Fakultät für Physik
Institut für Theorie der Kondensierten Materie

Universal non-equilibrium dynamics in quantum critical systems

PhD thesis

by

Pia Gagel

13.07.2018

Instructor: Prof. Dr. Jörg Schmalian
2nd Instructor: Priv-Doz. Dr. Boris Narozhny

Universelle Nichtgleichgewichts-Dynamik in quantenkritischen Systemen

Zur Erlangung des akademischen Grades eines

DOKTORS DER NATURWISSENSCHAFTEN

der Fakultät für Physik
des Karlsruher Instituts für Technologie

genehmigte

DISSERTATION

von

Dipl. Phys. Pia Gagel
aus Marburg

Tag der mündlichen Prüfung: 13.07.2018
Referent: Prof. Dr. Jörg Schmalian
Korreferent: Priv-Doz. Dr. Boris Narozhny

Für meine Familie

Introduction

In condensed matter physics, recent experiments on non-equilibrium dynamics of quantum systems have unveiled many novel insights [1–4]. In conjunction with conventional problems not fully understood yet in this field, they are raising many exciting questions. In the following, three of them are presented:

Far from equilibrium, so called prethermal plateaus have been observed, where the system is in a metastable, long-living state [5–7]. Key topics are the properties of such prethermal states, the typical time-scales limiting such a plateau and the conditions under which they can be reached.

An important second point is the question of thermalization, i. e. systems which relax and thermalize, and can be described by a thermal distribution function in the long-time limit. The question of thermalization is somewhat clearer for open systems, in particular for systems coupled to an external heat bath. However, even in open systems, memory effects may occur, leading to a significant slowing-down of thermalization. For isolated systems however, thermalization is an open and widely discussed question, see for example Ref. [8] and references therein. Integrable systems are known to never reach a thermal state, after being driven out of equilibrium [9]. However, it was also found, that isolated systems can relax to a steady state, which can be described by generalized Gibbs ensembles in the long-time limit [10].

And thirdly, there is the interplay of non-equilibrium dynamics and universality. Universal dynamics near a (quantum) critical point is well established in the Kibble-Zurek protocol [11, 12], where a system is adiabatically driven from the disordered into the ordered phase. In this scenario, the correlation length remains finite at each finite time, causing topological defects in the ordered phase. The number of those defects can be predicted with the quench-rate and universal equilibrium exponents [13, 14]. The basic argument in those predictions is, that at the beginning the system can adiabatically follow the change of the tuning parameters relative to the critical point. When the correlation time is of the order of the inverse quench rate, the system falls out of equilibrium by freezing out, i. e. the correlation length stays finite. The Kibble-Zurek scaling was for example observed in Ref. [15]. However, derivations from this prediction have been observed for fast quench rates [16]. In Ref. [17] it was shown, that the assumption of freezing out is not quiet valid, but that the correlation length continues to grow with a dynamical exponent. The concept of universal dynamics has also been reported for quasi-adiabatic relaxation close to a quantum critical point in Ref. [18], where the power-laws are given by equilibrium exponents. Both are examples for universal dynamics near equilibrium. On the other hand, the opposite protocol, the quantum quench, is a topic of many recent studies. In a quantum quench, the system is initially prepared in the ground state of a Hamiltonian H_i. At time $t = 0$ some parameters are instantaneously switched, such that the time evolution is governed by a new Hamiltonian H. A further open question is, if the quantum-classical mapping, well established in equilibrium also holds in an out-of-equilibrium setup [19–22]. Experimentally, such a quench can be performed by pump-probe experiments, where the system is exposed to a laser beam. Pump-probe experiments have been realized, in order to seek for far-from equilibrium superconductivity [3, 23, 24]. Of special interest are pump-probe experiments, with an induced energy of order twice the Bardeen-Cooper-Schrieffer gap [25]. Other quench protocols in combination with phase transitions have also been reported in spin

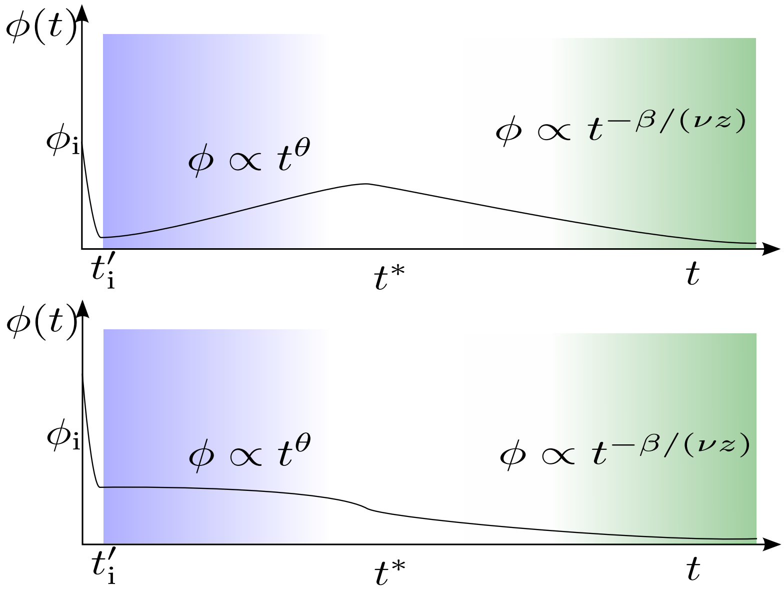


Figure 1: *Dynamics of the order parameter $\phi(t)$ after a quench from the ordered phase to the quantum critical point. (a) for $\theta > 0$ and (b) for $\theta < 0$. The time-scale t^* is the crossover time scale, separating the prethermal regime from the adiabatic long-time limit. Here, β is the equilibrium order parameter exponent, ν the correlation length exponent and z the dynamical exponent. As we will show in this thesis, the new exponent θ cannot be expressed in terms of equilibrium critical exponents.*

chain systems [7, 26] and cold atom gases [2, 27].
All three points have in common, that they are effects which originate from an interacting many body system. Thus, it is necessary to go beyond a mean-field approximation to find answers in those scenarios. This makes it necessary to develop new methods to predict the time evolution also in strongly correlated systems far from known equilibrium states.

In this thesis, the dynamics after a quantum quench to a quantum critical point (QCP) are analyzed to address those three points. This work is inspired by Janssen, Schaub and Schmittmann who found universal, prethermal dynamics after a quench in a classical system with white noise [28]. The extension to classical systems with colored noise was made in Ref. [29]. In this thesis, the question of post-quench universality is answered in open quantum systems [30, 31], as well as for nearly isolated systems. For perfectly isolated systems after a quantum quench, a non-thermal fixed point was found by Mitra et al. in Ref. [32].
Here, we address the question, under which conditions universal dynamics after a quantum quench can be expected, see table 1. We speak of universal dynamics if for example, the order-parameter dynamics can be described by an universal power-law in time. Two main regimes are distinguished, the prethermal regime, at intermediate times after the quench, and the long-time limit, where certain systems relax to the QCP. The main ingredient for non-equilibrium criticality is the light-cone growth of the correlation length:

$$\xi(t) \propto t^{1/z_d} . \tag{1}$$

This algebraic growth with the dynamic coarsening exponent z_d leads to power-laws in the order-parameter dynamics and the Green's function as well. In this thesis, the implications of these power-laws are analyzed at intermediate time-scales after the quench, in the prethermal regime and in the long-time limit. For clarity and to give the basic motivation for this thesis, the time evolution of the order parameter is discussed here, see also figure 1. In the prethermal regime, the power-law of ξ in Eq. (1) affects the dynamics of the order-parameter ϕ:

$$\phi(t \ll t^*) \propto t^{\theta}\,, \tag{2}$$

with a new, universal exponent θ. This exponent is calculated explicitly in this thesis. It turns out, that this exponent can be positive or negative for open quantum systems, depending on the external bath. For a positive exponent, the prethermal regime is characterized by the growth of order after a quench. This regime is limited by the cross-over timescale t^*, which will be shown to depend on details of the quench protocol and can be tuned to large values by performing a weak quench. In the long-time limit, equation Eq. (1) leads to a power-law decay to equilibrium in the order parameter:

$$\phi(t \gg t^*) \propto t^{-\beta/(\nu z)}\,. \tag{3}$$

In this limit, the non-equilibrium exponent θ enters in the universal relaxation amplitude, while the power-law in time is given by known equilibrium exponents β, ν and z. This universality can be observed in open quantum systems, while the question of thermalization is not answered yet for isolated quantum systems. In the long time limit a non-thermal distribution function $n(t, \omega)$ can be introduced, which is a generalization from the fluctuation-dissipation theorem. In table 1, an overview of the results is given for different kinds of systems as well as for the two different time regimes. Three kinds of systems are considered. The open system is coupled to an external heat bath and the dominant dynamics are given by relaxation processes via this external bath. Those systems equilibrate per construction, as the energy is not conserved. However, the power-laws in the long-time limit lead to a slow down of thermalization. Here, an universal prethermal regime can also be found. Perfectly isolated systems may thermalize due to some internal relaxation process, but this question is still open. Here, the considered dynamics are dominated by the ballistic, non-interacting term. The limit of a deep, i. e. a very strong quench, was considered by Refs. [32–34], where universality near a non-thermal fixed point was reported. Here, also the limit of a weak quench is considered, where the system is already prepared near the QCP. *Nearly isolated* refers to systems with small coupling to some external heat bath. This coupling is chosen such that it is irrelevant in the sense of scaling and renormalization, but still guarantees thermalization to the QCP in the long-time limit. In those systems, power-laws with a non-universal exponent have been found at intermediate times after the quench. In the long-time limit, the system equilibrates with a universal power-law given by the equilibrium exponents. Higher order scattering processes of the order-parameter field could possibly also be captured by such a bath coupling, however with thermalization to a finite temperature. This finite temperature should then be determined by the energy induced by the quench. If this is the case, universal relaxation can be found in the isolated system as well, if the finite temperature is not high, i. e. the system thermalizes near the QCP. However, in none of the scenarios of the isolated or nearly isolated system, the result found by the quantum-classical-mapping can be confirmed. The post-quench scenario seems thus to be an example where this mapping does not work.

In contrast to previous work in [30–34], the methods presented in this thesis to derive non-equilibrium universality are more general and can be applied on different models. This is done in chapters 5 and 6.

Table 1: *The existence of out-of-equilibrium universality after a quantum quench. Colored capital letters designate the answers found in this thesis. In other cases the corresponding reference is listed.*

	prethermal	long-time limit
Open quantum system, chapter 5	YES	YES
Perfectly isolated system, section 6.1		
deep quench	yes [32–34]	?
weak quench	NO	?
Nearly isolated system, section 6.2	NO	YES

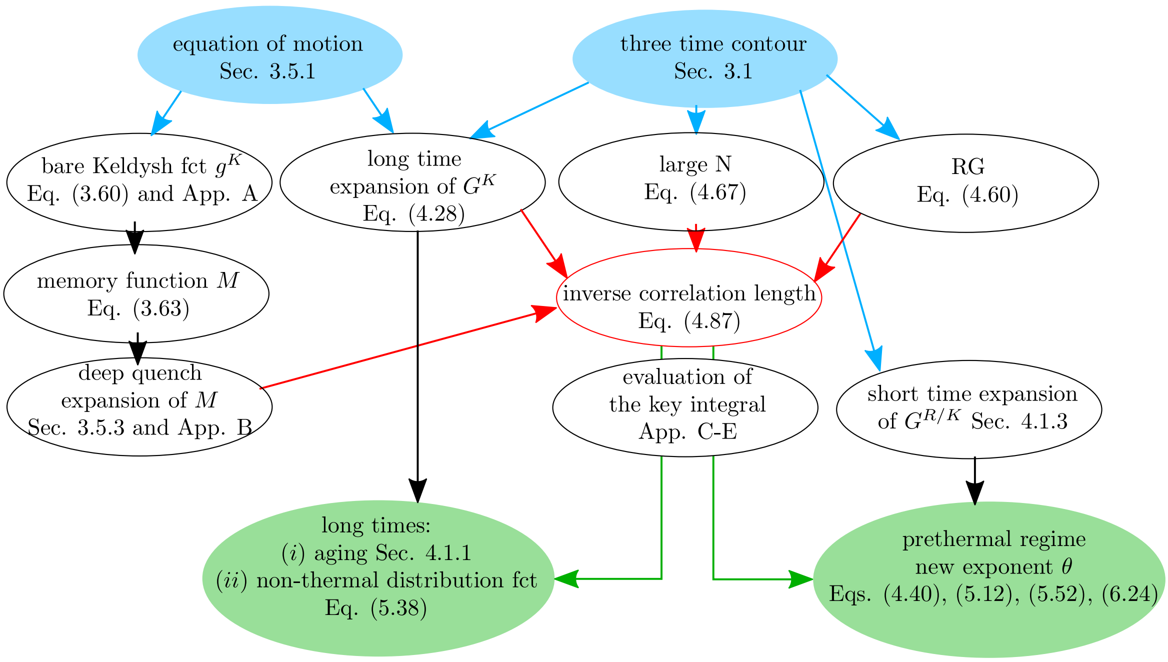


Figure 2: *Roadmap through the calculations done in this thesis.*
The non-equilibrium framework are the equations of motions to derive the bare Green's functions and the three-time-contour to include interactions (blue bubble). From this starting point, the large-N equation and the renormalization group (RG) equation are derived in an out-of-equilibrium version. Both lead to an equation for the inverse correlation length $r(t)$, which is solved self-consistently. Therefore, the bare Keldysh-function in the scaling limit of a deep quench is necessary, as well as the long-time expansion, derived with a Dyson equation. The final solution for $r(t)$ is given in Eq. (4.87). The two limits in time, large and intermediate times after the quench, are analyzed, and the final results are applied on different systems: open and isolated systems at zero temperature and classical systems at finite temperature.

In addition the classical limit can be easily reproduced, see section 5.6. The higher generality is reflected in more abstract calculations. To guide the reader through this jungle, a roadmap with references on the corresponding sections and equations is given in figure 2. The order of the thesis is the following:

In **chapter 1** the equilibrium properties of a system near a critical point are reviewed. Three methods to treat quantum critical systems are presented, the renormalization group (RG), the large-N or saddle point approximation and the quantum-classical mapping. The extension of those methods to a non-equilibrium setup is one goal of this thesis.

In **chapter 2** a scaling ansatz is used, to motivate the time evolution of the order parameter in Eq. (2), the correlation length and the Green's functions in the prethermal as well as in the long-time limit. Further, there is a discussion about the different time and frequency scales of diffusive and ballistic dynamics.

In **chapter 3** the quantum-field theoretical framework to confirm this scaling behavior with a real-time evolution is presented. Therefore a three-branch contour is introduced. An important point, to build up the perturbative expansion, is the knowledge of the bare, post-quench Green's functions. Here a memory ansatz is derived, to calculate those functions after the quench protocol.

In **chapter 4** the RG and the large-N equation are derived for the quench to the QCP. Both equations lead, in the appropriate limit, to the same result. Further, a general solution of this equation is derived.

In **chapter 5** the post-quench dynamics of an open system are analyzed, and the non-equilibrium exponent θ is evaluated. The scaling ansatz from chapter 2 can thus be confirmed and θ is evaluated for different kinds of bath-spectrums. Its impact on the Green's functions, the order parameter and a non-thermal distribution function in the long-time limit are analyzed. It is shown as well, that with the methods presented here, the known classical limits can be reproduced.

In **chapter 6** the post-quench dynamics of an isolated system and a nearly isolated system are analyzed. Here, results from the third method, the quantum-classical mapping, are derived for the isolated system and compared to the result from the real-time dynamics. In no scenario the result, found by the mapping can be confirmed.

Contents

1 Chapter 1

Universality in equilibrium near a quantum critical point

Condensed matter systems can undergo a transition between different phases. Such a transition is driven via tuning of a parameter e. g. by changing temperature, applying pressure or varying a magnetic field. Already Paul Ehrenfest classified two common types of phase transitions, first and second order. A first-order phase transition is characterized by a discontinuity in the first derivative of the free energy F with respect to some thermodynamic variable, e. g. the pressure. A second-order phase transition according Ehrenfest is characterized by a discontinuity in the second derivative of F. Modern classification schemes use the same nomenclature, but they distinguish between phase transitions with latent heat and continuous phase transitions [35]. An example for a first-order phase transition is the melting of ice. At the transition temperature, both phases, the solid and the liquid, coexist. As a consequence, the correlation length ξ stays finite in the system. Passing through the phase transition, the system either releases or absorbs energy, leading to a discontinuity in the entropy, i. e. the latent heat. In contrast, the second-order phase transition is continuous. Examples are magnetic, superfluid and superconducting transitions. Those systems are characterized by the continuous emergence of a symmetry-breaking order parameter. At the transition point, the system is in one phase, with infinite correlation length and zero order parameter. Near the phase transition, the system is characterized by strong correlations and large fluctuations. If the transition is controlled by thermal fluctuation, one refers to a classical critical point. In many cases, the temperature is the controlling parameter that drives the system through the critical point. At temperature $T = 0$, those thermal fluctuations freeze out. The system is now characterized uniquely by quantum fluctuations, originating from $\left[\boldsymbol{\varphi}, H\right] \neq 0$, where $\boldsymbol{\varphi}$ is the order parameter. They can also lead to a second-order phase transition. The parameter configuration, separating two such phases, is called QCP. It corresponds to a singularity in the ground state. Control parameters for driving the system from the symmetric into the symmetry broken phase are for example doping, applying pressure or electrical fields.

The divergence of the correlation length has a fundamental consequence, which makes systems in the vicinity of a critical point especially interesting: It implies that those systems are scale invariant, i. e. specific microscopic length scales as the lattice constant and other material parameters are insignificant. This leads to a very particular behavior of many thermodynamic variables. Their growth (or decay) with respect to the distance to the critical point can be described with universal exponents. For example the heat capacity grows with an universal exponent near the classical transition point. This universality in combination with the long range order of the fluctuations and the correlation length

ξ allows to describe critical systems with a phenomenological field theory, the φ^4-theory. Here, $\boldsymbol{\varphi}(x)$ are local fields, where the lowest order terms of the action in a gradient expansion are kept. By this approximation, only the long wavelength fluctuations, important for the phase transition are included. This leads to a φ^4-interaction term. Note that the long-range character of the fluctuation and the strong correlations prevent a simple perturbation expansion. With two different methods, the RG and the large-N expansion, one can include this φ^4-interaction term to obtain the critical exponents beyond a simple mean-field expansion. In this thesis, the post-quench dynamics of a φ^4 model will be analyzed. There exist also higher order phase transitions, which are continuous, but have no symmetry breaking order parameter, like the Berezinskii-Kosterlitz-Thouless (BKT)-transition. Those kinds of phase transitions will not be considered in this thesis.

This chapter is organized as follows: In the first section the φ^4-model and the Hamiltonian is introduced. For later convenience, two different kinds of systems are introduced: the closed (or isolated) system and the open system which is coupled to an external bath. In section 1.2 the spirit of scaling is presented. In section 1.3 and 1.4 two different methods for analyzing a system near a critical point are presented. In the last section the classical-quantum mapping is summarized, where a quantum, d-dimensional problem is mapped on a $(d+z)$-dimensional classical problem. Where z is the dynamical exponent, introduced in section 1.2. For example, for $z = 1$ the time is treated like one additional dimension. This method is well established in thermal equilibrium. We will discuss to what extend it can also be used for the description of the out-of-equilibrium dynamics.

1.1 The φ^4 model

To describe critical phenomena, the φ^4 model is commonly used. Here, $\boldsymbol{\varphi}(x) = \big(\varphi_1(x), \ldots, \varphi_N(x)\big)$ are N component continuous scalar fields. The Hamiltonian reads

$$H_s = \frac{1}{2}\int_x \left(\boldsymbol{\pi}^2 + \left(\nabla\boldsymbol{\varphi}\right)^2 + r_0\boldsymbol{\varphi}^2 + \frac{u(\boldsymbol{\varphi}\cdot\boldsymbol{\varphi})^2}{2N} - \mathbf{h}\cdot\boldsymbol{\varphi}\right). \tag{1.1}$$

Here, r_0 is the bare mass term and u the interaction parameter. The field $\mathbf{h}$ is an external field which couples to the order parameter. The vector-field $\boldsymbol{\pi}(x)$ is the canonically conjugated momentum to $\boldsymbol{\varphi}(x)$. They obey the commutator relation

$$\left[\varphi_l(x), \pi_{l'}\left(x'\right)\right]_- = \mathrm{i}\,\delta_{ll'}\delta\left(x - x'\right). \tag{1.2}$$

In this thesis, the reduced Planck constant $\hbar$ is set equal to one. The Hamiltonian H_s alone describes closed systems as e. g. cold atom systems [1, 2] where the quench protocol can be easily performed. Condensed matter systems are however generally in contact with an environment, e. g. via phonon coupling. Examples for experimental realizations of quantum phase transitions described via the φ^4 model are superfluid-insulator transitions in the Bose-Hubbard model, experimentally realized with cold atom gases [36]. Here, both cases can be studied, the isolated and the diffusive system. A bath coupling can be realized by considering two different species of atoms, where one acts like the heat bath. Another example are dissipative nanowires near the transition point to the superconducting state [37]. See also Ref. [31] for the experimental realization in the quantum dimer antiferromagnet $TlCuCl_3$, which can be driven through a quantum phase transition via changing the pressure [38] or an external magnetic field [39]. Such an environment can be included into the model by adding an external heat bath. The full Hamiltonian of such an open system consists of three parts:

$$H = H_s + H_b + H_{sb}, \tag{1.3}$$

where

$$\begin{aligned} H_b =& \frac{1}{2}\sum_j \int_x (\mathbf{P}_j^2 + \Omega_j^2 \mathbf{X}_j^2), \\ H_{sb} =& \sum_j c_j \int_x \mathbf{X}_j \cdot \boldsymbol{\varphi}. \end{aligned} \tag{1.4}$$

The part H_b describes the bath of harmonic oscillators. A single bath mode j is characterized by the canonical momentum operator $\mathbf{P}_j$, the position operator $\mathbf{X}_j$ and the frequency Ω_j. The operators $\mathbf{X}_i$ and $\mathbf{P}_j$ fulfill the Heisenberg-commutator relation:

$$\left[\boldsymbol{X}_i, \boldsymbol{P}_j\right] = \mathrm{i}\,\delta_{ij}. \tag{1.5}$$

Operators from different modes commute. Further, they commute with the system operators $\boldsymbol{\varphi}$ and $\boldsymbol{\pi}$.

The part of the Hamiltonian H_{sb} describes the coupling between the system and the bath, where c_j is the coupling between mode j and system. It will be shown explicitly in section 3.2 that an effective description of the bath can be achieved with the retarded bath Green's function η in Fourier space:

$$\eta(\omega) = -\sum_j \frac{c_j^2}{(\omega + \mathrm{i}\,0^+)^2 - \Omega_j^2}\,. \tag{1.6}$$

In this thesis, we assume that the bath is characterized by the low frequency dependence of the imaginary part of η, and by the exponent α:

$$\operatorname{Im}\eta(\omega) = \gamma\omega|\omega|^{\alpha-1}e^{-|\omega|/\omega_c}. \tag{1.7}$$

Here, the cutoff ω_c is used to control high frequencies [40]. The damping parameter γ controls the coupling between the system and the bath. For $\alpha = 1$, Eq. (1.7) describes an Ohmic bath, for $\alpha > 1$ a super-Ohmic and for $\alpha < 1$ a sub-Ohmic bath [40]. The real part of η can be obtained via Kramers-Kronig relation,

$$\begin{aligned} \delta\eta\left(\omega\right) &:= \eta\left(\omega\right) - \eta\left(0\right), \\ &= \gamma\left(-\cot\left(\frac{\pi\alpha}{2}\right) + \mathrm{i}\,\mathrm{sign}\left(\omega\right)\right)\left|\omega\right|^{\alpha}. \end{aligned} \tag{1.8}$$

The zero-frequency value $\eta\left(0\right) = \gamma\alpha\omega_c^{\alpha}/(2\pi)$ depends explicitly on the bath cutoff ω_c and the chosen cutoff procedure. However, as we will see, it merely shifts the bare mass r_0, and thus the the non-universal location of the critical point. It does not affect the critical exponents which are independent of specific bath configurations γ, ω_c and the regularization scheme.
Critical exponents do however depend on the choice for the bath exponent α. In chapter 3, the

Table 1.1: *Nomenclature of some critical exponents*

Correlation length	$\xi \sim \delta r^{-\nu}$
Susceptibility	$\chi \sim \delta r^{-\gamma}$
Order parameter	$\langle \varphi(\delta r, h=0) \rangle \sim \|\delta r\|^{\beta}$ $\langle \varphi(\delta r=0, h) \rangle \sim \|h\|^{1/\delta}$
Correlation function	$G(k) \sim k^{d-z-\eta}$

analytic continuation to the Matsubara axis is needed, to include the effects of finite temperature. For completeness it is given here:

$$\delta\eta^{M}(\omega_n) = -\frac{\gamma}{\sin\frac{\pi\alpha}{2}} |\omega_n|^{\alpha}. \tag{1.9}$$

It can be obtained via Kramers-Kronig $\omega \to \mathrm{i}\,\omega_n$ and is valid for frequencies that are small compared to the bath cut off ω_c.

Note, that all three kinds of baths are at temperature $T = 0$ non-Markovian, while in the classical limit fluctuations of the Ohmic bath are described in terms of white noise.

1.2 Universality, scaling and critical exponents

The existence of only one diverging length scale leads to a power law behavior of physical parameters, as for example the susceptibility, the order parameter and the correlation length [35]. In table 1.1, the notation of some of those exponents is presented. Here, δr refers to the distance to the QCP and h is an external field, which couples to the order-parameter $\langle \varphi \rangle$. To avoid problems with dimensionality, the variables in the table should be understood as dimensionless quantities. In general, one has to distinguish between the exponents for $\delta r > 0$, where the system is located in the symmetric phase and exponents for $\delta r < 0$, where the system is in the symmetric-broken phase. In cases of interest in this thesis, the exponents are the same in both phases, therefore this distinction is not made here. Only the non-universal proportionality constant will depend on the phase.

Experimentally, it turns out that the value of the respective exponents can be identical, even if the underlying phase transition is of different nature. For example, near a classical critical point in three dimensions, the specific heat exponent is found to be near 0.1 for the liquid-gas transition as well as for the easy axis antiferromagnetic transition. This again is a consequence of the scale invariance, where no typical length scale exists due to the divergence of ξ. Depending on the value of those exponents, the phase transition can be classified in different universality classes. In particular, the universality class depends on: the dimension d of the system, the number of components N of the order-parameter field and, for quantum systems, the dynamic exponent z. The dynamic exponent connects time and length under scaling, especially the correlation time

$$\tau \propto \delta r^{-z\nu} \propto \xi^{z}. \tag{1.10}$$

It turns out, that in addition to z, only two of this set of critical exponents are truly independent, implying the existence of scaling laws between different exponents. Those scaling laws can be derived by using thermodynamic relations and very few assumptions about the free energy F and the correlation function G. One assumption is, that the correlation function can be expressed by using a scaling function at $T = 0$:

$$G(k, \delta r, \omega) = \frac{1}{k^{2-\eta}} f_g(bk, b^y \delta r, b^{-z}\omega). \tag{1.11}$$

Here, b is some positive parameter which can be chosen freely. The exponents η and y are universal. In the literature, η is also referred to anomalous dimension. The parameter y is the scaling exponent of the distance to the QCP, determined below. The function f_g is also called the scaling function of G. The scaling form of Eq. (1.11) can be motivated from the Ginzburg-Landau theory or from the mean-field expansion of the φ^4-model of section 1.1, where the correlation function reads

$$\begin{aligned} G(k, \omega) =& \langle \varphi(k, \omega)\varphi(-k, -\omega)\rangle \\ =& \frac{1}{\delta r + k^2 + c^2\omega^{2/z}}. \end{aligned} \tag{1.12}$$

Now all lengths and times are rescaled by $x \to bx$ and $t \to b^{-z}t$. This corresponds to rescale the momentum k and the frequency ω according to $k \to k/b$, $\omega \to b^z\omega$. After the rescaling procedure, the correlation function reads:

$$G = b^2 f(bk, b^2\delta r, b^{-z}\omega). \tag{1.13}$$

Comparing Eq. (1.11) with Eq. (1.13) yields the mean-field values $\eta = 0$ and $y = 2$. Further, a connection between y and ν can be made. For this b is fixed to $b = \delta r^{-1/y}$, such that the second argument of the scaling function in Eq. (1.11) is equal to one. Using that the scaling function depends on a product of $k\xi$, one obtains

$$\nu = \frac{1}{y}. \tag{1.14}$$

With the mean-field value y obtained above, it holds $\nu = 1/2$.
To demonstrate the power of using scaling functions, one scaling law is derived explicitly:

$$(d + z)\nu = 2\beta + \gamma\,. \tag{1.15}$$

This law can be obtained by assuming a similar scaling form for the free energy, like in Eq. (1.11),

$$F(\delta r, h) = b^{-d-z} F\left(b^{1/\nu}\delta r, h\delta r^{-y_h}\right). \tag{1.16}$$

The prefactor b^{-d} originates from the fact, that the free energy is an extensive quantity, and thus has to scale with the volume. The prefactor b^{-z} is a consequence of the system being in equilibrium, i. e. the time independence. Thus this prefactor emerges if time or frequency are rescaled. This is not the case for classical systems, where per construction only the zeroth Matsubara mode is kept. Further, $y = 1/\nu$ is used, for rescaling the mass term. For rescaling the field h, the exponent y_h is introduced. Setting $b^{1/\nu}t = 1$, one finds

$$F(\delta r, h) = \delta r^{\nu(d+z)} F\left(1, h\delta r^{-\nu y_h}\right). \tag{1.17}$$

The order-parameter can be obtained via the first derivative of F with respect to the external field,

$$\begin{aligned}\langle\varphi\rangle &= \lim_{h\to 0}\frac{\partial F}{\partial h}\\ &=\delta r^{\nu(d+z)-\nu y_h}\partial_x F(1,x)\,.\end{aligned} \tag{1.18}$$

By comparing this with $\langle\varphi\rangle \propto \delta r^{\beta}$ in table 1.1, the scaling relation for β can be derived. Similar, considering the second derivative of F with respect to h yields for the susceptibility exponent γ:

$$\gamma = \nu(2-\eta). \tag{1.19}$$

The critical exponents have to fulfill simultaneously:

$$(d+z)\nu = \quad 2\beta+\gamma \tag{1.20a}$$

$$\gamma = \quad \nu(2-\eta). \tag{1.20b}$$

With those three relations, it can be shown that indeed only two of the five exponents in table 1.1 are independent. Those cannot be determined within a scaling analysis, but by applying a concrete model, for example a Landau analysis, if the Ginzburg-Landau criterion is fulfilled. A scaling law containing the dimension d is also called hyper-scaling. Within a mean-field calculation, the hyper-scaling laws are only fulfilled for $d < 4 - z$. The reason for this will become clear in section 1.4.

In the next chapter, the scaling of the order-parameter is of particular importance for our analysis. In equilibrium, the order parameter can be expressed by the following scaling function

$$\phi_{\text{eq}}\left(\delta r, h\right) = b^{-\beta/\nu}\phi_{\text{eq}}\left(b^{1/\nu}\delta r, b^{\beta\delta/\nu}h\right). \tag{1.21}$$

By setting $\delta r = 0$ and $b = h^{-\nu/(\beta\delta)}$, it can be checked that $\phi_{\text{eq}}\left(h\right) \propto h^{1/\delta}$. Alternatively for $h = 0$ and $b = \delta r^{\nu}$, that this scaling form indeed obeys the scaling relation given in table 1.1.

1.3 Renormalization group

There exist many different renormalization schemes. In this thesis the Kadanoff-Wilson scheme, also called the momentum shell RG is used [35, 41]. The basic idea of this renormalization scheme is to take advantage of the scale invariance at the critical point. Scale invariance at the critical point implies that the action remains the same under rescalation of all lengths x by some factor $1/b$. Instead of expressing the order-parameter field $\varphi(\boldsymbol{x}, t)$ in space, it is commonly Fourier-transformed to momentum space $\boldsymbol{k}$ with momentum cutoff Λ. The momentum shell RG analyzes how the parameter set (r, u) changes under the following procedure: The first step consists in integrating out modes with momentum $\boldsymbol{k}$ in a small shell with thickness Λ/b near the momentum cutoff Λ. The remaining modes have a new cutoff Λ/b. In a second step, every momentum $\boldsymbol{k} \to b\boldsymbol{k}$ is rescaled such that the action has again the old cutoff Λ but with modified parameters r' and u'. For infinitesimal small shells, or $b \sim 1 + l$ with $0 < l \ll 1$, one obtains differential equations for the parameter flow of $(r(l), u(l))$. The fixed point is reached, if (r, u) remains unchanged under the RG procedure. The action is thus self-similar under scaling. This corresponds to the system being right at the critical point. The flow equations describe how the system reacts near the fixed point. A second-order phase transition has two eigenvectors in the (r, u) plane, one pointing to the fixed point, one pointing away. The corresponding eigenvalues determine the universal exponents. It turns out, that the exponents are strongly sensitive to the dimension d and the dynamic exponent z. In the following, this procedure is presented in detail near the QCP at temperature $T = 0$.

1.3.1 RG equations at zero order

The first step in the RG is to split φ in slow(<) and fast(>) modes

$$\varphi_l(\boldsymbol{k},t) = \begin{cases} \varphi_l^<(\boldsymbol{k},t) & \text{for } |\boldsymbol{k}| < \Lambda/b, \\ \varphi_l^>(\boldsymbol{k},t) & \text{for } \Lambda/b \leq |\boldsymbol{k}| < \Lambda. \end{cases} \tag{1.22}$$

Inserting φ into the action S, one can split S into three parts. One part containing only slow fields $S^<(\varphi^<)$, one only fast fields $S^>(\varphi^>)$ and a mixing term $\delta S(\varphi^<,\varphi^>)$. At the lowest order in a perturbation in δS, one ignores the mixing term. This expansion is justified for a small coupling constant u, which will be shown below. The remaining integral over the fast fields is Gaussian, thus the fast fields can be integrated out. It remains the action for the slow modes,

$$\begin{aligned} S_0^< = & \int_k^{\Lambda/b}\int_\omega \varphi_i^<(-\boldsymbol{k},\omega) G_0^{-1}(\boldsymbol{k},\omega)\varphi_i^<(\boldsymbol{k},\omega) \\ & + \frac{u}{2N}\int_{k_1}^{\Lambda/b}\int_{\omega_1}\dots\int_{k_4}^{\Lambda/b}\int_{\omega_4}\delta\left(\boldsymbol{k_1}+\boldsymbol{k_2}+\boldsymbol{k_3}+\boldsymbol{k_4}\right)\delta\left(\omega_1+\omega_2+\omega_3+\omega_4\right) \\ & \quad \times \varphi_i^<(\boldsymbol{k_1},\omega_1)\varphi_i^<(\boldsymbol{k_2},\omega_2)\varphi_j^<(\boldsymbol{k_3},\omega_3)\varphi_j^<(\boldsymbol{k_4},\omega_4), \end{aligned} \tag{1.23}$$

where the sum is taken over $i,j=1\dots N$. Now, momentum and frequencies are rescaled, such that the old cutoff Λ is reobtained. By this procedure, the parameter r, u and γ become b-dependent. Momentum and frequency k and ω are rescaled according to

$$k \to k' = bk, \tag{1.24}$$

$$\omega \to \omega' = b^z\omega. \tag{1.25}$$

For a closed system, the dynamic exponent reads $z=1$. For an open system, z can be determined by analyzing the two different dynamic terms in the Green's function G_0:

$$G_0^{-1}(k,\omega) = c^{-2}\omega^2 + k^2 + r + \mathrm{i}\gamma\omega|\omega|^{\alpha-1} - \gamma\cot\left(\frac{\pi\alpha}{2}\right)|\omega|^\alpha. \tag{1.26}$$

For clarity, the velocity c is explicitly written in front of the ballistic term, in this section. For the rest of the thesis, c is set equal to one. Rescaling k and ω like in Eqs. (1.24), (1.25) and pulling b^{-2} out of G^{-1} yields

$$G_0^{-1}(k',\omega') = b^{-2}\left(b^{2-2z}c^{-2}\omega'^2 + k'^2 + b^2 r + b^{2-z\alpha}\mathrm{i}\gamma\omega'|\omega'|^{\alpha-1} - b^{2-z\alpha}\gamma\cot\left(\frac{\pi\alpha}{2}\right)|\omega'|^\alpha\right). \tag{1.27}$$

Since $b>1$, the ballistic term grows stronger than the dynamic term for $\alpha>2$. In this case, it is convenient to choose $z=1$ and consider only the ballistic part, since $\gamma(b)=b^{2-z\alpha}\gamma$ flows to zero. In the other case, $\alpha<2$, one chooses $z=2/\alpha$ such that γ remains unchanged under RG and the inverse velocity c^{-1} flows to zero. For $\alpha=2$, there are logarithmic divergent terms, dominating the ballistic part $c^{-2}\omega^2$. For $\alpha\geq 2$ the ballistic term is irrelevant and the dynamics are dominated by the coupling to the bath.

Going back to the action, the fields need to be rescaled:

$$\varphi(k) \to \varphi'(k') = b^{-\rho}\varphi(k). \tag{1.28}$$

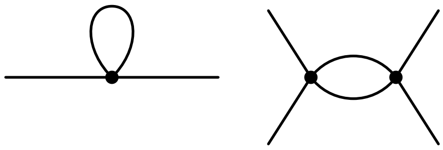


Figure 1.1: *One loop corrections for r (left) and u (right).*

The scaling-exponent ρ of the fields will be determined below. Putting everything together, the action reads after one rescaling procedure

$$\begin{aligned} S_0' =& b^{d+z-2\rho-2}\int_{k'}^{\Lambda}\int_{\omega'}\varphi_i'(-\boldsymbol{k}',\omega')G_0^{-1}(\boldsymbol{k}',\omega';r(b),c(b),\gamma(b))\varphi_i'(\boldsymbol{k}',\omega') \\ &+\frac{b^{4d+4z-4\rho}u}{2N}\int_{k_1'}^{\Lambda}\int_{\omega_1'}\dots\int_{k_4'}^{\Lambda}\int_{\omega_4'}\delta\left(\boldsymbol{k_1}'+\boldsymbol{k_2}'+\boldsymbol{k_3'}+\boldsymbol{k_4'}\right)\delta\left(\omega_1'+\omega_2'+\omega_3'+\omega_4'\right) \\ &\times\varphi_i'(\boldsymbol{k_1}',\omega_1')\varphi_i'(\boldsymbol{k_2}',\omega_2')\varphi_j'(\boldsymbol{k_3}',\omega_3')\varphi_j'(\boldsymbol{k_4}',\omega_4'). \end{aligned} \tag{1.29}$$

At the fixed point it holds $S_0' = S$. This implies that the coefficient in front of the quadratic term must vanish. This yields for the field exponent

$$\rho = \frac{d+z-2}{2}. \tag{1.30}$$

By substituting $b = 1 + l$, where l is a positive and small parameter, one can exponentiate $r(l)$, $u(l)$, $c^{-1}(l)$, $\gamma(l)$ and the flow equations for the system parameters read

$$\frac{dr}{dl} =2r, \tag{1.31}$$

$$\frac{du}{dl} =(4-d-z)u, \tag{1.32}$$

$$\frac{dc^{-1}}{dl} =(1-z)c^{-1}, \tag{1.33}$$

$$\frac{d\gamma}{dl} =(2-z\alpha)\gamma. \tag{1.34}$$

Under the RG procedure, the mass $r(l)$ will grow exponentially. For $d > 4-z$, the interaction parameter u flows to zero, making this mean-field description possible. For $d < 4-z$ however, interactions are getting more and more important under the RG procedure. Here, the assumption of neglecting the mixing term δS is no longer valid. The dimension $d_{uc} = 4-z$ is called the upper critical dimension, the parameter $\epsilon = 4-z-d$ measures the distance to d_{uc}. In the Kadanoff-Wilson RG, ϵ is the small parameter controlling the perturbative expansion.

1.3.2 RG equations including first order corrections

The mixing term δS can be included in a cumulant expansion. At lowest order this yields

$$\int\mathcal{D}\varphi^{>}e^{-\delta S(\varphi^{>},\varphi^{<})}\sim 1-\langle\delta S\rangle^{>}+\frac{1}{2}\left(\langle\delta S^2\rangle^{>}-\langle\delta S\rangle^{>2}\right), \tag{1.35}$$

where $\langle\rangle^{>}$ denotes the average over the fast modes. Further, starting with the RG flow in the symmetric phase, δS contains only terms that are quadratic in $\varphi^{>}$ and $\varphi^{<}$. If the RG starts in the ordered phase or for a finite external field h, also terms with odd exponents in $\varphi^{>}, \varphi^{<}$ can occur, but after averaging over fast modes, they will modify the RG flow of the external field and not the mass and the interaction parameter. The diagrammatic pictures are given in figure 1.1. The first diagram corresponds to $\langle\delta S\rangle^{>}$ in Eq. (1.35) and thus to a fluctuation of the slower fields. Physically, the second process in figure 1.1 corresponds to two slow fields, which can be excited by scattering processes for some short time to two fast modes, before relaxing again. Analyzing $\langle\delta S\rangle^{>}$ leads to a correction of the mass r. Here, one has to sum over all possible combinations for k_i. For a N-component vector field, there are $2(N+2)$ possible combinations. Further, one uses

$$\langle\varphi_i(k_1)\varphi_j(k_2)\rangle = \delta_{ij}\delta(k_1 - k_2)G(k_1). \tag{1.36}$$

This yields for the average of the fast modes

$$\langle\delta S\rangle^{>} = \frac{u}{4N}2(N+2)\int \mathrm{d}^d k_1 \ldots \int \mathrm{d}^d k_4 \langle\varphi^{>}(k_1)\phi^{>}(k_2)\rangle\varphi_i^{<}(k)\varphi_i^{<}(k). \tag{1.37}$$

The quadratic term can be simplified in the same manner. Note that all non-connected diagrams drop out due to the $-\langle\delta S\rangle^{>2}$, such that only diagrams of the type in figure 1.1 remain. Here, there are $8(N+8)$ possibilities to permute the eight different arguments. This yields:

$$\begin{aligned}\frac{1}{2}\left(\langle\delta S^2\rangle^{>} - \langle\delta S\rangle^{>2}\right) =& \frac{u^2}{16N^2}8(N+8)\int_k^{>}\int \mathrm{d}\omega\, G(k,\omega)G(-k,\omega) \\ & \times \varphi_i^{<}(\boldsymbol{k_1}',\omega_1')\varphi_i'(\boldsymbol{k_2}',\omega_2')\varphi_j'(\boldsymbol{k_3}',\omega_3')\varphi_j'(\boldsymbol{k_4}',\omega_4').\end{aligned} \tag{1.38}$$

From this one can read off the flow equations at the one loop level:

$$\frac{dr}{dl} = 2r + \big(N+2\big)\, uK_d \int_k^{>} \mathrm{d}k\, k^{d-1} \int \mathrm{d}\omega\, G(k,\omega)/l\,, \tag{1.39}$$

$$\frac{du}{dl} = \epsilon u - u^2\,\big(N+8\big)\,K_d \int_k^{>} \mathrm{d}k \int \mathrm{d}\omega\, G^2(k,\omega)/l\,. \tag{1.40}$$

Above the upper critical dimension, only one fixed point (r,u) exists: the Gaussian fixed point $(0,0)$, which is stable with respect to u. Below the critical dimension, this fixed point is unstable with respect to u, but a second fixed point at (r^*, u^*) emerges, with

$$r^* = -\frac{\epsilon}{2}\frac{N+2}{N+8}\Lambda^{2-\epsilon}, \tag{1.41}$$

$$u^* = \frac{1}{N+8}\frac{\epsilon\Lambda^{\epsilon}}{K_d}, \tag{1.42}$$

for a closed system. For $z \neq 1$, i. e. an open system, it holds:

$$r^* = -\frac{2(N+2)\epsilon}{(N+8)(2-z)}\Lambda^{2-\epsilon}, \tag{1.43}$$

$$u^* = \frac{4\epsilon\Lambda^{\epsilon}\gamma^{-2/z}}{(N+8)z(2-z)K_d}\left[\cos\left(\frac{\pi}{z}\right)\right]^{1-2/z}. \tag{1.44}$$

These fixed points are called Heisenberg fixed points. The values for (r^*, u^*) are obtained up to first order in an expansion for small ϵ. This fixed point is stable along the direction of one eigenvector $\boldsymbol{e_2}$ with eigenvalue $e_2 < 0$, but unstable with respect to the eigenvector $\boldsymbol{e_1}$ with eigenvalue $e_1 > 0$. This implies that small derivations from $\boldsymbol{e_2}$ grow exponentially under the RG flow with e_1. Thus, e_1 and e_2 determine the universal exponents. By linearization of the flow equations around the Heisenberg fixed point, one finds

$$\eta = 0 + \mathcal{O}\left(\epsilon^2\right), \tag{1.45}$$

$$\nu = \frac{1}{2} + \frac{N+2}{N+8}\frac{\epsilon}{8}. \tag{1.46}$$

The remaining universal exponents can be obtained by using scaling laws.

