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ON STABILITY OF THE CAUCHY FUNCTIONAL EQUATION IN GROUPOIDS

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Abstract. We give some stability results for the functional equation $a(xy) = a(x) + a(y)$, where $a\colon G \to E$, G being a groupoid and E a Banach space.

1. Introduction

Let G be a groupoid, i.e., G is a set and for all $x, y \in G$ we have a product $xy \in G$. Furthermore, let E be a Banach space; by θ we denote its zero element.

We consider the Cauchy equation

$$a(xy) = a(x) + a(y), \quad x, y \in G, \tag{1}$$

for functions $a\colon G \to E$; its solutions are called *additive* functions.

A subset V of E is *ideally convex* (Evgenij Arkad'evič Lifšic [4]), if for every bounded sequence $d_1, d_2, d_3, \dots$ in V and for every numerical sequence $\alpha_1, \alpha_2, \alpha_3, \ldots \geq 0$ such that $\sum\limits_{k=1}^{\infty} \alpha_k = 1$ we get $\sum\limits_{k=1}^{\infty} \alpha_k d_k \in V$.

Let us mention that a convex subset of E is ideally convex, provided it is closed, open or finite dimensional (cf. also Jacek Tabor [8], where the relation between ideally convex sets and stability of the Cauchy equation has been examined; for this relation see also Volkmann [11]). Thus closed and open balls

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in E are ideally convex. We denote them by $S(p;\rho)$ and $S^0(p;\rho)$, respectively ($p \in E$ being the centre and $\rho \geq 0$ the radius).

Now we consider triplets (G, E, V), where G, E, V essentially are as described before. More precisely, we introduce the following hypothesis:

(H) G is a groupoid, E a Banach space and V a bounded ideally convex subset of E.

Definition 1. For a triplet (G, E, V) according to (H) we say it has *property* (U), if for every $f\colon G \to E$ satisfying

$$f(xy) - f(x) - f(y) \in V, \quad x, y \in G, \tag{2}$$

there is an additive $a\colon G \to E$ such that

$$a(x) - f(x) \in V, \quad x \in G. \tag{3}$$

Remark 1. Concerning the special case $V = S(\theta;\varepsilon)$ (where $\varepsilon > 0$) we have:

Conditions (2), (3) can be written as

$$\|f(xy) - f(x) - f(y)\| \leq \varepsilon, \quad x, y \in G,$$

$$\|a(x) - f(x)\| \leq \varepsilon, \quad x, y \in G,$$

respectively, and (U) implies the Hyers–Ulam stability of the Cauchy equation (1) (in the sense of Zenon Moszner [5, Definition 1]; in fact, (U) is equivalent to the Hyers–Ulam stability of (1), which can be seen by using the Theorem 1 below). Finally property (U) for one $\varepsilon > 0$ implies already (U) for all $\varepsilon > 0$.

Definition 2. For $x \in G$, G being a groupoid, and $k = 0, 1, 2, \ldots$, the powers x^{2^k} are recursively defined by

$$x^{2^0} = x^1 = x, \quad x^{2^{k+1}} = x^{2^k} x^{2^k}.$$

The following result is a stability theorem for the functional equation $h(x^2) = 2h(x)$; see Volkmann [12] for the proof.

Theorem 1. *Consider* (G, E, V) *according to* (H) *and let* $f\colon G \to E$ *satisfy*

$$f(x^2) - 2f(x) \in V, \quad x \in G. \tag{4}$$

Then there is exactly one $h\colon G \to E$ *such that*

$$h(x^2) = 2h(x),\ \ h(x) - f(x) \in V, \quad x \in G, \tag{5}$$

namely

$$h(x) = \lim_{n\to\infty} \frac{1}{2^n} f(x^{2^n}), \quad x \in G. \tag{6}$$

In the next section we use this theorem for a characterization of property (U) and we give some applications. The third section will be devoted to direct products of groupoids and the last one to some concluding remarks.

