \mathcal{F}(R_{II,n}^{1,t}(w_{n_1}, \bar{w}_{n_2}, v_{n_3}))(\xi) = \sigma_n(\xi) \frac{w_{n_1} \bar{w}_{n_2} \hat{v}_{n_3}(\xi - n_1 + n_2)}{(\xi - n_1)(n_1 - n_2)} \,,$$

$$(46) \qquad \mathcal{F}(R_{III,n}^{1,t}(w_{n_1}, \bar{v}_{n_2}, w_{n_3}))(\xi) = \sigma_n(\xi) \frac{w_{n_1} \hat{\bar{v}}_{n_2}(\xi - n_1 - n_3) w_{n_3}}{(\xi - n_1)(\xi - n_3)} \,,$$

$$(47) \qquad \mathcal{F}(R_{IV,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, w_{n_3}))(\xi) = \sigma_n(\xi) w_{n_3} \int_{\mathbb{R}} \frac{\hat{v}_{n_1}(\xi_1) \hat{\bar{v}}_{n_2}(\xi - \xi_1 - n_3)}{(\xi - n_3)(\xi - \xi_1)} \, d\xi_1 \,,$$

$$(48) \qquad \mathcal{F}(R_{V,n}^{1,t}(v_{n_1}, \bar{w}_{n_2}, v_{n_3}))(\xi) = \sigma_n(\xi) \bar{w}_{n_2} \int_{\mathbb{R}} \frac{\hat{v}_{n_1}(\xi_1) \hat{v}_{n_3}(\xi - \xi_1 + n_2)}{(\xi - \xi_1)(\xi_1 - n_2)} \, d\xi_1 \,,$$

and observe that, if we let

$$(49) \qquad \hat{V}_n = e^{it\xi^2} \hat{v}_n, \ W_n = e^{itn^2} w_n \,,$$

then

$$(50) \qquad \mathcal{F}(\tilde{Q}_{I,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}))(\xi) = e^{-it\xi^2} \mathcal{F}(R_{I,n}^{1,t}(V_{n_1}, \bar{V}_{n_2}, V_{n_3}))(\xi) \,,$$

$$(51) \qquad \mathcal{F}(\tilde{Q}_{II,n}^{1,t}(w_{n_1}, \bar{w}_{n_2}, v_{n_3}))(\xi) = e^{-it\xi^2} \mathcal{F}(R_{II,n}^{1,t}(W_{n_1}, \bar{W}_{n_2}, V_{n_3}))(\xi) \,,$$

$$(52) \qquad \mathcal{F}(\tilde{Q}_{III,n}^{1,t}(w_{n_1}, \bar{v}_{n_2}, w_{n_3}))(\xi) = e^{-it\xi^2} \mathcal{F}(R_{III,n}^{1,t}(W_{n_1}, \bar{V}_{n_2}, W_{n_3}))(\xi) \,,$$

$$(53) \qquad \mathcal{F}(\tilde{Q}_{IV,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, w_{n_3}))(\xi) = e^{-it\xi^2} \mathcal{F}(R_{IV,n}^{1,t}(V_{n_1}, \bar{V}_{n_2}, W_{n_3}))(\xi)$$

$$(54) \qquad \mathcal{F}(\tilde{Q}_{V,n}^{1,t}(v_{n_1}, \bar{w}_{n_2}, v_{n_3}))(\xi) = e^{-it\xi^2} \mathcal{F}(R_{V,n}^{1,t}(V_{n_1}, \bar{W}_{n_2}, V_{n_3}))(\xi).$$

Also notice that, writing out the Fourier transforms of the functions inside the integral of (44)), it is not difficult to see

$$(55) \qquad R_{I,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3})(x) = \int_{\mathbb{R}^3} K_n^{(1)}(x, x_1, y, x_3) v_{n_1}(x) \bar{v}_{n_2}(y) v_{n_3}(x_3) \, dx_1 dy dx_3,$$

where

$$K_n^{(1)}(x, x_1, y, x_3) = \int_{\mathbb{R}^3} e^{i\xi_1(x-x_1)+i\eta(x-y)+i\xi_3(x-x_3)} \frac{\sigma_n(\xi_1 + \eta + \xi_3)}{(\eta + \xi_1)(\eta + \xi_3)} \, d\xi_1 d\eta d\xi_3 =$$

$$\mathcal{F}^{-1}\rho_n^{(1)}(x - x_1, x - y, x - x_3)$$

and

$$\rho_n^{(1)}(\xi_1, \eta, \xi_3) = \frac{\sigma_n(\xi_1 + \eta + \xi_3)}{(\eta + \xi_1)(\eta + \xi_3)}.$$

Remark 16. The operators $\tilde{Q}_{I,n}^{1,t}$ and $R_{I,n}^{1,t}$ are the same as the operators $\tilde{Q}_n^{1,t}$ and $R_n^{1,t}$ studied in [3, Lemma 12] and [13, Lemma 12]. Also notice that for $\hat{v}_{n_2} = w_{n_2}\delta_{n_2}$ and $\hat{v}_{n_1} = w_{n_1}\delta_{n_1}$ we have $R_{I,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3}) = R_{II,n}^{1,t}(w_{n_1}, \bar{w}_{n_2}, v_{n_3})$. Similar relations hold between $R_{I,n}^{1,t}$ and the remaining operators $R_{III,n}^{1,t}, R_{IV,n}^{1,t}$ and $R_{V,n}^{1,t}$.

Lemma 17. *For fixed* n, n_1, n_2, n_3, *the multilinear operators defined in* (44)–(48) *are bounded by*

$$\|R_{I,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3})\|_{L^2(\mathbb{R})} \lesssim \frac{\|v_{n_1}\|_2\|v_{n_2}\|_2\|v_{n_3}\|_2}{|n - n_1||n - n_3|},$$

$$\|R_{II,n}^{1,t}(w_{n_1}, \bar{w}_{n_2}, v_{n_3})\|_{L^2(\mathbb{R})} \lesssim \frac{|w_{n_1}||w_{n_2}|\|v_{n_3}\|_2}{|n - n_1||n - n_3|},$$

$$\|R_{III,n}^{1,t}(w_{n_1}, \bar{v}_{n_2}, w_{n_3})\|_{L^2(\mathbb{R})} \lesssim \frac{|w_{n_1}|\|v_{n_2}\|_2|w_{n_3}|}{|n - n_1||n - n_3|},$$

$$\|R_{IV,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, w_{n_3})\|_{L^2(\mathbb{R})} \lesssim \frac{\|v_{n_1}\|_2\|v_{n_2}\|_2|w_{n_3}|}{|n - n_1||n - n_3|},$$

and

$$\|R_{V,n}^{1,t}(v_{n_1}, \bar{w}_{n_2}, v_{n_3})\|_{L^2(\mathbb{R})} \lesssim \frac{\|v_{n_1}\|_2|w_{n_2}|\|v_{n_3}\|_2}{|n - n_1||n - n_3|}.$$

where the implicit constants do not depend on n, n_1, n_2, n_3.

Proof. As mentioned in Remark 16 the operator $R_{I,n}^{1,t}$ was estimated in [3] and [13].

For $R_{II,n}^{1,t}, R_{III,n}^{1,t}$ the estimate is obvious since $\xi \in \text{supp}(\sigma_n)$, otherwise the integrand is zero.

For $R_{IV,n}^{1,t}, R_{V,n}^{1,t}$ it suffices to estimate only $R_{IV,n}^{1,t}$ since for $R_{V,n}^{1,t}$ similar considerations apply. To bound $\|R_{IV,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, w_{n_3})\|_{L^2(\mathbb{R})}$ let $g \in L^2(\mathbb{R})$, $I_{n_1} = \text{supp}(\hat{v}_{n_1})$, $I_{n_2} = \text{supp}(\hat{\bar{v}}_{n_2})$,

and consider the duality pairing

$$\langle g, R_{IV,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, w_{n_3}) \rangle_{L^2(\mathbb{R})} = \Big| \int_{\mathbb{R}^2} \hat{g}(\xi) \, \sigma_n(\xi) \, \frac{w_{n_3} \hat{v}_{n_1}(\xi_1) \hat{\bar{v}}_{n_2}(\xi - \xi_1 - n_3)}{(\xi - n_3)(\xi - \xi_1)} \, d\xi_1 d\xi \Big|$$

$$= |w_{n_3}| \Big| \int_{\mathbb{R}^2} \hat{g}(\xi_1 + \eta + n_3) \, \sigma_n(\xi_1 + \eta + n_3) \, \frac{\hat{v}_{n_1}(\xi_1) \hat{\bar{v}}_{n_2}(\eta)}{(\eta + \xi_1)(\eta + n_3)} \, d\xi_1 d\eta \Big|$$

$$\lesssim \frac{\|\sigma_n\|_\infty |w_{n_3}|}{|n - n_1||n - n_3|} \int_{I_{n_1}} \int_{I_{n_2}} |\hat{g}(\xi_1 + \eta + n_3)||\hat{v}_{n_1}(\xi_1)||\hat{\bar{v}}_{n_2}(\eta)| \, d\xi_1 d\eta$$

$$\lesssim \frac{|w_{n_3}| \|v_{n_1}\|_2 \|v_{n_2}\|_2}{|n - n_1||n - n_3|} \|g\|_2 \, |I_{n_1}|^{\frac{1}{2}},$$

where we used that $\xi_1 \in I_{n_1}, -\eta \in I_{n_2}, \xi \in \mathrm{supp}(\sigma_n)$ and Hölder's inequality. $\qquad \square$

Remark 18. Notice that the same proof implies the following bounds

$$\|Q_{I,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3})\|_{L^2(\mathbb{R})} \lesssim \|v_{n_1}\|_2 \|v_{n_2}\|_2 \|v_{n_3}\|_2$$

$$\|Q_{II,n}^{1,t}(w_{n_1}, \bar{w}_{n_2}, v_{n_3})\|_{L^2(\mathbb{R})} \lesssim |w_{n_1}||w_{n_2}|\|v_{n_3}\|_2$$

$$\|Q_{III,n}^{1,t}(w_{n_1}, \bar{v}_{n_2}, w_{n_3})\|_{L^2(\mathbb{R})} \lesssim |w_{n_1}|\|v_{n_2}\|_2|w_{n_3}|$$

$$\|Q_{IV,n}^{1,t}(v_{n_1}, \bar{v}_{n_2}, w_{n_3})\|_{L^2(\mathbb{R})} \lesssim \|v_{n_1}\|_2 \|v_{n_2}\|_2|w_{n_3}|$$

$$\|Q_{V,n}^{1,t}(v_{n_1}, \bar{w}_{n_2}, v_{n_3})\|_{L^2(\mathbb{R})} \lesssim \|v_{n_1}\|_2|w_{n_2}|\|v_{n_3}\|_2,$$

which will be used later in Lemmata 23 and 25.

For the N_{21}^t operator the following bound holds

Lemma 19.

$$\|N_{21}^t(v)\|_{l^2(\mathbb{Z})L^2(\mathbb{R})} \lesssim N^{-\frac{1}{2}+}(\|v\|_{L^2(\mathbb{R})}^3 + \|w\|_{L^2(\mathbb{T})}\|v\|_{L^2(\mathbb{R})}^2 + \|w\|_{L^2(\mathbb{T})}^2\|v\|_{L^2(\mathbb{R})})$$

and

$$\|N_{21}^t(v) - N_{21}^t(u)\|_{l^2(\mathbb{Z})L^2(\mathbb{R})} \lesssim$$

$$N^{-\frac{1}{2}+}(\|v\|_{L^2(\mathbb{R})}^2 + \|u\|_{L^2(\mathbb{R})}^2 + \|w\|_{L^2(\mathbb{T})}(\|v\|_{L^2(\mathbb{R})} + \|u\|_{L^2(\mathbb{R})}) + \|w\|_{L^2(\mathbb{T})}^2)\|v - u\|_{L^2(\mathbb{R})}.$$

Proof. The sum

$$\sum_{A_N(n)^c} \tilde{Q}_{I,n}^{1,t}$$

was estimated in [13] Lemma 14 giving an upper bound of the form $N^{-\frac{1}{2}+}\|v\|_2^3$.