1.4 Large-N equation

This method is also called the self-consistent field, Hartree, or random phase approximation [35]. The basic idea is to replace one scalar product $(\boldsymbol{\phi}\cdot\boldsymbol{\phi})$ in the interaction term $u(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2$ by its expectation value $\langle\boldsymbol{\phi}\cdot\boldsymbol{\phi}\rangle$, and to determine this expectation value self-consistently. This yields to an effective mass $r_0 \to r$ with:

$$r = r_0 + u\int \mathrm{d}^d k \int \mathrm{d}\omega \frac{1}{\omega^2 + r + k^2 + \delta\eta^M(\mathrm{i}\omega)}. \tag{1.47}$$

Before deriving this equation in more detail, note that already here one can read off the role of the dimension and the correlation length exponent ν. The phase transition takes place at $r = 0$, resulting in the critical value for the bare mass:

$$r_0^* = -u\int \mathrm{d}^d k \int \mathrm{d}\omega \frac{1}{\omega^2 + k^2 + \delta\eta^M(\mathrm{i}\omega)}. \tag{1.48}$$

Below two dimensions, the integral is divergent at the lower boundary. This implies that fluctuations prevent a phase transition, see also the Mermin-Wagner theorem [42].
With the self-consistent equation, the correlation length $\xi^{-2} = r$ can be expressed as distance to the critical point,

$$\begin{aligned}\xi^{-2} =& r - r_0^* + uK_d\int_0^{\Lambda}\mathrm{d}k\int \mathrm{d}\omega\, k^{d-1}\left(\frac{1}{\omega^2+\xi^{-2}+k^2+\delta\eta^M(\mathrm{i}\omega)} - \frac{1}{\omega^2+k^2+\delta\eta^M(\mathrm{i}\omega)}\right)\\ =& r - r_0^* + \xi^{-2}uK_d\int_0^{\Lambda}\mathrm{d}k\int \mathrm{d}\omega\,\frac{k^{d-1}}{(\omega^2+\xi^{-2}+k^2+\delta\eta^M(\mathrm{i}\omega))(\omega^2+k^2+\delta\eta^M(\mathrm{i}\omega))}.\end{aligned} \tag{1.49}$$

The occurring integral is convergent for $\xi^{-2} \to 0$ in the denominator above $d_{uc} = 4 - z$ dimensions. In this case, it can be shown that the interaction term is small, and $\xi^{-2} \simeq r - r^*$. Thus above the upper critical dimension, the mean-field exponent $\nu = -1/2$ is reobtained. Below d_{uc} the integral is divergent for $\xi^{-2} \to 0$ or $r \to 0$ in the nominator. However, if the system is either ballistic or purely dissipative, it can be evaluated with Γ-functions. Neglecting the first term $r - r_0^*$ in Eq. (1.49) yields

$$\xi^{-1} \propto (r - r^*)^{-\frac{1}{d-2+z}}. \tag{1.50}$$

At $d = 4 - z$, the mean-field result is reproduced again, but with additional logarithmic corrections. Below d_{uc} fluctuations modify the critical exponents.
In the remaining section it will be shown, that the Hartree approximation is exact in the limit of an infinite number of components N. Therefore, the interaction term is decoupled via a Hubbard-Stratonovich field λ:

$$e^{-u(\boldsymbol{\varphi}\cdot\boldsymbol{\varphi})^2/(4N)} \propto \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathcal{D}\lambda e^{+Nu^2\lambda^2-\lambda\boldsymbol{\varphi}\cdot\boldsymbol{\varphi}}. \tag{1.51}$$

Here, the normalization constant is skipped. The remaining action is Gaussian in $\boldsymbol{\varphi}$, such that the order-parameter field can be integrated out, with an effective mass $r = r_0 + \lambda$,

$$\begin{aligned} Z &\propto \int \mathcal{D}\varphi_i \mathcal{D}\lambda \exp\left(-Nu^2 \int \mathrm{d}\omega \int \mathrm{d}x \lambda^2(x,\omega)\right) \\ &\quad \times \exp\left(-\int \mathrm{d}\omega \int \mathrm{d}^d x \left(r_0 + k^2 + \omega^2 + \eta(\omega) + \lambda\right) \boldsymbol{\varphi}\cdot\boldsymbol{\varphi}\right) \\ &\propto \int \mathcal{D}\lambda \exp\left(-Nu^2 \int \mathrm{d}\omega \int \mathrm{d}x \lambda^2(x,\omega) - N\log(\langle G(\omega, r_0+\lambda)\rangle)\right). \end{aligned} \tag{1.52}$$

In the limit $N \to \infty$ the integral can be evaluated by using the saddle point approximation. This gives the self-consistent equation:

$$\lambda = 4u \int \mathrm{d}^d k \frac{1}{\omega^2 + k^2 + \lambda + \eta(\omega)}. \tag{1.53}$$

By identifying the Hubbard-Stratonovich-field λ with the effective mass r, the same equation as in 1.47 is found, which has to be solved self-consistently. The $1/N$-expansion is a complementary ansatz to RG, valid near the upper critical dimension, but for arbitrary N.

1.5 Quantum-classical mapping

With both methods presented in this chapter, the RG and the $1/N$ expansion, it is possible to solve classical as well as quantum system for the simple φ^4-model. There exists a relationship between quantum and classical critical systems, which is well established in equilibrium: the quantum classical mapping or Euclidean mapping. It is based on the simple observation that one can replace the dimension d in a classical system by $(d+z)$, to obtain the correct values for the critical exponents of the quantum theory. This correspondence was already noticed in the 1940s by Feynman and shown explicitly for example for the Ising model in Ref. [43]. It can also be seen in an explicit path-integral formulation. Performing the $T \to 0$ limit in the Matsubara sum in the action, leads to an integration over frequencies. The basic idea of the mapping is to treat the frequency integration as supplementary z-dimensional integral over space, such that one obtains a $d+z$-dimensional classical theory [44]. In the real-time path-integral formalism, presented in chapter 3, this corresponds to analytic continuation from times t to imaginary times $\tau = \mathrm{i}t$. This path-integral approach is done in detail in section 6.3 for the post-quench dynamics. At this point, the analysis is therefore kept on the simple observation, that the basic integral in the $1/N$ expansion and in the first order correction of the mass in the RG flow,

$$\int \mathrm{d}^d k \int \mathrm{d}\omega\, G(q,\omega) = \int \mathrm{d}^d q \int \mathrm{d}\omega \, \frac{1}{k^2 + r + |\omega|^{2/z} f(z)}, \tag{1.54}$$

can indeed be evaluated by substituting $k' = |\omega|^{1/z}$ and $zk'^{z-1}dk' = d\omega$. Here, the function f was introduced to treat the isolated system with $z = 1$ and

$$f(1) = c^{-2}, \tag{1.55}$$

at the same time, with $z > 1$, where

$$f(z) = \gamma. \tag{1.56}$$

With the substitution one obtains:

$$\begin{aligned} \int \mathrm{d}^d k \int \mathrm{d}\omega\, G(q,\omega) &\propto \int \mathrm{d}k\, k^{d-1} z k'^{z-1} \mathrm{d}k' \frac{1}{k^2 + r + k'^2} \\ &= \int \mathrm{d}^{d+z} k \frac{1}{k^2 + r}\,. \end{aligned} \tag{1.57}$$

The Green's function in the last line is nothing else, than the Green's function of the classical $(d+z)$-dimensional problem. The same argumentation can be applied to the first order correction of u in the RG.

As it was pointed out in Ref. [44], there are already for the equilibrium situation many reasons why this quantum-classical mapping should be applied with care and one also needs a separate theory for the full quantum problem. Some concerns are:

- The geometry of the classical theory with $z \neq 1$, when one space direction scales differently, may be quiet special and artificial.
- A further problem can be, that the quantum phenomena have no classical analogy. An example in this thesis is the prominent role of boundary conditions and Heisenberg-commutator relations of the quantum system, which are not present in the high-temperature limit, see section 3.5.
- Sometimes, the back-transformation from imaginary to real time causes some problems if the dynamics and transport quantities are analyzed.

Thus, even if the mapping appears very simple and natural, it should be explicitly checked. Nevertheless, at least for the φ^4-model in equilibrium, this mapping works and leads to a great simplification, as for example the value of the universal exponents can be directly obtained by substituting d by $d+z$. As it was shown in Refs. [21, 22], this mapping can also lead to the correct results for $z \neq 1$ in more complex systems. It is therefore of great use, to check if such a mapping is possible, as it is very simple and instructive.

2 Chapter 2

Scaling after a quench towards the quantum critical point

While the first chapter concentrated solely on equilibrium criticality, the goal of this chapter is to perform a scaling analysis far from equilibrium. Scaling out-of-equilibrium is well known in the Kibble-Zurek limit, where a system is driven slowly through a QCP [11, 12]. Here, scaling after a complementary out-of-equilibrium protocol is analyzed, the quantum quench. Experimentally, after a quantum quench the system can reach states that are not accessible in equilibrium [3]. The dynamics cannot be captured within the Kibble-Zurek approach [16], but also non-equilibrium scaling has been observed recently in cold atom systems [45].

To perform a quantum quench, the system is prepared in the ground state $|\boldsymbol{\varphi}_{\mathrm{i}}\rangle$ of the initial Hamiltonian H_{i}. Here H_{i} is the Hamiltonian of a φ^4-model, introduced in section 1.1, given by the parameter configuration $R_{\mathrm{i}} = (r_{\mathrm{i}}, h_{\mathrm{i}}, u_{\mathrm{i}})$. The parameter r_{i} stands for the initial mass, h_{i} for the initial, external field and u_{i} is the initial interaction parameter. By instantly switching some parameters, e. g. applying a magnetic field or pressure, at time $t = 0$, the time evolution after the quench is governed by a different Hamiltonian H, with a new parameter configuration $R = (r_0, h, u)$, such that

$$|\boldsymbol{\varphi}(t)\rangle = e^{-\mathrm{i}Ht}|\boldsymbol{\varphi}_{\mathrm{i}}\rangle. \tag{2.1}$$

By applying this quench protocol, the system instantly falls out of equilibrium. Quantum quenches can be realized experimentally for example with cold atom systems [2]. Of particular interest is the quench to the QCP, where the final parameters are given by $R_c = (r^*, 0, u^*)$. After this quench protocol, the interplay between criticality and out-of-equilibrium dynamics can be studied in different time regimes. An important time regime is for example the long time limit. If the system is completely isolated, energy conservation after the quench implies, that it will never reach the QCP. If it thermalizes, it will rather end up in the critical cone at finite temperature, see the pink arrow in figure 2.1, but it might also reach a non-thermal steady state, which cannot be illustrated in this equilibrium phase-diagram. The question if and how such systems thermalize is still open. The situation is clearer for the open system, where the energy of the critical subsystem is not conserved. Due to the contact to the quantum bath, the system will thermalize and be located right at the critical point for infinite large times, see the blue arrow in figure 2.1. However, interaction effects in combination with criticality will lead to coarsening and a significant slowing down of thermalization, as it will be shown explicitly in chapter 5. Aging effects are well known from spin glasses and disordered systems [46–49].

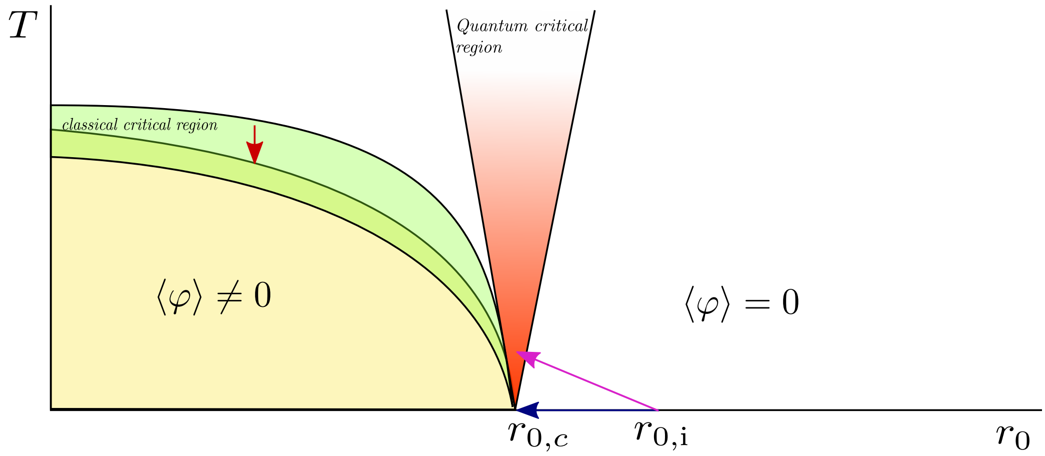


Figure 2.1: *Three different quench protocols in the $r-T$ plane. The red region indicates the quantum critical cone, the green region the classical phase transition. The blue arrow symbolizes a quench without energy conservation, thus for an open system which can indeed thermalize to the QCP. The pink arrow is a quench with energy conservation for $t>0$, thus the system might end up in the quantum critical region. The red arrow stands for a classical quench which can be treated with the methods of a quantum-classical crossover and has been analyzed recently in Refs. [28, 29].*

The influence of the QCP is also expected to affect the dynamics far from equilibrium at intermediate time scales, due to an universal grow of the correlation length with

$$\xi(t) \propto t^{1/\tilde{z}_d}\,, \tag{2.2}$$

where $\tilde{z}_d$ is the dynamic coarsening exponent of ξ. In equilibrium the power law of ξ was given with respect to the distance δr (or h, if the distance is controlled by an external field h) to the QCP, and thus the power laws in observables are power laws with respect to δr (h). Out of equilibrium, a very similar behavior can occur at intermediate times after the quench, in the so called prethermal regime. Here, universality in the correlation length in time will lead to a power-law in time of the order-parameter ϕ, the retarded function G^R and the Keldysh function G^K, defined as (see also section 3.2):

$$\mathrm{i}G^R(k,t,t') = \theta(t-t')\delta_{ij}\langle[\varphi_i(k,t),\varphi_j(k,t')]_-\rangle\,, \tag{2.3}$$

$$\mathrm{i}G^K(k,t,t') = \delta_{ij}\langle[\varphi_i(k,t),\varphi_j(k,t')]_+\rangle\,. \tag{2.4}$$

These power laws are described by a new, universal exponent θ, which is independent of the equilibrium universal exponents. Especially the following scaling form holds for the order parameter $\phi(t) = |\langle\boldsymbol{\varphi}(t)\rangle|$:

$$\phi(t) \propto t^{\theta}. \tag{2.5}$$

Depending on the sign of θ, the order parameter can also grow after a quench to the QCP, see Fig. 1. The exponent θ turns out to have a large impact on the Green's functions, whose scaling forms are given in Eqs. (2.23), (2.24). Those simple scaling forms of $\xi(t)$ and $\phi(t)$ will be confirmed by an explicit time evolution of the order parameter field and the expectation values in section 5.2. As this evolution is quiet technical, the main idea of post-quench universality is drawn here, motivated by generalizing

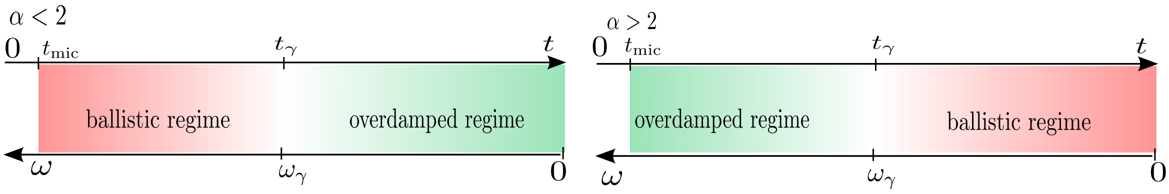


Figure 2.2: *The order of the two different dynamic regimes depends on the bath exponent $\alpha \lessgtr 2$. The time-scale separating those two regimes is $t_\gamma \propto \gamma^{1/(\alpha-2)}$. Only for times larger than the microscopic timescale t_{mic}, the φ^4 model can be applied, this corresponds equally to some cutoff frequency.*

the scaling arguments to an post-quench scenario.

This chapter is organized as follows: Before going into details of the scaling analysis, the hierarchy of the different timescales is introduced and discussed. One main question is whether the dynamics are characterized by a dynamic exponent $z = 1$ or $z \neq 1$. The order of both regimes is discussed in section 2.1 for different bath-exponents α. This discussion is probably also important for closed systems, where it is suggested that due to interactions the system could act as its own bath and thermalize. In section 2.2, the scaling analysis is performed for the order-parameter dynamics. In the same spirit, the scaling form of the correlation functions is extended to an out-of-equilibrium version in section 2.3.

2.1 Ballistic versus diffusive dynamics

In section 1.1 the Hamiltonian for the $\boldsymbol{\varphi}^4$ model was introduced, where $\boldsymbol{\varphi}(k,\omega)$ is the order-parameter field. The bare, retarded propagator $g^R(\omega,k) = \delta_{ij}\langle[\varphi_i(k,\omega),\varphi_j(-k,-\omega)]_-\rangle$ of an isolated system has a very simple form:

$$g^R_{\mathrm{iso}}(\omega,k) = \left(-\omega^2 + k^2 + r_0\right)^{-1}. \tag{2.6}$$

Here, the dynamics are purely given by the ballistic term ω^2, and the only time scale which separates long and short times is the mode depending eigenfrequency $\omega_k^2 = k^2 + r_0$. Coupling the non-interacting system to an external bath leads to a more complex picture of the different regimes. Now, the bare propagator is given by

$$g^R_{\mathrm{open}}(\omega) = \left(\omega^2 - k^2 - \bar{r_0} + \mathrm{i}\gamma\omega|\omega|^{\alpha-1} - \gamma|\omega|^\alpha \cot\left(\frac{\pi\alpha}{2}\right)\right)^{-1}. \tag{2.7}$$

The trivial shift of the bath spectral function $\eta(\omega = 0)$ is again already included into the bare mass $\bar{r_0}$, see also section 3.2. There are two different dynamic regimes. If $\alpha < 2$ and for small frequencies, the dynamics will be dominated by the coupling to the bath. This corresponds to large times. For short times, or large frequencies, the ballistic term is controlling the dynamics. The timescale t_γ which separates both regimes is given by the damping coefficient γ and the bath exponent α,

$$t_\gamma \propto \gamma^{\frac{1}{\alpha-2}}, \tag{2.8}$$

which follows directly from comparing the ballistic term ω^2 with $\eta(\omega) \propto \gamma\omega^\alpha$. For $\omega_\gamma = 1/t_\gamma$ both terms are equal. Via scaling or RG-flow arguments, it is possible to treat the dynamics at times $t < t_\gamma$ as purely ballistic, and at times $t > t_\gamma$ as purely diffusive. Further note that t_γ is inversely proportional to γ, leading to short t_γ for large friction parameters, and large t_γ for small γ. The characteristic mode frequency is still given by $k^2 + r$, which can be part of either the ballistic ($k^2 + r > \omega_\gamma$) or the diffusive ($k^2 + r < \omega_\gamma$) regime, depending on the parameter configuration r, γ, α and the mode k. This turns out to be a crucial point for the post-quench Keldysh function in section 3.5. The impact for the open system is discussed in section 5.1 and in section 6.2 for the isolated system. If $\alpha > 2$, the hierarchy of both regimes is reversed, leading to a dominantly diffusive regime at short times and a dominantly ballistic regime in the long time limit. In this limit, t_γ is proportional to the damping coefficient γ. The special case $\alpha = 2$ has a divergent real part in $\eta(\omega)$ and is therefore very different from the usual $z = 1$ problem of an isolated system. It will not be considered in this thesis.

If the φ^4-interaction is included, the situation is even more complicated, as at least one further time scale will appear. In a perturbative expansion in $1/N$, beyond the first order level, finite frequency-dependent terms can be generated. Those terms will lead to a further time scale t_{int}. If the system is strongly coupled to an external bath, and interactions are kept small, by some controlling parameter, there is no need to include this new time scale, as any further time-depending term will be overruled by the finite bath-damping γ. For a scaling ansatz like in Eq. (2.2), only the bath-dominated diffusive regime will be of interest. Higher order terms have however a strong impact for the dynamics of an isolated system. They can be effectively treated like the bath-spectral function η. Therefore, this is also referred to as an internal bath or that the system acts like its own bath. Even for systems, where this is not possible, it is important to keep in mind, that the ballistic regime will be either cut off by t_{int} or set in after t_{int}, depending on the concrete frequency dependence of this higher order interaction term.

Further time scales are given by the different cutoffs of the system, the inverse momentum cutoff $t_\Lambda = \Lambda^{-1}$, and for an open system, the inverse bath-cutoff $t_{\omega_c} = \omega_c^{-1}$. To make any scaling ansatz like in Eq. (2.2), it is not only necessary to be in the corresponding dynamic time regime. It is reasonable to be beyond those short-time scales, which are named t_{mic} in this section. The hierarchy of the different t_{mic} for an open system in combination with t_γ is discussed in section 5.5.

A last remark on the nomenclature of ballistic and diffusive dynamics. Without interactions, only the closed system is described by ballistic dynamics. On the other hand open systems are always governed by diffusive dynamics. Including interactions and especially scattering processes between different modes of the order-parameter field $\boldsymbol{\varphi}(k)$ will always lead to non-ballistic dynamics in the closed system. To make a difference between the two types of dynamics, ballistic refers to the dynamics dominated by ω^2, and diffusive to the dynamics dominated by η. Thus, for the open system and $\alpha < 2$ the short-time dynamics can be ballistically dominated. Besides for a closed system, if interaction effects can be approximately captured by η, some time regimes are dominated by diffusive dynamics.

2.2 Post-quench scaling of the order parameter

As discussed in section 1.2, the order parameter in equilibrium obeys the scaling relation,

$$\phi_{\text{eq}}\left(\delta r, h\right) = b^{-\beta/\nu}\phi_{\text{eq}}\left(b^{1/\nu}\delta r, b^{\beta\delta/\nu}h\right), \tag{2.9}$$

where δr is the distance to the critical point and h an external field. Out of equilibrium, the order parameter will not only depend on the final configuration $\delta r_{\mathrm{f}}, h_{\mathrm{f}}$, but additional arguments will emerge. First, the order parameter will obviously depend on time, which scales via

$$t \to b^{-z_d} t \tag{2.10}$$

with the dynamic coarsening exponent z_d, which can in general differ from the equilibrium dynamic exponent z. Like the equilibrium dynamic exponent, z_d connects typical time and length scales, and can be given by the coupling to the external bath. If the coupling to the bath is the dominant part of the dynamics, it is given up to leading order in both, RG and $1/N$, as $z_d = 2/\alpha$. On the other hand if the ballistic term ∂_t^2 dominates the dynamics, it is given as $z_d = 1$. Note, that in both cases $z_d = z$ is still the same as its equilibrium value, introduced in Chapter 1. This does not hold for the correlation length, where $\tilde{z}_d$ can take different values in the intermediate and the long time limit.

Second, the order parameter will also depend on the initial configuration $\delta r_{\mathrm{i}}, \delta h_{\mathrm{i}}$. This suggests to introduce a general scaling relation

$$\phi\left(t, R_{\mathrm{i}}(1), R_{\mathrm{f}}(1)\right) = b^{-\beta/\nu} \phi\left(b^{-z} t, R_{\mathrm{i}}(b), R_{\mathrm{f}}(b)\right). \tag{2.11}$$

Here, the short-hand notation with the parameter set R_{i} and R_{f} was used. They scale as

$$R_{\mathrm{f}}(b) = \left(b^{1/\nu} \delta r_{\mathrm{f}}, b^{\beta\delta/\nu} h_{\mathrm{f}}\right), \tag{2.12}$$

$$R_{\mathrm{i}}(b) = \left(b^{\kappa/\nu} \delta r_{\mathrm{i}}, b^{\kappa\beta\delta/\nu} h_{\mathrm{i}}\right). \tag{2.13}$$

The scaling form of both parameter sets, the final and the initial, will be discussed in detail below. In R_{f}, the scaling dimension of the order parameter ϕ and the final parameter δr_{f} and h_{f} take their equilibrium value. This is a reasonable assumption if the system is coupled to an external bath, as thermalization is expected in the long time limit. For an isolated system the question of thermalization is still open [8, 10]. This question is discussed in chapter 6. In R_{i} a new exponent κ is introduced. This is justified as follows: There is no reason why in a quench protocol, the initial parameter set R_{i} should also scale with its equilibrium exponents. In fact, similarities with critical surface scaling suggest to introduce one new exponent κ [50–52]. Via a simple scaling analysis, it is not completely clear why only one new exponent is sufficient to describe the scaling of both, initial distance δr_{i} and initial field h_{i}. It is motivated by the critical boundary scaling, discussed also in section 6.3.2. Here, the presence of some finite surface order is sufficient for the emergence of a new exponent at intermediate length scales, irrespective of how this ordered surface is achieved, by a finite surface field or by a spontaneous order. The existence of only one exponent is also a result of the explicit analysis of the time dependence of ϕ and the Green's functions, whose time dependence is derived in chapter 4 and applied for the open system in chapter 5. The fact that one new exponent occurs, can be understood in the RG picture in section 4.2, where the fixed point is now given by the so called deep quench limit. The deep quench corresponds to an infinite large quench amplitude, which can be achieved by either performing a pure mass quench $\delta r_{\mathrm{i}} \to \infty$ or a field quench $h_{\mathrm{i}} \to \infty$ or both. Interestingly, in this limit the system is independent of the quench amplitude and details of how this quench was performed, see also section 3.5.3. This leads to only one supplementary exponent.

Going back to the scaling ansatz in Eq. (2.11), one can now consider a critical quench, with $\delta r_{\mathrm{f}} = 0$ and $h_{\mathrm{f}} = 0$. To keep the discussion simple and as no further informations are stored in the scaling of h_{i}, also the initial field is set equal to zero. Thus the scaling equation reads

$$\phi\left(t, \delta r_{\mathrm{i}}\right) = b^{-\beta/\nu} \phi\left(b^{-z} t, b^{\kappa/\nu} \delta r_{\mathrm{i}}\right). \tag{2.14}$$

To analyze the short and long time behavior of the order parameter, the parameter b is chosen such that $b^{-z}t = t_{\text{mic}}$. Here t_{mic} is the largest microscopic timescale of the system. With this microscopic timescale, a scaling function $\Psi(x)$ can be introduced

$$\phi\left(t, \delta r_{\text{i}}\right) \propto t^{-\frac{\beta}{\nu z}} \Psi(t/t^*). \tag{2.15}$$

The time $t^* = t_{\text{mic}} \delta r_{\text{i}}^{-\frac{\nu z}{\kappa}}$ corresponds to the typical crossover timescale, which separates the prethermal regime from the adiabatic relaxation at large times. For times $t \gg t^*$, one expects $\Psi(x \gg 1) \to \text{const}$, such that the leading term decays adiabatically. It can be described by a power-law with equilibrium exponents

$$\phi\left(t \gg t^*\right) \propto t^{-\beta/(\nu z)}. \tag{2.16}$$

This power-law decay with equilibrium exponents is characteristic for the adiabatic long-time limit, where the correlation length grows with a scaling form $\xi \propto t^{1/z}$. Thus the order-parameter dynamics can also be expressed in terms of ξ:

$$\phi(\xi) \propto \xi^{-\beta/\nu}. \tag{2.17}$$

Note, that also if the exponents are the equilibrium ones, still memory effects of the initial state can be stored in the amplitude, which is a result of the explicit time evolution, see section 5.4.1. This effect is also known to occur in classical systems. Critical fluctuations slow down thermalization and lead to aging effects, where the system exhibits the memory over large time scales.

For times $t \ll t^*$ in the prethermal regime, one expects the order-parameter to scale like in the pre-quench setting with $\phi \propto \delta r_{\text{i}}^{\beta}$. Thus the expansion of the scaling function must obey $\Psi(x \ll 1) \propto x^{\kappa\beta/(\nu z)}$. This yields for the time dependence in the prethermal regime

$$\phi(t \ll t^*, \delta r_{\text{i}}) \propto t^{(\kappa-1)\beta/(\nu z)}. \tag{2.18}$$

In chapter 5 and 6 the value of κ will be determined for $z = 2/\alpha$ and $z = 1$ respectively. With Eq. (2.5) a connection between θ and the scaling exponent of the initial configuration can be made, it holds

$$\theta = (\kappa - 1)\,\beta/\left(\nu z\right). \tag{2.19}$$

It turns out, that κ strongly depends on z, and varies around 1, such that for some z an increasing order parameter in the prethermal regime is possible, while for other $\langle\varphi(t)\rangle$ decreases slowly. As $\kappa \sim 1$, the crossover timescale t^* is given roughly by the initial correlation length $\xi_{\text{i}} \propto r_{\text{i}}^{-1}$. This implies that t^* diverges for small quench amplitudes, and thus the prethermal regime can be extended to arbitrarily long times. Note however, that this does not imply, that the order parameter can grow to infinitely large values, as it is not possible with this simple analysis to gain any insight on times smaller than t_{mic}. Using these scaling arguments, it can be shown that there must be some collapse of the order parameter,

$$\phi(t_{\text{mic}}) \lesssim \phi_{\text{i}} \left(\frac{t_{\text{mic}}}{t^*}\right)^{\theta}, \tag{2.20}$$

such that $\phi(t^*)$ is of order of the initial magnetization ϕ_{i}. For the time dependence of the correlation length this break-down can be seen, with an explicit time evolution. The result is, $\xi(t^*) \sim \xi_{\text{i}}$, and as ξ grows after the quench, one can conclude that also the correlation length breaks down in the

microscopic time regime right after the quench.
This discussion is not restricted to a mass quench $\delta r_\mathrm{i} \to \delta r_\mathrm{f}$, but can easily be extended to a field quench $h_\mathrm{f} \to h_\mathrm{i}$ at $\delta r = 0$. The crossover time scale t^* separating the prethermal and the quasi-adiabatic regime is now determined by $t^* \propto h_\mathrm{i}^{-\frac{\nu z}{\beta \delta \kappa}}$. However, the time dependence of the order parameter $t \ll t^*$and $t \gg t^*$ are the same as in Eq. (2.18) and Eq. (2.16) respectively.

2.3 Scaling of the correlation functions

In this section the equilibrium scaling form will be extended to an out-of-equilibrium version. In equilibrium, it is not necessary to treat the correlation function G^K and the response function G^R separately, as they are connected via the fluctuation-dissipation theorem (FDT). However, out-of-equilibrium this theorem does not hold, making a separate analysis of both functions necessary.
In equilibrium, for a system coupled to an external bath and large time scales deep in the diffusive regime, the established scaling form reads:

$$G^R_\mathrm{eq}\left(k, t, t'\right) = \frac{1}{k^{2-z-\eta}} F^R_\mathrm{eq}\left(k^z\left(t - t'\right)\right), \tag{2.21}$$

$$G^K_\mathrm{eq}\left(k, t, t'\right) = \frac{1}{k^{2-z-\eta}} F^K_\mathrm{eq}\left(k^z\left(t - t'\right)\right). \tag{2.22}$$

$F^{R/K}_\mathrm{eq}(x)$ are the equilibrium scaling functions. If the dynamics are dominated by the bath coupling, the scaling function, as well as its arguments have a dimensionality of $\gamma^{-2/z}$. This factor is not explicitly written here, to keep the discussion general and to include also the closed system or ballistic dominated dynamics. Note that the scaling dimension $k^{2-z-\eta}$ is the same for the the response as the correlation function at the QCP. Near a classical critical point, they differ by a factor k^{-z}, due to the proportionality to temperature of the Keldysh function.
Out of equilibrium, one expects that G^R and G^K now depend on both time arguments t, t' and on the initial configuration. The simplest extension of the scaling form in Eq. (2.21) and Eq. (2.22) is to include the ratio t/t':

$$G^R\left(k, t, t'\right) = \left(\frac{t}{t'}\right)^\theta \frac{F^R\left(k^z t, t/t'\right)}{k^{2-z-\eta}}, \tag{2.23}$$

$$G^K\left(k, t, t'\right) = \left(\frac{t}{t'}\right)^{\theta'} \frac{F^K\left(k^z t, t/t'\right)}{k^{2-z-\eta}}. \tag{2.24}$$

The scaling function $F^{R/K}(x, t/t')$ is finite in the limit $t' \to 0$, such that possible singularities are included via the exponents θ, θ'.

To show a relation between the new exponents κ, θ and θ' and to obtain their value is task of an explicit analysis of the post-quench dynamics. This analysis will be performed in the subsequent chapters.

Chapter 3

3 Out-of-equilibrium quantum field theory for quantum quenches

In this chapter, the quantum field theory (QFT) is introduced in a way suitable to describe quantum quenches. For a real-time analysis of an out-of-equilibrium setup, the Schwinger-Keldysh formalism is an useful framework [53–55]. It can be applied to describe various protocols of systems driven out of equilibrium. There are many general introductions to this topic [56, 57]. The main idea of the Schwinger-Keldysh formalism is to double the number of degrees of freedom, such that fluctuation and dissipation of a system can be treated separately. This doubling results in the so called two-branch contour along the time-axis. To include quantum and thermal fluctuations of the initial state it can be extended to a three branch contour, where the third branch goes along the imaginary axis. Here, the three branch-contour will be directly applied for the quench protocol, introduced in section 3.1. In section 3.2 the QFT is derived along the three branch contour. In section 3.3 the FDT is presented. In time translational invariant systems, it connects fluctuations and dissipation via the distribution function. Section 3.4 gives a short introduction to double Laplace-transformations. This is a natural language to describe post-quench correlation functions, where both time arguments have to be larger than zero.

An important point, which cannot be overestimated, is the knowledge of the non-interacting Green's functions. In equilibrium, this knowledge is in the most cases trivial. For systems that are out-of-equilibrium, it is a complicated task on its own due to boundary conditions, imposed by the quench protocol. For quantum quenches further challenges arise, especially for the system coupled to a bath. Here, the ground state expectation values are a highly entangled mixture depending of the system and the bath and none of the expectation values can be neglected. This is qualitatively different from classical systems. The Heisenberg uncertainty leads to a further complication, as now some initial expectation values are divergent in the limit of infinite cutoff parameters. Those divergent terms have to be treated carefully. We show that, in the end, they add up to zero. Nevertheless, the knowledge of those bare correlation functions is crucial to build up a perturbative expansion. After a quench to the QCP, the typical energy scales of the pre-Keldysh function determine also the different time regimes. The equal-time, post-quench Keldysh function determines the value of the non-equilibrium exponent θ. As the free Keldysh and retarded Green's function have a high impact on the main results of this thesis, they are derived in detail in section 3.5.

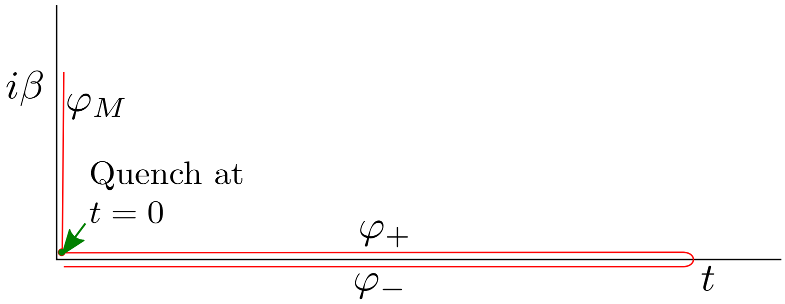


Figure 3.1: *The three-branch contour with the quench performed at $t = 0$.*

3.1 Quench protocol

To drive the system out of equilibrium the following protocol is used: Initially, the system is prepared in the ground state of an Hamiltonian H_{i}, with parameters $R_{\mathrm{i}} = (r_{0,\mathrm{i}}, u_{\mathrm{i}}, \mathbf{h}_{\mathrm{i}})$. At time $t = 0$ the parameter set R is switched to achieve its final value $R_{\mathrm{f}} = (r_{0,\mathrm{f}}, u_{\mathrm{f}}, \mathbf{h}_{\mathrm{f}})$, which is located right at the quantum critical point. The change of $R(t)$ takes place on a typical timescale τ_s. To speak of a quench, one takes the limit $\tau_s \to 0$. This leads to the time evolution given in Eq. (2.1). Experimentally, τ_s is limited by the concrete experimental setup. In numeric simulations and even in purely theoretically considerations, it is sometimes necessary to consider small but finite quench times. To apply the quench limit, it is sufficient that τ_s is of order of the microscopic time scales, given by section 2.1 and 5.5. For shorter time scales the considered φ^4-model, which is only a long wave-length description of the actual quantum mechanical system, is not valid. Hence to compare with an experiment or a numeric simulation, the requirement of switching instantaneously is not too meaningful. Thus, the quench-protocol can be described by

$$R(t) = R_{\mathrm{i}} + \theta(t) \left(R_{\mathrm{f}} - R_{\mathrm{i}}\right), \tag{3.1}$$

with the Heaviside step function $\theta(t)$.

3.2 The three-time-contour formalism

To describe an out-of-equilibrium set-up the closed-time-contour is of great use. It can be extended to the three branch contour, to include interactions and thermal fluctuation's of the initial state. This was originally done by Ref. [58] and by Ref. [59]. The concrete application for a quench protocol in a bosonic-system is presented in this section and in Ref. [31].

The partition function is defined as [56],

$$Z = \frac{\mathrm{Tr}\left(U_c \rho\right)}{\mathrm{Tr}\left(\rho\right)}, \tag{3.2}$$

where ρ is the initial density matrix. In a quench protocol the time evolution for $t > 0$ is governed by the finial Hamiltonian H_{f}. U_c is the corresponding time-evolution operator along the closed time or Schwinger-Keldysh contour SK, see the two horizontal branches in figure 3.1. U_c is given by

$$U_{\mathrm{SK}}(t, t') = e^{\mp \mathrm{i} H_{\mathrm{f}}(t - t')}, \tag{3.3}$$

where the minus sign refers to the forward branch and the plus sign to the backwards branch in time. The initial state at $t = 0$ is characterized by the ground state $\left|\Psi_0\right\rangle$ of H_i, thus it holds for the initial density matrix

$$\rho_\mathrm{i} = \left|\Psi_0\right\rangle\left\langle\Psi_0\right| . \tag{3.4}$$

More general, for finite initial temperatures it can also be written along the Matsubara contour

$$\rho_\mathrm{i} = \frac{1}{Z_\mathrm{i}} e^{-\beta H_\mathrm{i}}, \tag{3.5}$$

with the inverse temperature $\beta = 1/T$, and the pre-quench partition function Z_i. Evaluating the expectation value of some operator O at time t yields

$$\langle O(t)\rangle = \frac{1}{Z_\mathrm{i}}\langle e^{-\beta H_\mathrm{i}} U_\mathrm{SK}(0,t) O U_\mathrm{SK}(t,0)\rangle . \tag{3.6}$$

To handle the time-ordering from $\mathrm{i}\beta \to 0 \to t \to 0$ one can introduce a three branch contour $\mathcal{C}$, see figure 3.1, with the time ordering operator $T_\mathcal{C}$ along this contour. The expectation value of the operator O thus reads

$$\langle O(t)\rangle = \frac{\mathrm{tr}\left(T_\mathcal{C} e^{-\mathrm{i}\int_\mathcal{C} \mathrm{d}s\, H(s)} O\left(t\right)\right)}{\mathrm{tr}\left(T_\mathcal{C} e^{-\mathrm{i}\int_\mathcal{C} \mathrm{d}s\, H(s)}\right)} . \tag{3.7}$$

On the right hand side, the operator O is written with a time argument, to clarify at which point of the time-ordering it has to be inserted. In the denominator the property $1 = U(0,t)U(t,0)$ of the time evolution operator was used, to write the initial partition function Z_i as partition function Z along the full contour $\mathcal{C}$:

$$Z = \mathrm{tr}\left(T_\mathcal{C} \exp(-\mathrm{i}\int_\mathcal{C} \mathrm{d}s H)\right) . \tag{3.8}$$

In the same way, a two-time correlation function $G(t,t') = -\mathrm{i}\langle T_c \varphi(t)\varphi(t')\rangle$ can be written as

$$G(t,t') = -\mathrm{i}\frac{1}{Z}\mathrm{tr}\left(T_\mathcal{C} e^{-\mathrm{i}\int_\mathcal{C} \mathrm{d}s\, H(s)} \varphi(t)\varphi(t')\right) . \tag{3.9}$$

In this section, the spatial arguments of the field φ, as well as the component index $l = 1, \ldots, N$ are suppressed for easier reading. In quantum field theory, those expectation values are determined with the method of a generating functional along the contour $\mathcal{C}$:

$$W\left[\mathbf{h}\right] = \int D\boldsymbol{\varphi} D\mathbf{X} e^{\mathrm{i}\, S\left[\boldsymbol{\varphi},\mathbf{X}\right] - \mathrm{i}\int_x \int_\mathcal{C} \mathrm{d}t \mathbf{h}(t)\cdot\boldsymbol{\varphi}(t)} \tag{3.10}$$

with the action of the φ^4 model coupled to an external heat bath, introduced in section 1.1. This action consists of three parts,

$$S\left[\boldsymbol{\varphi},\mathbf{X}\right] = S_s\left[\boldsymbol{\varphi}\right] + S_b\left[\mathbf{X}\right] + S_{sb}\left[\boldsymbol{\varphi},\mathbf{X}\right] . \tag{3.11}$$

The $t \in \mathcal{C}$ is a short hand notation for the time-integral along $\mathcal{C}$. S_s refers to the action of the system

$$S_s = \frac{1}{2}\int_{x,t\in\mathcal{C}} \left\{\left(\partial_t \boldsymbol{\varphi}\left(t\right)\right)^2 - r_0\left(t\right)\boldsymbol{\varphi}\left(t\right)^2 - \left(\nabla\boldsymbol{\varphi}\left(t\right)\right)^2 - \frac{u\left(t\right)}{2N}\boldsymbol{\varphi}\left(t\right)^4\right\} , \tag{3.12}$$

and S_b to the bath

$$S_b = \frac{1}{2}\sum_j \int_{x,t\in\mathcal{C}} \left(\left(\partial_t \boldsymbol{X}_j(t)\right)^2 - \Omega_j^2 \boldsymbol{X}_j(t)^2 \right) . \tag{3.13}$$

S_{sb} describes the coupling between system and bath

$$S_{sb} = -\sum_j c_j \int_{x,t\in\mathcal{C}} \mathbf{X}_j(t)\cdot\boldsymbol{\varphi}(t) . \tag{3.14}$$

The action is Gaussian in the bath coordinates $\boldsymbol{X}$. Thus, they can be integrated out. The effects of the coupling to the bath can be captured exactly by a self-energy

$$\Delta(t-t') = -\sum_j c_j^2 \left(\partial_t^2 + \Omega_j^2\right)^{-1} \delta(t-t') , \tag{3.15}$$

which is non-local in time. This self-energy enters in the bare propagator

$$\mathcal{G}_0^{-1} = -\left(\partial_t^2 + r_0(t) - \nabla^2\right)\delta(t-t') + \Delta(t-t') \tag{3.16}$$

of the system. The effective action thus reads

$$S[\varphi] = \frac{1}{2}\int_{x;t,t'\in\mathcal{C}} \boldsymbol{\varphi}(t)\mathcal{G}_0^{-1}(t,t')\boldsymbol{\varphi}(t') - \frac{1}{4N}\int_{x;t\in\mathcal{C}} \left(\boldsymbol{\varphi}\cdot\boldsymbol{\varphi}\right)^2 . \tag{3.17}$$

In the Schwinger-Keldysh formalism, the four possible arrangements of t, t' along the $\pm$-contour are arranged in a 2×2 matrix, to handle the time integral in a more compact way. Here, this structure is extended to a 3×3 matrix, as also the Matsubara axis for the initial state is included [59]:

$$\mathcal{G} = \begin{pmatrix} \mathrm{i}G^M & \tilde{G}^< & \tilde{G}^< \\ \tilde{G}^> & G^T & G^< \\ \tilde{G}^> & G^> & G^{\bar{T}} \end{pmatrix} . \tag{3.18}$$

The individual matrix elements are discussed in detail in the following. The first element is the Matsubara Green's function of the initial state

$$G^M\left(\tau-\tau'\right) = -\left\langle T_\tau \varphi_M(\tau)\varphi_M\left(\tau'\right)\right\rangle . \tag{3.19}$$

Here, T_τ is the time-ordering along the imaginary axis form $\mathrm{i}\beta \to 0$ and $\varphi_M(\tau) = e^{\tau H_\mathrm{i}}\varphi e^{-\tau H_\mathrm{i}}$. The bare propagator along the Matsubara axis is given by

$$g_\mathrm{i}^M\left(\tau,\tau'\right) = \left[\left(\partial_\tau^2 - r_{0,\mathrm{i}} + \nabla^2\right)\delta\left(\tau-\tau'\right) + \eta^M\left(\tau-\tau'\right)\right]^{-1} . \tag{3.20}$$

The subscript i refers to the initial mass r_i. The bath-spectral function $\eta(\tau)$ was given in Eq. (1.9). The system prior to the quench is in equilibrium, thus g_i^M is time-translational invariant. The Fourier transform reads

$$g_\mathrm{i}^M\left(\omega_n\right) = \frac{1}{-\omega_n^2 - \bar{r}_{0,\mathrm{i}} - q^2 + \delta\eta^M\left(\omega_n\right)} . \tag{3.21}$$

The mass $\bar{r}_{0,\mathrm{i}} = r_{0,\mathrm{i}} - \eta^M(0)$ contains already the trivial shift due to the external bath, but no renormalization due to interaction effects, and is therefore called bare mass in the following.
The Green's function $G^{\lessgtr}$ describes the coupling across the quench. Without interactions, this coupling originates from two main effects. Firstly, memory effects stored in the heat bath. Secondly, from the concrete regularization chosen to write down the generating functional and to describe the quench. For an isolated system, only the second point leads to cross quench correlations. With the Heisenberg representation $\varphi_H(t) = e^{\mathrm{i}H_\mathrm{f}t}\varphi e^{-\mathrm{i}H_\mathrm{f}t}$ they are formally given by

$$\tilde{G}^{<}\left(\tau, t'\right) = \mathrm{i}\left\langle \varphi_M(\tau)\varphi_H(t')\right\rangle , \tag{3.22}$$

$$\tilde{G}^{>}\left(\tau, t'\right) = \mathrm{i}\left\langle \varphi_H(t')\varphi_M(\tau)\right\rangle . \tag{3.23}$$

The functions in the remaining 2×2 block are the usual Green's functions of the Schwinger-Keldysh formalism, with the time ordering operator T_t (anti-time ordering operator $\tilde{T}_t$) along the real time branch. They read

$$G^T\left(t, t'\right) = -\mathrm{i}\left\langle T_t\varphi_H(t)\varphi_H(t')\right\rangle , \tag{3.24}$$

$$G^{\tilde{T}}\left(t, t'\right) = -\mathrm{i}\left\langle \tilde{T}_t\varphi_H(t)\varphi_H(t')\right\rangle , \tag{3.25}$$

$$G^{>}\left(t, t'\right) = -\mathrm{i}\left\langle \varphi_H(t')\varphi_H(t)\right\rangle , \tag{3.26}$$

$$G^{<}\left(t, t'\right) = -\mathrm{i}\left\langle \varphi_H(t)\varphi_H(t')\right\rangle . \tag{3.27}$$

Those four Green's functions are known to contain redundant information. It is convenient in the usual Schwinger-Keldysh formalism to rotate to the quantum-classical basis, with

$$\varphi_H^{\mathrm{cl,q}} = \frac{1}{\sqrt{2}}\left(\varphi_H^{+} \pm \varphi_H^{-}\right). \tag{3.28}$$

After this rotation there are two independent Green's functions, the retarded function G^R and the Keldysh function G^K. Here, the rotation matrix for the 3×3 matrix block reads

$$L = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \tag{3.29}$$

where the sign of the first element was chosen such that $\det L = 1$. This yields

$$L^{-1}\mathcal{G}L = \begin{pmatrix} G^M & -\sqrt{2}\tilde{G}^{<} & 0 \\ -\sqrt{2}\tilde{G}^{>} & G^K & G^R \\ 0 & G^A & 0 \end{pmatrix}. \tag{3.30}$$

The two relevant Green's functions along the horizontal branch are given by

$$G^R\left(t, t'\right) = -\mathrm{i}\theta\left(t - t'\right)\left\langle \left[\varphi_H(t), \varphi_H(t')\right]_{-}\right\rangle \tag{3.31}$$

$$G^K\left(t, t'\right) = -\mathrm{i}\left\langle \left[\varphi_H(t), \varphi_H(t')\right]_{+}\right\rangle . \tag{3.32}$$

G^R measures the response of the order parameter at time t caused by an external field $\boldsymbol{h}$. This function has a retarded structure $t' < t$, with both time arguments after the quench. The advanced Green's function is given by

$$G^A\left(t,t'\right) = \mathrm{i}\theta\left(t'-t\right)\left\langle\left[\varphi_H(t),\varphi_H(t')\right]_-\right\rangle \tag{3.33}$$

and contains thus the same informations as $G^R(t,t') = G^A(t',t)$. Causality implies two important relations, $G^R(t,t) + G^A(t,t) = 0$ and $G^R(t,t) - G^A(t,t) = -\mathrm{i}$ [56]. Those are important in chapter 4, where they lead to a simplification for the number of Hubbard-Stratonovich fields, and for the out-of-equilibrium version of the RG flow equations. While the retarded Green's function contains informations about the spectrum, the Keldysh function G^K measures also order-parameter configurations. In addition to those usual Green's functions, correlations across the quench are taken into account via $\tilde{G}^{\lessgtr}$. Note that those cross-quench correlation functions couple only to the quantum component φ^{q} in the inverse Matrix $\mathcal{G}^{-1}$. For a non-interacting system the Matsubara fields φ_M can be formally integrated out. They couple only to the bare Keldysh Green's function g^K, thus only this function contains pre-quench information and displays memory effects. Those can be formally captured by

$$g^K = \int_0^t\int_0^{t'} \mathrm{d}s\,\mathrm{d}s'\, g^R(t,s)M(s,s')g^A(s',t')\,. \tag{3.34}$$

Here, the memory function M includes the effects of the pre-quench system. More details to the memory function, and a simple way to calculate it, are presented in section 3.5.2 below. The bare retarded function $g^R(t,t')$ does not couple to the pre-quench fields, thus it is given by the equilibrium value of the final parameter configuration R_{f}. Especially g^R depends only on the difference of the time arguments, thus its Fourier transform reads

$$g_{\mathrm{f}}^R(k,\omega) = \frac{1}{\omega^2 - \bar{r}_{0,f} - k^2 + \delta\eta(\omega)}\,. \tag{3.35}$$

Again, the trivial shift due to the bath coupling is taken into account via $\bar{r}_{0,f} = r_{0,f} - \eta(0)$. Including many-body interactions will however lead to aging in the retarded function as well.