2. A characterization of (U) and some consequences

The following result gives a characterization of property (U).

Theorem 2. *Consider* (G, E, V) *according to* (H) *and let* $f\colon G \to E$ *satisfy* (2). *Then the following assertions are equivalent:*

(A) *There is an additive* $a\colon G \to E$ *satisfying* (3).

(B) *The function* $h\colon G \to E$ *(given by* (6)*) is additive.*

Proof. (2) implies (4), and therefore we can apply Theorem 1: There is exactly one $h\colon G \to E$ satisfying (5), and this function is given by (6).

(A) $\Rightarrow$ (B): If (A) holds, then the additive function a has all the properties of h, which are stated in (5). The uniqueness of h gives $h = a$, and this proves (B).

(B) $\Rightarrow$ (A): If h is additive, then $a = h$ obviously leads to (A). $\square$

Remark 2. If the assertions (A), (B) of Theorem 2 are true, then $a = h$.

Theorem 3. *Consider* (G, E, V) *according to* (H).

I) *If* (U) *holds for* $(G, E, S(\theta; \varepsilon))$ $(\varepsilon > 0)$, *then* (U) *also holds for* (G, E, V).

II) *If* $\mathrm{Int} V \neq \emptyset$ *and* (U) *holds for* (G, E, V), *then* (U) *also holds for the triplet* $(G, E, S(\theta; \varepsilon))$ $(\varepsilon > 0)$.

Proof. I) Let $f\colon G \to E$ satisfy (2). We choose $\varepsilon > 0$ such that $V \subseteq S(\theta; \varepsilon)$, and we get

$$f(xy) - f(x) - f(y) \in S(\theta; \varepsilon), \quad x, y \in G. \tag{7}$$

Since (U) holds for $(G, E, S(\theta; \varepsilon))$, we can apply Theorem 2 with V replaced by $S(\theta; \varepsilon)$ to get the additivity of $h\colon G \to E$ given by (6). This finishes the proof of (U) for (G, E, V) (because of (B) $\Rightarrow$ (A) from Theorem 2).

II) We choose $p \in E$ and $\varepsilon > 0$ such that

$$S(p; \varepsilon) = p + S(\theta; \varepsilon) \subseteq V;$$

according to Remark 1 it is sufficient to keep the ε fixed and to show property (U) for $(G, E, S(\theta; \varepsilon))$. So let $f\colon G \to E$ satisfy (7). For

$$g(x) := f(x) - p, \quad x \in G,$$

we have

$$g(xy) - g(x) - g(y) = f(xy) - f(x) - f(y) + p \in S(p; \varepsilon) \subseteq V, \quad x, y \in G.$$

When using (U) for (G, E, V), we get by Theorem 2 the additivity of

$$k(x) = \lim_{n\to\infty} \frac{1}{2^n} g(x^{2^n}), \quad x \in G.$$

For $h\colon G \to E$ given by (6) we now have

$$h(x) = \lim_{n\to\infty} \frac{1}{2^n} f(x^{2^n}) = \lim_{n\to\infty} \frac{1}{2^n} \Big[g(x^{2^n}) + p \Big] = k(x),$$

hence h is an additive function, and from Theorem 2 we get property (U) for $(G, E, S(\theta; \varepsilon))$. $\square$

The next definition is taken from Roman Badora, Barbara Przebieracz, Volkmann [1]; we adopt the notation $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

Definition 3. A groupoid G is called *Tabor groupoid*, if for $x, y \in G$ there exists $k \in \mathbb{N}^*$ such that

$$(xy)^{2^k} = x^{2^k} y^{2^k}. \tag{8}$$

Groups satisfying this condition had been considered by Józef Tabor [9]; we call them *Tabor groups*. The special case $k = 1$ in (8), i.e.,

$$(xy)^2 = x^2 y^2, \quad x, y \in G, \tag{9}$$

had been called square-symmetry by Zsolt Páles, Volkmann, R. Duncan Luce [6]. Of course, (9) holds in commutative semigroups.

