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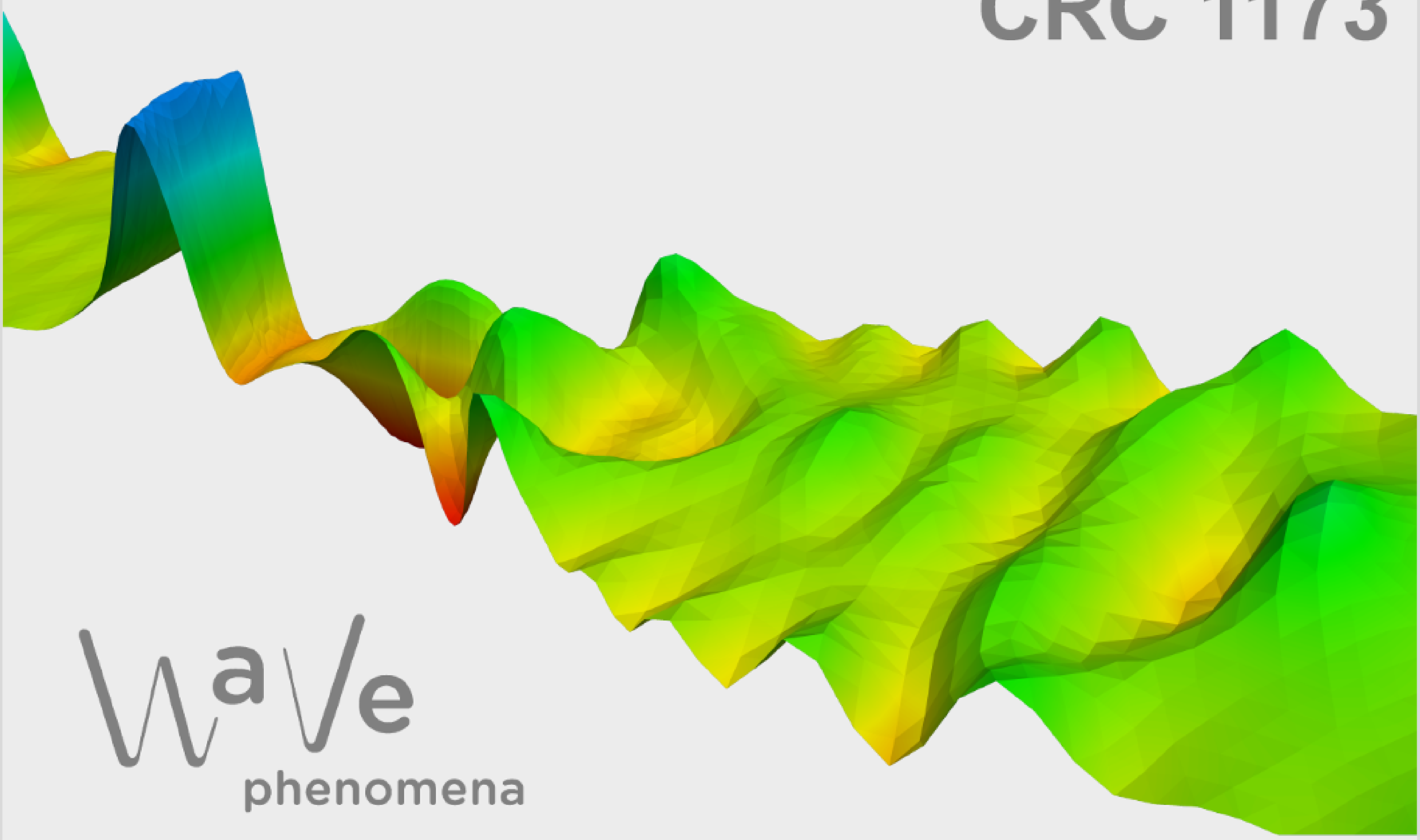

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# ASYMPTOTIC PRESERVING TRIGONOMETRIC INTEGRATORS FOR THE QUANTUM ZAKHAROV SYSTEM

SIMON BAUMSTARK AND KATHARINA SCHRATZ

Abstract. We present a new class of asymptotic preserving trigonometric integrators for the quantum Zakharov system. Their convergence holds in the strong quantum regime $\vartheta = 1$ as well as in the classical regime $\vartheta \to 0$ without imposing any step size restrictions. Moreover, the new schemes are asymptotic preserving and converge to the classical Zakharov system in the limit $\vartheta \to 0$ uniformly in the time discretization parameter. Numerical experiments underline the favorable error behavior of the new schemes with first- and second-order time convergence uniformly in $\vartheta$, first-order asymptotic convergence in $\vartheta$ and long time structure preservation properties.

## 1. Introduction

We consider the quantum Zakharov system

$$\begin{aligned} &i\partial_t E + \Delta E - \vartheta \Delta^2 E = Eu, \quad E(0,x) = E^0(x),\\ &\partial_{tt} u - \Delta u + \vartheta \Delta^2 u = \Delta |E|^2, \quad u(0,x) = u^0(x),\, \partial_t u(0,x) = {u'}^0(x), \end{aligned} \tag{1.1}$$

which describes the nonlinear interaction between quantum Langmuir waves and quantum ion-acoustic waves in an electron-ion dense quantum plasma. In the above model the quantum parameter $\vartheta > 0$ expresses the ratio between the ion plasmon energy and the electron thermal energy. We refer to [14, 17, 24, 37] and references therein for its physical motivation and derivation by a quantum fluid approach. Here, $z(t,x) \in \mathbb{C}$ denotes the varying envelope of the highly-oscillatory electric field and $n(t,x) \in \mathbb{R}$ is proportional to the ion-density fluctuation. Setting $\vartheta = 0$ in (1.1) yields the *classical Zakharov* system which is described by a coupled system of a Schrödinger equation for $z$ and a wave equation for $n$, see [8, 9, 10, 22, 40] and references therein.

The classical Zakharov system (that is $\vartheta = 0$ in (1.1)) is numerically extensively studied, see, e.g., [3, 4, 11, 12, 23, 18, 19, 27, 31, 36]. Various schemes have been proposed in case of $\vartheta = 0$ reaching from splitting methods up to trigonometric integrators. In context of the Klein–Gordon–Zakharov system we also refer to recent works [1, 2, 41]. However, up to our knowledge nothing is known so far in numerical analysis for the quantum Zakharov system (1.1), where the oscillations are driven by

$$\mathrm{e}^{\pm it\sqrt{-\Delta(1-\vartheta\Delta)}} \qquad \text{with} \qquad 0 \le \vartheta \le 1 \tag{1.2}$$

instead of the classical case $\mathrm{e}^{\pm it|\nabla|}$. While classical methods allow convergence bounds in the strong quantum regime $\vartheta = \mathcal{O}(1)$ and the classical regime $\vartheta = 0$,

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respectively, their error bounds do in general not hold uniformly with respect to the quantum parameter $\vartheta$. The latter is due to the fact that the highly oscillatory phases (1.2) are in general not preserved uniformly in $\vartheta$ under the discretisation. This phenomenon is illustrated in Figure 1, where soliton solutions ([39]) are simulated with the splitting method [31] generalized to the quantum setting (1.1). The simulation is carried out for different values of $\vartheta$ using the same time-step size: In the "strong quantum regime" (left picture) the shape of the solition is nicely preserved, whereas in the "classical regime" (right picture) the approximation fails.

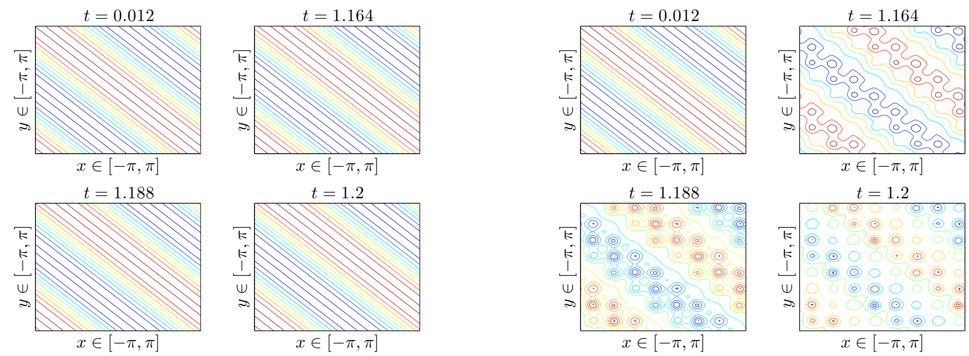


FIGURE 1. Contour plot: Simulation of soliton solution of the quantum Zakharov system (1.1) set on $\mathbb{T}^2$ with a classical splitting method (cf. [31]) for two different values of the quantum parameter $\vartheta$ using the same time-step size. Left picture: Strong quantum regime $\vartheta = 1$. Right picture: Classical regime $\vartheta = 0.01$.

The aim of this paper lies in the construction of a new class of *asymptotic preserving integrators* for the quantum Zakharov system (1.1) that

- converge in the strong quantum regime $\vartheta = \mathcal{O}(1)$ as well as the classical regime $\vartheta = 0$ without any $\vartheta$ dependent step size restriction, and
- converge asymptotically from the quantum to the classical Zakharov system without any step size restriction depending on the discretization parameter.

