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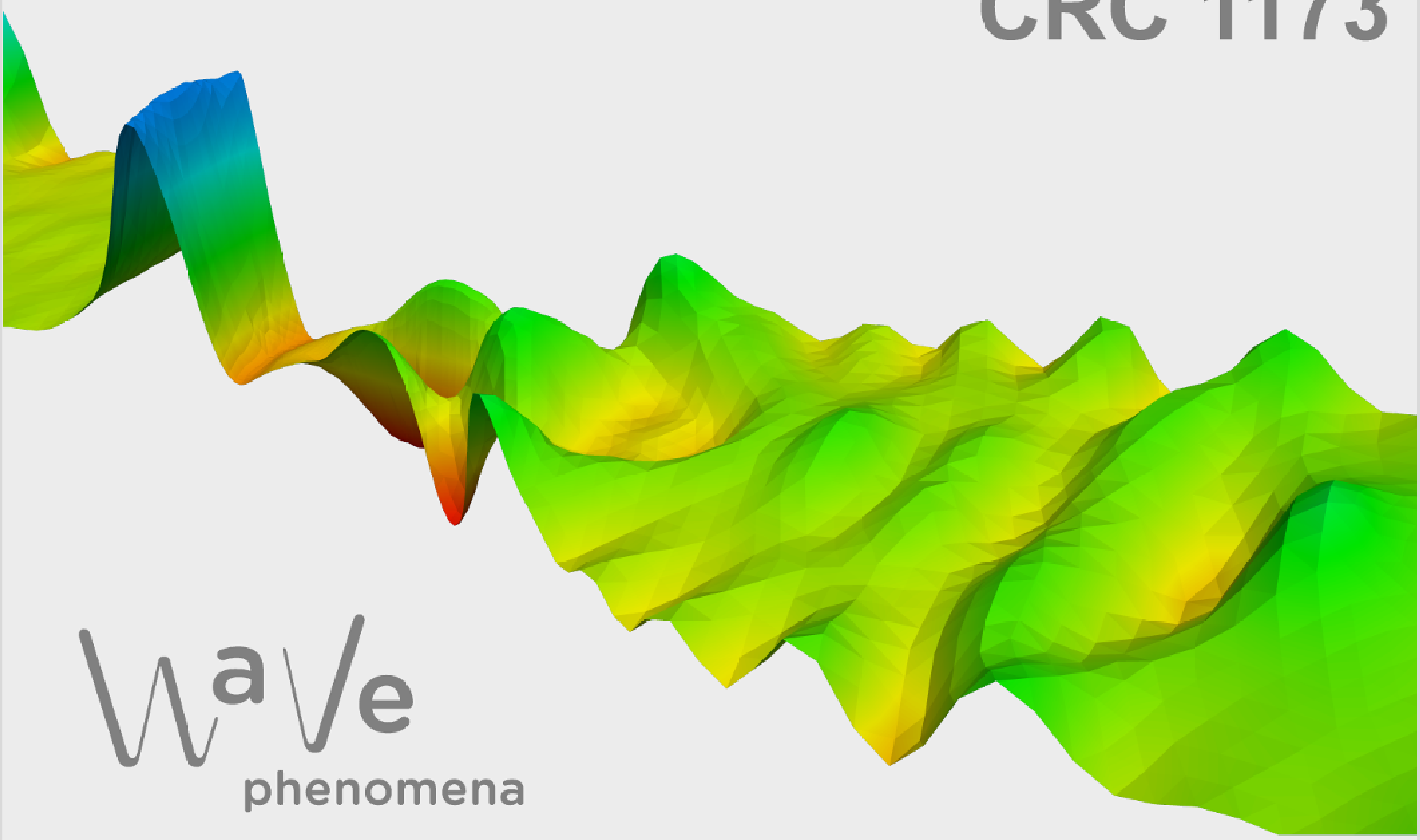

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Failure of the N-wave interaction approximation without imposing periodic boundary conditions

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Abstract

The N-wave interaction (NWI) system appears as an amplitude system in the description of N nonlinearly interacting and linearly transported wave packets in dispersive wave systems, such as the water wave problem or problems in nonlinear optics. The purpose of this paper is twofold. First we give a new simplified proof for the failure of the NWI approximation in case of resonances which are located at integer multiples of a basic wave number k_0 in the original dispersive wave system. Secondly, we give a first rigorous proof that an amplitude system fails in the description of an original system, without imposing periodic boundary conditions on the original system.

1 Introduction

Amplitude, modulation, or envelope equations, such as the Ginzburg-Landau, the KdV, or the NLS equation play a big role for the qualitative understanding of pattern forming systems or of dispersive wave systems in spatially unbounded domains. The last decades saw a big number of approximation results for these multiple scaling problems. It has been shown that amplitude equations make correct predictions about the dynamics of the original systems, cf. [Cra85a, SW00, SW02, Due12] for the KdV approximation, [Kal88, BSTU06, TW12, DSW16] for the NLS approximation, or [CE90, vH91, Sch94, Sch99, SZ13] for the Ginzburg-Landau approximation. For an introduction to the theory see [SU17, Chapters 10-12].

Only for a few examples [Sch95, Sch05] it has been known that amplitude equations can fail to make correct predictions. Therefore, in the last years, besides proving approximation results, we started to investigate the failure of amplitude equations more systematically, cf. [SSZ15, Sch16, BSSZ, dRHS]. It turned out that the question of validity of amplitude equations in many situations is really subtle. The amplitude equation can fail for Sobolev initial conditions, but can make correct predictions for analytic conditions, cf. [Sch95, DHSZ16]. It can fail for periodic boundary conditions, but can make correct predictions on the whole real line, cf. [DSS16].

The validity of an approximation $\varepsilon^\alpha \psi$ on a time scale $T = \varepsilon^\beta t$, with $0 < \varepsilon \ll 1$ a small perturbation parameter, for a system

$$\partial_t u = \Lambda u + B(u,u) + h.o.t.,$$

with linear operator Λ and symmetric bilinear operator B, is established by controlling the error $R = u - \varepsilon^\alpha \psi$ on the given $\mathcal{O}(1/\varepsilon^\beta)$-time scale. The difficulty lies in the fact that in general $\beta > \alpha$ such that a simple application of Gronwall's inequality will not be sufficient. The equation for the error is of the form

$$\partial_t R = \underbrace{\Lambda R}_{(1)} + \underbrace{2\varepsilon^\alpha B(\psi, R)}_{(2)} + \mathcal{O}(\varepsilon^{2\alpha} + R^2) + h.o.t..$$

By adding higher order corrections to the approximation $\varepsilon^\alpha \psi$, in general, it is possible to get rid of the terms indicated by $\mathcal{O}(\varepsilon^{2\alpha} + R^2) + h.o.t.$. Therefore, there are essentially two possibilities how an amplitude equation can fail, namely failure by linear instability via the terms indicated by **(1)** and failure by nonlinear dynamics via the terms indicated by **(2)**.

Examples for **(1)** are the failure of a number of modulation equations for the approximate description of the dynamics near unstable dispersive periodic waves [Sch16, §7.6] and the non-validity of a number of amplitude equations for the description of modulations of periodic waves at the Eckhaus boundary [dRHS, §8.4] in dissipative systems. Examples for **(2)** are the failure of the Newell-Whitehead equation for pattern forming systems via quadratic transverse instabilities [Sch95, §4], the failure of the NLS approximation for the water wave problem with suitably chosen small surface tension and periodic boundary conditions [SSZ15] via unstable resonances, and the failure of the NLS approximation for a modified Zakharov system [BSSZ, §4].

In this paper, with the failure of the N wave interaction (NWI) approximation for a special model problem we give a first example for **(2)**, without imposing periodic boundary conditions on the original system. The main ingredients of our construction are a periodic arrangement of resonant wave numbers and a finite speed of propagation in the original system. The construction goes in two steps.

First, in Section 4 we use this periodically arranged quadratic resonances to give a new simplified proof for the failure of the NWI approximation in case of suitably chosen periodic boundary conditions. The new proof avoids a number of unsatisfactory steps from the proof given in [SSZ15] for the possible failure of the NLS approximation in this situation. We remark that in case of periodic boundary conditions on the original system the NWI approximation with $N = 1$ and the NLS approximation coincide, of. Section 2.

