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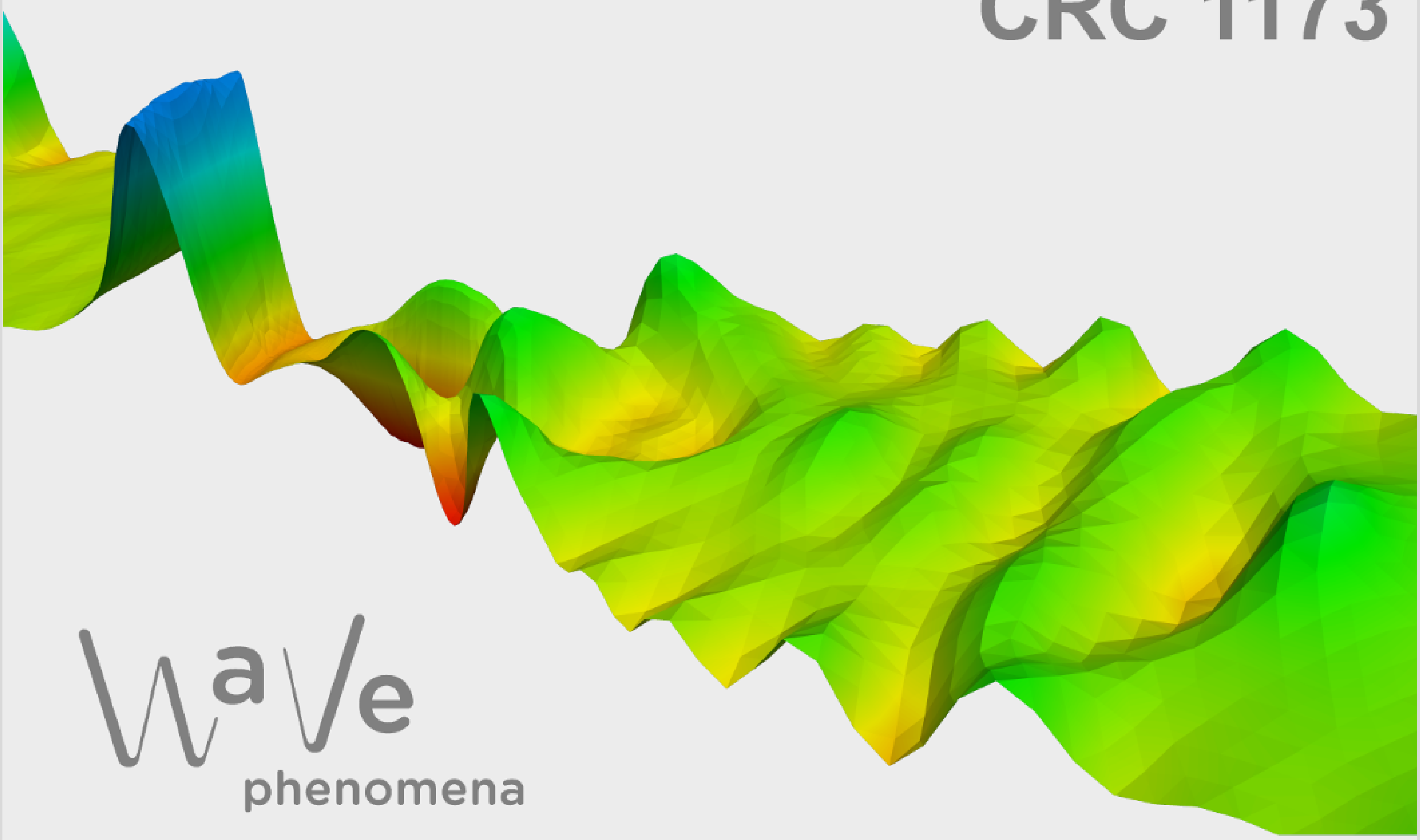

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ON HELMHOLTZ EQUATIONS AND COUNTEREXAMPLES TO STRICHARTZ ESTIMATES IN HYPERBOLIC SPACE

