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Roland Schnaubelt, Martin Spitz

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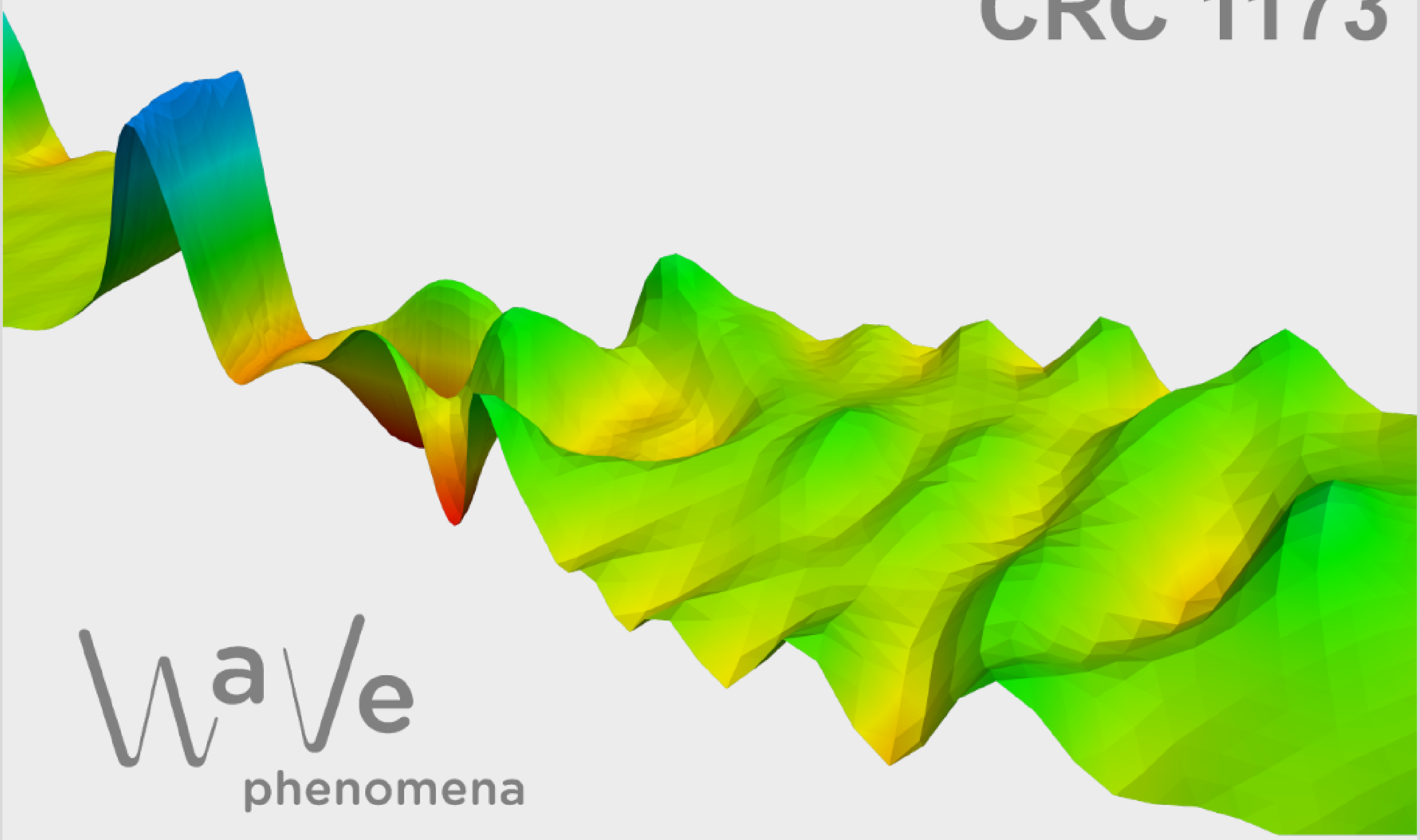

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LOCAL WELLPOSEDNESS OF QUASILINEAR MAXWELL EQUATIONS WITH CONSERVATIVE INTERFACE CONDITIONS

ROLAND SCHNAUBELT AND MARTIN SPITZ

ABSTRACT. We establish a comprehensive local wellposedness theory for the quasilinear Maxwell system with interfaces in the space of piecewise H^m-functions for $m \geq 3$. The system is equipped with instantaneous and piecewise regular material laws and perfectly conducting interfaces and boundaries. We also provide a blow-up criterion in the Lipschitz norm and prove the continuous dependence on the data. The proof relies on precise a priori estimates and the regularity theory for the corresponding linear problem also shown here.

1. INTRODUCTION

The Maxwell equations are the basis of electro-magnetic theory and thus one of the fundamental partial differential equations in physics. In the case of instantaneous nonlinear material laws, they form a symmetric quasilinear hyperbolic system under natural assumptions. For such systems on $\mathbb{R}^d$, in [16] Kato has established a satisfactory local wellposedness theory in $H^s(\mathbb{R}^d)$ for $s > 1 + \frac{d}{2}$. However, on a domain $G \neq \mathbb{R}^3$, the Maxwell system with the boundary conditions of a perfect conductor has a characteristic boundary and does not belong to the classes of hyperbolic systems for which one knows a wellposedness theory in H^3. The available results need much more regularity and exhibit a loss of derivatives in normal direction (encoded in weighted function spaces), see [12] or [22]. In the recent papers [24] and [25] by one of the authors, a comprehensive local wellposedness theory in H^m for $m \geq 3$ has been established for the boundary conditions of a perfect conductor. The main effort in these works is devoted to prove full regularity in normal direction at the boundary, heavily using the structure of the Maxwell system.

However, deriving boundary conditions for the Maxwell systems on a domain $G \subseteq \mathbb{R}^3$, one starts from the interface conditions (1.2) at ∂G and *assumes* that one knows the trace of the fields outside G, see Section I.4.2.2 of [8] or Section 7.12 in [11]. Moreover, in applications one often deals with composite materials in which the constitutive relations are only piecewise regular in $x \in G$. Here one has to treat the jumps in the material as interfaces. It is thus necessary to investigate interface problems in electro-magnetism, and not only (pure) boundary value problems.

In this work, we treat a (possibly unbounded) domain $G \subseteq \mathbb{R}^3$ being the disjoint union of two subdomains G_+ and G_- and the interface $\Sigma = \partial G_-$, where Σ and ∂G are smooth and have positive distance. Our results immediately extend to domains

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consisting of finitely many such components. We establish a comprehensive local wellposedness theory in H^m with $m \geq 3$ for the Maxwell system on G, given as

$$\begin{aligned}
\partial_t \boldsymbol{D}_\pm &= \operatorname{curl} \boldsymbol{H}_\pm - \boldsymbol{J}_\pm, && \text{for } x \in G_\pm, && t \in J,\\
\partial_t \boldsymbol{B}_\pm &= -\operatorname{curl} \boldsymbol{E}_\pm, && \text{for } x \in G_\pm, && t \in J,\\
\operatorname{div} \boldsymbol{D}_\pm &= \rho_\pm, \quad \operatorname{div} \boldsymbol{B}_\pm = 0, && \text{for } x \in G_\pm, && t \in J,\\
\boldsymbol{E}_+ \times \nu &= 0, \quad \boldsymbol{B}_+ \cdot \nu = 0, && \text{for } x \in \partial G, && t \in J,\\
\boldsymbol{E}_\pm(t_0) &= \boldsymbol{E}_{0,\pm}, \quad \boldsymbol{H}_\pm(t_0) = \boldsymbol{H}_{0,\pm}, && \text{for } x \in G_\pm,
\end{aligned} \tag{1.1}$$

for an initial time $t_0 \in \mathbb{R}$, $J = (t_0, T)$, and the unit outward normal vector ν of G_+. Here $\boldsymbol{E}_\pm(t,x), \boldsymbol{D}_\pm(t,x) \in \mathbb{R}^3$ are the electric and $\boldsymbol{H}_\pm(t,x), \boldsymbol{B}_\pm(t,x) \in \mathbb{R}^3$ the magnetic fields on $G_\pm$. It is known that the divergence equations and the magnetic boundary condition $\boldsymbol{B}_+ \cdot \nu = 0$ in (1.1) remain valid if they are satisfied by the initial fields. Here, the charge densities $\rho_\pm(t,x)$ are given by the initial charge and the current densities $\boldsymbol{J}_\pm(t,x) \in \mathbb{R}^3$ via

$$\rho_\pm(t) = \rho_\pm(t_0) - \int_{t_0}^t \operatorname{div} \boldsymbol{J}_\pm(s) ds$$

for all $t \geq t_0$ on $G_\pm$. (See Section I.4.2.2 in [8].) In (1.1) we have imposed the boundary conditions of a perfect conductor on ∂G. On Σ the Maxwell equations imply the interface conditions

$$[\boldsymbol{D} \cdot \nu] = -\rho_\Sigma, \qquad [\boldsymbol{B} \cdot \nu] = 0, \qquad [\boldsymbol{E} \times \nu] = 0, \qquad [\boldsymbol{H} \times \nu] = \boldsymbol{J}_\Sigma \tag{1.2}$$

for $x \in \Sigma$ and $t \in (t_0, T)$, see Section I.4.2.4 of [8], where $[\boldsymbol{D} \cdot \nu] = (\boldsymbol{D}_+ - \boldsymbol{D}_-) \cdot \nu$ etc. In (1.2) the charge density ρ_Σ on the interface is determined by

$$\rho_\Sigma(t) = \rho_\Sigma(t)(t_0) - \int_{t_0}^t (\operatorname{div}_\Sigma \boldsymbol{J}_\Sigma(s) - [\boldsymbol{J} \cdot \nu](s)) ds, \qquad t \in J,$$

and the equations for $\boldsymbol{D}$ and $\boldsymbol{B}$ are true if they are valid at $t = t_0$, see Lemma 8.1.

The system (1.1) has to be complemented by constitutive relations between the electric and magnetic fields, where we choose $\boldsymbol{E}_\pm$ and $\boldsymbol{H}_\pm$ as state variables. There are various classes of such material laws. In the so-called retarded ones the fields $\boldsymbol{D}_\pm$ and $\boldsymbol{B}_\pm$ depend also on the past of $\boldsymbol{E}_\pm$ and $\boldsymbol{H}_\pm$, see [3], [11], [19], or [21]. In dynamical material laws the material response is modelled by additional evolution equations, see [2], [9], [14], [15], or [19]. We concentrate on instantaneous material laws, see [6] or [11], where the fields $\boldsymbol{D}_\pm$ and $\boldsymbol{B}_\pm$ are given by

$$\boldsymbol{D}_\pm(t,x) = \theta_{1,\pm}(x, \boldsymbol{E}_\pm(t,x), \boldsymbol{H}_\pm(t,x)), \qquad \boldsymbol{B}_\pm(t,x) = \theta_{2,\pm}(x, \boldsymbol{E}_\pm(t,x), \boldsymbol{H}_\pm(t,x))$$

for regular functions $\theta_\pm = (\theta_{1,\pm}, \theta_{2,\pm})\colon G_\pm \times \mathbb{R}^6 \to \mathbb{R}^6$. The most prominent example is the so called Kerr nonlinearity $\boldsymbol{D}_\pm = \boldsymbol{E}_\pm + \vartheta_\pm |\boldsymbol{E}_\pm|^2 \boldsymbol{E}_\pm$ and $\boldsymbol{B}_\pm = \boldsymbol{H}_\pm$ with $\vartheta_\pm \colon G_\pm \to \mathbb{R}$. We further assume that the current density decomposes as

$$\boldsymbol{J}_\pm = \boldsymbol{J}_{0,\pm} + \tilde{\sigma}_\pm(\boldsymbol{E}_\pm, \boldsymbol{H}_\pm)\boldsymbol{E}_\pm, \tag{1.3}$$

where $\boldsymbol{J}_{\pm,0}$ is a given external current density and $\tilde{\sigma}_\pm$ denotes the conductivity on $G_\pm$. If we insert these material laws into (1.1) and formally differentiate, we derive

$$(\partial_t \boldsymbol{D}_\pm, \partial_t \boldsymbol{B}_\pm) = \partial_{(\boldsymbol{E}_\pm, \boldsymbol{H}_\pm)} \theta_\pm(x, \boldsymbol{E}_\pm, \boldsymbol{H}_\pm) \partial_t (\boldsymbol{E}_\pm, \boldsymbol{H}_\pm) = (\operatorname{curl} \boldsymbol{H}_\pm - \boldsymbol{J}_\pm, -\operatorname{curl} \boldsymbol{E}_\pm)$$

from (1.1). Our main structural assumption is that $\partial_{(\boldsymbol{E}_\pm, \boldsymbol{H}_\pm)} \theta_\pm$ is symmetric and positive definite, which is true for the Kerr law for small $\boldsymbol{E}_\pm$ (and globally if $\vartheta_\pm \geq 0$). Such assumptions are quite standard already for linear Maxwell equations.

The resulting equations form a symmetric quasilinear hyperbolic system of first order. In order to transform (1.1) into a standard form, we introduce the matrices

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$A_j^{\mathrm{co}} = \begin{pmatrix} 0 & -J_j \\ J_j & 0 \end{pmatrix}, \qquad j \in \{1,2,3\}. \tag{1.4}$$

Note that $J_1\partial_1 + J_2\partial_2 + J_3\partial_3 = \operatorname{curl}$. Writing $\chi_\pm = \partial_{(\boldsymbol{E}_\pm,\boldsymbol{H}_\pm)}\theta_\pm$, $f_\pm = (-\boldsymbol{J}_{\pm,0}, 0)$, $\sigma_\pm = \left(\begin{smallmatrix} \tilde\sigma_\pm & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and using $u_\pm = (\boldsymbol{E}_\pm, \boldsymbol{H}_\pm)$ as a new variable, we obtain the system

$$\chi_\pm(u_\pm)\partial_t u_\pm + \sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j u_\pm + \sigma_\pm(u_\pm)u_\pm = f_\pm, \qquad (t,x) \in J \times G_\pm. \tag{1.5}$$

To recast the electric boundary and interface conditions in (1.1) and (1.2), we set

$$B_\nu = \begin{bmatrix} 0 & \nu_3 & -\nu_2 \\ -\nu_3 & 0 & \nu_1 \\ \nu_2 & -\nu_1 & 0 \end{bmatrix}, \; B_{\partial G} = \begin{bmatrix} B_\nu & 0 \end{bmatrix}, \; B_\Sigma = \begin{bmatrix} B_\nu & 0 & -B_\nu & 0 \\ 0 & B_\nu & 0 & -B_\nu \end{bmatrix} \tag{1.6}$$

on ∂G respectively Σ, and put $g = (0, \boldsymbol{J}_\Sigma)^T$. System (1.1) is then equivalent to the symmetric quasilinear hyperbolic initial boundary value problem

$$\begin{cases} \chi_\pm(u_\pm)\partial_t u_\pm + \sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j u_\pm + \tilde\sigma_\pm(u_\pm)u_\pm = f_\pm, & x \in G_\pm, \quad t \in J; \\ B_{\partial G} u_+ = 0, & x \in \partial G, \quad t \in J; \\ B_\Sigma(u_+, u_-) = g, & x \in \Sigma, \quad t \in J; \\ u(t_0) = u_0, & x \in G. \end{cases} \tag{1.7}$$

On ∂G we could also allow for inhomogeneous boundary values, see [24]. As noted above, the magnetic boundary and interface conditions and the divergence relations in (1.1) and (1.2) are true if we impose corresponding conditions on u_0. (See Lemma 7.25 in [23] and Lemma 8.1.) We look for solutions u of (1.7) in the spaces

$$\mathcal{G}_m(J \times G) = \bigcap_{j=0}^{m} C^j(\overline{J}, \mathcal{H}^{m-j}(G)), \tag{1.8}$$
$$\mathcal{H}^k(G) = \{v \in L^2(G) \colon v_+ \in H^k(G_+), v_- \in H^k(G_-)\},$$

cf. [5, 20], where $k, m \in \mathbb{N}_0$ and $v_\pm$ are the restrictions of v to $G_\pm$. We assume that the coefficients and data are appropriately smooth and compatible (in the sense of (6.5)). Our main Theorem 7.3 then shows that

(1) the system (1.7) has a unique maximal solution $u \in \mathcal{G}_m(J \times G)$ with $m \geq 3$,
(2) finite existence time can be characterized by blow-up in the Lipschitz-norm,
(3) the solution depends continuously on the data.

These results are based on the detailed regularity theory in Theorem 3.1 for the corresponding nonautonomous linear system

$$\begin{cases} A_{0,\pm}\partial_t u_\pm + \sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j u_\pm + D_\pm u_\pm = f_\pm, & x \in G_\pm, \quad t \in J; \\ B_{\partial G} u_+ = 0, & x \in \partial G, \quad t \in J; \\ B_\Sigma(u_+, u_-) = g, & x \in \Sigma, \quad t \in J; \\ u(t_0) = u_0, & x \in G. \end{cases} \tag{1.9}$$

We follow the same strategy as for the pure initial boundary value problem in [24] and [25]. We freeze a map $\hat{u}$ in the nonlinearities of (1.7). The resulting linear problem (1.9) can be solved in $\mathcal{G}_0(J \times G)$ for Lipschitz coefficients using [10]. In a lengthy procedure one can first show a priori estimates for solutions in $\mathcal{G}_m(J \times G)$ and then prove that the $\mathcal{G}_0$–solution actually belongs to $\mathcal{G}_m(J \times G)$, provided that data and coefficients are regular enough and compatible. Here one has to inductively intertwine different results for the tangential, time, and normal directions. The normal part is the most difficult one due to the characteristic interface and boundary (i.e., $A_1^{\mathrm{co}}\nu_1 + A_2^{\mathrm{co}}\nu_2 + A_3^{\mathrm{co}}\nu_3$ is singular). Our treatment of the normal regularity heavily relies on the structure of the Maxwell system, see Proposition 4.3 and Lemma 5.1.

For these arguments one has to localize the system. In this procedure one at first loses many of the zeros in the coefficient matrices of (1.7), which also become non-constant. However, using an additional transformation described in (3.8), (3.9) and (3.12), we obtain localized systems with an unchanged space-independent matrix A_3^{co} and space-independent boundary matrices B_Σ and $B_{\partial G}$. This fact allows us to partly separate the treatment of the normal directions from the others. This achievement is crucial for our analysis.

The nonlinear problem is then solved by a contraction argument in Theorem 6.5, which is basically standard though one has to be very careful setting up the constants. Here one uses the precise form of the a priori estimate in Theorem 3.1. In the derivation of the blow-up criterion and the continuous dependence of the data, one has to use the localized problems and the structure of the system once more.

Fortunately, the methods developed in [24] and [25] for the pure boundary value problem work quite well in the present situation. Many arguments can be adapted with straightforward changes. These are omitted below. However, at several points the structure of the problem changes significantly because of the interface condition. In the first step one has to apply the basic linear L^2 results of [10] to the localized interface problem on $\mathbb{R}^3$. To this aim, one rewrites the Maxwell system as a 12×12 initial boundary value system on the positive half-space by reflecting the coefficients from the negative one. In this procedure extra signs arise due to the reflection and spoil the structure of the pure Maxwell system appearing in [25], see e.g. (3.6) and (4.4). However, the core parts of the proof concerning normal regularity heavily depend on cancellation properties of the arising (linear) Maxwell system. Similarly the structure of the new 12×12 Maxwell system is crucial in order to obtain constant coefficients A_3^{co} and B_Σ in the localization procedure. These and several other arguments are closely tied to the structure of the interface problem. They are thus worked out in detail, though they lead to lengthy and intricate calculations.

In the next section we introduce our basic notation and some auxiliary results. The localization procedure is discussed in Section 3. The core a priori estimates and regularity results for the linear problem are shown in Sections 4 and 5, respectively. The basic fixed point argument is included in Section 6, and the main local wellposedness theorem in Section 7.

2. Function spaces and linear compatibility conditions

Standing notation: Let $m \in \mathbb{N}_0$ and set $\tilde{m} = \max\{m, 3\}$. We work with domains G, G_+, and G_- in $\mathbb{R}^3$ such that G is the disjoint union of G_+, G_-, and $\Sigma := \partial G_-$. Moreover it is assumed that Σ and ∂G have a positive distance and are *tame uniform* $C^{\tilde m+2}$–boundaries, see Definitions 2.24 and 5.4 of [23]. This means that they are uniform $C^{\tilde m+2}$-boundaries (see e.g. [1]) and that there exist a smooth partition of unity $(\theta_i)_{i\in\mathbb{N}_0}$ of G_- respectively G subordinate to the locally finite covering $(U_i)_{i\in\mathbb{N}_0}$ (where $U_0 = G_-$ respectively $U_0 = G$), as well as test functions σ_i with $\sigma_i = 1$ on $\operatorname{supp}\theta_i$ and ω_i with $\omega_i = 1$ on $\varphi_i(\operatorname{supp}\sigma_i)$, which are all uniformly bounded in $C^{\tilde m+2}$. Of course, compact boundaries of class $C^{\tilde m+2}$ or halfspaces satisfy these assumptions.

Our solutions take values in domains $\mathcal{U}_+$ and $\mathcal{U}_-$ in $\mathbb{R}^6$. We further write $\mathcal{L}(\mathcal{A}_0,\dots,\mathcal{A}_3,\mathcal{D})$ or $\mathcal{L}(\mathcal{A}_j,\mathcal{D})$ for the differential operator $\sum_{j=0}^3 \mathcal{A}_j\partial_j + \mathcal{D}$ with the coefficients $\mathcal{A}_j$ and $\mathcal{D}$, where $\partial_0 = \partial_t$. By J we mean an open time interval and we set $\Omega = J\times\mathbb{R}^3_+$. The image of a function v is designated by $\operatorname{im} v$. For a function w in $\mathcal{H}^1(G)$, we denote by $\partial_j w$ the $L^2(G)$-function whose restriction to $G_\pm$ coincides with $\partial_j w_\pm$. In the localization procedure we employ the matrices

$$\mathcal{A}_j^{\mathrm{co}} = \begin{pmatrix} A_j^{\mathrm{co}} & 0\\ 0 & A_j^{\mathrm{co}}\end{pmatrix} \quad \text{for } \; j\in\{1,2,3\} \qquad \text{and} \qquad \tilde{\mathcal{A}}_3^{\mathrm{co}} = \begin{pmatrix} A_3^{\mathrm{co}} & 0\\ 0 & -A_3^{\mathrm{co}}\end{pmatrix}. \tag{2.1}$$

To introduce the necessary trace operators, take coefficients $A_j\in\mathcal{W}^{1,\infty}(J\times G)$, i.e., the restrictions $A_{j,\pm}$ belong to $W^{1,\infty}(J\times G_\pm)$. Let v_+ be an element of $L^2(J\times G_+)$ such that $\sum_{j=0}^3 A_{j,+}\partial_j v_+$ is contained in $L^2(J\times G_+)$. Then the product $A_+(\nu)v_+ = (\sum_{j=0}^3 A_{j,+}\nu_j)v_+$ has a trace on $J\times\partial G_+$ belonging to $H^{-1/2}(J\times\partial G_+)$, cf. [23, 25], for instance. Here ν denotes the unit outer normal of $J\times G_+$. We may restrict this trace to $J\times\Sigma$ and to $J\times\partial G$, respectively. Moreover, the corresponding trace operators $\operatorname{Tr}_{J\times\Sigma,+}$ and $\operatorname{Tr}_{J\times\partial G}$ are given by the standard ones $\operatorname{tr}_{\Sigma,+}$ and $\operatorname{tr}_{\partial G,+}$, respectively, if v_+ takes values in in $H^1(G_+)$. Here we can replace the subscript $+$ by $-$. We further set

$$\operatorname{Tr}_{J\times\Sigma,\pm}(A(\nu)u) = (\operatorname{Tr}_{J\times\Sigma,+}(A_+(\nu)u_+), \operatorname{Tr}_{J\times\Sigma,-}(A_-(\nu)u_-))$$

if $u\in L^2(J\times G)$ satisfies $\sum_{j=0}^3 A_{j,\pm}\partial_j u_\pm \in L^2(J\times G_\pm)$, respectively

$$\operatorname{tr}_{\Sigma,\pm} u = (\operatorname{tr}_{\Sigma,+} u_+, \operatorname{tr}_{\Sigma,-} u_-)$$

if $u\in\mathcal{H}^1(G)$. We define the trace $\operatorname{Tr}_{J\times\Sigma,+}(MA(\nu)u)$ by $M\operatorname{Tr}_{J\times\Sigma,+}(A(\nu)u)$ for matrix-functions $M\in\mathcal{W}^{1,\infty}(J\times G)$, and correspondingly for the other trace operators. Finally, tr_Σ is the usual trace at Σ for functions in $H^1(G)$ or $C(G)$. On $\mathbb{R}^3_+ = \{x\in\mathbb{R}^3 : x_3>0\}$ we use the trace operator $\operatorname{Tr}_{J\times\partial\mathbb{R}^3_+}$ as introduced in [25].

We will employ the same function spaces as in [25], but we have to add variants allowing discontinuities across the interface. For reasons of clarity, we introduce all the spaces here. Take a subdomain $\tilde G$ of $\mathbb{R}^3$. We have already encountered the spaces $\mathcal{G}_m(J\times G)$ and $\mathcal{H}^m(G)$ in (1.8). Their norms are given by

$$\|v\|_{\mathcal{G}_m(J\times G)} = \max_{j\in\{0,\dots,m\}} \|\partial_t^j v\|_{L^\infty(J,\mathcal{H}^{m-j}(G))},$$
$$\|v\|^2_{\mathcal{H}^m(G)} = \|v_+\|^2_{H^m(G_+)} + \|v_-\|^2_{H^m(G_-)}.$$

We also need the simpler version

$$G_m(J\times\tilde G) = \bigcap\nolimits_{j=0}^m C^j(J, H^{m-j}(\tilde G)).$$

Set $e_{-\gamma}(t) = e^{-\gamma t}$ for $\gamma\ge 0$ and $t\in\mathbb{R}$. We use the time-weighted norms

$$\|v\|_{G_{m,\gamma}(J\times\tilde G)} = \max_{j\in\{0,\dots,m\}} \|e_{-\gamma}\partial_t^j v\|_{L^\infty(J,H^{m-j}(\tilde G))}$$

for all $\gamma \geq 0$. If $\gamma = 0$, we also write $\|\cdot\|_{G_m(J\times\tilde{G})}$ instead of $\|\cdot\|_{G_{m,0}(J\times\tilde{G})}$. Other function spaces on $J \times \tilde{G}$ or $J \times G$ are treated analogously. We further set

$$\tilde{G}_m(J\times\tilde{G}) = \{v\in L^\infty(J,L^2(\tilde{G}))\colon \partial^\alpha v\in L^\infty(J,L^2(\tilde{G})) \text{ for all } \alpha\in\mathbb{N}_0^4 \text{ with } |\alpha| \leq m\},$$

and define $\tilde{\mathcal{G}}_m(J \times \tilde{G})$ in a similar way. These spaces are endowed with the same norms as $G_m(J \times \tilde{G})$ respectively $\mathcal{G}_m(J \times G)$.

The coefficients of the linear problem will be contained in

$$\begin{aligned}
F_{m,k}(J\times\tilde{G}) &= \{A \in W^{1,\infty}(J\times\tilde{G})^{k\times k}\colon \partial^\alpha A \in L^\infty(J,L^2(\tilde{G})) \text{ for all } \alpha\in\mathbb{N}_0^4 \\
&\qquad\qquad \text{with } 1 \leq |\alpha| \leq m\},\\
\|A\|_{F_m(J\times\tilde{G})} &= \max\{\|A\|_{W^{1,\infty}(J\times\tilde{G})}, \max_{1\leq|\alpha|\leq m} \|\partial^\alpha A\|_{L^\infty(J,L^2(\tilde{G}))}\};\\
\mathcal{F}_{m,k}(J\times G) &= \{A\in\mathcal{W}^{1,\infty}(J\times G)\colon A_+ \in F_{m,k}(J\times G_+), A_- \in F_{m,k}(J\times G_-)\},\\
\|A\|_{\mathcal{F}_m(J\times G)} &= \max\{\|A_+\|_{F_m(J\times G_+)}, \|A_-\|_{F_m(J\times G_-)}\}.
\end{aligned}$$

The regularity of time-evaluations is measured in the spaces

$$\begin{aligned}
F^0_{m,k}(\tilde{G}) &= \{A\in L^\infty(\tilde{G})^{k\times k}\colon \partial^\alpha A \in L^2(\tilde{G})^{k\times k} \text{ for all } \alpha\in\mathbb{N}_0^3 \text{ with } 1\leq|\alpha|\leq m\},\\
\|A\|_{F^0_m(\tilde{G})} &= \max\{\|A\|_{L^\infty(\tilde{G})}, \max_{1\leq|\alpha|\leq m} \|\partial^\alpha A\|_{L^2(\tilde{G})}\};\\
\mathcal{F}^0_{m,k}(G) &= \{A\in L^\infty(G)^{k\times k}\colon A_+\in F^0_{m,k}(G_+), A_-\in F^0_{m,k}(G_-)\},\\
\|A\|_{\mathcal{F}^0_m(G)} &= \max\{\|A_+\|_{F^0_m(G_+)}, \|A_-\|_{F^0_m(G_-)}\}.
\end{aligned}$$

The subscript η always designates the subspace of matrix-valued maps A with $A^T = A \geq \eta > 0$. By $\mathcal{F}^{\mathrm{cp}}_{m,k}(J\times G)$ we mean those $A \in \mathcal{F}_{m,k}(J\times G)$ which are constant outside of a compact subset of $\overline{J\times G}$, and by $\mathcal{F}^{\mathrm{cv}}_{m,k}(J\times G)$ those which have a limit as $|(t,x)| \to \infty$. The variants for F instead of $\mathcal{F}$ are defined analogously. We will only use the parameters $k \in \{1, 6, 12\}$. As it will be clear from the context which parameter we consider, we usually drop it from our notation.

After the localization procedure below, the coefficients in front of the spatial derivatives belong to the space

$$\begin{aligned}
F^{\mathrm{cp}}_{m,\mathrm{coeff}}(\mathbb{R}^3_+) = \{\mathcal{A}\in F^{\mathrm{cp}}_{m,12}(\Omega)\colon \exists\, \mu_1,\mu_2,\mu_3 \in F^{\mathrm{cp}}_{m,1}(\Omega) \text{ independent of time,}\\
\text{such that } \mathcal{A} = \sum\nolimits_{j=1}^3 \mathcal{A}^{\mathrm{co}}_j \mu_j\}.
\end{aligned} \tag{2.2}$$

Finally, we introduce the space for the data on the interface, namely

$$E_m(J\times\Sigma) = \bigcap\nolimits_{j=0}^m H^j(J, H^{m+\frac{1}{2}-j}(\Sigma)).$$

We next state several bilinear estimates, which will be ubiquitous in the following. One proves this result by applying Lemma 2.1 from [25] on G_- and on G_+.

Lemma 2.1. *Take $m_1, m_2 \in \mathbb{N}$ with $m_1 \geq m_2$ and $m_1 \geq 2$ and a parameter $\gamma \geq 0$.*

(1) Let $k \in \{0,\dots,m_1\}$, $f \in \tilde{\mathcal{G}}_{m_1-k}(J\times G)$, and $g \in \tilde{\mathcal{G}}_k(J\times G)$. Then

$$fg \in \tilde{\mathcal{G}}_0(J\times G) \quad \textit{and} \quad \|fg\|_{\mathcal{G}_{0,\gamma}(J\times G)} \leq C\|f\|_{\mathcal{G}_{m_1-k}(J\times G)}\|g\|_{\mathcal{G}_{k,\gamma}(J\times G)}.$$

(2) Let $f \in \tilde{\mathcal{G}}_{m_1}(J\times G)$ and $g\in\tilde{\mathcal{G}}_{m_2}(J\times G)$. Then $fg \in \tilde{\mathcal{G}}_{m_2}(J\times G)$ and

$$\begin{aligned}
\|fg\|_{\mathcal{G}_{m_2,\gamma}(J\times G)} \leq C\min\{&\|f\|_{\mathcal{G}_{m_1}(J\times G)}\|g\|_{\mathcal{G}_{m_2,\gamma}(J\times G)},\\
&\|f\|_{\mathcal{G}_{m_1,\gamma}(J\times G)}\|g\|_{\mathcal{G}_{m_2}(J\times G)}\}.
\end{aligned}$$

The result remains true if we replace $\tilde{\mathcal{G}}_{m_1}(J \times G)$ *by* $\mathcal{F}_{m_1}(J \times G)$ *and if we replace both* $\tilde{\mathcal{G}}_{m_1}(J \times G)$ *and* $\tilde{\mathcal{G}}_{m_2}(J \times G)$ *by* $\mathcal{F}_{m_1}(J \times G)$ *and* $\mathcal{F}_{m_2}(J \times G)$.

