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A Comparison of Markov Chain Methods for Reliability Estimation

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Two Markov chain Monte Carlo simulation methods for reliability estimation, subset simulation and the moving particles algorithm, are compared based on theoretical arguments and test cases. The differences in the efficiency between both algorithms are rather small. They seem to be well suited for off-the-shelf reliability estimations, but with a different setting for the most important parameters (proposal density and initial sample size).

1 Subset Simulation

Subset Simulation [1] is based on nested sets, $F_1 \supset F_2 \supset \ldots F_M$, where $F_M = \{\theta \in \mathbb{R}^n | g(\theta) < 0\}$ denotes the failure region and $g(\theta)$ is the performance function. If θ is a vector of random variables, the failure probability is given by

$$P_f = P_{F_1} \prod_{i=1}^{M-1} P(F_{i+1}|F_i). \tag{1}$$

The rather small failure probability is written as the product of larger probabilities that can be estimated with less effort. However, the estimation of the conditional probabilities requires the application of MCMC simulations, because the corresponding conditional probability density function is not known explicitly. In subset simulation, N_s parallel Markov chains are started from seeds that for step $i-1$ lie in F_i. The most influential parameters of subset simulation are the transition kernel of the MCMC algorithm and the sets F_i, for which sets of equal conditional probability $p_0 = P(F_{i+1}|F_i)$, for $i = 1, \ldots, M$, are preferred. Given p_0, the sets are obtained from a percentile estimation for the performance function. The failure probability is then $P_f = p_0^{M-1} \hat{P}_M$, where $\hat{P}_M$ is the estimate for $P(F_M|F_{M-1})$. The coefficient of variation of the estimator for the conditional probability $P_{i+1} = P(F_{i+1}|F_i)$ is given by $\sqrt{\frac{1-P_i}{NP_i}(1+\gamma_i)}$ [1], where the additional term γ_i is $\gamma_i = 2\sum_{k=1}^{N/N_s-1}\left(1-\frac{kN_s}{N}\right)\rho_i(k)$. $N_s = p_0 N$ is the number of samples that for step $i-1$ lie in F_i and constitute the initial values (seeds) of the Markov chains. $\rho_i(k)$ is the correlation coefficient of the series $I_{F_i}(\theta_{jk}^{(i-1)})$, where $\theta_{jk}^{(i-1)}$ is the kth sample of the j Markov chain that is generated in step $i-1$ and $I_{F_i}(\cdot)$ denotes the indicator function of F_i. A weak correlation of the samples produced by the Markov chain is thus necessary for a reduction of the coefficient of variation. The coefficient of variation for the failure probability estimated with subset simulation can be approximated by $\delta_{sub} \approx \sqrt{\frac{\log p_F}{\log p_0}\frac{(1-p_0)(1+\bar{\gamma})}{Np_0}}$, where $\bar{\gamma}$ is the average value of γ_i (averaged over the number of subsets) and $\frac{\log p_F}{\log p_0}$ represents the number of subsets. A typical value for $\bar{\gamma}$ is $\frac{1}{1+\bar{\gamma}} = 0.4$, cf. [2]. The average number of function evaluation is thus approximated by

$$N_{sub} = N\left(1 + (1-p_0)\frac{\log p_F}{\log p_0}\right), \tag{2}$$

and is composed by the initial Monte Carlo samples and the Markov chain samples (without burn-in) at each step.