For the latter, note that on the continues level the quantum Zakharov system (1.1) reduces to the classical Zakharov system

$$i\partial_t E + \Delta E = Eu, \quad \partial_{tt} u - \Delta u = \Delta |E|^2 \tag{1.3}$$

in the classical limit $\vartheta \to 0$. More precisely, solutions of the quantum Zakharov system (1.1) are approximated by the classical Zakharov system (1.3) with convergence rate $\mathcal{O}(\vartheta)$, where the latter holds for sufficiently smooth solutions. The new constructed trigonometric integrators are designed such that this approximative property is inherited in the numerical discretization. In particular, convergence of order $\mathcal{O}(\vartheta)$ from the quantum to the classical Zakharov approximation holds *uniformly in the time-step size* $\tau$ (see Section 4 for details). Our techniques allow us to develop asymptotic preserving schemes of arbitrary order. We will give details on the construction of a first- and second-order scheme. For the development and analysis of asymptotic preserving schemes in the context of highly oscillatory Klein–Gordon type equations we also refer to [5, 13, 16].

The main challenge in constructing such a robust method for system (1.1) originates from the presence of derivatives in the nonlinearity: Mild solutions are given by

$$
\begin{aligned}
E(t) =& \mathrm{e}^{-it\Omega_\vartheta^2}E(0) - i\int_0^t \mathrm{e}^{-i(t-\xi)\Omega_\vartheta^2}u(\xi)E(\xi)\mathrm{d}\xi,\\
u(t) =& \cos(t\Omega_\vartheta)u(0) + \frac{\sin(t\Omega_\vartheta)}{\Omega_\vartheta}u'(0) - \int_0^t \sin((t-\xi)\Omega_\vartheta)\Omega_\vartheta^{-1}\Delta|E(\xi)|^2\mathrm{d}\xi,
\end{aligned}
\tag{1.4}
$$

where we define the operator

$$
\Omega_\vartheta := \sqrt{-\Delta(1-\vartheta\Delta)}. \tag{1.5}
$$

With the aid of the Fourier expansion $f(x) = \sum_{k\in\mathbb{Z}^d}\hat{f}_k\mathrm{e}^{ikx}$ we can express the action of (1.5) as follows

$$
\Omega_\vartheta f(x) = \sum_{k\in\mathbb{Z}^d}\sqrt{|k|^2(1+\vartheta|k|^2)}\hat{f}_k\mathrm{e}^{ikx}, \tag{1.6}
$$

where

$$
|k|^2 = k_1^2 + \ldots k_d^2 \quad \text{and} \quad kx = k_1x_1 + \ldots + k_dx_d.
$$

Note that the coupling operator $\Omega_\vartheta^{-1}\Delta$ in (1.4) allows the following bounds: There exists a $c>0$ such that

$$
\forall\vartheta>0 \quad : \quad \|\Omega_\vartheta^{-1}\Delta f\|_s \le c\inf\{\vartheta^{-1/2}\|f\|_s, \|f\|_{s+1}\}. \tag{1.7}
$$

"Naively" estimating the solutions thus yields for $s>d/2$ and all $\vartheta>0$

$$
\begin{aligned}
\|E(t)\|_s &\le \|E(0)\|_s + c\int_0^t \|u(\xi)\|_s\|E(\xi)\|_s\mathrm{d}\xi,\\
\|u(t)\|_s &\le \|u(0)\|_s + \|u'(0)\|_{s-1} + c\int_0^t \|E(\xi)\|_{s+1}^2\mathrm{d}\xi,
\end{aligned}
\tag{1.8}
$$

respectively,

$$
\begin{aligned}
\|E(t)\|_s &\le \|E(0)\|_s + c\int_0^t \|u(\xi)\|_s\|E(\xi)\|_s\mathrm{d}\xi,\\
\|u(t)\|_s &\le \|u(0)\|_s + \|u'(0)\|_{s-1} + c\vartheta^{-1/2}\int_0^t \|E(\xi)\|_s^2\mathrm{d}\xi.
\end{aligned}
\tag{1.9}
$$

The estimate (1.9) seems the preferable choice as (1.8) amounts in a *loss of derivative*. However, the bound (1.9) yields an error constant involving the term $\exp(\vartheta^{-1/2}T)$ which explodes in the classical limit regime $\vartheta\to0$.

In order to avoid any CFL condition depending on $\vartheta$ nor the spatial discretization paramter $\Delta x$ we follow the techniques developed in the analytical work [35] on the well posedness analysis of the classical Zakharov system: We will reformulate the quantum Zakharov system as a system in $(E, \partial_t E, u, \partial_t u)$. This allows us to overcome the loss of derivative and derive robust and uniformly accurate trigonometric integrators for the quantum Zakharov system (1.1) without imposing any type of CFL condition – neither depending on the step size $\tau$ nor on $\vartheta$. This idea was recently applied to the classical Zakharov system (that is $\vartheta=0$ in (1.1)) where a new stable class of trigonometric integrators were introduced, see [27]. This approach was recently also successfully applied in the context of splitting schemes for the Zakharov system ([23]). In context of the quantum Zakharov system the analysis is however more involved as our aim lies in asymptotic preserving schemes which converge in the limit $\vartheta \to 0$ to solutions of the classical Zakharov system (1.3) without any step size restriction. In particular, and compared to the classical case $\vartheta = 0$ (see, e.g., [27]) the main task lies in establishing error bounds that hold uniformly in the quantum parameter $\vartheta$ and the asymptotic analysis $\vartheta \to 0$ (see Section 4). For further results on trigonometric (and exponential) integrators for semilinear wave equations we refer to [20, 28, 29] and the references therein. For practical implementation reasons we will impose periodic boundary conditions, i.e., $x \in \mathbb{T}^d$, hence both $E$ and $u$ are considered to be spatially periodic.

## 2. Trigonometric integrators for the quantum Zakharov system

Following the strategy in [27, 35] we reformulate the quantum Zakharov system (1.1) as follows

$$\begin{aligned} &i\partial_t F - \Omega_\vartheta^2 F = uF + \partial_t u \left( E(0) + \int_0^t F(\xi)\mathrm{d}\xi \right), \\ &\partial_{tt} u + \Omega_\vartheta^2 u = \Delta |E|^2, \\ &E = (\Omega_\vartheta^2 + 1)^{-1} \left\{ iF - (u-1)\left( E(0) + \int_0^t F(\xi)\mathrm{d}\xi \right) \right\}, \end{aligned} \tag{2.1}$$

where $F := \partial_t E$ and

$$F(0) = -i\big(\Omega_\vartheta^2 E(0) + u(0)E(0)\big),\ u(0) = u^0,\ \partial_t u(0) = {u'}^0,\ E(0) = E^0. \tag{2.2}$$

Let

$$\mathcal{I}_F(t) := E_0 + \int_0^t F(\lambda)\mathrm{d}\lambda.$$

Then the mild solutions of (2.1) at time $t_{n+1} = t_n + \tau$ with $t_0 = 0$ read

$$\begin{aligned} F(t_n+\tau) =& \mathrm{e}^{-i\tau\Omega_\vartheta^2} F(t_n) - i\int_0^\tau \mathrm{e}^{-i(\tau-\xi)\Omega_\vartheta^2}\Big( (uF + u'\mathcal{I}_F)(t_n+\xi)\mathrm{d}\xi \Big) \\ u(t_n+\tau) =& \cos(\tau\Omega_\vartheta)u(t_n) + \Omega_\vartheta^{-1}\sin(\tau\Omega_\vartheta)u'(t_n) \\ &+ \int_0^\tau \Omega_\vartheta^{-1}\sin((\tau-\xi)\Omega_\vartheta)\Delta|E(t_n+\xi)|^2\mathrm{d}\xi, \\ u'(t_n+\tau) =& -\Omega_\vartheta \sin(\tau\Omega_\vartheta)u(t_n) + \cos(\tau\Omega_\vartheta)u'(t_n) \\ &+ \int_0^\tau \cos((\tau-\xi)\Omega_\vartheta)\Delta|E(t_n+\xi)|^2\mathrm{d}\xi, \\ E(t_n+\tau) =& (\Omega_\vartheta^2+1)^{-1}\big\{ iF(t_n+\tau) - (u(t_n+\tau)-1)\mathcal{I}_F(t_n+\tau) \big\}. \end{aligned} \tag{2.3}$$

In order to construct a numerical scheme which preserves the oscillation (1.2) in the quantum parameter $\vartheta$ we approximate the exact solutions $(u, u', F, E)(t_n + \xi)$ appearing in the above integrals via Taylor series expansion. This allows us to integrate the oscillatory terms $\mathrm{e}^{-i\xi\Omega_\vartheta^2}$, $\cos(\xi\Omega_\vartheta)$ and $\sin(\xi\Omega_\vartheta)$ exactly. Furthermore, we use the following approximation for the integrals over $F$: For $0 \le \xi \le \tau$ we approximate

$$\int_0^{t_n+\xi} F(\lambda)\mathrm{d}\lambda \approx \tau \sum_{k=0}^{n} F(t_k).$$

