Modulation spaces and nonlinear Schrödinger equations

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1. Introduction

The topics of this thesis are perhaps approached best by the following model problem. Signal propagation in a single-mode Kerr-nonlinear optical fiber cable operated at a single carrier frequency is commonly modeled by the one-dimensional focusing cubic nonlinear Schrödinger equation (NLS), i.e.

$$\mathrm{i}\partial_t u = -\partial_{xx} u - |u|^2 u \qquad x, t \in \mathbb{R}. \tag{1.1}$$

In the equation above, physical constants have been eliminated by a change of coordinates; u corresponds to the complex amplitude of the mode, x to the retarded time and t to the position along the fiber (the uncommon names of the variables are chosen such that the above equation is easily recognized as the NLS). Of course, the above model is not complete without prescribing an initial condition $u(\cdot, t = 0) = u_0$, i.e. the signal being fed into the fiber at one of its ends. An initial condition corresponding to data transmission would have the form

$$u_0(x) = \sum_{n \in \mathbb{N}} f_n(x - n) \qquad \forall x \in \mathbb{R}, \tag{1.2}$$

where, for $n \in \mathbb{N}$, f_n is selected from a finite set of given functions and encodes the n-th transmitted symbol. In applications, it is of interest to study solutions of (1.1) with initial conditions (1.2) analytically or numerically to deduce their features. To that end, at least local, but better global, well-posedness of this Cauchy problem needs to be established in a suitable function space. Because functions in (1.2) are, in non-trivial cases, neither decaying nor periodic the well-developed theory in L^2-based Sobolev spaces on the real line or on the torus is not applicable to them. However, some modulation spaces seem to be good candidates for the well-posedness theory, because they include functions as in (1.2) and the Schrödinger propagator is a strongly continuous group on them, i.e. at least the linear problem is already solved. Today, global well-posedness of the NLS in modulation spaces has been shown only in a few cases and none of them covers the situation described above. In other words, the problem is interesting not only from a physics point of view, but also for a mathematician.

Studying the model problem described above, it makes sense to allow for more general nonlinearities F and arbitrary dimensions d, i.e. to consider

$$\begin{cases} \mathrm{i}u_t(x,t) = -\Delta u + F(u), & x \in \mathbb{R}^d,\ t \in \mathbb{R} \\ u(\cdot, 0) = u_0. \end{cases} \tag{1.3}$$

Literature survey

The body of literature covering the mentioned topics is of course huge. Hence the selection given below is far from exhaustive and is mainly based on the author's taste (at least for textbooks, overview publications and other well-known results).

Communication over optical glass fibers is treated in [Sch04]. This and other applications of the NLS are presented in [SS99], whereas [Tao06] has a purely mathematical perspective. Global well-posedness of the NLS is the subject of [Bou99]. Modern textbooks covering the NLS are [LP09] and [ET16].

Modulation spaces were introduced in [Fei83], which is still a good reference. For its reading, one probably should have [Fei80] at hand. Historical notes [Fei06] by the inventor of modulation spaces contain many references to recent literature. A modern textbook is [Grö01].

A book covering both modulation spaces and NLS is [WHHG11], but see also the overview article [RSW12].

Known results touching the model problem described above are as follows. In [WZG06] local well-posedness for the Cauchy problem for the NLS with a power nonlinearity on a certain modulation space is shown for arbitrary dimensions. This space does not include initial values of the form (1.2). More modulation spaces and general algebraic nonlinearites are covered in [BO09]. In fact, the latter result is applicable to the model problem. Moreover, in the article [Guo17] the cubic nonlinearity in one dimension is considered for different modulation spaces. Again, initial values of the form (1.2) are not covered. The same is true for the publications [Pat18] and [CHKP18] (the latter is co-authored by the PhD candidate).

In [WH07] some theorems concerning global well-posedness of the Cauchy problem for the NLS are derived. These results assume smallness of the initial data and neither include the cubic nonlinearity in one dimension nor initial values of the form (1.2). The same is true for their generalization in [Kat14]. In [CHKP17] (co-authored by the PhD candidate), cubic nonlinearity in one dimension is treated and a global well-posedness result is obtained. However, it is not applicable to initial values of the form (1.2).

Further literature is cited later, when the specific topic is touched. Also, more remarks on the already mentioned literature are made then.

Results obtained in this thesis and a conclusion

The contribution of the work at hand is as follows.

The well-posedness result from [BO09] is generalized to cover certain intersections of modulation spaces. The proof of the original theorem relies on the fact that certain modulation spaces are Banach *-algebras. As this is also true for the intersections considered here, the result follows immediately. The algebra property for the intersections is apparent from the proof of [STW11, Proposition 3.2]. However, to the best of the author's knowledge, neither the algebra property for the intersections nor the corresponding improved local well-posedness theorem has been published elsewhere.

A new Hölder-like inequality is obtained. Also, a characterization of modulation spaces via the Littlewood-Paley decomposition, which was not observed previously, is shown. With its help, a sufficient condition for some series to converge in certain modulation spaces is proven.

The main contribution is the extension of the global well-posedness result from [CHKP17] to cover a larger range of nonlinearities, more modulation spaces and arbitrary dimensions. Also, the proof is considerably simplified and the notion of a solution is made more precise. Finally, the underlying local well-posedness holds for a larger range of modulation spaces.

Initial values of the form (1.2) are not covered by the improved result. This was also not to be expected, as the proof is via a nonlinear interpolation argument and the required space is an endpoint of the scale. The classical approach of upgrading a local well-posedness to a global one is via a conserved quantity. However, it is not clear whether a suitable conservation law exists and how it could be connected to the modulation space norm. Hence, the global well-posedness for the model problem remains open.

Organization of this thesis

The remainder of the text at hand is structured as follows. The introductory chapter concludes with the explanation of the used notation. In the subsequent Chapter 2, modulation spaces are defined, their basic properties are presented and some embeddings needed later are shown. It also contains the aforementioned characterization of modulation spaces via the Littlewood-Paley decomposition and the resulting sufficient condition. Chapter 3 lays further the necessary ground for the well-posedness results. There, the Schrödinger group is shown to be strongly continuous on most of the modulation spaces, the classical Strichartz estimates are quoted and a nonlinear Strichartz estimate is derived from the sketched proof of the well-known global well-posedness of the mass-subcritical NLS in L^2. Chapter 4 contains the improved local well-posedness result and the algebra property it is based on. Also, the Hölder-like inequality is proven there. In the final Chapter 5, the improved global well-posedness result is stated and proven.

Notation

Only potentially uncommon notational choices are mentioned here, all others are documented in the appendix.

The duality pairing $\langle u, f\rangle = u(f)$ extends the L^2-duality and is linear in the *second* variable, i.e. $\langle u, f\rangle = \int \overline{u} f \mathrm{d}x$.

For two quantities A, B, the notation $A \lesssim_d B$ shall mean, that there is a constant $C > 0$ independent of A and B, but depending on another quantity d, such that $A \leq CB$. Another notation for such dependence shall be $C = C(d)$. Of course, $A \approx_d B$ shall mean that $A \lesssim_d B$ and $B \lesssim_d A$.

For Bessel potential spaces H^s_p, s shall be the regularity and p the integrability indices. The space of bounded continuous functions shall be denoted by C_{b} and the space of infinitely often differentiable functions with compact support by $\mathcal{D}$. The space of infinitely often differentiable functions such that each of its derivatives is bounded by a polynomial shall be denoted by C^∞_{pol}. Whether the continuous or the discrete norm is meant by $\|\cdot\|_p$ shall be apparent from its argument.

The Japanese bracket shall be defined by $\langle\cdot\rangle = \sqrt{1 + |\cdot|^2}$. For a countable index set I and a Banach space X, the space of $\langle\cdot\rangle^s$-weighted sequences in X shall be denoted by $l^q_s(I, X)$, where s is the regularity and q the summability index. More precisely, one has

$$\|(a_k)\|_{l^q_s} = \begin{cases} \left(\sum_{k\in I} \langle k\rangle^{qs} \|a_k\|_X^q\right)^{\frac{1}{q}}, & \text{if } q < \infty, \\ \sup_{k\in I} \langle k\rangle^s \|a_k\|_X, & \text{if } q = \infty. \end{cases}$$

The space of $\langle\cdot\rangle^s$-weighted sequences converging to zero shall be denoted by $c^0_s \subseteq l^\infty_s$.

The constants of the Fourier transform and its inverse are chosen symmetrically, i.e.

$$\begin{aligned} \hat{f}(\xi) &= (\mathcal{F}f)(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} f(x)\mathrm{d}x, \\ \check{g}(x) &= \left(\mathcal{F}^{(-1)} g\right)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\mathrm{i}\xi\cdot x} g(\xi)\mathrm{d}\xi. \end{aligned}$$

The operation of dilation shall be defined as $(\delta^a f)(y) = f(ay)$, right-shift by $S_x f(y) = f(y - x)$ and modulation by $(M_k f)(y) = e^{-\mathrm{i}k\cdot y} f(y)$.

2. Modulation spaces

Main purpose of this chapter is to introduce modulation spaces and their basic properties in a form suitable for the remainder of this thesis. Additionally, a characterization of modulation spaces via the Littlewood-Paley decomposition is proven. To the best of the author's knowledge, this characterization is new.

Modulation spaces were pioneered by Feichtinger in 1983 in his technical report [Fei83]. There, modulation spaces were defined in a quite abstract setting of locally compact Abelian groups. A modern textbook on modulation spaces on $\mathbb{R}^d$ is [Grö01], where they are introduced via the short-time Fourier transform. Another (equivalent) approach to modulation spaces, which is presented in [WH07, Section 2, 3] and [WHHG11, Section 6.2], is via the isometric decomposition operators. It clearly shows the similarity of Besov and modulation spaces — the former correspond to *dyadic* decomposition operators. For a general discussion of Banach spaces arising from decompositions see [FG85] and [Fei87]. Another example of such spaces are the Wiener amalgam spaces (see [Fei80]), which are closely connected to modulation spaces via the Fourier transform. For embeddings between modulation spaces and some more "classical" Banach spaces see [Grö92, Section G], [Oko04], [Tof04a] and [Tof04b], [WH07, Section 2] and [WHHG11, Section 6.3]. Of course, the compilation of the literature above is far from being exhaustive.

This chapter is structured as follows. In Section 2.1 modulation spaces are defined in terms of the isometric decomposition operators and their basic properties are proven, i.e. that they are Banach spaces not depending on the particular choice of the partition of unity. Also, most of their dual spaces and complex interpolation spaces are identified and some embeddings of modulation spaces into each other are proven. In Section 2.2, the short-time Fourier transform is introduced and modulation spaces are characterized in terms of it. Also, the modulation space norm of a complex Gaussian is calculated. Subsequently, in Section 2.3, the aforementioned characterization of modulation spaces via the Littlewood-Paley decomposition is presented. Also, a certain sufficient condition for a series to converge in a modulation space is shown. Finally, some embeddings for modulation spaces, needed in later chapters, are presented and proven in Section 2.4.

2.1. Definition via the isometric decomposition operators

Definition 2.1 (Isometric decomposition operators)**.** Let $d \in \mathbb{N}$. Put $Q_0 := \left[-\frac{1}{2}, \frac{1}{2}\right)^d$ and $Q_k := Q_0 + k$ for all $k \in \mathbb{Z}^d$. Assume that the sequence of functions (called *symbols*)

$(\sigma_k)_{k\in\mathbb{Z}^d} \in C^\infty(\mathbb{R}^d)^{\mathbb{Z}^d}$ satisfies the following conditions

(i) $\exists c > 0 : \forall k \in \mathbb{Z}^d : \forall \eta \in Q_k : |\sigma_k(\eta)| \geq c,$

(ii) $\forall k \in \mathbb{Z}^d : \operatorname{supp}(\sigma_k) \subseteq B_{\sqrt{d}}(k),$

(iii) $\sum_{k\in\mathbb{Z}^d} \sigma_k = 1$ and

(iv) $\forall m \in \mathbb{N}_0 : \exists C_m > 0 : \forall k \in \mathbb{Z}^d : \forall \alpha \in \mathbb{N}_0^d : |\alpha| \leq m \Rightarrow \|\partial^\alpha \sigma_k\|_\infty \leq C_m.$

Then the sequence of operators $(\Box_k)_{k\in\mathbb{Z}^d}$ on $S'(\mathbb{R}^d)$, which is defined by

$$\Box_k = \mathcal{F}^{(-1)}\sigma_k\mathcal{F} \qquad \forall k \in \mathbb{Z}^d,$$

is said to be a family of *isometric decomposition operators* (IDOs). Define also the formally adjoint IDOs $(\Box'_k)_{k\in\mathbb{Z}^d} = \left(\mathcal{F}^{(-1)}\overline{\sigma_k}\mathcal{F}\right)_{k\in\mathbb{Z}^d}$.

Let $(\Box_k)_{k\in\mathbb{Z}^d}$ be a families of IDOs. Observe, that for any $u \in \mathcal{S}'(\mathbb{R}^d)$ and any $f \in \mathcal{S}(\mathbb{R}^d)$ one has

$$\langle \Box_k u, f\rangle = \left\langle \mathcal{F}^{(-1)}\sigma_k\mathcal{F}u, f\right\rangle = \left\langle u, \mathcal{F}^{(-1)}\overline{\sigma_k}\mathcal{F}f\right\rangle = \langle u, \Box'_k f\rangle \qquad \forall k \in \mathbb{Z}^d$$

(operations on $\mathcal{S}'$ are given by Definition A.40). Because $(\Box'_k)_{k\in\mathbb{Z}^d}$ is a family of IDOs in its own right ($(\overline{\sigma_k})_{k\in\mathbb{Z}^d}$ satisfies the conditions of Definition 2.1), one also has

$$\left\langle \Box'_k u, f\right\rangle = \langle u, \Box_k f\rangle \qquad \forall k \in \mathbb{Z}^d.$$

Let $(\tilde{\Box}_k)_{k\in\mathbb{Z}^d}$ be another family of IDOs. For any $k \in \mathbb{Z}^d$ let σ_k and $\tilde{\sigma}_k$ denote the symbol of $\Box_k$ and $\tilde{\Box}_k$ respectively. Observe, that unless $|k-l| \leq 2\sqrt{d}$ one has $\operatorname{supp}(\sigma_k)\cap\operatorname{supp}(\tilde{\sigma}_l) = \emptyset$ by Property (ii) in Definition 2.1. For the rest of this section let $\Lambda(d) := \left\{l \in \mathbb{Z}^d \middle| \, |l| \leq 2\sqrt{d}\right\}$ be the set of *close indices.* By the above, one has

$$l \notin \Lambda(d) \Rightarrow \tilde{\Box}_k\Box_{k+l} = 0 \qquad \forall k, l \in \mathbb{Z}^d. \tag{2.1}$$

Finally, remark that for any $\xi \in \mathbb{R}^d$ there is exactly one $k(\xi) \in \mathbb{Z}^d$ such that $\xi \in Q_{k(\xi)}$. Unless $|k-l| \leq \frac{3}{2}\sqrt{d}$ one has $Q_k \cap \operatorname{supp}(\sigma_l) \subseteq \overline{B}_{\frac{\sqrt{d}}{2}}(k) \cap B_{\sqrt{d}}(l) = \emptyset$, again by Property (ii) in Definition 2.1. For the rest of this chapter define the set of *relevant indices* by $\Lambda'(d) := \left\{l \in \mathbb{Z}^d \middle| \, |l| \leq \frac{3}{2}\sqrt{d}\right\}$. By the above, one has

$$l \notin \Lambda'(d) \Rightarrow \xi \notin \operatorname{supp}(\sigma_{k(\xi)+l}) \qquad \forall \xi \in \mathbb{R}^d \, \forall l \in \mathbb{Z}^d. \tag{2.2}$$

Example 2.2 (Construction of the IDOs). Consider a $\rho \in \mathcal{D}(\mathbb{R})$ satisfying $\rho(x) = 1$, if $|x| \leq \frac{1}{2}$ and $\rho(x) = 0$, if $|x| \geq 1$. Put $\rho_0(\xi) := \rho\left(\frac{|\xi|}{\sqrt{d}}\right)$ and $\rho_k(\xi) := \rho_0(\xi - k)$ for any $\xi \in \mathbb{R}^d$ and any $k \in \mathbb{Z}^d$. Finally, set

$$\sigma_k(\xi) := \frac{\rho_k(\xi)}{\sum_{l\in\mathbb{Z}^d} \rho_l(\xi)} = \frac{\rho_k(\xi)}{\sum_{l\in\Lambda'(d)} \rho_{m(\xi)+l}(\xi)} \qquad \forall \xi \in \mathbb{R}^d \, \forall k \in \mathbb{Z}^d$$

(the fact that for a fixed $\xi \in \mathbb{R}^d$ the series above is just a finite sum is due to Implication (2.2)). Then $(\Box_k) = (\mathcal{F}^{(-1)}\sigma_k\mathcal{F})$ is a family of isometric decomposition operators. Observe, that $\sigma_k = S_k\sigma_0$ for any $k \in \mathbb{Z}^d$.

Definition 2.3 (Modulation space)**.** (Cf. [WH07, Proposition 2.1]). Let $d \in \mathbb{N}$, $p,q \in [1,\infty]$, $s \in \mathbb{R}$ and $(\Box_k)_{k\in\mathbb{Z}^d}$ a family of IDOs. Define the modulation space norm (w.r.t. the family of IDOs $(\Box_k)_{k\in\mathbb{Z}^d}$) by

$$\|u\|_{M^s_{p,q}(\mathbb{R}^d)} = \left\| \left(\langle k\rangle^s \|\Box_k u\|_{L^p(\mathbb{R}^d)} \right)_{k\in\mathbb{Z}^d} \right\|_{l^q(\mathbb{Z}^d)} \qquad \forall u \in \mathcal{S}'(\mathbb{R}^d). \tag{2.3}$$

Observe, that for every $k \in \mathbb{Z}^d$ there exists a unique $f_k \in C^\infty_{\text{pol}}(\mathbb{R}^d)$ such that $\Box_k u = \Phi f_k$ (as in Equation (A.24)) by Proposition A.44. This justifies taking the L^p-norm above (i.e. $\|\Box_k u\|_p = \|f_k\|_p$). The seminormed vector space

$$M^s_{p,q}(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d) \middle| \; \|u\|_{M^s_{p,q}(\mathbb{R}^d)} < \infty \right\} \tag{2.4}$$

shall be called *modulation space* (w.r.t. the family of IDOs $(\Box_k)_{k\in\mathbb{Z}^d}$) with *regularity index* s, *space index* p and *Fourier index* q.

One often shortens the notation to $M^s_{p,q} := M^s_{p,q}(\mathbb{R}^d)$ and $M_{p,q} =: M^0_{p,q}$. Finally, set

$$M^s_{p,0}(\mathbb{R}^d) := \left\{ u \in \mathcal{S}'(\mathbb{R}^d) \middle| \; \lim_{|k|\to\infty} \langle k\rangle^s \|\Box_k u\|_p = 0 \right\} \subseteq M^s_{p,\infty}(\mathbb{R}^d).$$

Shortly, it will be shown that the seminorm $\|\cdot\|_{M^s_{p,q}(\mathbb{R}^d)}$ is a norm on $M^s_{p,q}(\mathbb{R}^d)$ (Proposition 2.4), that the modulation spaces are Banach spaces (Proposition 2.11) and that different families of IDOs yield equivalent norms (Proposition 2.9). For the moment, consider a fixed family of IDOs $(\Box_k)_{k\in\mathbb{Z}^d}$.

Proposition 2.4 (Modulation spaces are normed spaces)**.** *Let $d \in \mathbb{N}$, $s \in \mathbb{R}$, $p,q \in [1,\infty]$ and $(\Box_k)_{k\in\mathbb{Z}^d}$ be a family of IDOs. Then $\left(M^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{M^s_{p,q}(\mathbb{R}^d)}\right)$ is a normed vector space.*

For the proof of the last proposition consider first the following

Lemma 2.5 ($\sum \Box_k$ converges strongly unconditionally to id in $\mathcal{S}$ and $\mathcal{S}'$)**.** *Let $d \in \mathbb{N}$, $f \in \mathcal{S}(\mathbb{R}^d)$ and $u \in \mathcal{S}'(\mathbb{R}^d)$. Then the series $\sum_{k\in\mathbb{Z}^d} \Box_k f$ converges unconditionally to f in $\mathcal{S}(\mathbb{R}^d)$ and $\sum_{k\in\mathbb{Z}^d} \Box_k u$ converges unconditionally to u in $\mathcal{S}'(\mathbb{R}^d)$.*

Recall, that unconditional convergence of a series $\sum_{k\in\mathbb{Z}^d} a_k$ in a Hausdorff topological vector space X means that for any ordering $(k_n)_{n\in\mathbb{N}}$ of $\mathbb{Z}^d$ (called *order of summation*) the series $\sum_{n=0}^\infty a_{k_n}$ converges in X to a value $s \in X$ and s does not depend on the order of summation.

Proof of Lemma 2.5. First, consider the case of convergence in $\mathcal{S}(\mathbb{R}^d)$. The Fourier transform and its inverse are continuous on $\mathcal{S}(\mathbb{R}^d)$ by Proposition A.34. Hence, by definition of

$\Box_k = \mathcal{F}^{(-1)}\sigma_k\mathcal{F}$, it is enough to show that $\sum_{k\in\mathbb{Z}^d}^{\infty}\sigma_k g$ unconditionally converges to g for any $g \in \mathcal{S}(\mathbb{R}^d)$. To that end consider any fixed order of summation $(k_n)_{n\in\mathbb{N}}$. For every $N \in \mathbb{N}$ define $I_N = \{k_1, \dots, k_N\}$,

$$M(N) = \overline{\left\{\xi \in \mathbb{R}^d \Big| \sum_{k\in I_N} \sigma_k(\xi) \neq 1\right\}}$$

and let $\alpha, \beta \in \mathbb{N}_0^d$. As $M(N)^c$ is open and $\sum_{n=1}^N \sigma_{k_n}(\xi) = 1$ for any $\xi \in M(N)^c$ one has

$$\begin{aligned}
\rho_{\alpha,\beta}\left(g - \sum_{k\in I_N}\sigma_k g\right) &= \sup_{\xi\in M(N)}\left|\xi^\alpha\partial^\beta\left(g - \sum_{k\in I_N}\sigma_k g\right)(\xi)\right| \\
&\leq \left(\sup_{\xi\in\mathbb{R}^d}\left|\langle\xi\rangle^2\xi^\alpha\partial^\beta g\right| + \sup_{\xi\in\mathbb{R}^d}\sum_{k\in\mathbb{Z}^d}\left|\langle\xi\rangle^2\xi^\alpha\partial^\beta(\sigma_k g)(\xi)\right|\right) \qquad (2.5)\\
&\quad\cdot\left(\sup_{\xi\in M(N)}\frac{1}{\langle\xi\rangle^2}\right).
\end{aligned}$$

Clearly, as N grows, the second factor above converges to zero. Hence, it suffices to show that the first factor is finite. The first supremum is indeed finite, as it is bounded above by a sum of seminorms of g.

For the second supremum, consider any fixed $\xi \in \mathbb{R}^d$. By the Leibnitz's rule (A.20), one has

$$\sum_{k\in\mathbb{Z}^d}\left|\langle\xi\rangle^2\xi^\alpha\partial^\beta(\sigma_k g)(\xi)\right| \leq \sum_{k\in\mathbb{Z}^d}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}|(\partial^\gamma\sigma_k)(\xi)|\left|\langle\xi\rangle^2\xi^\alpha(\partial^{\beta-\gamma}g)(\xi)\right|.$$

In the summation over k almost all summands vanish by the Implication (2.2). Hence, it may be replaced by the finite sum

$$\begin{aligned}
&\sum_{l\in\Lambda'(d)}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\left|(\partial^\gamma\sigma_{k(\xi)+l})(\xi)\right|\left|\langle\xi\rangle^2\xi^\alpha(\partial^{\beta-\gamma}g)(\xi)\right| \\
\leq\; &\sum_{l\in\Lambda'(d)}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}C_{|\beta|}\left|\langle\xi\rangle^2\xi^\alpha(\partial^{\beta-\gamma}g)(\xi)\right|,
\end{aligned}$$

where additionally Property (iv) in Definition 2.1 has been used. The right-hand side of the last inequality is bounded independently of ξ by a multiple of a finite sum of seminorms of g. This shows that the second supremum in (2.5) is finite and concludes the proof of convergence in $\mathcal{S}(\mathbb{R}^d)$.

For the convergence in $\mathcal{S}'(\mathbb{R}^d)$ consider again an arbitrary order of summation $(k_n)_{n\in\mathbb{N}}$. One has

$$\left\langle u - \sum_{n=1}^N \Box_{k_n}u, g\right\rangle = \langle u, g\rangle - \sum_{n=1}^N\langle\Box_{k_n}u, g\rangle = \left\langle u, g - \sum_{n=1}^N \Box'_{k_n}g\right\rangle$$

for every $N \in \mathbb{N}$ and $g \in \mathcal{S}(\mathbb{R}^d)$. As $(\overline{\Box'_k})_{k\in\mathbb{Z}^d}$ is a family of IDOs in its own right, the already proven unconditional convergence in $\mathcal{S}(\mathbb{R}^d)$ ensures that $\sum_{n=1}^N \Box'_{k_n}g \to g$ in $\mathcal{S}(\mathbb{R}^d)$

as $N \to \infty$ independently of the order of summation $(k_n)_{n\in\mathbb{N}}$. Recalling the definition of convergence in $\mathcal{S}'(\mathbb{R}^d)$ finishes the proof. □

Proof of Proposition 2.4. The only non-trivial property is the positive definiteness of $\|\cdot\|_{M^s_{p,q}}$. Consider a $u \in M^s_{p,q}(\mathbb{R}^d)$ with $\|u\|_{M^s_{p,q}} = 0$, i.e. $\Box_k u = 0$ for all $k \in \mathbb{Z}^d$. But then, by Lemma 2.5, $u = \sum_{k\in\mathbb{Z}^d} \Box_k u = 0$. □

How modulation spaces with different indices embed into one another is clarified in the following

Proposition 2.6 (Embeddings of modulation spaces into each other)**.** *(Cf. [WH07, Proposition 2.5]). Let $d \in \mathbb{N}$, $s_1, s_2 \in \mathbb{R}$ and $p_1, p_2, q_1, q_2 \in [1,\infty]$ satisfy*

$$s_1 \geq s_2, \qquad p_1 \leq p_2, \qquad q_1 \leq q_2.$$

Then

$$M^{s_1}_{p_1,q_1}(\mathbb{R}^d) \hookrightarrow M^{s_2}_{p_2,q_2}(\mathbb{R}^d). \tag{2.6}$$

Furthermore, if $q_2 < \infty$, then

$$M^{s_2}_{p_2,q_2}(\mathbb{R}^d) \subseteq M^{s_2}_{p_2,0}(\mathbb{R}^d). \tag{2.7}$$

For the proof consider first the following

Lemma 2.7 (IDOs on Lebesgue spaces)**.** *Let $d \in \mathbb{N}$, $(\Box_k)_{k\in\mathbb{Z}^d}$ a family of IDOs and $p_1, p_2 \in [1,\infty]$ satisfy $p_1 \leq p_2$. Then the family $(\Box_k)_{k\in\mathbb{Z}^d}$ is bounded in $\mathscr{L}(L^{p_1}(\mathbb{R}^d), L^{p_2}(\mathbb{R}^d))$ and this bound is independent of p_1 and p_2.*

Putting $p_1 = p_2 = p$ in the above lemma immediately yields the useful

Corollary 2.8 (IDOs on a Lebesgue space)**.** *Let $d \in \mathbb{N}$. Then for any family of IDOs $(\Box_k)_{k\in\mathbb{Z}^d}$ there exists a $C = C(d) > 0$ such that for any $p \in [1,\infty]$ one has*

$$\|\Box_k\|_{\mathscr{L}(L^p(\mathbb{R}^d))} \leq C \qquad \forall k \in \mathbb{Z}^d.$$

Proof of Lemma 2.7. By the Bernstein multiplier estimate from Corollary A.53, one immediately has

$$\|\Box_k\|_{\mathscr{L}(L^{p_1},L^{p_2})} \leq C(1 + |\operatorname{supp}(\sigma_k)|) \left(\|\sigma_k\|_\infty + \sum_{j=1}^{d} \left\| \partial^{de_j} \sigma_k \right\|_\infty \right) \tag{2.8}$$

for any $k \in \mathbb{Z}^d$. As, by Property (ii) in Definition 2.1, one has

$$|\operatorname{supp}(\sigma_k)| \leq \left| B_{\sqrt{d}}(k) \right| \lesssim_d 1 \qquad \forall k \in \mathbb{Z}^d$$

and, by Property (iv) (with C_d as there),

$$\left(\|\sigma_k\|_\infty + \sum_{j=1}^{d} \left\| \partial^{de_j} \sigma_k \right\|_\infty \right) \lesssim_d C_d \qquad \forall k \in \mathbb{Z}^d,$$

the right-hand side of (2.8) is bounded above independently of k. The proof is thus complete. □

Proof of Proposition 2.6. To prove the embedding (2.6) one may consider different indices separately, i.e. show

$$M^{s_1}_{p_1,q_1} \hookrightarrow M^{s_2}_{p_1,q_1} \hookrightarrow M^{s_2}_{p_1,q_2} \hookrightarrow M^{s_2}_{p_2,q_2}. \tag{2.9}$$

As $\langle \xi \rangle \geq 1$ one also has $\langle \xi \rangle^{s_2} \leq \langle \xi \rangle^{s_1}$ for any $\xi \in \mathbb{R}^d$, which together with the definition of the modulation space norm in Equation (2.3) implies the first embedding.

The second embedding follows from the well-known embedding of the sequence spaces $l^{q_1}(\mathbb{Z}^d) \hookrightarrow l^{q_2}(\mathbb{Z}^d)$ and the definition of the modulation space norm in Equation (2.3).

For the last embedding in (2.9) consider the identity (in $\mathcal{S}'(\mathbb{R}^d)$)

$$\square_k = \sum_{l \in \mathbb{Z}^d} \square_l \square_k = \sum_{l \in \Lambda(d)} \square_{k+l} \square_k \qquad \forall k \in \mathbb{Z}^d,$$

where Lemma 2.5 was used in the first equality and Implication (2.1) in the second. Hence, for any $u \in \mathcal{S}'(\mathbb{R}^d)$ one has

$$\|\square_k u\|_{p_2} = \left\| \sum_{l \in \Lambda(d)} \square_{k+l} \square_k u \right\|_{p_2} \leq \sum_{l \in \Lambda(d)} \|\square_{k+l} \square_k u\|_{p_2} \lesssim_d \|\square_k u\|_{p_1} \qquad \forall k \in \mathbb{Z}^d.$$

Here, Lemma 2.7 was used in the last estimate. Recalling the definition of the modulation space norm (Equation (2.3)), shows the last embedding.

To show the Inclusion (2.7), assume $q_2 < \infty$ and consider any $u \in M^{s_2}_{p_2,q_2}$. Then

$$\|u\|^{q_2}_{M^{s_2}_{p_2,q_2}} = \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{s_2 q_2} \|\square_k u\|^{q_2}_{p_2} < \infty$$

and hence $\lim_{|k| \to \infty} \langle k \rangle^{s_2} \|\square_k u\|_{p_2} = 0$, i.e. $u \in M^{s_2}_{p_2,0}$. This finishes the proof. □

Techniques used for the last proof can also be applied to show that different families of IDOs yield the same modulation spaces $M^s_{p,q}$. More precisely one has the following

Proposition 2.9 ($M^s_{p,q}$ is independent of the family of IDOs)**.** *Let $d \in \mathbb{N}$, $s \in \mathbb{R}$ and $p, q \in [1, \infty]$. Furthermore, let $(\square_k)_{k \in \mathbb{Z}^d}$ and $(\tilde{\square}_k)_{k \in \mathbb{Z}^d}$ be two families of IDOs. Then there is a constant $C > 0$ depending on $(\square_k)_{k \in \mathbb{Z}^d}$ and $(\tilde{\square}_k)_{k \in \mathbb{Z}^d}$ such that*

$$\frac{1}{C} \left\| \left(\langle k \rangle^s \|\tilde{\square}_k u\|_p \right)_{k \in \mathbb{Z}^d} \right\|_q \leq \left\| \left(\langle k \rangle^s \|\square_k u\|_p \right)_{k \in \mathbb{Z}^d} \right\|_q \leq C \left\| \left(\langle k \rangle^s \|\tilde{\square}_k u\|_p \right)_{k \in \mathbb{Z}^d} \right\|_q \tag{2.10}$$

for all $u \in \mathcal{S}'(\mathbb{R}^d)$. In particular, the modulation space $M^s_{p,q}(\mathbb{R}^d)$ as a set does not depend on the choice of the family of IDOs and any two modulation space norms on $M^s_{p,q}(\mathbb{R}^d)$ are equivalent.

Proof. It suffices to show only the first inequality in (2.10), as the second one follows by interchanging the roles of $(\square_k)$ and $(\tilde{\square}_k)$. To that end, let $k \in \mathbb{Z}^d$ and denote by σ_k the symbol of $\square_k$ and by $\tilde{\sigma_k}$ the symbol of $\tilde{\square}_k$. Consider any $u \in \mathcal{S}'(\mathbb{R}^d)$. One has

$$\tilde{\square}_k = \tilde{\square}_k \sum_{l \in \mathbb{Z}^d} \square_l = \sum_{l \in \Lambda(d)} \tilde{\square}_k \square_{k+l} \qquad \forall k \in \mathbb{Z}^d$$

by Lemma 2.5 and Implication (2.1). Hence,

$$\left\|\tilde{\square}_k u\right\|_p \le \sum_{l \in \Lambda(d)} \left\|\tilde{\square}_k \square_{k+l} u\right\|_p \lesssim_d \sum_{l \in \Lambda(d)} \|\square_{k+l} u\|_p \qquad \forall k \in \mathbb{Z}^d$$

by Corollary 2.8. Peetre's inequality (Lemma A.31) now implies

$$\left\| \left(\langle k \rangle^s \left\|\tilde{\square}_k u\right\|_p \right)_k \right\|_q \lesssim_d \sum_{l \in \Lambda(d)} \left\| \left(\langle k \rangle^s \|\square_{k+l} u\|_p \right)_k \right\|_q$$
$$\le \sum_{l \in \Lambda(d)} 2^{|s|} \langle l \rangle^{|s|} \left\| \left(\langle k+l \rangle^s \|\square_{k+l} u\|_p \right)_k \right\|_q \lesssim_{d,s} \left\| \left(\langle k \rangle^s \|\square_k u\|_p \right)_k \right\|_q .$$

This concludes the proof. $\square$

Proposition 2.10. *Let $d \in \mathbb{N}$, $s \in \mathbb{R}$, $p \in [1,\infty]$ and $q \in \{0\} \cup [1,\infty]$. Then*

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow M^s_{p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d).$$

Proof. Consider the first embedding. By Proposition 2.6, it is enough to show $\mathcal{S} \hookrightarrow M^s_{1,1}$. To that end, consider $u \in \mathcal{S}$ and observe that by Lemma A.51 one has

$$\|\square_k u\|_1 = \|\sigma_k \hat{u}\|_{\mathcal{F}L^1} \lesssim_d \|\sigma_k \hat{u}\|_{H^d} \qquad \forall k \in \mathbb{Z}^d.$$

(Above, any integer greater than $\frac{d}{2}$ could be used as the regularity index of the Bessel potential space instead of d.) Proposition A.52, together with the Leibniz' rule, further estimates

$$\|\sigma_k \hat{u}\|_{H^d} \lesssim_d \sum_{|\alpha| \le d} \|\partial^\alpha (\sigma_k \hat{u})\|_2 \le \sum_{|\alpha| \le d} \sum_{\beta \le \alpha} \left\| \left(\partial^{\alpha-\beta} \sigma_k\right) \left(\partial^\beta \hat{u}\right) \right\|_2 \qquad \forall k \in \mathbb{Z}^d.$$

Because of the compact support of σ_k (Property (ii) in Definition 2.1), one may estimate the L^2-norm by the L^∞-norm. Additionally using Property (iv) yields

$$\sum_{|\alpha| \le d} \sum_{\beta \le \alpha} \left\| \left(\partial^{\alpha-\beta} \sigma_k\right) \left(\partial^\beta \hat{u}\right) \right\|_2 \lesssim_d \sum_{|\alpha| \le d} \sum_{\beta \le \alpha} \sup_{\xi \in B_{\sqrt{d}}(k)} \left|\partial^\beta \hat{u}(\xi)\right| \qquad \forall k \in \mathbb{Z}^d.$$

By Peetre's inequality (see Lemma A.31) one has

$$\langle \xi \rangle^{-t} \le 2^t \langle \xi - k \rangle^t \langle k \rangle^{-t} \lesssim_{d,t} \langle k \rangle^{-t} \qquad \forall k \in \mathbb{Z}^d \, \forall \xi \in B_{\sqrt{d}}(k)$$

for any $t > 0$ (to be fixed later) and hence

$$\sup_{\xi \in B_{\sqrt{d}}(k)} \left|\partial^\beta \hat{u}(\xi)\right| \lesssim_{d,t} \langle k\rangle^{-t} \sup_{\xi\in\mathbb{R}^d} \langle \xi\rangle^t \left|\partial^\beta \hat{u}(\xi)\right| \qquad \forall k \in \mathbb{Z}^d.$$

Recalling the definition of the modulation space norm (Equation (2.3)) shows

$$\|u\|_{M^s_{1,1}} = \sum_{k\in\mathbb{Z}^d} \langle k\rangle^s \|\square_k u\|_1 \lesssim_{d,t} \sum_{|\alpha|\le d} \sum_{\beta\le\alpha} \sup_{\xi\in\mathbb{R}^d} \left[\langle\xi\rangle^t \left|\partial^\beta \hat{u}(\xi)\right|\right] \sum_{k\in\mathbb{Z}^d} \langle k\rangle^{s-t}.$$

Taking a large enough t (say $t > d + s$) makes the series above convergent, whereas the supremum is controllable by a finite sum of semi-norms of u due to the continuity of the Fourier transform on $\mathcal{S}$ (Proposition A.34). This shows the first embedding.

Consider the second embedding. By Proposition 2.6 it suffices to show $M^s_{\infty,\infty} \hookrightarrow \mathcal{S}'$. To that end, consider $u \in \mathcal{S}'$ and $f \in \mathcal{S}$. One has

$$\begin{aligned} |\langle u, f\rangle| &\le \sum_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d} |\langle \square_k u, \square_l f\rangle| = \sum_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d} \left|\langle \square_l' \square_k u, f\rangle\right| = \sum_{k\in\mathbb{Z}^d}\sum_{l\in\Lambda(d)} |\langle \square_k u, \square_{k+l} f\rangle| \\ &\le \sum_{l\in\Lambda(d)}\sum_{k\in\mathbb{Z}^d} \|\square_k u\|_\infty \|\square_{k+l} f\|_1 \lesssim_{d,s} \sum_{l\in\Lambda(d)}\sum_{k\in\mathbb{Z}^d} \langle k\rangle^s \|\square_k u\|_\infty \langle k+l\rangle^{-s} \|\square_{k+l} f\|_1 \\ &\lesssim_d \|u\|_{M^s_{\infty,\infty}} \|f\|_{M^{-s}_{1,1}}, \end{aligned}$$

where Lemma 2.5 was used for the first estimate, Implication (2.1) for the second equality, Hölder's inequality for the second and last estimate and Peetre's inequality (see Lemma A.31) for the third estimate. As $\|f\|_{M^{-s}_{1,1}}$ is finite by the first embedding, the proof is complete. □

Proposition 2.11 (Modulation spaces are Banach spaces). *Let $d \in \mathbb{R}^d$, $p, q \in [1, \infty]$. Then $\left(M^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{M^s_{p,q}(\mathbb{R}^d)}\right)$ is a Banach space. Moreover, $M^s_{p,0}(\mathbb{R}^d)$ is a closed linear subspace of $M^s_{p,\infty}(\mathbb{R}^d)$.*

For the proof of this proposition several provisions will be made.

Lemma 2.12 (Analysis operators). *Let $d \in \mathbb{N}$, $p, q \in [1, \infty]$ and $s \in \mathbb{R}$. Then the* analysis operator *$A^s_{p,q} : M^s_{p,q}(\mathbb{R}^d) \to l^q_s(\mathbb{Z}^d, L^p(\mathbb{R}^d))$, defined by*

$$A^s_{p,q} u := (\square_k u)_k \qquad \forall u \in M^s_{p,q}(\mathbb{R}^d), \tag{2.11}$$

is a linear isometry.