(3) *Let* $k \in \{0, \ldots, m_1\}$, $f \in \mathcal{H}^{m_1-k}(G)$, *and* $g \in \mathcal{H}^k(G)$. *Then* $fg \in L^2(G)$ *and*

$$\|fg\|_{L^2(G)} \le C\|f\|_{\mathcal{H}^{m_1-k}(G)}\|g\|_{\mathcal{H}^k(G)}.$$

(4) *Let* $f \in \mathcal{H}^{m_1}(G)$ *and* $g \in \mathcal{H}^{m_2}(G)$. *Then* $fg \in \mathcal{H}^{m_2}(G)$ *and*

$$\|fg\|_{\mathcal{H}^{m_2}(G)} \le C\|f\|_{\mathcal{H}^{m_1}(G)}\|g\|_{\mathcal{H}^{m_2}(G)}.$$

The result is also valid with $\mathcal{H}^{m_1}(G)$ *replaced by* $\mathcal{F}^0_{m_1}(G)$.

In assertions (1) and (2) one can also remove the tildes.

In Section 5 we develop a regularization procedure which needs the next approximation result for the coefficients, taken from Lemma 2.2 of [25]. (There it is stated for $k \in \{1, 6\}$, but the proof works componentwise and thus for all $k \in \mathbb{N}$, cf. [23, Lemma 2.21].)

Lemma 2.2. *Let* $m \in \mathbb{N}$. *Choose* $A \in F_m(\Omega)$. *Then there exists a family* $\{A_\varepsilon\}_{\varepsilon>0}$ *in* $C^\infty(\overline{\Omega})$ *satisfying*

(1) $\partial^\alpha A_\varepsilon \in F_m(\Omega)$ *for all* $\alpha \in \mathbb{N}_0^4$ *and* $\varepsilon > 0$,
(2) $\|A_\varepsilon\|_{W^{1,\infty}(\Omega)} \le C\|A\|_{W^{1,\infty}(\Omega)}$ *and* $\|\partial^\alpha A_\varepsilon\|_{L^\infty(J,L^2(\mathbb{R}^3_+))} \le C\|A\|_{F_m(\Omega)}$ *for all multiindices* $1 \le |\alpha| \le m$ *and* $\varepsilon > 0$,
(3) $A_\varepsilon \to A$ *in* $L^\infty(\Omega)$ *as* $\varepsilon \to 0$,
(4) $A_\varepsilon(0) \to A(0)$ *in* $L^\infty(\mathbb{R}^3_+)$, *and* $\partial^\alpha A$ *and* $\partial^\alpha A_\varepsilon$ *have a representative in the space* $C(\overline{J}, L^2(\mathbb{R}^3_+))$ *with* $\partial^\alpha A_\varepsilon(0) \to \partial^\alpha A(0)$ *in* $L^2(\mathbb{R}^3_+)$ *as* $\varepsilon \to 0$ *for all* $\alpha \in \mathbb{N}_0^4$ *with* $0 < |\alpha| \le m-1$.

If A *is independent of time, the same is true for* A_ε *for all* $\varepsilon > 0$. *If* A *additionally belongs to* $F_m^{\mathrm{cp}}(\Omega)$, $F_m^{\mathrm{cv}}(\Omega)$, $F_{m,\eta}(\Omega)$ *for a number* $\eta > 0$, *or the intersection of two of these spaces, then the same is true for* A_ε *for all* $\varepsilon > 0$.

In order to discuss the compatibility conditions both for the linear Maxwell system (1.9) and its localized variants, we look at (1.9) with variable, time-independent coefficients $A_1, A_2, A_3 \in \mathcal{F}_m(J \times G)$ for a moment. We further fix coefficients $A_0 \in \mathcal{F}_{m,\eta}(J \times G)$ and $D \in \mathcal{F}_m(J \times G)$, as well as data $f \in \mathcal{H}^m(J \times G)$, $g \in E_m(J \times \Sigma)$, and $u_0 \in \mathcal{H}^m(G)$. Given a solution u in $\mathcal{G}_m(J \times G)$ of (1.9), we can differentiate the differential equation in (1.9) up to $(m-1)$-times in time by means of Lemma 2.1, obtaining the identity

$$\partial_t^p u(t) = S_{G,m,p}(t, A_0, A_1, A_2, A_3, D, f, u(t)), \tag{2.3}$$

for all $t \in \overline{J}$ and $p \in \{0, \ldots, m-1\}$. Here we inductively define the maps $S_{G,m,p} = S_{G,m,p}(t_0, A_j, D, f, u_0) = S_{G,m,p}(t_0, A_0, A_1, A_2, A_3, D, f, u_0)$ by

$$\begin{aligned} &S_{G,m,0,\pm} = u_{0,\pm}, \\ &S_{G,m,p,\pm} = A_{0,\pm}(t_0)^{-1}\Big(\partial_t^{p-1} f_\pm(t_0) - \sum_{j=1}^3 A_{j,\pm}\partial_j S_{G,m,p-1,\pm} \\ &\qquad - \sum_{l=1}^{p-1}\binom{p-1}{l}\partial_t^l A_{0,\pm}(t_0) S_{G,m,p-l,\pm} - \sum_{l=0}^{p-1}\binom{p-1}{l}\partial_t^l D_\pm(t_0) S_{G,m,p-1-l,\pm}\Big), \end{aligned} \tag{2.4}$$

for $1 \leq p \leq m$. On the other hand, we can differentiate the boundary condition in (1.9) up to $(m-1)$-times in time and insert t. It follows the equation

$$B_\Sigma \operatorname{tr}_{\Sigma,\pm}(\partial_t^p u(t)) = \partial_t^p g(t) \tag{2.5}$$

on Σ for all $0 \leq p \leq m-1$ and $t \in \overline{J}$. We proceed on ∂G in the same way. For $t = t_0$ equations (2.3) and (2.5) yield the compatibility conditions of order m

$$\begin{aligned} B_\Sigma \operatorname{tr}_{\Sigma,\pm} S_{G,m,p}(t_0, A_0, \ldots, A_3, D, f, u_0) &= \partial_t^p g(t_0) \qquad \text{on } \Sigma \quad \text{for } 0 \leq p \leq m-1, \\ B_{\partial G} \operatorname{tr}_{\partial G} S_{G,m,p}(t_0, A_0, \ldots, A_3, D, f, u_0) &= 0 \qquad \text{on } \partial G \quad \text{for } 0 \leq p \leq m-1 \end{aligned} \tag{2.6}$$

for the coefficients and data. These conditions are thus necessary for the existence of a solution in $\mathcal{G}_m(J \times G)$. In Section 5 their sufficiency will be shown. We will also need them to treat the half-space problem arising from the localization procedure, where $G = \mathbb{R}^3_+$, $k = 12$, and A_j, D, and B_Σ are replaced by $\mathcal{A}_j$, $\mathcal{D}$, and B. We often suppress G in the notation.

As the maps $S_{G,m,p}$ appear frequently, the following estimates are indispensable. They follow from Lemma 2.3 of [25] applied on G_+ and on G_-.

Lemma 2.3. *Let $\eta > 0$, $m \in \mathbb{N}$, and $\tilde{m} = \max\{m, 3\}$. Pick $r_0 > 0$. Choose $A_0 \in \mathcal{F}_{\tilde{m},\eta}(J \times G)$, time-independent $A_1, A_2, A_3 \in \mathcal{F}_{\tilde{m}}(J \times G)$, and $D \in \mathcal{F}_{\tilde{m}}(J \times G)$ with*

$$\begin{aligned} &\|A_i(t_0)\|_{\mathcal{F}^0_{\tilde{m}-1}(G)} \leq r_0, \quad \|D(t_0)\|_{\mathcal{F}^0_{\tilde{m}-1}(G)} \leq r_0, \\ &\max_{1 \leq j \leq m-1} \|\partial_t^j A_0(t_0)\|_{\mathcal{H}^{\tilde{m}-1-j}(G)} \leq r_0, \quad \max_{1 \leq j \leq m-1} \|\partial_t^j D(t_0)\|_{\mathcal{H}^{\tilde{m}-1-j}(G)} \leq r_0 \end{aligned}$$

for all $i \in \{0, \ldots, 3\}$. Take $f \in \mathcal{H}^m(J \times G)$ and $u_0 \in \mathcal{H}^m(G)$. Let $0 \leq p \leq m$. Then the function $S_{G,m,p}(t_0, A_0, \ldots, A_3, D, f, u_0)$ is contained in $\mathcal{H}^{m-p}(G)$. Moreover, there exist constants $C_{m,p} = C_{m,p}(\eta, r_0) > 0$ such that

$$\|S_{G,m,p}\|_{\mathcal{H}^{m-p}(G)} \leq C_{m,p}\Big(\sum_{j=0}^{p-1} \|\partial_t^j f(t_0)\|_{\mathcal{H}^{m-1-j}(G)} + \|u_0\|_{\mathcal{H}^m(G)}\Big).$$

3. Localization

We first discuss the localization procedure. In fact, in the logical order of our reasoning this section should be placed after the linear part as in [23], but we decided to start with it as it determines the linear problems we have to study. The next theorem thus assumes that we can solve the arising linear problems on the half space, which will be shown in Sections 4 and 5.

Theorem 3.1. *Let $\eta > 0$, $m \in \mathbb{N}_0$, and $\tilde{m} = \max\{m, 3\}$. Fix $r \geq r_0 > 0$. Take a domain G as described at the beginning of Section 2. Choose $t_0 \in \mathbb{R}$, $T' > 0$, $T \in (0, T')$, and set $J = (t_0, t_0 + T)$. Take coefficients $A_0 \in \mathcal{F}^{\mathrm{cv}}_{\tilde{m},6,\eta}(J \times G)$ and $D \in \mathcal{F}^{\mathrm{cv}}_{\tilde{m},6}(J \times G)$ satisfying*

$$\begin{aligned} &\|A_0\|_{\mathcal{F}_{\tilde{m}}(J \times G)} \leq r, \quad \|D\|_{\mathcal{F}_{\tilde{m}}(J \times G)} \leq r, \\ &\max\{\|A_0(t_0)\|_{\mathcal{F}^0_{\tilde{m}-1}(G)}, \max_{1 \leq j \leq \tilde{m}-1} \|\partial_t^j A_0(t_0)\|_{\mathcal{H}^{\tilde{m}-j-1}(G)}\} \leq r_0, \\ &\max\{\|D(t_0)\|_{\mathcal{F}^0_{\tilde{m}-1}(G)}, \max_{1 \leq j \leq \tilde{m}-1} \|\partial_t^j D(t_0)\|_{\mathcal{H}^{\tilde{m}-j-1}(G)}\} \leq r_0. \end{aligned}$$

Choose data $f \in \mathcal{H}^m(J \times G)$, $g \in E_m(J \times \Sigma)$, and $u_0 \in \mathcal{H}^m(G)$ such that the tuples $(t_0, A_0, A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, D, B_\Gamma, f, g, u_0)$ fulfills the compatibility conditions (2.6) of order m on $\Gamma = \Sigma$ and on $\Gamma = \partial G$.

Then the linear initial boundary value problem (1.9) *has a unique solution* u *in* $\mathcal{G}_m(J \times G)$. *Moreover, there is a number* $\gamma_m = \gamma_m(\eta, r, T') \geq 1$ *such that*

$$\|u\|^2_{\mathcal{G}_{m,\gamma}(J\times G)} \leq (C_{m,0} + TC_m) e^{mC_1 T} \Big(\sum_{j=0}^{m-1} \|\partial_t^j f(t_0)\|^2_{\mathcal{H}^{m-1-j}(G)} + \|g\|^2_{E_{m,\gamma}(J\times\Sigma)} + \|u_0\|^2_{\mathcal{H}^m(G)} \Big) + \frac{C_m}{\gamma} \|f\|^2_{\mathcal{H}^m_\gamma(J\times G)} \qquad (3.1)$$

for all $\gamma \geq \gamma_m$, *where* $C_i = C_i(\eta, r, T') \geq 1$ *and* $C_{i,0} = C_{i,0}(\eta, r_0) \geq 1$ *for* $i \in \{1, m\}$.

Proof. Set $\mathbb{N}_{-1} = \{-1, 0\} \cup \mathbb{N}$. Fix a covering $(U_i)_{i\in\mathbb{N}_{-1}}$ of $\overline{G}$, a sequence of sets $(V_i)_{i\in\mathbb{N}_{-1}}$, and sequences of functions $(\varphi_i)_{i\in\mathbb{N}_{-1}}$, $(\theta_i)_{i\in\mathbb{N}_{-1}}$, $(\sigma_i)_{i\in\mathbb{N}_{-1}}$, and $(\omega_i)_{i\in\mathbb{N}_{-1}}$ as in Definition 5.4 in [23] for the tame uniform $C^{\tilde{m}+2}$-boundary Σ of G_- (complemented by a domain U_{-1} covering $\overline{G} \setminus \overline{G_-}$ and corresponding functions). We further take $\varphi_i = \mathrm{id}$ for $i \in \{-1, 0\}$. Here, $\varphi_i : U_i \to V_i$ is a chart, $(U_i)_{i\in\mathbb{N}}$ is a cover of Σ with positive distance to ∂G, the set U_0 covers $G_- \setminus \bigcup_{i=1}^\infty U_i$, while $\overline{G_+} \setminus \bigcup_{i=1}^\infty U_i$ is contained in U_{-1}. In particular, $(\theta_i)_{i\in\mathbb{N}_{-1}}$ is a smooth partition of unity on G. We recall that the maps ω_i equal 1 on the sets $K_i = \varphi_i(\operatorname{supp}\sigma_i)$ and that $\sigma_i = 1$ on $\operatorname{supp}\theta_i$ for all $i \in \mathbb{N}_{-1}$. Moreover, $\varphi_i(U_i \cap G_+) = \{y \in V_i : y_3 > 0\}$ and $\varphi_i(U_i \cap G_-) = \{y \in V_i : y_3 < 0\}$ for $i \in \mathbb{N}$. We use the same symbol for a function and its zero extensions.

I) In the first step we determine the coefficients of the localized problem on $\mathbb{R}^3_+$. To this aim, we write $\psi_i = \varphi_i^{-1} : V_i \to U_i$, and define the composition operators

$$\Phi_i : L^2(U_i) \to L^2(V_i), \quad v \mapsto v \circ \psi_i; \qquad \Phi_i^{-1} : L^2(V_i) \to L^2(U_i), \quad v \mapsto v \circ \varphi_i;$$

for all $i \in \mathbb{N}_{-1}$. Observe that φ_i, and thus Φ_i, are the identity for $i \in \{-1, 0\}$. The operators Φ_i and Φ_i^{-1} act componentwise on vector-valued functions. With a slight abuse of notation we also denote the composition with ψ_i on $L^2(J \times V_i)$ and $H^{-1}(J \times V_i)$ by Φ_i, and analogously for Φ_i^{-1}.

For $v \in L^2(J \times V_i)$ we introduce the differential operator

$$\begin{aligned} \mathfrak{A}^i_\pm v_\pm &:= \Phi_i \Big(A_{0,\pm}\partial_t + \sum_{j=1}^3 A_j^{\mathrm{co}} \partial_j + D_\pm \Big) \Phi_i^{-1} v_\pm \\ &= \Phi_i A_{0,\pm}\, \partial_t v_\pm + \sum_{l=1}^3 \Big(\sum_{j=1}^3 A_j^{\mathrm{co}} \Phi_i \partial_j \varphi_{i,l} \Big) \partial_l v_\pm + \Phi_i D_\pm\, v_\pm, \end{aligned} \qquad (3.2)$$

where $\varphi_{i,l}$ is the l-th component of φ_i for all $i \in \mathbb{N}$. Throughout, for a function v defined on V_i respectively $\mathbb{R}^3$ we write $v_\pm$ for the restrictions to $V_i \cap \mathbb{R}^3_\pm$ respectively to $\mathbb{R}^3_\pm$, where $\mathbb{R}^3_- = \{x \in \mathbb{R}^3 : x_3 < 0\}$. We define

$$\tilde{A}_0^i = \Phi_i A_0, \qquad \tilde{A}_l^i = \Phi_i \Big(\sum_{j=1}^3 A_j^{\mathrm{co}} \partial_j \varphi_{i,l} \Big), \qquad \tilde{D}^i = \Phi_i D \qquad (3.3)$$

on V_i for all $i \in \mathbb{N}$ and $l \in \{1, 2, 3\}$, as well as $\tilde{A}_0^0 = \Phi_0 A_0 = A_0$ and $\tilde{D}^0 = \Phi_0 D = D$ on U_0, and $\tilde{A}_0^{-1} = \Phi_{-1} A_0 = A_0$ and $\tilde{D}^{-1} = \Phi_{-1} D = D$ on U_{-1}. (This notation is only used if confusion with a matrix inverse is not possible.)

Lemma 5.1 in [23] yields numbers $z(i) \in \{1, 2, 3\}$ and $\tau \in (0, 1)$ with

$$|\partial_{z(i)} \varphi_{i,3}| \geq \tau \qquad \text{on } \; U_i \qquad (3.4)$$

for all $i \in \mathbb{N}$. We pick a point $y_i \in V_i$ for each $i \in \mathbb{N}$ and set

$$\begin{aligned} A_0^i &= \omega_i \tilde{A}_0^i + (1-\omega_i)\eta \qquad \text{for} \quad i \in \mathbb{N}_{-1}, \\ A_j^i &= \omega_i \tilde{A}_j^i + (1-\omega_i)\frac{\partial_{z(i)}\varphi_{i,3}}{|\partial_{z(i)}\varphi_{i,3}|}(\psi_i(y_i))A^{\mathrm{co}}_{z(i)} \qquad \text{for} \quad i \in \mathbb{N}, \quad j \in \{1,2,3\}, \\ D^i &= \omega_i \tilde{D}^i \qquad \text{for} \quad i \in \mathbb{N}_{-1}. \end{aligned} \tag{3.5}$$

These coefficients will only be multiplied with functions supported in the set where $\omega_i = 1$, but we need the above extensions in our reasoning. The differential operator $\mathfrak{A}^i$ can thus be extended to a differential operator on $\mathbb{R}^3$ by setting

$$\mathfrak{A}^i_{\pm} v_{\pm} = A^i_{0,\pm}\partial_t v_{\pm} + \sum\nolimits_{j=1}^3 A^i_{j,\pm}\partial_j v_{\pm} + D^i_{\pm} v_{\pm}$$

for all $v \in L^2(J \times \mathbb{R}^3)$ and $i \in \mathbb{N}$. To rewrite the interface problem on $\mathbb{R}^3$ as an boundary value problem on $\mathbb{R}^3_+$, we set

$$\breve{A}^i_{j,-}(\cdot, x_3) = A^i_{j,-}(\cdot, -x_3), \quad \breve{A}^i_{3,-}(\cdot, x_3) = -A^i_{3,-}(\cdot, -x_3), \quad \breve{D}^i_{-}(\cdot, x_3) = D^i_{-}(\cdot, -x_3)$$

for $j \in \{0,1,2\}$, and introduce the (12×12)-matrices

$$\mathcal{A}^i_j = \begin{pmatrix} A^i_{j,+} & 0 \\ 0 & \breve{A}^i_{j,-} \end{pmatrix} \qquad \text{and} \qquad \mathcal{D}^i = \begin{pmatrix} D^i_{+} & 0 \\ 0 & \breve{D}^i_{-} \end{pmatrix} \tag{3.6}$$

for all $j \in \{0, \dots, 3\}$ on $J \times \mathbb{R}^3_+$. Here the part of the equation on $\mathbb{R}^3_-$ is reflected to $\mathbb{R}^3_+$ and written in the new 6 lines. The minus in front of $A^i_{3,-}$ is needed to compensate the inner derivative when applying ∂_3.

We turn our attention to the interface condition. By Remark 5.2 in [23], the vector field $\nabla\varphi_{i,3}$ is normal to Σ, and hence there is a number $\kappa_i(x) \in \mathbb{R}$ with

$$\nabla\varphi_{i,3}(x) = \kappa_i(x)\nu(x)$$

for all $x \in \Sigma \cap U_i$ and $i \in \mathbb{N}$. In particular, $\kappa_i = \nabla\varphi_{i,3} \cdot \nu$ belongs to $C^{m+1}(\Sigma \cap U_i, \mathbb{R})$ for all $i \in \mathbb{N}$. Moreover, we can extend the product $\kappa_i \nu$ smoothly from $U_i \cap \Sigma$ to U_i by $\nabla\varphi_{i,3}$. Let $i \in \mathbb{N}$. We now introduce the interface matrices

$$\hat{B}^i = \omega_i \Phi_i(\kappa_i B_\Sigma) + (1-\omega_i)\frac{\partial_{z(i)}\varphi_{i,3}}{|\partial_{z(i)}\varphi_{i,3}|}(\psi_i(y_i))B^{\mathrm{co}}_{z(i)}, \qquad B^{\mathrm{co}}_j := B_\Sigma(e_j), \tag{3.7}$$

on $\mathbb{R}^3$ for $j \in \{1,2,3\}$, where e_j denotes the jth unit vector in $\mathbb{R}^3$ and $B_\Sigma(e_j)$ is given by the second line in (1.6) with $\nu = e_j$. Define the function $b_{z(i)} \colon \mathbb{R}^3 \to \mathbb{R}$ by

$$b_{z(i)} = \omega_i \Phi_i \partial_{z(i)}\varphi_{i,3} + (1-\omega_i)\frac{\partial_{z(i)}\varphi_{i,3}}{|\partial_{z(i)}\varphi_{i,3}|}(\psi_i(y_i)).$$

Since $\partial_{z(i)}\varphi_{i,3}$ does not change signs on U_i, estimate (3.4) implies the lower bound

$$|b_{z(i)}| = \omega_i|\Phi_i\partial_{z(i)}\varphi_{i,3}| + (1-\omega_i) \geq \tau\omega_i + 1 - \omega_i = 1 - (1-\tau)\omega_i \geq \tau$$

on $\mathbb{R}^3$ as $\tau \in (0,1)$. Consequently, the functions $b_{z(i)}$ and $b_{z(i)}^{-1}$ belong to $C^{m+1}(\mathbb{R}^3)$ and their restrictions to $\partial\mathbb{R}^3_+$ are elements of $C^{m+1}(\partial\mathbb{R}^3_+)$.

We next want to transform the coefficients $\mathcal{A}^i_3$ and $\hat{B}^i$ to constant coefficients similar to those in the original Maxwell system (1.9) on G. Here we only consider the case $z(i) = 3$ with $b_3 \geq \tau$ on $\mathbb{R}^3$. The other ones are treated analogously, cf. Section 5 of [23]. To rewrite $\mathcal{A}_3^i$, we use the matrices

$$\hat{A}_3^i = \begin{pmatrix} 0 & b_3^i & -\omega_i \Phi_i \partial_2 \varphi_{i,3} \\ -b_3^i & 0 & \omega_i \Phi_i \partial_1 \varphi_{i,3} \\ \omega_i \Phi_i \partial_2 \varphi_{i,3} & -\omega_i \Phi_i \partial_1 \varphi_{i,3} & 0 \end{pmatrix}$$

on $\mathbb{R}^3$. Let Q be the reflection operator defined by $Qv(\cdot, x_3) = v(\cdot, -x_3)$ for any $v \in L^2_{\mathrm{loc}}(J \times \mathbb{R}^3)$. The coefficient $\mathcal{A}_3^i$ can now be written as

$$\mathcal{A}_3^i = \begin{pmatrix} A_{3,+}^i & 0 \\ 0 & -QA_{3,-}^i \end{pmatrix} = \begin{pmatrix} 0 & \hat{A}_3^i & 0 & 0 \\ -\hat{A}_3^i & 0 & 0 & 0 \\ 0 & 0 & 0 & -Q\hat{A}_3^i \\ 0 & 0 & Q\hat{A}_3^i & 0 \end{pmatrix}.$$

Our main tool are the matrix-valued functions

$$\hat{G}_r^i = b_3^{i,-1/2} \begin{pmatrix} 1 & 0 & \omega_i \Phi_i \partial_1 \varphi_{i,3} \\ 0 & 1 & \omega_i \Phi_i \partial_2 \varphi_{i,3} \\ 0 & 0 & b_3^i \end{pmatrix}, \qquad \mathcal{G}_r^i = \begin{pmatrix} \hat{G}_r^i & 0 & 0 & 0 \\ 0 & \hat{G}_r^i & 0 & 0 \\ 0 & 0 & Q\hat{G}_r^i & 0 \\ 0 & 0 & 0 & Q\hat{G}_r^i \end{pmatrix} \tag{3.8}$$

on $\mathbb{R}^3$. Equation (2.1) then yields the first desired transformation

$$(\mathcal{G}_r^i)^T \mathcal{A}_3^i \mathcal{G}_r^i = \begin{pmatrix} A_3^{\mathrm{co}} & 0 \\ 0 & -A_3^{\mathrm{co}} \end{pmatrix} = \tilde{\mathcal{A}}_3^{\mathrm{co}}. \tag{3.9}$$

For the boundary condition, we note that

$$\hat{B}^i = \begin{pmatrix} \hat{B}_{3,\mathrm{bl}}^i & 0 & -\hat{B}_{3,\mathrm{bl}}^i & 0 \\ 0 & \hat{B}_{3,\mathrm{bl}}^i & 0 & -\hat{B}_{3,\mathrm{bl}}^i \end{pmatrix} \qquad \text{with} \quad \hat{B}_{3,\mathrm{bl}}^i := \hat{A}_3^i.$$

Setting $\hat{R}_3^i = (\hat{G}_r^i)^T$, we calculate

$$\hat{R}_3^i \hat{B}_{3,\mathrm{bl}}^i = b_3^{i,1/2} \begin{pmatrix} 0 & 1 & -\omega_i \Phi_i (\partial_2 \varphi_{i,3}) b_3^{i,-1} \\ -1 & 0 & \omega_i \Phi_i (\partial_1 \varphi_{i,3}) b_3^{i,-1} \\ 0 & 0 & 0 \end{pmatrix} =: \tilde{B}_{\mathrm{bl},3}^i$$

on $\partial\mathbb{R}^3_+$. Consequently,

$$R_3^i \hat{B}^i := \begin{pmatrix} \hat{R}_3^i & 0 \\ 0 & \hat{R}_3^i \end{pmatrix} \cdot \hat{B}^i = \begin{pmatrix} \tilde{B}_{\mathrm{bl},3}^i & 0 & -\tilde{B}_{\mathrm{bl},3}^i & 0 \\ 0 & \tilde{B}_{\mathrm{bl},3}^i & 0 & -\tilde{B}_{\mathrm{bl},3}^i \end{pmatrix}. \tag{3.10}$$

Delete in $\tilde{B}_{\mathrm{bl},3}^i$ the line of zeros and call the resulting matrix $B_{\mathrm{bl},3}^i$. We then introduce the boundary matrices

$$B_3^i = \begin{pmatrix} B_{\mathrm{bl},3}^i & 0 & -B_{\mathrm{bl},3}^i & 0 \\ 0 & B_{\mathrm{bl},3}^i & 0 & -B_{\mathrm{bl},3}^i \end{pmatrix}. \tag{3.11}$$

We next infer that

$$\begin{aligned} B_{\mathrm{bl},3}^i \hat{G}_r^i &= b_3^{i,1/2} \begin{pmatrix} 0 & 1 & -\omega_i \Phi_i (\partial_2 \varphi_{i,3}) b_3^{i,-1} \\ -1 & 0 & \omega_i \Phi_i (\partial_1 \varphi_{i,3}) b_3^{i,-1} \end{pmatrix} b_3^{i,-1/2} \begin{pmatrix} 1 & 0 & \omega_i \Phi_i \partial_1 \varphi_{i,3} \\ 0 & 1 & \omega_i \Phi_i \partial_2 \varphi_{i,3} \\ 0 & 0 & b_3^i \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} =: B_{\mathrm{bl}}. \end{aligned}$$

On the boundary $\partial\mathbb{R}^3_+$ we thus obtain the second crucial identity

$$B_3^i \cdot \mathcal{G}_r^i = \begin{pmatrix} B_{\mathrm{bl}} & 0 & -B_{\mathrm{bl}} & 0 \\ 0 & B_{\mathrm{bl}} & 0 & -B_{\mathrm{bl}} \end{pmatrix} =: \mathcal{B}^{\mathrm{co}}. \tag{3.12}$$

Finally, we define the matrices

$$C_{\mathrm{bl}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathcal{C}^{\mathrm{co}} = \begin{pmatrix} 0 & -C_{\mathrm{bl}} & 0 & -C_{\mathrm{bl}} \\ C_{\mathrm{bl}} & 0 & C_{\mathrm{bl}} & 0 \end{pmatrix} =: \mathcal{M}^{\mathrm{co}}.$$

Using (1.4), we then compute

$$\begin{aligned}
C_{\mathrm{bl}}^T \cdot B_{\mathrm{bl}} &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -J_3, \qquad B_{\mathrm{bl}}^T C_{\mathrm{bl}} = (-J_3)^T = J_3, \\
(\mathcal{C}^{\mathrm{co}})^T \mathcal{B}^{\mathrm{co}} &= \begin{pmatrix} 0 & C_{\mathrm{bl}}^T \\ -C_{\mathrm{bl}}^T & 0 \\ 0 & C_{\mathrm{bl}}^T \\ -C_{\mathrm{bl}}^T & 0 \end{pmatrix} \cdot \begin{pmatrix} B_{\mathrm{bl}} & 0 & -B_{\mathrm{bl}} & 0 \\ 0 & B_{\mathrm{bl}} & 0 & -B_{\mathrm{bl}} \end{pmatrix} \\
&= \begin{pmatrix} 0 & C_{\mathrm{bl}}^T B_{\mathrm{bl}} & 0 & -C_{\mathrm{bl}}^T B_{\mathrm{bl}} \\ -C_{\mathrm{bl}}^T B_{\mathrm{bl}} & 0 & C_{\mathrm{bl}}^T B_{\mathrm{bl}} & 0 \\ 0 & C_{\mathrm{bl}}^T B_{\mathrm{bl}} & 0 & -C_{\mathrm{bl}}^T B_{\mathrm{bl}} \\ -C_{\mathrm{bl}}^T B_{\mathrm{bl}} & 0 & C_{\mathrm{bl}}^T B_{\mathrm{bl}} & 0 \end{pmatrix}, \\
(\mathcal{B}^{\mathrm{co}})^T \mathcal{C}^{\mathrm{co}} &= \begin{pmatrix} 0 & -B_{\mathrm{bl}}^T C_{\mathrm{bl}} & 0 & -B_{\mathrm{bl}}^T C_{\mathrm{bl}} \\ B_{\mathrm{bl}}^T C_{\mathrm{bl}} & 0 & B_{\mathrm{bl}}^T C_{\mathrm{bl}} & 0 \\ 0 & B_{\mathrm{bl}}^T C_{\mathrm{bl}} & 0 & B_{\mathrm{bl}}^T C_{\mathrm{bl}} \\ -B_{\mathrm{bl}}^T C_{\mathrm{bl}} & 0 & -B_{\mathrm{bl}}^T C_{\mathrm{bl}} & 0 \end{pmatrix}.
\end{aligned}$$

We can now check certain algebraic conditions needed to apply [10], namely

$$\begin{aligned}
\operatorname{Re}\big((\mathcal{C}^{\mathrm{co}})^T \mathcal{B}^{\mathrm{co}}\big) &= \frac{1}{2}\Big((\mathcal{C}^{\mathrm{co}})^T \mathcal{B}^{\mathrm{co}} + (\mathcal{B}^{\mathrm{co}})^T \mathcal{C}^{\mathrm{co}}\Big) = \begin{pmatrix} 0 & -J_3 & 0 & 0 \\ J_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & J_3 \\ 0 & 0 & -J_3 & 0 \end{pmatrix} \\
&= \begin{pmatrix} A_3^{\mathrm{co}} & 0 \\ 0 & -A_3^{\mathrm{co}} \end{pmatrix} = \tilde{\mathcal{A}}_3^{\mathrm{co}}, \\
\mathcal{M}^{\mathrm{co}} \tilde{\mathcal{A}}_3^{\mathrm{co}} &= \mathcal{B}^{\mathrm{co}}.
\end{aligned} \tag{3.13}$$

To simplify the notation, we write B^i and R^i instead of $B^i_{z(i)}$ and $R^i_{z(i)}$ in the following. Observe that the restrictions of B^i and R^i to $\mathbb{R}^3_+$ belong to $C^{\tilde{m}+1}(\overline{\mathbb{R}^3_+})$. The rank of $\mathcal{B}^{\mathrm{co}}$ and $\mathcal{C}^{\mathrm{co}}$ is 4 and $R^i(x)$ is invertible for all $x \in \overline{\mathbb{R}^3_+}$. The inverse of R^i is as regular as R^i itself. Moreover, the transformed coefficients satisfy

$$\begin{aligned}
\tilde{\mathcal{A}}_0^i &:= (\mathcal{G}_r^i)^T \begin{pmatrix} A^i_{0,+} & 0 \\ 0 & QA^i_{0,-} \end{pmatrix} \mathcal{G}_r^i \;\in\; \mathcal{F}^{\mathrm{cp}}_{\tilde{m},\eta}(\Omega), \\
\tilde{\mathcal{A}}_j^i &:= (\mathcal{G}_r^i)^T \mathcal{A}_j^i \mathcal{G}_r^i \;\in\; \mathcal{F}^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff}}(\mathbb{R}^3_+) \qquad \text{for } j \in \{1,2\}, \\
\tilde{\mathcal{D}}^i &:= (\mathcal{G}_r^i)^T \mathcal{D}^i \mathcal{G}_r^i - \sum\nolimits_{j=1}^3 (\mathcal{G}_r^i)^T \mathcal{A}_j^i \mathcal{G}_r^i \partial_j (\mathcal{G}_r^i)^{-1} \mathcal{G}_r^i \;\in\; \mathcal{F}^{\mathrm{cp}}_{\tilde{m}}(\Omega),
\end{aligned} \tag{3.14}$$

where we reduced the size of η independently of i if necessary.

