Energy bounds for biharmonic wave maps in low dimensions
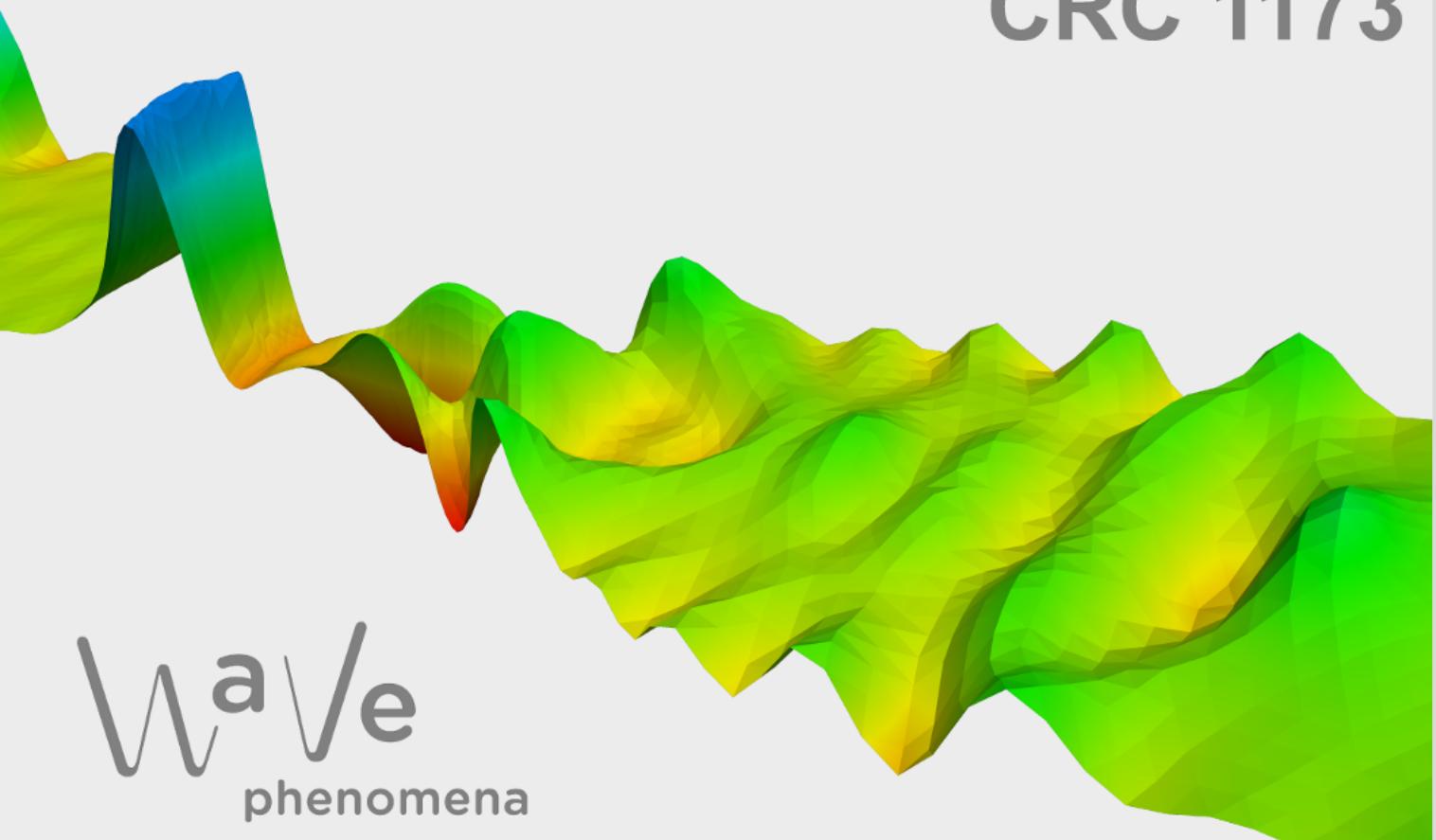
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ENERGY BOUNDS FOR BIHARMONIC WAVE MAPS IN LOW DIMENSIONS

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ABSTRACT. For compact, isometrically embedded Riemannian manifolds $N \hookrightarrow \mathbb{R}^L$, we introduce a fourth-order version of the wave map equation. By energy estimates, we prove an priori estimate for smooth local solutions in the energy subcritical dimension $n = 1, 2$. The estimate excludes blow-up of a Sobolev norm in finite existence times. In particular, combining this with an upcoming work of local well-posedness of the Cauchy problem, it follows that for smooth initial data with compact support, there exists a (smooth) unique global solution in dimension $n = 1, 2$. We also give a proof of the uniqueness of solutions that are bounded in these Sobolev norms.

1. INTRODUCTION

Let (N, h) be a (compact) Riemannian manifold, isometrically embedded (by Nash's theorem) into euclidean space $N \hookrightarrow \mathbb{R}^L$. For a Riemannian manifold (M, g), we introduce the action functional

$$\mathcal{L}(u) = \frac{1}{2} \int_0^T \int_M |\partial_t u(x,t)|^2 - |\Delta_{g(x)} u(x,t)|^2 \; dV_g(x) dt, \quad dV_g = \sqrt{\det g} \; dx$$

for (smooth) maps $u : M \times [0, T] \to N$. We call u a (extrinsic) biharmonic wave map, if $\mathcal{L}$ is critical in the following sense.

$$\frac{d}{d\delta} \mathcal{L}(u + \delta\Phi)_{|_{\delta=0}} = 0, \quad \Phi \in C_c^\infty(M \times [0,T), \mathbb{R}^L), \; \Phi(x,t) \in T_{u(x,t)}N, \quad (x,t) \in M \times [0,T].$$

In this case, u satisfies the condition

$$(1.1) \qquad \partial_t^2 u(x,t) + \Delta_g^2 u(x,t) \perp T_{u(x,t)}N, \; (x,t) \in M \times [0,T],$$

where Δ_g denotes the Laplace-Beltrami operator on (M, g). More explicitly, we use the fact that there exists a smooth familiy of orthogonal (linear) projector

$$P_p : \mathbb{R}^L \to T_p N, \; p \in N,$$

in order to expand (1.1) into the equation

$$(1.2) \quad \partial_t^2 u + \Delta_g^2 u = (I - P_u)(\partial_t^2 u + \Delta_g^2 u) = dP_u(\partial_t u, \partial_t u) + \Delta_g(\mathrm{tr}_g \, dP_u(\nabla u, \nabla u))$$
$$+ 2 \, \mathrm{div}_g(dP_u(\nabla u, \Delta_g u)) - dP_u(\Delta_g u, \Delta_g u).$$

In fact, the projector maps are derivatives of the metric distance (with respect to N) in $\mathbb{R}^L$, ie.

$$p = \Pi(p) + \frac{1}{2}\nabla_p(\mathrm{dist}^2(p, N)), \quad P_p = \nabla_p \Pi(p), \quad \mathrm{dist}(p, N) < \delta_0.$$

We note that via this representation, it is possible to extend this family smoothly to all of $\mathbb{R}^L$ in order to solve the Cauchy problem for (1.2) without restricting the coefficients a priori.