For the sum that contains $\tilde{Q}_{II,n}^{1,t}, \tilde{Q}_{III,n}^{1,t}$ it suffices to estimate

$$\sum_{A_N(n)^c} \Big\| \tilde{Q}_{II,n}^{1,t}(w_{n_1}, \bar{w}_{n_2}, v_{n_3}) \Big\|_{L^2(\mathbb{R})} = \sum_{A_N(n)^c} \Big\| R_{II,n}^{1,t}(W_{n_1}, \bar{W}_{n_2}, V_{n_3}) \Big\|_{L^2(\mathbb{R})},$$

where we used (51)), (14)) and (49)). By Lemma 17 and Hölder's inequality we obtain the upper bound

$$\sum_{A_N(n)^c} \frac{|W_{n_1}||W_{n_2}||V_{n_3}||_2}{|n-n_1||n-n_3|} \leq \Big(\sum_{A_N(n)^c} \frac{1}{|n-n_1|^2|n-n_3|^2} \Big)^{\frac{1}{2}} \Big(\sum_{A_N(n)^c} |W_{n_1}|^2|W_{n_2}|^2\|V_{n_3}\|_2^2 \Big)^{\frac{1}{2}}.$$

The first sum is estimated by $N^{-\frac{1}{2}+}$ with the use of (15)) and then by taking the $l^2(\mathbb{Z})$ norm and applying Young's inequality in $l^1(\mathbb{Z})$ we arrive at

$$N^{-\frac{1}{2}+}\Big(\sum_{n\in\mathbb{Z}} \sum_{A_N(n)^c} |W_{n_1}|^2|W_{n_2}|^2\|V_{n_3}\|_2^2 \Big)^{\frac{1}{2}} \leq N^{-\frac{1}{2}+}\|W\|_{L^2(\mathbb{T})}^2\|V\|_{L^2(\mathbb{R})} =$$

$$N^{-\frac{1}{2}+}\|w\|_{L^2(\mathbb{T})}^2\|v\|_{L^2(\mathbb{R})},$$

where we also used (14)).

For the sum that contains $\tilde{Q}_{IV,n}^{1,t}$ and $\tilde{Q}_{V,n}^{1,t}$ we use again Lemma 17 and a similar argument as above, we leave the details to the reader . $\qquad\square$

In order to use a similar strategy to bound the operator N_{22}^t, the last term in equation (43), we need to use equation (23)) for the terms where $\partial_t(w_n)$ appears and (33)) for the terms where $\partial_t(v_n)$ appears. Because of the nonlinearity $G(w,v)$ there will be 51 new operators in total. For example, the summand

$$\sum_{A_N(n)^c} \sigma_n(\xi) \int_{\mathbb{R}^2} \frac{e^{-2it(\xi-\xi_1)(\xi-\xi_3)}}{(\xi-\xi_1)(\xi-\xi_3)} \partial_t(\hat{v}_{n_1}(\xi_1))\hat{\tilde{v}}_{n_2}(\xi-\xi_1-\xi_3)\hat{v}_{n_3}(\xi_3) \, d\xi_1 d\xi_3$$

equals

$$\sum_{A_N(n)^c} \sigma_n(\xi) \int_{\mathbb{R}^2} \frac{e^{-2it(\xi-\xi_1)(\xi-\xi_3)}}{(\xi-\xi_1)(\xi-\xi_3)} \Big(\mathcal{F}(R_2^t(v)(n_1))(\xi_1)-\mathcal{F}(R_1^t(v)(n_1))(\xi_1) \Big)\hat{\tilde{v}}_{n_2}(\xi-\xi_1-\xi_3)\hat{v}_{n_3}(\xi_3) \, d\xi_1 d\xi_3$$

$$+ \sum_{A_N(n)^c} \sigma_n(\xi) \int_{\mathbb{R}^2} \frac{e^{-2it(\xi-\xi_1)(\xi-\xi_3)}}{(\xi-\xi_1)(\xi-\xi_3)} \mathcal{F}(N_1^t(v)(n_1))(\xi_1)\hat{\tilde{v}}_{n_2}(\xi-\xi_1-\xi_3)\hat{v}_{n_3}(\xi_3) \, d\xi_1 d\xi_3$$

the summand

$$\sum_{A_N(n)^c} \sigma_n(\xi) \frac{e^{-2it(\xi-n_1)(n_1-n_2)}}{(\xi-n_1)(n_1-n_2)} \partial_t(w_{n_1})\bar{w}_{n_2}\hat{v}_{n_3}(\xi-n_1+n_2)$$

equals

$$\sum_{A_N(n)^c} \sigma_n(\xi) \frac{e^{-2it(\xi-n_1)(n_1-n_2)}}{(\xi-n_1)(n_1-n_2)} \Big(\mathscr{R}_2^t(w)(n_1) - \mathscr{R}_1^t(w)(n_1) \Big)\bar{w}_{n_2}\hat{v}_{n_3}(\xi-n_1+n_2)$$

$$+ \sum_{A_N(n)^c} \sigma_n(\xi) \frac{e^{-2it(\xi-n_1)(n_1-n_2)}}{(\xi-n_1)(n_1-n_2)} \mathscr{N}_1^t(w)(n_1)\bar{w}_{n_2}\hat{v}_{n_3}(\xi-n_1+n_2),$$

and the summand

$$\sum_{A_N(n)^c} \sigma_n(\xi) \, \frac{e^{-2it(\xi-n_1)(n_1-n_2)}}{(\xi-n_1)(n_1-n_2)} \, w_{n_1} \bar{w}_{n_2} \partial_t(\hat{v}_{n_3}(\xi-n_1+n_2))$$

equals

$$\sum_{A_N(n)^c} \sigma_n(\xi) \frac{e^{-2it(\xi-n_1)(n_1-n_2)}}{(\xi-n_1)(n_1-n_2)} w_{n_1} \bar{w}_{n_2} \Big(\mathcal{F}(R_2^t(v)(n_3))(\xi-n_1+n_2) - \mathcal{F}(R_1^t(v)(n_3))(\xi-n_1+n_2) \Big)$$

$$+ \sum_{A_N(n)^c} \sigma_n(\xi) \, \frac{e^{-2it(\xi-n_1)(n_1-n_2)}}{(\xi-n_1)(n_1-n_2)} \, w_{n_1} \bar{w}_{n_2} \mathcal{F}(N_1^t(v)(n_3))(\xi-n_1+n_2) \,.$$

All summands that contain the resonant operators $\mathscr{R}_2^t(w), \mathscr{R}_1^t(w), \mathcal{F}(R_2^t(v)), \mathcal{F}(R_1^t(v))$ are good in the sense that they are controllable and all summands that contain the non-resonant operators $\mathscr{N}_1^t(w), \mathcal{F}(N_1^t(v))$ need to be decomposed further into "small" frequencies which give good operators and "big" frequencies using differentiation by parts.

In order to be able to consistently write all these summands in a closed form we need the tree notation similarly as it was introduced in [7], but with some modifications.

4. COLORED TREES AND THE INFINITE ITERATION PROCESS

A tree T is a finite, partially ordered set with the following properties:

- For any $a_1, a_2, a_3, a_4 \in T$ if $a_4 \leq a_2 \leq a_1$ and $a_4 \leq a_3 \leq a_1$ then $a_2 \leq a_3$ or $a_3 \leq a_2$.
- There exists a maximum element $r \in T$, that is $a \leq r$ for all $a \in T$ which is called the **root**.

We call the elements of T the **nodes** of the tree and in this content we will say that $b \in T$ is a **child** of $a \in T$ (or equivalently, that a is the **parent** of b) if $b \leq a, b \neq a$ and for all $c \in T$ such that $b \leq c \leq a$ we have either $b = c$ or $c = a$.

A node $a \in T$ is called **terminal** if it has no children. A **nonterminal** node $a \in T$ is a node with exactly 3 children a_1, the left child, a_2, the middle child, and a_3, the right child. We define the sets

(56) $$T^0 = \{\text{all nonterminal nodes}\},$$
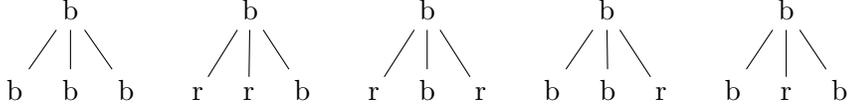
and

(57) $$T^\infty = \{\text{all terminal nodes}\}.$$

Obviously, $T = T^0 \cup T^\infty$, $T^0 \cap T^\infty = \emptyset$ and if $|T^0| = j \in \mathbb{Z}_+$ we have $|T| = 3j+1$ and $|T^\infty| = 2j+1$. We denote the collection of trees with j parental nodes by

(58) $$T(j) = \{T \text{ is a tree with } |T| = 3j+1\}.$$

So far, the notation agrees with the tree notation from [7]. In addition, we color the trees by assigning a specific color, **black** or **red**, to each one of the nodes of such a tree. Let us describe the procedure: The first generation of **colored** trees, $C(1)$, consists of the

following 5 trees

$$
\begin{array}{ccccc}
b & b & b & b & b \\
\diagup\,|\,\diagdown & \diagup\,|\,\diagdown & \diagup\,|\,\diagdown & \diagup\,|\,\diagdown & \diagup\,|\,\diagdown \\
b\;\;b\;\;b & r\;\;r\;\;b & r\;\;b\;\;r & b\;\;b\;\;r & b\;\;r\;\;b
\end{array}
$$

These trees describe all possible "patterns" of the non-linearity $G(w,v)$, namely all combinations of $|v|^2 v, |w|^2 v, w^2\bar{v}, |v|^2 w, v^2\bar{w}$ where v is black and w is red. There is also the red tree, which is not considered to belong to any generation, that plays an important role in the construction of the next generations and is simply given by

$$
\begin{array}{c}
r \\
\diagup\,|\,\diagdown \\
r\;\;r\;\;r
\end{array}
$$
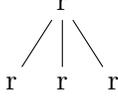
Next we assume that the Jth generation of colored trees, say $C(J)$, has been constructed, and we describe how the new generation $C(J+1)$ arises. Thus, let T_k^J be one of the trees of the $C(J)$ family. We look at each of the $2J+1$ terminal nodes of T_k^J:

- If one of these nodes is red then it gives rise to one new tree where this red node gave birth to three new red nodes. In other words, if a terminal node is red then attach the red tree to the tree T_k^J at the red node.
- If one of these nodes is black then it gives rise to five new trees where each one of them is born by attaching one of the trees of the first generation to the tree T_k^J at the black node.

We will denote by

$$
(59) \qquad\qquad\qquad N(J) := \operatorname{card}(C(J)).
$$

Moreover, for a tree $T = T_k^J \in C(J)$ let

$$
(60)
$$
$$
b_k^J = \text{number of black terminal nodes of } T_k^J, \;\; r_k^J = \text{number of red terminal nodes of } T_k^J
$$

and denote by

$$
(61) \qquad\qquad B_k^J = \{a \in T^\infty : a \text{ is black}\}, R_k^J = \{a \in T^\infty : a \text{ is red}\}.
$$

Obviously we have the relations $B_k^J \cup R_k^J = T^\infty, B_k^J \cap R_k^J = \emptyset, \operatorname{card}(B_k^J) = b_k^J, \operatorname{card}(R_k^J) = r_k^J$ and

$$
(62) \qquad b_k^J + r_k^J = 2J+1, \;\; \max_{1 \le k \le N(J)} b_k^J = 2J+1, \text{ and } \max_{1 \le k \le N(J)} r_k^J = 2J.
$$

The last two are true because there is at least one tree T_1^J that consists of only black nodes. Therefore, for such tree we have $b_1^J = 2J+1, r_1^J = 0$, and there is also at least one tree T_2^J with only one black terminal node, which implies $b_2^J = 1, r_2^J = 2J$. Also observe that by our construction there is no tree with only red terminal nodes.

We also define the quantities

$$
(63) \qquad\qquad\qquad b_J = \sum_{k=1}^{N(J)} b_k^J, \;\; r_J = \sum_{k=1}^{N(J)} r_k^J,
$$

which respectively give the total number of black and red terminal nodes of the colored family $C(J)$. Notice that the number of colored trees of the next generation $C(J+1)$ is given by the formula

$$N(J+1) = 5b_J + r_J \,. \tag{64}$$

This is because each one of the black nodes gives rise to 5 new trees and each one of the red nodes gives rise to just 1 new tree.