This yields for $n \geq 0$ the trigonometric time integration scheme

$$
\begin{aligned}
F^{n+1} &= \mathrm{e}^{-i\tau\Omega_\vartheta^2}F^n + i\tau\frac{1-\mathrm{e}^{-i\tau\Omega_\vartheta^2}}{-i\tau\Omega_\vartheta^2}\left(u^nF^n + u'^nE^0 + u'^n\big(\tau\sum_{k=0}^{n}F^k\big)\right),\\
u^{n+1} &= \cos(\tau\Omega_\vartheta)u^n + \Omega_\vartheta^{-1}\sin(\tau\Omega_\vartheta)u'^n + \tau\Omega_\vartheta^{-1}\frac{1-\cos(\tau\Omega_\vartheta)}{\tau\Omega_\vartheta}\Delta|E^n|^2,\\
u'^{n+1} &= -\Omega_\vartheta\sin(\tau\Omega_\vartheta)u^n + \cos(\tau\Omega_\vartheta)u'^n + \tau\frac{\sin(\tau\Omega_\vartheta)}{\tau\Omega_\vartheta}\Delta|E^n|^2,\\
E^{n+1} &= (\Omega_\vartheta^2+1)^{-1}\left\{iF^{n+1} - (u^{n+1}-1)\big(E^0 + (\tau\sum_{k=0}^{n}F^{k+1})\big)\right\}
\end{aligned}
\tag{2.4}
$$

equipped with the initial values (5). This scheme is an asymptotic preserving extension of the trigonometric integration scheme derived in [27] for the classical Zakharov system (1.3). The new scheme (2.4) in particular respects the oscillatory structure (1.2) which strongly depends on the quantum paramter $\vartheta$. In particular, in Section 4 below we will prove that in the limit $\vartheta \to 0$ it asymptotically converges towards the classical integration scheme derived in [27] without any step size restrictions (neither on $\tau$ nor on $\vartheta$).

## 3. Error analysis

In the following we set for $f(x) = \sum_{k\in\mathbb{Z}^d}\hat{f}(k)\mathrm{e}^{ik\cdot x}$ and $s \in \mathbb{R}$

$$
\langle\nabla\rangle^s f(x) := |\nabla|^s f(x) + \hat{f}(0), \quad \langle\Omega_\vartheta\rangle^s f(x) := \Omega_\vartheta^s f(x) + \hat{f}(0)
$$

and define

$$
\|f\|_s := \|\langle\nabla\rangle^s f\|_{L^2(\mathbb{T}^d)}.
$$

Furthermore, for $\alpha \in \mathbb{R}$ we set

$$
m_{s,\alpha,\vartheta}(T) := \sup_{0\leq t\leq T}\{\|\langle\Omega_\vartheta\rangle^{2\alpha+2}E(t)\|_s + \|\langle\Omega_\vartheta\rangle^{\alpha+1}u(t)\|_s + \|\langle\Omega_\vartheta\rangle^{\alpha}u'(t)\|_s\}. \tag{3.1}
$$

**Theorem 3.1.** *Fix $s > d/2$ and $\vartheta \geq 0$. For any $T > 0$ suppose that*

$$
m_{s,1,\vartheta}(T) = \sup_{0\leq t\leq T}\{\|\langle\Omega_\vartheta\rangle^4E(t)\|_s + \|\langle\Omega_\vartheta\rangle^2u(t)\|_s + \|\langle\Omega_\vartheta\rangle u'(t)\|_s\} < \infty. \tag{3.2}
$$

*Then there exists a $\tau_0 > 0$ such that for all $0 < \tau \leq \tau_0$ and $t_n \leq T$*

$$
\|\langle\Omega_\vartheta\rangle^2(E(t_n) - E^n)\|_s + \|\langle\Omega_\vartheta\rangle(u(t_n) - u^n)\|_s + \|u'(t_n) - u'^n\|_s \leq c\tau, \tag{3.3}
$$

*where $c$ depends only on $m_{s,1,\vartheta}(T)$ as well as on $T$, $s$ and $d$.*

*Remark* 3.2. Local-wellposedness results for the quantum Zakharov system (1.1) are given in [14]. For local-wellposedness of the classical Zakharov system in Sobolev spaces of low regularity on $\mathbb{T}^d$ we refer to [9, 38, 32]. Concerning the well-posedness theory on $\mathbb{R}^d$ we refer to [35, 10, 22, 7, 6] and references therein.

*Proof of Theorem 3.1.* Compared to the classical case $\vartheta = 0$ (see, e.g., [27]) the main task lies in establishing error bounds that hold uniformly in the quantum parameter $\vartheta$.

Fix $s > d/2$ and $\vartheta \geq 0$. In the following let $c$ denote a numerical constant that does not depend on $\vartheta$ nor $\tau, n$. Subtracting the numerical solutions (2.4) from the exact solutions (2.3) yields

$$\begin{aligned} F(t_{n+1}) - F^{n+1} &= \mathrm{e}^{-i\tau\Omega_\vartheta^2}(F(t_n) - F^n) \\ &\quad + i\tau\frac{1-\mathrm{e}^{-i\tau\Omega_\vartheta^2}}{-i\tau\Omega_\vartheta^2}\Big(u(t_n)(F(t_n)-F^n) + (u(t_n)-u^n)F^n \\ &\quad + (u'(t_n) - u'^n)E(0) + u'(t_n)\big(\tau\sum_{k=0}^n (F(t_k) - F^k)\big) \\ &\quad + (u'(t_n)-u'^n)(\tau\sum_{k=0}^n F^k)\Big) + L_F^n, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \langle\Omega_\vartheta\rangle(u(t_{n+1}) - u^{n+1}) =\ &\cos(\tau\Omega_\vartheta)\langle\Omega_\vartheta\rangle(u(t_n)-u^n) \\ &+ \sin(\tau\Omega_\vartheta)\frac{\langle\Omega_\vartheta\rangle}{\Omega_\vartheta}(u'(t_n)-u'^n) \\ &+ \tau\frac{1-\cos(\tau\Omega_\vartheta)}{\tau\Omega_\vartheta}\frac{\langle\Omega_\vartheta\rangle}{\Omega_\vartheta}\Delta\left(|E(t_n)|^2 - |E^n|^2\right) \\ &+ \langle\Omega_\vartheta\rangle L_u^n, \end{aligned} \tag{3.5}$$

as well as

$$\begin{aligned} u'(t_{n+1}) - u'^{n+1} =\ &-\sin(\tau\Omega_\vartheta)\frac{\Omega_\vartheta}{\langle\Omega_\vartheta\rangle}\langle\Omega_\vartheta\rangle(u(t_n)-u^n) \\ &+ \cos(\tau\Omega_\vartheta)(u'(t_n)-u'^n) \\ &+ \tau\frac{\sin(\tau\Omega_\vartheta)}{\tau\Omega_\vartheta}\Delta\left(|E(t_n)|^2-|E^n|^2\right) + L_{u'}^n, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} E(t_{n+1}) - E^{n+1} =\ &-(\Omega_\vartheta^2+1)^{-1}\Big\{i(F(t_{n+1}) - F^{n+1}) \\ &- (u(t_{n+1}) - u^{n+1})\big(E(0) + \tau\sum_{k=0}^n F^{k+1})\big) \\ &+ (1-u(t_{n+1}))\big(\tau\sum_{k=0}^n (F(t_{k+1}) - F^{k+1})\big) + \langle\Omega_\vartheta\rangle^2 L_E^n\Big\}. \end{aligned} \tag{3.7}$$

The local errors at time $t_n$ satisfy

$$\begin{aligned} \|L_F^n\|_s &= \Big\|\int_0^\tau \mathrm{e}^{-i(\tau-\xi)\Omega_\vartheta^2}\Big(u(t_n+\xi)F(t_n+\xi) - u(t_n)F(t_n) \\ &\qquad + u'(t_n+\xi)\mathcal{I}_F(t_n+\xi) - u'(t_n)\big(E(0) + \tau\sum_{k=0}^n F(t_k)\big)\Big)\mathrm{d}\xi\Big\|_s, \\ \|\langle\Omega_\vartheta\rangle L_u^n\|_s &= \Big\|\frac{\langle\Omega_\vartheta\rangle}{\Omega_\vartheta}\int_0^\tau \sin((\tau-\xi)\Omega_\vartheta)\Delta\left(|E(t_n+\xi)|^2 - |E(t_n)|^2\right)\mathrm{d}\xi\Big\|_s, \\ \|L_{u'}^n\|_s &= \Big\|\int_0^\tau \cos((\tau-\xi)\Omega_\vartheta)\Delta\left(|E(t_n+\xi)|^2 - |E(t_n)|^2\right)\mathrm{d}\xi\Big\|_s, \\ \|\langle\Omega_\vartheta\rangle^2 L_E^n\|_s &= \Big\|(1-u(t_{n+1}))\Big(\int_0^{t_n+\tau} F(\lambda)\mathrm{d}\lambda - \tau\sum_{k=0}^n F(t_{k+1})\Big)\Big\|_s. \end{aligned} \tag{3.8}$$

Set

$$m_{s,0,\vartheta}^n := \max_{0\le k\le n}\{\|\langle\Omega_\vartheta\rangle^2 E^n\|_s + \|F^n\|_s + \|\langle\Omega_\vartheta\rangle u^n\|_s + \|u'^n\|_s\}.$$

