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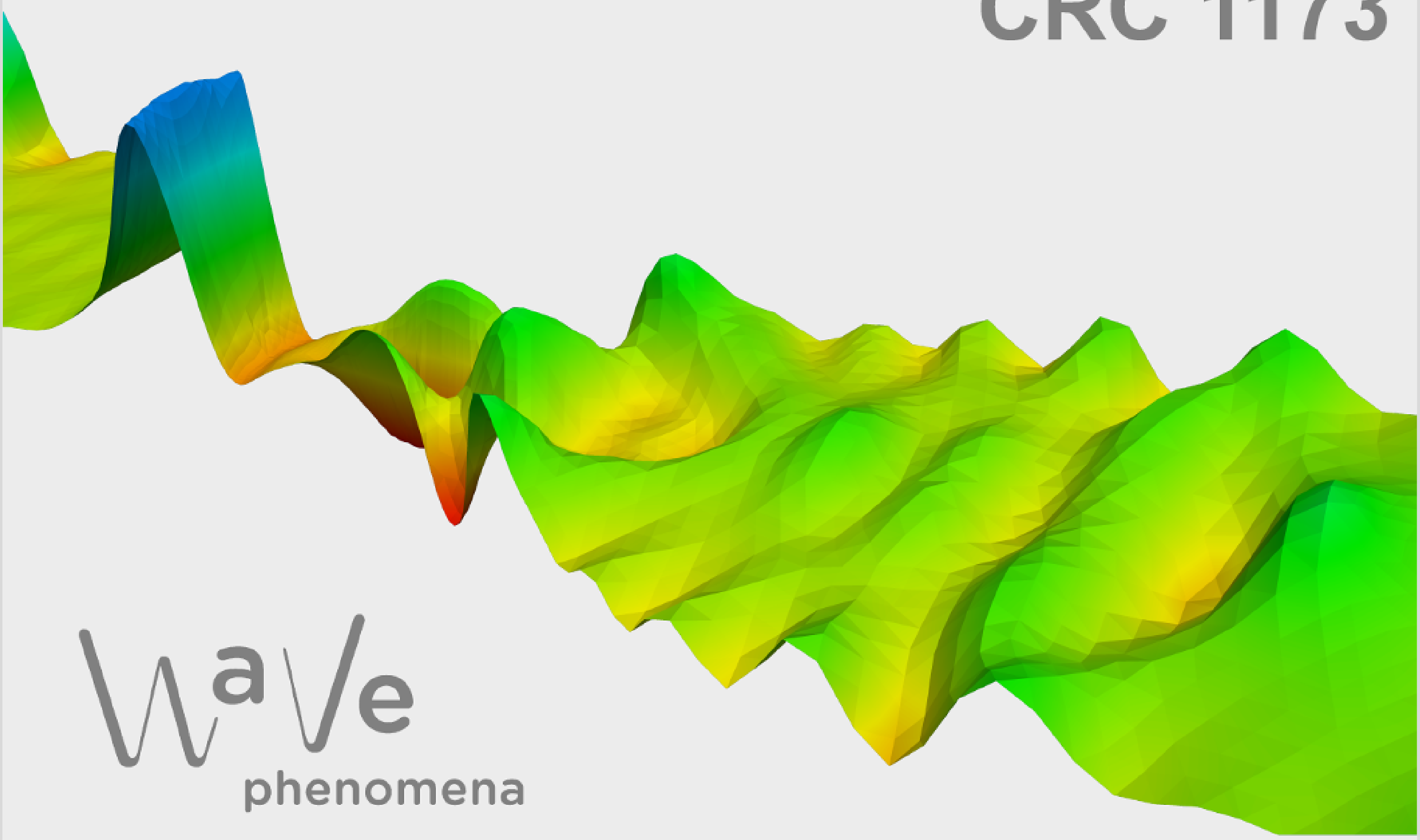

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The KdV approximation for a system with unstable resonances

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Abstract

The KdV equation can be derived via multiple scaling analysis for the approximate description of long waves in dispersive systems with a conservation law. In this paper we justify this approximation for a system with unstable resonances by proving estimates between the KdV approximation and true solutions of the original system. We expect that the approach will allow to handle more complicated systems without a detailed discussion of the resonances and without finding a suitable energy.

1 Introduction

We consider the Boussinesq-Klein-Gordon (BKG) system

$$\begin{aligned} \partial_t^2 u &= \alpha^2 \partial_x^2 u + \partial_t^2 \partial_x^2 u + \partial_x^2 (a_{uu} u^2 + 2 a_{uv} uv + a_{vv} v^2), & (1) \\ \partial_t^2 v &= \partial_x^2 v - v + b_{uu} u^2 + 2 b_{uv} uv + b_{vv} v^2, & (2) \end{aligned}$$

where $u = u(x,t), v = v(x,t), x, t \in \mathbb{R}$, and coefficients $\alpha > 0$, $a_{uu}, \ldots, b_{vv} \in \mathbb{R}$. Inserting the ansatz

$$\varepsilon^2 \psi_u^{\mathrm{KdV}}(x,t) = \varepsilon^2 A(\varepsilon(x - \alpha t), \varepsilon^3 t) \qquad \text{and} \qquad \varepsilon^2 \psi_v^{\mathrm{KdV}} = 0, \tag{3}$$

with small perturbation parameter $0 < \varepsilon^2 \ll 1$, into (1)-(2) yields the KdV equation

$$\partial_T A = \nu_1 \partial_X^3 A + \nu_2 \partial_X (A^2), \tag{4}$$

with coefficients ν_1, $\nu_2 \in \mathbb{R}$. The amplitude $A(X,T) \in \mathbb{R}$ depends on the long temporal variable $T = \varepsilon^3 t$ and on the long spatial variable $X = \varepsilon(x - \alpha t)$.

We are interested in the validity of the KdV approximation for the BKG system in case of unstable resonances, i.e., in case $\alpha > 2$, cf. Remark 1.6. We prove

Theorem 1.1. *Let A be a solution of the KdV equation* (4) *with*

$$\sup_{T \in [0,T_0]} \int_{\mathbb{R}} |\widehat{A}(K,T)| e^{\mu_A |K|} dK < \infty \tag{5}$$

for an $\mu_A > 0$. Then there exist $\varepsilon_0 > 0$, $T_1 \in (0, T_0]$, and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions (u, v) of (1)-(2) *with*

$$\sup_{t \in [0,T_1/\varepsilon^3]} \sup_{x \in \mathbb{R}} |(u,v)(x,t) - (\varepsilon^2 \psi_u^{\mathrm{KdV}}(x,t), 0)| \leq C\varepsilon^{7/2}.$$

Remark 1.2. Such an approximation result is nontrivial since solutions of order $\mathcal{O}(\varepsilon^2)$ have to be controlled on an $\mathcal{O}(1/\varepsilon^3)$ time scale.

Remark 1.3. The linearized problem is solved by

$$u(x,t) = e^{ikx \pm i\omega_u(k)t}, \qquad v(x,t) = e^{ikx \pm i\omega_v(k)t},$$

with

$$\omega_u(k) = \frac{\alpha k}{\sqrt{1+k^2}}, \qquad \omega_v(k) = \sqrt{1+k^2}.$$

We have $\omega_u(k) = \alpha k - \frac{1}{2}\alpha k^3 + \mathcal{O}(k^5)$. In Fourier space the KdV equation describes the modes in the u-equation which are strongly concentrated around the wave number $k = 0$, cf. Figure 1. We have $\nu_1 = -\frac{1}{2}\alpha$ in (4).

Remark 1.4. Historically, the KdV equation has been derived for the so called water wave problem first. Approximation results have been established in a number of papers. They are either based on energy estimates, cf. [Cra85, SW00, SW02, Due12], or on the use of analytic functions, cf. [KN86, Sch96].

Remark 1.5. Although the BKG system looks less complicated than the water wave problem, for the KdV approximation some features occur which are not present for the water wave problem over a flat bottom, namely the occurrence of quadratic resonances.