Lemma 2.13 (Synthesis operators). *Let $d \in \mathbb{N}$, $p, q \in [1, \infty]$ and $s \in \mathbb{R}$. Then the* synthesis operator *$S^s_{p,q} : l^q_s(\mathbb{Z}^d, L^p(\mathbb{R}^d)) \to M^s_{p,q}(\mathbb{R}^d)$, defined through*

$$S^s_{p,q}(u_k) := \sum_{k\in\mathbb{Z}^d}\sum_{l\in\Lambda(d)} \square_{k+l} u_k \qquad \forall (u_k) \in l^q_s(\mathbb{Z}^d, L^p(\mathbb{R}^d)), \tag{2.12}$$

is linear and continuous. More precisely, $u_k \in \mathcal{S}'(\mathbb{R}^d)$ in the sense of equation (A.24)*, the series above converges unconditionally in $\mathcal{S}'(\mathbb{R}^d)$ to an element of $M^s_{p,q}(\mathbb{R}^d)$ and its norm is controlled by the norm of the sequence (u_k).*

Proof. Consider an $f \in \mathcal{S}$ and observe

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^d}\sum_{l\in\Lambda} |\langle \Box_{k+l} u_k, f\rangle| &= \sum_{k\in\mathbb{Z}^d}\sum_{l\in\Lambda} |\langle \overline{u}_k, \Box'_{k+l} f\rangle| \le \sum_{k\in\mathbb{Z}^d} \|u_k\|_p \sum_{l\in\Lambda} \left\|\Box'_{k+l} f\right\|_{p'} \\
&\le \left\| \left(\langle k\rangle^s \|u_k\|_p\right)_k \right\|_q \left\| \sum_{l\in\Lambda} \left(\langle k\rangle^{-s} \left\|\Box'_{k+l} f\right\|_{p'}\right)_k \right\|_{q'} \\
&\lesssim_{d,s} \|(u_k)\|_{l^s_q L^p} \|f\|_{M^{-s}_{p',q'}},
\end{aligned}$$

where Hölder's inequality, first for the continuous and then for the discrete variable, was used for the first two estimates. Subsequently, Peetre's inequality (Lemma A.31) and equivalence of norms stemming from different families of IDOs (Proposition 2.9) were applied to obtain the last inequality. Proposition 2.10 shows that the last factor is bounded by a finite sum of seminorms of f and is hence finite. So the series defining $\langle S(u_k), f\rangle$ converges absolutely for any $f \in \mathcal{S}$. A fortiori, the series defining $S^s_{p,q}(u_k)$ converges unconditionally in $\mathcal{S}'$.

Furthermore, for every $n \in \mathbb{Z}^d$ one has

$$\Box_n S^s_{p,q}(u_k) = \Box_n \sum_{k\in\mathbb{Z}^d}\sum_{l\in\Lambda} \Box_{k+l} u_k = \sum_{l\in\Lambda}\sum_{k\in\mathbb{Z}^d} \Box_n \Box_k u_{k+l} = \sum_{l\in\Lambda}\sum_{k\in\Lambda} \Box_n \Box_{n+k} u_{n+k+l}.$$

Interchanging $\Box_n$ with the summation in the second equality is justified by $\Box_n$ being a continuous map on $\mathcal{S}'$ and the series being unconditionally convergent due to the argument above. For the last equality, Implication (2.1) and an index shift were used. Now, Corollary 2.8 and Peetre's inequality (Lemma A.31) imply

$$\begin{aligned}
\left\|S^s_{p,q}(u_k)\right\|_{M^s_{p,q}} &\le \left\| \left(\langle n\rangle^s \sum_{l\in\Lambda}\sum_{k\in\Lambda} \|\Box_n \Box_{n+k} u_{n+k+l}\|_p \right)_n \right\|_q \\
&\lesssim_{d,s} \left\| \left(\langle n\rangle^s \sum_{l\in\Lambda}\sum_{k\in\Lambda} \|u_{n+k+l}\|_p \right)_n \right\|_q \lesssim_{d,s} \|(u_n)\|_{l^q_s}.
\end{aligned}$$

As the linearity of $S^s_{p,q}$ is obvious, the proof is concluded. ☐

Lemma 2.14 **($A^s_{p,q}$ is a right inverse of $S^s_{p,q}$).** *Let $d \in \mathbb{N}$, $p, q \in [1,\infty]$ and $s \in \mathbb{R}$. Then $S^s_{p,q} \circ A^s_{p,q} = \mathrm{id}_{M^s_{p,q}(\mathbb{R}^d)}$ and $A^s_{p,q} \circ S^s_{p,q}$ is a continuous projection onto $\mathrm{Im}(A^s_{p,q})$.*

Proof. In fact,

$$(S^s_{p,q} \circ A^s_{p,q})(u) = \sum_{k\in\mathbb{Z}^d}\sum_{l\in\Lambda} \Box_{k+l}\Box_k u = \sum_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d} \Box_l \Box_k u = \sum_{k\in\mathbb{Z}^d} \Box_k u = u \qquad \forall u \in M^s_{p,q},$$

where the second equality is true by Implication (2.1) and the next two by Lemma 2.5. In other words one indeed has $S^s_{p,q} \circ A^s_{p,q} = \mathrm{id}_{M^s_{p,q}}$.

In particular, $S^s_{p,q}$ is surjective and hence $\mathrm{Im}(A^s_{p,q} \circ S^s_{p,q}) = \mathrm{Im}(A^s_{p,q})$. By the above, one has

$$A^s_{p,q} \circ S^s_{p,q} \circ A^s_{p,q} \circ S^s_{p,q} = A^s_{p,q} \circ \mathrm{id}_{M^s_{p,q}} \circ S^s_{p,q} = A^s_{p,q} \circ S^s_{p,q},$$

i.e. $A^s_{p,q} \circ S^s_{p,q}$ is a projection. Its continuity follows immediately from Lemmas 2.12 and 2.13 and finishes the proof. $\square$

Proof of Proposition 2.11. By Lemma 2.12, $M^s_{p,q}$ is isometrically isomorphic to $\operatorname{Im}(A^s_{p,q})$. Hence, to prove that $M^s_{p,q}$ is a Banach space, it suffices to show that $\operatorname{Im}(A^s_{p,q})$ is closed in $l^q_s(L^p)$. By Lemma 2.14, $A^s_{p,q} \circ S^s_{p,q}$ is a continuous projection with

$$\operatorname{Im}(A^s_{p,q}) = \operatorname{Im}(A^s_{p,q} \circ S^s_{p,q}) = \ker\left(\mathrm{id}_{l^q_s(L^p)} - A^s_{p,q} \circ S^s_{p,q}\right).$$

As $\mathrm{id}_{l^q_s(L^p)} - A^s_{p,q} \circ S^s_{p,q}$ is continuous, its kernel is indeed closed.

It remains to show that $M^s_{p,0}$ is a closed subspace of $M^s_{p,\infty}$. By Lemma 2.12 and Definition 2.3, $M^s_{p,0}$ is isometrically isomorphic to

$$\operatorname{Im}(A^s_{p,\infty}) \cap c^0_s(\mathbb{Z}^d, L^p) \subseteq l^\infty_s(\mathbb{Z}^d, L^p).$$

By the above and Proposition A.18, this intersection is a closed in l^∞_s. This finishes the proof. $\square$

Proposition 2.15 ($\mathcal{S}$ is dense in $M^s_{p,q}$ for finite p, q)**.** *Let $d \in \mathbb{N}$, $s \in \mathbb{R}$, $p \in [1, \infty)$ and $q \in \{0\} \cup [1, \infty)$. Then*

$$\overline{\mathcal{S}(\mathbb{R}^d)}^{M^s_{p,q}(\mathbb{R}^d)} = M^s_{p,q}(\mathbb{R}^d).$$

For the proof of Proposition 2.15 the following lemma will be used.

Lemma 2.16 ($\sum \square_k$ converges strongly unconditionally to id in $M^s_{p,q}$)**.** *Let $d \in \mathbb{N}$, $p \in [1, \infty]$, $q \in \{0\} \cup [1, \infty)$, $s \in \mathbb{R}$ and $u \in M^s_{p,q}(\mathbb{R}^d)$. Then the series $\sum_{k \in \mathbb{Z}^d} \square_k u$ converges unconditionally to u in $M^s_{p,q}(\mathbb{R}^d)$.*

Proof. Consider any fixed order of summation $(k_n)_{n \in \mathbb{N}}$ and set $I(M) := \{k_M, k_{M+1}, \ldots\}$ for every $M \in \mathbb{N}$. The sequence of partial sums $\left(\sum_{n=1}^N \square_{k_n} u\right)_{N \in \mathbb{N}}$ converges to u in $\mathcal{S}'$ by Lemma 2.5. As $M^s_{p,q}$ is a Banach space by Proposition 2.11, it suffices to show that $\left(\sum_{n=1}^N \square_{k_n} u\right)_{N \in \mathbb{N}}$ is a Cauchy sequence. To that end, consider any $M, N \in \mathbb{N}$ with $M \leq N$. Then, for any $l \in \mathbb{Z}^d$, one has

$$\left\| \square_l \sum_{n=M}^N \square_{k_n} u \right\|_p \leq \sum_{n=M}^N \|\square_l \square_{k_n} u\|_p \lesssim_d \begin{cases} \|\square_l u\|_p, & \text{if } l \in I(M) + \Lambda(d), \\ 0, & \text{otherwise.} \end{cases}$$

Above, Implication (2.1) was used in both cases and Corollary 2.8 in the first case.

Assume $q \in [1, \infty)$ for now. By the above, one has

$$\left\| \sum_{n=M}^N \square_{k_n} u \right\|^q_{M^s_{p,q}} = \sum_{l \in \mathbb{Z}^d} \langle l \rangle^{sq} \left\| \square_l \sum_{n=M}^N \square_{k_n} u \right\|^q_p \lesssim_{d,q} \sum_{l \in \mathbb{Z}^d} \mathbb{1}_{I(M)+\Lambda(d)}(l) \langle l \rangle^{sq} \|\square_l u\|^q_p.$$

The right-hand side above converges to zero as $M \to \infty$ by the dominated convergence theorem, which is applicable due to the assumption $u \in M^s_{p,q}$ and the fact that $l \notin I(M) + \Lambda(d)$ if $l \in \mathbb{Z}^d$ is fixed and M is large enough.

For $q = 0$ one has $\lim_{|k|\to\infty} \langle k\rangle^s \|\Box_k u\|_p = 0$. Hence, similarly to the case $q \in [1, \infty)$,

$$\left\| \sum_{n=M}^{N} \Box_{k_n} u \right\|_{M^s_{p,\infty}} = \sup_{l\in\mathbb{Z}^d} \left[\langle l\rangle^s \left\| \Box_l \sum_{n=M}^{N} \Box_{k_n} u \right\|_p \right] \lesssim_d \sup_{l\in I(M)+\Lambda(d)} \left[\langle l\rangle^s \|\Box_l u\|_p \right] \xrightarrow{M\to\infty} 0$$

follows. This completes the proof. □

Proof of Proposition 2.15. By Proposition 2.10 one has $\mathcal{S} \subseteq M^s_{p,q}$ and so taking its closure $\overline{\mathcal{S}}^{M^s_{p,q}}$ in $M^s_{p,q}$ makes sense and $\overline{\mathcal{S}}^{M^s_{p,q}} \subseteq M^s_{p,q}$ holds trivially.

To see the converse inclusion $M^s_{p,q} \subseteq \overline{\mathcal{S}}^{M^s_{p,q}}$, consider any $u \in M^s_{p,q}$. By Lemma 2.16, one may assume w.l.o.g. that $u = \sum_{|k|\le N} \Box_k u$ for some $N > 0$. But then Hölder's inequality implies

$$\|u\|_p \le \sum_{|k|\le N} \langle k\rangle^{-s} \langle k\rangle^s \|\Box_k u\|_p \lesssim_{N,q} \|u\|_{M^s_{p,q}} < \infty,$$

i.e. $u \in L^p$. Proposition A.33, which is applicable due to the assumption $p < \infty$, implies that for any $\varepsilon > 0$ there exists an $f \in \mathcal{S}$ such that $\|u - f\|_p < \varepsilon$. Put $g = \sum_{|k|\le N} \Box_k f$ and observe that $g \in \mathcal{S}$. In the case $q \in [1, \infty)$ one has

$$\|u - g\|^q_{M^s_{p,q}} = \sum_{|l|\le 2\sqrt{d}+N} \langle l\rangle^{qs} \left\| \Box_l \sum_{|k|\le N} \Box_k (u - f) \right\|^q_p \lesssim_{d,N,q,s} \|u - f\|_p < \varepsilon.$$

Above, Implication (2.1) was used for the equality and Corollary 2.8 for the first inequality. Similarly, for $q = 0$, one has

$$\|u - g\|_{M^s_{p,\infty}} = \sup_{|l|\le 2\sqrt{d}+N} \langle l\rangle^{s} \left\| \Box_l \sum_{|k|\le N} \Box_k (u - f) \right\|_p \lesssim_{d,N,s} \|u - f\|_p < \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, the proof is complete. □

Next, $(M^s_{p,q}(\mathbb{R}^d))^* \simeq M^{-s}_{p',q'}(\mathbb{R}^d)$ for finite p, q will be shown. More precisely, one has the following

Proposition 2.17 (Duals of modulation spaces)**.** *(Cf. [WH07, Theorem 3.1]). Let $d \in \mathbb{N}$, $p \in [1, \infty)$, $q \in \{0\} \cup [1, \infty)$ and $s \in \mathbb{R}$. Then the map $\Phi : M^{-s}_{p',q'}(\mathbb{R}^d) \to (M^s_{p,q}(\mathbb{R}^d))^*$ defined by*

$$(\Phi u)(v) = \sum_{l\in\Lambda(d)} \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \overline{\Box_{k+l} u} \Box_k v \mathrm{d}x \qquad \forall v \in M^s_{p,q}(\mathbb{R}^d) \tag{2.13}$$

is antilinear, bijective and continuous (for $q = 0$, set $q' := 1$ in this proposition and its proof).

The proof employs the following

Lemma 2.18 (Adjoints of the IDOs)**.** *Let $d \in \mathbb{N}$, $p_1, p_2 \in [1, \infty)$ satisfy $p_1 \leq p_2 < \infty$ and $(\Box_k)_{k\in\mathbb{Z}^d}$ be a family of IDOs. Then*

$$\Box_k^* = \Box_k'|_{L^{p_2'}(\mathbb{R}^d)} \qquad \forall k \in \mathbb{Z}^d.$$

Above, the left-hand side is understood as $L^{p_2'}(\mathbb{R}^d) \to L^{p_1'}(\mathbb{R}^d)$ instead of $(L^{p_2}(\mathbb{R}^d))^* \to (L^{p_1}(\mathbb{R}^d))^*$ and the right-hand side involves the embedding $\Phi : L^{p_2'}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ as in Equation (A.24).

Proof. The continuity of $\Box_k : L^{p_1} \to L^{p_2}$ for all $k \in \mathbb{Z}^d$ was established in Lemma 2.7. As $\Box_k = \mathcal{F}^{(-1)}\sigma_k\mathcal{F}$, one has $\Box_k g \in \mathcal{S}$ for any $g \in \mathcal{S}$. By definition one has

$$\left\langle \overline{\Box_k^* f}, g\right\rangle_{L^{p_1'}\times L^{p_1}} = \left\langle \overline{f}, \Box_k g\right\rangle_{L^{p_2'}\times L^{p_2}} = \langle \Phi f, \Box_k g\rangle_{\mathcal{S}'\times\mathcal{S}} = \left\langle \Box_k' \Phi f, g\right\rangle_{\mathcal{S}'\times\mathcal{S}} \qquad \forall g \in \mathcal{S}.$$

As $\mathcal{S}$ is dense in L^{p_1} by Proposition A.33 ($p_1 < \infty$), the claim follows. $\Box$

Proof of Proposition 2.17. One has for any $u \in M^{-s}_{p',q'}$ and any $v \in M^s_{p,q}$

$$\begin{aligned} |(\Phi u)(v)| &\leq \sum_{l\in\Lambda(d)}\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\left|\overline{\Box_{k+l}u}\Box_k v \mathrm{d}x\right| \leq \sum_{k\in\mathbb{Z}^d}\langle k\rangle^{-s}\|\Box_k u\|_{p'}\langle k\rangle^s\|\Box_k v\|_p \\ &\leq \left\|\left(\|\Box_k u\|_{p'}\right)_k\right\|_{l^{q'}_{-s}}\cdot\left\|\left(\|\Box_k v\|_p\right)_k\right\|_{l^q_s} = \|u\|_{M^{-s}_{p',q'}}\|v\|_{M^s_{p,q}} \end{aligned}$$

by Hölder's inequality. This shows that Φ is well-defined. As antilinearity is obvious, it also shows the continuity of Φ. For the proof of the injectivity of Φ, observe that

$$u(v) = \sum_{l\in\Lambda}\sum_{k\in\mathbb{Z}^d}(\Box_{k+l}u)(\Box_k v) = (\Phi u)(v) \qquad \forall v \in \mathcal{S} \tag{2.14}$$

by Lemma 2.5 and Implication (2.1). Hence, if $\Phi(u)(v) = u(v) = 0$ for all $v \in \mathcal{S}$, then $u = 0$ in $\mathcal{S}'$ and hence indeed $u = 0$ in $M^{-s}_{p',q'} \hookrightarrow \mathcal{S}'$ by Proposition 2.10.

To show surjectivity, consider any $u \in (M^s_{p,q})'$. By Lemma 2.14, one has $u = u \circ S^s_{p,q} \circ A^s_{p,q}$. Clearly, $u \circ S^s_{p,q} \in (l^q_s(L^p))'$ and hence, by Proposition A.19, there is a sequence $(u_l) \in l^{q'}_{-s}(L^{p'})$ such that

$$u(v) = \sum_{l\in\mathbb{Z}^d}\int \overline{u_l}\Box_l v\mathrm{d}x \qquad \forall v \in M^s_{p,q}.$$

Clearly, $u|_{\mathcal{S}} \in \mathcal{S}'$ by Proposition 2.10. Moreover, restricting the above formula to $v \in \mathcal{S}$ and applying Lemma 2.18 yields

$$u|_{\mathcal{S}}(v) = \sum_{l\in\mathbb{Z}^d}\int\overline{\Box_l^* u_l}v\mathrm{d}x = \left(\sum_{l\in\mathbb{Z}^d}\Box_l' u_l\right)(v) \qquad \forall v \in \mathcal{S},$$

where the last series converges unconditionally in $\mathcal{S}'$. It remains to show that $u|_{\mathcal{S}} \in M^{-s}_{p',q'}$. By Implication (2.1) and Corollary 2.8, one has

$$\|\square_k u|_{\mathcal{S}}\|_{p'} \le \sum_{l\in\Lambda(d)} \left\|\square_k \square'_{k+l} u_{k+l}\right\|_{p'} \lesssim_d \sum_{l\in\Lambda(d)} \|u_{k+l}\|_{p'} \qquad \forall k \in \mathbb{Z}^d.$$

Taking the $l^{q'}_{-s}$-norm in the k-variable and invoking Peetre's inequality (Lemma A.31) yields

$$\|u|_{\mathcal{S}}\|_{M^{-s}_{p',q'}} \lesssim_{d,s} \|(u_l)\|_{l^{q'}_{-s}(L^{p'})} < \infty,$$

i.e. indeed $u|_{\mathcal{S}} \in M^{-s}_{p',q'}$. Furthermore, $\Phi(u|_{\mathcal{S}})(v) = u|_{\mathcal{S}}(v)$ for any $v \in \mathcal{S}$ by equation (2.14). As $\mathcal{S}$ is dense in $M^s_{p,q}$ by Proposition 2.15 one has $\Phi(u|_{\mathcal{S}}) = u$ and the proof is complete. □

This section concludes with the following

Proposition 2.19 (Complex interpolation). *(Cf. [Fei83, Theorem 6.1 (D)]). Let $p_0, p_1 \in [1,\infty]$, and $q_0, q_1 \in \{0\} \cup [1,\infty]$ such that $q_0 \neq \infty$ or $q_1 \neq \infty$. Furthermore, let $s_0, s_1 \in \mathbb{R}$ and $\theta \in (0,1)$. Define $s = (1-\theta)s_1 + \theta s_2 \in \mathbb{R}$ and $p \in [1,\infty]$ via*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Finally, define $q \in \{0\} \cup [1,\infty)$ via

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

in the case $q_0 \neq 0$ and $q_1 \neq 0$. For the other cases, set

$$q := \begin{cases} \frac{q_0}{1-\theta} & \text{for } q_0 \neq \infty \text{ and } q_1 = 0, \\ \frac{q_1}{\theta} & \text{for } q_0 = 0 \text{ and } q_1 \neq \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$[M^{s_0}_{p_0,q_0}(\mathbb{R}^d), M^{s_1}_{p_1,q_1}(\mathbb{R}^d)]_\theta = M^s_{p,q}(\mathbb{R}^d),$$

where the equality above means the equality of sets and equivalence of norms.

Main idea of the proof is to reduce the interpolation problem to the well-known case of $[l^{q_0}_{s_0}(L^{p_0}), l^{q_1}_{s_1}(L^{p_1})]_\theta = l^q_s(L^p)$, i.e. to recognize the analysis operator to be the coretraction belonging to the synthesis operator (cf. [Tri78, Section 1.2.4]).

Proof of Proposition 2.19. By Proposition 2.11, $M^{s_i}_{p_i,q_i}$ are Banach spaces for $i \in \{0,1\}$. Furthermore, by Proposition 2.10, one has $M^{s_i}_{p_i,q_i} \hookrightarrow \mathcal{S}'$ for $i \in \{0,1\}$. As $\mathcal{S}'$ is a Hausdorff vector space, $\left\{M^{s_0}_{p_0,q_0}, M^{s_1}_{p_1,q_1}\right\}$ is an interpolation couple and the notion of the complex interpolation space $[M^{s_0}_{p_0,q_0}, M^{s_1}_{p_1,q_1}]_\theta$ makes sense.

Define for the rest of this proof the Banach spaces

$$\begin{aligned} \Delta_M &:= M^{s_0}_{p_0,q_0} \cap M^{s_1}_{p_1,q_1}, & & \Delta_L := l^{q_0}_{s_0}(L^{p_0}) \cap l^{q_1}_{s_1}(L^{p_1}), \\ \Sigma_M &:= M^{s_0}_{p_0,q_0} + M^{s_1}_{p_1,q_1}, & & \Sigma_L := l^{q_0}_{s_0}(L^{p_0}) + l^{q_1}_{s_1}(L^{p_1}), \\ I_M &:= [M^{s_0}_{p_0,q_0}, M^{s_1}_{p_1,q_1}]_\theta, & \text{and} \quad & I_L := [l^{q_0}_{s_0}(L^{p_0}), l^{q_1}_{s_1}(L^{p_1})]_\theta. \end{aligned}$$

Observe, that by Example A.64, one has $I_L = l^q_s(L^p)$. Moreover, by Proposition A.60, Δ_M is dense in I_M and Δ_L is dense in I_L.

The analysis operators $A^{s_0}_{p_0,q_0}$ and $A^{s_1}_{p_1,q_1}$ agree on Δ_M. Hence, they uniquely extend to the continuous linear operator $\overline{A} : \Sigma_M \to \Sigma_L$ given by

$$\overline{A}(u) = \overline{A}(v + w) := A^{s_0}_{p_0,q_0}(v) + A^{s_1}_{p_1,q_1}(w) = (\Box_k u)_k \qquad \forall u \in \Sigma_M.$$

In the same way, the synthesis operators $S^{s_0}_{p_0,q_0}$ and $S^{s_1}_{p_1,q_1}$ uniquely extend to the continuous operator $\overline{S} : \Sigma_L \to \Sigma_M$ given by

$$\overline{S}((u_k)_k) := \sum_{k\in\mathbb{Z}^d} \sum_{l\in\Lambda} \Box_{k+l} u_k \qquad \forall (u_k)_k \in \Sigma_L.$$

As I_M and I_L are values of the same complex interpolation functor, one has $A := \overline{A}|_{I_M} \in \mathscr{L}(I_M, I_L)$ and $S := \overline{S}|_{I_L} \in \mathscr{L}(I_L, I_M)$ by Proposition A.60.

One has $(S \circ A)|_{\Delta_M} = \mathrm{id}_{\Delta_M}$ by definition. Due to Δ_M being dense in I_M, $S \circ A = \mathrm{id}_{I_M}$ follows. But then $A \circ S \in \mathscr{L}(I_L, I_M)$ is a continuous projection. One concludes, as in Lemma 2.14, that $A : I_M \to \mathrm{Im}(A)$ is bijective and $\mathrm{Im}(A)$ is a closed subspace of $l^q_s(L^p)$. Hence, I_M and $\mathrm{Im}(A)$ are isomorphic via A by the open mapping theorem (Proposition A.15).

Now one is in the position to compare the norms $\|\cdot\|_{I_M}$ and $\|\cdot\|_{M^s_{p,q}}$ on Δ_M. By the above, one has

$$\|u\|_{I_M} \approx \|Au\|_{l^q_s(L^p)} = \|(\Box_k u)\|_{l^q_s(L^p)} = \|u\|_{M^s_{p,q}} \qquad \forall u \in \Delta_M.$$

It remains to show that $\Delta_M \subseteq M^s_{p,q}$ and Δ_M is dense in $M^s_{p,q}$, because then

$$I_M = \overline{\Delta_M}^{\|\cdot\|_{I_M}} = \overline{\Delta_M}^{\|\cdot\|_{M^s_{p,q}}} = M^s_{p,q}$$

follows.

For the inclusion $\Delta_M \subseteq M^s_{p,q}$, consider any $u \in \mathcal{S}'$. Then

$$\langle k\rangle^s \|\Box_k u\|_p \le \left[\langle k\rangle^{s_0} \|\Box_k u\|_{p_0}\right]^{1-\theta} \left[\langle k\rangle^{s_1} \|\Box_k u\|_{p_1}\right]^{\theta} \qquad \forall k \in \mathbb{Z}^d$$

by the definition of s and Littlewood's inequality (A.9). Assume $q_0 \neq 0$ and $q_1 \neq 0$ for now. Then $q \in [1, \infty)$ and another application of Littlewood's inequality with the exponents $\frac{q_0}{(1-\theta)q}, \frac{q_1}{\theta q} \in [1, \infty]$ shows that

$$\begin{aligned} \|u\|_{M^s_{p,q}} &= \left\| \left(\langle k\rangle^s \|\Box_k u\|_p\right)_k \right\|_q \le \left\| \left(\left[\langle k\rangle^{s_0} \|\Box_k u\|_{p_0}\right]^{1-\theta} \left[\langle k\rangle^{s_1} \|\Box_k u\|_{p_1}\right]^{\theta} \right)_k \right\|_q \\ &\le \|u\|^{1-\theta}_{M^{s_0}_{p_0,q_0}} \|u\|^{\theta}_{M^{s_1}_{p_1,q_1}} \end{aligned}$$

holds. Recalling that for $i \in \{0,1\}$ the space $M^{s_i}_{p_i,0}$ is just a closed subset of $M^{s_i}_{p_i,\infty}$ (i.e. the norm is the same), shows that the above inequality also holds in the case where $q_0 = 0$ or $q_1 = 0$. If $q = 0$ one has $q_0, q_1 \in \{0, \infty\}$ and $q_i = 0$ for at least one $i \in \{0,1\}$. By the above,

$$\sup_{|j|\geq N} \langle k\rangle^s \|\square_k u\|_p \leq \left[\sup_{|j|\geq N} \langle k\rangle^{s_0} \|\square_k u\|_{p_0}\right]^{1-\theta} \left[\sup_{|j|\geq N} \langle k\rangle^{s_1} \|\square_k u\|_{p_1}\right]^{\theta} \xrightarrow{N\to\infty} 0$$

follows. All in all this shows that $\Delta_M \subseteq M^s_{p,q}$.

To show that Δ_M is dense in $M^s_{p,q}$, assume first that $p \in [1,\infty)$ and recall that $q \in \{0\} \cup [1,\infty)$. Hence, in this case, $\mathcal{S}$ is dense in $M^s_{p,q}$ by Proposition 2.15. Moreover, Proposition 2.10 implies that $\mathcal{S} \subseteq M^{s_i}_{p_i,q_i}$ for $i \in \{0,1\}$. Hence, one has

$$M^s_{p,q} = \overline{\mathcal{S}}^{\|\cdot\|_{M^s_{p,q}}} \subseteq \overline{M^{s_0}_{p_0,q_0} \cap M^{s_1}_{p_1,q_1}}^{\|\cdot\|_{M^s_{p,q}}} \subseteq M^s_{p,q}$$

as claimed. In the other case $p = \infty$, one has that $p = p_0 = p_1 = \infty$. Define

$$D := \left\{u \in L^p \,\middle|\, \operatorname{supp}(\mathcal{F}u) \text{ is compact}\right\}$$

Then Corollary 2.8 implies that $D \subseteq M^{s_i}_{p,q_i}$ for $i \in \{0,1\}$. Moreover, D is dense in $M^s_{p,q}$ by Lemma 2.16. This concludes the proof. □

Observe, that the above proof relied only upon knowing the interpolation space I_L and the fact that Δ_M is dense in $M^s_{p,q}$. More interpolation spaces I_L are mentioned after the proof of Example A.64. Also $D \subseteq \Delta_M$ is dense in any modulation space $M^{\tilde{s}}_{\tilde{p},\tilde{q}}$ with $\tilde{q} < \infty$. One obtains the following result, which is not covered by Proposition 2.19. For $p \in [1,\infty]$, $s_0, s_1 \in \mathbb{R}$ with $s_0 \neq s_1$ and $\theta \in (0,1)$ one has

$$[M^{s_0}_{p,\infty}(\mathbb{R}^d), M^{s_1}_{p,\infty}(\mathbb{R}^d)]_\theta = M^s_{p,0}(\mathbb{R}^d).$$

Of course, if $s_0 = s_1 = s \in \mathbb{R}$, then

$$[M^{s_0}_{p,\infty}(\mathbb{R}^d), M^{s_0}_{p,\infty}(\mathbb{R}^d)]_\theta = M^s_{p,\infty}(\mathbb{R}^d)$$

by Proposition A.60.

2.2. Characterization via the Short-time Fourier-Transform

Suppose one is to study the "local" frequency distribution of a "nice" function f near a point $x \in \mathbb{R}^d$. One idea is to cut out a neighbourhood of x with a smooth window function $g \in \mathcal{S}(\mathbb{R}^d)$ and take the usual Fourier transform of the result, i.e. (see Figure 2.1).

$$\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(y)g(y-x)e^{-\mathrm{i}k\cdot y}\mathrm{d}y \qquad \forall x,k \in \mathbb{R}^d. \tag{2.15}$$

To make sense of this formula in $\mathcal{S}'(\mathbb{R}^d)$, consider the (continuous) *right-shift* and *modulation* operators on $\mathcal{S}(\mathbb{R}^d)$ defined via

$$(S_x f)(y) = f(y - x) \qquad \text{and} \qquad (M_k f)(y) = e^{-iky} f(y)$$

respectively, where $f \in \mathcal{S}(\mathbb{R}^d)$ and $k, x, y \in \mathbb{R}^d$. Ignoring the constant $(2\pi)^{-\frac{d}{2}}$ and the lack of complex conjugation on f, equation (2.15) leads to the following

Definition 2.20 (Short-time Fourier transform)**.** (Cf. [Grö01, Section 3.1]). Let $d \in \mathbb{N}$, $u \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. Define the *short-time Fourier transform* $V_g u : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ of u w.r.t. the *window function* g through

$$V_g u(x, k) = \langle u, M_k S_x g \rangle \qquad \forall x, k \in \mathbb{R}^d. \tag{2.16}$$

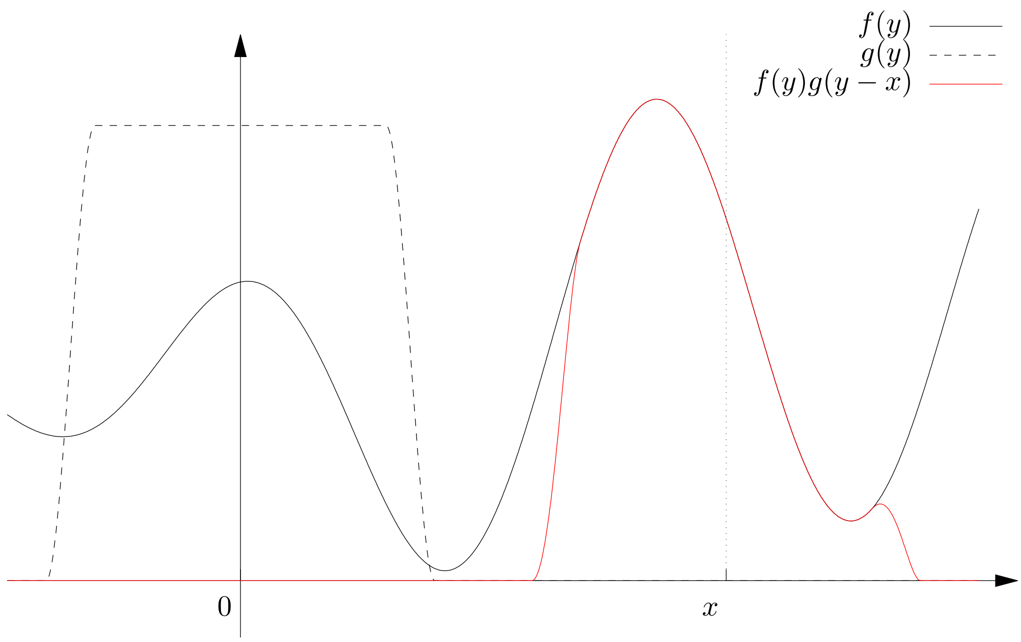


Figure 2.1.: Localization of functions.

Example 2.21 (STFT with a Gaussian window of a complex Gaussian). Let $d \in \mathbb{N}$. For any $\alpha \in \mathbb{C}$ define the complex Gaussian $f_\alpha : \mathbb{R}^d \to \mathbb{C}$ by

$$f_\alpha(x) = e^{-\frac{|x|^2}{2\alpha}} \qquad \forall x \in \mathbb{R}^d$$

and put $g \coloneqq f_1 \in \mathcal{S}(\mathbb{R}^d)$. Then, if $\mathrm{Re}(\alpha) > 0$, $f_\alpha \in \mathcal{S}(\mathbb{R}^d)$ and

$$(V_g f)(x, k) = \left(\sqrt{\frac{2\pi\alpha}{\alpha + 1}} \right)^d e^{-\frac{1}{2(\alpha+1)}|x|^2 - i\frac{\alpha}{\alpha+1}kx - \frac{\alpha}{2(\alpha+1)}|k|^2} \qquad \forall k, x \in \mathbb{R}^d. \tag{2.17}$$

Proof. Inserting f and g into the definition (2.16) confirms

$$\begin{aligned}(V_g f_\alpha)(x,k) &= \int_{\mathbb{R}^d} e^{-\frac{|y|^2}{2\alpha}} e^{-\mathrm{i}ky} e^{-\frac{|y-x|^2}{2}} \mathrm{d}y = \prod_{j=1}^d \int_{-\infty}^{\infty} e^{-\left(\frac{1}{2}+\frac{1}{2\alpha}\right)y_j^2+(x_j-\mathrm{i}k_j)y_j-\frac{x_j^2}{2}} \mathrm{d}y_j \\ &= \prod_{j=1}^d \int_{-\infty}^{\infty} e^{-\left(\sqrt{\frac{\alpha+1}{2\alpha}}y_j-\sqrt{\frac{\alpha}{2(\alpha+1)}}(x_j-\mathrm{i}k_j)\right)^2+\frac{\alpha}{2(\alpha+1)}(x_j-\mathrm{i}k_j)^2-\frac{x_j^2}{2}} \mathrm{d}y_j \\ &= \prod_{j=1}^d e^{-\frac{1}{2(\alpha+1)}x_j^2-\mathrm{i}\frac{\alpha}{\alpha+1}k_jx_j-\frac{\alpha}{2(\alpha+1)}k_j^2} I\left(\sqrt{\frac{\alpha+1}{2\alpha}}, -\sqrt{\frac{\alpha}{2(\alpha+1)}}(x_j-\mathrm{i}k_j)\right) \\ &= \left(\sqrt{\frac{2\pi\alpha}{\alpha+1}}\right)^d e^{-\frac{1}{2(\alpha+1)}|x|^2-\mathrm{i}\frac{\alpha}{\alpha+1}kx-\frac{\alpha}{2(\alpha+1)}|k|^2} \qquad \forall x,k\in\mathbb{R}^d,\end{aligned}$$

where formula (A.3) for Gaussian integrals was used in the last equality. □

Lemma 2.22 (Properties of $V_g u$)**.** *(See. [Grö01, Theorem 11.2.3]). Let $d \in \mathbb{N}$, $u \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. Then $V_g u \in C(\mathbb{R}^d \times \mathbb{R}^d)$ and there are $N \in \mathbb{N}_0$ and $C > 0$ such that*

$$|V_g u(x,k)| \le C\,(1+|x|+|k|)^N \qquad \forall x,k\in\mathbb{R}^d.$$

Definition 2.23 (Modulation spaces via STFT)**.** (See [Grö01, Definition 11.3.1]). Let $d \in \mathbb{N}$, $p,q \in [1,\infty]$, $s \in \mathbb{R}$ and $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. Define the *modulation space norm w.r.t. the STFT with window g* by

$$\|u\|_{\mathring{M}^s_{p,q}(\mathbb{R}^d)} = \left\| k \mapsto \langle k\rangle^s \left\|V_g u(\cdot,k)\right\|_p \right\|_q \qquad \forall u \in \mathcal{S}'(\mathbb{R}^d).$$

Observe, that $k \mapsto \left\|\mathbb{1}_{[-n,n]^d}(\cdot)V_g u(\cdot,k)\right\|_p$ are continuous and converge pointwise to $k \mapsto \left\|V_g u(\cdot,k)\right\|_p$ as $n \to \infty$. Hence, $k \mapsto \langle k\rangle^s \left\|V_g u(\cdot,k)\right\|_p$ is measureable and taking its L^q-norm is justified. Define the *modulation space w.r.t. the STFT with window g* through

$$\mathring{M}^s_{p,q}(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d) \middle| \, \|u\|_{\mathring{M}^s_{p,q}(\mathbb{R}^d)} < \infty \right\}.$$

Note that the modulation spaces, which include the initial values of the model problem from the introduction (see (1.2)) are those with $p = \infty$.

Proposition 2.24 ($\mathring{M}^s_{p,q}(\mathbb{R}^d)$ is independent of the window function)**.** *(Cf. [Grö01, Proposition 11.3.2 (c), Theorem 11.3.5(a)]). Let $d \in \mathbb{N}$, $p,q \in [1,\infty]$, $s \in \mathbb{R}$ and $g_1, g_2 \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. Denote, for $j \in \{1,2\}$, by X_j the modulation space w.r.t. the STFT with window g_j and by $\|\cdot\|_i$ its norm. Then $X_1 = X_2$ as sets and the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. More precisely, one has*

$$\|u\|_1 \le \frac{2^{|s|}}{\|g_2\|^2_{L^2(\mathbb{R}^d)}} \left\| (x,k) \mapsto \langle k\rangle^{|s|} \cdot (V_{g_1} g_2)(x,k) \right\|_{L^1(\mathbb{R}^{2d})} \|u\|_2 \qquad \forall u \in \mathring{M}^s_{p,q}(\mathbb{R}^d). \quad (2.18)$$

Finally, $\mathring{M}^s_{p,q}(\mathbb{R}^d)$ (equipped with any of the aforementioned norms) is a Banach space.