JEAN-BAPTISTE CASTERAS AND RAINER MANDEL

Abstract

In this paper, we study nonlinear Helmholtz equations

$$-\Delta_{\mathbb{H}^N} u - \frac{(N-1)^2}{4}u - \lambda^2 u = \Gamma|u|^{p-2}u \quad \text{in } \mathbb{H}^N,\ N \geq 2 \tag{NLH}$$

where $\Delta_{\mathbb{H}^N}$ denotes the Laplace-Beltrami operator in the hyperbolic space $\mathbb{H}^N$ and $\Gamma \in L^\infty(\mathbb{H}^N)$ is chosen suitably. Using fixed point and variational techniques, we find nontrivial solutions to (NLH) for all $\lambda > 0$ and $p > 2$. The oscillatory behaviour and decay rates of radial solutions is analyzed, with possible extensions to Cartan-Hadamard manifolds and Damek-Ricci spaces. Our results rely on a new Limiting Absorption Principle for the Helmholtz operator in $\mathbb{H}^N$. As a byproduct, we obtain simple counterexamples to certain Strichartz estimates.

1. Introduction

In this paper we are interested in nontrivial solutions of the Nonlinear Helmholtz Equation (NLH)

$$-\Delta_{\mathbb{H}^N} u - \frac{(N-1)^2}{4}u - \lambda^2 u = \Gamma|u|^{p-2}u \quad \text{in } \mathbb{H}^N \tag{1.1}$$

where $\Delta_{\mathbb{H}^N}$ denotes the Laplace-Beltrami operator in hyperbolic space $\mathbb{H}^N, N \geq 2$ and $\Gamma \in L^\infty(\mathbb{H}^N)$. As in the Euclidean setting, linear and nonlinear Helmholtz equations arise from a standing wave ansatz for the corresponding Schrödinger or wave equations that have attracted much interest in the last years [5, 6, 7, 9, 10, 11, 15, 27, 33, 39], especially concerning Strichartz estimates. In order to motivate our first result on the failure of Strichartz estimates in hyperbolic space and to provide the link to Helmholtz equations, let us first review the situation in $\mathbb{R}^N$.

For the homogeneous Schrödinger equation, it was Strichartz himself who proved the (global) Strichartz estimate

$$\begin{cases} i\partial_t \psi - \Delta\psi = 0 \quad \text{in } \mathbb{R}^N, \qquad \psi(0) = \psi_0 \\ \|\psi\|_{L^p(\mathbb{R};L^q(\mathbb{R}^N))} \leq C\|\psi_0\|_{L^r(\mathbb{R}^N)}, \end{cases} \tag{1.2}$$

for $r = 2, p = q = 2(N+2)/N$, see Corollary 1 [37]. The proof is based on Fourier restriction theory for paraboloids, which in turn relies on the Stein-Tomas theorem [42]. Since then, many generalizations of such estimates to more general r, p, q and other dispersive PDEs have been found. The topic being quite vast and intensively studied until today, we do not make any attempt to present a comprehensive list of related results. For a detailed treatment of the Schrödinger equation, we refer to Cazenave's book [16]. Let us only mention that the scaling invariance of the Schrödinger equation shows that in $\mathbb{R}^N$ the estimate (1.2) can only hold if $2/p + N/q = N/r$. Homogeneous Strichartz estimates (1.2) are known to hold for certain ranges of exponents p, q with $r \in (1, 2]$, but, up to the authors' knowledge, nothing is known for $r > 2$. For $r > 2N/(N-1)$, it follows from the theory of Helmholtz equations that no dispersive estimate and especially none of the above estimates (except for $p = \infty$) can hold. Indeed, the method of stationary phase shows that certain solutions to the Helmholtz equation $-\Delta\psi_0 - \omega\psi_0 = 0$ for $\omega > 0$, namely Herglotz waves given by a sufficiently smooth density over the sphere, decays exactly like $|x|^{(1-N)/2}$ as $|x| \to \infty$ (Theorem 1a [29], Proposition 1 [31]). In particular, $\psi(x,t) := e^{i\omega t}\psi_0(x)$ is a solution of the NLS with initial datum in $L^r(\mathbb{R}^N)$ precisely for $r > 2N/(N-1)$ that does not disperse as $t \to \infty$. We believe that it is an interesting open question, whether or not Strichartz estimates hold for initial data $\psi_0 \in L^r(\mathbb{R}^N)$ with $2 < r \le 2N/(N-1)$.