We are particularly interested in the following Cauchy problem for $M = \mathbb{R}^n$

(1.3)
$$\begin{cases} \partial_t^2 u(t,x) + \Delta^2 u(t,x) \perp T_{u(t,x)}N, & (t,x) \in (0,T) \times \mathbb{R}^n \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^n \\ (u_0, u_1) : \mathbb{R}^n \to TN, \quad u_1(x) \in T_{u_0(x)}N & x \in \mathbb{R}^n \end{cases}$$

We state the following result

Theorem 1.1. *Let $n \in \{1,2\}$ and $u \in C^\infty(\mathbb{R}^n \times [0,T), N)$ be a local solution of* (1.3). *Assume further*

$$u - u_0 \in C^0([0,T), H^{n+2}(\mathbb{R}^n)) \cap C^1([0,T), H^n(\mathbb{R}^n)).$$

Then there holds

(1.4)
$$\limsup_{t \nearrow T} (\|u_t(t)\|_{H^n} + \|\nabla u(t)\|_{H^{n+1}}) < \infty,$$

as long as $T < \infty$.

In an upcoming work [HLSS], the authors prove local wellposedness (in high regularity) and a blow up condition for the Cauchy problem (1.3), which (by the proof of Theorem 1.1) implies the following

Corollary 1.2. *Let $n \in \{1,2\}$ and $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}^L$, $u_0(x) \in N$, $u_1(x) \in T_{u_0(x)}N$ for $x \in \mathbb{R}^n$ and such that*

$$(\nabla u_0, u_1) \in H^k(\mathbb{R}^n) \times H^{k-1}(\mathbb{R}^n),$$

for $k \in \mathbb{N}$ sufficiently large. Then the Cauchy problem (1.3) *has a global solution $u : \mathbb{R}^n \times \mathbb{R} \to N$ with*

$$u - u_0 \in C^0(\mathbb{R}, H^k(\mathbb{R}^n)) \cap C^1(\mathbb{R}, H^{k-2}(\mathbb{R}^n)).$$

In particular, if u_0, u_1 are smooth and $supp(\nabla u_0), supp(u_1)$ are compact, then there exists a global smooth solution of (1.3).

We conclude this section with a few remarks.

In the sense explained above, (1.1) and (1.2) are higher order versions of the wave map equation

(1.5)
$$\Box_g u = \operatorname{tr}_g dP_u(\nabla u, \nabla u),$$

with the d'Alembert operator $\Box_g = \partial_t^2 - \Delta_g^2$. Equation (1.5) is the Euler Lagrange equation of the action functional

$$\mathcal{L}(u) = \int_0^T \int_M L(u) \, dV_g \, dt$$

on the Riemannian manifold (M,g) with Lagrangian $L(u) = \frac{1}{2}\tilde{g}^{\alpha\beta}\langle \frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial x_\beta}\rangle$, and where $\frac{\partial}{\partial x_0} = \frac{\partial}{\partial t}$ and $\tilde{g} = -dt^2 + g$. This wave equation has been studied intensively in the past, especially as a model problem for nonlinear dispersion and singularity formation. We refer to [SS00] and [GG17] for an overview over the wellposedness and singularity theory of the Cauchy problem for the wave map equation (1.5).

For the wave map problem, the action functional $\mathcal{L}$ is independent of the embedding $N \hookrightarrow \mathbb{R}^L$. In our case however, there is an intrinsic biharmonic wave map problem, arising from critical points of the (embedding independent) functional

$$\mathcal{L}_i(u) = \int_0^T \int_M |\partial_t u|_h^2 - |\operatorname{tr}_g(\nabla du)|_h^2 \, dV_g \, dt.$$

where ∇ denotes the Levi-Civita connection of the pullback bundle u^*TN endowed with the pullback metric u^*h and the energy potential is given by the tension field $\tau_g(u) = \mathrm{tr}_g(\nabla du)$ of u. Moreover, first variations are calculated intrinsically as follows.

$$\frac{d}{d\delta}\mathcal{L}_i(u^\delta)_{|_{\delta=0}} = 0, \quad u^{\cdot} \in C^\infty((-\delta_0, \delta_0) \times M \times [0, T), N), \ u = u^0$$

such that $\mathrm{supp}(u - u^\delta) \subset\subset M \times (0, T)$ for $|\delta| < \delta_0$.

Then the Euler-Lagrange equation, which has been calculated for static solutions e.g. in [Guo08], becomes

$$(1.6) \qquad\qquad \nabla_t \partial_t u + \Delta_{g,h}^2 u + R(u)(du, \Delta_{g,h}u)du = 0,$$

where R is the curvature tensor and in the covariant notation, we set $\Delta_{g,h}u = \mathrm{tr}_g(\nabla du)$, and use $\Delta_{g,h}^2 u = \Delta_{g,h}(\Delta_{g,h}u) = \mathrm{tr}_g(\nabla\tau_g(u))$.

Static solutions of (1.2) (and (1.6)) are extrinsic (and intrinsic) biharmonic maps, i.e. they are maps $u : (M, g) \to (N, h)$ between Riemannian manifolds that are critical for the (intrinsic or extrinsic) energy functional

$$F(u) = \frac{1}{2}\int_M |\mathrm{tr}_g(\nabla du)|_h^2 \ dV_g, \qquad E(u) = \frac{1}{2}\int_M |\Delta_g u|_h^2 \ dV_g, \ \text{respectively}$$

where the latter is defined subject to an isometric embedding $(N, h) \hookrightarrow \mathbb{R}^m$. Biharmonic maps (resp. the Euler Lagrange equation of E and F) and their heat flow has been studied intensively in the past.