Example 2.25 (Gaussians). Let $d \in \mathbb{N}$, $p, q \in [1, \infty]$ and $s \in \mathbb{R}$. Furthermore, let $\alpha \in \mathbb{C}$ such that $\mathrm{Re}(\alpha) > 0$ and the complex Gaussian f_α be as in Example 2.21. Then $f_\alpha \in \mathring{M}^s_{p,q}(\mathbb{R}^d)$ and

$$\|f_\alpha\|_{\mathring{M}^s_{p,q}(\mathbb{R}^d)} \approx_d |\alpha|^{\frac{d}{2}} |\alpha+1|^{d\left(\frac{1}{p}-\frac{1}{2}\right)} \mathrm{Re}(\alpha+1)^{-\frac{d}{2p}} \left\| \langle \cdot \rangle^s e^{-\frac{\mathrm{Re}(\alpha)}{\mathrm{Re}(\alpha)+1}\frac{|\cdot|^2}{2}} \right\|_q . \tag{2.19}$$

Proof. By Proposition 2.24, one may assume w.l.o.g. that $g = f_1$. For this case $V_g f_\alpha$ has been already calculated in example 2.21. To obtain $|V_g f_\alpha(x,k)|$ for $x, k \in \mathbb{R}^d$, one needs to figure out the real part of the exponent in equation 2.17. One has

$$\begin{aligned} &\mathrm{Re}\left[\frac{1}{2(\alpha+1)}|x|^2 + \mathrm{i}\frac{\alpha}{\alpha+1}kx + \frac{\alpha}{2(\alpha+1)}|k|^2\right] \\ &= \sum_{j=1}^d \left[\frac{\mathrm{Re}(\alpha)+1}{2|\alpha+1|^2}x_j^2 - \frac{\mathrm{Im}(\alpha)}{|\alpha+1|^2}k_j x_j + \frac{|\alpha|^2+\mathrm{Re}(\alpha)}{2|\alpha+1|^2}k_j^2\right] \end{aligned}$$

for any $x, k \in \mathbb{R}^d$. For the subsequent calculation of the L^p-norm in the variable x it is appropriate to complete the squares w.r.t. x_j which yields

$$\sum_{j=1}^d \frac{1}{2|\alpha+1|^2}\left[\left(\sqrt{\mathrm{Re}(\alpha)+1}\,x_j - \frac{\mathrm{Im}(\alpha)k_j}{\sqrt{\mathrm{Re}(\alpha)+1}}\right)^2 + \left(|\alpha|^2 + \mathrm{Re}(\alpha) - \frac{\mathrm{Im}(\alpha)^2}{\mathrm{Re}(\alpha)+1}\right)k_j^2\right].$$

The last summand above can be further simplified to

$$\begin{aligned} |\alpha|^2 + \mathrm{Re}(\alpha) - \frac{\mathrm{Im}(\alpha)^2}{\mathrm{Re}(\alpha)+1} &= |\alpha+1|^2 - \left(\mathrm{Re}(\alpha)+1+\frac{\mathrm{Im}(\alpha)^2}{\mathrm{Re}(\alpha)+1}\right) \\ &= |\alpha+1|^2\left(1 - \frac{1}{\mathrm{Re}(\alpha)+1}\right). \end{aligned}$$

Inserting this into (2.17) yields

$$|V_g f_\alpha(x,k)| = \left(2\pi\left|\frac{\alpha}{\alpha+1}\right|\right)^{\frac{d}{2}} \prod_{j=1}^d e^{-\frac{1}{2|\alpha+1|^2}\left(\sqrt{\mathrm{Re}(\alpha)+1}\,x_j - \frac{\mathrm{Im}(\alpha)}{\sqrt{\mathrm{Re}(\alpha)+1}}k_j\right)^2 - \frac{\mathrm{Re}(\alpha)}{\mathrm{Re}(\alpha)+1}\frac{k_j^2}{2}}$$

for all $x, k \in \mathbb{R}^d$. Hence, for $p = \infty$, one has

$$\|V_g f_\alpha(\cdot,k)\|_\infty = \left(2\pi\left|\frac{\alpha}{\alpha+1}\right|\right)^{\frac{d}{2}} e^{-\frac{\mathrm{Re}(\alpha)}{\mathrm{Re}(\alpha)+1}\frac{|k|^2}{2}},$$

whereas for $p < \infty$ one has

$$\|V_g f_\alpha(\cdot,k)\|_p = \|V_g f_\alpha(\cdot,k)\|_\infty \left(\prod_{j=1}^d \int_{-\infty}^{\infty} e^{-\frac{p}{2|\alpha+1|^2}\left(\sqrt{\mathrm{Re}(\alpha)+1}\,x_j - \frac{\mathrm{Im}(\alpha)}{\sqrt{\mathrm{Re}(\alpha)+1}}k_j\right)^2} \mathrm{d}x_j\right)^{\frac{1}{p}}$$

for any $k \in \mathbb{R}^d$. The integral above is a Gaussian integral (see example A.3) and has the value

$$\int_{-\infty}^{\infty} e^{-\frac{p}{2|\alpha+1|^2}\left(\sqrt{\mathrm{Re}(\alpha)+1}\,x_j - \frac{\mathrm{Im}(\alpha)}{\sqrt{\mathrm{Re}(\alpha)+1}}k_j\right)^2} \mathrm{d}x_j = |\alpha+1|\sqrt{\frac{2\pi}{p(\mathrm{Re}(\alpha)+1)}}.$$

Reinserting this number into the formula for $\|V_g f_\alpha(\cdot,k)\|_p$ and subsequently taking the weighted L^q-norm yields

$$\|f_\alpha\|_{\mathring{M}^s_{p,q}(\mathbb{R}^d)} = (2\pi)^{\frac{d}{2}\left(1+\frac{1}{p}\right)} p^{-\frac{d}{2p}} |\alpha|^{\frac{d}{2}} |\alpha+1|^{d\left(\frac{1}{p}-\frac{1}{2}\right)} \operatorname{Re}(\alpha+1)^{-\frac{d}{2p}} \left\| \langle\cdot\rangle^s e^{-\frac{\operatorname{Re}(\alpha)}{\operatorname{Re}(\alpha)+1}\frac{|\cdot|^2}{2}} \right\|_q .$$

Observing that the first two factors can be controlled independently of p finishes the proof. □

Observe the *fundamental identity of time-frequency analysis*

$$V_g f(x,k) = e^{\mathrm{i}k\cdot x} V_{\hat{g}} \hat{f}(k,-x), \tag{2.20}$$

where $k, x \in \mathbb{R}^d$ and the identity,

$$(V_g f)(x,k) = (\mathcal{F} S_k g f)(k) \qquad \forall x, k \in \mathbb{R}^d, \tag{2.21}$$

which is understood in the sense that for every fixed $x \in \mathbb{R}^d$ the tempered distribution on the right-hand side can be represented as a function given by the left-hand side.

Proposition 2.26 ($M^s_{p,q} = \mathring{M}^s_{p,q}$)**.** *(Cf. [WH07, Proposition 2.1]). Let $d \in \mathbb{N}$, $p, q \in [1,\infty]$, $s \in \mathbb{R}$ and $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. Then*

$$\|u\|_{M^s_{p,q}(\mathbb{R}^d)} \approx \|u\|_{\mathring{M}^s_{p,q}(\mathbb{R}^d)} \qquad \forall u \in \mathcal{S}'(\mathbb{R}^d) \tag{2.22}$$

and hence $M^s_{p,q} = \mathring{M}^s_{p,q}$ as Banach spaces.

Proof. Let $u \in \mathcal{S}'$, g denote the window function for $\mathring{M}^s_{p,q}$ and (σ_m) the family of IDOs for $M^s_{p,q}$. By Propositions 2.24, 2.9 and Example 2.2 one may assume w.l.o.g. that $\hat{g}$ has compact support, $\hat{g}(\xi) = 1$ for all $\xi \in \operatorname{supp}(\sigma_0) + Q_0$ and $\sigma_k = S_k \sigma_0$ for all $k \in \mathbb{Z}^d$, e.g. $\hat{g} = \sum_{l \in \Lambda''} \sigma_l$, where $\Lambda'' = \left\{ l \in \mathbb{Z}^d \middle| \, |l| \le \frac{5}{2}\sqrt{d} \right\}$.

Combining (2.20) and (2.21) yields

$$|V_g f(x,k)| = \left| (\mathcal{F}^{(-1)}(S_k \hat{g}) \mathcal{F} f)(x) \right| \qquad \forall k, x \in \mathbb{R}^d. \tag{2.23}$$

As $\hat{g}$ is compactly supported, there is a finite set $\Lambda''' \subseteq \mathbb{Z}^d$ such that $\sum_{l \in \Lambda'''} \sigma_l(\xi) = 1$ for any $\xi \in \operatorname{supp}(\hat{g}) + Q_0$. Thus, for any $m \in \mathbb{Z}^d$ and $k \in Q_m$ one has

$$\|V_g u(\cdot,k)\|_p = \left\| \mathcal{F}^{(-1)}(S_k \hat{g}) \mathcal{F}^{(-1)} \mathcal{F} \sum_{l \in \Lambda''} \sigma_{m+l} \mathcal{F} u \right\|_p \lesssim_d \sum_{l \in \Lambda''} \|\Box_{m+l} u\|_p , \tag{2.24}$$

where Bernstein multiplier estimate (Corollary A.53). Similarly, the converse estimate

$$\|\Box_m u\|_p = \left\| \mathcal{F}^{(-1)} \sigma_m S_k \hat{g} \mathcal{F} u \right\|_p \lesssim_d \|V_g u(\cdot,k)\|_p \qquad \forall m \in \mathbb{Z}^d, \forall k \in Q_m \tag{2.25}$$

holds.

Consider first the case $q < \infty$. Then

$$\|u\|_{\mathring{M}^s_{p,q}} = \left(\sum_{m\in\mathbb{Z}^d} \int_{Q_m} \langle k\rangle^{sq} \|V_g u(\cdot,k)\|_p^q \,\mathrm{d}k \right)^{\frac{1}{q}} \tag{2.26}$$
$$\approx_{d,s,q} \left(\sum_{m\in\mathbb{Z}^d} \langle m\rangle^{sq} \int_{Q_m} \|V_g u(\cdot,k)\|_p^q \,\mathrm{d}k \right)^{\frac{1}{q}}$$

by Peetre's inequality (Lemma A.31). Inserting (2.24), yields

$$\|u\|_{\mathring{M}^s_{p,q}} \lesssim_{d,s,q} \left(\sum_{m\in\mathbb{Z}^d} \langle m\rangle^{sq} \left(\sum_{l\in\Lambda'''} \|\Box_{m+l} u\|_p \right)^q \right)^{\frac{1}{q}}$$
$$\leq (\#\Lambda''')^{\frac{1}{q'}} \left(\sum_{l\in\Lambda'''} \sum_{m\in\mathbb{Z}^d} \langle m\rangle^{sq} \|\Box_{m+l} u\|_p^q \right)^{\frac{1}{q}}$$
$$\lesssim_{d,s,q} \left(\sum_{m\in\mathbb{Z}^d} \langle m\rangle^{sq} \|\Box_m u\|_p^q \right)^{\frac{1}{q}} = \|u\|_{M^s_{p,q}},$$

where Hölder's inequality for the sum over Λ''' was used for the second estimate and Peetre's and triangle inequalities for the last.

Inserting (2.25) into (2.26) immediately yields the converse estimate

$$\|u\|_{\mathring{M}^s_{p,q}} \gtrsim_{d,s,q} \left(\sum_{m\in\mathbb{Z}^d} \langle m\rangle^{sq} \|\Box_m u\|_p^q \right)^{\frac{1}{q}} = \|u\|_{M^s_{p,q}}.$$

For $q = \infty$ the equation (2.26) is replaced by

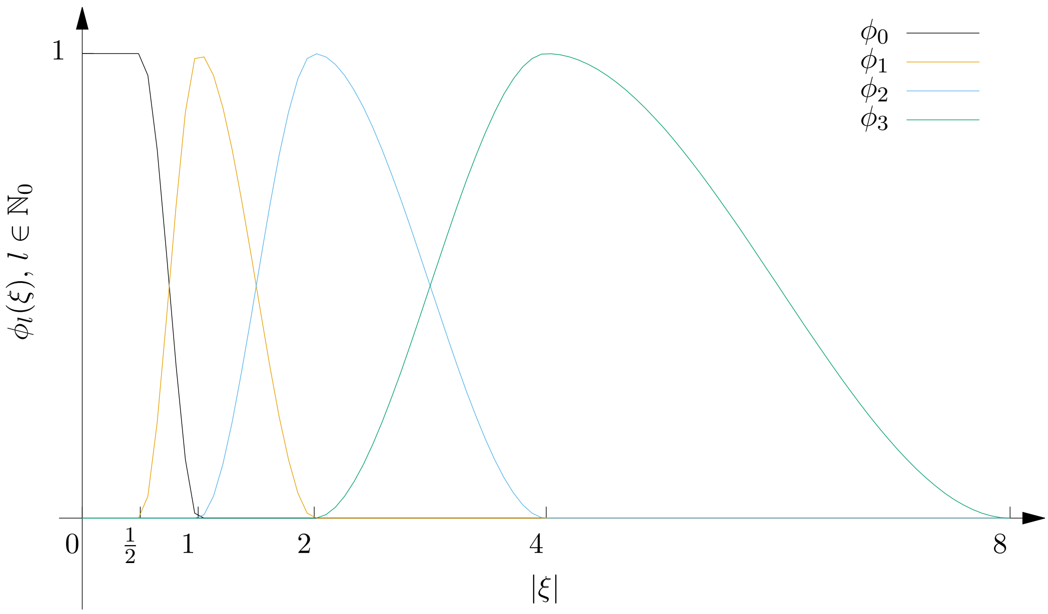
$$\|u\|_{\mathring{M}^s_{p,q}} = \sup_{m\in\mathbb{Z}^d} \sup_{k\in Q_m} \langle k\rangle^s \|V_g u(\cdot,k)\|_p \approx_{d,q,s} \sup_{m\in\mathbb{Z}^d} \langle m\rangle^s \sup_{k\in Q_m} \|V_g u(\cdot,k)\|_p .$$

Similarly to the case $q < \infty$, equation (2.24) together with Peetre's inequality and equation (2.25) yield the desired estimates. This concludes the proof. □

2.3. Characterization via the Littlewood-Paley decomposition

In this section, some ideas of the Littlewood-Paley decomposition for Sobolev spaces $H^s(\mathbb{R}^d)$ are carried over to modulation spaces $M^s_{p,q}(\mathbb{R}^d)$. The inspiration for this was [AG07, Chapter II].

Figure 2.2.: Symbols of the dyadic decomposition operators.



Definition 2.27 (Dyadic decomposition operators)**.** Let $d \in \mathbb{N}$ and $\phi_0 \in C_c^\infty(\mathbb{R}^d)$ with $\phi_0(\xi) = 1$ for all $|\xi| \leq \frac{1}{2}$ and $\mathrm{supp}(\phi_0) \subseteq B_1(0)$. Set $\phi_1 = \phi_0\left(\frac{\cdot}{2}\right) - \phi_0$ and $\phi_l = \phi_1\left(\frac{\cdot}{2^{l-1}}\right)$ for all $l \in \mathbb{N}$ (see figure 2.2). Observe, that for any $\xi \in \mathbb{R}^d$ one has

$$\sum_{l=0}^{\infty} \phi_l(\xi) = \phi_0(\xi) + \lim_{N\to\infty} \sum_{l=1}^{N} \left[\phi_1\left(\frac{\xi}{2^l}\right) - \phi_1\left(\frac{\xi}{2^{l-1}}\right)\right] = \lim_{N\to\infty} \phi_0\left(\frac{\xi}{2^N}\right) = 1,$$

i.e. $(\phi_l)_{l\in\mathbb{N}_0}$ is a smooth partition of unity. Then the sequence of operators $(\Delta_l)_{l\in\mathbb{N}_0}$ defined through

$$\Delta_l := \mathcal{F}^{(-1)} \phi_l \mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d) \qquad \forall l \in \mathbb{N}_0$$

is said to be a family of *dyadic decomposition operators* (DDOs).

For the rest of this section, set

$$A_0 = \left\{\xi \in \mathbb{R}^d |\ |\xi| \leq 1\right\}, \qquad A_l := \left\{\xi \in \mathbb{R}^d |\ 2^{l-2} \leq |\xi| \leq 2^l\right\}\ \forall l \in \mathbb{N}.$$

Observe, that $\mathrm{supp}(\phi_l) \subseteq A_l$ for any $l \in \mathbb{N}_0$. Hence, one has

$$|l - m| \geq 2 \Rightarrow \Delta_l \Delta_m = 0 \qquad \forall l, m \in \mathbb{N}_0 \tag{2.27}$$

analogously to Implication (2.1). Similarly to Lemma 2.5, one shows that the series $\sum_{l=0}^{\infty} \Delta_l$ converges strongly unconditionally to id in $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$. As for IDOs, one has that $\Delta_l u \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$ for any $u \in \mathcal{S}'(\mathbb{R}^d)$. Finally, one has the following equivalent of Corollary 2.8 for DDOs.

Lemma 2.28 (DDOs on a Lebesgue space). *Let $d \in \mathbb{N}$ and $p \in [1,\infty]$. Then for any family of DDOs $(\Delta_l)_{l\in\mathbb{N}_0}$ there exists a $C > 0$ such that for any $p \in [1,\infty]$ one has*

$$\|\Delta_l\|_{\mathscr{L}(L^p(\mathbb{R}^d))} \leq C \qquad \forall l \in \mathbb{N}_0.$$

Proof. By Lemma A.46 (put $p_1 = p_2 = p$ there), one immediately has

$$\|\Delta_l\|_{\mathscr{L}(L^p)} \leq \|\phi_l\|_{\mathcal{F}L^1} \qquad \forall l \in \mathbb{N}_0.$$

By the properties of the Fourier transform and change of variables one obtains

$$\|\phi_l\|_{\mathcal{F}L^1} = \left\|\mathcal{F}\left(\delta^{2^{-(l-1)}}\phi_1\right)\right\|_1 = \left\|2^{(l-1)}\delta^{2^{(l-1)}}\hat{\phi}_1\right\|_1 = \left\|\hat{\phi}_1\right\|_1 \qquad \forall l \in \mathbb{N}.$$

The right-hand side above is a finite number independent of l and so the proof is complete. □

The main result of this section is the following

Theorem 2.29 (Littlewood-Paley characterization of $M^s_{p,q}$). *Let $d \in \mathbb{N}$, $p, q \in [1,\infty]$ and $s \in \mathbb{R}$. Then*

$$\|u\| = \left\|\left(2^{ls}\|\Delta_l u\|_{M_{p,q}(\mathbb{R}^d)}\right)_{l\in\mathbb{N}_0}\right\|_q \qquad \forall u \in \mathcal{S}'(\mathbb{R}^d)$$

is an equivalent norm for $M^s_{p,q}(\mathbb{R}^d)$. The constants of the norm equivalence depend only on d and s.

Proof. Fix an $l \in \mathbb{N}_0$ and a $k \in \mathbb{Z}^d$. Recall, that $\operatorname{supp}(\phi_l) \subseteq A_l$ and $\operatorname{supp}(\sigma_k) \subseteq B_{\sqrt{d}}(k)$ and hence

$$k \notin A'_l := \left\{k' \in \mathbb{Z}^d \middle| \, |k'| \in \left(2^{l-2} - \sqrt{d}, 2^l + \sqrt{d}\right)\right\} \Rightarrow \Box_k \Delta_l = 0. \tag{2.28}$$

Peetre's inequality (Lemma A.31) implies

$$\langle k\rangle^t \approx_{d,t} 2^{lt}. \tag{2.29}$$

Finally, by definition of A'_l, one has

$$\sum_{l=0}^{\infty} \mathbb{1}_{A'_l}(k) \lesssim_d 1. \tag{2.30}$$

Fix a $u \in \mathcal{S}'(\mathbb{R}^d)$. In the following, $\|\cdot\| \lesssim \|\cdot\|_{M^s_{p,q}}$ will be shown. Consider first the case $q < \infty$. Then, one indeed has

$$\begin{aligned}\|u\|^q \quad &= \quad \sum_{l=0}^{\infty} 2^{lqs}\|\Delta_l u\|^q_{M_{p,q}} = \sum_{l=0}^{\infty} 2^{lqs}\sum_{k\in\mathbb{Z}^d}\|\Box_k\Delta_l u\|^q_p \lesssim \sum_{k\in\mathbb{Z}^d}\sum_{l=0}^{\infty}\mathbb{1}_{A'_l}(k)2^{lqs}\|\Box_k u\|^q_p \\ &\lesssim_{d,q,s} \sum_{k\in\mathbb{Z}^d}\langle k\rangle^{qs}\|\Box_k u\|^q_p = \|u\|^q_{M^s_{p,q}},\end{aligned}$$

where Implication 2.28 and Lemma 2.28, was used for the first estimate and equations (2.29) and (2.30) for the second. Similarly, for $q = \infty$, one has

$$\|u\| = \sup_{l\in\mathbb{N}_0} 2^{ls} \sup_{k\in\mathbb{Z}^d} \|\Delta_l \Box_k u\|_p \lesssim \sup_{k\in\mathbb{Z}^d} \sup_{l\in\mathbb{N}_0} \mathbb{1}_{A_l'}(k) 2^{ls} \|\Box_k u\|_p \lesssim_{d,s} \sup_{k\in\mathbb{Z}^d} \langle k\rangle^s = \|u\|_{M^s_{p,\infty}} .$$

It remains to show $\|\cdot\|_{M_{p,q}} \lesssim \|\cdot\|$. As mentioned above, $u = \sum_{l=0}^{\infty} \Delta_l u$ in $\mathcal{S}'$ and hence triangle inequality yields

$$\|u\|_{M^s_{p,q}} \leq \left\| \left(\langle k\rangle^s \sum_{l=0}^{\infty} \|\Box_k \Delta_l u\|_p \right)_k \right\|_q \lesssim_{d,s} \left\| \left(\sum_{l=0}^{\infty} 2^{ls} \mathbb{1}_{A_l'}(k) \|\Box_k \Delta_l u\|_p \right)_k \right\|_q ,$$

where additionally Implication (2.28), equation (2.29) were used for the second estimate. Consider again the case $q < \infty$ first. Then, Hölder's inequality for the variable l and the estimate (2.30), yield

$$\begin{aligned}
\left\| \left(\sum_{l=0}^{\infty} 2^{ls} \mathbb{1}_{A_l'}(k) \|\Box_k \Delta_l u\|_p \right)_k \right\|_q^q &= \sum_{k\in\mathbb{Z}^d} \left(\sum_{l=0}^{\infty} \mathbb{1}_{A_l'}(k) 2^{ls} \|\Box_k \Delta_l u\|_p \right)^q \\
&\lesssim_{d,q} \sum_{k\in\mathbb{Z}^d} \sum_{l=0}^{\infty} 2^{lqs} \|\Box_k \Delta_l u\|_p^q = \sum_{l=0}^{\infty} 2^{lqs} \|\Delta_l u\|_{M_{p,q}}^q \\
&= \|u\|^q .
\end{aligned}$$

Similarly, for $q = \infty$, one has

$$\begin{aligned}
\left\| \left(\sum_{l=0}^{\infty} 2^{ls} \mathbb{1}_{A_l'}(k) \|\Box_k \Delta_l u\|_p \right)_k \right\|_\infty &= \sup_{k\in\mathbb{Z}^d} \sum_{l=0}^{\infty} \mathbb{1}_{A_l'}(k) 2^{ls} \|\Box_k \Delta_l u\|_p \\
&\leq \sup_{k\in\mathbb{Z}^d} \left(\sum_{l=0}^{\infty} \mathbb{1}_{A_l'}(k) \right) \sup_{l'\in\mathbb{N}_0} 2^{l's} \|\Box_k \Delta_{l'} u\|_p \\
&\lesssim_d \sup_{l\in\mathbb{N}_0} 2^{ls} \sup_{k\in\mathbb{Z}^d} \|\Box_k \Delta_l u\|_p = \sup_{l\in\mathbb{N}_0} 2^{ls} \|\Delta_l u\|_{M_{p,\infty}} \\
&= \|u\|
\end{aligned}$$

due to the estimate (2.30).

Rechecking the implicit constants in the estimates above shows the claimed dependence on d and s only. This finishes the proof. □

The components $\Delta_l u$ of the Littlewood-Paley decomposition of $u \in \mathcal{S}'$ had their Fourier transform supported in "almost disjoint" dyadic annuli and Theorem 2.29 characterized elements of a modulation space $M^s_{p,q}$ by the decay of the $M_{p,q}$-norm of those components. The following lemma provides a sufficient condition for $u \in \mathcal{S}'$ to be an element of $M^s_{p,q}$ for any decomposition of u for which the Fourier transform of the individual components is supported in non-disjoint dyadic balls.

Lemma 2.30 (Sufficient condition)**.** *Let* $p \in [1, \infty]$*,* $q \in [1, \infty)$ *and* $s > 0$*. For each* $m \in \mathbb{N}_0$ *let* $u_m \in M_{p,q}(\mathbb{R}^d)$ *be such that* $\operatorname{supp}(\hat{u}_m) \subseteq B_m := \left\{\xi \in \mathbb{R}^d \middle| |\xi| \leq 2^m\right\}$ *and assume*

$$\left\| \left(2^{ms} \|u_m\|_{M_{p,q}(\mathbb{R}^d)}\right)_{m\in\mathbb{N}_0} \right\|_q < \infty. \tag{2.31}$$

Then the series $u := \sum_{m=0}^{\infty} u_m$ *converges in* $M^s_{p,q}(\mathbb{R}^d)$*. Moreover, there is a constant* $C = C(d, s)$ *such that*

$$\|u\|_{M^s_{p,q}} \leq C \left\| \left(2^{ms} \|u_m\|_{M_{p,q}(\mathbb{R}^d)}\right)_{m\in\mathbb{N}_0} \right\|_q . \tag{2.32}$$

If $2^{ms} \|u_m\|_{M_{p,\infty}(\mathbb{R}^d)} \xrightarrow{m\to\infty} 0$*, or if the series defining* u *converges in* $M^s_{p,\infty}(\mathbb{R}^d)$ *and* (2.31) *holds for* $q = \infty$*, then the above conclusions are true with* $q = \infty$*.*

Proof. Assume for now, that the series $\sum_{m=0}^{\infty} u_m$ converges in $M^s_{p,q}$. To show is the bound (2.32). Observe, that $A_l \cap B_m = \emptyset$ if $l > m + 2$. One has

$$\begin{aligned} \|u\|_{M^s_{p,q}} \quad &\approx_{d,s} \quad \left\| \left(2^{ls} \|\Delta_l u\|_{M_{p,q}}\right)_l \right\|_q \lesssim \left\| \left(2^{ls} \sum_{m=l}^{\infty} \|\Delta_l u_m\|_{M_{p,q}}\right)_l \right\|_q \\ &\lesssim \quad \left\| \left(2^{ls} \sum_{m=l}^{\infty} \|u_m\|_{M_{p,q}}\right)_l \right\|_q , \end{aligned} \tag{2.33}$$

where Theorem 2.29 was used for the first, triangle inequality and the above observation for the second and Lemma 2.28 for the last estimate. Assume for now that $q \in (1, \infty)$. Then

$$\begin{aligned} \left\| \left(2^{ls} \sum_{m=l}^{\infty} \|u_m\|_{M_{p,q}}\right)_l \right\|_q^q &= \sum_{l=0}^{\infty} \left(2^{ls} \sum_{m=l}^{\infty} \|u_m\|_{M_{p,q}}\right)^q \\ &= \sum_{l=0}^{\infty} \left(\sum_{m=l}^{\infty} 2^{\frac{(l-m)s}{q'}} \times 2^{\frac{(l-m)s}{q}} 2^{ms} \|u_m\|_{M_{p,q}}\right)^q . \end{aligned} \tag{2.34}$$

Fix an $l \in \mathbb{N}_0$. Then, by Hölder's inequality, one obtains

$$\left(\sum_{m=l}^{\infty} 2^{\frac{(l-m)s}{q'}} \times 2^{\frac{(l-m)s}{q}} 2^{ms} \|u_m\|_{M_{p,q}}\right)^q \leq \left(\sum_{m'=l}^{\infty} 2^{-(m'-l)s}\right)^{\frac{q}{q'}} \sum_{m=l}^{\infty} 2^{(l-m)s} 2^{mqs} \|u_m\|^q_{M_{p,q}} .$$

The first factor above is essentially the geometric series $\sum_{m'=0}^{\infty} 2^{-m's} = \frac{1}{1-2^{-s}}$. Reinserting the above estimate into (2.34) and interchanging the order of summation yields

$$\left\| \left(2^{ls} \sum_{m=l}^{\infty} \|u_m\|_{M_{p,q}}\right)_l \right\|_q \lesssim_s \left(\sum_{m=0}^{\infty} 2^{mqs} \|u_m\|^q_{M_{p,q}} \sum_{l=0}^{m} 2^{(l-m)s}\right)^{\frac{1}{q}} .$$

Because the sum over l is just a geometric sum $\sum_{l=0}^{m} 2^{(l-m)s} = \sum_{l=0}^{m} 2^{-ms} \leq \frac{1}{1-2^{-s}}$, the inequality (2.32) follows.

In the case $q = 1$, one can interchange the order of summation in (2.33) directly, which yields

$$\left\| \left(2^{ls} \sum_{m=l}^{\infty} \|u_m\|_{M_{p,1}} \right)_l \right\|_1 = \sum_{m=0}^{\infty} 2^{ms} \|u_m\|_{M_{p,1}} \sum_{l=0}^{m} 2^{-(m-l)s} \lesssim_s \sum_{m=0}^{\infty} 2^{ms} \|u_m\|_{M_{p,1}},$$

due to the sum over l being bounded above by a geometric series.

In the case $q = \infty$, (2.33) reads as

$$\begin{aligned} \left\| \left(2^{ls} \sum_{m=l}^{\infty} \|u_m\|_{M_{p,\infty}} \right)_l \right\|_\infty &= \sup_{l\in\mathbb{N}_0} \sum_{m=l}^{\infty} 2^{-(m-l)s} 2^{ms} \|u_m\|_{M_{p,\infty}} \\ &\leq \sup_{m'\in\mathbb{N}_0} 2^{m's} \|u_{m'}\|_{M_{p,\infty}} \sum_{l=0}^{\infty} 2^{-ls} \\ &\lesssim_s \sup_{m\in\mathbb{N}_0} 2^{ms} \|u_m\|_{M_{p,\infty}}, \end{aligned}$$

due to the sum over l being a geometric series.

It remains to show the convergence of $\sum_{m=0}^{\infty} u_m$. To that end define $u_M^N := \sum_M^N u_m \in M_{p,q}^s$, where $M, N \in \mathbb{N}_0$ and $M \leq N$. To show is $\|u_M^N\|_{M_{p,q}^s} \to 0$ for $M, N \to \infty$. By the already proven bound (2.32), one has

$$\|u_M^N\|_{M_{p,q}^s} \lesssim_{d,s} \left\| \left(2^{ms} \mathbb{1}_{[M,N]}(m) \|u_m\| \right)_{m\in\mathbb{N}_0} \right\|_q.$$

The right-hand side goes to zero for $M, N \to \infty$, either by the dominated convergence theorem for $q < \infty$, or by assumption for $q = \infty$. This finishes the proof. □

2.4. Some useful embeddings

Proposition 2.31. *(Cf. [WH07, Proposition 2.5 (2)]). Let $d \in \mathbb{N}$ and $p \in [1, \infty]$. Furthermore, let $q_1, q_2 \in [1, \infty]$ and $s_1, s_2 \in \mathbb{R}$ satisfy*

$$s_1 - s_2 > d \left(\frac{1}{q_2} - \frac{1}{q_1} \right) > 0. \tag{2.35}$$

Then $M_{p,q_1}^{s_1}(\mathbb{R}^d) \hookrightarrow M_{p,q_2}^{s_2}(\mathbb{R}^d)$.

Proof. Put $q = \left(\frac{1}{q_2} - \frac{1}{q_1} \right)^{(-1)}$. By assumptions on q_1, q_2, one has $q \in [1, \infty)$ and $\frac{1}{q_2} = \frac{1}{q_1} + \frac{1}{q}$. Hence, by Hölder's inequality,

$$\|u\|_{M_{p,q_2}^{s_2}} = \left\| k \mapsto \frac{\langle k \rangle^{s_1}}{\langle k \rangle^{s_1 - s_2}} \|\Box_k u\|_p \right\|_{q_2} \leq \left(\sum_{k\in\mathbb{Z}^d} \langle k \rangle^{q(s_2 - s_1)} \right)^{\frac{1}{q}} \|u\|_{M_{p,q_1}^{s_1}}$$

holds for any $u \in M^{s_1}_{p,q_1}$. Comparison of the series on the right-hand side with the corresponding integral in spherical coordinates yields

$$\sum_{k\in\mathbb{Z}^d} \langle k\rangle^{-q(s_1-s_2)} \approx_{q,s} \int_{\mathbb{R}^d} (1+|x|^2)^{-\frac{q(s_1-s_2)}{2}} \mathrm{d}x \lesssim_d 1 + \int_1^\infty r^{d-1-q(s_1-s_2)} \mathrm{d}r.$$

The integral over r is finite, if the exponent $d-1-q(s_1-s_2)$ is smaller than -1. As this is exactly the condition from (2.35), the proof is complete. □

Proposition 2.32 ($M_{\infty,1} \hookrightarrow C_{\mathrm{b}}$)**.** *(Cf. [WH07, Proposition 2.7]). Let $d \in \mathbb{N}$. Then*

$$M_{\infty,1}(\mathbb{R}^d) \hookrightarrow C_b(\mathbb{R}^d). \tag{2.36}$$

Proof. Denote by $\Phi : L^\infty \to \mathcal{S}'$ the embedding defined in equation (A.24). It will be shown that $M_{\infty,1} \subseteq \mathrm{Im}(\Phi)$ and that $\Phi^{(-1)}|_{M_{\infty,1}} \in \mathscr{L}(M_{\infty,1}, C_{\mathrm{b}})$ implements the embedding (2.36). To that end consider any $u \in M_{\infty,1}$. One has

$$\sum_{k\in\mathbb{Z}^d} \left\| \Phi^{(-1)}(\Box_k u) \right\|_\infty < \infty,$$

i.e. $\sum_{k\in\mathbb{Z}^d} \Phi^{(-1)}(\Box_k u)$ is absolutely convergent in L^∞, say to v. By the comment made in Definition 2.3, $\Phi^{(-1)}(\Box_k u) \in C$ for all $k \in \mathbb{Z}^d$. Hence, $v \in C_{\mathrm{b}}$ as a uniform limit of bounded continuous functions. By Lemma 2.5 and continuity of Φ one has

$$u = \sum_{k\in\mathbb{Z}^d} \Phi \circ \Phi^{(-1)}(\Box_k u) = \Phi(v).$$

This shows that $u \in \mathrm{Im}(\Phi)$ and $\Phi^{(-1)} u = v$. Furthermore, one has $\|v\|_\infty \le \|u\|_{M_{\infty,1}}$ by construction of v. As $u \in M_{\infty,1}$ was arbitrary, the proof is concluded. □

Lemma 2.33 ($M^s_{2,2} \simeq H^s$)**.** *Let $d \in \mathbb{N}$ and $s \in \mathbb{R}$. Then*

$$M^s_{2,2}(\mathbb{R}^d) \simeq H^s(\mathbb{R}^d). \tag{2.37}$$

Proof. As $\mathcal{S}$ is dense in H^s and $M^s_{2,2}$ it suffices to consider $u \in \mathcal{S}$. One indeed has

$$\begin{aligned} \|u\|^2_{H^s} &= \|\langle\cdot\rangle^s \hat{u}\|_2^2 \approx_s \left\| \sum_{k\in\mathbb{Z}^d} \langle k\rangle^s \sigma_k \hat{u} \right\|_2^2 = \sum_{k,l\in\mathbb{Z}^d} \langle \langle k\rangle^s \sigma_k \hat{u}, \langle l\rangle^s \sigma_l \hat{u}\rangle \\ &= \sum_{l\in\Lambda} \sum_{k\in\mathbb{Z}^d} \langle \langle k\rangle^s \sigma_k \hat{u}, \langle k+l\rangle^s \sigma_{k+l} \hat{u}\rangle \approx_{d,s} \sum_{k\in\mathbb{Z}^d} \langle k\rangle^{2s} \langle \sigma_k \hat{u}, \sigma_k \hat{u}\rangle = \|u\|^2_{M^s_{2,2}}, \end{aligned}$$

where Peetre's inequality (Lemma A.31) was used for the second and fifth equality and the compact support of σ_k for the third. □

By complex interpolation one obtains the following

Proposition 2.34. *Let $d \in \mathbb{N}$ and $p \in [2, \infty]$. Then $M_{p,p'}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$.*

Proof. The statement holds for $p = \infty$ by Proposition 2.32 and for $p = 2$ by Lemma 2.33. For any other $p \in (2, \infty)$, set $\theta = \frac{2}{p}$. Then $M_{p,p'} = [M_{\infty,1}(\mathbb{R}^d), M_{2,2}(\mathbb{R}^d)]_\theta$ by Proposition 2.19 and $L^p = [L^\infty(\mathbb{R}^d), L^2(\mathbb{R}^d)]_\theta$ by Example A.62. The claim follows by Proposition A.60 and the proof is thus complete. □

Proposition 2.35 (Isomorphism of $M^{r+s}_{p,q}$ and $M^s_{p,q}$)**.** *(Cf. [WH07, Proposition 2.4]). Let $d \in \mathbb{N}$, $r, s \in \mathbb{R}$ and $p, q \in [1, \infty]$. Then J^r, the Bessel potential of order $-r$ defined in equation* (A.26)*, maps as follows*

$$J^r : M^{r+s}_{p,q}(\mathbb{R}^d) \to M^s_{p,q}(\mathbb{R}^d)$$

and is an isomorphism.

Proof. Consider any $u \in M^{r+s}_{p,q}$. Then

$$\|J^r u\|_{M^s_{p,q}} = \left\| \left(\langle k \rangle^s \|\Box_k J^r u\|_p \right)_k \right\|_q .$$

Fix a $k \in \mathbb{Z}^d$ and put $\rho_k = \sum_{l \in \Lambda(d)} \sigma_{k+l}$. Due to Implication (2.1) and Property (iii) in Definition 2.1, one has $\rho_k(\xi) = 1$ for every $\xi \in \operatorname{supp}(\sigma_k)$. Furthermore, by Property (ii) in Definition 2.1, one has

$$\operatorname{supp}(\rho_k) \subseteq \bigcup_{l \in \Lambda} \operatorname{supp}(\sigma_{k+l}) \subseteq \bigcup_{l \in \Lambda} B_{\sqrt{d}}(k+l) \subseteq B_{3\sqrt{d}}(k)$$

and hence $|\operatorname{supp}(\rho_k)| \lesssim_d 1$. Define the multiplier $B_k := \mathcal{F}^{(-1)} \rho_k \langle \cdot \rangle^r \mathcal{F}$ and observe

$$\|\Box_k J^r u\|_p = \left\| \mathcal{F}^{(-1)} \sigma_k \langle \cdot \rangle^r \mathcal{F} u \right\|_p = \left\| \mathcal{F}^{(-1)} \rho_k \langle \cdot \rangle^r \sigma_k \mathcal{F} u \right\|_p \le \|B_k\|_{\mathscr{L}(L^p)} \|\Box_k u\|_p .$$

To show is $\|B_k\|_{\mathscr{L}(L^p)} \lesssim \langle k \rangle^r$, as then

$$\|J^r u\|_{M^s_{p,q}} \lesssim \left\| \left(\langle k \rangle^{r+s} \|\Box_k u\|_p \right)_k \right\|_q = \|u\|_{M^{r+s}_{p,q}}$$

follows, proving the continuity of J^r.

By a multiplier estimate from Corollary A.53 (with $p_1 = p_2 = p$), one has

$$\|B\|_{\mathscr{L}(L^p)} \lesssim_d \|\rho_k \langle \cdot \rangle^r\|_\infty + \sum_{j=1}^{d} \left\| \partial^{de_j} \left(\rho_k \langle \cdot \rangle^r \right) \right\|_\infty .$$

For the first summand above, one indeed has

$$\begin{aligned} \|\rho_k \langle \cdot \rangle^r\|_\infty &\le \|\rho_k\|_\infty \sup_{\xi \in B_{3\sqrt{d}}(k)} \langle \xi \rangle^r \le \sum_{l \in \Lambda} \|\sigma_{k+l}\|_\infty \sup_{\xi \in B_{3\sqrt{d}}(k)} \langle \xi - k + k \rangle^r \\ &\lesssim_{d,r} \langle k \rangle^r \sup_{\xi \in B_{3\sqrt{d}}(k)} \langle \xi - k \rangle^r \lesssim_{d,r} \langle k \rangle^r \end{aligned}$$

by Property (iv) in Definition 2.1 and Peetre's inequality (Lemma A.31). For the second summand, Leibnitz' rule (Lemma A.28) yields

$$\sum_{j=1}^{d} \left\| \partial^{de_j} \left(\rho_k \langle \cdot \rangle^r \right) \right\|_\infty \leq \sum_{j=1}^{d} \sum_{n=0}^{d} \binom{d}{n} \left\| \partial^{(d-n)e_j} \rho_k \right\|_\infty \sup_{\xi \in B_{3\sqrt{d}}(k)} \left| (\partial^{ne_j} \langle \cdot \rangle^r)(\xi) \right| .$$

For the first factor above, one again has

$$\left\| \partial^{(d-n)e_j} \rho_k \right\|_\infty \leq \sum_{l \in \Lambda} \left\| \partial^{(d-n)e_j} \sigma_{k+l} \right\|_\infty \lesssim_d 1$$

due to the Property (iv) in Definition 2.1. For the second factor, observe that for each $j, n \in \{1, \ldots, d\}$ and $\xi \in \mathbb{R}^d$ one has

$$\left| (\partial^{ne_j} \langle \cdot \rangle^r)(\xi) \right| \leq \sum_{0 \leq m_1 \leq m_2 \leq n} c_{m_1,m_2}^{(n)} \left| \xi_n \right|^{m_1} \langle \xi \rangle^{r-2m_2} \leq \sum_{0 \leq m_1 \leq m_2 \leq n} c_{m_1,m_2}^{(n)} \langle \xi \rangle^{r+m_1-2m_2}$$

for some coefficients $c_{m_1,m_2}^{(n)}$ (which additionally depend on r), due to the chain and product rules. This shows, again invoking Peetre's inequality, that

$$\sup_{\xi \in B_{3\sqrt{d}}(k)} \left| (\partial^{ne_j} \langle \cdot \rangle^r)(\xi) \right| \lesssim_{d,r} \langle k \rangle^r$$

proving $\|B_k\|_{\mathscr{L}(L^p)} \lesssim_{d,r} \langle k \rangle^r$ (i.e., by above, $J^r \in \mathscr{L}(M_{p,q}^{r+s}, M_{p,q}^s)$.