We next fix a constant M_1 as in Lemma 5.1 of [23] and constants M_2, M_3, and M_4 as in Definition 5.4 in [23] for the tame uniform $C^{\tilde{m}+2}$-boundary Σ of G_-. We put $M = \max_{i=1,\dots,4} M_i$ The construction of our extended coefficients then shows

$$\begin{aligned}
&\|\mathcal{A}_0^i\|_{F_m(\Omega)} \leq C(M_1, M_4)\|A_0\|_{\mathcal{F}_m(J\times G)} \leq R,\\
&\max\{\|\mathcal{A}_0^i(0)\|_{F^0_{\tilde{m}-1}(\mathbb{R}^3_+)}, \max_{1\leq j\leq \tilde{m}-1} \|\partial_t^j \mathcal{A}_0^i(0)\|_{H^{\tilde{m}-j-1}(\mathbb{R}^3_+)}\}\\
&\qquad \leq C(M_1, M_4)\max\{\|A_0(0)\|_{\mathcal{F}^0_{\tilde{m}-1}(G)}, \max_{1\leq j\leq \tilde{m}-1} \|\partial_t^j A_0(0)\|_{\mathcal{H}^{\tilde{m}-j-1}(G)}\} \leq R_0,\\
&\|\mathcal{A}_j^i\|_{F_{\tilde{m}}(\Omega)} \leq C(M_1, M_4) \leq R, \qquad (3.15)\\
&\|\mathcal{D}^i\|_{F_{\tilde{m}}(\Omega)} \leq C(M_1, M_4)\|D\|_{\mathcal{F}_m(J\times G)} \leq R,\\
&\max\{\|\mathcal{D}^i(0)\|_{F^0_{\tilde{m}-1}(\mathbb{R}^3_+)}, \max_{1\leq j\leq \tilde{m}-1} \|\partial_t^j \mathcal{D}^i(0)\|_{H^{\tilde{m}-j-1}(\mathbb{R}^3_+)}\}\\
&\qquad \leq C(M_1, M_4)\max\{\|D(0)\|_{\mathcal{F}^0_{\tilde{m}-1}(G)}, \max_{1\leq j\leq \tilde{m}-1} \|\partial_t^j D(0)\|_{\mathcal{H}^{\tilde{m}-j-1}(G)}\} \leq R,
\end{aligned}$$

for all $i \in \mathbb{N}$ and $j \in \{1, 2, 3\}$, and for constants $R = R(M, r)$ and $R_0 = R_0(M, r_0)$.

II) After introducing some notation, we relate the compatibility conditions of the localized problem to the given ones. Using the reflection operator Q from step I), we define the maps

$$\begin{aligned}
&\mathcal{R}_6\colon L^2_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^6) \to L^2_{\mathrm{loc}}(\mathbb{R}^3_+, \mathbb{R}^{12}), \quad v \mapsto (v_+, Qv_-),\\
&\mathcal{R}_{6\times 6}\colon L^2_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^{6\times 6}) \to L^2_{\mathrm{loc}}(\mathbb{R}^3_+, \mathbb{R}^{12\times 12}), \quad A \mapsto \begin{pmatrix} A_+ & 0 \\ 0 & QA_- \end{pmatrix},\\
&\hat{\mathcal{R}}_{6\times 6}\colon L^2_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{R}^{6\times 6}) \to L^2_{\mathrm{loc}}(\mathbb{R}^3_+, \mathbb{R}^{12\times 12}), \quad A \mapsto \begin{pmatrix} A_+ & 0 \\ 0 & -QA_- \end{pmatrix}.
\end{aligned}$$

As it will be clear from the context which operator we consider, we drop the index, and we put $\mathcal{R}_i = \mathrm{id}$ for $i \in \{-1, 0\}$ and $\mathcal{R}_i = \mathcal{R}$ for $i \in \mathbb{N}$.

In step IV) we determine the initial (boundary) value problem solved by the functions $\mathcal{R}_i\Phi_i(\theta_i u)$ on $J \times G$, $J \times \mathbb{R}^3$, respectively $J \times \mathbb{R}^3_+$. For given functions $v \in \mathcal{G}_m(J \times G)$ and $h \in \mathcal{H}^m(J \times G)$, then the transformed data

$$\begin{aligned}
f^i(h, v) &= \mathcal{R}_i\Phi_i(\theta_i h) + \mathcal{R}_i\Phi_i\Big(\sum_{j=1}^3 A_j^{\mathrm{co}}\partial_j\theta_i v\Big) \ \in \ H^m(\Omega),\\
g^i &= \big((\mathrm{tr}_{\partial\mathbb{R}^3_+} R^i)\tilde{\Phi}_i(\mathrm{tr}_\Sigma(\theta_i)\kappa_i g)\big)_{\alpha(i)} \ \in \ E_m(J \times \partial\mathbb{R}^3_+),\\
u_0^i &= \mathcal{R}_i\Phi_i(\theta_i u_0) \ \in \ H^m(\mathbb{R}^3_+), \qquad (3.16)
\end{aligned}$$

arise for $i \in \mathbb{N}_{-1}$ respectively $i \in \mathbb{N}$. Here $\alpha(i)$ denotes the 4-tuple obtained by removing $z(i)$ and $z(i) + 3$ from $(1, \dots, 6)$ and $\tilde{\Phi}_i$ the composition operator with the restriction of ψ_i to $U_i \cap \Sigma$.

Let $v \in \mathcal{G}_m(J \times G)$ be a map with $\partial_t^p v(0) = S_{G,m,p}(0, A_0, A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, D, f, u_0)$ for all $p \in \{0, \dots, m-1\}$, with the operators $S_{G,m,p}$ from (2.4). We abbreviate

$$\begin{aligned}
S^i_{m,p} &= S_{\mathbb{R}^3_+,m,p}(0, \mathcal{A}_0^i, \mathcal{A}_1^i, \mathcal{A}_2^i, \mathcal{A}_3^i, \mathcal{D}^i, f^i(f, v), u_0^i), \qquad (3.17)\\
S_{m,p} &= S_{G,m,p}(0, A_0, A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, D, f, u_0)
\end{aligned}$$

for all $p \in \{0, \dots, m\}$ and $i \in \mathbb{N}$. The maps $S^i_{m,p}$ and $S_{m,p}$ are well-defined due to the regularity of the coefficients and the data. Fix an index $i \in \mathbb{N}$. We claim that

$$S^i_{m,p} = \mathcal{R}\Phi_i(\theta_i S_{m,p}) \qquad \text{for all } \ p \in \{0, \dots, m\}. \qquad (3.18)$$

To show this assertion, we first note that

$$S^i_{m,0} = u^i_0 = \mathcal{R}\Phi_i(\theta_i u_0) = \mathcal{R}\Phi_i(\theta_i S_{m,0}).$$

Next, let the claim (3.18) be true for all $l \in \{0, \ldots, p-1\}$ and some $p \in \{1, \ldots, m\}$. The definition of the operators $S_{\mathbb{R}^3_+,m,p}$ then yields

$$\begin{aligned} S^i_{m,p} = \mathcal{A}^i_0(0)^{-1}\Big[&\partial_t^{p-1} f^i(f,v)(0) - \sum_{j=1}^{3} \mathcal{A}^i_j \partial_j S^i_{m,p-1} - \sum_{l=1}^{p-1} \binom{p-1}{l} \partial_t^l \mathcal{A}^i_0(0) S^i_{m,p-l} \\ &- \sum_{l=0}^{p-1} \binom{p-1}{l} \partial_t^l \mathcal{D}^i(0) S^i_{m,p-1-l}\Big]. \end{aligned} \tag{3.19}$$

The induction hypothesis implies that

$$\operatorname{supp} S^i_{m,p-l} = \operatorname{supp} \Phi_i(\theta_i S_{m,p}) \subseteq \operatorname{supp} \Phi_i \theta_i \subseteq K_i$$

for all $l \in \{1, \ldots, p\}$. Together with (3.5) and (3.6), we thus obtain

$$\mathcal{A}^i_j \partial_j S^i_{m,p-1} = \mathcal{R}(A^i_j)\partial_j S^i_{m,p-1} = \mathcal{R}(\tilde{A}^i_j)\partial_j \mathcal{R}\Phi_i(\theta_i S_{m,p-1}) = \mathcal{R}(\tilde{A}^i_j \partial_j \Phi_i(\theta_i S_{m,p-1}))$$

for $j \in \{1, 2\}$, as $\omega_i = 1$ on K_i. Similarly it follows

$$\mathcal{A}^i_3 \partial_3 S^i_{m,p-1} = \hat{\mathcal{R}}(A^i_3)\partial_3 \mathcal{R}\Phi_i(\theta_i S_{m,p-1}) = \mathcal{R}(\tilde{A}^i_3 \partial_3 \Phi_i(\theta_i S_{m,p-1})).$$

Using also (3.3), we next compute

$$\partial_j(\Phi_i(\theta_i S_{m,p-1})) = (\nabla(\theta_i S_{m,p-1})) \circ \psi_i \, \partial_j \psi_i = \sum_{l=1}^{3} \Phi_i(\partial_l(\theta_i S_{m,p-1}))\, \partial_j \psi_{i,l},$$

$$\begin{aligned} \mathcal{R}(\tilde{A}^i_j \partial_j \Phi_i(\theta_i S_{m,p-1})) &= \mathcal{R}\Big(\sum_{k=1}^{3} A^{\mathrm{co}}_k \Phi_i \partial_k \varphi_{i,j} \sum_{l=1}^{3} \Phi_i \partial_l(\theta_i S_{m,p-1}) \partial_j \psi_{i,l}\Big) \\ &= \mathcal{R}\Big(\sum_{k,l=1}^{3} A^{\mathrm{co}}_k \Phi_i \partial_l(\theta_i S_{m,p-1}) \Phi_i \partial_k \varphi_{i,j}\, \partial_j \psi_{i,l}\Big) \end{aligned}$$

for all $j \in \{1, 2, 3\}$. Applying Φ_i to the identity

$$\delta_{lk} = (\nabla \operatorname{id}_{U_i})_{lk} = (\nabla(\psi_i \circ \varphi_i))_{lk} = \sum_{j=1}^{3} \Phi_i^{-1} \partial_j \psi_{i,l}\, \partial_k \varphi_{i,j}$$

on U_i for all $k, l \in \{1, 2, 3\}$, we conclude

$$\begin{aligned} \sum_{j=1}^{3} \mathcal{A}^i_j \partial_j S^i_{m,p-1} &= \mathcal{R}\Big(\sum_{j,k,l=1}^{3} A^{\mathrm{co}}_k \Phi_i \partial_l(\theta_i S_{m,p-1}) \Phi_i \partial_k \varphi_{i,j}\, \partial_j \psi_{i,l}\Big) \\ &= \mathcal{R}\Big(\sum_{k,l=1}^{3} A^{\mathrm{co}}_k \Phi_i \partial_l(\theta_i S_{m,p-1}) \delta_{lk}\Big) = \mathcal{R}\Big(\sum_{k=1}^{3} A^{\mathrm{co}}_k \Phi_i \partial_k(\theta_i S_{m,p-1})\Big). \end{aligned}$$

Note that the support of every term in the brackets on the right hand side of (3.19) is contained in K_i and $\omega_i = 1$ on K_i. Proceeding as above, the induction hypothesis then yields that $S^i_{m,p}$ is equal to

$$\mathcal{R}\Phi_i A^i_0(0)^{-1}\Big[\mathcal{R}\Phi_i(\theta_i \partial_t^{p-1} f(0)) + \mathcal{R}\Phi_i\Big[\sum_{j=1}^{3} A^{\mathrm{co}}_j \partial_j \theta_i \partial_t^{p-1} v(0) - \sum_{j=1}^{3} A^{\mathrm{co}}_j \partial_j(\theta_i S_{m,p-1})\Big]$$

$$
\begin{aligned}
&- \sum_{l=1}^{p-1} c_{p,l} \mathcal{R}\Phi_i(\partial_t^l A_0^i(0)) \mathcal{R}\Phi_i(\theta_i S_{m,p-l}) - \sum_{l=0}^{p-1} c_{p,l} \mathcal{R}\Phi_i(\partial_t^l D^i(0)) \mathcal{R}\Phi_i(\theta_i S_{m,p-1-l})\Big] \\
&= \mathcal{R}\Phi_i \Big[\theta_i A_0(0)^{-1} \Big(\partial_t^{p-1} f(0) - \sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j S_{m,p-1} - \sum_{l=1}^{p-1} c_{p,l} \partial_t^l A_0(0) S_{m,p-l} \\
&\qquad\qquad - \sum_{l=0}^{p-1} c_{p,l} \partial_t^l D(0) S_{m,p-1-l}\Big)\Big], \\
&= \mathcal{R}\Phi_i(\theta_i S_{m,p}),
\end{aligned}
$$

where $c_{p,l} = \binom{p-1}{l}$ and we also employed that $\partial_t^{p-1} v(0) = S_{m,p-1}$. So (3.18) is true.

III) In this step we show that the tuple $(0, \mathcal{A}_0^i, \ldots, \mathcal{A}_3^i, \mathcal{D}^i, B^i, f^i(f,v), g^i, u_0^i)$ fulfills the linear compatibility conditions (2.6) on $G = \mathbb{R}^3_+$ of order m, where v is any function in $\mathcal{G}_m(J \times G)$ with $\partial_t^p v(0) = S_{m,p}$ for all $p \in \{0, \ldots, m-1\}$.

To that purpose, we exploit our assumption (2.6), i.e., $B_\Sigma \operatorname{tr}_{\Sigma,\pm} S_{m,p} = \partial_t^p g(0)$ for all $p \in \{0, \ldots, m-1\}$. Fix a number $p \in \{0, \ldots, m-1\}$. The trace operator commutes with multiplication by test functions and the composition with diffeomorphisms, so that (2.6) and (3.7) imply the identities

$$
\begin{aligned}
&\partial_t^p(\tilde{\Phi}_i(\operatorname{tr}_\Sigma(\theta_i)\kappa_i g))(0) = \tilde{\Phi}_i(\operatorname{tr}_\Sigma(\theta_i)\kappa_i \partial_t^p g(0)) = \tilde{\Phi}_i(\kappa_i B_\Sigma \operatorname{tr}_\Sigma(\theta_i) \operatorname{tr}_{\Sigma,\pm} S_{m,p}) \\
&= \operatorname{tr}_{\partial\mathbb{R}^3_+} \hat{B}^i \tilde{\Phi}_i \operatorname{tr}_{\Sigma,\pm}(\theta_i S_{m,p}) = \operatorname{tr}_{\partial\mathbb{R}^3_+} \hat{B}^i \operatorname{tr}_{\partial\mathbb{R}^3_+,\pm}(\Phi_i(\theta_i S_{m,p})) \\
&= \operatorname{tr}_{\partial\mathbb{R}^3_+} \hat{B}^i \operatorname{tr}_{\partial\mathbb{R}^3_+}(\mathcal{R}\Phi_i(\theta_i S_{m,p})) = \operatorname{tr}_{\partial\mathbb{R}^3_+}(\hat{B}^i S_{m,p}^i).
\end{aligned}
$$

Multiplying this equation with the trace of R^i, we arrive at

$$
\operatorname{tr}_{\partial\mathbb{R}^3_+}(R^i) \operatorname{tr}_{\partial\mathbb{R}^3_+}(\hat{B}^i S_{m,p}^i) = \partial_t^p(\operatorname{tr}_{\partial\mathbb{R}^3_+}(R^i)\tilde{\Phi}_i(\operatorname{tr}_\Sigma(\theta_i)\kappa_i g))(0). \tag{3.20}
$$

The $z(i)$-th and the $(z(i)+3)$-th components on the left-hand side are zero by (3.10), so that the same is true for the right-hand side. In view of formulas (3.10), (3.11) and (3.16), equation (3.20) thus yields the desired compatibility conditions

$$
\operatorname{tr}_{\partial\mathbb{R}^3_+}(B^i S_{m,p}^i) = \partial_t^p(\operatorname{tr}_{\partial\mathbb{R}^3_+}(R^i)\tilde{\Phi}_i(\operatorname{tr}_\Sigma(\theta_i)\kappa_i g))_{\alpha(i)}(0) = \partial_t^p g^i(0).
$$

IV) Let u be a solution in $\mathcal{G}_m(J \times G)$ of (1.9) with data f, g, and u_0. In this step we derive a priori estimates for u by applying a priori estimates on G_+ from [25], on $\mathbb{R}^3$ from [23], respectively on $\mathbb{R}^3_+$ from Theorem 5.9 below to $\theta_{-1}u$, $\theta_0 u$, respectively $\Phi_i(\theta_i u)$ for $i \in \mathbb{N}$. To that purpose, we first note that the properties of the functions φ_i, ψ_i, and θ_i imply the equivalences

$$
\begin{aligned}
u \in \mathcal{G}_m(J \times G) &\Longleftrightarrow \theta_{-1}u \in G_m(J \times G), \theta_0 u \in G_m(J \times \mathbb{R}^3) \\
&\qquad \text{and } \ \mathcal{R}\Phi_i(\theta_i u) \in G_m(J \times \mathbb{R}^3_+) \ \text{ for all } i \in \mathbb{N}, \\
f \in \mathcal{H}^m(J \times G) &\Longleftrightarrow \theta_{-1}u \in H^m(J \times G), \theta_0 f \in H^m(J \times \mathbb{R}^3) \\
&\qquad \text{and } \ \mathcal{R}\Phi_i(\theta_i u) \in H^m(J \times \mathbb{R}^3_+) \ \text{ for all } i \in \mathbb{N}, \\
g \in E_m(J \times \Sigma) &\Longleftrightarrow g^i \in E_m(J \times \partial\mathbb{R}^3_+) \ \text{ for all } i \in \mathbb{N},
\end{aligned} \tag{3.21}
$$

with corresponding bounds.

Fix an index $i \in \mathbb{N}$. Since $\operatorname{supp}\Phi_i(\theta_i u) \subseteq \operatorname{supp}\Phi_i\theta_i \subseteq K_i$, the definition of the extended coefficients in (3.6) as well as formulas (3.2) and (3.16) yield

$$\begin{aligned}
&\mathcal{A}_0^i\partial_t(\mathcal{R}\Phi_i(\theta_i u)) + \sum_{j=1}^3 \mathcal{A}_j^i\partial_j(\mathcal{R}\Phi_i(\theta_i u)) + \mathcal{D}^i\mathcal{R}\Phi_i(\theta_i u)\\
&\quad = \mathcal{R}\Phi_i\Big(A_{0,\pm}\partial_t(\theta_i u_\pm) + \sum_{j=1}^3 A_j^{\mathrm{co}}\partial_j(\theta_i u_\pm) + D_\pm(\theta_i u_\pm)\Big)\\
&\quad = \mathcal{R}\Phi_i(\theta_i f) + \mathcal{R}\Phi_i\Big(\sum_{j=1}^3 A_j^{\mathrm{co}}\partial_j\theta_i u\Big) = f^i(f,u)
\end{aligned}$$

on $J \times \mathbb{R}^3_+$. Since $\operatorname{Tr}_{J\times\Sigma}(B_\Sigma(u_+,u_-)) = g$ on $J\times\Sigma$, a similar computation as in step III) shows that

$$\begin{aligned}
\operatorname{Tr}_{J\times\partial\mathbb{R}^3_+}[\hat{B}^i\mathcal{R}\Phi_i(\theta_i u)] &= \operatorname{Tr}_{J\times\partial\mathbb{R}^3_+}[\Phi_i(\theta_i\kappa_i B_\Sigma(u_+,u_-))] = \tilde{\Phi}_i\operatorname{Tr}_{J\times\Sigma}[\theta_i\kappa_i B_\Sigma(u_+,u_-)]\\
&= \tilde{\Phi}_i(\operatorname{tr}_\Sigma(\theta_i)\kappa_i\ \operatorname{Tr}_{J\times\Sigma}[B_\Sigma(u_+,u_-)]) = \tilde{\Phi}_i(\operatorname{tr}_\Sigma(\theta_i)\kappa_i g).
\end{aligned}$$

Multiplying this equation with the trace of R^i and removing the $z(i)$-th and $z(i)+3$-th component of the result, we obtain

$$\begin{aligned}
\operatorname{Tr}_{J\times\partial\mathbb{R}^3_+}(B^i\mathcal{R}\Phi_i(\theta_i u)) &= \operatorname{Tr}_{J\times\partial\mathbb{R}^3_+}(R^i\hat{B}^i\mathcal{R}\Phi_i(\theta_i u))_{\alpha(i)}\\
&= (\operatorname{tr}_{\partial\mathbb{R}^3_+}(R^i)\tilde{\Phi}_i(\operatorname{tr}_\Sigma(\theta_i)\kappa_i g))_{\alpha(i)} = g^i,
\end{aligned}$$

cf. (3.10), (3.11) and (3.16). We conclude that the function $\mathcal{R}\Phi_i(\theta_i u)$ is a $G_m(J\times\mathbb{R}^3_+)$-solution of the initial boundary value problem

$$\begin{aligned}
\mathcal{A}_0^i\partial_t v + \textstyle\sum_{j=1}^3 \mathcal{A}_j^i\partial_j v + \mathcal{D}^i v &= f^i(f,u), && x\in\mathbb{R}^3_+, && t\in J;\\
B^i v &= g^i, && x\in\partial\mathbb{R}^3_+, && t\in J;\\
v(0) &= u_0^i, && x\in\mathbb{R}^3_+.
\end{aligned} \tag{3.22}$$

In the following we abbreviate $U_i\cap G$ by G_i for all $i\in\mathbb{N}_{-1}$. The spaces $\mathcal{H}^m(G_i)$, $\mathcal{H}^m(J\times G_i)$ and $\mathcal{G}_m(J\times G_i)$ are defined as their analogues on G.

To apply Theorem 5.9, we have to work with a constant boundary matrix $\mathcal{A}_3$ and a constant matrix B. As shown in step I), this is achieved via the multiplication with the matrices $\mathcal{G}_r^i$. We therefore recall, respectively define, the maps

$$\begin{aligned}
&\tilde{\mathcal{A}}_j^i = (\mathcal{G}_r^i)^T\mathcal{A}_j^i\mathcal{G}_r^i,\quad \tilde{\mathcal{B}}^i = B^i\mathcal{G}_r^i = \mathcal{B}^{\mathrm{co}},\quad \tilde{\mathcal{D}}^i = (\mathcal{G}_r^i)^T\mathcal{D}^i\mathcal{G}_r^i - \sum_{j=1}^3(\mathcal{G}_r^i)^T\mathcal{A}_j^i\mathcal{G}_r^i\partial_j(\mathcal{G}_r^i)^{-1}\mathcal{G}_r^i,\\
&\tilde{f}^i = (\mathcal{G}_r^i)^T f^i,\quad \tilde{g}^i = g^i,\quad \tilde{u}_0^i = (\mathcal{G}_r^i)^{-1}u_0^i
\end{aligned} \tag{3.23}$$

for all $j\in\{0,\dots,3\}$. Recall that $\tilde{\mathcal{A}}_3^i = \tilde{\mathcal{A}}_3^{\mathrm{co}}$ by (3.9). We claim that a function u^i belongs to $G_m(\Omega)$ and solves (3.22) if and only if the function $\tilde{u}^i = \mathcal{G}_r^{i,-1}u^i$ belongs to $G_m(\Omega)$ and solves the initial boundary value problem

$$\begin{aligned}
\tilde{\mathcal{L}}v := \tilde{\mathcal{A}}_0^i\partial_t v + \textstyle\sum_{j=1}^3 \tilde{\mathcal{A}}_j^i\partial_j v + \tilde{\mathcal{D}}^i v &= \tilde{f}^i, && x\in\mathbb{R}^3_+, && t\in J;\\
\mathcal{B}^{\mathrm{co}}v &= \tilde{g}^i, && x\in\partial\mathbb{R}^3_+, && t\in J;\\
v(0) &= \tilde{u}_0^i, && x\in\mathbb{R}^3_+.
\end{aligned} \tag{3.24}$$

To see this claim, we assume that u^i is a solution of (3.22). We then compute

$$\begin{aligned}\tilde{\mathcal{L}}\tilde{u}^i &= (\mathcal{G}_r^i)^T\Big[\mathcal{A}_0^i\partial_t u^i + \sum_{j=1}^3 \mathcal{A}_j^i\mathcal{G}_r^i\partial_j((\mathcal{G}_r^i)^{-1}u^i) + \mathcal{D}^i u^i - \sum_{j=1}^3 \mathcal{A}_j^i\mathcal{G}_r^i\partial_j(\mathcal{G}_r^i)^{-1}u^i\Big] \\ &= (\mathcal{G}_r^i)^T\Big[\mathcal{A}_0^i\partial_t u^i + \sum_{j=1}^3 \mathcal{A}_j^i\partial_j u^i + \mathcal{D}^i u^i\Big] = (\mathcal{G}_r^i)^T f^i = \tilde{f}^i, \\ \mathcal{B}^{\mathrm{co}}\tilde{u}^i &= B^i u^i = g^i = \tilde{g}^i, \\ \tilde{u}^i(0) &= (\mathcal{G}_r^i)^{-1}u^i(0) = (\mathcal{G}_r^i)^{-1}u_0^i = \tilde{u}_0^i.\end{aligned}$$

Analogously, one shows the other direction. We further note that the tuple $(0, \mathcal{A}_j^i, \mathcal{D}^i, B^i, f^i, g^i, u_0^i)$ fulfills the compatibility conditions of order m on $\partial\mathbb{R}^3_+$ if and only if the tuple $(0, \tilde{\mathcal{A}}_j^i, \tilde{\mathcal{D}}^i, \mathcal{B}^{\mathrm{co}}, \tilde{f}^i, \tilde{g}^i, \tilde{u}_0^i)$ fulfills the compatibility conditions of order m on $\partial\mathbb{R}^3_+$. To that purpose it is enough to show that

$$\tilde{S}^i_{m,p} = (\mathcal{G}_r^i)^{-1}S^i_{m,p}, \tag{3.25}$$

for all $0 \le p \le m$, where we use (3.23) and set, respectively recall,

$$\tilde{S}^i_{m,p} = S_{\mathbb{R}^3_+,m,p}(0, \tilde{\mathcal{A}}_j^i, \tilde{\mathcal{D}}^i, \tilde{f}^i, \tilde{u}_0^i), \qquad S^i_{m,p} = S_{\mathbb{R}^3_+,m,p}(0, \mathcal{A}_j^i, \mathcal{D}^i, f^i, u_0^i).$$

For $p = 0$ we have $\tilde{S}^i_{m,0} = \tilde{u}_0^i = (\mathcal{G}_r^i)^{-1}u_0^i = (\mathcal{G}_r^i)^{-1}S^i_{m,0}$. Next, let (3.25) be true for all $0 \le l \le p-1$. Inserting (3.23), we compute

$$\begin{aligned}\tilde{S}^i_{m,p} &= \tilde{\mathcal{A}}_0^{i,-1}\Big(\partial_t^{p-1}\tilde{f}^i(0) - \sum_{j=1}^3 \tilde{\mathcal{A}}_j^i\partial_j\tilde{S}^i_{m,p-1} - \sum_{l=1}^{p-1}\binom{p-1}{l}\partial_t^l\tilde{\mathcal{A}}_0^i(0)\tilde{S}^i_{m,p-l} \\ &\qquad - \sum_{l=0}^{p-1}\binom{p-1}{l}\partial_t^l\tilde{\mathcal{D}}^i(0)\tilde{S}^i_{m,p-1-l}\Big) \\ &= \mathcal{G}_r^{i,-1}\mathcal{A}_0^{i,-1}\mathcal{G}_r^{i,-T}\Big(\mathcal{G}_r^{i,T}\partial_t^p f^i(0) - \sum_{j=1}^3 \mathcal{G}_r^{i,T}\mathcal{A}_j^i\mathcal{G}_r^i\partial_j(\mathcal{G}_r^{i,-1}S^i_{m,p-1}) \\ &\quad - \sum_{l=1}^{p-1}\binom{p-1}{l}\partial_t^l(\mathcal{G}_r^{i,T}\mathcal{A}_0^i\mathcal{G}_r^i)(0)\mathcal{G}_r^{i,-1}S^i_{m,p-l} \\ &\quad - \sum_{l=0}^{p-1}\binom{p-1}{l}\partial_t^l\Big(\mathcal{G}_r^{i,T}\mathcal{D}^i\mathcal{G}_r^i - \sum_{j=1}^3 \mathcal{G}_r^{i,T}\mathcal{A}_j^i\mathcal{G}_r^i\partial_j\mathcal{G}_r^{i,-1}\mathcal{G}_r^i\Big)(0)\mathcal{G}_r^{i,-1}S^i_{m,p-1-l}\Big) \\ &= \mathcal{G}_r^{i,-1}\mathcal{A}_0^{i,-1}\Big(\partial_t^{p-1}f^i(0) - \sum_{j=1}^3 \mathcal{A}_j^i\partial_j S^i_{m,p-1} - \sum_{l=1}^{p-1}\binom{p-1}{l}\partial_t^l\mathcal{A}_0^i(0)S^i_{m,p-l} \\ &\quad - \sum_{l=0}^{p-1}\binom{p-1}{l}\partial_t^l\mathcal{D}^i(0)S^i_{m,p-1-l}\Big) \\ &= (\mathcal{G}_r^i)^{-1}S^i_{m,p},\end{aligned}$$

omitting some parentheses. The claim (3.25) is thus valid for all $0 \le p \le m$.

Consequently, we can apply Theorem 5.9 to this transformed problem and then obtain a solution of the same regularity of the original problem via the inverse transform. Also the a priori estimates carry over to the original problem with an additional constant $C(M_1)$. In order to simplify the notation, we suppress this transform in the following but assume that the matrices $\mathcal{A}_3^i$ and B^i are constant. Theorem 5.9 in combination with (3.16) and (3.21) then yield

$$\begin{aligned}
&\|\mathcal{R}\Phi_i(\theta_i u)\|_{G_{m,\gamma}(\Omega)}^2 \\
&\leq (C_{5.9,m,0} + TC_{5.9,m}) e^{mC_{5.9,1}T} \Big(\sum_{j=0}^{m-1} \|\partial_t^j f^i(f,u)(0)\|_{H^{m-1-j}(\mathbb{R}^3_+)}^2 \\
&\qquad + \|g^i\|_{E_{m,\gamma}(J\times\partial\mathbb{R}^3_+)}^2 + \|u_0^i\|_{H^m(\mathbb{R}^3_+)}^2 \Big) + C_{5.9,m} e^{mC_{5.9,1}T} \frac{1}{\gamma} \|f^i(f,u)\|_{H^m_\gamma(\Omega)}^2 \\
&\leq C(M_1)(C_{5.9,m,0} + TC_{5.9,m}) e^{mC_{5.9,1}T} \Big[\sum_{j=0}^{m-1} \|\theta_i \partial_t^j f(0)\|_{\mathcal{H}^{m-1-j}(G_i)}^2 \\
&\quad + \sum_{j=0}^{m-1} \sum_{k=1}^{3} \|\partial_k \theta_i S_{m,j}\|_{\mathcal{H}^{m-1-j}(G_i)}^2 + \|\operatorname{tr}_\Sigma(\theta_i)\, g\|_{E_{m,\gamma}(J\times\Sigma)}^2 + \|\theta_i u_0\|_{\mathcal{H}^m(G_i)}^2 \Big] \\
&\quad + C(M_1) \frac{C_{5.9,m}}{\gamma} e^{mC_{5.9,1}T} \Big(\|\theta_i f\|_{\mathcal{H}^m_\gamma(J\times G_i)}^2 + \sum_{k=1}^{3} \|\partial_k \theta_i u\|_{\mathcal{H}^m_\gamma(J\times G_i)}^2 \Big) \qquad (3.26)
\end{aligned}$$

for all $\gamma \geq \gamma_{5.9,m}$. Here we exploited that $\partial_t^j u(0) = S_{m,j}$ for all $j \in \{0, \ldots, m-1\}$, and $C_{5.9,m} = C_{5.9,m}(\eta, R, T')$, $C_{5.9,m,0} = C_{5.9,m,0}(\eta, R_0)$, and $\gamma_{5.9,m} = \gamma_{5.9,m}(\eta, R, T')$ are constants from Theorem 5.9. The estimates for $i \in \{-1, 0\}$ follow in the same way from Theorem 1.1 in [25] and Theorem 5.3 in [23] with corresponding constants $\tilde{C}_{m,0}$ and $\tilde{C}_m$.