Knowing the numbers b_k^J, r_k^J for each tree $T_k^J \in C(J), 1 \le k \le N(J)$ allows us to calculate the precise numbers b_{J+1} and r_{J+1} of the next generation by using the formulas

$$b_{J+1} = \sum_{k=1}^{N(J)} \left((5b_k^J + 4)b_k^J + r_k^J b_k^J \right) \tag{65}$$

$$r_{J+1} = 6b_J + 2r_J + \sum_{k=1}^{N(J)} \left(5b_k^J r_k^J + (r_k^J)^2 \right). \tag{66}$$

Indeed, each b_k^J gives rise to $9 + 5(b_k^J - 1)$ new black nodes and each red node r_k^J leaves the number of black nodes the same as before. Also, each black node b_k^J gives rise to $6 + 5r_k^J$ new red nodes and each red node r_k^J gives rise to $3 + r_k^J - 1$ new red nodes.

For our calculations it is important to know how fast the number $N(J)$ grows as J approaches infinity. Since we have to count trees, one expects a factorial growth and coloring the trees does not change this significantly:

Lemma 20. *For every $J \in \mathbb{N}$*

$$N(J) \le \frac{10^J \, \Gamma(J + \frac{1}{2})}{\sqrt{\pi}},$$

where Γ is the Gamma function.

Proof. By (64)) and (62)) we obtain

$$N(J+1) = 4b_J + N(J)(2J+1) \le 4(2J+1)N(J) + N(J)(2J+1) = 5(2J+1)N(J).$$

Let us define a sequence $A(J)$ by the recurrence relation $A(J+1) = 5(2J+1)A(J), A(1) = 5$. This can be solved explicitly in terms of the Gamma function using the equality $\Gamma(x+1) = x\Gamma(x), x > 0$ and gives the result

$$A(J) = \frac{10^J \, \Gamma(J + \frac{1}{2})}{\sqrt{\pi}}.$$

An easy induction argument shows that for all $J \in \mathbb{N}$ we have $N(J) \le A(J)$ which finishes the proof. $\square$

Using the equality

$$\Gamma(J + \frac{1}{2}) = \frac{(2J-1)!!}{2^J} \, \sqrt{\pi}, \tag{67}$$

$J \in \mathbb{N}$, where the double factorial $(2J-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2J-1)$, we obtain the bound

$$(68) \qquad\qquad N(J) \le 5^J (2J-1)!!,$$

for all $J \in \mathbb{N}$.

Given a colored tree $T = T_k^J$ of the $C(J)$ family we define an index function $n : T_k^J \to \mathbb{Z}$ such that

- If a is a black node in T^0 then $n_a \approx n_{a_1} - n_{a_2} + n_{a_3}$ (see (24)) where a_1, a_2, a_3 are the children of a.
- If a is a red node in T^0 then $n_a = n_{a_1} - n_{a_2} + n_{a_3}$, where a_1, a_2, a_3 are the children of a,
- $n_a \not\approx n_{a_1}$ and $n_a \not\approx n_{a_3}$ for all black nodes $a \in T^0$ and $n_a \neq n_{a_1}$ and $n_a \neq n_{a_3}$ for all red nodes $a \in T^0$.
- $|\mu_1| := 2|n_r - n_{r_1}||n_r - n_{r_3}| > N$, where r is the root of T_k^J.

We denote the collection of all such index functions by $\mathcal{R}(T_k^J)$.

Similar to what was done in [7], given a colored tree T in $C(J)$ and an index function $n \in \mathcal{R}(T)$, we need to keep track of the generations of frequencies. Consider the very first tree T_1, that is, the root r and its children r_1, r_2, r_3. We define the first generation of frequencies by

$$(n^{(1)}, n_1^{(1)}, n_2^{(1)}, n_3^{(1)}) := (n_r, n_{r_1}, n_{r_2}, n_{r_3}).$$

From the definition of the index function we have

$$n^{(1)} \approx n_1^{(1)} - n_2^{(1)} + n_3^{(1)}, \ n_1^{(1)} \not\approx n^{(1)} \not\approx n_3^{(1)},$$

since the root node is colored black. The tree T_2 of the second generation is obtained from T_1 by changing one of its terminal nodes $a = r_k \in T_1^\infty$ for some $k = 1, 2, 3$ into a nonterminal node. Then, the second generation of frequencies is defined by

$$(n^{(2)}, n_1^{(2)}, n_2^{(2)}, n_3^{(2)}) := (n_a, n_{a_1}, n_{a_2}, n_{a_3}).$$

Thus we have $n^{(2)} = n_k^{(1)}$ for some $k = 1, 2, 3$ and from the definition of the index function we get

$$n^{(2)} \approx n_1^{(2)} - n_2^{(2)} + n_3^{(2)}, \ n_1^{(2)} \not\approx n^{(2)} \not\approx n_3^{(2)}$$

if $n_k^{(1)}$ is black or

$$n^{(2)} = n_1^{(2)} - n_2^{(2)} + n_3^{(2)}, \ n_1^{(2)} \neq n^{(2)} \neq n_3^{(2)}$$

if $n_k^{(1)}$ is red. After $j-1$ steps, the tree T_j of the jth generation is obtained from T_{j-1} by changing one of its terminal nodes $a \in T_{j-1}^\infty$ into a nonterminal node. Then, the jth generation frequencies are defined as

$$(n^{(j)}, n_1^{(j)}, n_2^{(j)}, n_3^{(j)}) := (n_a, n_{a_1}, n_{a_2}, n_{a_3})$$

and we have $n^{(j)} = n_k^{(m)}(= n_a)$ for some $m = 1, 2, \ldots, j-1$ and $k = 1, 2, 3$, since this corresponds to the frequency of some terminal node in T_{j-1}. In addition, from the definition of the index function we have

$$n^{(j)} \approx n_1^{(j)} - n_2^{(j)} + n_3^{(j)}, \ n_1^{(j)} \not\approx n^{(j)} \not\approx n_3^{(j)}$$

if $n_k^{(m)}$ is black or

$$n^{(j)} = n_1^{(j)} - n_2^{(j)} + n_3^{(j)}, \ n_1^{(j)} \neq n^{(j)} \neq n_3^{(j)}$$

if $n_k^{(m)}$ is red.

We use μ_j to denote the corresponding phase factor introduced at the jth generation. That is,

(69) $$\mu_j = 2(n^{(j)} - n_1^{(j)})(n^{(j)} - n_3^{(j)}),$$

and we also introduce the quantities

(70) $$\tilde{\mu}_J = \sum_{j=1}^{J} \mu_j, \ \hat{\mu}_J = \prod_{j=1}^{J} \tilde{\mu}_j.$$

We should keep in mind that every time we apply differentiation by parts and split the operators, we need to control the new frequencies that arise from this procedure. For this reason, we need to define the sets

(71) $$C_J := \{|\tilde{\mu}_{J+1}| \leq (2J+3)^3 |\tilde{\mu}_J|^{1-\frac{1}{100}}\} \cup \{|\tilde{\mu}_{J+1}| \leq (2J+3)^3 |\mu_1|^{1-\frac{1}{100}}\}.$$

Let us denote by T_α all the nodes of the tree T that are descendants of the node $\alpha \in T^0$, i.e. $T_\alpha = \{\beta \in T : \beta \leq \alpha, \ \beta \neq \alpha\}$.

We also need to define the **principal and final "signs" of a node** $a \in T$ which are functions from the tree T into the set $\{\pm 1\}$:

(72) $$\mathrm{psgn}(a) = \begin{cases} +1, & a \text{ is not the middle child of his father} \\ +1, & a = r, \text{ the root node} \\ -1, & a \text{ is the middle child of his father} \end{cases}$$

(73) $$\mathrm{fsgn}(a) = \begin{cases} +1, & \mathrm{psgn}(a) = +1 \text{ and } a \text{ has an even number of middle predecessors} \\ -1, & \mathrm{psgn}(a) = +1 \text{ and } a \text{ has an odd number of middle predecessors} \\ -1, & \mathrm{psgn}(a) = -1 \text{ and } a \text{ has an even number of middle predecessors} \\ +1, & \mathrm{psgn}(a) = -1 \text{ and } a \text{ has an odd number of middle predecessors,} \end{cases}$$

where the root node $r \in T$ is not considered a middle father.

Next we define two "prototype" operators in the following way. Suppose that $T \in T(J)$ (see (58))) is a tree of only black nodes. Let $\tilde{q}_{T,\mathbf{n}}^{J,t}$ and $R_{T,\mathbf{n}}^{J,t}$ be related as

(74) $$\mathcal{F}(\tilde{q}_{T,\mathbf{n}}^{J,t}(\{v_{n_\beta}\}_{\beta \in T^\infty}))(\xi) = e^{-it\xi^2}\mathcal{F}(R_{T,\mathbf{n}}^{J,t}(\{e^{-it\partial_x^2}v_{n_\beta}\}_{\beta \in T^\infty}))(\xi),$$

where the operator $R_{T,\mathbf{n}}^{J,t}$ acts on the functions $\{v_{n_\beta}\}_{\beta \in T^\infty}$ as

(75) $$R_{T,\mathbf{n}}^{J,t}(\{v_{n_\beta}\}_{\beta \in T^\infty})(x) = \int_{\mathbb{R}^{2J+1}} K_T^{(J)}(x, \{x_\beta\}_{\beta \in T^\infty}) \left[\otimes_{\beta \in T^\infty} v_{n_\beta}(x_\beta) \right] \prod_{\beta \in T^\infty} dx_\beta,$$

and the Kernel $K_{T,\mathbf{n}}^{(J)}$ is defined as

$$(76) \qquad K_{T,\mathbf{n}}^{(J)}(x, \{x_\beta\}_{\beta\in T^\infty}) = \mathcal{F}^{-1}(\rho_{T,\mathbf{n}}^{(J)})(\{x - x_\beta\}_{\beta\in T^\infty}),$$

where the formula for the function $\rho_{T,\mathbf{n}}^{(J)}$ with $(|T^\infty| = 2J + 1)$-variables, ξ_β, $\beta \in T^\infty$ is

$$(77) \qquad \rho_{T,\mathbf{n}}^{(J)}(\{\xi_\beta\}_{\beta\in T^\infty}) = \Big[\prod_{\alpha\in T^0} \sigma_{n_\alpha} \Big(\sum_{\beta\in T^\infty\cap T_\alpha} \mathrm{fsgn}(\beta)\, \xi_\beta \Big) \Big] \frac{1}{\hat{\mu}_T}.$$

We denote by

$$(78) \qquad \hat{\mu}_T = \prod_{\alpha\in T^0} \tilde{\mu}_\alpha, \ \tilde{\mu}_\alpha = \sum_{\beta\in T^0\backslash T_\alpha} \mu_\beta,$$

and for $\beta \in T^0$ we have

$$(79) \qquad \mu_\beta = 2(\xi_\beta - \xi_{\beta_1})(\xi_\beta - \xi_{\beta_3}),$$

where we impose the relation $\xi_\alpha = \xi_{\alpha_1} - \xi_{\alpha_2} + \xi_{\alpha_3}$ for every $\alpha \in T^0$ that appears in the calculations, until we reach the terminal nodes of T^∞. This is due to the fact that in the definition of the function $\rho_T^{J,t}$ we need the variables "ξ" to be assigned only at the terminal nodes of the tree T. We use the notation μ_β similarly to μ_j of equation (69), because this is the "continuous" version of the discrete case. In addition, the variables $\xi_{\alpha_1}, \xi_{\alpha_2}, \xi_{\alpha_3}$ that appear in expression (77) are supported in such a way that $\xi_{\alpha_1} \approx n_{\alpha_1}, \xi_{\alpha_2} \approx n_{\alpha_2}, \xi_{\alpha_3} \approx n_{\alpha_3}$, due to the support properties of the cut–off functions σ_{n_α}. Therefore, $|\hat{\mu}_T| \sim |\hat{\mu}_J|$.