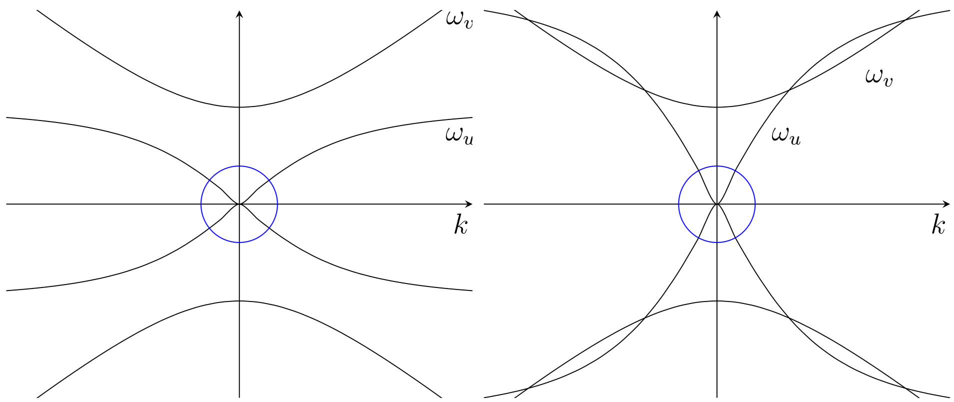

Figure 1: The curves of eigenvalues $\pm\omega_u$, $\pm\omega_v$ for the linearized BKG system plotted as a function over the Fourier wave numbers in case $\alpha^2 = 1$ (left) and $\alpha^2 = 5$ (right). The modes in the blue circles are described by the KdV approximation.

Remark 1.6. For $\alpha > 2$ the curves ω_u and ω_v intersect at two wave numbers k_1 and k_2, cf. the right panel of Figure 1. In [BCS17] it has been explained that there are $\frac{2\pi}{k_1}$-spatially periodic solutions of the form

$$\begin{aligned} u &= \varepsilon^2 A(\varepsilon^2 t) + \varepsilon^n A_1(\varepsilon^2 t) e^{i\omega_u(k_1)t} e^{ik_1 x} + \varepsilon^n A_{-1}(\varepsilon^2 t) e^{i\omega_u(-k_1)t} e^{-ik_1 x} + h.o.t., \\ v &= \varepsilon^n B_1(\varepsilon^2 t) e^{i\omega_v(k_1)t} e^{ik_1 x} + \varepsilon^n B_{-1}(\varepsilon^2 t) e^{-i\omega_v(-k_1)t} e^{-ik_1 x} + h.o.t., \end{aligned}$$

which satisfy $\partial_\tau^2 A = 0$ and

$$\partial_\tau \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = M \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}, \quad \text{with} \quad M = \frac{1}{i\omega_u(k_1)} \begin{pmatrix} -a_{uu}k_1^2 A & -a_{uv}k_1^2 A \\ b_{uu}A & b_{uv}A \end{pmatrix},$$

where $\tau = \varepsilon^2 t$. By suitably choosing the coefficients a_{uu}, a_{uv}, b_{uu}, and b_{uv}, the matrix M has eigenvalues with non-vanishing real part. Hence growth rates $e^{\beta\tau} = e^{\beta\varepsilon^2 t} = e^{\beta T/\varepsilon}$ with a $\beta > 0$ occur. These allow to bring $\varepsilon^n A_1$ and $\varepsilon^n B_1$, which are initially of order $\mathcal{O}(\varepsilon^n)$, to an order $\mathcal{O}(\varepsilon^2)$ at a time $T = \mathcal{O}((n-2)\varepsilon|\ln(\varepsilon)|) \ll 1$. Therefore, we have that $v = \mathcal{O}(\varepsilon^2)$ far before the natural time scale of the KdV equation. Hence, in this situation the KdV approximation makes wrong predictions. It can only make correct predictions if initially $\varepsilon^n A_1$ and $\varepsilon^n B_1$ are chosen exponentially small w.r.t. ε, cf. Assumption (5) in Theorem 1.1. Without excluding the possibility of unstable resonances the restriction to analytic solutions and to $T_1 \in (0, T_0]$ cannot be avoided.

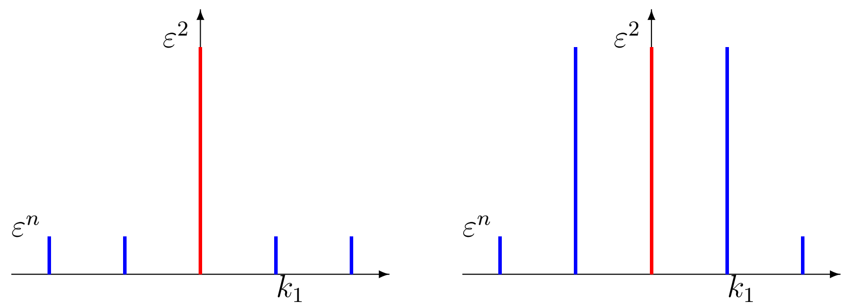

Figure 2: The mode distribution for $t = 0$ in the KdV case and the mode distribution for $t = \mathcal{O}(|\ln \varepsilon|/\varepsilon^2) \ll \mathcal{O}(1/\varepsilon^3)$. The KdV approximation is no longer valid in the right picture, since the modes at $\pm k_1$ are of the same order as the KdV modes at $k = 0$.

Remark 1.7. For the BKG system in [CS11] for $\alpha < 2$, i.e., in case of no additional quadratic resonances, i.e., in case $\omega_u(k) \neq \omega_v(k)$ for all $k \in \mathbb{R}$, a KdV approximation result has been established. It has been explained in [BCS17] how to establish an approximation result for all $\alpha \geq 2$ in case of stable quadratic resonances, i.e., in case that the eigenvalues of M are purely imaginary or have negative real part. Hence, it is the purpose of this paper to cover the case of unstable quadratic resonances, i.e, when M has at least one eigenvalue with positive real part.

Remark 1.8. The proof of Theorem 1.1 is based on a control of the solutions close to the wave number $k = 0$ by energy estimates and normal transformations. At the other wave numbers the solutions are solely controlled by working in spaces of analytic functions leading to some artificial damping. Functions which are analytic in a strip in the complex plane around the real axis of width $2\mu_A$ correspond in Fourier space to functions which decay as $e^{-\mu_A|K|}$ for $|K| \to \infty$, cf. Assumption (5) in Theorem 1.1. See [RS75].

Remark 1.9. The BKG system is a prototype model for a whole class of systems. Elements of this class are the poly-atomic FPU problem and the water wave problem over a periodic bottom. The transfer of the following analysis to these systems will be the subject of future research. The main strength of the approach of the present paper is, that it will allow to handle such more complicated systems without a detailed discussion of the resonances and without finding a suitable energy which will be different for every system. Except of very few exceptions [CS11, CCPS12, GMWZ14, BDS18] the

KdV approximation so far has only been justified for systems with a single pair of curves of eigenvalues $\pm i\omega_u$.

Notation. The Fourier transform of a function u is denoted by $\mathcal{F}u$ or $\widehat{u}$. Possibly different constants which can be chosen independently of the small perturbation parameter $0 < \varepsilon^2 \ll 1$ are denoted with the same symbol C.

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2 Derivation of the KdV approximation

Inserting the ansatz

$$\varepsilon^2 \psi_u^{\mathrm{KdV}}(x,t) = \varepsilon^2 A(\varepsilon(x-\alpha t), \varepsilon^3 t) \qquad \text{and} \qquad \varepsilon^2 \psi_v^{\mathrm{KdV}} = 0. \tag{6}$$

into the BKG system gives

$$\begin{aligned} \mathrm{Res}_u &= -\partial_t^2 u + \alpha^2 \partial_x^2 u + \partial_t^2 \partial_x^2 u + \partial_x^2 (a_{uu} u^2 + 2 a_{uv} uv + a_{vv} v^2) \\ &= \varepsilon^8 \partial_T^2 A, \\ \mathrm{Res}_v &= -\partial_t^2 v + \partial_x^2 v - v + b_{uu} u^2 + 2 b_{uv} uv + b_{vv} v^2 \\ &= \varepsilon^4 b_{uu} A^2, \end{aligned}$$

if we choose

$$-2\alpha \partial_T \partial_X A = \alpha^2 \partial_X^4 A + \partial_X^2 (a_{uu} A^2),$$

respectively

$$\partial_T A = -\frac{\alpha}{2} \partial_X^3 A - \frac{a_{uu}}{2\alpha} \partial_X (A^2). \tag{7}$$