By Definition 2.24 of [23] at most N of the sets U_i intersect at a given point, and we use the constants M_1 and M_2 introduced there and Definition 5.4 of [23]. The monotone convergence theorem thus implies that

$$\begin{aligned}
&\sum_{i=-1}^{\infty} \|\theta_i u_0\|_{\mathcal{H}^m(G_i)}^2 = \sum_{i=-1}^{\infty} \Big[\int_{G_+} \sum_{|\alpha|\leq m} |\partial^\alpha(\theta_i u_{0,+})|^2 dx + \int_{G_-} \sum_{|\alpha|\leq m} |\partial^\alpha(\theta_i u_{0,-})|^2 dx \Big] \\
&\leq C(m, M_2) \sum_{|\alpha|\leq m} \Big[\int_{G_+} \sum_{i=-1}^{\infty} \chi_{U_i} |\partial^\alpha u_{0,+}|^2 dx + \int_{G_-} \sum_{i=-1}^{\infty} \chi_{U_i} |\partial^\alpha u_{0,-}|^2 dx \Big] \\
&\leq C(m, M_2, N) \|u_0\|_{\mathcal{H}^m(G)}^2. \qquad (3.27)
\end{aligned}$$

Analogously, we treat the other terms on the right-hand side of (3.26). We set $C_m' = \max\{\tilde{C}_m, C_{5.9,m}\}$ and $C_{m,0}' = \max\{\tilde{C}_{m,0}, C_{5.9,m,0}\}$. Equation (3.26) then yields the inequality

$$\begin{aligned}
\|u\|_{\mathcal{G}_{m,\gamma}(J\times G)}^2 &\leq C(N) \sum_{i=-1}^{\infty} \|\theta_i u\|_{\mathcal{G}_{m,\gamma}(J\times G_i)}^2 \leq C(N, M_1) \sum_{i=-1}^{\infty} \|\mathcal{R}_i \Phi_i(\theta_i u)\|_{G_{m,\gamma}(\Omega)}^2 \\
&\leq C(m, N, M_1, M_2, \tau)(C_{m,0}' + TC_m') e^{mC_1' T} \Big(\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|_{\mathcal{H}^{m-1-j}(G)}^2 \\
&\qquad + \sum_{j=0}^{m-1} \|S_{m,j}\|_{\mathcal{H}^{m-1-j}(G)}^2 + \|g\|_{E_{m,\gamma}(J\times\Sigma)}^2 + \|u_0\|_{\mathcal{H}^m(G)}^2 \Big)
\end{aligned}$$

$$+ C(m, N, M_1, M_2) \frac{C_m'}{\gamma} e^{mC_1' T} \Big(\|f\|^2_{\mathcal{H}^m_\gamma(J\times G)} + \|u\|^2_{\mathcal{H}^m_\gamma(J\times G))} \Big)$$

for all $\gamma \geq \max\{\tilde{\gamma}_m, \gamma_{5.9,m}\}$. Choosing $\gamma_m = \gamma_m(\eta, \tau, N, M_1, M_2, r, T')$ large enough and using Lemma 2.3 we thus arrive at

$$\|u\|^2_{\mathcal{G}_{m,\gamma}(J\times G)} \leq (C_{m,0} + TC_m) e^{mC_1 T} \Big(\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|^2_{\mathcal{H}^{m-1-j}(G)} + \|g\|^2_{E_{m,\gamma}(J\times\Sigma)} + \|u_0\|^2_{\mathcal{H}^m(G)} \Big) + C_m e^{mC_1 T} \frac{1}{\gamma} \|f\|^2_{\mathcal{H}^m_\gamma(J\times G)}$$

for all $\gamma \geq \gamma_m$. Employing that $R = R(M, r)$ and $R_0 = R_0(M, r_0)$, we also deduce that the constants $C_{m,0}$ and C_m are of the claimed form (where we drop the dependence on M as G is fixed). We have thus shown the a priori estimates (3.1), which imply uniqueness of the $\mathcal{G}_m(J \times G)$-solution of (1.9).

V) To solve (1.9), we introduce the spaces

$$\mathcal{G}_{m,\mathrm{iv}}(J \times G) = \{v \in \mathcal{G}_m(J \times G) \colon \partial_t^j v(0) = S_{m,j},\ j \in \{0, \dots, m-1\}\},$$
$$\mathcal{H}^m_{\mathrm{iv},f}(J \times G) = \{\tilde{f} \in \mathcal{H}^m(J \times G) \colon \partial_t^j \tilde{f}(0) = \partial_t^j f(0),\ j \in \{0, \dots, m-1\}\}.$$

We point out that $\mathcal{G}_{m,\mathrm{iv}}(J\times G)$ is nonempty by Lemma 2.34 from [23] and $\mathcal{H}^m_{\mathrm{iv},f}(J\times G)$ is nonempty as $f \in \mathcal{H}^m_{\mathrm{iv},f}(J\times G)$. Because the time derivatives up to order $m-1$ in 0 of functions from $\mathcal{H}^m_{\mathrm{iv},f}(J \times G)$ respectively $\mathcal{G}_{m,\mathrm{iv}}(J \times G)$ coincide, we obtain

$$S_{\mathbb{R}^3_+,m,p}(0, \mathcal{A}_j^i, \mathcal{D}^i, f^i(\tilde{f}, \tilde{v}), u_0^i) = S_{\mathbb{R}^3_+,m,p}(0, \mathcal{A}_j^i, \mathcal{D}^i, f^i(f, v), u_0^i) = S^i_{m,p} \tag{3.28}$$

for all $\tilde{f} \in \mathcal{H}^m_{\mathrm{iv},f}(J \times G)$, $v, \tilde{v} \in \mathcal{G}_{m,\mathrm{iv}}(J \times G)$, $p \in \{0, \dots, m\}$, and $i \in \mathbb{N}$, cf. (3.17). The analogous equations for $i \in \{-1, 0\}$ are also true. Step III) thus implies that the tuple $(0, \mathcal{A}_j^i, \mathcal{D}^i, B^i, f^i(\tilde{f}, v), g^i, u_0^i)$ fulfills the compatibility conditions of order m for all $\tilde{f} \in \mathcal{H}^m_{\mathrm{iv},f}(J \times G)$, $v \in \mathcal{G}_{m,\mathrm{iv}}(J \times G)$, and $i \in \mathbb{N}$. As explained in step IV), we can now apply Theorem 5.9 which shows that the problem

$$\begin{aligned} \mathcal{A}_0^i \partial_t w + \sum\nolimits_{j=1}^3 \mathcal{A}_j^i \partial_j w + \mathcal{D}^i w &= f^i(\tilde{f}, v), && x \in \mathbb{R}^3_+, && t \in J; \\ B^i w &= g^i, && x \in \partial\mathbb{R}^3_+, && t \in J; \\ w(0) &= u_0^i, && x \in \mathbb{R}^3_+; \end{aligned} \tag{3.29}$$

has a unique solution $\mathcal{U}^i(\tilde{f}, v)$ in $G_m(\Omega)^{12}$ for all $\tilde{f} \in \mathcal{H}^m_{\mathrm{iv},f}(J\times G)$, $v \in \mathcal{G}_{m,\mathrm{iv}}(J\times G)$, and $i \in \mathbb{N}$. Moreover, Theorem 5.3 from [23] gives a function $\mathcal{U}^0(\tilde{f}, v)$ in $G_m(J \times \mathbb{R}^3)^6$ solving the initial value problem

$$\begin{aligned} A_0^0 \partial_t w + \sum\nolimits_{j=1}^3 A_j^{\mathrm{co}} \partial_j w + D^0 w &= f^0(\tilde{f}, v), && x \in \mathbb{R}^3, && t \in J; \\ w(0) &= u_0^0, && x \in \mathbb{R}^3; \end{aligned} \tag{3.30}$$

for all such $\tilde{f}$ and v. Finally, Theorem 1.1 and Remark 1.2 in [25] yield a solution $\mathcal{U}^{-1}(\tilde{f}, v)$ in $G_m(J \times G)^6$ of the initial boundary value problem

$$\begin{aligned} A_0^{-1} \partial_t w + \sum\nolimits_{j=1}^3 A_j^{\mathrm{co}} \partial_j w + D^{-1} w &= f^{-1}(\tilde{f}, v), && x \in G, && t \in J; \\ B_{\partial G} w &= 0, && x \in \partial G, && t \in J; \\ w(0) &= u_0^{-1}, && x \in G; \end{aligned} \tag{3.31}$$

for all such $\tilde{f}$ and v. We claim that there is a map $f^* = f^*(v)$ in $\mathcal{H}^m_{\mathrm{iv},f}(J\times G)$ with

$$f^* + \sum_{i=-1}^{\infty}\sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j\sigma_i\Phi_i^{-1}\mathcal{R}_i^{-1}\mathcal{U}^i(f^*,v) = f \tag{3.32}$$

for all $v\in\mathcal{G}_{m,\mathrm{iv}}(J\times G)$. To prove this claim, we define the operator

$$\Psi_v\colon \mathcal{H}^m_{\mathrm{iv},f}(J\times G)\to\mathcal{H}^m_{\mathrm{iv},f}(J\times G), \qquad \tilde{f}\longmapsto f - \sum_{i=-1}^{\infty}\sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j\sigma_i\Phi_i^{-1}\mathcal{R}_i^{-1}\mathcal{U}^i(\tilde{f},v)$$

for each $v\in\mathcal{G}_{m,\mathrm{iv}}(J\times G)$. We fix such a function v. The operator Ψ_v indeed takes values in $\mathcal{H}^m(J\times G)$ since $\Phi_i^{-1}\mathcal{R}^{-1}$ maps the $H^m(\Omega)$-function $\mathcal{U}^i(\tilde{f},v)$ into $\mathcal{H}^m(J\times U_i)$ for $i\in\mathbb{N}$, $\partial_j\sigma_i$ has compact support in U_i, and the covering $(U_i)_{i\in\mathbb{N}}$ is locally finite. We further compute

$$\begin{aligned}\partial_t^p\Psi_v(\tilde{f})(0) &= \partial_t^p f(0) - \sum_{i=-1}^{\infty}\sum_{j=1}^{3} A_j^{\mathrm{co}}\partial_j\sigma_i\Phi_i^{-1}\mathcal{R}_i^{-1}\partial_t^p\mathcal{U}^i(\tilde{f},v)(0)\\ &= \partial_t^p f(0) - \sum_{i=-1}^{\infty}\sum_{j=1}^{3} \mathcal{A}_j^{\mathrm{co}}\partial_j\sigma_i\Phi_i^{-1}\mathcal{R}_i^{-1}\mathcal{R}_i\Phi_i(\theta_i S_{m,p})\\ &= \partial_t^p f(0) - \sum_{i=-1}^{\infty}\sum_{j=1}^{3} \mathcal{A}_j^{\mathrm{co}}\partial_j\sigma_i\theta_i S_{m,p} = \partial_t^p f(0)\end{aligned}$$

for all $p\in\{0,\dots,m-1\}$ and $\tilde{f}\in\mathcal{H}^m_{\mathrm{iv},f}(J\times G)$, where we used (2.3), (3.28), (3.18), and that σ_i equals 1 on the support of θ_i for all $i\in\mathbb{N}_{-1}$. Therefore Ψ_v indeed maps $\mathcal{H}^m_{\mathrm{iv},f}(J\times G)$ into itself.

We observe that the difference $\mathcal{U}^i(f_1,v)-\mathcal{U}^i(f_2,v)$ solves a problem with zero initial and boundary data. Moreover, formula (3.16) and the initial conditions in the spaces $\mathcal{H}^m_{\mathrm{iv},f}(J\times G)$ and $\mathcal{G}_{m,\mathrm{iv}}(J\times G)$ imply that the time derivatives of the inhomogeneities $f^i(f_k,v)$ coincide at $t=0$. (Such facts are also used below without further notice.) Theorems 1.1 in [25], 5.3 in [23], and 5.9 then imply

$$\begin{aligned}&\|\Psi_v(f_1)-\Psi_v(f_2)\|^2_{\mathcal{H}^m_\gamma(J\times G)} \qquad (3.33)\\ &\le C(N,M_1,M_3)\Big(\|\mathcal{U}^{-1}(f_1,v)-\mathcal{U}^{-1}(f_2,v)\|^2_{H^m_\gamma(J\times G)}\\ &\quad + \|\mathcal{U}^0(f_1,v)-\mathcal{U}^0(f_2,v)\|^2_{H^m_\gamma(J\times\mathbb{R}^3)} + \sum_{i=1}^{\infty}\|\mathcal{U}^i(f_1,v)-\mathcal{U}^i(f_2,v)\|^2_{H^m_\gamma(\Omega)}\Big)\\ &\le \frac{C}{\gamma}\sum_{i=-1}^{\infty}\|\theta_i(f_1-f_2)\|^2_{\mathcal{H}^m_\gamma(J\times G_i)} \le \frac{C}{\gamma}\|f_1-f_2\|^2_{\mathcal{H}^m_\gamma(J\times G)}\end{aligned}$$

for all $\gamma\ge\max\{\gamma_{1.1,m},\gamma_{5.3,m},\gamma_{5.9,m}\}$, proceeding as in (3.27) in the last step and putting $C=C(m,\eta,\tau,N,M,r,T')$. We set

$$\gamma^* = \max\{\gamma_{1.1,m},\gamma_{5.3,m},\gamma_{5.9,m},4C_{3.33}\},$$

where $C_{3.33}$ denotes the constant on the right-hand side of (3.33). This estimate then leads to the bound

$$\|\Psi_v(f_1)-\Psi_v(f_2)\|_{\mathcal{H}^m_\gamma(J\times G)} \le \frac{1}{2}\|f_1-f_2\|_{\mathcal{H}^m_\gamma(J\times G)} \tag{3.34}$$

for all $\gamma \geq \gamma^*$. We conclude that Ψ_v is a strict contraction on $\mathcal{H}^m_{\mathrm{iv},f}(J\times G)$, and there thus exists a unique function $f^* = f^*(v)$ in $\mathcal{H}^m_{\mathrm{iv},f}(J\times G)$ satisfying equation (3.32).

We next define the operator

$$\mathcal{S}\colon \mathcal{G}_{m,\mathrm{iv}}(J\times G) \to \mathcal{G}_{m,\mathrm{iv}}(J\times G), \quad v \longmapsto \sum_{i=-1}^{\infty} \sigma_i \Phi_i^{-1}\mathcal{R}_i^{-1}\mathcal{U}^i(f^*(v), v).$$

Let $v \in \mathcal{G}_{m,\mathrm{iv}}(J\times G)$. We first check that $\mathcal{S}(v)$ indeed belongs to $\mathcal{G}_{m,\mathrm{iv}}(J\times G)$. Since $\mathcal{U}^i(f^*(v), v)$ is an element of $G_m(\Omega)$, the function $\Phi_i^{-1}\mathcal{R}^{-1}\mathcal{U}^i(f^*(v),v)$ belongs to $\mathcal{G}_m(J\times G_i)$ for $i \in \mathbb{N}$. Moreover, $\mathcal{U}^{-1}(f^*(v), v)$ is contained $G_m(J\times G)$ and $\mathcal{U}^0(f^*(v), v)$ in $G_m(J\times\mathbb{R}^3)$. Exploiting that σ_i has compact support in U_i, the a priori estimates for $\mathcal{U}^i$, and (3.27), we infer that $\mathcal{S}(v)$ belongs to $\mathcal{G}_m(J\times G)$. As $f^*(v) \in \mathcal{H}^m_{\mathrm{iv},f}(J\times G)$, we now combine formula (3.28) with (3.18) as well as $\sigma_i = 1$ on $\operatorname{supp}\theta_i$ for all $i \in \mathbb{N}_{-1}$, and compute

$$\begin{aligned}\partial_t^p \mathcal{S}(v)(0) &= \sum_{i=-1}^{\infty} \sigma_i \Phi_i^{-1}\mathcal{R}_i^{-1}\partial_t^p\mathcal{U}^i(f^*(v), v)(0)\\ &= \sigma_{-1}\theta_{-1}S_{m,p} + \sigma_0\theta_0 S_{m,p} + \sum_{i=1}^{\infty}\sigma_i\Phi_i^{-1}\mathcal{R}^{-1}\mathcal{R}\Phi_i(\theta_i S_{m,p}) = \sum_{i=-1}^{\infty}\theta_i S_{m,p} = S_{m,p}\end{aligned}$$

for all $p \in \{0,\dots,m\}$ and $v \in \mathcal{G}_{m,\mathrm{iv}}(J\times G)$. Hence, $\mathcal{S}$ maps into $\mathcal{G}_{m,\mathrm{iv}}(J\times G)$.

To show that $\mathcal{S}$ is a strict contraction, we take $v_1, v_2 \in \mathcal{G}_{m,\mathrm{iv}}(J\times G)$. Estimate (3.34) further yields

$$\begin{aligned}&\|f^*(v_1) - f^*(v_2)\|_{\mathcal{H}^m_\gamma(J\times G)} = \|\Psi_{v_1}(f^*(v_1)) - \Psi_{v_2}(f^*(v_2))\|_{\mathcal{H}^m_\gamma(J\times G)}\\ &\leq \|\Psi_{v_1}(f^*(v_1)) - \Psi_{v_1}(f^*(v_2))\|_{\mathcal{H}^m_\gamma(J\times G)} + \|\Psi_{v_1}(f^*(v_2)) - \Psi_{v_2}(f^*(v_2))\|_{\mathcal{H}^m_\gamma(J\times G)}\\ &\leq \frac{1}{2}\|f^*(v_1) - f^*(v_2)\|_{\mathcal{H}^m_\gamma(J\times G)} + \|\Psi_{v_1}(f^*(v_2)) - \Psi_{v_2}(f^*(v_2))\|_{\mathcal{H}^m_\gamma(J\times G)} \qquad (3.35)\end{aligned}$$

for all $\gamma \geq \gamma^*$. The definition of the operator Ψ_v, Theorems 1.1 in [25], 5.3 in [23], and 5.9, formula (3.16) and a variant of (3.27) imply

$$\begin{aligned}&\|\Psi_{v_1}(f^*(v_2)) - \Psi_{v_2}(f^*(v_2))\|^2_{\mathcal{H}^m_\gamma(J\times G)}\\ &\leq C(N, M_3)\sum_{i=-1}^{\infty}\|\Phi_i^{-1}\mathcal{R}_i^{-1}\mathcal{U}^i(f^*(v_2), v_1) - \Phi_i^{-1}\mathcal{R}_i^{-1}\mathcal{U}^i(f^*(v_2), v_2)\|^2_{\mathcal{H}^m_\gamma(J\times G_i)}\\ &\leq C(m,\eta,\tau,N,M,r,T')\frac{1}{\gamma}\sum_{i=-1}^{\infty}\Big\|\sum_{j=1}^{3}A_j^{\mathrm{co}}\partial_j\theta_i(v_1 - v_2)\Big\|^2_{\mathcal{H}^m_\gamma(J\times G)}\\ &\leq C(m,\eta,\tau,N,M,r,T')\frac{1}{\gamma}\|v_1 - v_2\|^2_{\mathcal{H}^m_\gamma(J\times G)} \qquad (3.36)\end{aligned}$$

for all $\gamma \geq \gamma^*$. We set $\gamma^{**} = \max\{\gamma^*, 16C_{3.36}\}$ and insert (3.36) into (3.35), where $C_{3.36}$ denotes the constant on the right-hand side of (3.36). We then arrive at

$$\|f^*(v_1) - f^*(v_2)\|_{\mathcal{H}^m_\gamma(J\times G)} \leq \frac{1}{2}\|v_1 - v_2\|_{\mathcal{H}^m_\gamma(J\times G)} \qquad \text{for all} \;\; \gamma \geq \gamma^{**}.$$

After these preparations, we can now estimate the difference of $\mathcal{S}(v_1)$ and $\mathcal{S}(v_2)$. Applying the a priori estimates from Theorem 1.1 in [25], Theorem 5.3 in [23], respectively Theorem 5.9 once more and recalling that v_1 and v_2 belong to $\mathcal{G}_{m,\mathrm{iv}}(J\times G)$, we infer as above

$$\begin{aligned}
&\|\mathcal{S}(v_1)-\mathcal{S}(v_2)\|^2_{\mathcal{G}_{m,\gamma}(J\times G)}\\
&\leq C(N,M_1,M_3)\sum_{i=-1}^{\infty}\|\Phi_i^{-1}\mathcal{R}_i^{-1}\big(\mathcal{U}^i(f^*(v_1),v_1)-\mathcal{U}^i(f^*(v_2),v_2)\big)\|^2_{\mathcal{G}_{m,\gamma}(J\times G)}\\
&\leq C(m,\eta,\tau,N,M,r,T')\frac{1}{\gamma}\Big(\|f^*(v_1)-f^*(v_2)\|^2_{\mathcal{H}^m_\gamma(J\times G)}+\|v_1-v_2\|^2_{\mathcal{H}^m_\gamma(J\times G)}\Big)\\
&\leq C(m,\eta,\tau,N,M,r,T')\frac{1}{\gamma}\cdot\frac{5}{4}\|v_1-v_2\|^2_{\mathcal{G}_{m,\gamma}(J\times G)} \qquad (3.37)
\end{aligned}$$

for all $\gamma\geq\gamma^{**}$. We finally set $\gamma_{\mathcal{S}}=\max\{\gamma^{**},5C_{3.37}\}$, for the constant $C_{3.37}$ on the right-hand side of (3.37). It follows

$$\|\mathcal{S}(v_1)-\mathcal{S}(v_2)\|_{\mathcal{G}_{m,\gamma}(J\times G)}\leq\frac{1}{2}\|v_1-v_2\|_{\mathcal{G}_{m,\gamma}(J\times G)}$$

for all $\gamma\geq\gamma_{\mathcal{S}}$. There thus exists a unique fixed point $u\in\mathcal{G}_{m,\mathrm{iv}}(J\times G)$ of $\mathcal{S}$.

VI) We claim that the fixed point u of $\mathcal{S}$ is a solution of (1.9). To verify this assertion, we first compute for $u_\pm=\mathcal{S}(u)_\pm$

$$\begin{aligned}
\mathcal{L}_\pm u_\pm &:= A_{0,\pm}\partial_t u_\pm+\sum_{j=1}^{3}A_j^{\mathrm{co}}\partial_j u_\pm+D_\pm u_\pm\\
&=\sum_{i=-1}^{\infty}\sigma_{i,\pm}\Big(A_{0,\pm}\partial_t(\Phi_i^{-1}\mathcal{R}_i^{-1}\mathcal{U}^i(f^*(u),u))_\pm+\sum_{j=1}^{3}A_j^{\mathrm{co}}\partial_j(\Phi_i^{-1}\mathcal{R}_i^{-1}\mathcal{U}^i(f^*(u),u))_\pm\\
&\qquad+D_\pm(\Phi_i^{-1}\mathcal{R}_i^{-1}\mathcal{U}^i(f^*(u),u))_\pm\Big)+\sum_{i=-1}^{\infty}\sum_{j=1}^{3}A_j^{\mathrm{co}}\partial_j\sigma_{i,\pm}(\Phi_i^{-1}\mathcal{R}_i^{-1}\mathcal{U}^i(f^*(u),u))_\pm
\end{aligned}$$

on $J\times G_\pm$. Recalling (3.3), (3.5), (3.6), and that $\omega_i=1$ on $\varphi_i(\operatorname{supp}\sigma_i)$, on $G_+\cap\operatorname{supp}\sigma_i$ we have

$$\begin{aligned}
&\sum_{j=1}^{3}A_j^{\mathrm{co}}\partial_j(\Phi_i^{-1}\mathcal{R}^{-1}v)=\sum_{j=1}^{3}A_j^{\mathrm{co}}\partial_j v_{(1,\ldots,6)}(\varphi_i(x))\\
&=\sum_{j,l=1}^{3}A_j^{\mathrm{co}}\partial_l v_{(1,\ldots,6)}(\varphi_i(x))\partial_j\varphi_{i,l}(x)=\sum_{l=1}^{3}\Phi_i^{-1}(A_l^i)(x)\partial_l v_{(1,\ldots,6)}(\varphi_i(x))\\
&=\Phi_i^{-1}\mathcal{R}^{-1}\Big(\sum_{l=1}^{3}\mathcal{A}_l^i\partial_l v\Big),
\end{aligned}$$

whereas on $G_-\cap\operatorname{supp}\sigma_i$, we deduce

$$\begin{aligned}
&\sum_{j=1}^{3}A_j^{\mathrm{co}}\partial_j(\Phi_i^{-1}\mathcal{R}^{-1}v)=\sum_{j=1}^{3}A_j^{\mathrm{co}}\partial_j v_{(7,\ldots,12)}(\varphi_{i,1}(x),\varphi_{i,2}(x),-\varphi_{i,3}(x))\\
&=\sum_{j=1}^{3}A_j^{\mathrm{co}}\nabla v_{(7,\ldots,12)}(\varphi_{i,1}(x),\varphi_{i,2}(x),-\varphi_{i,3}(x))\cdot(\partial_j\varphi_{i,1}(x),\partial_j\varphi_{i,2}(x),-\partial_j\varphi_{i,3}(x))
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j,l=1}^{3} A_j^{\mathrm{co}} \partial_l v_{(7,\dots,12)}(\varphi_{i,1}(x), \varphi_{i,2}(x), -\varphi_{i,3}(x)) \partial_j \varphi_{i,l}(x)(-1)^{\delta_{3l}} \\
&= \sum_{l=1}^{3} \Phi_i^{-1}(A_{l,-}^i (-1)^{\delta_{3l}} Q \partial_l v_{(7,\dots,12)}) = \sum_{l=1}^{3} \Phi_i^{-1} Q(\breve{A}_{l,-}^i \partial_l v_{(7,\dots,12)}) \\
&= \sum_{l=1}^{3} \Phi_i^{-1} Q(\mathcal{A}_l^i \partial_l v)_{(7,\dots,12)} = \Phi_i^{-1}\mathcal{R}^{-1}\Big(\sum_{l=1}^{3} \mathcal{A}_l^i \partial_l v\Big)
\end{aligned}
$$

for all $v \in L^2(V_i \cap \mathbb{R}^3_+)^{12}$. Since also $A_{0,\pm} = (\Phi_i^{-1}\mathcal{R}^{-1}\mathcal{A}_0^i)_\pm$ and $D_\pm^i = (\Phi_i^{-1}\mathcal{R}^{-1}\mathcal{D}^i)_\pm$ (where we put $\mathcal{A}_0^i = A_0$ and $\mathcal{D}^i = D$ for $i \in \{-1, 0\}$) on $\operatorname{supp}\sigma_i$ for all $i \in \mathbb{N}_{-1}$, the definition of the maps $\mathcal{U}^i(f^*(u), u)$ and (3.16) imply the equality

$$
\begin{aligned}
\mathcal{L}_\pm u_\pm &= \sum_{i=-1}^{\infty} \sigma_{i,\pm}\Big(\Phi_i^{-1}\mathcal{R}_i^{-1}\Big(\mathcal{A}_0^i \partial_t \mathcal{U}^i(f^*(u), u) + \sum_{j=1}^{3} \mathcal{A}_j^i \partial_j \mathcal{U}^i(f^*(u), u) \\
&\qquad + \mathcal{D}^i \mathcal{U}^i(f^*(u), u)\Big)\Big)_\pm + \sum_{i=-1}^{\infty}\sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j \sigma_{i,\pm}(\Phi_i^{-1}\mathcal{R}_i^{-1}\mathcal{U}^i(f^*(u), u))_\pm \\
&= \sum_{i=-1}^{\infty}\Big[\sigma_{i,\pm}(\Phi_i^{-1}\mathcal{R}_i^{-1} f^i(f^*(u), u))_\pm + \sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j \sigma_{i,\pm}(\Phi_i^{-1}\mathcal{R}^{-1}\mathcal{U}^i(f^*(u), u))_\pm\Big] \\
&= \sum_{i=-1}^{\infty}\Big[\sigma_{i,\pm}\theta_{i,\pm} f^*(u)_\pm + \sum_{j=1}^{3} A_j^{\mathrm{co}}\Big[\sigma_{i,\pm}\partial_j \theta_{i,\pm} u_\pm + \partial_j \sigma_{i,\pm}(\Phi_i^{-1}\mathcal{R}_i^{-1} w^i)_\pm\Big]\Big].
\end{aligned}
$$

where $w^i := \mathcal{U}^i(f^*(u), u))$. Employing that $\sigma_i = 1$ on the support of θ_i, that $(\theta_i)_{i\in\mathbb{N}_{-1}}$ is a partition of unity, and the defining property of $f^*(u)$, i.e. (3.32), we deduce

$$
\begin{aligned}
\mathcal{L}_\pm u_\pm &= \sum_{i=-1}^{\infty}\Big[\theta_{i,\pm} f^*(u)_\pm + \sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j \theta_{i,\pm} u_\pm + \sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j \sigma_{i,\pm}(\Phi_i^{-1}\mathcal{R}_i^{-1}\mathcal{U}^i(f^*(u), u))_\pm\Big] \\
&= f^*(u)_\pm + \sum_{i=-1}^{\infty}\sum_{j=1}^{3} A_j^{\mathrm{co}} \partial_j \sigma_{i,\pm}(\Phi_i^{-1}\mathcal{R}_i^{-1}\mathcal{U}^i(f^*(u), u))_\pm = f_\pm.
\end{aligned}
$$

Since the covering $(U_i)_{i\in\mathbb{N}_{-1}}$ is locally finite, we can compute

$$
\begin{aligned}
\operatorname{Tr}_{J\times\Sigma,\pm}(B_\Sigma u) &= \operatorname{Tr}_{J\times\Sigma}(B_\Sigma \cdot (\mathcal{S}(u)_+, \mathcal{S}(u)_-)) = \operatorname{Tr}_{J\times\Sigma}\Big[B_\Sigma \sum_{i=1}^{\infty} \sigma_i \Phi_i^{-1}\mathcal{U}^i(f^*(u), u)\Big] \\
&= \sum_{i=1}^{\infty} \operatorname{tr}_\Sigma \sigma_i \operatorname{Tr}_{J\times\Sigma}(B_\Sigma \Phi_i^{-1}\mathcal{U}^i(f^*(u), u)) \\
&= \sum_{i=1}^{\infty} \operatorname{tr}_\Sigma(\sigma_i)\kappa_i^{-1} \operatorname{Tr}_{J\times\Sigma}\Big(\Phi_i^{-1}\big(\omega_i \Phi_i(\kappa_i B_\Sigma)\mathcal{U}^i(f^*(u), u)\big)\Big),
\end{aligned}
$$

using $\Phi_i^{-1}\omega_i = 1$ on $\operatorname{supp}\sigma_i$. The identity $\hat{B}^i = \omega_i \Phi_i(\kappa_i B)$ on $\operatorname{supp}\sigma_i$ then yields

$$
\operatorname{Tr}_{J\times\Sigma}(B_\Sigma u) = \sum_{i=1}^{\infty} \operatorname{tr}_\Sigma(\sigma_i)\kappa_i^{-1} \operatorname{Tr}_{J\times\Sigma}\Big(\Phi_i^{-1}\big(\hat{B}^i\, \mathcal{U}^i(f^*(u), u)\big)\Big)
$$

$$= \sum_{i=1}^{\infty} \operatorname{tr}_\Sigma(\sigma_i)\kappa_i^{-1}\tilde{\Phi}_i^{-1} \operatorname{Tr}_{J\times\partial\mathbb{R}^3_+}((R^i)^{-1}R^i\hat{B}^i\,\mathcal{U}^i(f^*(u),u)).$$

Because $\mathcal{U}^i(f^*(u),u)$ solves the initial boundary value problem (3.29) with the boundary value g^i defined in (3.16) for every $i \in \mathbb{N}$, we arrive at

$$\begin{aligned}
\operatorname{Tr}_{J\times\Sigma}(Bu) &= \sum_{i=1}^{\infty} \operatorname{tr}_\Sigma(\sigma_i)\kappa_i^{-1}\tilde{\Phi}_i^{-1} \operatorname{Tr}_{J\times\partial\mathbb{R}^3_+}\Big((R^i)^{-1}R^i\hat{B}^i\,\mathcal{U}^i(f^*(u),u)\Big) \\
&= \sum_{i=1}^{\infty} \operatorname{tr}_\Sigma(\sigma_i)\kappa_i^{-1}\tilde{\Phi}_i^{-1}\Big(\operatorname{tr}_{\partial\mathbb{R}^3_+}((R^i)^{-1})\, g^i_{z(i)\to 0}\Big) \\
&= \sum_{i=1}^{\infty} \operatorname{tr}_\Sigma(\sigma_i)\kappa_i^{-1}\tilde{\Phi}_i^{-1}\Big(\operatorname{tr}_{\partial\mathbb{R}^3_+}((R^i)^{-1})\ \operatorname{tr}_{\partial\mathbb{R}^3_+}(R^i)\tilde{\Phi}_i(\operatorname{tr}_\Sigma(\theta_i)\kappa_i g)\Big) \\
&= \sum_{i=1}^{\infty} \operatorname{tr}_\Sigma(\sigma_i\theta_i) g = \sum_{i=1}^{\infty} \operatorname{tr}_\Sigma(\theta_i) g = g,
\end{aligned}$$

where $g^i_{z(i)\to 0}$ denotes the vector we get by adding a zero in the $z(i)$-th and $z(i)+3$-th component of g^i. Moreover, we get

$$\operatorname{Tr}_{J\times\partial G}(B_{\partial G} u) = \operatorname{Tr}_{J\times\partial G}(B_{\partial G}\mathcal{S}(u)) = \operatorname{Tr}_{J\times\partial G}(B_{\partial G}\mathcal{U}^{-1}(f^*(u),u)) = 0$$

as $\mathcal{U}^{-1}(f^*(u),u)$ solves the problem (3.31). Similarly it follows

$$\begin{aligned}
u(0) = \mathcal{S}(u)(0) &= \sum_{i=-1}^{\infty} \sigma_i\Phi_i^{-1}\mathcal{R}_i^{-1}\mathcal{U}^i(f^*(u),u)(0) = \sum_{i=-1}^{\infty} \sigma_i\Phi_i^{-1}\mathcal{R}_i^{-1}u_0^i \\
&= \sum_{i=-1}^{\infty} \sigma_i\Phi_i^{-1}\mathcal{R}_i^{-1}\mathcal{R}_i\Phi_i(\theta_i u_0) = \sum_{i=-1}^{\infty} \sigma_i\theta_i u_0 = \sum_{i=-1}^{\infty} \theta_i u_0 = u_0.
\end{aligned}$$

We conclude that u is a solution of (1.9) in $\mathcal{G}_m(J\times G)$. □

4. A priori estimates for the linear problem

In the previous section we have reduced (1.9) to the system

$$\begin{cases}
\mathcal{A}_0\partial_t u + \sum_{j=1}^{3}\mathcal{A}_j\partial_j u + \mathcal{D}u = f, & x\in\mathbb{R}^3_+, \quad t\in J; \\
Bu = g, & x\in\partial\mathbb{R}^3_+, \quad t\in J; \\
u(0) = u_0, & x\in\mathbb{R}^3_+;
\end{cases} \tag{4.1}$$

on $\mathbb{R}^3_+$ with $\mathcal{A}_3 = \tilde{\mathcal{A}}_3^{\mathrm{co}}$, $B = \mathcal{B}^{\mathrm{co}}$, and $\mathcal{A}_1, \mathcal{A}_2 \in F^{\mathrm{cp}}_{m,\mathrm{coeff}}(\mathbb{R}^3_+)$, cf. (2.1) and (3.12). Here we fix $T' > 0$ and assume that $J = (0,T)$ for a time $T \in (0,T')$.