The next theorem has two parts: Part I) is from Volkmann [12]; Part II) is similar to a result of Jürg Rätz [7].

Theorem 4. *A triplet* (G,E,V) *according to* (H) *satisfies* (U) *in the following two cases:*

I) *G is a Tabor groupoid.*

II) *For every $x \in G$ the set $\{x, x^2, x^4, x^8, \dots\}$ is finite.*

Proof. Let $f\colon G \to E$ satisfy (2). According to Theorem 2 it is sufficient to show the additivity of $h\colon G \to E$ (given by (6)). In Case I) this can be done by the procedure of Józef Tabor [9]. In Case II) we simply get $h(x) \equiv \theta$. $\square$

Remark 3. For commutative semigroups G, Part I) goes back to Jacek Tabor [8].

Remark 4. Condition II) is equivalent to the following: For every $x \in G$ there are $m, n \in \mathbb{N}$, $m \neq n$ such that $x^{2^m} = x^{2^n}$.

Remark 5. In groupoids G, Rätz [7] uses the "left" powers, here for $x \in G$ written as $x^{(n)} := x(x^{(n-1)})$ $(n \in \mathbb{N} \setminus \{0,1\})$, where $x^{(1)} := x$. By Rätz' Theorem 2, a triplet (G,E,V) according to (H) also satisfies (U) in the following case:

III) *For every $x \in G$ the set $\{x^{(1)}, x^{(2)}, x^{(3)}, \dots\}$ is finite.*

Let us give an example of a groupoid G, where I), II) hold but III) does not hold. We take $G = \mathbb{N}$, equipped with the product

$$x \circ y = \begin{cases} 0 & \text{if } x = y, \\ x + y + 1 & \text{if } x \neq y, \quad x, y \in \mathbb{N}. \end{cases}$$

I), II) are obviously satisfied, but

$$1^{(2)} = 1 \circ 1 = 0, \quad 1^{(3)} = 1 \circ 0 = 2, \quad 1^{(4)} = 1 \circ 2 = 4,$$

$$1^{(n)} = 2(n-2), \quad n \geq 2.$$

$\{1^{(1)}, 1^{(2)}, 1^{(3)}, \dots\}$ is an infinite set, hence III) does not hold.

3. Direct products of groupoids

Let G, H be groupoids. By the direct product of them we understand (as usual) $G \times H$ equipped with the coordinate-wise defined product, i.e.,

$$(x, y)(\bar{x}, \bar{y}) = (x\bar{x}, y\bar{y}), \quad x, \bar{x} \in G;\ y, \bar{y} \in H.$$

A basic question is the following: Let furthermore V be an ideally convex set in a Banach space E, and suppose (G, E, V), (H, E, V) to have property (U). Under which conditions is it true that $(G \times H, E, V)$ also has property (U)?

Concerning this question, Badora, Przebieracz, Volkmann [2] observed that $G \times H$ is a Tabor groupoid, provided G, H have this property and in at least one of them the square-symmetry (9) holds. This fact also follows from Theorem 5 below, which gives a necessary and sufficient condition for $G \times H$ to be a Tabor groupoid.

Definition 4. For groupoids G and $x, y \in G$ we set

$$T_G(x, y) = \{k | k \in \mathbb{N}^*, (xy)^{2^k} = x^{2^k} y^{2^k}\}.$$

Remark 6. A groupoid G is a Tabor groupoid if and only if

$$T_G(x, y) \neq \emptyset, \quad x, y \in G,$$

and in square-symmetric groupoids G we have

$$T_G = \mathbb{N}^*, \quad x, y \in G.$$

Theorem 5. *Let G, H be groupoids. Then $G \times H$ is a Tabor groupoid if and only if $T_G(x, y) \cap T_H(a, b) \neq \emptyset$ $(x, y \in G;\ a, b \in H)$. In particular G and H are Tabor groupoids in this case.*