In hyperbolic space, homogeneous Strichartz estimates of the form

$$\begin{cases} i\partial_t\psi - \Delta_{\mathbb{H}^N}\psi = 0 \quad \text{in } \mathbb{H}^N, \qquad \psi(0) = \psi_0 \\ \|\psi\|_{L^p(\mathbb{R};L^q(\mathbb{H}^N))} \le C\|\psi_0\|_{L^r(\mathbb{H}^N)} \end{cases} \tag{1.3}$$

hold for $r = 2$ and $p \in [2,\infty), q \in [2,\infty]$ with $\frac{2}{p} + \frac{N}{q} \ge \frac{N}{r}$, see Theorem 3.6 [5]. In particular, restricting the attention to the classical case $r = 2$, one sees that Strichartz estimates hold for more exponents than in the Euclidean setting. Again, nothing seems to be known for $r > 2$. Given the above considerations in $\mathbb{R}^N$, one way of disproving the validity of Strichartz estimates is to determine the decay rate of solutions of the linear homogeneous Helmholtz equation in $\mathbb{H}^N$. Since we will prove that these solutions lie in $L^r(\mathbb{H}^N)$ for all $r \in (2,\infty]$, we infer the following.

Theorem 1.1. *Let $p \in [1,\infty), q, r \in [1,\infty]$. Then the homogeneous Strichartz estimate for the Schrödinger equation in $\mathbb{H}^N$ (1.3) can only hold provided $1 \le r \le 2$.*

Remark 1.1. *The analogous statement holds for the initial value problem for the wave equation in $\mathbb{H}^N$ and thereby complements the results on Strichartz estimates from* [6, 33, 35, 39].

Next we present our results for the Nonlinear Helmholtz Equation (1.1), which has not been considered in the literature so far. For $-\lambda^2$ replaced by $+\lambda^2$, results on positive and sign-changing solutions can for instance be found in [21, 22, 30]. We stress that the techniques used in these papers are in spirit close to their Euclidean counterparts and the latter change drastically according to the sign in front of λ^2. A much more helpful reference are the papers by Gutiérrez [23], Evequoz, Weth [18, 20] and [31, 32] where the Nonlinear Helmholtz Equation was studied in Euclidean space. In this setting the operator $-\Delta_{\mathbb{H}^N} - (N-1)^2/4$ is replaced by the negative Euclidean Laplacian so that 0 is again the bottom of the essential spectrum. Our intention is to demonstrate that (1.1) can be handled much more easily compared to its Euclidean analogue, which is due to a stronger Limiting Absorption Principle for the Helmholtz operator $L - \lambda^2$ that we will prove in Section 2. Using these results we can follow the lines of [23, 31] and prove the existence of uncountably many small solutions via a fixed point argument.

Theorem 1.2. *Let $N \ge 2, \Gamma \in L^\infty(\mathbb{H}^N), \lambda > 0$ and $p > 2$. Then (1.1) has uncountably many small solutions lying in $W^{2,r}(\mathbb{H}^N)$ for all $r \in (2,\infty)$.*

This result actually holds under much weaker assumptions on the nonlinearity. In fact, in the proof we will replace $\Gamma|u|^{p-2}u$ by any function $f \in C^1(\mathbb{R}_+ \times \mathbb{R})$ satisfying $|f(x,z)| + |z||f_z(x,z)| \le C|z|^{q-1}$ for some $C > 0$ and all $x \in \mathbb{H}^N, |z| \le 1$. We mention that no upper bound for p is necessary. The solutions from the previous theorem are parametrized by hyperbolic Herglotz waves and therefore can even be shown to form a continuum in the above-mentioned Sobolev spaces as in [31]. Large solutions of (1.1) can be constructed using a dual variational approach as in [19, 20]. Note that classical variational approaches are not suitable since solutions are not expected to lie in $L^2(\mathbb{H}^N)$, as we will show further below. The dual variational method yields the following.

Theorem 1.3. *Let $N \geq 2, \lambda > 0, \Gamma \in L^\infty(\mathbb{H}^N)$ satisfy $\Gamma \geq 0, \Gamma \not\equiv 0$ and $2 < p < 2^*$. Then (1.1) has a nontrivial solution $u \in W^{2,r}(\mathbb{H}^N)$ for all $r \in (2,\infty)$ provided*

(i) $\Gamma(x) \geq \Gamma_0$ *where* $\Gamma_0 = \lim_{d(0,x)\to\infty} \Gamma(x)$.

Under the assumptions

(ii) $\Gamma(x) \to 0$ *as* $d(0,x) \to \infty$ *or*

(iii) Γ *is radially symmetric about some point in* $\mathbb{H}^N$

there is a sequence of nontrivial solutions $u_n \in W^{2,r}(\mathbb{H}^N)$ for all $r \in (2,\infty)$ that is unbounded in $L^p(\mathbb{H}^N)$. In the case (iii), the solutions are radial.