Notice that if $\{\beta_1, \ldots, \beta_{2J+1}\} = T^\infty$, then we may rewrite (75) as

$$(80) \qquad R_{T,\mathbf{n}}^{J,t}(v_{n_{\beta_1}}, \ldots, v_{n_{\beta_{2J+1}}})(x) =$$

$$\int_{\mathbb{R}} e^{ix\xi} \Big(\int_{\mathbb{R}^{2J}} \rho_{T,\mathbf{n}}^{(J)}(\xi_{\beta_1}, \ldots, \xi_{\beta_{2J}}, \xi - \sum_{k=1}^{2J} \xi_{\beta_k}) \prod_{k=1}^{2J} \hat{v}_{n_{\beta_k}}(\xi_{\beta_k})\, \hat{v}_{n_{\beta_{2J+1}}}(\xi - \sum_{k=1}^{2J} \xi_{\beta_k}) \prod_{k=1}^{2J} d\xi_{\beta_k} \Big) d\xi$$

which implies

$$(81) \qquad \mathcal{F}(R_{T,\mathbf{n}}^{J,t}(v_{n_{\beta_1}}, \ldots, v_{n_{\beta_{2J+1}}}))(\xi) =$$

$$\int_{\mathbb{R}^{2J}} \rho_{T,\mathbf{n}}^{(J)}(\xi_{\beta_1}, \ldots, \xi_{\beta_{2J}}, \xi - \sum_{k=1}^{2J} \xi_{\beta_k}) \prod_{k=1}^{2J} \hat{v}_{n_{\beta_k}}(\xi_{\beta_k})\, \hat{v}_{n_{\beta_{2J+1}}}(\xi - \sum_{k=1}^{2J} \xi_{\beta_k}) \prod_{k=1}^{2J} d\xi_{\beta_k}.$$

Such an operator was studied in [3] Lemma 21 and in [13] Lemma 21.

Our goal is to define the operators $\tilde{q}_{T,\mathbf{n}}^{J,t}$ and $R_{T,\mathbf{n}}^{J,t}$ for any colored tree T_k^J of the $C(J)$ family. From (61) we know that $B_k^J \cup R_k^J = (T_k^J)^\infty$. If $R_k^J = \emptyset$ then the tree is black and the operators have already been defined by (81). Thus, assume $R_k^J = \{r_m\}_{m=1}^{r_k^J} \neq \emptyset$, $B_k^J = \{b_m\}_{m=1}^{b_k^J}$ and consider functions $\{v_{n_{b_m}}\}_{m=1}^{b_k^J}$ and Fourier coefficients $\{w_{n_{r_{\tilde{m}}}}\}_{\tilde{m}=1}^{r_k^J}$. Let $\{v_{n_{r_{\tilde{m}}}}\}$ be defined by

$$(82) \qquad \hat{v}_{n_{r_{\tilde{m}}}}(\xi) = w_{n_{r_{\tilde{m}}}} \delta_{n_{r_{\tilde{m}}}}(\xi),$$

for all $\tilde{m} \in \{1, \ldots, r_k^J\}$. Then the operator $R_{T_k^J, \mathbf{n}}^{J,t}$ is defined as

$$
(83) \qquad \mathcal{F}(R_{T_k^J, \mathbf{n}}^{J,t}(\{v_{n_{b_m}}\}_{m=1}^{b_k^J}, \{w_{n_{r_{\tilde{m}}}}\}_{\tilde{m}=1}^{r_k^J}))(\xi) =
$$

$$
\mathcal{F}(R_{T, \mathbf{n}}^{J,t}(\{v_{n_{b_m}}\}_{m=1}^{b_k^J}, \{v_{n_{r_{\tilde{m}}}}\}_{\tilde{m}=1}^{r_k^J}))(\xi)
$$

and

$$
(84) \qquad \mathcal{F}(\tilde{q}_{T_k^J, \mathbf{n}}^{J,t}(\{v_{n_{b_m}}\}_{m=1}^{b_k^J}, \{w_{n_{r_{\tilde{m}}}}\}_{\tilde{m}=1}^{r_k^J}))(\xi) =
$$

$$
e^{-it\xi^2} \mathcal{F}(R_{T_k^J, \mathbf{n}}^{J,t}(\{e^{-it\partial_x^2} v_{n_{b_m}}\}_{m=1}^{b_k^J}, \{e^{itn_{r_{\tilde{m}}}^2} w_{n_{r_{\tilde{m}}}}\}_{\tilde{m}=1}^{r_k^J}))(\xi).
$$

For these operators the following holds.

Lemma 21.

$$
\|R_{T_k^J, \mathbf{n}}^{J,t}(\{v_{n_{b_m}}\}_{m=1}^{b_k^J}, \{w_{n_{r_{\tilde{m}}}}\}_{\tilde{m}=1}^{r_k^J})\|_{L^2(\mathbb{R})} \lesssim \frac{\prod_{m=1}^{b_k^J} \|v_{n_{b_m}}\|_2 \; \prod_{\tilde{m}=1}^{r_k^J} |w_{n_{r_{\tilde{m}}}}|}{|\hat{\mu}_{T_k^J}|}
$$

Proof. The proof of the above bound is similar to the strategy of the proof of Lemma 17: a repeated use of duality and Hölder's inequality. We leave the details to the reader. □

Next, given a colored tree $T = T_k^J$ of the $C(J)$ family and $\alpha \in T^\infty$ we define the operators $\mathbf{R}_2^{t,\alpha} - \mathbf{R}_1^{t\alpha}, \mathbf{N}_1^{t,\alpha}$ by

$$
(85) \qquad \mathbf{R}_2^{t,\alpha} - \mathbf{R}_1^{t,\alpha} = \begin{cases} R_2^t - R_1^t & , \; \alpha \in B_k^J \\ \mathscr{R}_2^t - \mathscr{R}_1^t & , \; \alpha \in R_k^J \end{cases}, \qquad \mathbf{N}_1^{t,\alpha} := \begin{cases} N_1^t & , \; \alpha \in B_k^J \\ \mathscr{N}_1^t & , \; \alpha \in R_k^J \end{cases}.
$$

Next, for such a tree $T = T_k^J$, index function $\mathbf{n} \in \mathcal{R}(T)$, $\alpha \in T^\infty$ and set of functions $\{v_{n_{b_m}}\}_{m=1}^{b_k^J}, \{w_{n_{r_{\tilde{m}}}}\}_{\tilde{m}=1}^{r_k^J}$ we define the action of the operator $\mathbf{N}_1^{t,\alpha}$ onto the set of functions to be the same set as before but with the difference that we have substituted the function $f_{n_\alpha} := v_{n_\alpha} \chi_{B_k^J}(\alpha) + w_{n_\alpha} \chi_{R_k^J}(\alpha)$ with $\mathbf{N}_1^{t,\alpha}(f_{n_\alpha})$. Similarly, we define the action of the operator $\mathbf{R}_2^{t,\alpha} - \mathbf{R}_1^{t,\alpha}$ onto the set of functions $\{v_{n_{b_m}}\}_{m=1}^{b_k^J}, \{w_{n_{r_{\tilde{m}}}}\}_{\tilde{m}=1}^{r_k^J}$.

The operator of the Jth step, $J \geq 2$, that we want to estimate, is given by the formula

$$
(86) \qquad N_2^{(J)}(v)(n) := \sum_{T \in C(J-1)} \sum_{\alpha \in T^\infty} \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \tilde{q}_{T, \mathbf{n}}^{J-1,t}(\mathbf{N}_1^{t,\alpha}(\{v_{n_{b_m}}\}_{m=1}^{b_k^J}, \{w_{n_{r_{\tilde{m}}}}\}_{\tilde{m}=1}^{r_k^J})).
$$

Applying differentiation by parts on the Fourier side, keeping in mind that from the splitting procedure we are on the sets $A_N(n)^c, C_1^c, \ldots, C_{J-1}^c$, we obtain the expression

$$
(87) \qquad N_2^{(J)}(v)(n) = \partial_t(N_0^{(J+1)}(v)(n)) + N_r^{(J+1)}(v)(n) + N^{(J+1)}(v)(n),
$$

where

$$
(88) \qquad N_0^{(J+1)}(v)(n) := \sum_{T \in C(J)} \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \tilde{q}_{T, \mathbf{n}}^{J,t}(\{v_{n_{b_m}}\}_{m=1}^{b_k^J}, \{w_{n_{r_{\tilde{m}}}}\}_{\tilde{m}=1}^{r_k^J}),
$$

and

$$(89) \quad N_r^{(J+1)}(v)(n) := \sum_{T \in C(J)} \sum_{\alpha \in T^\infty} \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \tilde{q}_{T,\mathbf{n}}^{J,t}((\mathbf{R}_2^{t,\alpha} - \mathbf{R}_1^{t,\alpha})(\{v_{n_{b_m}}\}_{m=1}^{b_k^J}, \{w_{n_{r_{\tilde{m}}}}\}_{\tilde{m}=1}^{r_k^J})),$$

and

$$(90) \quad N^{(J+1)}(v)(n) := \sum_{T \in C(J)} \sum_{\alpha \in T^\infty} \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \tilde{q}_{T,\mathbf{n}}^{J,t}(\mathbf{N}_1^{t,\alpha}(\{v_{n_{b_m}}\}_{m=1}^{b_k^J}, \{w_{n_{r_{\tilde{m}}}}\}_{\tilde{m}=1}^{r_k^J})).$$

We also split the operator $N^{(J+1)}$ as the sum

$$(91) \qquad\qquad N^{(J+1)} = N_1^{(J+1)} + N_2^{(J+1)},$$

where $N_1^{(J+1)}$ is the restriction of $N^{(J+1)}$ onto C_J and $N_2^{(J+1)}$ onto C_J^c.

First we estimate the operators $N_0^{(J+1)}$ and $N_r^{(J+1)}$.

Lemma 22.

$$\|N_0^{(J+1)}(v)\|_{l^2(\mathbb{Z})L^2(\mathbb{R})} \lesssim N^{-\frac{J}{2}+\frac{(J-1)}{200}+}(\|v\|_{L^2(\mathbb{R})} + \|w\|_{L^2(\mathbb{T})})^{2J+1}$$

and

$$\|N_0^{(J+1)}(v) - N_0^{(J+1)}(u)\|_{l^2(\mathbb{Z})L^2(\mathbb{R})} \lesssim N^{-\frac{J}{2}+\frac{(J-1)}{200}+}(\|v\|_{L^2(\mathbb{R})} + \|u\|_{L^2(\mathbb{R})} + \|w\|_{L^2(\mathbb{T})})^{2J}\|v-u\|_{L^2(\mathbb{R})}.$$

$$\|N_r^{(J+1)}(v)\|_{l^2(\mathbb{Z})L^2(\mathbb{R})} \lesssim N^{-\frac{J}{2}+\frac{(J-1)}{200}+}(\|v\|_{L^2(\mathbb{R})} + \|w\|_{L^2(\mathbb{T})})^{2J+3}$$

and

$$\|N_r^{(J+1)}(v) - N_r^{(J+1)}(u)\|_{l^2(\mathbb{Z})L^2(\mathbb{R})} \lesssim N^{-\frac{J}{2}+\frac{(J-1)}{200}+}(\|v\|_{L^2(\mathbb{R})} + \|u\|_{L^2(\mathbb{R})} + \|w\|_{L^2(\mathbb{T})})^{2J+2}\|v-u\|_{L^2(\mathbb{R})}.$$

Proof. By (15) for fixed $n^{(j)}$ and μ_j there are at most $o(|\mu_j|^+)$ many choices for $n_1^{(j)}, n_2^{(j)}, n_3^{(j)}$. In addition, let us observe that μ_j is determined by $\tilde{\mu}_1, \ldots, \tilde{\mu}_j$ and $|\mu_j| \lesssim \max(|\tilde{\mu}_{j-1}|, |\tilde{\mu}_j|)$, since $\mu_j = \tilde{\mu}_j - \tilde{\mu}_{j-1}$. Then, for a fixed tree $T = T_k^J \in C(J)$, by Lemma 21 the estimate for the operator $\tilde{q}_{T,\mathbf{n}}^{J,t}$ is as follows (remember that $|\hat{\mu}_T| \sim |\hat{\mu}_J| = \prod_{k=1}^J |\tilde{\mu}_k|$)

$$\sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \|\tilde{q}_{T,\mathbf{n}}^{J,t}(\{v_{n_{b_m}}\}_{m=1}^{b_k^J}, \{w_{n_{r_{\tilde{m}}}}\}_{\tilde{m}=1}^{r_k^J})\|_2 \lesssim$$

$$\sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \left(\prod_{m=1}^{b_k^J} \|v_{n_{b_m}}\|_2 \prod_{\tilde{m}=1}^{r_k^J} |w_{n_{r_{\tilde{m}}}}| \right) \left(\prod_{k=1}^J \frac{1}{|\tilde{\mu}_k|} \right),$$

and, by Hölder's inequality, this is bounded from above by

$$(92) \quad \left(\sum_{\substack{|\mu_1|>N \\ |\tilde{\mu}_j|>(2j+1)^3 N^{1-\frac{1}{100}} \\ j=2,\ldots,J}} \prod_{k=1}^J \frac{1}{|\tilde{\mu}_k|^2} |\mu_k|^+ \right)^{\frac{1}{2}} \left(\sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \prod_{m=1}^{b_k^J} \|v_{n_{b_m}}\|_2^2 \prod_{\tilde{m}=1}^{r_k^J} |w_{n_{r_{\tilde{m}}}}|^2 \right)^{\frac{1}{q}}.$$

The first sum behaves like $N^{-\frac{J}{2}+\frac{(J-1)}{200}+}$ and for the remaining part we take the $l^2(\mathbb{Z})$ norm in n and by the use of Young's inequality we obtain the upper bound of

$$N^{-\frac{J}{2}+\frac{(J-1)}{200}+}\|v\|_{L^2(\mathbb{R})}^{b_k^J}\|w\|_{L^2(\mathbb{T})}^{r_k^J}.$$

Collecting terms, one sees that this proves the bound for $\|N_0^{(J+1)}(v)\|_{l^2(\mathbb{Z})L^2(\mathbb{R})}$.

