ON STABILITY OF THE CAUCHY FUNCTIONAL EQUATION IN GROUPOIDS

IMKE TOBORG, PETER VOLKMANN

Abstract. We give some stability results for the functional equation $a(xy) = a(x) + a(y)$, where $a: G \rightarrow E$, $G$ being a groupoid and $E$ a Banach space.

1. Introduction

Let $G$ be a groupoid, i.e., $G$ is a set and for all $x, y \in G$ we have a product $xy \in G$. Furthermore, let $E$ be a Banach space; by $\theta$ we denote its zero element.

We consider the Cauchy equation

$$a(xy) = a(x) + a(y), \quad x, y \in G,$$

for functions $a: G \rightarrow E$; its solutions are called additive functions.

A subset $V$ of $E$ is ideally convex (Evgenij Arkad’evič Lifšic [4]), if for every bounded sequence $d_1, d_2, d_3, \ldots$ in $V$ and for every numerical sequence $\alpha_1, \alpha_2, \alpha_3, \ldots \geq 0$ such that $\sum_{k=1}^{\infty} \alpha_k = 1$ we get $\sum_{k=1}^{\infty} \alpha_k d_k \in V$.

Let us mention that a convex subset of $E$ is ideally convex, provided it is closed, open or finite dimensional (cf. also Jacek Tabor [5], where the relation between ideally convex sets and stability of the Cauchy equation has been examined; for this relation see also Volkmann [11]). Thus closed and open balls...
in $E$ are ideally convex. We denote them by $S(p; \rho)$ and $S^0(p; \rho)$, respectively ($p \in E$ being the centre and $\rho \geq 0$ the radius).

Now we consider triplets $(G, E, V)$, where $G, E, V$ essentially are as described before. More precisely, we introduce the following hypothesis:

(H) $G$ is a groupoid, $E$ a Banach space and $V$ a bounded ideally convex subset of $E$.

**Definition 1.** For a triplet $(G, E, V)$ according to (H) we say it has property $(U)$, if for every $f : G \to E$ satisfying

\[
(2) \quad f(xy) - f(x) - f(y) \in V, \quad x, y \in G,
\]

there is an additive $a : G \to E$ such that

\[
(3) \quad a(x) - f(x) \in V, \quad x \in G.
\]

**Remark 1.** Concerning the special case $V = S(\theta; \varepsilon)$ (where $\varepsilon > 0$) we have:

Conditions (2), (3) can be written as

\[
\|f(xy) - f(x) - f(y)\| \leq \varepsilon, \quad x, y \in G, \\
\|a(x) - f(x)\| \leq \varepsilon, \quad x, y \in G,
\]

respectively, and $(U)$ implies the Hyers–Ulam stability of the Cauchy equation \([\text{1}]\) (in the sense of Zenon Moszner \([\text{5}]\) Definition 1); in fact, $(U)$ is equivalent to the Hyers–Ulam stability of \([\text{1}]\), which can be seen by using the Theorem \([\text{1}]\) below). Finally property $(U)$ for one $\varepsilon > 0$ implies already $(U)$ for all $\varepsilon > 0$.

**Definition 2.** For $x \in G$, $G$ being a groupoid, and $k = 0, 1, 2, \ldots$, the powers $x^{2^k}$ are recursively defined by

\[
x^{2^0} = x^1 = x, \quad x^{2^{k+1}} = x^{2^k} x^{2^k}.
\]

The following result is a stability theorem for the functional equation

\[
h(x^2) = 2h(x); \quad \text{see Volkmann \([\text{12}]\) for the proof.}
\]

**Theorem 1.** Consider $(G, E, V)$ according to (H) and let $f : G \to E$ satisfy

\[
(4) \quad f(x^2) - 2f(x) \in V, \quad x \in G.
\]
Then there is exactly one $h: G \rightarrow E$ such that
\begin{equation}
(5) \quad h(x^2) = 2h(x), \quad h(x) - f(x) \in V, \quad x \in G,
\end{equation}
namely
\begin{equation}
(6) \quad h(x) = \lim_{n \to \infty} \frac{1}{2^n} f(x^{2^n}), \quad x \in G.
\end{equation}

In the next section we use this theorem for a characterization of property (U) and we give some applications. The third section will be devoted to direct products of groupoids and the last one to some concluding remarks.