2. Related work and local wellposedness in high regularity

In [HLS18], the authors pove the existence of a global weak solution into round spheres $\mathbb{S}^{L-1} \subset \mathbb{R}^L$. This is done by a penalization functional of Ginzburg Landau type, which then gives a uniform energy bound in the penalty parameter. To prove convergence of such approximations, the authors depend on the geometry of the sphere, more precisely, the equation can be rewritten in divergence form. This argument has been used for the wave map equation (1.5) with $N = \mathbb{S}^{L-1}$ and $M = \mathbb{R}^n$ in [Sha88] and further the divergence form has been used in [Str03], in order to prove weak compactness of the class of stationary solutions of (1.2) on the domain $M = \mathbb{R}^4$.

As mentioned above, in the upcoming work [HLSS], the authors prove local wellposedness of the Cauchy problem (1.3). More precisely, let $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}^L$, $u_0(x) \in N$, $u_1(x) \in T_{u_0(x)}N$ for $\mathcal{L}^n$ a.e. $x \in \mathbb{R}^n$ with

$$(\nabla u_0, u_1) \in H^{k-1}(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n), \quad k > \left\lfloor\frac{n}{2}\right\rfloor + 2, \ k \in \mathbb{N}.$$

Then there exists a $T > 0$ and a (unique) solution $u : \mathbb{R}^n \times [0, T) \to N$ of (1.3) with

$$u - u_0 \in C^0([0, T), H^k(\mathbb{R}^n)) \cap C^1([0, T), H^{k-2}(\mathbb{R}^n)).$$

From this, we note that in particular we obtain Corollary 1.2 from a blow up condition contained in [HLSS]. In the following, note that the energy functional

$$(2.1) \qquad\qquad E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^n} |u_t|^2 + |\Delta u|^2 dx,$$

is formally conserved along solutions u. This implies the bound

$$(2.2) \qquad\qquad \frac{d}{dt}\int_{\mathbb{R}^n} |\nabla u|^2 dx \leq E(u(0)).$$

Both, (2.1) and (2.2), will be used in the following for smooth solutions. We further note that below in section 4, we inculde a short argument for the uniqueness of such solutions in dimension $n = 1, 2, 3$.

3. Proof of Theorem 1.1

Since for solutions u of (1.2), resp. the Cauchy problem (1.3), the term $\partial_t^2 u + \Delta^2 u$ is a section over the normal bundle of $u^*(TN)$, we let $\mathrm{codim}(N) = L - l$ for $l \in \mathbb{N}$, $l \leq L$ and first assume the normalbundle $T^\perp N$ of $N \subset \mathbb{R}^L$ is parallelizable. This means there exists a frame of (smooth) orthogonal vectorfields $\{\nu_1(p), \ldots, \nu_{L-l}(p)\} \subset \mathbb{R}^L$, $p \in N$ with $\nu_i(p) \perp T_p N$ for every $p \in N$.

In this case, for any local solution u, we have an explicit representation for the nonlinearity in terms of $\nu_i(u)$.

$$(3.1) \qquad \partial_t^2 u + \Delta^2 u =: \sum_{i=1}^{L-l} G^i(u)\nu_i(u) =: G^i(u)\nu_i(u),$$

where $G_i(u) = \langle \partial_t^2 u + \Delta^2 u, \nu_i(u) \rangle$. We thus calculate

$$\langle \partial_t^2 u, \nu_i(u) \rangle = -\langle u_t, d\nu_i(u)u_t \rangle,$$
$$\langle \Delta^2 u, \nu_i(u) \rangle = -3\langle \nabla\Delta u, d\nu_i(u)\nabla u \rangle - \langle \nabla u, d\nu_i(u)\nabla\Delta u \rangle$$
$$\quad - \langle \nabla u, d^3\nu_i(u)(\nabla u)^3 + 2d^2\nu_i(u)(\nabla u, \nabla^2 u) + d^2\nu_i(u)(\nabla u, \Delta u) \rangle$$
$$\quad - 2\langle \nabla^2 u, d^2\nu_i(u)(\nabla u)^2 + d\nu_i(\nabla^2 u) \rangle - \langle \Delta u, d^2\nu_i(u)(\nabla u)^2 + d\nu_i(\Delta u) \rangle,$$

where we denote by $d^k \nu_i$ the kth order differential of ν_i on N and write $(\nabla u)^2$, $(\nabla u)^3$ for products of first order derivatives of u with eiter two order three factors, respectively. The precise product, e.g. $\partial_{x_j} u \cdot \partial^{x_j} u$ or $\partial_{x_i} u \cdot \partial^{x_j} u \cdot \partial_{x_j} u$ will become clear in the terms of the expansion. The result in Theorem 1.1 is known for $N = \mathbb{S}^{L-1}$ and $n \leq 2$ thanks to [FO10].

Case: n = 2 We apply $\Delta = \partial_i \partial^i$ on both sides of (3.1). Then, testing the differentiated equation by Δu_t, we infer

$$(3.2) \qquad \frac{d}{2dt} \int_{\mathbb{R}^n} (|\Delta u_t|^2 + |\Delta^2 u|^2)dx = \int_{\mathbb{R}^n} \Delta(G^i(u)\nu_i(u))\Delta u_t dx.$$

Since $G^i(u)$ contains derivatives of order three, we can not proceed by the Hölder inequality. Instead, we follow [FO10], where the authors showed that the highest order derivative cancel in the case $N = \mathbb{S}^{L-1}$, $\nu(u) = u$. Since

$$\Delta(G^i(u)\nu_i(u))\Delta u_t = \Delta(G^i(u))\nu_i(u)\Delta u_t + 2\nabla(G^i(u)) \cdot \nabla(\nu_i(u))\Delta u_t + G^i(u)\Delta\nu_i(u)\Delta u_t,$$

and

$$0 = \Delta(\nu_i(u)u_t) = 2d\nu_i(u)(\nabla u) \cdot \nabla u_t + \nu_i(u)\Delta u_t + d^2\nu_i(u)(\nabla u)^2 u_t + d\nu_i(u)(\Delta u)u_t,$$

it follows

$$\Delta(G^i(u)\nu_i(u))\Delta u_t = -\Delta G^i(u)\left(2d\nu_i(u)(\nabla u) \cdot \nabla u_t + d^2\nu_i(u)(\nabla u)^2 u_t + d\nu_i(u)(\Delta u)u_t\right)$$
$$\quad + 2\nabla G^i(u) \cdot d\nu_i(u)(\nabla u)\Delta u_t$$
$$\quad + G^i(u)\left(d^2\nu_i(u)(\nabla u)^2 + d\nu_i(u)\Delta u\right)\Delta u_t.$$