Note that there is an extra factor $\sim J$ when we estimate the differences $N_0^{(J+1)}(v) - N_0^{(J+1)}(w)$ since $|a^{2J+1} - b^{2J+1}| \lesssim (\sum_{j=1}^{2J+1} a^{2J+1-j}b^{j-1})|a-b|$ has $O(J)$ many terms. Also, we have $N(J) = \text{card}(C(J))$ many summands in the operator $N_0^{(J+1)}$ since there are $N(J)$ many trees of the Jth generation. However, these observations do not cause any problem since the constant that we obtain from estimating the first sum of (92) decays like a fractional power of a double factorial in J, or to be more precise, with the use of (68) we have the following behaviour in J

(93)
$$t\frac{5^J \cdot (2J-1)!!}{(2J-1)!!^{\frac{3}{2}}} = \frac{5^J}{(2J-1)!!^{\frac{1}{2}}}.$$

For the operator $N_r^{(J+1)}$ the proof is the same but in addition we use Lemma 13 and Remark 14 for the operator $\mathbf{R}_2^t - \mathbf{R}_1^t$. $\qquad\square$

Then the estimate for the operator $N_1^{(J+1)}$ is the following.

Lemma 23.

$$\|N_1^{(J+1)}(v)\|_{l^2(\mathbb{Z})L^2(\mathbb{R})} \lesssim N^{-\frac{J-1}{2}+\frac{(J-2)}{200}+}(\|v\|_{L^2(\mathbb{R})} + \|w\|_{L^2(\mathbb{T})})^{2J+3}$$

and

$$\|N_1^{(J+1)}(v) - N_1^{(J+1)}(u)\|_{l^2(\mathbb{Z})L^2(\mathbb{R})} \lesssim N^{-\frac{J-1}{2}+\frac{(J-2)}{200}+}(\|v\|_{L^2(\mathbb{R})}+\|u\|_{L^2(\mathbb{R})}+\|w\|_{L^2(\mathbb{T})})^{2J+2}\|v-u\|_{L^2(\mathbb{R})}.$$

Proof. As before, for fixed $n^{(j)}$ and μ_j there are at most $o(|\mu_j|^+)$ many choices for $n_1^{(1)}, n_2^{(1)}, n_3^{(1)}$ and note that μ_j is determined by $\tilde{\mu}_1, \ldots, \tilde{\mu}_j$.

Let us assume that $|\tilde{\mu}_{J+1}| = |\tilde{\mu}_J + \mu_{J+1}| \lesssim (2J+3)^3|\tilde{\mu}_J|^{1-\frac{1}{100}}$ holds in (71). Then, $|\mu_{J+1}| \lesssim |\tilde{\mu}_J|$ and for fixed $\tilde{\mu}_J$ there are at most $o(|\tilde{\mu}_J|^{1-\frac{1}{100}})$ many choices for $\tilde{\mu}_{J+1}$ and therefore, also for $\mu_{J+1} = \tilde{\mu}_{J+1} - \tilde{\mu}_J$. For a fixed tree $T = T_k^J \in C(J), \alpha \in B_k^J \subset T^\infty$, Lemma 21, Remark 18 and the definition of the operator $N_1^t(v)$, see (32), we estimate $\tilde{q}_{T,\mathbf{n}}^{J,t}$ as follows (remember that $|\hat{\mu}_T| \sim |\hat{\mu}_J| = \prod_{k=1}^J |\tilde{\mu}_k|$)

$$\sum_{\substack{\mathbf{n}\in\mathcal{R}(T) \\ \mathbf{n}_r=n}} \|\tilde{q}_{T,\mathbf{n}}^{J,t}(\mathbf{N}_1^{t,\alpha}(\{v_{n_{b_m}}\}_{m=1}^{b_k^J}, \{w_{n_{r_{\tilde{m}}}}\}_{\tilde{m}=1}^{r_k^J}))\|_2 \lesssim$$

$$\sum_{\substack{\mathbf{n}\in\mathcal{R}(T) \\ \mathbf{n}_r=n}} \left(\left[\|v_{n_{\alpha_1}}\|_2\|v_{n_{\alpha_2}}\|_2\|v_{n_{\alpha_3}}\|_2 + \|v_{n_{\alpha_1}}\|_2|w_{n_{\alpha_2}}|\|v_{n_{\alpha_3}}\|_2 + \|v_{n_{\alpha_1}}\|_2\|v_{n_{\alpha_2}}\|_2|w_{n_{\alpha_3}}| + \right.\right.$$

$$\Big|w_{n_{\alpha_1}}\Big|\|v_{n_{\alpha_2}}\|_2\big|w_{n_{\alpha_3}}\big| + \big|w_{n_{\alpha_1}}\big|\|_2\big|w_{n_{\alpha_2}}\big|\|v_{n_{\alpha_3}}\|_2\Big] \prod_{\beta \in B_k^J \setminus \{\alpha\}} \|v_{n_\beta}\|_2 \prod_{\tilde{m}=1}^{r_k^J} |w_{n_{r_{\tilde{m}}}}|\Big)\Big(\prod_{k=1}^J \frac{1}{|\tilde{\mu}_k|}\Big).$$

Then for the $\|v_{n_{\alpha_1}}\|_2\|v_{n_{\alpha_2}}\|_2\|v_{n_{\alpha_3}}\|_2$ term, the same calculations work for the other terms, we apply Hölder's inequality and obtain the upper bound

(94)
$$\Big(\sum_{\substack{|\mu_1|>N \\ |\tilde{\mu}_j|>(2j+1)^3 N^{1-\frac{1}{100}} \\ j=2,\dots,J}} |\tilde{\mu}_J|^{1-\frac{1}{100}+} \prod_{k=1}^J \frac{1}{|\tilde{\mu}_k|^2}|\mu_k|^+\Big)^{\frac{1}{2}}$$

$$\Big(\sum_{\substack{\mathbf{n}\in\mathcal{R}(T) \\ \mathbf{n}_r=n}} \|v_{n_{\alpha_1}}\|_2^2\|v_{n_{\alpha_2}}\|_2^2\|v_{n_{\alpha_3}}\|_2^2 \prod_{\beta \in B_k^J \setminus \{\alpha\}} \|v_{n_\beta}\|_2^2 \prod_{\tilde{m}=1}^{r_k^J} |w_{n_{r_{\tilde{m}}}}|^2\Big)^{\frac{1}{2}}.$$

An easy calculation shows that the first sum behaves like $N^{-\frac{J-1}{2}+\frac{(J-2)}{200}+}$ and then by taking the $l^2(\mathbb{Z})$ norm and use Young's inequality we arrive at

$$N^{-\frac{J-1}{2}+\frac{(J-2)}{200}+}\|v\|_{L^2(\mathbb{R})}^{b_k^J+2}\|w\|_{L^2(\mathbb{T})}^{r_k^J}.$$

Similar considerations apply in the case that $\alpha \in R_k^J \subset T^\infty$ and give the upper bound

$$N^{-\frac{J-1}{2}+\frac{(J-2)}{200}+}\|v\|_{L^2(\mathbb{R})}^{b_k^J}\|w\|_{L^2(\mathbb{T})}^{r_k^J+2}.$$

If $|\tilde{\mu}_{J+1}| \lesssim (2J+3)^3|\mu_1|^{1-\frac{1}{100}}$ holds in (71), then for fixed μ_j, $j = 1,\dots,J$, there are at most $O(|\mu_1|^{1-\frac{1}{100}})$ many choices for μ_{J+1}. The same argument as above leads us to exactly the same expressions as in (94) but with the first sum replaced by the following

$$\Big(\sum_{\substack{|\mu_1|>N \\ |\tilde{\mu}_j|>(2j+1)^3 N^{1-\frac{1}{100}} \\ j=2,\dots,J}} |\mu_1|^{1-\frac{1}{100}} \prod_{k=1}^J \frac{1}{|\tilde{\mu}_k|^2}|\mu_k|^+\Big)^{\frac{1}{2}},$$

which again is bounded from above by $N^{-\frac{J-1}{2}+\frac{(J-2)}{200}+}$ and the proof is complete. $\square$

Remark 24. For $s > 0$ we have to observe that all previous lemmata hold true if we replace the $l^2 L^2$ norm by the $l_s^2 L^2$ norm and the $L^2(\mathbb{R})$ norm by the $H^s(\mathbb{R})$ norm. To see this, consider $n^{(j)}$ large. Then there exists at least one of $n_1^{(j)}, n_2^{(j)}, n_3^{(j)}$ such that $|n_k^{(j)}| \geq \frac{1}{3}|n^{(j)}|$, $k \in \{1,2,3\}$, since we have the relation $n^{(j)} \approx n_1^{(j)} - n_2^{(j)} + n_3^{(j)}$. Therefore, in the estimates of the Jth generation, there exists at least one frequency $n_k^{(j)}$ for some $j \in \{1,\dots,J\}$ with the property

$$\langle n\rangle^s \leq 3^{js}\langle n_k^{(j)}\rangle^s \leq 3^{Js}\langle n_k^{(j)}\rangle^s.$$

This exponential growth does not affect our calculations due to the double factorial decay in the denominator of (93).

Before we finish this section let us state a lemma about the behaviour of the remainder operator $N_2^{(J)}$ as $J \to \infty$.