To show that J^r is an isomorphism, observe that, as $r, s \in \mathbb{R}$ were arbitrary, one has $J^{-r} \in \mathscr{L}(M_{p,q}^s, M_{p,q}^{s+r})$. But clearly, $J^{-r} \circ J^r = \mathrm{id}_{M_{p,q}^{r+s}}$ and $J^r \circ J^{-r} = \mathrm{id}_{M_{p,q}^s}$, i.e. $(J^r)^{(-1)} = J^{-r} \in \mathscr{L}(M_{p,q}^s, M_{p,q}^{r+s})$. This finishes the proof. □

3. Estimates for the Schrödinger propagator

This chapter covers the boundedness of the Schrödinger propagator on modulation spaces and some classical Strichartz estimates. This lays the foundation for the local and global well-posedness results treated in this thesis.

The boundedness of the Schrödinger group on modulation spaces was first shown for a special case in [WZG06]. More spaces and more general operator groups are treated in [BGOR07]. The sharpness of the time exponent in these estimates was proven in [CN09]. See [WHHG11, Section 6.4] for a monographic, coherent account.

Strichartz estimates mathematically measure dispersion and are typically used to prove local well-posedness of dispersive equations. An example is [Tsu87, Lemma 3.1], where the mass-subcritical nonlinear Schrödinger equation is treated in L^2. In the aforementioned paper, global well-posedness follows by mass conservation. Strichartz estimates go back to [Str77] and have been generalized and adapted to different settings in a multitude of works, but see [KT98] and [Tag10] for maybe most noteable abstract results and, for example, [LP09, Section 4.2] for a textbook presentation. Strichartz estimates for modulation spaces are available, see [WH07, Proposition 5.3] or [WHHG11, Section 6.4], but did not give rise to any well-posedness theorems of this thesis.

This chaper is structured as follows. In Section 3.1 the Schrödinger propagator is defined and its boundedness on modulation spaces is proven. Moreover it is shown, that for modulation spaces with finite Fourier index the Schrödinger propagator is a strongly continous group on it. Subsequently, in Section 3.2, the classical homogeneous and inhomogeneous Strichartz estimates for the Schrödinger propagator are presented. Finally, the aforementioned global well-posedness result of Tsutsumi is stated and its proof is sketched in Section 3.3. A non-linear version of the homogeneous Strichartz estimate, which will be of importance for the global well-posedness result of this thesis, is observed and proven.

3.1. Free Schrödinger propagator on modulation spaces

Consider for any $d \in \mathbb{N}$ the Cauchy problem for the free Schrödinger equation

$$\begin{cases} \mathrm{i}\dfrac{\partial u}{\partial t}(x,t) = -\Delta u(x,t) & (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ u(\cdot,0) = u_0. \end{cases} \tag{3.1}$$

Formally taking the Fourier transform in the x-variable of (3.1) yields

$$\partial_t \hat{u}(\xi,t) = -\mathrm{i}\,|\xi|^2\,\hat{u}(\xi,t), \qquad \hat{u}(\xi,0)=\hat{u}_0(\xi) \qquad (\xi,t)\in\mathbb{R}^d\times\mathbb{R},$$

which is an ordinary differential equation for each $\xi\in\mathbb{R}^d$ with the solution given by $\hat{u}(\xi,t) = e^{-\mathrm{i}t|\xi|^2}\hat{u}_0(\xi)$ for $t\in\mathbb{R}$. This gives rise to the following

Definition 3.1 (Free Schrödinger propagator)**.** Let $d\in\mathbb{N}$. The family of operators $(e^{\mathrm{i}t\Delta})_{t\in\mathbb{R}}$ in $\mathcal{S}'(\mathbb{R}^d)$ defined by

$$e^{\mathrm{i}t\Delta} = \mathcal{F}^{(-1)}e^{-\mathrm{i}t|\cdot|^2}\mathcal{F} \qquad \forall t\in\mathbb{R} \tag{3.2}$$

is called the *(free) Schrödinger propagator.*

If and in which sense $t\mapsto e^{\mathrm{i}t\Delta}u_0$ solves (3.1) will be clarified after Proposition 3.5. For the moment, observe the generalization of the fact that $e^{\mathrm{i}t\Delta}$ is unitary on $L^2 = M^0_{2,2}$ for any $t\in\mathbb{R}$.

Lemma 3.2 (Adjoints of Schrödinger propagators in modulation spaces)**.** *Let $d\in\mathbb{N}$, $p,q\in[1,\infty)$ and $s,t\in\mathbb{R}$. Then $e^{\mathrm{i}t\Delta}\in\mathscr{L}(M^s_{p,q}(\mathbb{R}^d))$ and*

$$(e^{\mathrm{i}t\Delta})^* = e^{-\mathrm{i}t\Delta}\in\mathscr{L}\left(M^{-s}_{p',q'}(\mathbb{R}^d)\right).$$

Proof. The fact $e^{\mathrm{i}t\Delta}\in\mathscr{L}(M^s_{p,q})$ and $e^{-\mathrm{i}t\Delta}\in\mathscr{L}(M^{-s}_{p',q'})$ has been proven in Theorem 3.4. As $(M^s_{p,q})^*\cong M^{-s}_{p',q'}$ by Proposition 2.17, one may view $(e^{\mathrm{i}t\Delta})^*$ as an element of $\mathscr{L}(M^{-s}_{p',q'})$. In view of equation (2.13) it remains to show that

$$\sum_{l\in\Lambda(d)}\sum_{k\in\mathbb{Z}^d}\int\overline{\Box_{k+l}u}\,\Box_k e^{\mathrm{i}t\Delta}v\,\mathrm{d}x = \sum_{l\in\Lambda(d)}\sum_{k\in\mathbb{Z}^d}\int\overline{\Box_{k+l}e^{-\mathrm{i}t\Delta}u}\,\Box_k v\,\mathrm{d}x$$

holds for any $u\in M^{-s}_{p',q'}$ and any $v\in M^s_{p,q}$. As $p,q<\infty$, one may assume w.l.o.g. that $v\in\mathcal{S}$ by Proposition 2.15. Fix $l\in\Lambda(d)$ and $k\in\mathbb{Z}^d$. Then $\Box_k e^{\mathrm{i}t\Delta}v\in\mathcal{S}$ and hence indeed

$$\begin{aligned}\int\overline{\Box_{k+l}u}\,\Box_k e^{\mathrm{i}t\Delta}v\,\mathrm{d}x &= \langle\Box_{k+l}u,\Box_k e^{\mathrm{i}t\Delta}v\rangle_{\mathcal{S}'\times\mathcal{S}} = \left\langle\mathcal{F}^{(-1)}\sigma_{k+l}\mathcal{F}u,\mathcal{F}^{(-1)}\sigma_k e^{-\mathrm{i}t|\cdot|^2}\mathcal{F}v\right\rangle_{\mathcal{S}'\times\mathcal{S}}\\ &= \left\langle\sigma_{k+l}e^{\mathrm{i}t|\cdot|^2}\mathcal{F}u,\sigma_k\mathcal{F}v\right\rangle_{\mathcal{S}'\times\mathcal{S}} = \langle\Box_{k+l}e^{-\mathrm{i}t\Delta}u,\Box_k v\rangle_{\mathcal{S}'\times\mathcal{S}}\\ &= \int\overline{\Box_{k+l}e^{-\mathrm{i}t\Delta}u}\,\Box_k v\,\mathrm{d}x\end{aligned}$$

by the definition of the operations on $\mathcal{S}'$ (Definition A.40). This concludes the proof. □

Example 3.3 (Gaussian wave packet). Let $d\in\mathbb{N}$. Consider $u_0\in\mathcal{S}(\mathbb{R}^d)$ given by

$$u_0(x) = e^{-\frac{|x|^2}{2}} \qquad \forall x\in\mathbb{R}^d.$$

Then $e^{\mathrm{i}t\Delta}u_0$ is given by

$$u(x,t) = \frac{1}{(1+2\mathrm{i}t)^{\frac{d}{2}}}e^{-\frac{|x|^2}{2(1+2\mathrm{i}t)}} = \alpha(-t)^{-\frac{d}{2}}f_{\alpha(t)}(x) \qquad \forall x\in\mathbb{R}^d\,\forall t\in\mathbb{R}, \tag{3.3}$$

where $\alpha(t) = 1-2\mathrm{i}t$ and $f_{\alpha(t)}$ is as in Example 2.21.

Proof. Recall from Example A.26 that $\hat{u}_0 = u_0$. Using (A.3) one immediately confirms that

$$\begin{aligned} u(x,t) &= \left(\mathcal{F}^{(-1)} e^{-\mathrm{i}t|\cdot|^2} \mathcal{F} u_0\right)(x,t) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\mathrm{i}kx} e^{-\mathrm{i}t|k|^2} e^{-\frac{|k|^2}{2}} \mathrm{d}k \\ &= \prod_{j=1}^{d} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_i^2}{2(1+2\mathrm{i}t)}} \int_{-\infty}^{\infty} e^{-\left(\sqrt{\frac{1}{2}+\mathrm{i}t}k_i - \frac{\mathrm{i}x_i}{2\sqrt{\frac{1}{2}+\mathrm{i}t}}\right)^2} \mathrm{d}k_i \\ &= e^{-\frac{|x|^2}{2(1+2\mathrm{i}t)}} \prod_{j=1}^{d} \frac{1}{\sqrt{2\pi}} I\left(\sqrt{\frac{1}{2}+\mathrm{i}t}, \frac{\mathrm{i}x_i}{2\sqrt{\frac{1}{2}+\mathrm{i}t}}\right) \\ &= \frac{1}{(1+2\mathrm{i}t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{2(1+2\mathrm{i}t)}} \end{aligned}$$

holds for all $x \in \mathbb{R}^d$ and $t \in \mathbb{R}$. □

Theorem 3.4 (Schrödinger propagator bound)**.** *(Cf. [WZG06, Proposition 5.5], [BGOR07, Corollary 18] and [CN09, Proposition 4.1]). Let $d \in \mathbb{N}$, $p, q \in [1,\infty]$ and $s \in \mathbb{R}$. Then there is a constant $C = C(d,s)$ such that*

$$\left\| e^{\mathrm{i}t\Delta} \right\|_{\mathscr{L}(M_{p,q}^s(\mathbb{R}^d))} \leq C \langle t \rangle^{d\left|\frac{1}{2}-\frac{1}{p}\right|} \tag{3.4}$$

holds for all $t \in \mathbb{R}$. Furthermore, the exponent of the time dependence is sharp.

The fact that the Schrödinger propagator is bounded on $M_{p,q}^s$ was first observed in [WZG06, Proposition 5.5] for the case $p = 2$. This was improved to $p, q \in [1,\infty]$, in [BGOR07, Corollary 18]. In fact, the last paper treats the more general multipliers with symbols of the form $e^{\mathrm{i}|\xi|^\alpha}$, where $\alpha \in [0,2]$. The sharpness of the time exponent for $p \in [1,2]$ was shown in [CN09, Proposition 4.1]. Adding a duality argument and treating the remaining case $M_{\infty,1}$ (which is not a dual or a predual of another modulation space) yields a

Proof of Theorem 3.4. Fix a $t \in \mathbb{R}$. By Proposition 2.24 and 2.26, one may assume the norm on $M_{p,q}^s$ to be defined in terms of the STFT w.r.t. the window function $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$.

Suppose in the following that $g = g_0$, where $g_0(x) = e^{-\frac{|x|^2}{2}}$ for any $x \in \mathbb{R}^d$. From Example 3.3 and 2.25 one obtains

$$\left\| e^{-\mathrm{i}t\Delta} g_0 \right\|_{M_{p,q}^s} = |\alpha(t)|^{-\frac{d}{2}} \left\| f_{\alpha(-t)} \right\|_{M_{p,q}^s} \approx_{d,p,q,s} |\alpha(-t)+1|^{d\left(\frac{1}{p}-\frac{1}{2}\right)} \approx_{d,p} \langle t \rangle^{d\left(\frac{1}{p}-\frac{1}{2}\right)}, \tag{3.5}$$

where $\alpha(t) = 1 - 2\mathrm{i}t$ and $f_\alpha(x) = e^{-\frac{|x|^2}{2\alpha}}$ are as in the examples above. This shows that

$$\left\| e^{\mathrm{i}t\Delta} \right\|_{\mathscr{L}(M_{p,q}^s(\mathbb{R}^d))} \gtrsim_{d,p,q,s} \langle t \rangle^{d\left(\frac{1}{p}-\frac{1}{2}\right)},$$

i.e. the time exponent in (3.4) is indeed optimal for $p \in [1,2]$.

Now, drop the assumption $g = g_0$ and consider any $u_0 \in M^s_{p,q}$. Then

$$\left\|e^{\mathrm{it}\Delta}u_0\right\|_{M^s_{p,q}} = \left\|k \mapsto \langle k\rangle^s \left\|(V_g e^{\mathrm{it}\Delta}u_0)(\cdot,k)\right\|_p\right\|_q . \tag{3.6}$$

For every $k, x \in \mathbb{R}^d$ one has

$$(V_g e^{\mathrm{it}\Delta}u_0)(x,k) = \left\langle \mathcal{F}^{(-1)} e^{-\mathrm{it}|\cdot|^2}\mathcal{F}u_0, M_k S_x g\right\rangle = \left\langle u_0, \mathcal{F}^{(-1)} e^{\mathrm{it}|\cdot|^2}\mathcal{F}M_k S_x g\right\rangle .$$

Observe, that

$$(\mathcal{F}^{(\pm 1)}M_l h)(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{\mp \mathrm{i}(\xi \pm l)z}h(z)\mathrm{d}z = (S_{\mp l}\mathcal{F}^{(\pm 1)}h)(\xi)$$

and

$$(\mathcal{F}^{(\pm 1)}S_y h)(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{\mp \mathrm{i}\xi(z+y)}h(z)\mathrm{d}z = (M_{\pm y}\mathcal{F}^{(\pm 1)}h)(\xi)$$

holds for any $h \in \mathcal{S}(\mathbb{R}^d)$ and $l, y, \xi \in \mathbb{R}^d$. This implies

$$(V_g e^{\mathrm{it}\Delta}u_0)(x,k) = \left\langle u_0, \mathcal{F}^{(-1)} e^{\mathrm{it}|\cdot|^2} S_{-k}M_x\mathcal{F}g\right\rangle = \left\langle u_0, \mathcal{F}^{(-1)} S_{-k} e^{\mathrm{it}|\cdot-k|^2} M_x\mathcal{F}g\right\rangle .$$

Furthermore, as $e^{\mathrm{i}t|\xi-k|^2} = e^{\mathrm{it}|\xi|^2}e^{-2\mathrm{it}\xi k}e^{-\mathrm{it}|k|^2}$ holds for every $\xi \in \mathbb{R}^d$,

$$\begin{aligned}(V_g e^{\mathrm{it}\Delta}u_0)(x,k) &= e^{-\mathrm{it}|k|^2}\left\langle u_0, \mathcal{F}^{(-1)} S_{-k}M_{2tk}M_x e^{\mathrm{it}|\cdot|^2}\mathcal{F}g\right\rangle = e^{-\mathrm{it}|k|^2}\left\langle u_0, M_k S_{x+2tk}e^{\mathrm{it}\Delta}g\right\rangle \\ &= (V_{e^{\mathrm{it}\Delta}g}u_0)(x+2tk,k)\end{aligned}$$

follows. Inserting this into equation (3.6) shows that the Schrödinger time evolution of u_0 corresponds to changing the window function from g to $e^{\mathrm{it}\Delta}g$. Changing it back to g via equation (2.18) yields

$$\begin{aligned}\left\|e^{\mathrm{it}\Delta}u_0\right\|_{M^s_{p,q}} &= \left\|k \mapsto \langle k\rangle^s \left\|(V_{e^{\mathrm{it}\Delta}g}u_0)(\cdot,k)\right\|_p\right\|_q \\ &\leq \frac{2^{|s|}}{\|g\|_2^2}\left\|(x,k)\mapsto \langle k\rangle^{|s|}(V_{e^{\mathrm{it}\Delta}g}g)(x,k)\right\|_{L^1(\mathbb{R}^{2d})}\|u_0\|_{M^s_{p,q}} .\end{aligned}$$

Choose now $g = e^{-\mathrm{it}\Delta}g_0$. Then

$$\|g\|_2^2 = \|g_0\|_2^2 = \int_{\mathbb{R}^d} e^{-|x|^2}\mathrm{d}x = \pi^{\frac{d}{2}} \approx_d 1$$

and

$$(V_{e^{\mathrm{it}\Delta}g}g)(x,k) = (V_{g_0}e^{-\mathrm{it}\Delta}g_0)(x,k) \qquad \forall k, x \in \mathbb{R}^d,$$

i.e., if one assumes g_0 as the window function for $M^{|s|}_{1,1}$,

$$\left\|(x,k)\mapsto \langle k\rangle^{|s|}(V_{e^{\mathrm{it}\Delta}g}g)(x,k)\right\|_{L^1(\mathbb{R}^{2d})} \approx \left\|e^{-\mathrm{it}\Delta}g_0\right\|_{M^{|s|}_{1,1}} .$$

Estimate (3.5) with $p = q = 1$ and regularity index $|s|$ proves the bound (3.4) for $p \in \{1, \infty\}$.

For $p=2$, observe that by Plancherel theorem (Proposition A.35) one has

$$\left\|\square_k e^{\mathrm{i}t\Delta}u_0\right\|_2 = \left\|\sigma_k e^{-\mathrm{i}t|\cdot|^2}\mathcal{F}u_0\right\|_2 = \|\sigma_k \mathcal{F}u_0\|_2 = \|\square_k u_0\|_2 \qquad \forall k \in \mathbb{Z}^d,$$

i.e. $\left\|e^{\mathrm{i}t\Delta}u_0\right\|_{M^s_{2,q}} = \|u_0\|_{M^s_{2,q}}$ by the defintion of the modulation space norm in equation (2.3). This proves the bound claimed in equation (3.4) and, once again, the optimality of the time exponent in this case.

Complex interpolation between the cases $p=1$, $p=2$ and $p=\infty$ proves the bound (3.4) in the remaining cases. More precisely, suppose that $p \in (1,2)$. Then

$$M^s_{p,q} = [M^s_{1,q}, M^s_{2,q}]_\theta \qquad \text{with} \qquad \theta = 2\left(1-\frac{1}{p}\right)$$

holds by Proposition 2.19. As the complex interpolation functor is exact and of type θ,

$$\left\|e^{\mathrm{i}t\Delta}\right\|_{\mathscr{L}(M^s_{p,q})} \leq \left\|e^{\mathrm{i}t\Delta}\right\|^{1-\theta}_{\mathscr{L}(M^s_{1,q})} \left\|e^{\mathrm{i}t\Delta}\right\|^{\theta}_{\mathscr{L}(M^s_{2,q})} \lesssim_{d,s} \langle t\rangle^{d\left(\frac{1}{p}-\frac{1}{2}\right)}$$

follows, i.e. the bound (3.4) holds in that case. Similarly, interpolating between $p=2$ and $p=\infty$ shows (3.4) for $p \in (2,\infty)$.

It remains to prove optimality of the time exponent for $p>2$. If additionally $q>1$, then $p'<2$, $q'<\infty$ and $e^{\mathrm{i}t\Delta} \in \mathscr{L}(M^s_{p,q})$ is the dual operator of $e^{-\mathrm{i}t\Delta} \in \mathscr{L}(M^{-s}_{p',q'})$ by Lemma 3.2. As $\left|\frac{1}{2}-\frac{1}{p'}\right| = \left|\frac{1}{2}-\frac{1}{p}\right|$, one has

$$\langle t\rangle^{d\left|\frac{1}{2}-\frac{1}{p}\right|} \approx_{d,p,q,s} \left\|e^{-\mathrm{i}t\Delta}\right\|_{\mathscr{L}(M^{-s}_{p',q'})} = \left\|(e^{\mathrm{i}t\Delta})^*\right\|_{(M^s_{p,q})^*} = \left\|e^{\mathrm{i}t\Delta}\right\|_{\mathscr{L}(M^s_{p,q})}$$

by the already known case, where additionally Proposition A.16 was used for the last equality.

A similar duality argument applies if $q=1$ and $p<\infty$.

For the last case $p=\infty$ and $q=1$, assume that the time exponent $\frac{d}{2}$ is not optimal, i.e. there is an $\varepsilon>0$ such that

$$\left\|e^{\mathrm{i}t\Delta}\right\|_{\mathscr{L}(M^s_{\infty,1})} \lesssim_{d,s} \langle t\rangle^{\frac{d}{2}-\varepsilon} \qquad \forall t \in \mathbb{R}.$$

But then interpolating between the cases $p=2$ and $p=\infty$ yields the bound

$$\left\|e^{\mathrm{i}t\Delta}\right\|_{\mathscr{L}(M^s_{p,1})} \leq \left\|e^{\mathrm{i}t\Delta}\right\|^{\frac{2}{p}}_{\mathscr{L}(M^s_{2,1})} \left\|e^{\mathrm{i}t\Delta}\right\|^{1-\frac{2}{p}}_{\mathscr{L}(M^s_{\infty,1})} \lesssim_{d,s} \langle t\rangle^{d\left(\frac{1}{2}-\frac{1}{p}\right)-\varepsilon\left(1-\frac{2}{p}\right)} \qquad \forall t \in \mathbb{R}$$

for any $p \in (2,\infty)$. This contradicts the already proven optimality of the time exponent for these p and finishes the proof. □

Proposition 3.5 (($e^{\mathrm{i}t\Delta}$) is a strongly continuous group)**.** *Let $d \in \mathbb{N}$, $p \in [1,\infty]$, $q \in [1,\infty)$ and $s \in \mathbb{R}$. Then $(e^{\mathrm{i}t\Delta})_{t\in\mathbb{R}}$ is a C_0-group in $M^s_{p,q}(\mathbb{R}^d)$. Its generator A is given by*

$$\operatorname{dom}(A) = M^{s+2}_{p,q}(\mathbb{R}^d), \qquad Au = \mathcal{F}^{(-1)}(-\mathrm{i}\,|\cdot|^2)\hat{u} = \mathrm{i}\Delta u \qquad \forall u \in \operatorname{dom}(A). \tag{3.7}$$

In the situation of Proposition 3.5 consider the Cauchy problem (3.1). By [EN00, Proposition II.6.2]) one has that if $u_0 \in M^{s+2}_{p,q}(\mathbb{R}^d)$ then $e^{\mathrm{i}t\Delta}u_0$ is the unique *classical* solution of the Cauchy problem (3.1), i.e. $e^{\mathrm{i}t\Delta} \in C^1(\mathbb{R}, M^s_{p,q})$, $e^{\mathrm{i}t\Delta} \in M^{s+2}_{p,q}$ for all $t \in \mathbb{R}$ and (3.1) holds. By [EN00, II.6.4] one has that for a general $u_0 \in M^s_{p,q}(\mathbb{R}^d)$, $e^{\mathrm{i}t\Delta}u_0$ is the unique *mild* solution of (3.1), i.e. $e^{\mathrm{i}t\Delta}u_0 \in C(\mathbb{R}, M^s_{p,q}(\mathbb{R}^d))$, $\int_0^t e^{\mathrm{i}s\Delta}u_0 \mathrm{d}s \in M^{s+2}_{p,q}$ for all $t \in \mathbb{R}$ and

$$u(\cdot, t) = u_0 + \mathrm{i}\Delta \int_0^t u(s)\mathrm{d}s \qquad \forall t \in \mathbb{R} \tag{3.8}$$

holds. The integral above is understood as the Riemann integral in $M^s_{p,q}$.

For $q = \infty$, the situation is more subtle. In fact, $(e^{\mathrm{i}t\Delta})_{t\in\mathbb{R}}$ is no longer a C_0-group in $M^s_{p,\infty}$, but only a bi-continuous group. See [Kun18] for this case.

Proof of Proposition 3.5. First it will be shown that $\Delta = \mathcal{F}^{(-1)}(-|\cdot|^2)\mathcal{F} \in \mathscr{L}(M^{s+2}_{p,q}, M^s_{p,q})$. The proof of this is very close to the proof of Proposition 2.35. Consider any $u \in M^{s+2}_{p,q}$. One has

$$\|\Delta u\|_{M^s_{p,q}} = \left(\sum_{k\in\mathbb{Z}^d} \langle k\rangle^{qs} \|\Box_k \Delta u\|_p^q\right)^{\frac{1}{q}}.$$

Fix a $k \in \mathbb{Z}^d$ and define $\rho_k := \sum_{l\in\Lambda} \sigma_{k+l}$, where $\Lambda = \left\{l \in \mathbb{Z}^d | \ |l| \leq 2\sqrt{d}\right\}$ is the set of close indices as in chapter 2. One has $\mathrm{supp}(\rho_k) \subseteq B_{3\sqrt{d}}(k)$ and hence $|\mathrm{supp}(\rho_k)| \lesssim_d 1$ holds. Define further the operator $B_k := \mathcal{F}^{(-1)}\langle\cdot\rangle^2\mathcal{F}$. Then, because $\rho_k(\xi) = 1$ for any $\xi \in \mathrm{supp}(\sigma_k)$, one has

$$\|\Box_k \Delta u\|_p = \left\|\mathcal{F}^{(-1)}\sigma_k(-|\cdot|^2)\mathcal{F}u\right\|_p = \left\|\mathcal{F}^{(-1)}\rho_k |\cdot|^2 \sigma_k\mathcal{F}u\right\|_p \leq \|B_k\|_{\mathscr{L}(L^p)} \|\Box_k u\|_p$$

and so it suffices to show that $\|B_k\|_{\mathscr{L}(L^p)} \lesssim \langle k\rangle^2$. Bernstein multiplier estimate from Corollary A.53 (with $p_1 = p_2 = p$) shows

$$\|B_k\|_{\mathscr{L}(L^p)} \lesssim_d \|\rho_k\langle\cdot\rangle^2\|_\infty + \sum_{m=1}^d \left\|\partial^{de_m}(\rho_k\langle\cdot\rangle^2)\right\|_\infty.$$

Leibnitz's rule (Lemma A.28) yields

$$\|\partial^\alpha(\rho_k\langle\cdot\rangle^2)\|_\infty \leq \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \left\|\partial^\beta\rho_k\right\|_\infty \sup_{\xi\in B_{3\sqrt{d}}(k)} \left|(\partial^{\alpha-\beta}\langle\cdot\rangle^2)(\xi)\right|$$

for any $|\alpha| \leq d$. Due to the properties of the symbols of IDOs (Definition 2.1), the first factor is bounded independently of β and k. For the second factor, observe that any derivative of $\langle\cdot\rangle^2$ is either $\langle\cdot\rangle^2$, $\xi \mapsto \xi_n$ for an $n \in \{1, \ldots, d\}$, 2 or 0 and the absolute value of all these functions is bounded above pointwise by $2\langle\cdot\rangle^2$. Hence

$$\sup_{\xi\in B_{3\sqrt{d}}(k)} \left|(\partial^{\alpha-\beta}\langle\cdot\rangle^2)(\xi)\right| \lesssim \sup_{\xi\in B_{3\sqrt{d}}(k)} \langle\xi\rangle^2 \lesssim_d \langle k\rangle^2,$$

where Peetre's inequality was used for the last step. This shows the sufficient condition $\|B_k\|_{\mathscr{L}(L^p)} \lesssim_d \langle k \rangle^2$.

The fact that $(T(t))_{t\in\mathbb{R}} := (e^{\mathrm{i}t\Delta})_{t\in\mathbb{R}}$ is a family of operators on $M^s_{p,q}$ has already been proven in Theorem 3.4. The group property (A.17) of $(T(t))_{t\in\mathbb{R}}$ is obvious.

For the strong continuity of $(T(t))$ in $t = 0$, let $|t| \in (0, 1]$ and consider any $u \in M^s_{p,q}$. To show is

$$\|T(t)u - u\|_{M^s_{p,q}} = \left(\sum_{k\in\mathbb{Z}^d} \langle k \rangle^{qs} \|\Box_k (T(t)u - u)\|_p^q \right)^{\frac{1}{q}} \xrightarrow{t\to 0} 0. \tag{3.9}$$

By the triangle inequality and the boundedness of the Schrödinger group on $M^s_{p,q}$ (see Theorem 3.4), one has

$$\begin{aligned} \|T(t)u - u\|_{M^s_{p,q}} &\leq \|T(t)(u - v)\|_{M^s_{p,q}} + \|u - v\|_{M^s_{p,q}} + \|T(t)v - v\|_{M^s_{p,q}} \\ &\lesssim_{d,s} \|u - v\| + \|T(t)v - v\|_{M^s_{p,q}} \qquad \forall u, v \in M^s_{p,q}. \end{aligned}$$

Hence, it suffices to show (3.9) for a dense subset D_0 of $M^s_{p,q}$. Let $j \in \{0, 1\}$ (Objects with index $j = 1$ will be used later, while determining the generator A. For the sake of brevity, they are treated already here). Define and observe

$$\begin{aligned} D_j &:= \left\{ v \in M^{s+2j}_{p,q}(\mathbb{R}^d) \,\middle|\, \operatorname{supp}(\hat{v}) \text{ is compact} \right\} \\ &= \left\{ v \in M^{s+2j}_{p,q}(\mathbb{R}^d) \,\middle|\, \exists M \in \mathbb{N} : \, v = \sum_{|k|\leq M} \Box_k v \right\}. \end{aligned}$$

By Lemma 2.16, D_j is dense in $M^{s+2j}_{p,q}$. Moreover,

$$\operatorname{supp}\left(\mathcal{F}(T(t)v)\right) = \operatorname{supp}\left(e^{-\mathrm{i}t|\cdot|^2} \hat{v}\right) = \operatorname{supp}(\hat{v}) \qquad \forall v \in \mathcal{S}'. \tag{3.10}$$

This implies that for $u \in D_0$ the series in (3.9) is just a finite sum and it is hence enough to show that

$$\|\Box_k (T(t)u - u)\|_p \xrightarrow{t\to 0} 0 \qquad \forall k \in \mathbb{Z}^d.$$

Fix for the following a $k \in \mathbb{Z}^d$. Define the multipliers $B^t_{k,j} := \mathcal{F}^{(-1)} \rho_k \omega^t_j \mathcal{F}$ (as mentioned above, $j = 1$ will be used later), where

$$\omega^t_0 := e^{-\mathrm{i}t|\cdot|^2} - 1, \qquad \omega^t_1 := \frac{e^{-\mathrm{i}t|\cdot|^2} - 1}{t} + \mathrm{i}\,|\cdot|^2$$

and ρ_k is as in the defintion of the operator B_k. Because of

$$\begin{aligned} \|\Box_k(T(t)u - u)\|_p &= \left\| \mathcal{F}^{(-1)} \sigma_k \left(e^{-\mathrm{i}t|\cdot|^2} - 1\right) \mathcal{F}u \right\|_p = \left\| \mathcal{F}^{(-1)} \rho_k \omega^t_0 \sigma_k \mathcal{F}u \right\|_p \\ &\leq \left\| B^t_{k,0} \right\|_{\mathscr{L}(L^p)} \|\Box_k u\|_p \,, \end{aligned}$$

it is enough to show that $\left\|B_{k,0}^t\right\|_{\mathscr{L}(L^p)} \to 0$ as $t \to 0$, which is done in the following using the same techniques as for the operator B_k. Bernstein multiplier estimate from Corollary A.53 yields

$$\left\|B_{k,j}^t\right\|_{\mathscr{L}(L^p)} \lesssim_d \left\|\rho_k \omega_j^t\right\|_\infty + \sum_{m=1}^d \left\|\partial^{de_m}(\rho_k \omega_j^t)\right\|_\infty .$$

Applying the Leibnitz's rule shows that

$$\left\|\partial^\alpha(\rho_k \omega_j)\right\|_\infty \le \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \left\|\partial^\beta \rho_k\right\|_\infty \sup_{\xi\in B_{3\sqrt{d}}(k)} \left|\omega_j^t(\xi)\right|$$

for any $|\alpha| \le d$. The first factor is again bounded independently of β and k. For the second factor, observe that

$$\omega_j^t = t\sum_{n=j+1}^\infty \frac{(-\mathrm{i})^n}{n!} t^{n-1} |\xi|^{2n} \qquad \forall \xi \in \mathbb{R}^d.$$

As the series above defines a real analytic function on $\mathbb{R}^d$, one has that

$$\sup_{\xi\in B_{3\sqrt{d}}(k)} \left|\omega_j^t\right| \lesssim_{d,k} |t|$$

for any $|\alpha| \le N$ and $\beta \le \alpha$. All in all this showed that $\left\|B_{k,j}^t\right\|_{\mathscr{L}(L^p)} \to 0$ as $t \to 0$. This implies by the above that $(T(t))$ is indeed strongly continuous in $t = 0$.

To characterize the generator A of $(T(t))$, assume first that $u_0 \in D_1$. Then there exists an $M \in \mathbb{N}$ such that

$$\left\|\frac{1}{t}\left(e^{\mathrm{i}t\Delta}u_0 - u_0\right) - \mathrm{i}\Delta u_0\right\|_{M_{p,q}^s} = \left(\sum_{|k|\le M} \langle k\rangle^{qs} \left\|\Box_k\left(\frac{1}{t}\left(e^{\mathrm{i}t\Delta}u_0 - u_0\right) - \mathrm{i}\Delta u_0\right)\right\|_p^q\right)^{\frac{1}{q}} .$$

for any $t \neq 0$. Fix for the following $|t| \in (0, 1]$ and $k \in \mathbb{Z}^d$. One has

$$\begin{aligned}\left\|\Box_k\left(\frac{1}{t}\left(e^{\mathrm{i}t\Delta}u_0 - u_0\right) - \mathrm{i}\Delta u_0\right)\right\|_p &= \left\|\mathcal{F}^{(-1)}\sigma_k\left(\frac{e^{-\mathrm{i}t|\cdot|^2} - 1}{t} + \mathrm{i}\,|\cdot|^2\right)\mathcal{F}u_0\right\|_p \\ &= \left\|\mathcal{F}^{(-1)}\rho_k\omega_1^t\sigma_k\mathcal{F}u_0\right\|_p \le \left\|B_{k,1}^t\right\|_{\mathscr{L}(L^p)} \left\|\Box_k u_0\right\|_p .\end{aligned}$$

As shown above, $\left\|B_{k,1}^t\right\|_{M_{p,q}^s} \to 0$ as $t \to 0$. This implies that $D_1 \subseteq \operatorname{dom}(A)$ and $Au = \mathrm{i}\Delta u$ for all $u \in D_1$.

To complete the proof of $A = \mathrm{i}\Delta$, consider the following. By Proposition A.23 and equation (3.10), D_1 is a core for A, i.e. $\overline{D_1}^{\|\cdot\|_A} = \operatorname{dom}(A)$. Because D_1 is dense in $M_{p,q}^{s+2}$, it suffices to show that $\|u\|_A \approx \|u\|_{M_{p,q}^{s+2}}$ for all $u \in D_1$. On the one hand, one immediately has

$$\|u\|_A = \|u\|_{M_{p,q}^s} + \|Au\|_{M_{p,q}^s} = \|u\|_{M_{p,q}^s} + \|\Delta u\|_{M_{p,q}^s} \lesssim_d \|u\|_{M_{p,q}^s} + \|u\|_{M_{p,q}^{s+2}} \lesssim \|u\|_{M_{p,q}^{s+2}}$$

for any $u \in D_1$ by the above proof of the boundedness of $\Delta \in \mathscr{L}(M^{s+2}_{p,q}, M^s_{p,q})$. On the other hand, Proposition 2.35 implies

$$\|u\|_{M^{s+2}_{p,q}} \approx_d \left\|J^2 u\right\|_{M^s_{p,q}} = \|(I-\Delta)u\|_{M^s_{p,q}} \leq \|u\|_{M^s_{p,q}} + \|\Delta u\|_{M^s_{p,q}} = \|u\|_A \qquad \forall u \in D_1.$$

The proof is now complete. □

3.2. Two classical Strichartz estimates

Definition 3.6 (Admissible pairs)**.** (Cf. [KT98, Definition 1.1]) Let $2 \leq q, r \leq \infty$, $d \in \mathbb{N}$. The pair (r, q) is called *admissible* , if

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} \tag{3.11}$$

and $(r, q, d) \neq (\infty, 2, 2)$. Put $q_{\mathrm{a}}(d, r) \coloneqq \frac{2}{d\left(\frac{1}{2}-\frac{1}{r}\right)}$.

Proposition 3.7 (Homogeneous Strichartz estimate)**.** *(Cf. [KT98, Corollary 1.4]) Let $d \in \mathbb{N}$ and $(r, q_{\mathrm{a}}(d, r))$ be admissible. Then there is a constant $C = C(d, r) > 0$ such that for any $T > 0$ and any $u_0 \in L^2(\mathbb{R}^d)$ the following* homogeneous Strichartz estimate

$$\left\|e^{\mathrm{i}t\Delta} u_0\right\|_{L^{q_{\mathrm{a}}(d,r)}([0,T], L^r(\mathbb{R}^d))} \leq C \|u_0\|_{L^2(\mathbb{R}^d)} \tag{3.12}$$

holds.

To formulate the inhomogeneous Strichartz estimate the geometric notation of Kato shall be introduced (cf. [Kat89, Section 2]). Consider the points

$$A = \left(\frac{1}{2}, 0\right), \quad B = \begin{cases} (0, \frac{1}{2}) & \text{if } d = 1, \\ (\frac{1}{2} - \frac{1}{d}, 1) & \text{if } d \geq 2, \end{cases} \quad C = \begin{cases} (0, 0) & \text{if } d = 1, \\ (\frac{1}{2} - \frac{1}{d}, 0) & \text{if } d \geq 2, \end{cases}$$

$$A' = \left(\frac{1}{2}, 1\right), \quad B' = \begin{cases} (1, \frac{1}{2}) & \text{if } d = 1, \\ (\frac{1}{2} + \frac{1}{d}, 0) & \text{if } d \geq 2, \end{cases} \quad C' = \begin{cases} (1, 1) & \text{if } d = 1, \\ (\frac{1}{2} + \frac{1}{d}, 1) & \text{if } d \geq 2 \end{cases}$$

and the triangles $\widehat{T}(d) = \Delta(A, B, C)$ and $\widehat{T}'(d) = \Delta(A', B', C')$, which are open, except that $A \in \widehat{T}(d)$ and $A' \in \widehat{T}'(d)$ (cf. figure 3.1).

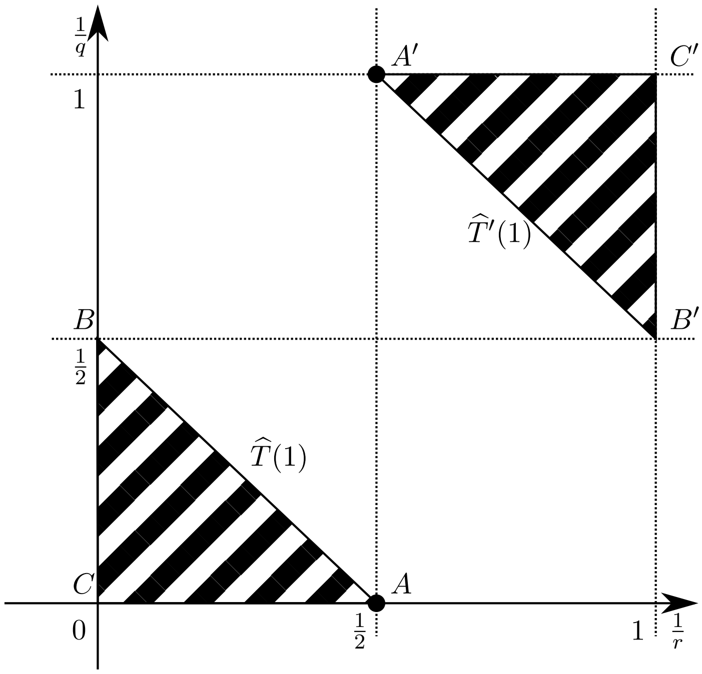
Proposition 3.8 (Inhomogeneous Strichartz estimates)**.** *(Cf. [Kat94, Theorem 2.1]) Let $d \in \mathbb{N}$, $1 \leq q, r, \gamma, \rho \leq \infty$ such that $\left(\frac{1}{r}, \frac{1}{q}\right) \in \widehat{T}(d)$, $\left(\frac{1}{\rho}, \frac{1}{\gamma}\right) \in \widehat{T}'(d)$ and*

$$\left(\frac{2}{\gamma} + \frac{d}{\rho}\right) - \left(\frac{2}{q} + \frac{d}{r}\right) = 2. \tag{3.13}$$

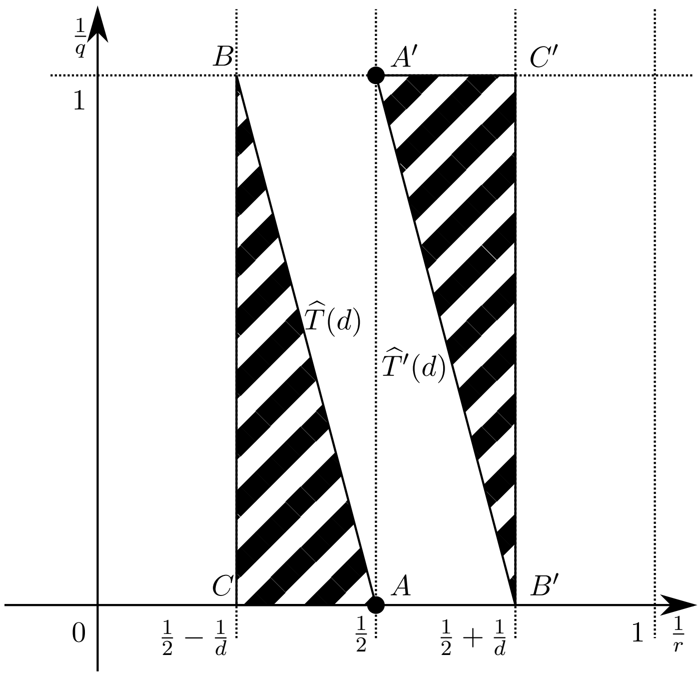
Then there is a constant $C = C(d, q, r, \rho) > 0$ such that for any $T > 0$ and any $F \in L^\gamma([0,T], L^\rho(\mathbb{R}^d))$ the following inhomogeneous Strichartz estimate

$$\left\|\int_0^t e^{\mathrm{i}(t-\tau)\Delta} F(\cdot, \tau)\right\|_{L^q([0,T], L^r(\mathbb{R}^d))} \leq C \|F\|_{L^\gamma([0,T], L^\rho(\mathbb{R}^d))} \tag{3.14}$$

holds.