Secondly, in Section 5 we use the result from the first step and the finite speed of propagation of the original system to give the first rigorous proof that an amplitude system fails in the description of the original system without imposing periodic boundary conditions on the original system. As preparation, we provide some background about the NWI approximation and about stable and unstable quadratic resonances.

We refrain from greatest generality and restrict ourselves to a particular equation which we introduce at the beginning of Section 3.1. We expect that the ideas of the present paper apply to the water wave problem without surface tension over a suitably chosen periodic bottom, cf. Remark 6.3.

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2 Some background

In this section we provide some background which is necessary for the understanding of the subsequent sections. These are the basics of the NWI approximation and the notion of stable and unstable quadratic resonances.

2.1 The NWI approximation

There are essentially two consistent descriptions of modulated oscillating wave packets for dispersive wave systems by amplitude equations. These are the Nonlinear Schrödinger (NLS) description

$$\varepsilon\psi_{\mathrm{NLS}}(\varepsilon,x,t)=\varepsilon A(\varepsilon(x+ct),\varepsilon^2t)e^{i(k_1x+\omega_1t)}+c.c.+\mathcal{O}(\varepsilon^2), \tag{1}$$

and the N wave interaction (NWI) description

$$\varepsilon\psi_{\mathrm{NWI}}(\varepsilon,x,t)=\sum_{j\in I_N}\varepsilon A_j(\varepsilon^2x,\varepsilon^2t)e^{i(k_jx+\omega_jt)}+c.c.+\mathcal{O}(\varepsilon^2), \tag{2}$$

with index set $I_N=\{1,\ldots,N\}$, cf. [AS81]. Herein, we have the small perturbation parameter $0<\varepsilon\ll1$, the amplitude functions $A(\xi,\tau)$, $A_j(X,\tau)\in\mathbb{C}$, and the wave numbers $k_j=-k_{-j}\in\mathbb{R}$ and $\omega_j=-\omega_{-j}\in\mathbb{R}$ which are related via the linear dispersion relation of the original system. The spatial scalings in the NLS and NWI ansatz are different, see Figure 1.

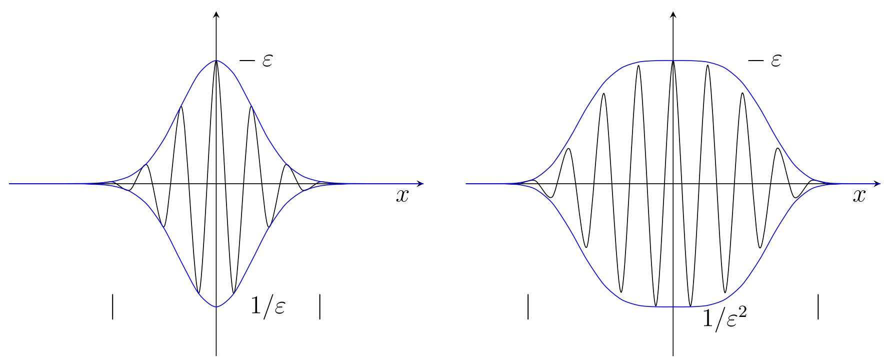

Figure 1: Left panel: The NLS scaling: An amplitude function of size $\mathcal{O}(\varepsilon)$ modulates on an $\mathcal{O}(1/\varepsilon)$ spatial scale the underlying carrier wave. Right panel: The NWI scaling: An amplitude function of size $\mathcal{O}(\varepsilon)$ modulates on an $\mathcal{O}(1/\varepsilon^2)$ spatial scale the underlying carrier wave in case $N=1$.

The amplitude function A in the NLS description varies via $\xi=\varepsilon(x+ct)$ on a spatial scale of order $\mathcal{O}(1/\varepsilon)$. It is transported with velocity c and is affected by dispersion on the $\mathcal{O}(1/\varepsilon^2)$-time scale via the subsequent $i\nu_1\partial_\xi^2A$-term in the NLS equation

$$\partial_\tau A=i\nu_1\partial_\xi^2A+i\nu_2A|A|^2, \tag{3}$$

with coefficients $\nu_j \in \mathbb{R}$. In lowest order the amplitude function A satisfies the NLS equation (3).

In contrast, the amplitude functions A_j in the NWI description vary via $X = \varepsilon^2 x$ on a spatial scale of order $\mathcal{O}(1/\varepsilon^2)$. They are transported via the subsequent terms $c_j \partial_X A_j$ in (4), and up to the order of preciseness of the ansatz they are not affected by dispersion on the $\mathcal{O}(1/\varepsilon^2)$-time scale. Plugging in the NWI ansatz into the original system and equating the coefficients of $\varepsilon^3 e^{i(k_j x+\omega_j t)}$ to zero yields in general the (non-resonant) NWI system

$$\partial_\tau A_j = c_j \partial_X A_j + i \sum_{l \in I_N} d_{j,l} |A_l|^2 A_j, \tag{4}$$

with group velocities $c_j = \frac{d\omega}{dk}|_{k=k_j,\omega=\omega_j}$ and coefficients $d_{j,l} \in \mathbb{R}$, in case the wave numbers k_j and ω_j are not in quadratic or cubic resonance, see (5) and (8).

In case that four spatial wave numbers $k_j, k_{j_1}, k_{j_2}, k_{j_3} \in \mathbb{R}$ together with their associated temporal wave numbers $\omega_j, \omega_{j_1}, \omega_{j_2}, \omega_{j_3} \in \mathbb{R}$ satisfy

$$k_j + k_{j_1} + k_{j_2} + k_{j_3} = 0 \qquad \text{and} \qquad \omega_j + \omega_{j_1} + \omega_{j_2} + \omega_{j_3} = 0, \tag{5}$$

we obtain the resonant NWI system

$$\partial_\tau A_j = c_j \partial_X A_j + i \sum_{l \in I_N} d_{jl} |A_l|^2 A_j + i \sum_{\text{(5) is satisfied}} d^j_{j_1 j_2 j_3} \overline{A_{j_1} A_{j_2} A_{j_3}}, \tag{6}$$

with additional coefficients $d^j_{j_1 j_2 j_3} \in \mathbb{R}$. Resonant NWI systems appear in a number of physical situations. They are used as a model for the description of gravity driven surface water waves, cf. [AS81, Cra85b], and they are expected to be important in the description of so called freak waves in deep sea, cf. [Kar11].

In case of no quadratic resonances, see (8), the justification of the NLS and NWI approximations via error estimates can be done very similarly, cf. [SZ05]. In contrast to the NLS approximation the NWI approximation changes so slowly in space and time, that resonances of the original system have enough time to destroy the NWI approximation property. For a more detailed discussion see Section 5 and Remark 6.1.

In case of $2\pi/k_0$-periodic boundary conditions the (multi-) NLS and NWI approximation coincide

$$\varepsilon \psi^{per}_{\mathrm{NLS}}(\varepsilon, x, t) = \varepsilon \psi^{per}_{\mathrm{NWI}}(\varepsilon, x, t) = \sum_{j \in I_N} \varepsilon A_j(\varepsilon^2 t) e^{i(k_j x + \omega_j t)} + c.c. + \mathcal{O}(\varepsilon^2). \tag{7}$$

We remark that periodic boundary conditions on the original system correspond to ξ-, respectively X-independent solutions of (3) and (4).

2.2 Stable and unstable quadratic resonances

In systems with a quadratic nonlinearity spatial wave numbers $\widetilde{k}_1$, $\widetilde{k}_2$, and $\widetilde{k}_3$ with associated temporal wave numbers $\widetilde{\omega}_1$, $\widetilde{\omega}_2$, and $\widetilde{\omega}_3$ are called quadratically resonant if

$$\widetilde{k}_1 + \widetilde{k}_2 + \widetilde{k}_3 = 0 \quad \text{and} \quad \widetilde{\omega}_1 + \widetilde{\omega}_2 + \widetilde{\omega}_3 = 0. \tag{8}$$

The dynamics of the resonant modes

$$\varepsilon \psi_{\mathrm{TWI}}(\varepsilon, x, t) = \sum_{j=1,2,3} \varepsilon B_j(\varepsilon t) e^{i(\widetilde{k}_j x + \widetilde{\omega}_j t)} + c.c.,$$

with $\widetilde{k}_j = -\widetilde{k}_{-j}$, is approximately described by the resonant three wave interaction (TWI) system

$$\partial_T B_1 = i\gamma_1 \overline{B}_2 \overline{B}_3, \qquad \partial_T B_2 = i\gamma_2 \overline{B}_1 \overline{B}_3, \qquad \partial_T B_3 = i\gamma_3 \overline{B}_1 \overline{B}_2,$$

with amplitudes $B_j(T) = \overline{B_{-j}(T)} \in \mathbb{C}$ and coefficients $\gamma_j \in \mathbb{R}$, cf. [Cra85b]. The TWI approximation is fundamentally different from the NWI approximation, because it describes quadratic resonances which are explicitly excluded among the wave numbers k_j of the NWI approximation, i.e., in the following we will have $k_3 = \widetilde{k}_3$, but $|\widetilde{k}_1|, |\widetilde{k}_2| \notin \{k_1, \ldots, k_N\}$. As a consequence of the quadratic resonance the dynamics of the TWI modes happens on a much shorter time scale than the NWI dynamics.