3.3 Fluctuation-dissipation theorem

In equilibrium, there is a strong link between fluctuations and the response of a system to an external force, called fluctuation-dissipation theorem FDT. A short derivation is given in this section, for more details see for example Refs. [35, 40, 60].

In thermal equilibrium the Green's functions depend only on the time difference $t-t'$ of their arguments. Therefore it is convenient to express them into Fourier space

$$G^{R/K}(k,\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, e^{\mathrm{i}\omega t} G^{R/K}(k,t). \tag{3.36}$$

The FDT states that the equilibrium fluctuations are proportional to the imaginary part of a response function:

$$G^K(k,\omega) = 2\mathrm{i}\coth\left(\frac{\omega}{2T}\right)\mathrm{Im}\, G^R(\omega,k). \tag{3.37}$$

A first version of the classical FDT was derived by Einstein, when he analyzed the Brownian motion [61]. Thereby he noticed a connection between the diffusive constant D in Fick's law with the inverse mobility. The full quantum version was given by Callen-Welton in 1952 [62]. This form can be derived from the general structure of the Keldysh contour [56]. In general, even in an out-of-equilibrium situation the following relation holds

$$G^K(k,t,t') = F \circ G^R + G^A \circ F, \tag{3.38}$$

where $\circ$ stands for the convolution in time. This relation follows from the Hermitian property of the Keldysh function $G^K(t,t') = G^K(t',t)$. This version is sometimes also called generalized FDT in the literature [41], even if, far from equilibrium, the function F is not given by the Bose-Einstein distribution function n_B. In this thesis, FDT refers explicitly to the relation in Eq. (3.37), like in Ref [56].
Note, that for $T \to \infty$ in the classical system a prefactor of $2T/\omega$ emerges. Thus, the scaling dimension of G^K and G^R will differ by the characteristic frequency. In contrast, in the quantum limit $T \to 0$, $\coth(\omega/(2T)) \to \operatorname{sign}\omega$, and G^K and G^R have the same scaling dimension, but different parity with respect to ω.
In section 5.4.2 it is shown explicitly that in the quasi-adiabatic long time limit for an open system a relation like in Eq. (3.37) holds between G^R and G^K, and thus the Wigner transform of the function F is given explicitly.

3.4 Double Laplace transformations

Laplace transformations are the natural framework to describe retarded time evaluations in equilibrium, as all times have to be larger than zero. A natural generalization to non-equilibrium situations are double Laplace transformations. Here, the Laplace transformation (LT) of the function $d(\omega)$ is defined with $\exp(i\omega t)$ as

$$d(\omega) = \int_0^\infty \mathrm{d}t\, e^{\mathrm{i}(\omega+\mathrm{i}0^+)t} d(t)\,. \tag{3.39}$$

The infinitesimal small $\mathrm{i}0^+$ is introduced to keep the LT of d finite. The advantage of performing the LT with $\exp(i\omega t)$, is the simple inverse transformation, if $d(\omega)$ is a retarded function, , e. g. it has no poles in the upper half of the complex plane. In this case the inverse transformation reads:

$$d(t) = 2\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{1}{2\pi} \mathrm{Im}\, d(\omega) e^{-\mathrm{i}\omega t}. \tag{3.40}$$

This relation can be verified by using Kramers-Kronig relations for $d(\omega)$ and $\frac{1}{\omega+\mathrm{i}0^+} = -\mathrm{i}\int_0^\infty \mathrm{d}t\, e^{\mathrm{i}(\omega+\mathrm{i}0^+)t}$.

Of further importance is the double LT, as in the post-quench system the Green's functions depend in general on both time arguments t, t' individually. The double LT is defined as

$$f(\omega,\omega') = \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}t' e^{\mathrm{i}(\omega+\mathrm{i}0^+)t} e^{\mathrm{i}(\omega'+\mathrm{i}0^+)t'} f(t,t')\,. \tag{3.41}$$

To derive the post-quench Keldysh-function, one also needs the double LT of the pre-quench Keldysh function. This function describes a system in equilibrium, thus $G^K_{\mathrm{i}}(t,t') = G^K_{\mathrm{i}}(|t-t'|)$. The subscript i is kept here, to make clear, that this relation is a property only of the initial, equilibrium Green's functions. The double LT of the Keldysh function obeys:

$$G_{\mathrm{i}}^{K}\left(q,\omega,\omega'\right)=\mathrm{i}\frac{G_{\mathrm{i}}^{K}\left(q,\omega\right)+G_{\mathrm{i}}^{K}\left(q,\omega'\right)}{\omega+\omega'+\mathrm{i}0^{+}}, \tag{3.42}$$

with $G_{\mathrm{i}}^{K}\left(q,\omega\right)$ being the Laplace transform of the equilibrium Keldysh function $G_{\mathrm{i}}^{K}(|t-t'|)$ of the initial state. This relation is obtained by using the definition of the LT and performing a variable transformation $G_{\mathrm{i}}^{K}(t,t')=G_{\mathrm{i}}^{K}(|t-t'|)$, to use the equilibrium properties of G_{i}^{K}.

Below, not only the double LT of the initial, equilibrium Keldysh function is needed, but also the simple LT of $G_{\mathrm{i}}^{K}(|t-t'|)$. Using the initial Keldysh function g_{i}^{K} given by the FDT in Fourier space (see section 3.3) yields:

$$G_{\mathrm{i}}^{K}(k,t)=\mathrm{i}\int_{-\infty}^{\infty}\frac{\mathrm{d}\epsilon}{\pi}e^{-\mathrm{i}\epsilon t}\coth\left(\frac{\epsilon}{2T}\right)\mathrm{Im}\,G_{\mathrm{i}}^{R}(k,\epsilon)\,. \tag{3.43}$$

In the quantum limit $T\to 0$, the $\coth\left(\frac{\epsilon}{2T}\right)$ can be replaced by $\mathrm{sign}(\epsilon)$. Inserting the FDT into the Laplace-transformation of $G_{\mathrm{i}}^{K}(k,t)$ yields

$$\begin{aligned}G_{\mathrm{i}}^{K}(k,\omega)&=\int_{0}^{t}\mathrm{d}t\,e^{\mathrm{i}(\omega+\mathrm{i}0^{+})t}G_{\mathrm{i}}^{K}(k,t)\\&=-\int_{-\infty}^{\infty}\frac{\mathrm{d}\epsilon}{\pi}\frac{\mathrm{sign}(\epsilon)\mathrm{Im}\,G_{\mathrm{i}}^{R}(k,\epsilon)}{\omega-\epsilon+\mathrm{i}0^{+}}\,.\end{aligned} \tag{3.44}$$

By using the identity $\delta(\omega-\epsilon)=-\frac{1}{\pi}\mathrm{Im}\frac{1}{\omega-\epsilon+\mathrm{i}0^{+}}$, the imaginary part of G_{i}^{K} is found:

$$\mathrm{Im}\,G_{\mathrm{i}}^{K}(k,\omega)=\mathrm{sign}(\epsilon)\mathrm{Im}G_{\mathrm{i}}^{R}(\omega). \tag{3.45}$$

The real part is given by a principal value integral

$$\mathrm{Re}\,G_{\mathrm{i}}^{K}(k,\omega)=-\int_{-\infty}^{\infty}\frac{\mathrm{d}\epsilon}{\pi}\,\mathrm{sign}(\epsilon)\mathrm{Im}\,G_{\mathrm{i}}^{R}(k,\epsilon)\frac{\omega-\epsilon}{(\omega-\epsilon)^{2}+\mathrm{i}0^{+2}}\,. \tag{3.46}$$

To avoid worrying about this principal value integral, one can use the following trick, to bring the Keldysh-function in a more convenient form. Consider the Kramers-Kronig relation for the retarded Green's function

$$G_{\mathrm{i}}^{R}(\omega)=-\int_{-\infty}^{\infty}\frac{\mathrm{d}\epsilon}{\pi}\frac{\mathrm{Im}\,G_{\mathrm{i}}^{R}(k,\epsilon)}{\omega-\epsilon+\mathrm{i}0^{+}}\,. \tag{3.47}$$

With this relation the sum and the difference of G_{i}^{R} and G_{i}^{K} can be derived:

$$G_{\mathrm{i}}^{R}(k,\omega)+G_{\mathrm{i}}^{K}(k,\omega)=-\,2\int_{0}^{\infty}\frac{\mathrm{d}\epsilon}{\pi}\frac{\mathrm{Im}\,G_{\mathrm{i}}^{R}(k,\epsilon)}{\omega-\epsilon+\mathrm{i}0^{+}}\,, \tag{3.48}$$

$$G_{\mathrm{i}}^{R}(k,\omega)-G_{\mathrm{i}}^{K}(k,\omega)=2\int_{0}^{\infty}\frac{\mathrm{d}\epsilon}{\pi}\frac{\mathrm{Im}\,G_{\mathrm{i}}^{R}(k,\epsilon)}{\omega+\epsilon+\mathrm{i}0^{+}}\,. \tag{3.49}$$

Thus, for $\omega > 0$ one uses the differences of those two functions, while for $\omega < 0$ the sum. In both cases, the remaining integral has no poles. Written in a compact way for all ω one finally finds

$$G_{\mathrm{i}}^{K}(k,\omega) = \mathrm{sign}\,(\omega)\left(G_{\mathrm{i}}^{R}\left(k,\omega\right) - g_{\mathrm{i}}\left(k,\omega\right)\right), \tag{3.50}$$

with the real and finite function g_{i}

$$g_{\mathrm{i}}\left(k,\omega\right) = 2\int_0^\infty \frac{\mathrm{d}\epsilon}{\pi}\frac{\mathrm{Im}\,G_{\mathrm{i}}^{R}\left(k,\epsilon\right)}{|\omega|+\epsilon}. \tag{3.51}$$

This is the Laplace transformed version of the FDT at $T = 0$. Together with Eq. (3.42), it is now possible to express the double LT of the equilibrium Keldysh function in terms of the retarded function and the temperature.

3.5 Bare post-quench correlation functions

In this section the bare, post-quench Green's functions are derived. The knowledge is fundamental to build up any perturbative approach where interactions are included via for example a Dyson equation. Evaluating the post-quench correlation function analytically, often boundary conditions for a free harmonic oscillator are used to derive the bare or full post-quench Keldysh function, like in Ref. [63]. However, it will be shown, that also small derivations from the free harmonic oscillator in the initial state can lead to a completely different post-quench Keldysh function. Therefore in this section, a more general method for obtaining g^K is presented . The quench protocol is introduced in section 3.1: a parameter-quench at time $t = 0$ from $R_{\mathrm{i}} = (r_{\mathrm{i}}, h_{\mathrm{i}}, 0)$ to the finial parameters $R = (r_0, h, 0)$. In this section $u = 0$ as only the bare Green's functions are considered. For a post-quench system the bare retarded function is unchanged compared to its equilibrium value as it has no memory of the pre-quench system, see also Eq. (3.35). Again, it is convenient to express g^R in Laplace space:

$$g_{\mathrm{f}}^{R}(k,\omega) = \frac{1}{\omega^2 + \eta(\omega) - k^2 - r_0}\,. \tag{3.52}$$

The subscript f refers to the final mass r_0, while the subscript i refers to the initial mass, as in this section the distinction between pre and post-quench mass is crucial.

The derivation of the Keldysh function is more challenging. In this section the Heisenberg equations of motion (EOM) are used to obtain an analytic expression for g^K. One could also try to directly derive the bare Keldysh function with the framework given in section 3.2, but there one has to face a technical subtlety: the regularization. Especially one must keep in mind, that the time steps along the Keldysh contour must be small compared to any physical timescale, including the quench time τ. Hence regularization problems can occur, as known for example from the bath-regularization Itô versus Stratonovich [56] in classical non-equilibrium dynamics. The three-branch contour is however useful in chapter 4 to include interaction effects. To use a kinematic equation or a quantum Boltzmann equation is neither constructive, as it is not possible to include the boundary conditions in a simple way. Further, it is not guaranteed that a scale separation between total and relative time is valid and thus a gradient expansion of the Wigner transformation is not possible. To avoid this subtleties and to gain more physical insight, the Heisenberg EOM are used in this section.

This section is therefore organized as follows, first the EOM for the vector-field $\boldsymbol{\varphi}$ are derived for the post-quench problem in section 3.5.1. With this solution and a memory-function ansatz the bare

Keldysh function is derived. The general structure of the memory function is derived in section 3.5.2. The memory function is also evaluated for the quench from a free harmonic oscillator in appendix A.3, to show that the results given in Ref. [63] can be reproduced. In section 3.5.3 the memory and the Keldysh function are evaluated in the deep quench limit, which is an essential ingredient for critical quenches.

3.5.1 Equations of motion for φ

The EOM for the bare vector field is given by

$$\left(\partial_t^2 + r_{0,\mathrm{f}} + k^2\right) \boldsymbol{\varphi}\left(k,t\right) = \int_0^\infty \mathrm{d}s\, \eta\left(t-s\right) \boldsymbol{\varphi}\left(k,s\right) + \boldsymbol{\Xi}\left(k,t\right) + \mathbf{h}\left(t\right). \tag{3.53}$$

An external field which couples to $\boldsymbol{\varphi}$ is denoted by $\mathbf{h}$. The source operator $\boldsymbol{\Xi}(k,t)$ is given in terms of the initial bath operators $\mathbf{X}_j^0 = \mathbf{X}_j\left(k, t=0\right)$ and $\mathbf{P}_j^0 = \mathbf{P}_j\left(k, t=0\right)$:

$$\boldsymbol{\Xi}\left(k,t\right) = -\sum_j c_j \left(\mathbf{X}_j^0\left(k\right)\cos\left(\Omega_j t\right) + \frac{1}{\Omega_j}\mathbf{P}_j^0\left(k\right)\sin\left(\Omega_j t\right)\right). \tag{3.54}$$

It can also be expressed in terms of the bath spectral function η and the pre-quench state φ_i:

$$\boldsymbol{\Xi}(k,t) = -\int_{-\infty}^0 \mathrm{d}s\, \eta(t-s)\boldsymbol{\varphi}_\mathrm{i}(k,s). \tag{3.55}$$

Technical details how to obtain this result are given in the appendix A.1.
Eq. (3.53) can be formally solved via Laplace transformation, yielding

$$\boldsymbol{\varphi}\left(k,\omega\right) = \boldsymbol{F}\left(k,\omega\right) g_\mathrm{f}^R\left(k,\omega\right). \tag{3.56}$$

Here, the force operator $\boldsymbol{F}$ is introduced, as well as the retarded Green's function g_f^R. The Green's function g^R is given in Eq. (3.52). One can double check, that this is indeed the correct retarded Green's function also given in Eq. (3.35), by using the definition as anti-commutator of $\mathrm{i}g^R = \langle[\varphi_0, \varphi_0]_-\rangle$. The force operator $\boldsymbol{F}$ is defined as

$$\boldsymbol{F}\left(k,\omega\right) = \boldsymbol{\pi}_\mathrm{i}\left(k\right) - \mathrm{i}\,\omega\boldsymbol{\varphi}_\mathrm{i}\left(k\right) + \boldsymbol{\Xi}\left(k,\omega\right) + \mathbf{h}\left(\omega\right). \tag{3.57}$$

Here, the subscript in the operators $\boldsymbol{\varphi}$ and $\boldsymbol{\pi}$ indicates, that those operators are evaluated at time $t=0$ by their pre-quench value. Inserting this formal solution for the vector field φ in the definition of g^K, one can express the bare Keldysh-function as commutator of two force fields:

$$\mathrm{i}g^K(k,\omega,\omega') = \delta_{ij}\langle\left[\varphi_i(k,\omega), \varphi_j(k,\omega')\right]_+\rangle \tag{3.58}$$

$$= \delta_{ij}\langle\left[F_i(k,\omega), F_j(k,\omega')\right]_+\rangle g_\mathrm{f}^R(k,\omega) g_\mathrm{f}^R(k,\omega') \tag{3.59}$$

$$= M_\mathrm{i}(k,\omega,\omega') g_\mathrm{f}^R(k,\omega) g_\mathrm{f}^R(k,\omega'). \tag{3.60}$$

In what follows, the force-force commutator is also called the memory function $M_\mathrm{i}(k,\omega,\omega')$, as only via this function pre-quench informations enter in g^K.

3.5.2 Memory function for g^K

To obtain the memory function, two basic approaches are used, that lead to the same result. Firstly, by evaluating explicitly every expectation value, which occurs in the force-force commutator, e. g. $\langle[\varphi,\varphi]_-\rangle$, $\langle[\varphi,X_i]_-\rangle$, and so on. This procedure is long and tedious, as some of those expectation values diverge with the bath cutoff. For example $\langle[\pi,\pi]_-\rangle$ is divergent, as well as parts of the momentum-bath-momentum expectation value $\langle\pi,P_i\rangle$. However, in the final result all those divergences cancel, such that the memory function itself is well defined in the limit of $\Lambda\to\infty$ and $\omega_c\to\infty$. This way is presented in the appendix A.2. The second approach is easier. Note, that M is given uniquely by the initial pre-quench expectation values at time $t=0$, thus in a state where the system is still in equilibrium. Therefore, it is also possible to determine M for an equilibrium system without quench. Here, the final parameter configuration is given by R_i. In this scenario the double Laplace transformed Keldysh function must have the same formal structure

$$g_\mathrm{i}^K\left(k,\omega,\omega'\right)=M_\mathrm{i}\left(k,\omega,\omega'\right)g_\mathrm{i}^R\left(k,\omega\right)g_\mathrm{i}^R\left(k,\omega'\right), \tag{3.61}$$

where the subscript i denotes that g^K is given by the initial parameter configuration R_i. In this no-quench scenario, g_i^K is the LT of the equilibrium Keldysh function. Also note, that the structure Mg^Rg^R in Laplace space looks very similar to the usual Fourier expression of the Keldysh function, $g^K=g^R[g^K]^{-1}g^A$. Indeed, M is nothing else than the inverse Keldysh component for a quench problem, which one would naturally obtain if the Matsubara-fields in section 3.2 are integrated out. Within the large-N approximation, one can show, that this structure also holds for interactions at the one-loop-level, and probably also beyond.

The crucial point is now, that the memory function in Eq. (3.61) must be exactly the same function as in Eq. (3.60) in the limit of an infinitely fast quench. This is a consequence of the retarded Green's function g^R taking instantaneously its final, equilibrium form. This immediately yields,

$$M_\mathrm{i}(k,\omega,\omega')=\frac{g_\mathrm{i}^K(k,\omega)+g_\mathrm{i}^K(k,\omega')}{\omega+\omega'+\mathrm{i}0^+}\left[g_\mathrm{i}^R(k,\omega)\right]^{-1}\left[g_\mathrm{i}^R(k,\omega)\right]^{-1}, \tag{3.62}$$

where the double LT equilibrium Keldysh function of Eq. (3.42) was used. This is exactly the same result that has been obtained by evaluating directly the pre-quench expectation values in appendix A.2. Using the result of Eq. (3.50), the initial Keldysh function is found to be,

$$g_\mathrm{i}^K(k,\omega)=\mathrm{sign}\left(\omega\right)\left(g_\mathrm{i}^R\left(k,\omega\right)-g_\mathrm{i}\left(k,\omega\right)\right), \tag{3.63}$$

with the real and finite function g_i

$$g_\mathrm{i}\left(k,\omega\right)=2\int_0^\infty\frac{\mathrm{d}\epsilon}{\pi}\frac{\mathrm{Im}\,g_\mathrm{i}^R\left(k,\epsilon\right)}{|\omega|+\epsilon}. \tag{3.64}$$

Inserting this result into Eq. (3.62), the final result for the memory function is given by

$$M(k,\omega,\omega')=\frac{\mathrm{sign}\left(\omega\right)n\left(\omega,\omega',\omega_\mathrm{i}\right)+\mathrm{sign}\left(\omega'\right)n\left(\omega',\omega,\omega_\mathrm{i}\right)}{\omega+\omega'+\mathrm{i}0^+}, \tag{3.65}$$

with the initial frequency $\omega_\mathrm{i}=\sqrt{k^2+r_\mathrm{i}}$ and

$$n\left(\omega,\omega',\omega_\mathrm{i}\right)=g_\mathrm{i}^R(k,\omega')^{-1}-g_\mathrm{i}(k,\omega)g_\mathrm{i}^R(k,\omega)^{-1}g_\mathrm{i}^R(k,\omega')^{-1}. \tag{3.66}$$

This function n is evaluated explicitly for the quench in an isolated, non-interacting system in appendix A.3 and in the following section in the limit of a large quench amplitude.

3.5.3 Deep quench expansion

The deep quench limit is of significant importance for quenches to a QCP. To achieve scale invariance and universality, it is necessary that the post-quench Keldysh function, or at least parts of it, obey a scaling form. Therefore, it is not only necessary, that G^K is given in either the diffusive or the ballistic dynamical regime, but also that it is independent of the energy scale ω_i given by the quench amplitude. In this section, the limit of an infinite large quench amplitude $\omega_\mathrm{i} \to \infty$ is taken, which corresponds to a Taylor expansion for small $1/\omega_\mathrm{i}$ in the memory function M. It will be shown, that this deep quench expansion yields for the equal-time Keldysh function G^K:

$$G^K(k,t,t) = \frac{1}{k^{2-z}} f^K(k^z t) + \omega_\mathrm{i}^{2-2/\alpha} C_\alpha^{(1)} g_\mathrm{f}^R(k,t) g_\mathrm{f}^R(k,t) + \frac{\omega_\mathrm{i}^{2-4/\alpha} k^{-4+3z}}{t^2} C_\alpha^{(2)} . \tag{3.67}$$

The first term is the scaling part g_sc^K, with the scaling function f^K given in Eq. (3.73). By including interactions, this term will lead to the universal post-quench dynamics. Its impact is analyzed in chapter 4. The second term $\omega_\mathrm{i}^{2-2/\alpha} C_\alpha^{(1)} g^R(k,t) g^R(k,t)$ is irrelevant in the deep quench limit for $\alpha < 1$. For $\alpha > 1$, this term is divergent, but it is shown in appendix B.1 that this first order term does not affect the effective mass $r(t)$ near the upper critical dimension $d_\mathrm{uc} = 4 - z$. The coefficient $C_\alpha^{(1)}$ is given in Eq. (3.78). The last term in Eq. (3.67) is irrelevant in the limit $\omega_\mathrm{i} \to \infty$ for $\alpha < 2$. It does however play a crucial role for the isolated system, see section 6.2. The coefficient $C_\alpha^{(2)}$ is given in Eq. (B.28). In this section, the function n defined in Eq. (3.66) is explicitly analyzed in the deep quench limit, to derive Eq. (3.67). The conditions for the deep quench limit and technical details are discussed in appendix B. Let us start with the function $g_\mathrm{i}(k,\omega)$ in the limit $\omega_\mathrm{i} \to \infty$ and with over-damped dynamics $\eta(\omega)$ which are large compared to the ballistic term. In the following, the subscript i is dropped for easier reading, and only introduced, when a difference between initial and final mass is necessary. In this scenario, g can be expressed with the scaling form

$$g_{\alpha<2}(\omega_\mathrm{i}, \omega) = \frac{1}{\omega_\mathrm{i}^2} \phi\left(\gamma^{z/2} \omega_\mathrm{i}^{-z} |\omega|\right) , \tag{3.68}$$

where the scaling function $\phi(x)$ is given as

$$\phi(x) = -\frac{2}{\pi} \int_0^\infty \mathrm{d}y \, \frac{1}{x+y} \frac{y^{2/z}}{\left(1 + \cot\frac{\pi}{z} y^{2/z}\right)^2 + y^{4/z}} . \tag{3.69}$$

It holds $\phi(0) = -1$ for all z, which allows to perform the scaling limit $\omega_\mathrm{i} \to \infty$. This also holds if the dynamics are given in the ballistic regime. In this case, the retarded Green's function is just the sum of two δ-functions, making the evaluation of g for $z = 1$ straightforward:

$$g_\mathrm{ballistic}(\omega, \omega_\mathrm{i}) = -\frac{1}{\omega_\mathrm{i}} \frac{1}{|\omega| + \omega_\mathrm{i}} . \tag{3.70}$$

The subscript ballistic means, that this result holds strictly only for the non-interacting, isolated, pre-quench system. The first non-vanishing term in an expansion in ω_i^{-1} would be $g(\omega, \omega_\mathrm{i} \to \infty) \approx -\omega_\mathrm{i}^{-2} + \mathcal{O}(\omega_\mathrm{i}^{-3})$, like in the diffusive regime. Inserting $g_\mathrm{i} = -\omega_\mathrm{i}^{-2}$ into the function n leads to the deep-quench result

$$n_\mathrm{dq}(\omega, \omega', k) = \omega'^2 - \eta(\omega'). \tag{3.71}$$

Substituting this back into the memory function and performing the inverse LT, one can show, that it leads indeed to a scaling form of the Keldysh function

$$g_{\rm sc}^K(k,t,t) = \frac{1}{k^{2-z}} f^K(k^z t), \tag{3.72}$$

with the scaling function

$$f^K(x) = \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{d}y' \, \frac{\mathrm{sign}(y) n_{\rm dq}(y,y',1) + \mathrm{sign}(y') n_{\rm dq}(y',y,1)}{y+y'} f^R(y) f^R(y'). \tag{3.73}$$

The retarded scaling function reads for the isolated system:

$$f^R(y) = (1+y^2)^{-1}, \tag{3.74}$$

and

$$f^R(y) = \gamma^{-z/2}(1+\cot(2\pi\alpha)|y|^\alpha + \mathrm{i}\,\mathrm{sign}\, y|y|^\alpha)^{-1}, \tag{3.75}$$

for the diffusive system with $\alpha < 2$. This is the first term on the right hand side in Eq. (3.67).

However, in the function n, the function g will be multiplied with $\omega_{\rm i}^4$. This makes it necessary to go beyond the zeroth order in an expansion for small $\omega_{\rm i}^{-2/\alpha}|\omega|$ for $\alpha > 1$, or for small $\omega/\omega_{\rm i}$ in the ballistic regime. The next order terms of this expansion in g generate the other terms in Eq. (3.67). To include higher order terms, the function g must be analyzed in more detail. Here, the case $\alpha < 2$ is considered:

$$\begin{aligned} g_\alpha(\omega_{\rm i},\omega) =& 2\int_0^\infty \frac{\mathrm{d}\epsilon}{\pi} \frac{1}{|\omega|+\epsilon} \mathrm{Im}\left[\frac{-1}{\epsilon^2-\omega_{\rm i}^2+\eta(\epsilon)}\right], \\ \approx& 2\int_0^{\omega_\gamma} \frac{\mathrm{d}\epsilon}{\pi} \frac{1}{|\omega|+\epsilon} \mathrm{Im}\left[\frac{1}{\omega_{\rm i}^2+\eta(\epsilon)}\right] + 2\int_{\omega_\gamma}^\infty \frac{\mathrm{d}\epsilon}{\pi} \frac{1}{|\omega|+\epsilon} \mathrm{Im}\left[\frac{1}{\epsilon^2-\omega_{\rm i}^2+\mathrm{i}0^+}\right]. \end{aligned} \tag{3.76}$$

The low frequency integral $\epsilon < \omega_\gamma$ refers to the diffusive regime, the high frequency integral $\epsilon > \omega_\gamma$ to the ballistic regime. For $\alpha > 2$ the order of the two dynamic regimes is inverted, leading to a ballistic dominated part at low frequencies, and a diffusive dominated part at high frequencies. For both integral parts, the scaling function reads

$$\phi(x) \simeq -1 + |x| C_\alpha^{(1)}, \tag{3.77}$$

with

$$C_\alpha^{(1)} = \begin{cases} \theta(\omega_{\rm i}-\omega_\gamma) & \text{in the ballistic regime,} \\ 2\gamma^{1/\alpha}\int_0^\infty \mathrm{d}y\, y^{\alpha-2}\left((1+\coth(2\pi\alpha))^2 + x^{2\alpha}\right)^{-1} & \text{in the overdamped regime.} \end{cases} \tag{3.78}$$

The dimensionless variable x is given by $\omega/\omega_{\rm i}$ in the ballistic regime, and $\gamma^{1/\alpha}\omega/\omega_{\rm i}^{2/\alpha}$ in the diffusive regime. In the diffusive regime, the limit of large ω_γ was taken, to send the upper bound of the integral to infinity. This is a reasonable limit, for $\alpha < 2$ and times $t \gg t_\gamma$, where the ballistic part can be ignored. In appendix B.1, the inverse LT of those terms is performed, yielding

$$g_1^K(k,t,t)\omega_{\rm i}^{2-2/\alpha} C_\alpha^{(1)} g_{\rm f}^R(k,t) g_{\rm f}^R(k,t)\,, \tag{3.79}$$

in the equal-time Keldysh function. It is also shown, that those first order terms have no impact on the effective mass $r(t) \propto \int \mathrm{d}^d k \, g^K(k,t,t)$, for all dynamic exponents z near the upper quantum critical dimension $d_{\mathrm{uc}} = 4 - z$. Thus, for $\alpha < 2$ the deep quench limit can be taken in $r(t)$, even if the bare post-quench Keldysh function has divergent terms in the limit $\omega_{\mathrm{i}} \to \infty$.

For $\alpha > 2$ and for the ballistic dominated part of g, it is necessary to include second order terms proportional to $\omega_{\mathrm{i}}^{-4/\alpha}$. The time dependence of those second-order terms is derived in appendix B.2. They contribute via

$$G^{K(2)}(k,t,t) = C_{\alpha}^{(2)}(\omega_{\mathrm{i}}, \gamma) \times \frac{k^{-4+3z}}{t^2}, \tag{3.80}$$

in the equal-time Keldysh function, where $C_{\alpha}^{(2)}(\omega_{\mathrm{i}}, \gamma)$ is given in Eq. (B.28). The impact of those terms is discussed for both dynamic regimes separately, in section 5.1 for the open system and in section 6.2 for the isolated system. For the diffusive systems $z \neq 1$, they do not affect the scaling form of the effective mass $r_{\mathrm{sc}}(t) = at^{-2/z}$, as those terms lead to corrections in $r(t)$ going with t^{-2}. However, they have a strong impact for $z = 1$.

It is important to note, that the deep-quench limit corresponds to times $t, t' > t_{\mathrm{i}} = (\gamma/\omega_{\mathrm{i}}^2)^{z/2}$, see also section 5.1 for the open system and section 6.2.1 for the isolated system. Thus for times larger than t_{i} the deep quench limit can always be taken, even if the quench amplitude is nominally not large.

Chapter 4

4 Non-equilibrium dynamics in post-quench systems, general formalism

With the QFT along the three branch contour and the bare Green's function derived in the last chapter, the effects of the φ^4-interaction term can now be studied. Especially, it can be shown, that including interactions leads to aging of the Keldysh as well as the retarded Green's function. After a quench to the QCP, the φ^4-interaction leads to an inverse correlation length, which obeys scaling:

$$r(t) = \frac{a}{t^{2/z}}. \tag{4.1}$$

Here z is the dynamical exponent given by $z = 1$ for the isolated system and $z = 2/\alpha$, with the characteristic bath-exponent α, for the dissipative system. In analogy to the equilibrium field theory, this inverse correlation length will be called effective mass in the following. This effective mass modifies the Green's function at intermediate times after the quench, where $G^{R/K}$ obeys the following scaling form

$$G^{R(K)}(k,t,t') = \frac{1}{k^{2-z}} \left(\frac{t}{t'}\right)^{\theta(\theta')} F^{R(K)}\left(k^z t, t'/t\right) . \tag{4.2}$$

This form was motivated in Eqs. (2.21) and (2.22) in chapter 2. Here, this form can be derived with the Dyson equation, see section 4.1.3 under the assumption of Eq. (4.1). The singularity for $t' \to 0$ is uniquely captured by the exponent θ. It holds for a quantum system $\theta = \theta'$, in contrast to classical systems [28, 29], where $\theta' = \theta - 1$ for $z < 2$ and $\theta' = \theta - 2/z$ for $z > 2$. A further result from the Dyson equation is the connection of the light cone amplitude a of Eq. (4.1) with the dimensionless exponent θ. In addition, in the quasi-adiabatic long-time limit the thermalization is significantly slowed down by the presence of interactions, see section 4.1.2:

$$G^{R/K}(k,t,t') = G^{R/K}_{\mathrm{eq}}(k,t-t') + r\left(\frac{t+t'}{2}\right) C^{R/K}_z(k,t-t'). \tag{4.3}$$

This result is only meaningful if the system is known to thermalize, as it is based on the assumption of equilibration to the QCP for $t, t' \to \infty$. For a non-interacting system, thermalization takes place exponentially fast. Including interactions leads to a power law in the absolute time $t + t'$ via the effective mass given in Eq. (4.1), while the coefficient $C^{R/K}_z$ relaxes exponentially with the relative time $t - t'$.

The goal of this chapter is to calculate the universal exponent θ. Therefore a combination of the methods presented in chapter 1 with the out-of-equilibrium formalism of chapter 3 is used to derive an out-of-equilibrium version of the RG (see section 4.2) and large-N equations (see section 4.3). The discussion here is kept as general as possible, the application to the two models, the open and the isolated φ^4-model, and a detailed discussion with physical implications is given in the chapters 5 and 6, respectively. Inspired by the similarity between the RG and $1/N$ equation, it is shown explicitly in section 4.4 that both methods indeed lead to the same result for a, if the system is self-similar after the quench. With a combination of both methods, i. e. a simultaneous expansion in ϵ and $1/N$ and by including the long-time behavior in Eq. (4.3), it is possible to *derive* the scaling form of the time-dependent mass $r(t)$ in Eq. (4.1). The amplitude a is given by the equilibrium fixed point u^* and the difference between the out-of-equilibrium Keldysh function and its equilibrium value, making θ indeed a truly new universal exponent.

4.1 Dyson equation

In this section, the φ^4-interaction term along the two-branch contour is considered. At this point, it is possible to neglect the effect of the Matsubara branch, as the one-loop correction has only one time argument. Thus correlations across the quench are not included via the interactions, but only via the memory of the external bath. This naive physical argumentation will be confirmed in section 4.3 by an explicit calculation along the three-branch contour.
After the Keldysh rotation, the interaction term along the real-time contour reads

$$S_{int}[\varphi] = \frac{u}{4N}\sum_{i,j=1}^{N}\int \mathrm{d}^d x \int_0^\infty \mathrm{d}t\, \varphi_i^{\mathrm{cl}}(x,t)\varphi_i^{\mathrm{q}}(x,t)\left(\varphi_j^{\mathrm{cl}}(x,t)\varphi_j^{\mathrm{cl}}(x,t)+\varphi_j^{\mathrm{q}}(x,t)\varphi_j^{\mathrm{q}}(x,t)\right). \tag{4.4}$$

Due to doubling the degrees of freedom by introducing two independent fields $\varphi^{\mathrm{cl/q}}$, now two types of vertices occur, $u_1\varphi_{\mathrm{cl}}^3\varphi_{\mathrm{q}}$ and $u_3\varphi_{\mathrm{cl}}\varphi_{\mathrm{q}}^3$. However, including only one-loop corrections, as it will be done below, only the vertex u_1 enters, therefore this difference is not made in this section. In the following, the component index l of $\boldsymbol{\varphi}$ is omitted for better reading. A natural way to include this interaction term is to use the Dyson equation. Therefore the self-energy is introduced, which reads up to first order:

$$\Sigma(t) = \begin{pmatrix} 0 & 0 \\ 0 & u(N+2)\int \mathrm{d}^d k\, G^K(k,t,t) \end{pmatrix}. \tag{4.5}$$

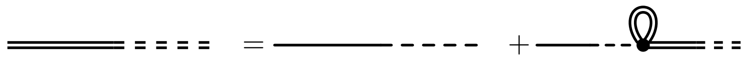
In general, also components with $\Sigma^{R/A}$ can occur, but at one-loop they cannot be generated, due to the causality $G^R(x,t,t)+G^A(x,t,t)=0$. The self-energy Σ has the same causal structure as G^{-1}. With this self-energy, the full Green's functions can be written as

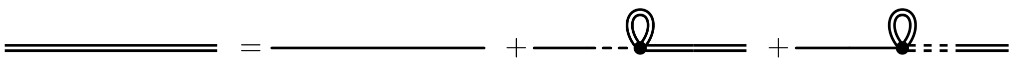
$$G(k,t,t') = g(k,t,t') + \int \mathrm{d}s\, g(k,t,s)\Sigma(s)G(k,s,t'), \tag{4.6}$$

where g refers to the bare Green's function matrix and G to the full Green's function matrix. Both have the usual bosonic structure [56],

$$g = \begin{pmatrix} g^K & g^R \\ g^A & 0 \end{pmatrix} \text{ and } G = \begin{pmatrix} G^K & G^R \\ G^A & 0 \end{pmatrix}. \tag{4.7}$$

In this section, only Green's functions with the final bare mass $r_{0,\mathrm{f}}$ are considered, thus, the subscript f is dropped in the following. The Dyson equation can also be interpreted graphically by using Feynman diagrams:


(4.8)


(4.9)

where the double lines stands for the full or dressed Green's function and the single lines for the bare Green's function, the dashed lines for the quantum and the plain lines for the classical fields. Thus Eq. (4.8) stands for the Dyson equation of G^R and Eq. (4.9) for the Dyson equation of G^K. As the advanced Green's function does not include further information, in the following only G^R and G^K are considered. This approach is especially useful, if an expansion in the self-energy and thus in a small interaction parameter u is justified. In a non-equilibrium set-up, it may be useful, to expand around a known stationary state $G^K_{\mathrm{stat}}(k, 0)$ by introducing

$$r(t) = \Sigma(t) - r\,, \tag{4.10}$$

where

$$r = u(N+2) \int \mathrm{d}^d k\, G^K_{\mathrm{stat}}(k, 0), \tag{4.11}$$

is the effective mass of the full stationary state. This leads in the Dyson equations to corrections up to first order:

$$G^R(k,t,t') = G^R_{\mathrm{stat}}(k,t,t') + \int \mathrm{d}s\, G^R_{\mathrm{stat}}(k,t,s) r(s) G^K_{\mathrm{stat}}(k,s,t'), \tag{4.12}$$

$$G^K(k,t,t') = G^R_{\mathrm{stat}}(k,t,t') + \int \mathrm{d}s \left(G^R_{\mathrm{stat}}(k,t,s) r(s) G^K_{\mathrm{stat}}(k,s,t') + G^K_{\mathrm{stat}}(k,t,s) r(s) G^A_{\mathrm{stat}}(k,s,t') \right). \tag{4.13}$$

This expansion turns out to be useful to obtain the long-time limit of the quench problem, where the stationary state is given by the equilibrium Green's functions at the critical point.

The expansion in the self-energy fails in the prethermal regime after a quench to the QCP, if the effective mass obeys scaling in Eq. (4.1). In this case, a simple expansion in a small effective mass is not justified, but it still gives some insight how the amplitude of the effective mass should be interpreted. Especially it gives a relation between a and θ, see Eq. (4.40).

4.1.1 Solution of the equation of motion at large times

In the long-time limit the open system is expected to thermalize. Under this assumption the Dyson equation can be used to build up a perturbative approach around the equilibrium state $|\varphi_{\mathrm{eq}}\rangle$ and the

equilibrium Green's function G_{eq}. Note, that neither $|\varphi_{\text{eq}}\rangle$ nor G_{eq} corresponds to the bare system but to the full interacting system. In the language introduced above, the $g^{R/K}$-functions are given by the full equilibrium correlation functions. Here, the control parameter of the perturbative expansion is the effective mass $r(t)$ which approaches zero for $t \to \infty$, when the system is equilibrated. An equivalent expansion in $r(t)$ can be performed within the EOM for the field $\boldsymbol{\varphi}$, leading to the same results. This is the way followed in this section. This ansatz is not fruitful, for an isolated system where it is not clear to which final state it might equilibrate. This is also discussed in chapter 6. Note, that in this section, it is not necessary to assume the scaling form of $r(t)$, but that the aging effects in Eq. (4.3) follow from the assumption of thermalization to the QCP.
The equations of motion for the vector field $\boldsymbol{\varphi}$ in the limit of large times can be written as

$$\left(\partial_t^2 + r\left(t\right) + k^2\right) \boldsymbol{\varphi}\left(k,t\right) = \int_{-\infty}^{t} \mathrm{d}s\, \delta\eta(t-s)\boldsymbol{\varphi}(k,s), \tag{4.14}$$

where it was used that the quench is performed to the QCP, thus $r_{\text{eq}} = 0$. The long-time limit enters by sending the lower limit in the convolution integral to $-\infty$ without making a difference between pre- and post-quench order parameter fields. At intermediate times after the quench the pre-quench part $\int_{-\infty}^{0} \mathrm{d}s\, \delta\eta(t-s)\boldsymbol{\varphi}_{\text{i}}(q,s)$ must be treated as an inhomogeneity to the differential equation. In the long-time limit it can be neglected, since $\delta\eta(t-s)$ decays fast enough for large arguments. This is the reason why in the following the lower limit of the integral will be ignored. At large times after the quench the time dependence of the effective mass will vanish $r(t \to \infty) = 0$. This yields for the equilibrium part

$$\left(\partial_t^2 + k^2\right) \boldsymbol{\varphi}_{\text{eq}}\left(k,t\right) = \int_{-\infty}^{t} \mathrm{d}s\, \delta\eta(t-s)\boldsymbol{\varphi}_{\text{eq}}(k,s) \tag{4.15}$$

Note, that in general $\boldsymbol{\varphi}_{\text{eq}}$ is not the solution of the interaction free problem, but of the full system in equilibrium with the effective mass at the critical point $r_{\text{eq}} = 0$.

Around this solution $\boldsymbol{\varphi}_{\text{eq}}$, the perturbation ansatz will be build up, where the small control parameter is the amplitude a in the time-dependent mass $r(t)$. Thus one expands the field via

$$\boldsymbol{\varphi}(k,t) = \boldsymbol{\varphi}_{\text{eq}}(k,t) + \boldsymbol{\varphi}_1(k,t) + \mathcal{O}(a^2), \tag{4.16}$$

where $\boldsymbol{\varphi}_1(k,t)$ are corrections linear in a. Inserting this ansatz in the equation of motion immediately yields

$$\boldsymbol{\varphi}_1(k,t) = -\int^{t} \mathrm{d}s\, G^R_{\text{eq}}(k,t-s)r(s)\boldsymbol{\varphi}_{\text{eq}}(k,s)\,, \tag{4.17}$$

where $G^R_{\text{eq}}(t-t') = -\mathrm{i}\theta(t-t')\langle[\varphi_{\text{eq}}(t), \varphi_{\text{eq}}(t')]\rangle$.