2. A characterization of (U) and some consequences

The following result gives a characterization of property (U).

**Theorem 2.** Consider $(G, E, V)$ according to (H) and let $f: G \rightarrow E$ satisfy (2). Then the following assertions are equivalent:

(A) There is an additive $a: G \rightarrow E$ satisfying (3).

(B) The function $h: G \rightarrow E$ (given by (6)) is additive.

**Proof.** (2) implies (4), and therefore we can apply Theorem 1. There is exactly one $h: G \rightarrow E$ satisfying (5), and this function is given by (6).

(A) $\Rightarrow$ (B): If (A) holds, then the additive function $a$ has all the properties of $h$, which are stated in (5). The uniqueness of $h$ gives $h = a$, and this proves (B).

(B) $\Rightarrow$ (A): If $h$ is additive, then $a = h$ obviously leads to (A). $\square$

**Remark 2.** If the assertions (A), (B) of Theorem 2 are true, then $a = h$.

**Theorem 3.** Consider $(G, E, V)$ according to (H).

I) If (U) holds for $(G, E, S(\theta; \varepsilon))$ ($\varepsilon > 0$), then (U) also holds for $(G, E, V)$.

II) If Int$V \neq \emptyset$ and (U) holds for $(G, E, V)$, then (U) also holds for the triplet $(G, E, S(\theta; \varepsilon))$ ($\varepsilon > 0$).

**Proof.** I) Let $f: G \rightarrow E$ satisfy (2). We choose $\varepsilon > 0$ such that $V \subseteq S(\theta; \varepsilon)$, and we get
\begin{equation}
(7) \quad f(xy) - f(x) - f(y) \in S(\theta; \varepsilon), \quad x, y \in G.
\end{equation}
Since (U) holds for \((G, E, S(\theta; \varepsilon))\), we can apply Theorem 2 with \(V\) replaced by \(S(\theta; \varepsilon)\) to get the additivity of \(h: G \to E\) given by (6). This finishes the proof of (U) for \((G, E, V)\) (because of (B) \(\Rightarrow\) (A) from Theorem 2).

II) We choose \(p \in E\) and \(\varepsilon > 0\) such that \(S(p; \varepsilon) = p + S(\theta; \varepsilon) \subseteq V\); according to Remark it is sufficient to keep the \(\varepsilon\) fixed and to show property (U) for \((G, E, S(\theta; \varepsilon))\). So let \(f: G \to E\) satisfy (7). For

\[
g(x) := f(x) - p, \quad x \in G,
\]

we have

\[
g(xy) - g(x) - g(y) = f(xy) - f(x) - f(y) + p \in S(p; \varepsilon) \subseteq V, \quad x, y \in G.
\]

When using (U) for \((G, E, V)\), we get by Theorem 2 the additivity of

\[
k(x) = \lim_{n \to \infty} \frac{1}{2^n} g(x^{2^n}), \quad x \in G.
\]

For \(h: G \to E\) given by (6) we now have

\[
h(x) = \lim_{n \to \infty} \frac{1}{2^n} f(x^{2^n}) = \lim_{n \to \infty} \frac{1}{2^n} \left[ g(x^{2^n}) + p \right] = k(x),
\]

hence \(h\) is an additive function, and from Theorem 2 we get property (U) for \((G, E, S(\theta; \varepsilon))\).

The next definition is taken from Roman Badora, Barbara Przebieracz, Volkmann [1]; we adopt the notation \(\mathbb{N} = \{0, 1, 2, \ldots\}\), \(\mathbb{N}^* = \mathbb{N} \setminus \{0\}\).