The resonant wave number $\widetilde{k}_1$ is called stable if the invariant subspace $M_1 = \{(B_1, 0, 0) : B_1 \in \mathbb{C}\}$ for the TWI system is stable. It is called unstable, if M_1 is unstable. There is a simple stability criterion. M_1 is stable, if $\gamma_2\gamma_3 < 0$, and unstable, if $\gamma_2\gamma_3 > 0$. It is an easy exercise to see that, independent of the signs of the γ_js, at least one of the subspaces M_j is unstable, say the one to the wave number $\widetilde{k}_3$.

In the following this unstable resonance is used as follows. For our purposes we restrict ourselves to the NWI approximation with $N = 1$ and associated wave number k_*. We suppose that $k_* = \widetilde{k}_3$ is quadratically resonant with two other wave numbers $\widetilde{k}_1$ and $\widetilde{k}_2$, where all the $\widetilde{k}_j$ are integer multiples of a basic wave number $k_0 > 0$. In such a situation the NWI approximation in case of $2\pi/k_0$-periodic boundary conditions will fail to make correct predictions about the dynamics of the original system.

The reason is as follows. Since $k_* = \widetilde{k}_3$ is an unstable resonance, the solution at $\tilde{k}_1$ and $\tilde{k}_2$ will grow. If the $\widetilde{k}_3$-mode is of order $\mathcal{O}(\varepsilon)$, like for the NWI approximation, the growth happens with a rate of order $\mathcal{O}(e^{\varepsilon t})$. Since the solution at $\tilde{k}_1$ and $\tilde{k}_2$ is small initially in terms of ε, it takes a little bit longer than $\mathcal{O}(1/\varepsilon)$ for these modes to grow to the size $\mathcal{O}(\varepsilon)$ of the NWI approximation, but this will happen far before the end of the NWI $\mathcal{O}(1/\varepsilon^2)$-time scale. Since the modes to $\tilde{k}_1$ and $\tilde{k}_2$ belong to the error and not to the NWI approximation, the NWI system cannot make correct predictions on its natural time scale, see Figure 2. In Section 4 these heuristic arguments are made rigorous.

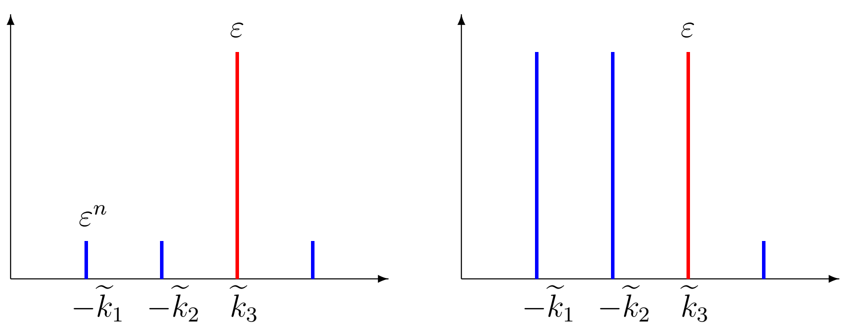

Figure 2: The mode distribution for $t = 0$ and the mode distribution for $t = \mathcal{O}(|\ln \varepsilon|/\varepsilon) \ll \mathcal{O}(1/\varepsilon^2)$. The NWI approximation is no longer valid in the right picture, since the modes at $\pm\widetilde{k}_1$ and $\pm\widetilde{k}_2$ are of the same order w.r.t. powers of ε as the NWI mode at $k = k_* = \widetilde{k}_3$.

3 The result

In this section we construct our model problem, a scalar PDE on the real line, we derive the associated TWI and NWI system, and state our non-approximation results.

3.1 The model problem

For showing that the NWI approximation can make wrong predictions, without imposing periodic boundary conditions on the original system, we use two ingredients, namely a periodic arrangement of the quadratically resonant wave numbers and a finite speed of propagation in the original system. In Section 4 we use these periodically arranged quadratic resonances to prove the failure of the NWI approximation in case of suitably chosen periodic boundary conditions. In Section 5 we use the result from Section 4 and the finite speed of propagation of our chosen original system to give a first proof that an amplitude system fails in the description of the original system without imposing periodic boundary conditions on the original system.

In order to construct an original system with these properties we proceed as follows. First, the resonant wave numbers $\widetilde{k}_1$, $\widetilde{k}_2$, and $\widetilde{k}_3$ will be chosen as $\widetilde{k}_1 = -k_0$, $\widetilde{k}_2 = -2k_0$, and $\widetilde{k}_3 = 3k_0$ for a suitably chosen $k_0 > 0$. Then we choose a dispersion relation which gives a finite speed of propagation, i.e., $\sup_{k\in\mathbb{R}} |\frac{d\omega}{dk}| \leq C$ and for which $\widetilde{\omega}_1 + \widetilde{\omega}_2 + \widetilde{\omega}_3 = 0$, with $\widetilde{\omega}_3 = \omega(k_3)$, $\widetilde{\omega}_2 = -\omega(k_2)$, and $\widetilde{\omega}_1 = -\omega(k_1)$. For instance

$$\omega^2 = 1 - \frac{k^2}{k^2+1} + 80\frac{(\frac{k}{5})^2}{(\frac{k}{5})^2+1} \tag{9}$$

is such a dispersion relation. This can be seen easily from the left panel of Figure 3. We choose $\omega(k)$ to be the positive root.

The resonance function

$$r(k) = \omega(3k) - \omega(k) - \omega(2k) \tag{10}$$

has two positive zeroes. See the right panel of Figure 3. In the following w.l.o.g. let k_0 be the larger of the two positive zeroes of r.

Therefore, as already stated above, we have $\omega_1 + \omega_2 + \omega_3 = 0$ for $\omega_3 = \omega(k_3)$, $\omega_2 = -\omega(k_2)$, and $\omega_1 = -\omega(k_1)$.

In order to construct a PDE having the dispersion relation (9), we use that (9) can be written as

$$\omega^2(k^2+1)((\frac{k}{5})^2+1) = (k^2+1)((\frac{k}{5})^2+1) - k^2((\frac{k}{5})^2+1) + 80(\frac{k}{5})^2(k^2+1).$$

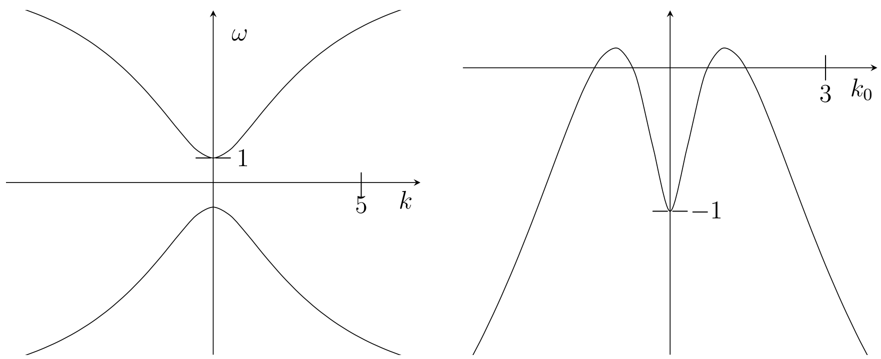

Figure 3: Left panel: The curves of eigenvalues $k \mapsto \pm\omega(k)$ for the linearized model equation (9) plotted as a function over the Fourier wave numbers. Right panel: Plot of the resonance function $r = r(k)$, cf. (10), which has two positive zeroes.