Proof. Consider $x, y \in G$ and $a, b \in H$. The theorem follows from the formula

$$T_{G\times H}((x, a), (y, b)) = T_G(x, y) \cap T_H(a, b),$$

which is easily shown: For $k \in T_{G\times H}((x, a), (y, b))$ we have

$$\begin{aligned} ((xy)^{2^k}, (ab)^{2^k}) &= (xy, ab)^{2^k} = ((x, a)(y, b))^{2^k} \\ &= (x, a)^{2^k}(y, b)^{2^k} = (x^{2^k}, a^{2^k})(y^{2^k}, b^{2^k}) = (x^{2^k} y^{2^k}, a^{2^k} b^{2^k}), \end{aligned}$$

hence $k \in T_G(x,y) \cap T_H(a,b)$. On the other hand, for $k \in T_G(x,y) \cap T_H(a,b)$, we get

$$((x,a)(y,b))^{2^k} = (xy,ab)^{2^k} = ((xy)^{2^k},(ab)^{2^k}) =$$
$$= (x^{2^k}y^{2^k}, a^{2^k}b^{2^k}) = (x^{2^k},a^{2^k})(y^{2^k},b^{2^k}) = (x,a)^{2^k}(y,b)^{2^k},$$

hence $k \in T_{G\times H}((x,a),(y,b))$. □

Theorem 6. *Let (G,E,V) satisfy* (U). *Let Σ be a groupoid with an element $\sigma = \sigma^2 \in \Sigma$ such that for every $\xi \in \Sigma$ there exists $m \in \mathbb{N}$ yielding $\xi^{2^m} = \sigma$. Then $(G\times\Sigma,E,V)$ also has property* (U).

Proof. Let $f\colon G\times\Sigma \to E$ satisfy

$$f(xy,\xi\eta) - f(x,\xi) - f(y,\eta) \in V, \quad x,y \in G; \xi,\eta \in \Sigma. \tag{10}$$

Then $y = x, \eta = \xi$ leads to

$$f(x^2,\xi^2) - 2f(x,\xi) \in V, \quad x \in G, \xi \in \Sigma,$$

and Theorem 1 applied to $(G\times\Sigma,E,V)$ shows the existence of exactly one function $h\colon G\times\Sigma \to E$ such that

$$h(x^2,\xi^2) = 2h(x,\xi),\ h(x,\xi) - f(x,\xi) \in V, \quad x \in G, \xi \in \Sigma. \tag{11}$$

This function is given by

$$h(x,\xi) = \lim_{n\to\infty} \frac{1}{2^n} f(x^{2^n},\xi^{2^n}), \quad x \in G, \xi \in \Sigma.$$

The choice $\xi = \eta = \sigma$ in (10) leads to

$$f(xy,\sigma) - f(x,\sigma) - f(y,\sigma) \in V, \quad x,y \in G,$$

and since (G,E,V) has the property (U), we get an additive $a\colon G \to E$ such that

$$a(x) - f(x,\sigma) \in V, \quad x \in G.$$

By Theorem 2 and Remark 2 we now have

$$a(x) = \lim_{n\to\infty} \frac{1}{2^n} f(x^{2^n},\sigma) = h(x,\sigma), \quad x \in G. \tag{12}$$

For $\xi \in \Sigma$ let $m \in \mathbb{N}$ be such that $\xi^{2^m} = \sigma$. Then we get by (11), (12) for $x \in G$:

$$\begin{aligned} h(x,\xi) &= \frac{1}{2}h(x^2,\xi^2) = \dots \\ &= \frac{1}{2^m}h(x^{2^m},\sigma) = \frac{1}{2^m}a(x^{2^m}) = a(x). \end{aligned}$$