The residuals Res_u and Res_v contain the terms which do not cancel after inserting the approximation into the BKG system. For our subsequent error estimates we need $\mathrm{Res}_u = \mathcal{O}(\varepsilon^8)$ and $\mathrm{Res}_v = \mathcal{O}(\varepsilon^8)$. In order to achieve this goal, we have to extend our approximation of v by higher order terms. Therefore, our final approximation is given by

$$\begin{aligned} \varepsilon^2 \psi_u(x,t) &= \varepsilon^2 A(\varepsilon(x-\alpha t), \varepsilon^3 t), \\ \varepsilon^4 \psi_v(x,t) &= \varepsilon^4 B_1(\varepsilon(x-\alpha t), \varepsilon^3 t) + \varepsilon^6 B_2(\varepsilon(x-\alpha t), \varepsilon^3 t). \end{aligned}$$

For this improved approximation we find

$$\begin{aligned}\mathrm{Res}_u &= \mathcal{O}(\varepsilon^8),\\ \mathrm{Res}_v &= \varepsilon^4(-B_1 + b_{uu}A^2) + \varepsilon^6(-B_2 + \partial_X^2 B_1 + 2b_{uv}AB_1) + \mathcal{O}(\varepsilon^8) = \mathcal{O}(\varepsilon^8),\end{aligned}$$

if we choose B_1 and B_2 to satisfy

$$-B_1 + b_{uu}A^2 = 0 \quad \text{and} \quad -B_2 + \partial_X^2 B_1 + 2b_{uv}AB_1 = 0.$$

Due to $(\int_{\mathbb{R}} |u(\varepsilon x)|^2 dx)^{1/2} = \varepsilon^{-1/2}(\int_{\mathbb{R}} |u(X)|^2 dX)^{1/2}$ we lose a factor $\varepsilon^{-1/2}$ when computing the magnitude of the residual w.r.t. powers of ε in L^2-based spaces. Therefore, we have

Lemma 2.1. *Fix $s_A - s \geq 8$, $s > 1/2$, and $T_0 > 0$. Let $A \in C([0,T_0], H^{s_A})$ be a solution of the KdV equation (7) and $\varepsilon^2\psi_u$ and $\varepsilon^4\psi_v$ be defined as above. For this approximation then there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ we have*

$$\sup_{T\in[0,T_0]} \|Res_u\|_{H^s} \leq C\varepsilon^{15/2} \qquad \text{and} \qquad \sup_{T\in[0,T_0]} \|Res_v\|_{H^s} \leq C\varepsilon^{15/2}.$$

Proof. Counting the powers of ε is straightforward. The term which loses most regularity is $\partial_T^2 B_2$ which can be expressed in terms of $\partial_T^2(AB_1)$ and $\partial_T^2\partial_X^2 B_1$. Since B_1 can be expressed in terms of A^2 it is sufficient to control $\partial_T^2\partial_X^2(A^2)$. We have that $\partial_T A$ can be expressed via the right-hand side of the KdV equation in terms of $A,\ldots,\partial_X^3 A$. Differentiating the KdV equation w.r.t. T shows that $\partial_T^2(A^2)$ can be expressed in terms of $A,\ldots,\partial_X^6 A$. Therefore, $\partial_T^2\partial_X^2 B$ can be expressed in terms of $A,\ldots,\partial_X^8 A$. □

Then writing the equations for the error, obtained from the BKG system (1)-(2), as a first order system, a term $\partial_x^{-1}\mathrm{Res}_u$ occurs.

Lemma 2.2. *Under the assumption of Lemma 2.1 we have the estimate*

$$\sup_{T\in[0,T_0]} \|\partial_x^{-1} Res_u\|_{H^{s+1}} \leq C\varepsilon^{13/2}.$$

Proof. The loss of ε^{-1} comes from $\partial_x^{-1} = \varepsilon^{-1}\partial_X^{-1}$. It is not obvious that $\partial_X^{-1}\mathrm{Res}_u$ is again in L^2. This is obvious for all terms which have a derivative ∂_X in front. There is only one term in the residual for which this is not obvious, namely $\partial_T^2 A$. We have

$$\partial_T^2 A = \partial_T(\partial_T A) = \partial_T(\nu_1\partial_X^3 A + \nu_2\partial_X(A^2)) = \partial_X\partial_T(\nu_1\partial_X^2 A + \nu_2(A^2)).$$

Therefore, we are done. □

3 The equations for the error

The error functions $(\varepsilon^{7/2}R_u, \varepsilon^{7/2}R_v)$, defined by

$$u = \varepsilon^2\psi_u + \varepsilon^{7/2}R_u, \qquad u = \varepsilon^2\psi_u + \varepsilon^{7/2}R_u,$$

satisfy

$$\begin{aligned} \partial_t^2 R_u &= \alpha^2\partial_x^2 R_u + \partial_t^2\partial_x^2 R_u + 2\varepsilon^2\partial_x^2(a_{uu}\psi_u R_u + a_{uv}\psi_u R_v) + \varepsilon^3\partial_x^2 f_u, \quad (8) \\ \partial_t^2 R_v &= \partial_x^2 R_v - R_v + 2\varepsilon^2(b_{uu}\psi_u R_u + b_{uv}\psi_u R_v) + \varepsilon^3 f_v, \quad (9) \end{aligned}$$

where

$$\begin{aligned} \varepsilon^3\partial_x^2 f_u &= \varepsilon^{7/2}\partial_x^2(a_{uu}R_u^2 + 2a_{uv}R_uR_v + a_{vv}R_v^2) \\ &\quad +2\varepsilon^4\partial_x^2(a_{uv}\psi_v R_u + a_{vv}\psi_v R_v) + \varepsilon^{-7/2}\mathrm{Res}_u, \\ \varepsilon^3 f_v &= \varepsilon^{7/2}(b_{uu}R_u^2 + 2b_{uv}R_uR_v + b_{vv}R_v^2) \\ &\quad +2\varepsilon^4(b_{uv}\psi_v R_u + b_{vv}\psi_v R_v) + \varepsilon^{-7/2}\mathrm{Res}_v. \end{aligned}$$

This system is written as first order system

$$\begin{aligned} \partial_t R_u &= i\omega_u \widetilde{R}_u, \\ \partial_t \widetilde{R}_u &= i\omega_u R_u + 2\varepsilon^2 i\omega_u(a_{uu}\psi_u R_u + a_{uv}\psi_u R_v) + \varepsilon^3 i\omega_u f_u, \\ \partial_t R_v &= i\omega_v \widetilde{R}_v, \\ \partial_t \widetilde{R}_v &= i\omega_v R_v + 2\varepsilon^2(i\omega_v)^{-1}\left(b_{uu}\psi_u R_u + b_{uv}\psi_u R_v\right) + \varepsilon^3(i\omega_v)^{-1} f_v. \end{aligned}$$

After diagonalization

$$R_{u,1} = \frac{1}{\sqrt{2}}(R_u + \widetilde{R}_u), \qquad R_{u,-1} = \frac{1}{\sqrt{2}}(R_u - \widetilde{R}_u)$$

and

$$R_{v,2} = \frac{1}{\sqrt{2}}(R_v + \widetilde{R}_v), \qquad R_{v,-2} = \frac{1}{\sqrt{2}}(R_v - \widetilde{R}_v)$$

of the linear part we obtain

$$\begin{aligned} \partial_t R_{u,1} &= i\omega_u R_{u,1} + \frac{1}{\sqrt{2}}\varepsilon^3 i\omega_u f_u \\ &\quad +\varepsilon^2 i\omega_u(a_{uu}\psi_u(R_{u,1} + R_{u,-1}) + a_{uv}\psi_u(R_{v,2} + R_{v,-2})), \\ \partial_t R_{v,2} &= i\omega_v R_{v,2} + \frac{1}{\sqrt{2}}\varepsilon^3 (i\omega_v)^{-1} f_v \\ &\quad +\varepsilon^2(i\omega_v)^{-1}(b_{uu}\psi_u(R_{u,1} + R_{u,-1}) + b_{uv}\psi_u(R_{v,2} + R_{v,-2})), \end{aligned}$$

and similar for $R_{u,-1}$ and $R_{v,-2}$.