Lemma 25. *Suppose that w is a smooth periodic solution of* (6) *in $L^2(\mathbb{T})$ such that its Fourier coefficients $\{w_m\}_{m \in \mathbb{Z}} \in l^1(\mathbb{Z})$ and v is a smooth solution of* (7) *such that $v \in M_{2,1}(\mathbb{R}) \subset L^2(\mathbb{R})$. Then*

$$\lim_{J \to \infty} \|N_2^{(J+1)}(v)\|_{l^2(\mathbb{Z})L^2(\mathbb{R})} = 0.$$

Proof. Obviously,

$$\|N_2^{(J+1)}(v)\|_2 \leq \sum_{T \in C(J)} \sum_{\alpha \in T^\infty} \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \|\tilde{q}_{T,\mathbf{n}}^{J,t}(\mathbf{N}_1^{t,\alpha}(\{v_{n_{b_m}}\}_{m=1}^{b_k^J}, \{w_{n_{r_{\tilde{m}}}}\}_{\tilde{m}=1}^{r_k^J}))\|_2.$$

For a fixed tree $T = T_k^J \in C(J)$ assume that $\alpha \in B_k^J$. Using Lemma 21 we have the upper bound

$$\sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \prod_{\beta \in B_k^J \setminus \{\alpha\}} \|v_{n_\beta}\|_2 \frac{\|N_1^t(v)(n_\alpha)\|_2}{\prod_{k=1}^J |\tilde{\mu}_k|} \prod_{\tilde{m}=1}^{r_k^J} |w_{n_{r_{\tilde{m}}}}|.$$

By the definition of the operator $N_1^t(v)$, see (32), and Remark 18, we bound this further

$$\sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \prod_{\beta \in B_k^J \setminus \{\alpha\}} \|v_{n_\beta}\|_2 \prod_{\tilde{m}=1}^{r_k^J} |w_{n_{r_{\tilde{m}}}}| \Big(\sum_{\substack{n_\alpha \approx n_{\alpha_1} - n_{\alpha_2} + n_{\alpha_3} \\ n_{\alpha_1} \not\approx n_\alpha \not\approx n_{\alpha_3}}} \|v_{n_{\alpha_1}}\|_2 \|v_{n_{\alpha_2}}\|_2 \|v_{n_{\alpha_3}}\|_2 +$$

$$\|v_{n_{\alpha_1}}\|_2 |w_{n_{\alpha_2}}| \|v_{n_{\alpha_3}}\|_2 + \|v_{n_{\alpha_1}}\|_2 \|v_{n_{\alpha_2}}\|_2 |w_{n_{\alpha_3}}| + |w_{n_{\alpha_1}}| \|v_{n_{\alpha_2}}\|_2 |w_{n_{\alpha_3}}| +$$

$$|w_{n_{\alpha_1}}| \|v_{n_{\alpha_2}}\| |v_{n_{\alpha_3}}\|_2 \Big) \frac{1}{\prod_{k=1}^J |\tilde{\mu}_k|}.$$

Let us treat only the sum that contains the quantity $\|v_{n_{\alpha_1}}\|_2 |w_{n_{\alpha_2}}| \|v_{n_{\alpha_3}}\|_2$, the remaining terms can be treated in a similar manner. As in the proof of Lemma 22, Hölder's inequality implies the upper bound

$$\frac{1}{(2J-1)!!^{\frac{3}{2}}} \Big(\sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \prod_{\beta \in B_k^J \setminus \{\alpha\}} \|v_{n_\beta}\|_2^2 \prod_{\tilde{m}=1}^{r_k^J} |w_{n_{r_{\tilde{m}}}}|^2 \Big(\sum_{\substack{n_\alpha \approx n_{\alpha_1} - n_{\alpha_2} + n_{\alpha_3} \\ n_{\alpha_1} \not\approx n_\alpha \not\approx n_{\alpha_3}}} \|v_{n_{\alpha_1}}\|_2 |w_{n_{\alpha_2}}| \|v_{n_{\alpha_3}}\|_2 \Big)^2 \Big)^{\frac{1}{2}}.$$

Then by taking the $l^2(\mathbb{Z})$ norm we arrive at

$$\frac{1}{(2J-1)!!^{\frac{3}{2}}} \Big(\sum_{n \in \mathbb{Z}} \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \prod_{\beta \in B_k^J \setminus \{\alpha\}} \|v_{n_\beta}\|_2^2 \prod_{\tilde{m}=1}^{r_k^J} |w_{n_{r_{\tilde{m}}}}|^2 (\{\|v_{n_{\alpha_1}}\|_2\} * \{|w_{n_{\alpha_2}}|\} * \{\|v_{n_{\alpha_3}}\|_2\})^2 (n_\alpha) \Big)^{\frac{1}{2}},$$

applying Young's inequality in $l^1(\mathbb{Z})$ for $2J+1$ sequences we get

$$\frac{1}{(2J-1)!!^{\frac{3}{2}}} \|v\|_{L^2(\mathbb{R})}^{b_k^J - 1} \|w\|_{L^2(\mathbb{T})}^{r_k^J} \|\{\|v_{n_{\alpha_1}}\|_2\} * \{|w_{n_{\alpha_2}}|\} * \{\|v_{n_{\alpha_3}}\|_2\}\|_{l^2},$$

and, again using Young's inequality together with the embedding $M_{2,1}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$ and the assumption that the Fourier coefficients of w are in $l^1(\mathbb{Z})$, this implies the upper bound

$$\frac{1}{(2J-1)!!^{\frac{3}{2}}} \|v\|_{M_{2,1}}^{b_k^J-1} \|\{w_m\}_{m\in\mathbb{Z}}\|_{l^1(\mathbb{Z})}^{r_k^J} \|v\|_{M_{2,1}}^2 \|\{w_m\}_{m\in\mathbb{Z}}\|_{l^1(\mathbb{Z})} = \frac{\|v\|_{M_{2,1}}^{b_k^J+1} \|\{w_m\}_{m\in\mathbb{Z}}\|_{l^1}^{r_k^J+1}}{(2J-1)!!^{\frac{3}{2}}}.$$

Similar estimates apply in the case $\alpha \in R_k^J$.

Finally, by adding up all these expressions for every different colored tree $T \in C(J)$, see (68), we get

$$\|N_2^{(J+1)}(v)\|_{l^2(\mathbb{Z})L^2(\mathbb{R})} \lesssim \frac{5^J}{(2J-1)!!^{\frac{1}{2}}} \left(\|v\|_{M_{2,1}} + \|\{w_m\}_{m\in\mathbb{Z}}\|_{l^1(\mathbb{Z})}\right)^{2J+3},$$

which goes to zero as $J \to \infty$. So the proof is complete. $\qquad\square$

5. EXISTENCE OF WEAK SOLUTIONS IN THE EXTENDED SENSE

In this subsection we prove Theorem 6. The calculations are the similar as in [7], [3], [13], however, with the additional difficulty that we have to handle mixed continuous and discrete variables. For this reason we only mention the basic steps of the argument, concentrating mainly on the important differences.

We start by defining the partial sum operator $\Gamma_{v_0}^{(J)}$ as

$$(95) \qquad \Gamma_{v_0}^{(J)} v(t) = v_0 + \sum_{j=2}^{J} N_0^{(j)}(v)(n) - \sum_{j=2}^{J} N_0^{(j)}(v_0)(n)$$

$$+ \int_0^t R_1^\tau(v)(n) + R_2^\tau(v)(n) + \sum_{j=2}^{J} N_r^{(j)}(v)(n) + \sum_{j=1}^{J} N_1^{(j)}(v)(n) \ d\tau,$$

where we have $N_1^{(1)} := N_{11}^t$ from (36), $N_0^{(2)} := N_{21}^t$ from (43) and $v_0 \in H^{s_1}(\mathbb{R})$ is our initial data. Here we assume that we have smooth solutions (see Section 2 so that all calculations of Sections 3 and 4 are applicable. Moreover, let us state that all operators appearing in the definition of $\Gamma_{v_0}^{(J)} v(t)$ depend also on the fixed function $w \in X_{T_0}(\mathbb{T}) = C([0,T_0], H^{s_2}(\mathbb{T}))$ that is the solution of (6) with initial data $w_0 \in H^{s_2}(\mathbb{T})$. For this w we know that

$$(96) \qquad \|w\|_{X_{T_0}(\mathbb{T})} \lesssim \|w_0\|_{H^{s_2}(\mathbb{T})}.$$

In the following we will denote by $X_T(\mathbb{R}) = C([0,T], H^{s_1}(\mathbb{R}))$. Our goal is to show that the series appearing on the RHS of (95) converge absolutely in $X_T(\mathbb{R})$ for sufficiently small $T > 0$, if $v \in X_T(\mathbb{R})$, even for $J = \infty$. Indeed, by Lemmata 13, 15, 22, and 23 we obtain

$$(97) \quad \|\Gamma_{v_0}^{(J)} v\|_{X_T(\mathbb{R})} \le \|v_0\|_{H^{s_1}(\mathbb{R})} + C \sum_{j=2}^{J} N^{-\frac{j-1}{2}+\frac{j-2}{200}} (\|v\|_{X_T(\mathbb{R})}^{2j-1} + \|v_0\|_{H^{s_1}(\mathbb{R})}^{2j-1} + \|w\|_{X_{T_0}(\mathbb{T})}^{2j-1})$$

$$+ CT \left[\|v\|_{X_T(\mathbb{R})}^3 + \|w\|_{X_{T_0}(\mathbb{T})}^3 + \sum_{j=2}^{J} N^{-\frac{j-1}{2}+\frac{j-2}{200}} (\|v\|_{X_T(\mathbb{R})}^{2j+1} + \|w\|_{X_{T_0}(\mathbb{T})}^{2j+1}) + \right.$$

$$N^{\frac{1}{2}+}(\|v\|_{X_T(\mathbb{R})}^3 + \|w\|_{X_{T_0}(\mathbb{T})}^3) + \sum_{j=2}^{J} N^{-\frac{j-2}{2}+\frac{j-3}{200}+}(\|v\|_{X_T(\mathbb{R})}^{2j+1} + \|w\|_{X_{T_0}(\mathbb{T})}^{2j+1})\Big].$$

From (96) we estimate $\|w\|_{X_{T_0}(\mathbb{T})}$ by $\|w_0\|_{H^{s_2}(\mathbb{T})}$ and assuming that the sum $\|v_0\|_{H^{s_1}(\mathbb{R})} + \|w_0\|_{H^{s_2}(\mathbb{T})} \leq R$ and $\|v\|_{X_T(\mathbb{R})} \leq \tilde{R}$, with $\tilde{R} \geq R \geq 1$ we may continue from (97) in exactly the same way as in [7], [3], [13] to show that for sufficiently large N and sufficiently small $T = T(\|v_0\|_{H^{s_1}(\mathbb{R})} + \|w_0\|_{H^{s_2}(\mathbb{T})}) > 0$ the partial sum operators $\Gamma_{v_0}^{(J)}$ are well defined in $X_T(\mathbb{R})$, for every $J \in \mathbb{N} \cup \{\infty\}$. We will write Γ_{v_0} for $\Gamma_{v_0}^{(\infty)}$.

Our next step is, given an initial data $u_0 = v_0 + w_0 \in H^{s_1}(\mathbb{R}) + H^{s_2}(\mathbb{T})$, to construct a solution u with the properties claimed in Theorem 6. We start with the periodic part w_0. As it was done in [7] we approximate w_0 by smooth initial data $w_0^{(m)} \in H^{\infty}(\mathbb{T})$ with

$$(98) \qquad \lim_{m\to\infty} w_0^{(m)} = w_0, \text{ in } H^{s_2}(\mathbb{T}).$$

For such initial data $w_0^{(m)}$ we know that we can find smooth solution $w^{(m)}$ of NLS (6) in $C([0,T], H^{s_2}(\mathbb{T}))$ that satisfies Duhamel's formulation

$$(99) \qquad w^{(m)} = w_0^{(m)} \pm \int_0^t S(-\tau)[|S(\tau)w^{(m)}|^2 S(\tau)w^{(m)}] \, d\tau$$

and from [7] it follows that there is a common time of existence $T_0 = T_0(\|w_0\|_{H^{s_2}(\mathbb{T})})$ for all solutions $w^{(m)}$. In addition, they show that the sequence $\{w^{(m)}\}_{m\in\mathbb{N}}$ is Cauchy in $X_{T_0}(\mathbb{T}) = C([0,T_0], H^{s_2}(\mathbb{T}))$ and that the limit function $w \in X_{T_0}(\mathbb{T})$ satisfies NLS (6) in the sense of Definition 3.