So we have

$$h(x,\xi) = a(x), \quad x \in G, \xi \in \Sigma,$$

and therefore the function $h\colon G \times \Sigma \to E$ occuring in (11) is additive. Indeed, for $x, y \in G$ and $\xi, \eta \in \Sigma$ we get

$$\begin{aligned} h((x,\xi)(y,\eta)) &= h(xy,\xi\eta) = a(xy) \\ &= a(x) + a(y) = h(x,\xi) + h(y,\eta). \end{aligned}$$

This finishes the proof of (U) for $(G \times \Sigma, E, V)$. $\square$

Remark 7. When taking $G = \{0\}$ (a singleton) and observing $\Sigma \cong \{0\} \times \Sigma$, we see that (Σ, E, V) has property (U). In fact, Σ is a Tabor groupoid (a proof is easy), which in the semigroup-case already is known from Badora, Przebieracz, Volkmann [2, Theorem 3, Case I)].

The next theorem is trivial (we omit the proof), but it may be useful in applications.

Theorem 7. *Let (G_1, E_1, V_1), (G_2, E_2, V_2) have property (U) and let the function $f\colon G_1 \times G_2 \to E_1 \times E_2$ be given by*

$$f(x_1, x_2) = (f_1(x_1), f_2(x_2)), \quad (x_1, x_2) \in G_1 \times G_2,$$

where $f_j\colon G_j \to E_j$ $(j = 1, 2)$. Suppose

$$f(xy) - f(x) - f(y) \in V_1 \times V_2, \quad x, y \in G_1 \times G_2.$$

Then there exists an additive $a\colon G_1 \times G_2 \to E_1 \times E_2$ such that

$$a(x) - f(x) \in V_1 \times V_2, \quad x \in G_1 \times G_2.$$

Remark 8. If $E_1 \times E_2$ is normed by $\|(x_1, x_2)\| = \max\{\|x_1\|, \|x_2\|\}$ and V_1, V_2 are closed (or open) ε-balls centered at θ, then $V_1 \times V_2$ also is a closed (or open) ε-ball centered at θ, i.e.,

$$S_{E_1}(\theta;\varepsilon) \times S_{E_2}(\theta;\varepsilon) = S_{E_1 \times E_2}(\theta;\varepsilon),$$

$$S^0_{E_1}(\theta;\varepsilon) \times S^0_{E_2}(\theta;\varepsilon) = S^0_{E_1 \times E_2}(\theta;\varepsilon).$$

Of course, this remark concerns in particular the Hyers–Ulam stability mentioned in Remark 1.

Let us conclude this section by recalling some known results for groups.

Theorem 8. *Let G be a group.*

I) *If every element of G has odd order, then G is a Tabor group.*

II) *If every element of G has an order 2^n (where $n \in \mathbb{N}$), then G is a Tabor group.*

III) *If $G \cong G_1 \times G_2$ with groups G_1, G_2 as in* I), II) *(respectively), then G is a Tabor group.*

IV) *Any finite Tabor group G has the form given in* III).

Remark 9. I), II) follow from Badora, Przebieracz, Volkmann [2], concerning II) cf. Remark 7; III), IV) are from Toborg [10].

4. Final remarks

Let $F(a, b)$ be the free group with two generators and let $\mathbb{R}$ denote the space of the reals. Gian Luigi Forti [3] has shown that the triplet $(F(a, b), \mathbb{R}, [-1, 1])$ does not have property (U). Thus $F(a, b)$ is not a Tabor group. Now the question is of interest, whether there exist torsion free non commutative Tabor groups.

Finally let us mention that all groupoids with two elements are Tabor groupoids; this can be easily checked. On the other hand, there is a groupoid $G = \{a, b, c\}$ which is not a Tabor groupoid: It is sufficient to require $a^2 = a, ab = c, b^2 = c^2 = b \neq c$. Indeed, assume for some $k \in \mathbb{N}^*$ that

$$(ab)^{2^k} = a^{2^k} b^{2^k}.$$

We get $c^{2^k} = ab$, hence $b = c$, which is a contradiction.

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