In this section we derive a priori estimates for $G_m(\Omega)$-solutions of (4.1). A (weak) *solution* of (4.1) is a function $u \in C(\overline{J}, L^2(\mathbb{R}^3_+))$ with $\mathcal{L}(\mathcal{A}_0,\ldots,\mathcal{A}_3,\mathcal{D})u = f$ in the weak sense, $\operatorname{Tr}_{J\times\partial\mathbb{R}^3_+}(Bu) = g$ on $J\times\partial\mathbb{R}^3_+$, and $u(0) = u_0$.

We first state the basic wellposedness result on L^2-level which directly follows from Proposition 5.1 in [10] because of the formulas (3.13). The precise form of the constants is a consequence of the proof in [10].

Lemma 4.1. *Let $\eta > 0$ and $r \geq r_0 > 0$. Take $\mathcal{A}_0 \in F^{\mathrm{cp}}_{0,\eta}(\Omega)$, $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{F}^{\mathrm{cp}}_{0,\mathrm{coeff}}(\mathbb{R}^3_+)$ with $\|\mathcal{A}_i\|_{W^{1,\infty}(\Omega)} \leq r$ and $\|\mathcal{A}_i(0)\|_{L^\infty(\mathbb{R}^3_+)} \leq r_0$ for all $i \in \{0,1,2\}$, and $\mathcal{A}_3 = \tilde{\mathcal{A}}_3^{\mathrm{co}}$. Let $\mathcal{D} \in L^\infty(\Omega)$ with $\|\mathcal{D}\|_{L^\infty(\Omega)} \leq r$ and $B = \mathcal{B}^{\mathrm{co}}$. Choose data $f \in L^2(\Omega)$, $g \in L^2(J, H^{1/2}(\partial\mathbb{R}^3_+))$, and $u_0 \in L^2(\mathbb{R}^3_+)$. Then (4.1) has a unique solution u in $C(\overline{J}, L^2(\mathbb{R}^3_+))$, and there exists a number $\gamma_0 = \gamma_0(\eta, r) \geq 1$ such that we obtain*

$$\sup_{t\in J}\|e^{-\gamma t}u(t)\|^2_{L^2(\mathbb{R}^3_+)} + \gamma\|u\|^2_{L^2_\gamma(\Omega)} \leq C_{0,0}\|u_0\|^2_{L^2(\mathbb{R}^3_+)} + C_{0,0}\|g\|^2_{L^2_\gamma(J,H^{1/2}(\partial\mathbb{R}^3_+))} + \frac{C_0}{\gamma}\|f\|^2_{L^2_\gamma(\Omega)} \tag{4.2}$$

for all $\gamma \geq \gamma_0$, where $C_0 = C_0(\eta, r)$ and $C_{0,0} = C_{0,0}(\eta, r_0)$.

The a priori estimates for the αth tangential and time derivatives of a regular solution of (4.1) now follow in a standard way: These derivatives satisfy (4.1) with new data f_α, g_α and $u_{0,\alpha}$, where f_α also contains commutator terms involving $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{D}$. On the resulting problem one can apply the L^2-estimate (4.2). The differentiated system has the same structure as the corresponding problem (3.4) in [25], and hence the proof of the next result is analogous to that given there. It is thus omitted. We use the space $H^m_{\mathrm{ta}}(\Omega)$ of those maps $v \in L^2(\Omega)$ with $\partial^\alpha v \in L^2(\Omega)$ for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| \leq m$ and $\alpha_3 = 0$. It is equipped with its natural norm.

Lemma 4.2. *Let $\eta > 0$, $r \geq r_0 > 0$, $m \in \mathbb{N}$, and $\tilde{m} = \max\{m, 3\}$. Take $\mathcal{A}_0 \in F^{\mathrm{cp}}_{\tilde{m},\eta}(\Omega)$, $\mathcal{A}_1, \mathcal{A}_2 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff}}(\mathbb{R}^3_+)$, $\mathcal{A}_3 = \tilde{\mathcal{A}}_3^{\mathrm{co}}$, $\mathcal{D} \in F^{\mathrm{cp}}_{\tilde{m}}(\Omega)$, and $B = \mathcal{B}^{\mathrm{co}}$ with*

$$\|\mathcal{A}_i\|_{F_{\tilde{m}}(\Omega)} \leq r, \quad \|\mathcal{D}\|_{F_{\tilde{m}}(\Omega)} \leq r,$$
$$\max\{\|\mathcal{A}_i(0)\|_{F^0_{\tilde{m}-1}(\mathbb{R}^3_+)}, \max_{1\leq j\leq m-1}\|\partial_t^j\mathcal{A}_0(0)\|_{H^{\tilde{m}-1-j}(\mathbb{R}^3_+)}\} \leq r_0,$$
$$\max\{\|\mathcal{D}(0)\|_{F^0_{\tilde{m}-1}(\mathbb{R}^3_+)}, \max_{1\leq j\leq m-1}\|\partial_t^j\mathcal{D}(0)\|_{H^{\tilde{m}-1-j}(\mathbb{R}^3_+)}\} \leq r_0,$$

for all $i \in \{0,1,2\}$. Choose data $f \in H^m_{\mathrm{ta}}(\Omega)$, $g \in E_m(J \times \partial\mathbb{R}^3_+)$, and $u_0 \in H^m(\mathbb{R}^3_+)$. Assume that the solution u of (4.1) belongs to $G_m(\Omega)$. Then there exists a parameter $\gamma_m = \gamma_m(\eta, r) \geq 1$ such that u satisfies

$$\sum_{\substack{|\alpha|\leq m\\ \alpha_3=0}}\|\partial^\alpha u\|^2_{G_{0,\gamma}(\Omega)} + \gamma\|u\|^2_{H^m_{\mathrm{ta},\gamma}(\Omega)} \leq C_{m,0}\Big[\sum_{j=0}^{m-1}\|\partial_t^j f(0)\|^2_{H^{m-1-j}(\mathbb{R}^3_+)} + \|g\|^2_{E_{m,\gamma}(J\times\partial\mathbb{R}^3_+)} + \|u_0\|^2_{H^m(\mathbb{R}^3_+)}\Big] + \frac{C_m}{\gamma}\Big[\|f\|^2_{H^m_{\mathrm{ta},\gamma}(\Omega)} + \|u\|^2_{G_{m,\gamma}(\Omega)}\Big],$$

for all $\gamma \geq \gamma_0$, where $C_m = C_m(\eta, r, T')$, and $C_{m,0} = C_{m,0}(\eta, r_0)$.

The full H^m-norm of solutions u to (4.1) cannot be controlled in this way since normal derivatives destroy the boundary condition. From the system (4.1) itself one can read off regularity of normal derivatives of the tangential components of u because of the structure of the boundary matrix $\mathcal{A}_3 = \tilde{\mathcal{A}}_3^{\mathrm{co}}$. The remaining four components will be recovered by means of cancellation properties of the Maxwell equations which imply that the 'generalized divergence' $\mathrm{Div}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ of the Maxwell operator only contains first order derivatives.

To define this concept, take $\mathcal{A}_1, \mathcal{A}_2 \in F^{\mathrm{cp}}_{0,\mathrm{coeff}}(\mathbb{R}^3_+)$ and $\mathcal{A}_3 = \tilde{\mathcal{A}}^{\mathrm{co}}_3$. In particular, there are functions $\mu_{lj} \in F^{\mathrm{cp}}_{0,1}(\Omega)$ such that

$$\mathcal{A}_j = \sum_{l=1}^{3} \mathcal{A}^{\mathrm{co}}_l \mu_{lj} \quad \text{for} \quad j \in \{1,2\} \qquad \text{and} \qquad \mu_{13} = \mu_{23} = 0, \quad \mu_{33} = 1, \tag{4.3}$$

see (2.2) and (2.1). We now set

$$\mu = (\mu_{lj})_{l,j=1}^3, \qquad \hat{\mu} = \begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & -\mu_{33} \end{pmatrix}, \qquad \tilde{\mu} = \begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \hat{\mu} & 0 \\ 0 & 0 & 0 & \hat{\mu} \end{pmatrix}, \tag{4.4}$$

and for $h \in L^2(\mathbb{R}^3_+)^{12}$ we define

$$\operatorname{Div}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)h = \sum_{k=1}^{3} \Big((\tilde{\mu}^T \nabla h)_{kk}, (\tilde{\mu}^T \nabla h)_{(k+3)k}, (\tilde{\mu}^T \nabla h)_{(k+6)k}, (\tilde{\mu}^T \nabla h)_{(k+9)k} \Big). \tag{4.5}$$

In view of the iteration and regularization process below, in the next proposition we treat solutions and data which are a bit less regular than needed in this section and we consider the initial value problem

$$\begin{cases} \mathcal{L}(\mathcal{A}_0, \ldots, \mathcal{A}_3, \mathcal{D})u = f, & x \in \mathbb{R}^3_+, \qquad t \in J; \\ \qquad\qquad u(0) = u_0, & x \in \mathbb{R}^3_+. \end{cases} \tag{4.6}$$

A *solution* of (4.6) is a function $u \in C(\overline{J}, L^2(\mathbb{R}^3_+))$ with $u(0) = u_0$ in $L^2(\mathbb{R}^3_+)$ and $\mathcal{L}u = f$ in $H^{-1}(\Omega)$. The following result is the core step in our regularity theory.

Proposition 4.3. *Let $T' > 0$, $\eta > 0$, $\gamma \geq 1$, and $r \geq r_0 > 0$. Take coeffcients $\mathcal{A}_0 \in F^{\mathrm{cp}}_{0,\eta}(\Omega)$, $\mathcal{A}_1, \mathcal{A}_2 \in F^{\mathrm{cp}}_{0,\mathrm{coeff}}(\mathbb{R}^3_+)$, $\mathcal{A}_3 = \tilde{\mathcal{A}}^{\mathrm{co}}_3$, and $\mathcal{D} \in F^{\mathrm{cp}}_0(\Omega)$ with*

$$\|\mathcal{A}_i\|_{W^{1,\infty}(\Omega)} \leq r, \quad \|\mathcal{D}\|_{W^{1,\infty}(\Omega)} \leq r,$$
$$\|\mathcal{A}_i(0)\|_{L^\infty(\mathbb{R}^3_+)} \leq r_0, \quad \|\mathcal{D}(0)\|_{L^\infty(\mathbb{R}^3_+)} \leq r_0$$

for all $i \in \{0,1,2\}$. Choose data $f \in G_0(\Omega)$ with $\operatorname{Div}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)f \in L^2(\Omega)$ and $u_0 \in H^1(\mathbb{R}^3_+)$. Let u solve (4.6) and assume that u is an element of $C^1(\overline{J}, L^2(\mathbb{R}^3_+)) \cap C(\overline{J}, H^1_{\mathrm{ta}}(\mathbb{R}^3_+)) \cap L^\infty(J, H^1(\mathbb{R}^3_+))$. Then u belongs to $G_1(\Omega)$ and there are constants $C_{1,0} = C_{1,0}(\eta, r_0) \geq 1$ and $C_1 = C_1(\eta, r, T') \geq 1$ such that it satisfies

$$\|\nabla u\|^2_{G_{0,\gamma}(\Omega)} \leq e^{C_1 T} \Big((C_{1,0} + TC_1) \Big(\sum_{j=0}^{2} \|\partial_j u\|^2_{G_{0,\gamma}(\Omega)} + \|f\|^2_{G_{0,\gamma}(\Omega)} + \|u_0\|^2_{H^1(\mathbb{R}^3_+)} \Big) + \frac{C_1}{\gamma} \|\operatorname{Div}(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) f\|^2_{L^2_\gamma(\Omega)} \Big). \tag{4.7}$$

If f is even contained in $H^1(\Omega)$, we obtain

$$\|\nabla u\|^2_{G_{0,\gamma}(\Omega)} \leq e^{C_1 T} \Big((C_{1,0} + TC_1) \Big(\sum_{j=0}^{2} \|\partial_j u\|^2_{G_{0,\gamma}(\Omega)} + \|f(0)\|^2_{L^2(\mathbb{R}^3_+)} + \|u_0\|^2_{H^1(\mathbb{R}^3_+)} \Big) + \frac{C_1}{\gamma} \|f\|^2_{H^1_\gamma(\Omega)} \Big). \tag{4.8}$$

Finally, if f merely belongs to $L^2(\Omega)$ with $\operatorname{Div}(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)f \in L^2(\Omega)$, we still have

$$\|\nabla u\|_{L^2_\gamma(\Omega)}^2 \le e^{C_1 T}\Big((C_{1,0}+TC_1)\Big(\sum_{j=0}^{2}\|\partial_j u\|_{L^2_\gamma(\Omega)}^2 + \|f\|_{L^2_\gamma(\Omega)}^2 + \|u_0\|_{H^1(\mathbb{R}^3_+)}^2\Big) + \frac{C_1}{\gamma}\|\operatorname{Div}(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)f\|_{L^2_\gamma(\Omega)}^2\Big). \tag{4.9}$$

Proof. We have to show that $\partial_3 u \in C(\overline{J}, L^2(\mathbb{R}^3_+))$ and that inequalities (4.7) to (4.9) are true. We employ the matrix $\tilde{\mu}$ from (4.4). Recall that the coefficients $\mathcal{A}_l$ are given by (4.3) and $\mathcal{A}_3 = \tilde{A}_3^{\mathrm{co}}$, $\mathcal{A}_l^{\mathrm{co}}$ and $\tilde{A}_3^{\mathrm{co}}$ by (2.1), as well as A_l^{co} and J_l by (1.4), for $l \in \{1,2,3\}$ Morever, $J_{l;mn} = -\varepsilon_{lmn}$ for all $l,m,n \in \{1,2,3\}$ and the Levi-Civita symbol, i.e.,

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}, \\ -1 & \text{if } (i,j,k) \in \{(3,2,1),(2,1,3),(1,3,2)\}, , \\ 0 & \text{else.} \end{cases}$$

Since the coefficients are Lipschitz, we can differentiate

$$\begin{aligned} \partial_t(\tilde{\mu}^T\mathcal{A}_0\nabla u) &= \tilde{\mu}^T\partial_t\mathcal{A}_0\nabla u + \tilde{\mu}^T\mathcal{A}_0\partial_t\nabla u \\ &= \tilde{\mu}^T\partial_t\mathcal{A}_0\nabla u + \tilde{\mu}^T\mathcal{A}_0\nabla\Big(\mathcal{A}_0^{-1}\Big(f - \sum_{j=1}^{3}\mathcal{A}_j\partial_j u - \mathcal{D}u\Big)\Big) \\ &= \tilde{\mu}^T\partial_t\mathcal{A}_0\nabla u + \tilde{\mu}^T\mathcal{A}_0\nabla\mathcal{A}_0^{-1}\Big(f - \sum_{j=1}^{3}\mathcal{A}_j\partial_j u - \mathcal{D}u\Big) \\ &\quad + \tilde{\mu}^T\nabla f - \tilde{\mu}^T\sum_{j=1}^{2}\nabla\mathcal{A}_j\partial_j u - \tilde{\mu}^T\nabla\mathcal{D}u - \tilde{\mu}^T\mathcal{D}\nabla u - \tilde{\mu}^T\sum_{j=1}^{3}\mathcal{A}_j\nabla\partial_j u \\ &=: \Lambda - \tilde{\mu}^T\sum_{j=1}^{3}\mathcal{A}_j\nabla\partial_j u \end{aligned} \tag{4.10}$$

in $L^\infty(J, H^{-1}(\mathbb{R}^3_+))$. Here we use (4.6) and write $((\nabla\mathcal{A}_0^{-1})h)_{jk} := \sum_{l=1}^{12}\partial_k\mathcal{A}_{0;jl}^{-1}h_l$ etc. Note that Λ only contains first order spatial derivatives of u. We next compute

$$\begin{aligned} \sum_{k=1}^{3}\Big(\tilde{\mu}^T\sum_{j=1}^{3}\mathcal{A}_j\nabla\partial_j u\Big)_{kk} &= \sum_{j,k=1}^{3}\sum_{l,p=1}^{12}\tilde{\mu}_{kl}^T\mathcal{A}_{j;lp}\partial_k\partial_j u_p = \sum_{j,k,l=1}^{3}\sum_{p=1}^{12}\mu_{lk}\mathcal{A}_{j;lp}\partial_k\partial_j u_p \\ &= \sum_{j,k,l,n,p=1}^{3}\mu_{lk}A^{\mathrm{co}}_{n;l(p+3)}\mu_{nj}\partial_k\partial_j u_{p+3} = \sum_{j,k,l,n,p=1}^{3}\varepsilon_{nlp}\mu_{lk}\mu_{nj}\partial_k\partial_j u_{p+3} && (4.11) \\ &= \sum_{j,k,l,n,p=1}^{3}\varepsilon_{lnp}\mu_{nj}\mu_{lk}\partial_j\partial_k u_{p+3} = -\sum_{j,k,l,n,p=1}^{3}\varepsilon_{nlp}\mu_{lk}\mu_{nj}\partial_k\partial_j u_{p+3}\,, && (4.12) \end{aligned}$$

exchanging the indices l and n as well as k and j in the penultimate step. Equations (4.11) and (4.12) yield

$$\sum_{k=1}^{3}\Big(\tilde{\mu}^T\sum_{j=1}^{3}\mathcal{A}_j\nabla\partial_j u\Big)_{kk} = 0. \tag{4.13}$$

Analogously, it follows

$$\sum_{k=1}^{3}\Big(\tilde{\mu}^T\sum_{j=1}^{3}\mathcal{A}_j\nabla\partial_j u\Big)_{(k+3)k}=0. \tag{4.14}$$

In the other components we take care of the extra signs in (4.4) and (2.1), calculating

$$\begin{aligned}
\sum_{k=1}^{3}\Big(\tilde{\mu}^T\sum_{j=1}^{3}\mathcal{A}_j\nabla\partial_j u\Big)_{(k+6)k} &= \sum_{j,k=1}^{3}\sum_{l,p=1}^{12}\tilde{\mu}^T_{(k+6)l}\mathcal{A}_{j;lp}\partial_k\partial_j u_p\\
&= \sum_{j,k,l=1}^{3}\sum_{p=1}^{12}\hat{\mu}_{lk}\mathcal{A}_{j;(l+6)p}\partial_k\partial_j u_p = \sum_{j,k,l,p=1}^{3}\hat{\mu}_{lk}\mathcal{A}_{j;(l+6)(p+9)}\partial_k\partial_j u_{p+9}\\
&= \sum_{j,k,l,n,p=1}^{3}\mu_{lk}(-1)^{\delta_{3l}\delta_{3k}}A^{\mathrm{co}}_{n;l(p+3)}\mu_{nj}(-1)^{\delta_{3j}\delta_{3n}}\partial_k\partial_j u_{p+9}\\
&= \sum_{j,k,l,n,p=1}^{3}\varepsilon_{nlp}(-1)^{\delta_{3l}\delta_{3k}}(-1)^{\delta_{3n}\delta_{3j}}\mu_{lk}\mu_{nj}\partial_k\partial_j u_{p+9} \qquad (4.15)\\
&= \sum_{j,k,l,n,p=1}^{3}\varepsilon_{lnp}(-1)^{\delta_{3n}\delta_{3j}}(-1)^{\delta_{3l}\delta_{3k}}\mu_{nj}\mu_{lk}\partial_j\partial_k u_{p+9}\\
&= -\sum_{j,k,l,n,p=1}^{3}\varepsilon_{nlp}(-1)^{\delta_{3l}\delta_{3k}}(-1)^{\delta_{3n}\delta_{3j}}\mu_{lk}\mu_{nj}\partial_k\partial_j u_{p+9}. \qquad (4.16)
\end{aligned}$$

Comparing the expressions (4.15) and (4.16), we infer

$$\sum_{k=1}^{3}\Big(\tilde{\mu}^T\sum_{j=1}^{3}\mathcal{A}_j\nabla\partial_j u\Big)_{(k+6)k}=0. \tag{4.17}$$

Proceeding similarly, we derive

$$\sum_{k=1}^{3}\Big(\tilde{\mu}^T\sum_{j=1}^{3}\mathcal{A}_j\nabla\partial_j u\Big)_{(k+9)k}=0. \tag{4.18}$$

Integrating in time, the formulas (4.10), (4.13) (4.14), (4.17) and (4.18) imply the identities

$$\sum_{k=1}^{3}(\tilde{\mu}^T\mathcal{A}_0\nabla u)_{(k+i)k}(t)=\sum_{k=1}^{3}(\tilde{\mu}^T\mathcal{A}_0\nabla u)_{(k+i)k}(0)+\sum_{k=1}^{3}\int_0^t\Lambda_{(k+i)k}(s)ds$$

in $H^{-1}(\mathbb{R}^3_+)$ for all $t\in\overline{J}$ and $i\in\{0,3,6,9\}$. The function Λ is integrable with values in $L^2(\mathbb{R}^3_+)$ so that the equality holds in $L^2(\mathbb{R}^3_+)$ for all $t\in\overline{J}$. Let $t\in\overline{J}$. We denote the k-th row respectively the k-th column of a matrix N by $N_{k\cdot}$ respectively $N_{\cdot k}$, and we set

$$F_{13+l}(t)=\sum_{k=1}^{3}(\tilde{\mu}^T\mathcal{A}_0\nabla u)_{(k+3l)k}(0)+\sum_{k=1}^{3}\int_0^t\Lambda_{(k+3l)k}(s)ds-\sum_{k=1}^{2}(\tilde{\mu}^T\mathcal{A}_0)_{(k+3l)\cdot}\partial_k u(t),$$

$$(F_1,\ldots,F_{12})^T=f-\sum_{j=0}^{2}\mathcal{A}_j\partial_j u-\mathcal{D}u$$

for $l \in \{0,1,2,3\}$. The map $F = (F_1, \dots, F_{16})^T$ belongs to $C(\overline{J}, L^2(\mathbb{R}^3_+))$ and

$$\breve{\mu}\partial_3 u = F, \qquad \text{setting } \ \breve{\mu} = \begin{pmatrix} \mathcal{A}_3 \\ (\tilde{\mu}^T\mathcal{A}_0)_{3\cdot} \\ (\tilde{\mu}^T\mathcal{A}_0)_{6\cdot} \\ (\tilde{\mu}^T\mathcal{A}_0)_{9\cdot} \\ (\tilde{\mu}^T\mathcal{A}_0)_{12\cdot} \end{pmatrix} \in F_0(\Omega)^{16\times 12}. \tag{4.19}$$

Let $\zeta = \tilde{\mu}^T\mathcal{A}_0$ and the matrix G_1 be equal to

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-\zeta_{3,5} & \zeta_{3,4} & 0 & \zeta_{3,2} & -\zeta_{3,1} & 0 & \zeta_{3,11} & -\zeta_{3,10} & 0 & -\zeta_{3,8} & \zeta_{3,7} & 0 & 1 & 0 & 0 & 0 \\
-\zeta_{6,5} & \zeta_{6,4} & 0 & \zeta_{6,2} & -\zeta_{6,1} & 0 & \zeta_{6,11} & -\zeta_{6,10} & 0 & -\zeta_{6,8} & \zeta_{6,7} & 0 & 0 & 1 & 0 & 0 \\
\zeta_{9,5} & -\zeta_{9,4} & 0 & -\zeta_{9,2} & \zeta_{9,1} & 0 & -\zeta_{9,11} & \zeta_{9,10} & 0 & \zeta_{9,8} & -\zeta_{9,7} & 0 & 0 & 0 & -1 & 0 \\
\zeta_{12,5} & -\zeta_{12,4} & 0 & -\zeta_{12,2} & \zeta_{12,1} & 0 & -\zeta_{12,11} & \zeta_{12,10} & 0 & \zeta_{12,8} & -\zeta_{12,7} & 0 & 0 & 0 & 0 & -1
\end{pmatrix}.$$

We derive the crucial identity

$$G_1\breve{\mu} = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_{3,3} & 0 & 0 & \alpha_{3,6} & 0 & 0 & \alpha_{3,9} & 0 & 0 & \alpha_{3,12} \\
0 & 0 & \alpha_{6,3} & 0 & 0 & \alpha_{6,6} & 0 & 0 & \alpha_{6,9} & 0 & 0 & \alpha_{6,12} \\
0 & 0 & \alpha_{9,3} & 0 & 0 & \alpha_{9,6} & 0 & 0 & \alpha_{9,9} & 0 & 0 & \alpha_{9,12} \\
0 & 0 & \alpha_{12,3} & 0 & 0 & \alpha_{12,6} & 0 & 0 & \alpha_{12,9} & 0 & 0 & \alpha_{12,12}
\end{pmatrix},$$

$$\alpha_{kn} := \zeta_{kn} = \sum_{l=1}^{12} \tilde{\mu}^T_{kl}\mathcal{A}_{0;ln} = \mathcal{A}_{0;kn} \qquad \text{for } \ k \in \{3,6\},$$

$$\alpha_{kn} := -\zeta_{kn} = -\sum_{l=1}^{12} \tilde{\mu}^T_{kl}\mathcal{A}_{0;ln} = \mathcal{A}_{0;kn} \qquad \text{for } \ k \in \{9,12\},$$

where $n \in \{3, 6, 9, 12\}$. Here we use $\tilde{\mu}_{lk} = 1$ for $l = k$ and $\tilde{\mu}_{lk} = 0$ for $l \neq k$, if $k \in \{3, 6\}$, as well as $\tilde{\mu}_{lk} = -1$ for $l = k$ and $\tilde{\mu}_{lk} = 0$ for $l \neq k$, if $k \in \{9, 12\}$. Since

$$\begin{pmatrix} \alpha_{3,3} & \alpha_{3,6} & \alpha_{3,9} & \alpha_{3,12} \\ \alpha_{6,3} & \alpha_{6,6} & \alpha_{6,9} & \alpha_{6,12} \\ \alpha_{9,3} & \alpha_{9,6} & \alpha_{9,9} & \alpha_{9,12} \\ \alpha_{12,3} & \alpha_{12,6} & \alpha_{12,9} & \alpha_{12,12} \end{pmatrix} = \begin{pmatrix} \mathcal{A}_{0;3,3} & \mathcal{A}_{0;3,6} & \mathcal{A}_{0;3,9} & \mathcal{A}_{0;3,12} \\ \mathcal{A}_{0;6,3} & \mathcal{A}_{0;6,6} & \mathcal{A}_{0;6,9} & \mathcal{A}_{0;6,12} \\ \mathcal{A}_{0;9,3} & \mathcal{A}_{0;9,6} & \mathcal{A}_{0;9,9} & \mathcal{A}_{0;9,12} \\ \mathcal{A}_{0;12,3} & \mathcal{A}_{0;12,6} & \mathcal{A}_{0;12,9} & \mathcal{A}_{0;12,12} \end{pmatrix} \geq \eta,$$

this matrix has an inverse β bounded by $C(\eta)$. Setting $G_2 = \left(\begin{smallmatrix} I_{12\times 12} & 0 \\ 0 & \beta \end{smallmatrix}\right)$, we compute

$$G_2 G_1 \breve{\mu} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} =: \tilde{M}. \tag{4.20}$$

Equations (4.19) and (4.20) yield

$$\tilde{M} \partial_3 u = G_2 G_1 F. \tag{4.21}$$

The formulas in (4.3) imply the inequality

$$\|G_2 G_1\|_{L^\infty(\Omega)} \leq C(\eta)(1 + c_0)^2 \quad \text{with} \quad c_0 := \max\{\max_{j=0,\ldots,3} \|\mathcal{A}_j\|_{L^\infty(\Omega)}, \|\mathcal{D}\|_{L^\infty(\Omega)}\}.$$

Since the matrix $\tilde{M}$ has rank 12, equation (4.21) shows that $\partial_3 u$ is contained in $C(\overline{J}, L^2(\mathbb{R}^3_+))$ and bounded by

$$\|\partial_3 u(t)\|_{L^2(\mathbb{R}^3_+)} \leq C(\eta)(1 + c_0)^2 \|F(t)\|_{L^2(\mathbb{R}^3_+)}. \tag{4.22}$$

This estimate is analogous to (3.29) in the proof of Proposition 3.3 in [25], where a comparable function F was involved. The remaining arguments are the same as in [25] and therefore omitted. They mainly consist of straightforward estimates and an application of Gronwall's inequality. □

We can now combine Lemma 4.1, Lemma 4.2 and Proposition 4.3 in an iteration argument to establish the desired a priori estimates of arbitrary order. This is done as in the proof of Theorem 4.4 in [25], also using the auxiliary results from Section 2. Here the different structure in (4.1) arising from the interface condition does not play a role. So we do not give the proof.