Here, 0 stands for the origin in hyperbolic space and $d(x,y)$ denotes the geodesic distance of two points $x, y \in \mathbb{H}^N$. Radially symmetric functions only depend on the geodesic distance to one particular point in $\mathbb{H}^N$, which we denote by r in the following. Let us mention that the dual variational method is flexible enough to treat also higher order problems as in [13] or negative Γ as in [32].

In our final result, we restrict our attention to radially symmetric solutions. One motivation is our interest in exact pointwise decay properties of solutions to (1.1), which do not seem to be availabe in general. Let us point out that, using the ideas of Lemma 2.9 in [20], it is possible to prove that the solutions constructed in (1.3) decay at least like $e^{(1-N)r/2}$ as $r \to \infty$ provided that $p > 4$. Not being convinced in the optimality of this result, we dispense with a proof. In the radial case we can show with elementary means that the solutions decay exactly like $e^{(1-N)r/2}$ as $r \to \infty$ and, in particular, do not lie in $H^1(\mathbb{H}^N)$. We show this to hold without any restriction on p. As a consequence those solutions can not be found with classical variational methods, as we mentioned above. Let us point out that finite energy solutions of nonresonant problems in $\mathbb{H}^N$ as in [30] decay faster, see Remark 3.8 in that paper. Moreover, we prove that radial solutions oscillate. Another interesting feature is that we can analyze radial solutions on manifolds M which are much more general than $\mathbb{H}^N$ such as Damek-Ricci spaces [4, 8, 35]. Indeed, it is known that the radial part of the Laplace-Beltrami operator on such a manifold is given by

$$\partial_{rr} + \frac{f'(r)}{f(r)}\partial_r \qquad \text{where } f(r) = \sinh^{m+k}(\frac{r}{2})\cosh^k(\frac{r}{2}),$$

see (2.11) [4]. Here, $m, k \in \mathbb{N}$ and the dimension of the manifold is $m+k+1$. The corresponding formula also holds in hyperbolic space ($f(r) = \sinh(r)^{N-1}$) and Euclidean space ($f(r) = r^{N-1}$) and even more general classes of rotationally symmetric Cartan-Hadamard manifolds. Each of these examples satisfies assumption (H1) that we will need. The full set of conditions reads as follows:

(H1) $f \in C^1(\mathbb{R})$ with $f' > 0$ and such that $\log(f)'(r) \to \kappa$, $\log(f)''(r) \to 0$ as $r \to \infty$ and $(\log(f)')^2 - \kappa^2$ is integrable near infinity for some $\kappa \in [0,\infty)$.

(H2) $V \in C^1(\mathbb{R}), V > 0$ with $V(r) \to V_\infty > \kappa^2/4$ and $V' \in L^1(\mathbb{R}_+)$

(H3) $\Gamma \in C^1(\mathbb{R}), \Gamma \geq 0$ with $\Gamma(r) \to \Gamma_\infty \geq 0$ and $|\Gamma'|/\Gamma \in L^1(\mathbb{R}_+)$ or $\Gamma \equiv 0$.

Under these conditions we prove the following:

Theorem 1.4. *Assume (H1),(H2),(H3) and $p > 2$. Then the solution u_γ of*

$$-u''(r) - \frac{f'(r)}{f(r)}u(r) - V(r)u = \Gamma(r)|u|^{p-2}u \quad \text{on } [0,\infty), \quad u(0) = \gamma,\ u'(0) = 0$$

has infinitely many zeros and satisfies for all $r \geq 0$

$$\begin{aligned} &|u_\gamma(r)|^2 + |u_\gamma'(r)|^2 \leq C(V(0)\gamma^2 + \Gamma(0)|\gamma|^p)(1 + f(r)), \\ &|u_\gamma(r)|^2 + |u_\gamma'(r)|^2 \geq c(V(0)\gamma^2 + \Gamma(0)|\gamma|^p)(1 + f(r)) \end{aligned} \tag{1.4}$$

where $c, C > 0$ *are independent of* γ.

The proof of Theorem 1.4 partly generalizes and improves Theorem 1.2 [32] given that we can allow for a quite large class of functions f and that the bounds on the right hand side in (1.4) are more explicit. Moreover, though being similar, the proof is much shorter. As in Theorem 1.2 or Theorem 2.10 in [32] the method of proof is also suitable for more general nonlinearities, including $\Gamma|u|^{p-2}u$ with negative Γ. Note that in this case one can show that radial solutions are unbounded if $|\gamma|$ is large and bounded for small $|\gamma|$ provided some mild additional assumptions on f, V, Γ are satisfied.