(a) Triangles $\widehat{T}(1)$ and $\widehat{T}'(1)$.



(b) Triangles $\widehat{T}(d)$ and $\widehat{T}'(d)$ for $d \geq 2$.

Figure 3.1.: Geometric notation of Kato.

3.3. Global well-posedness of the mass-subcritical NLS in L^2

Set for all $u, v, w \in \mathbb{C}$

$$G(u,v,w) = |u+v|^{\nu-1}(u+v) - |u+w|^{\nu-1}(u+w) \tag{3.15}$$

and observe the following

Lemma 3.9 (Size estimate)**.** *Let $\nu > 1$. Then the following* size estimate

$$|G(u,v,w)| \leq \nu \max\left\{1, 2^{\nu-1}\right\}\left(|u|^{\nu-1} + |v|^{\nu-1} + |w|^{\nu-1}\right)|v-w| \tag{3.16}$$

holds for any $u, v, w \in \mathbb{C}$.

Proof. W.l.o.g. $u+v$ and $u+w$ are not colinear (as elements of $\mathbb{R}^2 \cong \mathbb{C}$). Then, by the fundamental theorem of calculus,

$$\begin{aligned}
& \left| |u+v|^{\nu-1}(u+v) - |u+w|^{\nu-1}(u+w) \right| \\
\leq\ & \int_0^1 \left| \frac{\partial}{\partial \tau}\left[|u+\tau v + (1-\tau)w|^{\nu-1}(u+\tau v+(1-\tau)w) \right] \right| \mathrm{d}\tau \\
\leq\ & \nu |v-w| \int_0^1 \left(|u| + \tau |v| + (1-\tau)|w|\right)^{\nu-1} \mathrm{d}\tau.
\end{aligned}$$

Clearly the function $f : (0,\infty) \to (0,\infty)$, $x \mapsto f(x) = x^{\nu-1}$ is strictly convex for $\nu > 2$ and subadditive for $1 < \nu \leq 2$. The first case is seen by taking the second derivative. For the second case, consider any $x, y > 0$ and set $a = \frac{y}{x}$. One has

$$\begin{aligned}
& f(x+y) = x^{\nu-1}(1+a)^{\nu-1} \leq x^{\nu-1}\left(1+a^{\nu-1}\right) = f(x) + f(y) \\
\Leftrightarrow\quad & (1+a)^{\nu-1} \leq 1 + a^{\nu-1}.
\end{aligned}$$

Last inequality is Bernoulli's inequality, which can be proven by observing that it is true for $a = 0$ and considering the derivatives of the left and right-hand sides.

Hence, the integrand above satisfies

$$2^{\nu-1}\left(\frac{1}{2}|u| + \frac{\tau}{2}|v| + \frac{1-\tau}{2}|w|\right)^{\nu-1} \leq \begin{cases} |u|^{\nu-1} + |v|^{\nu-1} + |w|^{\nu-1} & \text{if } 1 < \nu \leq 2, \\ 2^{\nu-2}\left(|u|^{\nu-1} + |v|^{\nu-1} + |w|^{\nu-1}\right) & \text{if } \nu > 2, \end{cases}$$

for all $\tau \in [0,1]$. This concludes the proof. □

One has the following

Lemma 3.10 (Strichartz estimate for a Banach contraction mapping argument)**.** *Let $d \in \mathbb{N}$, $\nu \in \left(1, 1+\frac{4}{d}\right)$ and $\frac{1}{r} \in \left(\max\left\{0, \frac{1}{2}-\frac{1}{d}\right\}, \frac{1}{2}\right]$. Then there is a constant $C = C(d,\nu,r) > 0$ such that for any $T > 0$ and any $v, w \in L^{q_a(d,\nu+1)}([0,T], L^{\nu+1}(\mathbb{R}^d))$ the estimate*

$$\begin{aligned} &\left\| \int_0^t e^{\mathrm{i}(t-\tau)\Delta} G(u,v,w) \mathrm{d}\tau \right\|_{L^{q_a(d,r)}([0,T],L^r(\mathbb{R}^d))} \\ \leq\ & CT^{1-\frac{d}{4}(\nu-1)} \Big[\|u\|^{\nu-1}_{L^{q_a(d,\nu+1)}([0,T],L^{\nu+1}(\mathbb{R}^d))} \\ &+ \|v\|^{\nu-1}_{L^{q_a(d,\nu+1)}([0,T],L^{\nu+1}(\mathbb{R}^d))} + \|w\|^{\nu-1}_{L^{q_a(d,\nu+1)}([0,T],L^{\nu+1}(\mathbb{R}^d))} \Big] \\ &\cdot \|v-w\|_{L^{q_a(d,\nu+1)}([0,T],L^{\nu+1}(\mathbb{R}^d))} \end{aligned}$$

holds.

Proof. Proposition 3.8 yields

$$\left\| \int_0^t e^{\mathrm{i}(t-\tau)\Delta} G(u,v,w) \mathrm{d}\tau \right\|_{L^{q_a(d,r)}([0,T],L^r(\mathbb{R}^d))} \lesssim_{d,r,\rho} \|G(u,v,w)\|_{L^\gamma([0,T],L^\rho(\mathbb{R}^d))},$$

if $\left(\frac{1}{\rho}, \frac{1}{\gamma}\right) \in \widehat{T}'(d)$ satisfies condition 3.13, i.e.

$$\frac{1}{\rho} \in \left[\frac{1}{2}, \min\left\{1, \frac{1}{2}+\frac{1}{d}\right\}\right) \qquad \text{and} \qquad \frac{1}{\gamma} = 1 - \frac{d}{2}\left(\frac{1}{\rho}-\frac{1}{2}\right).$$

By the size estimate from Lemma 3.9 one has

$$\begin{aligned} &\|G(u,v,w)\|_{L^\gamma L^\rho} \\ \lesssim_\nu &\left\| u^{(\nu-1)}(v-w) \right\|_{L^\gamma L^\rho} + \left\| v^{(\nu-1)}(v-w) \right\|_{L^\gamma L^\rho} + \left\| w^{(\nu-1)}(v-w) \right\|_{L^\gamma L^\rho}. \end{aligned}$$

Consider $f \in \{u, v, w\}$. Applying Hölder's inequality to the functions $(v-w)$, $f^{\nu-1}$ and 1 yields

$$\left\| f^{(\nu-1)} |v-w| \right\|_{L^\gamma([0,T],L^\rho(\mathbb{R}^d))} \leq T^{\frac{1}{q_3}} \|f\|^{(\nu-1)}_{L^{(\nu-1)q_2}([0,T],L^{(\nu-1)r_2}(\mathbb{R}^d))} \|v-w\|_{L^{q_1}([0,T],L^{r_1}(\mathbb{R}^d))},$$

where all but the last of the exponents $r_1, r_2, q_1, q_2, q_3 \in [1,\infty]$ are already fixed by the norm indices to

$$\begin{aligned} r_1 &= \nu+1, \qquad & q_1 &= q_a(d,\nu+1) = (\nu+1)\frac{4}{d}\frac{1}{\nu-1}, \\ r_2 &= \frac{\nu+1}{\nu-1}, \qquad & q_2 &= \frac{q_a(d,\nu+1)}{\nu-1} = \frac{\nu+1}{\nu-1}\frac{4}{d}\frac{1}{\nu-1} \end{aligned}$$

and need to satisfy the Hölder conditions

$$\frac{1}{\rho} = \frac{1}{r_1} + \frac{1}{r_2} \qquad \text{and} \qquad \frac{1}{\gamma} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3}.$$

This immediately fixes $\frac{1}{\rho} = \frac{\nu}{\nu+1}$, $\frac{1}{\gamma} = 1 - \frac{1}{\nu+1}\frac{d}{4}(\nu-1)$ and $\frac{1}{q_3} = 1 - \frac{d}{4}(\nu-1)$. A short calculation confirms that indeed $\frac{1}{\rho} \in \left(\frac{1}{2}, \min\left\{1, \frac{1}{2}+\frac{1}{d}\right\}\right)$ and that all Hölder exponents lie in the interval $[1,\infty]$.

Summing over $f \in \{u, v, w\}$ finishes the proof. □

The aforementioned local well-posedness result is stated in

Theorem 3.11 (Local well-posedness in L^2)**.** *(Cf. [LP09, Theorem 5.2]) Let $d \in \mathbb{N}$, $\nu \in \left(1, 1+\frac{4}{d}\right)$ and $v_0 \in L^2(\mathbb{R}^d)$. Then there exists a $C = C(d,\nu) > 0$ such that the Cauchy problem for the mass-subcritical NLS, i.e.*

$$\begin{cases} \mathrm{i}v_t(x,t) + \Delta v(x,t) \pm \left(|v|^{(\nu-1)}\, v\right)(x,t) = 0 & (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ v(\cdot,0) = v_0, & \end{cases} \tag{3.17}$$

has a unique mild solution in

$$C([0,\delta], L^2(\mathbb{R}^d)) \cap L^{q_a(d,\nu+1)}([0,\delta], L^{\nu+1}(\mathbb{R}^d)) \quad \textit{provided} \quad \delta \le C\, \|v_0\|_{L^2(\mathbb{R}^d)}^{-\frac{1}{\frac{1}{\nu-1}-\frac{d}{4}}}.$$

The NLS with $\nu \in \left(1, 1+\frac{4}{d}\right)$ as in the theorem above is called *mass-subcritical* (cf. [KV13, Section 1.]).

Observe, that the uniqueness is claimed in the space $L^\infty L^2 \cap L^{q_a(d,\nu+1)} L^{\nu+1}$ only (*conditional uniqueness*). In fact, it is not immediately clear how to make sense of the nonlinearity in (3.17) for a general $u \in L^\infty L^2$.

Proof. For $\delta, R > 0$ set

$$\begin{aligned} X(\delta) &= C([0,\delta], L^2(\mathbb{R}^d)) \cap L^{q_a(d,\nu+1)}([0,\delta], L^{\nu+1}(\mathbb{R}^d)) \quad \text{and} \\ M(R,\delta) &= \left\{ f \in X(\delta) \,\middle|\, \|f\|_{X(\delta)} \le R \right\}, \end{aligned}$$

where $\|f\|_{X(\delta)} = \sup_{0\le t\le\delta} \|f(\cdot,t)\|_2 + \|f\|_{L^{q_a(d,\nu+1)}([0,\delta],L^{\nu+1}(\mathbb{R}^d))}$ for any $f \in X(\delta)$.

The Cauchy problem for the NLS is formulated as the corresponding integral equation

$$v(\cdot,t) = e^{\mathrm{i}t\Delta} v_0 \pm \mathrm{i} \int_0^t e^{\mathrm{i}(t-\tau)\Delta}\, |v|^{\nu-1}\, v \mathrm{d}\tau := (\mathcal{T} v)(\cdot,t)$$

and it is to show that for some $\delta, R > 0$ its right-hand side defines a contractive self-mapping $\mathcal{T} : M(R,\delta) \to M(R,\delta)$.

To fix R, consider the self-mapping property of $\mathcal{T}$ first. For the linear evolution part the estimate

$$\left\| e^{\mathrm{i}t\Delta} v_0 \right\|_{X(\delta)} \le C(d,\nu)\, \|v_0\|_2 \tag{3.18}$$

holds by Proposition 3.7 (the pair $(2,\infty)$ is also admissible). This suggests the choice $R = 2C(d,\nu)\|v_0\|_2$.

The integral part is estimated via Lemma 3.10 (observe that the assumptions on r are satisfied for $r=2$ and $r=\nu+1$) against

$$\begin{aligned}\left\|\int_0^t e^{\mathrm{i}(t-\tau)\Delta}|v|^{\nu-1}v\mathrm{d}\tau\right\|_{X(\delta)} &= \left\|\int_0^t e^{\mathrm{i}(t-\tau)\Delta}G(0,v,0)\mathrm{d}\tau\right\|_{X(\delta)} \\ &\lesssim_{d,\nu} \delta^{1-\frac{d}{4}(\nu-1)}\|v\|^{\nu}_{L^{q_{\mathrm{a}}(d,\nu)}([0,\delta],L^{\nu+1}(\mathbb{R}^d))} \\ &\leq \delta^{1-\frac{d}{4}(\nu-1)}R^{\nu} \leq R,\end{aligned} \tag{3.19}$$

which holds, provided that $\delta \lesssim_{d,\nu} \|v\|_2^{-\frac{1}{\frac{1}{\nu-1}-\frac{d}{4}}}$ (as assured by the assumptions).

The contraction property is shown in the same manner (i.e. via Lemma 3.10), only potentially making the implicit constant above smaller. This finishes the proof. □

The fact that the time-step δ in the last theorem depends on $\|v_0\|_2$ only, together with the conservation law $\|v(\cdot,t)\|_2 = \|v_0\|_2$ allows one to extend the local solution v globally. This has first been proven by Tsutsumi in 1987, cf. [Tsu87].

Proposition 3.12 (Global L^2 solutions). *(Cf. [LP09, Theorem 6.1]) Let $d \in \mathbb{N}$, $\nu \in \left(1, 1+\frac{4}{d}\right)$ and $v_0 \in L^2(\mathbb{R}^d)$. Then the Cauchy problem* (3.17) *has a unique mild solution v in*

$$C(\mathbb{R}, L^2(\mathbb{R}^d)) \cap L^{q_a(d,\nu+1)}_{loc}(\mathbb{R}, L^{\nu+1}(\mathbb{R}^d)).$$

The given proof of Theorem 3.11 implies also the following

Corollary 3.13 (Nonlinear Strichartz estimate). *(Cf. [LP09, Corollary 5.1]) Let $d \in \mathbb{N}$, $\nu \in \left(1, 1+\frac{4}{d}\right)$, $v_0 \in L^2(\mathbb{R}^d)$ and v be the global solution of the Cauchy problem* (3.17) *as in Proposition 3.12. Furthermore let $\frac{1}{r} \in \left[\frac{1}{2}, \max\left\{0, \frac{1}{2}-\frac{1}{d}\right\}\right)$. Then*

$$v \in L^{q_a(d,r)}_{loc}(\mathbb{R}, L^r(\mathbb{R}^d)).$$

More precisely, there is a constant $C = C(d,\nu,r) > 0$ such that the estimate

$$\|v\|_{L^{q_a(d,r)}([t,t+\delta],L^r(\mathbb{R}^d))} \leq \frac{1}{C}\|v(\cdot,t_0)\|_{L^2(\mathbb{R}^d)} \tag{3.20}$$

holds for any $t_0 \in \mathbb{R}$, provided

$$\delta \leq C\|v(\cdot,t_0)\|_{L^2(\mathbb{R}^d)}^{-\frac{\nu-1}{1-\frac{d}{4}(\nu-1)}}.$$

Proof. Consider $v(\cdot,t_0) \in L^2(\mathbb{R}^d)$ as the initial value in Theorem 3.11 and denote the unique solution constructed there by $\tilde{v} \in C([0,\delta], L^2(\mathbb{R}^d)) \cap L^{q_{\mathrm{a}}(d,\nu+1)}([0,\delta], L^{\nu+1}(\mathbb{R}^d))$. Observe, that the assumptions on r in Proposition 3.7 and Lemma 3.10 allow one to replace $X(\delta)$ by

$L^{q_{\mathrm{a}}(d,r)}([0,\delta], L^r(\mathbb{R}^d))$ in the inequality (3.18) and (3.19) (but the respective constant now additionally depends on r). This shows

$$\|\tilde{v}\|_{L^{q_{\mathrm{a}}(d,r)}([0,\delta],L^r(\mathbb{R}^d))} \lesssim_{d,\nu,r} \|v(\cdot,t_0)\|_2 .$$

Recalling that, by uniqueness of v, one has $v(\cdot,t) = \tilde{v}(\cdot,t-t_0)$ for $t \in [t_0, t_0+\delta]$ finishes the proof. $\square$

4. A local well-posedness result

Local well-posedness of the Cauchy problem for the nonlinear Schrödinger equation with an algebraic nonlinearity on a certain intersection of modulation spaces is presented in this chapter. A weaker version of this result, covering the range of parameters for which no intersection is neccessary, is in [CHKP16, Theorem 1]. Furthermore, this chapter contains a new Hölder-like inequality, which is also in [CHKP16, Theorem 3].

The aforementioned local well-posedness relies on the fact that the Schrödinger propagator is a strongly continuous group and the algebra property of the intersection. From [Fei83, Proposition 6.9 and Remark 6.4], i.e. since the very beginning of modulation spaces, it is known that certain modulation spaces are Banach *-algebras. The fact that the same is true for particular intersections of modulation spaces seems to be known in the community, see e.g. [STW11, remark before Proposition 3.2]. However, even a modern monograph like [WHHG11, Theorem 6.2] contains only a version of the local well-posedness result from [BO09, Theorem 1.1], which is weaker than the one in the thesis at hand.

This chapter is structured as follows. In Section 4.1 the algebra property of the intersection is stated and shown. Also, the Hölder-like inequality for certain modulation spaces is observed and proven. Subsequently, in Section 4.2, the notion of an algebraic nonlinearity is defined and the local well-posedness is derived. The chapter concludes with Section 4.3, which contains some comments on this and comparable local well-posedness results in the current literature.

4.1. Algebra property of $M^s_{p,q} \cap M_{\infty,1}$

Recall, that each $u \in M_{\infty,1}$ has a unique representation in C_{b} by Proposition 2.32. This allows a meaningful definition of multiplication and complex conjugation of elements of $M_{\infty,1}$. As $C_{\mathrm{b}} \hookrightarrow \mathcal{S}'$, the question whether $uv \in M^s_{p,q}$ or $\overline{u} \in M^s_{p,q}$ holds is also meaningful. Consider first the following technical

Lemma 4.1. *(Cf. [WZG06, Proof of Lemma 4.1]). Let $d \in \mathbb{N}$ and $u, v \in M_{\infty,1}(\mathbb{R}^d)$. Then there is a constant $C = C(d) > 0$ such that*

$$\|\Box_k(uv)\|_p \leq C \sum_{l\in\Lambda} \sum_{m\in\mathbb{Z}^d} \|(\Box_{k+l-m}u)(\Box_m v)\|_p \,, \tag{4.1}$$

where $\Lambda = \left\{ l \in \mathbb{Z}^d \middle|\ |l| < 3\sqrt{d} \right\}$.

Proof. One has

$$\Box_k(uv) = \Box_k \left[\left(\sum_{l\in\mathbb{Z}^d} \Box_l u \right) \cdot \left(\sum_{m\in\mathbb{Z}^d} \Box_m v \right) \right] = \sum_{l,m\in\mathbb{Z}^d} \Box_k \left[(\Box_{l-m} u) \cdot (\Box_m v) \right] \tag{4.2}$$

for any $k \in Z^d$. Above, the series in l and m are absolutely convergent in L^∞ which justifies taking the Cauchy product. Interchanging $\Box_k$ with the series is due to the continuity of the IDOs on $\mathcal{S}'$. Moreover, applying Proposition A.45, one has

$$\begin{aligned}(2\pi)^{\frac{d}{2}} \Box_k[(\Box_{l-m}u)\cdot(\Box_m v)] &= \mathcal{F}^{(-1)}\sigma_k \mathcal{F}[(\mathcal{F}^{(-1)}\sigma_{l-m}\hat{u})\cdot(\mathcal{F}^{(-1)}\sigma_m \hat{v})] \\ &= \mathcal{F}^{(-1)}\sigma_k[(\sigma_{l-m}\hat{u}) * (\sigma_m \hat{v})]\end{aligned}$$

for any $k, l, m \in \mathbb{Z}^d$. By Proposition A.43 one has

$$\operatorname{supp}(\sigma_k[(\sigma_{l-m}\hat{u}) * (\sigma_m\hat{v})]) \subseteq B_{\sqrt{d}}(k) \cap [\operatorname{supp}(\sigma_{l-m}) + \operatorname{supp}(\sigma_m)] \subseteq B_{\sqrt{d}}(k) \cap B_{2\sqrt{d}}(l),$$

where the right-hand side is the empty set, unless $k - l \in \Lambda$. This means that many summands in the double series over $l, m \in \mathbb{Z}^d$ in (4.2) vanish. More precisely, one has

$$\Box_k(uv) = \sum_{l\in\Lambda} \sum_{m\in\mathbb{Z}^d} \Box_k[(\Box_{k+l-m}u)\cdot(\Box_m v)]$$

for all $k \in \mathbb{Z}^d$. Taking the L^p-norm, invoking the triangle inequality and applying Corollary 2.8 shows (4.1) and finishes the proof. □

A Banach *-algebra X shall be a Banach algebra over $\mathbb{C}$ on which a continuous *involution* $*$ is defined, i.e. $(x+y)^* = x^* + y^*$, $(\lambda x)^* = \overline{\lambda} x^*$, $(xy)^* = y^* x^*$ and $(x^*)^* = x$ for any $x, y \in X$ and $\lambda \in \mathbb{C}$. It is neither required that X has a unit element nor that C in the continuity estimates

$$\|x \cdot y\| \leq C \|x\| \|y\|, \quad \|x^*\| \leq C \|x\| \qquad \forall x, y \in X \tag{4.3}$$

is equal to one. The proof of [STW11, Proposition 3.2] implies the following stronger statetement.

Proposition 4.2 (Algebra property)**.** *Let $d \in \mathbb{N}$, $p, q \in [1, \infty]$ and $s \geq 0$. Then $M^s_{p,q}(\mathbb{R}^d) \cap M_{\infty,1}(\mathbb{R}^d)$ is a Banach *-algebra w.r.t. pointwise multiplication and complex conjugation.*

Proof. Only the closedness of $M^s_{p,q} \cap M_{\infty,1}$ under pointwise multiplication and the continuity of this operation will be shown here. This is because all other properties of a Banach *-algebra are easily verified for $M^s_{p,q} \cap M_{\infty,1}$.

To that end consider two elements $u, v \in M^s_{p,q} \cap M_{\infty,1}$. Let Λ be as in Lemma 4.1. Observe that by Lemmas A.31 and A.30 one has

$$\langle k \rangle^s \lesssim_{d,s} \langle k + l \rangle^s \lesssim_s \langle k + l - m \rangle^s + \langle m \rangle^s$$

for any $k, m \in \mathbb{Z}^d$ and any $l \in \Lambda$. Inserting equation (4.1) into the definition of the modulation space norm, employing the estimate above and applying the triangle inequality yields

$$\begin{aligned} \|uv\|_{M^s_{p,q}} &= \left\| \left(\langle k \rangle^s \|\Box_k(uv)\|_p \right)_k \right\|_q \qquad (4.4) \\ &\lesssim_{d,s} \left\| \left(\sum_{l\in\Lambda} \sum_{m\in\mathbb{Z}^d} \langle k+m-l \rangle^s \|(\Box_{k+l-m}u)(\Box_m v)\|_p \right)_k \right\|_q \\ &\quad + \left\| \left(\sum_{l\in\Lambda} \sum_{m\in\mathbb{Z}^d} \langle m \rangle^s \|(\Box_{k+l-m}u)(\Box_m v)\|_p \right)_k \right\|_q . \end{aligned}$$

Consider the first summand. By Hölder's inequality, one has

$$\|(\Box_{k+l-m}u)(\Box_m v)\|_p \leq \|\Box_{k+l-m}u\|_p \|\Box_m v\|_\infty \qquad (4.5)$$

for any $k, l, m \in \mathbb{Z}^d$. Inserting this estimate into the first summand above and subsequently invoking Young's inequality yields

$$\begin{aligned} &\left\| k \mapsto \sum_{l\in\Lambda} \sum_{m\in\mathbb{Z}^d} \langle k+m-l \rangle^s \|(\Box_{k+l-m}u)(\Box_m v)\|_p \right\|_q \\ \lesssim_d &\left\| \left(\langle k \rangle^s \|(\Box_k u\|_p \right)_k * (\|\Box_m v\|_\infty)_m \right\|_q \leq \|u\|_{M^s_{p,q}} \|v\|_{M_{\infty,1}} , \end{aligned}$$

where "$*$" denotes the discrete convolution of sequences.

The other summand is estimated in the same way, i.e.

$$\left\| k \mapsto \sum_{l\in\Lambda} \sum_{m\in\mathbb{Z}^d} \langle m \rangle^s \|(\Box_{k+l-m}u)(\Box_m v)\|_p \right\|_q \lesssim_d \|u\|_{M_{\infty,1}} \|v\|_{M^s_{p,q}} ,$$

which yields

$$\|uv\|_{M^s_{p,q}} \lesssim_{d,s} \left(\|u\|_{M^s_{p,q}} \|v\|_{M_{\infty,1}} + \|u\|_{M_{\infty,1}} \|v\|_{M^s_{p,q}} \right) \lesssim \|u\|_{M^s_{p,q}\cap M_{\infty,1}} \|v\|_{M^s_{p,q}\cap M_{\infty,1}} .$$

For $s = 0$, $p = \infty$ and $q = 1$ this shows $\|uv\|_{M_{\infty,1}} \lesssim_{d,s} \|u\|_{M_{\infty,1}} \|v\|_{M_{\infty,1}}$ and hence implies

$$\|uv\|_{M^s_{p,q}\cap M_{\infty,1}} \lesssim_{d,s} \|u\|_{M^s_{p,q}\cap M_{\infty,1}} \|v\|_{M^s_{p,q}\cap M_{\infty,1}}$$

completing the proof. $\square$

If $q = 1$ and $s \geq 0$ or if $q > 1$ and $s > \frac{d}{q'}$, then, by Proposition 2.31, $M^s_{p,q} \hookrightarrow M_{\infty,1}$ and the intersection in Proposition 4.2 is superfluous. In this case one even has the following

Theorem 4.3 (Hölder-like inequality)**.** *Let $d \in \mathbb{N}$ and $p, p_1, p_2, \in [1,\infty]$ satisfy the* Hölder condition $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. *Furthermore, let $q \in [1,\infty]$. For $q = 1$ let $s \geq 0$, for $q > 1$ let $s > \frac{d}{q'}$. Then there is a constant $C = C(d,q,s) > 0$ such that for any $u \in M^s_{p_1,q}(\mathbb{R}^d)$ and any $v \in M^s_{p_2,q}(\mathbb{R}^d)$ one has $uv \in M^s_{p,q}(\mathbb{R}^d)$ and*

$$\|uv\|_{M^s_{p,q}(\mathbb{R}^d)} \leq C \|u\|_{M^s_{p_1,q}(\mathbb{R}^d)} \|v\|_{M^s_{p_2,q}(\mathbb{R}^d)} . \tag{4.6}$$

Proof. As in the proof of Proposition 4.2 one arrives at the inequality (4.4). In contrast to (4.5), one chooses the Hölder exponents differently, namely

$$\|(\Box_{k+l-m}u)(\Box_m v)\|_p \leq \|\Box_{k+l-m}u\|_{p_1} \|\Box_m v\|_{p_2} .$$

Inserting this estimate into (4.4) and invoking Young's inequality shows

$$\|uv\|_{M^s_{p,q}} \lesssim_{d,s} \|u\|_{M^s_{p_1,q}} \|v\|_{M_{p_2,1}} + \|u\|_{M_{p_1,1}} \|v\|_{M^s_{p_2,q}} .$$

As $\|v\|_{M_{p_2,1}} \lesssim_{d,q,s} \|v\|_{M^s_{p_2,q}}$ and $\|u\|_{M_{p_1,1}} \lesssim_{d,q,s} \|u\|_{M^s_{p_1,q}}$ by Proposition 2.31, the inequality (4.6) holds and the proof is completed. □

This result easily generalizes to $N \in \mathbb{N}$ factors and $0 < p, p_1, \ldots, p_N \leq \infty$. Hence, it extends the multilinear estimate [BO09, eqn. (2.4)] to the case $q_0 = \ldots = q_m \geq 1$.

4.2. Local well-posedness for algebraic nonlinearities

Definition 4.4 (Algebraic nonlinearities)**.** Let X be a Banach *-algebra and $(c_k) \in \mathbb{C}^{\mathbb{N}}$ such that $\lim_{k\to\infty} \sqrt[k]{|c_k|} = 0$. A mapping $F : X \to X$ given by

$$F(u) \coloneqq \sum_{k=1}^{\infty} c_k (uu^*)^k u \qquad \forall u \in X$$

is called an *algebraic nonlinearity* on X.

Lemma 4.5. *Let X be a Banach *-algebra and F an algebraic nonlinearity on X. Then F is locally Lipschitz continuous, i.e. for any $R > 0$ there is an $L = L(R) > 0$ such that for any $u, v \in X$ with $\|u\|, \|v\| \leq R$ the inequality*

$$\|F(u) - F(v)\| \leq L \|u - v\| \tag{4.7}$$

holds.

Proof. Let $R > 0$ and consider any $u, v \in X$ with $\|u\|, \|v\| \leq R$. Then

$$\|F(u) - F(v)\| \leq \sum_{k=1}^{\infty} |c_k| \left\| u^{k+1}(u^*)^k - v^{k+1}(v^*)^k \right\| \tag{4.8}$$

Fix a $k \in \mathbb{N}$. One has

$$\left\| u^{k+1}(u^*)^k - v^{k+1}(v^*)^k \right\| \leq \int_0^1 \left\| \frac{\partial}{\partial \tau}(v+\tau(u-v))^{k+1}(v^*+\tau(u^*-v^*))^k \right\| \mathrm{d}\tau,$$

where for every $\tau \in [0,1]$

$$\begin{aligned} & \frac{\partial}{\partial \tau}(v+\tau(u-v))^{k+1}(v^*+\tau(u^*-v^*))^k \\ = \; & (k+1)(u-v)(v+\tau(u-v))^k(v^*+\tau(u^*-v^*))^k \\ & +k(u^*-v^*)(v+\tau(u-v))^{k+1}(v^*+\tau(u^*-v^*))^{k-1} \end{aligned}$$

holds. Taking the norm and using the continuity estimates (4.3) yields

$$\begin{aligned} \left\| \frac{\partial}{\partial \tau}(v+\tau(u-v))^{k+1}(v^*+\tau(u^*-v^*))^k \right\| & \leq \|u-v\| \, C^{2k+2}(2k+1)(\|v\| + \tau \|u-v\|)^{2k} \\ & \leq \|u-v\| \, C^2(2k+1)(3CR)^{2k} \end{aligned}$$

for every $\tau \in [0,1]$. Reinserting the above into (4.8) shows

$$\|F(u)-F(v)\| \leq \|u-v\| \, C^2 \sum_{k=1}^{\infty}(2k+1) \, |c_k| \, (3CR)^{2k}.$$

As series above converges due to the decay assumption on (c_k), (4.7) holds and the proof is complete. □

Theorem 4.6 (Local well-posedness)**.** *Let $d \in \mathbb{N}$, $p \in [1,\infty]$, $q \in [1,\infty)$ and $s \geq 0$. Moreover, let F be an algebraic nonlinearity on $X := M^s_{p,q}(\mathbb{R}^d) \cap M_{\infty,1}(\mathbb{R}^d)$ and $u_0 \in X$. Then the Cauchy problem (1.3) for the NLS in X has a unique maximal mild solution $u : C((-a,b),X)$, where $a,b > 0$. The blow-up alternative holds, i.e. if $a < \infty$, then* $\liminf_{t\to a+} \|u(t)\| = \infty$ *and if $b < \infty$, then* $\liminf_{t \to b-} \|u(t)\| = \infty$. *Finally, the map $u_0 \mapsto u$ is locally Lipschitz continuous.*

Proof. By Proposition 3.5, $e^{\mathrm{i}t\Delta}$ is a C_0-group on $M^s_{p,q}$ and on $M_{\infty,1}$ and hence it is also a C_0-group on X. Let A denote its generator. Clearly, $C := M^{s+2}_{p,q} \cap M^2_{\infty,1} \subseteq \mathrm{dom}(A)$ and

$$Au = \mathrm{i}\Delta \qquad \forall u \in C.$$

Also, C is dense in X by Lemma 2.16 and $e^{\mathrm{i}t\Delta}u \in C$ for all $t \in \mathbb{R}$ and $u \in C$ by Theorem 3.4. Proposition A.23 hence implies that C is a core for A. As

$$\begin{aligned} \|u\|_A & = \|u\|_X + \|\mathrm{i}\Delta u\|_X = \|u\|_{M^s_{p,q}} + \|\mathrm{i}\Delta u\|_{M^s_{p,q}} + \|u\|_{M_{\infty,1}} + \|\mathrm{i}\Delta u\|_{M_{\infty,1}} \\ & \approx \|u\|_{M^{s+2}_{p,q}} + \|u\|_{M^2_{\infty,1}} = \|u\|_C \qquad \forall u \in C, \end{aligned}$$

it follows that $C = \mathrm{dom}(A)$ and $A = \mathrm{i}\Delta$.

Hence, the integral equation corresponding to the given Cauchy problem is indeed

$$u = e^{\mathrm{i}t\Delta}u_0 + \int_0^t e^{\mathrm{i}(t-s)\Delta}F(u(s))\mathrm{d}s.$$

As F is locally Lipschitz continuous by Proposition 4.2 and Lemma 4.5, the claim follows by Proposition A.24 and hence the proof is complete. □

4.3. Comments

The proof of the algebra property above is largely inspired by the proofs of [WZG06, Lemma 4.4] and [STW11, Proposition 3.2]. The former work also includes a version of Theorem 4.6 for the space $M_{2,1}$. A version of the local well-posedness for $M_{p,1}$ from [BO09, Theorem 1.1] is proven there via the theory of pseudo-differential operators.

Observe, that Theorem 4.6 immediately generalizes to other dispersive equations, for which the respective group is strongly continuous on $M^s_{p,q} \cap M_{\infty,1}$. Examples include the nonlinear wave and the nonlinear Klein-Gordon equations (cf. [BO09, Theorem 1.2 and 1.3]).

Other results of local well-posedness of a nonlinear Schrödinger equation for initial data in a modulation space include the following. In [Guo17, Theorem 1.4] local well-posedness of the cubic NLS in one dimension in the space $M_{2,q}$ with $q \in [2, \infty)$ is shown.

The same equation with the same nonlinerity is treated in [Pat18] in the space $M^s_{2,q}$. There, existence is obtained for $q \in [1, 2]$ and $s \geq 0$. For $q \in \left[1, \frac{3}{2}\right]$ and $s \geq 0$ or $q \in \left(\frac{3}{2}, 2\right]$ and $s > \frac{3}{2} - \frac{1}{q}$ even unconditional well-posedness holds.

This result is generalized in [CHKP18] in the following way. For $q \in [1, 2]$, $s \geq 0$ and $p \in \left[2, \frac{10q'}{q'+6}\right)$ existence is obtained in $M^s_{p,q}$. For $q \in \left[1, \frac{3}{2}\right]$, $s \geq 0$ and $p \in [2, 3]$ or $q \in \left(\frac{3}{2}, \frac{18}{11}\right]$, $s > \frac{2}{3} - \frac{1}{q}$ and $p \in \left[2, \frac{10q'}{10q'+6}\right)$ even unconditional well-posedness holds.

In [STW11, Theorem 4.2] nonlinearities of the form $F(u)$ are treated in the Banach *-algebra $M^s_{p,2}$ ($s > \frac{d}{2}$), where $F : \mathbb{R} \to \mathbb{R}$ is sufficiently often continuously differentiable and sufficiently many derivatives of F vanish in the origin. Due to an upper bound on p in terms of the number of the derivatives of F to vanish in the origin, this does not give rise to a well-posedness result interesting for the model problem from the introduction, which would require $p = \infty$ due to the form of the initial values stated in Equation (1.2).

Negative results concerning the construction of nonlinearities in $M_{p,q}$ include [RSTT09, Theorem 2.4 and 2.6].

5. A Global well-posedness result

In this chapter a global well-posedness result stated in Theorem 5.4 is presented. It deals with the Cauchy problem

$$\begin{cases} \mathrm{i}u_t(x,t) + \Delta u(x,t) \pm \left(|u|^{(\nu-1)}\,u\right)(x,t) = 0 \qquad (x,t) \in \mathbb{R}^d \times \mathbb{R}, \\ \qquad\qquad\qquad\qquad\qquad u(\cdot,0) = u_0, \end{cases} \tag{5.1}$$

for the mass-subcritical (i.e. $\nu \in \left(1, 1+\frac{4}{d}\right)$) NLS and $u_0 \in M_{p,p'}(\mathbb{R}^d)$ and its mild solutions, i.e. solutions to the corresponding integral equation

$$u(\cdot,t) = e^{\mathrm{i}t\Delta}u_0 \pm \mathrm{i}\int_0^t e^{\mathrm{i}(t-\tau)\Delta}\left(|u|^{\nu-1}\,u\right)(\tau)\mathrm{d}\tau. \tag{5.2}$$

A much weaker version (with $d = 1$, $\nu = 3$ and a smaller range of allowed p's) of the theorem has already been published in [CHKP17].

This work is inspired by the article [HT12] of Hyakuna and Tsutsumi, where, motivated by the work [VV01] of Vargas and Vega, they successfully adapt Bourgain's *high-low frequency decomposition* (HLFD) method (see e.g. [Bou99, Section IV.3], [Tao06, Section 3.9] and [ET16, Section 4.2]) to initial data in $\widehat{L}^p(\mathbb{R})$ for p sufficiently close to 2.

At the heart of the HLFD is the following idea: Consider a splitting of the initial datum u_0 into a good part $v_0 \in X_0$ ("low frequencies") and a bad part $w_0 \in Y_0$ ("high frequencies"). Assume, that local well-posedness of the NLS with IV in X_0 is already known in a space $X \subseteq C(I, X_0)$ (here, $I \ni 0$ denotes a time interval). Assume further that linear theory in the space $Y \subseteq C(\mathbb{R}, Y_0)$ has already been developed. Using nonlinear smoothing (i.e. control of the X-norm of the integral in (5.2)) show local well-posedness of (5.2) in $X+Y$.

To show global existence, assume that global existence of (5.2) for initial values in X_0 is known in X and relies on a conserved quantity $M(v(t)) = M(v_0)$ (say, mass conservation). Try to construct solutions u of the form $u = (v+w) + e^{\mathrm{i}t\Delta}w_0$, where $v \in X$ is the nonlinear time evolution of v_0, $e^{\mathrm{i}t\Delta}w_0 \in Y$ is the linear evolution of w_0 and $w \in Y$ is their nonlinear interaction term. Using interpolation theory (i.e. assuming that $u_0 \in (X_0, Y_0)_{\theta,\infty}$), argue that $\|w_0\|_{Y_0}$ can be made arbitrarily small resulting in $v + w$ being close to v in X and the quantity $M((v+w)(t))$, although no longer conserved, growing slowly enough to yield a global solution.

The remainder of this chapter is structured as follows: In Section 5.1 the splitting of the initial data is made precise, the notion of a solution to (5.1) is fixed and the results of local well-posedness (Theorem 5.3) and global existence (Theorem 5.4) are formulated. The proof of the former theorem is given in Section 5.2. Existence and properties of the perturbation w, which was introduced above, are provided in Section 5.3. Globality of solutions is proven in Section 5.4. The chapter concludes with Section 5.5, which includes a literature survey and a comparison of the achieved result to other works.