Such a polynomial relation comes from the linear PDE

$$
\begin{aligned}
-\partial_t^2(-\partial_x^2+1)(-\frac{1}{25}\partial_x^2+1)u &= (-\partial_x^2+1)(-\frac{1}{25}\partial_x^2+1)u \\
&\quad +\partial_x^2(-\frac{1}{25}\partial_x^2+1)u - 80\frac{1}{25}\partial_x^2(-\partial_x^2+1)u
\end{aligned}
$$

which can be rewritten as

$$P(\partial_x)\partial_t^2 u = Q(\partial_x)u.$$

with

$$P(\partial_x) = 1 - \frac{26}{25}\partial_x^2 + \frac{1}{25}\partial_x^4 \qquad \text{and} \qquad Q(\partial_x) = -1 + \frac{81}{25}\partial_x^2 - \frac{16}{5}\partial_x^4.$$

For our purposes we add some quadratic terms to this linear PDE. So, we finally choose

$$P(\partial_x)\partial_t^2 u = Q(\partial_x)u + P(\partial_x)u^2. \tag{11}$$

By this choice the system can be written as

$$\partial_t^2 u = \omega^2(-i\partial_x)u + u^2. \tag{12}$$

As a consequence also the nonlinear terms lead to a finite speed of propagation, since they have no derivatives in front. Moreover, this choice in the following allows a simple calculation of all nonlinear coefficients.

Remark 3.1. Since ω is a bounded operator in H^s, the Picard-Lindelöf theorem applies. Therefore, there is local existence and uniqueness of solutions. Moreover, $u \in C([0,T_0],H^s)$ for $s \geq 1$ for solutions u implies $u \in C^n([0,T_0],H^s)$ for all $n \in \mathbb{N}$.

3.2 Derivation of the NWI and TWI approximation

In this section we compute the resonant TWI approximation and the NWI approximation for our model equation (12). We start with the resonant TWI approximation and make the ansatz

$$u(x,t) \approx \varepsilon\psi_{\mathrm{TWI}}(\varepsilon,x,t) = \sum_{j=1}^{3} \varepsilon B_j(\varepsilon t)\, e^{i\left(\widetilde{k}_j x + \widetilde{\omega}_j t\right)} + c.c., \tag{13}$$

where $\widetilde{k}_1 + \widetilde{k}_2 + \widetilde{k}_3 = 0$, with $\widetilde{k}_3 > 0$, $\widetilde{k}_1 < 0$, and $\widetilde{k}_2 < 0$, and $\widetilde{\omega}_1 + \widetilde{\omega}_2 + \widetilde{\omega}_3 = 0$, with $\widetilde{\omega}_3 > 0$, $\widetilde{\omega}_1 < 0$, and $\widetilde{\omega}_2 < 0$. By equating the coefficient of $e^{i\left(\widetilde{k}_1 x + \widetilde{\omega}_1 t\right)}$ to zero, we find

$$-\varepsilon i\widetilde{\omega}_1^2 B_1 + \varepsilon^2 2i\widetilde{\omega}_1 \partial_T B_1 + \varepsilon^3 \partial_T^2 B_1 = -\varepsilon i\widetilde{\omega}_1^2 B_1 + 2\varepsilon^2 \overline{B_2}\,\overline{B_3},$$

and similar for B_2 and B_3. Hence, at ε^2 we find

$$\partial_T B_1 = i\gamma_1 \overline{B_2 B_3}, \qquad \partial_T B_2 = i\gamma_2 \overline{B_1 B_3}, \qquad \partial_T B_3 = i\gamma_3 \overline{B_1 B_2}, \tag{14}$$

where $\gamma_j = -\dfrac{1}{\widetilde{\omega}_j}$. Therefore, we have $\gamma_3 < 0$, $\gamma_1 > 0$, and $\gamma_2 > 0$.

As a consequence the B_3-subspace of the resonant TWI system (14) is unstable and so we choose $k_* = \widetilde{k}_3 = 3k_0$ as basic wave number for the derivation of the NWI approximation with $N = 1$. We make the ansatz

$$\begin{aligned} u(x,t) \approx \psi_{NWI}(\varepsilon,x,t) \;=\;& \left(\varepsilon A_1\left(\varepsilon^2 x, \varepsilon^2 t\right) e^{i(k_* x + \omega_3 t)} + c.c.\right) + \varepsilon^2 A_{0,0}\left(\varepsilon^2 x, \varepsilon^2 t\right) \\ & + \left(\varepsilon^2 A_{2,0}\left(\varepsilon^2 x, \varepsilon^2 t\right) e^{2i(k_* x + \omega_3 t)} + c.c.\right). \end{aligned}$$

With $\omega_j^2 = \omega^2(jk_0)$ and $\omega_j > 0$, this gives by equating the coefficient of $\varepsilon^3 e^{i\left(\widetilde{k}_* x + \omega_3 t\right)}$, of ε^2, and of $\varepsilon^2 e^{2i\left(\widetilde{k}_* x + \omega_3 t\right)}$ to zero, that

$$\begin{aligned} 2i\omega_3 \partial_\tau A_1 &= \nu \partial_X A_1 + 2A_1 A_{0,0} + 2A_{2,0} A_{-1}, \\ 0 &= -\omega_0^2 A_{0,0} + 2A_1 A_{-1}, \\ -4\omega_3^2 A_{2,0} &= -\omega_6^2 A_{2,0} + A_1^2. \end{aligned}$$

Eliminating $A_{0,0}$ and $A_{2,0}$ in the first equation via the second and third equation yields the NWI system

$$2i\omega_3\partial_T A_1 = \nu\partial_X A_1 + \gamma A_1 \left|A_1\right|^2, \tag{15}$$

with

$$\gamma = \frac{2}{\omega_0^2} + \frac{1}{\omega_6^2 - 4\omega_3^2} \qquad \text{and} \qquad \frac{\nu}{2i\omega_3} = \frac{d\omega}{dk}\Big|_{k=k_*}.$$

In case of $\frac{2\pi}{k_0}$-spatially periodic boundary conditions on (11), the NWI system (15) degenerates into the ODE

$$2i\omega_3\partial_T A_1 = \gamma A_1 \left|A_1\right|^2, \tag{16}$$

and the approximation is given by

$$\begin{aligned} u\left(x,t\right) \approx \psi_{NWI}^{\mathrm{per}}(\varepsilon,x,t) &= (\varepsilon A_1\left(\varepsilon^2 t\right) e^{i(k_* x+\omega_3 t)} + c.c.) + \varepsilon^2 A_{0,0}\left(\varepsilon^2 t\right) \\ &\quad +(\varepsilon^2 A_{2,0}\left(\varepsilon^2 t\right) e^{2i(k_* x+\omega_3 t)} + c.c.). \end{aligned}$$

3.3 The non-approximation results

It is the purpose of this section to state our non-approximation results, i.e., to give a precise statement for the fact that the NWI approximation fails to predict the dynamics of our model problem (12) for small values of the perturbation parameter $0 < \varepsilon \ll 1$ on the natural time scale $\mathcal{O}(1/\varepsilon^2)$ of the NWI approximation. We start with the non-approximation result in case of $2\pi/k_0$-periodic boundary conditions.