Hence we observe, by integration by parts for the first summand,

$$\int_{\mathbb{R}^n} \Delta(G^i(u)\nu_i(u))\Delta u_t \, dx = \int_{\mathbb{R}^n} \nabla G^i(u) \cdot [3d^2\nu_i(u)(\nabla u)^2\nabla u_t + 3d\nu_i(u)(\Delta u)\nabla u_t]dx$$

$$+ \int_{\mathbb{R}^n} \nabla G^i(u) \cdot [4d\nu_i(u)(\nabla u)\Delta u_t + d^3\nu_i(u)(\nabla u)^3 u_t]dx$$

$$+ \int_{\mathbb{R}^n} \nabla G^i(u) \cdot [3d^2\nu_i(u)(\Delta u, \nabla u)u_t + d\nu_i(u)(\nabla\Delta u)u_t]dx$$

$$+ \int_{\mathbb{R}^n} G^i(u)(d^2\nu_i(u)(\nabla u)^2 u_t + d\nu_i(u)(\Delta u))\Delta u_t \, dx.$$

Instead of deducing bounds for this terms that depend on the normal frame $\{\nu_1, \ldots \nu_{L-l}\}$, we turn to the general case and use the normal projector $I - P_u : \mathbb{R}^L \to (T_u N)^\perp$ along the map $u : \mathbb{R}^n \times [0, T] \to N$ in order to represent the nonlinearity in (3.1) as

$$(3.3) \qquad \partial_t^2 u + \Delta^2 u = (I - P_u)(\partial_t^2 u + \Delta^2 u).$$

Here, we proceed similarly, ie. we use

$$(3.4) \qquad \Delta((I - P_u)(\partial_t^2 u + \Delta^2 u))\Delta u_t = \Delta((I - P_u)^2(\partial_t^2 u + \Delta^2 u))\Delta u_t,$$

and hence

$$\Delta((I - P_u)^2(\partial_t^2 u + \Delta^2 u))\Delta u_t = \Delta[(I - P_u)]((I - P_u)(\partial_t^2 u + \Delta^2 u))\Delta u_t$$

$$+ 2\nabla(I - P_u) \cdot \nabla((I - P_u)(\partial_t^2 + \Delta^2 u))\Delta u_t$$

$$+ (\Delta[(I - P_u)(\partial_t^2 u + \Delta^2 u)])(I - P_u)\Delta u_t.$$

In order to treat the last summand, we expand

$$0 = \Delta((I - P_u)u_t) = (I - P_u)\Delta u_t - d^2 P_u((\nabla u)^2, u_t) - dP_u(\Delta u, u_t) - 2dP_u(\nabla u, \nabla u_t).$$

Hence, as before, integration by parts yields

$$\int_{\mathbb{R}^n} \Delta((I - P_u)(\partial_t^2 u + \Delta^2 u))\Delta u_t$$

$$= -\int_{\mathbb{R}^n} d^2 P_u((\nabla u)^2, (I - P_u)(\partial_t^2 u + \Delta^2 u))\Delta u_t \, dx$$

$$-\int_{\mathbb{R}^n} dP_u(\Delta u, (I - P_u)(\partial_t^2 u + \Delta^2 u))\Delta u_t \, dx$$

$$- 2\int_{\mathbb{R}^n} dP_u(\nabla u, \nabla[(I - P_u)(\partial_t^2 u + \Delta^2 u)])\Delta u_t \, dx$$

$$-\int_{\mathbb{R}^n} \nabla[(I - P_u)(\partial_t^2 u + \Delta^2 u)] \cdot \nabla[dP_u(\Delta u, u_t) + 2dP_u(\nabla u, \nabla u_t) + d^2 P_u((\nabla u)^2, u_t)] \, dx.$$

We first note the pointwise bounds

$$(3.5)$$
$$|(I - P_u)(\partial_t^2 u + \Delta^2 u)| \lesssim \quad |u_t|^2 + |\nabla^2 u|^2 + |\nabla^2 u||\nabla u|^2 + |\nabla^3 u||\nabla u| + |\nabla u|^4$$

$$(3.6)$$
$$|\nabla[(I - P_u)(\partial_t^2 u + \Delta^2 u)]| \lesssim \quad |\nabla u_t||u_t| + |\nabla u||u_t|^2 + |\Delta^2 u||\nabla u|$$
$$+ |\nabla^3 u|(|\nabla^2 u| + |\nabla u|^2) + |\nabla u||\nabla^2 u|^2 + |\nabla u|^3|\nabla^2 u| + |\nabla u|^5,$$

where the constants only depend on the supremum norm

$$\|dP\|_{C_b^3} = \|dP\|_{C_b(N)} + \|d^2 P\|_{C_b(N)} + \|d^3 P\|_{C_b(N)} + \|d^4 P\|_{C_b(N)}.$$