**Definition 3.** A groupoid \(G\) is called **Tabor groupoid**, if for \(x, y \in G\) there exists \(k \in \mathbb{N}^*\) such that

\[
(xy)^{2^k} = x^{2^k} y^{2^k}.
\]

Groups satisfying this condition had been considered by Józef Tabor [9]; we call them **Tabor groups**. The special case \(k = 1\) in (8), i.e.,

\[
(xy)^2 = x^2 y^2, \quad x, y \in G,
\]

had been called square-symmetry by Zsolt Páles, Volkmann, R. Duncan Luce [6]. Of course, (9) holds in commutative semigroups.
The next theorem has two parts: Part I) is from Volkmann [12]; Part II) is similar to a result of Jürg Rätz [7].

**Theorem 4.** A triplet \((G, E, V)\) according to \((H)\) satisfies \((U)\) in the following two cases:

I) \(G\) is a Tabor groupoid.
II) For every \(x \in G\) the set \(\{x, x^2, x^4, x^8, \ldots\}\) is finite.

**Proof.** Let \(f : G \to E\) satisfy \((2)\). According to Theorem 2 it is sufficient to show the additivity of \(h : G \to E\) (given by \((6)\)). In Case I) this can be done by the procedure of Józef Tabor [9]. In Case II) we simply get \(h(x) \equiv \theta\). \(\square\)

**Remark 3.** For commutative semigroups \(G\), Part I) goes back to Jacek Tabor [8].

**Remark 4.** Condition II) is equivalent to the following: For every \(x \in G\) there are \(m, n \in \mathbb{N}\), \(m \neq n\) such that \(x^{2^m} = x^{2^n}\).

**Remark 5.** In groupoids \(G\), Rätz [7] uses the “left” powers, here for \(x \in G\) written as \(x^{(n)} := x(x^{(n-1)})\) \((n \in \mathbb{N} \setminus \{0, 1\})\), where \(x^{(1)} := x\). By Rätz’ Theorem 2, a triplet \((G, E, V)\) according to \((H)\) also satisfies \((U)\) in the following case:

III) For every \(x \in G\) the set \(\{x^{(1)}, x^{(2)}, x^{(3)}, \ldots\}\) is finite.

Let us give an example of a groupoid \(G\), where I), II) hold but III) does not hold. We take \(G = \mathbb{N}\), equipped with the product

\[
x \circ y = \begin{cases} 
0 & \text{if } x = y, \\
x + y + 1 & \text{if } x \neq y, \quad x, y \in \mathbb{N}.
\end{cases}
\]

I), II) are obviously satisfied, but

\[
1^{(2)} = 1 \circ 1 = 0, \quad 1^{(3)} = 1 \circ 0 = 2, \quad 1^{(4)} = 1 \circ 2 = 4,
\]

\[
1^{(n)} = 2(n - 2), \quad n \geq 2.
\]

\(\{1^{(1)}, 1^{(2)}, 1^{(3)}, \ldots\}\) is an infinite set, hence III) does not hold.
3. Direct products of groupoids

Let $G, H$ be groupoids. By the direct product of them we understand (as usual) $G \times H$ equipped with the coordinate-wise defined product, i.e.,

$$(x, y)(\bar{x}, \bar{y}) = (x\bar{x}, y\bar{y}), \quad x, \bar{x} \in G; \ y, \bar{y} \in H.$$  

A basic question is the following: Let furthermore $V$ be an ideally convex set in a Banach space $E$, and suppose $(G, E, V), (H, E, V)$ to have property (U). Under which conditions is it true that $(G \times H, E, V)$ also has property (U)?

Concerning this question, Badora, Przebieracz, Volkmann [2] observed that $G \times H$ is a Tabor groupoid, provided $G, H$ have this property and in at least one of them the square-symmetry (9) holds. This fact also follows from Theorem 5 below, which gives a necessary and sufficient condition for $G \times H$ to be a Tabor groupoid.

**Definition 4.** For groupoids $G$ and $x, y \in G$ we set

$$T_G(x, y) = \{k | k \in \mathbb{N}^*, (xy)^{2^k} = x^{2^k}y^{2^k}\}.$$  

**Remark 6.** A groupoid $G$ is a Tabor groupoid if and only if

$$T_G(x, y) \neq \emptyset, \quad x, y \in G,$$

and in square-symmetric groupoids $G$ we have

$$T_G = \mathbb{N}^*, \quad x, y \in G.$$

**Theorem 5.** Let $G, H$ be groupoids. Then $G \times H$ is a Tabor groupoid if and only if $T_G(x, y) \cap T_H(a, b) \neq \emptyset$ $(x, y \in G; \ a, b \in H)$. In particular $G$ and $H$ are Tabor groupoids in this case.