Since $\varepsilon^2\psi_u$ is strongly concentrated at $k=0$ we separate $\varepsilon^2\psi_u$ into a part concentrated close to $k=0$ and into the rest. For $\delta>0$ we define the mode projection E_δ via $\widehat{E_\delta u}=\widehat{E_\delta}\widehat{u}$, where $\widehat{E_\delta}=1$ for $|k|\leq\delta$ and $\widehat{E_\delta}=0$ elsewhere. Moreover, we define E_δ^c via $\widehat{E_\delta^c}(k)=1-\widehat{E_\delta}(k)$. Since $E_\delta^c\psi_u$ is $\mathcal{O}(\varepsilon^{s_A})$-small, for instance w.r.t. the sup-norm if A is s_A-times continuously differentiable, see Remark A.6 in the appendix, we write the equations for the error as

$$\begin{aligned}
\partial_t R_{u,1} &= i\omega_u R_{u,1}+\frac{1}{\sqrt{2}}\varepsilon^3 i\omega_u g_u \\
&\quad +\varepsilon^2 i\omega_u(a_{uu}(E_\delta\psi_u)(R_{u,1}+R_{u,-1})+a_{uv}(E_\delta\psi_u)(R_{v,2}+R_{v,-2})),\\
\partial_t R_{v,2} &= i\omega_v R_{v,2}+\frac{1}{\sqrt{2}}\varepsilon^3 (i\omega_v)^{-1} g_v \\
&\quad +\varepsilon^2 (i\omega_v)^{-1}(b_{uu}(E_\delta\psi_u)(R_{u,1}+R_{u,-1})+b_{uv}(E_\delta\psi_u)(R_{v,2}+R_{v,-2})),
\end{aligned}$$

and similar for $R_{u,-1}$ and $R_{v,-2}$, where

$$\begin{aligned}
\varepsilon^3 g_u &= \varepsilon^3 f_u+2\varepsilon^2(a_{uu}(E_\delta^c\psi_u)(R_{u,1}+R_{u,-1})+a_{uv}(E_\delta^c\psi_u)(R_{v,2}+R_{v,-2})),\\
\varepsilon^3 g_v &= \varepsilon^3 f_v+2\varepsilon^2(b_{uu}(E_\delta^c\psi_u)(R_{u,1}+R_{u,-1})+b_{uv}(E_\delta^c\psi_u)(R_{v,2}+R_{v,-2})).
\end{aligned}$$

4 The functional analytic set-up

In order to control the unstable resonances we introduce a number of function spaces. By $\langle\cdot,\cdot\rangle$ we denote the Euclidean inner product and by $|\cdot|$ the associated Euclidean norm in $\mathbb{R}^d$. The Fourier transform is denoted by

$$\mathcal{F}(u)(k)=\widehat{u}(k)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ikx}u(x)dx.$$

For $m\geq 0$ we define the Sobolev spaces

$$H^m=\{u\in L^2(\mathbb{R}):(1+|\cdot|^2)^{\frac{m}{2}}\widehat{u}\in L^2(\mathbb{R})\},$$

endowed with the inner product

$$\langle u,v\rangle_{H^m}=\langle\widehat{u},\widehat{v}\rangle_{L^2_m}=\int_{\mathbb{R}}\left(1+|k|^2\right)^m\langle\widehat{u}(k),\widehat{v}(k)\rangle\,dk.$$

For any $m \in \mathbb{N}$, the induced norm is equivalent to the usual H^m-norm. Finally, for $m \geq 0$ we introduce

$$W^m := \left\{ u : u = \mathcal{F}^{-1}(\widehat{u}), \widehat{u} \in L^1(\mathbb{R}), \|u\|_{W_m} = \int_{\mathbb{R}} (1 + |k|^m) |\widehat{u}(k)| \, dk < \infty \right\}.$$

By Sobolev's embedding theorem the space $H^{m+\delta}(\mathbb{R})$ is continuously embedded into W^m for each $\delta > 1/2$. Moreover, every $u \in W^m$ is $\lfloor m \rfloor$-times continuously differentiable with finite $C_b^{\lfloor m \rfloor}(\mathbb{R})$-norm.

In order to control the positive growth rates, occuring at the resonances, we work in the space

$$H^\infty_{\mu,m} = \{u \in L^2(\mathbb{R}) : e^{\mu|\cdot|}(1 + |\cdot|^2)^{\frac{m}{2}} \widehat{u} \in L^2(\mathbb{R})\},$$

equipped with the norm

$$\|u\|_{H^\infty_{\mu,m}} = \left(\int_{\mathbb{R}} |\widehat{u}(k)|^2 e^{2\mu|k|} (1 + |k|^2)^m dk \right)^{\frac{1}{2}},$$

where $\mu \geq 0$ and $m \geq 0$. Functions $u \in H^\infty_{\mu,0}$ can be extended to functions that are analytic on the strip $\{z \in \mathbb{C}\colon |\mathrm{Im}(z)| < \mu\}$. Similarly, we define the spaces $W^\infty_{\mu,m}$.

In our notations of the spaces and norms we do not distinguish between scalar and vector-valued functions. The spaces $H^\infty_{\mu,m}$ are closed under pointwise multiplication for every $\mu \geq 0$ and $m > 1/2$ and the spaces $W^\infty_{\mu,m}$ for every $\mu \geq 0$ and $m \geq 0$. See the appendix.

5 Some first estimates

In this section we collect various estimates which are necessary for the proof of Theorem 1.1. We start by rewriting Lemma 2.1 and Lemma 2.2 into $H^\infty_{\mu,s}$-spaces.

Lemma 5.1. *Fix $\mu_A \geq \mu \geq 0$, $s_A - s \geq 8$, $s > 1/2$, and $T_0 > 0$. Let $A \in C([0, T_0], H^\infty_{\mu_A, s_A})$ be a solution of the KdV equation (7), and let $\varepsilon^2 \psi_u$ and $\varepsilon^4 \psi_v$ be defined as above. For this approximation then there exist $\varepsilon_0 > 0$ and $C_{\mathrm{res}} > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\sup_{T \in [0,T_0]} (\|Res_u\|_{H^\infty_{\mu,s}} + \|Res_v\|_{H^\infty_{\mu,s}} + \varepsilon^2 \|\partial_x^{-1} Res_u\|_{H^\infty_{\mu,s+1}}) \leq C_{\mathrm{res}} \varepsilon^{15/2}.$$

Proof. Using Lemma A.1 from the appendix the proof goes line for line as the proofs of Lemma 2.1 and Lemma 2.2. $\square$

Since ω_u is a bounded operator in $H^\infty_{\mu,s}$ and since $(\omega_v)^{-1}$ is a bounded operator from $H^\infty_{\mu,s}$ to $H^\infty_{\mu,s+1}$ we have, using Lemma 5.1, that

$$\begin{aligned}
&\|\varepsilon^3 i\omega_u f_u\|_{H^\infty_{\mu,s}} + \|\varepsilon^3 (i\omega_v)^{-1} f_v\|_{H^\infty_{\mu,s+1}} \\
&\quad \leq C\varepsilon^{7/2}(\|R_u\|^2_{H^\infty_{\mu,s}} + \|R_v\|^2_{H^\infty_{\mu,s}}) \\
&\qquad + C\varepsilon^4(\|\psi_v\|_{W^\infty_{\mu,s}}\|R_u\|_{H^\infty_{\mu,s}} + \|\psi_v\|_{W^\infty_{\mu,s}}\|R_v\|_{H^\infty_{\mu,s}}) + C_{\mathrm{res}}\varepsilon^3.
\end{aligned}$$

With these estimates and Lemma A.5 we find

$$\begin{aligned}
\varepsilon^3\|i\omega_u g_u\|_{H^\infty_{\mu,s}} &\leq \varepsilon^3\|f_u\|_{H^\infty_{\mu,s}} + C\varepsilon^2 C_\psi \varepsilon^{s_A-s}(\|R_{u,1}\|_{H^\infty_{\mu,s}} + \|R_{u,-1}\|_{H^\infty_{\mu,s}}) \\
&\quad + C\varepsilon^2 C_\psi \varepsilon^{s_A-s}(\|R_{v,2}\|_{H^\infty_{\mu,s}} + \|R_{v,-2}\|_{H^\infty_{\mu,s}}), \\
\varepsilon^3\|(i\omega_v)^{-1} g_v\|_{H^\infty_{\mu,s+1}} &\leq \varepsilon^3\|f_v\|_{H^\infty_{\mu,s+1}} + C\varepsilon^2 C_\psi \varepsilon^{s_A-s}(\|R_{u,1}\|_{H^\infty_{\mu,s}} + \|R_{u,-1}\|_{H^\infty_{\mu,s}}) \\
&\quad + C\varepsilon^2 C_\psi \varepsilon^{s_A-s}(\|R_{v,2}\|_{H^\infty_{\mu,s}} + \|R_{v,-2}\|_{H^\infty_{\mu,s}})
\end{aligned}$$