Theorem 3.2. *Assume the situation decrsibed in Section 3.2 and consider* (12) *with periodic boundary conditions $u_{per}(x,t) = u_{per}(x + 2\pi/k_0, t)$ for all $x \in \mathbb{R}$. Let $A \in C([0,T_0],\mathbb{C})$ be a solution of the NWI equation* (16)*. Then there exist $\varepsilon_0 > 0$, $C_1 > 0$, and $C_2 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ there is an open set of initial conditions in H^1_{per} for* (12) *with*

$$\|u_{per}(\cdot,0) - \varepsilon\psi_{\mathrm{NWI}}^{per}(\varepsilon,\cdot,0)\|_{H^1_{per}} + \|\partial_t u_{per}(\cdot,0) - \varepsilon\partial_t\psi_{\mathrm{NWI}}^{per}(\varepsilon,\cdot,0)\|_{H^1_{per}} \leq C_1\varepsilon^2,$$

for which the associated solutions satisfy

$$\sup_{t\in[0,1/\varepsilon^{3/2}]}\sup_{x\in\mathbb{R}}|u_{per}(x,t) - \varepsilon\psi_{\mathrm{NWI}}^{per}(\varepsilon,x,t)| \geq C_2\varepsilon.$$

This means that the error made by the NWI approximation is of the same order as the solution $u(\cdot,t)$ and the NWI approximation $\varepsilon\psi_{\mathrm{NWI}}^{\mathrm{per}}(\cdot,t)$ far before the end of the natural approximation time of the NWI approximation, although the initial condition $u_{\mathrm{per}}(\cdot,0)$ of the original system and the initial NWI approximation $\varepsilon\psi_{\mathrm{NWI}}^{\mathrm{per}}(\cdot,0)$ are close together. Therefore, the dynamics of the NWI system in general can not be used to predict the dynamics of the original system (11) in case of $2\pi/k_0$-periodic boundary conditions.

The theorem can be improved in various directions. First of all the initial condition $u_{\mathrm{per}}(\cdot,0)$ can be chosen much closer to an higher order NWI approximation, cf. Section A. Every polynomial order $\mathcal{O}(\varepsilon^n)$ with $n\in\mathbb{N}$ w.r.t. the small perturbation parameter $0<\varepsilon\ll 1$ would be fine. Secondly, from the proof given in Section 4 it will be clear that failure of the approximation is not the exception, but the rule, for our set-up.

In Section 5 we use Theorem 3.2 and the finite speed of propagation of the original system (11) to give a rigorous proof that the NWI approximation fails in the description of the original system (11) also in case without periodic boundary conditions on (11). In detail we prove:

Theorem 3.3. *Assume the situation decrsibed in Section 3.2 and consider* (12) *with $x\in\mathbb{R}$. There is a solution $A\in C([0,T_0],H^s)$ with $s\geq 2$ arbitrary, but fixed, of the NWI equation* (15) *for which there exist $\varepsilon_0>0$, $C_1>0$, and $C_2>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$ there are solutions of* (12) *satisfying initially*

$$\|u(\cdot,0)-\varepsilon\psi_{\mathrm{NWI}}(\varepsilon,\cdot,0)\|_{C_b^1}+\|\partial_t u(\cdot,0)-\varepsilon\partial_t\psi_{\mathrm{NWI}}(\varepsilon,\cdot,0)\|_{C_b^1}\leq C_1\varepsilon^2,$$

for which

$$\sup_{t\in[0,1/\varepsilon^{3/2}]}\sup_{x\in\mathbb{R}}|u(x,t)-\varepsilon\psi_{\mathrm{NWI}}(\varepsilon,x,t)|\geq C_2\varepsilon.$$

The assertion of Theorem 3.3 is much weaker than the assertion of Theorem 3.2. This is due to the construction of the solution A of (15), cf. Section 5. However, in the class of solutions constructed in Section 5, the previous remarks apply for the problem on the real line, too.

4 A new proof in case of periodic b.c.

Unstable quadratic resonances in dispersive wave systems have been used before to prove that NLS and NWI approximations can fail in case of $2\pi/k_0$-periodic boundary conditions assuming that the three resonant wave numbers

$\widetilde{k}_1$, $\widetilde{k}_2$, and $\widetilde{k}_3$ are integer multiples of k_0. They have been used in [SSZ15] for proving the failure of the NLS approximation for the water wave problem with suitably chosen small surface tension and periodic boundary conditions. The proof given in [SSZ15] is unsatisfactory in the sense that an approximation theorem beyond the natural time scale for an extended TWI system has to be established and the qualitative behavior of this high-dimensional amplitude system has to be discussed.

The first purpose of this paper is thus to develop an alternative strategy which is based on the instability proof of spectrally unstable fixed points and which avoids the previous unsatisfactory steps. Therefore, the rest of this section contains the proof of Theorem 3.2.

4.1 The functional analytic set-up

We consider $2\pi/k_0$-spatially periodic solutions of (12) which is expanded in Fourier modes

$$u_{\text{per}}(x,t) = \sum_{j\in\mathbb{Z}} u_j(t) e^{jk_0 x}.$$

Inserting this Fourier expansion into our original system (12) yields for the Fourier coefficients

$$\partial_t^2 u_j(t) = -\omega_j^2 u_j(t) + \sum_{m\in\mathbb{Z}} u_{j-m}(t) u_m(t). \tag{17}$$

It is well known that the Fourier transform $u \mapsto \widehat{u}$ with $\widehat{u} = (u_j)_{j\in\mathbb{Z}}$ is an isomorphism between H^1_{per} and

$$\ell^2_1 = \{\widehat{u} : \mathbb{Z} \to \mathbb{C} : \|\widehat{u}\|^2_{\ell^2_1} = \sum_{j\in\mathbb{Z}} |u_j|^2 (1+j^2) < \infty\}.$$

The norm in $(\ell^2_1)^m$ is denoted with $\|\cdot\|_{\ell^2_1}$, too. Since H^1_{per} is closed under multiplication, the space ℓ^2_1 is closed under convolution.

System (17) is written as first order system by introducing $\widetilde{u}_j$ through $\partial_t u_j = i\omega_j \widetilde{u}_j$. After diagonalization

$$v_{j,+} = \frac{1}{\sqrt{2}}(u_j + \widetilde{u}_j), \qquad v_{j,-} = \frac{1}{\sqrt{2}}(u_j - \widetilde{u}_j)$$

we obtain

$$\begin{aligned}\partial_t v_{j,\pm}(t) &= i\omega_{j,\pm} v_{j,\pm}(t) \\ &\quad + \rho_{j,\pm} \sum_{m\in\mathbb{Z}} (v_{j-m,+}(t) + v_{j-m,-}(t))(v_{m,+}(t) + v_{m,-}(t)),\end{aligned} \tag{18}$$

where $\rho_{j,\pm} = \mp\frac{1}{2\sqrt{2}i\omega_j}$ and $\omega_{j,\pm} = \pm\omega_j$. We introduce $\widehat{v} = (v_{j,+}, v_{j,-})_{j\in\mathbb{Z}} \in (\ell_1^2)^2$. Since we have $\sup_{j\in\mathbb{Z}} |\omega_j| < \infty$, the right hand side of (18) is locally Lipschitz-continuous in $(\ell_1^2)^2$, and so there is the local existence and uniqueness of solutions with the Picard-Lindelöf theorem in $(\ell_1^2)^2$, cf. Remark 3.1.

4.2 The normal form transformation

By near identity transformations, such as

$$w_{j,s_j} = v_{j,s_j} + \sum_{j_1,j_2\in\mathbb{Z}, s_{j_1}, s_{j_2}\in\{+,-\}} \alpha_{jj_1j_2}^{s_j s_{j_1} s_{j_2}} v_{j_1,s_{j_1}} v_{j_2,s_{j_2}} \tag{19}$$

with

$$\alpha_{jj_1j_2}^{s_j s_{j_1} s_{j_2}} = \frac{\delta_{j,j_1+j_2}\rho_{j,s_j}}{\omega_{j,s_j} - \omega_{j_1,s_{j_1}} - \omega_{j_2,s_{j_2}}} \in \mathbb{C} \tag{20}$$

all terms $v_{j_1,s_{j_1}} v_{j_2,s_{j_2}}$ can be eliminated if the non-resonance condition $\omega_{j,s_j} \neq \omega_{j_1,s_{j_1}} + \omega_{j_2,s_{j_2}}$ is satisfied. Due to the resonances constructed in Section 3.1 we set

$$\alpha_{3,-1,-2}^{+--} = \alpha_{-1,3,-2}^{-+-} = \alpha_{-2,3,-1}^{-+-} = \alpha_{3,-2,-1}^{+--} = \alpha_{-1,-2,3}^{--+} = \alpha_{-2,-1,3}^{--+} = 0$$

and

$$\alpha_{-3,1,2}^{-++} = \alpha_{1,-3,2}^{+-+} = \alpha_{2,-3,1}^{+-+} = \alpha_{-3,2,1}^{-++} = \alpha_{1,2,-3}^{++-} = \alpha_{2,1,-3}^{++-} = 0,$$

but choose (20) otherwise.