Let us give a short outline of this paper and comment on the notation that we will employ. In Section 2 we will prove resolvent estimates for Helmholtz operators in $\mathbb{H}^N$ and use them for the proof of a Limiting Absorption Principle. This will be used to prove (very quickly) Theorem 1.1 in Section 3 and Theorem 1.2 via a fixed point argument in Section 4. In Section 5 we implement the dual variational method following [20] in order to prove Theorem 1.3. In the final section, we prove Theorem 1.4. In the following, C denotes a generic constant that may change from line to line. The N-dimensional hyperbolic space $\mathbb{H}^N = \{x \in \mathbb{R}^N : x_N > 0\}$ is considered in the half space model with geodesic distance $d(x,y) = 2\arcsin(|x-y|/(2\sqrt{x_N y_N}))$ and volume element $dV = x_N^{-2}((dx')^2 + dx_N^2) = \sinh(r)^{N-1}\,dr\,d\theta$. Its Laplace-Beltrami operator is given by $\Delta_{\mathbb{H}^N} := x_N^2(\partial_1^2 + \ldots + \partial_N^2) - (N-2)x_N\partial_N$.

2. Resolvent estimates

In this section we discuss resolvent estimates for the operators $L - (\lambda + i\mu)^2$ for $\lambda > 0, \mu \neq 0$ and

$$L := -\Delta_{\mathbb{H}^N} - \frac{(N-1)^2}{4}.$$

It is well-known that the spectrum of L is given by $\sigma(L) = [0,\infty)$ so that the resolvent of $L - \lambda^2$ does not exist in the classical sense. However, as in the Euclidean case, it is possible to prove a Limiting Absorption Principle which yields a solution of the linear Helmholtz equation $Lu - \lambda^2 u = f$ for functions f that decay sufficiently fast at infinity, see Theorem I.4.2 [28]. In this approach the resolvents of $L-(\lambda+i\mu)^2$ are studied and function spaces are identified, in which the limits of the resolvent operators persist as their imaginary parts tend to zero from the right respectively from the left. These operators will in the following be denoted by $(L-\lambda^2-i0)^{-1}$ respectively $(L-\lambda^2+i0)^{-1}$. In the Euclidean setting, estimates for $(-\Delta-\lambda^2-i0)^{-1}$ from weighted Lebesgue spaces $L^{2,s}(\mathbb{R}^N)$ to $L^{2,-s}(\mathbb{R}^N)$ ($s > \frac{1}{2}$) or even from $B(\mathbb{R}^N)$ to its dual $B^*(\mathbb{R}^N)$ are due to Ikebe and Saito [26] (Theorem 1.2) as well as Agmon and Hörmander, see Theorem 4.1 in [2], [1] and Theorem 3.1 in [3]. Here,

$$L^{2,s}(\mathbb{R}^N) = \big\{v \in L^2_{loc}(\mathbb{R}^N) : |\cdot|^s v \in L^2(\mathbb{R}^N)\big\},$$

$$B(\mathbb{R}^N) = \Big\{v \in L^2_{loc}(\mathbb{R}^N) : \sum_{j=1}^{\infty} 2^{\frac{j-1}{2}}\Big(\int_{\{2^{j-1}<|x|<2^j\}} |u(x)|^2\,dx\Big)^{1/2} < \infty\Big\}.$$

The counterpart for the latter estimate in the hyperbolic case was proved by Perry, see Theorem 5.1 in [34] or Theorem I.4.2 in [28]. In Theorem 1.2 of [25] Huang and Sogge proved that these operators may as well be defined as bounded linear operators from $L^p(\mathbb{H}^N)$ to $L^q(\mathbb{H}^N)$, where p, q satisfy

$$\frac{1}{p} - \frac{1}{q} = \frac{2}{N} \quad \text{and} \quad \min\big\{\big|\frac{1}{p} - \frac{1}{2}\big|, \big|\frac{1}{q} - \frac{1}{2}\big|\big\} > \frac{1}{2N}.$$

We stress that these restrictions are essentially due to the authors' focus on uniform estimates with respect to λ within the range $|\lambda|^2 \geq 1$. For our purposes such a uniform behaviour is not needed, which allows us to modify and adapt some of the ideas from [25] in order to obtain resolvent estimates for larger ranges of exponents. This extension is based on recent results by Chen and Hassell [17]. For $\sigma_p > 0$ given by

$$\sigma_p := \frac{2N}{p} + N - 1 \qquad \text{if } 1 \le p \le \frac{2(N+1)}{N+3},$$
$$\sigma_p := \frac{N-1}{p} - \frac{N-1}{2} \qquad \text{if } \frac{2(N+1)}{N+3} \le p < 2,$$

their result about the spectral resolution $\mathbb{R} \ni \lambda \mapsto E_P(\lambda)$ of the selfadjoint operator $P := \sqrt{L}$ reads as follows.