6 From analytic to Sobolev functions

In order to control the unstable resonances, we solve the equations for the error in $H^\infty_{\mu,s}$-spaces with $s \geq 1$ and $\mu = \mu(t)$ decreasing in time. In detail, we choose

$$\mu(t) = \mu_A/\varepsilon - \beta\varepsilon^2 t$$

for $0 \leq t \leq T_1/\varepsilon^3$ with $T_1 = \mu_A/\beta$. With respect to this time-dependent norm the unstable resonant modes are damped artificially. Since the solutions of the KdV equation satisfy

$$\sup_{T\in[0,T_0]} \|A(T)\|_{W^\infty_{\mu_A,s_A}} \leq C_\psi$$

we have

$$\sup_{T\in[0,T_0/\varepsilon^3]} \left\|\varepsilon^2\psi_u(t)\right\|_{W^\infty_{\mu_A,s_A}} \leq C_\psi\varepsilon^2.$$

In order to work in usual Sobolev we introduce

$$R_j(t) = S_\omega(t) R_{u,v,j}(t),$$

with $S_\omega(t)$ a multiplication operator defined in Fourier space by

$$\widehat{S}_\omega(k,t) = e^{(\mu_A/\varepsilon - \beta\varepsilon^2 t)|k|}.$$

As a direct consequence of the definitions we have:

Lemma 6.1. *For $t \in [0, \mu_A/(\beta\varepsilon^3)]$ the linear mapping $S_\omega(t) : H^\infty_{\mu(t),s} \to H^s$, resp. $S_\omega(t) : W^\infty_{\mu(t),s} \to W^s$, with $\mu(t) = (\mu_A - \eta\varepsilon^3 t)/\varepsilon$, is bijective and bounded with bounded inverse.*

The new variables satisfy

$$\begin{aligned}
\partial_t R_1 &= -\beta\varepsilon^2 |k|_{op} R_1 + i\omega_u R_1 + \frac{1}{\sqrt{2}}\varepsilon^3 i\omega_u S_\omega(t) g_u \\
&\quad + \varepsilon^2 i\omega_u S_\omega(t)(a_{uu}(S_\omega^{-1}(t)E_\delta\psi_u)S_\omega^{-1}(t)(R_1+R_{-1}) \\
&\qquad\qquad + a_{uv}(S_\omega^{-1}(t)E_\delta\psi_u)S_\omega^{-1}(t)(R_2+R_{-2})), \\
\partial_t R_2 &= -\beta\varepsilon^2 |k|_{op} R_2 + i\omega_v R_2 + \frac{1}{\sqrt{2}}\varepsilon^3 (i\omega_v)^{-1} S_\omega(t) g_v \\
&\quad + \varepsilon^2 (i\omega_v)^{-1} S_\omega(t)(b_{uu}(S_\omega^{-1}(t)E_\delta\psi_u)S_\omega^{-1}(t)(R_1+R_{-1}) \\
&\qquad\qquad + b_{uv}(S_\omega^{-1}(t)E_\delta\psi_u)S_\omega^{-1}(t)(R_2+R_{-2})),
\end{aligned}$$

and similar for R_{-1} and R_{-2}. The operator $|k|_{op}$ is defined via its operation in Fourier space $\widehat{|k|_{op}R}(k) = |k|\widehat{R}(k)$.

Using Lemma 6.1, Lemma A.1 and the previous estimates we find

$$\begin{aligned}
&\varepsilon^3 \|i\omega_u S_\omega(t) g_u\|_{H^s} + \varepsilon^3 \|(i\omega_v)^{-1} S_\omega(t) g_v\|_{H^{s+1}} \\
&\quad \le C\varepsilon^{7/2}(\|R_1\|^2_{H^s} + \|R_2\|^2_{H^s}) \\
&\qquad + C\varepsilon^4(\|\psi_v\|_{W^\infty_{\mu,s}}\|R_1\|_{H^s} + \|\psi_v\|_{W^\infty_{\mu,s}}\|R_2\|_{H^s}) + C_{\mathrm{res}}\varepsilon^3 \\
&\qquad\quad + C\varepsilon^2 C_\psi \varepsilon^{s_A - s}(\|R_1\|_{H^s} + \|R_2\|_{H^s}).
\end{aligned}$$

7 The normal form transformation

The modes to wave numbers bounded away from zero, are controlled with the sectorial operator $-\beta|k|_{op}$. At $k=0$ this operator is of no use and so normal transformations and energy estimates have to be used at $k=0$.

For wave numbers in a δ_0-neighborhood of the origin the error equations are simplified by a number of normal form transformations, i.e., by a number of near identity change of variables.

• The term $E_{\delta_0} i\omega_u S_\omega(t)(a_{uv}(S_\omega^{-1}(t)E_\delta\psi_u)S_\omega^{-1}(t)(R_2+R_{-2}))$ in the R_1-equation can be written in Fourier space as

$$\int q_{1,2}(k,k-l,l)\widehat{\psi_u}(k-l)\widehat{R_2}(l)dl + \int q_{1,-2}(k,k-l,l)\widehat{\psi_u}(k-l)\widehat{R_{-2}}(l)dl,$$

with

$$q_{1,2}(k,k-l,l) = \widehat{E_{\delta_0}}(k) i\widehat{\omega_u}(k)\widehat{S}_\omega(k,t)a_{uv}(\widehat{S_\omega^{-1}}(k-l,t)\widehat{E_\delta}(k-l)S_\omega^{-1}(l,t)$$

and similarly for $q_{1,-2}(k,k-l,l)$. It is well known that this term can be eliminated by a near identity change of variables

$$\widehat{R_3} = \widehat{R_1} + \int b_{1,2}(k,k-l,l)\widehat{\psi_u}(k-l)\widehat{R_2}(l)dl + \int b_{1,-2}(k,k-l,l)\widehat{\psi_u}(k-l)\widehat{R_{-2}}(l)dl,$$

where

$$b_{1,\pm 2}(k,k-l,l) = \frac{q_{1,\pm 2}(k,k-l,l)}{i\omega_u(k) - i\omega_u(k-l) \mp i\omega_v(l) - \beta\varepsilon^2(|k|-|k-l|-|l|)}$$

For $|k| \leq \delta_0$ and $|k-l| \leq \delta$ the denominator is bounded away from zero if $\delta_0 > 0$ and $\delta > 0$ are sufficiently small.

• The term $E_{\delta_0}(i\omega_v)^{-1}S_\omega(t)(b_{uu}(S_\omega^{-1}(t)E_\delta\psi_u)S_\omega^{-1}(t)(R_1+R_{-1}))$ and the term $E_{\delta_0}(i\omega_v)^{-1}S_\omega(t)(b_{uv}(S_\omega^{-1}(t)E_\delta\psi_u)S_\omega^{-1}(t)(R_{-2}))$ in the R_2-equation can be written in Fourier space as

$$\int q_{2,1}(k,k-l,l)\widehat{\psi_u}(k-l)\widehat{R_1}(l)dl + \int q_{2,-1}(k,k-l,l)\widehat{\psi_u}(k-l)\widehat{R_{-2}}(l)dl$$
$$+ \int q_{2,-2}(k,k-l,l)\widehat{\psi_u}(k-l)\widehat{R_{-2}}(l)dl,$$

with

$$q_{2,1}(k,k-l,l) = \widehat{E_{\delta_0}}(k)(i\widehat{\omega_v}(k))^{-1}\widehat{S}(k,t)b_{uu}(\widehat{S^{-1}}(k-l,t)\widehat{E_\delta}(k-l)S^{-1}(l,t)$$

and similarly for $q_{2,-1}(k,k-l,l)$ and $q_{2,-2}(k,k-l,l)$. It is well known that this term can be eliminated by a near identity change of variables

$$\begin{aligned}\widehat{R_4} &= \widehat{R_2} + \int b_{2,1}(k,k-l,l)\widehat{\psi_u}(k-l)\widehat{R_1}(l)dl \\ &\quad + \int b_{2,-1}(k,k-l,l)\widehat{\psi_u}(k-l)\widehat{R_{-2}}(l)dl + \int b_{2,-2}(k,k-l,l)\widehat{\psi_u}(k-l)\widehat{R_{-2}}(l)dl\end{aligned}$$

where

$$b_{2,\pm 1}(k,k-l,l) = \frac{q_{2,\pm 1}(k,k-l,l)}{i\omega_v(k) - i\omega_u(k-l) \mp i\omega_u(l) - \beta\varepsilon^2(|k|-|k-l|-|l|)}$$

and

$$b_{2,-2}(k,k-l,l) = \frac{q_{2,-2}(k,k-l,l)}{i\omega_v(k) - i\omega_u(k-l) + i\omega_v(l) - \beta\varepsilon^2(|k|-|k-l|-|l|)}.$$