Lemma 4.1. *There exists an $r > 0$ such that for all $v \in \ell_1^2$ with $\|v\|_{\ell_1^2} < r$ the near identity change of variables* (19) *is a smooth invertible mapping $v \mapsto w$ with $\|w - v\|_{\ell_1^2} \leq C\|v\|_{\ell_1^2}^2$.*

Proof. Young's convolution inequality implies

$$\begin{aligned}&\Big(\sum_{j\in\mathbb{Z}, s_j\in\{+,-\}} \Big(\sum_{j_1,j_2\in\mathbb{Z}, s_{j_1}, s_{j_2}\in\{+,-\}} \alpha_{jj_1j_2}^{s_j s_{j_1} s_{j_2}} v_{j_1,s_{j_1}} v_{j_2,s_{j_2}}\Big)^2\Big)^{1/2} \\ \leq\ & 4 \sup_{j,j_1,j_2\in\mathbb{Z}, s_j, s_{j_1}, s_{j_2}\in\{+,-\}} |\alpha_{jj_1j_2}^{s_j s_{j_1} s_{j_2}}| \quad \|v\|_{\ell_1^2}^2\end{aligned}$$

where we used Sobolev's embedding $\ell_1^2 \subset \ell^1$. Since $v \mapsto w$ is a polynomial and $w \mapsto v$ a convergent power series in a small ball of ℓ_1^2 the smoothness is obvious. $\square$

After the near identity transformations the new system is given by

$$\begin{aligned} \partial_t w_{-1,-} &= -i\omega_1 w_{-1,-} + 2\rho_{-1,-}\overline{w_{3,+}w_{-2,-}} +, g_{-1,-} \\ \partial_t w_{-2,-} &= -i\omega_2 w_{-2,-} + 2\rho_{-2,-}\overline{w_{3,+}w_{-1,-}} + g_{-2,-}, \\ \partial_t w_{3,+} &= i\omega_3 w_{3,+} + 2\rho_{3,+}\overline{w_{-2,-}w_{-1,-}} + g_{3,+}, \end{aligned} \tag{21}$$

and similar for the index pairs $(1,+)$, $(2,+)$, and $(3,-)$, where we used $w_{j,+} = \overline{w_{-j,-}}$. For all other index pairs we have

$$\partial_t w_{j,s} = i\omega_{j,s} w_{j,s} + g_{j,s}.$$

The nonlinear terms $g_{j,s}$ satisfy

$$\|g\|_{\ell_1^2} \leq C\|w\|_{\ell_1^2}^3$$

where $g = (g_{j,s})_{j,s\in\mathbb{Z}\times\{+,-\}} : (\ell_1^2)^2 \to (\ell_1^2)^2$ is an analytic function for $\|v\|_{\ell_1^2} < r$ for $r > 0$ sufficiently small, cf. Lemma 4.1. The system for $\widehat{w} = (w_{j,+}, w_{j,-})_{j\in\mathbb{Z}}$ is abbreviated as

$$\partial_t \widehat{w} = \Lambda\widehat{w} + N(\widehat{w}). \tag{22}$$

Remark 4.2. For System (21) the TWI system can be derived by making the ansatz $w_{-1,-}(t) = \varepsilon B_1(\varepsilon t)e^{-i\omega_1 t}$, $w_{-2,-}(t) = \varepsilon B_2(\varepsilon t)e^{-i\omega_2 t}$, $w_{3,+}(t) = \varepsilon B_3(\varepsilon t)e^{i\omega_3 t}$, similar for the index pairs $(1,+)$, $(2,+)$, and $(3,-)$, and $w_{j,s}(t) = 0$ for all other index pairs. In lowest order the B_j satisfy the TWI system (14).

4.3 The NWI approximation

The NWI approximation to the unstable subspace of the TWI approximation can be derived by making the ansatz

$$w_{3,+}(t) = \varepsilon\psi_{3,+}(t)e^{i\omega_3 t} = \varepsilon A_1(\varepsilon^\beta t)e^{i\omega_3 t},$$

similar for $w_{-3,-}(t)$, and $w_{j,s}(t) = 0$ for all other index pairs. With $\tau = \varepsilon^\beta t$ we find

$$\partial_\tau A_1 = \mathcal{O}(\varepsilon^{2-\beta})A_1|A_1|^2.$$

Before we had chosen $\beta = 2$, but since the failure will happen on an $\mathcal{O}(|\ln(\varepsilon)|/\varepsilon)$-time scale for our purposes it is sufficient to choose a $\beta \in (1, 2)$, for instance $\beta = 3/2$, what we will do in the following. Thus, on the $\mathcal{O}(1/\varepsilon^{3/2})$-time scale, the NWI approximation can be considered to be stationary, in detail we have

$$|\varepsilon\psi_{3,+}(t) - \varepsilon\psi_{3,+}(0)| \leq C\varepsilon^3 t. \tag{23}$$

For our subsequent estimates we need that the residual for the NWI approximation is small. The residual

$$\mathrm{Res}(\widehat{w}) = -\partial_t \widehat{w} + \Lambda \widehat{w} + N(\widehat{w}).$$

contains all terms which do not cancel after inserting the NWI approximation into the equation. In Section A we recall the construction of an approximation $\varepsilon\psi$ with $w_{\pm 3,\pm}(t) = \varepsilon\psi_{\pm 3,\pm}(t)$, but now only with $|w_j(t)| \leq \mathcal{O}(\varepsilon^3)$ instead of $w_j(t) = 0$ for $t \in [0, 1/\varepsilon^2]$ for all other index pairs and

$$\sup_{t \in [0, 1/\varepsilon^{3/2}]} \|\mathrm{Res}(\varepsilon\psi)\|_{\ell^2_1} \leq C\varepsilon^{n+2} \tag{24}$$

for a chosen fixed $n \in \mathbb{N}$. For the statement of the theorem $n = 2$ is sufficient. We remark that only finitely many other $w_{j,\pm}$ are non-zero and all of them are integer multiples of the index $j = 3$, cf. Section A.

4.4 Estimates for the unstable sector

The solution $\widehat{w}$ of (22) is a sum of the NWI approximation $\varepsilon\psi$ and an error $\widehat{R}$. We set

$$\begin{aligned} w_{-1,-}(t) &= R_{-1,-}(t) e^{-i\omega_1 t}, \\ w_{-2,-}(t) &= R_{-2,-}(t) e^{-i\omega_2 t}, \\ w_{3,+}(t) &= \varepsilon\psi_{3,+}(t) e^{i\omega_3 t} + R_{3,+}(t) e^{i\omega_3 t} \end{aligned}$$

and similar for the index pairs $(1,+)$, $(2,+)$, and $(3,-)$. For all other index pairs we set

$$w_{j,\pm}(t) = \varepsilon^2 \psi_{j,\pm}(t) e^{\pm i\omega_j t} + R_{j,\pm}(t) e^{\pm i\omega_j t},$$

with $\psi_{j,\pm} = \mathcal{O}(1)$ for $j \in 3\mathbb{Z}$ and $\psi_{j,\pm} = 0$ else. Using (23) the error satisfies

$$\begin{aligned} \partial_t R_{-1,-} &= 2\varepsilon\rho_{-1,-} \overline{\psi_{3,+}|_{t=0} R_{-2,-}} + h_{-1,-}, \\ \partial_t R_{-2,-} &= 2\varepsilon\rho_{-2,-} \overline{\psi_{3,+}|_{t=0} R_{-1,-}} + h_{-2,-}, \\ \partial_t R_{3,+} &= h_{3,+}, \end{aligned}$$

and similar for the index pairs $(1,+)$, $(2,+)$, and $(3,-)$. For all other index pairs we find

$$\partial_t R_{j,\pm} = h_{j,\pm}.$$

The nonlinear terms $h_{j,\pm}$ satisfy

$$\|h\|_{\ell^2_1} \leq C(\varepsilon^2 + \varepsilon^3 t)\|R\|_{\ell^2_1} + C\|R\|^2_{\ell^2_1} + C\|\mathrm{Res}\|_{\ell^2_1},$$

for $\|R\|_{\ell^2_1} \leq 1$, where $h = (h_{j,s})_{(j,s)\in\mathbb{Z}\times\{+,-\}} : (\ell^2_1)^2 \to (\ell^2_1)^2$ is an analytic function in a sufficiently small, but ε-independent, ball.