Theorem 4.4. *Let $T' > 0$, $\eta > 0$, $r \geq r_0 > 0$, $m \in \mathbb{N}$, and $\tilde{m} = \max\{m, 3\}$. Pick $T \in (0, T']$ and set $J = (0, T)$. Take coefficients $\mathcal{A}_0 \in F^{\text{cp}}_{\tilde{m},\eta}(\Omega)$, $\mathcal{A}_1, \mathcal{A}_2 \in$*

$F^{\mathrm{cp}}_{\tilde m,\mathrm{coeff}}(\mathbb{R}^3_+)$, $\mathcal{A}_3 = \tilde{\mathcal{A}}^{\mathrm{co}}_3$, $\mathcal{D} \in F^{\mathrm{cp}}_{\tilde m}(\Omega)$, and $B = \mathcal{B}^{\mathrm{co}}$ satisfying

$$\|\mathcal{A}_i\|_{F_{\tilde m}(\Omega)} \leq r, \quad \|\mathcal{D}\|_{F_{\tilde m}(\Omega)} \leq r,$$
$$\max\{\|\mathcal{A}_i(0)\|_{F^0_{\tilde m-1}(\mathbb{R}^3_+)}, \max_{1\leq j\leq \tilde m-1} \|\partial_t^j \mathcal{A}_0(0)\|_{H^{\tilde m-j-1}(\mathbb{R}^3_+)}\} \leq r_0,$$
$$\max\{\|\mathcal{D}(0)\|_{F^0_{\tilde m-1}(\mathbb{R}^3_+)}, \max_{1\leq j\leq \tilde m-1} \|\partial_t^j \mathcal{D}(0)\|_{H^{\tilde m-j-1}(\mathbb{R}^3_+)}\} \leq r_0$$

for all $i \in \{0,1,2\}$. Choose data $f \in H^m(\Omega)$, $g \in E_m(J\times\partial\mathbb{R}^3_+)$, and $u_0 \in H^m(\mathbb{R}^3_+)$. Assume that the solution u of (4.1) belongs to $G_m(\Omega)$. Then there is a number $\gamma_m = \gamma_m(\eta, r, T') \geq 1$ such that u satisfies

$$\|u\|^2_{G_{m,\gamma}(\Omega)} \leq (C_{m,0} + TC_m)e^{mC_1T}\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(0)\|^2_{H^{m-1-j}(\mathbb{R}^3_+)} + \|g\|^2_{E_{m,\gamma}(J\times\partial\mathbb{R}^3_+)}$$
$$+ \|u_0\|^2_{H^m(\mathbb{R}^3_+)}\Big) + \frac{C_m}{\gamma}\|f\|^2_{H^m_\gamma(\Omega)}$$

for all $\gamma \geq \gamma_m$, where $C_m = C_m(\eta, r, T') \geq 1$, $C_{m,0} = C_{m,0}(\eta, r_0) \geq 1$, and $C_1 = C_1(\eta, r, T')$ is a constant independent of m.

5. Regularity of solutions to the linear problem

In this section we prove that the $G_0(\Omega)$-solution u of (4.1) actually belongs to $G_m(\Omega)$ if the data and the coefficients are accordingly smooth and compatible. To this aim, different regularizing techniques in normal, tangential, and time direction are used. We first show that regularity in time and in tangential directions implies regularity in normal direction. This is the crucial step in the regularization argument, and it heavily relies on the structure of the Maxwell system. As in Proposition 4.3, we only look at the linear initial value problem (4.6).

Lemma 5.1. *Let $\eta > 0$, $m \in \mathbb{N}$, and $\tilde m = \max\{m, 3\}$. Take coefficients $\mathcal{A}_0 \in F^{\mathrm{cp}}_{\tilde m,\eta}(\Omega)$, $\mathcal{A}_1, \mathcal{A}_2 \in F^{\mathrm{cp}}_{\tilde m,\mathrm{coeff}}(\mathbb{R}^3_+)$, $\mathcal{A}_3 = \tilde{\mathcal{A}}^{\mathrm{co}}_3$, and $\mathcal{D} \in F^{\mathrm{cp}}_{\tilde m}(\Omega)$. Choose data $f \in H^m(\Omega)$ and $u_0 \in H^m(\mathbb{R}^3_+)$. Let u be a solution of (4.6) for these coefficients and data. Assume that u belongs to $\bigcap_{j=1}^m C^j(\overline{J}, H^{m-j}(\mathbb{R}^3_+))$.*

Take $k \in \{1, \ldots, m\}$ and a multi-index $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$, $\alpha_0 = 0$, and $\alpha_3 = k$. Suppose that $\partial^\beta u$ is contained in $G_0(\Omega)$ for all $\beta \in \mathbb{N}_0^4$ with $|\beta| = m$ and $\beta_3 \leq k-1$. Then $\partial^\alpha u$ is an element of $G_0(\Omega)$.

Proof. I) We begin with several preparations. Let M_ε, $\varepsilon > 0$, be a standard mollifier on $\mathbb{R}^3$ with kernel $\rho \geq 0$. Let $\delta > 0$. We introduce the translation operator

$$T_\delta v(x) = v(x_1, x_2, x_3 + \delta) \quad \text{for} \quad v \in L^1_{\mathrm{loc}}(\mathbb{R}^3_+) \text{ and a.e. } x \in \mathbb{R}^2 \times (-\delta, \infty). \tag{5.1}$$

Notice that T_δ maps $W^{l,p}(\mathbb{R}^3_+)$ continuously into $W^{l,p}(\mathbb{R}^2 \times (-\delta, \infty))$ and that $\partial^{\tilde\alpha} T_\delta v = T_\delta \partial^{\tilde\alpha} v$ for all $v \in W^{l,p}(\mathbb{R}^3_+)$, $\tilde\alpha \in \mathbb{N}_0^4$ with $|\tilde\alpha| \leq l$, $l \in \mathbb{N}_0$, and $1 \leq p \leq \infty$. If $v \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, we further define $T_\delta v$ by formula (5.1) for all $\delta \in \mathbb{R}$.

Functions which are only defined on a subset of $\mathbb{R}^3$ will be identified with their zero-extensions. Moreover, restrictions of a map v to a subset are also denoted by v. We extend the translations T_δ to continuous operators on $H^{-1}(\mathbb{R}^3_+)$ by setting

$$\langle T_\delta v, \psi\rangle_{H^{-1}(\mathbb{R}^3_+)\times H^1_0(\mathbb{R}^3_+)} = \langle v, T_{-\delta}\psi\rangle_{H^{-1}(\mathbb{R}^3_+)\times H^1_0(\mathbb{R}^3_+)}$$

for all $\psi \in H^1_0(\mathbb{R}^3_+)$ and $\delta > 0$. It is then straighforward to check that $\partial_j T_\delta v = T_\delta \partial_j v$ for all $v \in L^2(\mathbb{R}^3_+)$ and $\delta > 0$.

We want to apply M_ε to functions in $L^1_{\mathrm{loc}}(\mathbb{R}^3_+)$ without obtaining singularities at the boundary in limit processes. To that purpose, we take $0 < \varepsilon < \delta$ and look at the regularization $M_\varepsilon T_\delta v$ for $v \in L^1_{\mathrm{loc}}(\mathbb{R}^3_+)$. If v and $\partial_j v$ belong to $L^1_{\mathrm{loc}}(\mathbb{R}^3_+)$, then also $M_\varepsilon T_\delta v$ has a weak derivative in $\mathbb{R}^3_+$ and $\partial_j M_\varepsilon T_\delta v = M_\varepsilon T_\delta \partial_j v$ for all $j \in \{1,2,3\}$.

We set $\tilde{\rho}(x) = \rho(-x)$ for all $x \in \mathbb{R}^3$ and denote the corresponding mollifier by $\tilde{M}_\varepsilon$. A straightforward computation shows that

$$\langle M_\varepsilon T_\delta v, \psi\rangle_{H^{-1}(\mathbb{R}^3_+)\times H^1_0(\mathbb{R}^3_+)} = \langle v, T_{-\delta}\tilde{M}_\varepsilon \psi\rangle_{H^{-1}(\mathbb{R}^3_+)\times H^1_0(\mathbb{R}^3_+)} \tag{5.2}$$

for all $v \in L^2(\mathbb{R}^3_+)$ and $\psi \in H^1_0(\mathbb{R}^3_+)$. As $T_{-\delta}\tilde{M}_\varepsilon$ maps $H^1_0(\mathbb{R}^3_+)$ continuously into itself, the mapping $M_\varepsilon T_\delta$ continuously extends to an operator on $H^{-1}(\mathbb{R}^3_+)$ via formula (5.2). We deduce the identity

$$\partial_j M_\varepsilon T_\delta v = M_\varepsilon \partial_j T_\delta v = M_\varepsilon T_\delta \partial_j v$$

by duality for all $j \in \{1,2,3\}$ and $v \in L^2(\mathbb{R}^3_+)$. Finally, for $A \in W^{1,\infty}(\mathbb{R}^3_+)$ and $v \in H^{-1}(\mathbb{R}^3_+)$ we obtain $(T_\delta A)T_\delta v = T_\delta(Av)$ in $H^{-1}(\mathbb{R}^3_+)$.

II) Let $0 < \varepsilon < \delta$. We abbreviate the differential operators $\mathcal{L}(T_\delta \mathcal{A}_j, T_\delta \mathcal{D})$ by $\mathcal{L}_\delta$ and $\mathrm{Div}(T_\delta\mathcal{A}_1, T_\delta\mathcal{A}_2, T_\delta\mathcal{A}_3)$ by Div_δ. (Recall (4.5).) Let $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$, $\alpha_0 = 0$, and $\alpha_3 = k$. We set $\alpha' = \alpha - e_3 \in \mathbb{N}_0^4$. The derivative $\partial^{\alpha'} u$ belongs to $G_0(\Omega)$ by assumption. Because of the mollifier, the map $M_\varepsilon T_\delta \partial^{\alpha'} u$ is contained in $C^1(\overline{J}, H^2(\mathbb{R}^3_+)) \hookrightarrow G_1(\Omega)$, $M_\varepsilon T_\delta \partial^{\alpha'} u_0$ in $H^1(\mathbb{R}^3_+)$, $\mathcal{L}_\delta M_\varepsilon T_\delta \partial^{\alpha'} u$ in $G_0(\Omega)$, and $\mathrm{Div}_\delta\, \mathcal{L}_\delta M_\varepsilon T_\delta \partial^{\alpha'} u$ in $L^2(\Omega)$. To show convergence of $\partial_3 M_\varepsilon T_\delta \partial^{\alpha'} u$ as $\varepsilon \to 0$, we want to apply the a priori estimate (4.7). Therefore, we have to study the convergence properties of the functions $\mathcal{L}_\delta M_\varepsilon T_\delta \partial^{\alpha'} u$ and $\mathrm{Div}_\delta\, \mathcal{L}_\delta M_\varepsilon T_\delta \partial^{\alpha'} u$ as $\varepsilon \to 0$. We focus on the latter as this is the more difficult one.

We use the maps μ_{kl}, $\hat{\mu}$, and $\tilde{\mu}$ from (4.4). Exploiting step I), we compute

$$\begin{aligned}
&(T_\delta \tilde{\mu})^T \nabla \mathcal{L}_\delta M_\varepsilon T_\delta \partial^{\alpha'} u \qquad\qquad (5.3)\\
&= \sum_{j=0}^{2} (T_\delta\tilde{\mu})^T (T_\delta \nabla \mathcal{A}_j) \partial_j M_\varepsilon T_\delta \partial^{\alpha'} u + (T_\delta \tilde{\mu})^T (T_\delta \nabla \mathcal{D}) M_\varepsilon T_\delta \partial^{\alpha'} u \\
&\quad + T_\delta(\tilde{\mu}^T \mathcal{A}_0) \nabla M_\varepsilon T_\delta \partial_t \partial^{\alpha'} u + T_\delta(\tilde{\mu}^T \mathcal{D}) \nabla M_\varepsilon T_\delta \partial^{\alpha'} u + \sum_{j=1}^{3} T_\delta(\tilde{\mu}^T \mathcal{A}_j) \nabla \partial_j M_\varepsilon T_\delta \partial^{\alpha'} u \\
&=: \Lambda^{\delta,\varepsilon} + \sum_{j=1}^{3} T_\delta(\tilde{\mu}^T \mathcal{A}_j) \nabla \partial_j M_\varepsilon T_\delta \partial^{\alpha'} u.
\end{aligned}$$

The cancellation properties of $\mathcal{L}_\delta$ established in formulas (4.13), (4.14), (4.17) and (4.18) show that

$$\sum_{k=1}^{3}\sum_{j=1}^{3} (T_\delta(\tilde{\mu}^T \mathcal{A}_j) \nabla \partial_j M_\varepsilon T_\delta \partial^{\alpha'} u)_{(k+3l)k} = 0$$

for all $l \in \{0,1,2,3\}$. Equation (5.3) thus leads to

$$\mathrm{Div}_\delta\, \mathcal{L}_\delta M_\varepsilon T_\delta \partial^{\alpha'} u = \sum_{k=1}^{3} \Big(\Lambda^{\delta,\varepsilon}_{kk}, \Lambda^{\delta,\varepsilon}_{(k+3)k}, \Lambda^{\delta,\varepsilon}_{(k+6)k}, \Lambda^{\delta,\varepsilon}_{(k+9)k}\Big). \tag{5.4}$$

We rewrite $\Lambda^{\delta,\varepsilon}$ in the form

$$\begin{aligned}\Lambda^{\delta,\varepsilon} &= \sum_{j=0}^{2} [T_\delta(\tilde{\mu}^T \nabla \mathcal{A}_j), M_\varepsilon] \partial_j T_\delta \partial^{\alpha'} u + [T_\delta(\tilde{\mu}^T \nabla \mathcal{D}), M_\varepsilon] T_\delta \partial^{\alpha'} u \\ &\quad + [T_\delta(\tilde{\mu}^T \mathcal{A}_0), M_\varepsilon] \nabla T_\delta \partial_t \partial^{\alpha'} u + [T_\delta(\tilde{\mu}^T \mathcal{D}), M_\varepsilon] \nabla T_\delta \partial^{\alpha'} u \\ &\quad + M_\varepsilon T_\delta \Big(\sum_{j=0}^{2} \tilde{\mu}^T \nabla \mathcal{A}_j \partial_j \partial^{\alpha'} u + \tilde{\mu}^T \nabla \mathcal{D} \partial^{\alpha'} u + \tilde{\mu}^T \mathcal{A}_0 \nabla \partial_t \partial^{\alpha'} u + \tilde{\mu}^T \mathcal{D} \nabla \partial^{\alpha'} u \Big).\end{aligned}$$

In view of the terms with m space derivatives in the last line, we introduce the map

$$\begin{aligned}\tilde{f}_{\alpha'} &= \sum_{0<\beta\le\alpha'} \binom{\alpha'}{\beta} \partial^\beta (\tilde{\mu}^T \mathcal{A}_0) \nabla \partial^{\alpha'-\beta} \partial_t u + \sum_{0<\beta\le\alpha'} \binom{\alpha'}{\beta} \partial^\beta (\tilde{\mu}^T \mathcal{D}) \nabla \partial^{\alpha'-\beta} u \\ &\quad + \sum_{j=0}^{2} \sum_{0<\beta\le\alpha'} \binom{\alpha'}{\beta} \partial^\beta (\tilde{\mu}^T \nabla \mathcal{A}_j) \partial^{\alpha'-\beta} \partial_j u + \sum_{0<\beta\le\alpha'} \binom{\alpha'}{\beta} \partial^\beta (\tilde{\mu}^T \nabla \mathcal{D}) \partial^{\alpha'-\beta} u.\end{aligned}$$

As u and $\partial_t u$ are contained in $C(\overline{J}, H^{m-1}(\mathbb{R}^3_+))$, Lemma 2.1 implies that $\tilde{f}_{\alpha'}$ is an element of $L^2(\Omega)$. It follows

$$\begin{aligned}\Lambda^{\delta,\varepsilon} &= \sum_{j=0}^{2} [T_\delta(\tilde{\mu}^T \nabla \mathcal{A}_j), M_\varepsilon] \partial_j T_\delta \partial^{\alpha'} u + [T_\delta(\tilde{\mu}^T \nabla \mathcal{D}), M_\varepsilon] T_\delta \partial^{\alpha'} u \\ &\quad + [T_\delta(\tilde{\mu}^T \mathcal{A}_0), M_\varepsilon] \nabla T_\delta \partial_t \partial^{\alpha'} u + [T_\delta(\tilde{\mu}^T \mathcal{D}), M_\varepsilon] \nabla T_\delta \partial^{\alpha'} u + \partial^{\alpha'} M_\varepsilon T_\delta (\tilde{\mu}^T \nabla f) \\ &\quad - M_\varepsilon T_\delta \tilde{f}_{\alpha'} - \sum_{j=1}^{3} \partial^{\alpha'} M_\varepsilon T_\delta (\tilde{\mu}^T \mathcal{A}_j \nabla \partial_j u) \\ &=: \tilde{\Lambda}^{\delta,\varepsilon} - \sum_{j=1}^{3} \partial^{\alpha'} M_\varepsilon T_\delta (\tilde{\mu}^T \mathcal{A}_j \nabla \partial_j u).\end{aligned}$$

Equations (4.13), (4.14), (4.17) and (4.18) also yield that

$$\sum_{k=1}^{3} \Big(\Lambda^{\delta,\varepsilon}_{kk}, \Lambda^{\delta,\varepsilon}_{(k+3)k}, \Lambda^{\delta,\varepsilon}_{(k+6)k}, \Lambda^{\delta,\varepsilon}_{(k+9)k} \Big) = \sum_{k=1}^{3} \Big(\tilde{\Lambda}^{\delta,\varepsilon}_{kk}, \tilde{\Lambda}^{\delta,\varepsilon}_{(k+3)k}, \tilde{\Lambda}^{\delta,\varepsilon}_{(k+6)k}, \tilde{\Lambda}^{\delta,\varepsilon}_{(k+9)k} \Big).$$

By means of (5.4), we arrive at the core identity

$$\operatorname{Div}_\delta \mathcal{L}_\delta M_\varepsilon T_\delta \partial^{\alpha'} u = \sum_{k=1}^{3} \Big(\tilde{\Lambda}^{\delta,\varepsilon}_{kk}, \tilde{\Lambda}^{\delta,\varepsilon}_{(k+3)k}, \tilde{\Lambda}^{\delta,\varepsilon}_{(k+6)k}, \tilde{\Lambda}^{\delta,\varepsilon}_{(k+9)k} \Big). \tag{5.5}$$

Starting from its counterpart (4.7) in [25], the rest of the reasoning is now the same as in the proof of Lemma 4.1 in this paper. One uses that $M_\varepsilon T_\delta \partial^{\alpha'} u$ solves the initial value problem (4.6) with differential operator $\mathcal{L}_\delta$, inhomogeneity $\mathcal{L}_\delta M_\varepsilon T_\delta \partial^{\alpha'} u$ and initial value $M_\varepsilon T_\delta u_0$. In these data and in (5.5), one can pass to the limit in L^2 as $\varepsilon \to 0$ employing estimates for the commutators of the mollifier and the coefficients. The estimate (4.7) from Proposition 4.3 then allows to bound $\nabla T_\delta \partial^{\alpha'} u$ in $G_0(\Omega)$, uniformly in $\delta > 0$, see (4.15) in [25]. One can then let $\delta \to 0$ obtaining the result. We omit the details. $\square$

Replacing estimate (4.7) from Proposition 4.3 by inequality (4.9) in the above proof, one derives the following variant of Lemma 5.1, cf. Corollary 4.2 in [25].

Corollary 5.2. *Let $\eta > 0$, $m \in \mathbb{N}$, and $\tilde{m} = \max\{m, 3\}$. Take coefficients $\mathcal{A}_0 \in F^{\mathrm{cp}}_{\tilde{m},\eta}(\Omega)$, $\mathcal{A}_1, \mathcal{A}_2 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff}}(\mathbb{R}^3_+)$, $\mathcal{A}_3 = \tilde{\mathcal{A}}^{\mathrm{co}}_3$, and $\mathcal{D} \in F^{\mathrm{cp}}_{\tilde{m}}(\Omega)$. Choose data $f \in H^m(\Omega)$ and $u_0 \in H^m(\mathbb{R}^3_+)$. Let u be a solution of the initial value problem* (4.6) *with these coefficients and data. Assume that u belongs to $\bigcap_{j=1}^m C^j(\overline{J}, H^{m-j}(\mathbb{R}^3_+))$.*

Take $k \in \{1, \dots, m\}$ and a multi-index $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$, $\alpha_0 = 0$, and $\alpha_3 = k$. Suppose that $\partial^\beta u$ is contained in $L^2(\Omega)$ for all $\beta \in \mathbb{N}_0^4$ with $|\beta| = m$ and $\beta_3 \leq k - 1$. Then $\partial^\alpha u$ is an element of $L^2(\Omega)$.

Based on Lemma 5.1 and Corollary 5.2, the regularization arguments in tangential and time direction are analogous to the proofs of Lemma 4.4 and 4.5 in [25]. One first studies the solution u mollified in (x_1, x_2). The regularized solution u_ε satisfies the Maxwell system with modified data (as in (4.20) of [25]). It then crucially enters into the bound of u in a family of weighted tangential Sobolev norms, taken from Section 1.7 and Section 2.4 in [13]. The a priori estimate from Lemma 4.1 allows us to control u_ε in G_0. It is then possible to take the limit $\varepsilon \to 0$. The results from [13] require smooth coefficients so that temporarily we have to assume this extra regularity.

In the time direction one looks at the problem solved by the time derivative v of u, cf. (4.32) in [25]. Integration with respect to time yields a function which coincides with u, implying the required time regularity. Here the compatibility conditions are needed. In these arguments the new features of the problem (4.1) do not play a role and one can follow the lines of the proofs of [25]. We thus only state the results.

Lemma 5.3. *Let $\eta > 0$, $m \in \mathbb{N}$, and $\tilde{m} = \max\{m, 3\}$. Take coefficients $\mathcal{A}_0 \in F^{\mathrm{cp}}_{\tilde{m},\eta}(\Omega)$, $\mathcal{A}_1, \mathcal{A}_2 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff}}(\mathbb{R}^3_+)$, $\mathcal{A}_3 = \tilde{\mathcal{A}}^{\mathrm{co}}_3$, $\mathcal{D} \in F^{\mathrm{cp}}_{\tilde{m}}(\Omega)$ and $B = \mathcal{B}^{\mathrm{co}}$. We further assume that these coefficients belong to $C^\infty(\overline{\Omega})$. Let u be the weak solution of* (4.1) *with data $f \in H^m_{\mathrm{ta}}(\Omega)$, $g \in E_m(J \times \partial\mathbb{R}^3_+)$, and $u_0 \in H^m_{\mathrm{ta}}(\mathbb{R}^3_+)$. Suppose that u belongs to $\bigcap_{j=1}^m C^j(\overline{J}, H^{m-j}(\mathbb{R}^3_+))$. Pick a multi-index $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = m$ and $\alpha_0 = \alpha_3 = 0$. Then $\partial^\alpha u$ is an element of $C(\overline{J}, L^2(\mathbb{R}^3_+))$.*

Lemma 5.4. *Let $\eta > 0$. Take coefficients $\mathcal{A}_0 \in F^{\mathrm{cp}}_{3,\eta}(\Omega)$, $\mathcal{A}_1, \mathcal{A}_2 \in F^{\mathrm{cp}}_{3,\mathrm{coeff}}(\mathbb{R}^3_+)$, $\mathcal{A}_3 = \tilde{\mathcal{A}}^{\mathrm{co}}_3$, $\mathcal{D} \in F^{\mathrm{cp}}_3(\Omega)$, and $B = \mathcal{B}^{\mathrm{co}}$. Choose data $u_0 \in H^1(\mathbb{R}^3_+)$, $g \in E_1(J \times \partial\mathbb{R}^3_+)$, and $f \in H^1(\Omega)$. Assume that the tuple $(0, \mathcal{A}_0, \dots, \mathcal{A}_3, \mathcal{D}, B, f, g, u_0)$ fulfills the compatibility conditions* (2.6) *on $G = \mathbb{R}^3_+$ of order* 1. *Let $u \in C(\overline{J}, L^2(\mathbb{R}^3_+))$ be the weak solution of* (4.1) *with data f, g, and u_0. Assume that $u \in C^1(\overline{J'}, L^2(\mathbb{R}^3_+))$ implies $u \in G_1(J' \times \mathbb{R}^3_+)$ for every open interval $J' \subseteq J$. Then u belongs to $G_1(\Omega)$.*

To iterate the previous result, we need a relation between the operators $S_{m,p}$ of different order stated in the next lemma. It follows from a straightforward computation based on definition (2.4) of $S_{m,p}$ as in Lemma 4.8 of [23].

Lemma 5.5. *Let $\eta > 0$, $m \in \mathbb{N}$ and $\tilde{m} = \max\{m, 3\}$. Take coefficients $\mathcal{A}_0 \in F^{\mathrm{cp}}_{\max\{m+1,3\},\eta}(\Omega)$ with $\partial_t \mathcal{A}_0 \in F^{\mathrm{cp}}_{\tilde{m}}(\Omega)$, $\mathcal{A}_1$, $\mathcal{A}_2 \in F^{\mathrm{cp}}_{\max\{m+1,3\},\mathrm{coeff}}(\mathbb{R}^3_+)$, $\mathcal{A}_3 = \tilde{\mathcal{A}}^{\mathrm{co}}_3$, $\mathcal{D} \in F^{\mathrm{cp}}_{\max\{m+1,3\}}(\Omega)$, and $B = \mathcal{B}^{\mathrm{co}}$. Choose data $t_0 \in \overline{J}$, $u_0 \in H^{m+1}(\mathbb{R}^3_+)$, $g \in E_{m+1}(J \times \partial\mathbb{R}^3_+)$, and $f \in H^{m+1}(\Omega)$. Assume that $u \in G_m(\Omega)$ solves* (4.1) *with initial time t_0. Set $u_1 = S_{m+1,1}(t_0, \mathcal{A}_0, \dots, \mathcal{A}_3, \mathcal{D}, f, u_0)$ and $f_1 = \partial_t f - \partial_t \mathcal{D} u$. Let $p \in \{0, \dots, m-1\}$. We then obtain*

$$S_{m,p}(t_0, \mathcal{A}_0, \dots, \mathcal{A}_3, \partial_t \mathcal{A}_0 + \mathcal{D}, f_1, u_1) = S_{m+1,p+1}(t_0, \mathcal{A}_0, \dots, \mathcal{A}_3, \mathcal{D}, f, u_0).$$

Combining the above results with an iteration argument, we derive the desired regularity of the solution u provided the coefficients are smooth.

Proposition 5.6. *Let $\eta > 0$, $m \in \mathbb{N}$, and $\tilde{m} = \max\{m, 3\}$. Take coefficients $\mathcal{A}_0 \in F^{\mathrm{cp}}_{\tilde{m},\eta}(\Omega)$ with $\partial_t \mathcal{A}_0 \in F^{\mathrm{cp}}_{\max\{m-1,3\}}(\Omega)$, $\mathcal{A}_1, \mathcal{A}_2 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff}}(\mathbb{R}^3_+)$, $\mathcal{A}_3 = \tilde{\mathcal{A}}^{\mathrm{co}}_3$, $\mathcal{D} \in F^{\mathrm{cp}}_{\tilde{m}}(\Omega)$, and $B = \mathcal{B}^{\mathrm{co}}$. Assume that these coefficients are contained in $C^\infty(\overline{\Omega})$. Choose data $f \in H^m(\Omega)$, $g \in E_m(J \times \partial\mathbb{R}^3_+)$, and $u_0 \in H^m(\mathbb{R}^3_+)$ such that the tuple $(0, \mathcal{A}_0, \ldots, \mathcal{A}_3, \mathcal{D}, B, f, g, u_0)$ satisfies the compatibility conditions* (2.6) *on $G = \mathbb{R}^3_+$ of order m. Let u be the weak solution of* (4.1) *Then u belongs to $G_m(\Omega)$.*

Proof. Lemma 5.4, Lemma 5.3, and Lemma 5.1 show the assertion for $m = 1$. Let the claim be true for some $m \in \mathbb{N}$ and let the assumptions be fulfilled for $m + 1$. The weak solution u of (4.1) hence belongs to $G_m(\Omega)$, and $\partial_t u$ satisfies

$$\begin{cases} \mathcal{L}_{\partial_t} v = \partial_t f - \partial_t \mathcal{D} u =: f_1, & x \in \mathbb{R}^3_+, \quad t \in J; \\ Bv = \partial_t g, & x \in \partial\mathbb{R}^3_+, \quad t \in J; \\ v(0) = S_{m+1,1}(0, \mathcal{A}_0, \ldots, \mathcal{A}_3, \mathcal{D}, f, u_0) =: u_1, & x \in \mathbb{R}^3_+, \end{cases}$$

where we write $\mathcal{L}_{\partial_t}$ for $\mathcal{L}(\mathcal{A}_0, \ldots, \mathcal{A}_3, \partial_t \mathcal{A}_0 + \mathcal{D})$. The initial field u_1 belongs to $H^m(\mathbb{R}^3_+)$ by Lemma 2.3, the inhomogeneity f_1 to $H^m(\Omega)$ by Lemma 2.1, and $\partial_t g$ to $E_m(J \times \partial\mathbb{R}^3_+)$. The coefficients satisfy the conditions of Lemma 5.5 and $\partial_t \mathcal{A}_0 + \mathcal{D}$ is an element of $F^{\mathrm{cp}}_{\tilde{m}}(\Omega) \cap C^\infty(\overline{\Omega})$. Lemma 5.5 thus shows the compatibility conditions (2.6) of order m for the tuple $(0, \mathcal{A}_0, \ldots, \mathcal{A}_3, \partial_t \mathcal{A}_0 + \mathcal{D}, f_1, \partial_t g, u_1)$. By the induction hypothesis, the function $\partial_t u$ is contained in $G_m(\Omega)$, so that u belongs to $\bigcap_{j=1}^{m+1} C^j(\overline{J}, H^{m+1-j}(\mathbb{R}^3_+))$. Lemma 5.3 and Lemma 5.1 then imply that the solution u is an element of $G_{m+1}(\Omega)$. □

It remains to remove the extra regularity assumptions. Lemma 2.2 provides suitable approximations of the given coefficients. However, after this procedure the compatibility conditions can be violated. To overcome this difficulty, we modify the initial fields appropriately in Lemma 5.8. The proof of this result is based on the next fact which again relies on the algebraic structure of the coefficient matrices.