We also approximate v_0 by smooth functions $v_0^{(m)} \in H^{s_1}(\mathbb{R})$, so that

$$(100) \qquad \lim_{m\to\infty} v_0^{(m)} = v_0, \text{ in } H^{s_1}(\mathbb{R}),$$

and by Section 2 we may find smooth solutions $v^{(m)}$ of (7) in $X_T(\mathbb{R}) = C([0,T], H^{s_1}(\mathbb{R}))$ that satisfy Duhamel's formulation

$$(101) \qquad v^{(m)} = v_0^{(m)} \pm \int_0^t S(-\tau)[G(S(\tau)w^{(m)}, S(\tau)v^{(m)})] \, d\tau =$$

$$v_0^{(m)} + \sum_{j=2}^{\infty} N_0^{(j)}(v^{(m)})(n) - \sum_{j=2}^{\infty} N_0^{(j)}(v_0^{(m)})(n)$$

$$+ \int_0^t R_1^\tau(v^{(m)})(n) + R_2^\tau(v^{(m)})(n) + \sum_{j=2}^{\infty} N_r^{(j)}(v^{(m)})(n) + \sum_{j=1}^{\infty} N_1^{(j)}(v^{(m)})(n) \, d\tau = \Gamma_{v_0^{(m)}} v^{(m)},$$

where we used Lemma 25, namely that the remainder operator goes to zero as $J \to \infty$. From this, following exactly the same arguments as in [7], [3], [13] we can prove that (101) holds in $X_{T_0}(\mathbb{R})$ for the same time $T_0 = T_0(R) > 0$ independent of $m \in \mathbb{N}$ and also that

$$(102) \quad \|v^{(m_1)} - v^{(m_2)}\|_{X_{T_0}(\mathbb{R})} = \|\Gamma_{v_0^{(m_1)}} v^{(m_1)} - \Gamma_{v_0^{(m_2)}} v^{(m_2)}\|_{X_{T_0}(\mathbb{R})} \leq c \, \|v_0^{(m_1)} - v_0^{(m_2)}\|_{H^{s_1}(\mathbb{R})}$$

for some constant $c > 0$. Therefore, the sequence $\{v^{(m)}\}_{m \in \mathbb{N}}$ is Cauchy in the Banach space $X_{T_0}(\mathbb{R})$, we denote by v^∞ its limit in $X_{T_0}(\mathbb{R})$.

We will show that $V^\infty = S(t)v^\infty$ satisfies NLS (7) in the sense of Definition 5. For convenience, we drop the superscript ∞ and write V, and v. In addition, let $V^{(m)} := S(t)v^{(m)}, W^{(m)} = S(t)w^{(m)}$ and $W = S(t)w$. Obviously, $V^{(m)} \to V$ in $X_{T_0}(\mathbb{R})$, because $v^{(m)} \to v$ in $X_{T_0}(\mathbb{R})$, and similarly $W^{(m)} \to W$ in $X_{T_0}(\mathbb{T})$ since $w^{(m)} \to w$ in $X_{T_0}(\mathbb{T})$. Thus, $\partial_x V^{(m)} \to \partial_x V, \partial_t V^{(m)} \to \partial_t V$ and $\partial_x W^{(m)} \to \partial_x W$, $\partial_t W^{(m)} \to \partial_t W$ in the sense of distributions. Since $V^{(m)}$ satisfies (7) and $W^{(m)}$ satisfies (6) for every $m \in \mathbb{N}$, we have that

$$(103) \qquad \mathcal{N}(W^{(m)}) = |W^{(m)}|^2 W^{(m)} = -i\partial_t W^{(m)} + \partial_x^2 W^{(m)}$$

converges to some distribution $\tilde{w}$, which is equal to $\mathcal{N}(W)$ interpreted in the sense of Definition 3, as it was shown in [7]. and

$$(104)$$
$$G(W^{(m)}, V^{(m)}) = |W^{(m)} + V^{(m)}|^2 (W^{(m)} + V^{(m)}) - |W^{(m)}|^2 W^{(m)} = -i\partial_t V^{(m)} + \partial_x^2 V^{(m)}$$

converges to some distribution $\tilde{v}$. Our claim is the following.

Proposition 26. Let $\tilde{v}$ be the limit of $G(W^{(m)}, V^{(m)})$ in the sense of distributions as $m \to \infty$. Then $\tilde{v} = G(W, V)$ where $G(W, V)$ is to be interpreted in the sense of Definition 5.

Proof. Consider a sequence of Fourier cutoff multipliers $\{T_N\}_{N \in \mathbb{N}}$ as in Definition 1. We will prove that

$$\lim_{N \to \infty} G(T_N W, T_N V) = \tilde{v}$$

in the sense of distributions. Let ϕ be a test function and $\epsilon > 0$ a fixed given number. Our goal is to find $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ we have

$$(105) \qquad |\langle \tilde{v} - G(T_N W, T_N V), \phi \rangle| < 3\epsilon.$$

The LHS can be estimated by

$$(106) \qquad |\langle \tilde{v} - G(W^{(m)}, V^{(m)}), \phi \rangle| + |\langle G(W^{(m)}, V^{(m)}) - G(T_N W^{(m)}, T_N V^{(m)}), \phi \rangle| +$$

$$|\langle G(T_N W^{(m)}, T_N V^{(m)}) - G(T_N W, T_N V), \phi \rangle|.$$

The first term is estimated very easily since by the definition of $\tilde{v}$ we have that

$$(107) \qquad |\langle \tilde{v} - G(W^{(m)}, V^{(m)}), \phi \rangle| < \epsilon,$$

for sufficiently large $m \in \mathbb{N}$.

To continue, we consider the second summand of (106) for fixed m. Writing out the difference, we see that we have to estimate five expressions

$$\langle |V^{(m)}|^2 V^{(m)} - |T_N V^{(m)}|^2 T_N V^{(m)}, \phi \rangle + \langle (W^{(m)})^2 \overline{V^{(m)}} - (T_N W^{(m)})^2 \overline{T_N V^{(m)}}, \phi \rangle$$

$$+ \langle (V^{(m)})^2 \overline{W^{(m)}} - (T_N V^{(m)})^2 \overline{T_N W^{(m)}}, \phi \rangle + 2 \langle |W^{(m)}|^2 V^{(m)} - |T_N W^{(m)}|^2 T_N V^{(m)}, \phi \rangle$$

$$+ 2 \langle |V^{(m)}|^2 W^{(m)} - |T_N V^{(m)}|^2 T_N W^{(m)}, \phi \rangle.$$

The first was estimated in [3] and [13]. For the second term we note

$$\left| \int \int (W^{(m)})^2 \overline{(Id - T_N)V^{(m)}}\phi + \overline{T_N V^{(m)}}(W^{(m)} - T_N W^{(m)})(W^{(m)} + T_N W^{(m)})\phi \right|$$

$$\leq \|W^{(m)}\|_{L^\infty_{T,x}}^2 \|(Id - T_N)V^{(m)}\|_{L^2_{T,x}} \|\phi\|_{L^2_{T,x}} + \|T_N V^{(m)}\|_{L^\infty_{T,x}} \|W^{(m)}$$

$$+ T_N W^{(m)}\|_{L^\infty_{T,x}} \int \int |W^{(m)} - T_N W^{(m)}||\phi|.$$

The integral term can be written as

$$\int_0^T \sum_{k \in \mathbb{Z}} \int_k^{k+1} |W^{(m)} - T_N W^{(m)}||\phi| \leq \int_0^T \sum_{k \in \mathbb{Z}} \|(Id - T_N)W^{(m)}\|_{L^2(k,k+1)} \|\phi\|_{L^2(k,k+1)}$$

$$= \int_0^T \|(Id - T_N)W^{(m)}\|_{L^2(\mathbb{T})} \sum_{k \in \mathbb{Z}} \|\phi\|_{L^2(k,k+1)},$$

which is bounded from above by

$$\|(Id - T_N)W^{(m)}\|_{L^2_{T,x}} \|t \to \sum_{k \in \mathbb{Z}} \|\phi(t, \cdot)\|_{L^2(k,k+1)}\|_{L^2(0,T)}.$$

Therefore, for the second term we have the estimate

$$(108) \qquad C_{\phi,m}\left(\|(Id - T_N)V^{(m)}\|_{L^2([0,T],L^2(\mathbb{R}))} + \|(Id - T_N)W^{(m)}\|_{L^2([0,T],L^2(\mathbb{T}))} \right)$$

which tends to zero as $N \to \infty$ by the definition of the Fourier cutoff operators and the Dominated Convergence Theorem. For the third term we have to consider the quantities

$$\left| \int \int (V^{(m)})^2 \overline{W^{(m)} - T_N W^{(m)}}\phi + \overline{T_N W^{(m)}}(V^{(m)} - T_N V^{(m)})(V^{(m)} + T_N V^{(m)})\phi \right|.$$

Doing the same as for the previous term, we obtain an expression analog to (108). We treat the forth and fifth terms similarly. This allows us to choose $N_0 = N_0(m) > 0$ with the property

$$(109) \qquad C_{\phi,m}\left(\|(Id - T_N)V^{(m)}\|_{L^2([0,T],L^2(\mathbb{R}))} + \|(Id - T_N)W^{(m)}\|_{L^2([0,T],L^2(\mathbb{T}))} \right) < \epsilon,$$

for all $N \geq N_0$.

For the last term of (106) we need to observe two things. Firstly, by applying the iteration process (see also [7], [3] and [13]) that we described in Sections 3 and 4 we see that $\{G(W^{(m)}, V^{(m)})\}_{m \in \mathbb{N}}$ is Cauchy in $S'((0,T) \times \mathbb{R})$ as $m \to \infty$ for each fixed N, since the sequences $V^{(m)}, W^{(m)}$ are Cauchy in $C([0,T], H^{s_1}(\mathbb{R}))$ and $C([0,T], H^{s_2}(\mathbb{T}))$ respectively. Because the multipliers m_N of T_N are uniformly bounded we conclude that this convergence is uniform in N.

Secondly, for fixed N, $T_N V \in C([0,T], H^\infty(\mathbb{R}))$ and $T_N W \in C([0,T], H^\infty(\mathbb{T}))$ since $V \in H^{s_1}(\mathbb{R}), W \in H^{s_2}(\mathbb{T})$ and the multiplier m_N of T_N is compactly supported. Hence

$$G(T_N W, T_N V) = |T_N V|^2 T_N V + (T_N W)^2 \overline{T_N V} + (T_N V)^2 \overline{T_N W} + 2|T_N W|^2 T_N V + 2|T_N V|^2 T_N W$$

makes sense as a function. Then we have to estimate the following five summands

$$\langle |T_N V^{(m)}|^2 T_N V^{(m)} - |T_N V|^2 T_N V, \phi\rangle + \langle (T_N W^{(m)})^2 \overline{T_N V^{(m)}} - (T_N W)^2 \overline{T_N V}, \phi\rangle +$$

$$\langle (T_N V^{(m)})^2 \overline{T_N W^{(m)}} - (T_N V)^2 \overline{T_N W}, \phi\rangle + 2\langle |T_N W^{(m)}|^2 T_N V^{(m)} - |T_N W|^2 T_N V, \phi\rangle +$$

$$2\langle |T_N V^{(m)}|^2 T_N W^{(m)} - |T_N V|^2 T_N W, \phi\rangle.$$

The first term was estimated in [3] and [13]. For the second term we have to bound

$$\left| \int \int (T_N W^{(m)})^2 \overline{(T_N(V^{(m)} - V))}\phi + \overline{T_N V}(T_N W^{(m)} - T_N W)(T_N W^{(m)} + T_N W)\phi \right| \leq$$

$$\|T_N W^{(m)}\|_{L_{T,x}^\infty}^2 \|T_N(V^{(m)} - V)\|_{L_{T,x}^2}\|\phi\|_{L_{T,x}^2} +$$

$$\int_0^T \sum_{k\in\mathbb{Z}} \int_k^{k+1} \left| T_N V(T_N W^{(m)} - T_N W)(T_N W^{(m)} + T_N W)\phi \right|.$$

The second expression is bounded from above by

$$\int_0^T \sum_{k\in\mathbb{Z}} \|T_N V\|_{L^4(k,k+1)}\|T_N(W^{(m)} - W)\|_{L^4(k,k+1)}\|T_N W^{(m)} + T_N W\|_{L^4(k,k+1)}\|\phi\|_{L^4(k,k+1)}$$

which is less than

$$\int_0^T \|T_N V\|_{L^4(\mathbb{R})}\|T_N(W^{(m)} - W)\|_{L^4(\mathbb{T})}\|T_N W^{(m)} + T_N W\|_{L^4(\mathbb{T})} \sum_{k\in\mathbb{Z}} \|\phi\|_{L^4(k,k+1)} \leq$$

$$\|T_N V\|_{L_{T,x}^4}\|T_N W^{(m)} + T_N W\|_{L_{T,x}^4}\|T_N(W^{(m)} - W)\|_{L_{T,x}^4}\|t \to \sum_{k\in\mathbb{Z}} \|\phi\|_{L^4(k,k+1)}\|_{L^4(0,T)}.$$

Then we use Hölder's inequality in the interval $(0, T)$ to pass from the L^4 norm to the L^∞ norm and in the space variable an application of Parseval's identity, together with the fact that the multiplier operators T_N have compactly supported symbols m_N, implies the bound

$$C_{\phi,\|V\|_{X_T(\mathbb{R})},\|W\|_{X_T(\mathbb{T})}} M^{\frac{3}{4}} T^{\frac{3}{4}} \|W^{(m)} - W\|_{X_T(\mathbb{T})} < \epsilon,$$

where the number $M = M(N) > 0$ is chosen so that $\text{supp} m_N \subset [-M, M]$. For the third term we have to estimate the quantity

$$\left| \int \int (T_N V^{(m)})^2 \overline{(T_N(W^{(m)} - W))}\phi + \overline{T_N W}(T_N(V^{(m)} - V))(T_N V^{(m)} + T_N V)\phi \right|,$$

for which similar bounds apply as for the previous term. The same holds for the forth and fifth terms.