Theorem 2.1 (Theorem 1.6 [17]). *Let $N \ge 2$ and $1 \le p < 2$. Then there a $C > 0$ such that the following estimate holds for all $\lambda > 0$:*

$$\|\frac{d}{d\lambda}E_P(\lambda)\|_{L^p(\mathbb{H}^N)\to L^{p'}(\mathbb{H}^N)} \le C\lambda^2 \qquad (0 < \lambda \le 1)$$
$$\|\frac{d}{d\lambda}E_P(\lambda)\|_{L^p(\mathbb{H}^N)\to L^{p'}(\mathbb{H}^N)} \le C\lambda^{\sigma_p} \qquad (\lambda \ge 1)$$

As we will show below, this estimate may be used in order to prove the resolvent estimates along the lines of [25]. Before going on with this, we recall some useful information about the Green's function associated with the operator $L - (\lambda + i\mu)^2$ given by $(x, y) \mapsto G_{\lambda+i\mu}(d(x, y))$. In other words, for all $f \in C_0^\infty(\mathbb{H}^N)$, we have

$$(L - (\lambda + i\mu)^2)^{-1} f = G_{\lambda+i\mu} * f := \int_{\mathbb{H}^N} G_{\lambda+i\mu}(d(x, y))f(y)\, dV(y).$$

For notational convenience, we will in the following assume $\mu > 0$. It is known ([41] p.125) that there are complex constants $c_N \neq 0$ such that, for odd space dimensions N, we have

$$G_{\lambda+i\mu}(t) = \frac{c_N}{i\lambda - \mu}\left(\frac{1}{\sinh t}\frac{\partial}{\partial t}\right)^{\frac{N-1}{2}}\left[e^{(i\lambda-\mu)t}\right], \tag{2.1}$$

whereas in the case of even N we have

$$G_{\lambda+i\mu}(t) = \frac{c_N}{i\lambda - \mu}\int_t^\infty \frac{\sinh s}{\sqrt{\cosh s - \cosh t}}\left(\frac{1}{\sinh s}\frac{\partial}{\partial s}\right)^{\frac{N}{2}}\left[e^{(i\lambda-\mu)s}\right] ds. \tag{2.2}$$

The properties of the Green's function are summarized in the following proposition.

Proposition 2.1. *Let $N \in \mathbb{N}, N \ge 2$ and let $G_{\lambda+i\mu}$ be given by (2.1) for odd N and by (2.2) for even N. Then for all $\Lambda > 0$ there is a constant $C > 0$ such that*

$$|G_{\lambda+i\mu}(t)| \le C \max\{t^{2-N}, |\log(t)|\} \qquad (|t| \le 1),$$
$$|G_{\lambda+i\mu}(t)| \le Ce^{(\frac{1-N}{2}-|\mu|)t} \qquad (|t| \ge 1)$$

for all $\lambda \in [\Lambda^{-1}, \Lambda]$ and $\mu \in [-\Lambda, \Lambda] \setminus \{0\}$.

We shall also use the formula from (4.4) in [25] that allows to write the convolution $G_{\lambda+i\mu} * f$ in a different way. It reads

$$(L - (\lambda + i\mu)^2)^{-1} f = -\frac{1}{i\lambda - \mu}\int_0^\infty e^{(i\lambda-\mu)t}\cos(tP)f\, dt \tag{2.3}$$

where the function $\cos(tP)$ is defined via functional calculus. Note that this formula is a consequence of the fact that for any given test function f the function $u(t) :=$