We now estimate, using (3.5) and(3.6),

$$\|d^2 P_u((\nabla u)^2, (I - P_u)(\partial_t^2 u + \Delta^2 u))\Delta u_t\|_{L^1}$$

$$\lesssim \|\Delta u_t\|_{L^2} \|\nabla u\|_{L^\infty}^2 \Big[\|u_t\|_{L^\infty} \|u_t\|_{L^2} + \|\nabla^2 u\|_{L^\infty} \|\Delta u\|_{L^2} + \|\nabla u\|_{L^\infty}^2 \|\Delta u\|_{L^2}$$

$$+ \|\nabla \Delta u\|_{L^2} \|\nabla u\|_{L^\infty} + \|\nabla u\|_{L^4}^2 \|\nabla u\|_{L^\infty}^2 \Big],$$

$$\|dP_u(\Delta u, (I - P_u)(\partial_t^2 u + \Delta^2 u))\Delta u_t\|_{L^1}$$

$$\lesssim \|\Delta u_t\|_{L^2} \|\Delta u\|_{L^\infty} \Big[\|u_t\|_{L^\infty} \|u_t\|_{L^2} + \|\nabla^2 u\|_{L^\infty} \|\Delta u\|_{L^2} + \|\nabla u\|_{L^\infty}^2 \|\Delta u\|_{L^2}$$

$$+ \|\nabla \Delta u\|_{L^2} \|\nabla u\|_{L^\infty} + \|\nabla u\|_{L^4}^2 \|\nabla u\|_{L^\infty}^2 \Big]$$

$$= \|\Delta u_t\|_{L^2} \|\Delta u\|_{L^\infty} \big[\|u_t\|_{L^\infty} \|u_t\|_{L^2} + \|\nabla^2 u\|_{L^\infty} \|\Delta u\|_{L^2} \big]$$

$$+ h(t)^2 \|\Delta u_t\|_{L^2} \|\Delta u\|_{L^\infty} \big[\|\Delta u\|_{L^2} + \|\nabla u\|_{L^4}^2 \big]$$

$$+ h(t) \|\Delta u_t\|_{L^2} \|\Delta u\|_{L^\infty} \|\nabla \Delta u\|_{L^2} .$$

where we set $h(t) := \|\nabla u(t)\|_{L^\infty}$. We note further that the equality is up to the constant from the estimate. We hence proceed by estimating

$$\|dP_u(\nabla u, \nabla[(I - P_u)(\partial_t^2 u + \Delta^2 u)])\Delta u_t\|_{L^1}$$

$$\lesssim \|\Delta u_t\|_{L^2} \|\nabla u\|_{L^\infty} \Big[\|u_t\|_{L^\infty} \|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^2} \|u_t\|_{L^\infty}^2 + \|\Delta^2 u\|_{L^2} \|\nabla u\|_{L^\infty}$$

$$+ \|\nabla \Delta u\|_{L^2} (\|\nabla^2 u\|_{L^\infty} + \|\nabla u\|_{L^\infty}^2) + \|\nabla u\|_{L^\infty} \|\Delta u\|_{L^2} \|\nabla^2 u\|_{L^\infty} + \|\Delta u\|_{L^2} \|\nabla u\|_{L^\infty}^3$$

$$+ \|\nabla u\|_{L^2} \|\nabla u\|_{L^\infty}^4 \Big]$$

$$= h(t) \|\Delta u_t\|_{L^2} \big[\|u_t\|_{L^\infty} \|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^2} \|u_t\|_{L^\infty}^2 + \|\nabla \Delta u\|_{L^2} \|\nabla^2 u\|_{L^\infty} + \|\Delta u\|_{L^2} \|\nabla u\|_{L^\infty}^3 \big]$$

$$+ h^2(t) \|\Delta u_t\|_{L^2} \big[\|\Delta^2 u\|_{L^2} + \|\nabla u\|_{L^\infty} \|\nabla \Delta u\|_{L^2} + \|\Delta u\|_{L^2} \|\nabla^2 u\|_{L^\infty} + \|\nabla u\|_{L^2} \|\nabla u\|_{L^\infty}^3 \big].$$

We calculate

$$\nabla[dP_u(\Delta u, u_t) + 2dP_u(\nabla u, \nabla u_t) + d^2 P_u((\nabla u)^2, u_t)]$$

$$= d^2 P_u(\nabla u, \Delta u, u_t) + dP_u(\nabla \Delta u, u_t) + dP_u(\Delta u, \nabla u_t) + 2d^2 P_u((\nabla u)^2, \nabla u_t)$$

$$+ 2dP_u(\nabla^2 u, \nabla u_t) + 2dP_u(\nabla u, \nabla^2 u_t) + d^3 P_u((\nabla u)^3, u_t) + 2d^2 P_u(\nabla u, \nabla^2 u, u_t) + d^2 P_u((\nabla u)^2, \nabla u_t),$$

and hence

$$\|\nabla[dP_u(\Delta u, u_t) + 2dP_u(\nabla u, \nabla u_t) + d^2 P_u((\nabla u)^2, u_t)] \cdot \nabla[(I - P_u)(\partial_t^2 u + \Delta^2 u)]\|_{L^1}$$

$$\lesssim \big(\|\Delta u\|_{L^2} \|\nabla u\|_{L^\infty} \|u_t\|_{L^\infty} + \|\nabla\Delta u\|_{L^2} \|u_t\|_{L^\infty} + (\|\Delta u\|_{L^\infty} + \|\nabla u\|_{L^\infty}^2) \|\nabla u_t\|_{L^2}$$

$$+ \|\Delta u_t\|_{L^2} \|\nabla u\|_{L^\infty} + \|\nabla u\|_{L^\infty}^3 \|u_t\|_{L^2} \big) \Big[\|u_t\|_{L^\infty} \|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^2} \|u_t\|_{L^\infty}^2 + \|\Delta^2 u\|_{L^2} \|\nabla u\|_{L^\infty}$$

$$+ \|\nabla\Delta u\|_{L^2} (\|\nabla^2 u\|_{L^\infty} + \|\nabla u\|_{L^\infty}^2) + \|\nabla u\|_{L^\infty} \|\Delta u\|_{L^2} \|\nabla^2 u\|_{L^\infty}$$

$$+ \|\Delta u\|_{L^2} \|\nabla u\|_{L^\infty}^3 + \|\nabla u\|_{L^2} \|\nabla u\|_{L^\infty}^4 \Big]$$

$$= (\|\nabla\Delta u\|_{L^2} \|u_t\|_{L^\infty} + \|\Delta u\|_{L^\infty} \|\nabla u_t\|_{L^2}) \big[\|u_t\|_{L^\infty} \|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^2} \|u_t\|_{L^\infty}^2$$

$$+ \|\nabla\Delta u\|_{L^2} \|\nabla^2 u\|_{L^\infty} + \|\Delta u\|_{L^2} \|\nabla u\|_{L^\infty}^3 \big]$$