*(i) Error in $F$:* Note that for all $\tau\in\mathbb{R}$

$$\|\mathrm{e}^{i\tau\Omega_\vartheta^2}\|_s \le 1, \quad \|(i\tau\Omega_\vartheta^2)^{-1}(1-\mathrm{e}^{i\tau\Omega_\vartheta^2})\|_s \le 2. \tag{3.9}$$

Furthermore, for all $\delta,\gamma\ge 0$ we have

$$\begin{aligned}\|f\|_{s+\delta} &= \sum_{k\in\mathbb{Z}^d}|k|^{2s+2\delta}|\hat f_k|^2 + |\hat f_0|^2 \le \sum_{k\in\mathbb{Z}^d}|k|^{2s+2\delta}(1+\vartheta|k|)^{2\delta}|\hat f_k|^2 + |\hat f_0|^2\\ &\le \|\langle\Omega_\vartheta\rangle^\delta f\|_s \le \|\langle\Omega_\vartheta\rangle^{\delta+\gamma} f\|_s.\end{aligned} \tag{3.10}$$

Thus, plugging the stability bound (3.9) into the error recursion (3.4) yields that

$$\begin{aligned}\|F(t_{n+1}) - F^{n+1}\|_s &\le \big(1+\tau c t_n m_{s,0,\vartheta}(t_n)\big)\max_{0\le k\le n}\|F(t_k)-F^k\|_s\\ &+ c\tau\big(m_{s,0,\vartheta}(0) + t_n m^n_{s,0,\vartheta}\big)\left(\|\langle\Omega_\vartheta\rangle(u(t_n)-u^n)\|_s + \|u'(t_n)-u'^n\|_s\right) + \|L_F^n\|_s.\end{aligned} \tag{3.11}$$

*(ii) Error in $E$:* The error recursion (3.7) together with (3.10) implies that

$$\begin{aligned}\|\langle\Omega_\vartheta\rangle^2 E(t_n) - E^n\|_s &\le (1+ct_n m_{s,0,\vartheta}(t_n))\max_{0\le k\le n}\|F(t_k)-F^k\|_s\\ &+ c\big(m_{s,0,\vartheta}(0)+t_n m^n_{s,0,\vartheta}\big)\|\langle\Omega_\vartheta\rangle(u(t_n)-u^n)\|_s + \|\langle\Omega_\vartheta\rangle^2 L_E^{n-1}\|_s.\end{aligned} \tag{3.12}$$

*(iii) Error in $(\langle\nabla\rangle u, u')$:* Formula (3.6) and (3.5) imply that

$$\begin{aligned}\begin{pmatrix}\langle\Omega_\vartheta\rangle(u(t_{n+1})-u^{n+1})\\ u'(t_{n+1})-u'^{n+1}\end{pmatrix} &= O_\tau\begin{pmatrix}\langle\Omega_\vartheta\rangle(u(t_n)-u^n)\\ u'(t_n)-u'^n\end{pmatrix}\\ &+\tau\begin{pmatrix}\frac{1-\cos(\tau\Omega_\vartheta)}{\tau\Omega_\vartheta}\frac{\langle\Omega_\vartheta\rangle}{\Omega_\vartheta}\\ \frac{\sin(\tau\Omega_\vartheta)}{\tau\Omega_\vartheta}\end{pmatrix}\Delta\left(|E(t_n)|^2 - |E^n|^2\right) + \begin{pmatrix}\langle\Omega_\vartheta\rangle L_u^n\\ L_{u'}^n\end{pmatrix},\end{aligned} \tag{3.13}$$

with the rotation matrix

$$O_\tau = \begin{pmatrix}\cos(\tau\Omega_\vartheta) & \sin(\tau\Omega_\vartheta)\frac{\langle\Omega_\vartheta\rangle}{\Omega_\vartheta}\\ -\sin(\tau\Omega_\vartheta)\frac{\Omega_\vartheta}{\langle\Omega_\vartheta\rangle} & \cos(\tau\Omega_\vartheta)\end{pmatrix}. \tag{3.14}$$

Note that the error recursion (3.12) together with (3.10) yields that

$$\begin{aligned}\|E(t_n) - E^n\|_{s+2} &\le (1+ct_n m_{s,0,\vartheta}(t_n))\max_{0\le k\le n}\|F(t_k)-F^k\|_s\\ &+ c\big(m_{s,0,\vartheta}(0)+t_n m^n_{s,0,\vartheta}\big)\|\langle\Omega_\vartheta\rangle(u(t_n)-u^n)\|_s + \|\langle\Omega_\vartheta\rangle^2 L_E^{n-1}\|_s.\end{aligned} \tag{3.15}$$

Also note that the stability result [27, Lemma 3.5.] holds true as we have that

$$O_\tau = V^{-1}\mathrm{diag}(e^{i\tau\Omega_\vartheta}, e^{-i\tau\Omega_\vartheta})V + Z_\tau, \tag{3.16}$$

where

$$V = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix},\; Z_\tau := \begin{pmatrix}0 & \tau\frac{\sin(\tau\Omega_\vartheta)}{\tau\Omega_\vartheta}(\langle\Omega_\vartheta\rangle - \Omega_\vartheta)\\ \frac{\sin(\tau\Omega_\vartheta)}{\tau\langle\nabla\rangle}(\langle\Omega_\vartheta\rangle - \Omega_\vartheta) & 0\end{pmatrix}$$

sucht that the action of $Z_\tau$ is nothing but multiplication by $\tau$ of the zero mode of the second component weights with the sinc-like function $\frac{\sin(\tau\Omega_\vartheta)}{\tau\langle\nabla\rangle}$. Using (3.15) as well as (3.16) in (3.13) thus yields that

$$
\begin{aligned}
&\|\langle\Omega_\vartheta\rangle(u(t_{n+1}) - u^{n+1})\|_s + \|u'(t_{n+1}) - u'^{n+1}\|_s \\
&\le \Big(ct_n\big(m_{s,0,\vartheta}(t_n) + m^n_{s,0,\vartheta}\big)\big\{(1 + t_n m_{s,0,\vartheta}(t_n)) \max_{0\le k\le n} \|F(t_k) - F^k\|_s \\
&\quad + \max_{0\le k\le n} \|\langle\Omega_\vartheta\rangle^2 L^k_E\|_s\big\} + 2n \max_{0\le k\le n} (\|\langle\Omega_\vartheta\rangle L^k_u\|_s + \|L^k_{u'}\|_s)\Big)\mathrm{e}^{t_n\left(1+q_n\right)},
\end{aligned}
\tag{3.17}
$$

where $q_n \le c(1 + t_n)(m_{s,0,\vartheta}(t_n) + m^n_{s,0,\vartheta})^2$.

Collecting the results in (3.11), (3.12) and (3.17) together with the local error bounds given in Lemma 3.3 below yields the assertion by a bootstrap argument. □

**Lemma 3.3.** *Let $s > d/2$. Assume that*

$$m_{s,1,\vartheta}(t_{n+1}) < \infty.$$

*Then the local errors defined in (3.8) satisfy*

$$\max_{0\le k\le n} \{\|L^k_F\|_s + \|\langle\Omega_\vartheta\rangle L^k_u\|_s + \|L^k_{u'}\|_s + \tau\|\langle\Omega_\vartheta\rangle^2 L^k_E\|_s\} \le c\tau^2,$$

*where $c$ depends on $m_{s,1,\vartheta}(t_{n+1})$.*

*Proof.* The proof follows the line of argumentation to the proof of Lemma 3.4 in [27] by replacing the operator $(-\Delta)^{\frac{\alpha}{2}}$ by $\Omega^\alpha_\vartheta$ in the local error analysis. □

*Remark* 3.4. The quantum Zakharov system (1.1) has a Hamiltonian structure and if $\partial_t u(0, x)$ has mean zero,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} H(E, u) \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^d} u|E|^2 + ||\nabla|E|^2 + \vartheta|\Delta E|^2 + \frac{1}{2}\big(u^2 + \vartheta||\nabla|u|^2 + ||\nabla|^{-1}\partial_t u|^2\big)\mathrm{d}x = 0,
\end{aligned}
\tag{3.18}
$$

where $|\nabla| := \sqrt{-\Delta}$. Theorem 3.1 implies first-order convergence of the trigonometric integrator (2.4) in the corresponding energy space for sufficiently smooth data.