4.1.2 Keldysh function at large times

With the solution of the equation of motion given in the section above and the definition of the Green's functions, the relaxation to equilibrium of G^R and G^K can be analyzed. As the equilibration of the equal-time Keldysh function is of special interest, this function will be determined in detail. As sketched in the previous section, it is possible to expand around the equilibrium functions, which will be reached in the limit $t \to \infty$. The Keldysh function G^K will be determined up to linear order in $r(t)$, like the order parameter field φ. It is not necessary to assume the scaling form of $r(t)$ in Eq. (4.1), but only thermalization to the QCP. However, to derive this scaling form below, an expansion in small a is necessary, therefore Eq. (4.1) is used for $r(t)$, and a is the small parameter controlling the expansion. Up to linear order in a, the Green's function G_r^K reads:

$$G_r^K(k,t,t') = G_{\text{eq}}^K(k,t-t') + \delta G_r^K\left(k,t,t'\right) + \mathcal{O}(a^2). \tag{4.18}$$

Inserting $\varphi = \varphi_{\text{eq}} + \varphi_1$, where the component index is suppressed again, this yields for δG_r^K,

$$\begin{aligned}\delta G_r^K\left(k,t,t'\right) &= -\frac{\mathrm{i}}{2}\left(\langle[\varphi_1(k,t),\varphi_{\text{eq}}(k,t')]_+\rangle + \langle[\varphi_1(k,t'),\varphi_{\text{eq}}(k,t)]_+\rangle\right)\\ &= -\int^t \mathrm{d}s\, G_{\text{eq}}^R(k,t-s)r(s)G_{\text{eq}}^K(k,s-t') - \int^{t'} \mathrm{d}s\, G_{\text{eq}}^R(k,t'-s)r(s)G_{\text{eq}}^K(k,s-t),\end{aligned} \tag{4.19}$$

where the definition of the equilibrium Keldysh function was used. The same result can also be obtained directly for the Dyson equation given in Eq. (4.13). This expression can be simplified by expressing the equilibrium Green's functions in Fourier space. For the retarded Green's function there is, due to the special LT introduced in 3.4, no difference between LT and Fourier transformation. So the correction to the equilibrium function reads

$$\begin{aligned}\delta G_r^K\left(q,t,t'\right) =& -\mathrm{i}\int \frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{2\pi^2}\left[G_{\text{eq}}^R(\omega)\right]G_{\text{eq}}^K(\omega')e^{-\mathrm{i}\omega'(t-t')}\int^t \mathrm{d}s\, r(s)e^{-\mathrm{i}(\omega-\omega')(t-s)}\\ &-\mathrm{i}\int \frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{2\pi^2}\left[G_{\text{eq}}^R(\omega)\right]G_{\text{eq}}^K(\omega')e^{-\mathrm{i}\omega'(t-t')}\int^{t'} \mathrm{d}s\, r(s)e^{-\mathrm{i}(\omega-\omega')(t'-s)}.\end{aligned} \tag{4.20}$$

To evaluate the time integral for t, t' being the largest scale of the problem, while $t-t'$ is small, the following approximation can be made: Note that in general the integrand of s is highly oscillating except at the upper boundary. Therefore, the integral can be expanded around the upper boundary t

$$\int^t \mathrm{d}s\, e^{-\mathrm{i}\left(\omega-\omega'\right)(t-s)} r\left(s\right) \approx \frac{\mathrm{i}\, r(t)}{\omega-\omega'}\,. \tag{4.21}$$

Further, as t, t' are both large, but the difference between both times is small, it holds approximately that $r(t) \simeq r(t') \simeq r((t+t')/2)$, in the long time limit, where $r(t) \to 0$. The frequency integration over ω can be done by using the Kramers-Kronig relation:

$$G^R(k,\omega') = \mathrm{i}⨍ \frac{\mathrm{d}\omega}{\pi}\frac{G^R(k,\omega)}{\omega-\omega'}\,, \tag{4.22}$$

where $⨍$ refers to the principal value integral. This finally yields

$$G_r^K\left(k,t,t'\right) = G_{\text{eq}}^K(k,t-t') + 2\mathrm{i}r((t+t')/2)\int \frac{\mathrm{d}\omega}{2\pi}G_{\text{eq}}^R(\omega)G_{\text{eq}}^K(\omega)e^{-\mathrm{i}\omega(t-t')}\,. \tag{4.23}$$

First note, that for $r(t) = at^{-2/z}$, first order corrections lead to δG^K which decays with a power law, thus critical fluctuations lead to a significant slowing down of thermalization. This effect is also called aging. In the long-time limit it is possible to separate the time dependence of the relative time $t-t'$ and the absolute time, passed since the quench. In the following, the equal time correlation function plays an important role. Therefore the coefficient for $t = t'$ is analyzed in more detail. The integral $\int \frac{d\omega}{2\pi} G^R_{\mathrm{eq}}(\omega) G^K_{\mathrm{eq}}(\omega)$ can be expressed in terms of the RG fixed point value of u^*

$$u^* = \frac{\epsilon}{(N+8)K_d} \left[\int \frac{d\omega}{2\pi} G^R_{\mathrm{eq}}(\Lambda,\omega) G^K_{\mathrm{eq}}(\Lambda,\omega) \right]^{-1} . \tag{4.24}$$

This integral can also be evaluated using the FDT

$$\int \frac{d\omega}{2\pi} G^R_{\mathrm{eq}}(k,\omega) G^K_{\mathrm{eq}}(k,\omega) = \int \frac{d\omega}{2\pi} \frac{1}{\omega^2 + \eta(\omega) + k^2} \frac{\mathrm{sign}(\omega) 2\mathrm{i}\, \mathrm{Im}\, \eta(\omega)}{(\omega^2 + \mathrm{Re}\, \eta(\omega) + k^2)^2 + \mathrm{Im}\, \eta(\omega)^2} . \tag{4.25}$$

In the dissipative scaling limit where $\eta(\omega) \gg \omega^2$ one finally finds

$$\int \frac{d\omega}{2\pi} G^R_{\mathrm{eq}}(k,\omega) G^K_{\mathrm{eq}}(k,\omega) \simeq \frac{1}{k^{4-z} \gamma^{z/2}} \frac{z\,(2-z) \sin^{z/2}(\pi/z)}{4 \sin(\pi z/2)} . \tag{4.26}$$

And in the limit $\eta(\omega) \ll \omega^2$ it holds

$$\int \frac{d\omega}{2\pi} G^R_{\mathrm{eq}}(k,\omega) G^K_{\mathrm{eq}}(k,\omega) = \frac{1}{2k^3} . \tag{4.27}$$

In both cases, the quasi-adiabatic limit of the equal time Keldysh function can be brought to the form

$$G^K(k,t,t) \simeq G^K_{\mathrm{eq}}(k,0) + 2\mathrm{i}\, r(t) \frac{C^K_z}{k^{4-z}} , \tag{4.28}$$

with the coefficient

$$C^K_z = \begin{cases} z\,(2-z) \sin^{z/2}(\pi/z)/(\gamma^{z/2} 4 \sin(\pi z/2)) & \text{for } z > 1 , \\ 1/2 & \text{for } z = 1 . \end{cases} \tag{4.29}$$

The coefficients are positive for both types of dynamics, $z \neq 1$ and $z = 1$, which is important for our later discussion. Note that C^K_z approaches zero for $z \to 4$. This is to be expected, as the upper critical dimension vanishes in this limit.

A similar discussion can be made for the retarded Green's function. The final result is

$$G^R_r(k,t,t') = G^R_{\mathrm{eq}}(k,t-t') + 2\mathrm{i} r\Big((t+t')/2\Big)\, C^R_z(k,t-t'). \tag{4.30}$$

Here, the coefficient is given by $C^R_z(k,t) = \int \frac{d\omega}{2\pi} G^R_{\mathrm{eq}}(k,\omega) G^K_{\mathrm{eq}}(k,\omega) e^{-\mathrm{i}\omega t}$.

4.1.3 Logarithmically divergent terms at short times

In this section the scaling ansatz for the effective mass

$$r(t) = \frac{a}{t^{2/z}} \tag{4.31}$$

will be used to show a relation between the amplitude a and the universal exponent θ, introduced in chapter 2. The Dyson equation leads to logarithmically divergent corrections to the bare Green's functions. This has two key implications: Firstly, one cannot use a simple expansion in the interaction parameter, as those corrections will grow at sufficiently long times, making any initially small parameter large. Better approaches are to formulate an out-of-equilibration version of the RG-equations or to use a $1/N$ expansion, as it is done in sections 4.2 and 4.3, respectively. Secondly, it still gives a beautiful interpretation of the light cone amplitude a, as a new universal exponent, as predicted by the simple scaling analysis given in chapter 2. Especially, it shows that $\theta = \theta'$ in the scaling function ansatz of G^R and G^K. This is in contrast to the classical systems [28, 29] and the dynamics in the isolated quantum system [33, 34, 64], where it holds $\theta' = \theta + 1$. The difference 1 between θ and θ' in those systems can be traced back to the fact, that the temperature (for classical systems) or the quench amplitude (for the isolated system) are the largest scale of the system, and the dynamics are dominated by the low-frequency range. The quantum post-quench Keldysh function has however a scaling part, independent of the quench amplitude. Thus the known quantum result, that response and correlations have the same scaling dimension, holds also out of equilibrium.

Using the Dyson Eqs. (4.12) and (4.13) and the scaling ansatz for $r(t)$ yields for the corrections $\delta G^{R/K}$ to the bare functions $g^{R/K}$:

$$\delta G_r^R\left(k,t,t'\right) \approx \int_{t_{\text{mic}}}^{t} \mathrm{d}s\, g^R\left(k,t-s\right) \frac{a}{s^{2/z}} g^R\left(k,s-t'\right), \tag{4.32}$$

$$\delta G_r^K\left(k,t,t'\right) \approx \int_{t_{\text{mic}}}^{t} \mathrm{d}s\, g^R\left(k,t-s\right) \frac{a}{s^{2/z}} g^K\left(k,s,t'\right). \tag{4.33}$$

Here, the lower boundary t_{mic} indicates that the scaling ansatz can only be used for times larger than some microscopic time after the quench. Both times t and t' have to be larger than this microscopic timescale. The set of Eqs. (4.32) will be analyzed in the short-time limit of each mode k, thus $t \ll k^{-z}$. Without loss of generality, it is further assumed that $t \gg t'$. First, the time dependence of δG^R is analyzed in this short-time limit. The condition $t \ll k^{-z}$ corresponds to large frequencies $\omega \gg k^z$ in Laplace space. In this regime, the bare function is entirely given by the dynamic part

$$g^R(k,\omega) \approx \frac{1}{\delta\eta}, \tag{4.34}$$

for the open system, if $\omega \ll \omega_\gamma$ and $\alpha < 2$. For the closed system it is given by:

$$g^R(k,\omega) \approx \frac{1}{\omega^2}. \tag{4.35}$$

Note that it is a natural consequence of the limit $\omega \gg k^z$, that g^R is independent of the momentum k. In both cases, the back-transformed function reads

$$g^R(k,t) \approx C_z t^{2/z-1}, \tag{4.36}$$

with $C_z = -\sin(\pi/z)/(\gamma\Gamma(2/z))$ for $z > 1$, and $C_1 = 1$ for $z = 1$. Inserting this ansatz in Eq. (4.32), one finds

$$\delta G_r^R\left(k,t,t'\right) = aC_z^2 \int_{t'}^{t} \mathrm{d}s\, \frac{(t-s)^{2/z-1}\left(s-t'\right)^{2/z-1}}{s^{2/z}}. \tag{4.37}$$

This integral can be evaluated analytically. It holds for the leading term in an expansion $t \gg t'$:

$$\delta G_r^R\left(k,t,t'\right) \approx a C_z^2 t^{2/z-1} \log\frac{t}{t'}\,. \tag{4.38}$$

Thus, in the limit $t \gg t'$ the zeroth order and first correction of the retarded Green's function reads

$$G_r^R\left(k,t,t'\right) = g^R\left(k,t-t'\right)\left(1+\theta\log\frac{t}{t'}+\dots\right), \tag{4.39}$$

with

$$\theta = -aC_z\,. \tag{4.40}$$

This result implies, that first, G^R displays aging effects, e. g. depends on both time arguments separately. The scaling form in $r(t)$ leads to a logarithmic correction in the retarded Green's function. If $\theta \ll 1$, the logarithm can be exponentiated, leading to a scaling form introduced in Eq. (2.21), with exponent θ. Note, that θ is indeed dimensionless, for both dynamic regimes, the closed as well as the diffusive. In the latter case, $z \neq 1$, the amplitude a must have the dimension γ in the ansatz for the time-dependent mass in Eq. (4.1). Including higher order terms in the Dyson equation, makes it possible to go beyond this first order result. It leads to higher powers of the logarithm, with appropriate coefficients, making a reexponentiation possible also for larger values of θ. Including the k-dependence of each mode, makes the analysis more complicated. However, in the relevant regime $k^{-z} > t \gg t' \gg t_{\text{mic}}$, the logarithm is the dominant term.

The same analysis can be performed for the Keldysh function in Eq. (4.13). Like for the retarded Green's function, the hierarchy $t_{\text{mic}} \ll t' \ll t \ll k^{-z}$ is considered here. With the memory ansatz, presented in section 3.5, the short-time behavior of $g^K(k,t,t')$ can be evaluated. In the deep quench limit, the double LT Keldysh function reads:

$$g^K(k,\omega,\omega') = \mathrm{i}\frac{\operatorname{sign}(\omega)\left(\omega^2+\delta\eta(\omega)\right)+\operatorname{sign}(\omega')\left(\omega'^2+\delta\eta(\omega')\right)}{\omega+\omega'+\mathrm{i}\,0^+}g^R(k,\omega)g^R(k,\omega')\,. \tag{4.41}$$

The limit $t \gg t'$ corresponds to $\omega' \gg \omega$. Again, either the ballistic or the overdamped regime is considered. In both cases, is $\omega'^{2/z} \gg \omega^{2/z}$. Thus, only this leading term can be kept in the memory function. It cancels with $g^R(k,\omega')$, such that the bare Keldysh function can be simplified to:

$$g^K(k,\omega,\omega' \gg \omega) \simeq \frac{\mathrm{i}}{|\omega'|}g^R(k,\omega)\,. \tag{4.42}$$

Using that the back-transformed function of $|\omega'|^{-1}$ is a constant, one obtains the same time dependence as for the retarded Green's function in Eq. (4.36):

$$g^K(k,t) \approx C_z t^{2/z-1}. \tag{4.43}$$

As the Keldysh and the retarded Green's functions obey the same short-time behavior and the corresponding Dyson equation differs only in the bare values of those functions, one can immediately conclude that the scaling form is the same, thus $\theta = \theta'$. This is in contrast to the scaling form after a classical quench or a quench protocol in an isolated system like in Ref. [64], where the bare Keldysh function vanishes in the limit $t' \to 0$ with $t'^{2/z}$. Including interactions does not change this behavior. In a quantum system, the scaling part of the free Keldysh function is a constant in the limit $t' \to 0$, including interactions leads thus to a divergent contribution for small t'. This divergence is cutoff like for G^R by some microscopic timescale.

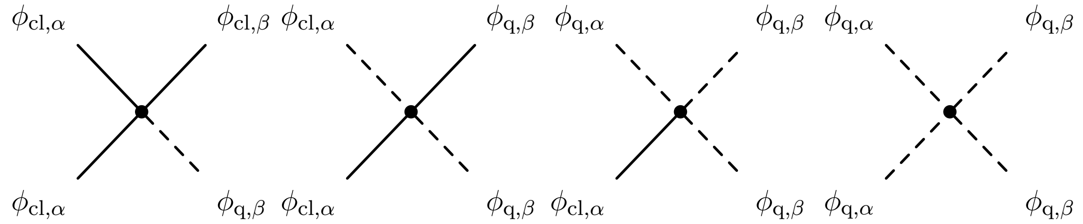


Figure 4.1: *Four different types of vertices, which must be treated separately in an out-of-equilibrium RG. The index* cl/q *refers to the classical or quantum field, α/β to the component of the vector. All those vertices can be generated, if one starts the RG with the usual $\varphi_{\mathrm{cl}}^3\varphi_{\mathrm{q}}$ and $\varphi_{\mathrm{cl}}\varphi_q^3$ vertices. Note that the generation of a vertex type of the form $\varphi_{\mathrm{c},\alpha}^2\varphi_{\mathrm{q},\beta}^2$ under the RG flow, is impossible due to causality. In the classical limit, only the first vertex $\varphi_{\mathrm{cl}}^3\varphi_{\mathrm{q}}$ is relevant under the RG flow.*

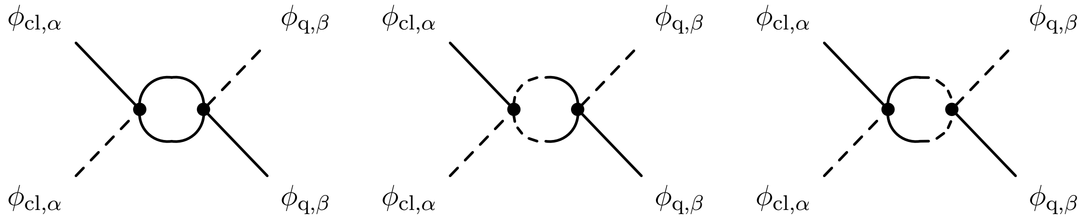


Figure 4.2: *Three examples for the generation of a new vertex type during the RG flow. This generation is only possible out of equilibrium, where the FDT does not hold. In equilibrium at temperature $T = 0$, the sum of those three diagrams is zero.*

4.2 Renormalization group equations

To derive the out-of-equilibrium version of the RG-equations, one starts in full analogy to the equilibrium momentum shell RG presented in section 1.3.2. Supplementary to the equilibrium RG, now new parameters emerge: the initial configuration $R_{\mathrm{i}} = (r_{\mathrm{i}}, u_{\mathrm{i}}, h_{\mathrm{i}})$. For simplicity, only a mass quench $R_{\mathrm{i}} = (r_{\mathrm{i}}, 0, 0)$ is considered in this section. This formulation for of the RG equations for a quench to the QCP was presented in Ref. [30]. To start the RG, we need the action along the two branch-Schwinger-Keldysh contour. The non-interacting part S_0 reads

$$S_0[\varphi] = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int_0^\infty \mathrm{d}t\, \mathrm{d}t' \times \left(\varphi^{\mathrm{cl}}(-k,t), \varphi^{q}(-k,t)\right) \begin{pmatrix} 0 & \left[g^R\right]^{-1}(k,t,t') \\ \left[g^A\right]^{-1}(k,t,t') & \left[g^K\right]^{-1}(k,t,t') \end{pmatrix} \begin{pmatrix} \varphi^{\mathrm{cl}}(k,t') \\ \varphi^{\mathrm{q}}(k,t') \end{pmatrix}, \tag{4.44}$$

with the bare post-quench Green's functions g^R and g^K, calculated in section 3.5. Information about the quench are captured uniquely in the Keldysh function, g^R is the same as in equilibrium. The interaction part reads

$$S_{\mathrm{int}}[\varphi] = \frac{u}{4N} \int \mathrm{d}^d x \int_0^\infty \mathrm{d}t\, \varphi_i^{\mathrm{cl}}(x,t)\varphi_i^{\mathrm{q}}(x,t) \left(\varphi_j^{\mathrm{cl}}(x,t)\varphi_j^{\mathrm{cl}}(x,t) + \varphi_j^{\mathrm{q}}(x,t)\varphi_j^{\mathrm{q}}(x,t)\right). \tag{4.45}$$

For the RG flow one must generally treat the two different vertex types $\left(\varphi^q\right)^3\varphi^{\mathrm{cl}}$ and $\varphi^q\left(\varphi^{\mathrm{cl}}\right)^3$ separately, as they flow differently under RG. Further, every possible combination of vertex types should be included, with $u_n\left(\varphi^{\mathrm{cl}}\right)^{4-n}\left(\varphi^{\mathrm{q}}\right)^n$, with $1 \leq n \leq 4$. In figure 4.1 the four possible vertex types are depicted. Even if such a vertex like for example u_2 is not included initially, it can be generated out-of-equilibrium by one-loop corrections under the RG-flow, see for example figure 4.2). For a detailed analysis of the out-of-equilibrium flow equations for the different u_{i}'s see for example Ref. [65]. In this thesis, the out-of-equilibrium flow equations for the different interactions parameters are not considered in detail. Below, it will be shown that the time dependence for u enters at least at order ϵ^2, resulting at this level of perturbation theory in the unchanged equilibrium results for $u(t) = u^*$. Therefore, different vertices are not introduced here to keep the notation as simple as possible.

To apply the momentum shell RG concept, one formally integrates out fast modes with momentum in a shell with thickness Λ/b around the cutoff Λ. In the next step, the slow modes are rescaled according to

$$k \to k' = bk, \tag{4.46}$$

$$t \to t' = b^{-z}t, \tag{4.47}$$

$$\varphi_i^{\mathrm{q/cl}} \to \varphi_i'^{\mathrm{q/cl}}(k', t') = b^{-\frac{d-z+2}{2}}\varphi_i^{\mathrm{q/cl}}(bk, b^{-z}t). \tag{4.48}$$

In general, the scaling exponent of the quantum and classical fields have not to be the same. Here, ρ is chosen such that $\rho = \frac{d-z-2}{2}$ for both fields, to obtain the results for the equilibrium quantum critical point at large times. Compared to Eq. (1.30), z has to be replaced by $-z$, as here the fields are expressed in time-space and not by their frequency dependence. The flow equation for r_{i} can be read off in the bare Keldysh function. The leading term for large r_{i} reads

$$r_{\mathrm{i}}' = b^2 r_{\mathrm{i}} \tag{4.49}$$

independently of the value of z. The initial mass r_{i} is thus strongly relevant and grows rapidly under the RG flow. Its fixed point value $r_{\mathrm{i}}^* = \infty$ is the justification for the deep quench limit. With this rescaling procedure the zero order results are obtained for r and u, where both parameters now depend on the time and the initial mass

$$r'(t', r_{\mathrm{i}}') = b^2 r(b^z t, b^{-2} r_{\mathrm{i}}), \tag{4.50}$$

$$u'(t', r_{\mathrm{i}}') = b^{\epsilon} u(b^z t, b^{-2} r_{\mathrm{i}}). \tag{4.51}$$

The parameter $\epsilon = 4 - d - z$ again indicates that below the upper critical dimension, first order terms have to be included, as u flows away from the Gaussian fix point. One-loop corrections to r can be included with the same diagrammatic language as in equilibrium, however, now the Green's function is time dependent.

$$r'\left(t, r_{\mathrm{i}}\right) = b^2 r\left(b^z t, b^{-2} r_{\mathrm{i}}\right) + u\frac{N+2}{2}\int^{>}\frac{\mathrm{d}^d k}{(2\pi)^d}\mathrm{i}\,G^K\left(b^{-2}r_{\mathrm{i}}, k, t, t\right). \tag{4.52}$$

The flow equation of u is more complex, as each vertex must be treated separately. However, the one-loop correction has a similar structure as in equilibrium, it is a convolution of two Green's functions with a prefactor of u^2, e. g.

$$u'\left(t, r_{\mathrm{i}}\right) = b^{4-d-z}u - u^2\int^{>}\frac{\mathrm{d}^d k}{(2\pi)^d}G^R(t, t')G^K\left(b^{-2}r_{\mathrm{i}}, k, t', t\right) + \ldots, \tag{4.53}$$

where the dots refer to terms which can be generated from different types of vertices. Here are two remarks: First, the time dependence of r and u is a consequence of the broken time invariance due to the quench. In the following an ϵ expansion around the QCP with its fixed point values (r^*, u^*) will be performed. Since the time dependence of u is at least of order ϵ^2 it will be ignored. In section 4.4 it is shown via a different approach that this assumption is indeed valid and $u(t)$ approaches its equilibrium value u^* exponentially fast. Therefore u is replaced by its equilibrium fixed point value

$$u^* = \frac{\Lambda^\epsilon \epsilon}{(N+8)K_d} \times \begin{cases} 1 & \text{for a closed system,} \\ \frac{4\gamma^{-2/z}}{z(2-z)} \left[\cos\left(\frac{\pi z}{2}\right)\right]^{1-2/z} & \text{for an overdamped system.} \end{cases} \tag{4.54}$$

A second remark concerns the initial mass. In general r is now also a function of the initial mass r_i. Here, the deep quench limit $r_\mathrm{i} \to \infty$ is taken in the Keldysh function. As it was shown in section 3.5 this limit is well defined for $z > 1$ and near the upper critical dimension $d_{uc} = 4 - z$. In the remaining section, the initial mass r_i will be dropped in the argument of r, the deep quench limit in the Keldysh function is taken. A more detailed analysis how to take and interpret the deep quench limit in the isolated system is given in chapter 6.

By introducing a small parameter l via $b \simeq 1 + l$, the shell integral can approximately evaluated. Further, only the scaling part of the Keldysh function

$$G^K_{sc}(k,t) = \frac{1}{k^{2-z}} F^K(k^z t) \tag{4.55}$$

will be considered. This scaling part is responsible for the scaling form of the Green's functions, and only this part will create log-divergent terms, as discussed in the next two chapters. With this simplification the scaling equation can be written as

$$r'(t) = e^{2l} r\left(e^{zl} t\right) + (N+2) K_d u^* \Lambda^2 F^K(\Lambda^z t) l. \tag{4.56}$$

From this equation the flow equation for the mass can be derived

$$\frac{dr}{dl} = 2r(t) + zt\frac{dr(t)}{dt} + (N+2)K_d u^* \Lambda^2 F^K\left(\Lambda^z t\right). \tag{4.57}$$

At the fixed point it holds $\frac{dr^*(t)}{dl} = 0$, leading to a linear, inhomogeneous, first order differential equation for $r^*(t)$,

$$2r^*(t) + zt\frac{dr^*(t)}{dt} + (N+2)K_d u^* \Lambda^2 F^K\left(\Lambda^z t\right) = 0, \tag{4.58}$$

with the solution

$$r^*(t) = \frac{a}{t^{2/z}} - \frac{(N+2)K_d u^* \epsilon \Lambda^2}{z t^{2/z}} \int_{t_0}^{t} \mathrm{d}t'\, t'^{\frac{2-z}{z}} F^K(\Lambda^z t'). \tag{4.59}$$

Here a is an integration constant, which will be evaluated below. Note that if the system is coupled to the bath, it will equilibrate and the scaling function F^K approaches its equilibrium value $F^K(\infty) = F^K_{\mathrm{eq}}$ at large times. Within a perturbative RG approach a long-time decay of the mass fixed point should not emerge, and r^* should reach its fixed point value $r^*_{\mathrm{eq}} = (N+2)K_d u^* \epsilon$ exponentially fast. This results in a condition for the integration parameter a:

$$a = \frac{(N+2)u^*}{2z} \int_0^\infty \mathrm{d}x \left(F^K(x) - F^K_{\mathrm{eq}}\right) x^{\frac{2-z}{z}}. \tag{4.60}$$

From the condition for a the value of the universal short-time exponent θ can be derived.

4.3 Large-N equations

In this section the large-N expansion, presented in section 1.4 is extended to the three-branch contour, introduced in section 3.2. The action along the three-branch contour can be evaluated within a saddle point approximation, which becomes correct in the limit $N \to \infty$. The controlling parameter in this expansion is the inverse number of field components $1/N$. The strategy for deriving the self-consistent equations is the same as in equilibrium, however now with the HS-transformation performed along the three-branch contour. Instead of introducing one Hubbard-Stratonovich field, one field ρ_j for each branch $j = M, +, -$ of the contour is necessary. After the Keldysh rotation one thus obtains $\boldsymbol{\rho} = (\rho_M, \rho_c, \rho_q)^T$. The same must be done for the order parameter field h, which is now a vector along the contour $\boldsymbol{h} = (h_M, h_c, h_q)^T$. After integrating out the fields $(\varphi_M, \varphi_c, \varphi_q)$ along the contour, this leads to three effective masses (r_M, r_c, r_q) for each contour respectively. Performing the saddle point approximation, nine saddle point equations are found. Those equations are coupled, not only due to the external bath, but also due to the φ^4-interaction term. This derivation is done in detail in Ref. [31] and is therefore not repeated here. The final result is the following:
Along the Matsubara axis the self-consistent equations for the field h_M and the effective mass r_M are given by,

$$h_M = r_M \phi_M \,, \tag{4.61}$$

$$r_M = \bar{r}_{0,\mathrm{i}} + \frac{u_\mathrm{i}}{2}\phi_M^2 + u_\mathrm{i} G_r^M \left(x, \tau; x, \tau\right), \tag{4.62}$$

with the Matsubara-Green's function $G_M^{-1} = \omega_M^2 + r + k^2 + \eta(\omega_M)$, derived in section 3.2. This set of equations, which has to be solved self-consistently, is the usual equilibrium result. It is, as causality implies, independent of the post-quench system. From those equations one obtains the known equilibrium exponents $h \propto \delta r^\beta$, with $\beta = 1/2$ and $\xi \propto \delta r^\nu$ with $\nu = 1/(d + z - 2)$ below the upper critical dimension d_{uc}. Above d_{uc} all exponents take their mean field value.
For the post-quench system, only the classical component of the effective mass r_c and the field h_c is non-zero, while for the quantum components only the trivial solution $h_q = r_q = 0$ is possible. This is again a consequence of causality. For the classical components of field and mass one finds

$$h_\mathrm{c} = \int \mathrm{d}t' \, G_r^{R-1} \left(t, t'\right) \phi_\mathrm{c} \left(t'\right) + \phi_M \int_\tau \tilde{\eta} \left(\mathrm{i}\,\tau, t\right), \tag{4.63}$$

$$r\left(t\right) = \bar{r}_{0,\mathrm{f}} + \frac{u_\mathrm{f}}{2}\phi_\mathrm{c}^2\left(t\right) + \frac{u}{2}\mathrm{i}\, G_r^K \left(x, t; x, t\right). \tag{4.64}$$

Here $\int_\tau \tilde{\eta}\left(\mathrm{i}\,\tau, t\right)$ is the coupling across the quench due to the external bath. G_r^R is the retarded function, which now, due to the time-dependent mass also displays aging effects:

$$G_r^{R^{-1}}(t, t', k) = -\left(\partial_t^2 + r(t) + k^2\right)\delta(t - t') + \delta\eta(t - t'). \tag{4.65}$$

The Keldysh function is given by

$$G^K(t, t', k) = \int_{s,s'} G_r^R(t, s, k) M_{r_M}(s, s', k) G_r^A(s', t', k). \tag{4.66}$$

This function has still the same structure as in Eq. (3.60), but now, the bare Green's functions as well as the memory function have to be replaced by the dressed ones.

By expressing the final mass $r_{0,\mathrm{f}}$ via the equilibrium Keldysh function G^K_{eq} and using the critical point property $r_\mathrm{f} = 0$, the large-N equation can be brought to a more convenient form

$$r(t) = \frac{u}{2}\int_k \Big(G^K(r_t, t, k) - G^K_{\mathrm{eq}}(0, t)\Big)\,. \tag{4.67}$$

This equation has the same structure as the RG result in Eq. (4.60). By using the scaling form of $G^K(r_t, t, k)$, one can show that both equations indeed result in the same condition for a. The relation of the RG and the large-N equation is discussed in the next section.

4.4 Connection between RG and large-N

The time-dependent mass equations obtained via RG in Eq. (4.60) and $1/N$ in Eq. (4.67) look very similar. Indeed, it can be shown via scaling arguments, that they lead to the same value for a, of course, taken in the appropriate limit $N \to \infty$ and near the upper critical dimension. Taking the scaling form of the Keldysh-function yields for Eq. (4.67):

$$\begin{aligned} r(t) =& \frac{u}{2}\int_k \Big(G^K(r_t, t, k) - G^K_{\mathrm{eq}}(0, t)\Big) \\ =& \frac{uK_d}{2zt^{(2-\epsilon)/z}}\int_0^{\Lambda t} \mathrm{d}x\, x^{\frac{2-z-\epsilon}{z}}\Big(F^K(x) - F^K_{\mathrm{eq}}\Big)\,. \end{aligned} \tag{4.68}$$

This equation is in the limit $\epsilon \to 0$ identical to the condition for a, derived via the RG-flow equation for the fixed point r^*. The basic ingredient is the assumption of a scaling form $G^K(k, t, t) = k^{-2+z}F^K(k^z t)$. The goal of this section is to explicitly derive the RG flow equations, starting from the self-consistent equation obtained in the large-N limit. This is instructive, as it allows to derive a flow equation for u which shows that $u(t)$ approaches its equilibrium fixed point value exponentially fast. It further highlights once more the scaling connection between momentum and time and the role of reaching a stationary state for large times.

The large-N Eq. (4.64) for a quench starting in the symmetric phase reads:

$$r(t) = r_{0,\mathrm{f}} + \frac{u}{2}\int^{\Lambda} \frac{\mathrm{d}^d k}{(2\pi)^d}\mathrm{i}\, G^K(r(t), k, t, t)\,. \tag{4.69}$$

In this section, the short notation $r(t) = r_t$ and for the equal-time Keldysh function $G^K(r_t, q, t, t) = G^K(r_t, q, t)$ is used. Further, as the deep quench limit can be taken in the Keldysh function, the irrelevant parameter r_i^{-1} is skipped in the argument of the effective mass. In the RG spirit, the momentum integral is split into a spherical zone with slow modes with $|k| < \Lambda/b$ and a thin shell with fast modes $\Lambda/b \le |k| < \Lambda$. The thickness of the shell can also approximately be expressed via the small parameter $l = \log b$. This yields

$$\int^{\Lambda} \frac{\mathrm{d}^d k}{(2\pi)^d}\mathrm{i}\, G^K(r_t, k, t) = \int^{\Lambda/b} \frac{\mathrm{d}^d k}{(2\pi)^d}\mathrm{i}\, G^K(r_t, k, t) + K_d \Lambda^d \mathrm{i}\, G^K(r_t, \Lambda, t) l \tag{4.70}$$

Note, that the Keldysh function $G^K(r_t, \Lambda, t)$ is evaluated at the cutoff $k = \Lambda$. As $t \gg \Lambda^{-1}$, it is possible to use the long-time expansion derived in section 4.1.2. This corresponds equally to a Taylor expansion of G^K around $r(t) = 0$, yielding

$$\begin{aligned} r(t) \simeq & r_{0,\mathrm{f}} + \frac{u}{2} b^{2-z-d} \int^{\Lambda} \frac{\mathrm{d}^d k}{(2\pi)^d} \mathrm{i}\, G^K \left(r'(t'), k, t' \right) \\ & + \frac{u K_d \Lambda^d}{2} \left(\mathrm{i}\, G^K \left(0, \Lambda, t\right) + \frac{\partial \mathrm{i}\, G^K \left(r, \Lambda, t\right)}{\partial r} \bigg|_{r=0} r(t) \right) l \\ \simeq & \frac{r_{0,\mathrm{f}} + \frac{u}{2} b^{2-z-d} \int^{\Lambda} \frac{\mathrm{d}^d k}{(2\pi)^d} \mathrm{i}\, G^K \left(r'(t'), k, t' \right) + \frac{u K_d \Lambda^d}{2} \mathrm{i}\, G^K \left(0, \Lambda, t\right) l}{1 - \frac{u K_d}{2} \frac{\partial \mathrm{i}\, G^K \left(r, \Lambda, t\right)}{\partial r} |_{r=0}\, l} . \end{aligned} \tag{4.71}$$

In the last line, the expansion for small l and

$$r(t) \simeq r_{0,\mathrm{f}} + \frac{u}{2} b^{2-z-d} \int^{\Lambda} \frac{\mathrm{d}^d k}{(2\pi)^d} \mathrm{i}\, G^K \left(r'(t'), k, t' \right), \tag{4.72}$$

was used, to write the self-consistent equation in a more convenient way. Now, every term can be rescaled according to the appropriate scaling law, especially $b^2 r(b^z t) = r'(t')$. As $r_{0,\mathrm{f}}$ will become time dependent under this procedure, one needs to write $r_{0,\mathrm{f}} = r_0(t)$. For the same reason, it is also necessary to consider $u = u(t)$. It holds

$$r'(t') = \frac{b^2 r_{0,\mathrm{f}}(t) + \frac{u(t)}{2} b^{4-d-z} \int^{\Lambda} \mathrm{d}^d q\, \mathrm{i}\, G^K(r'(t'), q, t') + \frac{u(t) K_d \Lambda^d}{2} \mathrm{i}\, G^K(0, \Lambda, t) l}{1 - \frac{u(t) K_d \Lambda^d}{2} \frac{\partial \mathrm{i}\, G^K(r, \Lambda, t)}{\partial r} |_{r=0} l} . \tag{4.73}$$

At this point, it is possible to derive the flow equations for the different parameters $r(t)$ and $u(t)$. For the time-dependent mass the differential equation of section 4.2 is obtained:

$$\frac{dr}{dl} = 2r + \frac{\partial r}{\partial t} t z + \frac{u K_d \Lambda^d}{2} \mathrm{i}\, G^K(r(t), \Lambda, t)\,, \tag{4.74}$$

where again only terms up to linear order for small l have been included. For the rescaled interaction parameter the scaling equation is given by:

$$\begin{aligned} u'(t') = & \frac{u(t) b^{4-d-z}}{1 - \frac{u K_d \Lambda^d}{2} \frac{\partial \mathrm{i}\, G^K(r, \Lambda, t)}{\partial r} \bigg|_{r=0} l} \\ \simeq & u(t) b^{4-d-z} + \frac{u^2(t) K_d \Lambda^d}{2} \frac{\partial \mathrm{i}\, G^K(r, \Lambda, t)}{\partial r} \bigg|_{r=0} l. \end{aligned} \tag{4.75}$$

This results in a flow equation for u

$$\frac{du(t)}{dl} = \epsilon u(t) + \frac{du}{dt} z t + \frac{u^2(t) K_d \Lambda^d}{2} \frac{\partial \mathrm{i}\, G^K(r_t, \Lambda, t)}{\partial r} \bigg|_{r=0} . \tag{4.76}$$

At the fixed point it holds $\frac{du^*}{dl} = 0$, which leads to a nonlinear differential equation for $u^*(t)$. This differential equation can however be solved by using the long-time expansion for $G^K(r_t, \Lambda, t)$ which was derived in Eq. (4.23). The assumption for the long-time expansion $t\Lambda^z \gg 1$ is always fulfilled in the φ^4-model, as Λ^{-1} is one of the microscopic timescales, below which the model is no longer valid. The final result of the long-time expansion is

$$G^K(r_t, \Lambda, t) \simeq G^K(0, \Lambda, t) + 2\mathrm{i}r(t)c_K, \tag{4.77}$$

with the coefficient $c_K = \int \frac{\mathrm{d}\omega}{2\pi} G^R_{\mathrm{eq}}(\omega) G^K_{\mathrm{eq}}(\omega)$. Inserting this result into Eq. (4.76) one finds

$$\epsilon u^*(t) + \frac{du^*}{dt} zt - u^{*2}(t) K_d \Lambda^d c^K = 0\,. \tag{4.78}$$

Before writing the solution for $u(t)$, it is helpful to note, that the equilibrium fixed point $u^*_{\mathrm{eq}} = \epsilon/(K_d \Lambda^d C^K)$ fulfills this non-linear differential equation. The differential equation has the general solution

$$u^*(t) = \frac{\epsilon}{K_d \Lambda^d c^K} \frac{1}{1 + e^{-\epsilon\alpha} t^{\epsilon/z}/(K_d c^K \Lambda^d)}, \tag{4.79}$$

where α is an integration constant, which has to be fixed by the boundary conditions. An open system will thermalize due to the coupling to the external bath. In this case, one expects $u^*(t \to \infty) = u^*_{\mathrm{eq}}$, which is only possible for $\alpha \to \infty$. For any finite α, the time dependence $t^{-\epsilon/z}$ will lead to $u^*(t \to \infty) = 0$. Thus, with $\alpha \to \infty$, u^* must reach its fixed point value on microscopic time-scales, which is only possible for an exponential fast decay right after the quench. In contrast to the differential equation for $r(t)$ no further conditions and constraints are stored in the differential equation for u. Thus only one new exponent is sufficient to describe the post-quench dynamics. The interaction parameter will reach its equilibrium value exponentially fast on timescales smaller than t_{mic}.

4.5 Formal solution for the time-depended mass $r(t)$

In this section, the RG-Eq. (4.60) and the self-consistent Eq. (4.67) for the time-dependent mass $r(t)$ will be solved. To handle both cases at the same time, the appropriate limit of small ϵ and $N \to \infty$ is taken. The extension to finite N is straightforward. The result of this section is:

$$r(t) = \frac{c_K C_0 \epsilon}{2z} \frac{1}{t^{2/z}}. \tag{4.80}$$

With the coefficient

$$c_K = \int \frac{\mathrm{d}\omega}{2\pi} G^R_{\mathrm{eq}}(\omega) G^K_{\mathrm{eq}}(\omega), \tag{4.81}$$

which is an universal coefficient from the fixed point value of u^* and

$$C_0 = \int_0^\infty \mathrm{d}x\, x^{2/z-1} \left(\mathrm{i}\, f^K(x,1) - \mathrm{i}\, F^K_{\mathrm{eq}} \right) . \tag{4.82}$$

The universal constant C_0 is the difference between the bare post-quench Keldysh function and its equilibrium value. The emergence of the factor guarantees that the exponent θ is indeed new in the sense that it cannot be expressed by equilibrium exponents. The result also shows that θ is universal, as it can be expressed uniquely with dimensionless integrals over scaling functions. Being proportional to the distance ϵ to the upper critical dimension justifies a controlled expansion in $r(t)$, as it was done

e. g. in the Dyson equation. And finally it shows, that the scaling ansatz is indeed a self-consistent solution to the post-quench problem.

To solve the equation

$$r(t) = \frac{uK_d}{2}\int_0^\infty \mathrm{d}k\, k^{3-z-\epsilon}\left(\mathrm{i}\, G^K(k,t,t) - \mathrm{i}G^K_{\mathrm{eq}}(k)\right) \tag{4.83}$$

self-consistently, one further expansion is needed, an expansion for small ϵ, hence near the upper critical dimensions. Since the interaction parameter u is of order ϵ, see also section 4.4. At the first glance, it would be sufficient to expand the Keldysh function around $r(t) = 0$, and thus replace $G^K(q,t,t)$ in the integral by its bare value $g^K(q,t,t)$. This integral is exponentially convergent for any open system. For discussions of an isolated system see chapter 6. But already the simple long-time expansion with the Dyson equation shows, that thermalization is significantly slowed down by interactions, leading to corrections going with $G^K_1(k,t,t) \simeq r(t)k^{-4+z}$. Such corrections generate $1/\epsilon$-terms under the d-dimensional momentum integral,

$$\int_{k_0}^{\Lambda} \mathrm{d}k\, \frac{1}{k^{1+\epsilon}} = -\frac{1}{\epsilon}\left(\Lambda^{-\epsilon} - k_0^{-\epsilon}\right), \tag{4.84}$$

where k_0 is some lower cutoff, introduced below. Multiplied with $ur(t)$ those corrections are also linear in ϵ and have thus to be included. Thus, up to first order in a controlled ϵ-expansion, one has to include not only the bare post-quench Green's function, but also the first order corrections from the long-time limit. As the long-time expansion is well controlled for $t \gg k^z$, and for short times the Green's functions are independent of the momentum k, see section 4.1.3, no further terms can generate $1/\epsilon$ under the integral. Here, it is useful to interpret large times, as $k \gg t^{-1/z}$, which leads to a natural lower bound of the momentum integral. Thus, by expanding $G^K_r(k,t,t) = g^K(k,t,t) + G^K_1(k,t,t)$, where G^K_1 contains only first order terms generating $1/\epsilon$ under the integral, one can solve the equation for the time-dependent mass self-consistently

$$r(t) = \frac{uK_d}{2}\int_0^\infty \mathrm{d}k\, k^{3-z-\epsilon}\left(\mathrm{i}\, g^K(k,t,t) - \mathrm{i}G^K_{\mathrm{eq}}(k,0)\right) + \frac{uK_d r(t)}{c_K\epsilon\gamma^{z/2}}\left(\Lambda^{-\epsilon} - \left(\frac{\gamma^{1/2}}{t^{1/z}}\right)^{-\epsilon}\right). \tag{4.85}$$

This equation must hold for all times after the quench. Thus, the time dependence of terms going with $r(t)$ and with some different time dependence, the momentum integral over $g^K(k,t,t) - iG^K_{\mathrm{eq}}(k,0)$ and $r(t)t^{-\epsilon/z}$, have to cancel separately. This leads to two different conditions, which have to be fulfilled at the same time, one for the interaction parameter and one for the time-dependent mass. Note, that this form is completely general and holds for both dynamic regimes. It is only based on the assumption of thermalization in the long time limit. In Ref. [30] and Ref. [31], the scaling form of the Keldysh function was used, to extract those two conditions. Going over to the scaling form, makes it more clear, which terms have to cancel, but it is not necessary to solve the $1/N$-equation in full generality. For the RG solution it is however elementary to be near the deep-quench fixed point.
The condition for the interaction parameter is fixed by the long-time expansion of the Dyson equation:

$$u = \frac{c_k\epsilon\Lambda^\epsilon}{K_d}. \tag{4.86}$$

This is again the equilibrium fixed point value u^*. It shows, how strongly the assumption of equilibration at large times is related with the fixed point value of u^*. Within the large-N approach, it is not necessary

to tune u to this precise value u^* to observe non-equilibrium scaling. $u = u^*$ is a consequence of using the long-time limit of G^K and an ϵ-expansion.
The time-dependent mass is given by

$$r(t) = \frac{c_K \epsilon t^{\epsilon/z}}{2} \int_0^\infty \mathrm{d}k\, k^{3-z-\epsilon} \left(i g^K(k,t,t) - \mathrm{i}\, G^K_{\mathrm{eq}}(k,0) \right) . \tag{4.87}$$

As mentioned above, it is not necessary to assume in the $1/N$ equation a pure scaling form in g^K. Neither it is necessary that the deep-quench limit is well defined. In contrast, the amplitude dependent parts are still relevant in the prethermal regime, for the closed as well as for the open system. However, using only the deep-quench scaling part of the bare post-quench Keldysh function, derived in section 3.5.3, one can introduce a dimensionless constant

$$C_0 = \int_0^\infty \mathrm{d}x\, x^{2/z-1} \left(i f^K(x,1) - i F^K_{\mathrm{eq}} \right) . \tag{4.88}$$

And one finally finds

$$r(t) = \frac{c_K C_0 \epsilon}{2 z t^{2/z}} . \tag{4.89}$$

The physical implications of this effective mass are discussed in the next two chapters for two different models.

4.6 Comparison of the developed non-equilibrium methods

Three different methods have been presented in this chapter to analyze the post-quench dynamics of an interacting system. The Dyson equation and thus an expansion in a small self-energy is a useful approach to obtain the long-time limit of the Green's function. Essential is however the knowledge of this final, equilibrium state. With this approach, it was shown that also in the quasi-adiabatic long-time limit the Green's function displays aging effects. Those aging effects originate from the thermalization to the QCP, independently of the assumed decay in time of $r(t)$. For short times, this approach fails, as the scaling solution r_{sc} leads to logarithmic corrections, and thus more effective methods must be developed to sum up those logarithmic singularities in a controlled way. We presented two such methods, the extension of the RG and the large-N method to the non-equilibrium post-quench dynamics. The result in the RG leads to

$$\begin{aligned} 0 =& \frac{a}{t^{2/z}} - \frac{(N+2) K_d u^* \epsilon \Lambda^2}{z t^{2/z}} \int_{t_0}^{t} \mathrm{d}t'\, t'^{\frac{2-z}{z}} \left(F^K(\Lambda^z t') - F^K_{\mathrm{eq}} \right) \\ =& \frac{a}{t^{2/z}} - \frac{(N+2) K_d u^* \epsilon \Lambda^{4-z}}{z t^{2/z}} \int_{t_0}^{t} \mathrm{d}t'\, t'^{\frac{2-z}{z}} \left(G^K(\Lambda, t', t') - G^K_{\mathrm{eq}}(\Lambda, 0) \right) . \end{aligned} \tag{4.90}$$

The parameter a is an integration constant. Here, a scaling form for the Keldysh function is used, $G^K(\Lambda,t,t) = \Lambda^{-2+z} F^K(\Lambda^z t)$, to make the connection to the large-N result visible:

$$r(t) = \frac{u}{2} \int_{|\boldsymbol{k}|<\Lambda} \frac{\mathrm{d}^d k}{(2\pi)^d} \left(G^K(r_t, t, k) - G^K_{\mathrm{eq}}(0,t) \right) . \tag{4.91}$$

Both equations lead to the same condition for a, by taking the appropriate limit $N \to \infty$ and $\epsilon \ll 1$, if G^K obeys a scaling form. It is important, that it is not necessary to assume a scaling form $r(t) = a t^{-2/z}$.

On the contrary this effective mass is an explicit result of the post-quench dynamics. Thus, it is also possible to calculate the dynamics for systems where the ansatz for $r(t)$ is less clear from scaling arguments, for example at the upper critical dimension [31]. It is also not necessary to know the full solution of the EOM with $r(t)$, but we present a method how to include the effective mass perturbatively, by paying attention to the generation of $1/\epsilon$ terms. The price is to perform also an ϵ-expansion in the $1/N$ equation, which is in general not necessary. Here however, complications arise from the interplay of the coupling to the bath and the time-dependent mass. Thus the full solution for the equation of motion for φ is not known, making a further expansion in a small interaction parameter u and thus small $r(t)$ necessary. The final result for the effective mass reads:

$$r(t) = \frac{c_K \epsilon t^{\epsilon/z}}{2} \int_0^\infty \mathrm{d}k\, k^{3-z-\epsilon} \left(\mathrm{i}\, g^K_{\mathrm{sc}}(k,t,t) - \mathrm{i}\, G^K_{\mathrm{eq}}(k,0) \right) . \tag{4.92}$$

Where g^k_{sc} refers to the part of the Keldysh function which obeys scaling, thus where the deep-quench limit can be performed. This equation is evaluated for the open system in chapter 5 and for the isolated system in chapter 6.