$$+ h(t)(\|\nabla\Delta u\|_{L^2} \|u_t\|_{L^\infty} + \|\Delta u\|_{L^\infty} \|\nabla u_t\|_{L^2}) \big[\|\Delta^2 u\|_{L^2} + \|\nabla u\|_{L^\infty} \|\nabla\Delta u\|_{L^2}$$

$$+ \|\Delta u\|_{L^2} \|\nabla^2 u\|_{L^\infty} + \|\nabla u\|_{L^2} \|\nabla u\|_{L^\infty}^3 \big]$$

$$+ h(t)(\|\Delta u\|_{L^2} \|u_t\|_{L^\infty} + \|\nabla u_t\|_{L^2} \|\nabla u\|_{L^\infty} + \|\Delta u_t\|_{L^2} + \|\nabla u\|_{L^\infty}^2 \|u_t\|_{L^2}) \big[\|u_t\|_{L^\infty} \|\nabla u_t\|_{L^2}$$

$$+ \|\nabla u\|_{L^2} \|u_t\|_{L^\infty}^2 + \|\nabla\Delta u\|_{L^2} \|\nabla^2 u\|_{L^\infty} + \|\Delta u\|_{L^2} \|\nabla u\|_{L^\infty}^3 \big]$$

$$+ h^2(t)(\|\Delta u\|_{L^2} \|u_t\|_{L^\infty} + \|\nabla u_t\|_{L^2} \|\nabla u\|_{L^\infty} + \|\Delta u_t\|_{L^2} + \|\nabla u\|_{L^\infty}^2 \|u_t\|_{L^2}) \big[\|\Delta^2 u\|_{L^2}$$

$$+ \|\nabla u\|_{L^\infty} \|\nabla\Delta u\|_{L^2} + \|\Delta u\|_{L^2} \|\nabla^2 u\|_{L^\infty} + \|\nabla u\|_{L^2} \|\nabla u\|_{L^\infty}^3 \big].$$

We note that the energy is conserved, ie. for $t \in [0, T)$

$$(3.7) \qquad 2E(u(t)) = \|\Delta u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2 = \|\Delta u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2 = 2E(u_0, u_1),$$

and further, this implies the bounds

$$(3.8) \qquad \sup_{t \in [0,T)} \|\nabla u(t)\|_{L^2} \lesssim \sqrt{1+T}(\sqrt{E(u_0, u_1)} + \|\nabla u_0\|_{L^2}), \quad \text{and}$$

$$(3.9) \qquad \sup_{t \in [0,T)} \|u(t) - u_0\|_{L^2} \lesssim T\sqrt{E(u_0, u_1)}.$$

We recall the following cases of Gagliardo-Nirenberg's interpolation for $n = 2$

$$(3.10) \quad \|\Delta u\|_{L^\infty} + \|\nabla\Delta u\|_{L^2} \lesssim \|\Delta^2 u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}}, \quad \|u_t\|_{L^\infty} \lesssim \|\Delta u_t\|_{L^2}^{\frac{1}{2}} \|u_t\|_{L^2}^{\frac{1}{2}},$$

$$(3.11) \qquad \|\nabla u\|_{L^\infty} \lesssim \|\Delta^2 u\|_{L^2}^{\frac{1}{3}} \|\nabla u\|_{L^2}^{\frac{2}{3}}, \quad \|\nabla u\|_{L^4} \lesssim \|\Delta^2 u\|_{L^2}^{\frac{1}{6}} \|\nabla u\|_{L^2}^{\frac{5}{6}}, \quad \text{and}$$

$$(3.12) \qquad \|\nabla u_t\|_{L^4} \lesssim \|\Delta u_t\|_{L^2}^{\frac{3}{4}} \|u_t\|_{L^2}^{\frac{1}{4}}.$$

Setting

$$\mathcal{E}(u(t)) := \|\Delta u_t(t)\|_{L^2} + \|\Delta^2 u(t)\|_{L^2}, \ t \in [0, T),$$

by (3.10), (3.11) and the estimates above, there exists a constant $C(T) = C(N, u_0, u_1)(1 + T)^\alpha$ for some $\alpha > 0$, such that $C(N, u_0, u_1)$ only depends on the norm $\|dP\|_{C_b^3}$, the optimal Sobolev constant in Gagliardo-Nirenberg's interpolation and $E(u_0, u_1)$, $\|\nabla u_0\|_{L^2}$ and such that the following holds.

$$(3.13) \qquad \frac{d}{dt}\mathcal{E}^2(u(t)) \leq C(T)(1 + h(t) + h^2(t))(\mathcal{E}(t) + \mathcal{E}^2(t))$$

$$\leq C(T)(1 + h^2(t))(1 + \mathcal{E}^2(t)), \ t \in [0, T).$$

Using the idea from [FO10], we now apply the sharp Sobolev inequality of Brezis-Gallouet-Wainger from [BW80] in order to bound (we assume u is not a constant)

$$(3.14) \qquad h(t) \leq \tilde{C} \left\| \nabla u(t) \right\|_{H^1} \left(1 + \log^{\frac{1}{2}} \left(1 + \frac{\left\| \nabla u(t) \right\|_{H^2}^2}{\left\| \nabla u(t) \right\|_{H^1}^2} \right) \right), \quad t \in [0, T).$$

Thus, using (3.10), (3.8) and (3.7),

$$(3.15) \qquad h^2(t) \leq C(T) \left(1 + \log \left(1 + \mathcal{E}^2(t) \right) \right), \quad t \in [0, T),$$

and hence

$$(3.16) \qquad \frac{d}{dt} (e + \mathcal{E}^2(u(t))) \leq C(T) \log \left(e + \mathcal{E}^2(t) \right) (e + \mathcal{E}^2(t)), \ t \in [0, T).$$

This suffices for a Gronwall-type inequality for $\log(e + \mathcal{E}^2(t))$ and hence by (3.7) and (3.10), (3.11) and (3.12), we have

$$\limsup_{t \to T} (\|u_t\|_{H^2}^2 + \|\nabla u\|_{H^3}^2) < \infty,$$

as long as $T < \infty$.