**Proof.** Consider $x, y \in G$ and $a, b \in H$. The theorem follows from the formula

$$T_{G \times H}((x, a), (y, b)) = T_G(x, y) \cap T_H(a, b),$$

which is easily shown: For $k \in T_{G \times H}((x, a), (y, b))$ we have

$$(xy)^{2^k}, (ab)^{2^k} = (xy, ab)^{2^k} = ((x, a)(y, b))^{2^k} = (x^{2^k}, a^{2^k})(y^{2^k}, b^{2^k}) = (x^{2^k}y^{2^k}, a^{2^k}b^{2^k}),$$
hence \( k \in T_G(x, y) \cap T_H(a, b) \). On the other hand, for \( k \in T_G(x, y) \cap T_H(a, b) \), we get
\[
((x, a)(y, b))^{2k} = (xy, ab)^{2k} = ((xy)^{2k}, (ab)^{2k}) =
\]
\[
= (x^{2k} y^{2k}, a^{2k} b^{2k}) = (x^{2k}, a^{2k})(y^{2k}, b^{2k}) = (x, a)^{2k} (y, b)^{2k},
\]
hence \( k \in T_{G \times H}((x, a), (y, b)). \)

**Theorem 6.** Let \((G, E, V)\) satisfy (U). Let \( \Sigma \) be a groupoid with an element \( \sigma = \sigma^2 \in \Sigma \) such that for every \( \xi \in \Sigma \) there exists \( m \in \mathbb{N} \) yielding \( \xi^{2m} = \sigma \). Then \((G \times \Sigma, E, V)\) also has property (U).

**Proof.** Let \( f: G \times \Sigma \to E \) satisfy
\[
(10) \quad f(xy, \xi \eta) - f(x, \xi) - f(y, \eta) \in V, \quad x, y \in G; \xi, \eta \in \Sigma.
\]
Then \( y = x, \eta = \xi \) leads to
\[
f(x^2, \xi^2) - 2f(x, \xi) \in V, \quad x \in G, \xi \in \Sigma,
\]
and Theorem 1 applied to \((G \times \Sigma, E, V)\) shows the existence of exactly one function \( h: G \times \Sigma \to E \) such that
\[
(11) \quad h(x^2, \xi^2) = 2h(x, \xi), \quad h(x, \xi) - f(x, \xi) \in V, \quad x \in G, \xi \in \Sigma.
\]
This function is given by
\[
h(x, \xi) = \lim_{n \to \infty} \frac{1}{2^n} f(x^{2^n}, \xi^{2^n}), \quad x \in G, \xi \in \Sigma.
\]
The choice \( \xi = \eta = \sigma \) in (10) leads to
\[
f(xy, \sigma) - f(x, \sigma) - f(y, \sigma) \in V, \quad x, y \in G,
\]
and since \((G, E, V)\) has the property (U), we get an additive \( a: G \to E \) such that
\[
a(x) - f(x, \sigma) \in V, \quad x \in G.
\]
By Theorem 2 and Remark 2 we now have
\[
(12) \quad a(x) = \lim_{n \to \infty} \frac{1}{2^n} f(x^{2^n}, \sigma) = h(x, \sigma), \quad x \in G.
\]
For $\xi \in \Sigma$ let $m \in \mathbb{N}$ be such that $\xi^{2m} = \sigma$. Then we get by (11), (12) for $x \in G$:

$$h(x, \xi) = \frac{1}{2} h(x^2, \xi^2) = \ldots = \frac{1}{2^m} h(x^{2^m}, \sigma) = \frac{1}{2^m} a(x^{2^m}) = a(x).$$

So we have

$$h(x, \xi) = a(x), \quad x \in G, \xi \in \Sigma,$$

and therefore the function $h: G \times \Sigma \to E$ occurring in (11) is additive. Indeed, for $x, y \in G$ and $\xi, \eta \in \Sigma$ we get

$$h((x, \xi)(y, \eta)) = h(xy, \xi\eta) = a(xy) = a(x) + a(y) = h(x, \xi) + h(y, \eta).$$