5 Chapter 5

Post-quench dynamics in an open system

In this chapter, the post-quench dynamics of an open system are analyzed. Therefore, the result for the inverse correlation length $r(t)$, derived in the previous chapter in Eq. (4.80), is evaluated explicitly. Here, the φ^4-model is coupled to an external bath of harmonic oscillators (see section 1.1). Due to this external bath, the system can dissipate the energy induced by the quench. The bath is expected to stay in equilibrium at zero temperature for all times after the quench, thus the QCP is reached in the limit $t \to \infty$. In this chapter, only $z > 1$, and thus bath exponents $\alpha < 2$ are considered, as only for those kinds of baths the equilibrium dynamics are dominated by the diffusive term.
The non-equilibrium dynamics of an open system quenched to the QCP are analyzed in Refs. [30, 31]. Especially, it will be shown, that the scaling form of the effective mass,

$$r_{\mathrm{sc}}(t) \propto t^{1/z}\,, \tag{5.1}$$

allows to analytically determine the time dependence of the order parameter as well as the Green's functions in two different time regimes: the prethermal regime at intermediate times and a quasi-adiabatic decay to equilibrium. The basic ingredient to obtain the effective mass in Eq. (5.1) is the deep-quench limit. The conditions for taking this deep-quench limit is discussed in section 5.1. It will be shown, that this scaling term is always present for times larger than some microscopic time scale. However also quench-amplitude dependent terms emerge, which lead to a ballistic growth in the correlation length in the prethermal regime. In section 5.2 the Heisenberg equation of motion for the order-parameter $\langle \boldsymbol{\varphi} \rangle(t)$ are derived. This equation is evaluated in both time regimes. In the prethermal regime the scaling form, postulated in chapter 2 is confirmed. Especially, the order-parameter dynamics are given by

$$\langle |\boldsymbol{\varphi}(t)| \rangle \propto t^{\theta}\,, \tag{5.2}$$

with universal exponent θ, which is connected to the light-cone amplitude a by Eq. (4.40). The value of the θ can directly be evaluated with Eq. (4.80). For an Ohmic bath we find analytically $\theta = \epsilon/4$, with $\epsilon = 4 - d - z$. The general z-dependence of θ can be evaluated numerically and is depicted in figure 5.2. The implications of the universal exponent θ/ϵ in the prethermal regime are discussed in section 5.3. For the long time dynamics the known equilibrium exponents are recovered in the order-parameter dynamics, with θ entering in an universal amplitude. Such effects are also known in classical systems [66]. Further, it is possible to extend the FDT in the quasi-adiabatic long time limit to calculate time depended corrections to the Bose-Einstein distribution function n_B. In section 5.5

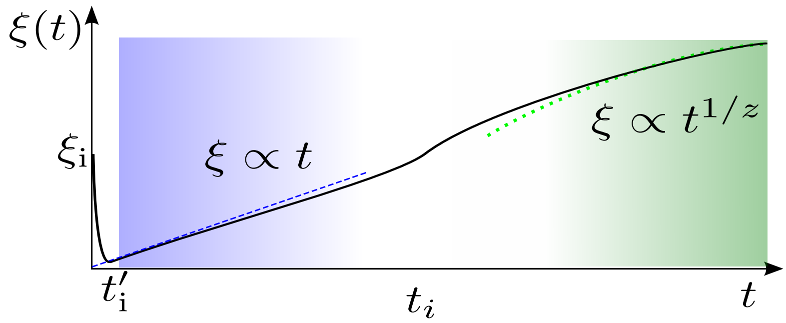


Figure 5.1: *The growth of the correlation length $\xi(t)$ after the quench. For times $t < t_{\rm i} = \omega_{\rm i}^{-z}\gamma^{z/2}$, the correlation length ξ grows approximately with a light-cone (blue dashes), while for $t > t_{\rm i}$, in the long time limit, ξ grows approximately with the predicted scaling form $t^{1/z}$ (green dots).*

the hierarchy of the different short-time scales is discussed. In the last section 5.6, the classical limit is taken, where the results from Refs. [28, 29] are recovered. This limit is not only a double check for the results in the quantum limit, but is also instructive to understand the difference between classical and quantum quenches.

5.1 The deep quench limit and the correlation length

The bare post-quench Keldysh function is the essential ingredient for analyzing the post-quench dynamics, as its equal time value $g^K(k,t,t)$ determines the effective mass $r(t)$. To obtain the scaling form

$$r_{\rm sc}(t) = at^{-2/z}\,, \tag{5.3}$$

with quench amplitude independent coefficient a, it is necessary to take the so called deep quench limit in g^K, where the quench amplitude $\omega_{\rm i}$ is formally sent to infinity. In this limit, there are terms in $g^K(k,t,t)$, which are independent of the quench amplitude and independent of $\omega_{\rm i} \lessgtr \omega_\gamma$. Those terms generate the scaling form of $r_{\rm sc}(t)$. But in the expansion around $\omega_{\rm i}^{-1} \to 0$, there are also terms in $g^K(k,t,t)$, which are relevant and strongly dependent on the quench amplitude. At short times after the quench, those terms can be larger than the scaling solution and thus determine up to leading order $r(t)$. They are analyzed in appendix B, where it is shown, that for times $t \gg t_{\rm i} = \omega_{\rm i}^{-z}\gamma^{z/2}$ the deep-quench limit can always be performed, without divergent terms for $\omega_{\rm i} \to 0$. In this limit, the correlation length, $\xi(t) \propto r(t)^{-1/2}$ is given by its scaling form:

$$\xi\,(t \gg t_{\rm i}) \propto t^{-1/z}\,. \tag{5.4}$$

For times smaller than $t_{\rm i}$ second order corrections emerge supplementary to the scaling part in $r(t)$, which is still present. The prefactor of those corrections can be larger than the light-cone amplitude. The time dependence however of those dominant terms goes always with

$$\xi\,(t \ll t_{\rm i}) \propto t\,, \tag{5.5}$$

independent of z. The time-evolution of ξ is presented in figure 5.1. By comparing the time-scale $t_{\rm i}$ with the time-scale limiting the prethermal regime t^*, it can be shown, that $t_{\rm i} \ll t^*$, at least for positive exponents θ and small quench amplitudes. Thus somewhere in the prethermal regime, the time evolution of the correlation length changes its power-law exponent. On the other hand, the scaling form is always present, which generates the power-laws in the order-parameter and in the Green's functions. In the remaining chapter, only the impact of the scaling form $r_{\rm sc}(t)$ will be considered.

5.2 Order parameter dynamics, general formalism

In the sections 4.2 and 4.3, only quenches starting in the symmetric phase with $\langle \boldsymbol{\varphi} \rangle = 0$ have been considered. In Ref. [31] we derived the $1/N$-equations in great detail along the three-branch contour. They include already the effects of a finite initial order parameter $\langle \boldsymbol{\varphi} \rangle = \phi_{\rm i}$. To include a finite order parameter within the RG-equations, one needs to consider a supplementary flow equation for $\phi(t) = \langle \boldsymbol{\varphi}(t) \rangle$, leading to a differential equation in time for the order parameter dynamics. Here, only the $1/N$ equation is considered, as the extension to finite N is straightforward, if $G^K(k,t,t)$ obeys a scaling form.

A finite initial order parameter can be achieved by either switching off an initial, external field $h_{\rm i}$ at $t = 0$. Or by an initial mass parameter $r_{\rm i} < 0$ located in the symmetric broken phase. Both protocols lead the same result within the deep-quench limit. This is similar to the classical, critical boundary scaling [50, 52], where there exists a analogy between the ordinary phase transition with finite external field and the extraordinary phase transition. In this section, the general formalism to describe such a quench protocol is derived. A spatially homogeneous order parameter $\phi(x) = \phi$ is considered. It is evaluated in the prethermal regime in section 5.3.2 as well as in the quasi-adiabatic long time limit in section 5.4.1.

Consider the $1/N$-equations for a quench starting in the symmetric phase, with finite initial magnetization $\phi_{\rm i}$. In this case, the $1/N$-equation is modified by a supplementary term:

$$r(t) = r_{0,\rm f} + u_{\rm f} \int_q G^K(q,t,t) + \frac{u_{\rm f}}{2}\phi^2(t). \tag{5.6}$$

The order parameter obeys the following $1/N$-equation:

$$\phi_{\rm i} \int_{-\infty}^{0} {\rm d}s\, \delta\eta(t-s) = r(t)\phi(t) - \int_0^t {\rm d}s\, \delta\eta(t-s)\phi(s) - \partial_t^2 \phi(t). \tag{5.7}$$

For times $t \gg t_\gamma$ the second derivative can be neglected compared to the bath spectral function η. First, the effective mass in Eq. (5.6) is analyzed. The bare post-quench Keldysh function G^K defined via the anticommutator of $\boldsymbol{\varphi}(t)$, is unchanged compared to the Keldysh function for a quench starting in the disordered phase. Thus, it can be obtained by exactly the same methods, as presented in section 3.5. Further, it can be evaluated in the same manner like for a quench with $\phi_{\rm i} = 0$, where the Keldysh function is expanded for small $r(t)$. Including also terms from the long time limit yields

$$r(t) = \frac{u_{\rm f}}{2}\phi^2(t) + u_{\rm f} \int_q \left(g^K(k,t,t) - G^K_{eq}(k) \right) + \frac{u_{\rm f}}{u^*} r(t) \left[1 - \left(\frac{\Lambda^z t}{\gamma^{z/2}} \right)^{\epsilon/z} \right]. \tag{5.8}$$

This equation is fulfilled for all times after the quench if

$$u_{\mathrm{f}} = u^*, \tag{5.9}$$

$$r(t) = r_{\mathrm{sym}}(t) + \frac{c_K \epsilon \gamma^{z/2}}{2K_d} \left(\frac{\gamma^{z/2}}{t} \right)^{\epsilon/z} \phi^2(t), \tag{5.10}$$

where $r_{\mathrm{sym}}(t)$ is the effective mass for a quench starting in the symmetric phase, derived in section 4.5. The value of u is not affected by the quench direction. But the finite order parameter $\phi(t)$ changes the effective mass by a supplementary term. This effective mass can be inserted into Eq. (5.7), to obtain a non-linear differential equation in time for $\phi(t)$:

$$r_{\mathrm{sym}}(t)\phi(t) + \frac{c_K \epsilon \gamma^{z/2}}{2K_d} \left(\frac{\gamma^{z/2}}{t} \right)^{\epsilon/z} \phi^3(t) - \int_0^t \mathrm{d}s\, \delta\eta(t-s)\phi(s) - \partial_t^2 \phi(t) = \phi_{\mathrm{i}} \int_{-\infty}^0 \mathrm{d}s\, \delta\eta(t-s) \,. \tag{5.11}$$

This differential equation can be solved approximately in two limits, in the prethermal regime at intermediate times and in the quasi-adiabatic long time limit where $r(t) \simeq 0$. Note that the supplementary term does not change the scaling solution $r_{\mathrm{sc}}(t) = at^{-2/z}$ in the prethermal regime, if $\phi^2(t) \neq t^{\epsilon/(z)} t^{-2/z}$. This will be confirmed in the next section for the prethermal regime. In the quasi-adiabatic limit, $\phi \propto t^{-(2+\epsilon)/(2z)}$ turns out to be the only non trivial solution.

5.3 Prethermal regime

In this section, the prethermal regime is analyzed for the open system for quenches starting in the disordered as well as in the symmetric broken phase. The quench direction does not affect the scaling form of the effective mass, as well as the scaling form of the Green's functions. This effect is like in equilibrium, where the exponents are found to be dependent only on the absolute value of the distance to the critical point and independent of the system being in the ordered or disordered phase. In the prethermal regime, the dynamics are described by a new, universal exponent θ, which will be determined for different dynamical exponents $z > 1$. The implications for the Green's functions are shortly summarized, as they are derived in detail in section 4.1.3. Further, the order parameter dynamics are analyzed for a quench starting in the symmetric-broken phase.

5.3.1 Determination of the exponent

The non-equilibrium exponent $\theta = aC_z$ is given by (see section 4.5),

$$\theta = c_K C_0 C_z \epsilon, \tag{5.12}$$

with the different coefficients:

$$C_z = -\frac{\sin(\pi/z)}{\gamma\Gamma(2/z)}, \tag{5.13}$$

$$c_K = \gamma^{z/2} \frac{z\,(2-z)\sin^{z/2}(\pi/z)}{4\sin(\pi z/2)}, \tag{5.14}$$

$$C_0 = \gamma^{1-z/2} \int_0^\infty \mathrm{d}x\, x^{2/z-1} \left(\mathrm{i}\, f_{\mathrm{sc}}^K(x,1) - \mathrm{i}\, F_{\mathrm{eq}}^K \right) . \tag{5.15}$$

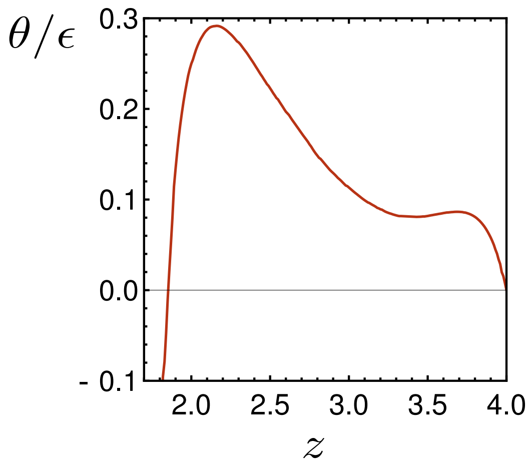


Figure 5.2: *The value of the universal exponent θ for different z.*

The three coefficients will be discussed separately. The first one, C_z enters from the short-time evolution of G^R. It is the factor connecting the light cone amplitude a with the universal exponent θ, see Eq. (4.40). This coefficient is the same for the classical, as well as for the quantum system. The factor c_K originates from the long-time expansion of the ultra-fast modes $k \lesssim \Lambda$, which equilibrate on timescales t_{mic}. It also fixes the interaction parameter u to its fixed point value u^*, see Eq. (4.29). The coefficient C_0 is given by the difference between post-quench and equilibrium Keldysh function, expanded around $r_{sc}(t) = 0$, see Eq. (4.82). The subscript sc indicates that only the part of the Keldysh function is taken which obeys scaling.
Note that due to the interplay of the three coefficients, θ is indeed independent of γ and thus dimensionless.
The integral in C_0 can be evaluated numerically for $1 < z < 4$ and is analytically calculated for $z = 2$ in appendix C, see also Refs. [30, 31]. The result for θ/ϵ is depicted in figure 5.2, for $z = 2$ one explicitly finds $\theta(z = 2) = \epsilon/4$. The z-dependence of θ is very particular. For example, the exponent θ vanishes for $z \to 4$. This is due the upper critical dimension being $4 - z$, thus $z = 4$ corresponds to evaluating a zero-dimensional system. At $z = 4$ there is a sign change in the long-time coefficient c_z^K. For the classical system c_z^K decays monotonically to zero with increasing z, see also figure 5.5.

Note that for sub-Ohmic bath spectra, $z \lesssim 1.8$, there is a sign change in the exponent. This sign change is due to the coefficient C_0, as the other two coefficients are positive for all z. This is due to quantum oscillation in the scaling function $F^K(t)$. The sign change indicates a crossover from a damped oscillator behavior to overdamped dynamics. Such an effect is not present for a quench to the classical critical point, where the post-quench dynamics are always overdamped.

5.3.2 Order parameter dynamics and Green's functions

The naive scaling argumentation in chapter 2 suggested that the order parameter grows in the prethermal regime with $\phi(t) \propto t^{\theta}$. This picture can be confirmed by inserting the scaling solution $r_{\text{sc}}(t) = at^{-2/z}$ in the equation of motion for ϕ in Eq. (5.7). In the prethermal regime, this equation can be

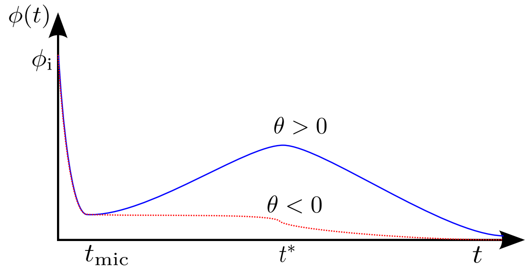


Figure 5.3: *Order parameter dynamics in the prethermal regime for $\theta > 0$ (blue) and $\theta < 0$(red dots). In the prethermal regime, $\phi(t)$ grows/decays with t^θ.*

simplified to:

$$\frac{\gamma a}{t^{2/z}}\phi(t) - \int_{t_{\text{mic}}}^{t} \mathrm{d}s\, \delta\eta(t-s)\phi(s) = \phi_{\text{i}} \int_{-\infty}^{0} \mathrm{d}s\, \delta\eta(t-s) + \int_{0}^{t_{\text{mic}}} \mathrm{d}s\, \phi(s)\delta(t-s). \tag{5.16}$$

The assumption of neglecting the ϕ^3-term, which also enters due to the effective mass given in Eq. (5.10), and which renders the differential equation non-linear, is only reasonable for small ϕ. It will be justified below. As the scaling ansatz for $r(t)$ is only valid beyond the microscopic time scales, the time evolution on very short times is taken into account by introducing $\phi'_{\text{i}} = \phi(t_{\text{mic}})$.
The equation 5.16 can be solved via LT. It reads

$$g^{R^{-1}}(k=0,\omega)\phi(\omega) = \int_{-\infty}^{\infty} \mathrm{d}\omega'\, r(\omega')\phi(\omega-\omega') + \phi'_{\text{i}}\frac{\mathrm{i}\,\delta\eta(\omega)}{\omega} \tag{5.17}$$

where $g^R(k=0,\omega) = \delta\eta(\omega)$. The LT has been introduced in section 3.4. To obtain this form, it is crucial to recall that the bath spectral function η has a retarded structure with $\eta(t<0) = 0$. Also note, that there is no regularization problem at the lower boundary of the time integral ds, as $\eta((t-s)\to\infty) = 0$, due to the bath cutoff. This equation can be solved by an expansion for small amplitudes a in $\phi(\omega) \approx \phi_0(\omega) + a\phi_1(\omega)$. As a is of order ϵ, this expansion is justified near the upper critical dimension. Inserting this expansion in Eq. (5.17) yields:

$$\phi_0 = \phi'_{\text{i}}\frac{i}{\omega}\,, \tag{5.18}$$

$$a\phi_1 = \int_{-\infty}^{\infty} \mathrm{d}\omega'\, r(\omega')\phi_0(\omega-\omega')g^R(k=0,\omega)\,. \tag{5.19}$$

Transformed back into time-space, this yields

$$\phi(t) \simeq \phi'_{\text{i}} + \phi'_{\text{i}}\int_{t_{\text{mic}}}^{t} \mathrm{d}s\, g^R(t,s)\frac{\gamma a}{s^{2/z}}. \tag{5.20}$$

This integral is similar to the one obtained in the short-time expansion of the Dyson equation for g^R. It has a dominant contribution which is logarithmically divergent

$$\phi(t) \approx \phi'_{\text{i}}\left(1 + \theta\log\frac{t}{t_{\text{mic}}}\right). \tag{5.21}$$

As θ is a small parameter and by including higher order corrections, which also show logarithmically divergent behavior with appropriate coefficients, the logarithm can be exponentiated:

$$\phi(t) = \phi_{\rm i}' \left(\frac{t}{t_{\rm mic}}\right)^{\theta} . \tag{5.22}$$

Note that θ is the same exponent as in the Green's functions $G^{R/K}$. It is given by Eq. (5.12), which was evaluated above. For positive θ, the order parameter grows after a quench to the QCP, while it decays for $\theta < 0$, or $z \lesssim 1.8$, see figure 5.3. Still the exponent θ is of order ϵ and thus the effect of growing/decaying of the order parameter is small, therefore it is not in contradiction with the language of a prethermal plateau, used in literature [5].
Further, note that φ is indeed small, thus the assumption of neglecting the φ^3-term is indeed valid. And the time dependence of φ is such that it will not affect the scaling-part of the effective mass, going with $r_{\rm sc}(t) = a/t^{2/z}$.

The time dependence of the Green's function has already been evaluated within the Dyson equation in section 4.1.3. As it was shown above, the time dependence of $\phi(t)$ will not affect the scaling form of the effective mass, and thus does not change the generation of logarithms in an expansion in $r(t)$. The main result thus remains unchanged:

$$G^{R/K}(k,t,t') = \frac{1}{k^{2-z}} \left(\frac{t}{t'}\right)^{\theta} F^{R/K}\left(k^z t, t'/t\right), \tag{5.23}$$

where the scaling function $F^{R/K}(x,y)$ is well defined in the limit $y \to 0$. This result was obtained for $t \gg t'$ and in the short-time limit of mode $t < k^z$. It was used, that in the quantum limit, the singularity for $t' \to 0$ is captured by the same exponent θ, in contrast to the classical limit.

5.3.3 Timescales of the prethermal regime

The emergence of the prethermal regime in $\phi(t)$ is characterized by different timescales. To apply the $\boldsymbol{\varphi}^4$-model, it is necessary to be at times larger than the microscopic timescales $t_{\rm mic}$, which are discussed in section 5.5. It is limited from above by the cross-over time-scale t^*. A scaling argumentation for the order of t^* was given in chapter 2, in this section, t^* is derived from the order parameter dynamics and is compared to $t_{\rm i}$.

One basic assumption to solve the order parameter differential Eq. (5.7), was that the ϕ^2-term in the effective mass can be neglected. This is reasonable for small ϕ. The smallness of ϕ is guaranteed by a small initial $\phi_{\rm i}$, thus a small quench amplitude, and the collapse of the order parameter right after the quench. This collapse follows from the result of the correlation length, which recovers at t^* to its initial pre-quench value $\xi_{\rm i}$.
The order parameter grows for $z \gtrsim 1.8$, making at some time t^* the assumption of neglecting the ϕ^3-term in the differential Eq. (5.16) no longer valid. The timescale t^* can thus be defined if the quadratic term in the effective mass is of the same order as the term originating from the Keldysh function. In the case of an open system with $z > 1$ it was argued in section 5.1 that only the scaling form is a relevant contribution to $r(t)$, thus t^* is defined via

$$\frac{a}{t^{*2/z}} = \left(\phi_{\rm i}' t^{*\theta}\right)^2 \frac{c_K \epsilon \gamma^{z/2}}{2K_d} \left(\frac{\gamma^{z/2}}{t^*}\right)^{\epsilon/z} . \tag{5.24}$$

At this point it is useful to recall, that a has in this picture the dimension of γ. Considering the dimensionless part $\tilde{a} = a/\gamma$, one finds

$$t^* = \left(\frac{2|\tilde{a}|K_d t_\gamma^{2\theta}}{c_K \gamma^{2/z-1+\epsilon/2}} \right)^{z/(2+2z\theta-\epsilon)} \phi_{\mathrm{i}}'^{-\nu z/(\beta+\theta\nu z)} . \tag{5.25}$$

The order parameter $\phi_{\mathrm{i}}' = \phi(t_{\mathrm{mic}})$ is related to the initial order parameter, via some coefficient, describing the break-down. By performing a weak quench, the crossover timescale t^* can be tuned to large values. Note that taking the mean-field values of the exponents, $\theta = 0$, one finds the main proportionality $r_{\mathrm{i}} \propto \phi_{\mathrm{i}}^{\nu z/\beta}$. This is the value of the time-scale t_{i}, where the time dependence of the correlation length changes. Thus, for positive θ, it holds $t_{\mathrm{i}} \ll t^*$, while for negative θ $t_{\mathrm{i}} \gg t^*$. This implies, that for negative exponents, the correlation length grows ballistically in the complete prethermal regime and that the dominant part of the effective mass decays with t^{-2}.

5.4 Adiabatic regime

With the solution of the time-dependent mass, it is also possible to obtain the quasi-adiabatic relaxation to the QCP at large times, where the system thermalizes. Corrections to the equilibrium values are the dynamics described by the equilibrium exponents, similar to the Kibble-Zurek protocol. However, interaction effects lead to a slowing down of thermalization and thus to aging. The non-equilibrium exponent θ enters as universal amplitude of those aging coefficients [31]. It is also possible to connect the long-time expansion of the retarded and the Keldysh-Green's functions. This results in an extension of the FDT, with a time-dependent distribution function $n(t, \omega)$.

5.4.1 Aging in the order parameter and the Green's function

At large times, the order parameter is expected to relax to zero for a quench right to the critical point. Thus, for times larger than t^*, the order parameter varies only slowly in time, making it possible to neglect the time derivatives of $\phi(t)$ in Eq. (5.7):

$$\int \mathrm{d}t' \, \delta\eta(t-t')\phi(t') \approx \phi(t) \int_0^\infty \mathrm{d}t' \, \delta\eta(t-t') = 0 \, . \tag{5.26}$$

The inhomogeneous part $\phi_{\mathrm{i}} \int_{-\infty}^0 \mathrm{d}s \, \delta\eta(t-t')$ may also be neglected, as $\eta(t)$ decays to zero for large times. With those assumptions the order parameter equation of Eq. (5.7) reduces to

$$r(t \gg t^*)\phi(t) = 0 \, . \tag{5.27}$$

For a non-trivial solution, the effective mass must be equal to zero. This yields for the order parameter dynamics

$$\begin{aligned} \phi(t \gg t^*) &= \left(\frac{2\tilde{a}K_d}{\epsilon\gamma^{2/z-1+\epsilon/2} c_K} \right)^{\frac{1}{2}} t^{-(2-\epsilon)/(2z)} \\ &= \left(\frac{2\tilde{a}\Lambda^\epsilon}{u^*\gamma^{\epsilon/2}} \right)^{\beta} t^{-\beta/(\nu z)} \, , \end{aligned} \tag{5.28}$$

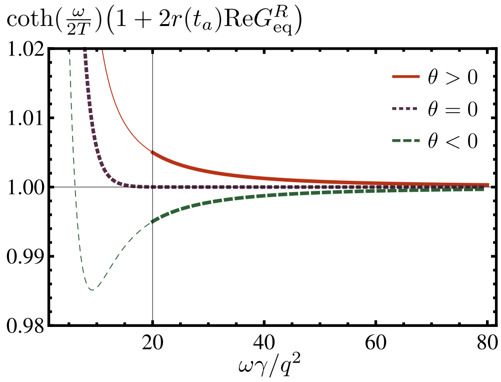


Figure 5.4: *The correction $\delta n(t_\mathrm{a}, \omega)$ to the Bose-distribution function n_B, depending on frequencies and for different dynamical exponents z.*

where the time dependence of ϕ was expressed via the large-N value of the equilibrium exponents β and ν, and the dimensionless light-cone amplitude $\tilde{a} = a/\gamma$ was used. This form confirms the scaling argumentation of section 2. A further insight of this argumentation is, that the universal amplitude a/γ enters with the coefficient u^*, leading to an universal relaxation amplitude.

Within the Dyson equation, the long-time limit of the retarded and the Keldysh Green's functions, was derived in section 4.1.2. For times t, t', where both times t, t' are larger than the mode time k^{-z}, but the time difference is still small, $t - t' \ll k^{-z}$, the result was:

$$G^R_r(k, t, t') = G^R_{\mathrm{eq}}(k, t - t') - 2\mathrm{i}\,\theta\left(t - t'\right) r\left(\frac{t + t'}{2}\right) C^R(t - t')\,, \tag{5.29}$$

$$G^K_r(k, t, t') = G^K_{\mathrm{eq}}(k, t - t') - 4\mathrm{i}\, r\left(\frac{t + t'}{2}\right) C^K_z(k, t - t')\,, \tag{5.30}$$

with the coefficients

$$C^R(k, t) = \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{Re}\, G^R_{\mathrm{eq}}(k, \omega) \mathrm{Im}\, G^R_{\mathrm{eq}}(k, \omega) e^{-\mathrm{i}\omega t}\,, \tag{5.31}$$

$$C^K(k, t) = \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{Re}\, G^R_{\mathrm{eq}}(k, \omega) G^K_{\mathrm{eq}}(k, \omega) e^{-\mathrm{i}\omega t} \tag{5.32}$$

For a quench from the disordered phase, the scaling solution $r(t) = at^{-1/(\nu z)}$ can be used to express the corrections to equilibrium via the large-N or RG values of the equilibrium exponents. For a quench from the ordered phase, the solution of the order parameter equation suggests either $\phi(t \gg t^*) = 0$ or $r(t \gg t^*) = 0$. In the first case, the same result as for the opposite quench direction is obtained, but at the cost of a trivial order parameter for $t \gg t^*$. In the second case, the relaxation in the Green's function seems to go faster for $t, t' \gg t^*$, making it necessary to include higher order corrections in the order-parameter dynamics, to see derivations from equilibrium.

5.4.2 Fluctuation dissipation theorem

The similarity between G^R and G^K in the long-time expansion suggests to seek for a FDT-version out-of-equilibrium with the distribution function $n(t, \omega)$ depending on time [31]. We recall the equilibrium

FDT:

$$G^K_{\rm eq}(k,\omega) = 2{\rm i}\,\coth\left(\frac{\omega}{2T}\right){\rm Im}\,G^R_{\rm eq}(k,\omega)\,. \tag{5.33}$$

To generalize the FDT to the post-quench regimes, the absolute time $t_a = (t+t')/2$ is introduced. Further, the Wigner-transformation of Eqs. 5.29 and 5.30 is performed. By expressing $G^K_{\rm eq}(k,\omega)$ by $G^R_{\rm eq}(k,\omega)$, one finds

$$G^K_r(k,t_a,\omega) = 2{\rm i}\,\coth\left(\frac{\omega}{2T}\right)\left[1-2r(t_a){\rm Re}\,G^R_{\rm eq}(k,\omega)\right]{\rm Im}\,G^R_{\rm eq}(k,\omega). \tag{5.34}$$

This form implies that G^K_r can also be written

$$G^K_r(k,t_a,\omega) \simeq \coth\left(\frac{\omega}{2T}\right)\frac{2{\rm i}\,{\rm Im}\,\eta(\omega)}{\left[k^2+r(t_a)+{\rm Re}\,\eta(\omega)\right]^2+\left[{\rm Im}\,\eta(\omega)\right]^2}\,, \tag{5.35}$$

where an expansion around a small mass $r(t)$ is used. This result demonstrates that the limit $t_a \gg k^{-z}$ and $t-t' \ll k^{-z}$ corresponds indeed to the limit of a quasi-adiabatic relaxation. Further, one can use Eq. (5.29), to introduce a distribution function $n(t_a,\omega)$:

$$G^K_r(k,t_a,\omega) = 2{\rm i}\,\coth\left(\frac{\omega}{2T}\right)\left[1+2r(t_a){\rm Re}\,G^R_{\rm eq}(k,\omega)\right]{\rm Im}\,G^R_r(k,t_a,\omega) \tag{5.36}$$

$$= 2{\rm i}\,\left[2n(t_a,\omega)+1\right]{\rm Im}\,G^R_r(k,t_a,\omega)\,. \tag{5.37}$$

With the Bose function n_B, the post-quench distribution function can be written as $n(t_a,\omega) = n_B(\omega)+\delta n$, where the correction to equilibrium is given by

$$\delta n(t_a,\omega) = \coth\left(\frac{\omega}{2T}\right)\frac{\theta\Gamma\left(2/z\right)}{(|\omega|t_a)^{2/z}}\left[\cos\left(\frac{\pi}{z}\right)+\frac{k^2}{\gamma|\omega|^{2/z}}\sin\left(\frac{\pi}{z}\right)\right]. \tag{5.38}$$

Note that this result was obtained in the limit $t_a \gg k^{-z} \gg t-t'$. This implies that $(\omega t_a)^{-2/z} \ll 1$, as well as $\frac{k^2}{\gamma|\omega|^{2/z}} \ll 1$, thus the dominant contribution for $z \neq 2$ originates from the cosine part, going with $\delta n \propto \omega^{2/z}$. For the Ohmic bath holds $\delta n \propto \omega^2$. This slow algebraic decay shows that the system is non-thermal and a time-dependent temperature cannot be introduced, as in this case the decay would be exponential. The amplitude of δn is proportional to the non-equilibrium exponent θ. Via θ the sign of δn can change. It is positive for $z \gtrsim 1.8$, indicating that for Ohmic and sub-Ohmic baths there is even at large times after the quench an increased number of excitations. For super-Ohmic baths δn becomes negative, showing that the density matrix is non-diagonal in the energy basis. Those off-diagonal terms come from quantum coherence and prevent an interpretation of δn as distribution function. The frequency dependence of $\delta n(t_{\rm a},\omega)$ is depicted in figure 5.4.

5.5 Hierarchy of different short time scales

The post-quench dynamics are not only characterized by t_γ, $t_{\rm int}$ and k^2+r_0, introduced in section 2.1, but also by some microscopic time scales. In this section, the damping coefficient γ was assumed to be large, thus t_γ also plays the role of a short time scale. One further time scale is the momentum cutoff Λ,

corresponding to the time $t_\Lambda = \Lambda^{-1}$. Below this time, the assumption of a φ^4-model is not reasonable. For an open system also the bath cutoff ω_c plays a role, with its corresponding timescale $t_c = 1/\omega_c$. In this section, the hierarchy of those microscopic time scales is discussed. Note that in contrast to the timescales limiting the prethermal regime, they do not depend on the quench amplitude.
To assume a spectral function $\eta(\omega) \propto \omega^\alpha$, one must clearly be at frequencies lower than the bath cutoff ω_c, or alternatively at times larger than t_c. Thus, for the post-quench dynamics derived in this thesis, t_c must be one of the smallest timescales.
The scaling form for the effective mass can only be obtained for timescales beyond t_0, defined via

$$t_0 = \gamma^{z/2}/\Lambda^z. \tag{5.39}$$

For times smaller than t_0 the long-time expansion of the mode on the cut-off shell is not possible. The timescale t_0 corresponds further to $\Lambda^2 \sim \gamma t_0^{2/z}$, where the largest possible momentum contribution Λ^2 is of the order of the damping term. Clearly it is

$$t_0 > t_c, \tag{5.40}$$

as else the assumption of the low-frequency dependence of η would not be possible. This condition can be translated into

$$\gamma\omega_c^{2/z} > \Lambda^2. \tag{5.41}$$

This indicates that even the largest momentum contribution Λ^2 has to be smaller than the largest possible damping term.
This hierarchy can be brought into combination with t_γ, the timescale after which (for $\alpha < 2$ or $z > 1$) the ballistic term ω^2 can be ignored and the dynamics are purely dissipative. t_γ was defined as

$$t_\gamma = \gamma^{-\frac{z}{2(z-1)}} \tag{5.42}$$

and can be made small by considering a large damping coefficient γ. Obviously, it is $t_\gamma > t_c$, as else again the assumption of the low-frequency dependence of η would not be possible. This indicates

$$\gamma\omega_c^{2/z} < \omega_c^2. \tag{5.43}$$

Combining this condition with Eq. (5.41) yields

$$\Lambda < \omega_c. \tag{5.44}$$

This condition is reasonable, as it implies that for all modes the assumption of the low-frequency dependence of η is possible. It also implies that $t_\Lambda > t_c$. The prethermal regime considered in this section can only set in after $(t_c, t_\Lambda, t_0, t_\gamma)$. t_c was identified to be the smallest timescale. It follows that t_Λ always lays between t_0, t_γ. As t_γ is the only cut-off independent timescale the following hierarchy is reasonable

$$\omega_c^{-1} \ll t_0 \ll t_\Lambda \ll t_\gamma. \tag{5.45}$$

In this section only timescales where the bath dominates the dynamics have been considered.

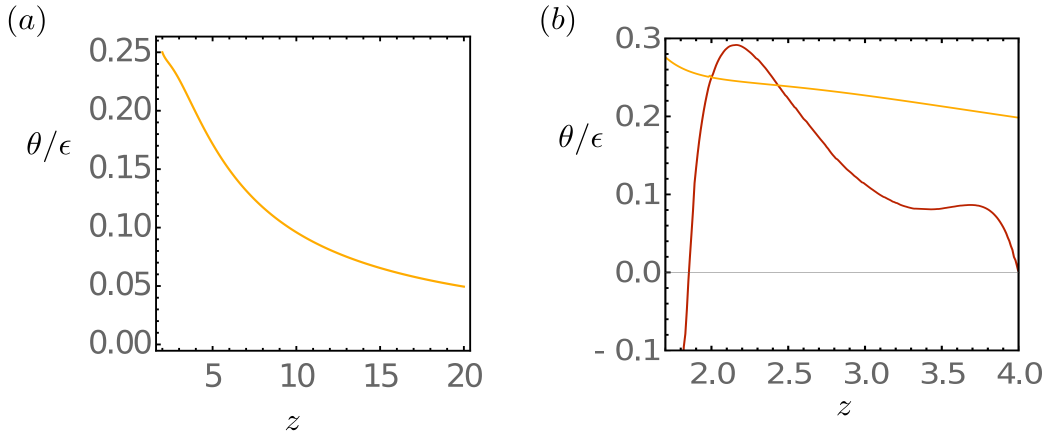


Figure 5.5: *(a): The value θ_c/ϵ_c in the classical limit, (b): Comparing θ/ϵ for different dynamical exponents z in the quantum (red curve) as well as in the classical (yellow curve) limit. Note that $\epsilon = 4 - d - z$ in the quantum limit, and $\epsilon_c = 4 - d$ in the classical limit.*

5.6 Classical limit

With the quantum field theoretical framework derived in chapters 3 and 4, one can also reproduce the results for a classical system quenched to the critical point. The results were obtained by Janssen, Schaub and Schmittmann in Ref. [28] for an Ohmic bath and by Bonart, Cugliandolo and Gambassi in Ref. [29] for colored noise. In this section the main steps to reproduce this known results are presented shortly. The key difference arise from the bare post-quench Keldysh function and its equilibrium value. The logarithmic correction of the retarded Green's function at short times due to the presence of a scaling form of the effective mass, as well as the large-N analysis and the order parameter dynamics can be analyzed in exactly the same manner as for the quantum system. Especially, $r_{\text{cl}}(t) = a_{\text{cl}} t^{-2/z}$ has the same time dependence for the classical as well as for the quantum system, but with a different amplitude a_{cl}. Before determining a_{cl} self-consistently, one short recall, how the crossover from quantum to classical dynamics is taken in equilibrium within the RG-approach. At finite temperatures, one further scale has to be considered: the flow of the temperature T, leading to the following flow equation

$$\frac{\mathrm{d}T(l)}{\mathrm{d}l} = zT. \tag{5.46}$$

Thus, temperature is relevant under the RG-flow and grows to larger values. The first order corrections to r and u are given by a frequency integral over the $uG^K(\Lambda,\omega)$ and $uG^R(\Lambda,\omega)G^K(\Lambda,\omega)$, respectively. In equilibrium, G^K is given by the FDT, where temperature enters via $\coth\big(\omega/(2T)\big)$. If the temperature is the largest scale in the system, the coth can be expanded for small arguments, leading to a prefactor of $T/(2\omega)$, at each point, where the Keldysh function enters. To handle the relevant parameter T, one can introduce a new interaction parameter $g(l) = u(l)T(l)$, and consider its flow under RG. This is the usual procedure to describe the crossover of quantum and classical dynamics. It leads to the following flow equation for g

$$\frac{\mathrm{d}g}{\mathrm{d}l} = \big(4-d\big)\, g - g^2(N+8)K_d \int^{>} \mathrm{d}k\, k^{d-1} G^2(k, \mathrm{i}\omega_n = 0)\,. \tag{5.47}$$

Note that now the upper critical dimension is 4, independent of z. Thus, the small parameter controlling the expansion is $\epsilon_c = 4 - d$ for a classical system in equilibrium.

The quench to a classical critical point can be performed analogously to a quantum quench by suddenly changing the mass parameter, the interaction parameter or an external field, such that the final parameter configuration corresponds to the parameters of the critical point. The difference now is that the external bath is and stays during the whole time evolution, at some large but finite temperature T. The classical limit is taken by considering T as the largest scale of the system. Consider now the $1/N$-equation, obtained for the effective mass $r(t)$ in Eq. (4.85) near four dimensions:

$$r(t) = \frac{uK_d}{2}\int_0^\infty \mathrm{d}k\, k^{3-\epsilon_c}\left(\mathrm{i}\, g^K(k,t,t) - \mathrm{i}\, G^K_{\mathrm{eq}}(k)\right) + \frac{TuK_d r(t)}{c_K^{\mathrm{cl}}\epsilon_c\gamma^{z/2}}\left(\Lambda^{-\epsilon_c} - \left(\frac{\gamma^{1/2}}{t^{1/z}}\right)^{-\epsilon_c}\right).$$

This equation has been obtained under the assumption of equilibration for the fast modes. This assumption is still valid, however, the equilibrium Green's functions are given now at finite temperature T, leading to a prefactor

$$\begin{aligned} c_K^{\mathrm{cl}}(z) =& 4k^4\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{1}{\omega}\mathrm{Im}\, G^R_{\mathrm{eq}}(k,\omega)G^R_{\mathrm{eq}}(k,\omega) \\ =& 1\,, \end{aligned} \tag{5.48}$$

in the long-time expansion. Here, the FDT was used to express $G^K = \mathrm{i}4T/\omega\, \mathrm{Im}G^R(\omega)$. In contrast to the quantum problem, c_K^{cl} has no z dependence, leading to a z-independent fixed point value of g^*. It also affects the bare post-quench Keldysh function g^K, which turns out to be also proportional to T. In analogy to the quantum-classical crossover, this suggests to introduce a new interaction parameter $g = Tu$. The control parameter of the expansion is now ϵ_{cl}, which is small near the classical upper critical dimension $d_{\mathrm{uc}} = 4$. For the $1/N$-equation to hold at all times, one can derive a condition for g:

$$g = g^* = \frac{\epsilon_{\mathrm{cl}}\Lambda^{\epsilon_{\mathrm{cl}}}}{(N+8)K_d c_K^{\mathrm{cl}}}. \tag{5.49}$$

And for the time-dependent mass:

$$\begin{aligned} r(t) =& \frac{c_K^{\mathrm{cl}}\epsilon t^{\epsilon_{\mathrm{cl}}/z}}{2}\int_0^\infty \mathrm{d}k\, k^{3-\epsilon_{\mathrm{cl}}}\left(\frac{\mathrm{i}\, g^K(k,t,t) - \mathrm{i}\, G^K_{\mathrm{eq}}(k,0)}{T}\right) \\ =& \frac{c_K^{\mathrm{cl}}\epsilon}{2zt^{2/z}}\int \mathrm{d}x\, x^{2/z-1}\left(f^K_{0,\mathrm{cl}}(x) - 1\right). \end{aligned} \tag{5.50}$$

The temperature will cancel out for the equilibrium part, as well as for the post-quench Keldysh function. The scaling form of the bare Keldysh function $g^K(k,t,t) = 2Tk^{-2}f^K_{0,\mathrm{cl}}(x)$, was used, to explicitly derive the integral. To obtain this Keldysh scaling function, one can directly solve the Heisenberg EOM and evaluate the corresponding expectation values. This is easier as in the quantum case, because in the high temperature limit only the zeroth Matsubara mode has to be considered and thus no divergent terms occur. The memory function and thus the post-quench Keldysh function is evaluated in appendix D. It holds for the difference between the post-quench and the equilibrium Keldysh functions:

$$\begin{aligned} \delta G^K =& g^K(k,t,t) - G^K_{\mathrm{eq}}(k,0,0) \\ =& \frac{T}{k^2}\frac{1}{4}E^2_{2/z}\left(-\sin(\pi/z)\left[k^z t/\gamma^{z/2}\right]^{2/z}\right)\,, \end{aligned} \tag{5.51}$$

with the Mittag-Leffler function $E_\alpha(x)$. This function is positive for all times, since for positive z the Mittag-Leffler function must be real. The integral in Eq. (5.50) can now be evaluated, see appendix D. One obtains the known result for the exponent:

$$\theta_{\mathrm{cl}} = \frac{N+2}{N+8} \frac{d(2/z)}{4\Gamma\left(2/z\right)} \epsilon_{\mathrm{cl}} \,, \tag{5.52}$$

with $d(\alpha) = 2\int_0^\infty \mathrm{d}x \, E_\alpha^2(-x)$. The dependence of $\theta_{\mathrm{cl}}(z)$ is depicted in figure 5.5 for the classical as well as for the quantum system. Note that the small parameter is $\epsilon_{\mathrm{cl}} = 4-d$ for the classical system and $\epsilon = 4-d-z$ for the quantum system. Both prefactors take however the same value for the Ohmic bath $z=2$ with $\theta/\epsilon = \theta_{\mathrm{cl}}/\epsilon_{\mathrm{cl}} = 1/4$. This seems to be a coincidence, as the prefactors $c_K^{(\mathrm{cl})}$ and C_0 take different values for an Ohmic system at $T=0$ and in the high temperature limit. Two major differences occur. Firstly, the quantum value vanishes for $z>4$, in contrast to the classical system where $\theta_{\mathrm{cl}}(z\to\infty)\to 0$. This is due to the upper critical dimension being z-dependent only for the quantum system and vanishing for $d\to 4$. This is reflected in the vanishing coefficient of u^*, and thus in θ. This effect enters via c_K, which determines the fixed point value of u^* in the quantum limit and g^* in the classical limit. The coefficient c_K strongly depends on z for $T=0$, and is a constant in the high temperature limit. Secondly, the classical value for θ is monotonic with z and positive for all z. For the quantum value of $\theta(z)$, the curve is not monotonic. A maximum for $z\approx 2.1$ occurs. For $z \lesssim 1.8$ the sign of the exponent changes. The sign change enters via C_0, where due to quantum oscillations also negative values are possible, again in contrast to classical systems, where δG^K and thus C_0 are positive for $z>1$. Also note that in the classical limit the correlation length is always given by $\xi(t) \propto t^{1/z}$. Here, there are never divergent terms for $\omega_{\mathrm{i}}\to\infty$, independently of the bath exponent and the order of limits.

6 Chapter 6

Post-quench dynamics in an isolated system

With the results of chapter 4 it is possible to analyze the dynamics of a closed system after a quench to a critical point. Such an analysis was already done for the quench to a non-thermal fixed point in Refs. [33, 34, 64]. In this chapter the picture is completed by analyzing also the weak quench limit and studying the effects of a small but finite coupling to an external heat bath. Those results are compared with the results obtained by an Euclidean mapping.

Compared to the post-quench dynamics in the open system, two main differences occur: First, due to energy conservation it is impossible for a perfectly isolated system to reach the quantum critical point after a quench, and the question if and how the system will thermalize is still open [8, 10]. The question of thermalization will not be addressed here. Second, due to the missing relaxation processes induced by an external bath leading to overdamped behavior, the dynamics will show oscillations [9, 67]. Post quench dynamics in isolated systems have been realized recently in cold atoms systems [27, 45, 68]. Here universal scaling functions have been observed already short times after a quench. The question is, whether this is an universality class of a non-thermal fixed point [69], or whether the non-equilibrium scaling is influenced by the underlying QCP. In the first case, it cannot be assumed, that the quantum-classical mapping will give the correct results. In the second case, the mapping might work. The post-quench dynamics for an isolated φ^4-model was analyzed theoretically and numerically by Refs. [33, 34, 64]. The Hamiltonian is given by Eq. (1.1). They used the following quench protocol: The system was prepared initially in a non-interacting ground state with some finite mass. At $t = 0$ the interactions are switched on and the mass was quenched to a critical value. By making the scaling ansatz for the effective mass,

$$r(t) = \frac{a}{t^2}\,, \tag{6.1}$$

they observe scaling functions and a new universal exponent θ on intermediate timescales, and determine the light cone amplitude a self-consistently. The quench amplitude ω_i plays the role similar to a temperature in the quantum classical crossover. Due to this similarity, this quench protocol corresponds to a quench to a non-thermal fixed point. It is briefly reviewed in section 6.1.1 with the methods derived in chapter 4. In section 6.1.2 it is shown, that pre-thermal scaling cannot be found if a weak quench near the QCP is performed.