Case: n = 1 Here, by Gagliardo-Nirenberg's estimate, we infer the bound

$$(3.17) \qquad \|\nabla u\|_{L^\infty} \lesssim \left\| \nabla^2 u \right\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}}.$$

Hence, the a priori bound is derived similarly for $(\nabla u(t), u_t(t)) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$. We note

$$\frac{d}{dt} \int_{\mathbb{R}} |\nabla u_t|^2 + |\nabla \Delta u|^2 \ dx = - \int_{\mathbb{R}} dP_u(\nabla u, (I - P_u)(\partial_t^2 u + \Delta^2 u)) \cdot \nabla u_t$$

$$- \int_{\mathbb{R}} (I - P_u)(\partial_t^2 u + \Delta^2 u)) \cdot (d^2 P_u((\nabla u)^2, u_t) + dP_u(\nabla^2 u, u_t) + dP_u(\nabla u, \nabla u_t)) dx.$$

Thus we estimate, as before

$$\|dP_u(\nabla u, (I - P_u)(\partial_t^2 u + \Delta^2 u)) \nabla u_t\|_{L^1}$$

$$\lesssim \|\nabla u_t\|_{L^2} \|\nabla u\|_{L^\infty} \Big[\|u_t\|_{L^\infty} \|u_t\|_{L^2} + \left\| \nabla^2 u \right\|_{L^\infty} \left\| \nabla^2 u \right\|_{L^2} + \|\nabla u\|_{L^\infty}^2 \left\| \nabla^2 u \right\|_{L^2}$$

$$+ \left\| \nabla^3 u \right\|_{L^2} \|\nabla u\|_{L^\infty} + \|\nabla u\|_{L^4}^2 \|\nabla u\|_{L^\infty}^2 \Big], \quad \text{and}$$

$$\|(d^2 P_u((\nabla u)^2, u_t) + dP_u(\nabla^2 u, u_t) + dP_u(\nabla u, \nabla u_t))[(I - P_u)(\partial_t^2 u + \Delta^2 u)]\|_{L^1}$$

$$\lesssim (\|u_t\|_{L^2} \|\nabla u\|_{L^\infty}^2 + \left\| \nabla^2 u \right\|_{L^2} \|u_t\|_{L^\infty} + \|\nabla u_t\|_{L^2} \|\nabla u\|_{L^\infty}) \Big[\|u_t\|_{L^\infty} \|u_t\|_{L^2}$$

$$+ \left\| \nabla^2 u \right\|_{L^\infty} \left\| \nabla^2 u \right\|_{L^2} + \|\nabla u\|_{L^\infty}^2 \left\| \nabla^2 u \right\|_{L^2} + \left\| \nabla^3 u \right\|_{L^2} \|\nabla u\|_{L^\infty} + \|\nabla u\|_{L^4}^2 \|\nabla u\|_{L^\infty}^2 \Big]$$

Hence from the interpolation estimates (3.17),

$$(3.18) \qquad \left\| \nabla^2 u \right\|_{L^\infty} \lesssim \left\| \nabla^4 u \right\|_{L^2}^{\frac{1}{4}} \left\| \nabla^2 u \right\|_{L^2}^{\frac{3}{4}}, \quad \|u_t\|_{L^\infty} \lesssim \|\nabla u_t\|_{L^2}^{\frac{1}{2}} \|u_t\|_{L^2}^{\frac{1}{2}},$$

and (3.7), (3.8), there holds (for $C(T) > 0$ as before)

$$(3.19) \qquad \frac{d}{dt} (1 + \mathcal{E}(t)) \leq C(T)(1 + \mathcal{E}(t)), \ t \in [0, T)$$

which suffices to use a Gronwall argument in order to conclude the proof.

4. A UNIQUENESS ARGUMENT

We now give a short argument for the uniqueness of solutions $u : \mathbb{R}^n \times [0, T) \to N$, $n = 1, 2, 3$ with

(4.1) $$u - u(0) \in C^0([0, T), H^4(\mathbb{R}^n)) \cap C^1([0, T), H^2(\mathbb{R}^n))$$

Setting $w = u - v$ for solutions u, v of (1.3) in the class (4.1) with $u(0) = v(0)$, $u_t(0) = v_t(0)$, we provide a Gronwall type argument in the energy space, i.e. more precisely for the norm $\|w_t\|_{L^2}^2 + \|w\|_{H^2}^2$. We note the interpolation estimate

$$\|w\|_{L^\infty} \lesssim \|\Delta w\|_{L^2}^{\frac{n}{4}} \|w\|_{L^2}^{\frac{4-n}{4}}, \quad n = 1, 2, 3,$$

and the identity

(4.2) $$\frac{d}{2dt} \left(\int_{\mathbb{R}^n} |w_t|^2 + |\Delta w|^2 \right) dx = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{\mathbb{R}^n} \left[dP_u(u_t u_t + 4\nabla u \cdot \nabla \Delta u + \Delta u \Delta u + 2\nabla^2 u \cdot \nabla^2 u) + d^3 P_u(\nabla u)^4 \right.$$
$$\left. + d^2 P_u(2(\nabla u)^2 \Delta u + 4(\nabla u)^2 \cdot \nabla^2 u) \right] (P_v - P_u) w_t \; dx$$
$$+ \int_{\mathbb{R}^n} \left[(dP_u - dP_v)(u_t u_t + 4\nabla u \cdot \nabla \Delta u + \Delta u \Delta u + 2\nabla^2 u \cdot \nabla^2 u) \right.$$
$$\left. + (d^3 P_u - d^3 P_v)(\nabla u)^4 + (d^2 P_u - d^2 P_v)(2(\nabla u)^2 \Delta u + 4(\nabla u)^2 \cdot \nabla^2 u) \right] (P_u - P_v) u_t \; dx$$

$$I_2 = \int_{\mathbb{R}^n} \left[dP_v(u_t w_t + w_t v_t + 4\nabla w \cdot \nabla \Delta u + \Delta w \Delta u + \Delta v \Delta w \right.$$
$$+ 2\nabla^2 w \cdot \nabla^2 u + 2\nabla^2 v \cdot \nabla^2 w) + d^3 P_v(\nabla w \cdot (\nabla u)^3 + \nabla w \cdot (\nabla u)^2 \nabla v + \nabla w \cdot (\nabla v)^2 \nabla u + \nabla w \cdot (\nabla v)^3)$$
$$+ d^2 P_v(2\nabla w \cdot \nabla u \Delta u + 2\nabla v \cdot \nabla w \Delta u + 2(\nabla v)^2 \Delta w + 4\nabla w \cdot \nabla u \cdot \nabla^2 u$$
$$\left. + 4\nabla v \cdot \nabla w \cdot \nabla^2 u + 4(\nabla v)^2 \cdot \nabla^2 w \right] (P_u - P_v) u_t \; dx$$

$$I_3 = \int_{\mathbb{R}^n} 4 dP_v(\nabla v, \nabla \Delta w)(P_u - P_v) u_t \; dx.$$

This follows from

$$(I - P_u)(\partial_t^2 u - \Delta^2 u) - (I - P_v)(\partial_t^2 v - \Delta^2 v)$$
$$= (P_v - P_u)[(I - P_u)(\partial_t^2 u - \Delta^2 u)]$$
$$+ (I - P_v)[(I - P_u)(\partial_t^2 u - \Delta^2 u) - (I - P_v)(\partial_t^2 v - \Delta^2 v)],$$

and

$$(I - P_v) w_t = (I - P_v) u_t = (P_u - P_v) u_t.$$

We further note

$$\int_{\mathbb{R}^n} dP_v(\nabla v, \nabla \Delta w)(P_u - P_v) u_t \; dx = -\int_{\mathbb{R}^n} \left[d^2 P_v((\nabla v)^2, \Delta w) + dP_v(\Delta v, \Delta w) \right] (P_u - P_v) u_t \; dx$$
$$- \int_{\mathbb{R}^n} dP_v(\nabla v, \Delta w)(P_u - P_v)\nabla u_t + dP_v(\nabla v, \Delta w)(dP_u - dP_v)(\nabla u, u_t) \; dx$$
$$- \int_{\mathbb{R}^n} dP_v(\nabla v, \Delta w) dP_v(\nabla w, u_t) \; dx.$$

Hence, we estimate

$$I_1 \lesssim (\|w_t\|_{L^2} \|w\|_{L^\infty} + \|w\|_{L^\infty}^2 \|u_t\|_{L^2})(\|u_t\|_{L^2} \|u_t\|_{L^\infty} + \|\nabla \Delta u\|_{L^2} \|\nabla u\|_{L^\infty} + \|\nabla u\|_{L^2} \|\nabla u\|_{L^\infty}^3$$
$$+ \|\Delta u\|_{L^2} \|\nabla u\|_{L^\infty}^2 + \|\Delta u\|_{L^2} \|\nabla^2 u\|_{L^\infty}),$$

$$I_2 \lesssim \|w\|_{L^\infty} \|u_t\|_{L^2} \left(\|\nabla w\|_{L^2} \|\nabla u\|_{L^\infty}^3 + \|\nabla w\|_{L^2} \|\nabla v\|_{L^\infty} \|\nabla u\|_{L^\infty}^2 + \|\nabla w\|_{L^2} \|\nabla v\|_{L^\infty}^2 \|\nabla u\|_{L^\infty} \right.$$

$$+ \|\nabla w\|_{L^2} \|\nabla v\|_{L^\infty}^3 + \|\nabla w\|_{L^2} \|\nabla u\|_{L^\infty} \|\nabla^2 u\|_{L^\infty} + \|\nabla w\|_{L^2} \|\nabla v\|_{L^\infty} \|\nabla^2 u\|_{L^\infty}$$

$$+ \|\nabla v\|_{L^\infty}^2 \|\Delta w\|_{L^2} + \max\{\|u_t\|_{L^\infty}, \|v_t\|_{L^\infty}\} \|w_t\|_{L^2} + \max\{\|\nabla^2 u\|_{L^\infty}, \|\nabla^2 v\|_{L^\infty}\} \|\Delta w\|_{L^2} \right)$$

$$+ \|w\|_{L^\infty} \|u_t\|_{L^\infty} \|\nabla \Delta u\|_{L^2} \|\nabla w\|_{L^2}$$

and

$$I_3 \lesssim \|w\|_{L^\infty} \|\nabla u_t\|_{L^2} \|\Delta w\|_{L^2} \|\nabla v\|_{L^\infty} + \|w\|_{L^\infty} \|u_t\|_{L^2} \|\Delta w\|_{L^2} \|\nabla v\|_{L^\infty} \|\nabla u\|_{L^\infty}$$

$$+ \|\Delta w\|_{L^2} \|\nabla w\|_{L^2} \|u_t\|_{L^\infty} \|\nabla v\|_{L^\infty} + \|w\|_{L^\infty} \|u_t\|_{L^2} \|\Delta w\|_{L^2} \|\nabla v\|_{L^\infty}^2$$

$$+ \|w\|_{L^\infty} \|u_t\|_{L^\infty} \|\Delta w\|_{L^2} \|\Delta v\|_{L^2} .$$

We set

$$\mathcal{E}^2(t) := \|w_t\|_{L^2}^2 + \|w\|_{H^2}^2 .$$

Using the aforementioned interpolation inequality, we obtain in particular $\|w\|_{L^\infty} \lesssim \mathcal{E}(t)$. Since also

$$(4.3) \qquad \frac{d}{dt} \int_{\mathbb{R}^n} |\nabla w|^2 \, dx \leq \|w_t\|_{L^2}^2 + \|\Delta w\|_{L^2}^2 \leq \mathcal{E}^2(t), \quad \text{and}$$

$$(4.4) \qquad \frac{d}{dt} \int_{R^n} |w|^2 \, dx \leq \|w\|_{L^2}^2 + \|w_t\|_{L^2}^2 \leq \mathcal{E}^2(t),$$

estimating (4.2) gives

$$(4.5) \qquad \frac{d}{dt} \mathcal{E}(t)^2 \lesssim (1 + \|\nabla u\|_{H^3}^4 + \|u_t\|_{H^2}^4 + \|\nabla v\|_{H^3}^4 + \|v_t\|_{H^2}^4) \mathcal{E}^2(t) =: C(u,v) \mathcal{E}^2(t)$$

This suffices for uniqueness, as long as $C(u,v)$ stays bounded in time. We also remark that in $n = 1$, in order to conclude uniqueness from similar arguments, it suffices for a smooth solution u to stay bounded in $u(t) \in H^3(\mathbb{R})$, $\partial_t u(t) \in H^1(\mathbb{R})$.

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