For $|k| \leq \delta_0$ and $|k-l| \leq \delta$ the denominator is bounded away from zero if $\delta_0 > 0$ and $\delta > 0$ are sufficiently small. Estimates such as

$$\| \int b(k,k-l,l)\psi_u(k-l)R(l)dl\|_{L^2_s(dk)} \leq (\sup_{k,l} |b(k,k-l,l)|)\|\psi_u\|_{W^s}\|R\|_{H^s}$$

then imply:

Lemma 7.1. *There exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0,\varepsilon_0]$ the transformation $(R_1,R_2) \mapsto (R_3,R_4)$ is bijective with*

$$\|R_1 - R_3\|_{H_s} + \|R_2 - R_4\|_{H_s} \leq C\varepsilon^2(\|R_1\|_{H^s} + (\|R_2\|_{H^s}).$$

After the transformation our system is of the form

$$\begin{aligned}
\partial_t R_3 &= -\beta\varepsilon^2|k|_{op}R_3 + i\omega_u R_3 + \frac{1}{\sqrt{2}}\varepsilon^3 i\omega_u S_\omega(t)g_3 \\
&\quad +\varepsilon^2 i\omega_u S(t)(a_{uu}(S_\omega^{-1}(t)E_\delta\psi_u)S_\omega^{-1}(t)(R_3+R_{-3})) \\
&\qquad +\varepsilon^2 i\omega_u S_\omega(t)E_\delta^c(a_{uv}(S_\omega^{-1}(t)E_\delta\psi_u)S_\omega^{-1}(t)(R_4+R_{-4})), \\
\partial_t R_4 &= -\beta\varepsilon^2|k|_{op}R_4 + i\omega_v R_4 + \frac{1}{\sqrt{2}}\varepsilon^3(i\omega_v)^{-1}S(t)g_4 \\
&\quad +\varepsilon^2(i\omega_v)^{-1}S_\omega(t)E_\delta^c(b_{uu}(S_\omega^{-1}(t)E_\delta\psi_u)S_\omega^{-1}(t)(R_3+R_{-3})) \\
&\qquad +\varepsilon^2(i\omega_v)^{-1}S_\omega(t)(b_{uv}(S_\omega^{-1}(t)E_\delta\psi_u)S_\omega^{-1}(t)(R_4)) \\
&\qquad\quad +\varepsilon^2(i\omega_v)^{-1}S_\omega(t)E_\delta^c(b_{uv}(S_\omega^{-1}(t)E_\delta\psi_u)S_\omega^{-1}(t)(R_{-4})),
\end{aligned}$$

and similar for R_{-3} and R_{-4}. The terms with E_δ^c in front will be controlled with $-\beta\varepsilon^2|k|_{op}R_j$ by choosing $\beta = \mathcal{O}(1)$ sufficiently large. The remaining terms of order $\mathcal{O}(\varepsilon^2)$ at $k=0$ have to be estimated differently. The terms with g_3 and g_4 are coming from g_u and g_v and from higher order terms obtained via the normal form transformation from terms not contained in g_u and g_v. Therefore, using $s_A - s \geq 2$, the terms with g_3 and g_4 obey

$$\begin{aligned}
&\varepsilon^3 \| i\omega_u S_\omega(t) g_3 \|_{H^s} + \varepsilon^3 \| (i\omega_v)^{-1} S_\omega(t) g_4 \|_{H^{s+1}} \\
&\quad \leq C\varepsilon^{7/2} (\|R_3\|_{H^s}^2 + \|R_4\|_{H^s}^2) \\
&\qquad + C_1 \varepsilon^4 (\|R_3\|_{H^s} + \|R_4\|_{H^s}) + C_{\mathrm{res}} \varepsilon^3,
\end{aligned} \tag{10}$$

where C_1 is a constant independent of $0 < \varepsilon \ll 1$, solely depending on $\|A\|_{W^\infty_{\mu_A, s_A}}$.

8 The terms at $k = 0$

Ignoring at the moment all terms which are of higher order or which have some E_δ^c in front, our system is of the form

$$\begin{aligned}
\partial_t R_3 &= -\beta\varepsilon^2 |k|_{op} R_3 + i\omega_u R_3 \\
&\quad + \varepsilon^2 i\omega_u S_\omega(t) (2 a_{uu} (S_\omega^{-1}(t) E_\delta \psi_u) S_\omega^{-1}(t)(R_3 + R_{-3})) + \ldots, \\
\partial_t R_4 &= -\beta\varepsilon^2 |k|_{op} R_4 + i\omega_v R_4 \\
&\quad + \varepsilon^2 (i\omega_v)^{-1} S_\omega(t) (b_{uv} (S_\omega^{-1}(t) E_\delta \psi_u) S_\omega^{-1}(t)(R_4)) + \ldots .
\end{aligned}$$

Hence, the R_3-equation and the R_4-equation decouple up to the terms hidden in $\ldots$. Due to the $i\omega_u$-operator in front of the term in the second line of the R_3-equation, in Fourier space this term vanishes at the wave number $k = 0$. Therefore, in the end also this term will be handled subsequently with the damping term $-\beta\varepsilon^2 |k|_{op} R_3$. The same is true, if we write an E_δ^c-operator in front of the term in the second line of the R_4-equation, and so it remains to discuss

$$\begin{aligned}
\partial_t R_4 &= -\beta\varepsilon^2 |k|_{op} R_4 + i\omega_v R_4 \\
&\quad + \frac{1}{2} \varepsilon^2 (i\omega_v)^{-1} S_\omega(t) E_\delta (b_{uv} (S_\omega^{-1}(t) E_\delta \psi_u) S_\omega^{-1}(t)(R_4)) + \ldots .
\end{aligned}$$

We find

$$\frac{d}{dt} \int \omega_v |R_4|^2 dx = \frac{d}{dt} \int \omega_v (R_4 R_{-4}) dx = -\beta\varepsilon^2 \int \omega_v ||k|_{op} R_4|^2 dx + 0 + \varepsilon^2 s_4 + \ldots$$

where

$$\begin{aligned}
\widehat{s_4} &= 2i \int\int \widehat{R_{-4}}(k)\widehat{S_\omega}(k,t)\widehat{E_\delta}(k)b_{uv}\widehat{S_\omega^{-1}}(k-l,t)\widehat{E_\delta}(k-l)\widehat{\psi_u}(k-l)\widehat{S_\omega^{-1}}(l,t)\widehat{R_4}(l)dldk \\
&\quad -2i \int\int \widehat{R_4}(k)\widehat{S}_\omega(k,t)\widehat{E_\delta}(k)\overline{b_{uv}}\widehat{S_\omega^{-1}}(k-l,t)\widehat{E_\delta}(k-l)\overline{\widehat{\psi_u}(k-l)}\widehat{S_\omega^{-1}}(l,t)\widehat{R_{-4}}(l)dldk \\
&= 2i \int\int \widehat{R_{-4}}(k)\widehat{S}_\omega(k,t)\widehat{E_\delta}(k)b_{uv}\widehat{S_\omega^{-1}}(k-l,t)\widehat{E_\delta}(k-l)\widehat{\psi_u}(k-l)\widehat{S_\omega^{-1}}(l,t)\widehat{R_4}(l)dldk \\
&\quad -2i \int\int \widehat{R_4}(l)\widehat{S_\omega}(l,t)\widehat{E_\delta}(k)\overline{b_{uv}}\widehat{S_\omega^{-1}}(l-k,t)\widehat{E_\delta}(l-k)\overline{\widehat{\psi_u}(l-k)}\widehat{S_\omega^{-1}}(k,t)\widehat{R_{-4}}(k)dldk \\
&= 2i \int\int q_0(k,k-l,l)\widehat{R_{-4}}(k)\widehat{\psi_u}(k-l)\widehat{R_4}(l)dldk,
\end{aligned}$$

with

$$\begin{aligned}
q_0(k,k-l,l) &= 2i\widehat{S}_\omega(k,t)\widehat{E_\delta}(k)b_{uv}\widehat{S_\omega^{-1}}(k-l,t)\widehat{E_\delta}(k-l)\widehat{S_\omega^{-1}}(l,t) \\
&\quad -2i\widehat{S_\omega}(l,t)\widehat{E_\delta}(k)\overline{b_{uv}}\widehat{S_\omega^{-1}}(l-k,t)\widehat{E_\delta}(l-k)\widehat{S_\omega^{-1}}(k,t),
\end{aligned}$$

where we used $\overline{\widehat{\psi_u}(l-k)} = \widehat{\psi_u}(k-l)$ which holds due to the fact that ψ_u is real-valued. By definition we have

$$b_{uv}\widehat{S_\omega^{-1}}(k-l,t)\widehat{E_\delta}(k-l) = b_{uv}\widehat{S_\omega^{-1}}(l-k,t)\widehat{E_\delta}(l-k) \in \mathbb{R},$$

and so $q_0(k,0,k) = 0$ for all $k \in \mathbb{R}$. Since we have a compact set of wave numbers this implies $|q_0(k,k-l,l)| \leq C|k-l|$. As a consequence, we can apply Corollary A.5 and obtain

$$\int\int q_0(k,k-l,l)\widehat{R_{-4}}(k)\widehat{\psi_u}(k-l)\widehat{R_4}(l)dldk = \mathcal{O}(\varepsilon),$$

resp. $\varepsilon^2 s_4 = \mathcal{O}(\varepsilon^3)$.