A further question is the influence of some small but finite contact to an external environment, as condensed matter systems are in general in contact with some environment. By analyzing the bare post-quench Keldysh function, one result of this thesis is that even a small damping rate of the initial state has a high impact on certain time regimes. This scenario is analyzed in section 6.2 for different bath exponents α and time regimes. The advantage is here, that the oscillations are naturally cut off by the finite damping rate. Thermalization to the QCP can be enforced in the long time limit. For the effective mass, the scaling solution of Eq. (6.1) is confirmed, but with a quench-amplitude depending prefactor a. This implies that quantum fluctuations are always strong and recover the memory of the initial state.

In the last section 6.3, the universal exponent θ is determined via the Euclidean mapping approach. The result is compared with the real-time result for an isolated and a nearly isolated system. For the isolated system, the Euclidean mapping completely fails. This is reasonable, as the system is quenched to a non-thermal fixed point, not reachable in equilibrium. For the nearly isolated system the method recovers the universal part of the light-cone amplitude a, but fails to predict the dominant non-universal part. This shows, that there is no analogy between out-of-equilibrium quantum fluctuations and thermal fluctuations. In quantum systems memory effects can be restored over large timescales. This seems to be an example where the mapping does not work and care has to be taken when applying it to the post-quench scenario.

6.1 A perfectly isolated system

A basic ingredient to solve the $1/N$ and RG equation self-consistently was the assumption of thermalization in the long-time limit or equally for the fast modes near the cutoff $k \lesssim \Lambda$. Those modes, which are near their equilibrium value, lead to corrections of order $1/\epsilon$ in the large-N equation and a scaling form can be assumed. With those corrections two conditions could be extracted, one fixing the interaction parameter u to its fixed point value u^*, and one for the time dependence and the coefficient in the effective mass $r(t)$. If such thermalization terms do not exist, such a long time expansion cannot be performed. It is not clear, whether this also implies that a straightforward expansion in $G^K = g^K + G_1^K$ is possible in the integral determining the effective mass.
However, like in Ref. [33] the full system can be analyzed analytically, assuming that $r(t)$ obeys the scaling form:

$$r(t) = \frac{a}{t^2}. \tag{6.2}$$

The corresponding Heisenberg equation of motion for each mode reads

$$\left(\partial_t^2 + k^2 + r(t)\right) \varphi(k,t) = 0\,, \tag{6.3}$$

with the solution

$$\varphi(k,t) = A(k)\sqrt{t} J_\alpha(kt) + B(k)\sqrt{t} J_{-\alpha}(kt)\,. \tag{6.4}$$

Here, $\alpha = \sqrt{1/4 - a}$. A and B are operators, whose expectation values have to be fixed via boundary conditions. $J_\alpha(x)$ is the Bessel function of the first kind, which has the following long and short time expansion

$$J_\alpha(x) \simeq \begin{cases} x^\alpha / \Gamma(1+\alpha) & \text{for } x \ll 1\,, \\ \cos\left(x - \alpha\pi/2 - \pi/4\right) \left(2/(\pi x)\right)^{1/2} & \text{for } x \gg 1\,. \end{cases} \tag{6.5}$$

To determine the expectation values of the operators A and B, the Keldysh function is needed. This Green's function can be determined with the anticommutator of φ:

$$\begin{aligned} \mathrm{i}\, G^K(k,t,t') =& \left\langle \left[\varphi(k,t), \varphi(k,t')\right]_+ \right\rangle \\ =& \sqrt{tt'} J_\alpha(kt) J_\alpha(kt') \left\langle A^2(k) \right\rangle + \sqrt{tt'} J_{-\alpha}(kt) J_{-\alpha}(kt') \left\langle B^2(k) \right\rangle \\ &+ \sqrt{tt'} \left(J_\alpha(kt) J_{-\alpha}(kt') + J_{-\alpha}(kt) J_\alpha(kt') \right) \left\langle \left[A(k), B(k)\right]_+ \right\rangle . \end{aligned} \tag{6.6}$$

Similarly, the retarded Green's function can be obtained from the commutator of φ. To fix the boundary conditions, the Keldysh function plays the more prominent role, therefore the retarded Green's function is only quoted below in Eq. (6.12). Choosing as boundary conditions the ground state of a free harmonic oscillator with the initial frequency $\omega_\mathrm{i}^2 = k^2 + r_\mathrm{i}$, like in Ref. [63], yields

$$\mathrm{i}\, G_\mathrm{i}^K(k, t=0) = \frac{1}{2\omega_\mathrm{i}(k)} , \tag{6.7a}$$

$$\mathrm{i}\langle[\dot\varphi(k,0), \dot\varphi(k,0)]_+\rangle = \frac{\omega_\mathrm{i}(k)}{2} , \tag{6.7b}$$

$$\langle\left[\varphi(k,0), \dot\varphi(k,0)\right]_-\rangle = \mathrm{i} . \tag{6.7c}$$

Here $\dot\varphi(k,t) = \partial_t \varphi(k,t)$. Note that those boundary conditions are only correct for a non-interacting initial state. Experimentally, this protocol can be achieved for example in cold atom systems by performing the quench from ground state and switching on the interaction at $t = 0$. One problem with those boundary conditions is, that within the time regime $0 < t < \Lambda^{-1}$ the φ^4-model is not applicable. This problem does not occur in the open system, where the final state for $t \to \infty$ is given due to equilibration with the heat bath. Thus, for the open system, the boundary conditions are fixed at $t = \infty$, and not at the boundary to the non-accessible microscopic time regime. In the isolated system, the problem can be approximately solved by fixing the boundary conditions at $t = \Lambda^{-1}$ [33], and ignoring what might have happened within this short-time regime immediately after the quench. Using the short-time expansion of Eq. (6.5) and simultaneously expanding $\alpha \simeq 1/2 - a$ for small a yields

$$\mathrm{i}\, G^K(k, \Lambda^{-1}, \Lambda^{-1}) \simeq \frac{k^{1-2a}\langle A^2(k)\rangle}{\Gamma\left(3/2-a\right)} \Lambda^{-2+2a} + \frac{k^{-1+2a}\langle B^2(k)\rangle}{\Gamma\left(1/2+a\right)} \Lambda^{-2a} + \frac{2\langle\left[A(k),B(k)\right]_+\rangle}{\Gamma\left(3/2-a\right)\Gamma\left(1/2+a\right)} \Lambda^{-1} \tag{6.8}$$

$$\langle[\dot\varphi(k,0), \dot\varphi(k,0)]_+\rangle \simeq \frac{k^{1-2a}\langle A^2(k)\rangle}{\Gamma\left(3/2-a\right)} \Lambda^{2a} + \mathcal{O}(a) . \tag{6.9}$$

The expansion in small a is motivated in the next subsection. For modes k far from the cutoff Λ, terms going with $1/\Lambda$ and higher orders can be neglected. Using the boundary conditions of Eq. (6.7) yields

$$\langle A^2(k)\rangle \simeq \frac{\omega_\mathrm{i}(k)\Lambda^{-2a}\Gamma^2\left(3/2-a\right)}{2k^{1-2a}} , \tag{6.10a}$$

$$\langle B^2(k)\rangle \simeq \frac{\Lambda^{2a}k^{1-2a}\Gamma\left(1/2-a\right)}{2\omega_\mathrm{i}(k)} , \tag{6.10b}$$

$$\langle\left[A,B\right]_-\rangle \simeq \Gamma\left(1/2-a\right)\Gamma\left(3/2+a\right) . \tag{6.10c}$$

The set of equations, obtained via those boundary conditions can be analyzed in the limit of a deep or a weak quench, which will be done in the next two subsections. It corresponds to the boundary conditions found numerically in [33].

6.1.1 Deep-quench limit: Scaling near a non-thermal fixed point

In the limit of a deep quench $r_\mathrm{i} \to \infty$, the k dependence of $\omega_\mathrm{i} = \sqrt{r_\mathrm{i} + k^2}$ can be neglected. Keeping only the leading order in r_i yields for the Green's functions

$$\mathrm{i}\, G^K(k,t,t') = \frac{\sqrt{r_\mathrm{i}}\Lambda^{-2a}\Gamma^2\left(3/2-a\right)}{2k^{2-2a}}\sqrt{k^2tt'}J_\alpha(kt)J_\alpha(kt')\,, \tag{6.11}$$

$$\mathrm{i}\, G^R(k,t,t') = \theta(t-t')\frac{\pi\sqrt{k^2tt'}}{2k\sin\left(\pi\alpha\right)}\Big(J_\alpha(kt)J_{-\alpha}(kt') - J_{-\alpha}(kt)J_\alpha(kt')\Big)\,. \tag{6.12}$$

The retarded Green's function depends on the light-cone amplitude a, but not on the quench amplitude r_i. Indeed, the approximation $\omega_\mathrm{i} \approx \sqrt{r_\mathrm{i}}$ is not necessary to obtain the retarded function. This is a direct consequence of the anticommutator $\langle\big[\varphi(k,0),\dot\varphi(k,0)\big]_-\rangle$, and thus $\langle\big[A,B\big]_-\rangle$ being independent of ω_i. By using the short-time expansion of $J_\alpha(x)$ it can also be shown, that $G^R(t,t')$ fulfills a scaling form like in Eq. (2.23) with exponent $\theta = 1/2 - \alpha \approx a$. However, r_i enters as prefactor in the Keldysh function. This form suggests to handle the quench-amplitude depending prefactor $\sqrt{r_\mathrm{i}}$ like in the quantum-classical crossover. Thus, a perturbation theory is built up with small parameter $g = \sqrt{r_\mathrm{i}}\Lambda^{-2a}\Gamma^2\left(3/2-a\right)u$ and $\sqrt{r_\mathrm{i}}\Lambda^{2a}$ is treated similar to a high temperature T. Note however that the scaling dimension in k of $G^K \propto k^{2-2a}$ is for $a \neq 0$ different to the one of a classical, thermal Keldysh function $G^K_{\mathrm{cl,eq}}(k,t=0) \propto T/(k^2)$. This directly shows the limitations of the analogy between the quench amplitude and the temperature, as it would require a mode-depending temperature $T_k \sim \sqrt{r_\mathrm{i}}\Lambda^{-2a}k^{2a}$. This is not reasonable for the concepts of statistical physics. One should rather interpret the scaling form of G^K and G^R as a scaling near a non-thermal fixed point. In Ref. [33] the full solution of Eqs. (6.11), (6.12) was used, to determine the parameter a self-consistently. In the first paper this was done within a non-equilibrium RG-approach, in the second with a non-equilibrium formulation of the $1/N$ expansion. Alternatively one can also perform an analysis similar to section 5.6. All three methods find in a simultaneous expansion in ϵ and $1/N$:

$$a = \frac{\epsilon}{4}\,, \tag{6.13}$$

with $\epsilon = 4-d$, like for a classical system. The method derived in section 4.5 combined with section 5.6 will be shortly applied on this model. At three points the argumentation has to be slightly modified: First, the scaling form for $a = 0$ suggests to perform the expansion around the upper critical dimension $d_\mathrm{uc} = 4$, similar to the quantum-classical crossover. As mentioned above, this results in an expansion for small $g = \sqrt{r_\mathrm{i}}\Lambda^{-2a}\Gamma^2\left(3/2-a\right)u$. Second, in contrast to the post-quench dynamics in open systems, a turns out to be of order ϵ. The non-thermal scaling dimension k^{2-2a} in Eq. (6.14) must be kept also in the bare Keldysh-function. The long-time expansion must also be performed around a Keldysh-function which obeys this non-thermal scaling form. Together with the d-dimensional integral, this leads to terms going with $t^{-2+\epsilon-2a}$. And last, compared to the classical diffusive system, also the integral in C_0, determining the exponent is highly oscillating. Those oscillations can be handled by introducing some cutoff procedure $e^{-\eta x}$, where $x = kt$ is a dimensionless variable and η is a damping constant. The limit $\eta \to 0$ can be taken after evaluating the integral determining the constant C_0. The first two points are of purely technical nature. They reflect the fact, that the system is near a non-thermal, but still classical-like fixed point. Those points will modify the coefficient c_K of the long-time expansion. The last point, contains the assumption about some infinitesimal damping process which obeys a scaling form. This might only occur in higher order in $1/N$, as it corresponds to an imaginary term going with $\mathrm{i}\eta k/\omega$ in the retarded Green's function.

For the post-quench Keldysh function in Eq. (6.11) the following scaling form can be introduced,

$$G^K(k,t,t) = \frac{\sqrt{r_\mathrm{i}}\Lambda^{-2a}\Gamma^2\left(3/2-a\right)}{2k^{2-2a}} F_a^K(kt), \tag{6.14}$$

where the function $F_a^K(x) = \pi x J^2_{1/2-a}(x)$ is the light-cone depending scaling function. For the bare value $a = 0$ of the scaling function, one finds

$$F_0^K(x) = 1 - \cos(2x)\,. \tag{6.15}$$

Note, that this function does not equilibrate for t and hence $x \to \infty$ is not time-translational invariant, but highly oscillating. Equilibration can be enforced by a cutoff scheme that was mentioned above,

$$F_{0,\eta}^K(x) = 1 - e^{-\eta x}\cos(2x)\,, \tag{6.16}$$

and taking the limit $\eta \to 0$ at the end of the calculation.
The equation for the effective mass is given by:

$$r(t) = \frac{g}{4}\int_0^\Lambda \mathrm{d}k\, k^{d-1}k^{-2+2a}\left(\mathrm{i}\,F_a^K(kt) - \mathrm{i}\,F_\mathrm{eq}^K(0)\right), \tag{6.17}$$

where the coefficient $g = u\sqrt{r_\mathrm{i}}\Lambda^{-2a}\Gamma^2\left(3/2-a\right)$ was introduced. Eq. (6.17) can be solved by expanding F^K for small a near $d = 4-\epsilon$ dimensions. One has to keep the $1/\epsilon$ divergent first-order terms generated from the long-time expansion. By expanding the Bessel function in F^K for $kt \gg 1$ one finds for the long-time limit of the scaling function:

$$F^K(x) = 1 - \cos(2x) - \frac{a}{2x^2} - a\frac{\sin(2x)}{x} + a\sin(2x) - \frac{a}{2x^2}\cos(2x) + \mathcal{O}(a^2). \tag{6.18}$$

The $\sin(2x)$-part does not contribute near $d = 4$ dimensions, while the contributions with $\sin(x)/x$ or $\cos(2x)/x^2$ will not generate $1/\epsilon$ under the k-integral, as those terms converge. The same result can be obtained by evaluating the Green's functions around the thermal value given by $G^K_\mathrm{thermal}(k,t,t) = \frac{\sqrt{r_\mathrm{i}}\Lambda^{-2a}\Gamma^2\left(3/2-a\right)}{2k^{2-2a}}$ and determining the long-time constant $c_k = \int \mathrm{d}\omega\, G^R(\omega) G^K_\mathrm{termal}(\omega)$.
For the constant C_0 one finds in $d = 4-\epsilon$ dimensions:

$$\begin{aligned} C_0 &= \int_0^\infty \mathrm{d}x\, \left(\mathrm{i}\,F_0^K(x) - \mathrm{i}\,F_\mathrm{eq}^K\right) \\ &= -\int_0^\infty \mathrm{d}x\, x\cos\left(2x\right) \end{aligned} \tag{6.19}$$

This integral does not converge, but with $F_{0,\eta}^K$ one finds

$$\begin{aligned} C_0 &= -\lim_{\eta\to 0}\int_0^\infty \mathrm{d}x x\cos\left(2x\right)e^{-\eta x} \\ &= +\frac{1}{4}\,. \end{aligned} \tag{6.20}$$

Inserting F of Eq. (6.19) into Eq. (6.17) and using the result of Eq. (6.20) yields

$$\begin{aligned} r(t) =& \frac{g}{16} t^{-2-2a+\epsilon} + \frac{g}{8t^2} \int_{1/t}^{\Lambda} \mathrm{d}k\, k^{d-5+2a} \\ =& \frac{g}{16} t^{-2-2a+\epsilon} + \frac{ga}{8(\epsilon - 2a)t^2} \left[(\Lambda t)^{\epsilon-2a} - 1 \right] . \end{aligned} \tag{6.21}$$

By comparing this result with the ansatz $r(t) = a/t^2$ one finds for g and a:

$$g = g^* = 8\Lambda^{\epsilon/2}\epsilon/2\,, \tag{6.22}$$

$$a = \frac{\epsilon}{4}\,. \tag{6.23}$$

There is also a second solution $a = \epsilon/2$ with $\epsilon = 3 - d$. This solution can be found with the scaling function in Eq. (6.18), where now the term $\sin(2x)/x$ enters, while the $\cos(2x)$-part vanishes. This leads to terms going with t^{-1} and t^{-2} in the effective mass. However, this results in a fixed point value for $g^* = 8\Lambda$ which does not vanish at the upper critical dimension. Therefore no controlled calculation seems possible for this solution.
Thus, the exponent θ reads

$$\theta = \frac{N+2}{N+8}\frac{\epsilon_{\mathrm{cl}}}{4}\,, \tag{6.24}$$

with the classical $\epsilon = 4 - d$.

6.1.2 Weak quench limit

Instead of analyzing the deep quench limit of Eqs. (6.7), the scenario of a very weak quench can also be considered. In this limit, only very slow modes are affected drastically by the quench. Most of the k modes do not sense the quench directly, but only via the effective mass $r(t)$. Physically, this corresponds to a slightly perturbed system, which is still near its equilibrium values at the QCP. However, those boundary conditions lead to a contradiction with the ansatz $r(t) = a/t^2$ in the equation for the effective mass, thus, in this case no universal post-quench dynamics exists. This result is in contrast to the quantum-to-classical crossover, described in Ref. [70] using numeric simulations.

In the limit of a weak quench $\omega_{\mathrm{i}} \simeq k$, but with an effective mass going with $r(t) = a/t^2$, the boundary condition for A and B are given by:

$$\langle A^2(k)\rangle \simeq \frac{1}{2} k^{2a} \Lambda^{-2a} \Gamma^2\left(3/2 - a\right)\,, \tag{6.25a}$$

$$\langle B^2(k)\rangle \simeq \frac{1}{2} k^{-2a} \Lambda^{2a} \Gamma\left(1/2 + a\right)\,, \tag{6.25b}$$

$$\langle [A,B]_{-}\rangle \simeq \Gamma\left(1/2 - a\right)\Gamma\left(3/2 + a\right)\,. \tag{6.25c}$$

Two remarks are in order. First, the expectation value of the anticommutator of $[A, B]$ is unchanged, thus, it will lead to the same retarded Green's function as in Eq. (6.12). This is reasonable, as G^R does not depend on the boundary conditions of the EOM, but only on the effective mass. Second, $\langle A^2(k)\rangle$ and $\langle B^2(k)\rangle$ are quench-amplitude independent. Which of them is larger, is not tuned by the quench-amplitude, but only by the cutoff Λ and the sign of a.

For $a > 0$ the B-part of $\varphi(t)$ is thus the dominant one. This yields

$$G_B^K(k,t,t) = \frac{\Lambda^{2a}\Gamma^2\left(1/2+a\right)}{2k^{1-2a}} F_a^K(kt), \tag{6.26}$$

with the scaling function $F_{a,B}^K(x) = xJ_{-1/2+a}^2(x)$. Note, that here the Bessel function $J_{-\alpha}$ enters. This scaling function has the long-time expansion:

$$F_{a,B}^K(x \gg 1) \simeq \frac{1}{\pi}\left(1-\cos(2x)\right) - \frac{a}{2\pi x^2}\left[1-\cos(2x)\right] + \frac{a}{\pi x}\sin(2x). \tag{6.27}$$

Evaluating the equation for the effective mass near $d = 3 - \epsilon$ yields

$$r(t) = \frac{\tilde{u}\pi}{2}\int_{k_{\text{mic}}}^{\Lambda} \mathrm{d}k\, k^{1-\epsilon} F_{a,B}^K(kt), \tag{6.28}$$

where the effective interaction parameter $\tilde{u} = uK_d\Gamma^2(1/2+a)/\pi$ was introduced. The weak quench limit can be performed for momentum modes $k > k_{\text{mic}}$. The deep quench limit should be taken for modes $k \leq k_{\text{mic}}$. Those terms will not generate any singularities near $d = 3$, as they obey a scaling form like in Eq. (6.14), but are far from equilibrium. Thus, they will lead to a term going with $\sqrt{r_{\text{i}}}$, which can be neglected in a perturbative expansion for small $\sqrt{r_{\text{i}}}$. Inserting the long-time expansion of the scaling function $F_{a,B}^K$ for modes $k > 1/t$ yields

$$r(t) = \frac{\tilde{u}}{8t^{2+\epsilon+2a}} - \frac{\tilde{u}a}{4t^2}\frac{\Lambda^{-\epsilon-2a} - t^{\epsilon+2a}}{\epsilon+2a}. \tag{6.29}$$

Here, the same cutoff procedure like in Eq. (6.20) was used. In fact, the supplementary k in the scaling dimension of G^k compensates with the system being near $d = 3$ dimensions, such that the integrals are the same. The terms going with $a\sin(2x)/x$ in F^K vanish at the upper critical dimension and with an exponential cutoff scheme. By comparing this with the ansatz $r(t) = a/t^2$, the values of $\tilde{u}$ and a are determined:

$$\tilde{u} = u^*_{z=1}, \tag{6.30}$$

$$a = -\frac{\epsilon}{4}. \tag{6.31}$$

The value of $a < 0$ is in contradiction with the above assumption. The same analysis can be done with the assumption of negative a and thus the A-part in the Keldysh function being the dominant one. In this case, the self-consistent equation leads to a positive value $a = \epsilon/4$, thus again to a contradiction with the assumption. The second solution, $a = \epsilon/2$ leads also to a contradiction, as it requires a $\log(\Lambda/t)$-depending interaction parameter u. To conclude, only the solution $a = 0$ seems reasonable, and thus no post-quench universality exists in the limit of a weak quench near the QCP. The solution, found numerically in [70] suggests an exponent $\theta = \epsilon/4$, with the quantum $\epsilon = 3 - d$, in the limit of a weak quench. This result seams not to include the full solution of the EOMin Eq. (6.4) with the two Bessel functions.

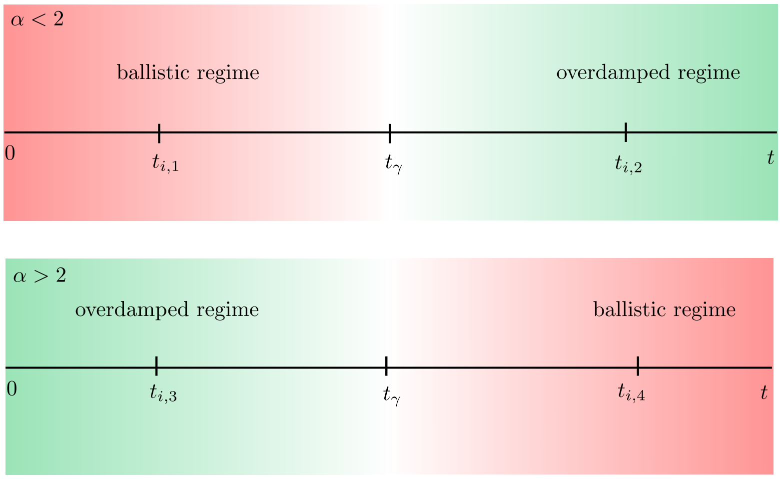


Figure 6.1: *The hierarchy of the two different dynamic limits in Laplace space. The crossover time scale is given by* $t_\gamma = \gamma^{1/(\alpha-2)}$. *For times in the ballistic regime and* $t_{\mathrm{i},1}, t_{\mathrm{i},4}$ *one finds the bare Keldysh-function given by the boundary conditions in 6.10 and the implications discussed in section 6.1. For times in the diffusive regime, one recovers for any quench-amplitude* $t_{\mathrm{i},j}$ *with* $j = 1, \ldots, 4$, *the post-quench dynamics discussed in chapter 5. In section 6.2 times in the ballistic regime, but* t_{i} *in the diffusive regime is analyzed.*

6.2 Nearly isolated systems

The boundary conditions of Eqs. (6.7) correspond to a non-interacting initial state. It was shown in the previous section, that under those conditions, one can quench the system to a non-thermal fixed point, but it is not possible to see non-equilibrium universality influenced by the QCP. Therefore it seems necessary that the system thermalizes to the QCP. However, as it was shown in section 3.5, the initial Keldysh function has a strong impact on the post-quench dynamics. Even an infinitesimal small damping rate can lead to a completely different behaviour of G^K in certain time regimes, where the boundary-conditions of Eq. (6.7) fail. Therefore, in this section some small, but finite damping is assumed to be present in the pre-quench state. Such damping can originate internally from higher order scattering processes or externally from a coupling to a heat bath. In the latter case, it is clear, that the system will thermalize to the QCP, and the Hamiltonian is the same as in chapter 5, but now with ballistically dominated dynamics. This case will be considered in the following. Probably also the internal bath can be treated with those methods in certain time regimes, but this is not the subject of this thesis.

6.2.1 Different time regimes

Before going into detail, a short reminder of the hierarchy of the time and frequency scales is presented. The hierarchy of the different regimes was also discussed in section 2.1, here this discussion is combined with the result for the post-quench Keldysh function derived with the memory ansatz in section 3.5. The influence of a bath can be included by adding the bath-spectral function into the Green's function, as it was introduced in section 1.1. The bath exponent α plays a crucial role for the order of the different dynamical regimes, while the coupling strength γ defines the typical crossover frequency.

For $\alpha < 2$ the dynamics of the system will be at first in the microscopic regime, where the φ^4-model cannot be applied. Then, they will be dominated by the ballistic term, and for times larger than $t_\gamma \propto \gamma^{1/(\alpha-2)}$ they will be dominated by the diffusive term. By considering a small bath coupling constant γ this crossover time can be tuned to large values, and it is thus not reasonable to include it into the set of microscopic timescales. In chapter 5, the dynamics are analyzed for times being deep in the diffusive regime. Still the typical timescale of the initial Keldysh function ω_i can be independent of the considered dynamic regime either in the ballistic or in the diffusive regime, see figure 6.1. For the scaling solution of the effective mass in Eq. (4.87), it has no impact if $t_\mathrm{i} \lessgtr t_\gamma$, with $t_\mathrm{i} = 1/\omega_\mathrm{i}$ and thus it has no influence on the value of the exponent θ. However, in both cases second order terms of the deep-quench expansion in g_i, going with t^{-2} were found, see appendix B. For $t_\mathrm{i} > t_\gamma$ the amplitude of those terms is large, leading to a light-cone growth of the correlation length. For $t_\mathrm{i} < t_\gamma$ the second order terms of the bare Keldysh function lead to a term going with t^{-2} but with a quench-amplitude independent prefactor of the effective mass. This term is smaller than the scaling solution and has thus a negligible impact on the correlation length.
For $\alpha > 2$ the order of the ballistic and the diffusive regime is inverted. Thus, after the microscopic regime, the system first undergoes dynamics dominated by the coupling to the bath, and at large times after the quench the dynamics become ballistic. In this case, a small bath coupling γ leads to a short diffusive regime. The deep quench limit in the post-quench Keldysh function must be taken carefully, if the typical frequency $\omega_\mathrm{i} < \omega_\gamma$ starts within the diffusive dominated regime. For the solution of the effective mass, second order terms in Eq. (B.1) going with $(\gamma\omega_\mathrm{i}^{\alpha-2})^{2/\alpha}/t^2$ play an important role, as they are dominant in the deep quench limit. For $\omega_\mathrm{i} < \omega_\gamma$ one finds again for the Keldysh function the result of Sotiriadis and Cardy in [63].

In both cases, $\alpha \lessgtr 2$, those corrections have no effect on the scaling solution and thus they have no effect on the value of the exponent θ for times in the overdamped regime. In this section, the dynamics within the ballistic regime are analyzed. If the timescale ω_i lays also in the ballistic limit, one finds the post-quench Keldysh function which fulfills the boundary conditions given in Eq. (6.7), see also appendix A.3. This limit is analyzed in the previous section, where a non-thermal fixed point is found, but no universality near the QCP. Therefore, the post-quench Keldysh function is considered now, where $\omega_\mathrm{i} > \omega_\gamma$ lays in the diffusive regime, see also appendix B. For $\alpha < 2$ this corresponds to $\omega_\mathrm{i} \ll \omega_\gamma$ and for $\alpha > 2$ to $\omega_\mathrm{i} \gg \omega_\gamma$.

For any value of both parameters, the system will thermalize to the QCP for large times after the quench.

6.2.2 Self-consistent solution without second order terms of the deep quench expansion

As it was argued above, the post-quench Keldysh function will be given with ω_i in the diffusive regime, which will be explained below. Recall first, that the memory function in Eq. (3.65) was given by

$$M(\omega,\omega') = \frac{\mathrm{sign}\,(\omega)\, n\left(\omega,\omega',\omega_\mathrm{i}\right) + \mathrm{sign}\left(\omega'\right) n\left(\omega',\omega,\omega_\mathrm{i}\right)}{\omega+\omega'+\mathrm{i}0^+}\,, \tag{6.32}$$

with

$$n\left(\omega,\omega',\omega_\mathrm{i}\right) = g^R(\omega_\mathrm{i},\omega')^{-1} - g_\mathrm{i}(k,\omega) g^R(\omega_\mathrm{i},\omega)^{-1} g^R(\omega_\mathrm{i},\omega')^{-1}\,. \tag{6.33}$$

The function $g_\mathrm{i}(k,\omega)$ arises from the LT-version of the FDT. It reads:

$$g_\mathrm{i}\left(\omega_\mathrm{i},\omega\right) = 2\int_0^\infty \frac{\mathrm{d}\omega'}{\pi}\frac{1}{|\omega|+\omega'}\frac{\mathrm{Im}\,\eta(\omega')}{(-\omega_\mathrm{i}^2+\mathrm{Re}\,\eta+\omega'^2)^2+\gamma^2\omega'^{2\alpha}}\,. \tag{6.34}$$

This function can be evaluated within the deep quench limit, presented in section 3.5.3 and the appendix B. In the appendix, in Eq. (B.8) (for $\alpha < 2$) and Eq. (B.9)(for $\alpha > 2$) it is shown that the integral in g_i can be split into the two dynamic regimes. We speak of ω_i in the diffusive regime, if $\omega_\mathrm{i} < \omega_\gamma$ for $\alpha < 2$ or $\omega_\mathrm{i} > \omega_\gamma$ for $\alpha > 2$, thus if the main contribution in g_i originates from the diffusive part of the integral.
Taking the deep quench limit $\omega_\mathrm{i} \to \infty$ (but still for $\alpha < 2$ with $\omega_\mathrm{i} \ll \omega_\gamma$), one finds

$$g_\mathrm{i}\left(\omega_\mathrm{i},\omega\right) \simeq \frac{1}{\omega_\mathrm{i}^2}\left(-1+\frac{\gamma^{1/\alpha}|\omega|}{\omega_\mathrm{i}^{-2/\alpha}c_\alpha^{(1)}}-\frac{\gamma^{2/\alpha}\omega^2}{\omega_\mathrm{i}^{4/\alpha}}c_\alpha^{(2)}+\dots\right). \tag{6.35}$$

The coefficients $c_\alpha^{(1)}$ and $c_\alpha^{(2)}$ are given by the first and the second terms in a Taylor expansion of the integral in g_i:

$$c_\alpha^{(1)} = 2\int_0^\infty \frac{\mathrm{d}x}{\pi}\frac{x^{\alpha-2}}{(1+\cot(2\pi\alpha)x^\alpha)^2+x^{2\alpha}}\,, \tag{6.36}$$

$$c_\alpha^{(2)} = 2\int_0^\infty \frac{\mathrm{d}x}{\pi}\frac{x^{\alpha-3}}{(1+\cot(2\pi\alpha)x^\alpha)^2+x^{2\alpha}}\,. \tag{6.37}$$

The first term in Eq. (6.35) in the brackets refers to the zero order term, the second to the first order and the last to the second order term in an expansion in $1/\omega_\mathrm{i}$. The solution for $r(t)$ originating from the first order term will be discussed below. It will be shown, that this term leads to a scaling form of the effective mass going with t^{-2}. The contribution from the first order term vanishes here, as the system is at the upper critical dimension, see also appendix B.1. The important second order term is discussed in section 6.2.3.

Evaluating the function n with the zeroth order term yields

$$n^{(0)}(\omega_\mathrm{i},\omega) = -\omega^2 - \mathrm{i}\,\eta\,. \tag{6.38}$$

With this result one finds for the memory function

$$M^{(0)}(\omega,\omega') = \frac{\mathrm{sign}(\omega)\left(-\omega^2-\mathrm{i}\,\eta\right)+\mathrm{sign}(\omega')\left(-\omega'^2-\mathrm{i}\,\eta'\right)}{\omega+\omega'}\,. \tag{6.39}$$

This memory function is completely independent of the quench amplitude ω_i. Due to the presence of the bath, the system is expected to thermalize to the QCP. For $\alpha > 2$ the dynamics in the long-time limit will be dominated by the ballistic term. In this case it is reasonable to assume that the Green's functions will equilibrate with the scaling form $G_\mathrm{eq}^{R/K}(k,t,t') = k^{-1}F_\mathrm{eq}^{R/K}(k(t-t'))$. This form suggests in contrast to section 6.1, that also out of equilibrium, the scaling form for G^K is given by:

$$G^K(k,t,t) = k^{-1}F^K(kt)\,. \tag{6.40}$$

This scaling form can be derived from the memory function, by assuming η to be infinitesimal small and introducing dimensionless variables $y = \omega/k$ for the inverse Laplace transformation. With Eq. (6.40) the results from section 4.5 can directly be applied. One finds

$$C_0 = \int_0^\infty \mathrm{d}x\, x^{2/z-1} \left(\mathrm{i}\, f^K(x,1) - \mathrm{i}\, F_{\mathrm{eq}}^K\right) = \frac{1}{4}. \tag{6.41}$$

The integral is evaluated in appendix E. By solving the self-consistent equation for all times, it must hold for the fixed point of the interaction parameter

$$u^* = \frac{4\epsilon}{K_d} \Lambda^\epsilon\,. \tag{6.42}$$

This is the fixed point of the ballistic system. The effective mass in the large-N limit is given by:

$$r(t) = \frac{2\epsilon}{t^2} I_0 = \frac{\epsilon}{2t^2}\,. \tag{6.43}$$

With $a^{(0)} = \epsilon/2$, one finds for the exponent θ:

$$\theta^{(0)} = \frac{\epsilon}{2} \tag{6.44}$$

with $\epsilon = 3 - d$. The index $^{(0)}$ refers to the fact, that only the zeroth order contribution of the deep-quench expansion is kept. By using the RG scheme, one finds for an arbitrary N-component vector field $\boldsymbol{\varphi}$:

$$\theta^{(0)} = \frac{N+2}{N+8}\frac{\epsilon}{2}\,. \tag{6.45}$$

This solution will be modified by second order terms, which are derived in the next section.

6.2.3 Full solution for the effective mass

In the same way, the influence of the second order term of the deep quench expansion in Eq. (6.34) can be analyzed. This second order term results in a further term in the bare Keldysh function:

$$G^{K(2)}(k,t,t') = \omega_{\mathrm{i}}^{2-4/\alpha} \gamma^{2/\alpha} k^{-1} c_\alpha^{(2)} \int \mathrm{d}y \int \mathrm{d}y'\, \mathrm{Im}\, g(y) \mathrm{Im}\, g(y') \frac{\mathrm{sign}\, y|y|^2 + \mathrm{sign}\, y'|y'|^2}{y+y'} e^{-\mathrm{i}kt} e^{-\mathrm{i}kt'}\,. \tag{6.46}$$

Here, $g(y) = (y^2+1+\tilde{\eta})^{-1}$ is the scaling form of the free retarded Green's function and $\tilde{\eta}(y) = k^{-2}\eta(\omega)$. To demonstrate the impact of this term, the large-N equation for the effective mass is considered. However, the influence of this term can also be analyzed for the RG equations. Note that second order terms always go with t^{-2}, thus vanish in the limit $t \to \infty$ and does not affect the coefficient c_K of the long-time expansion. Performing the k- integral over $G^{K(2)}(k,t,t')$ leads to a supplementary t-dependent term:

$$\int_0^\Lambda \mathrm{d}k\, k^{d-1} G^{K(2)}(k,t,t) = \frac{(\omega_{\mathrm{i}}^{\alpha-2}\gamma)^{2/\alpha} c_\alpha^{(2)}}{t^{2-\epsilon}} \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{d}y'\, \mathrm{Im}\, g(y) \mathrm{Im}\, g(y') \frac{\mathrm{sign}\, y|y|^2 + \mathrm{sign}\, y'|y'|^2}{(y+y'+i0^+)(y+y'-\mathrm{i}0^+)^2}\,. \tag{6.47}$$

Here, it is used the fact that the system is at the upper critical dimension $d = d_{\mathrm{uc}} = 3 - \epsilon$. The ϵ-dependence is important for the time dependence, but not in the integral over dimensionless variables, where it was set to zero. The integral is finite for the reason of symmetry and thus this term does not vanish like the one originating from the first order term. Including this term into the effective mass, with the solution of the zeroth order term derived above, yields:

$$r(t) = \frac{\epsilon}{2t^2}\left(1 + \left(\gamma\omega_{\mathrm{i}}^{\alpha-2}\right)^{2/\alpha} \times C\right). \tag{6.48}$$

Here,

$$C = 2c_{\alpha}^{(2)} \int_{-\infty}^{\infty} \mathrm{d}y\,\mathrm{d}y'\,\mathrm{Im}\,g(y)\mathrm{Im}\,g(y')\frac{\mathrm{sign}\,y|y|^2 + \mathrm{sign}\,y'|y'|^2}{(y+y'+i0^+)(y+y'-\mathrm{i}0^+)^2}, \tag{6.49}$$

is a constant under the scaling procedure. This effective mass has a light-cone form, where the amplitude consists of two parts: an universal, quench-amplitude independent part, and a part containing informations about the specific bath parameters and the quench amplitude. To determine which term is larger, consider first $\alpha > 2$. The condition that the initial Keldysh function was given within the overdamped time regime corresponds to $\omega_{\mathrm{i}} \gg \omega_\gamma$. Using $\omega_\gamma \propto \gamma^{1/(2-\alpha)}$ thus immediately yields

$$\left(\gamma\omega_{\mathrm{i}}^{\alpha-2}\right)^{2/\alpha} \gg \left(\gamma\gamma^{-1}\right)^{2/\alpha} = 1. \tag{6.50}$$

Thus, this term is dominant over the 1 originating from the zeroth order term. This can also be derived from Eq. (B.9). The correction of the deep-quench expansion arises only for $\omega_{\mathrm{i}} > \omega_\gamma$, else the post-quench Keldysh-function is given by the result of Sotiriadis and Cardy in [63] and one finds the non-equilibrium universality discussed in section 6.1. The same analysis can be done for $\alpha < 2$. This case is more delicate from the point of view of thermalization and fixing the boundary conditions, as the relaxation regime is located after the ballistic regime, see figure 6.1. Ignoring this point, it must hold $\omega_{\mathrm{i}} \ll \omega_\gamma$, such that the Keldysh function is determined within the diffusive regime. This immediately yields that also in this case it holds $\gamma\omega_{\mathrm{i}}^{\alpha-2} \gg 1$. The bath-depending contribution is thus always relevant, as it requires that for the damping term holds $\gamma\omega_{\mathrm{i}}^{\alpha} \gg \omega_{\mathrm{i}}^2$. Per construction it holds $\left(\gamma\omega_{\mathrm{i}}^{\alpha-2}\right)^{2/\alpha} \gg 1$ for any positive value of α. This reflects that the influence of the bath is essential to build up a scaling form of the effective mass near the QCP. Thus the second, quench-amplitude dependent term always dominates the universal, quench amplitude independent part. If the parameter condition $\gamma\omega_{\mathrm{i}}^{\alpha} \gg \omega_{\mathrm{i}}^2$ can be reached in the prethermal regime, this implies, that the non-equilibrium exponent θ is non-universal. It implies also for $\alpha > 2$ that in the quasi-adiabatic long-time limit, the order parameter relaxes with a ω_{i} dependent amplitude to its equilibrium value. Thus in contrast to the dynamics dominated by the bath, the memory of the initial state configuration is still present even on very large time scales. Further for $\alpha > 2$ the memory never gets completely lost.

6.3 Failure of the Euclidean mapping

In this section the critical exponent θ will be calculated via a mapping to a $(d+z)$-dimensional, classical boundary-layer problem. This mapping of the post-quench dynamics was recently discussed and used in Refs. [19, 20]. This mapping is well established in equilibrium, where it is indeed possible to treat quantum fluctuations like z supplementary dimensions of a classical system. However, it is known, that

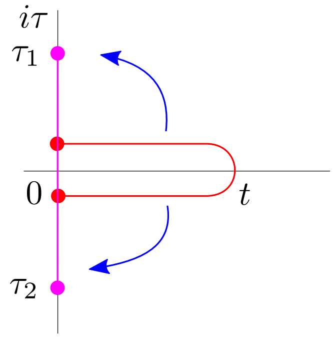


Figure 6.2: *The Euclidean mapping. The Keldysh contour in real times is analytically continued to imaginary times* $\tau_{1/2}$.

the analytic continuation and integral-path deformation which are typically done in such mappings can lead to wrong results due to singularities in the complex plane. At least for quenches going beyond the critical point, e. g. from the ordered in the symmetric phase, such singularities are known to emerge [26]. On the other hand, if such a mapping works, the time dependence of the operators can much easier be evaluated. This was checked by evaluating the dynamics for $d = 1$ [19]. Here, an example is presented where the Euclidean mapping does not work, because quantum fluctuations restore the memory of the initial state even in the presence of a small coupling to a bath.