Furthermore, Theorem 3.1 together with the observation in (3.10) implies the following uniform convergence bound:

**Corollary 3.5.** *Fix $s > d/2$. For any $T > 0$ suppose that*

$$m_{s,1,*}(T) := \sup_{0\le\vartheta\le 1} m_{s,1,\vartheta}(T) < \infty. \tag{3.19}$$

*Then there exists a $\tau_0 > 0$ such that for all $0 < \tau \le \tau_0$, $t_n \le T$ and all $0 \le \vartheta \le 1$ we have*

$$\|E(t_n) - E^n\|_{s+2} + \|u(t_n) - u^n\|_{s+1} + \|u'(t_n) - u'^n\|_s \le c\tau, \tag{3.20}$$

*where $c$ depends only on $m_{s,1,*}(T)$ as well as on $T$, $s$ and $d$, but can be chosen uniformly with respect to $\vartheta$.*

## 4. Asymptotic convergence in the limit $\vartheta \to 0$

The classical Zakharov system (that is $\vartheta = 0$ in (1.1)) is approximated by the quantum Zakharov system for $\vartheta \to 0$. More precisely, the approximation holds with order $\mathcal{O}(\vartheta)$ for sufficiently smooth solutions. Denote by $(E_\vartheta, u_\vartheta, u'_\vartheta)$ the solutions of the quantum Zakharov system (1.1) and accordingly by $(E^n_\vartheta, u^n_\vartheta, u'^n_\vartheta)$ its numerical approximation by the trigonometric integration scheme defined in (2.4). Thus, in particular let $(E_0, u_0, u'_0)$ denote the exact solutions of the classical Zakharov system (1.3) and $(E^n_0, u^n_0, u'^n_0)$ their numerical approximation defined in (2.4) by setting $\vartheta = 0$. Then, for $s > d/2$ and sufficiently smooth solutions the following approximation holds

$$
\begin{aligned}
&\|E^n_0 - E^n_\vartheta\|_{s+2} + \|u^n_0 - u^n_\vartheta\|_{s+1} + \|u'^n_0 - u'^n_\vartheta\|_s \\
&\leq \|E_0(t_n) - E^n_0\|_{s+2} + \|u_0(t_n) - u^n_0\|_{s+1} + \|u'_0(t_n) - u'^n_0\|_s \\
&+ \|E_\vartheta(t_n) - E^n_\vartheta\|_{s+2} + \|u_\vartheta(t_n) - u^n_\vartheta\|_{s+1} + \|u'_\vartheta(t_n) - u'^n_\vartheta\|_s \\
&+ \|E_0(t_n) - E_\vartheta(t_n)\|_{s+2} + \|u_0(t_n) - u_\vartheta(t_n)\|_{s+1} + \|u'_0(t_n) - u'_\vartheta(t_n)\|_s \\
&\leq c\big(\tau + \vartheta + \|E_0(0) - E_\vartheta(0)\|_{s+2} + \|u_0(0) - u_\vartheta(0)\|_{s+1} + \|u'_0(0) - u'_\vartheta(0)\|_s\big).
\end{aligned} \tag{4.1}
$$

This naturally implies that for time steps $\tau \leq \vartheta$ the quantum approximation convergences to the classical approximation with order $\mathcal{O}(\vartheta)$. However, the even more desirable property of *asymptotic preservation* holds true for the new schemes (2.4).

**Theorem 4.1.** *Fix $s > d/2$. For any $T > 0$ and $\varepsilon > 0$ suppose that*

$$
M_{s,*}(T) = \sup_{0\leq t\leq T} \{\|E(t)\|_{s+6+2\varepsilon} + \|u(t)\|_{s+5+\varepsilon} + \|u'(t)\|_{s+4+\varepsilon}\} < \infty. \tag{4.2}
$$

*Then there exists a $\tau_0 > 0$ such that for all $0 < \tau \leq \tau_0$, $t_n \leq T$ and all $0 \leq \vartheta \leq 1$*

$$
\|\langle\Omega_\vartheta\rangle^2(E^n_0 - E^n_\vartheta)\|_s + \|\langle\Omega_\vartheta\rangle(u^n_0 - u^n_\vartheta)\|_s + \|u'^n_0 - u'^n_\vartheta\|_s \tag{4.3}
$$

$$
\leq c\big(\vartheta + \|\langle\Omega_\vartheta\rangle^2(E_0(0) - E_\vartheta(0))\|_s + \|\langle\Omega_\vartheta\rangle(u_0(0) - u_\vartheta(0))\|_s + \|u'_0(0) - u'_\vartheta(0)\|_s\big), \tag{4.4}
$$

*where $c$ depends only on $M_{s,*}(T)$ as well as on $T, s$ and $d$.*

*Proof.* First note that Theorem 3.1 together with the regularity assumptions (4.2) imply that there exists a $\tau_0 > 0$ such that for all $\tau \leq \tau_0$ and $t_n \leq T$ we have

$$
m^n_s := \sup_{0\leq\vartheta\leq 1} \max_{0\leq k\leq n} \Big\{\|\langle\Omega_\vartheta\rangle^2 E^k_\vartheta\|_s + \|F^k_\vartheta\|_s + \|\langle\Omega_\vartheta\rangle u^k_\vartheta\|_s + \|u'^k_\vartheta\|_s\Big\} < \infty, \tag{4.5}
$$

$$
m^n_{s,*} := \max_{0\leq k\leq n} \Big\{\|E^k_0\|_{s+6} + \|F^k_0\|_{s+4} + \|u^k_0\|_{s+5} + \|u'^k_0\|_{s+4}\Big\} < \infty. \tag{4.6}
$$

Note that

$$
\sup_{0\leq\vartheta\leq 1} \|\Omega^4_\vartheta f\|_s \leq 2\|f\|_{s+6}.
$$

Therefore the regularity assumptions (4.2) imply that $m_{s,1,\vartheta}(T) < \infty$. Hence, the first claim (4.5) follows by Theorem 3.1. The second claim (4.6) follows by replacing $s$ with $s + 4$ in Theorem 3.1. in [27].

*(i) Error in $F_0^n - F_\vartheta^n$:* The definition of the numerical solutions in (2.4) together with the stability bound (3.9) implies that

$$\begin{aligned}
&\|F_0^{n+1} - F_\vartheta^{n+1}\|_s \\
&\le (1+\tau m_s^n t_n c) \sup_{0\le k\le n} \|F_0^k - F_\vartheta^k\|_s + \tau c m_s^n \big(\|u_0^n - u_\vartheta^n\|_s + \|u_0'^n - u_\vartheta'^n\|_s\big) \\
&+ \| \left(\mathrm{e}^{-i\tau\Omega_\vartheta^2} - \mathrm{e}^{i\tau\Delta}\right) F_0^n\|_s \\
&+ \| \left(\frac{1-\mathrm{e}^{-i\tau\Omega_\vartheta^2}}{-i\tau\Omega_\vartheta^2} - \frac{1-\mathrm{e}^{i\tau\Delta}}{i\tau\Delta}\right) \left(u_0^n F_0^n + u_0'^n E_0^0 + u_0'^n \big(\tau \sum_{k=0}^n F_0^k\big)\right)\|_s.
\end{aligned} \tag{4.7}$$

The definition of $\Omega_\vartheta$ in (1.5) implies that

$$\| \left(\mathrm{e}^{-i\tau\Omega_\vartheta^2} - \mathrm{e}^{i\tau\Delta}\right) f\|_s = \| \left(1 - \mathrm{e}^{i\tau(\Delta+\Omega_\vartheta^2)}\right) f\|_s \le \tau\|(\Delta+\Omega_\vartheta^2)f\|_s \le \tau\vartheta\|f\|_{s+4}. \tag{4.8}$$

Similarly, we have that

$$\| \left(\frac{1-\mathrm{e}^{-i\tau\Omega_\vartheta^2}}{-i\tau\Omega_\vartheta^2} - \frac{1-\mathrm{e}^{i\tau\Delta}}{i\tau\Delta}\right) f\|_s \le \tau\|(\Delta+\Omega_\vartheta^2)f\|_s \le \tau\vartheta\|f\|_{s+4}. \tag{4.9}$$

Plugging (4.8) and (4.9) into (4.7) yields

$$\begin{aligned}
\|F_0^{n+1} - F_\vartheta^{n+1}\|_s &\le (1+\tau m_s^n t_n c) \sup_{0\le k\le n} \|F_0^k - F_\vartheta^k\|_s \\
&\quad + \tau c m_s^n \big(\|u_0^n - u_\vartheta^n\|_s + \|u_0'^n - u_\vartheta'^n\|_s\big) + c\tau\vartheta m_{s,*}^n.
\end{aligned} \tag{4.10}$$

*(ii) Error in $E_0^n - E_\vartheta^n$:* By (2.4) we have that

$$\begin{aligned}
\Omega_\vartheta^2 E_\vartheta^{n+1} &= iF_\vartheta^{n+1} - u_\vartheta^{n+1}\big(E_\vartheta^0 + \tau \sum_{k=0}^n F_\vartheta^{k+1}\big), \\
-\Delta E_0^{n+1} &= iF_0^{n+1} - u_0^{n+1}\big(E_0^0 + \tau \sum_{k=0}^n F_0^{k+1}\big).
\end{aligned}$$

Using the bound

$$\| \left(\frac{\Omega_\vartheta^2}{\Delta} + 1\right) f\|_s = \| \left(\Omega_\vartheta^2 + \Delta\right) f\|_{s-2} \le \vartheta\|f\|_{s+2}$$

we therefore obtain that

$$\begin{aligned}
\|\langle\Omega_\vartheta\rangle^2(E_0^{n+1} - E_\vartheta^{n+1})\|_s &\le (1+c m_s^n t_n) \sup_{0\le k\le n} \|F_0^{k+1} - F_\vartheta^{k+1}\|_s \\
&\quad + c(m_s^0 + m_s^n t_n)\|u_0^{n+1} - u_\vartheta^{n+1}\|_s + c\vartheta m_{s,*}^n.
\end{aligned} \tag{4.11}$$