9 The final energy estimates

We have now all ingredients to perform the final energy estimates. We define an operator Ω via the multiplier $\widehat{\Omega}(k) = \min(\omega_v(k), 4)$ in Fourier space. On the one hand this operator allows to perform the energy estimates from the last section for k close to 0, on the other hand this operator allows to work with the same regularity for R_3 and R_4.

We start now to estimate the time derivative of

$$E_s = \|R_3\|_{H^s}^2 + \left\|\Omega^{1/2} R_4\right\|_{H^s}^2 .$$

We compute

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E_s &= (R_3, -\beta\varepsilon^2|k|_{op}R_3)_{H^s} + (\Omega^{1/2}R_4, -\beta\varepsilon^2|k|_{op}\Omega^{1/2}R_4)_{H^s} \\ &\quad + s_1 + s_2 + s_3 + \ldots + s_8,\end{aligned}$$

where

$$\begin{aligned}
s_1 &= (R_3, i\omega_u R_3)_{H^s} + \left(\Omega^{1/2}R_4, i\omega_v \Omega^{1/2} R_4\right)_{H^s},\\
s_2 &= \left(R_3, \frac{1}{\sqrt{2}}\varepsilon^3 i\omega_u S_\omega(t) g_3\right)_{H^s} + \left(\Omega R_4, \frac{1}{\sqrt{2}}\varepsilon^3 (i\omega_v)^{-1} S(t) g_4\right)_{H^s},\\
s_3 &= \left(R_3, \varepsilon^2 i\omega_u S_\omega(t)(a_{uu}(S_\omega^{-1}(t)E_\delta\psi_u)S_\omega^{-1}(t)(R_3 + R_{-3}))\right)_{H^s},\\
s_4 &= \left(R_3, \varepsilon^2 i\omega_u S_\omega(t)E_\delta^c(a_{uv}(S_\omega^{-1}(t)E_\delta\psi_u)S_\omega^{-1}(t)(R_4 + R_{-4}))\right)_{H^s},\\
s_5 &= \left(\Omega R_4, \varepsilon^2 (i\omega_v)^{-1} S_\omega(t)E_\delta^c(b_{uu}(S_\omega^{-1}(t)E_\delta\psi_u)S_\omega^{-1}(t)(R_3 + R_{-3}))\right)_{H^s},\\
s_6 &= \left(\Omega R_4, \varepsilon^2 (i\omega_v)^{-1} S_\omega(t)E_\delta(b_{uv}(S_\omega^{-1}(t)E_\delta\psi_u)S_\omega^{-1}(t)(R_4))\right)_{H^s},\\
s_7 &= \left(\Omega R_4, \varepsilon^2 (i\omega_v)^{-1} S_\omega(t)E_\delta^c(b_{uv}(S_\omega^{-1}(t)E_\delta\psi_u)S_\omega^{-1}(t)(R_4))\right)_{H^s},\\
s_8 &= \left(\Omega R_4, \varepsilon^2 (i\omega_v)^{-1} S_\omega(t)E_\delta^c(b_{uv}(S_\omega^{-1}(t)E_\delta\psi_u)S_\omega^{-1}(t)(R_{-4}))\right)_{H^s}.
\end{aligned}$$

In the following we give the final estimates. The detailed calculations to obtain the estimates can be found in Appendix B.

We start with the good terms, namely

$$\begin{aligned}&(R_3, -\beta\varepsilon^2|k|_{op}R_3)_{H^s} + (\Omega^{1/2}R_4, -\beta\varepsilon^2|k|_{op}\Omega^{1/2}R_4)_{H^s}\\ = \; & -\beta\varepsilon^2 \left\||k|^{1/2} R_3\right\|_{H^s} - \beta\varepsilon^2 \left\||k|^{1/2}\Omega^{1/2}R_4\right\|_{H^s}.\end{aligned}$$

Using the skew-symmetry of $i\omega_u$ and $i\omega_v$ yields

$$s_1 = 0.$$

Using the Cauchy-Schwarz inequality and (10) yields

$$|s_2| \leq C\varepsilon^3 E_s + C\varepsilon^{7/2}E_s^{3/2} + C\varepsilon^3.$$

The terms s_4, s_5, s_7, and s_8 have some E^c_δ in front and can be estmated by the good terms

$$-\beta\varepsilon^2 \left\| |k|^{1/2} R_3 \right\|_{H^s} - \beta\varepsilon^2 \left\| |k|^{1/2} \Omega^{1/2} R_4 \right\|_{H^s}$$

which for the E^c_δ-terms is a lower bound for

$$-\beta\varepsilon^2 \left\| \delta^{1/2} R_3 \right\|_{H^s} - \beta\varepsilon^2 \left\| \delta^{1/2}\Omega^{1/2} R_4 \right\|_{H^s}.$$

The term s_3 can be estimated by the good terms, too, using the fact that $|\widehat{\omega_u}(k)| \leq C|k|$ for $k \to 0$. Finally, the term s_6, as explained in Section 8, is of order $\mathcal{O}(\varepsilon^3)$.

Summarizing all estimates gives

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}E_s &\leq (-\beta\varepsilon^2 + C\varepsilon^2 + C\varepsilon^{7/2}E_s^{1/2})(\left\| |k|^{1/2} R_3 \right\|_{H^s} + \left\| |k|^{1/2} \Omega^{1/2} R_4 \right\|_{H^s}) \\
&\quad + C\varepsilon^3 E_s + \varepsilon^{7/2}E_s^{3/2} + C\varepsilon^3 \\
&\leq C\varepsilon^3 E_s + \varepsilon^{7/2}E_s^{3/2} + C\varepsilon^3 \\
&\leq (C+1)\varepsilon^3 E_s + C\varepsilon^3,
\end{aligned}$$

if

$$-\beta + C + C\varepsilon^{3/2}E_s^{1/2} < 0$$

and $\varepsilon^{1/2}E_s^{1/2} \leq 1$. Gronwall's inequality then implies

$$E_s(t) \leq Cte^{(C+1)t} \leq CT_0e^{(C+1)T_0} =: M.$$

We are done, if we choose $\varepsilon_0 > 0$ so small that $\varepsilon_0^{1/2}M^{1/2} \leq 1$ and then $\beta > 0$ so big that

$$-\beta + C + C\varepsilon_0^{3/2}M^{1/2} < 0.$$

A Some technical estimates

In this section we collect a number of estimates which we used in previous sections. Together with their proofs they can be found as Lemma A.4, Corollary A.5, Corollary A.6, Lemma A.9, and Corollary A.10 in [HdRS18]. We start with some estimates for the nonlinear terms.

Lemma A.1. *The spaces $H^{\infty}_{\mu,s}$ are Banach algebras for $\mu \geq 0$ and $s > \frac{1}{2}$. In detail, there exists a μ-independent constant C_s such that*

$$\|uv\|_{H^{\infty}_{\mu,s}} \leq C_s \|u\|_{H^{\infty}_{\mu,s}} \|v\|_{H^{\infty}_{\mu,s}}$$

for all $u, v \in H^{\infty}_{\mu,s}$.

For our error estimates we need the following tame estimates version of this lemma.