In order to compute the eigenvalues $\varepsilon\mu_\pm$ of the $R_{\pm1,\pm}$ and $R_{\pm2,\pm}$ part, we differentiate the first linearized equation and insert the second equation. We obtain

$$\partial_t^2 R_{-1,-} = 4\varepsilon^2 \rho_{-1,-}\rho_{-2,-}|\psi_{3,+}|_{t=0}|^2 R_{-1,-},$$

and so

$$\mu_\pm = \pm 2(\rho_{-1,-}\rho_{-2,-})^{1/2}|\psi_{3,+}|_{t=0}|.$$

We diagonalize the $R_{\pm1,\pm}$ and $R_{\pm2,\pm}$ part and obtain new equations

$$\begin{aligned}\partial_t R_u &= \mu\varepsilon R_u + h_u,\\ \partial_t R_s &= -\mu\varepsilon R_s + h_s,\end{aligned}$$

with h_u and h_s obeying the same properties as the h_j.

We introduce the quantities

$$E_u = |R_u|^2 \qquad \text{and} \qquad E_s = |R_s|^2 + \Sigma_{\mathrm{other}(j,\pm)}\,|R_{j,\pm}|^2\,(1+j^2).$$

For $E = E_u - E_s$ we find

$$\begin{aligned}\frac{d}{dt}E &= 2\mu\varepsilon E_u + 2\mu\varepsilon E_s - \Sigma_{\mathrm{other}(j,\pm)}(\overline{R_{j,\pm}}h_{j,\pm} + \overline{h_{j,\pm}}R_{j,\pm})(1+j^2)\\ &\quad +2\mathrm{Re}(\overline{R}_u h_u - \overline{R}_s h_s)\\ &\geq 2\mu\varepsilon E_u - |\Sigma_{\mathrm{other}(j,\pm)}(\overline{R_{j,\pm}}h_{j,\pm} + \overline{h_{j,\pm}}R_{j,\pm})(1+j^2)|\\ &\quad -2|\overline{R}_u h_u| - 2|\overline{R}_s h_s|\\ &\geq 2\mu\varepsilon E_u - 2\|R\|_{\ell^2_1}(C(\varepsilon^2+\varepsilon^3 t)\|R\|_{\ell^2_1} + C\|R\|^2_{\ell^2_1} + C\|\mathrm{Res}\|_{\ell^2_1})\\ &\geq 2\mu\varepsilon E_u - C_1(\varepsilon^2+\varepsilon^3 t)E_u - C_1(\varepsilon^2+\varepsilon^3 t)E_s - C_2E_u^{3/2} - C_2E_s^{3/2} - C_3\|\mathrm{Res}\|_{\ell^2_1}\\ &\geq \mu\varepsilon E_u - \mu\varepsilon E_s - C_3\|\mathrm{Res}\|_{\ell^2_1}\\ &\geq \mu\varepsilon E - C_3\|\mathrm{Res}\|_{\ell^2_1} \geq \frac{1}{2}\mu\varepsilon E,\end{aligned}$$

with positive constants C_1, C_2, and C_3, under the assumptions

$$C_1(\varepsilon + \varepsilon^2 t) \leq \mu/2, \tag{25}$$
$$C_2 E_u^{1/2} \leq \mu\varepsilon/2, \tag{26}$$
$$C_2 E_s^{1/2} \leq \mu\varepsilon/2, \tag{27}$$
$$C_3 \|\mathrm{Res}\|_{\ell_1^2} \leq \mu\varepsilon E/2. \tag{28}$$

Define

$$t_* = \inf\{t : E(t) \geq \mu\varepsilon/2\}.$$

We are done if we prove $t_* \leq 1/\varepsilon^{3/2}$.

On the time interval $[0, 1/\varepsilon^{3/2}]$ the assumption (25) can be satisfied by choosing $\varepsilon_0 > 0$ so small that

$$C_1(\varepsilon_0 + \varepsilon_0^{1/2}) \leq \mu/2.$$

In order to satisfy (28) we recall that in (24) we assumed a NWI approximation $\varepsilon\psi$ with

$$\|\mathrm{Res}\|_{\ell_1^2} = \mathcal{O}(\varepsilon^{n+2}),$$

cf. Section A for details about the construction. Assumption (28) will follow from

$$C_3 \|\mathrm{Res}\|_{\ell_1^2} \leq \mu\varepsilon E(0)/2. \tag{29}$$

Assumption (29) holds for $\varepsilon_0 > 0$ sufficiently small since $E_u(0) = \mathcal{O}(\varepsilon^n)$.

If the assumptions (26) and (27) are not satisfied for a $t \in [0, 1/\varepsilon^{3/2}]$ we are done. Hence, we assume in the following that (26) and (27) are satisfied.

Due to continuity assumption (29) implies that (28) holds also for all $t > 0$ in a neighborhood of $t = 0$. Therefore, the assumptions (25)-(28) are satisfied in this neighborhood and there we we find

$$E(t) \geq E(0)e^{\mu\varepsilon t/2}.$$

With the same argument assumption (28) holds for all $t \in [0, t_*]$ and so $E(t) \geq E(0)e^{\mu\varepsilon t/2}$ for all $t \in [0, t_*]$. We remark that by construction and continuity of $E(t)$ we then have $E_s(t) \leq E_u(t)$. From $E_u(0) = \mathcal{O}(\varepsilon^n)$ it follows $E(t) = \mathcal{O}(\varepsilon) = e^{\mu\varepsilon t}\mathcal{O}(\varepsilon^n)$ for $t = \mathcal{O}((n-1)|\ln(\varepsilon)|/\varepsilon) \ll 1/\varepsilon^{3/2}$ contradicting the assumptions (26) and (27). Therefore, $t^* \leq 1/\varepsilon^{3/2}$.

Therefore, we are done. □

5 Failure without imposing periodic b.c.

In this section we give the proof of Theorem 3.3. In order to prove that the NWI approximation $\varepsilon\psi_{NWI}$ also fails for (11) without imposing periodic boundary conditions, we use the failure of the NWI approximation for (11) with $2\pi/k_0$-spatially periodic boundary conditions, such as stated in Theorem 3.2.

Model (11) possesses a finite speed of propagation whose absolute value is bounded by c_g, i.e., the value of $u(x,t)$ only depends on $u(y,t)$ with $|y-x| \leq c_g(t_2-t_1)$ if $t_2 \geq t_1$. With the help of $u_{\text{per}}(x,t)$ and $\varepsilon\psi_{NWI}^{\text{per}}(x,t)$ we construct a solution $u(x,t)$, respectively initial condition $u(x,0)$ for (11) for which the NWI approximation ψ_{NWI} fails to make correct predictions about the dynamics without imposing periodic boundary conditions on (11), i.e. we can choose

$$\lim_{|x|\to\infty} u(x,t) = 0 \qquad \text{and} \qquad \lim_{|x|\to\infty} \varepsilon\psi(x,t) = 0.$$

We write the solution $u_{per}(x,t)$ from Theorem 3.2 as

$$u_{\text{per}}(x,t) = u_{3,+}(t)e^{i(k_3x+\omega_3t)} + u_{-1,-}(t)e^{-(k_1x+i\omega_1t)} + u_{-2,-}(t)e^{-(k_2x+i\omega_2t)} + c.c. + u_{rest}(x,t),$$

with $\sup_{t\in[0,1/\varepsilon^{3/2}]}\sup_{x\in\mathbb{R}}|u_{rest}(x,t)| \leq C\varepsilon^2$. Based on this representation we define

$$\widetilde{u}_{j,s}(\xi,0) = \begin{cases} u_{j,s}(0), & \text{for } |\xi| \leq 3, \\ 0, & \text{for } |\xi| \geq 5, \\ r_j(\xi), & \xi \text{ else}, \end{cases}$$

with $|r_j(\xi)| \leq |u_{j,s}(0)|$ such that $\widetilde{u}_{j,s} \in C_0^\infty$ for $(j,s) \in \{(\pm 3,\pm),(\pm 2,\pm),(\pm 1,\pm)\}$. Then, we define

$$\varepsilon\psi_{\text{NWI}}(x,0) = \widetilde{u}_{3,+}(\varepsilon^2 x,0)e^{ik_3x} + c.c.$$

and set

$$u(x,0) = \widetilde{u}_{3,+}(\varepsilon^2 x,0)e^{ik_3x} + \widetilde{u}_{-1,-}(\varepsilon^2 x,0)e^{-ik_1x} + \widetilde{u}_{-2,-}(\varepsilon^2 x,0)e^{-ik_2x} + c.c.$$