*(iii) Error in $(u_0^n - u_\vartheta^n, u_0'^n - u_\vartheta'^n)$:* (2.4) implies that

$$\begin{aligned}
\begin{pmatrix} \langle\Omega_\vartheta\rangle(u_0^{n+1} - u_\vartheta^{n+1}) \\ u_0'^{n+1} - u_\vartheta'^{n+1} \end{pmatrix} &= O_\tau \begin{pmatrix} \langle\Omega_\vartheta\rangle(u_0^n - u_\vartheta^n) \\ u_0'^n - u_\vartheta'^n \end{pmatrix} \\
&\quad + \tau \begin{pmatrix} \frac{1-\cos(\tau\Omega_\vartheta)}{\tau\Omega_\vartheta} \frac{\langle\Omega_\vartheta\rangle}{\Omega_\vartheta} \\ \frac{\sin(\tau\Omega_\vartheta)}{\tau\Omega_\vartheta} \end{pmatrix} \Delta\left(|E_0^n|^2 - |E_\vartheta^n|^2\right) + \mathcal{R}(\tau,\vartheta,m_{s,*}^n),
\end{aligned} \tag{4.12}$$

where

$$O_\tau = \begin{pmatrix} \cos(\tau\Omega_\vartheta) & \sin(\tau\Omega_\vartheta)\frac{\langle\Omega_\vartheta\rangle}{\Omega_\vartheta} \\ -\sin(\tau\Omega_\vartheta)\frac{\Omega_\vartheta}{\langle\Omega_\vartheta\rangle} & \cos(\tau\Omega_\vartheta) \end{pmatrix} \tag{4.13}$$

and the remainder satisfies

$$\begin{aligned} &\|\mathcal{R}(\tau,\vartheta,m^n_{s,*})\|_s \\ &\quad\le \|\Omega_\vartheta\big(\cos(\tau\Omega_\vartheta)-\cos(\tau|\nabla|)\big)u_0^n\|_s + \|\big(\Omega_\vartheta\sin(\tau|\nabla|)-|\nabla|\sin(\tau\Omega_\vartheta)\big)u_0^n\|_s \\ &\quad+\|\big(\sin(\tau\Omega_\vartheta)-\sin(\tau|\nabla|)\big)u_0'^n\|_s + \|\big(\cos(\tau\Omega_\vartheta)-\cos(\tau|\nabla|)\big)u_0'^n\|_s \\ &\quad+\tau\|\left(\frac{\Omega_\vartheta}{|\nabla|}\frac{1-\cos(\tau|\nabla|)}{\tau|\nabla|}-\frac{1-\cos(\tau\Omega_\vartheta)}{\tau\Omega_\vartheta}\right)\Delta|E_0^n|^2\|_s \\ &\quad+\tau\|\left(\frac{\sin(\tau\Omega_\vartheta)}{\tau\langle\Omega_\vartheta\rangle}-\frac{\sin(\tau|\nabla|)}{\tau|\nabla|}\right)\Delta|E_0^n|^2\|_s. \end{aligned}$$

Note that for $\alpha = 1, \frac{1}{2}$ we have that

$$\begin{aligned} \|\big(\sqrt{1-\vartheta\Delta}-1\big)f\|_s &= \|\frac{\big(\sqrt{1-\vartheta\Delta}-1\big)\big(\sqrt{1-\vartheta\Delta}+1\big)}{\big(\sqrt{1-\vartheta\Delta}+1\big)}f\|_s \le \|\frac{\vartheta\Delta}{\big(\sqrt{1-\vartheta\Delta}+1\big)}f\|_s \\ &\le \vartheta^\alpha\|f\|_{s+2\alpha}, \end{aligned}$$

which in particular implies that for $\alpha = 1, \frac{1}{2}$

$$\|\big(\langle\Omega_\vartheta\rangle-\langle\nabla\rangle\big)f\|_s \le \vartheta^\alpha\|f\|_{s+1+2\alpha}, \quad \|\sin\big(\tau(\Omega_\vartheta-|\nabla|)\big)f\|_s \le \tau\vartheta^\alpha\|f\|_{s+1+2\alpha}.$$

With the aid of the relations

$$\begin{aligned} \cos(\tau\Omega_\vartheta)-\cos(\tau|\nabla|) &= -2\sin\big(\tau/2(\Omega_\vartheta+|\nabla|)\big)\sin\big(\tau/2(\Omega_\vartheta-|\nabla|)\big), \\ \sin(\tau\Omega_\vartheta)-\sin(\tau|\nabla|) &= 2\cos\big(\tau/2(\Omega_\vartheta+|\nabla|)\big)\sin\big(\tau/2(\Omega_\vartheta-|\nabla|)\big) \end{aligned}$$

we thus obtain the following bound on the remainder

$$\begin{aligned} &\|\mathcal{R}(\tau,\vartheta,m^n_{s,*})\|_s \\ &\le 4\|\big(\Omega_\vartheta-|\nabla|\big)\sin\big(\tau/2(\Omega_\vartheta-|\nabla|)\big)u_0^n\|_s + 4\||\nabla|\sin\big(\tau/2(\Omega_\vartheta-|\nabla|)\big)u_0^n\|_s \\ &+4\|\sin\big(\tau/2(\Omega_\vartheta-|\nabla|)\big)u_0'^n\|_s + 4\tau\|\big(\Omega_\vartheta-|\nabla|\big)|E_0^n|^2\|_{s+1} \\ &+2\|\frac{1}{|\nabla|\Omega_\vartheta}\Big(\big(\Omega_\vartheta-|\nabla|\big)(1-\cos(\tau\Omega_\vartheta))+\Omega_\vartheta\big(\cos(\tau\Omega_\vartheta)-\cos(\tau|\nabla|)\big)\Big)|E_0^n|^2\|_{s+2} \\ &+2\|\frac{1}{|\nabla|\Omega_\vartheta}\Big(\big(|\nabla|-\Omega_\vartheta\big)\sin(\tau\Omega_\vartheta)+|\nabla|\big(\sin(\tau\Omega_\vartheta)-\sin(\tau|\nabla|)\big)\Big)|E_0^n|^2\|_{s+2} \\ &\le c\tau\vartheta\big(\|u_0^n\|_{s+4}+\|u_0'^n\|_{s+3}+\|E_0^n\|_{s+4}\big) \\ &\le c\tau\vartheta m^n_{s,*}. \end{aligned} \tag{4.14}$$

Collecting the results in (4.10), (4.11), (4.13) and (4.14) yields the assertion by an inductive argument. □

## 5. Second-order asymptotic preserving trigonometric integrators

Following the above approach we can also derive a second-order trigonometric integration scheme for the quantum Zakharov system (1.1). To achieve an asymptotic preserving scheme we discretize the mild solution (2.3) thereby as follows: In the approximation of $F$ we employ the second-order Taylor series expansion

$$(uF+u'\mathcal{I}_F)(t_n+\xi) = (uF+u'\mathcal{I}_F)(t_n)+\xi(uF+u'\mathcal{I}_F)'(t_n)+\mathcal{O}(\xi^2)$$

while integrating the appearing oscillatory phases (cf. (1.2)) exactly by using that

$$\int_0^\tau \mathrm{e}^{-i(\tau-\xi)\Omega_\vartheta^2}\mathrm{d}\xi = \frac{1}{i\Omega_\vartheta^2}\left(\mathrm{e}^{-i\tau\Omega_\vartheta^2}-1\right)$$
$$\int_0^\tau \mathrm{e}^{-i(\tau-\xi)\Omega_\vartheta^2}\xi\mathrm{d}\xi = \frac{1}{i\Omega_\vartheta^2}\left(\tau-\frac{1}{i\Omega_\vartheta^2}\left(1-\mathrm{e}^{-i\tau\Omega_\vartheta^2}\right)\right).$$

For the approximation of the wave part $(u,u')$ we apply the trapezoidal rules

$$\int_0^\tau \Omega_\vartheta^{-1}\sin((\tau-\xi)\Omega_\vartheta)\Delta|E(t_n+\xi)|^2\mathrm{d}\xi = \frac{\tau}{2}\Omega_\vartheta^{-1}\sin(\tau\Omega_\vartheta)\Delta|E(t_n)|^2+\mathcal{O}(\tau^3)$$
$$\int_0^\tau \cos((\tau-\xi)\Omega_\vartheta)\Delta|E(t_n+\xi)|^2\mathrm{d}\xi = \frac{\tau}{2}\Delta\left(|E(t_n+\tau)|^2+\cos(\tau\Omega_\vartheta)|E(t_n)|^2\right)+\mathcal{O}(\tau^3).$$