This finishes the proof of (U) for $(G \times \Sigma, E, V)$. \qed

**Remark 7.** When taking $G = \{0\}$ (a singleton) and observing $\Sigma \cong \{0\} \times \Sigma$, we see that $(\Sigma, E, V)$ has property (U). In fact, $\Sigma$ is a Tabor groupoid (a proof is easy), which in the semigroup-case already is known from Badora, Przebieracz, Volkmann [2, Theorem 3, Case I].

The next theorem is trivial (we omit the proof), but it may be useful in applications.

**Theorem 7.** Let $(G_1, E_1, V_1)$, $(G_2, E_2, V_2)$ have property (U) and let the function $f: G_1 \times G_2 \to E_1 \times E_2$ be given by

$$f(x_1, x_2) = (f_1(x_1), f_2(x_2)), \quad (x_1, x_2) \in G_1 \times G_2,$$

where $f_j: G_j \to E_j$ $(j = 1, 2)$. Suppose

$$f(xy) - f(x) - f(y) \in V_1 \times V_2, \quad x, y \in G_1 \times G_2.$$

Then there exists an additive $a: G_1 \times G_2 \to E_1 \times E_2$ such that

$$a(x) - f(x) \in V_1 \times V_2, \quad x \in G_1 \times G_2.$$
Remark 8. If $E_1 \times E_2$ is normed by $\|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}$ and $V_1, V_2$ are closed (or open) $\varepsilon$-balls centered at $\theta$, then $V_1 \times V_2$ also is a closed (or open) $\varepsilon$-ball centered at $\theta$, i.e.,

$$S_{E_1}(\theta; \varepsilon) \times S_{E_2}(\theta; \varepsilon) = S_{E_1 \times E_2}(\theta; \varepsilon),$$

$$S_{E_1}^0(\theta; \varepsilon) \times S_{E_2}^0(\theta; \varepsilon) = S_{E_1 \times E_2}^0(\theta; \varepsilon).$$

Of course, this remark concerns in particular the Hyers–Ulam stability mentioned in Remark 1.

Let us conclude this section by recalling some known results for groups.

Theorem 8. Let $G$ be a group.

I) If every element of $G$ has odd order, then $G$ is a Tabor group.

II) If every element of $G$ has an order $2^n$ (where $n \in \mathbb{N}$), then $G$ is a Tabor group.

III) If $G \cong G_1 \times G_2$ with groups $G_1, G_2$ as in I), II) (respectively), then $G$ is a Tabor group.

IV) Any finite Tabor group $G$ has the form given in III).

Remark 9. I), II) follow from Badora, Przebieracz, Volkmann [2], concerning II) cf. Remark 7; III), IV) are from Toborg [10].

4. Final remarks

Let $F(a, b)$ be the free group with two generators and let $\mathbb{R}$ denote the space of the reals. Gian Luigi Forti [3] has shown that the triplet $(F(a, b), \mathbb{R}, [-1, 1])$ does not have property (U). Thus $F(a, b)$ is not a Tabor group. Now the question is of interest, whether there exist torsion free non commutative Tabor groups.

Finally let us mention that all groupoids with two elements are Tabor groupoids; this can be easily checked. On the other hand, there is a groupoid $G = \{a, b, c\}$ which is not a Tabor groupoid: It is sufficient to require $a^2 = a, ab = c, b^2 = c^2 = b \neq c$. Indeed, assume for some $k \in \mathbb{N}^*$ that

$$(ab)^{2^k} = a^{2^k}b^{2^k}.$$

We get $c^{2^k} = ab$, hence $b = c$, which is a contradiction.
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References


Institut für Mathematik
Naturwissenschaftliche Fakultät II
Martin-Luther-Universität
Halle-Wittenberg
06099 Halle (Saale)
Germany

e-mail: imke.toborg@mathematik.uni-halle.de

Institut für Analysis
KIT
Martin-Luther-Universität
Halle-Wittenberg
76128 Karlsruhe
Germany