2 Moving Particles Algorithm

The moving particles algorithm can be considered as subset simulation with a maximum number of steps. For each step, the values $g(\theta_j)$, $j = 1, \ldots, N_s$, of the N_s samples are ranked. The sample with the maximum value of the performance function is moved: MCMC is carried out starting from one of the remaining samples and the final state of the Markov chain is accepted, if the value of the performance function could be reduced. Otherwise, the sample is simply replaced by the seed of the Markov chain. Instead of computing the probability of failure from eq. (1), each initial sample is moved until it reaches the failure region and the number of moves is count. As has been shown in [3], the number of moves to get an initial sample into the failure region follows a Poisson distribution with parameter $\lambda = \log \frac{1}{p_F}$. The estimator for the parameter of the Poisson distribution is $\hat{\lambda} = \frac{\sum_{j=1}^{N_s} M_j}{N_s}$, where M_j denotes the number of moves until sample j reaches the failure state. In order to obtain an unbiased estimate, it is mandatory that the trajectories of the Poisson process generated from the initial samples remain independent until the samples finally reach the failure domain. A burn-in period for the Markov chain should ensure the independence

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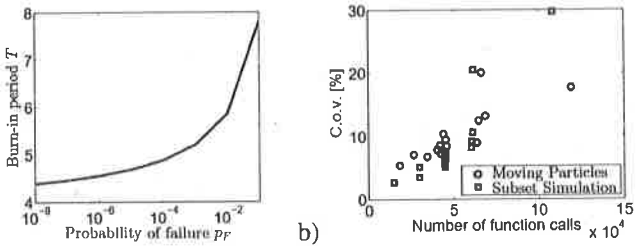
of the candidate and the seed of the Markov chain. The coefficient of variation for the failure probability estimated with the moving particles algorithm is given by $\delta_{mp} = \sqrt{\frac{-\log p_F}{N_m}}$, cf. [3], and the average number of function evaluations is

$$N_{mp} = N_m(1 - T\log p_F), \tag{3}$$

where the first term accounts for the initial Monte Carlo simulation and the second term for the Markov chain samples (with burn-in period T). A parallel version of the algorithm is easily obtained, if the k samples with highest values of the performance function are moved in parallel.

3 Comparison

The algorithms differ in the following points: (1) In subset simulation, the number of steps is rather small; however, the number of steps in the moving particles algorithm is maximal. (2) In subset simulation, only p_0 (usually 10%) of the samples are retained in each step and serve as seed for the Markov chains. In contrast, only one sample is resampled in each step of the moving particles algorithm, and the seed can be selected among the other samples. (3) The moving particles algorithm has a clear interpretation from Poisson process theory, but requires that all initial samples finally reach the failure region.



a) b)

Fig. 1: a) Burn-in period, such that the number of performance function evaluations of subset simulation and the moving particles algorithm is approximately the same. b) Efficiency of subset simulation and the moving particles algorithm.

The efficiency of both algorithms can be compared by setting $N_m = 0.1N$ for the number of initial samples and $p_0 = 0.1$ as before. In this case, the coefficient of variation of both algorithms will be nearly the same. From equations (2) and (3), one obtains the relationship

$$\frac{N_{sub}}{N_{mp}} = \frac{10(1 - 0.39\log p_F)}{1 - T\log p_F} \tag{4}$$

By setting this expression to one, a burn-in period T can be obtained as a function of the failure probability for which both algorithms would require approximately the same amount of function evaluations. This relationship is depicted in Fig. 1 a). It can be seen that the obtained burn-in period is in the range of values that has been found to be sufficient and thus the number of function evaluations for both algorithms is of the same order of magnitude. For small failure probabilities, subset simulation becomes slightly more efficient than the moving particles algorithm, while for larger probabilities of failure, the opposite is the case. The efficiency of both algorithms has been compared for the test cases summarized in [5], Table 1. Figure 1 b), underlines that both algorithms lead to a similar efficiency. However, while the number of function evaluations is approximately the same, the coefficient of variation for the results from the moving particles algorithm is slightly higher than for subset simulation. As a burn-in period of $T = 5$ has been applied for the moving particles algorithm and the failure probabilities are in the range from 10^{-4} to 10^{-6}, this confirms the results of Figure 1 a). Finally, both algorithms have been applied to high dimensional test cases, a single-degree-of-freedom oscillator with 1500 random variables and a paraboloid with varying curvatures. Both algorithms yielded results of similar accuracy and efficiency. Acceptance rates were similar as for the low-dimensional test cases.

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