On the one hand we have

$$\begin{aligned} & \sup_{x\in\mathbb{R}}|u(x,0) - \varepsilon\psi_{NWI}(x,0)| \\ \leq\ & \sup_{|x|\leq 3/\varepsilon^2}|u_{\text{per}}(x,0) - \varepsilon\psi_{NWI}^{\text{per}}(x,0)| \\ & + \sup_{3/\varepsilon^2\leq|x|\leq 5/\varepsilon^2}\Big|\sum_{|j|\in\{1,2\}} r_j(\varepsilon^2 x,0)e^{ik_jx}\Big| \leq 3\varepsilon^2 \end{aligned}$$

and similar for the time derivatives. But on the other hand we have

$$\sup_{t\in[0,1/\varepsilon^{3/2}]}\sup_{x\in\mathbb{R}}|u(x,t)-\varepsilon\psi_{NWI}(x,t)| \geq \sup_{t\in[0,1/\varepsilon^{3/2}]}\sup_{|x|\leq 1/\varepsilon^2}|u_{\text{per}}(x,t)-\varepsilon\psi^{\text{per}}_{NWI}(x,t)| \geq C\varepsilon,$$

since $u(x,t)$ for $|x|\leq 1/\varepsilon^2$ is only affected by $u(y,0)$ for

$$|y-x| < c_g t \leq c_g/\varepsilon^{3/2} \ll 1/\varepsilon^2.$$

See the left panel of Figure 4.

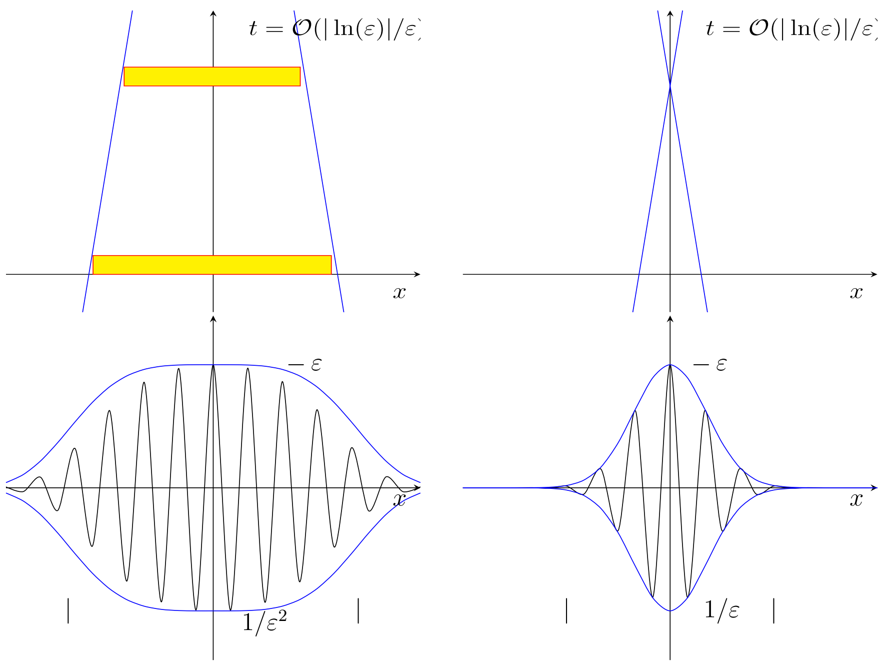

Figure 4: Left panel: For the NWI spatial scaling of order $\mathcal{O}(1/\varepsilon^2)$ the transport of velocity $\mathcal{O}(1)$ is too slow to hinder the resonances to grow. If the solution u_{per} and u coincide in the yellow rectangle fot $t=0$, they still coincide for $t=\mathcal{O}(|\ln(\varepsilon)|/\varepsilon)$. Right panel: For the NLS spatial scaling of order $\mathcal{O}(1/\varepsilon)$ the transport of velocity $\mathcal{O}(1)$ is sufficiently fast to hinder the resonances to grow.

6 Discussion

We close this paper with a number of remarks about possible extensions of the previous constructions.

Remark 6.1. As already said, there are essentially two consistent descriptions of oscillating wave packets for dispersive wave systems by amplitude equations, namely the NLS description and the NWI description. The previous construction in Section 5 is not possible for the NLS approximation. The failure in the periodic situation happens at a time of order $\mathcal{O}(|\ln \varepsilon|/\varepsilon)$. Giving up the periodicity the spatial domain of the NLS approximation is $\mathcal{O}(1/\varepsilon)$ and thus smaller than the spatial domain of size $\mathcal{O}(|\ln \varepsilon|/\varepsilon)$ which is influenced by the neighboring points on the $\mathcal{O}(|\ln \varepsilon|/\varepsilon)$ time scale. See the right panel of Figure 4. In fact we expect that due to the different group velocities at the resonant wave numbers the NLS approximation is valid for spatially localized solutions, even in case of k_1 in (1) belonging to an unstable resonance. See [Sch05, §4] or [MN13] for more explanations and [DHSZ16] for NLS approximation properties in case of unstable resonances and analytic initial conditions.

Remark 6.2. For original systems with an unbounded speed of propagation or resonant wave numbers which are not integer multiples of a basic wave number k_0, the main difficulty in giving a proof of failure for the NWI approximation are bounds on the derivatives of the solutions of an extended TWI system on the $\mathcal{O}(|\ln(\varepsilon)|/\varepsilon)$-time scale. These bounds are necessary for making the residual small. So far we could only establish these for the lowest order part of the approximation by an explicit representation of the solutions. In [Sch05] the NLS approximation has been justified in case that the wave number k_1 of the underlying oscillatory wave packet in (1) belongs to a stable resonance.

Remark 6.3. Although (11) is a rather artificial scalar PDE problem we strongly expect that in principle the same construction will be possible for the water wave problem without surface tension over a suitably chosen periodic bottom. The water wave problem without surface tension has a finite speed of propagation for solutions of order $\mathcal{O}(\varepsilon)$, and we expect that the periodic bottom allows to arrange the resonances in the way necessary for the previous construction. This will be subject of future research.

A Higher order NWI approximations

In this section we explain how to construct higher order NWI respectively NLS approximations for (11) in case $N = 1$, basic wave number $k_* = \widetilde{k}_3$, and $2\pi/k_*$-periodic boundary conditions, cf. (7). In order to achieve for the residual terms

$$\mathrm{Res}(\varepsilon\psi_\beta) = \mathcal{O}(\varepsilon^\beta),$$

where $\mathrm{Res}(u) = -\partial_t^2 u + \omega^2(\partial_x)u + u^2$, we choose

$$\varepsilon\psi_\beta = \sum_{|m|=1,2,\ldots,2N+1} \sum_{n=1}^{\tilde\beta(m)} \varepsilon^{\alpha(m)+n} A_{mn}(T)E^m,$$

with N and $\tilde\beta(m)$ sufficiently big, where $E = e^{i(k_*x+\omega_3 t)}$, $\alpha(m) = ||m|-1|$, and $T = \varepsilon^2 t$. As before A_{11} satisfies the NWI equation (16), the A_{1n} for $n \geq 2$ linearized NWI equations, and the A_{mn} for $m \neq \pm 1$ algebraic equations which can be solved w.r.t. A_{mn}, since $m\omega(k_*) \neq \omega(mk_*)$ for all $m \in \mathbb{N} \setminus \{-1, 1\}$. The estimates for the residual are trivial and follow line for line the estimates of the non-periodic situation, cf. [SU17, §11], where in the spatially periodic case all expansions of curves of eigenvalues and kernels of nonlinear terms can be skipped.

Remark A.1. We remark explicitly that the derivation of the NWI system and the estimate for the residual on the natural NWI $\mathcal{O}(1/\varepsilon^2)$-time scale are trivial. In contrast, in the approach used in [SSZ15] an extended TWI approximation system has to be constructed whose validity has to be shown beyond the natural $\mathcal{O}(1/\varepsilon)$ TWI time scale. This turned out to be highly non-trivial.

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