Lemma 5.7. *Let $\eta > 0$, $p \in \mathbb{N}_0$, and $m, k \in \mathbb{N}$ with $m \geq 3$ and $k \leq m - 1$. Take $\mathcal{A}_0 \in F_{m,12,\eta}(\Omega)$ and $\mathcal{A}_3 = \tilde{\mathcal{A}}^{\mathrm{co}}_3$. Choose $r > 0$ such that $\|\mathcal{A}_0(0)\|_{F^0_{m-1}(\mathbb{R}^3_+)} \leq r$. Take an approximating family $\{\mathcal{A}_{0,\varepsilon}\}_{\varepsilon>0}$ provided by Lemma 2.2. Let $v_{0,\varepsilon}$ be maps in $H^k(\mathbb{R}^3_+)^{12}$ for $\varepsilon > 0$. Then there exists a number $\varepsilon_0 > 0$, a constant $C = C(\eta, r)$, and a family of functions $\{v_{p,\varepsilon}\}_{0<\varepsilon<\varepsilon_0}$ in $H^k(\mathbb{R}^3_+)^{12}$ such that*

$$\mathcal{A}_3(\mathcal{A}_{0,\varepsilon}(0)^{-1}\mathcal{A}_3)^p v_{p,\varepsilon} = \mathcal{A}_3 v_{0,\varepsilon} \qquad \text{and} \qquad \|v_{p,\varepsilon}\|_{H^k(\mathbb{R}^3_+)} \leq C \|v_{0,\varepsilon}\|_{H^k(\mathbb{R}^3_+)}$$

for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. I) By Lemma 2.2 there is a number $\varepsilon_0 > 0$ such that

$$\|\mathcal{A}_{0,\varepsilon}(0)\|_{F^0_{m-1}(\mathbb{R}^3_+)} \leq 2r \tag{5.6}$$

for all $\varepsilon \in (0, \varepsilon_0)$. Let $\varepsilon \in (0, \varepsilon_0)$. We introduce the invertible matrices

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathcal{Q} = \begin{pmatrix} Q & 0 \\ 0 & -Q \end{pmatrix}$$

and note that

$$\mathcal{A}_3 \mathcal{Q} = \tilde{A}_3^{\mathrm{co}} \mathcal{Q} = \begin{pmatrix} J_{\mathrm{bl}} & 0 & 0 & 0 \\ 0 & J_{\mathrm{bl}} & 0 & 0 \\ 0 & 0 & J_{\mathrm{bl}} & 0 \\ 0 & 0 & 0 & J_{\mathrm{bl}} \end{pmatrix}, \quad \text{where} \quad J_{\mathrm{bl}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\mathcal{A}_{0,\varepsilon} \geq \eta$, also the matrix

$$\Theta_\varepsilon = \begin{pmatrix} \mathcal{A}_{0,\varepsilon;3,3} & \mathcal{A}_{0,\varepsilon;3,6} & \mathcal{A}_{0,\varepsilon;3,9} & \mathcal{A}_{0,\varepsilon;3,12} \\ \mathcal{A}_{0,\varepsilon;6,3} & \mathcal{A}_{0,\varepsilon;6,6} & \mathcal{A}_{0,\varepsilon;6,9} & \mathcal{A}_{0,\varepsilon;6,12} \\ \mathcal{A}_{0,\varepsilon;9,3} & \mathcal{A}_{0,\varepsilon;9,6} & \mathcal{A}_{0,\varepsilon;9,9} & \mathcal{A}_{0,\varepsilon;9,12} \\ \mathcal{A}_{0,\varepsilon;12,3} & \mathcal{A}_{0,\varepsilon;12,6} & \mathcal{A}_{0,\varepsilon;12,9} & \mathcal{A}_{0,\varepsilon;12,12} \end{pmatrix},$$

satisfies $\Theta_\varepsilon \geq \eta$ on Ω. In particular, Θ_ε has an inverse with

$$\|\Theta_\varepsilon^{-1}(0)\|_{F^0_{m-1}(\mathbb{R}^3_+)} \leq C(\eta, r) \qquad \text{for all } \ \varepsilon \in (0, \varepsilon_0). \tag{5.7}$$

II) Let $w_0 \in H^k(\mathbb{R}^3_+)^{12}$. We can define scalar functions $h_{1,\varepsilon}, \ldots, h_{4,\varepsilon}$ by

$$(h_{1,\varepsilon}, \ldots, h_{4,\varepsilon}) = -\Theta_\varepsilon^{-1}(0)(\mathcal{A}_{0,\varepsilon}(0) w_0)_{(3,6,9,12)},$$

where we write $\zeta_{(3,6,9,12)} = (\zeta_3, \zeta_6, \zeta_9, \zeta_{12})$ for any vector $\zeta \in \mathbb{R}^{12}$. Lemma 2.1 and the inequalities (5.6) and (5.7) imply that

$$\|(h_{1,\varepsilon}, \ldots, h_{4,\varepsilon})\|_{H^k(\mathbb{R}^3_+)} \leq C(\eta, r) \|w_0\|_{H^k(\mathbb{R}^3_+)}. \tag{5.8}$$

We next set

$$\hat{w}_\varepsilon = \mathcal{Q} \tilde{w}_\varepsilon, \qquad \tilde{w}_\varepsilon = -\mathcal{A}_{0,\varepsilon}(0)\Big(w_0 + h_{1,\varepsilon} e_3 + h_{2,\varepsilon} e_6 + h_{3,\varepsilon} e_9 + h_{4,\varepsilon} e_{12}\Big). \tag{5.9}$$

Lemma 2.1, (5.6), and (5.8) again provide a constant $C(\eta, r)$ such that

$$\|\hat{w}_\varepsilon\|_{H^k(\mathbb{R}^3_+)} \leq C(\eta, r) \|w_0\|_{H^k(\mathbb{R}^3_+)}. \tag{5.10}$$

Observe that

$$(\tilde{w}_\varepsilon)_{(3,6,9,12)} = (-\mathcal{A}_{0,\varepsilon}(0) w_0)_{(3,6,9,12)} - \Theta_\varepsilon(0)(h_{1,\varepsilon}, \ldots, h_{4,\varepsilon}) = 0,$$

and hence $\mathcal{A}_3 \mathcal{Q} \tilde{w}_\varepsilon = \tilde{w}_\varepsilon$. We thus compute

$$\mathcal{A}_3(-\mathcal{A}_{0,\varepsilon}(0)^{-1} \mathcal{A}_3) \hat{w}_\varepsilon = \mathcal{A}_3(-\mathcal{A}_{0,\varepsilon}(0)^{-1}) \tilde{w}_\varepsilon = \mathcal{A}_3 w_0 \tag{5.11}$$

using (5.9) and $\ker \mathcal{A}_3 = \operatorname{span}\{e_3, e_6, e_9, e_{12}\}$.

III) To show the assertion of the lemma, we proceed inductively. We claim that for all $p \in \mathbb{N}_0$, $\varepsilon \in (0, \varepsilon_0)$, and $w \in H^k(\mathbb{R}^3_+)^{12}$ there is a function $w_{p,\varepsilon}(w)$ in $H^k(\mathbb{R}^3_+)^{12}$ and a constant $C_p = C_p(\eta, r)$ such that

$$\mathcal{A}_3(-\mathcal{A}_{0,\varepsilon}(0)^{-1} \mathcal{A}_3)^p w_{p,\varepsilon}(w) = \mathcal{A}_3 w, \tag{5.12}$$

$$\|w_{p,\varepsilon}(w)\|_{H^k(\mathbb{R}^3_+)} \leq C_p \|w\|_{H^k(\mathbb{R}^3_+)}. \tag{5.13}$$

We can simply set $w_{0,\varepsilon}(w) = w$. Let the claim be true for a number $p \in \mathbb{N}_0$. Fix $\varepsilon \in (0, \varepsilon_0)$ and $w \in H^k(\mathbb{R}^3_+)^{12}$. Step II) applied with $w_0 = w$ yields a function $\tilde{w}_{p,\varepsilon} \in H^k(\mathbb{R}^3_+)^{12}$ satisfying

$$\mathcal{A}_3(-\mathcal{A}_{0,\varepsilon}(0)^{-1}\mathcal{A}_3)\tilde{w}_{p,\varepsilon} = \mathcal{A}_3 w \quad \text{and} \quad \|\tilde{w}_{p,\varepsilon}\|_{H^k(\mathbb{R}^3_+)} \leq C(\eta, r)\|w\|_{H^k(\mathbb{R}^3_+)}. \tag{5.14}$$

We now define $w_{p+1,\varepsilon}(w) = w_{p,\varepsilon}(\tilde{w}_{p,\varepsilon})$ for each $\varepsilon \in (0, \varepsilon_0)$. The map $w_{p+1,\varepsilon}(w)$ then is contained in $H^k(\mathbb{R}^3_+)^{12}$, and we compute

$$\begin{aligned}\mathcal{A}_3(-\mathcal{A}_{0,\varepsilon}(0)^{-1}\mathcal{A}_3)^{p+1} w_{p+1,\varepsilon}(w) &= \mathcal{A}_3(-\mathcal{A}_{0,\varepsilon}(0)^{-1})\mathcal{A}_3(-\mathcal{A}_{0,\varepsilon}(0)^{-1}\mathcal{A}_3)^p w_{p,\varepsilon}(\tilde{w}_{p,\varepsilon}) \\ &= \mathcal{A}_3(-\mathcal{A}_{0,\varepsilon}(0)^{-1})\mathcal{A}_3 \tilde{w}_{p,\varepsilon} = \mathcal{A}_3 w,\end{aligned}$$

where we employed the induction hypothesis (5.12) and (5.14). Combining (5.13) with (5.10), we further obtain

$$\|w_{p+1,\varepsilon}(w)\|_{H^k(\mathbb{R}^3_+)} = \|w_{p,\varepsilon}(\tilde{w}_{p,\varepsilon})\|_{H^k(\mathbb{R}^3_+)} \leq C_p\|\tilde{w}_{p,\varepsilon}\|_{H^k(\mathbb{R}^3_+)} \leq C\|w\|_{H^k(\mathbb{R}^3_+)},$$

where $C = C(\eta, r)$. The claim now follows by induction.

We obtain the assertion of the lemma by setting $v_{p,\varepsilon} = w_{p,\varepsilon}(v_{0,\varepsilon})$. □

Lemma 5.8. *Let $\eta > 0$, $m \in \mathbb{N}$, and $\tilde{m} = \max\{m, 3\}$. Take coefficients $\mathcal{A}_0 \in F^{\mathrm{cp}}_{\tilde{m},\eta}(\Omega)$, $\mathcal{A}_1, \mathcal{A}_2 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff}}(\mathbb{R}^3_+)$, $\mathcal{A}_3 = \tilde{\mathcal{A}}_3^{\mathrm{co}}$, $\mathcal{D} \in F^{\mathrm{cp}}_{\tilde{m}}(\Omega)$, and $B = \mathcal{B}^{\mathrm{co}}$. Choose data $f \in H^m(\Omega)$, $g \in E_m(J \times \partial\mathbb{R}^3_+)$, and $u_0 \in H^m(\mathbb{R}^3_+)$ which fulfill the compatibility conditions* (2.6) *on $G = \mathbb{R}^3_+$ of order m in $t_0 \in \overline{J}$. Let $\{\mathcal{A}_{i,\varepsilon}\}_{\varepsilon>0}$ and $\{\mathcal{D}_\varepsilon\}_{\varepsilon>0}$ be the families of functions provided by Lemma 2.2 for $\mathcal{A}_i$ and $\mathcal{D}$ respectively for $i \in \{0, 1, 2\}$. Then there exists a number $\varepsilon_0 > 0$ and a family $\{u_{0,\varepsilon}\}_{0<\varepsilon<\varepsilon_0}$ in $H^m(\mathbb{R}^3_+)$ such that the compatibility conditions for the tuple $(t_0, \mathcal{A}_{0,\varepsilon}, \mathcal{A}_{1,\varepsilon}, \mathcal{A}_{2,\varepsilon}, \mathcal{A}_3, \mathcal{D}_\varepsilon, B, f, g, u_{0,\varepsilon})$ of order m are satisfied and $u_{0,\varepsilon}$ tends to u_0 in $H^m(\mathbb{R}^3_+)$ as $\varepsilon \to 0$.*

Proof. Without loss of generality we assume $t_0 = 0$. We set $u_{0,\varepsilon} = u_0 + h_\varepsilon$ and look for functions $h_\varepsilon \in H^m(\mathbb{R}^3_+)$ with $h_\varepsilon \to 0$ in $H^m(\mathbb{R}^3_+)$ such that the compatibility conditions are fulfilled. Since $B = \mathcal{M}\mathcal{A}_3$ for a constant matrix $\mathcal{M} = \mathcal{M}^{\mathrm{co}}$ by (3.13), it suffices to find h_ε with

$$\mathcal{A}_3 S_{m,p}(0, \mathcal{A}_{0,\varepsilon}, \mathcal{A}_{1,\varepsilon}, \mathcal{A}_{2,\varepsilon}, \mathcal{A}_3, \mathcal{D}_\varepsilon, f, u_0 + h_\varepsilon) = \mathcal{A}_3 S_{m,p}(0, \mathcal{A}_0, \ldots, \mathcal{A}_3, \mathcal{D}, f, u_0)$$

for all $0 \leq p \leq m-1$ on $\partial\mathbb{R}^3_+$. Using Lemma 5.7 one can now repeat steps I) and II) of the proof of Lemma 4.8 of [25] in which the structure arising from the interface problem does not play a role. We thus omit the details. □

We can now deduce the differentiability theorem by applying Proposition 5.6 to the solutions of the approximating initial boundary value problems with coefficients and data from Lemma 5.8. Compared to [25], again the specific structure of our problem does not enter the reasoning, and thus we do not give a proof and refer to Theorem 4.10 of [25] for the details.

Theorem 5.9. *Let $\eta > 0$, $m \in \mathbb{N}$, and $\tilde{m} = \max\{m, 3\}$. Take coefficients $\mathcal{A}_0 \in F^{\mathrm{cp}}_{\tilde{m},\eta}(\Omega)$, $\mathcal{A}_1, \mathcal{A}_2 \in F^{\mathrm{cp}}_{\tilde{m},\mathrm{coeff}}(\mathbb{R}^3_+)$, $\mathcal{A}_3 = \tilde{\mathcal{A}}_3^{\mathrm{co}}$, $\mathcal{D} \in F^{\mathrm{cp}}_{\tilde{m}}(\Omega)$, and $B = \mathcal{B}^{\mathrm{co}}$. Choose data $f \in H^m(\Omega)$, $g \in E_m(J \times \partial\mathbb{R}^3_+)$, and $u_0 \in H^m(\mathbb{R}^3_+)$ such that the tuple $(0, \mathcal{A}_0, \ldots, \mathcal{A}_3, \mathcal{D}, B, f, g, u_0)$ satisfies the compatibility conditions* (2.6) *on $G = \mathbb{R}^3_+$ of order m. Then the weak solution u of* (4.1) *belongs to $G_m(\Omega)$.*

Remark 5.10. Recall that Theorem 3.1 is valid for coefficients A_0 and D which have merely a limit as $|(t,x)| \to \infty$. Also all intermediate results extend to such coefficients. In particular, Proposition 4.3, Theorem 4.4, and Theorem 5.9 are still true if $\mathcal{A}_0$ and $\mathcal{D}$ only have a limit as $|(t,x)| \to \infty$, cf. the proof of Theorem 4.13 in [23].

6. Local existence and uniqueness of the nonlinear system

In this section we prove existence and uniqueness of a solution of (1.7) by a fixed point argument based on the a priori estimates and the regularity theory from Sections 4 and 5 for the corresponding linear problem. We define a solution of (1.7) to be a function u belonging to $\bigcap_{j=0}^m C^j(I, \mathcal{H}^{m-j}(G))$ with $\overline{\operatorname{im} u_\pm} \subseteq \mathcal{U}_\pm$ for all $t \in I$ and satisfying (1.7). Here I is an interval with $t_0 \in I$. We further allow more general functions σ than arising from the model (1.3). The specific structure of the interface conditions does not enter very much in the proofs from now on. For this reason we can be more brief in this part of the paper and often refer the reader to the article [24], where the initial boundary value problem was treated in detail. We first introduce the spaces

$$
\begin{aligned}
&\mathcal{ML}^{m,n}(G,\mathcal{U}_\pm) \qquad (6.1)\\
&= \{\theta \colon (G_+ \times \mathcal{U}_+) \cup (G_- \times \mathcal{U}_-) \to \mathbb{R}^{n\times n} \text{ with } \theta_\pm \in C^m(G_\pm \times \mathcal{U}_\pm, \mathbb{R}^{n\times n}) \text{ and}\\
&\qquad \sup_{(x,y)\in G_\pm \times \mathcal{U}_{\pm,1}} |\partial^\alpha \theta(x,y)| < \infty \text{ for all } \alpha \in \mathbb{N}_0^9 \text{ with } |\alpha| \le m \text{ and } \mathcal{U}_{\pm,1} \Subset \mathcal{U}_\pm\},\\
&\mathcal{ML}^{m,n}_{\mathrm{pd}}(G,\mathcal{U}_\pm) = \{\theta \in \mathcal{ML}^{m,n}(G,\mathcal{U}_\pm) \colon \text{There exists } \eta > 0 \text{ with } \theta = \theta^T \ge \eta\\
&\qquad\qquad \text{on } G_\pm \times \mathcal{U}_\pm\}
\end{aligned}
$$

for our nonlinearities. Here θ_+ and θ_- denote the restrictions of θ to $G_+ \times \mathcal{U}_+$ respectively $G_- \times \mathcal{U}_-$. Moreover, by writing $G_\pm \times \mathcal{U}_\pm$ we address the two sets $G_+ \times \mathcal{U}_+$ and $G_- \times \mathcal{U}_-$. Actually, we only need the dimensions $n = 1$ or $n = 6$.

We often have to control compositions $\theta(v)$ in higher regularity in terms of v. In Lemma 2.1 and Corollary 2.2 of [24] the necessary formulas and estimates have been provided for functions defined on a single domain. Our interface case can then be treated by applying these facts to the subsets $G_\pm$ separately. Since the proofs below are only sketched, we do not repeat the modified versions of these rather lengthy auxiliary results.

As in the linear case discussed in Section 2, regular solutions of (1.7) have to satisfy compatibility conditions. To express them, we first introduce the operators that give the initial values of the time differentiated version of (1.7), cf. (2.4).

Definition 6.1. *Let $J \subseteq \mathbb{R}$ be an open interval, $m \in \mathbb{N}$, $\chi \in \mathcal{ML}^{m,6}_{\mathrm{pd}}(G,\mathcal{U}_\pm)$, and $\sigma \in \mathcal{ML}^{m,6}(G,\mathcal{U}_\pm)$. We inductively define the operators*

$$S_{\chi,\sigma,G,m,p} \colon \overline{J} \times \mathcal{H}^{\max\{m,3\}}(J\times G) \times \mathcal{H}^{\max\{m,2\}}(G,\mathcal{U}) \to \mathcal{H}^{m-p}(G)$$

by $S_{\chi,\sigma,G,m,0,\pm}(t_0, f_\pm, u_{0,\pm}) = u_{0,\pm}$ *and*

$$
\begin{aligned}
&S_{\chi,\sigma,G,m,p,\pm}(t_0, f_\pm, u_{0,\pm}) \qquad (6.2)\\
&\quad = \chi_\pm(u_{0,\pm})^{-1}\Big(\partial_t^{p-1} f_\pm(t_0) - \sum_{j=1}^3 A_j^{\mathrm{co}} \partial_j S_{\chi,\sigma,G,m,p-1,\pm}(t_0, f_\pm, u_{0,\pm})
\end{aligned}
$$

$$- \sum_{l=1}^{p-1} \binom{p-1}{l} M_{1,\pm}^l(t_0, f_\pm, u_{0,\pm}) S_{\chi,\sigma,G,m,p-l,\pm}(t_0, f_\pm, u_{0,\pm})$$
$$- \sum_{l=0}^{p-1} \binom{p-1}{l} M_{2,\pm}^l(t_0, f_\pm, u_{0,\pm}) S_{\chi,\sigma,G,m,p-1-l,\pm}(t_0, f_\pm, u_{0,\pm})\Big),$$
$$M_{k,\pm}^p = \sum_{1 \le j \le p} \sum_{\substack{\gamma_1,\dots,\gamma_j \in \mathbb{N}_0^4 \setminus \{0\} \\ \sum \gamma_i = (p,0,0,0)}} \sum_{l_1,\dots,l_j=1}^{6} C((p,0,0,0), \gamma_1, \dots, \gamma_j)$$
$$\cdot (\partial_{y_{l_j}} \cdots \partial_{y_{l_1}} \theta_{k,\pm})(u_{0,\pm}) \prod_{i=1}^{j} S_{\chi,\sigma,G,m,|\gamma_i|,\pm}(t_0, f_\pm, u_{0,\pm})_{l_i} \quad (6.3)$$

for $1 \le p \le m$, $k \in \{1,2\}$, where $\theta_1 = \chi$, $\theta_2 = \sigma$, $M_{2,\pm}^0 = \sigma_\pm(u_{0,\pm})$, and C is a combinatorical constant, cf. Lemma 2.1 and (2.8) of [24]. *By $\mathcal{H}^{\max\{m,2\}}(G,\mathcal{U})$ we mean those functions $u_0 \in \mathcal{H}^{\max\{m,2\}}(G)$ with $\overline{\operatorname{im} u_{0,\pm}} \subseteq \mathcal{U}_\pm$.*

Lemma 2.4 of [24] shows that the operators $S_{\chi,\sigma,G,m,p}$ indeed map into $\mathcal{H}^{m-p}(G)$ and it provides corresponding estimates. (One applies it to the subsets $G_\pm$ separately.) Using Lemma 2.1 of [24], we can differentiate (1.7) p-times and obtain

$$\partial_t^p u(t_0) = S_{\chi,\sigma,G,m,p}(t_0, f, u_0) \quad \text{for all } p \in \{0, \dots, m\} \quad (6.4)$$

if $u \in \mathcal{G}_m(J \times G)$ is a solution of (1.7) with data $f \in \mathcal{H}^m(J \times G)$, $u_0 \in \mathcal{H}^m(G)$, and $g \in E_m(J \times \Sigma)$. Proceeding similarly with the interface and boundary condition, equation (6.4) leads to the identities

$$B_\Sigma S_{\chi,\sigma,G,m,p}(t_0, f, u_0) = \partial_t^p g(t_0) \quad \text{on } \Sigma, \quad (6.5)$$
$$B_{\partial G} S_{\chi,\sigma,G,m,p}(t_0, f, u_0) = 0 \quad \text{on } \partial G \qquad \text{for all } \ p \in \{0, \dots, m-1\},$$

which are necessary for the existence of a $\mathcal{G}_m(J \times G)$-solution of (1.7). We say that the data tuple $(\chi, \sigma, t_0, B_\Sigma, B_{\partial G}, f, g, u_0)$ fulfills the *compatibility conditions* of order m if $\overline{\operatorname{im} u_{0,\pm}} \subseteq \mathcal{U}_\pm$ and the equations (6.5) are true.

Remark 6.2. Analogously to Remark 1.2 in [25], the linear theory allows for coefficients in $\mathcal{W}^{1,\infty}(J \times G)$ whose derivatives up to order m on $G_\pm$ are contained in $L^\infty(J, L^2(G_\pm)) + L^\infty(J \times G_\pm)$. In view of Lemma 2.1 in [24], we can thus apply the linear theory with coefficients $\chi(\hat{u})$ and $\sigma(\hat{u})$ and $\hat{u} \in \tilde{\mathcal{G}}_{\tilde{m}}(J \times G)$. However, the part of the derivatives in $L^\infty(J \times G)$ is easier to treat so that we concentrated on coefficients from $\mathcal{F}_m(J \times G)$ in Sections 4 and 5. The same is true for the nonlinear problem. In the proofs we will thus assume without loss of generality that χ and σ from $\mathcal{ML}^{m,6}(G, \mathcal{U}_\pm)$ have decaying space derivatives as $|x| \to \infty$. More precisely, for all multiindices $\alpha \in \mathbb{N}_0^9$ with $\alpha_4 = \ldots = \alpha_9 = 0$ and $1 \le |\alpha| \le m$, $R > 0$, $\mathcal{U}_{1,\pm} \Subset \mathcal{U}_\pm$, and $v \in L^\infty(J, L^2(G))$ with $\operatorname{im} v_\pm \subseteq \mathcal{U}_{1,\pm}$ and $\|v\|_{L^\infty(J,L^2(G))} \le R$ we require

$$(\partial^\alpha \chi_\pm)(v_\pm), (\partial^\alpha \sigma_\pm)(v_\pm) \in L^\infty(J, L^2(G_\pm)),$$
$$\|(\partial^\alpha \chi_\pm)(v_\pm)\|_{L^\infty(J,L^2(G_\pm))} + \|(\partial^\alpha \sigma_\pm)(v_\pm)\|_{L^\infty(J,L^2(G_\pm))} \le C, \quad (6.6)$$

where $C = C(\chi, \sigma, m, R, \mathcal{U}_{1,\pm})$. With this assumption we obtain from Lemma 2.1 in [24] that $\chi(\hat{u})$ and $\sigma(\hat{u})$ belong to $\mathcal{F}_m(J \times G)$.

Finally, we note that for unbounded G the above considerations are unnecessary since then $L^2(G_\pm) + L^\infty(G_\pm) = L^2(G_\pm)$.

The next lemma relates the maps $S_{\chi,\sigma,G,m,p}$ to their linear counterparts in (2.4).

Lemma 6.3. *Let $J \subseteq \mathbb{R}$ be an open interval, $t_0 \in \overline{J}$, and $m \in \mathbb{N}$ with $m \geq 3$. Take $\chi \in \mathcal{ML}^{m,6}_{\mathrm{pd}}(G,\mathcal{U}_\pm)$ and $\sigma \in \mathcal{ML}^{m,6}(G,\mathcal{U}_\pm)$. Choose data $f \in \mathcal{H}^m(J \times G)$ and $u_0 \in \mathcal{H}^m(G)$ with $\overline{\operatorname{im} u_{0,\pm}} \subseteq \mathcal{U}_\pm$. Let $r > 0$. Assume that f and u_0 satisfy*

$$\|u_0\|_{\mathcal{H}^m(G)} \leq r, \qquad \max_{0 \leq j \leq m-1} \|\partial_t^j f(t_0)\|_{\mathcal{H}^{m-j-1}(G)} \leq r,$$
$$\|f\|_{\mathcal{G}_{m-1}(J\times G)} \leq r, \quad \|f\|_{\mathcal{H}^m(J\times G)} \leq r.$$

(1) Let $\hat{u} \in \tilde{\mathcal{G}}_m(J \times G)$ with $\partial_t^p \hat{u}(t_0) = S_{\chi,\sigma,G,m,p}(t_0, f, u_0)$ for $0 \leq p \leq m-1$. Then $\hat{u}$ fulfills the equations

$$S_{G,m,p}(t_0, \chi(\hat{u}), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(\hat{u}), f, u_0) = S_{\chi,\sigma,G,m,p}(t_0, f, u_0) \tag{6.7}$$

for all $p \in \{0, \ldots, m\}$.

(2) There is a constant $C(\chi, \sigma, m, r, \mathcal{U}_{1,\pm}) > 0$ and a function u in $\mathcal{G}_m(J \times G)$ realizing the initial conditions

$$\partial_t^p u(t_0) = S_{\chi,\sigma,G,m,p}(t_0, f, u_0)$$

for all $p \in \{0, \ldots, m\}$ and it is bounded by

$$\|u\|_{\mathcal{G}_m(J\times G)} \leq C(\chi, \sigma, m, r, \mathcal{U}_{1,\pm})\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(t_0)\|_{\mathcal{H}^{m-j-1}(G)} + \|u_0\|_{\mathcal{H}^m(G)}\Big).$$

Here $\mathcal{U}_{1,\pm}$ denote compact subsets of $\mathcal{U}_\pm$ with $\operatorname{im} u_{0,\pm} \subseteq \mathcal{U}_{1,\pm}$.

Proof. Assertion (1) can be shown by induction using the definitions of the operators $S_{G,m,p}$ in (2.4) and of $S_{\chi,\sigma,G,m,p}$ in (6.2), as well as Lemma 2.1 in [24].

Since $S_{\chi,\sigma,G,m,p}(t_0, f, u_0)$ belongs to $\mathcal{H}^{m-p}(G)$ for all $p \in \{0, \ldots, m\}$, an extension theorem (see e.g. Lemma 2.34 in [23] applied on G_+ and G_- separately) yields the existence of a function u in $\mathcal{G}_m(J \times G)$ with $\partial_t^p u(t_0) = S_{\chi,\sigma,G,m,p}(t_0, f, u_0)$ and

$$\|u\|_{\mathcal{G}_m(J\times G)} \leq C \sum_{p=0}^{m} \|S_{\chi,\sigma,G,m,p}(t_0, f, u_0)\|_{\mathcal{H}^{m-p}(G)}$$

for all $p \in \{0, \ldots, m\}$. Lemma 2.4 of [24] then implies assertion (2). □

We introduce slightly strengthened assumptions on our material laws χ and σ to guarantee that $\chi(\hat{u})$ and $\sigma(\hat{u})$ converge at infinity, as required in Theorem 3.1.

$$\begin{aligned}\mathcal{ML}^{m,n,\mathrm{cv}}(G,\mathcal{U}_\pm) = \{\theta \in \mathcal{ML}^{m,n}(G,\mathcal{U}_\pm) \colon & \exists A \in \mathbb{R}^{n\times n} \text{ such that for all} \\ & (x_k, y_k)_k \in (G \times \mathcal{U})^{\mathbb{N}} \text{ with } |x_k| \to \infty \text{ and } y_k \to 0 : \\ & \theta(x_k, y_k) \to A \text{ as } k \to \infty\}, \\ \mathcal{ML}^{m,n,\mathrm{cv}}_{\mathrm{pd}}(G,\mathcal{U}_\pm) = \mathcal{ML}^{m,n}_{\mathrm{pd}}(G,\mathcal{U}_\pm) \cap & \mathcal{ML}^{m,n,\mathrm{cv}}(G,\mathcal{U}_\pm).\end{aligned}$$

The space $\mathcal{ML}^{m,n,\mathrm{cv}}(G,\mathcal{U}_\pm)$ coincides with $\mathcal{ML}^{m,n}(G,\mathcal{U}_\pm)$ in (6.1) if G is bounded.

The next result provides the uniqueness of solutions of (1.7). Its proof is an obvious modification of Lemma 7.1 in [24] and therefore omitted.

Lemma 6.4. *Let $t_0 \in \mathbb{R}, T > 0$, $J = (t_0, t_0 + T)$, and $m \in \mathbb{N}$ with $m \geq 3$. Take material laws $\chi \in \mathcal{ML}^{m,6,\mathrm{cv}}_{\mathrm{pd}}(G,\mathcal{U}_\pm)$ and $\sigma \in \mathcal{ML}^{m,6,\mathrm{cv}}(G,\mathcal{U}_\pm)$. Choose data $f \in \mathcal{H}^m(J \times G)$, $g \in E_m(J \times \Sigma)$, and $u_0 \in \mathcal{H}^m(G)$. Let u_1 and u_2 be two solutions in $\mathcal{G}_m(J \times G)$ of (1.7) with initial time t_0. Then $u_1 = u_2$.*

We now show the basic local existence theorem for (1.7) by a contraction argument. To close the argument, one has to take great care of the constants. In particular, the structure of the a priori estimate in Theorem 3.1 is crucial here.

Theorem 6.5. *Let $t_0 \in \mathbb{R}$, $T > 0$, $J = (t_0, t_0 + T)$, and $m \in \mathbb{N}$ with $m \geq 3$. Take $\chi \in \mathcal{ML}^{m,6,\mathrm{cv}}_{\mathrm{pd}}(G,\mathcal{U}_\pm)$ and $\sigma \in \mathcal{ML}^{m,6,\mathrm{cv}}(G,\mathcal{U}_\pm)$. Let B_Σ and $B_{\partial G}$ be given by* (1.6)*. Choose data $f \in \mathcal{H}^m(J \times G)$, $g \in E_m(J \times \Sigma)$, and $u_0 \in \mathcal{H}^m(G)$ with $\overline{\operatorname{im} u_{0,\pm}} \subseteq \mathcal{U}_\pm$ such that the tuple $(\chi, \sigma, t_0, B_\Sigma, B_{\partial G}, f, g, u_0)$ fulfills the nonlinear compatibility conditions* (6.5) *of order m. Pick a radius $r > 0$ satisfying*

$$\sum_{j=0}^{m-1} \|\partial_t^j f(t_0)\|^2_{\mathcal{H}^{m-1-j}(G)} + \|g\|^2_{E_m(J\times\Sigma)} + \|u_0\|^2_{\mathcal{H}^m(G)} + \|f\|^2_{\mathcal{H}^m(J\times G)} \leq r^2. \quad (6.8)$$

Take a number $\kappa > 0$ with

$$\operatorname{dist}(\{u_{0,\pm}(x) \colon x \in G_\pm\}, \partial\mathcal{U}_\pm) > \kappa.$$

Then there exists a time $\tau = \tau(\chi, \sigma, m, T, r, \kappa) > 0$ such that the nonlinear initial boundary value problem (1.7) *with data f, g, and u_0 has a unique solution u on $[t_0, t_0 + \tau]$ which belongs to $\mathcal{G}_m(J_\tau \times G)$, where $J_\tau = (t_0, t_0 + \tau)$.*

Proof. Without loss of generality we assume $t_0 = 0$ and that (6.6) holds true for χ and σ, cf. Remark 6.2. Let $\tau \in (0, T]$. We set $J_\tau = (0, \tau)$ and

$$\mathcal{U}_{\kappa,\pm} = \{y \in \mathcal{U}_\pm \colon \operatorname{dist}(y, \partial\mathcal{U}_\pm) \geq \kappa\} \cap \overline{B}_{2C_{\mathrm{Sob}}r}(0), \quad (6.9)$$

where C_{Sob} is the norm of the Sobolev embedding $\mathcal{H}^2(G) \hookrightarrow L^\infty(G)$. The sets $\mathcal{U}_{\kappa,\pm}$ are compact and contain $\overline{\operatorname{im} u_{0,\pm}}$.