$\cos(tP)f$ is the unique solutions of the initial value problem $\partial_{tt}u+P^2u=0$, $u(0)=f$, $\partial_t u(0)=0$. So $L=P^2$ yields

$$\begin{aligned}
&(L-(\lambda+i\mu)^2)\left(-\frac{1}{i\lambda-\mu}\int_0^\infty e^{(i\lambda-\mu)t}\cos(tP)f\,dt\right)\\
&=-\frac{1}{i\lambda-\mu}\int_0^\infty e^{(i\lambda-\mu)t}(P^2-(\lambda+i\mu)^2)u(t)\,dt\\
&=-\frac{1}{i\lambda-\mu}\int_0^\infty e^{(i\lambda-\mu)t}\left(-\partial_{tt}u(t)+(i\lambda-\mu)^2u(t)\right)\,dt\\
&=-\frac{1}{i\lambda-\mu}\left[e^{(i\lambda-\mu)t}\big(-\partial_t u(t)+(i\lambda-\mu)u(t)\big)\right]_0^\infty\\
&=u(0)=f.
\end{aligned}$$

With these preparations, we can now prove the resolvent estimates for L.

Theorem 2.2. *Let $N\geq 2$ and $\Lambda>0$. Then there exists a constant C such that*

$$\|(L-(\lambda+i\mu)^2)^{-1}f\|_{L^q(\mathbb{H}^N)}\leq C\|f\|_{L^p(\mathbb{H}^N)}$$

for all $\lambda\in[\Lambda^{-1},\Lambda]$ and $\mu\in[-\Lambda,\Lambda]\backslash\{0\}$ provided that $1\leq p<2<q$ and $\frac{1}{p}-\frac{1}{q}\leq\frac{2}{N}$ with $(p,q)\neq(1,\frac{N}{N-2}),(\frac{N}{2},\infty)$.

Proof. As above we only discuss the case $\mu>0$. Let $\beta\in C_0^\infty((1/2,2))$ be a nonnegative function such that $\sum_{k\in\mathbb{Z}}\beta(2^{-k}t)=1$ for all $t>0$ and set

$$\beta_k(t):=\beta(2^{-k}t)\quad(k\geq 1),\qquad \beta_0(t):=1-\sum_{k=1}^\infty\beta_k(t).$$

In particular, we have $\beta_0(t)=1$ for $t\in[0,1]$, $\beta_0(t)=0$ for $t\geq 2$ and the β_k are supported on annuli with inner resp. outer radius $2^{k-1},2^{k+1}$. Recalling (2.1) and (2.2), we define, for all $k\in\mathbb{N}_0$ and odd space dimensions N, the function

$$S_k(t):=\frac{c_N}{i\lambda-\mu}\left(\frac{1}{\sinh t}\frac{\partial}{\partial t}\right)^{\frac{N-1}{2}}\left[\beta_k(t)e^{(i\lambda-\mu)t}\right].$$

For even N the corresponding definition is

$$S_k(t):=\frac{c_N}{i\lambda-\mu}\int_t^\infty\frac{\sinh s}{\sqrt{\cosh s-\cosh t}}\left(\frac{1}{\sinh s}\frac{\partial}{\partial s}\right)^{\frac{N}{2}}\left[\beta_k(s)e^{(i\lambda-\mu)s}\right]ds.$$

These definitions and Proposition 2.1 give $\sum_{k=0}^\infty S_k=G_{\lambda+i\mu}$ so that we have to estimate the integrals S_k*f for $f\in L^p(\mathbb{H}^N)$.

We start with the estimates for S_0*f. By definition of β_0,S_0 and Proposition 2.1 we have

$$S_0(t)=0\quad\text{for } t\geq 2,\qquad |S_0(t)|\leq C\max\{t^{2-N},|\log(t)|\}\quad\text{for } 0<t\leq 2.$$

So $S_0\in L^r(\mathbb{H}^N)$ for $1\leq r<\frac{N}{N-2}$ as well as $S_0\in L^{\frac{N}{N-2},\infty}(\mathbb{H}^N)$ if $N\geq 3$. Using the Weak Young Inequality in $\mathbb{H}^N$ (see Theorem 6.2.3 [36] and the following remarks) we obtain

$$S_0*f\in L^q(\mathbb{H}^N)\quad\text{if } 1+\frac{1}{q}=\frac{1}{r}+\frac{1}{p},\ 1\leq r<\frac{N}{N-2}\text{ or } r=\frac{N}{N-2},1<p,q<\infty.$$

In other words, we have

$$S_0*f\in L^q(\mathbb{H}^N)\quad\text{if } 0\leq\frac{1}{p}-\frac{1}{q}<\frac{2}{N}\ \text{ or }\ \frac{1}{p}-\frac{1}{q}=\frac{2}{N},\ 1<p,q<\infty.$$

Since these conditions are satisfied by our assumptions on p,q, it remains to estimate the integrals S_k*f for $k\geq 1$.