5.1. Statement of the results

The splitting of the initial data is done via the following

Proposition 5.1. *Let $d \in \mathbb{N}$, $r > 2$ and $p \in (2, r)$. Then there exists a constant $C = C(d, p, r)$ such that for any $u \in M_{p,p'}(\mathbb{R}^d)$ and $N > 0$ there are $v \in L^2(\mathbb{R}^d)$ and $w \in M_{r,r'}(\mathbb{R}^d)$ satisfying*

$$u = v + w, \quad \|v\|_{L^2(\mathbb{R}^d)} \leq C \|u\|_{M_{p,p'}(\mathbb{R}^d)} N^\alpha \quad \textit{and} \quad \|w\|_{M_{r,r'}(\mathbb{R}^d)} \leq C \|u\|_{M_{p,p'}(\mathbb{R}^d)} \frac{1}{N},$$

where $\alpha = \frac{\frac{1}{2} - \frac{1}{p}}{\frac{1}{p} - \frac{1}{r}}$ shall be called the trading exponent.

Proof. By Proposition 2.19 one has

$$M_{p,p'} = [L^2, M_{r,r'}]_\theta \qquad \text{for} \qquad \theta = \frac{\frac{1}{2} - \frac{1}{p}}{\frac{1}{2} - \frac{1}{r}}.$$

Furthermore, by Theorem A.63,

$$[L^2, M_{r,r'}]_\theta \hookrightarrow \left(L^2, M_{r,r'}\right)_{(\theta,\infty)}$$

holds. Given a $u \in M_{p,p'}$, recall equation (A.29) to obtain

$$\|u\|_{\left(L^2, M_{r,r'}\right)_{(\theta,\infty)}} = \sup_{t>0} \inf_{\substack{u=v+w \\ v \in L^2, w \in M_{r,r'}}} \left(t^{-\theta} \|v\|_2 + t^{1-\theta} \|w\|_{M_{r,r'}} \right).$$

Given an $N > 0$ consider $t = N^{\frac{1}{1-\theta}}$ in the formula above. Observing that

$$\frac{\theta}{1-\theta} = \frac{\frac{\frac{1}{2}-\frac{1}{p}}{\frac{1}{2}-\frac{1}{r}}}{1 - \frac{\frac{1}{2}-\frac{1}{p}}{\frac{1}{2}-\frac{1}{r}}} = \frac{\frac{1}{2} - \frac{1}{p}}{\frac{1}{p} - \frac{1}{r}} = \alpha$$

shows that there are indeed $v \in L^2$ and $w \in M_{r,r'}$ such that $u = v + w$ and

$$N^{-\alpha} \|v\|_2 + N \|w\|_{M_{r,r'}} \lesssim \|u\|_{(L^2, M_{r,r'})_{(\theta,\infty)}} \lesssim \|u\|_{[L^2, M_{r,r'}]_\theta}.$$

Rearranging this inequality shows the required estimates and finishes the proof. □

Next, the notion of a solution to (5.1) needs to be fixed. This is done in the following

Definition 5.2 (Mild solutions of the NLS)**.** Let $d \in \mathbb{N}$, $\nu \in \left(1, 1 + \frac{4}{d}\right)$. Consider $u_0 = v_0 + w_0 \in L^2(\mathbb{R}^d) + M_{(\nu+1),(\nu+1)'}(\mathbb{R}^d)$ and a $T > 0$. A function $u : \mathbb{R}^d \times [0, T) \to \mathbb{C}$ is said to be a (*mild*) solution of (5.1) up to time T, if it satisfies the corresponding integral equation (5.2) in $C([0, T'], L^2(\mathbb{R}^d)) \cap L^{q_a(d,\nu+1)}([0, T'], L^{\nu+1}(\mathbb{R}^d)) + C([0, T'], M_{(\nu+1),(\nu+1)'}(\mathbb{R}^d))$ for any $T' \in (0, T)$. The supremum of all such T is called *maximal time of existence* T_*.

A solution is called *global* if $T_* = \infty$. It is called *unique*, if any other solution u_1 up to time T_1 satisfies $u_1|_{T'} = u|_{T'}$ for any $T' \in [0, \min\{T_1, T_*\})$.

The choice of $M_{r,r'} = M_{(\nu+1),(\nu+1)'}$ in the definition above is such that the integral in the Duhamel's formula (5.2) makes sense. This will become more clear in the proof of the local well-posedness formulated in

Theorem 5.3 (Local well-posedness for initial values in $L^2 + M_{(\nu+1),(\nu+1)'}$)**.** *Let $d \in \mathbb{N}$, $\nu \in \left(1, 1 + \frac{4}{d}\right)$ and $u_0 \in L^2(\mathbb{R}^d) + M_{(\nu+1),(\nu+1)'}(\mathbb{R}^d)$. Then, there exists a unique maximal mild solution u of* (5.1) *(in the sense of Definition 5.2) and the* blow-up alternative

$$T_* < \infty \qquad \Rightarrow \qquad \limsup_{t \to T_*-} \|u(\cdot, t)\|_{L^2(\mathbb{R}^d) + M_{(\nu+1),(\nu+1)'}(\mathbb{R}^d)} = \infty$$

holds. Moreover, there is a $C = C(d, \nu) > 0$ such that

$$T_* \geq C \, \|u_0\|_{L^2(\mathbb{R}^d) + M_{(\nu+1),(\nu+1)'(\mathbb{R}^d)}}^{-\frac{\nu-1}{1-\frac{d}{4}(\nu-1)}}.$$

Finally, there is a $T' \in (0, T_*)$ and a neighborhood V of u_0 in $L^2(\mathbb{R}^d) + M_{(\nu+1),(\nu+1)'}(\mathbb{R}^d)$, such that the initial-data-to-solution-map

$$V \to C([0, T'], L^2(\mathbb{R}^d)) \cap L^{q_a(d,\nu+1)}([0, T'], L^{\nu+1}(\mathbb{R}^d)) + C([0, T'], M_{(\nu+1),(\nu+1)'}(\mathbb{R}^d))$$

is Lipschitz continuous.

Local well-posedness for IVs in $L^2(\mathbb{R}^d) + M_{(\nu+1),(\nu+1)'}(\mathbb{R}^d)$ implies uniqueness for smaller spaces such as $M_{p,p'}(\mathbb{R}^d)$. These are used to construct global solutions in

Theorem 5.4 (Global well-posedness for initial values in $M_{p,p'}$)**.** *(Cf. [CHKP17, Theorem 3]) Let $d \in \mathbb{N}$, $\nu \in \left(1, 1 + \frac{4}{d}\right)$ and $p \in (2, p_{max})$, where*

$$p_{max} = \begin{cases} 2 + \frac{2}{\nu} - \frac{d}{2}\left(1 - \frac{1}{\nu}\right) & \text{if } \nu > \frac{1}{2} - \frac{d}{4} + \sqrt{2 + \left(\frac{1}{2} + \frac{d}{4}\right)^2}, \\ \nu + 1 & \text{otherwise.} \end{cases} \tag{5.3}$$

Then the Cauchy problem (5.1) with initial data $u_0 \in M_{p,p'}(\mathbb{R}^d)$ has a unique global solution (in the sense of Definition 5.2).

Observe, that the uniqueness in the two theorems above is a conditional one. That means that the solutions are not guaranteed to be unique in

$$C(M_{p,p'}) \hookrightarrow L^\infty(L^2 + M_{(\nu+1),(\nu+1)'}) = L^\infty L^2 + L^\infty M_{(\nu+1),(\nu+1)'},$$

but in the space $L^\infty L^2 \cap L^{q_{\mathrm{a}}(d,\nu+1)} L^{\nu+1} + L^\infty M_{(\nu+1),(\nu+1)'}$ only, which is smaller than the right-hand side of the formula above.

While this is similar to the situation of Theorem 3.11 (it is again not obvious how to even make sense of the nonlinearity for a general $u \in C(M_{p,p'})$), the solutions now (at least possibly) lack persistence in the sense that it is not clear that $u \in C(M_{p,p'})$.

5.2. Proof of the local well-posedness

The proof is similar to that of Theorem 3.11. The linear evolution of initial data u_0 poses no problem, as it splits as the initial data. The integral part is handled using the embedding $L^\infty M_{(\nu+1),(\nu+1)'} \hookrightarrow L^{q_{\mathrm{a}}(d,\nu+1)} L^{\nu+1}$ allowing for the usual Strichartz estimates. Recall the notation $G(u,v,w) = |u+v|^{\nu-1} v - |u+w|^{\nu-1} w$ from equation (3.15). Consider the

Proof of Theorem 5.3. For $\delta, R > 0$ set

$$\begin{aligned} X_1(\delta) &= C([0,\delta], L^2(\mathbb{R}^d)) \cap L^{q_{\mathrm{a}}(d,\nu+1)}([0,\delta], L^{\nu+1}(\mathbb{R}^d)), \\ X_2(\delta) &= C([0,\delta], M_{(\nu+1),(\nu+1)'}(\mathbb{R}^d)), \\ X(\delta) &= X_1(\delta) + X_2(\delta) \quad \text{and} \\ M(R,\delta) &= \left\{ f \in X(\delta) \,\middle|\, \|f\|_{X(\delta)} \le R \right\}, \end{aligned}$$

where

$$\|f\|_{X(\delta)} = \inf_{\substack{f=g+h \\ g\in X_1(\delta), \\ h\in X_2(\delta)}} \left(\|g\|_{X_1(\delta)} + \|h\|_{X_2(\delta)} \right)$$

for any $f \in X(\delta)$. Consider an arbitrary decomposition of $u_0 = v_0 + w_0$, where $v_0 \in L^2$ and $w_0 \in M_{(\nu+1),(\nu+1)'}$, and of $u = v + w \in X(\delta)$, where $v \in X_1(\delta)$ and $w \in X_2(\delta)$. It is to show that the right-hand side of (5.2) defines a contractive self-mapping $\mathcal{T} : M(R,\delta) \to M(R,\delta)$ for some $\delta, R > 0$.

To fix R, consider the self-mapping property of $\mathcal{T}$ fist. For the linear evolution part one has the estimate

$$\begin{aligned} \left\| e^{\mathrm{i}t\Delta} u_0 \right\|_{X(\delta)} &\le \left\| e^{\mathrm{i}t\Delta} v_0 \right\|_{X_1(\delta)} + \left\| e^{\mathrm{i}t\Delta} w_0 \right\|_{X_2(\delta)} \\ &\lesssim_{d,\nu} \|v_0\|_2 + (1+\delta)^{d\left(\frac{1}{2} - \frac{1}{\nu+1}\right)} \|w_0\|_{M_{(\nu+1),(\nu+1)'}} \\ &\lesssim_d \|v_0\|_2 + \|w_0\|_{M_{(\nu+1),(\nu+1)'}}, \end{aligned}$$

where the first inequality is due to the fact that $X(\delta)$ is the sum of $X_1(\delta)$ and $X_2(\delta)$, the second to Proposition 3.7 and Theorem 3.4 and the last one to the assumption $\delta \le 1$ (which is made here w.l.o.g.). As the decomposition $u_0 = v_0 + w_0$ was arbitrary, it follows that

$$\left\| e^{\mathrm{i}t\Delta} u_0 \right\|_{X(\delta)} \le C(d,\nu) \|u_0\|_{L^2 + M_{(\nu+1),(\nu+1)'}}. \tag{5.4}$$

This suggests the choice $R = 2C(d,\nu)\left\|u_0\right\|_{L^2+M_{(\nu+1),(\nu+1)'}}$. From now on, set

$$R(r) \coloneqq 2C(d,\nu)r \qquad \forall r > 0.$$

Before considering the integral part, observe that $M_{(\nu+1),(\nu+1)'} \hookrightarrow L^{\nu+1}$ by Proposition 2.34. Hence, by Hölder's inequality for the time variable and the assumption that $\delta \leq 1$,

$$\|w\|_{L^{q_a(d,\nu+1)}([0,\delta],L^{\nu+1}(\mathbb{R}^d))} \leq \delta^{\frac{1}{q_a(d,\nu+1)}} \|w\|_{L^\infty([0,\delta),M_{(\nu+1),(\nu+1)'}(\mathbb{R}^d)} \leq \|w\|_{X_2(\delta)}$$

follows, which in its turn implies

$$\|u\|_{L^{q_a(d,\nu+1)}L^{\nu+1}} \leq \|v\|_{L^{q_a(d,\nu+1)}L^{\nu+1}} + \|w\|_{L^{q_a(d,\nu+1)}L^{\nu+1}} \lesssim_d \|v\|_{X_1(\delta)} + \|w\|_{X_2(\delta)}.$$

As the decomposition $u = v + w$ was arbitrary, $\|u\|_{L^{q_a(d,\nu+1)}L^{\nu+1}} \lesssim_d \|u\|_{X(\delta)}$ follows.

The integral part is estimated by Lemma 3.10 (put $r \in \{\nu+1, 2\}$ there) against

$$\begin{aligned}
\left\|\int_0^t e^{i(t-\tau)\Delta}\left(|u|^{\nu-1}u\right)(\tau)\mathrm{d}\tau\right\|_{X_1(\delta)} &= \left\|\int_0^t e^{i(t-\tau)\Delta}G(0,u,0)\mathrm{d}\tau\right\|_{X_1(\delta)} \\
&\lesssim_{d,\nu} \delta^{1-\frac{d}{4}(\nu-1)}\|u\|^{\nu}_{L^{q_a(d,\nu+1)}L^{\nu+1}} \\
&\lesssim_d \delta^{1-\frac{d}{4}(\nu-1)}\|u\|^{\nu}_{X(\delta)} \leq \delta^{1-\frac{d}{4}(\nu-1)}R^{\nu},
\end{aligned}$$

where the penultimate inequality is due to the observation above and G is defined just before Lemma 3.9. The estimates of the linear evolution and of the integral part show that the self-mapping property holds, if

$$\delta \lesssim_{d,\nu} \|u_0\|_{L^2+M_{(\nu+1),(\nu+1)'}}^{-\frac{\nu-1}{1-\frac{d}{4}(\nu-1)}}. \tag{5.5}$$

For the contraction property of $\mathcal{T}$ apply Lemma 3.10 again to observe that

$$\begin{aligned}
\|\mathcal{T}(u) - \mathcal{T}(v)\|_{X_1(\delta)} &= \left\|\int_0^t e^{i(t-\tau)\Delta}G(0,u,v)\mathrm{d}\tau\right\|_{X_1(\delta)} \\
&\lesssim_{d,\nu} \delta^{1-\frac{d}{4}(\nu-1)}\left(\|u\|^{\nu-1}_{L^{q_a(d,\nu+1)}L^{\nu+1}} + \|v\|^{\nu-1}_{L^{q_a(d,\nu+1)}L^{\nu+1}}\right) \\
&\quad \cdot \|u-v\|_{L^{q_a(d,\nu+1)}L^{\nu+1}} \\
&\lesssim_d \delta^{1-\frac{d}{4}(\nu-1)}R^{\nu-1}\|u-v\|_{X(\delta)}.
\end{aligned}$$

This means that the condition on δ sufficient for $\mathcal{T}$ to be contractive only imposes an additional smallness assumption on the implicit constant in (5.5). From now on, set

$$\delta(r) \coloneqq \min\left\{C(d,\nu)r^{-\frac{\nu-1}{1-\frac{d}{4}(\nu-1)}}, 1\right\} \qquad \forall r > 0,$$

where $C(d,\nu)$ is chosen so small, that all previous requirements are fulfilled.

By the above and Banach's fixed-point theorem, there is exactly one $u \in M(R, \delta)$ such that $u = \mathcal{T}(u)$. That means that this u is a solution of (5.2) in the sense of Definition 5.2 and $u = v$ for any other solution $v : [0, \delta] \to C([0, \delta], M_{(\nu+1),(\nu+1)'} + C([0, \delta], M_{(\nu+1),(\nu+1)'}$, if $\|v\|_{L^\infty(L^2) \cap L^{q_a(\nu+1)}(L^{\nu+1}) + L^\infty(M_{(\nu+1),(\nu+1)'})} \leq R \approx_{d,\nu} \|u_0\|_{L^2 + M_{(\nu+1),(\nu+1)'}}$.

To show uniqueness, assume $u^1 \in X(T_1)$ and $u^2 \in X(T_2)$ both satisfy (5.2) for some $T_1, T_2 > 0$. One has $u^1(\cdot, 0) = u^2(\cdot, 0) = u_0$ and hence

$$T := \sup \left\{ S \in [0, \min\{T_1, T_2\}) \middle| \forall t \in [0, S] : u^1(\cdot, t) = u^2(\cdot, t) \right\} \geq 0.$$

Assume that $T < \min\{T_1, T_2\}$. By continuity, one has $u^1(\cdot, T) = u^2(\cdot, T) =: u_1$. Hence, u^1 and u^2 both solve the time-shifted version of (5.2)

$$u(\cdot, t) = e^{\mathrm{i}(t-T)\Delta} u_1 \pm \mathrm{i} \int_T^t e^{\mathrm{i}(t-\tau)\Delta} \left(|u|^{\nu+1} u \right)(\tau) \mathrm{d}\tau \tag{5.6}$$

on $[T, \min T_1, T_2)$. Furthermore, for any $i \in \{1, 2\}$,

$$\left\| u^i|_{[T,T+\varepsilon]} \right\|_{X(\min\{T_1,T_2\})} \xrightarrow{\varepsilon \to 0+} \|u_1\|_{L^2 + M_{(\nu+1),(\nu+1)'}}$$

by the dominated convergence theorem (for the norm in $L^{q_a(\nu+1)}(L^{\nu+1})$) and continuity (for the norms in $L^\infty(L^2)$ and $L^\infty(M_{(\nu+1),(\nu+1)'})$). Hence, for both $i \in \{1, 2\}$, one has $\left\| u^i|_{[T,T+\varepsilon]} \right\|_{X(\min\{T_1,T_2\})} \lesssim \|u_1\|_{L^2 + M_{(\nu+1),(\nu+1)'}}$ for some $\varepsilon > 0$. This fact allows one to apply the uniqueness statement of the Banach's fixed-point theorem from above to (5.6) to conclude that $u^1(\cdot, t) = u^2(\cdot, t)$ for all $t \in [T, T + \varepsilon]$, if $\varepsilon > 0$ is sufficiently small. This contradicts the definition of T and hence $T = \min\{T_1, T_2\}$ follows.

To show the blow-up alternative, let u now denote the maximal solution, which is unique by the above, and let $T_* < \infty$. Assume that $\limsup_{t \to T_*-} \|u(\cdot, t)\|_{L^2 + M_{(\nu+1),(\nu+1)'}} < \infty$, i.e.

$$\sup_{t \in [0, T_*]} \|u(\cdot, t)\|_{L^2 + M_{(\nu+1),(\nu+1)'}} =: S < \infty.$$

But then, given any time point $T \in [0, T_*)$, the solution u is defined at least up to $t + \delta(S)$ by the Banach's fixed-point theorem applied to (5.6). This contradicts $T_* < \infty$.

For the local Lipschitz continuity, fix any $\varepsilon > 0$, put $r := \varepsilon + \|u_0\|_{L^2 + M_{(\nu+1),(\nu+1)'}}$ and consider any v_0, w_0 with $\|v_0\|_{L^2 + M_{(\nu+1),(\nu+1)'}}, \|w_0\|_{L^2 + M_{(\nu+1),(\nu+1)'}} \leq r$. Denote by v and w the unique maximal solution of (5.2) with initial value v_0 and w_0, respectively. Observe that by the above, v and w are defined at least on $[0, T']$, where $T' := \delta(r)$. Moreover, $v|_{[0,T']}, w|_{[0,T']} \in M(R(r), \delta(r))$ and hence

$$\begin{aligned} \|v - w\|_{X(T')} &= \left\| e^{\mathrm{i}t\Delta} v_0 - e^{\mathrm{i}t\Delta} w_0 + \mathcal{T}(v) - \mathcal{T}(w) \right\|_{X(T')} \\ &\leq C(d, \nu) \|v_0 - w_0\|_{L^2 + M_{(\nu+1),(\nu+1)'}} + C_c \|v - w\|_{X(T')}, \end{aligned}$$

where $C(d, \nu) > 0$ is the constant from Equation (5.4) and $C_c < 1$ the contraction constant of $\mathcal{T}$. Collecting terms containing $\|v - w\|_{X(T')}$ shows

$$\|v - w\|_{X(T')} \lesssim_{d,\nu} \|v_0 - w_0\|_{L^2 + M_{(\nu+1),(\nu+1)'}}$$

for any

$$v_0, w_0 \in V := B_\varepsilon(u_0) \subseteq L^2 + M_{(\nu+1),(\nu+1)'},$$

i.e. the claimed local Lipschitz continuity. This concludes the proof. $\square$

5.3. Construction and properties of the perturbation

Global existence follows by constructing a solution of (5.2) of a special form which is suitable to exploit the mass conservation. This will be done using the following

Lemma 5.5 (Strichartz for perturbation)**.** *Let $d \in \mathbb{N}$, $\nu \in \left(1, 1+\frac{4}{d}\right)$ and consider $\frac{1}{r} \in \left(\max\left\{0, \frac{1}{2}-\frac{1}{d}\right\}, \frac{1}{2}\right]$. Then there is a constant $C = C(d,\nu,r) > 0$ such that for any $T > 0$, any $v \in L^{q_a(d,\nu+1)}([0,T], L^{\nu+1}(\mathbb{R}^d))$ and any $w \in L^\infty([0,T], L^{\nu+1}(\mathbb{R}^d)$ the estimate*

$$\begin{aligned}
&\left\| \int_0^t e^{\mathrm{i}(t-\tau)\Delta} G(v,w,0) \mathrm{d}\tau \right\|_{L^{q_a(d,r)}([0,T],L^r(\mathbb{R}^d))} \\
&\leq C\Big[T^{1-\frac{\nu}{\nu+1}\frac{d}{4}(\nu-1)} \|v\|^{\nu-1}_{L^{q_a(d,\nu+1)}([0,T],L^{\nu+1}(\mathbb{R}^d))} \|w\|_{L^\infty([0,T],L^{\nu+1}(\mathbb{R}^d))} \\
&\quad + T^{1-\frac{1}{\nu+1}\frac{d}{4}(\nu-1)} \|w\|^{\nu}_{L^\infty([0,T],L^{\nu+1}(\mathbb{R}^d))} \Big]
\end{aligned}$$

holds.

Proof. The proof is very similar to that of Lemma 3.10. Again, Proposition 3.8 yields

$$\left\| \int_0^t e^{\mathrm{i}(t-\tau)\Delta} G(v,w,0) \mathrm{d}\tau \right\|_{L^{q_a(d,r)}([0,T],L^r(\mathbb{R}^d))} \lesssim_{d,r,\rho} \|G(v,w,0)\|_{L^\gamma([0,T],L^\rho(\mathbb{R}^d))},$$

if $\left(\frac{1}{\rho}, \frac{1}{\gamma}\right) \in \widehat{T}'(d)$ satisfies condition (3.13), i.e.

$$\frac{1}{\rho} \in \left[\frac{1}{2}, \min\left\{1, \frac{1}{2}+\frac{1}{d}\right\}\right) \qquad \text{and} \tag{5.7}$$

$$\frac{1}{\gamma} = 1 - \frac{d}{2}\left(\frac{1}{\rho}-\frac{1}{2}\right). \tag{5.8}$$

By the size estimate from Lemma 3.9 one has

$$\|G(v,w,0)\|_{L^\gamma L^\rho} \lesssim_\nu \left\| v^{(\nu-1)} w \right\|_{L^\gamma L^\rho} + \|w^\nu\|_{L^\gamma L^\rho}.$$

Consider the term $v^{(\nu-1)}w$ first. Applying Hölder's inequality to the functions w, $v^{\nu-1}$ and 1 yields

$$\left\| v^{(\nu-1)} w \right\|_{L^\gamma([0,T],L^\rho(\mathbb{R}^d))} \leq T^{\frac{1}{q_3}} \|v\|^{(\nu-1)}_{L^{(\nu-1)q_2}([0,T],L^{(\nu-1)r_2}(\mathbb{R}^d))} \|w\|_{L^{q_1}([0,T],L^{r_1}(\mathbb{R}^d))},$$

where all but the last of the exponents $r_1, r_2, q_1, q_2, q_3 \in [1,\infty]$ are already fixed (by comparison with the indices of the norm of the same term in the claim) to

$$\begin{aligned} r_1 &= \nu+1, & q_1 &= \infty, \\ r_2 &= \frac{\nu+1}{\nu-1}, & q_2 &= \frac{q_{\mathrm{a}}(d,\nu+1)}{\nu-1} = \frac{\nu+1}{\nu-1}\frac{4}{d}\frac{1}{\nu-1} \end{aligned}$$

and need to satisfy the Hölder conditions

$$\frac{1}{\rho} = \frac{1}{r_1} + \frac{1}{r_2} \quad \text{and} \tag{5.9}$$

$$\frac{1}{\gamma} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3}. \tag{5.10}$$

Equation (5.9) immediately fixes $\frac{1}{\rho} = \frac{\nu}{\nu+1}$. Inserting ρ into (5.8) yields $\frac{1}{\gamma} = 1 - \frac{1}{\nu+1}\frac{d}{4}(\nu-1)$. Inserting γ, q_1 and q_2 into (5.10) shows $\frac{1}{q_3} = 1 - \frac{\nu}{\nu+1}\frac{d}{4}(\nu-1)$.

Considering $\frac{1}{\rho} = 1 - \frac{1}{\nu+1}$ and inserting the upper and lower bounds on ν yields

$$\frac{1}{\rho} \in \left(\frac{1}{2}, \frac{1}{2} + \frac{1}{2+d}\right),$$

which shows that (5.7) is satisfied. The fact that all Hölder exponents lie in the interval $[1,\infty]$ is apparent from their values and the bounds on ν.

As the term w^ν has already been treated in the proof of Lemma 3.10 (put $f = v$ there), the proof is concluded. □

Theorem 5.3 already implies the uniqueness of solutions u for initial values $\theta \in L^2 + M_{(\nu+1),(\nu+1)'}$. To show that in the case $\theta \in M_{p,p'}(\mathbb{R}^d)$ these unique solutions are global, consider their special decomposition into $u = (v+w) + e^{\mathrm{i}t\Delta}\psi$, where $\theta = \phi + \psi \in L^2(\mathbb{R}^d) + M_{(\nu+1),(\nu+1)'}$ and v is the unique solution to the NLS with initial value ϕ (see the introduction to this chapter). Inserting this ansatz into (5.2) yields

$$v + w = e^{\mathrm{i}t\Delta}\phi \pm \mathrm{i}\int_0^t e^{\mathrm{i}(t-\tau)\Delta}\left(\left|v+w+e^{\mathrm{i}\tau\Delta}\psi\right|^{\nu-1}(v+w+e^{\mathrm{i}\tau\Delta}\psi)\right)\mathrm{d}\tau,$$

which, after subtracting the equation for v, transforms into

$$\begin{aligned} w &= \pm\mathrm{i}\int_0^t e^{\mathrm{i}(t-\tau)\Delta}\left(\left|v+w+e^{\mathrm{i}\tau\Delta}\psi\right|^{\nu-1}(v+w+e^{\mathrm{i}\tau\Delta}\psi) - |v|^{\nu-1}v\right)\mathrm{d}\tau \\ &= \pm\mathrm{i}\int_0^t e^{\mathrm{i}(t-\tau)\Delta}G(v+e^{\mathrm{i}\tau\Delta}\psi, w, 0)\mathrm{d}\tau \pm \mathrm{i}\int_0^t e^{\mathrm{i}(t-\tau)\Delta}G(v, e^{\mathrm{i}\tau\Delta}\psi, 0)\mathrm{d}\tau \end{aligned} \tag{5.11}$$

as the governing equation for the perturbation w. Existence of solutions to (5.11) is established in the following

Proposition 5.6 (Local well-posedness for the perturbation)**.** *Let $d \in \mathbb{N}$, $\nu \in \left(1, 1+\frac{4}{d}\right)$, $\phi \in L^2(\mathbb{R}^d)$ and $\psi \in M_{(\nu+1),(\nu+1)'}(\mathbb{R}^d)$. Denote by v the global L^2-solution from Proposition 3.12 for initial value ϕ and by $e^{it\Delta}\psi$ the free propagation of ψ in $C([0,\infty), M_{(\nu+1),(\nu+1)'}(\mathbb{R}^d))$. Then there exists a constant $C = C(d,\nu) > 0$ such that the integral equation (5.11) has a (unique) solution $w \in L^{q_a(d,\nu+1)}([0,\delta], L^{\nu+1}(\mathbb{R}^d))$ provided*

$$\delta \leq 1, \tag{5.12}$$

$$\delta \leq C\left(\|\phi\|_2 + \|\psi\|_{M_{(\nu+1),(\nu+1)'}}\right)^{-\frac{\nu-1}{1-\frac{d}{4}(\nu-1)}}, \tag{5.13}$$

$$\delta \leq C\|\psi\|_{M_{(\nu+1),(\nu+1)'}}^{-\frac{\nu-1}{1-\frac{2}{\nu+1}\frac{d}{4}(\nu-1)}}. \tag{5.14}$$

Proof. Assume, w.l.o.g. that δ is the minimum of the right-hand sides of (5.12), (5.13) and (5.14). For $R > 0$ set

$$X(\delta) = L^{q_a(d,\nu+1)}([0,\delta], L^{\nu+1}(\mathbb{R}^d)) \qquad \text{and} \qquad M(R,\delta) = \left\{ f \in X(\delta) \mid \|f\|_{X(\delta)} \leq R \right\}.$$

One has to show that after fixing the constant C and some $R > 0$ the right-hand side of (5.11) defines a contractive self-mapping $\mathcal{T} : M(R,\delta) \to M(R,\delta)$ (Banach's contraction mapping principle).

To fix R, consider the integral not involving w first. By Lemma 5.5 one has

$$\left\| \int_0^t e^{i(t-\tau)\Delta} G(v, e^{i\tau\Delta}\psi, 0) d\tau \right\|_{X(\delta)} \lesssim_{d,\nu} \delta^{1-\frac{\nu}{\nu+1}\frac{d}{4}(\nu-1)} \|v\|_{X(\delta)}^{\nu-1} \left\|e^{it\Delta}\psi\right\|_{L^\infty L^{\nu+1}} \tag{5.15}$$
$$+\delta^{1-\frac{1}{\nu+1}\frac{d}{4}(\nu-1)} \left\|e^{it\Delta}\psi\right\|_{L^\infty L^{\nu+1}}^{\nu}.$$

For the first summand, observe that $1 - \frac{\nu}{\nu+1}\frac{d}{4}(\nu-1) = \frac{1}{\nu+1}\frac{d}{4}(\nu-1) + 1 - \frac{d}{4}(\nu-1)$ and hence

$$\delta^{1-\frac{\nu}{\nu+1}\frac{d}{4}(\nu-1)} \|v\|_{X(\delta)}^{\nu-1} = \delta^{\frac{1}{\nu+1}\frac{d}{4}(\nu-1)} \delta^{1-\frac{d}{4}(\nu-1)} \|v\|_{X(\delta)}^{\nu-1} \lesssim_{d,\nu} \delta^{\frac{1}{\nu+1}\frac{d}{4}(\nu-1)}$$

by Corollary 3.13 (justified by assumption (5.13)). Furthermore,

$$\left\|e^{it\Delta}\psi\right\|_{L^\infty L^{\nu+1}} \lesssim_d \left\|e^{it\Delta}\psi\right\|_{L^\infty M_{(\nu+1),(\nu+1)'}} \lesssim_d \|\psi\|_{M_{(\nu+1),(\nu+1)'}}$$

by Proposition 2.34 and Theorem 3.4 under the assumption (5.12). This suggests the choice

$$R = \frac{3}{C(d,\nu)} \delta^{\frac{1}{\nu+1}\frac{d}{4}(\nu-1)} \|\psi\|_{M_{(\nu+1),(\nu+1)'}}, \tag{5.16}$$

so only the constant $C(d,\nu)$ (the same as in conditions (5.13) and (5.14)) remains to be fixed.

For the second summand, observe that $1 - \frac{1}{\nu+1}\frac{d}{4}(\nu-1) = \frac{1}{\nu+1}\frac{d}{4}(\nu-1) + 1 - \frac{2}{\nu+1}\frac{d}{4}(\nu-1)$ and hence

$$\begin{aligned}
&\delta^{1-\frac{1}{\nu+1}\frac{d}{4}(\nu-1)} \left\|e^{it\Delta}\psi\right\|_{L^\infty L^{\nu+1}}^{\nu} \\
\lesssim_{d,\nu}\ &\delta^{\frac{1}{\nu+1}\frac{d}{4}(\nu-1)} \|\psi\|_{M_{(\nu+1),(\nu+1)'}} \delta^{1-\frac{2}{\nu+1}\frac{d}{4}(\nu-1)} \|\psi\|_{M_{(\nu+1),(\nu+1)'}}^{\nu-1} \\
\lesssim_{d,\nu}\ &\delta^{\frac{1}{\nu+1}\frac{d}{4}(\nu-1)} \|\psi\|_{M_{(\nu+1),(\nu+1)'}},
\end{aligned}$$

where additionally assumption (5.14) was used for the last inequality. Comparing the last expression with (5.16) and choosing $C(d,\nu)$ small enough shows that the right-hand side above can be estimated by $\frac{R}{3}$.

The integral involving w is estimated by Lemma 3.10 against

$$\begin{aligned}\left\|\int_0^t e^{\mathrm{i}(t-\tau)\Delta}G(v+e^{\mathrm{i}\tau\Delta}\psi,w,0)\mathrm{d}\tau\right\|_{X(\delta)} \\ \lesssim_{d,\nu} \delta^{1-\frac{d}{4}(\nu-1)}\left(\left\|v+e^{\mathrm{i}t\Delta}\psi\right\|_{X(\delta)}^{\nu-1}+\|w\|_{X(\delta)}^{\nu-1}\right)\|w\|_{X(\delta)}.\end{aligned} \tag{5.17}$$

For the first summand, observe that

$$\left\|v+e^{\mathrm{i}t\Delta}\psi\right\|_{X(\delta)} \leq \|v\|_{X(\delta)}+\left\|e^{\mathrm{i}t\Delta}\psi\right\|_{X(\delta)} \lesssim_{d,\nu} \|\phi\|_2+\|\psi\|_{M_{(\nu+1),(\nu+1)'}},$$

where again Corollary 3.13 was used for the estimate on v, whereas the estimate on w follows by embedding $L^\infty \hookrightarrow L^{q_a(d,\nu+1)}$, Proposition 2.34 and Theorem 3.4. Making C in (5.13) small enough then implies

$$\delta^{1-\frac{d}{4}(\nu-1)}\left\|v+e^{\mathrm{i}t\Delta}\psi\right\|_{X(\delta)}^{\nu-1} \leq \frac{1}{3}.$$

Recalling that $w \in M(R,\delta)$ hence shows that

$$\delta^{1-\frac{d}{4}(\nu-1)}\left\|v+e^{\mathrm{i}t\Delta}\psi\right\|_{X(\delta)}^{\nu-1}\|w\|_{X(\delta)} \leq \frac{R}{3}.$$

For the second summand, estimate w in the same spirit to obtain

$$\delta^{1-\frac{d}{4}(\nu-1)}\|w\|_{X(\delta)}^{\nu} \lesssim_{d,\nu} \delta^{1-\frac{d}{4}(\nu-1)}\|\psi\|_{M_{(\nu+1),(\nu+1)'}}^{\nu-1}R \leq \frac{R}{3},$$

under yet another smallness assumption of $C(d,\nu)$ in (5.13) for the last inequality.

All in all this shows that

$$\|\mathcal{T}w\|_{X(\delta)} \leq R \qquad \forall w \in M(R,\delta),$$

i.e. the self-mapping property of $\mathcal{T}$.

Contractivity of $\mathcal{T}$ is shown in the same way (i.e. via Lemma 3.10), possibly enforcing an even smaller constant $C(d,\nu)$, and finishing the proof. □

The L^2-norm of the perturbation is controllable by Lemma 5.5. This is stated in the following

Corollary 5.7 (L^2-norm increase)**.** *There exists a constant $C = C(d,\nu) > 0$ such that the solution w of* (5.11) *constructed under the assumptions of Proposition 5.6 satisfies*

$$\|w\|_{L^\infty([0,\delta],L^2(\mathbb{R}^d))} \leq C\delta^{\frac{1}{\nu+1}\frac{d}{4}(\nu-1)}\|\psi\|_{M_{(\nu+1),(\nu+1)'}}.$$

Proof. The proof is similar to that of Corollary 3.13. As w solves (5.11), one may work with its right-hand side. Both its summands are estimated in the $X(\delta)$-norm in inequalities (5.15) and (5.17). In both of them the norm on the left-hand side may be replaced by the $L^\infty L^2$-norm as the pair $(2,\infty)$ is admissible and hence Lemma 5.5 and 3.10 respectively are still applicable. This shows that indeed

$$\begin{aligned}\|w\|_{L^\infty L^2} &\leq \left\|\int_0^t e^{\mathrm{i}(t-\tau)\Delta}G(v,e^{\mathrm{i}\tau\Delta}\psi,0)\mathrm{d}\tau\right\|_{L^\infty L^2} \\ &\quad+\left\|\int_0^t e^{\mathrm{i}(t-\tau)\Delta}G(v+e^{\mathrm{i}\tau\Delta}\psi,w,0)\mathrm{d}\tau\right\|_{L^\infty L^2} \\ &\leq R \lesssim_{d,\nu} \delta^{\frac{1}{\nu+1}\frac{d}{4}(\nu-1)}\,\|\psi\|_{M_{(\nu+1),(\nu+1)'}}\end{aligned}$$

and finishes the proof. □

5.4. Proof of global existence

Now, all ingredients are at hand and the main theorem of this chapter can finally be proven.

Proof of Thm. 5.4. Assume, that for an initial datum $u_0 \in M_{p,p'}$ the unique maximal solution w from Theorem 5.3 is not global, i.e. $T_* < \infty$. This will be shown to be a contradiction by constructing a solution $\tilde{u}$ on a larger time interval.

To that end, recall from Proposition 5.1 that there is a constant C_1 such that for any $N > 0$ there are $v_0 \in L^2$ and $w_0 \in M_{(\nu+1),(\nu+1)'}$ such that

$$\|v_0\|_2 \leq C_1 N^\alpha \qquad \text{and} \qquad \|w_0\|_{M_{(\nu+1),(\nu+1)'}} \leq C_1\frac{1}{N}, \tag{5.18}$$

where $\alpha = \frac{\frac{1}{2}-\frac{1}{p}}{\frac{1}{p}-\frac{1}{\nu+1}}$. Observe, that α is strictly increasing as a function of p. Hence, the prerequisite $p \in (2, p_{\max})$ translates to the equivalent condition $\alpha \in (0, A(d,\nu))$, where $A(d,\nu)$ is calculated by substituting p in the formula for α by $p_{\max}$ (see equation (5.3)). To that end, observe that

$$\frac{1}{p_{\max}} = \begin{cases} \frac{\nu}{2}\cdot\frac{1}{\nu+1-\frac{d}{4}(\nu-1)} & \text{if } \nu > \frac{1}{2}-\frac{d}{4}+\sqrt{2+\left(\frac{1}{2}+\frac{d}{4}\right)^2}, \\ \frac{1}{\nu+1} & \text{otherwise.}\end{cases}$$

The value of $\frac{1}{p_{\max}}$ from the first of the two cases distinguished above yields

$$\frac{\frac{1}{2}-\frac{\nu}{2}\cdot\frac{1}{\nu+1-\frac{d}{4}(\nu-1)}}{\frac{\nu}{2}\cdot\frac{1}{\nu+1-\frac{d}{4}(\nu-1)}-\frac{1}{\nu+1}} = \frac{1-\frac{\nu}{\nu+1-\frac{d}{4}(\nu-1)}}{\frac{\nu}{\nu+1-\frac{d}{4}(\nu-1)}-\frac{2}{\nu+1}} = \frac{1-\frac{d}{4}(\nu-1)}{\nu-2\frac{\nu+1-\frac{d}{4}(\nu-1)}{\nu+1}} = \frac{1-\frac{d}{4}(\nu-1)}{\nu-2+\frac{d}{2}\frac{\nu-1}{\nu+1}}$$

as the corresponding value of $A(d,\nu)$. In fact, the denominator of the last term is positive if and only if the condition corresponding to the first case holds, i.e.

$$\begin{aligned} \nu - 2 + \frac{d}{2}\frac{\nu-1}{\nu+1} > 0 \quad &\Leftrightarrow \quad (\nu-2)(\nu+1) + (\nu-1)\frac{d}{2} > 0 \\ &\Leftrightarrow \quad \left((\nu-1) + \left(\frac{1}{2} + \frac{d}{4}\right)\right)^2 > 2 + \left(\frac{1}{2} + \frac{d}{4}\right)^2 \\ &\Leftrightarrow \quad \nu > \frac{1}{2} - \frac{d}{4} + \sqrt{2 + \left(\frac{1}{2} + \frac{d}{4}\right)^2}. \end{aligned}$$

In the other case $p_{\max} = \nu + 1$ and so ∞ is the corresponding value of $A(d,\nu)$. All in all, one obtains

$$A(d,\nu) := \begin{cases} \frac{1-\frac{d}{4}(\nu-1)}{\nu-2+\frac{\nu-1}{\nu+1}\frac{d}{2}} & \text{if } \nu - 2 + \frac{\nu-1}{\nu+1}\frac{d}{2} > 0, \\ \infty & \text{otherwise.} \end{cases} \tag{5.19}$$

Denote the constant from Proposition 5.6 by $C_2 = C_2(d,\nu)$ and put

$$\delta = \delta(N) = C_2 \left(3C_1 N^{\alpha}\right)^{-\frac{\nu-1}{1-\frac{d}{4}(\nu-1)}} \xrightarrow{N\to\infty} 0 \tag{5.20}$$

(any number greater than 1 instead of 3 works).