This motivates us to define the following second-order trigonometric integration scheme

$$\begin{aligned}
F^{n+1} &= \mathrm{e}^{-i\tau\Omega_\vartheta^2}F^n + i\tau\mathcal{D}_1(-i\tau\Omega_\vartheta^2)\big(u^nF^n+u'^n\mathcal{I}_F^n\big)+\tau\mathcal{D}_2(-i\tau\Omega_\vartheta^2)\\
&\quad\Big(2u'^nF^n+iu^n(-\Omega_\vartheta^2F^n-u^nF^n-u'^n\mathcal{I}_F^n)+\mathcal{I}_F^n(-\Omega_\vartheta^2u^n+\Delta|E^n|^2)\Big),\\
u^{n+1} &= \cos(\tau\Omega_\vartheta)u^n+\Omega_\vartheta^{-1}\sin(\tau\Omega_\vartheta)u'^n+\frac{\tau}{2}\Omega_\vartheta^{-1}\sin(\tau\Omega_\vartheta)\Delta|E^n|^2,\\
M^{n+1} &= M^n+u^nF^n+u'^n(E_0+S_F^n),\\
\mathcal{I}_F^{n+1} &= E_0-\mathcal{D}_1(-i\tau\Omega_\vartheta^2)S_F^n+\tau\mathcal{D}_2(-i\tau\Omega_\vartheta^2)M^{n+1},\\
E^{n+1} &= (\Omega_\vartheta^2+1)^{-1}\left(iF^{n+1}-(u^{n+1}-1)\mathcal{I}_F^{n+1}\right),\\
u'^{n+1} &= -\Omega_\vartheta\sin(\tau\Omega_\vartheta)u^n+\cos(\tau\Omega_\vartheta)u'^n+\frac{\tau}{2}\left(\Delta|E^{n+1}|^2+\cos(\tau\Omega_\vartheta)\Delta|E^n|^2\right),\\
S_F^{n+1} &= S_F^n+\tau F^{n+1}
\end{aligned}\tag{5.1}$$

with $\Omega_\vartheta$ defined in (1.5), the operators

$$\mathcal{D}_1(-i\tau\Omega_\vartheta^2)=\frac{1}{-i\tau\Omega_\vartheta^2}\left(1-\mathrm{e}^{-i\tau\Omega_\vartheta^2}\right),\quad \mathcal{D}_2(-i\tau\Omega_\vartheta^2)=\frac{1}{-\Omega_\vartheta^2}\left(1+\mathcal{D}_1(-i\tau\Omega_\vartheta^2)\right)\tag{5.2}$$

and initial conditions (cf. (5))

$$\begin{aligned}
&F^0=-i\big(\Omega_\vartheta^2E(0)+u(0)E(0)\big),\quad u^0=u(0),\quad {u'}^0=\partial_tu(0),\quad E^0=E(0),\\
&M^0=0,\quad S_F^0=\tau F^0,\quad \mathcal{I}_F^0=E_0.
\end{aligned}$$

The scheme (5.1) is second-order convergent in time without any $\vartheta-$ dependent step size restriction. In addition, it asymptotically converges to the the classical Zakharov system (1.3) in the limit $\vartheta\to 0$. The analysis follows the line of argumentation to the first order scheme and will therefore be neglected here. The convergence properties are underlined in the numerical experiments, see Section 6.

## 6. Numerical experiments

In this section we numerically underline the theoretical convergence results derived in the previous sections. For the spatial discretization we use a pseudo spectral method in which we choose the highest Fourier mode $K = 2^8$. This corresponds to a spatial mesh-size $\Delta x = 0.0245$. Furthermore, we chose the initial values

$$\begin{aligned} &E(0,x) = (2-\cos(x)\sin(2x))^{-1}\sin(2x)\cos(4x) + i\sin(2x)\cos(x),\\ &u(0,x) = (2-\sin(2x)^2)^{-1}\sin(x)\cos(2x), \quad \partial_t u(0,x) = (2-\cos(2x)^2)^{-1}\sin(x). \end{aligned} \tag{6.1}$$

*Example* 6.1 (First-order convergence rate in time). We numerically test the first-order convergence rate in time of the trigonometric integrator (2.4) towards the exact solutions of the (quantum) Zakharov system (1.1). The convergence rate holds uniformly with respect to the quantum parameter $\vartheta$, i.e., for sufficiently smooth solutions (such that $m_{s,1,\vartheta}(T) < \infty$ cf. (3.2)) there exist $C > 0$ such that for all $t_n \leq T$

$$\|\langle\Omega_\vartheta\rangle^2(E_\vartheta(t_n) - E_\vartheta^n)\|_s + \|\langle\Omega_\vartheta\rangle(u_\vartheta(t_n) - u_\vartheta^n)\|_s + \|u'_\vartheta(t_n) - u'^n_\vartheta\|_s \leq C_2\tau, \tag{6.2}$$

see Theorem 3.1. The convergence bound (6.2) is numerically confirmed in Figure 2 with the initial values (6.1) normalized in $\|\langle\Omega_\vartheta\rangle^4\cdot\|_0, \|\langle\Omega_\vartheta\rangle^2\cdot\|_0$ and $\|\langle\Omega_\vartheta\rangle\cdot\|_0$, respectively.

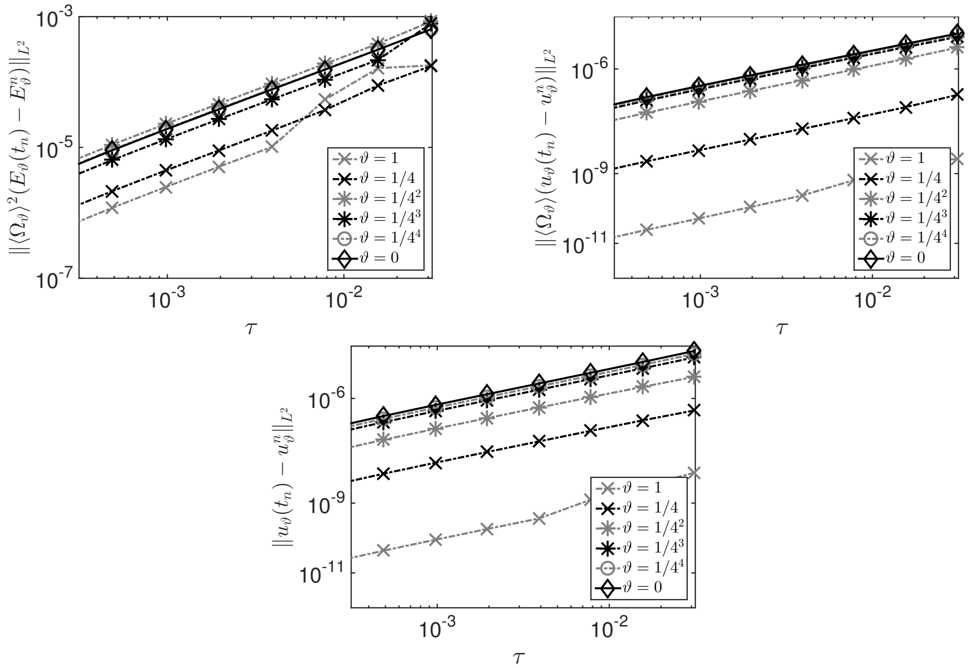


FIGURE 2. Orderplot (double logarithmic). Convergence rate of the first-order trigonometric integrator (2.4) towards the exact solutions of the quantum Zakharov system (1.1) for different values of $\vartheta$.

*Example* 6.2 (Second-order convergence rate in time). In Figure 3 we illustrate the uniform convergence in $\vartheta$ of the second-order scheme (5.1) with initial values (6.1).

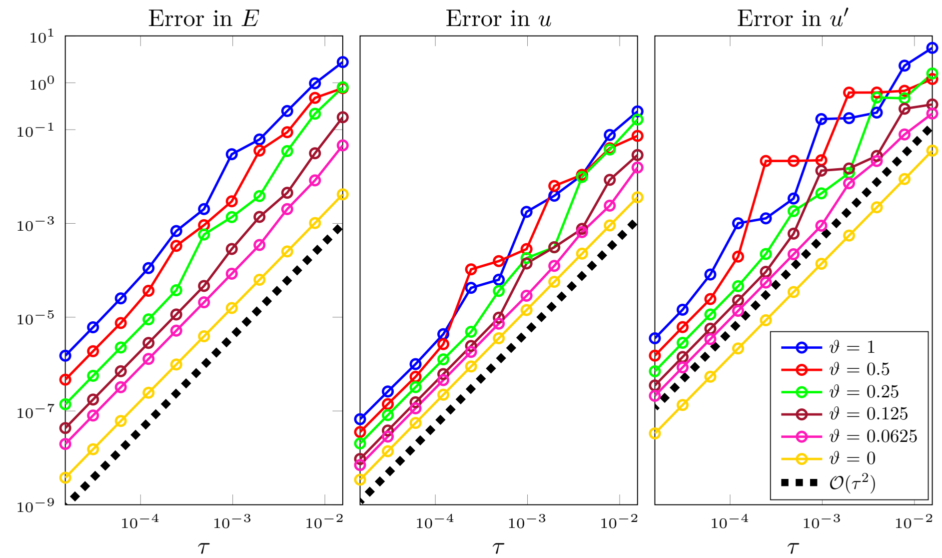


FIGURE 3. Orderplot (double logarithmic). Convergence rate of the second-order integrator (5.1) for different values of $\vartheta$.

*Example* 6.3 (From the quantum to the classical approximation). We numerically test the convergence rate from the quantum to the classical approximation of the trigonometric integrator of first order (2.4) and second order (5.1) for $\vartheta \to 0$ using the initial values (6.1) normalized in $H^2, H^1$ and $L^2$, respectively. The first- and second-order trigonometric integrators allow a convergence rate of order $\vartheta$ independently of the time-step size, see Figure 4 and Theorem 4.1.