6.3.1 Euclidean mapping

The quench protocol is performed in two steps. Initially, the system is prepared in the ground state $|\varphi_\mathrm{i}\rangle$ of an initial Hamiltonian H_i. At time $t = 0$, the parameter set in the Hamiltonian is suddenly switched, such that the time evolution is governed by a new Hamiltonian H. Here, H is the Hamiltonian of an isolated φ^4-model,

$$H = \frac{1}{2}\int_x \left(\boldsymbol{\pi}^2 + \left(\nabla\boldsymbol{\varphi}\right)^2 + r_0\boldsymbol{\varphi}^2 + \frac{u(\boldsymbol{\varphi}\cdot\boldsymbol{\varphi})^2}{2N} - \mathbf{h}\cdot\boldsymbol{\varphi} \right) . \tag{6.51}$$

For simplicity and to keep the notation as simple as possible, here only a scalar field φ is considered. The extension to a N-component vector field is however straightforward. The Hamiltonian given above yields for the time evolution of the initial state

$$|\varphi(t)\rangle = e^{-\mathrm{i}\,Ht}|\varphi_\mathrm{i}\rangle . \tag{6.52}$$

The expectation value of a physical operator O reads

$$\langle O(t)\rangle = \langle\varphi_\mathrm{i}|e^{\mathrm{i}\,Ht} O\, e^{-\mathrm{i}\,Ht}|\varphi_\mathrm{i}\rangle . \tag{6.53}$$

Instead of analyzing the time dependence of $\langle O(t)\rangle$, one can also perform the limit to the imaginary time $\tau = -\mathrm{i}\,t$, and analyze the Euclidean operator $\langle O(\tau)\rangle_E$:

$$\langle O(\tau)\rangle_E = \langle\varphi_\mathrm{i}|e^{-H\tau} O e^{H\tau}|\varphi_\mathrm{i}\rangle . \tag{6.54}$$

Note that as usual, H consists of a kinetic part $T = \frac{1}{2}\int_x \hat{\pi}^2$ and a potential part

$$V = \frac{1}{2}\int_x \left(r\hat{\varphi}^2 + (\nabla\hat{\varphi})^2 + u\hat{\varphi}^4/2 \right) . \tag{6.55}$$

The evolution of the operator will be sketched below, this analysis was also done in detail in Ref. [19], and the desired real time expectation value will be obtained by an analytic continuation $\langle O\rangle(t) = \langle O(-\mathrm{i}\,t)\rangle_E$. To start, the Euclidean partition function $Z(\tau)$ is evaluated:

$$Z(\tau) = \langle\varphi_\mathrm{i}|e^{-H\tau}|\varphi_\mathrm{i}\rangle. \tag{6.56}$$

With the usual steps of evaluating functional integrals, the time axis τ is discretized into N small time steps $\delta\tau = \tau/N$. At the same time, the time evolution can also be split into small parts $e^{-H\tau} = e^{-\delta\tau H}\dots e^{-\delta\tau H}$. After each exponential, an 1 is inserted in form of the eigenstates of the scalar field operator $\hat{\varphi}$: $1 = \int d\varphi_j|\varphi_j\rangle\langle\varphi_j|$, with $j = 1,\dots,N$. The expectation value $\langle\varphi_j|e^{\delta\tau H}|\varphi_{j+1}\rangle$ can be evaluated by Taylor-expanding $e^{\delta\tau H}$ in the limit of small $\delta\tau$ (or equally $N\to\infty$). The expectation values of the potential part of the Hamiltonian can be directly evaluated, as it contains only the scalar field operator $\hat{\varphi}$. The remaining expectation value of the kinetic part can be evaluated by inserting $1 = \int \mathrm{d}\pi_j|\pi_j\rangle\langle\pi_j|$. By using the scalar product between φ and π, reexponentiating and finally taking the continuum limit $N\to\infty$, one finds

$$Z(\tau) = \int \mathcal{D}\phi\mathcal{D}\pi\langle\varphi_\mathrm{i}|\varphi(\tau)\rangle e^{-S[\varphi,\pi]}\langle\varphi(\tau=0)|\varphi_\mathrm{i}\rangle, \tag{6.57}$$

with the action

$$S[\varphi,\pi] = \int_x\int_0^\tau \mathrm{d}\tau'\left(-\mathrm{i}\pi\partial_{\tau'}\varphi + H[\varphi,\pi]\right). \tag{6.58}$$

To obtain the expectation value $\langle O(\tau)\rangle_E$, one needs to apply the above logic twice. This leads to a two branch contour along the imaginary axis, similar to the Schwinger-Keldysh contour along the real axis. The expectation value thus reads

$$\langle O(t)\rangle_E = \frac{1}{Z(2\delta)}\langle\varphi_\mathrm{i}|e^{\mathrm{i}\,H(t+\mathrm{i}\,\delta)}Oe^{-\mathrm{i}\,H(t-\mathrm{i}\,\delta)}|\varphi_\mathrm{i}\rangle, \tag{6.59}$$

where δ is some parameter to distinguish between both contours. At the end, the limit $\delta\to 0$ must be taken. It is convenient to introduce two times $\tau_i = -\mathrm{i}(t+\mathrm{i}\,\delta)$ and $\tau_2 = -\mathrm{i}\,t+\delta$ to handle the differences of the real part of $\tau_{1/2}$. The expectation value thus reads

$$\langle O(\tau_1,\tau_2)\rangle_E = \frac{1}{Z(2\delta)}\langle\varphi_\mathrm{i}|e^{-H\tau_2}Oe^{H\tau_1}|\varphi_\mathrm{i}\rangle. \tag{6.60}$$

To evaluate this expectation value, one further simplification is made by assuming $\tau_{1/2}$ to be real. This is only true for $t = 0$, but any finite time only leads to a shift along the imaginary axis for τ. This simplification allows to write

$$\langle O(\tau_1,\tau_2)\rangle_E = \frac{1}{Z(2\delta)}\langle\varphi_\mathrm{i}|e^{-H\tau_2}Oe^{-H|\tau_1|}|\varphi_\mathrm{i}\rangle, \tag{6.61}$$

and to discretize the imaginary time axis. Here, it is split into time steps with $\delta\tau = \tau_1/N_1 = \tau_2/N_2$ and the time axis is discretized according to $\delta\tau_j = \tau_1 + \frac{j}{N_1+N_2}(\tau_2-\tau_1)$, with $j = 0,\dots,N_1+N_2$. By evaluating $\langle\varphi_j|e^{\delta\tau H}|\varphi_{j+1}\rangle$, taking the continuum limit $N_1+N_2\to\infty$ and performing the integral over π, one finds

$$\langle O(\tau_1,\tau_2)\rangle_E = \frac{1}{Z(2\delta)}\int \mathcal{D}\varphi\langle\varphi_\mathrm{i}|\varphi(\tau_2)\rangle\langle\varphi(\tau_1)|\varphi_\mathrm{i}\rangle e^{-S[\varphi]}\langle\varphi(0)|O|\varphi(0)\rangle. \tag{6.62}$$

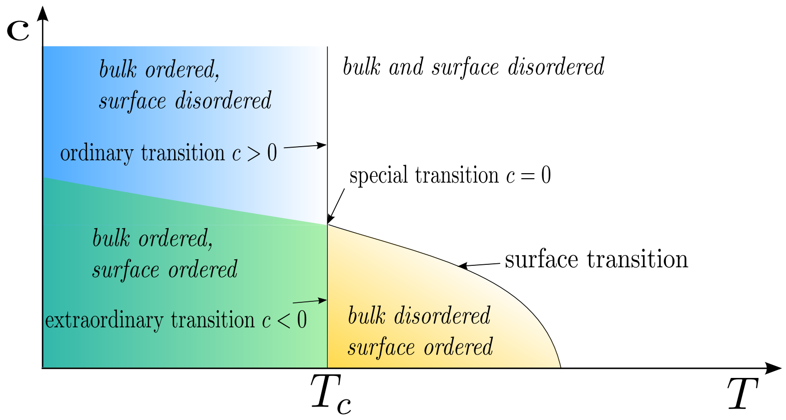


Figure 6.3: *Different types of phase transitions depending on the surface parameter c. The bulk phase transition takes place at $T = T_c$, and is affected of the surface-universality class.*

The operator O is evaluated at $\tau = 0$, at the discrete time step $\delta\tau_{N_1} = 0$. The action S is given as

$$S[\varphi] = \frac{1}{2}\int_x \int_{\tau_1}^{\tau_2} \mathrm{d}\tau_2 \left(\left(\partial_\tau \varphi\right)^2 + r\varphi^2 + \left(\nabla\varphi\right)^2 + \frac{u}{2}\varphi^4 \right). \tag{6.63}$$

Note that at this point, the restriction of τ_j to be real with $j = 1, 2$ is not necessary. It was a simplification to build up the path integral. Compared to the usual field integral, the boundary conditions at τ_1 and τ_2 enter via $\langle\varphi(\tau_1)|\varphi_\mathrm{i}\rangle\langle\varphi_\mathrm{i}|\varphi(\tau_2)\rangle$. Such a boundary condition can be enforced by adding the following term into the action:

$$S_\mathrm{boundary} = \sum_{j=1,2} \int_x \left(\frac{c}{2}\varphi^2(x,\tau_j) - h_\mathrm{i}\varphi(x,\tau_j) \right). \tag{6.64}$$

The constant c is here given by the initial mass of the pre-quench state, and h_i is given by an initial external field. The form in Eq. (6.64) is useful to make a connection with critical boundary scaling. The structure of S_boundary can be obtained equally with the assumption that the initial density, $\rho_\mathrm{i} = |\varphi_\mathrm{i}\rangle\langle\varphi_\mathrm{i}|$, can be written in a Gaussian form

$$\langle\varphi|\rho_\mathrm{i}|\varphi'\rangle = \exp\left[-\int_x \left(\frac{c}{2}\varphi^2(x,\tau_1) - h_\mathrm{i}\varphi(x,\tau_1) \right) - \int_x \left(\frac{c}{2}\varphi'^2(x,\tau_2) - h_\mathrm{i}\varphi'(x,\tau_2) \right) \right]. \tag{6.65}$$

Via scaling arguments, it can be shown that indeed only terms up to quadratic order in $\varphi(\tau_j)$ in S_boundary are relevant. Consider further terms going with $u_{s,n}/n\varphi^n(\tau_j)$, and $n \geq 3$. The cubic term can be eliminated by shifting it into the $\varphi^4(\tau_i)$ term, but powers going with $n \geq 4$ are irrelevant above $d = 2$ dimensions.

6.3.2 Critical boundary scaling

The total action of Eq. (6.63) and Eq. (6.64),

$$S_\mathrm{tot} = S[\varphi] + S_\mathrm{boundary}, \tag{6.66}$$

corresponds to a $(d+1)$-dimensional field theory with two boundary conditions along one space direction [52, 71]. Thus the quench to the QCP in real time corresponds to critical boundary scaling near surfaces. The bulk at the critical temperature T_c, and the influence of boundary conditions on the surface is analyzed. In this section the surface criticality, relevant for the quench problem, is shortly reviewed. The first assumption which is made, is that the distance between the two opposite surfaces τ_1 and τ_2 is sufficiently large to treat them separately and to neglect interference effects between them. This assumption may be not valid for bulk-critical systems, where the correlation length ξ diverges and thus the condition $|\tau_1 - \tau_2| \gg \xi$ is not fulfilled. Nevertheless, under this assumption it is possible to concentrate on one semi-infinite system, where the surface criticality is analyzed in detail in Ref. [50]. The notation of this section and nomenclature of the exponents is the same as in Ref. [52].
The Lagrangian of one surface is up to some constant given by,

$$L_s[\varphi(\tau_j)] = \frac{c}{2}\left(\varphi - m_s\right)^2 , \tag{6.67}$$

where $m_0 = h/c$. For one bulk universality class, three different universality classes are possible for the bulk-phase transition at $T = T_c$, see figure 6.3. These universality classes depend on the the value of c. If $c > 0$, this corresponds to a reduced critical temperature on the surface, thus the bulk can be ordered, while the surface is already disordered. Under the RG-flow c is a relevant parameter, flowing to infinity. This transition is called ordinary transition. If $c < 0$ or some finite field h_i is applied on the surface, the situation can be reversed. This corresponds to an enhanced critical temperature on the surface, where the bulk can be disordered, while the surface is still ordered. In this case, the bulk transition is called the extraordinary transition. The case $c = 0$ refers to the special phase transition.

The universal exponents with respect to the distance to the surface are independent of the bulk exponents but depend on the type of surface transition. The ordinary phase transition is characterized by the fixed point $c = \infty$ and c^{-1} is a dangerously irrelevant parameter in the RG-sense. For the bulk magnetization m, the following scaling form is assumed

$$m(\tau, r, h, c) = b^{-\beta/\nu} m\left(b^{-1}\tau, b^{1/\nu} r, h b^{\Delta_1}, b^{\Phi/\nu} c\right) , \tag{6.68}$$

where r is the distance to the bulk critical point and τ the distance to the surface. β and ν are the bulk exponents whose nomenclature is given in table 1.1. The surface scaling exponents Δ_1 and Φ are the corresponding exponents of the external field h_i and the suppression on the surface due to $c > 0$, respectively. The scaling dimension of the distance to the surface is the usual length-rescaling factor b^{-1}, where b is some freely chosen parameter.

For the quench problem, the interesting protocol is $c > 0$ for a quench starting in the disordered phase. This corresponds to the ordinary phase transition. If the quench starts in the ordered phase, $c < 0$, and the quench protocol corresponds thus to the extraordinary phase transition. An initial finite magnetization can also be achieved by $c > 0$ and a finite external field h_i on the surface. As the origin of the finite order on the surface should not matter [50], it was argued that the extraordinary exponents can be obtained as well from the ordinary exponents with a finite field h_i. This argumentation was partially confirmed for the exponents by a $1/N$ expansion, but the singularities of some scaling functions suggest more care [72]. For the analysis here only the universal exponents play a role, so the extraordinary phase transition will not be discussed separately.

Further, for the quench to the QCP, this corresponds to the bulk being critical with $r = 0$. As c^{-1} is a dangerously irrelevant parameter, one can introduce instead of c and h_i a scaling form containing the relevant field $\tilde{h} = h_{\text{i}}/c^y$. Here, $y = (\Delta_1^{\text{sp}} - \Delta_1^{\text{ord}})/\Phi$ is the scaling exponent of $\tilde{h}$, Δ_1^{sp} the surface-field exponent of the special transition, and Δ_1^{ord} the surface-field exponent of the ordinary transition. The magnetization behaves as

$$m(\tau, 0, \tilde{h}) = b^{-\beta/\nu} m\left(b^{-1}\tau, b^{\Delta_1^{\text{ord}}/\nu}\tilde{h}\right). \tag{6.69}$$

By setting $b^{-1}\tau = 1$, one can express the magnetization depending on the distance to the surface and obtains a new scaling function $Y(\tau/\xi_h) = m(1, \tau^{\Delta_1^{\text{ord}}/\nu}\tilde{h})$:

$$m(\tau, 0, \tilde{h}) = \tau^{\beta/\nu} Y(\tau/\xi_h), \tag{6.70}$$

with the length $x_h = \tilde{h}^{-\nu/\Delta_1^{\text{ord}}}$. Similar to the scaling in chapter 2, on large length scales $\tau \gg \xi_h$ one expects the magnetization to be independent of ξ_h, thus $Y(x >> 1) \to \text{const}$. This leads to a decay of the magnetization with the bulk-exponents β and ν:

$$m(\tau) \propto \tau^{\beta/\nu}, \qquad \text{for } \tau \gg \xi_h\,. \tag{6.71}$$

In the other limit $\tau \ll \xi_h$, one expects a strong influence of the surface field h_{i}. As the susceptibility $\partial m/\partial\tilde{h}|_{\tilde{h}\to 0}$ is finite on the surface, one can Taylor expand $Y(x)$ for small arguments, which immediately yields

$$m\left(\tau, 0, \tilde{h}\right) \propto \tau^{(\Delta_1^{\text{ord}} - \beta)/\nu}\tilde{h}, \qquad \text{for } \tau \ll \xi_h\,. \tag{6.72}$$

Making the connection to the quantum-quench problem and the notation introduced in chapter 2, the non-equilibrium exponent θ can be directly read off

$$\theta = (\Delta_1^{\text{ord}} - \beta)/\nu\,. \tag{6.73}$$

The value of Δ_1^{ord} is given in Ref. [52] in an ϵ expansion

$$\Delta_1^{\text{ord}} = \frac{1}{2} - \frac{4-N}{4(N+8)}\epsilon, \tag{6.74}$$

with $\epsilon = 4 - (d+1) = 3 - d$, where d is the dimension parallel to the surface and thus the bulk has the dimension $d+1$. Together with the known bulk exponents

$$\nu = \frac{1}{2} + \frac{N+2}{4(N+8)}\epsilon\,, \tag{6.75}$$

$$\beta = \frac{1}{2} - \frac{3}{2(N+8)}\epsilon\,, \tag{6.76}$$

it holds for θ

$$\theta = \frac{N+2}{2(N+8)}\epsilon\,. \tag{6.77}$$

This is the same result one would obtain by assuming equilibration to the critical point and using simultaneously the time-dependence of the Cardy-Sotiriadis post-quench Keldysh function:

$$G^K(k, t, t) = \frac{1}{k}\left(1 - \cos\left(kt\right)\right). \tag{6.78}$$

However, it was shown in this thesis, that this kind of Keldysh function can not be obtained after a quench to the QCP. Within neither the deep nor the weak quench limit the scaling form of G^K is the same as in equilibrium. Including the effect of a small, but finite damping leads to a non-universal part in θ which is always dominant compared to the universal one. This non-universal part cannot be reproduced within the methods of critical boundary scaling. The post-quench scenario is thus an example where the dynamics cannot be reproduced with a simple Euclidean mapping approach.

Table 6.1: *Results for the post-quench dynamics in isolated or nearly isolated systems*

No bath:	Full solution for the EOM of φ with ansatz $r(t) = a/t^2$. Fixing boundary conditions at times $t = \Lambda^{-1}$ after the quench, this leads to the Cardy-Sotiriadis result for the bare g^K.
Deep-quench limit:	Non-thermal fixed point, prethermal universality, see A. Mitra et al. $\theta = \epsilon/4$ in the limit $\epsilon \ll 1$ and $N \to \infty$.
Weak-quench limit:	Contradiction for a, no prethermal universality near the QCP.
With bath: $\alpha > 2$	Knowledge of the final, thermal state at $t = \infty$. Thermalization to the QCP, oscillations are naturally damped. Effective mass can be obtained approximately in the limit of small ϵ. Relaxation to equilibrium with universal exponents.
$\omega_\mathrm{i} < \omega_\gamma$	No ω_i independent terms, Cardy-Sotiriadis result for the bare g^K. Prethermal universality only for the deep quench limit, see above.
$\omega_\mathrm{i} > \omega_\gamma$	No prethermal universality, as θ is found to be: $\theta = \epsilon/2 \left(1 + \left(\gamma \omega_\mathrm{i}^{\alpha-2}\right)^{2/\alpha} \times C\right)$, C is given by Eq. (6.49).
Euclidean mapping:	$\theta = \epsilon/2$, this result was found in none of the presented scenarios.

6.4 Summary

In this chapter the post-quench dynamics of an isolated and a nearly isolated system are analyzed. Taking the deep-quench limit in the isolated system, without considering higher order scattering terms or finite coupling to some external heat bath, leads to prethermal universality near a non-thermal fixed point. The methods used to describe this non-thermal fixed point are similar to the quantum-to-classical crossover, where the quench-amplitude acts like the temperature. The upper critical dimension is found to be four, like in a classical, critical system. However a major difference to the classical scenario is the non-thermal scaling dimension of the Keldysh function. Those kinds of systems have been analyzed in great detail by A. Mitra and A. Gambassi et al., here their results have been reproduced with the methods developed in this thesis. The question of universality in the long-time limit is not well defined here, as without any further terms or assumptions, those systems will not thermalize.
In the limit of a weak quench, we could not find prethermal universality, since the assumption of a scaling form in the effective mass leads to a contradiction. This is in contrast to numeric simulations reported in Ref. [70].
Considering the influence of some small, but finite damping, which is irrelevant in the sense of scaling, the power-law decay to the QCP is found in the long-time limit. However, also in this scenario, no universal prethermal regime exists, as the self-consistent solution for the exponent θ has always a relevant, quench-amplitude or bath-coupling dependent term. This also implies, that even in the quasi-adiabatic limit, the amplitudes will be non-universal, in contrast to the behaviour found in dissipative systems. Thus, in none of the presented quench protocols, it is possible to reproduce the results obtained by the Euclidean mapping, which predicted an universal, short-time exponent with $\theta = (3 - d)/2$.

The main results for the isolated system and the different systems and quench protocols are summarized in table 6.1.

Conclusion

In this thesis, the dynamics after a quench to the QCP are analyzed for the isolated system as well as for a system coupled to an external heat bath. The central point is the out-of-equilibrium version of the large-N equation in Eq. (4.67):

$$r(t) = \frac{u}{2} \int_{|\mathbf{k}|<\Lambda} \mathrm{d}^d k \, G^K_{r(t)}(k,t,t) \,. \tag{6.1}$$

This equation can be solved self-consistently. To find non-equilibrium universality it is essential that the Keldysh function G^K is scale invariant. This scale invariance implies that G^K must be taken in a limit, where it is independent of the quench amplitude ω_i. In general this is not the case on intermediate timescales after the quench. In an open system, the system needs time to dissipate the induced energy to the heat bath. In an isolated system energy conservation implies that the system will heat up, if it thermalizes, by the amount of the induced energy. We showed, that in the deep-quench limit, $\omega_\mathrm{i} \to \infty$, a part of the Keldysh function is scale invariant for open systems. The opposite limit of a weak quench leads to a contradiction in Eq. (6.1), i. e. there is only the trivial solution $r(t) = 0$. By expanding the bare Keldysh function in this deep-quench limit, it was shown, that parts of the inverse correlation length obey scaling:

$$r_\mathrm{sc}(t) = \frac{a}{t^{2/z}}, \tag{6.2}$$

with an universal amplitude a, and the dynamic exponent z. This scaling form causes many physical results, which are summarized in the following.

In the **open quantum system** in chapter 5, the scaling form of $r(t)$ leads to an universal, prethermal regime. Here, the order parameter grows with a new, universal exponent θ:

$$\phi(t \ll t^*) \propto t^\theta \,. \tag{6.3}$$

The exponent θ also captures the singularities in the Green's functions for $t > t'$:

$$G^R(k,t,t') = \left(\frac{t}{t'}\right)^\theta \frac{1}{k^{2-z}} F^R(k^z t, t'/t) \,, \tag{6.4}$$

$$G^K(k,t,t') = \left(\frac{t}{t'}\right)^\theta \frac{1}{k^{2-z}} F^K(k^z t, t'/t) \,. \tag{6.5}$$

In contrast to classical systems, the singularity t/t' for $t' \to 0$ enters in both, the retarded and the Keldysh Green's function with the same exponent. An other difference to classical systems is the leading term of the correlation length $\xi(t) \propto r^{-1}(t)$. In an open quantum system, the correlation length grows with a light cone in the prethermal regime, and the scaling part is only a small correction to this ballistic light cone. Furthermore, the z-dependence of θ is particular and reflects the quantum nature not only by taking into account the upper critical dimension, $z \leq 4$, but also negative exponents are possible. Those negative exponents are due to quantum oscillations, where the bare scaling function $f^K(k^z t)$ shows damped oscillations in time for $z < 2$, while for larger z some crossover to overdamped dynamics can be observed. The z-dependence of θ is given in figure 5.2. For an Ohmic bath we found

$$\theta = \frac{N+2}{N+8}\frac{\epsilon}{4}\,, \tag{6.6}$$

with $\epsilon = 4 - d - z$. The time-scale t^*, limiting this prethermal regime is given by the inverse quench-amplitude and can thus be tuned to large values for weak quenches.
In the limit of large times, the universal relaxation to equilibrium was confirmed. In the long-time limit, the inverse correlation length is only given by the scaling part $r(t) = at^{-2/z}$. The relaxation to the QCP leads to aging effects in the Green's functions with adiabatic decay instead of an exponentially fast decay to their equilibrium values $G^{R/K}_{\text{eq}}$:

$$G^R_r(k,t,t') = G^R_{\text{eq}}(k,t-t') - 2\mathrm{i}\,\theta\left(t-t'\right) r\left(\frac{t+t'}{2}\right) C^R(t-t')\,, \tag{6.7}$$

$$G^K_r(k,t,t') = G^K_{\text{eq}}(k,t-t') - 4\mathrm{i}\, r\left(\frac{t+t'}{2}\right) C^K_z(k,t-t')\,. \tag{6.8}$$

The coefficients $C^{R/K}$ are given by a convolution of the equilibrium Green's functions. Thus, the aging amplitude is given by the coefficient a and therefore by the non-equilibrium exponent θ. In analogy to the FDT in equilibrium, the retarded and the Keldysh Green's functions can be connected by introducing a time-dependent distribution function $n(\omega,t)$. The aging effects lead to a correction δn to the usual Bose-Einstein distribution function n_B, given by:

$$\delta n(\omega,t) = \coth\left(\frac{\omega}{2T}\right)\frac{\theta\Gamma\left(2/z\right)}{(|\omega|t)^{2/z}}\left[\cos\left(\frac{\pi}{z}\right) + \frac{k^2}{\gamma|\omega|^{2/z}}\sin\left(\frac{\pi}{z}\right)\right]. \tag{6.9}$$

This correction shows the non-thermal nature of this relaxation process, as it is not possible to introduce an effective, time-dependent temperature. Also, for negative exponents, this leads to a negative δn, making an interpretation as distribution function impossible. To conclude, in the adiabatic limit, thermalization is slowed down with an algebraic decay and with θ entering as universal amplitude.

The isolated quantum system is analyzed in chapter 6. Here two different kinds of systems have been distinguished, the **perfectly isolated** and the nearly isolated system. In the perfectly isolated systems no contact to an external bath is considered, and the initial state at time $t = 0$ is the ground state of the non-interacting Hamiltonian. This point is crucial, as even small corrections due to interactions of the initial state can have large impact in certain time regimes after the quench, and thus completely change the dynamics in the post-quench Keldysh function. The methods developed in this thesis in chapter 4 can be straightforwardly extended to this scenario. Here an important point are the boundary conditions when the equations of motion are solved. In contrast to the open system, which thermalizes by construction, those boundary conditions have to be fixed for the isolated system at a microscopic timescale after the quench. We confirmed the result of [32–34] in the limit of a deep quench. In contrast to the open quantum system, this deep quench limit corresponds not to a quench to the QCP, but to a non-thermal fixed point. As argued above, the Keldysh function depends strongly

on the quench amplitude ω_i. Similar to the quantum-classical crossover, the quench amplitude can be interpreted as temperature, however a k-mode dependent one. The universal dynamics due to this fixed point are also characterized by an universal, prethermal regime with $\theta = \epsilon_\mathrm{cl}/4$, with the classical value $\epsilon_\mathrm{cl} = 4 - d$. Like in the open system, θ characterizes the growth of the order parameter, as well as the singularities in the Green's functions in the prethermal regime. In the long-time limit it is not clear if and how this system thermalizes.
In analogy to the quantum-to-classical crossover, also the limit of a weak quench $\omega_\mathrm{i} \to 0$ was analyzed. Here, we find a different result than in [70]. The boundary conditions in the limit of a weak quench lead to a contradiction in the equation for $r(t)$. Thus, we conclude that no prethermal universality can be found for the perfectly isolated system.

We also introduced the **nearly isolated** system, where the dynamics are dominated by the ballistic term but with a supplementary small but finite coupling to an external heat bath. The diffusive dynamics due to the bath are chosen such that they are irrelevant in the sense of scaling. However, this bath coupling guarantees the thermalization to the QCP, which seems crucial to observe a prethermal regime influenced by it. With small damping in the Keldysh functions, but ballistically dominated dynamics, we found that the value of θ consists of two parts, an universal one and a quench-amplitude dependent one. The quench-amplitude dependent part is always large compared to the universal one. This non-universal part originates from the deep-quench expansion of the Keldysh function. Therefore, no prethermal universality exists and this deep-quench expansion fails. However, due to the bath, it is possible to recover the algebraic decay to equilibrium with equilibrium exponents β, ν and z. In contrast to the diffusive dominated dynamics of the open system, the aging amplitude is now non-universal.

The results for the isolated system have been compared to the result from the **quantum-classical mapping** or Euclidean mapping. Here the time is treated like a supplementary dimension, and the quench protocol corresponds to a classical surface problem. The corresponding universality class predicts an exponent

$$\theta = \frac{N+2}{N+8}\frac{\epsilon}{2} \tag{6.10}$$

with $\epsilon = 3 - d$. This result was found in none of the considered scenarios of an isolated system. Therefore, this seems to be an example where the quantum-classical mapping does not work. However, to understand why this mapping fails, why the deep-quench expansion of G^K is not possible, and how the energy induced by the quench dissipates into the bath, further studies are required. It would also be interesting in this context, to include higher order terms of the $1/N$ expansion and check if they could be treated like an external bath, to gain a better understanding of the process of thermalization.

Acknowledgments

“Will das Glück nach seinem Sinn,
Dir was Gutes schenken,
Sage Dank und nimm es hin,
ohne viel Bedenken.”
Wilhem Busch

This work was written not only due to chance, but many people had spend hard work on it, so my thanks goes to them.

First of all, I want to thank my *Doktorvater* Jörg Schmalian for the opportunity writing this thesis. He taught me the physics of phase transitions, especially I liked the motto “Erskalieren wir uns das Problem”. His office door was always open for my many questions and he spend a lot of his time with this work, which I appreciate. I am full of gratitude for being supervised by you, thank you!

I thank Boris Narozhny, who agreed to be my second instructor. He gave me very useful suggestions for my PhD defense.

I thank my collaborator Peter Orth, who guided me through my diploma thesis and PhD thesis. He has been of great help by teaching me how to give talks and how to supervise students. I have been benefitting immensely from his input and his motivation.

I also want to thank Michael Schütt and Tim Ludwig for our many fruitful discussions. Without their input, I would have understood less about non-equilibrium physics, bath coupling and thermalization.

My proofreaders Janina Klier, Julia Link, Tim Ludwig, Matthias Bard, Peter Orth, Florian Gagel and Matthias von Laer made many helpful suggestions and improvements, I owe them a lot. I also thank my office mates Mareike Hoyer, Mattias Bard, Nikolaos Kanaris, Parik Holbil and Egor Kiselev for the many discussions and the nice atmosphere. Fabienne Flatter, Carmen Dörflinger and Sonja König guided me through the administrative aspects and my personal chaos. I thank all members of the TKM for the nice and friendly working atmosphere. And the skat-and-coffee-team for the good tradition of playing after lunch.

A big thank goes to my friends and my family! Especially, I thank Matthias, for his patience in reviewing the thesis and his encouragement. Without him, I would not have written this thesis in this way. I thank Marion and my mother, who supported me by taking care of the children and my father for the many advices. And my two sons, who make me laugh even in hard times. The eldest, who gave me great timeouts with Lego-Duplo and who agreed that I was “writing a book”. The youngest, for his many smiles and the will and the force to finish this project.

List of publications

In the following a list of the publications of the author of this thesis is presented:

1. P. Gagel, *Nichtgleichgewichtsdynamik in der Nähe eines Quantenkritischen Punktes*, diploma thesis (2013).
2. P. Gagel, P. P. Orth and J. Schmalian, *Universal post-quench prethermalization at a quantum critical point*, Phys. Rev. Lett. 113, 220401 (2014). Ref. [30] in the bibliography
3. P. Gagel, P. P. Orth and J. Schmalian, *Universal post-quench coarsening and quantum aging at a quantum critical point*, Phys. Rev. **B** 92, 115121 (2015). Ref. [31] in the bibliography
4. X. Yang, C. Vaswani, C. Sundahl, M. Mootz, P. Gagel, L. Luo, J.H. Kang, P. P. Orth, I. E. Perakis, C. B. Eom and J. Wang, *Discovery of a Prethermalized Gapless Quantum State with Persisting Coherent Transport by Terahertz Quench of Superconductivity*, accepted in Nature Materials, Ref. [25] in the bibliography

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List of Figures

Acronyms

BKT Berezinskii-Kosterlitz-Thouless. 2

EOM equations of motion. 29, 30, 38, 52, 65, 72, 73, 84, 109

FDT fluctuation-dissipation theorem. 19, 21, 26–29, 40, 53, 60–62, 64, 65, 76, 86, 106, 115, 127

LT Laplace transformation. 27–29, 31, 33, 39, 42, 58, 76, 111, 112, 117, 119, 124, 127, 130

QCP quantum critical point. viii, ix, xi, 1, 4–6, 13, 14, 19, 21, 32, 35, 37–39, 43, 45, 51, 53, 59, 60, 67, 68, 72–76, 78, 82–87, 101

QFT quantum field theory. 21, 35

RG renormalization group. xi, 2, 6–12, 16, 17, 26, 36, 40, 41, 43–45, 47, 49–51, 55, 61, 64, 68, 70, 77, 82, 101, 102, 117, 120, 121

Notations and conventions

Here we present a list of notations and conventions used throughout this thesis.

1. We use units where $\hbar = k_B = c = 1$, where $\hbar$ is the reduced Planck's quantum, k_B is Boltzmann's constant and c is the velocity of light.

2. The symbol $\int_k$ denotes the d-dimensional integration over momenta:

$$\int_k \cdots = \int \frac{\mathrm{d}^d k}{2\pi} \cdots \ .$$

3. The volume of the d-dimensional unity sphere is given by

$$K_d = \frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{d/2}(2\pi)^d} \, . \tag{6.1}$$

4. In order to distinguish between bare and full Green's- and scaling-functions, functions including interaction effects are denoted with a capital G, whereas bare functions with a small g.

5. The Pauli matrices are defined as

$$\sigma_0 = \tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad \sigma_x = \tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$
$$\sigma_y = \tau_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_z = \tau_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

where the symbol σ refers to the microscopic spin of the electron and τ refers to pseudospin.

6. The commutator $[\cdot,\cdot]$ and anticommutator $\{\cdot,\cdot\}$ are defined as

$$[\hat{A},\hat{B}] := \hat{A}\hat{B} - \hat{B}\hat{A} \ , \qquad \{\hat{A},\hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A} \, ,$$

for two operators $\hat{A}$ and $\hat{B}$.

7. The Bose-Einstein distribution function is denoted by

$$n_B(\epsilon) = \frac{1}{\exp \frac{\epsilon-\mu}{k_B T} - 1} \, .$$

8. In this thesis different types of Green's functions occur. Here, we distinguish between three kinds, the retarded function G^R and the Keldysh function G^K. Regarding the Laplace transformation of the equilibrium Green's function G^K_{eq} and the fluctuation-dissipation theorem, a supplementary function g occurs, compared to the usual FDT given in Fourier space. This function is called g and given by a principal value integral over G^R.
   Further, to distinguish between bare and full Green's functions, full Green's functions are written with a capital $G^{R/K}$, and the bare ones with a small $g^{R/K}$. The function g is considered only for the free system.
   All three Green's functions depend on two time arguments t and t', a mode k and a mass r. To make the initial Green's function more clear, they have a subscript i, which means they should be evaluated with the initial mass r_{i}. Further subscripts are $G^{R/K}_{\mathrm{eq}}$ for the equilibrium Green's functions and $G^{R/K}_{\mathrm{stat}}$, for a stationary, non-thermal state.

Next we present the basic notation used in this thesis:

T	temperature
t	time
d	dimension
d_{uc}	upper critical dimension
k	momentum
$\eta(\omega)$	bath spectral function
α	bath-exponent characterizing the spectral function
z	dynamical exponent
ν	correlation length exponent
β	order-parameter exponent
θ	non-equilibrium exponent
$\boldsymbol{\varphi}$	bosonic order-parameter field
$r(t)$	time depended effective mass
Λ	momentum cutoff
ω_c	high frequency cutoff of the bath
$\omega_\gamma = \gamma^{1/(2-\alpha)}$	frequency separating the ballistic from the diffusive regime
$t_\gamma = \gamma^{1/(\alpha-2)}$	time separating the ballistic from the diffusive regime
u	interaction parameter
N	number of components of the bosonic field
$\simeq$	approximately
$\propto$	proportional
$\sim$	asymptotic
Re and Im	real and imaginary part
$\mathcal{P}$	principal value
$\delta(x)$	delta function
δ_{ij}	Kronecker delta
$\Gamma(x)$	Euler Gamma function
$\Theta(x)$	Heaviside step function
$\mathrm{sign}(x)$	signum function
$E_\alpha(x)$	Mittag-Leffler function
$J_\alpha(x)$	Bessel function of the first kind

A Appendix A

Bare Keldysh and memory function

In this appendix some more details are given concerning the bare post-quench Keldysh and memory function.

A.1 Derivation of the bare equations of motion for φ

We are considering the model introduced in section 1.1, with $H = H_s + H_b + H_{sb}$. Here, the interaction parameter is set to zero, $u = 0$, as the EOM for the bare order-parameter field are derived. Further, the component index is omitted in this section for better readability. To solve the Heisenberg equation of motion, we need the commutators between the different systems and bath-operators. The only non-vanishing commutators are:

$$[\varphi, \pi] = \mathrm{i} \qquad \text{and } [X_i, P_j] = \mathrm{i}\,\delta_{ij}. \tag{A.1}$$

Next, we use the Heisenberg equation

$$\mathrm{i}\frac{\partial A}{\partial t} = [A, H], \tag{A.2}$$

to determine the time evolution of the different operators:

$$\frac{\partial^2 \varphi(k,t)}{\partial t^2} = -\,\omega^2 \varphi(k,t) - \sum_j c_j X_j\,, \tag{A.3}$$

$$\frac{\partial^2 \pi(k,t)}{\partial t^2} = -\,\omega^2 \pi(k,t) - \sum_j c_j P_j\,, \tag{A.4}$$

$$\frac{\partial^2 X_i}{\partial t^2} = -\,\Omega_i^2 X_i - c_i \varphi(k,t)\,, \tag{A.5}$$

$$\frac{\partial^2 P_i}{\partial t^2} = -\,\Omega_i^2 P_i - c_i \pi(k,t)\,. \tag{A.6}$$

Here, the frequency ω is given by $\omega^2 = r + k^2$. The mass term r is in general a function of time: $r = r(t)$, due to the quench, but also if interaction effects are included via a time-dependent self energy. We want to solve those coupled differential equations under the condition that at time $t \to -\infty$, we couple the system and the bath. The differential equation for the bath coordinates X_i can be solved by introducing a Green's function G_b:

$$(\partial_t^2 + \Omega_i^2)G_b(t) = \delta(t) \Rightarrow G(t) = \frac{\theta(t)}{\Omega_i}\sin(\Omega_i t). \tag{A.7}$$

If follows

$$\begin{aligned} X_i &= -\int_{-\infty}^{t} \mathrm{d}t'\, G(t-t')c_i\phi(t') \\ &= -\int_{-\infty}^{t} \mathrm{d}t'\, \frac{c_i}{\Omega_i}\sin\Big(\Omega_i(t-t')\Big)\,\phi(t'). \end{aligned} \tag{A.8}$$

The lower integration limit is $t = -\infty$, where we couple the system to the bath. At times $t \lesssim 0$ the system and the bath are in equilibrium, with the pre-quench mass r_i. At $t = 0$ the quench is performed. We are only interested in the dynamics after the quench, therefore we want to shift all lower integration limits up to zero. To do so, we have to introduce the initial values of the bath operators $X_i(t=0) = X_i^0$ and $P_i(t=0) = P_i^0$. It holds:

$$\begin{aligned} X_i &= -\int_{-\infty}^{0} \mathrm{d}t'\, \frac{c_i}{\Omega_i}\sin\Big(\Omega_i(t-t')\Big)\,\phi(t') - \int_{0}^{t} \mathrm{d}t'\, \frac{c_i}{\Omega_i}\sin\Big(\Omega_i(t-t')\Big)\,\phi(t') \\ &= -\int_{-\infty}^{0} \mathrm{d}t'\, \frac{c_i}{\Omega_i}(\sin(\Omega_i t)\cos(-\Omega_i t') + \cos(\Omega_i t)\sin(-\Omega_i t'))\phi(t') - \int_{0}^{t} \mathrm{d}t'\, \frac{c_i}{\Omega_i}\sin\Big(\Omega_i(t-t')\Big)\,\phi(t') \\ &= -\int_{0}^{t} \mathrm{d}t'\, \frac{c_i}{\Omega_i}\sin\Big(\Omega_i(t-t')\Big)\,\phi(t') + X_i^0\cos(\Omega_i t) + \frac{P_i^0}{\Omega_i}\sin(\Omega_i t). \end{aligned} \tag{A.9}$$

Next, this result can be used to derive the equation of motion for φ:

$$\begin{aligned} (\partial_t^2 + r(t) + k^2)\varphi(k,t) &= \sum_j \int \mathrm{d}t' \frac{c_j^2}{\Omega_j}\theta(t-t')\sin\Big(\Omega_j(t-t')\Big)\,\varphi(k,t') - \sum_j c_j X_j^0\cos(\Omega_j t) \\ &\quad - \sum_j c_j \frac{P_j^0}{\Omega_j}\sin(\Omega_j t) \\ &= -\int_{-\infty}^{\infty} \mathrm{d}t'\eta(t-t')\varphi(k,t') + \Xi(t). \end{aligned} \tag{A.10}$$

Here, we introduced the function

$$\Xi(t) = -\sum_j c_j(X_j^0\cos(\Omega_j t) + P_j^0/\Omega_j\, \sin(\Omega_j t)), \tag{A.11}$$

which contains the memory of the bath about the pre-quench state X_j and P_j. The spectral function $\eta(t)$ of the bath is given by:

$$\eta(t) = -\theta(t)\int \frac{\mathrm{d}\omega'}{\pi}\eta(\omega')\sin(\omega' t). \tag{A.12}$$

$\eta(\omega)$ is given by Eq. (1.6).

A.2 Derivation of the memory function by explicitly analyzing the force-force anticommutator

In this section, the expectation values in $M(\omega,\omega') = \delta_{ij}\langle\Big[F_i(k,\omega), F_j(k,\omega')\Big]_+\rangle$ are evaluated explicitly. $\boldsymbol{F}$ is given by

$$\boldsymbol{F}\left(k,\omega\right) = \boldsymbol{\pi}_{\rm i}\left(k\right) - {\rm i}\,\omega\boldsymbol{\varphi}_{\rm i}\left(k\right) + \boldsymbol{\Xi}\left(k,\omega\right), \tag{A.13}$$

$$\boldsymbol{\Xi}(k,t) = -\int_{-\infty}^{0} {\rm d}s\,\eta(t-s)\boldsymbol{\varphi}_{\rm i}(k,s) \tag{A.14}$$

In this section, the external field $\boldsymbol{h}$ is set equal to zero, to simplify the calculations. Further, the component index of φ and π is skipped, as only one component enters in the memory function due to the Kronecker δ_{ij}. Inserting the explicit form of F into M yields

$$\begin{aligned} M(\omega,\omega') = {} & \langle[\pi_{\rm i},\pi_{\rm i}]_+\rangle - \omega\omega'\langle[\varphi_{\rm i},\varphi_{\rm i}]\rangle + {\rm i}\omega\langle[\varphi_{\rm i},\pi_{\rm i}]_+\rangle + {\rm i}\omega'\langle[\pi_{\rm i},\varphi_{\rm i}]_+\rangle \\ & + {\rm i}\omega'\langle[\Xi(\omega),\varphi_{\rm i}]_+\rangle + \langle[\Xi(\omega),\pi_{\rm i}]_+\rangle + {\rm i}\omega\langle[\varphi_{\rm i},\Xi(\omega')]_+\rangle + \langle[\pi_{\rm i},\Xi(\omega')]\rangle \\ & + \langle[\Xi(\omega,\Xi(\omega')]_+\rangle. \end{aligned} \tag{A.15}$$

As those expectation values are taken in equilibrium, of the pre-quench system, they can be evaluated with the Matsubara technique, where it holds for two bosonic operators A and B,

$$\begin{aligned} \langle A(0)B(0)\rangle = & {\rm i}\int_{-\infty}^{\infty} {\rm d}\omega n_{\rm B}(\omega)\left(G^R(\omega+{\rm i}\delta) - G^A(\omega-{\rm i}\delta)\right) \\ = & T\sum_{\omega_n} G({\rm i}\omega_n), \end{aligned} \tag{A.16}$$

with the corresponding retarded Green's function ${\rm i}G^R(t,t') = \theta(t-t')\langle[A(t),B(t')]_-\rangle = {\rm i}G^A(t',t)$ and the Matsubara Green's function $G({\rm i}\omega_n)$. $n_{\rm B}$ is the Bose-Einstein distribution function. This yields for the zero-time expectation values in $\langle\Big[F_i(k,\omega), F_i(k,\omega')\Big]_+\rangle$:

$$\langle\varphi_{\rm i}\varphi_{\rm i}\rangle = T\sum_n \frac{1}{\omega_n^2+\omega_{\rm i}^2+\eta^M({\rm i}\omega_n)}, \tag{A.17}$$

$$\langle\pi_{\rm i}\pi_{\rm i}\rangle = T\sum_n \frac{\omega_{\rm i}^2+\eta^M({\rm i}\omega_n)}{\omega_n^2+\omega_{\rm i}^2+\eta^M({\rm i}\omega_n)}, \tag{A.18}$$

$$\langle\varphi_{\rm i}\pi_i\rangle = T\sum_n \frac{-\omega_n}{\omega_n^2+\omega_{\rm i}^2+\eta^M({\rm i}\omega_n)}. \tag{A.19}$$

The expectation value of $\langle\pi\pi\rangle$ is divergent in the limit of an infinite large bath cutoff $\omega_{\rm c}$. The term in the last line vanishes, as $G({\rm i}\omega_n)$ is a symmetric function in the sum. The LT of Ξ must be performed with some care. It holds

$$\langle \Xi(\omega)\varphi\rangle = T\sum_n G(\mathrm{i}\omega_n)\frac{\mathrm{i}\omega}{\omega^2-\omega_n^2}\left(\eta(\omega)-\eta^M(\mathrm{i}\omega)\right), \tag{A.20}$$

$$\langle \Xi(\omega)\pi\rangle = -T\sum_n G(\mathrm{i}\omega_n)\frac{\omega_n^2}{\omega^2-\omega_n^2}\left(\eta(\omega)-\eta^M(\mathrm{i}\omega)\right), \tag{A.21}$$

$$\langle \Xi(\omega)\Xi(\omega')\rangle = \frac{1}{2}\nu(\omega,\omega') + T\sum_n G(\mathrm{i}\omega_n)\frac{\left(\omega\omega'-\omega_n^2\right)}{(\omega^2-\omega_n^2)(\omega'^2-\omega_n^2)}\left[\eta(\omega)-\eta^M(\mathrm{i}\omega_n)\right]\left[\eta(\omega')-\eta^M(\mathrm{i}\omega_n)\right]. \tag{A.22}$$

The function $\nu(\omega,\omega')$ is the double LT of the Keldysh component of the bath-spectral function. The expectation values of $\langle\Xi(\omega)\pi\rangle$ are also divergent in the limit of an infinite large bath cutoff ω_c. Taking the anticommutator, all terms can be inserted into M. This yields

$$\begin{aligned} M(\omega,\omega') =& \nu(\omega,\omega') + 2T\sum_n G(\mathrm{i}\omega_n)\left(\omega_\mathrm{i}^2+\eta^M(\mathrm{i}\omega_n)-\omega\omega'-\mathrm{i}(\omega+\omega')\omega_n\right) \\ &+ 2T\sum_n G(\mathrm{i}\omega_n)\frac{-\omega'\omega-\omega_n^2}{-\omega_n^2+\omega^2}\left(\eta(\omega)-\eta^M(\mathrm{i}\omega)\right) \\ &+ 2T\sum_n G(\mathrm{i}\omega_n)\frac{-\omega'\omega-\omega_n^2}{-\omega_n^2+\omega'^2}\left(\eta(\omega')-\eta^M(\mathrm{i}\omega')\right) \\ &+ 2T\sum_n G(\mathrm{i}\omega_n)\frac{\left(\omega\omega'-\omega_n^2\right)}{(\omega^2-\omega_n^2)(\omega'^2-\omega_n^2)}\left[\eta(\omega)-\eta^M(\mathrm{i}\omega_n)\right]\left[\eta(\omega')-\eta^M(\mathrm{i}\omega_n)\right]. \end{aligned} \tag{A.23}$$

To bring the memory function in the desired form, it is necessary to transform all terms to the common denominator $((\omega-\mathrm{i}\omega_n)(\omega'+\mathrm{i}\omega_n))^{-1}$. This term can be simplified:

$$\frac{1}{(\omega-\mathrm{i}\omega_n)(\omega'+\mathrm{i}\omega_n)} = \frac{1}{\omega+\omega'}\left(\frac{1}{\omega-\mathrm{i}\omega_n}+\frac{1}{\omega'-\mathrm{i}\omega_n}\right). \tag{A.24}$$

Further, one uses

$$G^{R^{-1}}(\omega) = -\omega^2+\omega_\mathrm{i}^2+\eta(\omega). \tag{A.25}$$

This yields

$$M(\omega,\omega') = 2T\sum_n \frac{G(\mathrm{i}\omega_n)}{\omega+\omega'}\left(\frac{1}{\omega-\mathrm{i}\omega_n}+\frac{1}{\omega'-\mathrm{i}\omega_n}\right)G^{R^{-1}}(\omega)G^{R^{-1}}(\omega') \tag{A.26}$$

With the LT of the Keldysh function

$$G^K(\omega) = 2T\sum_n \frac{G(\mathrm{i}\omega_n}{\omega-\mathrm{i}\omega_n}, \tag{A.27}$$

one finally finds the result in Eq. (3.62).

A.3 The bare post-quench Keldysh function in the limit of Sotiriadis and Cardy

In this section the bare post-quench Keldysh function obtained by the boundary conditions in Eq. (6.10) is derived via the memory ansatz. If the system is initially a non-interacting, purely ballistic system, g is given by

$$g_{\text{ballistic}}(\omega, \omega_{\text{i}}) = -\frac{1}{\omega_{\text{i}}} \frac{1}{|\omega| + \omega_{\text{i}}}. \tag{A.28}$$

This follows directly from evaluating the function g and using $\operatorname{Im} g^R(\omega_{\text{i}}, \omega) = \frac{\pi}{2\omega_{\text{i}}} \delta(\omega - \omega_{\text{i}})$. Inserting this result into n gives:

$$\begin{aligned} n\left(\omega, \omega', \omega_{\text{i}}\right) =& g^R(\omega_{\text{i}}, \omega')^{-1} - g_{\text{i}}(\omega_{\text{i}}, \omega) g^R(\omega_{\text{i}}, \omega)^{-1} g^R(\omega_{\text{i}}, \omega')^{-1} \\ =& \frac{|\omega|}{\omega_{\text{i}}} \left(\omega'^2 - \omega_{\text{i}}^2 + \text{i}0^+\right). \end{aligned} \tag{A.29}$$

Inserting n into the memory function yields

$$\begin{aligned} M(\omega, \omega', \omega_{\text{i}}) =& \frac{1}{\omega_{\text{i}}} \frac{\omega\left(\omega'^2 - \omega_{\text{i}}^2 + \text{i}0^+\right) + \omega'\left(\omega^2 - \omega_{\text{i}}^2 + \text{i}0^+\right)}{\omega + \omega' + 2\text{i}0^+} \\ =& \frac{-\omega_{\text{i}}^2 + \omega\omega'}{\omega_{\text{i}}}. \end{aligned} \tag{A.30}$$

Note that M is real and has no poles. Thus the back-transformation of the Keldysh function can straightforwardly be performed by

$$\begin{aligned} G^K(t, t') =& \text{i}^2 \int \frac{\text{d}\omega \text{d}\omega'}{\pi^2} \text{Im} g^R(\omega_f, \omega) \text{Im} g^R(\omega_f, \omega') M(\omega, \omega', \omega_{\text{i}}) e^{-\text{i}\omega t} e^{-\text{i}\omega' t'} \\ =& -\frac{1}{4\omega_f^2} \Big(M(\omega_f, \omega_f, \omega_{\text{i}}) e^{-\text{i}\,\omega_f (t+t')} + M(-\omega_f, -\omega_f, \omega_{\text{i}}) e^{\text{i}\,\omega_f (t+t')} \\ & -M(\omega_f, -\omega_f, \omega_{\text{i}}) e^{-\text{i}\,\omega_f (t-t')} + M(-\omega_f, \omega_f, \omega_{\text{i}}) e^{\text{i}\,\omega_f (t-t')} \Big). \end{aligned} \tag{A.31}$$

Evaluating the memory function and using the exponential expression for the cosine one finds the result:

$$G^K(t, t') = \frac{\omega_{\text{i}}^2 - \omega_{\text{f}}^2}{\omega_{\text{i}} \omega_{\text{f}}^2} \cos\left(\omega_f [t + t']\right) - \frac{\omega_{\text{i}}^2 + \omega_{\text{f}}^2}{2\omega_{\text{i}} \omega_{\text{f}}^2} \cos\left(\omega_{\text{f}} [t - t']\right). \tag{A.32}$$

This result was derived by Sotiriadis and Cardy in Ref. [63] by using the boundary conditions at $t = 0$ of a free harmonic oscillator with eigenfrequency ω_{i}.