Let $R > 0$. As in step I of the proof of Theorem 3.3 in [24] one checks that

$$B_R(J_\tau) := \{v \in \tilde{\mathcal{G}}_m(J_\tau \times G) \colon \|v\|_{\mathcal{G}_m(J_\tau\times G)} \leq R,\ \|v - u_0\|_{L^\infty(J_\tau\times G)} \leq \kappa/2,$$
$$\partial_t^j v(0) = S_{\chi,\sigma,G,m,j}(0, f, u_0) \text{ for } 0 \leq j \leq m-1\}$$

is a complete metric space when endowed with $d(v_1, v_2) = \|v_1 - v_2\|_{\mathcal{G}_{m-1}(J_\tau\times G)}$. It is non-empty thanks to Lemma 6.3 and the choice of R and τ below.

Let $\hat{u} \in B_R(J_\tau)$. We have $\chi \geq \eta$ for some $\eta > 0$. The map $\chi(\hat{u})$ is contained in $\mathcal{F}^{\mathrm{cv}}_{m,\eta}(J_\tau \times G)$ and $\sigma(\hat{u})$ in $\mathcal{F}^{\mathrm{cv}}_m(J_\tau \times G)$ by Lemma 2.1 in [24], Remark 6.2, and Sobolev's embedding. Lemma 6.3 and the assumptions imply that the tuple $(t_0, \chi(\hat{u}), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(\hat{u}), B_\Sigma, B_{\partial G}, f, g, u_0)$ fulfills the linear compatibility conditions (2.6). Theorem 3.1 then yields a solution $u \in \mathcal{G}_m(J_\tau \times G)$ of the linear sytem (1.9) with differential operator $L(\chi(\hat{u}), A_1^{\mathrm{co}}, A_2^{\mathrm{co}}, A_3^{\mathrm{co}}, \sigma(\hat{u}))$ and data f, g, and u_0. In this way one defines a mapping $\Phi \colon \hat{u} \mapsto u$ from $B_R(J_\tau)$ to $\mathcal{G}_m(J_\tau \times G)$. We are now looking for a radius $R > 0$ and a (small) time $\tau > 0$ such that Φ leaves invariant $B_R(J_\tau)$.

For this purpose take numbers $\tau \in (0, T]$ and $R > C_{6.3}(\chi, \sigma, m, r, \mathcal{U}_{\kappa,\pm})(m+1)r$ which will be fixed below. Let $\hat{u} \in B_R(J_\tau)$. Lemma 2.4 in [24] and (6.8) imply that

$$\|S_{\chi,\sigma,G,m,p}(0, f, u_0)\|_{\mathcal{H}^{m-p}(G)} \leq C_{2.4,[24]}(\chi, \sigma, m, r, \mathcal{U}_{\kappa,\pm}) \quad (6.10)$$

for all $p \in \{0, \ldots, m\}$ and a constant $C_{2.4,[24]}$. From Lemma 2.1 of [24] we infer

$$\|\chi(\hat{u})(0)\|_{\mathcal{F}^0_{m-1}(G)}, \|\sigma(\hat{u})(0)\|_{\mathcal{F}^0_{m-1}(G)} \leq C_{2.1,[24]}(\chi, \sigma, m, r, \mathcal{U}_{\kappa,\pm}),$$

using (6.8) and $\chi(\hat{u})(0) = \chi(u_0)$, for instance. Note that $\overline{\operatorname{im} \hat{u}_\pm}$ is contained in the compact set

$$\tilde{\mathcal{U}}_{\kappa,\pm} = \mathcal{U}_{\kappa,\pm} + \overline{B}(0, \kappa/2) \subseteq \mathcal{U}_\pm$$

as $\hat{u} \in B_R(J_\tau)$. Lemma 2.1 in [24] and estimate (6.10) lead to the bounds

$$\begin{aligned}
\|\partial_t^l \chi(\hat{u})(0)\|_{\mathcal{H}^{m-l-1}(G)} &\leq C_{2.1,[24]}(\chi, m, \mathcal{U}_{\kappa,\pm})(1 + \max_{0\leq k\leq l} \|\partial_t^k \hat{u}(0)\|_{\mathcal{H}^{m-k-1}(G)})^{m-1} \\
&= C_{2.1,[24]}(\chi, m, \mathcal{U}_{\kappa,\pm})(1 + \max_{0\leq k\leq l} \|S_{\chi,\sigma,G,m,k}(0, f, u_0)\|_{\mathcal{H}^{m-k-1}(G)})^{m-1} \\
&\leq C_{2.1,[24]}(\chi, m, \mathcal{U}_{\kappa,\pm})(1 + C_{2.4,[24]}(\chi, \sigma, m, r, \mathcal{U}_{\kappa,\pm}))^{m-1}, \\
\|\partial_t^l \sigma(\hat{u})(0)\|_{\mathcal{H}^{m-l-1}(G)} &\leq C_{2.1,[24]}(\sigma, m, \mathcal{U}_{\kappa,\pm})(1 + C_{2.4,[24]}(\chi, \sigma, m, r, \mathcal{U}_{\kappa,\pm}))^{m-1}
\end{aligned}$$

for all $l \in \{1, \ldots, m-1\}$. We thus find a radius $r_0 = r_0(\chi, \sigma, m, r, \kappa)$ such that

$$\begin{aligned}
\max\{\|\chi(\hat{u})(0)\|_{\mathcal{F}^0_{m-1}(G)}, \max_{1\leq l\leq m-1} \|\partial_t^l \chi(\hat{u})(0)\|_{\mathcal{H}^{m-l-1}(G)}\} \leq r_0, \\
\max\{\|\sigma(\hat{u})(0)\|_{\mathcal{F}^0_{m-1}(G)}, \max_{1\leq l\leq m-1} \|\partial_t^l \sigma(\hat{u})(0)\|_{\mathcal{H}^{m-l-1}(G)}\} \leq r_0.
\end{aligned}$$

Since $\hat{u}$ belongs to $B_R(J_\tau)$, Lemma 2.1 in [24] yields the inequality

$$\|\chi(\hat{u})\|_{\mathcal{F}_m(J\times G)}, \|\sigma(\hat{u})\|_{\mathcal{F}_m(J\times G)} \leq C_{2.1,[24]}(\chi, \sigma, m, \tilde{\mathcal{U}}_{\kappa,\pm})(1 + R)^m.$$

Hence, there is a radius $R_1 = R_1(\chi, \sigma, m, R, \kappa)$ with

$$\|\chi(\hat{u})\|_{\mathcal{F}_m(J\times G)} \leq R_1 \qquad \text{and} \qquad \|\sigma(\hat{u})\|_{\mathcal{F}_m(J\times G)} \leq R_1.$$

We next define the constant $C_{m,0} = C_{m,0}(\chi, \sigma, r, \kappa)$ by

$$C_{m,0}(\chi, \sigma, r, \kappa) = C_{3.1,m,0}(\eta(\chi), r_0(\chi, \sigma, m, r, \kappa)),$$

where $C_{3.1,m,0}$ denotes the constant $C_{m,0}$ from Theorem 3.1. The radius $R = R(\chi, \sigma, m, r, \kappa)$ for $B_R(J_\tau)$ is now fixed as

$$R = \max\Big\{\sqrt{6\, C_{m,0}(\chi, \sigma, r, \kappa)}\, r,\ C_{6.3}(\chi, \sigma, m, r, \mathcal{U}_{\kappa,\pm})(m+1)r + 1\Big\}. \tag{6.11}$$

We further introduce the constants

$$\begin{aligned}
\gamma_m = \gamma_m(\chi, \sigma, T, r, \kappa) &:= \gamma_{3.1,m}(\eta(\chi), R_1(\chi, \sigma, m, R(\chi, \sigma, m, r, \kappa), \kappa), T), \\
C_m = C_m(\chi, \sigma, T, r) &:= C_{3.1,m}(\eta(\chi), R_1(\chi, \sigma, m, R(\chi, \sigma, m, r, \kappa), \kappa), T),
\end{aligned}$$

where $\gamma_{3.1,m}$ and $C_{3.1,m}$ are the corresponding constants from Theorem 3.1. Let $C_{2.2,[24]}(\theta, m, R, \tilde{\mathcal{U}}_{\kappa,\pm})$ be the constant that arises when applying Corollary 2.2 of [24] to the components of $\theta \in \mathcal{ML}^{m,6}(G, \mathcal{U}_\pm)$. We now define the parameter $\gamma = \gamma(\chi, \sigma, m, T, r, \kappa)$ and the time step $\tau = \tau(\chi, \sigma, m, T, r, \kappa)$ by

$$\begin{aligned}
\gamma &= \max\Big\{\gamma_m,\ C_{m,0}^{-1} C_m\Big\}, \\
\tau &= \min\Big\{T,\ (2\gamma + mC_{3.1,1})^{-1} \log 2,\ C_m^{-1} C_{m,0}, (2C_{\mathrm{Sob}} R)^{-1}\kappa, \\
&\qquad [32 R^2 C_{m,0} C_P^2 (C^2_{2.2,[24]}(\chi, m, R, \tilde{\mathcal{U}}_\kappa) + C^2_{2.2,[24]}(\sigma, m, R, \tilde{\mathcal{U}}_\kappa))]^{-1}\Big\},
\end{aligned} \tag{6.12}$$

where C_P denotes the constant from Lemma 2.1.

>From now on the reasoning follows the lines of steps III)–V) of the proof of Theorem 3.3 in [24]. The above choice of constants and the linear results of our paper imply that Φ is a strict contraction on $B_R(J_\tau)$ which yields the assertion. □

Remark 6.6. Using time reversion and adapting coefficients and data accordingly, we can transfer the result of Theorem 6.5 to the negative time direction, cf. Remark 7.12 in [23].

We assume that the conditions of Theorem 6.5 are valid and that the functions f and g belong to the spaces $\mathcal{H}^m((-T,T)\times G)$ respectively $E_m((-T,T)\times\Sigma)$, for all $T>0$. We now define the *maximal existence times* by

$$\begin{aligned} T_+(m,t_0,f,g,u_0) &= \sup\{\tau\geq t_0\colon \exists\, \mathcal{G}_m\text{-solution of (1.7) on } [t_0,\tau]\},\\ T_-(m,t_0,f,g,u_0) &= \inf\{\tau\leq t_0\colon \exists\, \mathcal{G}_m\text{-solution of (1.7) on } [\tau,t_0]\}. \end{aligned} \tag{6.13}$$

The interval $(T_-(m,t_0,f,g,u_0),T_+(m,t_0,f,g,u_0)) =: I_{max}(m,t_0,f,g,u_0)$ is called the *maximal interval of existence*. These notions are modified in a straightforward way if the inhomogeneities are given on an open interval $J\subseteq\mathbb{R}$ with $t_0\in\overline{J}$. By standard methods we can extend the solution given by Theorem 6.5 and Remark 6.6 to a *maximal solution* $u\in\bigcap_{j=0}^m C^j(I_{max},\mathcal{H}^{m-j}(G))$ of (1.7) on I_{max} which cannot be extended beyond this interval. More precisely, we obtain the following basic blow-up criterion, cf. Lemma 4.1 of [24].

Proposition 6.7. *Let $t_0\in\mathbb{R}$ and $m\in\mathbb{N}$ with $m\geq 3$. Take $\chi\in\mathcal{ML}^{m,6,\mathrm{cv}}_{\mathrm{pd}}(G,\mathcal{U}_\pm)$ and $\sigma\in\mathcal{ML}^{m,6,\mathrm{cv}}(G,\mathcal{U}_\pm)$. Choose data $f\in\mathcal{H}^m((-T,T)\times G)$, $g\in E_m((-T,T)\times\Sigma)$, and $u_0\in\mathcal{H}^m(G)$ for all $T>0$ and define B_Σ and $B_{\partial G}$ as in* (1.6)*. Assume that the tuple $(\chi,\sigma,t_0,B_\Sigma,B_{\partial G},f,g,u_0)$ fulfills the compatibility conditions* (6.5) *of order m. Let u be the maximal solution of* (1.7) *on I_{max} introduced above. If $T_+=T_+(m,t_0,f,g,u_0)<\infty$, then one of the following blow-up properties*

(1) $\liminf_{t\nearrow T_+}\operatorname{dist}(\{u_+(t,x)\colon x\in G_+\},\partial\mathcal{U}_+)=0$ or correspondingly for u_-,
(2) $\lim_{t\nearrow T_+}\|u(t)\|_{\mathcal{H}^m(G)}=\infty$

occurs. The analogous result is true for $T_-(m,t_0,f,g,u_0)$.

7. Local well-posedness

The blow-up criterion in Proposition 6.7 can be improved. By Theorem 7.3, if $T_+<\infty$ (and the solution does not come arbitrarily close to $\partial\mathcal{U}_+$ or $\partial\mathcal{U}_-$), then the spatial Lipschitz norm of the solution has to blow up as $t\to T_+$, see Theorem 7.3 below. Similar blow-up criteria have been established for several quasilinear hyperbolic systems both on the full space and on domains, see e.g. [4, 5, 17, 18]. For this improvement over the $\mathcal{H}^m(G)$-norm, one has to exploit that a solution u of the nonlinear problem (1.7) solves the linear problem (1.9) with coefficients $\chi(u)$ and $\sigma(u)$, and then use Moser-type estimates. Lemma 4.2 from [24] provides a version of these estimates suited to our setting in which we admit space dependent nonlinearities. We can apply this lemma to the subdomains $G_\pm$ separately.

The next proposition is the main step towards the improved blow-up condtion. In its proof one differentiates (1.7) and applies the basic L^2-estimate (4.2) to the derivative of u. For the tangential and time derivatives, the Moser-type estimates allow us to treat the arising inhomogeneities in such a way that the Gronwall lemma yields the desired estimate. In order to bound the normal derivatives of u, we have to combine the above approach with Proposition 4.3. Once more the reasoning is parallel to that in [24], making use of the linear results of the present paper. For details we thus refer to the proof of Proposition 4.4 in [24].

Proposition 7.1. *Let $m\in\mathbb{N}$ with $m\geq 3$ and $t_0\in\mathbb{R}$. Take nonlinearities $\chi\in\mathcal{ML}^{m,6,\mathrm{cv}}_{\mathrm{pd}}(G,\mathcal{U}_\pm)$ and $\sigma\in\mathcal{ML}^{m,6,\mathrm{cv}}(G,\mathcal{U}_\pm)$. Let B_Σ and $B_{\partial G}$ be defined as in* (1.6)*. Choose data $u_0\in\mathcal{H}^m(G)$, $g\in E_m((-T,T)\times\Sigma)$, and $f\in\mathcal{H}^m((-T,T)\times G)$ for all $T>0$ such that the tuple $(\chi,\sigma,t_0,B_\Sigma,B_{\partial G},f,g,u_0)$ fulfills the compatibility conditions* (6.5) *of order m. Let u denote the maximal solution of* (1.7) *on*

(T_-, T_+). We introduce the quantity

$$\omega(T) = \sup_{t \in (t_0, T)} \|u(t)\|_{\mathcal{W}^{1,\infty}(G)}$$

for every $T \in (t_0, T_+)$. We further take $r > 0$ with

$$\sum_{j=0}^{m-1} \|\partial_t^j f(t_0)\|_{\mathcal{H}^{m-j-1}(G)} + \|g\|_{E_m((t_0,T_+)\times\Sigma)} + \|u_0\|_{\mathcal{H}^m(G)} + \|f\|_{\mathcal{H}^m((t_0,T_+)\times G)} \leq r.$$

We set $T^* = T_+$ if $T_+ < \infty$ and take any $T^* > t_0$ if $T_+ = \infty$. Let $\omega_0 > 0$ and let $\mathcal{U}_{1,\pm}$ be compact subsets of $\mathcal{U}_\pm$.

Then there exists a constant $C = C(\chi, \sigma, m, r, \omega_0, \mathcal{U}_{1,\pm}, T^* - t_0)$ such that

$$\|u\|^2_{\mathcal{G}_m((t_0,T)\times G)} \leq C\Big(\sum_{j=0}^{m-1} \|\partial_t^j f(t_0)\|^2_{\mathcal{H}^{m-1-j}(G)} + \|u_0\|^2_{\mathcal{H}^m(G)} + \|g\|^2_{E_m((t_0,T)\times\Sigma)} + \|f\|^2_{\mathcal{H}^m((t_0,T)\times G)}\Big)$$

for all times $T \in (t_0, T^*)$ which have the property that $\omega(T) \leq \omega_0$ and $\operatorname{im} u_\pm(t) \subseteq \mathcal{U}_{1,\pm}$ for all $t \in [t_0, T]$. The analogous result is true on (T_-, t_0).

The main missing part of the final local wellposedness theorem is the continuous dependence on initial data. Here a loss of derivatives occurs since the difference of two solutions satisfies an equation with a less regular right-hand side. The next lemma shows the core fact in this context. It improves the convergence of solutions u_n by one level of regularity, assuming uniform bounds of u_n and convergence of the data in the higher norm. In the proof one uses that derivatives of the solutions satisfy a system with modified forcing terms. These problems are then splitted in one with fixed inhomogeneities (arising from the limit data) and one with right-hand sides tending to 0 (up to to an error term treated in a Gronwall argument). Such techniques were developed for the full space (see e.g. [4]). We combine this approach with our linear results to prevent a loss of normal regularity at the characteristic boundary. Here again the structure of Maxwell's equations is crucially used. The proof is a combination of that of Lemma 5.2 in [24] with the theorems of the previous sections. It is thus omitted.

Lemma 7.2. *Let $J' \subseteq \mathbb{R}$ be an open and bounded interval, $t_0 \in \overline{J'}$, and $m \in \mathbb{N}$ with $m \geq 3$. Take functions $\chi \in \mathcal{ML}^{m,6,\mathrm{cv}}_{\mathrm{pd}}(G, \mathcal{U}_\pm)$ and $\sigma \in \mathcal{ML}^{m,6,\mathrm{cv}}(G, \mathcal{U}_\pm)$. Let B_Σ and $B_{\partial G}$ be defined by (1.6). Choose data $f_n, f \in \mathcal{H}^m(J' \times G)$, $g_n, g \in E_m(J' \times \Sigma)$, and $u_{0,n}, u_0 \in \mathcal{H}^m(G)$ for all $n \in \mathbb{N}$ with*

$$\|u_{0,n} - u_0\|_{\mathcal{H}^m(G)} \longrightarrow 0, \quad \|g_n - g\|_{E_m(J'\times\Sigma)} \longrightarrow 0, \quad \|f_n - f\|_{\mathcal{H}^m(J'\times G)} \longrightarrow 0,$$

as $n \to \infty$. We further assume that the system (1.7) with data $(t_0, f_n, g_n, u_{0,n})$ and (t_0, f, g, u_0) has $\mathcal{G}_m(J' \times G)$-solutions u_n and u for all $n \in \mathbb{N}$, that there are compact subsets $\tilde{\mathcal{U}}_{1,\pm}$ of $\mathcal{U}_\pm$ with $\operatorname{im} u_\pm(t) \subseteq \tilde{\mathcal{U}}_{1,\pm}$ for all $t \in J'$, that $(u_n)_n$ is bounded in $\mathcal{G}_m(J' \times G)$, and that $(u_n)_n$ converges to u in $\mathcal{G}_{m-1}(J' \times G)$. Then the functions u_n tend to u in $\mathcal{G}_m(J' \times G)$.

Finally, we can prove the full local wellposedness theorem. In the following we will write $B_M(x, r)$ for the ball of radius r around a point x from a metric space M. For times $t_0 < T$ we further define the data space

$$M_{\chi,\sigma,m}(t_0, T) = \{(\tilde{f}, \tilde{g}, \tilde{u}_0) \in \mathcal{H}^m((t_0,T)\times G) \times E_m((t_0,T)\times\Sigma) \times \mathcal{H}^m(G):$$

$(\chi, \sigma, t_0, B_\Sigma, B_{\partial G}, \tilde{f}, \tilde{g}, \tilde{u}_0)$ is compatible of order $m\}$,

and endow it with the metric

$$d((\tilde{f}_1, \tilde{g}_1, \tilde{u}_{0,1}), (\tilde{f}_2, \tilde{g}_2, \tilde{u}_{0,2})) = \max\{\|\tilde{f}_1 - \tilde{f}_2\|_{\mathcal{H}^m((t_0,T)\times G)}, \|\tilde{g}_1 - \tilde{g}_2\|_{E_m((t_0,T)\times\Sigma)}, \|\tilde{u}_{0,1} - \tilde{u}_{0,2}\|_{\mathcal{H}^m(G)}\}.$$

Theorem 7.3. *Let $m \in \mathbb{N}$ with $m \geq 3$ and fix $t_0 \in \mathbb{R}$. Take functions $\chi \in \mathcal{ML}^{m,6,\mathrm{cv}}_{\mathrm{pd}}(G, \mathcal{U}_\pm)$ and $\sigma \in \mathcal{ML}^{m,6,\mathrm{cv}}(G, \mathcal{U}_\pm)$. Let B_Σ and $B_{\partial G}$ be defined by (1.6). Choose data $u_0 \in \mathcal{H}^m(G)$, $g \in E_m((-T,T)\times\Sigma)$, and $f \in \mathcal{H}^m((-T,T)\times G)$ for all $T > 0$ such that $\overline{\operatorname{im} u_{0,\pm}} \subseteq \mathcal{U}_\pm$ and the tuple $(\chi, \sigma, t_0, B_\Sigma, B_{\partial G}, f, g, u_0)$ fulfills the compatibility conditions (6.5) of order m.*

Then the maximal existence times $T_\pm = T_\pm(m, t_0, f, g, u_0)$ from (6.13) do not depend on $k \in \{3, \dots, m\}$. Moreover, the following assertions are true.

(1) There exists a unique maximal solution u of (1.7) which belongs to the function space $\bigcap_{j=0}^m C^j((T_-, T_+), \mathcal{H}^{m-j}(G))$.

(2) If $T_+ < \infty$, then
  (a) the restriction u_+ leaves every compact subset of $\mathcal{U}_+$ or u_- leaves every compact subset of $\mathcal{U}_-$, or
  (b) $\limsup_{t\nearrow T_+} \max\{\|\nabla u_+(t)\|_{L^\infty(G_+)}, \|\nabla u_-(t)\|_{L^\infty(G_-)}\} = \infty$.

  The analogous result holds for T_-.

(3) Fix $T \in (t_0, T_+)$ and take $T' \in (T, T_+)$. Then there is a number $\delta > 0$ such that for all data $(\tilde{f}, \tilde{g}, \tilde{u}_0) \in B_{M_{\chi,\sigma,m}(t_0,T')}((f, g, u_0), \delta)$ the maximal existence time satisfies $T_+(m, t_0, \tilde{f}, \tilde{g}, \tilde{u}_0) > T$. We denote by $u(\cdot; \tilde{f}, \tilde{g}, \tilde{u}_0)$ the corresponding maximal solution of (1.7). The flow map

$$\Psi\colon B_{M_{\chi,\sigma,m}(t_0,T')}((f,g,u_0),\delta) \to \mathcal{G}_m((t_0,T)\times G), \quad (\tilde{f}, \tilde{g}, \tilde{u}_0) \longmapsto u(\cdot; \tilde{f}, \tilde{g}, \tilde{u}_0),$$

is continuous, and there is a constant $C = C(\chi, \sigma, m, r, T_+ - t_0, \kappa_0)$ such that

$$\begin{aligned} &\|\Psi(\tilde{f}_1, \tilde{g}_1, \tilde{u}_{0,1}) - \Psi(\tilde{f}_2, \tilde{g}_2, \tilde{u}_{0,2})\|_{\mathcal{G}_{m-1}((t_0,T)\times G)} \\ &\leq C \sum_{j=0}^{m-1} \|\partial_t^j \tilde{f}_1(t_0) - \partial_t^j \tilde{f}_2(t_0)\|_{\mathcal{H}^{m-j-1}(G)} + C\|\tilde{g}_1 - \tilde{g}_2\|_{E_{m-1}((t_0,T)\times\Sigma)} \\ &\quad + C\|\tilde{u}_{0,1} - \tilde{u}_{0,2}\|_{\mathcal{H}^m(G)} + C\|\tilde{f}_1 - \tilde{f}_2\|_{\mathcal{H}^{m-1}((t_0,T)\times G)} \end{aligned}$$

for all $(\tilde{f}_1, \tilde{g}_1, \tilde{u}_{0,1}), (\tilde{f}_2, \tilde{g}_2, \tilde{u}_{0,2}) \in B_{M_{\chi,\sigma,m}(t_0,T')}((f,g,u_0),\delta)$, where $\kappa_0 = \operatorname{dist}(\operatorname{im} u_{0,\pm}, \partial\mathcal{U}_\pm)$. The analogous result is true for T_-.

Sketch of the proof. We note that in part (3) one may extend $\tilde{f}$ and $\tilde{g}$ to the time interval $\mathbb{R}$ to be in the framework of the previous parts of the theorem. Except for part (3), the assertions easily follow from Propositions 6.7 and 7.1. In the context of part (3) we set $\tilde{u} = u(\cdot; \tilde{f}, \tilde{g}, \tilde{u}_0)$. If this solution exists on an interval $[t_0, t']$ with $\mathcal{G}_m$–norm less than R', Theorem 3.1 and the results of Section 2 in [24] allow us to bound $u - \tilde{u}$ in $\mathcal{G}_{m-1,\gamma}((t_0, t') \times G)$ by analogous norms of the differences of the data, if $\gamma(R')$ is large enough. We next use a time step τ as in (6.12) and a radius R as in (6.11) in the proof of Theorem 6.5, where we have fixed a sufficiently large radius $r > 0$ for the data. If $\delta > 0$ is small enough, this theorem then yields a solution $\tilde{u}$ of (1.7) in $\mathcal{G}_m((t_0, t + \tau) \times G)$ with norm less or equal R, for data $(\tilde{f}, \tilde{g}, \tilde{u}_0)$. Using the bound in $\mathcal{G}_{m-1,\gamma}((t_0, t') \times G)$ just mentioned and Lemma 7.2, we obtain the continuity of the flow map on $\mathcal{G}_m((t_0, t+\tau)\times G)$. Decreasing $\delta > 0$ if necessary, one can then deduce assertion (3) iteratively. The details are analogous to the proof of Theorem 5.3 in [24] which only uses different linear results. □

8. Appendix

In this appendix we show that the interface conditions for $\boldsymbol{D}$ and $\boldsymbol{B}$ are preserved.

Lemma 8.1. *Let $t_0, T \in \mathbb{R}$ with $t_0 < T$ and set $J = (t_0, T)$. Let $(\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}, \boldsymbol{B})$ in $C(\overline{J}, \mathcal{H}^1(G)) \cap C^1(\overline{J}, L^2(G))$ be a solution of the Maxwell system (1.1) with $\boldsymbol{J} \in L^2(J, \mathcal{H}(\operatorname{div}, G))$ and $\boldsymbol{J}_\Sigma \in L^2(J, H(\operatorname{div}, \Sigma))$ satisfying $[\boldsymbol{E} \times \nu] = 0$ and $[\boldsymbol{H} \times \nu] = \boldsymbol{J}_\Sigma$ on $J \times \Sigma$. Set $\rho_\Sigma(t) = \rho_{\Sigma,0} - \int_{t_0}^t (\operatorname{div}_\Sigma \boldsymbol{J}_\Sigma - [\boldsymbol{J} \cdot \nu])(s) ds$ for all $t \in \overline{J}$.*

(1) If $[\boldsymbol{B} \cdot \nu](t_0) = 0$ on Σ, then $[\boldsymbol{B} \cdot \nu] = 0$ on $J \times \Sigma$.
(2) If $[\boldsymbol{D} \cdot \nu](t_0) = -\rho_{\Sigma,0}$, then $[\boldsymbol{D} \cdot \nu] = -\rho_\Sigma$ on $J \times \Sigma$.

Proof. (1) Since $\partial_t \boldsymbol{B}_\pm$ belongs to $H(\operatorname{div}, G_\pm)$, these fields have a normal trace in $H^{-1/2}(\Sigma)$ for each $t \in \overline{J}$. Employing that also $\operatorname{curl} \boldsymbol{E}_\pm \in H(\operatorname{div}, G_\pm)$, we compute

$$\begin{aligned}
&\langle \partial_t [\boldsymbol{B} \cdot \nu](t), \varphi \rangle_{H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)} = \langle [\partial_t \boldsymbol{B} \cdot \nu](t), \varphi \rangle_{H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)} \\
&\quad = \langle [-\operatorname{curl} \boldsymbol{E} \cdot \nu](t), \varphi \rangle_{H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)} \\
&\quad = \langle -\operatorname{curl} \boldsymbol{E}_+(t) \cdot \nu, \varphi \rangle_{H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)} + \langle \operatorname{curl} \boldsymbol{E}_-(t) \cdot \nu, \varphi \rangle_{H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)} \\
&\quad = -\int_{G_+} \operatorname{div} \operatorname{curl} \boldsymbol{E}_+(t) \varphi \, dx - \int_{G_+} \operatorname{curl} \boldsymbol{E}_+(t) \cdot \nabla \varphi \, dx - \int_{G_-} \operatorname{div} \operatorname{curl} \boldsymbol{E}_-(t) \varphi \, dx \\
&\qquad - \int_{G_-} \operatorname{curl} \boldsymbol{E}_-(t) \cdot \nabla \varphi \, dx \\
&\quad = -\int_{G_+} \boldsymbol{E}_+(t) \cdot \operatorname{curl} \nabla \varphi \, dx + \langle \boldsymbol{E}_+(t) \times \nu, \nabla \varphi \rangle_{H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)} \\
&\qquad - \int_{G_-} \boldsymbol{E}_-(t) \cdot \operatorname{curl} \nabla \varphi \, dx + \langle \boldsymbol{E}_-(t) \times (-\nu), \nabla \varphi \rangle_{H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)} \\
&\quad = \langle [\boldsymbol{E} \times \nu](t), \nabla \varphi \rangle_{H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)} = 0
\end{aligned}$$

for all $t \in J$ and $\varphi \in C_c^\infty(G)$. Since $\operatorname{tr}_\Sigma H_0^1(G) = H^{1/2}(\Sigma)$, we infer that $\partial_t [\boldsymbol{B} \cdot \nu] = 0$ on $\overline{J} \times G$. As $[\boldsymbol{B} \cdot \nu](t_0) = 0$ on Σ, we arrive at $[\boldsymbol{B} \cdot \nu] = 0$ on $J \times \Sigma$.

(2) We proceed as in part (1). Using the assumptions on $\boldsymbol{J}$, we compute

$$\begin{aligned}
&\langle \partial_t [\boldsymbol{D} \cdot \nu](t), \varphi \rangle_{H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)} = \langle [\partial_t \boldsymbol{D} \cdot \nu](t), \varphi \rangle_{H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)} \\
&\quad = \langle [(\operatorname{curl} \boldsymbol{H} - \boldsymbol{J}) \cdot \nu](t), \varphi \rangle_{H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)} \\
&\quad = -\langle [\boldsymbol{J} \cdot \nu](t), \varphi \rangle_{H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)} - \langle [\boldsymbol{H} \times \nu](t), \nabla \varphi \rangle_{H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)} \\
&\quad = -\langle [\boldsymbol{J} \cdot \nu](t), \varphi \rangle_{H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)} - \langle \boldsymbol{J}_\Sigma(t), \nabla \varphi \rangle_{H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)}
\end{aligned}$$

for all $\varphi \in C_c^\infty(G)$ and almost all $t \in J$. Since $J_\Sigma = [\boldsymbol{H} \times \nu]$, the boundary current density $\boldsymbol{J}_\Sigma$ is tangent to Σ, i.e., $\boldsymbol{J}_\Sigma = \pi_\Sigma \boldsymbol{J}_\Sigma$, where $\pi_\Sigma = \pi_{\Sigma,x}$ denotes the orthogonal projection on the tangent space at $x \in \Sigma$. We infer that

$$\begin{aligned}
&\langle \boldsymbol{J}_\Sigma(t), \nabla \varphi \rangle_{H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)} = \langle \pi_\Sigma \boldsymbol{J}_\Sigma(t), \pi_\Sigma \nabla \varphi \rangle_{H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)} \\
&\quad = \langle \boldsymbol{J}_\Sigma(t), \nabla_\Sigma \varphi \rangle_{H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)} = -\langle \operatorname{div}_\Sigma \boldsymbol{J}_\Sigma(t), \varphi \rangle_{H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)},
\end{aligned}$$

where we refer to Definition 2.2 of ∇_Σ and $\operatorname{div}_\Sigma$ in [7]. We conclude that

$$\langle \partial_t [\boldsymbol{D} \cdot \nu](t), \varphi \rangle_{H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)} = \langle \operatorname{div}_\Sigma \boldsymbol{J}_\Sigma - [\boldsymbol{J} \cdot \nu](t), \varphi \rangle_{H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)}$$

for all $\varphi \in C_c^\infty(G)$ and almost all $t \in J$. Arguing as in (1), we derive claim (2). □

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E-mail address: `schnaubelt@kit.edu, martin.spitz@kit.edu`

Department of Mathematics, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany.