

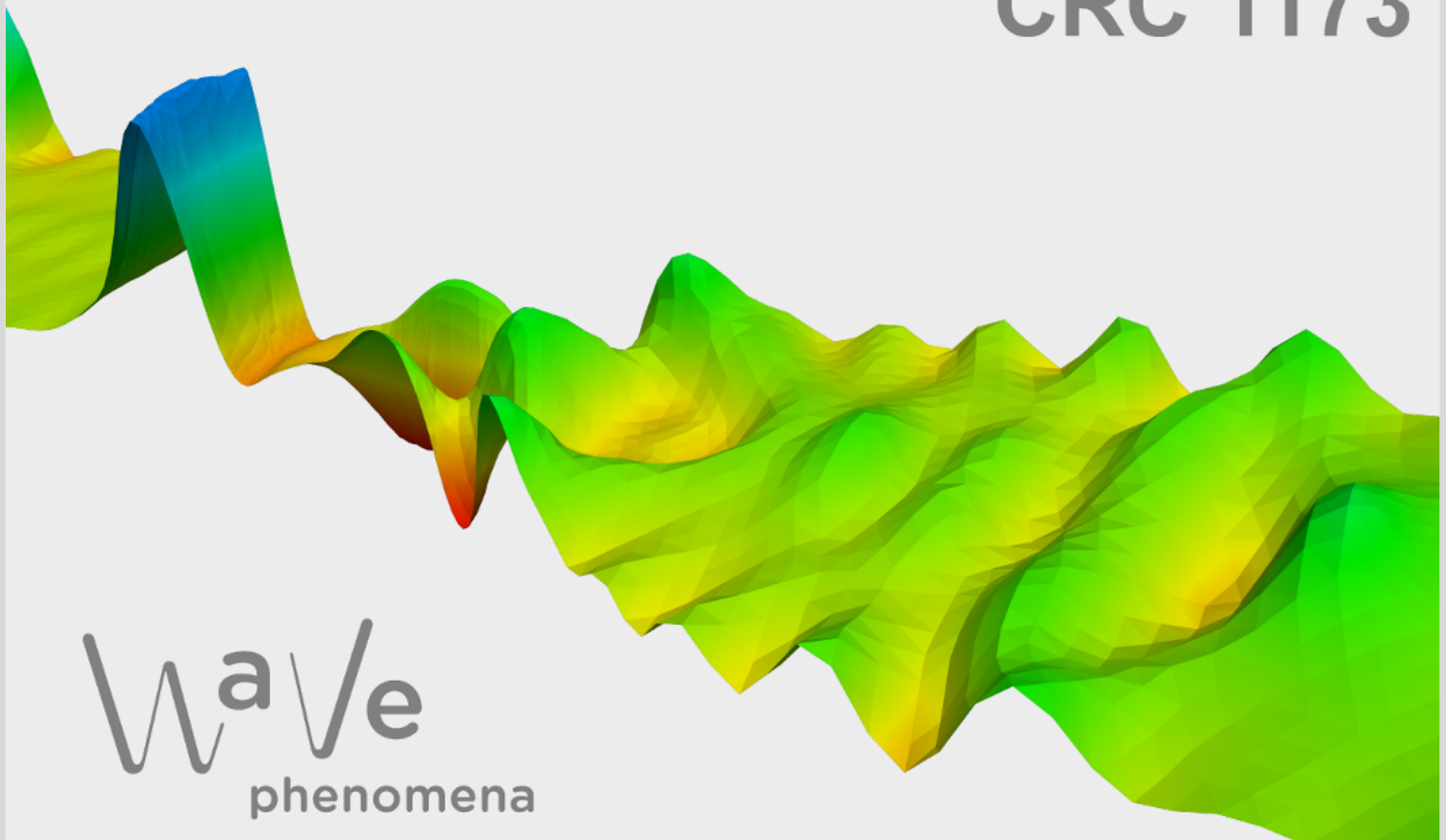
# General class of optimal Sobolev inequalities and nonlinear scalar field equations

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# GENERAL CLASS OF OPTIMAL SOBOLEV INEQUALITIES AND NONLINEAR SCALAR FIELD EQUATIONS

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ABSTRACT. We find a class of optimal Sobolev inequalities

$$\left( \int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{N}{N-2}} \geq C_{N,G} \int_{\mathbb{R}^N} G(u) dx, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N), N \geq 3,$$

where the nonlinear function  $G : \mathbb{R} \rightarrow \mathbb{R}$  satisfies general assumptions in the spirit of the fundamental works of Berestycki and Lions involving zero, positive as well as infinite mass cases. We show that any minimizer is radial up to a translation, moreover, up to a dilation, it is a least energy solution of the nonlinear scalar field equation

$$-\Delta u = g(u) \quad \text{in } \mathbb{R}^N, \quad \text{with } g = G'.$$

In particular, if  $G(u) = u^2 \log |u|$ , then the sharp constant is  $C_{N,G} := 2^* \left(\frac{N}{2}\right)^{2^*} e^{\frac{2(N-1)}{N-2}} (\pi)^{\frac{N}{N-2}}$  and  $u_\lambda(x) = e^{\frac{N-1}{2} - \frac{\lambda^2}{2}|x|^2}$  with  $\lambda > 0$  constitutes the whole family of minimizers up to translations. The above optimal inequality provides a simple proof of the classical logarithmic Sobolev inequality.

Moreover, if  $N \geq 4$ , then there is at least one nonradial solution and if, in addition,  $N \neq 5$ , then there are infinitely many nonradial solutions of the nonlinear scalar field equation. The energy functional associated with the problem may be infinite on  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  and is not Fréchet differentiable in its domain. We present a variational approach to this problem based on a new variant of Lions' lemma in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ .

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**Key words:** Nonlinear scalar field equations, logarithmic Sobolev inequality, cubic-quintic effect, critical point theory, nonradial solutions, concentration compactness, Lions' lemma, Pohozaev manifold, zero mass case, infinite mass case.

## INTRODUCTION

In view of the classical Sobolev inequality one can show that there is a constant  $C_{N,G} > 0$  such that the following inequality

$$(1.1) \quad \left( \int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{N}{N-2}} \geq C_{N,G} \int_{\mathbb{R}^N} G(u) dx$$

holds for all  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ , where  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  stands for the completion of  $C_0^\infty(\mathbb{R}^N)$  with respect to the norm  $\|u\| = \left( \int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{1}{2}}$ ,  $N \geq 3$ , and  $G$  satisfies the following assumptions

- (g0)  $g : \mathbb{R} \rightarrow \mathbb{R}$  is continuous,  $g(0) = 0$ ,  $G(s) = \int_0^s g(t) dt$ ,  $G_+(s) = \int_0^s \max\{g(t), 0\} dt$  for  $s \geq 0$  and  $G_+(s) = \int_s^0 \max\{-g(t), 0\} dt$  for  $s < 0$ .
- (g1)  $\lim_{s \rightarrow 0} G_+(s)/|s|^{2^*} = 0$ , where  $2^* = \frac{2N}{N-2}$ .
- (g2) There exists  $\xi_0 > 0$  such that  $G(\xi_0) > 0$ .
- (g3)  $\lim_{|s| \rightarrow \infty} G_+(s)/|s|^{2^*} = 0$  and  $\limsup_{|s| \rightarrow \infty} |g(s)|/|s|^{2^*-1} < \infty$ .

We show that (1.2) is optimal, that is the equality holds in (1.1) for some  $u \neq 0$ , and then  $u$  is called a *minimizer*. Observe that, if  $u$  is a minimizer, then  $u(\lambda \cdot)$  and  $u(\cdot + y)$  are minimizers for any  $\lambda > 0$  and  $y \in \mathbb{R}^N$ . The first main result reads as follows.

**Theorem 1.1.** *Suppose that (g0)–(g3) are satisfied.*

(a) *There is a radially symmetric solution  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  of*

$$(1.2) \quad -\Delta u = g(u) \quad \text{in } \mathbb{R}^N$$

*such that  $u \in \mathcal{M}$  and  $J(u) = \inf_{\mathcal{M}} J > 0$ , where  $J$  is the associated energy functional*

$$(1.3) \quad J(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} G(u) dx,$$

*and*

$$(1.4) \quad \mathcal{M} = \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} |\nabla u|^2 = 2^* \int_{\mathbb{R}^N} G(u) dx \right\}.$$

*If in addition  $g$  is odd, then  $u$  is positive.*

(b) *If  $u \in \mathcal{M}$  and  $J(u) = \inf_{\mathcal{M}} J$ , then  $u$  is a radial (up to a translation) solution of (1.2).*

(c) *The optimal constant in (1.1) is*

$$C_{N,G} = 2^* \left( \frac{1}{2} - \frac{1}{2^*} \right)^{-\frac{2}{N-2}} (\inf_{\mathcal{M}} J)^{\frac{2}{N-2}}.$$

*Moreover, if  $u \in \mathcal{M}$  and  $J(u) = \inf_{\mathcal{M}} J$ , then  $u$  is a minimizer of (1.1). If  $u$  is a minimizer of (1.1), then  $u(\lambda \cdot) \in \mathcal{M}$  and  $J(u(\lambda \cdot)) = \inf_{\mathcal{M}} J$  for a unique  $\lambda > 0$ . In particular, there is a radially symmetric solution of (1.2) such that the equality holds in (1.1).*

Using standard arguments we show that any (weak) solution  $u$  of (1.2) such that  $|G(u)| \in L^1(\mathbb{R}^N)$  satisfies the Pohozaev identity

$$(1.5) \quad \int_{\mathbb{R}^N} |\nabla u|^2 = 2^* \int_{\mathbb{R}^N} G(u) dx,$$

see Proposition 3.1. Hence  $\mathcal{M}$  contains all nontrivial finite energy solutions, and  $u$  obtained in Theorem 1.1 (a) is a *least energy solution*. Moreover if, in addition,

$$G(s) \leq -c_1 s^2 + c_2 s^{2^*}$$

for some constants  $c_1, c_2 > 0$ , for instance in the positive mass case below (1.6), then (1.5) implies that  $u \in \mathcal{M} \subset H^1(\mathbb{R}^N)$ .

If  $g$  is odd, then positive and radially symmetric solutions of (1.2) have been considered by Berestycki and Lions in their fundamental papers [5, 6] and multiplicity of radial solutions

have been given in [6, 7]. In fact, by the strong maximum principle we can solve (1.2) under the following more general growth assumption introduced in [7]:

(g3') Let  $\xi_1 := \inf\{\xi > 0 : G(\xi) > 0\}$ . If  $g(s) > 0$  for all  $s > \xi_1$ , then

$$\lim_{s \rightarrow \infty} G(s)/s^{2^*} = 0, \text{ and } \limsup_{s \rightarrow \infty} g(s)/s^{2^*-1} < \infty.$$

Namely, suppose that  $g$  is odd and satisfies (g0)–(g2) and (g3'). Similarly as in [5], we modify  $g$  in the following way. If  $g(s) > 0$  for all  $s > \xi_0$ , then  $\tilde{g} = g$ . Otherwise we set  $\xi_1 := \inf\{\xi \geq \xi_0 : g(\xi) \leq 0\}$ ,

$$\tilde{g}(s) = \begin{cases} g(s) & \text{if } 0 \leq s \leq \xi_1, \\ g(\xi_1) & \text{if } s > \xi_1, \end{cases}$$

and  $\tilde{g}(s) = -\tilde{g}(-s)$  for  $s < 0$ . Hence  $\tilde{g}$  satisfies assumptions (g0)–(g3) of Theorem 1.1 and by the strong maximum principle if  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  solves  $-\Delta u = \tilde{g}(u)$ , then  $|u(x)| \leq \xi_1$  and  $u$  is a solution of (1.2). However, it is not clear whether  $J(u) = \inf_{\mathcal{M}} J$  and  $u$  is a least energy solution. So far, a positive, radially symmetric and *least energy* solution has been obtained in [5][Theorem 3] in the *positive mass case* for the modified nonlinearity  $\tilde{g}$ . Namely, instead of (g1), we have

$$(1.6) \quad -\infty < \liminf_{s \rightarrow 0} g(s)/s \leq \limsup_{s \rightarrow 0} g(s)/s = -m < 0,$$

and after the above modification of  $g$ , in fact, it has been assumed that

$$(1.7) \quad \lim_{|s| \rightarrow \infty} g(s)/|s|^{2^*-1} = 0,$$

also in other works [17, 18, 24]. The latter condition excludes some important examples, which are taken into account in our assumptions (g0)–(g3). Indeed, take

$$(1.8) \quad g(s) = |s|^{p-2}s - |s|^{2^*-2}s - ms, \quad 2 < p < 2^*,$$

and note that  $g$  satisfies (g0)–(g3) if and only if  $m \in (0, m_0)$ , where

$$m_0 := \frac{(N-2)(2^*-p)}{N(p-2)} \left( \frac{N(p-2)}{2p} \right)^{\frac{2^*-2}{2^*-p}}.$$

Therefore we get the following result.

**Theorem 1.2.** *Suppose that  $g$  is given by (1.8).*

(a) *For any  $\omega \in (0, m_0)$  there is a positive and radially symmetric solution  $u$  of (1.2) minimizing  $J$  on  $\mathcal{M} \subset H^1(\mathbb{R}^N)$ , which is also a minimizer of (1.1).*

(b) *If  $\omega \notin (0, m_0)$ , then (1.2) has only trivial finite energy solution.*

In a particular case  $N = 3$  and  $p = 4$  we solve the cubic-quintic problem recently studied by Killip et al. in [19][Theorem 2.2.(i)]. Theorem 1.2 provides an additional information about this solution, that is,  $J(u) = \inf_{\mathcal{M}} J$  and  $u$  is a minimizer of (1.1). If  $N \geq 4$ , we also show below the existence of nonradial solutions and their multiplicity.

The relation between solutions of (1.2) and minimizers of (1.1) allows to provide a new and simple proof of the classical logarithmic Sobolev inequality given in [33]:

$$(1.9) \quad \frac{N}{4} \log \left( \frac{2}{\pi e N} \int_{\mathbb{R}^N} |\nabla u|^2 dx \right) \geq \int_{\mathbb{R}^N} |u|^2 \ln(|u|) dx, \quad \text{for } u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^2 dx = 1,$$

which is also equivalent to the Gross inequality [15]. Indeed, note that the following nonlinearity

$$(1.10) \quad G(s) = s^2 \log |s| \quad \text{for } s \neq 0, \text{ and } G(0) = 0$$

is in the *infinite mass case* and satisfies (g0)–(g3). In view of Theorem 1.1 there is a positive and radially symmetric solution of (1.2) with  $g(s) = 2s \log |s| + s$ . The Gausson [8]

$$(1.11) \quad u_1(x) = e^{\frac{N-1}{2} - \frac{1}{2}|x|^2}$$

solves (1.2) and in view of Serrin and Tang [28] (cf. [12]),  $u_1$  is a unique positive and radial solution of (1.2) up to a translation. Thus, one easily verifies that  $J(u_1) = \left(\frac{1}{2} - \frac{1}{2^*}\right) e^{N-1} \frac{N}{2} (\pi)^{\frac{N}{2}} = \inf_{\mathcal{M}} J$  and by Theorem 1.1 (c)

$$(1.12) \quad C_{N,G} := 2^* \left(\frac{N}{2}\right)^{2^*} e^{\frac{2(N-1)}{N-2}} (\pi)^{\frac{N}{N-2}}.$$

Moreover  $u_1$  is a unique minimizer of (1.1) solving (1.2) up to a translation. Now observe that (1.1) is equivalent to

$$(1.13) \quad \left( \int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{N}{N-2}} \geq C_{N,G} \max_{\alpha \in \mathbb{R}} \left\{ e^{-\alpha/2^*} \int_{\mathbb{R}^N} G(e^\alpha u) dx \right\}, \quad \text{for } u \in \mathcal{D}^{1,2}(\mathbb{R}^N),$$

and the equality holds if and only if  $u = e^\beta u_1(\lambda \cdot)$  for some  $\beta \in \mathbb{R}$ ,  $\lambda > 0$  and up to a translation. Assuming that  $\int_{\mathbb{R}^N} u^2 dx = 1$ , the maximum of the right hand side of (1.13) is attained at  $\alpha = \frac{N-2}{4} - \int_{\mathbb{R}^N} G(u) dx$ . Hence, taking into account (1.12) we verify that (1.13) is equivalent to (1.9) provided that  $\int_{\mathbb{R}^N} |u|^2 dx = 1$ . Moreover, (1.9) is sharp and the family  $\lambda^{\frac{N}{2}} u_1(\lambda \cdot)$ ,  $\lambda > 0$  are unique minimizers up to translations.

Recall that the optimality of (1.9) and the characterization of minimizers have been already proved by Carlen [10] in the context of the Gross inequality as well as by del Pino and Dolbeault [13] for  $L^p$ -Sobolev logarithmic inequality. A generalization of the optimal Gross inequality in the context of Orlicz spaces is given by Adams [1]. The optimal inequality (1.1) can be also regarded as a generalization of (1.9) and note that we do not need any structural assumptions in the Orlicz setting as in [1]. We would like to also mention that Wang and Zhang [32] have recently provided another proof of the logarithmic Sobolev inequality due to Lieb and Loss [20] based on an approximation by minimizers of the classical Sobolev inequalities.

In order to solve (1.2) under the above assumptions (g0)–(g3), we consider the associated energy functional  $J : \mathcal{D}^{1,2}(\mathbb{R}^N) \rightarrow \mathbb{R} \cup \{\infty\}$  given by (1.3) and observe that  $J$  may be infinite on a dense subset of  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ . We look for weak solutions of (1.2), i.e.  $J'(u)(v) = 0$  for any  $v \in C_0^\infty(\mathbb{R}^N)$ , however,  $J$  cannot be Fréchet differentiable and this is the first main difficulty in comparison to the positive mass case (1.6) studied e.g. in [5, 6, 17, 18, 24]. Note that in the positive mass case and under assumption (1.7),  $J$  is well-defined, of class  $\mathcal{C}^1$  on  $H^1(\mathbb{R}^N)$

and Jeanjean and Tanaka [17] showed that the least energy solution obtained in [5] minimizes the energy on the *Pohozaev manifold*  $\mathcal{M}$  defined by (1.4) in  $H^1(\mathbb{R}^N)$ . This result has been proved directly in [24] by a critical point theory developed therein. In Theorem 1.1 (a) we prove that there is a least energy solution minimizing  $J$  on the Pohozaev manifold  $\mathcal{M}$  under more general assumptions (g0)–(g3) including also the zero mass case ( $m = 0$ ) as well as the infinite mass case ( $m = \infty$ ), e.g. (1.10). We also present a new and simple approach of finding minimizers on  $\mathcal{M}$ , see Lemma 3.3, which is equivalent to finding minimizers of (1.1).

Note that in [24] we have indeed studied the positive mass case, and if  $N \geq 4$  we have found nonradial solutions and answered to the open problem [6][Section 10.8] concerning the existence and multiplicity of nonradial solutions of (1.2). Moreover Jeanjean and Lu [18] have recently provided a mountain pass approach and reproved the main results from [24] based on the monotonicity trick [16]. Therefore, our next aim is to show that the similar results hold under assumptions (g0)–(g3) and we give an answer to this problem also in the zero mass case as well as in the infinite mass case.

Namely, let  $N \geq 4$  and similarly as in [4], let us fix  $\tau \in \mathcal{O}(N)$  such that  $\tau(x_1, x_2, x_3) = (x_2, x_1, x_3)$  for  $x_1, x_2 \in \mathbb{R}^m$  and  $x_3 \in \mathbb{R}^{N-2m}$ , where  $x = (x_1, x_2, x_3) \in \mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{N-2m}$  and  $2 \leq m \leq N/2$ . We define

$$(1.14) \quad X_\tau := \{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) : u(x) = -u(\tau x) \text{ for all } x \in \mathbb{R}^N\}.$$

Clearly, if  $u \in X_\tau$  is radial, i.e.  $u(x) = u(\rho x)$  for any  $\rho \in \mathcal{O}(N)$ , then  $u = 0$ . Hence  $X_\tau$  does not contain nontrivial radial functions. Then  $\mathcal{O}_1 := \mathcal{O}(m) \times \mathcal{O}(m) \times \text{id} \subset \mathcal{O}(N)$  acts isometrically on  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  and let  $\mathcal{D}_{\mathcal{O}_1}^{1,2}(\mathbb{R}^N)$  denote the subspace of invariant functions with respect to  $\mathcal{O}_1$ .

**Theorem 1.3.** *If  $N \geq 4$ , then there is a solution  $u \in \mathcal{M} \cap X_\tau \cap \mathcal{D}_{\mathcal{O}_1}^{1,2}(\mathbb{R}^N)$  of (1.2) such that*

$$(1.15) \quad J(u) = \inf_{\mathcal{M} \cap X_\tau \cap \mathcal{D}_{\mathcal{O}_1}^{1,2}(\mathbb{R}^N)} J > 2 \inf_{\mathcal{M}} J.$$

Clearly, we infer that problem (1.2) with (1.8) or with (1.10) has a nonradial solution for  $N \geq 4$ . If, in addition,  $N \neq 5$ , then we find infinitely many nonradial solutions. Indeed, we may assume that  $N - 2m \neq 1$  and let us consider  $\mathcal{O}_2 := \mathcal{O}(m) \times \mathcal{O}(m) \times \mathcal{O}(N - 2m) \subset \mathcal{O}(N)$  acting isometrically on  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  with the subspace of invariant function denoted by  $\mathcal{D}_{\mathcal{O}_2}^{1,2}(\mathbb{R}^N)$ .

**Theorem 1.4.** *If  $N \geq 4$  and  $N \neq 5$ , then the following statements hold.*

(a) *There is a solution  $u \in \mathcal{M} \cap X_\tau \cap \mathcal{D}_{\mathcal{O}_2}^{1,2}(\mathbb{R}^N)$  of (1.2) such that*

$$(1.16) \quad J(u) = \inf_{\mathcal{M} \cap X_\tau \cap \mathcal{D}_{\mathcal{O}_2}^{1,2}(\mathbb{R}^N)} J \geq \inf_{\mathcal{M} \cap X_\tau \cap H_{\mathcal{O}_1}^1(\mathbb{R}^N)} J.$$

(b) *If, in addition,  $g$  is odd, then there is an infinite sequence of solutions  $(u_n) \subset \mathcal{M} \cap X_\tau \cap \mathcal{D}_{\mathcal{O}_2}^{1,2}(\mathbb{R}^N)$  of (1.2) such that  $J(u_n) \rightarrow \infty$  as  $n \rightarrow \infty$ .*

Note that there is little work on the problem (1.2) involving the zero or infinite mass case expressed by general assumptions without Ambrosetti-Rabinowitz-type condition [2], or

any monotonicity behaviour. The first difficulty is that  $J$  may be infinite and is not Fréchet differentiable in its domain. The second one is related with the lack of compactness of the problem in  $\mathbb{R}^N$ ; even if we find a Palais-Smale sequence, we do not know whether the sequence is bounded and contains a (weakly) convergent subsequence. Berestycki and Lions in [5] minimized  $u \mapsto \int_{\mathbb{R}^N} |\nabla u|^2 dx$  on the constraint of radial functions such that  $|G(u)| \in L^1(\mathbb{R}^N)$  and  $\int_{\mathbb{R}^N} G(u) dx = 1$ . In order to get multiplicity of solutions they approximated the zero mass case  $g$  by suitable functions  $g_\varepsilon$  in the positive mass case, i.e.  $-g'_\varepsilon(0) > 0$  and  $g_\varepsilon \rightarrow g$  uniformly on compact subsets of  $\mathbb{R}$  as  $\varepsilon \rightarrow 0^+$ . Using results of [6] they solved the approximated problem in the positive mass case. Letting  $\varepsilon \rightarrow 0$ , a sequence of radial solutions of (1.2) have been obtained. Another approach based on approximations of  $\mathcal{D}_{\mathcal{O}(N)}^{1,2}(\mathbb{R}^N)$  by  $\{u \in \mathcal{D}_{\mathcal{O}(N)}^{1,2}(\mathbb{R}^N) : u(x) = 0 \text{ for } |x| \geq L\}$  for  $L \rightarrow \infty$  is due to Struwe [30]. Observe that in all these works the radial symmetry plays an important role, since one gets the uniform decay at infinity of functions from  $\mathcal{D}_{\mathcal{O}(N)}^{1,2}(\mathbb{R}^N)$  (see [5][Radial Lemma A.III]) and the compactness lemma of Strauss [5][Lemma A.I] is applicable. In the nonradial setting these arguments are no longer available.

Now we sketch our approach with a new and simple approximation  $J_\varepsilon$  of  $J$ . Let  $g_+(s) = G'_+(s)$ ,  $g_-(s) := g_+(s) - g(s)$  and  $G_-(s) := G_+(s) - G(s) \geq 0$  for  $s \in \mathbb{R}$ . In view of (g3),  $G_+(u) \in L^1(\mathbb{R}^N)$  for  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N)$ , however  $G_-(u)$  may not be integrable unless  $G_-(u) \leq c|u|^{2^*}$  for some  $c > 0$ . In order to overcome this problem, for any  $\varepsilon \in [0, 1)$  let us take any even function  $\varphi_\varepsilon : \mathbb{R} \rightarrow [0, 1]$  of class  $\mathcal{C}^1$  such that  $\varphi_\varepsilon(s) = \frac{1}{2^*}|s|^{2^*}$  for  $|s| \leq \varepsilon$ ,  $\varphi_\varepsilon(s) = 1$  for  $|s| \geq 2\varepsilon$ , and  $|\varphi'_\varepsilon(s)| \leq \varepsilon^{2^*-1}$  for any  $s \in \mathbb{R}$ . Moreover we may assume that  $\varphi_\varepsilon(s) \geq \varphi_{1/2}(s)$  for  $s \in \mathbb{R}$  and  $\varepsilon \in [0, 1/2]$ . We introduce a new functional

$$(1.17) \quad J_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} \varphi_\varepsilon(u) G_-(u) dx - \int_{\mathbb{R}^N} G_+(u) dx,$$

and now observe that  $\varphi_\varepsilon(u) G_-(s) \leq c_\varepsilon |s|^{2^*}$  for any  $s \in \mathbb{R}$  and some constant  $c_\varepsilon > 0$  depending on  $\varepsilon > 0$ . Hence, for  $\varepsilon \in (0, 1)$ ,  $J_\varepsilon$  is well-defined on  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  and we easily check that  $J_\varepsilon$  is of class  $\mathcal{C}^1$ . We show that any minimizing sequence of  $J_\varepsilon$  on the following Pohozaev manifold

$$(1.18) \quad \mathcal{M}_\varepsilon = \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} |\nabla u|^2 = 2^* \int_{\mathbb{R}^N} G_+(u) - \varphi_\varepsilon(u) G_-(u) dx \right\}$$

converges to a nontrivial critical point  $u_\varepsilon$  of  $J_\varepsilon$  up to a subsequence and up to a translation – see Lemma 3.3. The last argument requires the following variant of the classical Lions' lemma [22], [34][Lemma 1.21] applied to  $\Psi = G_+$  satisfying (1.20).

**Lemma 1.5.** *Suppose that  $(u_n) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$  is bounded and for some  $r > 0$*

$$(1.19) \quad \lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y,r)} |u_n|^2 dx = 0.$$

*Then*

$$\int_{\mathbb{R}^N} \Psi(u_n) dx \rightarrow 0 \quad \text{as } n \rightarrow \infty$$



for any continuous  $\Psi : \mathbb{R} \rightarrow [0, \infty)$  satisfying

$$(1.20) \quad \lim_{s \rightarrow 0} \frac{\Psi(s)}{|s|^{2^*}} = \lim_{|s| \rightarrow \infty} \frac{\Psi(s)}{|s|^{2^*}} = 0.$$

Note that concentration-compactness arguments in the zero mass case have been considered so far in more restrictive settings e.g. in [11][Lemma 3.5] or [3][Lemma 2], where one has to require that  $\Psi(s) \leq c \min\{|s|^p, |s|^q\}$  for some  $2 < p < 2^* < q$  and constant  $c > 0$ . Condition (1.20) seems to be optimal and we prove Lemma 1.5 in Section 2, see also Lemma 2.1.

Having found a critical point  $u_\varepsilon \in \mathcal{M}_\varepsilon$  of the approximated functional  $J_\varepsilon$  we let  $\varepsilon \rightarrow 0$  and passing to a subsequence we obtain a solution of (1.2) in Theorem 1.1. Next, repeating the similar arguments, we prove Theorem 1.3 as well as Theorem 1.4 (a) in the nonradial setting. Note that this is a simpler approach in comparison to [18, 24] and it seems that we cannot argue directly as in these papers, since we do not require (1.6) and (1.7), which are crucial for decompositions of Palais-Smale sequences in [18] and for the variant of Palais-Smale condition [24][ $(M)_\beta(i)$ ]. We expect that our approach based on minimization on a Pohozaev manifold with Lemma 3.3 as well as Lions' type results in the spirit of Lemma 1.5 allows to study other nonlinear elliptic problems involving general nonlinearities.

In order to prove the multiplicity result in Theorem 1.4 (b), we employ the critical point theory from [24][Section 2]. Namely we observe that there is a homeomorphism  $m : \mathcal{U} \rightarrow \mathcal{M}_\varepsilon$  such that

$$\mathcal{U} := \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u|^2 dx = 1 \text{ and } \int_{\mathbb{R}^N} G_+(u) - \varphi_\varepsilon(u)G_-(u) dx > 0 \right\}.$$

We show that  $J_\varepsilon \circ m : \mathcal{U} \rightarrow \mathbb{R}$  is still of class  $\mathcal{C}^1$ . The advantage of working with  $J_\varepsilon \circ m$  is that  $\mathcal{U}$  is an open subset of a manifold of class  $\mathcal{C}^{1,1}$  and we can use a critical point theory based on the deformation lemma involving a Cauchy problem on  $\mathcal{U}$ . This is not feasible on  $\mathcal{M}_\varepsilon$ , since  $\mathcal{M}_\varepsilon$  need not be of class  $\mathcal{C}^{1,1}$ . We show that  $J_\varepsilon \circ m$  satisfies the Palais-Smale condition in  $\mathcal{U} \cap \mathcal{D}_{\mathcal{O}_2}^{1,2}(\mathbb{R}^N)$  and we find an unbounded sequence of critical points. This requires a next approximation of  $J_\varepsilon$  described in Section 4. Similarly as above, letting  $\varepsilon \rightarrow 0$  we prove Theorem 1.4 (b). Based on this work, under assumptions (g0)–(g3) one can obtain an unbounded sequence of radial solutions in  $\mathcal{M} \cap \mathcal{D}_{\mathcal{O}(N)}^{1,2}(\mathbb{R}^N)$ , which was considered in [7, 30], however by means of different techniques, in particular without the radial lemma of Strauss [5, 29] – we leave details for the reader.

## 2. CONCENTRATION-COMPACTNESS IN SUBSPACES OF $\mathcal{D}^{1,2}(\mathbb{R}^N)$

**Lemma 2.1.** *Suppose that  $(u_n) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$  is bounded. Then  $u_n(\cdot + y_n) \rightarrow 0$  in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  for any  $(y_n) \subset \mathbb{Z}^N$  if and only if*

$$(2.1) \quad \int_{\mathbb{R}^N} \Psi(u_n) dx \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for any continuous  $\Psi : \mathbb{R} \rightarrow [0, \infty)$  satisfying (1.20).

*Proof.* Let  $(u_n) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$  be such that  $u_n(\cdot + y_n) \rightharpoonup 0$  in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  for any  $(y_n) \subset \mathbb{Z}^N$ . Take any  $\varepsilon > 0$  and  $2 < p < 2^*$  and suppose that  $\Psi$  satisfies (1.20). Then we find  $0 < \delta < M$  and  $c_\varepsilon > 0$  such that

$$\begin{aligned}\Psi(s) &\leq \varepsilon |s|^{2^*} && \text{for } |s| \leq \delta, \\ \Psi(s) &\leq \varepsilon |s|^{2^*} && \text{for } |s| > M, \\ \Psi(s) &\leq c_\varepsilon |s|^p && \text{for } |s| \in (\delta, M].\end{aligned}$$

Let us define  $w_n(x) := |u_n(x)|$  for  $|u_n(x)| > \delta$  and  $w_n(x) := |u_n(x)|^{2^*/2} \delta^{1-2^*/2}$  for  $|u_n(x)| \leq \delta$ . Then  $(w_n)$  is bounded in  $H^1(\mathbb{R}^N)$  and by the Sobolev inequality one has

$$\begin{aligned}\int_{\Omega+y} \Psi(u_n) dx &= \int_{(\Omega+y) \cap \{\delta < |u_n| \leq M\}} \Psi(u_n) dx + \int_{(\Omega+y) \cap (\{|u_n| > M\} \cup \{|u_n| \leq \delta\})} \Psi(u_n) dx \\ &\leq c_\varepsilon \int_{(\Omega+y) \cap \{\delta < |u_n| \leq M\}} |w_n|^p dx + \varepsilon \int_{(\Omega+y) \cap (\{|u_n| > M\} \cup \{|u_n| \leq \delta\})} |u_n|^{2^*} dx \\ &\leq c_\varepsilon C \left( \int_{\Omega+y} |\nabla w_n|^2 + |w_n|^2 dx \right) \left( \int_{\Omega+y} |w_n|^p dx \right)^{1-2/p} + \varepsilon \int_{\Omega+y} |u_n|^{2^*} dx,\end{aligned}$$

for every  $y \in \mathbb{R}^N$ , where  $\Omega = (0, 1)^N$  and  $C > 0$  is a constant. Then we sum the inequalities over  $y \in \mathbb{Z}^N$  and we get

$$\int_{\mathbb{R}^N} \Psi(u_n) dx \leq c_\varepsilon C \left( \int_{\mathbb{R}^N} |\nabla w_n|^2 + |w_n|^2 dx \right) \left( \sup_{y \in \mathbb{Z}^N} \int_{\Omega} |w_n(\cdot + y)|^p dx \right)^{1-2/p} + \varepsilon \int_{\mathbb{R}^N} |u_n|^{2^*} dx.$$

Let us take  $(y_n) \subset \mathbb{Z}^N$  such that

$$\sup_{y \in \mathbb{Z}^N} \int_{\Omega} |w_n(\cdot + y)|^p dx \leq 2 \int_{\Omega} |w_n(\cdot + y_n)|^p dx$$

for any  $n \geq 1$ . Note that  $u_n(\cdot + y_n) \rightharpoonup 0$  in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  and passing to a subsequence we obtain  $u_n(\cdot + y_n) \rightarrow 0$  in  $L^p(\Omega)$ . Since  $|w_n(x)| \leq |u_n(x)|$ , we infer that  $w_n(\cdot + y_n) \rightarrow 0$  in  $L^p(\Omega)$ . Therefore

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \Psi(u_n) dx \leq \varepsilon \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{2^*} dx,$$

and since  $\varepsilon > 0$  is arbitrary, we conclude (2.1). On the other hand, suppose that  $u_n(\cdot + y_n)$  does not converges to 0 for some  $(y_n) \subset \mathbb{Z}^N$  and (2.1) holds. We may assume that  $u_n(\cdot + y_n) \rightarrow u_0 \neq 0$  in  $L^p(\Omega)$  for some bounded domain  $\Omega \subset \mathbb{R}^N$  and  $2 < p < 2^*$ . Take any  $\varepsilon > 0$ ,  $q > 2^*$  and  $\Psi(s) := \min\{|s|^p, \varepsilon^{p-q}|s|^q\}$  for  $s \in \mathbb{R}$ . Then

$$\begin{aligned}\int_{\mathbb{R}^N} \Psi(u_n) dx &\geq \int_{\Omega \cap \{|u_n| \geq \varepsilon\}} |u_n|^p dx + \int_{\Omega \cap \{|u_n| \leq \varepsilon\}} \varepsilon^{q-p} |u_n|^q dx \\ &= \int_{\Omega} |u_n|^p dx + \int_{\Omega \cap \{|u_n| \leq \varepsilon\}} \varepsilon^{p-q} |u_n|^q - |u_n|^p dx \\ &\geq \int_{\Omega} |u_n|^p dx - 2\varepsilon^p |\Omega|.\end{aligned}$$

Thus we get  $u_n \rightarrow 0$  in  $L^p(\Omega)$  and this contradicts  $u_0 \neq 0$ .  $\square$

*Proof of Lemma 1.5.* Suppose that there is  $(y_n) \subset \mathbb{Z}^N$  such that  $u_n(\cdot + y_n)$  does not converge weakly to 0 in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$ . Since  $u_n(\cdot + y_n)$  is bounded, then there is  $u_0 \neq 0$  such that, up to a subsequence,

$$u_n(\cdot + y_n) \rightharpoonup u_0$$

as  $n \rightarrow \infty$ . We find  $y \in \mathbb{R}^N$  such that  $u_0 \chi_{B(y,r)} \neq 0$  in  $L^2(B(y,r))$ . Note that passing to a subsequence  $u_n(\cdot + y_n) \rightarrow u_0$  in  $L^2(B(y,r))$ . Then, in view of (1.19)

$$\int_{B(y,r)} |u_n(\cdot + y_n)|^2 dx = \int_{B(y_n+y,r)} |u_n|^2 dx \rightarrow 0$$

as  $n \rightarrow \infty$ , which contradicts the fact  $u_n(\cdot + y_n) \rightarrow u_0 \neq 0$  in  $L^2(B(y,r))$ . Therefore  $u_n(\cdot + y_n) \rightharpoonup 0$  in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  for any  $(y_n) \subset \mathbb{Z}^N$  and by Lemma 2.1 we conclude.  $\square$

Let us consider  $x = (x^1, x^2, x^3) \in \mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{N-2m}$  with  $2 \leq m \leq N/2$  such that  $x^1, x^2 \in \mathbb{R}^m$  and  $x^3 \in \mathbb{R}^{N-2m}$ . Let  $\mathcal{O}_1 = \mathcal{O}(m) \times \mathcal{O}(m) \times \text{id} \subset \mathcal{O}(N)$ . Then for  $\mathcal{O}_1$  invariant functions we get the following corollary, which proof is postponed to Appendix and follows from Proposition A.2.

**Corollary 2.2.** *Suppose that  $(u_n) \subset \mathcal{D}_{\mathcal{O}_1}^{1,2}(\mathbb{R}^N)$  is bounded,  $r_0 > 0$  is such that for all  $r \geq r_0$*

$$(2.2) \quad \lim_{n \rightarrow \infty} \sup_{z \in \mathbb{R}^{N-2m}} \int_{B((0,0,z),r)} |u_n|^2 dx = 0.$$

Then

$$\int_{\mathbb{R}^N} \Psi(u_n) dx \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for any continuous function  $\Psi : \mathbb{R} \rightarrow [0, \infty)$  such that (1.20) holds.

### 3. PROOFS OF THEOREM 1.1 AND THEOREM 1.2

We prove the following Pohozaev type result using a truncation argument due to Kavian, cf. [31][Lemma 3.5] and [34][Theorem B.3].

**Proposition 3.1.** *Let  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  be a weak solution of (1.2). Then  $u \in W_{loc}^{2,q}(\mathbb{R}^N)$  for any  $q < +\infty$ , and*

$$(3.1) \quad \int_{\mathbb{R}^N} |\nabla u|^2 dx = 2^* \int_{\mathbb{R}^N} G(u) dx$$

provided that  $G_-(u) \in L^1(\mathbb{R}^N)$ .

*Proof.* Since

$$|g(u)| \leq c(1 + |u|^{2^*-1})$$

for  $u \in \mathbb{R}$  and for some constant  $c > 0$ , by Brezis and Kato theorem [9] we infer that  $u \in W_{loc}^{2,q}(\mathbb{R}^N)$  for any  $q < +\infty$ . Let  $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$  be such that  $0 \leq \varphi \leq 1$ ,  $\varphi(r) = 1$  for  $r \leq 1$

and  $\varphi(r) = 0$  for  $r \geq 2$ . Similarly as in [34][Theorem B.3] we define  $\varphi_n \in \mathcal{C}_0^\infty(\mathbb{R}^N)$  by the following formula

$$\varphi_n(x) = \varphi\left(\frac{|x|^2}{n^2}\right).$$

Then there exists  $C > 0$  such that

$$\varphi_n(x) \leq C, \text{ and } |x||\nabla\varphi_n(x)| \leq C$$

for every  $n$  and  $x \in \mathbb{R}^N$ . Recall that

$$\begin{aligned} \Delta u \varphi_n \langle x, \nabla u \rangle &= \operatorname{div} \left( \varphi_n (\nabla u \langle x, \nabla u \rangle - x \frac{|\nabla u|^2}{2}) \right) + \frac{N-2}{2} \varphi_n |\nabla u|^2 \\ &\quad - \langle \nabla \varphi_n, \nabla u \rangle \langle x, \nabla u \rangle + \langle \nabla \varphi_n, x \rangle \frac{|\nabla u|^2}{2}. \end{aligned}$$

Then by the divergence theorem it is standard to show that

$$\begin{aligned} \frac{N-2}{2} \int_{\mathbb{R}^N} \varphi_n |\nabla u|^2 dx &= \int_{\mathbb{R}^N} -\langle \nabla \varphi_n, \nabla u \rangle \langle x, \nabla u \rangle + \langle \nabla \varphi_n, x \rangle \frac{|\nabla u|^2}{2} dx \\ &\quad + N \int_{\mathbb{R}^N} \varphi_n G(u) dx + \int_{\mathbb{R}^N} \langle \nabla \varphi_n, x \rangle G(u) dx. \end{aligned}$$

Since  $\langle \nabla \varphi_n, x \rangle$  is bounded,  $\langle \nabla \varphi_n, x \rangle \rightarrow 0$  as  $n \rightarrow \infty$  and  $G(u) \in L^1(\mathbb{R}^N)$ , then by the Lebesgue dominated convergence theorem we get

$$\int_{\mathbb{R}^N} -\langle \nabla \varphi_n, \nabla u \rangle \langle x, \nabla u \rangle + \langle \nabla \varphi_n, x \rangle \frac{|\nabla u|^2}{2} dx + \int_{\mathbb{R}^N} \langle \nabla \varphi_n, x \rangle G(u) dx \rightarrow 0$$

as  $n \rightarrow \infty$ . Since  $\varphi_n(x) \rightarrow 1$  and we get the required equality.  $\square$

Let  $X := \mathcal{D}^{1,2}(\mathbb{R}^N)$  and we set  $G_\varepsilon(s) := G_+(s) - \varphi_\varepsilon(u)G_-(s)$ ,  $g_\varepsilon(s) := G'_\varepsilon(s)$  for  $s \in \mathbb{R}$ . Note that there is  $c > 0$  such that  $|g_\varepsilon(s)| \leq c|s|^{2^*-1}$  for  $s \in \mathbb{R}$ , which implies that  $J_\varepsilon$  is of class  $\mathcal{C}^1$ . Moreover let

$$\begin{aligned} \mathcal{M}_\varepsilon &:= \left\{ u \in X : \int_{\mathbb{R}^N} |\nabla u|^2 - 2^* \int_{\mathbb{R}^N} G_\varepsilon(u) dx = 0 \right\}, \\ \mathcal{S} &:= \left\{ u \in X : \|u\| = 1 \right\}, \\ \mathcal{P} &:= \left\{ u \in X : \int_{\mathbb{R}^N} G_\varepsilon(u) dx > 0 \right\}, \\ \mathcal{U} &:= \mathcal{S} \cap \mathcal{P}. \end{aligned}$$

**Proposition 3.2.** *The following holds for  $\varepsilon > 0$ .*

(i)  $\mathcal{P}$  is open and nonempty. Moreover there is a map  $m_{\mathcal{P}} : \mathcal{P} \rightarrow \mathcal{M}_\varepsilon$  such that  $m_{\mathcal{P}}(u) = u(r \cdot) \in \mathcal{M}_\varepsilon$  with

$$(3.2) \quad r = r(u) = \frac{\left( 2^* \int_{\mathbb{R}^N} G_\varepsilon(u) dx \right)^{1/2}}{\|u\|} > 0.$$

(ii)  $m := m_{\mathcal{P}}|_{\mathcal{U}} : \mathcal{U} \rightarrow \mathcal{M}_\varepsilon$  is a homeomorphism with the inverse  $m^{-1}(u) = u(\|u\|^{\frac{2}{N-2}} \cdot)$ ,  $J_\varepsilon \circ m_{\mathcal{P}} : \mathcal{P} \rightarrow \mathbb{R}$  is of class  $\mathcal{C}^1$  and

$$\begin{aligned} (J_\varepsilon \circ m_{\mathcal{P}})'(u)(v) &= J'_\varepsilon(m_{\mathcal{P}}(u))(v(r(u) \cdot)) \\ &= r(u)^{2-N} \int_{\mathbb{R}^N} \langle \nabla u, \nabla v \rangle dx - r(u)^{-N} \int_{\mathbb{R}^N} g_\varepsilon(u)v dx \end{aligned}$$

for  $u \in \mathcal{P}$  and  $v \in X$ .

(iii)  $J_\varepsilon$  is coercive on  $\mathcal{M}_\varepsilon$ , i.e. for  $(u_n) \subset \mathcal{M}_\varepsilon$ ,  $J_\varepsilon(u_n) \rightarrow \infty$  as  $\|u_n\| \rightarrow \infty$ , and

$$(3.3) \quad c_\varepsilon := \inf_{\mathcal{M}_\varepsilon} J_\varepsilon = \inf_{\mathcal{U}} J_\varepsilon \circ m > 0.$$

(iv) If  $u_n \rightarrow u$ ,  $u_n \in \mathcal{U}$  and  $u \in \partial\mathcal{U}$ , where the boundary of  $\mathcal{U}$  is taken in  $\mathcal{S}$ , then  $(J_\varepsilon \circ m)(u) \rightarrow \infty$  as  $n \rightarrow \infty$ .

*Proof.* Similarly as in [5][page 325] or in [24][Remark 4.2] we check that  $\mathcal{P} \neq \emptyset$ . Next, we easily verify (i)–(iv), e.g. arguing as in the positive mass case in [24][Proposition 4.1].  $\square$

The following lemma is crucial and allows to avoid the analysis of decompositions of Palais-Smale sequences required in [18, 24].

**Lemma 3.3.** *Suppose that  $(u_n) \subset \mathcal{M}_\varepsilon$ ,  $J_\varepsilon(u_n) \rightarrow c_\varepsilon$  and*

$$u_n \rightharpoonup \tilde{u} \neq 0 \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^N), \quad u_n(x) \rightarrow \tilde{u}(x) \text{ for a.e. } x \in \mathbb{R}^N$$

*for some  $\tilde{u} \in X$ . Then  $u_n \rightarrow \tilde{u}$ ,  $\tilde{u}$  is a critical point of  $J_\varepsilon$  and  $J_\varepsilon(\tilde{u}) = c_\varepsilon$ .*

*Proof.* Take any  $v \in X$  and observe that by the Vitaly convergence theorem

$$\begin{aligned} (3.4) \quad & \liminf_{n \rightarrow \infty} \lim_{t \rightarrow 0} \frac{1}{t} \left( \left( 2^* \int_{\mathbb{R}^N} G_\varepsilon(u_n + tv) dx \right)^{\frac{N-2}{N}} - \left( 2^* \int_{\mathbb{R}^N} G_\varepsilon(u_n) dx \right)^{\frac{N-2}{N}} \right) \\ &= \liminf_{n \rightarrow \infty} \frac{N-2}{N} \left( 2^* \int_{\mathbb{R}^N} G_\varepsilon(u_n) dx \right)^{-\frac{2}{N}} \left( 2^* \int_{\mathbb{R}^N} g_\varepsilon(u_n)(v) dx \right) \\ &\geq \liminf_{n \rightarrow \infty} \frac{N-2}{N} \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \right)^{-\frac{2}{N}} \left( 2^* \int_{\mathbb{R}^N} g_\varepsilon(\tilde{u})(v) dx \right) \\ &= \frac{N-2}{N} \left( \frac{1}{2} - \frac{1}{2^*} \right)^{\frac{2}{N}} c_\varepsilon^{-\frac{2}{N}} \left( 2^* \int_{\mathbb{R}^N} g_\varepsilon(\tilde{u})v dx \right), \end{aligned}$$

where the last equality holds, since  $u_n \in \mathcal{M}_\varepsilon$  and

$$J_\varepsilon(u_n) = \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \rightarrow c_\varepsilon.$$

Moreover

$$(3.5) \quad c_\varepsilon^{\frac{2}{N}} \left( 2^* \int_{\mathbb{R}^N} G(u_n) dx \right)^{\frac{N-2}{N}} = c_\varepsilon^{\frac{2}{N}} \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \right)^{\frac{N-2}{N}} = \left( \frac{1}{2} - \frac{1}{2^*} \right)^{\frac{2}{N}} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + o(1).$$

Then we find  $t_n \rightarrow 0$  such that  $u_n + t_n v \in \mathcal{P}$ , and observe that  $J_\varepsilon(m_{\mathcal{P}}(u_n + t_n v)) \geq c_\varepsilon$ , that is

$$r(u_n + t_n v)^{2-N} \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |\nabla(u_n + t_n v)|^2 dx \geq c_\varepsilon.$$

Hence

$$\left( \frac{1}{2} - \frac{1}{2^*} \right)^{\frac{2}{N}} \int_{\mathbb{R}^N} |\nabla(u_n + t_n v)|^2 dx \geq c_\varepsilon^{\frac{2}{N}} \left( 2^* \int_{\mathbb{R}^N} G_\varepsilon(u_n + t_n v) dx \right)^{\frac{N-2}{N}}$$

and by (3.4), (3.5) we obtain

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \left( \frac{1}{2} - \frac{1}{2^*} \right)^{\frac{2}{N}} \left( 2 \int_{\mathbb{R}^N} \langle \nabla u_n, \nabla v \rangle dx + t_n \int_{\mathbb{R}^N} |\nabla v|^2 dx \right) \\ & \geq \liminf_{n \rightarrow \infty} \frac{1}{t_n} c_\varepsilon^{\frac{2}{N}} \left( \left( 2^* \int_{\mathbb{R}^N} G(u_n + t_n v) dx \right)^{\frac{N-2}{N}} - \left( 2^* \int_{\mathbb{R}^N} G(u_n) dx \right)^{\frac{N-2}{N}} \right) \\ & \geq \liminf_{n \rightarrow \infty} \frac{N-2}{N} \left( \frac{1}{2} - \frac{1}{2^*} \right)^{\frac{2}{N}} \left( 2^* \int_{\mathbb{R}^N} g_\varepsilon(\tilde{u})v dx \right). \end{aligned}$$

Thus

$$\int_{\mathbb{R}^N} \langle \nabla \tilde{u}, \nabla v \rangle dx \geq \int_{\mathbb{R}^N} g_\varepsilon(\tilde{u})v dx$$

for any  $v \in X$  and we infer that  $\tilde{u}$  is a critical point of  $J_\varepsilon$ . In view of the Pohozaev identity (cf. Proposition 3.1),  $\tilde{u} \in \mathcal{M}_\varepsilon$ ,  $m_{\mathcal{P}}(\tilde{u}) = \tilde{u}$  and

$$c_\varepsilon \leq J(\tilde{u}) = \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 dx \leq \liminf_{n \rightarrow \infty} \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |\nabla u_n|^2 dx = c_\varepsilon.$$

Therefore  $\|u_n\| \rightarrow \|\tilde{u}\|$  and  $u_n \rightarrow \tilde{u}$ .  $\square$

*Proof of Theorem 1.1.* (a) Let  $(u_n) \subset \mathcal{M}_\varepsilon$  be a minimizing sequence of  $J_\varepsilon$ . i.e.  $J_\varepsilon(u_n) \rightarrow c_\varepsilon$ . Since  $J_\varepsilon$  is coercive on  $\mathcal{M}_\varepsilon$ ,  $(u_n)$  is bounded. Observe that

$$(3.6) \quad \begin{aligned} 2^* \int_{\mathbb{R}^N} G_+(u_n) dx & \geq \int_{\mathbb{R}^N} |\nabla u_n|^2 dx = \left( \frac{1}{2} - \frac{1}{2^*} \right)^{-1} c_\varepsilon + o(1), \\ \lim_{s \rightarrow 0} G_+(s)/s^{2^*} & = \lim_{|s| \rightarrow \infty} G_+(s)/s^{2^*} = 0, \end{aligned}$$

and in view of Lemma 1.5, (1.19) is not satisfied. Therefore, passing to a subsequence, we find  $u_\varepsilon \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  and  $(y_n) \subset \mathbb{R}^N$  such that

$$u_n(\cdot + y_n) \rightharpoonup u_\varepsilon \neq 0 \quad \text{and} \quad u_n(x + y_n) \rightarrow u_\varepsilon(x)$$

for a.e.  $x \in \mathbb{R}^N$  as  $n \rightarrow \infty$ . By Lemma 3.3 we infer that  $u_\varepsilon \in \mathcal{M}_\varepsilon$  is a critical point of  $J_\varepsilon$  at level  $c_\varepsilon$ . Now we let  $\varepsilon \rightarrow 0$  and in order to avoid confusion with notation, we denote the dependence of  $\mathcal{P}$  and  $m_{\mathcal{P}}$  on  $\varepsilon$  by  $\mathcal{P}_\varepsilon$  and  $m_{\mathcal{P}_\varepsilon}$  respectively. Take any  $u \in \mathcal{M}$  and observe that

$$(3.7) \quad \begin{aligned} J_\varepsilon(u_\varepsilon) & \leq J_\varepsilon(m_{\mathcal{P}_\varepsilon}(u)) = \left( \frac{1}{2} - \frac{1}{2^*} \right) \left( \int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{N}{2}} \left( 2^* \int_{\mathbb{R}^N} G_+(u) - \varphi_\varepsilon(u) G_-(u) dx \right)^{-\frac{N-2}{2}} \\ & \leq \left( \frac{1}{2} - \frac{1}{2^*} \right) \left( \int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{N}{2}} \left( 2^* \int_{\mathbb{R}^N} G(u) dx \right)^{-\frac{N-2}{2}} \\ & = J(u) \end{aligned}$$

Hence

$$J_\varepsilon(u_\varepsilon) \leq \inf_{\mathcal{M}} J$$

and

$$(3.8) \quad \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 dx \leq \left(\frac{1}{2} - \frac{1}{2^*}\right)^{-1} \inf_{\mathcal{M}} J.$$

Moreover,

$$J_\varepsilon(u_\varepsilon) = J_\varepsilon(m_{\mathcal{P}_\varepsilon}(u_\varepsilon)) \geq J_{1/2}(m_{\mathcal{P}_{1/2}}(u_\varepsilon)) \geq J_{1/2}(u_{1/2})$$

and we obtain

$$2^* \int_{\mathbb{R}^N} G_+(u_\varepsilon) dx \geq \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 dx \geq \left(\frac{1}{2} - \frac{1}{2^*}\right)^{-1} J_{1/2}(u_{1/2})$$

for  $\varepsilon \in (0, 1/2]$ . Since  $u_\varepsilon$  is bounded in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  and  $\int_{\mathbb{R}^N} G_+(u_\varepsilon) dx$  is bounded away from 0, in view of Lemma 1.5 we infer that (1.19) does not hold. Therefore, passing to a subsequence and up to a translation, we may assume that  $u_\varepsilon \rightharpoonup u_0 \neq 0$  and  $u_\varepsilon(x) \rightarrow u_0(x)$  for a.e.  $x \in \mathbb{R}^N$  as  $\varepsilon \rightarrow 0$ . Observe that for any  $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$  one has

$$\begin{aligned} J'_\varepsilon(u_\varepsilon)(\psi) &= \int_{\mathbb{R}^N} \langle u_\varepsilon, \psi \rangle dx - \int_{\mathbb{R}^N} g_+(u_\varepsilon)\psi dx + \int_{\mathbb{R}^N} \varphi_\varepsilon(u_\varepsilon)g_-(u_\varepsilon)\psi + \varphi'_\varepsilon(u_\varepsilon)G_-(u_\varepsilon)\psi dx \\ &\rightarrow J'(u_0)(\psi), \end{aligned}$$

hence  $u_0$  is a nontrivial weak solution of (1.2). Since

$$2^* \int_{\mathbb{R}^N} G_-(u_\varepsilon) dx = 2^* \int_{\mathbb{R}^N} G_+(u_\varepsilon) dx - \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 dx$$

is bounded, we infer that  $G_-(u_0) \in L^1(\mathbb{R}^N)$  and by the Pohozaev identity in Proposition 3.1,  $u_0 \in \mathcal{M}$ . Taking into account (3.8),

$$\begin{aligned} J(u_0) &= \left(\frac{1}{2} - \frac{1}{2^*}\right) \int_{\mathbb{R}^N} |\nabla u_0|^2 \leq \left(\frac{1}{2} - \frac{1}{2^*}\right) \liminf_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 dx \\ &\leq \inf_{\mathcal{M}} J, \end{aligned}$$

hence  $J(u_0) = \inf_{\mathcal{M}} J$ . Now suppose that  $g$  is odd. Then  $G_+$  and  $G_-$  are even. Observe that for the minimizing sequence  $(u_n) \subset \mathcal{M}_\varepsilon$  we can consider  $(|u_n|(\cdot r_n)) \subset \mathcal{M}_\varepsilon$  with suitable  $r_n \geq 1$  and then

$$J_\varepsilon(|u_n|(\cdot r_n)) = r_n^{2-N} \left(\frac{1}{2} - \frac{1}{2^*}\right) \int_{\mathbb{R}^N} |\nabla |u_n||^2 dx \leq J_\varepsilon(u_n).$$

Hence  $(|u_n|(\cdot r_n))$  is a minimizing sequence of  $J_\varepsilon$  and therefore we can assume that  $u_\varepsilon \geq 0$ . Hence  $u_0 \geq 0$  and in view of the strong maximum principle  $u_0 > 0$ .

(b) Suppose that  $J(u) = c := \inf_{\mathcal{M}} J$ . Note that  $G(u+v) \in L^1(\mathbb{R}^N)$  for any  $v \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ . Let us fix  $v \in \mathcal{C}_0^\infty(\mathbb{R}^N)$  and similarly as in proof of Lemma 3.3 we show that by the Vitaly convergence theorem

$$\begin{aligned} &\lim_{t \rightarrow 0} \frac{1}{t} \left( \left( 2^* \int_{\mathbb{R}^N} G(u+tv) dx \right)^{\frac{N-2}{N}} - \left( 2^* \int_{\mathbb{R}^N} G(u) dx \right)^{\frac{N-2}{N}} \right) \\ &= \frac{N-2}{N} \left( \frac{1}{2} - \frac{1}{2^*} \right)^{\frac{2}{N}} c^{-\frac{2}{N}} \left( 2^* \int_{\mathbb{R}^N} g(u)v dx \right). \end{aligned}$$

Note that

$$\int_{\mathbb{R}^N} G(u + tv) dx > 0$$

if  $|t|$  is sufficiently small. Hence  $(u + tv)(r \cdot) \in \mathcal{M}$  for  $r = \left(2^* \int_{\mathbb{R}^N} G(u + tv) dx\right)^{1/2} / \|u\|$ ,  $J((u + tv)(r \cdot)) \geq c$ , i.e.

$$\left(\frac{1}{2} - \frac{1}{2^*}\right)^{\frac{2}{N}} \int_{\mathbb{R}^N} |\nabla(u + tv)|^2 dx \geq c^{\frac{2}{N}} \left(2^* \int_{\mathbb{R}^N} G(u + tv) dx\right)^{\frac{N-2}{N}}.$$

Similarly as in proof of Lemma 3.3 we show that  $J'(u)(v) = 0$ . Therefore  $u$  is a weak solution of (1.2). Take  $\lambda := \int_{\mathbb{R}^N} G(u) dx = \frac{1}{2^*} \|u\|^2 > 0$ . Then, for any  $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  such that

$$(3.9) \quad \int_{\mathbb{R}^N} G(v) dx = \lambda$$

we get  $v(r \cdot) \in \mathcal{M}$  for  $r := (2^* \lambda)^{1/2} / \|v\|$ . Hence  $J(v(r \cdot)) \geq J(u)$ ,

$$\left(\frac{1}{2} - \frac{1}{2^*}\right) r^{2-N} \|v\|^2 \geq \left(\frac{1}{2} - \frac{1}{2^*}\right) \|u\|^2,$$

and we get

$$\|v\|^2 \geq \|u\|^2.$$

Therefore  $u$  is a minimizer of the functional  $\mathcal{D}^{1,2}(\mathbb{R}^N) \ni v \mapsto \|v\|^2 \in \mathbb{R}$  under the constraint (3.9). In view of Mariş [23][Theorem 2],  $u$  is radial up to a translation.

(c) Take any  $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  such that  $\int_{\mathbb{R}^N} G(u) dx > 0$ . Then  $u(r \cdot) \in \mathcal{M}$  for some  $r > 0$  and the inequality  $J(u(r \cdot)) \geq \inf_{\mathcal{M}} J$  is equivalent to (1.1) with  $C_{N,G} = 2^* \left(\frac{1}{2} - \frac{1}{2^*}\right)^{-\frac{2}{N-2}} (\inf_{\mathcal{M}} J)^{\frac{2}{N-2}}$ . Clearly, if  $u \in \mathcal{M}$  and  $J(u) = \inf_{\mathcal{M}} J$ , then  $u$  is a minimizer of (1.1).

Now let  $u$  be a minimizer of (1.1). Then  $\int_{\mathbb{R}^N} G(u) dx > 0$  and  $u(\lambda \cdot) \in \mathcal{M}$  for a unique  $\lambda > 0$  and  $J(u(\lambda \cdot)) = \inf_{\mathcal{M}} J$ .  $\square$

*Proof of Theorem 1.2.* (a) follows from Theorem 1.1 (a).

(b) Observe that  $G(s)$  has nonpositive values for  $m \geq m_0$  and in view of (1.5), (1.2) does not have any nontrivial solutions. Similarly combining (1.5) with  $J'(u)(u) = 0$  we infer that there are nontrivial solutions also for  $m \leq 0$ .  $\square$

#### 4. PROOFS OF THEOREM 1.3 AND THEOREM 1.4

Now, let us consider  $\mathcal{O}_1$ -invariant functions.

*Proof of Theorem 1.3.* Assume that  $X := \mathcal{D}_{\mathcal{O}_1}^{1,2}(\mathbb{R}^N) \cap X_\tau$  and  $2 \leq m < N/2$ . Let  $(u_n) \subset \mathcal{M}_\varepsilon \cap X$  be a sequence such that  $J_\varepsilon(u_n) \rightarrow \beta$  with

$$\beta := \inf_{\mathcal{M}_\varepsilon \cap X} J_\varepsilon.$$



Since  $J_\varepsilon$  is coercive on  $\mathcal{M}_\varepsilon$ ,  $(u_n)$  is bounded. Observe that

$$2^* \int_{\mathbb{R}^N} G_+(u_n) dx \geq \int_{\mathbb{R}^N} |\nabla u_n|^2 dx = \left(\frac{1}{2} - \frac{1}{2^*}\right)^{-1} \beta + o(1)$$

and in view of Corollary 2.2, passing to a subsequence, we find  $(y_n) \subset \{0\} \times \{0\} \times \mathbb{R}^{N-2m}$  such that

$$u_n(\cdot + y_n) \rightharpoonup u_\varepsilon \neq 0 \quad \text{and} \quad u_n(x + y_n) \rightarrow u_\varepsilon(x)$$

for a.e.  $x \in \mathbb{R}^N$  as  $n \rightarrow \infty$ . Similarly as in proof of Lemma 3.3 we show that  $u_\varepsilon$  is a critical point of  $J_\varepsilon|_X$  and by the Palais principle of symmetric criticality [26],  $J'_\varepsilon(u_\varepsilon) = 0$ . By the Pohozaev identity (cf. Proposition 3.1),  $u_\varepsilon \in \mathcal{M}_\varepsilon \cap X$ ,  $m_{\mathcal{P}}(u_\varepsilon) = u_\varepsilon$  and

$$\beta \leq J_\varepsilon(u_\varepsilon) = \left(\frac{1}{2} - \frac{1}{2^*}\right) \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 dx \leq \liminf_{n \rightarrow \infty} \left(\frac{1}{2} - \frac{1}{2^*}\right) \int_{\mathbb{R}^N} |\nabla u_n(\cdot + y_n)|^2 dx = \beta.$$

Letting  $\varepsilon \rightarrow 0$  as in proof of Theorem 1.1, we find a critical point  $u \in \mathcal{M} \cap X$  of  $J$  such that

$$J(u) = \inf_{\mathcal{M} \cap X} J.$$

In view of the Palais principle of symmetric criticality [26],  $u$  solves (1.2). Let

$$\begin{aligned} \Omega_1 &:= \{x \in \mathbb{R}^N : |x_1| > |x_2|\}, \\ \Omega_2 &:= \{x \in \mathbb{R}^N : |x_1| < |x_2|\}. \end{aligned}$$

Since  $u \in X_\tau \cap \mathcal{D}_{\mathcal{O}_1}^{1,2}(\mathbb{R}^N)$ , we get  $\chi_{\Omega_1} u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$  and  $\chi_{\Omega_2} u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ . Moreover  $\chi_{\Omega_1} u \in \mathcal{M}$  and

$$J(u) = J(\chi_{\Omega_1} u) + J(\chi_{\Omega_2} u) = 2J(\chi_{\Omega_1} u) \geq 2 \inf_{\mathcal{M}} J.$$

Suppose that  $J(u) = 2 \inf_{\mathcal{M}} J$ . Then

$$J(\chi_{\Omega_1} u) = \inf_{\mathcal{M}} J$$

and in view of Theorem 1.1 (b),  $\chi_{\Omega_1} u$  is radial (up to a translation), which is a contradiction. This completes proof of (1.15). The remaining case  $2 \leq m = N/2$  is contained in Theorem 1.4. □

Now let us consider  $\mathcal{O}_2$ -invariant functions. In order to get the multiplicity of critical points, we need to modify  $J_\varepsilon$  in order to ensure that (4.1) and (4.5) below are satisfied. Take any even function  $\psi_\lambda : \mathbb{R} \rightarrow [0, 1]$  of class  $\mathcal{C}^1$  such that  $\psi_\lambda(s) = 1$  for  $\lambda \leq |s| \leq 1/\lambda$  and  $\text{supp}(\psi_\lambda)$  is compact and does not contain 0 for  $\lambda \in (0, 1]$ . We set  $\psi_0 \equiv 1$ . Let  $G_{+,\lambda}(s) := \psi_\lambda(s)G_+(s)$  and instead of  $G_\varepsilon$  we consider now

$$G_{(\varepsilon,\lambda)}(s) := G_{+,\lambda}(s) - \lambda|s|^{2^*} - \varphi_\varepsilon(s)G_-(s).$$

Take  $g_{+,\lambda}(s) := (\psi_\lambda(s)G_+(s))'$  and we check that

$$(4.1) \quad \lim_{s \rightarrow 0} g_{+,\lambda}(s)/|s|^{2^*-1} = \lim_{|s| \rightarrow \infty} g_{+,\lambda}(s)/|s|^{2^*-1} = 0.$$

Let us introduce the following functional

$$J_{(\varepsilon,\lambda)}(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} G_{(\varepsilon,\lambda)}(u) dx$$

for  $\varepsilon \in (0, 1/2]$  and  $\lambda \in [0, 1]$ . Clearly, Proposition 3.2 holds if we replace  $J_\varepsilon$ ,  $g_\varepsilon$  and  $G_\varepsilon$  by  $J_{(\varepsilon,\lambda)}$ ,  $g_{(\varepsilon,\lambda)} := G'_{(\varepsilon,\lambda)}$  and  $G_{(\varepsilon,\lambda)}$  respectively and  $\lambda > 0$  is sufficiently small, i.e. there is  $\lambda_0 \in (0, 1]$  such that  $G_{(0,\lambda)}(\xi_0) > 0$  for  $\lambda \in [0, \lambda_0]$ . We may also assume that  $\psi_\lambda(s) \geq \psi_{\lambda_0}(s)$ , hence  $G_{(0,\lambda)}(s) \geq G_{(0,\lambda_0)}(s)$  for any  $s \in \mathbb{R}$  and  $\lambda \in [0, \lambda_0]$ . Here and what follows  $\mathcal{P}$ ,  $\mathcal{U}$ ,  $m$  depend on  $\varepsilon$  and  $\lambda$ , and are given in Proposition 3.2, where  $J_\varepsilon$ ,  $g_\varepsilon$  and  $G_\varepsilon$  are replaced by  $J_{(\varepsilon,\lambda)}$ ,  $g_{(\varepsilon,\lambda)}$  and  $G_{(\varepsilon,\lambda)}$  respectively.  $\mathcal{M}_{(\varepsilon,\lambda)}$  stands for the Pohozaev manifold for  $J_{(\varepsilon,\lambda)}$ .

**Lemma 4.1.** *Suppose that  $X := \mathcal{D}_{\mathcal{O}_2}^{1,2}(\mathbb{R}^N) \cap X_\tau$  and  $(u_n) \subset \mathcal{U} \cap X$  is a  $(PS)_\beta$ -sequence of  $(J_{(\varepsilon,\lambda)}|_X \circ m|_{\mathcal{U} \cap X})$  at level  $\beta \in \mathbb{R}$ , i.e.*

$$(J_{(\varepsilon,\lambda)}|_X \circ m|_{\mathcal{U} \cap X})'(u_n) \rightarrow 0 \text{ and } (J_{(\varepsilon,\lambda)}|_X \circ m|_{\mathcal{U} \cap X})(u_n) \rightarrow \beta.$$

- (i) *Then, passing to a subsequence,  $u_n \rightarrow u_0$  for some  $u_0 \in \mathcal{U} \cap X$ .*
- (ii)  *$J'_{(\varepsilon,\lambda)}(m(u_0)) = 0$  provided that  $\lambda \in (0, \lambda_0]$ .*

*Proof.* Note that, if  $\beta = \inf_{\mathcal{M}_{\varepsilon \cap X}} J_{(\varepsilon,\lambda)}$ , then we can argue as in Lemma 3.3. Let  $(u_n) \subset \mathcal{U} \cap X$  be a sequence such that  $(J_{(\varepsilon,\lambda)}|_X \circ m|_{\mathcal{U} \cap X})'(u_n) \rightarrow 0$  and  $(J_{(\varepsilon,\lambda)}|_X \circ m|_{\mathcal{U} \cap X})(u_n) \rightarrow \beta$ . Observe that  $\beta \geq \inf_{\mathcal{M}_{\varepsilon \cap X}} J_{(\varepsilon,\lambda)} > 0$ . Since  $J_{(\varepsilon,\lambda)}$  is coercive on  $\mathcal{M}_{(\varepsilon,\lambda)}$ ,  $(m(u_n))$  is bounded and, passing to subsequence, we may assume that  $m(u_n) \rightharpoonup \tilde{u}$  and  $m(u_n)(x) \rightharpoonup \tilde{u}(x)$  for a.e.  $x \in \mathbb{R}^N$ . In view of Lemma A.1 (b) we infer that

$$(4.2) \quad \int_{\mathbb{R}^N} G_{+,\lambda}(m(u_n)) dx \rightarrow \int_{\mathbb{R}^N} G_{+,\lambda}(\tilde{u}) dx$$

as  $n \rightarrow \infty$ . If  $\tilde{u} = 0$ , then we get a contradiction with the following inequality

$$2^* \int_{\mathbb{R}^N} G_{+,\lambda}(m(u_n)) dx \geq \int_{\mathbb{R}^N} |\nabla m(u_n)|^2 dx = \left(\frac{1}{2} - \frac{1}{2^*}\right)^{-1} \beta + o(1).$$

Therefore  $\tilde{u} \neq 0$  and we easily check that  $r(u_n)$  given by (3.2) is bounded and bounded away from 0. For any  $v \in X$  we set  $v_n := v(r(u_n)^{-1})$  and we find the following decomposition

$$v_n = \left( \int_{\mathbb{R}^N} \langle \nabla u_n, \nabla v_n \rangle dx \right) u_n + \tilde{v}_n$$

with

$$\tilde{v}_n \in T_{u_n} \mathcal{S} := \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \langle \nabla u_n, \nabla u \rangle dx = 0 \right\}.$$

Clearly  $(\tilde{v}_n) \subset X$  is bounded and  $(J_{(\varepsilon,\lambda)}|_X \circ m|_{\mathcal{U} \cap X})'(u_n)(\tilde{v}_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Since

$$\int_{\mathbb{R}^N} \langle \nabla u_n, \nabla v_n \rangle dx = r(u_n)^{N-2} \int_{\mathbb{R}^N} \langle \nabla m(u_n), \nabla v \rangle dx \rightarrow 0$$

for any  $v \in X$  such that  $\int_{\mathbb{R}^N} \langle \nabla \tilde{u}, \nabla v \rangle dx = 0$ , we get

$$\begin{aligned} (J_{(\varepsilon, \lambda)}|_X \circ m|_{\mathcal{U} \cap X})'(u_n)(v(r(u_n)^{-1} \cdot)) &= \left( \int_{\mathbb{R}^N} \langle \nabla u_n, \nabla v_n \rangle dx \right) (J_{(\varepsilon, \lambda)}|_X \circ m|_{\mathcal{U} \cap X})'(u_n)(u_n) \\ &\quad + (J_{(\varepsilon, \lambda)}|_X \circ m|_{\mathcal{U} \cap X})'(u_n)(\tilde{v}_n) \\ &\rightarrow 0. \end{aligned}$$

By Proposition (3.2) (ii) we obtain

$$(4.3) \quad J'_{(\varepsilon, \lambda)}(\tilde{u})(v) = \lim_{n \rightarrow \infty} J'_{(\varepsilon, \lambda)}(m(u_n))(v) = \lim_{n \rightarrow \infty} (J_{(\varepsilon, \lambda)}|_X \circ m|_{\mathcal{U} \cap X})'(u_n)(v(r(u_n)^{-1} \cdot)) = 0$$

for  $v \in X$  such that  $\int_{\mathbb{R}^N} \langle \nabla \tilde{u}, \nabla v \rangle dx = 0$ . Now we define a linear map  $\xi : X \rightarrow \mathbb{R}$  by the following formula

$$\begin{aligned} \xi(v) &= \int_{\mathbb{R}^N} \langle \nabla \tilde{u}, \nabla v \rangle dx - \int_{\mathbb{R}^N} g_{(\varepsilon, \lambda)}(\tilde{u})v dx \\ &\quad - \left( \int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 dx - \int_{\mathbb{R}^N} g_{(\varepsilon, \lambda)}(\tilde{u})\tilde{u} dx \right) \|\tilde{u}\|^{-2} \int_{\mathbb{R}^N} \langle \nabla \tilde{u}, \nabla v \rangle dx \end{aligned}$$

and observe that  $\xi(\tilde{u}) = 0$ . Since any  $v \in X$  has the following decomposition

$$v = \left( \int_{\mathbb{R}^N} \langle \nabla \tilde{u}, \nabla v \rangle dx \right) \|\tilde{u}\|^{-2} \tilde{u} + \tilde{v}, \quad \text{where } \int_{\mathbb{R}^N} \langle \nabla \tilde{u}, \nabla \tilde{v} \rangle dx = 0,$$

in view of (4.3) we infer that  $\xi \equiv 0$ . Hence by the Palais principle of symmetric criticality [26],  $\tilde{u}$  is a weak solution of the problem

$$(4.4) \quad -\theta \Delta \tilde{u} = g_{(\varepsilon, \lambda)}(\tilde{u})$$

with

$$\theta = 1 - \left( \int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 dx - \int_{\mathbb{R}^N} g_{(\varepsilon, \lambda)}(\tilde{u})\tilde{u} dx \right) \|\tilde{u}\|^{-2} = \|\tilde{u}\|^{-2} \int_{\mathbb{R}^N} g_{(\varepsilon, \lambda)}(\tilde{u})\tilde{u} dx.$$

Moreover, similarly as above we define linear maps  $\xi_n : X \rightarrow \mathbb{R}$  by the following formula

$$\begin{aligned} \xi_n(v) &= \int_{\mathbb{R}^N} \langle \nabla m(u_n), \nabla v \rangle dx - \int_{\mathbb{R}^N} g_{(\varepsilon, \lambda)}(m(u_n))v dx \\ &\quad - \left( \int_{\mathbb{R}^N} |\nabla m(u_n)|^2 dx - \int_{\mathbb{R}^N} g_{(\varepsilon, \lambda)}(m(u_n))m(u_n) dx \right) \|m(u_n)\|^{-2} \int_{\mathbb{R}^N} \langle \nabla m(u_n), \nabla v \rangle dx, \end{aligned}$$

and we show that  $\xi_n \rightarrow 0$  in  $X^*$ . Hence, passing to a subsequence

$$\begin{aligned} \theta_n &:= 1 - \left( \int_{\mathbb{R}^N} |\nabla m(u_n)|^2 dx - \int_{\mathbb{R}^N} g_{(\varepsilon, \lambda)}(m(u_n))m(u_n) dx \right) \|m(u_n)\|^{-2} \\ &= \|m(u_n)\|^{-2} \int_{\mathbb{R}^N} g_{(\varepsilon, \lambda)}(m(u_n))m(u_n) dx \end{aligned}$$

converges to  $\theta$ . Since (4.1) holds, in view of Lemma A.1 and (A.3) we infer that

$$\int_{\mathbb{R}^N} g_{+, \lambda}(m(u_n))m(u_n) dx \rightarrow \int_{\mathbb{R}^N} g_{+, \lambda}(\tilde{u})\tilde{u} dx$$

and by the Fatou's lemma

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} g_{(\varepsilon, \lambda)}(m(u_n))m(u_n) dx \leq \int_{\mathbb{R}^N} g_{(\varepsilon, \lambda)}(\tilde{u})\tilde{u} dx.$$

Since  $\theta_n \rightarrow \theta$ , we conclude that  $\|m(u_n)\| \rightarrow \|\tilde{u}\|$  and therefore  $m(u_n) \rightarrow \tilde{u}$  and  $\tilde{u} \in \mathcal{M}_{(\varepsilon, \lambda)}$ . By Proposition 3.2 (ii),  $u_n \rightarrow u_0 := m^{-1}(\tilde{u})$ . We show that  $\theta \neq 0$  provided that  $\lambda > 0$ . By a contradiction, suppose that  $\theta = 0$ , then  $g_{(\varepsilon, \lambda)}(\tilde{u}(x)) = 0$  for a.e.  $x \in \mathbb{R}^N$ . Take  $\Sigma := \{x \in \mathbb{R}^N : g_{(\varepsilon, \lambda)}(\tilde{u}(x)) = 0\}$  and clearly  $\mathbb{R}^N \setminus \Sigma$  has measure zero and let  $\Omega := \{x \in \Sigma : \tilde{u}(x) \neq 0\}$ . Suppose that  $\delta := \inf_{x \in \Omega} |\tilde{u}(x)| > 0$ . Since  $\tilde{u} \in L^6(\mathbb{R}^N) \setminus \{0\}$ , we infer that  $\Omega$  has finite positive measure,  $\tilde{u} \in H^1(\mathbb{R}^N)$  and note that

$$\int_{\mathbb{R}^N} |\tilde{u}(x+h) - \tilde{u}(x)|^2 dx \geq \delta \int_{\mathbb{R}^N} |\chi_\Omega(x+h) - \chi_\Omega(x)|^2 dx \quad \text{for any } h \in \mathbb{R}^N,$$

where  $\chi_\Omega$  is the characteristic function of  $\Omega$ . In view of [35][Theorem 2.1.6] we infer that  $\chi_\Omega \in H^1(\mathbb{R}^N)$ , hence we get a contradiction. Therefore we find a sequence  $(x_n) \subset \mathbb{R}^N$  such that  $\tilde{u}(x_n) \rightarrow 0$ ,  $\tilde{u}(x_n) \neq 0$  and  $g_{(\varepsilon, \lambda)}(\tilde{u}(x_n)) = 0$ . Again we get a contradiction, since

$$(4.5) \quad \limsup_{s \rightarrow 0^+} g_{(\varepsilon, \lambda)}(s)/s^{2^*-1} \leq -\lambda < 0.$$

Therefore  $\theta \neq 0$  and in view of the Pohozaev identity (cf. Proposition 3.1) we obtain that  $\theta = 1$ , since  $\tilde{u} \in \mathcal{M}_{(\varepsilon, \lambda)}$ . Hence (ii) holds.  $\square$

*Proof of Theorem 1.4.*

(a) Assume that  $X := \mathcal{D}_{\mathcal{O}_2}^{1,2}(\mathbb{R}^N) \cap X_\tau$ . Similarly as in proof of Theorem 1.1 we find a critical point  $u \in \mathcal{M} \cap X$  of  $J|_X$  such that

$$J(u) = \inf_{\mathcal{M} \cap X} J$$

and by the Palais principle of symmetric criticality [26],  $u$  solves (1.2).

(b) *Step 1.* For any  $\varepsilon \in (0, 1/2]$  and  $\lambda \in (0, \lambda_0]$ , we show the existence of a sequence  $(u_{(\varepsilon, \lambda)}^k)$  of critical points of  $J_{(\varepsilon, \lambda)}$  such that  $J_{(\varepsilon, \lambda)}(u_{(\varepsilon, \lambda)}^k)$  as  $k \rightarrow \infty$ . Let us fix  $\lambda \in [0, \lambda_0]$ . In view of [6][Theorem 10], for any  $k \geq 1$  we find an odd continuous map

$$\tau : S^{k-1} \rightarrow H_0^1(B(0, R)) \cap L^\infty(B(0, R))$$

such that  $\tau(\sigma)$  is a radial function and  $\tau(\sigma) \neq 0$  for all  $\sigma \in S^{k-1}$ , where  $S^{k-1}$  is the unit sphere in  $\mathbb{R}^k$ . Moreover, since  $G_{(0, \lambda)}(\xi_0) > 0$ , we may find some constants  $c_2, c_3 > 0$  independent on  $R$  such that

$$\int_{B(0, R)} G_{(0, \lambda)}(\tau(\sigma)) dx \geq c_2 R^N - c_3 R^{N-1}$$

for any  $\sigma \in S^{k-1}$ . As in [24][Remark 4.2] we define a map

$$\tilde{\tau} : S^{k-1} \rightarrow H_0^1(B(0, R)) \cap L^\infty(B(0, R))$$

such that  $\tilde{\tau}(\sigma)(x_1, x_2, x_3) = \tau(\sigma)(x_1, x_2, x_3)\varphi(|x_1| - |x_2|)$  and  $\varphi : \mathbb{R} \rightarrow [0, 1]$  is an odd and smooth function such that  $\varphi(x) = 1$  for  $x \geq 1$ ,  $\varphi(x) = -1$  for  $x \leq -1$ . If  $\lambda = \lambda_0$ , then we

denote this map by  $\tilde{\tau}_{\lambda_0}$ . Observe that  $\tilde{\tau}(\sigma) \in X$  and, again as in [24][Remark 4.2], we show that

$$\int_{B(0,R)} G_{(0,\lambda)}(\tilde{\tau}(\sigma)) dx \geq \int_{B(0,R)} G_{(0,\lambda)}(\tau(\sigma)) dx - c_1 R^{N-1}$$

for  $\sigma \in S^{k-1}$  and some constant  $c_1 > 0$ . Therefore, for sufficiently large  $R = R(\lambda)$

$$(4.6) \quad \int_{B(0,R)} G_{(\varepsilon,\lambda)}(\tilde{\tau}(\sigma)) dx \geq \int_{B(0,R)} G_{(0,\lambda)}(\tilde{\tau}(\sigma)) dx > 0$$

for any  $\varepsilon \in [0, 1/2]$  and  $\lambda \in [0, \lambda_0]$ . Hence  $\tilde{\tau}(\sigma) \in \mathcal{P} \cap X$  if  $\varepsilon > 0$ . Taking  $p(u) := u/\|u\|$  we obtain that

$$(4.7) \quad \gamma\left(p(\tilde{\tau}(S^{k-1}))\right) \geq k,$$

where  $\gamma$  stands for the Krasnoselskii genus for closed and symmetric subsets of  $X$ . Therefore the Lusternik-Schnirelman values

$$(4.8) \quad \beta_{(\varepsilon,\lambda)}^k := \inf \{ \beta \in \mathbb{R} : \gamma(\Phi_{(\varepsilon,\lambda)}^\beta) \geq k \}$$

are finite, where  $\Phi_{(\varepsilon,\lambda)} := J_{(\varepsilon,\lambda)} \circ m|_X : \mathcal{U} \cap X \rightarrow \mathbb{R}$  and  $\Phi_{(\varepsilon,\lambda)}^\beta := \{u \in \mathcal{U} \cap X : \Phi_{(\varepsilon,\lambda)}(u) \leq \beta\}$  for any  $\varepsilon \in (0, 1/2]$  and  $\lambda \in [0, \lambda_0]$ . Recall that  $\mathcal{P}, \mathcal{U}, m$  depend on  $\varepsilon$  and  $\lambda$ . Moreover, observe that

$$\Phi_{(\varepsilon,\lambda)}(u) = J_{(\varepsilon,\lambda)}(m(u)) = \left(\frac{1}{2} - \frac{1}{2^*}\right) \left(2^* \int_{\mathbb{R}^N} \psi_\lambda(u) G_+(u) - \varphi_\varepsilon(u) G_-(u) - \frac{\lambda}{2^*} |u|^{2^*} dx\right)^{-\frac{N-2}{2}},$$

and in view of (4.6) we obtain the following estimates

$$(4.9) \quad \begin{aligned} \beta_{(1/2,0)}^k &\leq \beta_{(\varepsilon,0)}^k \leq \beta_{(\varepsilon,\lambda)}^k \leq \beta_{(\varepsilon,\lambda_0)}^k \\ &\leq M^k := \sup_{u \in p(\tilde{\tau}_{\lambda_0}(S^{k-1}))} \left(\frac{1}{2} - \frac{\lambda}{2^*}\right) \left(2^* \int_{B(0,R(\lambda_0))} G_{(0,\lambda_0)}(u) dx\right)^{-\frac{N-2}{2}}, \end{aligned}$$

for any  $\varepsilon \in (0, 1/2]$  and  $\lambda \in [0, \lambda_0]$ . Since Lemma 4.1 holds, in view of [24][Theorem 2.2 (c)] we get an infinite sequence of critical points, namely  $(\beta_{(\varepsilon,\lambda)}^k)_{k \geq 1}$  are critical values provided that  $\varepsilon \in (0, 1/2]$  and  $\lambda \in (0, \lambda_0]$ . It is standard to show that the sequence is unbounded. Indeed, as in [24, 27] we show that  $\beta_{(\varepsilon,\lambda)}^1 < \beta_{(\varepsilon,\lambda)}^2 < \dots < \beta_{(\varepsilon,\lambda)}^k < \dots$  is an increasing sequence of critical values, due to Lemma 4.1 and  $\Phi_{(\varepsilon,\lambda)}(u) \rightarrow \infty$  as  $u \rightarrow u_0$  for some  $u_0 \in \partial(\mathcal{U} \cap X)$ . Suppose that  $\bar{\beta} := \lim_{k \rightarrow \infty} \beta_{(\varepsilon,\lambda)}^k < \infty$ . Note that

$$\mathcal{K}^{\bar{\beta}} := \{u \in \mathcal{U} \cap X : \Phi'_{(\varepsilon,\lambda)}(u) = 0 \text{ and } \Phi_{(\varepsilon,\lambda)}(u) = \bar{\beta}\}$$

is compact and  $\gamma(\text{cl} B(\mathcal{K}^{\bar{\beta}}, \delta)) = \gamma(\mathcal{K}^{\bar{\beta}}) < \infty$  for some small  $\delta > 0$ . Similarly as in proof of [24][Theorem 2.2] we construct a continuous and odd map  $h : \Phi_{(\varepsilon,\lambda)}^{\bar{\beta}+\eta} \setminus B(\mathcal{K}^{\bar{\beta}}, \delta) \rightarrow \Phi_{(\varepsilon,\lambda)}^{\bar{\beta}-\eta}$  for sufficiently small  $\eta > 0$  such that

$$\Phi_{(\varepsilon,\lambda)}^{\bar{\beta}+\eta} \setminus (B(\mathcal{K}^{\bar{\beta}}, \delta) \cup \Phi_{(\varepsilon,\lambda)}^{\bar{\beta}-\eta})$$

does not contain any critical point. Hence

$$\begin{aligned} \gamma(\Phi_{(\varepsilon,\lambda)}^{\bar{\beta}+\eta}) &\leq \gamma(\text{cl } B(\mathcal{K}^{\bar{\beta}}, \delta)) + \gamma(\Phi_{(\varepsilon,\lambda)}^{\bar{\beta}+\eta} \setminus B(\mathcal{K}^{\bar{\beta}}, \delta)) \\ &\leq \gamma(\text{cl } B(\mathcal{K}^{\bar{\beta}}, \delta)) + \gamma(\Phi_{(\varepsilon,\lambda)}^{\bar{\beta}-\eta}) =: l < \infty. \end{aligned}$$

We obtain a contradiction with  $\gamma(\Phi_{(\varepsilon,\lambda)}^{\bar{\beta}+\eta}) \geq \gamma(\Phi_{(\varepsilon,\lambda)}^{\beta_{(\varepsilon,\lambda)}^{l+2}}) \geq l+1$ . Therefore  $\Phi_{(\varepsilon,\lambda)}$  has a sequence of critical points  $(u_{(\varepsilon,\lambda)}^k) \subset \mathcal{S}$  with

$$\Phi_{(\varepsilon,\lambda)}((u_{(\varepsilon,\lambda)}^k)) = \beta_{(\varepsilon,\lambda)}^k \rightarrow \infty$$

as  $k \rightarrow \infty$ , for  $\varepsilon \in (0, 1/2]$  and  $\lambda \in (0, \lambda_0]$ . Hence, by Lemma 4.1 (ii),  $J_{(\varepsilon,\lambda)}$  has an unbounded sequence of critical points  $(m(u_{(\varepsilon,\lambda)}^k))$  for  $\varepsilon \in (0, 1/2]$  and  $\lambda \in (0, \lambda_0]$ .

*Step 2.* We show the existence a sequence  $(u_\varepsilon^k)$  of critical points of  $J_\varepsilon$  for any  $\varepsilon \in (0, 1/2]$  such that  $J_\varepsilon(u_\varepsilon^k) \rightarrow \infty$  as  $k \rightarrow \infty$ . Indeed, take  $\lambda_n \in (0, \lambda_0]$  such that  $\lambda_n \rightarrow 0$  as  $n \rightarrow \infty$  and in view of (4.9),  $v_n := m(u_{(\varepsilon,\lambda_n)}^k)$  is bounded. Passing to a subsequence,  $v_n \rightharpoonup v_0$  and  $v_n(x) \rightarrow v_0(x)$  for a.e.  $x \in \mathbb{R}^N$ . Since  $J'_{(\varepsilon,\lambda_n)}(v_n) = 0$ , we obtain that  $J'_\varepsilon(v_0) = 0$  and by Lemma A.1 (b)

$$\int_{\mathbb{R}^N} G_+(v_n) dx \rightarrow \int_{\mathbb{R}^N} G_+(v_0) dx$$

as  $n \rightarrow \infty$ . If  $v_0 = 0$ , then we get a contradiction since

$$0 < \liminf_{n \rightarrow \infty} \Phi_{(\varepsilon,\lambda_n)}(u_{(\varepsilon,\lambda_n)}^k) \leq \liminf_{n \rightarrow \infty} \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} G_+(v_n) - \varphi_\varepsilon(v_n) G_-(v_n) dx \leq 0.$$

Therefore  $v_0 \in \mathcal{M}_\varepsilon$  and  $u_\varepsilon^k := m^{-1}(v_0)$  is a critical point of  $\Phi_{(\varepsilon,0)}$ . Moreover by Fatou's lemma

$$\begin{aligned} \|v_0\|^2 + 2^* \int_{\mathbb{R}^N} \varphi_\varepsilon(v_0) G_-(v_0) dx &\leq 2^* \liminf_{n \rightarrow \infty} \left( \|v_n\|^2 + 2^* \int_{\mathbb{R}^N} \varphi_\varepsilon(v_n) G_-(v_n) dx \right) \\ &\leq 2^* \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} G_+(v_n) dx = 2^* \int_{\mathbb{R}^N} G_+(v_0) dx \\ &= \|v_0\|^2 + 2^* \int_{\mathbb{R}^N} \varphi_\varepsilon(v_0) G_-(v_0) dx, \end{aligned}$$

hence  $v_n \rightarrow v_0$ . Therefore  $u_{(\varepsilon,\lambda_n)}^k \rightarrow u_\varepsilon^k$  and  $\beta_{(\varepsilon,\lambda_n)}^k \rightarrow \Phi_\varepsilon(u_\varepsilon^k)$  as  $n \rightarrow \infty$ . Moreover  $J_\varepsilon(m(u_\varepsilon^k)) = \Phi_\varepsilon(u_\varepsilon^k) \geq \beta_{(\varepsilon,0)}^k \rightarrow \infty$  as  $k \rightarrow \infty$ .

*Step 3.* We show the existence of an unbounded sequence of critical point of  $J$  with finite energy. Take  $\varepsilon_n \in (0, 1/2]$  such that  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ . Again, in view of (4.9) and passing to a subsequence, we may assume that  $v_n := m(u_{\varepsilon_n}^k) \rightharpoonup v^k$  and  $v_n(x) \rightarrow v^k(x)$  for a.e.  $x \in \mathbb{R}^N$ . Since  $J'_{\varepsilon_n}(v_n) = 0$ , we obtain that  $J'(v^k)(\psi) = 0$  for any  $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ , and by Lemma A.1

$$\int_{\mathbb{R}^N} G_+(v_n) dx \rightarrow \int_{\mathbb{R}^N} G_+(v^k) dx$$

as  $n \rightarrow \infty$ . If  $v^k = 0$ , then

$$\beta_{(1/2,0)}^k \leq \liminf_{n \rightarrow \infty} \Phi_{(\varepsilon_n,0)}(u_{\varepsilon_n}^k) = \liminf_{n \rightarrow \infty} \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} G_+(v_n) - \varphi_{\varepsilon_n}(v_n) G_-(v_n) dx \leq 0$$

and we get a contradiction since  $\beta_{(1/2,0)}^k$  is a critical value and by (3.3),

$$\beta_{(1/2,0)}^k \geq \inf_{\mathcal{M}_{1/2}} J_{1/2} > 0.$$

By the Fatou's lemma

$$\|v^k\|^2 + 2^* \int_{\mathbb{R}^N} G_-(v^k) dx \leq \liminf_{n \rightarrow \infty} \left( \|v_n\|^2 + 2^* \int_{\mathbb{R}^N} \varphi_{\varepsilon_n}(v_n) G_-(v_n) dx \right) = G_+(v^k) dx$$

and  $G_-(v^k) \in L^1(\mathbb{R}^N)$ . In view of Proposition 3.1, we obtain that  $v^k \in \mathcal{M}$ , i.e. the equality holds above, hence  $\|v_n\| \rightarrow \|v^k\|$ . Therefore  $v_n \rightarrow v^k$  and

$$J(v^k) \geq \beta_{(1/2,0)}^k \rightarrow \infty$$

as  $k \rightarrow \infty$ . □

## APPENDIX A. CONVERGENCE RESULTS AND PROFILE DECOMPOSITIONS

In our variational approach, the following lemma replaces compactness results of Strauss for radial functions [5][Lemma A.I, Lemma A.III] and allows to consider a wider class of symmetric functions. Recall that  $\mathcal{O} \subset \mathcal{O}(N)$  is a subgroup such that  $\mathbb{R}^N$  is *compatible with*  $\mathcal{O}$  (in the sense of [34][Definition 1.23], cf. [21]), if for some  $r > 0$

$$\lim_{|y| \rightarrow \infty} m(y, r) = \infty,$$

where

$m(y, r) := \sup \{n \in \mathbb{N} : \text{there exist } g_1, \dots, g_n \in \mathcal{O} \text{ such that } B(g_i y, r) \cap B(g_j y, r) = \emptyset \text{ for } i \neq j\}$   
and  $y \in \mathbb{R}^N$ . For instance  $\mathbb{R}^N$  is compatible with  $\mathcal{O}(N)$  and with  $\mathcal{O}_2$ .

**Lemma A.1.** *Suppose that  $(u_n) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$  is bounded and  $u_n(x) \rightarrow u_0(x)$  for a.e.  $x \in \mathbb{R}^N$ . (a) Then*

$$(A.1) \quad \lim_{n \rightarrow \infty} \left( \int_{\mathbb{R}^N} \Psi(u_n) dx - \int_{\mathbb{R}^N} \Psi(u_n - u_0) dx \right) = \int_{\mathbb{R}^N} \Psi(u_0) dx$$

for any function  $\Psi : \mathbb{R} \rightarrow \mathbb{R}$  of class  $\mathcal{C}^1$  such that  $|\Psi'(u_n)| \leq C|s|^{2^*-1}$  for any  $s \in \mathbb{R}$  and some constant  $C > 0$ .

(b) Suppose that  $\mathbb{R}^N$  is compatible with  $\mathcal{O} \subset \mathcal{O}(N)$  and assume that each  $u_n$  is  $\mathcal{O}$ -invariant. If, in addition,  $s \mapsto |\Psi(s)|$  satisfies (1.20), then

$$(A.2) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \Psi(u_n) dx = \int_{\mathbb{R}^N} \Psi(u_0) dx,$$

and if  $s \mapsto |\Psi'(s)s|$  satisfies (1.20), then

$$(A.3) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \Psi'(u_n)u_n dx = \int_{\mathbb{R}^N} \Psi'(u_0)u_0 dx.$$

*Proof.* (a) Observe that by Vitali's convergence theorem

$$\begin{aligned} \int_{\mathbb{R}^N} \Psi(u_n) - \Psi(u_n - u_0) dx &= \int_{\mathbb{R}^N} \int_0^1 -\frac{d}{ds} \Psi(u_n - su_0) ds dx = \int_{\mathbb{R}^N} \int_0^1 \Psi'(u_n - su_0) u_0 ds dx \\ &\rightarrow \int_0^1 \int_{\mathbb{R}^N} \Psi'(\tilde{u}_0 - su_0) u_0 dx ds = \int_{\mathbb{R}^N} \int_0^1 -\frac{d}{ds} \Psi(\tilde{u}_0 - su_0) ds dx \\ &= \int_{\mathbb{R}^N} \Psi(u_0) dx \end{aligned}$$

as  $n \rightarrow \infty$ .

(b) Suppose that  $\mathbb{R}^N$  is compatible with  $\mathcal{O}$  and then

$$m(y, r) \int_{B(y, r)} |u_n - u_0|^{2^*} dx \leq \int_{\mathbb{R}^N} |u_n - u_0|^{2^*} dx$$

is bounded. Observe that

$$\int_{B(y, r)} |u_n - u_0|^2 dx \leq C \left( \int_{B(0, r)} |(u_n - u_0)(\cdot + y)|^{2^*} dx \right)^{2/2^*} \leq C |u_n - u_0|_{2^*}^2 m(y, r)^{-2/2^*}$$

for some constant  $C > 0$ . Take any  $\varepsilon > 0$  and note that we find  $R > 0$  such that

$$C |u_n - u_0|_{2^*}^2 m(y, r)^{-2/2^*} < \varepsilon$$

for  $|y| \geq R$  and

$$\int_{B(y, r)} |u_n - u_0|^2 dx \leq \int_{B(0, r+R)} |u_n - u_0|^2 dx < \varepsilon$$

for  $|y| < R$  and sufficiently large  $n$ . Therefore (1.19) holds for  $u_n - u_0$  and in view of Lemma 1.5 we get

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \Psi(u_n - u_0) dx = 0$$

and (A.2) holds. Now observe that for any  $\varepsilon > 0$ ,  $2 < p < 2^* < q$  we find  $0 < \delta < M$  and  $c_\varepsilon > 0$  such that

$$|\Psi'(s)| \leq \varepsilon |s|^{2^*-1} \quad \text{for } |s| < \delta \text{ and } |s| > M,$$

and

$$|\Psi'(s)| \leq c_\varepsilon \min \left\{ |s|^{2^*(1-\frac{1}{p})}, |s|^{2^*(1-\frac{1}{q})} \right\} \quad \text{for } \delta \leq |s| \leq M.$$

Then, by the Vitali convergence theorem and by (A.2) applied to  $\tilde{\Psi}(s) = \min\{|s|^p, |s|^q\}$  and  $(u_n - u_0)$  we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}^N} \Psi'(u_n) u_n - \Psi(u_0) u_0 dx \right| &\leq \int_{\mathbb{R}^N} |\Psi'(u_n) - \Psi'(u_0)| |u_0| dx \\ &\quad + \int_{\mathbb{R}^N} |\Psi'(u_n)| |u_n - u_0| dx = o(1) + \int_{\mathbb{R}^N} |\Psi'(u_n)| |u_n - u_0| dx \\ &\leq o(1) + \varepsilon |u_n|_{2^*}^{2^*-1} |u_n - u_0|_{2^*} + \int_{|u_n - u_0| > 1} |u_n|^{2^*(1-\frac{1}{p})} |u_n - u_0| dx \end{aligned}$$



$$\begin{aligned}
 & + \int_{|u_n - u_0| \leq 1} |u_n|^{2^*(1-\frac{1}{q})} |u_n - u_0| dx \\
 & \leq o(1) + \varepsilon |u_n|_{2^*}^{2^*-1} |u_n - u_0|_{2^*} + |u_n|_{2^*}^{2^*(1-\frac{1}{p})} \left( \int_{\mathbb{R}^N} \tilde{\Psi}(u_n - u_0) dx \right)^{\frac{1}{p}} \\
 & \quad + |u_n|_{2^*}^{2^*(1-\frac{1}{q})} \left( \int_{\mathbb{R}^N} \tilde{\Psi}(u_n - u_0) dx \right)^{\frac{1}{q}} \\
 & \leq o(1) + \varepsilon |u_n|_{2^*}^{2^*-1} |u_n - u_0|_{2^*}.
 \end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary we infer that

$$\int_{\mathbb{R}^N} \Psi'(u_n) u_n dx \rightarrow \int_{\mathbb{R}^N} \Psi'(u_0) u_0 dx.$$

□

**Proposition A.2.** *Let  $\mathcal{O} = \mathcal{O}' \times \text{id} \subset \mathcal{O}(N)$  such that  $\mathcal{O}' \subset \mathcal{O}(M)$  and  $\mathbb{R}^M$  is compatible with  $\mathcal{O}'$  for some  $0 \leq M \leq N$ . Suppose that  $(u_n) \subset \mathcal{D}_{\mathcal{O}}^{1,2}(\mathbb{R}^N)$  is bounded,  $r_0 > 0$  is such that for all  $r \geq r_0$*

$$(A.4) \quad \lim_{n \rightarrow \infty} \sup_{z \in \mathbb{R}^{N-M}} \int_{B((0,z),r)} |u_n|^2 dx = 0.$$

Then

$$\int_{\mathbb{R}^N} \Psi(u_n) dx \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for any continuous function  $\Psi : \mathbb{R} \rightarrow [0, \infty)$  such that (1.20) holds.

*Proof.* Suppose that

$$(A.5) \quad \int_{B((y_n, z_n), r_1)} |u_n|^2 dx \geq c > 0$$

for some sequence  $(y_n, z_n) \subset \mathbb{R}^M \times \mathbb{R}^{N-M}$  and a constant  $c$ , where  $r_1$  is such that

$$\lim_{|y| \rightarrow \infty, y \in \mathbb{R}^M} m(y, r_1) = \infty.$$

Then  $\int_{B((y_n, z_n), r_1)} |u_n|^{2^*} dx$  is bounded away from 0. Since  $(u_n)$  is bounded in  $L^{2^*}(\mathbb{R}^N)$  and in the family  $\{B(gy_n, r_1)\}_{g \in \mathcal{O}'}$  we find an increasing number of disjoint balls as  $|y_n| \rightarrow \infty$ , we infer that  $|y_n|$  must be bounded. Then for sufficiently large  $r$  one obtains

$$\int_{B((0, z_n), r)} |u_n|^2 dx \geq \int_{B((y_n, z_n), r_1)} |u_n|^2 dx \geq c > 0$$

and we get a contradiction with (A.4). Therefore (1.19) is satisfied with  $r = r_1$  and by Lemma 1.5 we conclude. □

At the end of this section we would like to mention that the above variant of Brezis-Lieb lemma (A.1) and Lemma 1.5 allow to obtain the following profile decomposition theorem in  $\mathcal{D}^{1,2}(\mathbb{R}^N)$  in the spirit of Gérard [14], cf. [25].

**Theorem A.3.** *Suppose that  $(u_n) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$  is bounded. Then there are sequences  $(\tilde{u}_i)_{i=0}^\infty \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ ,  $(y_n^i)_{i=0}^\infty \subset \mathbb{R}^N$  for any  $n \geq 1$ , such that  $y_n^0 = 0$ ,  $|y_n^i - y_n^j| \rightarrow \infty$  as  $n \rightarrow \infty$  for  $i \neq j$ , and passing to a subsequence, the following conditions hold for any  $i \geq 0$ :*

$$(A.6) \quad \begin{aligned} & u_n(\cdot + y_n^i) \rightharpoonup \tilde{u}_i \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^N) \text{ as } n \rightarrow \infty, \\ & \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx = \sum_{j=0}^i \int_{\mathbb{R}^N} |\nabla \tilde{u}_j|^2 dx + \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla v_n^i|^2 dx, \end{aligned}$$

where  $v_n^i := u_n - \sum_{j=0}^i \tilde{u}_j(\cdot - y_n^j)$  and

$$(A.7) \quad \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \Psi(u_n) dx = \sum_{j=0}^i \int_{\mathbb{R}^N} \Psi(\tilde{u}_j) dx + \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \Psi(v_n^i) dx$$

for any function  $\Psi : \mathbb{R} \rightarrow \mathbb{R}$  of class  $\mathcal{C}^1$  such that  $|\Psi'(s)| \leq C|s|^{2^*-1}$  for any  $s \in \mathbb{R}$  and some constant  $C > 0$ . Moreover, if in addition  $\Psi$  satisfies (1.20), then

$$(A.8) \quad \lim_{i \rightarrow \infty} \left( \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \Psi(v_n^i) dx \right) = 0.$$

*Proof.* In order to prove (A.6)–(A.8), we follow arguments of proof of [24][Theorem 1.4] with some modifications. Namely, let  $(u_n) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$  be a bounded sequence and  $\Psi$  as above. Applying Lemma 1.5 we find  $K \in \mathbb{N} \cup \{\infty\}$  and there is a sequence  $(\tilde{u}_i)_{i=0}^K \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ , for  $0 \leq i < K + 1$  ( $K = \infty$  then  $K + 1 = \infty$  as well), there are sequences  $(v_n^i) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ ,  $(y_n^i) \subset \mathbb{R}^N$  and positive numbers  $(c_i)_{i=0}^K, (r_i)_{i=0}^K$  such that  $y_n^0 = 0$ ,  $r_0 = 0$  and, up to a subsequence, for any  $n$  and  $0 \leq i < K + 1$  one has

$$\begin{aligned} & u_n(\cdot + y_n^i) \rightharpoonup \tilde{u}_i \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^N) \text{ and } \int_{B(0,n)} |u_n(\cdot + y_n^i) - \tilde{u}_i|^2 dx \rightarrow 0 \text{ as } n \rightarrow \infty, \\ & \tilde{u}_i \neq 0 \text{ for } 1 \leq i < K + 1, \\ & |y_n^i - y_n^j| \geq n - r_i - r_j \text{ for } j \neq i, 0 \leq j < K + 1, \\ & v_n^i := v_n^{i-1} - \tilde{u}_i(\cdot - y_n^i), \\ & \int_{B(y_n^i, r_i)} |v_n^{i-1}|^2 dx \geq c_i \geq \frac{1}{2} \sup_{y \in \mathbb{R}^N} \int_{B(y, r_i)} |v_n^{i-1}|^2 dx \\ & \geq \frac{1}{4} \sup_{r > 0, y \in \mathbb{R}^N} \int_{B(y, r)} |v_n^{i-1}|^2 dx > 0, r_i \geq \max\{i, r_{i-1}\} \text{ for } i \geq 1, \end{aligned}$$

and (A.6) is satisfied. Next, we prove that (A.7) holds for every  $i \geq 0$  by applying (A.1). If there is  $i \geq 0$  such that

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y, r)} |v_n^i|^2 dx = 0$$

for every  $r \geq \max\{i, r_i\}$ , then  $K = i$ . If, in addition, (1.20) holds, then in view of Lemma 1.5 we obtain that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \Psi(v_n^i) dx = 0$$

and we finish the proof by setting  $\tilde{u}_j = 0$  for  $j > i$ . Otherwise we have  $K = \infty$  and we prove (A.8) similarly as in [24][Theorem 1.4].  $\square$

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