Consider the finite sequences $v_1, v_2, \ldots, v_K$ and $w^1, w^2, \ldots, w^K$ constructed using the following

Algorithm 1 Iterative Procedure

```
1: k ← 0
2: while kδ ≤ T_* and φ = v_k, ψ = e^{ikδΔ} w_0, δ satisfy (5.12), (5.13), (5.14) do
3:     k ← k + 1
4:     w^k ← w from Proposition 5.6 {applicable by conditions of the loop}
5:     v_k ← w(·, kδ) + v(·, kδ) {v from the same proposition}
6: end while
7: K ← k
```

Put v^{k+1} for $k \in \{0, \ldots, K-1\}$ to be the NLS evolution of IV v_k and observe, that by construction

$$\tilde{u}(\cdot,t) := v^{k+1}(\cdot, t-k\delta) + w^{k+1}(\cdot, t-k\delta) + e^{\mathrm{i}t\Delta}w_0 \qquad \text{if } t \in [k\delta, (k+1)\delta], k \in \{0, \ldots, K-1\}$$

for any $t \in [0, K\delta]$ defines a solution of (5.1). Hence, it remains to show that the iterative procedure terminates with $K\delta > T_*$ for sufficiently large N. Consider, to that end, the conditions in line 2 of the algorithm above.

The smallness condition (5.12) is satisfied independently of k for large N by definition of δ in (5.20).

Now consider the condition (5.14) on the modulation space norm. By Theorem 3.4 and the second inequality in (5.18) one has

$$\left\|e^{\mathrm{i}t\Delta}w_0\right\|_{L^\infty([0,T_*+1],M_{(\nu+1),(\nu+1)'})} \lesssim_{d,T_*} \|w_0\|_{M_{(\nu+1),(\nu+1)'}} \lesssim_d \frac{1}{N} \xrightarrow{N\to\infty} 0. \tag{5.21}$$

Inserting this estimate ($\psi = e^{\mathrm{i}k\delta\Delta}w_0$) into the right-hand side of (5.14) yields

$$\|\psi\|_{M_{(\nu+1),(\nu+1)'}}^{-\frac{\nu-1}{1-\frac{2}{\nu+1}\frac{d}{4}(\nu-1)}} \gtrsim_{d,T_*} N^{\frac{\nu-1}{1-\frac{\nu-1}{\nu+1}\frac{d}{2}}} \xrightarrow{N\to\infty} \infty.$$

As this lower bound is independent of k and $\delta \xrightarrow{N\to\infty} 0$, this condition is also satisfied for large N.

That means that $K\delta > T_*$ or the condition (5.13) involving both norms fails in the last iteration step $k = K$, i.e.

$$3C_1N^\alpha < \|v_K\|_2 + \left\|e^{\mathrm{i}K\delta\Delta}w_0\right\|_{M_{(\nu+1),(\nu+1)'}}.$$

By inequality (5.21) the second summand of the right-hand side is smaller that C_1N^α for large N and hence

$$2C_1N^\alpha < \|v_K\|_2\,. \tag{5.22}$$

By definition, mass conservation and the first inequality in (5.18) one has

$$\begin{aligned}
\|v_K\|_2 &\le \left\|v^K\right\|_{L^\infty L^2} + \left\|w^K\right\|_{L^\infty L^2} = \|v_{K-1}\|_2 + \left\|w^K\right\|_{L^\infty L^2} \\
&\le \left\|v^{K-1}\right\|_{L^\infty L^2} + \left\|w^{K-1}\right\|_{L^\infty L^2} + \left\|w^K\right\|_{L^\infty L^2} = \|v_{K-2}\|_2 + \sum_{k=K-1}^{K}\left\|w^k\right\|_{L^\infty L^2} \\
&\le \ldots \le \|v_0\|_2 + \sum_{k=1}^{K}\left\|w^k\right\|_{L^\infty L^2} \le C_1N^\alpha + \sum_{k=1}^{K}\left\|w^k\right\|_{L^\infty L^2}.
\end{aligned}$$

The sum $\sum_{k=1}^{K}$ is further estimated by Corollary 5.7 and inequality (5.21) against

$$\sum_{k=1}^{K}\left\|w^k\right\|_{L^\infty L^2} \lesssim_{d,\nu} \delta^{\frac{1}{\nu+1}\frac{d}{4}(\nu-1)}\sum_{k=0}^{K-1}\left\|e^{\mathrm{i}k\delta\Delta}w_0\right\|_{M_{(\nu+1),(\nu+1)'}} \lesssim_{d,T_*} K\delta^{\frac{1}{\nu+1}\frac{d}{4}(\nu-1)}\frac{1}{N}.$$

Inserting the estimate on v_K into the right-hand side of (5.22) and recalling (5.20) yields

$$\begin{aligned}
K\delta &\gtrsim_{d,\nu,T_*} N^{1+\alpha}\delta^{1-\frac{1}{\nu+1}\frac{d}{4}(\nu-1)} \approx_{d,\nu,u_0} N^{1+\alpha\left(1-\frac{(\nu-1)\left(1-\frac{1}{\nu+1}\frac{d}{4}(\nu-1)\right)}{1-\frac{d}{4}(\nu-1)}\right)} \\
&= N^{1-\alpha\frac{(\nu-1)-\left(1-\frac{2}{\nu+1}\frac{d}{4}(\nu-1)\right)}{1-\frac{d}{4}(\nu-1)}}.
\end{aligned}$$

By the prerequisite on α from equation (5.19) the exponent of N above is positive. Hence, for sufficiently large N, one has $K\delta > T_*$ in each case. This concludes the proof. □

5.5. Comments

The precursor of Theorem 5.4, [CHKP17, Theorem 3], was (to the best of the author's knowledge) the first global well-posedness result for the NLS with IVs in a modulation space $M_{p,q}(\mathbb{R}^d)$ which required no smallness condition of the initial data. The former theorem extends the latter result to arbitrary dimensions, arbitrary powers of the nonlinearity (such that the NLS is still mass-subcritical) and a larger range of the index p. The last fact is due to the estimate (5.21) of the modulation space norm being more careful than the original one [CHKP17, p. 4438]. Also, the notion of a solution to (5.1) is clearly stated in Definition 5.2 and the proof of their uniqueness (Theorem 5.3) is much more elaborated than in the original publication.

Other known global well-posedness results for the NLS with initial data in a modulation space are [WH07, Theorem 1.1] (see also [RSW12, Theorem 4.11] and [WHHG11, Theorem 6.3]) and [Kat14, Theorem 1.1]. Both require smallness of the initial data and none cover the cubic nonlinearity in dimension one.

Further results involving initial data with infinite L^2-norm are [VV01, Theorem 2] ($u_0 \in L^2 + Y_{3,6}$ with trading exponent $\alpha < 1$), [Grü05, Theorem 1.5] ($u_0 \in \widehat{H^r_s}$ where $s \geq \frac{1}{2}$ and $r \in (1,2]$) and [HT12, Theorem 2] ($u_0 \in \widehat{L}^p$ where p is sufficiently close to 2). In fact, as

$$Y_{p,q} = \left\{ \phi \in \mathcal{S}(\mathbb{R})' \middle| \ \|\phi\|_{Y_{p,q}} := \sup \left\{ \left\| e^{\mathrm{i}t\Delta}\phi \right\|_{L^p(I,L^q(\mathbb{R}))} \middle| \, I \text{ interval of length } 1 \right\} \right\}$$

and

$$\left\| e^{\mathrm{i}t\Delta}\phi \right\|_{L^3(I,L^6(\mathbb{R}))} \leq \left\| e^{\mathrm{i}t\Delta}\phi \right\|_{L^\infty(I,L^6(\mathbb{R}))} \lesssim \left\| e^{\mathrm{i}t\Delta}\phi \right\|_{L^\infty(I,M_{6,\frac{6}{5}}(\mathbb{R}))} \lesssim \|\phi\|_{M_{6,\frac{6}{5}}(\mathbb{R})}$$

by Theorem 3.4, one recognizes that the aforementioned theorem by Vargas and Vega applies to $u_0 \in M_{p,p'}(\mathbb{R})$ for $p \in (2,3)$. However, their result does not guarantee persistence in $L^2 + M_{6,\frac{6}{5}}$.

A. Appendix

Complex Analysis

Lemma A.1. *(Standard estimate, cf. [FL12, proposition I.5.4])*
Let γ be a complex path of integration and $f : \mathrm{Tr}(\gamma) \to \mathbb{C}$ continuous. Then

$$\left| \int_\gamma f(z)\mathrm{d}z \right| \leq L(\gamma) \max_{z\in(\gamma)} |f(z)| . \tag{A.1}$$

Theorem A.2. *(Cauchy's integral theorem, cf. [FL12, section IV.1])*
Let $\Omega \subseteq \mathbb{C}$ be a domain. Then the following statements are equivalent:

(a) Ω is simply connected.

(b) For every f holomorphic on Ω and every closed, piecewise continuously differentiable path in Ω it is

$$\int_\gamma f(z)\mathrm{d}z = 0. \tag{A.2}$$

Example A.3 (Gaussian integrals). Let $\alpha, \beta \in \mathbb{C}$ such that $\mathrm{Re}(\alpha^2) > 0$ and define the *Gaussian integral*

$$I(\alpha, \beta) = \int_{\mathbb{R}} e^{-(\alpha t+\beta)^2}\mathrm{d}t.$$

Then the integral above is absolutely convergent and

$$I(\alpha, \beta) = I(\alpha, 0) = \frac{1}{\alpha} I(1, 0) = \frac{\sqrt{\pi}}{\alpha}. \tag{A.3}$$

Proof. Let α, β be as above. Observe, that

$$\begin{aligned} \left|\exp(-(\alpha t + \beta)^2)\right| &= \exp(-\mathrm{Re}((\alpha t + \beta)^2)) \\ &= \exp(-\mathrm{Re}(\beta^2)) \exp\left(-t^2 \left(\mathrm{Re}(\alpha^2) + \frac{2\,\mathrm{Re}(\alpha\beta)}{t}\right)\right) \\ &\leq \exp(-\mathrm{Re}(\beta^2)) \exp\left(-t^2 \frac{\mathrm{Re}(\alpha^2)}{2}\right) \end{aligned}$$

for all $|t| > 4\frac{|\mathrm{Re}(\alpha\beta)|}{\mathrm{Re}(\alpha^2)}$. This establishes the absolute convergence of the integral.

For the first equality consider

$$I(\alpha, \beta) = \lim_{R\to\infty} \int_{-R}^{R} e^{-(\alpha t+\beta)^2} \mathrm{d}t = \frac{1}{\alpha} \lim_{R\to\infty} \int_{\gamma_{\alpha,\beta}^{R,1}} e^{-z^2} \mathrm{d}z,$$

where $\gamma_{\alpha,\beta}^{R,1}$ is as in figure A.1a. For any fixed $R > 0$ it is

$$\int_{\gamma_{\alpha,\beta}^{R,1}} e^{-z^2} \mathrm{d}z + \int_{\gamma_{\alpha,\beta}^{R,2}} e^{-z^2} \mathrm{d}z - \int_{\gamma_{\alpha,0}^{R,1}} e^{-z^2} \mathrm{d}z + \int_{\gamma_{\alpha,\beta}^{R,3}} e^{-z^2} \mathrm{d}z = 0$$

by Cauchy's integral theorem formula A.2. The standard estimate A.1 yields

$$\int_{\gamma_{\alpha,\beta}^{R,2}} e^{-z^2} \mathrm{d}z, \int_{\gamma_{\alpha,\beta}^{R,3}} e^{-z^2} \mathrm{d}z \xrightarrow{R\to\infty} 0$$

and hence

$$I(\alpha, \beta) = \frac{1}{\alpha} \lim_{R\to\infty} \int_{\gamma_{\alpha,\beta}^{R,1}} e^{-z^2} \mathrm{d}z = \frac{1}{\alpha} \lim_{R\to\infty} \int_{\gamma_{\alpha,0}^{R,1}} e^{-z^2} \mathrm{d}z = I(\alpha, 0).$$

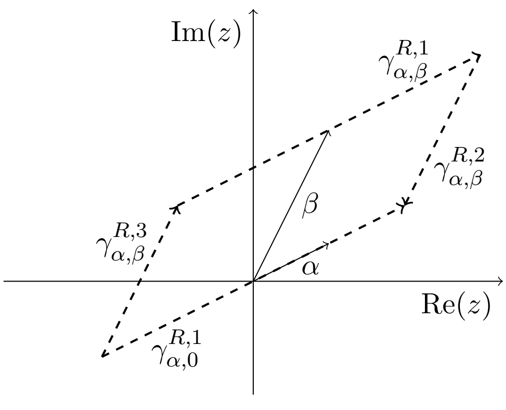
For the second equality observe, that

$$I(\alpha, 0) = \int_{-\infty}^{\infty} e^{-\alpha^2 t^2} \mathrm{d}t = 2 \int_{0}^{\infty} e^{-\alpha^2 t^2} \mathrm{d}t = \frac{2}{\alpha} \lim_{R\to\infty} \int_{\gamma_{\alpha}^{R,1}} e^{-z^2} \mathrm{d}z,$$

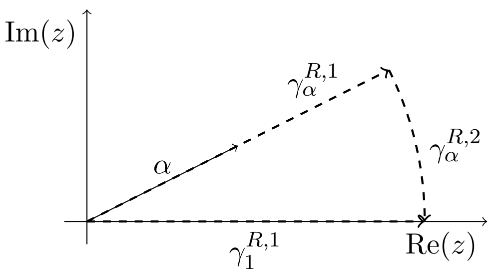
where $\gamma_{\alpha}^{R,1}$ is as in figure A.1b. Same arguments as for the first equality yield

$$\lim_{R\to\infty} \int_{\gamma_{\alpha}^{R,2}} e^{-z^2} \mathrm{d}z = 0 \qquad \text{and} \qquad I(\alpha, 0) = \frac{2}{\alpha} \lim_{R\to\infty} \int_{0}^{R} e^{-t^2} \mathrm{d}t = \frac{1}{\alpha} I(1, 0).$$

The last equality is the well-known value $I(1, 0) = \int_{\mathbb{R}} e^{-t^2} \mathrm{d}t = \sqrt{\pi}$.



(a) Independence of β.



(b) Independence of α.

Figure A.1.: Paths of integration used in proof of example A.3.

□

Measure and Integration

Theorem A.4 (Continuity of parameter integrals). *(Cf. [Els11, Chapter IV, Theorem 5.6]). Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, (M, d) a metric space and $x_0 \in M$. Furthermore, let $f : M \times \Omega \to \mathbb{F}$ satisfy the following conditions:*

(a) For each $x \in M$ it is $f(x, \cdot) \in L^1(\Omega)$.

(b) For μ-a.e. $\omega \in \Omega$ the function $f(\cdot, \omega)$ is continuous in x_0.

(c) There is a neighborhood U of x_0 and a non-negative function $g \in L^1(\Omega)$ s.th. for all $x \in U$ it is $|f(x, \omega)| \leq g(\omega)$ for μ-a.e. $\omega \in \Omega$.

Then the function $F : M \to \mathbb{F}$ defined by $F(x) = \int_\Omega f(x, \omega)\mu(\mathrm{d}\omega)$ for every $x \in M$ is continuous in x_0, i.e.

$$\lim_{x \to x_0} \int_\Omega f(x, \omega)\mu(\mathrm{d}\omega) = \int_\Omega \lim_{x \to x_0} f(x, \omega)\mu(\mathrm{d}\omega).$$

Theorem A.5 (Differentiation under the integral sign). *(Cf. [Els11, Chapter IV, Theorem 5.7]). Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, $\vec{x}_0 \in U \subseteq \mathbb{R}^d$ open. Furthermore, let $f : U \times \Omega \to \mathbb{F}$ satisfy the following conditions:*

(a) For each $\vec{x} \in U$ it is $f(\vec{x}, \cdot) \in L^1(\Omega)$.

(b) For some $i \in \{1, \dots, d\}$ the partial derivative $\frac{\partial f}{\partial x_i}(\vec{x}, \omega)$ exists for all $\omega \in \Omega$ and all $\vec{x} \in U$.

(c) There is a non-negative function $g \in L^1(\Omega)$ s.th. $\left|\frac{\partial f}{\partial x_i}\right|(\vec{x}, \omega) \leq g(\omega)$ for all $\vec{x} \in U$ and all $\omega \in \Omega$.

Then the function $F : U \to \mathbb{F}$ defined by $F(\vec{x}) = \int_\Omega f(\vec{x}, \omega)\mu(\mathrm{d}\omega)$ for every $\vec{x} \in U$ is partially differentiable in $\vec{x}_0$ w.r.t. x_i, $\frac{\partial f}{\partial x_i}(\vec{x}_0, \cdot) \in L^1(\Omega)$ and

$$\frac{\partial F}{\partial x_i}(\vec{x}_0) = \int_\Omega \frac{\partial f}{\partial x_i}(\vec{x}_0, \omega)\mu(\mathrm{d}\omega).$$

Theorem A.6 (Order of integration). *(Cf. [Els11, Chapter V, Theorem 2.1])*
Let $(\Omega_1, \mathcal{A}_1, \mu)$ and $(\Omega_2, \mathcal{A}_2, \nu)$ be measure spaces where μ and ν are σ-finite. Then:

(i) For each $\mathcal{A}_1 \otimes \mathcal{A}_2$-measureable $f : \Omega_1 \times \Omega_2 \to \mathbb{R}_0^+ \cup \{\infty\}$

- *$\Omega_1 \ni \omega_1 \mapsto \int_{\Omega_2} f(\omega_1, \omega_2)\nu(\mathrm{d}\omega_2)$ is $\mathcal{A}_1$-measureable,*

- *$\Omega_2 \ni \omega_2 \mapsto \int_{\Omega_1} f(\omega_1, \omega_2)\mu(\mathrm{d}\omega_1)$ is $\mathcal{A}_2$-measureable* *and*

$$\int_{\Omega_1}\int_{\Omega_2} f \mathrm{d}\nu \mathrm{d}\mu = \int_{\Omega_2}\int_{\Omega_1} f \mathrm{d}\mu \mathrm{d}\nu = \int\limits_{\Omega_1\times\Omega_2} f \mathrm{d}\mu \otimes \nu \tag{A.4}$$

*(*Tonelli's theorem*).*

(ii) For each $\mu \otimes \nu$-integrable $f : \Omega_1 \times \Omega_2 \to \mathbb{F}$

- *the function $f(\omega_1, \cdot)$ is ν-integrable for μ-a.e. $\omega_1 \in \Omega_1$, the (μ-a.e. defined) mapping $\Omega_1 \ni \omega_1 \mapsto \int_{\Omega_2} f(\omega_1, \omega_2)\nu(\mathrm{d}\omega_2)$ is μ-integrable;*
- *the function $f(\cdot, \omega_2)$ is μ-integrable for ν-a.e. $\omega_2 \in \Omega_2$, the (ν-a.e. defined) mapping $\Omega_2 \ni \omega_2 \mapsto \int_{\Omega_1} f(\omega_1, \omega_2)\mu(\mathrm{d}\omega_1)$ is ν-integrable;*

*and (A.4) holds (*Fubini's Theorem*).*

Theorem A.7 (Change of variables)**.** *(Cf. [Els11, Chapter V, Theorem 4.2]). Let $d \in \mathbb{N}$, $X, Y \subseteq \mathbb{R}^d$, $\Phi \in C^1(X, Y)$ bijective such that $\Phi^{(-1)} \in C^1(Y, X)$ and $f : Y \to \mathbb{C}$ measureable. Then $f \in L^1(Y)$ if and only if $(f \circ \Phi) |\det(\nabla\Phi)| \in L^1(X)$. In that case one has*

$$\int_{\Phi(X)} f(y)\mathrm{d}y = \int_X (f \circ \Phi)(x) |\det(\nabla\Phi)| (x)\mathrm{d}x. \tag{A.5}$$

Example A.8. Let $d \in \mathbb{N}$, $p \in [1, \infty]$, $f : \mathbb{R}^d \to \mathbb{C}$ measureable, $A \in \mathrm{GL}(d, \mathbb{R})$, $y_0 \in \mathbb{R}^d$ and $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ defined by $\Phi(x) = Ax + y_0$ for all $x \in \mathbb{R}^d$. Then

$$\|f \circ \Phi\|_p = |A|^{-\frac{1}{p}} \|f\|_p . \tag{A.6}$$

Lemma A.9 (Surface of the $d-1$-sphere, volume of a d-ball)**.** *(Cf. [Els11, Chapter V, Example 1.8]) Let $d \in \mathbb{N}$, $r > 0$ and denote by $B = \left\{x \in \mathbb{R}^d \middle| |x| \leq 1\right\}$ the unit ball in $\mathbb{R}^d$. Then*

$$\lambda^d(B) = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)} \qquad \text{and} \qquad \sigma^{d-1}(\partial B) = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)}. \tag{A.7}$$

Definition A.10. (*Dual exponent*)
Let $1 \leq p \leq \infty$. Define the dual exponent $1 \leq p' \leq \infty$ via the formula

$$\frac{1}{p} + \frac{1}{p'} = 1, \qquad \text{i.e.} \qquad p' = \begin{cases} \infty & \text{for } p = 1, \\ \frac{p}{p-1} & \text{for } 1 < p < \infty, \\ 1 & \text{for } p = \infty. \end{cases}$$

Lemma A.11 (Hölder's inequality)**.** *(Cf. [Els11, Chapter VI, Theorem 1.5])*
Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, $f, g : \Omega \to \mathbb{F}$ measureable and $1 \leq p \leq \infty$. Then the so-called Hölder's inequality

$$\|fg\|_1 \leq \|f\|_p \|g\|_{p'} \tag{A.8}$$

holds.

Corollary A.12 (Littlewood's inequality). *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, $f : \Omega \to \mathbb{F}$ measureable, $1 \leq p_0, p_1 \leq \infty$, $\theta \in (0,1)$ and $1 \leq p \leq \infty$ such that*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Then the so-called Littlewood's inequality

$$\|f\|_p \leq \|f\|_{p_0}^{1-\theta} \|f\|_{p_1}^{\theta} \tag{A.9}$$

holds.

Theorem A.13 (Convolution). *(Cf. [LL01, Theorem 4.2]) Let $d \in \mathbb{N}$ and $p, q, r \in [1, \infty]$ satisfy*

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}. \tag{A.10}$$

Furthermore, let $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$. Then the integral

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y) g(y) \mathrm{d}y$$

*exists for almost all $x \in \mathbb{R}^d$ and defines a measureable function $f * g$ (*convolution *of f and g). Futhermore,* Young's inequality

$$\|f * g\|_r \leq \|f\|_p \|g\|_q \tag{A.11}$$

holds.

Functional Analysis

Definition A.14 (Complemented subspace). (Cf. [Bre11, Section 2.4]) Let X be a Banach space and $U \subseteq X$ one of its closed subspaces. A subspace $V \subseteq X$ is called a *complement* of U in X, if

(i) V is closed,

(ii) $U \cap V = \emptyset$ and

(iii) $X = U + V$.

If U has at least one complement, it is called *complemented.*

Proposition A.15 (Open mapping Theorem). *(See [Bre11, Corollary 2.7]) Let X, Y be Banach spaces and $T \in \mathscr{L}(X, Y)$ be bijective. Then $T^{-1} \in \mathscr{L}(Y, X)$.*

Proposition A.16 (Adjoint operators)**.** *Let X be a Banach space and $T \in \mathscr{L}(X)$. Then its adjoint $T^* \in \mathscr{L}(X^*)$ satisfies*

$$\|T^*\| = \|T\|\,, \tag{A.12}$$

where the norm above is the operator norm on X and X^* respectively.

Proof. One indeed has

$$\begin{aligned} \|T\| &= \sup_{\|x\|_X=1} \|Tx\|_X = \sup_{\|x\|_X=1} \sup_{\|x^*\|_{X^*}=1} \left|\langle x^*, Tx\rangle_{X^*\times X}\right| \\ &= \sup_{\|x^*\|_{X^*}=1} \sup_{\|x\|_X=1} \left|\langle T^*x^*, x\rangle_{X^*\times X}\right| = \sup_{\|x^*\|_{X^*}=1} \|T^*x^*\|_{X^*} \\ &= \|T^*\|\,, \end{aligned}$$

where the second equality holds by the Hahn-Banach theorem and all others by definition. □

Definition A.17 (Sequence spaces)**.** (Cf. [WZG06, Proof of Proposition 3.2]) Let $d \in \mathbb{N}$, $p, q \in [1, \infty]$, and $s \in \mathbb{R}$. Define

$$\|(f_k)_{k\in\mathbb{Z}^d}\|_{l^q_s(\mathbb{Z}^d, L^p(\mathbb{R}^d))} = \begin{cases} \left(\sum_{k\in\mathbb{Z}^d} \langle k\rangle^{qs} \|u_k\|_p^q\right)^{\frac{1}{q}} & \text{if } q < \infty, \\ \sup_{k\in\mathbb{Z}^d} \langle k\rangle \|u_k\|_p & \text{if } q = \infty \end{cases} \qquad \forall (f_k)_{k\in\mathbb{Z}^d} \in \mathcal{M}, \tag{A.13}$$

where

$$\begin{aligned} \mathcal{M} &= \left\{(f_k)_{k\in\mathbb{Z}^d} \middle| \forall k \in \mathbb{Z}^d : f_k \in L^0(\mathbb{R}^d)\right\}, \\ L^0(\mathbb{R}^d) &= \left\{f : \mathbb{R}^d \to \mathbb{C} \middle| f \text{ measureable}\right\} / N \text{ and} \\ N &= \left\{f : \mathbb{R}^d \to \mathbb{C} \middle| f \text{ measureable and } f = 0 \text{ almost everywhere}\right\}. \end{aligned}$$

Furthermore, define

$$\begin{aligned} l^q_s(\mathbb{Z}^d, L^p(\mathbb{R}^d)) &:= \left\{(f_k)_{k\in\mathbb{Z}^d} \in \mathcal{M} \middle| \|(f_k)_{k\in\mathbb{Z}^d}\|_{l^q_s(\mathbb{Z}^d, L^p(\mathbb{R}^d))} < \infty\right\}, \\ c^0_s(\mathbb{Z}^d, L^p(\mathbb{R}^d)) &:= \left\{(f_k)_{k\in\mathbb{Z}^d} \in \mathcal{M} \middle| \lim_{|k|\to\infty} \langle k\rangle^s \|f_k\|_p = 0\right\} \subseteq l^\infty_s(\mathbb{Z}^d, L^p(\mathbb{R}^d)) \qquad \text{and} \\ c^{00}(\mathbb{Z}^d, L^p(\mathbb{R}^d)) &:= \left\{(f_k)_{k\in\mathbb{Z}^d} \in l^q_0(\mathbb{Z}^d, L^p(\mathbb{R}^d)) \middle| \exists K \in \mathbb{N} : \forall |k| > K : f_k = 0\right\}. \end{aligned}$$

Often, the notation is shortened to $l^q_s(L^p) := l^q_s(\mathbb{Z}^d, L^p(\mathbb{R}^d))$ and $l^q(L^p) := l^q_0(L^p)$. Furthermore, $l^0_s(\mathbb{Z}^d, L^p(\mathbb{R}^d)) := c^0_s(L^p) := c^0_s(\mathbb{Z}^d, L^p(\mathbb{R}^d))$.

Proposition A.18. *Let $d \in \mathbb{N}$, $p, q \in [1, \infty]$ and $s \in \mathbb{R}$. Then space $l^q_s(\mathbb{Z}^d, L^p(\mathbb{R}^d))$ endowed with the norm $\|\cdot\|_{l^q_s(\mathbb{Z}^d, L^p(\mathbb{R}^d))}$ is a Banach space. Moreover, $c^0_s(\mathbb{Z}^d, L^p(\mathbb{R}^d))$ is a closed subset of $l^\infty_s(\mathbb{Z}^d, L^p(\mathbb{R}^d))$. Furthermore, $c^{00}(\mathbb{Z}^d, L^p(\mathbb{R}^d))$ is dense in $c^0_s(\mathbb{Z}^d, L^p(\mathbb{R}^d))$ and, if $q < \infty$, in $l^q_s(\mathbb{Z}^d, L^p(\mathbb{R}^d))$.*

A proof of the last proposition is not given here, but see [Wer18, Beispiel I.1 (f,g)] for the scalar case $l^p(\mathbb{N},\mathbb{C})$, which is very similar.

Proposition A.19. *Let $d \in \mathbb{N}$, $p \in [1,\infty)$, $q \in \{0\} \cup [1,\infty)$ and $s \in \mathbb{R}$. Then the map $\Phi : l^{q'}_{-s}(\mathbb{Z}^d, L^{p'}(\mathbb{R}^d)) \to \left(l^q_s(\mathbb{Z}^d, L^p(\mathbb{R}^d))\right)'$ defined by*

$$(\Phi(u_k)_{k\in\mathbb{Z}^d})\,(v_k)_{k\in\mathbb{Z}^d} = \sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\overline{u}_k(x)v_k(x)\mathrm{d}x \qquad \forall (v_k)_{k\in\mathbb{Z}^d} \in l^q_s(\mathbb{Z}^d, L^p(\mathbb{R}^d)) \tag{A.14}$$

is antilinear, bijective and isometric (for $q=0$, set $q'=1$ in this proposition and its proof).

Proof. For any $(u_k) \in l^{q'}_{-s}(L^{p'})$ and any $(v_k) \in l^q(L^p)$ one has

$$\begin{aligned}
|(\Phi(u_k)_k)(v_k)_k| &\le \sum_{k\in\mathbb{Z}^d}\int |u_k|\,|v_k|\,\mathrm{d}x \le \sum_{k\in\mathbb{Z}^d}\langle k\rangle^{-s}\|u_k\|_{p'}\,\langle k\rangle^{s}\|v_k\|_p \\
&\le \|(u_k)\|_{l^{q'}_{-s}(L^{p'})}\,\|(v_k)\|_{l^q_s(L^p)}
\end{aligned}$$

by Hölder's inequality. This shows that $\Phi : l^{q'}_{-s}(L^{p'}) \to (l^q_s(L^p))'$ and

$$\|\Phi(u_k)\| \le \|u_k\|_{l^{q'}_{-s}(L^{p'})}. \tag{A.15}$$

The antilinearity of Φ is immediately clear from Equation (A.14). It remains to show the converse inequality of (A.15) (which implies the injectivity of Φ), and the fact that Φ is surjective.

To show the surjectivity of Φ, consider a $\phi \in (l^q_s(L^p))'$ and fix a $k \in \mathbb{Z}^d$. Then the linear functional $v \mapsto \phi((\delta_{km}v)_m)$ is continuous on L^p and hence there exists a unique $u_k \in L^{p'}$ such that

$$\phi((\delta_{km}v)_m) = \int \overline{u_k}v\mathrm{d}x \qquad \forall v \in L^p$$

(see [Bre11, Theorem 4.11] for $p>1$ and [Bre11, Theorem 4.13] for $p=1$). Furthermore, $\|v \mapsto \phi((\delta_{km}v)_m)\| = \|u_k\|_{p'}$ and thus, for any $\varepsilon > 0$, there is a $v_k = v_k(\varepsilon) \in L^p$ with $\|v_k\|_p = 1$ such that

$$\int \overline{u_k}v_k\mathrm{d}x = \left|\int \overline{u_k}v_k\mathrm{d}x\right| \ge \|u_k\|_{p'}(1-\varepsilon).$$

To show is that $(u_k)_k \in l^{q'}_{-s}(L^{p'})$, i.e. $(\|u_k\|)_k \in l^{q'}_{-s}$. Arguments similar to those above applied to the space of complex sequences $l^{q'}(\mathbb{Z}^d,\mathbb{C})$ show that for any $\varepsilon > 0$ there is a (real) sequence $(\alpha_k)_{k\in\mathbb{Z}^d} = (\alpha_k(\varepsilon))_{k\in\mathbb{Z}^d} \in c^{00}$ with $\|(\langle k\rangle^s\alpha_k)\|_q = 1$ such that

$$\sum_{k\in\mathbb{Z}^d}\alpha_k\|u_k\|_{p'} = \left|\sum_{k\in\mathbb{Z}^d}\alpha_k\|u_k\|_{p'}\right| \ge \|(u_k)_k\|_{l^{q'}_{-s}(L^{p'})}(1-\varepsilon).$$

Fix any $\varepsilon > 0$ and choose the corresponding (v_k) and (α_k). Linearity of ϕ implies that $\phi((w_k)_k) = \sum_{k\in\mathbb{Z}^d} \int \overline{u_k} w_k$ for any $(w_k) \in c^{00}(L^p)$. In particular

$$\phi((\alpha_k v_k)_k) = \sum_{k\in\mathbb{Z}^d} \alpha_k \int \overline{u_k} v_k \mathrm{d}x \geq (1-\varepsilon) \sum_{k\in\mathbb{Z}^d} \alpha_k \left\|u_k\right\|_{p'} \geq (1-\varepsilon)^2 \left\|(u_k)\right\|_{l^{q'}_{-s}(L^{p'})}.$$

Due to the continuity of ϕ, the left-hand side is bounded from above by

$$|\phi((\alpha_k v_k)_k)| \leq \|\phi\| \left\|(\alpha_k v_k)\right\|_{l^q_s(L^p)} = \|\phi\|.$$

Hence, passing to the limit $\varepsilon \to 0+$ shows that

$$\left\|(u_k)_k\right\|_{l^{q'}_{-s}(L^{p'})} \leq \|\phi\| < \infty. \tag{A.16}$$

Thus $\Phi((u_k))$ is defined and $(\Phi(u_k))(v_k) = \phi(v_k)$ for any $(v_k) \in c^{00}(L^p)$. As $c^{00}(L^p)$ is dense in $l^q_s(L^p)$ by Proposition A.18, $\Phi((u_k)) = \phi$ follows and finishes the proof of the surjectivity of Φ.

Observing that the choice of (u_k) was unambiguous shows that ϕ may be replaced by $\Phi(u_k)$ in (A.16), which proves the converse inequality of (A.15) and finishes the proof. □

Strongly continuous groups

Definition A.20 (C_0-group)**.** (Cf. [EN00, Definition I.5.1]). Let X be Banach space and $(T(t))_{t\in\mathbb{R}} \in \mathscr{L}(X)^{\mathbb{R}}$ be a family of bounded operators on X. If $(T(t))_{t\in\mathbb{R}}$ satisfies the functional equation

$$T(t+s) = T(t)T(s) \tag{A.17}$$

for all $t, s \in \mathbb{R}$ and the initial condition $T(0) = \mathrm{id}_X$, then it is called a *group*. If in addition $(T(t))_{t\in\mathbb{R}}$ is strongly continuous, i.e. for any $x \in X$ the *orbit map* $t \mapsto T(t)x$ is continuous, then $(T(t))_{t\in\mathbb{R}}$ is called a *C_0-group*.

Definition A.21 (Generator of a C_0-group)**.** (Cf. [EN00, Definition II.1.2]). Let $(T(t))_{t\in\mathbb{R}}$ be a C_0-group on the Banach space X. The linear operator $A : \mathrm{dom}(A) \to X$ defined by

$$\begin{aligned} \mathrm{dom}(A) &= \left\{ x \in X \middle| \lim_{h\to 0+} \frac{1}{h}\left(T(h)x - x\right) \text{ exists} \right\} \qquad \text{and} \\ Ax &= \lim_{h\to 0+} \frac{1}{h}\left(T(h)x - x\right) \qquad \forall x \in \mathrm{dom}(A) \end{aligned}$$

is called the *generator* of $(T(t))_{t\in\mathbb{R}}$.

Definition A.22 (Core for a linear operator)**.** (Cf. [EN00, II.1.6]). Let X be a Banach space, $C \subseteq \mathrm{dom}(A)$ linear subspaces of X and $A : \mathrm{dom}(A) \to X$ a linear operator. C is a *core* for A, if $\overline{C}^{\|\cdot\|_A} = \mathrm{dom}(A)$, where

$$\|x\|_A := \|x\| + \|Ax\| \qquad \forall x \in \mathrm{dom}(A)$$

is the *graph norm*.

Proposition A.23 (Core criterion for generators)**.** *(Cf. [EN00, Proposition II.1.7]). Let $(T(t))_{t\in\mathbb{R}}$ be a C_0-group in the Banach space X and $A : \mathrm{dom}(A) \to X$ its generator. Furthermore, let $C \subseteq \mathrm{dom}(A)$ be a linear subspace of X. If C is dense in X and $T(t)x \in C$ for any $x \in C$ and $t \in \mathbb{R}$, then C is a core for A.*

Proposition A.24. *(Cf. [Paz92, Theorem 6.1.4]). Let $(T(t))_{t\in\mathbb{R}}$ be a C_0-group with generator A. Moreover, let $F : X \to X$ be locally Lipschitz continuous and $u_0 \in X$. Then the integral equation*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(u(s))\mathrm{d}s$$

has a unique maximal solution $u \in C((-a, b), X)$, where $a, b > 0$. If $a < \infty$, then $\liminf_{t\to a+} \|u(t)\| = \infty$. Similarly, if $b < \infty$, then $\liminf_{t\to b-} \|u(t)\| = \infty$. Finally, the map $u_0 \mapsto u$ is locally Lipschitz continuous.

Fourier Analysis

Definition A.25 (Fourier transform on $L^1(\mathbb{R}^d)$)**.** (Cf. [Gra08, Definition 2.2.8]). Let $d \in \mathbb{N}$ and $f \in L^1(\mathbb{R}^d)$. The *Fourier transform* $\mathcal{F}f := \hat{f}$ of f is given by

$$\hat{f}(k) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x)e^{-\mathrm{i}k\cdot x}\mathrm{d}x \qquad \forall k \in \mathbb{R}^d. \tag{A.18}$$

The *inverse Fourier transform* $\mathcal{F}^{(-1)}f := \check{f}$ of f is defined by $\check{f}(x) = \hat{f}(-x)$ for all $x \in \mathbb{R}^d$.

Example A.26 (Fourier transform of a Gaussian). Let $d \in \mathbb{N}$ and $g \in L^1(\mathbb{R}^d)$ be given by

$$g(x) = e^{-\frac{|x|^2}{2}} \qquad \forall x \in \mathbb{R}^d. \tag{A.19}$$

Then $\hat{g} = g$.

Proof. Using (A.3) one immediately confirms that

$$\begin{aligned} \hat{g}(k) &= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\mathrm{i}kx} e^{-\frac{|x|^2}{2}} = \prod_{j=1}^{d} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x_j}{\sqrt{2}} + \mathrm{i}\frac{k_j}{\sqrt{2}}\right)^2 - \frac{k_j^2}{2}} \mathrm{d}x \\ &= e^{-\frac{|k|^2}{2}} \prod_{j=1}^{d} \frac{I\left(\frac{1}{\sqrt{2}}, \frac{\mathrm{i}}{\sqrt{2}}\right)}{\sqrt{2\pi}} = g(k) \end{aligned}$$

holds for all $k \in \mathbb{R}^d$. □

Definition A.27 (Multi-indices)**.** Let $d \in \mathbb{N}$. A tuple $\alpha \in \mathbb{N}_0^d$ is called *multi-index*. Its *size* $\sum_{i=1}^d \alpha_i$ is denoted by $|\alpha|$. For $x \in \mathbb{R}^d$ set $x^\alpha = \prod_{i=1}^d x_i^{\alpha_i}$. For $f \in C^{|\alpha|}(\mathbb{R}^d)$ set $\partial^\alpha f = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1}\cdots\partial^{\alpha_d}} f$. For another multi-index $\beta \in \mathbb{N}_0^d$ define

$$\beta \leq \alpha \Leftrightarrow \forall i \in \{1, \ldots, d\} : \beta_i \leq \alpha_i.$$

Of course, if $\beta \leq \alpha$, then their *difference* $\alpha - \beta = (\beta_1 - \alpha_1, \ldots, \beta_d - \alpha_d)$ is again a multi-index. Finally, define the *binomial coefficient* $\binom{\alpha}{\beta} = \prod_{i=1}^d \binom{\alpha_i}{\beta_i}$.