Appendix B

B The deep quench expansion

The post-quench, bare Keldysh function was obtained via the memory ansatz presented in section 3.5.2. Typically, at finite times after a quench the Keldysh function will depend on the quench amplitude ω_{i} for the isolated as well as for the open system. However, to achieve a scaling form, it is necessary that the bare Keldysh function $g^K(k,t,t')$ does not depend on any further energy scale than k^z. In chapter 3.5.3, the relevant limit of an infinite large quench amplitude is considered. A crucial role plays the function $g_{\mathrm{i}}(k,\omega)$, which arises as supplementary term of the FDT in Laplace-space. It is shown, that the memory function M is independent of ω_{i}, if only the zeroth order term in an expansion in $1/\omega_{\mathrm{i}}$ in g_{i} is considered. However, second order terms in this expansion are relevant in section 6.2. Details of the expansion $\omega_{\mathrm{i}} \to \infty$ of g_{i} are considered in this section. The final result of this expansion is

$$g_{\mathrm{i}}(\omega_{\mathrm{i}} \to \infty, \omega) = -\frac{1}{\omega_{\mathrm{i}}^2}\left(1 + \frac{|\omega|}{\omega_{\mathrm{i}}^{z'}} C_\alpha^{(1)} + \frac{\omega^2}{\omega_{\mathrm{i}}^{2z'}} C_\alpha^{(2)} + \mathcal{O}\left(\left(\omega/\omega_{\mathrm{i}}^{z'}\right)^3\right)\right) . \tag{B.1}$$

The coefficients $C_\alpha^{(1)/(2)}$ and the exponent z' are derived in this section, as well as their impact on the effective mass, given in Eq. (4.87),

$$r(t) = \frac{c_K \epsilon t^{\epsilon/z}}{2} \int_0^\infty \mathrm{d}k\, k^{3-z-\epsilon}\left(i g^K(k,t,t) - \mathrm{i}\, G^K_{\mathrm{eq}}(k,0)\right) . \tag{B.2}$$

The memory function M can be obtained from the function g_{i} via Eq. (3.65) and Eq. (3.66):

$$M(k,\omega,\omega') = \frac{\mathrm{sign}\,(\omega)\, n\left(\omega,\omega',\omega_{\mathrm{i}}\right) + \mathrm{sign}\left(\omega'\right) n\left(\omega',\omega,\omega_{\mathrm{i}}\right)}{\omega + \omega' + \mathrm{i}0^+} , \tag{B.3}$$

$$n\left(\omega,\omega',\omega_{\mathrm{i}}\right) = g_{\mathrm{i}}^R(k,\omega')^{-1} - g_{\mathrm{i}}(k,\omega) g_{\mathrm{i}}^R(k,\omega)^{-1} g_{\mathrm{i}}^R(k,\omega')^{-1} . \tag{B.4}$$

With those equations, the Keldysh function can be obtained via $G^K(\omega,\omega') = M(\omega,\omega') g^R(\omega) g^R(\omega')$. Let us start with the function $g_{\mathrm{i}}(k,\omega)$, arising from the FDT version in Laplace space:

$$g_{\mathrm{i}}(k,\omega) = 2\int_0^\infty \frac{\mathrm{d}\epsilon}{\pi} \frac{1}{|\omega| + \epsilon} \mathrm{Im}\, G_{\mathrm{i}}^R(\epsilon) . \tag{B.5}$$

This function is analyzed in the deep quench limit $\omega_{\mathrm{i}} \to \infty$. The pre-quench Green's function G_{i}^R does not obey a scaling form, but still the integral can be split approximately into the two dynamic regimes.

For $\alpha < 2$ the function g is given by:

$$g_{\mathrm{i}}(k,\omega) \approx 2\int_0^{\omega_\gamma} \frac{\mathrm{d}\epsilon}{\pi}\frac{1}{|\omega|+\epsilon}\frac{-\gamma\epsilon^\alpha}{(\omega_{\mathrm{i}}^2+\gamma\coth(2\pi\alpha)\,\epsilon^\alpha)^2+\gamma^2\epsilon^{2\alpha}} + 2\int_{\omega_\gamma}^{\infty} \frac{\mathrm{d}\epsilon}{\pi}\frac{1}{|\omega|+\epsilon}\frac{-0^+}{(\epsilon^2-\omega_{\mathrm{i}}^2)^2+0^{+2}}\,, \tag{B.6}$$

and for $\alpha > 2$:

$$g_{\mathrm{i}}(k,\omega) \approx 2\int_0^{\omega_\gamma} \frac{\mathrm{d}\epsilon}{\pi}\frac{1}{|\omega|+\epsilon}\frac{-0^+}{(\epsilon^2-\omega_{\mathrm{i}}^2)^2+0^{+2}} + \int_{\omega_\gamma}^{\infty} \frac{\mathrm{d}\epsilon}{\pi}\frac{1}{|\omega|+\epsilon}\frac{-\gamma\epsilon^\alpha}{(\omega_{\mathrm{i}}^2+\gamma\coth(2\pi\alpha)\,\epsilon^\alpha)^2+\gamma^2\epsilon^{2\alpha}}\,. \tag{B.7}$$

By using the property of the Dirac delta function $\delta(x) = \pi\lim_{\epsilon\to 0}\epsilon/(\epsilon^2+x^2)$, the integrals can be simplified further, yielding for $\alpha < 2$:

$$g_{\mathrm{i}}(k,\omega) \approx 2\int_0^{\omega_\gamma} \frac{\mathrm{d}\epsilon}{\pi}\frac{1}{|\omega|+\epsilon}\frac{-\gamma\epsilon^\alpha}{(\omega_{\mathrm{i}}^2+\gamma\coth(2\pi\alpha)\,\epsilon^\alpha)^2+\gamma^2\epsilon^{2\alpha}} - 2\theta(\omega-\omega_\gamma)\frac{1}{|\omega|+\omega_{\mathrm{i}}}\,, \tag{B.8}$$

and for $\alpha > 2$:

$$g_{\mathrm{i}}(k,\omega) \approx -2\theta(\omega_\gamma-\omega)\frac{1}{|\omega|+\omega_{\mathrm{i}}} + \int_{\omega_\gamma}^{\infty} \frac{\mathrm{d}\epsilon}{\pi}\frac{1}{|\omega|+\epsilon}\frac{-\gamma\epsilon^\alpha}{(\omega_{\mathrm{i}}^2+\gamma\coth(2\pi\alpha)\,\epsilon^\alpha)^2+\gamma^2\epsilon^{2\alpha}}\,. \tag{B.9}$$

The two cases $\omega_{\mathrm{i}} \gg \omega_\gamma$ and $\omega_{\mathrm{i}} \ll \omega_\gamma$ will be discussed in the following for $\alpha \lessgtr 2$.

Consider first $\omega_{\mathrm{i}} \to \infty$, where the quench amplitude is the largest scale of the system. In this case, the integral part $\int_{\omega_\gamma}^{\infty}$ is the dominant contribution in g. In this integral, the limit $\omega_\gamma \to 0$ can be approximately taken, keeping in mind the correct order of the limits. This can be seen explicitly, here for $\alpha < 2$. If $\omega_{\mathrm{i}} \gg \omega_\gamma$, the first integral on the right hand side in Eq. (B.8) reads

$$-2\int_0^{\omega_\gamma} \frac{\mathrm{d}\epsilon}{\pi}\frac{1}{|\omega|+\epsilon}\frac{\gamma\epsilon^\alpha}{(\omega_{\mathrm{i}}^2+\gamma\coth(2\pi\alpha)\,\epsilon^\alpha)^2+\gamma^2\epsilon^{2\alpha}} \approx -\frac{2\gamma}{\omega_{\mathrm{i}}^4}\int_0^{\omega_\gamma}\frac{\mathrm{d}\epsilon}{\pi}\frac{\epsilon^{2/z}}{|\omega|+\epsilon}\,. \tag{B.10}$$

A further approximation can be made by neglecting the ω-dependence in the integral, as only frequencies $\omega \ll \omega_\gamma$ are considered, and the integral is convergent at the lower boundary. Thus this term is, multiplied with ω_{i}^4, some non-universal constant which decays exponentially fast in the post-quench Keldysh function. The scaling-form important terms going with $\gamma\omega^{2/z}$ are not generated by this integral. For $\alpha > 2$, i. e. a dynamic exponent $z = 1$, only the second integral on the right hand side in Eq. (B.9) can contribute, thus here no further terms occur.
With those approximations, in the deep quench limit, a scaling function $g = \phi(|\omega|/\omega_{\mathrm{i}}^{z'})/\omega_{\mathrm{i}}^2$ can be introduced. Note, that this scaling form has a counter intuitive dynamical exponent, $z' = 1$ for $\alpha < 2$, as here the dominant part in Eq. (B.8) originates from the ballistic dominated contribution for $\omega_{\mathrm{i}} \to \infty$. For $\alpha > 2$, it holds $z' = 2/\alpha$, as the dominant part in Eq. (B.9) is the diffusive one.

The highest order term in an expansion for $|\omega|\omega_{\mathrm{i}}^{z'} \ll 1$ yields

$$\lim_{\omega_{\mathrm{i}}\to\infty} g_{\mathrm{i}}(k,\omega) \approx \frac{-1}{\omega_{\mathrm{i}}^2} \tag{B.11}$$

for all z and α. The universal exponent θ is determined with this scaling part in appendix C and E. However, higher order terms have to be considered, as the function g will be multiplied with ω_{i}^4, see also sections B.1 and B.2.

The condition for ω_{i} being the largest scale of the system is a strong one, especially if weak quenches are considered. In the RG-sense it will always be fulfilled for $\alpha < 2$ after some time t_{i}, see section B.3. A scaling form of the effective mass can already be expected below this time-scale, due to the properties of g. Now, the scaling form of g for $\omega_{\mathrm{i}} \ll \omega_\gamma$ is analyzed. First consider the case $\alpha < 2$. For $\omega_{\mathrm{i}} \ll \omega_\gamma$, the function g reads:

$$g(\omega_{\mathrm{i}},\omega) \approx 2\int_0^{\omega_\gamma} \frac{\mathrm{d}\epsilon}{\pi} \frac{1}{|\omega|+\epsilon} \frac{-\gamma\epsilon^\alpha}{(\omega_{\mathrm{i}}^2+\gamma\coth{(2\pi\alpha)}\,\epsilon^\alpha)^2+\gamma^2\epsilon^{2\alpha}}. \tag{B.12}$$

The upper boundary of the integral can be sent to infinity, as ω_γ is now the largest scale of the system. This yields

$$g(\omega_{\mathrm{i}},\omega) \approx \frac{\phi(|\omega|/\omega_{\mathrm{i}}^z)}{\omega_{\mathrm{i}}^2}. \tag{B.13}$$

Thus, the scaling function of g has the same dynamic exponent as the system dynamics. This scaling function reads

$$\phi(x) = -\frac{2}{\pi}\int_0^\infty \mathrm{d}y\, \frac{1}{\gamma^{z/2}x+y} \frac{y^{2/z}}{\left(1+\cot\frac{\pi}{z}y^{2/z}\right)^2+y^{4/z}}. \tag{B.14}$$

Again, $\phi(0) = -1$, leading to the scaling form in the memory function. Higher order terms in the expansion for small $\gamma^{z/2}|\omega|/\omega_{\mathrm{i}}^z \ll 1$ have to be included, as they are multiplied with ω_{i}^4. The first order corrections have no impact at the upper critical dimension, for the same reason as in section B.1. Second order corrections are now important as well, as they go with

$$\begin{aligned}\gamma\omega_{\mathrm{i}}^{2-2z} &\gg \gamma^z\omega_\gamma^{2(1-z)}\\ &\gg 1,\end{aligned} \tag{B.15}$$

where $\omega_\gamma = \gamma^{z/(2(z-1))}$ has been used, as well as $z > 1$, thus the exponent of ω_{i} is negative. Note however, that the time dependence of those second order terms goes with t^{-2}, and thus does not affect the time dependence and the coefficient of the scaling form. For systems with $\alpha > 2$, only the ballistic part of the integral has to be considered, as the integral $\omega_\gamma^\infty d\epsilon\epsilon^{-1}\mathrm{Im}G_{\mathrm{i}}^R$ leads to a constant which decays exponentially in time in the post-quench Keldysh function. In this order of limits, the function g reads

$$g(\omega_{\mathrm{i}},\omega) \simeq \frac{1}{\omega_{\mathrm{i}}}\frac{1}{\omega_{\mathrm{i}}+|\omega|}. \tag{B.16}$$

Inserting this function into the Keldysh function, and performing the inverse LT, one finds the result derived by Cardy and Sotiriadis in Ref. [63] and in section A.3.

B.1 First order terms at the upper critical dimension

In this section, ω_{i} is the largest scale of the system. If this condition is not fulfilled, one has to take the appropriate other limit, by inverting $\alpha \lessgtr 2$.

Performing a Taylor expansion for large quench amplitudes ω_{i}, it is shown in section 3.5.3, that the scaling function in $g(\omega, \omega_{\mathrm{i}}) = \phi(|\omega|\omega_{\mathrm{i}}^{z'})/\omega_{\mathrm{i}}^2$ reads:

$$\phi(x) \simeq -1 + \frac{|x|}{\omega_{\mathrm{i}}^{z'}} C_\alpha^{(1)} \,, \tag{B.17}$$

with

$$C_\alpha^{(1)} = \begin{cases} 1 & \text{for the closed, non-interacting system,} \\ 1 & \text{for } \alpha < 2, \\ 2\gamma^{1/\alpha} \int \mathrm{d}y \, y^{\alpha-2} \left((1 + \coth(2\pi\alpha) x^\alpha)^2 + x^{2\alpha} \right)^{-1} & \text{for } \alpha > 2. \end{cases} \tag{B.18}$$

Note, that again, the ballistic limit $z' = 1$ is given for $\alpha < 2$, while the overdamped limit $z' = 2/\alpha$ corresponds to $\alpha > 2$. In this section, corrections in the Keldysh function due to this first order terms are calculated. It will be shown, that those terms have no impact on the effective mass $r(t)$ near the upper critical dimension. Inserting the first order correction, proportional to $C_\alpha^{(1)}$, into the function g yields

$$g^{(1)}(\omega_{\mathrm{i}}, \omega) = \frac{|\omega|}{\omega_{\mathrm{i}}^{2+2/\alpha}} C_\alpha^{(1)} . \tag{B.19}$$

In n the function g will be multiplied with $[g^R]^{-1}[g^R]^{-1}$. Thus, the leading correction to the deep-quench result reads

$$\begin{aligned} n^{(1)}(\omega, \omega', \omega_{\mathrm{i}}) =& - \omega_{\mathrm{i}}^4 g^{(1)}(\omega_{\mathrm{i}}, \omega) \\ =& |\omega| \omega_{\mathrm{i}}^{2-2/\alpha} C_\alpha^{(1)} . \end{aligned} \tag{B.20}$$

For $\alpha \leq 2$ this term is relevant in the limit $\omega_{\mathrm{i}} \to \infty$. The correction to the memory function from this part,

$$M^{(1)}(\omega, \omega') = \omega_{\mathrm{i}}^{2-2/\alpha} C_\alpha^{(1)}, \tag{B.21}$$

is independent of ω and ω'. The Keldysh function originating from this term can be evaluated directly:

$$G^K_{(1)}(k, t, t') = \omega_{\mathrm{i}}^{2-2/\alpha} C_\alpha^{(1)} g^R(k, t) g^R(k, t') \,. \tag{B.22}$$

To consider corrections of this term in the effective mass, the k-integral over $G^K_{(1)}(k, t, t')$ is needed. Here, the dimension is directly expressed by $d = d_{\mathrm{uc}} - \epsilon = 4 - z - \epsilon$. Further, as only the bare retarded function enters, one can use the scaling form $g^R(k, t) = k^{-2+z} f^R(k^z t)$ to rewrite the k-integral as integral over time. This yields

$$\begin{aligned}\int \mathrm{d}^d k G^K_{(1)}(k,t,t) =& K_d \omega_\mathrm{i}^{2-2/\alpha} C_1(\alpha) \int \mathrm{d}k\, k^{3-z-\epsilon} g^R(k,t) g^R(k,t) \\ =& \frac{K_d \omega_\mathrm{i}^{2-2/\alpha} C_\alpha^{(1)}}{zt} \int_0^\infty \mathrm{d}x\, F^R(x) F^R(x) \\ =& K_d \frac{k^{4-d}}{zt} \int \mathrm{d}t'\, G^K_{(1)}(k,t',t'). \end{aligned} \tag{B.23}$$

The integral over time of $G^K_{(1)}(k,t',t')$ can again be expressed by using the double LT properties of $G^K_{(1)}$:

$$\begin{aligned}\int \mathrm{d}t'\, G^K_{(1)}(k,t',t') =& \omega_\mathrm{i}^{2-2/\alpha} C_\alpha^{(1)} \int_0^\infty \mathrm{d}t' \int_{-\infty}^\infty \frac{\mathrm{d}\omega \mathrm{d}\omega'}{4\pi^2} \mathrm{Im} g^R(k,\omega) \mathrm{Im} g^R(k,\omega') e^{-\mathrm{i}(\omega+\omega'-\mathrm{i}0^+)t} \\ =& \omega_\mathrm{i}^{2-2/\alpha} C_\alpha^{(1)} \int_{-\infty}^\infty \frac{\mathrm{d}\omega \mathrm{d}\omega'}{4\pi^2} \mathrm{Im} g^R(k,\omega) \mathrm{Im} g^R(k,\omega') \frac{\mathrm{i}}{\omega+\omega'}\,. \end{aligned} \tag{B.24}$$

This integral vanishes since it is odd under the transformation $\omega \to -\omega$, $\omega' \to -\omega'$. In summary, the deep quench limit can be taken because the system is near the upper critical dimension and in this case terms with $G^K \propto \omega_\mathrm{i}^{2-z}$ vanish under the integral in a controlled ϵ-expansion.

B.2 Second order terms at the upper critical dimension

For the second order terms the correction in the scaling function ϕ in g reads

$$\phi_2(y) = -\frac{2|y|^2}{\pi} \int_0^\infty \mathrm{d}x \frac{x^{\alpha-3}}{(1+x^\alpha)^2 + x^{2\alpha}}, \tag{B.25}$$

if g is given in the diffusive regime, and

$$\varphi_2(y) = -\frac{|\omega|^2}{\omega_\mathrm{i}^2}, \tag{B.26}$$

if g is given within the ballistic regime. Thus in g this leads to a second correction

$$g^{(2)}(\omega,\omega_\mathrm{i}) = -\frac{|\omega|^2}{\omega_\mathrm{i}^{2+2z}} C_\alpha^{(2)}(\omega_\mathrm{i},\gamma) \tag{B.27}$$

with the constant

$$C_\alpha^{(2)}(\omega_\mathrm{i},\gamma) = \begin{pmatrix} 1 & \text{if } \omega_\mathrm{i} \text{ is in the ballistic regime,} \\ 2\gamma^{2/\alpha}/\pi \int_0^\infty \mathrm{d}x \frac{x^{\alpha-3}}{(1+x^\alpha)^2+x^{2\alpha}} & \text{if } \omega_\mathrm{i} \text{ is in the diffusive regime.} \end{pmatrix} \tag{B.28}$$

Multiplied with ω_i^4 in the function n this term results in a second order correction

$$n^{(2)}(\omega,\omega') = -|\omega|^2 \omega_\mathrm{i}^{2-2z} C_\alpha^{(2)}(\omega_\mathrm{i},\gamma). \tag{B.29}$$

Note, that this second order term is always important if g_i is determined in the overdamped limit, since it holds $\gamma\omega_\mathrm{i}^\alpha \gg \omega_\mathrm{i}^2$. The back-transformation of this term into the Keldysh-function yields

$$g^K_{(2)}(k,t,t') = \omega_\mathrm{i}^{2-2z} C^{(2)}_\alpha(\omega_\mathrm{i},\gamma) \int \frac{\mathrm{d}\omega \mathrm{d}\omega'}{\pi^2} \mathrm{Im} g^R(\omega) \mathrm{Im} g^R(\omega') \frac{\mathrm{sign}(\omega)|\omega|^2 + \mathrm{sign}(\omega')|\omega'|^2}{\omega+\omega'} e^{-\mathrm{i}\,\omega t} e^{-\mathrm{i}\,\omega' t'} . \tag{B.30}$$

By using the scaling form of g^R it can be expressed as

$$g^K_{(2)}(k,t,t') = \omega_\mathrm{i}^{2-2z} C^{(2)}_\alpha(\omega_\mathrm{i},\gamma) k^{-4+3z} \int \frac{\mathrm{d}y \mathrm{d}y'}{\pi^2} \mathrm{Im} g(y) \mathrm{Im} g(y') \frac{\mathrm{sign} y|y|^2 + \mathrm{sign} y'|y'|^2}{y+y'} e^{-\mathrm{i}\,q^z t} e^{-\mathrm{i}\,q^z t'} . \tag{B.31}$$

In the effective mass, the $\boldsymbol{k}$-integral over $g^K_{(2)}(k,t,t')$ is needed. Like for the first order term, the integral is evaluated at the upper critical dimension. This yields

$$\int \mathrm{d}^d k\, g^K_{(2)}(k,t,t') = \frac{\omega_\mathrm{i}^{2-2z} C^{(2)}_\alpha(\omega_\mathrm{i},\gamma)}{z t^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}y \mathrm{d}y'}{\pi^2} \mathrm{Im}\, g(y) \mathrm{Im}\, g(y') \frac{\mathrm{sign}(y)|y|^2 + \mathrm{sign}(y')|y'|^2}{(y+y'+\mathrm{i}0^+)(y+y'-\mathrm{i}0^+)^2} . \tag{B.32}$$

This integral is finite. It can be brought on the form

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{\mathrm{d}y \mathrm{d}y'}{\pi^2} \mathrm{Im}\, g(y) \mathrm{Im}\, g(y') \frac{\mathrm{sign}(y)|y|^2 + \mathrm{sign}(y')|y'|^2}{(y+y'+\mathrm{i}0^+)(y+y'-\mathrm{i}0^+)^2} \\ =& \int_0^\infty \int_0^\infty \frac{\mathrm{d}y \mathrm{d}y'}{\pi^2} \mathrm{Im}\, g(y) \mathrm{Im}\, g(y') \frac{(y^2+y'^2)}{(y+y')^3} - ⨍_0^\infty \frac{\mathrm{d}y \mathrm{d}y'}{\pi^2} \mathrm{Im}\, g(y) \mathrm{Im}\, g(y') \frac{(y-y')(y^2-y'^2)}{(y+y')^4} , \end{aligned} \tag{B.33}$$

where $⨍$ is the principal value.
Interestingly the time-dependence of this term goes like the scaling solution of the ballistic system with t^{-2} even for $z \neq 1$.

B.3 Time-scales of the deep quench expansion

The deep-quench limit can be taken after a various number of different times, depending on the exponent α and the function g, as well as the interplay of the different parameters γ and ω_i. In this section those time-scales are summed up.

From the RG-flow, the time-scale t_RG can be derived, and it follows $\omega_\mathrm{i} > \omega_\gamma$. In this time regime, the deep-quench condition, $\omega \ll \omega_\mathrm{i}$ for $\alpha < 2$, is per construction always fulfilled. Therefore the flow of $\omega_\mathrm{i}(l)$ is considered (here for $\alpha < 2$):

$$\omega_\gamma = \omega_\mathrm{i} e^{l^*} . \tag{B.34}$$

l^* can be expressed by a time, using $t_\mathrm{RG} = t_\mathrm{mic} e^{-z l^*}$. This yields

$$t_\mathrm{RG} = t_\mathrm{mic} \left(\frac{\omega_\mathrm{i}}{\omega_\gamma}\right)^{-z} . \tag{B.35}$$

The dependence of t_{RG} of the quench-amplitude corresponds to the mean-field value of the cross-over timescale t^*. Thus, beyond the prethermal regime, the deep-quench limit can always be taken.

For times within the prethermal regime, one has to consider the deep-quench conditions

$$\gamma^{z/2}|\omega|/\omega_{\mathrm{i}}^{z} \ll 1\,. \tag{B.36}$$

This corresponds to times $t \gg \omega_{\mathrm{i}}^{-z}\gamma^{z/2}$. Also note, that $\omega_{\mathrm{i}} = \sqrt{k^2 + r_{0,\mathrm{i}}}$, thus for modes near the cutoff or with $\gamma^{z/2}k^{-z}t > 1$, the deep-quench condition is always fulfilled, and the limit (with some corrections arising from the slow modes) can be taken already on microscopic times.

For $\alpha > 2$, one must include the flow of γ, and thus $\omega_\gamma(l) = \omega_\gamma(l = 0)e^l$. Both frequencies flow equally under the RG. The deep quench condition in the ballistic part reads:

$$t \gg \frac{1}{\omega_{\mathrm{i}}}\,. \tag{B.37}$$

Appendix C

C Derivation of the integral in C_0 for the open system

The amplitude of the effective mass and thus the value of the exponent θ, are determined by the coefficient C_0 in Eq. (4.82). In this chapter the key integral in C_0 is evaluated for the open quantum system, discussed in chapter 5. The integral in C_0 can be evaluated numerically for $z > 1$ and analytically for $z = 2$. Here, we show how to bring the integral in C_0 on a scaling form, see Eq. (C.20). This enables the application of numerical analysis, and the explicit evaluation of the integral for an Ohmic bath. The final result for θ is depicted in figure 5.2.

The effective mass is given by Eq. (4.87),

$$r(t) = \frac{c_K \epsilon t^{\epsilon/z}}{2} \int_0^\infty \mathrm{d}k\, k^{3-z-\epsilon} \left(i g^K(k,t,t) - \mathrm{i}\, G^K_{\mathrm{eq}}(k,0) \right) . \tag{C.1}$$

To determine the key integral

$$I(t) = \int_0^\infty \mathrm{d}k\, k^{3-z-\epsilon} \left(i g^K(k,t,t) - \mathrm{i}\, G^K_{\mathrm{eq}}(k,0) \right) , \tag{C.2}$$

one can use the scaling properties of $g^K(k,t,t)$ and the equilibrium Keldysh function $G^K_{\mathrm{eq}}(k,0)$, to write this as integration over time:

$$I(t) = t^{-2/z-\epsilon/z} k^{4-z} I(k) , \tag{C.3}$$

$$I(k) = \int_0^\infty \mathrm{d}t\, t^{(2-z)/z} \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}s'\, \delta M(s,s') g^R(k,t-s) g^R(k,t-s') . \tag{C.4}$$

By evaluating the integral $I(k)$, the coefficient $C_0 = k^{4-z} I(k)$ can be obtained.
The bare post-quench Keldysh function in $I(k)$ can be written as

$$\mathrm{i}\, g^K(k,t,t) = \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}s'\, M(s,s',\omega_\mathrm{i}) g^R(k,t-s) g^R(k,t-s') , \tag{C.5}$$

where the memory function introduced in section 3.5.2 was used. Using this memory function, to express the equilibrium Keldysh function, one finds

$$I(k) = \int_0^\infty \mathrm{d}t\, t^{(2-z)/z} \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}s'\, \delta M(s,s') g^R(k,t-s) g^R(k,t-s') , \tag{C.6}$$

with $\delta M(s,s') = M(s,s',\omega_\mathrm{i}) - M(s,s',k)$. To calculate the t-integral, the retarded Green's function is expressed as integral over frequencies, yielding

$$I(k) = -\int \frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{\pi^2}\mathrm{Im}\,g^R(k,\omega)\mathrm{Im}\,g^R(k,\omega')\int_0^\infty \mathrm{d}s\int_0^\infty \mathrm{d}s'\,\delta M(s,s')e^{\mathrm{i}\omega s+\mathrm{i}\omega' s'}\int_0^\infty \mathrm{d}t\,t^{(2-z)/z}e^{-\mathrm{i}(\omega+\omega')t}\,. \tag{C.7}$$

The integral over t can now be calculated analytically:

$$\int_0^\infty \mathrm{d}t\,t^{(2-z)/z}e^{-\mathrm{i}(\omega+\omega')t} = \frac{\Gamma\left(2/z\right)}{\left(\mathrm{i}(\omega+\omega')\right)^{2/z}}\,. \tag{C.8}$$

The integral over s, s' in Eq. (C.7) leads to the double Laplace transform of δM. Thus, it follows

$$I(k) = -\frac{\Gamma\left(2/z\right)}{\mathrm{i}^{2/z}}\int \frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{\pi^2}\mathrm{Im}\,g^R(k,\omega)\mathrm{Im}\,g^R(k,\omega')\delta M(\omega,\omega')\,. \tag{C.9}$$

The difference of the post-quench memory function and its equilibrium value can be evaluated with the explicit form given in Eq. (3.65). By using this explicit form and the LT of the equilibrium Keldysh function, δM can be written as follows:

$$\delta M(\omega,\omega') = \frac{\mathrm{sign}\,\omega\,\delta n(\omega,\omega') + \mathrm{sign}\,\omega'\,\delta n(\omega',\omega)}{\omega+\omega'}\,, \tag{C.10}$$

with

$$\delta n(\omega,\omega' = n(\omega,\omega',\omega_\mathrm{i}) - n(\omega,\omega',k)\,. \tag{C.11}$$

The function n was derived in Eq. (3.66), which yields

$$\begin{aligned}\delta n(\omega,\omega' =& [g^R(\omega_\mathrm{i},\omega')]^{-1} - g(\omega,\omega_\mathrm{i})[g^R(\omega_\mathrm{i},\omega')]^{-1}[g^R(\omega_\mathrm{i},\omega)]^{-1}\\ &- [g^R(k,\omega')]^{-1} - g(\omega,k)[g^R(k,\omega')]^{-1}[g^R(k,\omega)]^{-1}\,.\end{aligned} \tag{C.12}$$

Here, to achieve a scaling form in the Keldysh function, only the zeroth order term in the deep-quench limit $\omega_\mathrm{i}\to\infty$ of $g(\omega,\omega_\mathrm{i})$ is considered. As the integral is evaluated explicitly in the over-damped limit, the ω^2 in g^R can be neglected. With this approximations, δn reads

$$\delta n(\omega,\omega') = k^2 - \eta(\omega) - \eta(\omega') + g(\omega,k)\left(k^2-\eta(\omega)\right)\left(k^2-\eta(\omega')\right)\,. \tag{C.13}$$

At this point, it is useful to change over to dimensionless variables $y^{(')} = \gamma^{z/2}\omega^{(')}/k^z$. It holds

$$\delta n(\omega,\omega') = k^2\delta\mu\left(\gamma^{z/2}\omega/k^z,\gamma^{z/2}\omega'/k^z\right)\,. \tag{C.14}$$

Here, the function μ is introduced, which can be split into two parts

$$\mu(y,y') = \delta\mu_a(y,y') + \delta\mu_b(y,y')\,, \tag{C.15}$$

where

$$\delta\mu_b(y,y') = 1 - h(y) - h(y') \tag{C.16}$$

$$\delta\mu_a(y,y') = \phi(y)\left(1-h(y)\right)\left(1-h(y')\right)\,. \tag{C.17}$$

The function ϕ is the scaling function of $g(k,\omega)$, given in Eq. (B.14). The function

$$h(y) = |y|^{2/z}(-\cot(\pi/z) + \mathrm{i}\,\mathrm{sign}\, y)\,, \tag{C.18}$$

is the dimensionless version of $\eta(\omega)$. Note, that $\delta\mu_b$ is symmetric in y, y'. To write the integral in C_0, the scaling function of the retarded Green's function is needed, given as

$$f^R(y) = \frac{1}{h(y)-1}\,. \tag{C.19}$$

In total, C_0 reads

$$\begin{aligned} C_0 =& \frac{\Gamma\left(2/z\right)}{\mathrm{i}^{2/z}} \int_{-\infty}^{\infty} \frac{\mathrm{d}y\,\mathrm{d}y'}{\pi^2} \mathrm{Im}\, f^R(y) \mathrm{Im}\, f^R(y') \frac{1}{y+y'} \\ &\times \left[(\mathrm{sign}\, y + \mathrm{sign} y')\delta\mu_b(y,y') + \mathrm{sign}\, y \delta\mu_a(y,y') + \mathrm{sign} y' \delta\mu_a(y',y) \right]\,. \end{aligned} \tag{C.20}$$

This integral is real, as $h(-y) = h^*(y)$, where h^* is the complex conjugate. This integral can be evaluated numerically for $z > 1$. For $z = 2$ it is also possible to derive the result analytically. Therefore, parts of the integral originating from μ_a refer to I_a and parts from μ_b to I_b.
Explicitly for $z = 2$, the imaginary part of f^R reads

$$\mathrm{Im} f^R(y) = \frac{y}{1+y^2}\,, \tag{C.21}$$

and the scaling function ϕ of g:

$$\phi(y) = -\frac{1 + \frac{2}{\pi}|y| \ln|y|}{1+y^2}\,. \tag{C.22}$$

Firstly, the integral part I_b is evaluated:

$$\begin{aligned} I_b =& -\int_{-\infty}^{\infty} \frac{\mathrm{d}y\,\mathrm{d}y'}{\pi^2} \frac{y}{1+y^2} \frac{y'}{1+y'^2} \frac{1}{y+y'} \\ =& -\frac{2}{\pi}\,. \end{aligned} \tag{C.23}$$

And the part of I_a

$$\begin{aligned} I_a =& -4\int_0^{\infty} \frac{\mathrm{d}y\,\mathrm{d}y'}{\pi^2} \frac{y}{1+y^2} \frac{y'}{1+y'^2} \frac{\phi(y)}{y+y'} \\ & -4 ⨍_0^{\infty} \frac{\mathrm{d}y\,\mathrm{d}y'}{\pi^2} \frac{y}{1+y^2} \frac{y'}{1+y'^2} \frac{\phi(y)(y-y')}{(y-y')^2}\,, \end{aligned} \tag{C.24}$$

where in the last line, $⨍$ stands for the principal value integration. The first integral gives $2/(3\pi)$, while the second leads to $1/(3\pi)$, such that in total $I_a + I_b = -1/\pi$. This finally yields for the exponent:

$$\theta(z=2) = \frac{N+2}{N+8}\frac{\epsilon}{4}\,. \tag{C.25}$$

Appendix D

Classical limit of the integral in C_0

The amplitude of the effective mass and thus the value of the exponent θ, are determined by the coefficient C_0 in Eq. (4.82). In this appendix, the key integral in C_0 is evaluated for the open classical systems, discussed in section 5.6. Those results have been reported by using a different approach in Ref. [28], for an Ohmic bath and in Ref. [29] for colored noise.
The first step, is to derive the classical limit of the bare, post-quench Keldysh function, or

$$\delta G^K(k,t) = g^K(k,t,t) - G^K_{\text{eq,cl}}(k,0,0)\,. \tag{D.1}$$

Therefore, the classical limit in the memory function is taken. It is straightforward, as only the zeroth Matsubara mode has to be kept, see Eqs. (A.17-A.22), yielding

$$M(\omega,\omega') = \nu(\omega,\omega') - 2T + 2T\frac{\eta\left(\omega\right)\eta\left(\omega'\right)}{\omega_{\text{i}}^2\omega\omega'}\,, \tag{D.2}$$

where the ω^2 terms have been neglected compared to η. The function $\nu(\omega,\omega')$ is the double LT of the bath-Keldysh part, which can be obtained via the FDT. The equilibrium memory function at the critical point can be obtained by replacing ω_{i} with k. Thus, the difference of the post-quench and the equilibrium memory function reads in the deep quench limit:

$$\delta M(\omega,\omega') = -\frac{2T}{k^2}\frac{\eta(\omega)\eta(\omega')}{\omega\omega'}\,. \tag{D.3}$$

And the difference of the corresponding Keldysh functions reads:

$$\delta G^K(k,t) = \frac{2T}{k^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{\pi^2}\text{Im}\,g^R(\omega)\text{Im}\,g^R(\omega')\frac{\eta(\omega)\eta(\omega')}{\omega\omega'}e^{\mathrm{i}(\omega+\omega')t}\,. \tag{D.4}$$

By introducing $\delta f^K(k^z t/\gamma^{z/2}) = k^2/(2T)\delta G^K(k,t,t)$, the integrals can be expressed with dimensionless variables,

$$\delta f^K(x) = \int_{-\infty}^{\infty}\frac{\mathrm{d}y\,\mathrm{d}y'}{\pi^2}\frac{h(y)h(y')}{yy'}\text{Im}\,\frac{1}{1+h(y)}\text{Im}\,\frac{1}{1+h(y')}e^{\mathrm{i}\,yx}e^{\mathrm{i}\,y'x}\,, \tag{D.5}$$

with $h(y)$ given in Eq. (C.18). The two integrals over y, y' are the inverse LT, with imaginary frequencies, as introduced in section 3.4. Performing the LT with $\mathrm{i}\omega$, it was convenient to evaluate the integrals of the quantum problem, where the back-transformation with the retarded Green's functions was straightforward. Here, to avoid taking care for the real and imaginary part of h, it is easier to substitute $y \to \mathrm{i}y$. The bath spectral function η has already been calculated along the imaginary axis in Eq. (1.9), and thus one part of the integrand can be written as

$$\frac{y^{2/z}/\sin\left(\pi/z\right)}{y}\frac{1}{1+y^{2/z}/\sin\left(\pi/z\right)}=\frac{1}{y}\sum_j\left(-\sin\left(\pi/z\right)y^{-2/z}\right)^j\,, \tag{D.6}$$

where $1/(1+y^{-2/z}\sin\left(\pi/z\right))$ was formally expanded. The back-transformation of $y^{-\beta-1}$ is given by x^β, yielding

$$\begin{aligned}\delta f^K(x)&=\left[\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}y}{\pi}e^{xy}\sum_j\frac{(-1)^j\sin^j\left(\pi/z\right)}{y^{1+j2/z}}\right]^2\\&=\left[\sum_j\frac{\left(-\sin\left(\pi/z\right)x^{2/z}\right)^j}{\Gamma\left(j2/z+1\right)}\right]^2\\&=E^2_{2/z}\left(-\sin(\pi/z)x^{2/z}\right)\,,\end{aligned} \tag{D.7}$$

with the Mittag-Leffler function $E_\alpha(x)=\sum_j x^j/\Gamma(\alpha j+1)$. Inserting this result into C_0 in Eq. (5.50), yields

$$\begin{aligned}C_0&=\frac{1}{2z}\gamma\int\mathrm{d}x\,x^{2/z-1}E^2\left(-\sin(\pi/z)x^{2/z}\right)\\&=\frac{\gamma}{4\sin\left(\pi/z\right)}\int\mathrm{d}x\,E^2(-x)\,.\end{aligned} \tag{D.8}$$

The coefficient $C_z=-\sin(\pi/z)/(\gamma\Gamma(2/z))$ is the same as in the quantum limit. With $c_K=1$ in Eq. (5.48), one finds:

$$\theta=\frac{N+2}{N+8}\frac{d(2/z)}{4\Gamma\left(2/z\right)}\epsilon_{\mathrm{cl}}\,. \tag{D.9}$$

Appendix E

E Derivation of the integral in C_0 for the ballistic system

The amplitude of the effective mass and thus the value of the exponent θ, are determined by the coefficient C_0 in Eq. (4.82). In this appendix, the key integral in C_0 is evaluated for the nearly isolated system, discussed in section 6.2.2. To evaluated the integral in C_0, one can follow the steps presented in the first appendix of Ref. [30]. The same notation is used here as in Ref. [30]. One needs to calculate the integral $I(k)$, which determines the exponent θ. Therefore the difference between the initial and final memory function is needed:

$$\begin{aligned}\delta M(\omega,\omega') &= M_{\mathrm{i}}(\omega,\omega') - M(\omega,\omega') \\ &= \frac{\mathrm{sign}(\omega)\delta n(\omega,\omega') + \mathrm{sign}(\omega')\delta n(\omega',\omega)}{\omega+\omega'},\end{aligned} \tag{E.1}$$

with

$$\begin{aligned}\delta n(\omega,\omega') &= n_{\mathrm{i}}(\omega,\omega') - n_k(\omega,\omega') \\ &= -\omega^2 - \omega'^2 + k^2 - \mathrm{i}\eta(\omega) - \mathrm{i}\eta(\omega') + \frac{(-k^2+\omega^2+\mathrm{i}\eta(\omega))(-k^2+\omega'^2+\mathrm{i}\eta(\omega'))}{k(|\omega|+k)} \\ &= \mu_a(\omega,\omega') + \mu_b(\omega,\omega').\end{aligned} \tag{E.2}$$

μ_a and μ_b are defined as

$$\mu_a(\omega,\omega') = -\omega^2 - \omega'^2 + k^2 - \mathrm{i}\eta(\omega) - \mathrm{i}\eta(\omega'), \tag{E.3}$$

$$\mu_b(\omega,\omega') = \frac{(-k^2+\omega^2+\mathrm{i}\eta(\omega))(-k^2+\omega'^2+\mathrm{i}\eta(\omega'))}{k(|\omega|+k)}. \tag{E.4}$$

Inserting δn in C_0 yields

$$C_0 = \frac{k^3}{i^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{\pi^2}\frac{\mathrm{Im}G^R(\omega)\mathrm{Im}G^R(\omega')}{(\omega+\omega'-\mathrm{i}0^+)^2}\frac{\mathrm{sign}(\omega)\delta n(\omega,\omega')+\mathrm{sign}(\omega')\delta n(\omega,\omega')}{\omega+\omega'+\mathrm{i}0^+}. \tag{E.5}$$

First the integral over μ_a is evaluated in detail. μ_a is symmetric in ω, ω', so only terms with $\operatorname{sign}\omega = \operatorname{sign}\omega'$ contribute

$$\begin{aligned} I_a &= \frac{4k^3}{\mathrm{i}^2} \int_0^\infty \frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{\pi^2} \frac{\pi^2}{4k^2} \delta(\omega - k)\delta(\omega' - k) \frac{\mu_a(\omega, \omega')}{(\omega + \omega')^3} \\ &= -k \frac{\mu_a(k,k)}{8k^3} \\ &= +\frac{1}{8} \,. \end{aligned} \tag{E.6}$$

To perform the integral over μ_b it is useful to note that $\mu_b(\pm k, \omega) = \mu_b(\omega, \pm k) = 0 + \mathcal{O}(\gamma)$. Hence, the part of the integral with $\operatorname{sign}(\omega) = \operatorname{sign}(\omega')$ is zero in the ballistic regime and only the part with $\operatorname{sign}(\omega) = -\operatorname{sign}(\omega')$ can contribute via the singular term $1/(\omega + \omega')^3$. By analyzing the μ_b-part of δM one finds:

$$\begin{aligned} \delta M_b &= \frac{\operatorname{sign}(\omega)\mu_b(\omega,\omega') + \operatorname{sign}(\omega')\mu_b(\omega',\omega)}{\omega + \omega'} \\ &= \frac{(-k^2 + \omega^2 + \mathrm{i}\eta(\omega))(-k^2 + \omega'^2 + \mathrm{i}\eta(\omega'))}{k} \frac{(\frac{\operatorname{sign}(\omega)}{|\omega|+k} + \frac{\operatorname{sign}(\omega')}{|\omega'|+k})}{\omega + \omega'} \\ &= \frac{(-k^2 + \omega^2 + \mathrm{i}\eta(\omega))(-k^2 + \omega'^2 + \mathrm{i}\eta(\omega'))}{k} \frac{\operatorname{sign}(\omega)(\frac{1}{|\omega|+k} - \frac{1}{|\omega'|+k})}{\omega + \omega'} \\ &= \frac{(-k^2 + \omega^2 + \mathrm{i}\eta(\omega))(-k^2 + \omega'^2 + \mathrm{i}\eta(\omega'))}{k(|\omega|+k)(|\omega'|+k)} \frac{\operatorname{sign}(\omega)(|\omega'| - |\omega|)}{\omega + \omega'} \\ &= -\frac{(-k^2 + \omega^2 + \mathrm{i}\eta(\omega))(-k^2 + \omega'^2 + \mathrm{i}\eta(\omega'))}{k(|\omega|+k)(|\omega'|+k)} \,. \end{aligned} \tag{E.7}$$

For $\omega = k$, $\omega' = -k$ this yields with $\mathrm{i}\eta(\omega) = \mathrm{i}\omega 0^+$

$$\delta M_b(k, -k) = \delta M_b(-k, k) = \frac{(\mathrm{i}k0^+)(-\mathrm{i}k0^+)}{4k^3} \,. \tag{E.8}$$

This choice for the bath exponent corresponds to an Ohmic bath. Per construction, the infinitesimal imaginary part from the Laplace transformation has the same units as η/ω. For a different type of bath, the infinitesimal element of the LT has to be adapted correspondingly, leading to the same result. Inserting into I_b yields:

$$\begin{aligned} I_b &= \frac{k^3}{\mathrm{i}^2} \int_{-\infty}^\infty \frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{\pi^2} \frac{\pi^2}{4k^2} \Big(-\delta(k+\omega)\delta(k-\omega) - \delta(k+\omega')\delta(k-\omega)\Big) \frac{\delta M_b(\omega,\omega')}{(\omega + \omega' - \mathrm{i}0^2)^2} \\ &= +\frac{k}{4} \frac{2k^2 0^{+2}}{4k^3} \frac{1}{(-\mathrm{i}0^+)^2} \\ &= \frac{1}{8} \,. \end{aligned} \tag{E.9}$$

The integral over μ_b is also $+\frac{1}{8}$, so in total it holds:

$$C_0 = I_a + I_b = \frac{1}{4} \,. \tag{E.10}$$

This yields for the non-equilibrium exponent $\theta = \epsilon/2$.