Corollary A.2. *For $\delta > 0$, $\mu \geq 0$ and $s > 1/2$ we have*

$$\|u^2\|_{H^{\infty}_{\mu,s}} \leq C_s \|u\|_{H^{\infty}_{\mu,1/2+\delta}} \|u\|_{H^{\infty}_{\mu,s}}$$

for all $u \in H^{\infty}_{\mu,s}$.

Corollary A.3. *For $\mu \geq 0$ and $s \geq 0$ we have*

$$\|uv\|_{H^{\infty}_{\mu,s}} \leq C_s \|u\|_{W_{\mu,s}} \|v\|_{H^{\infty}_{\mu,s}}$$

for all $u \in W^{\infty}_{\mu,s}$ and $v \in H^{\infty}_{\mu,s}$.

The expansion of the kernels in the multilinear maps can be estimated with the following lemma.

Lemma A.4. *Let $\vartheta_0 \geq 0$, $\vartheta_\infty \in \mathbb{R}$, and let $g : \mathbb{R} \to \mathbb{C}$ satisfy*

$$|g(k)| \leq C \min(|k|^{\vartheta_0}, (1 + |k|)^{\vartheta_\infty}).$$

Then for the associated multiplication operator $g_{op} = \mathcal{F}^{-1} g \mathcal{F}$ the following holds. For i) $\mu_1 > \mu_2$ and $m_1, m_2 \geq 0$ or ii) $\mu_1 = \mu_2$ and $m_2 - m_1 \geq \max(\vartheta_0, \vartheta_\infty)$ we have

$$\|g_{op} A(\varepsilon \cdot)\|_{H^{\infty}_{\mu_1/\varepsilon, m_1}} \leq C \varepsilon^{\vartheta_0 - 1/2} \|A(\cdot)\|_{H^{\infty}_{\mu_2, m_2}}$$

for all $\varepsilon \in (0, 1)$.

In $W^{\infty}_{\mu,m}$-spaces there is no $\varepsilon^{-1/2}$ loss due to the scaling invariance of the norm and so we have as a direct consequence:

Corollary A.5. *Let $\vartheta_0 \geq 0$, $\vartheta_\infty \in \mathbb{R}$, and let $g(k)$ satisfy*

$$|g(k)| \leq C \min(|k|^{\vartheta_0}, (1+|k|)^{\vartheta_\infty}).$$

Then for the associated operator $g_{op} = \mathcal{F}^{-1} g \mathcal{F}$ the following holds. For i) $\mu_1 > \mu_2$ and $m_1, m_2 \geq 0$ or ii) $\mu_1 = \mu_2$ and $m_2 - m_1 \geq \max(\vartheta_0, \vartheta_\infty)$ we have

$$\|g_{op} A(\varepsilon \cdot)\|_{W_{\mu_1/\varepsilon, m_1}} \leq C \varepsilon^{\vartheta_0} \|A(\cdot)\|_{W_{\mu_2, m_2}}$$

for all $\varepsilon \in (0,1)$.

Remark A.6. This corollary is used for instance to estimate E^c_δ. Since $\widehat{E}^\delta_c(k)$ is identical zero in a neighborhood of the origin we have $|\widehat{E}^\delta_c(k)| \leq C|k|^r$ for every $r \in \mathbb{N}$.

B Estimates for the s_j

In this section we give the detailed calculations which are necessary for obtaining the bounds in Section 9.

a) We start with the bound on the linear terms $(R_3, -\beta\varepsilon^2 |k|_{op} R_3)_{H^s}$ and $(\Omega^{1/2} R_4, -\beta\varepsilon^2 |k|_{op} \Omega^{1/2} R_4)_{H^s}$. Using the Fourier representation of $|k|_{op}$ gives

$$(R_3, -\beta\varepsilon^2 |k|_{op} R_3)_{H^s} = -\beta\varepsilon^2 (\widehat{R}_3, |k| \widehat{R}_3)_{L^2_s} = -\beta\varepsilon^2 (|k|^{1/2} \widehat{R}_3, |k|^{1/2} \widehat{R}_3)_{L^2_s}$$

and similar for $(\Omega^{1/2} R_4, -\beta\varepsilon^2 |k|_{op} \Omega^{1/2} R_4)_{H^s}$.

The problem is now, that these 'good' terms do not allow to estimate terms at the wave number $k=0$. We have to use the fact, that the terms s_3, s_4, s_5, s_7, and s_8 vanish at the wave number $k=0$, too.

b1) The term s_3 can be estimated by the 'good' terms using the fact that $|\widehat{\omega_u}(k)| \leq C|k|$ for $k \to 0$. The last estimate implies that the symbol of $\vartheta = |k|^{-1/2}_{op} \omega_u$ is bounded at the wave number $k=0$. We find

$$\begin{aligned}
|s_3| &= |\left(R_3, \varepsilon^2 i \omega_u S_\omega(t)(a_{uu}(S^{-1}_\omega(t) E_\delta \psi_u) S^{-1}_\omega(t)(R_3 + R_{-3}))\right)_{H^s}| \\
&= |\left(|k|^{1/2}_{op} R_3, \varepsilon^2 i |k|^{-1/2}_{op} \omega_u S_\omega(t)(a_{uu}(S^{-1}_\omega(t) E_\delta \psi_u) S^{-1}_\omega(t)(R_3 + R_{-3}))\right)_{H^s}| \\
&\leq C\varepsilon^2 \||k|^{1/2}_{op} R_3\|_{H^s} \|\vartheta S_\omega(t)(a_{uu}(S^{-1}_\omega(t) E_\delta \psi_u) S^{-1}_\omega(t)(R_3 + R_{-3}))\|_{H^s} \\
&\leq C\varepsilon^2 \||k|^{1/2}_{op} R_3\|_{H^s} \\
&\qquad \times (\|\vartheta^{1/2} \psi_u\|_{W_s} \|R_3\|_{H^s} + \|\psi_u\|_{W_s} \|\vartheta^{1/2} R_3\|_{H^s}) \\
&\leq C\varepsilon^{5/2} \||k|^{1/2}_{op} R_3\|_{H^s} \|R_3\|_{H^s} + C\varepsilon^2 \||k|^{1/2}_{op} R_3\|_{H^s} \||k|^{1/2}_{op} R_3\|_{H^s} \\
&\leq C(\varepsilon^2 \||k|^{1/2}_{op} R_3\|^2_{H^s} + \varepsilon^3 \|R_3\|^2_{H^s})
\end{aligned}$$

where we used that $\|\vartheta^{1/2}\psi_u\|_{W_s} = \mathcal{O}(\varepsilon^{1/2})$ due to Corollary A.5 applied to $|\widehat{\vartheta}(k)| \leq C|k|^{1/2}$. In the last line we used $\varepsilon^{5/2}ab \leq \varepsilon^2 a^2 + \varepsilon^3 b^2$.

b2) The term s_4 can be estimated exactly the same as the term s_3. The last lines have to be modified into

$$\begin{aligned}
|s_4| &= C\varepsilon^2 \||k|_{op}^{1/2} R_3\|_{H^s} \\
&\qquad \times(\|\vartheta^{1/2}\psi_u\|_{W_s}\|R_4\|_{H^s} + \|\psi_u\|_{W_s}\|\vartheta^{1/2}R_4\|_{H^s}) \\
&\leq C\varepsilon^{5/2}\||k|_{op}^{1/2}R_3\|_{H^s}\|R_4\|_{H^s} + C\varepsilon^2\||k|_{op}^{1/2}R_3\|_{H^s}\||k|_{op}^{1/2}R_4\|_{H^s} \\
&\leq C(\varepsilon^2\||k|_{op}^{1/2}R_3\|_{H^s}^2 + \varepsilon^2\||k|_{op}^{1/2}\Omega^{1/2}R_4\|_{H^s}^2 + \varepsilon^3\|\Omega^{1/2}R_4\|_{H^s}^2).
\end{aligned}$$

b3) The remaining terms s_5, s_7, and s_8 can be estimated by the 'good' terms using the fact that they have a E_δ^c in front which vanishes at the wave number $k = 0$, too. We finally obtain

$$\begin{aligned}
|s_5| + |s_7| + |s_8| \leq\; & C(\varepsilon^2\||k|_{op}^{1/2}R_3\|_{H^s}^2 + \varepsilon^2\||k|_{op}^{1/2}\Omega^{1/2}R_4\|_{H^s}^2 \\
& + \varepsilon^3\|R_3\|_{H^s}^2 + \varepsilon^3\|\Omega^{1/2}R_4\|_{H^s}^2).
\end{aligned}$$

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