Lemma A.28 (Leibnitz' rule)**.** *(Cf. [Gra08, equation 2.2.4]). Let $d, m \in \mathbb{N}_0$ and $f, g \in C^m(\mathbb{R}^d)$. Then for any $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq m$ one has the multidimensional* Leibntiz' rule

$$\partial^\alpha(fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g). \tag{A.20}$$

Definition A.29 (Japanese bracket)**.** (Cf. [Tao06, Preface]) Let $d \in \mathbb{N}$ and $\xi \in \mathbb{R}^d$. Denote by

$$\langle \xi \rangle = \left(1 + |\xi|^2\right)^{\frac{1}{2}}$$

the *Japanese bracket* of ξ.

Lemma A.30 (Quasi-subadditivity of $\langle \cdot \rangle^s$)**.** *Let $d \in \mathbb{N}$ and $s \geq 0$. Then the inequality*

$$\langle x + y \rangle^s \leq 2^s \left(\langle x \rangle^s + \langle y \rangle^s \right) \tag{A.21}$$

holds for all $x, y \in \mathbb{R}^d$.

Proof. Observe, that because $s \geq 0$ the function $a \mapsto a^s$ is increasing on $[0, \infty)$. Assume w.l.o.g. that $|x| \leq |y|$. Then one indeed has

$$\frac{\langle x + y \rangle^s}{\langle x \rangle^s + \langle y \rangle^s} \leq \left(\frac{\langle x + y \rangle^2}{\langle y \rangle^2} \right)^{\frac{s}{2}} \leq \left(\frac{1 + (|x| + |y|)^2}{1 + |y|^2} \right)^{\frac{s}{2}} \leq \left(\frac{1 + 4\,|y|^2}{1 + |y|^2} \right)^{\frac{s}{2}} \leq 2^s.$$

□

Lemma A.31 (Peetre's inequality)**.** *(Cf. [RT10, Proposition 3.3.31]). Let $d \in \mathbb{N}$, $s \in \mathbb{R}$ and ξ, $\eta \in \mathbb{R}^d$. Then*

$$\langle \xi + \eta \rangle^s \leq 2^{|s|} \langle \xi \rangle^s \langle \eta \rangle^{|s|}. \tag{A.22}$$

Definition A.32 (Schwartz space)**.** (Cf. [Gra08, Definition 2.2.1]) Let $d \in \mathbb{N}$. For $\alpha, \beta \in \mathbb{N}_0^d$ and $f \in \mathbb{C}^\infty(\mathbb{R}^d)$ consider the *Schwartz seminorm* $\rho_{\alpha,\beta}(f) = \sup_{x \in \mathbb{R}^d} \left| x^\alpha (\partial^\beta f)(x) \right|$. Denote by

$$\mathcal{S}(\mathbb{R}^d) := \left\{ f \in \mathbb{C}^\infty(\mathbb{R}^d) \middle| \, \forall \alpha, \beta \in \mathbb{N}_0^d : \rho_{\alpha,\beta}(f) < \infty \right\}$$

the so-called *Schwartz space.* The topology on $\mathcal{S}(\mathbb{R}^d)$ is induced by the family of seminorms $(\rho_{\alpha,\beta})$, i.e.

$$f_k \xrightarrow{k \to \infty} f \Leftrightarrow \forall \alpha, \beta \in \mathbb{N}_0^d : \rho_{\alpha,\beta}(f_k - f) \xrightarrow{k \to \infty} 0.$$

Proposition A.33. *(Cf. [Gra08, Proposition 2.2.6] and [Wer18, remark after Definition V.2.3]) Let $d \in \mathbb{N}$ and $p \in [1, \infty]$. Then $\mathcal{S}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$. If $p < \infty$, then $\mathcal{S}(\mathbb{R}^d)$ is even dense in $L^p(\mathbb{R}^d)$.*

Proposition A.34 (Fourier transform on $\mathcal{S}$)**.** *(Cf. [Gra08, Corollary 2.2.15]) The Fourier transform is bijective and continuous on $\mathcal{S}(\mathbb{R}^d)$. Its inverse is continuous and is given by the inverse Fourier transform.*

Proposition A.35 (Parseval's theorem). *(Cf. [Gra08, Theorem 2.2.14]) Let $d \in \mathbb{N}$ and $f, g \in \mathcal{S}(\mathbb{R}^d)$. Then $\left\langle \hat{f}, \hat{g} \right\rangle = \langle f, g \rangle$. In particular, the Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{(-1)}$ uniquely extend to isometries on $L^2(\mathbb{R}^d)$ satisfying $\mathcal{F} \circ \mathcal{F}^{(-1)} = \mathcal{F}^{(-1)} \circ \mathcal{F} = \mathrm{id}_{L^2(\mathbb{R}^d)}$.*

Proposition A.36 (Hausdorff-Young inequality). *(Cf. [Gra08, Proposition 2.2.16]) Let $d \in \mathbb{N}$ and $p \in [1, 2]$. Then the Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{(-1)}$ uniquely extend to continuous linear operators from $L^p(\mathbb{R}^d)$ to $L^{p'}(\mathbb{R}^d)$ satisfying*

$$\|\mathcal{F}f\|_{p'} = \left\|\mathcal{F}^{(-1)}f\right\|_{p'} \leq \|f\|_p \qquad \forall f \in L^p(\mathbb{R}^d). \tag{A.23}$$

Proposition A.37 (Fourier transform of a convolution in $\mathcal{S}$). *(Cf. [Gra08, Proposition 2.2.11]) Let $d \in \mathbb{N}$ and $f, g \in \mathcal{S}(\mathbb{R}^d)$. Then $f * g \in \mathcal{S}(\mathbb{R}^d)$, $fg \in \mathcal{S}(\mathbb{R}^d)$,*

$$\widehat{f * g} = (2\pi)^{\frac{d}{2}} \hat{f} \cdot \hat{g} \qquad \text{and} \qquad \widehat{fg} = \frac{1}{(2\pi)^{\frac{d}{2}}} \hat{f} * \hat{g}.$$

Definition A.38 (Functions of moderate growth). Let $d \in \mathbb{N}$. A smooth function $f \in C^\infty(\mathbb{R}^d)$ shall be of *moderate growth*, if every of its derivatives grows at most polynomially. Denote the space of such functions by

$$C^\infty_{\mathrm{pol}}(\mathbb{R}^d) = \left\{ f \in C^\infty(\mathbb{R}^d) \middle| \, \forall \alpha \in \mathbb{N}_0^d : \exists C > 0, n \in \mathbb{N}_0 : \forall x \in \mathbb{R}^d \, |(\partial^\alpha f)(x)| \leq C(1 + |x|^n) \right\}.$$

Definition A.39 (Tempered distributions). (Cf. [Gra08, Definition 2.3.3, Proposition 2.3.4]) Let $d \in \mathbb{N}$. Denote by $\mathcal{S}'(\mathbb{R}^d)$ the space of continuous linear functionals on $\mathcal{S}(\mathbb{R}^d)$ (so-called *tempered distributions*), i.e. for a linear functional $u : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ one has

$$u \in \mathcal{S}'(\mathbb{R}^d) \Leftrightarrow \exists C > 0, k, m \in \mathbb{N} : \forall f \in \mathcal{S}(\mathbb{R}^d) : |\langle u, f \rangle| \leq C \sum_{\substack{|\alpha| \leq m, \\ |\beta| \leq k}} \rho_{\alpha,\beta}(f).$$

The topology on $\mathcal{S}'(\mathbb{R}^d)$ is the weak *-topology, so

$$u_k \xrightarrow{k \to \infty} u \Leftrightarrow \forall f \in \mathcal{S}(\mathbb{R}^d) : \langle u_k - u, f \rangle \xrightarrow{k \to \infty} 0.$$

For a suitable measureable function $f : \mathbb{R}^d \to \mathbb{C}$ one obtains a tempered distribution Φf defined by

$$\langle \Phi f, g \rangle = \int_{\mathbb{R}^d} \overline{f(x)} g(x) \mathrm{d}x \qquad \forall g \in \mathcal{S}(\mathbb{R}^d), \tag{A.24}$$

if the integral above exists and can be controlled by the sum of finitely many Schwartz seminorms of g. For example $\Phi : L^p(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ for any $p \in [1, \infty]$ by Lemma A.11 and Proposition A.33. In such cases one often identifies $f = \Phi f$.

Furthermore, given an operation A on functions one tries to consistently extend it to an operation $\tilde{A}$ on tempered distributions, i.e. $\Phi A f = \tilde{A} \Phi f$. If this is possible one again often identifies $A = \tilde{A}$. Some examples relevant to the thesis at hand are shown below.

Definition A.40 (Operations on $\mathcal{S}'(\mathbb{R}^d)$)**.** (Cf. [Gra08, Definitions 2.3.6, .7, .11, .15]) Let $d \in \mathbb{N}$. For $u \in \mathcal{S}'(\mathbb{R}^d)$ define

(a) the α-th derivative $\partial^\alpha u$ of u, for $\alpha \in \mathbb{N}_0^d$, by

$$\langle \partial^\alpha u, f \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha f \rangle \qquad \forall f \in \mathcal{S}(\mathbb{R}^d),$$

(b) the *reflection* $\tilde{u}$ of u by

$$\langle \tilde{u}, f \rangle = \left\langle u, \tilde{f} \right\rangle \qquad \forall f \in \mathcal{S}(\mathbb{R}^d),$$

(c) the *Fourier transform* $\mathcal{F}u$ of u by

$$\langle \mathcal{F}u, f \rangle \coloneqq \langle \hat{u}, f \rangle \coloneqq \left\langle u, \mathcal{F}^{(-1)} f \right\rangle \qquad \forall f \in \mathcal{S}(\mathbb{R}^d),$$

(d) *multiplication* gu with a function of moderate growth $g \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$ of u by

$$\langle gu, f \rangle = \langle u, \overline{g} f \rangle \qquad \forall f \in \mathcal{S}(\mathbb{R}^d),$$

(e) *convolution* $g * u$ with a Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$ of u by

$$\langle g * u, f \rangle = \left\langle u, \overline{\tilde{\overline{f}} * g} \right\rangle \qquad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

Proposition A.41. *(Cf. [Gra08, Proposition 2.3.22]) Let $d \in \mathbb{N}$, $f \in \mathcal{S}(\mathbb{R}^d)$ and $u \in \mathcal{S}'(\mathbb{R}^d)$. Then*

$$\widehat{f * u} = (2\pi)^{\frac{d}{2}} \hat{f} \cdot \hat{u} \qquad \text{and} \qquad \widehat{fu} = \frac{1}{(2\pi)^{\frac{d}{2}}} \hat{f} * \hat{u}.$$

Definition A.42 (Support of $u \in \mathcal{S}'(\mathbb{R}^d)$)**.** (Cf. [Gra08, Definition 2.3.16]) Let $d \in \mathbb{N}$ and $u \in \mathcal{S}'(\mathbb{R}^d)$. The *support* of u is defined as the intersection of all closed sets $K \subseteq \mathbb{R}^d$ which satisfy

$$\mathrm{supp}(f) \subseteq K^c \Rightarrow \langle u, f \rangle = 0 \qquad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

The set of (tempered) distributions with compact support shall be denoted by $\mathcal{E}'(\mathbb{R}^d)$.

Proposition A.43 (Convolution $\mathcal{E}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$)**.** *(Cf. [Vla02, subsections 4.2.7, 5.6.1]). Let $d \in \mathbb{N}$, $u \in \mathcal{S}'(\mathbb{R}^d)$ and $v \in \mathcal{E}'(\mathbb{R}^d)$. Then*

$$\langle v * u, f \rangle \coloneqq \langle u, y \mapsto \langle S_y v, f \rangle \rangle \qquad \forall f \in \mathcal{S}(\mathbb{R}^d)$$

defines a tempered distribution $u * v \in \mathcal{S}'(\mathbb{R}^d)$, which is called the convolution *of u with v. If $v = \Phi g$ for a function $g \in \mathcal{D}(\mathbb{R}^d)$, then $v * u = g * u$ as given in Definition A.40. Moreover,*

$$\mathrm{supp}(u * v) \subseteq \overline{\mathrm{supp}(u) + \mathrm{supp}(v)} \tag{A.25}$$

holds, so in particular $u * v \in \mathcal{E}'(\mathbb{R}^d)$, if $u, v \in \mathcal{E}'(\mathbb{R}^d)$.

Proposition A.44. *(Cf. [Gra08, Theorem 2.3.21]) Let $d \in \mathbb{N}$, $u \in \mathcal{S}'(\mathbb{R}^d)$ and $\hat{u} \in \mathcal{E}'(\mathbb{R}^d)$. Then u can be uniquely represented by a function $f \in C^\infty_{pol}(\mathbb{R}^d)$ as in (A.24), which is given by*

$$f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \overline{\hat{u}\left(\sigma e^{-\mathrm{i}x\cdot}\right)} \qquad \forall x \in \mathbb{R}^d,$$

where $\sigma \in \mathcal{D}(\mathbb{R}^d)$ satisfying $\sigma(y) = 1$ for all $y \in \operatorname{supp}(\hat{u})$ is arbitrary. Moreover, f has a holomorphic extension to $\mathbb{C}^d$.

Proposition A.45. *(Cf. [Vla02, Section 6.5]). Let $d \in \mathbb{N}$, $u, v \in \mathcal{S}'(\mathbb{R}^d)$ and $\hat{v} \in \mathcal{E}'(\mathbb{R}^d)$. Then*

$$\mathcal{F}(fu) = \frac{1}{(2\pi)^{\frac{d}{2}}} \hat{v} * \hat{u},$$

where $f \in C^\infty_{pol}(\mathbb{R}^d)$ denotes the unique representation of v as in Proposition A.44.

Lemma A.46. *Let $d \in \mathbb{N}$, $\sigma \in \mathcal{S}(\mathbb{R}^d)$ (*symbol*) and $p_1, p_2 \in [1, \infty]$ satisfy $p_1 \le p_2$. Then the* multiplier Operator *$T_\sigma = \mathcal{F}^{(-1)}\sigma\mathcal{F}$ is bounded from $L^{p_1}(\mathbb{R}^d)$ to $L^{p_2}(\mathbb{R}^d)$ and $\|T_\sigma\|_{\mathscr{L}(L^{p_1}, L^{p_2})} \le \|\hat{\sigma}\|_{L^r}$ for*

$$\frac{1}{r} = 1 - \left(\frac{1}{p_1} - \frac{1}{p_2}\right).$$

Proof. One has $L^{p_1}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ via Equation (A.24). As Φ by construction commutes with all operations from Definition A.40,

$$T_\sigma \Phi f = \widetilde{\mathcal{F}\sigma\widetilde{\mathcal{F}}\Phi f} = \frac{1}{(2\pi)^{\frac{d}{2}}} \widetilde{\hat{\sigma} * \Phi\tilde{f}} = \frac{1}{(2\pi)^{\frac{d}{2}}} \Phi\left(\check{\sigma} * f\right)$$

follows (i.e. $T_\sigma f$ is understood as $\frac{1}{(2\pi)^{\frac{d}{2}}} \check{\sigma} * f$ for $f \in L^{p_1}$). By the prerequisites $r \in [1, \infty]$ and $1 + \frac{1}{p_2} = \frac{1}{r} + \frac{1}{p_1}$. Hence, one has

$$\|\check{\sigma} * f\|_{p_2} \le \|\check{\sigma}\|_r \|f\|_{p_1} \qquad \forall f \in L^{p_1}(\mathbb{R}^d)$$

by Young's inequality (A.11). As $\sigma \in \mathcal{S}$ one also has $\check{\sigma} \in \mathcal{S}$ by Proposition A.34 and so

$$\|T_\sigma\|_{\mathscr{L}(L^{p_1}, L^{p_2})} \lesssim_d \|\check{\sigma}\|_r < \infty,$$

where finiteness of the right-hand side follows by Proposition A.33 finishing the proof. □

The operator norm bound in the multiplier estimate above gives rise to the following

Definition A.47 (Fourier-Lebesgue spaces)**.** (Cf. [PTT10, Equation (1.2)]) Let $d \in \mathbb{N}$ and $p \in [1, \infty]$. Then the *Fourier-Lebesgue* space $\mathcal{F}L^p(\mathbb{R}^d)$ is defined as

$$\mathcal{F}L^p(\mathbb{R}^d) = \left\{u \in \mathcal{S}'(\mathbb{R}^d) | \, \mathcal{F}u \in L^p(\mathbb{R}^d)\right\}.$$

Here, $\mathcal{F}u \in L^p(\mathbb{R}^d)$ means that $\mathcal{F}u = \Phi f$ as in Equation (A.24) and $f \in L^p(\mathbb{R}^d)$. Another notation for the Fourier-Lebesgue spaces is (note the dual exponent) $\widehat{L}^{p'}(\mathbb{R}^d) := \mathcal{F}L^p(\mathbb{R}^d)$.

Fourier-Lebesgue spaces equipped with the norm

$$\|u\|_{\mathcal{F}L^p(\mathbb{R}^d)} := \|\mathcal{F}u\|_p$$

are Banach spaces.

By the Hausdorff-Young inequality (Proposition A.36) one immediately obtains

Lemma A.48. *Let $d \in \mathbb{N}$ and $r \in [2, \infty]$. Then $L^{r'}(\mathbb{R}^d) \hookrightarrow \mathcal{F}L^r(\mathbb{R}^d)$.*

For $r \in [1, 2)$, the situation is more subtle.

Definition A.49 (Bessel potential of order s)**.** (Cf. [BL76, Section 6.2].) Let $d \in \mathbb{N}$ and $s \in \mathbb{R}$. Define the operator $J^s := (I - \Delta)^{\frac{s}{2}} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ through

$$J^s u := \mathcal{F}^{(-1)} \langle \cdot \rangle^s \mathcal{F}u \qquad \forall u \mathcal{S}'(\mathbb{R}^d). \tag{A.26}$$

The operator J^s is called *Bessel potential* of order $-s$.

Definition A.50 (Bessel potential spaces)**.** (Cf. [BL76, Definition 6.2.2]). Let $d \in \mathbb{N}$, $p \in [1, \infty]$ and $s \in \mathbb{R}$. Then the *Bessel potential space* $H^s_p(\mathbb{R}^d)$ is defined as

$$H^s_p(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d) |\, J^s u \in L^p(\mathbb{R}^d) \right\}.$$

Here, $J^s u \in L^p(\mathbb{R}^d)$ means that $\mathcal{F}^{(-1)} \langle \cdot \rangle^s \mathcal{F}u = \Phi f$ as in Equation (A.24) and $f \in L^p(\mathbb{R}^d)$. Another common name for the Bessel potential spaces is *(generalized) Sobolev spaces.*

Bessel potential spaces equipped with the norm

$$\|u\|_{H^s_p(\mathbb{R}^d)} := \|J^s \mathcal{F}u\|_p$$

are Banach spaces. In the special case $p = 2$ they are Hilbert spaces and are denoted by $H^s(\mathbb{R}^d)$.

Lemma A.51. *Let $d \in \mathbb{N}$, $r \in [1, 2)$ and $s > d\left(\frac{1}{r} - \frac{1}{2}\right)$. Then $H^s(\mathbb{R}^d) \hookrightarrow \mathcal{F}L^r(\mathbb{R}^d)$ and the implicit constant depends on r and s only (i.e. not on d).*

Proof. By Hölder's inequality one has

$$\|u\|_{\mathcal{F}L^r} = \left\| \frac{\langle \cdot \rangle^s}{\langle \cdot \rangle^s} \hat{u} \right\|_r \le \left\| \langle \cdot \rangle^{-s} \right\|_p \|u\|_{H^s(\mathbb{R}^d)},$$

for $\frac{1}{p} + \frac{1}{2} = \frac{1}{r}$. The first factor is estimated using hyperspherical coordinates against

$$\begin{aligned} \left\| \langle \cdot \rangle^{-s} \right\|_p &= \left(\int_{\mathbb{R}^d} \frac{1}{(1 + |\xi|^2)^{\frac{sp}{2}}} \mathrm{d}\xi \right)^{\frac{1}{p}} \approx \left(\int_0^\infty \frac{\rho^{d-1}}{(1 + \rho^2)^{\frac{sp}{2}}} \right)^{\frac{1}{p}} \mathrm{d}\rho \\ &\le \left(1 + \int_1^\infty \rho^{-(sp-d)-1} \mathrm{d}\rho \right)^{\frac{1}{p}} = \left(1 - (sp - d) \left[\rho^{-(sp-d)} \right]_{\rho=1}^\infty \right)^{\frac{1}{p}} \end{aligned}$$

and is hence finite, if $s > \frac{d}{p} = d\left(\frac{1}{r} - \frac{1}{2}\right)$. This is true by the prerequisites.

The implicit constant in the first estimate does not depend on d, as the maximum of the surface measure of the unit $d-1$-sphere is attained for $d = 7$. Hence, one has

$$\|u\|_{\mathcal{F}L^r} \lesssim (1 + sp - d)^{\frac{1}{p}} \|u\|_{H^s} \leq \left(1 + \frac{s}{\frac{1}{r} - \frac{1}{2}}\right)^{\frac{1}{\frac{1}{r}-\frac{1}{2}}} \|u\|_{H^s}.$$

As the right-hand side does not depend on d, the proof is concluded. □

Observe, that for $d \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^d$ and $f \in \mathcal{S}(\mathbb{R}^d)$, one has

$$(\partial^\alpha f)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} (\mathrm{i}k)^\alpha e^{\mathrm{i}kx} \hat{f}(k) \mathrm{d}k = \left(\mathcal{F}^{(-1)} (\mathrm{i}\cdot)^\alpha \mathcal{F} f\right)(x) \qquad \forall x \in \mathbb{R}^d,$$

due to Proposition A.34 and Theorem A.5. Together with Definition A.40 this proves

$$\partial^\alpha u = \mathcal{F}^{(-1)} (\mathrm{i}\cdot)^\alpha \mathcal{F} u \qquad \forall u \in \mathcal{S}'(\mathbb{R}^d). \tag{A.27}$$

Proposition A.52 (Characterization of H^s_p via derivatives)**.** *(Cf. [BL76, Theorem 6.2.3.]) Let $d \in \mathbb{N}$, $p \in (1, \infty)$ and $s \in \mathbb{N}_0$. Then*

$$H^s_p(\mathbb{R}^d) = \left\{ f \in L^p(\mathbb{R}^d) \middle| \, \forall j \in \{1, \ldots, d\} : \partial^{se_j} f \in L^p(\mathbb{R}^d) \right\}$$

and

$$\|f\|_{H^s_p(\mathbb{R}^d)} \approx_{d,p,s} \|f\|_p + \sum_{j=1}^d \|\partial^{se_j} f\|_p \qquad \forall f \in H^s_p(\mathbb{R}^d).$$

Here, of course, $f \in L^p(\mathbb{R}^d)$ is identified with $\Phi f \in \mathcal{S}'(\mathbb{R}^d)$ and $\partial^{se_j} f$ is then understood in the sense of (A.27)*. Moreover, the fact that $\partial^{se_j} f = \Phi g \in \mathcal{S}'(\mathbb{R}^d)$ for a $g \in L^p(\mathbb{R}^d)$ is implied and $\|\partial^{se_j} f\|_p$ is understood as $\|g\|_p$. For $f \in \mathcal{S}(\mathbb{R}^d)$, $\partial^{se_j} f$ is the usual derivative, i.e. $g = \partial^{se_j} f \in \mathcal{S}(\mathbb{R}^d)$.*

In the thesis at hand the following corollary of Lemma A.46 is heavily used.

Corollary A.53 (Bernstein multiplier estimate)**.** *(Cf. [WH07, Proposition 1.9]). Let $d \in \mathbb{N}$, $\sigma \in \mathcal{D}(\mathbb{R}^d)$ and $p_1, p_2 \in [1, \infty]$ satisfy $p_1 \leq p_2$. Then the multiplier $T_\sigma = \mathcal{F}^{(-1)} \sigma \mathcal{F}$ is bounded from $L^{p_1}(\mathbb{R}^d)$ to $L^{p_2}(\mathbb{R}^d)$ and there is a constant $C = C\left(d, \frac{1}{p_1} - \frac{1}{p_2}\right)$ such that*

$$\|T_\sigma\|_{\mathscr{L}(L^{p_1}, L^{p_2})} \leq C(1 + |\mathrm{supp}(\sigma)|) \left(\|\sigma\|_\infty + \sum_{j=1}^d \left\| \partial^{de_j} \sigma \right\|_\infty \right). \tag{A.28}$$

Proof. Define $r \in [0, 1]$ through

$$\frac{1}{r} = 1 - \left(\frac{1}{p_1} - \frac{1}{p_2}\right).$$

By Lemma A.46, one immediately has

$$\|T_\sigma\|_{\mathscr{L}(L^{p_1},L^{p_2})} \le \|\sigma\|_{\mathcal{F}L^r}\,.$$

Distinguish between the cases $r \ge 2$ (i.e. $\frac{1}{r'} \ge \frac{1}{2}$) and $r < 2$ (i.e. $\frac{1}{r} > \frac{1}{2}$).

In the first case, one immediately has

$$\|\sigma\|_{\mathcal{F}L^r} \le \|\sigma\|_{r'} \le |\mathrm{supp}(\sigma)|^{\frac{1}{r'}}\,\|\sigma\|_\infty \le (1+|\mathrm{supp}(\sigma)|)\,\|\sigma\|_\infty$$

by the Hausdorff-Young inequality from Proposition A.36. This shows (A.28) with $C = 1$.

For the second case, put $s \coloneqq d$. As $\left(\frac{1}{r}-\frac{1}{2}\right) < 1$, one has $s > d\left(\frac{1}{r}-\frac{1}{2}\right)$ and hence Lemma A.51 applies. Together with Proposition A.52 (for $p = 2$ and s as above) this yields

$$\begin{aligned}\|\sigma\|_{\mathcal{F}L^r} &\lesssim_{d,r} \|\sigma\|_{H^d} \lesssim_d \|\sigma\|_2 + \sum_{j=1}^{d}\left\|\partial^{de_j}\sigma\right\|_2 \le |\mathrm{supp}(\sigma)|^{\frac{1}{2}}\left(\|\sigma\|_\infty + \sum_{|\alpha|\le d+1}\|\partial^\alpha\sigma\|_\infty\right)\\ &\le (1+|\mathrm{supp}(\sigma)|)\left(\|\sigma\|_\infty + \sum_{|\alpha|\le d+1}\|\partial^\alpha\sigma\|_\infty\right)\end{aligned}$$

and hence shows (A.28) with $C = C(d,r) = C\left(d, \frac{1}{p_1}-\frac{1}{p_2}\right)$. The proof is complete. □

Interpolation theory

Definition A.54 (Interpolation couple, intermediate space)**.** (Cf. [Tri78, Subsection 1.2.1].) Let X, Y be complex Banach spaces. If there is a topological Hausdorff vector space $\mathcal{V}$ such that $X, Y \subseteq \mathcal{V}$ and $X, Y \hookrightarrow \mathcal{V}$, then $\{X, Y\}$ is said to be an *interpolation couple*. In this case $X \cap Y$ equipped with the norm

$$\|z\|_{X\cap Y} \coloneqq \max\{\|z\|_X, \|z\|_Y\} \qquad \forall z \in X \cap Y$$

and $X + Y = \left\{z \in \mathcal{V} \,\middle|\, \exists x \in X\ \exists y \in Y : z = x + y\right\}$ equipped with the norm

$$\|z\| \coloneqq \inf_{\substack{z=x+y\\ x\in X, y\in Y}} [\|x\|_X + \|y\|_Y] \qquad \forall z \in X + Y$$

are Banach spaces. Any Banach space Z satisfying $X \cap Y \subseteq Z \subseteq X + Y$ and

$$X \cap Y \hookrightarrow Z \hookrightarrow X + Y$$

is called an *intermediate space* (w.r.t. the interpolation couple $\{X, Y\}$).

Definition A.55 (Interpolation functor, interpolation space)**.** (Cf. [Tri78, Definition 1.2.2/1].) An *interpolation functor* F is any “procedure” which, given an interpolation couple $\{X_0, X_1\}$, produces an intermediate space $F(\{X_0, X_1\})$ such that for any other interpolation couple

$\{Y_0, Y_1\}$ and any $T \in \mathscr{L}(X_0 + X_1, Y_0 + Y_1)$, which satisfies $T|_{X_i} \in \mathscr{L}(X_i, Y_i)$ for $i \in \{0, 1\}$, one has

$$T|_{F(\{X_0,X_1\})} \in \mathscr{L}(F(\{X_0, X_1\}, F(\{Y_0, Y_1\})).$$

If for a Banach space X there exists an interpolation functor F such that $X = F(\{X_0, X_1\})$, then X is called an *interpolation space* (w.r.t. the interpolation couple $\{X_0, X_1\}$.

Real interpolation

Definition A.56 (K-functional)**.** (See [Tri78, Subsection 1.3.1].) Let $\{X, Y\}$ be an interpolation couple. Define the functional $K : (0, \infty) \times (X + Y) \to \mathbb{R}_0^+$ by

$$K(t, z; X, Y) := \inf_{\substack{z=x+y \\ x\in X, y\in Y}} (\|x\|_X + t\,\|y\|_Y) \tag{A.29}$$

for any $t > 0$ and any $z \in X+Y$. One often shortens the notation to $K(t, z) := K(t, z; X, Y)$.

Definition A.57 (Real interpolation spaces $(X, Y)_{\theta,q}$)**.** (See [Tri78, Definition 1.3.2].) Let $\{X, Y\}$ be an interpolation couple, $\theta \in (0, 1)$ and $q \in [1, \infty]$. Define the *real interpolation space*

$$\begin{aligned} (X, Y)_{\theta,q} &:= \left\{ z \in X + Y \middle| \, \|z\|_{(X,Y)_{\theta,q}} < \infty \right\}, \quad \text{where} \\ \|z\|_{(X,Y)_{\theta,q}} &:= \begin{cases} \left(\int_0^\infty \left[t^{-\theta} K(t, z)\right]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} & \text{for } q < \infty, \\ \sup\limits_{t>0} \left[t^{-\theta} K(t, z)\right] & \text{for } q = \infty. \end{cases} \end{aligned}$$

Proposition A.58. *(Cf. [Tri78, Theorem 1.3.3].) Let $\theta \in (0, 1)$ and $q \in [1, \infty]$. Then the mapping $\{X, Y\} \mapsto (X, Y)_{\theta,q}$ defines an interpolation functor.*

Complex interpolation

Definition A.59 (Complex interpolation spaces $[X, Y]_\theta$)**.** (Cf. [Tri78, Definition 1.9.2].) Let $\{X, Y\}$ be an interpolation couple and $\theta \in (0, 1)$. Set $S = \left\{z \in \mathbb{C} \middle| 0 < \mathrm{Re}(z) < 1\right\}$, $\overline{S} = \left\{z \in \mathbb{C} \middle| 0 \le \mathrm{Re}(z) \le 1\right\}$ and let $F(\{X, Y\})$ denote the set of functions $f : \overline{S} \to (X+Y)$ satisfying

(i) $f \in C_\mathrm{b}(\overline{S}, X + Y)$ and f is analytic on S with values in $X + Y$,

(ii) $t \mapsto f(\mathrm{i}t) \in C_\mathrm{b}(\mathbb{R}, X)$, $t \mapsto f(1 + \mathrm{i}t) \in C_\mathrm{b}(\mathbb{R}, Y)$.

Define the *complex interpolation space*

$$[X, Y]_\theta := \left\{z \in X + Y \middle| \exists f \in F(\{X, Y\}) : z = f(\theta)\right\} \tag{A.30}$$

equipped with the norm

$$\|z\|_{[X,Y]_\theta} := \inf_{\substack{z=f(\theta),\\ f\in F(\{X,Y\})}} \max\left\{\sup_{s\in\mathbb{R}} \|f(\mathrm{i}s)\|_X, \sup_{t\in\mathbb{R}} \|f(1+\mathrm{i}t)\|_Y\right\}. \tag{A.31}$$

Proposition A.60. *(Cf. [Tri78, Theorem 1.9.3].) Let $\theta \in (0,1)$. Then the mapping $\{X,Y\} \mapsto [X,Y]_\theta$ is an interpolation functor. Moreover, $X \cap Y$ is dense in $[X,Y]_\theta$ and if $X = Y$ one has $[X,Y]_\theta = X = Y$.*

Lemma A.61. *Let $\{X,Y\}$ be an interpolation couple, $a,b > 0$ and $\theta \in (0,1)$. Then*

$$[aX, bY]_\theta = a^{1-\theta} b^\theta [X,Y]_\theta,$$

where the equality above means not only the equality of sets but also the equality of norms.

Proof. As $F := F(\{aX,bY\}) = F(\{X,Y\})$, the equality $Z := [aX,bY]_\theta = a^{1-\theta}b^\theta[X,Y]_\theta$ as sets is apparent from Equation (A.30).

For the equality of norms, fix any $z \in Z$ and consider any $\varepsilon > 0$. By Equation (A.31), there is an $f_0 \in F$ such that $f_0(\theta) = z$ and

$$a^{1-\theta}b^\theta \max\left\{\sup_{s\in\mathbb{R}} \|f_0(\mathrm{i}s)\|_X, \sup_{t\in\mathbb{R}} \|f_0(1+\mathrm{i}t)\|_Y\right\} \le \|z\|_{a^{1-\theta}b^\theta[X,Y]_\theta} + \varepsilon.$$

Set $g_0 := \left(\frac{a}{b}\right)^{z-\theta} f_0$ and observe that $g_0 \in F$, $g_0(\theta) = f_0(\theta) = z$,

$$\begin{aligned}\sup_{s\in\mathbb{R}} \|g_0(\mathrm{i}s)\|_{aX} &= a \sup_{s\in\mathbb{R}} \left[\left|\left(\frac{a}{b}\right)^{\mathrm{i}s-\theta}\right| \|f_0(\mathrm{i}s)\|_X\right] = a \sup_{s\in\mathbb{R}} \left[\left(\frac{a}{b}\right)^{\mathrm{Re}(\mathrm{i}s-\theta)} \|f_0(\mathrm{i}s)\|_X\right] \\ &= a^{1-\theta}b^\theta \sup_{s\in\mathbb{R}} \|f_0(\mathrm{i}s)\|_X \le \|z\|_{a^{1-\theta}b^\theta[X,Y]_\theta} + \varepsilon\end{aligned}$$

and similarly

$$\sup_{t\in\mathbb{R}} \|g_0(1+\mathrm{i}t)\|_{bY} \le \|z\|_{a^{1-\theta}b^\theta[X,Y]_\theta} + \varepsilon.$$

Hence

$$\|z\|_{[aX,bY]_\theta} \le \max\left\{\sup_{s\in\mathbb{R}} \|g_0(\mathrm{i}s)\|_{aX}, \sup_{t\in\mathbb{R}} \|g_0(1+\mathrm{i}t)\|_{bY}\right\} \le \|z\|_{a^{1-\theta}b^\theta[X,Y]_\theta} + \varepsilon$$

and thus $\|z\|_{[aX,bY]_\theta} \le \|z\|_{a^{1-\theta}b^\theta[X,Y]_\theta}$ for any $z \in Z$, because $\varepsilon > 0$ was arbitrary.

To show the converse inequality, set $\tilde{X} := aX$, $\tilde{Y} := bY$, $\tilde{a} := \frac{1}{a}$ and $\tilde{b} := \frac{1}{b}$. Then, $X = \tilde{a}\tilde{X}$, $Y = \tilde{b}\tilde{Y}$ and, by the above, one has

$$\|z\|_{a^{1-\theta}b^\theta[X,Y]_\theta} = a^{1-\theta}b^\theta \|z\|_{[\tilde{a}\tilde{X},\tilde{b}\tilde{Y}]_\theta} \le a^{1-\theta}b^\theta \|z\|_{\tilde{a}^{1-\theta}\tilde{b}^\theta[\tilde{X},\tilde{Y}]_\theta} = \|z\|_{[aX,bY]_\theta} \qquad \forall z \in Z.$$

This finishes the proof. ☐

Example A.62 (Riesz-Thorin). (Cf. [Tri78, Theorem 1.18.6/2 and the proof of Theorem 1.18.7/1]). Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space, $p_0, p_1 \in [1, \infty]$ and $\theta \in (0,1)$. Define $p \in (1, \infty)$ via

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Then

$$[L^{p_0}(\Omega), L^{p_1}(\Omega)]_\theta = L^p(\Omega),$$

where the equality above means not only the equality of sets but also the equality of norms.

Theorem A.63. *(Cf. [Tri78, Definition 1.10.1 and Theorem 1.10.3/1].) Let $\{X, Y\}$ be an interpolation couple and $\theta \in (0,1)$. Then $[X,Y]_\theta \hookrightarrow (X,Y)_{(\theta,\infty)}$.*

Example A.64 (Complex interpolation of $l^q_s(\mathbb{Z}^d, L^p(\mathbb{R}^d))$ spaces). Let $d \in \mathbb{N}$, $p_0, p_1 \in [1, \infty]$ and $q_0, q_1 \in \{0\} \cup [1, \infty]$ such that $q_0 \neq \infty$ or $q_1 \neq \infty$. Furthermore, let $s_0, s_1 \in \mathbb{R}$ and $\theta \in (0,1)$. Define $s \coloneqq (1-\theta)s_0 + \theta s_1 \in \mathbb{R}$ and $p \in [1, \infty]$ via

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Finally, define $q \in \{0\} \cup [1, \infty)$ via

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

in the case $q_0 \neq 0$ and $q_1 \neq 0$. For the other cases, set

$$q \coloneqq \begin{cases} \frac{q_0}{1-\theta} & \text{for } q_0 \neq \infty \text{ and } q_1 = 0, \\ \frac{q_1}{\theta} & \text{for } q_0 = 0 \text{ and } q_1 \neq \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$[l^{q_0}_{s_0}(\mathbb{Z}^d, L^{p_0}(\mathbb{R}^d)), l^{q_1}_{s_1}(\mathbb{Z}^d, L^{p_1}(\mathbb{R}^d))]_\theta = l^q_s(\mathbb{Z}^d, L^p(\mathbb{R}^d)),$$

where the equality above means not only the equality of sets but also the equality of norms.

Proof. Observe, that L^0 is canonically equipped with the topology of local convergence in measure and $\mathcal{M} = \prod_{z \in \mathbb{Z}^d} L^0$ from Definition A.17 with the corresponding product topology. With this topology, $\mathcal{M}$ is a Hausdorff vector space and $l^{q_i}_{s_i}(L^{p_i}) \hookrightarrow \mathcal{M}$, for $i \in \{0,1\}$. Hence, $\{l^{q_0}_{s_0}(\mathbb{Z}^d, L^{p_0}(\mathbb{R}^d)), l^{q_1}_{s_1}(\mathbb{Z}^d, L^{p_1}(\mathbb{R}^d))\}$ is an interpolation couple and the notion of the complex interpolation space $[l^{q_0}_{s_0}(\mathbb{Z}^d, L^{p_0}(\mathbb{R}^d)), l^{q_1}_{s_1}(\mathbb{Z}^d, L^{p_1}(\mathbb{R}^d))]_\theta$ makes sense.

For every $k \in \mathbb{Z}^d$ set $A_k \coloneqq \langle k \rangle^{s_0} L^{p_0}$, i.e. $A_k = L^{p_0}(\mathbb{R}^d)$ as sets and

$$\|f\|_{A_k} = \langle k \rangle^{s_0} \|f\|_{p_0} \qquad \forall f \in A_k.$$

Define in the same way $B_k \coloneqq \langle k \rangle^{s_1} L^{p_1}$ for every $k \in \mathbb{Z}^d$. By [Tri78, Theorem 1.18.1 and Remarks 1.18.1/1-3] one has that

$$[l^{q_0}_{s_0}(\mathbb{Z}^d, L^{p_0}(\mathbb{R}^d)), l^{q_1}_{s_1}(\mathbb{Z}^d, L^{p_1}(\mathbb{R}^d))]_\theta = [l^{q_0}(A_k), l^{q_1}(B_k)]_\theta = l^q([A_k, B_k]_\theta), \tag{A.32}$$

where the equalities above also mean the equalities of norms.

Observe, that by Lemma A.61 and Example A.62 one has

$$[A_k, B_k]_\theta = [\langle k\rangle^{s_0} L^{p_0}, \langle k\rangle^{s_1} L^{p_1}]_\theta = \langle k\rangle^s [L^{p_0}, L^{p_1}]_\theta = \langle k\rangle^s L^p \qquad \forall k \in \mathbb{Z}^d,$$

where, again, the equalities above also mean the equalities of norms. Inserting this equality into Equation (A.32) finishes the proof. □

For the case $q_0 = q_1 = \infty$, which is not covered by Example A.64, more can be said if additionally $p_0 = p_1 = p \in [1, \infty]$. One has for $s_0, s_1 \in \mathbb{R}$ with $s_0 \neq s_1$, $\theta \in (0,1)$ and $s = (1-\theta)s_0 + \theta s_1$ that

$$[l^\infty_{s_0}(L^p), l^\infty_{s_1}(L^p)]_\theta = l^0_s(L^p)$$

(the proof of this statement is along the lines of the proof of [Tri78, Equation (1.18.1/16)]). Of course, if $s_0 = s_1 = s \in \mathbb{R}$, then

$$[l^\infty_{s_0}(L^p), l^\infty_{s_1}(L^p)]_\theta = l^\infty_s(L^p)$$

by Proposition A.60.

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