From these observations we derive that $G(T_N W^{(m)}, T_N V^{(m)}) \to G(T_N W, T_N V)$ in the space $S'((0, T) \times \mathbb{R})$ as $m \to \infty$ uniformly in N. Equivalently,

$$(110) \qquad\qquad |\langle G(T_N W^{(m)}, T_N V^{(m)}) - G(T_N W, T_N V), \phi\rangle| < \epsilon,$$

for all large m, uniformly in N. Therefore, (105) follows by choosing m sufficiently large so that (107) and (110) hold, and then choosing $N_0 = N_0(m)$ such that (109) holds.

$\square$

Finally, we have shown that the function $V = V^\infty$ is a solution of NLS (7) in the sense of Definition 5.

6. UNCONDITIONAL UNIQUENESS OF SOLUTIONS

In this section we prove Theorem 7. Let us assume that the initial condition $u_0 = v_0 + w_0 \in H^s(\mathbb{R}) + H^{\frac{1}{2}+\epsilon}(\mathbb{T})$ where $\frac{1}{6} \leq s \leq \frac{1}{2}$ and $\epsilon > 0$. Notice that for such s we have the embeddings $H^s(\mathbb{R}) \hookrightarrow L^3(\mathbb{R})$ and $H^{\frac{1}{2}+\epsilon}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$. Therefore, if V is a solution of NLS (7) in $C([0,T], H^s(\mathbb{R}))$, then V and hence $v = e^{it\partial_x^2}V$ are elements of $C([0,T], H^s(\mathbb{R})) \hookrightarrow C([0,T], L^3(\mathbb{R}))$. Similarly, for W being a solution of NLS (6) in $C([0,T], H^{\frac{1}{2}+\epsilon}(\mathbb{T}))$, we have $w = e^{it\partial_x^2}W \in C([0,T], H^{\frac{1}{2}+\epsilon}(\mathbb{T})) \hookrightarrow C([0,T], L^\infty(\mathbb{T}))$.

Therefore, the nonlinearity $G(w,v)$ makes sense as a function in $L^1(\mathbb{R}) + L^2(\mathbb{R})$ since $|v|^2v \in L^1(\mathbb{R}), w^2\bar{v}, |w|^2v \in L^3(\mathbb{R}) \cap L^2(\mathbb{R})$ and $v^2\bar{w}, |v|^2w \in L^1(\mathbb{R}) \cap L^{\frac{3}{2}}(\mathbb{R})$.

As a consequence of this, its box operator $\square_n G(w,v) \in L^2(\mathbb{R})$ and from the PDE

$$(111) \qquad i\partial_t v_n = S(t)\square_n G(S(-t)w, S(-t)v),$$

which is true in the sense of distributions $(C^\infty([0,T], S(\mathbb{R})))'$, we infer $\partial_t v_n \in C([0,T], L^2(\mathbb{R}))$.

This, together with $v_n \in C([0,T], L^2(\mathbb{R}))$, already implies $v_n \in C^1([0,T], L^2(\mathbb{R}))$. Indeed, to obtain this it suffices to know that if two space–time distributions S and $T \in (C^\infty([0,T], S(\mathbb{R})))'$ have the same time derivatives, $\partial_t S = \partial_t T$, then there is distribution c, acting only on the space variable, such that $S = T + c$. This can be found, for example, in [17, Section 3.3].

Thus, we can rewrite the the PDE in the integral form

$$(112) \qquad v_n = v_n(0) + i \int_0^t S(\tau)\square_n G(S(-\tau)w, S(-\tau)v) \, d\tau,$$

which means that we can continue with the differentiation by parts technique, as it was described in Sections 3 and 4, directly for the function v without having to approximate it by smooth solutions, as done in the previous Section 5. The next lemma justifies the interchange of time differentiation and space integration

Lemma 27. *Let $f(t,x), \partial_t f(t,x) \in C([0,T], L^1(\mathbb{R}^d))$ and define the distribution $\int_{\mathbb{R}^d} f(\cdot, x)dx$ by*

$$\left\langle \int_{\mathbb{R}^d} f(\cdot, x)dx, \phi \right\rangle = \int_\mathbb{R} \int_{\mathbb{R}^d} f(t,x)\phi(t)dxdt,$$

with $\phi \in C_c^\infty(\mathbb{R})$. Then, $\partial_t \int_{\mathbb{R}^d} f(\cdot, x)dx = \int_{\mathbb{R}^d} \partial_t f(\cdot, x)dx$.

Proof. By definition

$$\left\langle \partial_t \int_{\mathbb{R}^d} f(\cdot, x)dx, \phi \right\rangle = -\left\langle \int_{\mathbb{R}^d} f(\cdot, x)dx, \phi' \right\rangle = -\int_\mathbb{R} \int_{\mathbb{R}^d} f(t,x)\phi'(t)dxdt$$

and, since $f \in C([0,T], L^1(\mathbb{R}^d))$, we can change the order of integration by Fubini's Theorem to obtain

$$-\int_{\mathbb{R}^d} \int_\mathbb{R} f(t,x)\phi'(t)dtdx = \int_{\mathbb{R}^d} \int_\mathbb{R} \partial_t f(t,x)\phi(t)dtdx = \int_\mathbb{R} \int_{\mathbb{R}^d} \partial_t f(t,x)\phi(t)dxdt,$$

where in the first equality we used the definition of the weak derivative of f and in the second equality Fubini's Theorem with the fact that $\partial_t f \in C([0,T], L^1(\mathbb{R}^d))$. The last integral is equal to

$$\left\langle \int_{\mathbb{R}^d} \partial_t f(\cdot, x) dx, \phi \right\rangle$$

and the proof is complete. $\hspace{5cm}\square$

Consider now the expressions (38), (41) and (42) for fixed n and ξ. We want to apply Lemma 27 to each one of the following functions

$$f_1(t, \xi_1, \xi_3) = \sigma_n(\xi) \frac{e^{-2it(\xi-\xi_1)(\xi-\xi_3)}}{-2i(\xi-\xi_1)(\xi-\xi_3)} \, \hat{v}_{n_1}(\xi_1)\hat{\bar{v}}_{n_2}(\xi-\xi_1-\xi_3)\hat{v}_{n_3}(\xi_3),$$

$$f_2(t, \xi_1) = \sigma_n(\xi) w_{n_3} \frac{e^{-2it(\xi-n_3)(\xi-\xi_1)}}{-2i(\xi-n_3)(\xi-\xi_1)} \hat{v}_{n_1}(\xi_1)\hat{\bar{v}}_{n_2}(\xi-\xi_1-n_3),$$

$$f_3(t, \xi_1) = \sigma_n(\xi) \bar{w}_{n_2} \frac{e^{-2it(\xi-\xi_1)(\xi_1-n_2)}}{-2i(\xi-\xi_1)(\xi_1-n_2)} \hat{v}_{n_1}(\xi_1)\hat{v}_{n_3}(\xi-\xi_1+n_2),$$

where $\xi \approx n, \xi_1 \approx n_1, \xi_3 \approx n_3, \xi-\xi_1-\xi_3 \approx -n_2$ and $(n, n_1, n_2, n_3) \in A_N(n)^c$ given by (35)). With the use of Young's inequality and the fact that for all n, $\hat{v}_n, \partial_t \hat{v}_n$ are compactly supported functions in $L^2(\mathbb{R})$, it is not hard to obtain that $f_1, \partial_t f_1 \in C([0,T], L^1(\mathbb{R}^2))$ and $f_2, f_3, \partial_t f_2, \partial_t f_3 \in C([0,T], L^1(\mathbb{R}))$. Thus, for f_1, and similarly for f_2, f_3,

$$\partial_t \left[\int_{\mathbb{R}^2} \sigma_n(\xi) \frac{e^{-2it(\xi-\xi_1)(\xi-\xi_3)}}{-2i(\xi-\xi_1)(\xi-\xi_3)} \, \hat{v}_{n_1}(\xi_1)\hat{\bar{v}}_{n_2}(\xi-\xi_1-\xi_3)\hat{v}_{n_3}(\xi_3) d\xi_1 d\xi_3 \right]$$

$$= \int_{\mathbb{R}^2} \sigma_n(\xi) \partial_t \left[\sigma_n(\xi) \frac{e^{-2it(\xi-\xi_1)(\xi-\xi_3)}}{-2i(\xi-\xi_1)(\xi-\xi_3)} \, \hat{v}_{n_1}(\xi_1)\hat{\bar{v}}_{n_2}(\xi-\xi_1-\xi_3)\hat{v}_{n_3}(\xi_3) \right] d\xi_1 d\xi_3$$

$$= \int_{\mathbb{R}^2} \sigma_n(\xi) \partial_t \left[\frac{e^{-2it(\xi-\xi_1)(\xi-\xi_3)}}{-2i(\xi-\xi_1)(\xi-\xi_3)} \right] \hat{v}_{n_1}(\xi_1)\hat{\bar{v}}_{n_2}(\xi-\xi_1-\xi_3)\hat{v}_{n_3}(\xi_3) d\xi_1 d\xi_3$$

$$+ \int_{\mathbb{R}^2} \sigma_n(\xi) \frac{e^{-2it(\xi-\xi_1)(\xi-\xi_3)}}{-2i(\xi-\xi_1)(\xi-\xi_3)} \partial_t \left[\hat{v}_{n_1}(\xi_1)\hat{\bar{v}}_{n_2}(\xi-\xi_1-\xi_3)\hat{v}_{n_3}(\xi_3) \right] d\xi_1 d\xi_3.$$

In the second equality we used the product rule which is applicable since $\hat{v}_n \in C^1([0,T], L^2(\mathbb{R}))$.

Finally it remains to justify the interchange of differentiation in time and summation in the discrete variable but this is done in exactly the same way as in [7] (Lemma 5.1). Similar arguments justify the interchange on the Jth step of the infinite iteration procedure.

Thus, we obtain the following expression in $C([0,T], H^s(\mathbb{R}))$ for the solution v of NLS (111) with initial data v_0

$$(113) \qquad\qquad v = \Gamma_{v_0} v + \lim_{J \to \infty} \int_0^t N_2^{(J+1)}(v) d\tau,$$

where the limit is an element of $C([0,T], H^s(\mathbb{R}))$. Its existence follows from the fact that the operators $\Gamma_{v_0}^{(J)} v$ converge to $\Gamma_{v_0} v$ in the norm of $C([0,T], H^s(\mathbb{R}))$ as $J \to \infty$. The important estimate about the remainder operator $N_2^{(J)}$ is the following

Lemma 28.

$$\lim_{J\to\infty} \|N_2^{(J)}(v)\|_{l^\infty L^2(\mathbb{R})} = 0.$$

The proof is very similar to the one given in [13], Lemma 28, where we have to consider the cases $\partial_t v_n, \partial_t w_n$ with similar arguments. This lemma implies that $\lim_{J\to\infty} \int_0^t N_2^{(J+1)}(v)d\tau$ is equal to 0 in $X(T) = C([0, T], H^s(\mathbb{R}))$. From this we obtain the uniqueness of NLS (111) since if there are two solutions v_1 and v_2 with the same initial datum v_0 we obtain by (102)

$$\|v_1 - v_2\|_{X_T} = \|\Gamma_{v_0} v_1 - \Gamma_{v_0} v_2\|_{X_T} \lesssim \|v_0 - v_0\|_{H^s(\mathbb{R})} = 0.$$

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