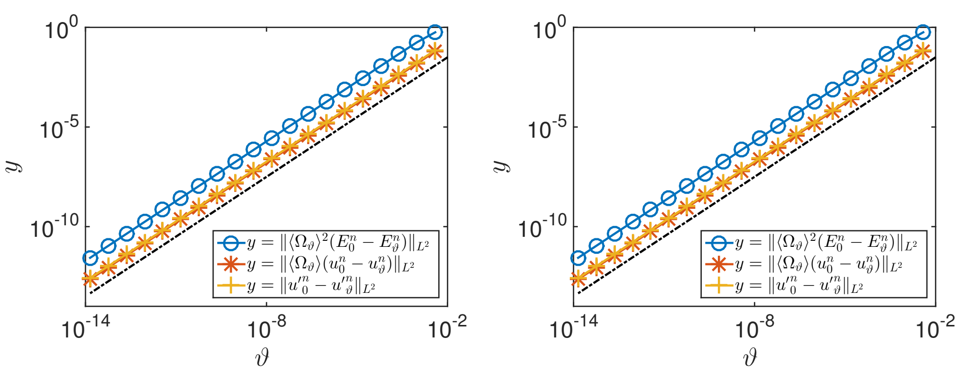


FIGURE 4. Orderplot (double logarithmic). Convergence rate from the quantum to the classical approximation of the trigonometric integrator (2.4) for $\vartheta \to 0$. Left picture: time step-size $\tau = 10^{-2}$ Right picture: time step-size $\tau = 10^{-3}$. The slope of the dashed line is one.

*Example* 6.4 (Energy conservation). In Figure 6, 7 and 8 we simulate the numerical energy $H(E_\vartheta^n, u_\vartheta^n, u_v^{'n})$ (with $H$ defined in (3.18)) as well as the $L^2-$norm of $E_\vartheta^n$

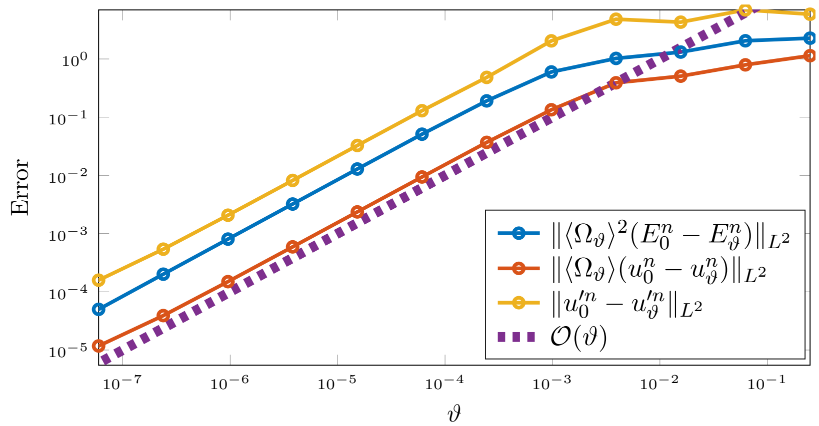


FIGURE 5. Orderplot (double logarithmic). Convergence rate from the quantum to the classical approximation for $\vartheta \to 0$ with the second-order trigonometric integration scheme (5.1). All errors are measured in $L^2$. Time step-size $\tau = 1.191 \cdot 10^{-6}$. Slope of the dashed line is one

in the quantum regime $\vartheta = \mathcal{O}(1)$ as well as in the classical regime $\vartheta = 0$ for the trigonometric integrator of first order (2.4) and second order (5.1).

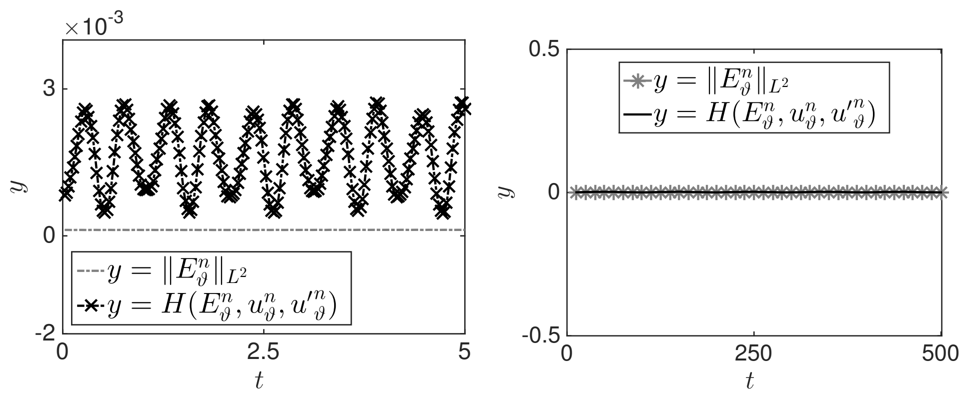


FIGURE 6. Energy conservation of the first-order scheme (2.4) for $\vartheta = 1$. Initial values (6.1) normalized in $\|\langle\Omega_\vartheta\rangle^4 \cdot \|_0$, $\|\langle\Omega_\vartheta\rangle^2 \cdot \|_0$ and $\|\langle\Omega_\vartheta\rangle \cdot \|_0$, respectively. Time-step size: $\tau = 0.025$. Left picture: Time-scale $[0, 5]$. Right picture: Time-scale $[0, 500]$.

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## References

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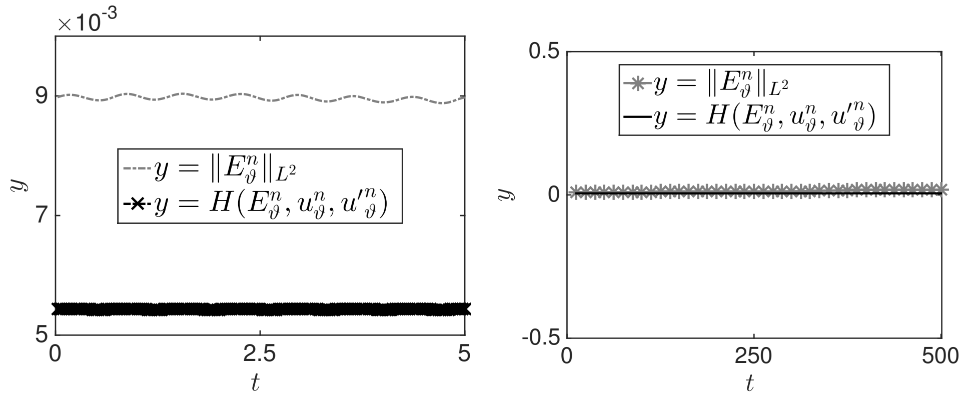


FIGURE 7. Energy conservation of the first-order scheme (2.4) for $\vartheta = 0$. Initial values (6.1) normalized in $\|\cdot\|_4, \|\cdot\|_2$ and $\|\cdot\|_1$, respectively. Time-step size: $\tau = 0.025$. Left picture: Time-scale $[0, 5]$. Right picture: Time-scale $[0, 500]$.

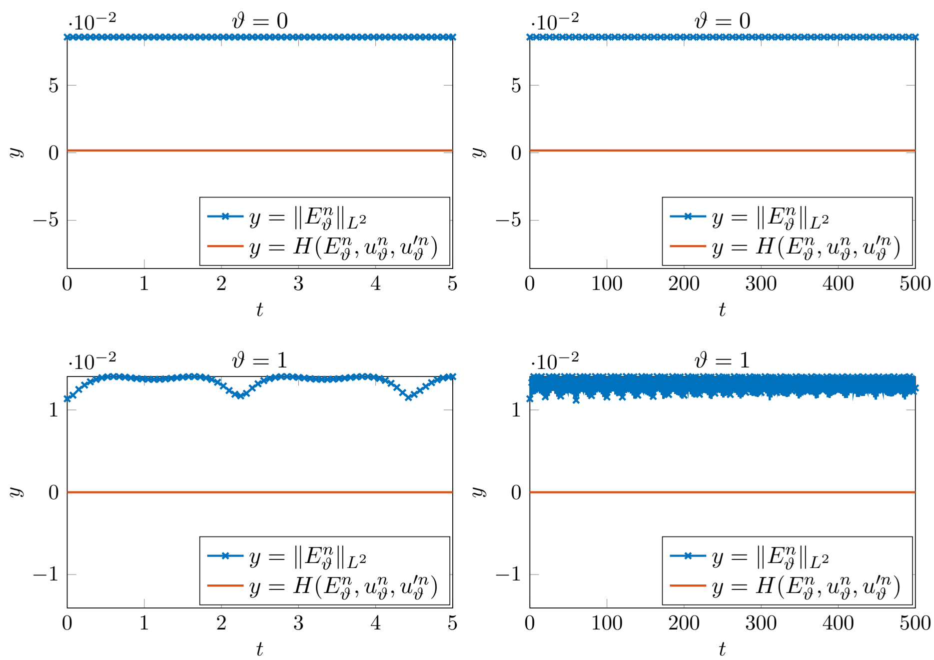


FIGURE 8. Energy conservation of the second-order scheme (5.1). Upper row: $\vartheta = 0$. Lower row: $\vartheta = 1$. Initial values (6.1) normalized in $\|\cdot\|_4, \|\cdot\|_2$ and $\|\cdot\|_1$, respectively. Time-step size: $\tau = 0.025$. Left picture: Time-scale $[0, 5]$. Right picture: Time-scale $[0, 500]$.

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