Next, we are going to show that, for $k \geq 1$,

$$\|S_k * f\|_{L^q(\mathbb{H}^N)} \leq C 2^{-k} \|f\|_{L^p(\mathbb{H}^N)}.$$

First, we prove the corresponding inequality for $p = 1, q = \infty$. In (4.14) [25] it is shown that for any given $M > 0$ there is a $C_M > 0$ such that

$$\|S_k * f\|_{L^\infty(\mathbb{H}^N)} \leq C_M 2^{-kM} \|f\|_{L^1(\mathbb{H}^N)}. \tag{2.4}$$

This is a consequence of the uniform pointwise exponential decay of $G_{\lambda+i\mu}$ at infinity, see Proposition 2.1. In order to prove the inequality for all $q > 2$ and $p = 2$, we make use the formula

$$S_k * f = \frac{c_N}{i\lambda - \mu} \int_0^\infty \beta_k(t) e^{(i\lambda-\mu)t} \cos(tP) f \, dt$$

from p.4655 [25] (which can be proved just as (2.3)). We define

$$\begin{aligned} \psi_k(s) &:= \int_{\mathbb{R}} \beta_k(t) e^{(i\lambda-\mu)t} \cos(ts) \, dt \\ &= \frac{1}{2} \mathcal{F} \left(\beta_k(\cdot) e^{(i\lambda-\mu)\cdot} + \beta_k(-\cdot) e^{-(i\lambda-\mu)\cdot} \right)(s) \end{aligned} \tag{2.5}$$

where $\mathcal{F}$ denotes the one-dimensional Fourier transform. For any given $r > 2$ we choose $M \in \mathbb{N}$ such that $\sigma_{r'} \leq 2M$. We then obtain for all $k \in \mathbb{N}$

$$\begin{aligned}
&\|S_k * f\|^2_{L^2(\mathbb{H}^N)} \\
&= \left\| \int_{\mathbb{R}} \beta_k(t) e^{(i\lambda-\mu)t} \cos(tP) \, dt f \right\|^2_{L^2(\mathbb{H}^N)} \\
&= \|\psi_k(P) f\|^2_{L^2(\mathbb{H}^N)} \\
&= \int_{\mathbb{R}} |\psi_k(s)|^2 \, d\langle E_P(s) f, f\rangle \\
&\leq \int_{\mathbb{R}} |\psi_k(s)|^2 \|\frac{d}{ds} E_P(s) f\|_{L^r(\mathbb{H}^N)} \|f\|_{L^{r'}(\mathbb{H}^N)} \, ds \\
&\leq C \int_0^\infty |\psi_k(s)|^2 \left(s^2 1_{[0,1]}(s) + s^{\sigma_{r'}} 1_{[1,\infty)}(s) \right) \|f\|^2_{L^{r'}(\mathbb{H}^N)} \, ds \\
&\leq C \|f\|^2_{L^{r'}(\mathbb{H}^N)} \cdot \int_0^\infty |\psi_k(s)|^2 (s^2 + \ldots + s^{2M}) \, ds \\
&= C \|f\|^2_{L^{r'}(\mathbb{H}^N)} \cdot \int_{\mathbb{R}} |(\mathcal{F}^{-1}\psi_k)'(s)|^2 + \ldots + |(\mathcal{F}^{-1}\psi_k)^{(M)}(s)|^2 \, ds \\
&\overset{(2.5)}{=} C \|f\|^2_{L^{r'}(\mathbb{H}^N)} \cdot \int_0^\infty \left| \frac{d}{ds} \left(\beta(2^{-k}s) e^{(i\lambda-\mu)s} \right) + \ldots + \frac{d^M}{ds^M} \left(\beta(2^{-k}s) e^{(i\lambda-\mu)s} \right) \right|^2 ds \\
&\leq C \|f\|^2_{L^{r'}(\mathbb{H}^N)} \cdot \int_0^\infty \left(\beta(2^{-k}s)^2 + \ldots + \beta^{(M)}(2^{-k}s)^2 \right) e^{-2\mu s} \, ds \\
&\leq C 2^k \|\beta\|^2_{H^M(\mathbb{R})} \|f\|^2_{L^{r'}(\mathbb{H}^N)}.
\end{aligned}$$

Taking the square root of this estimate we obtain by duality (note that $f \mapsto S_k * f$ is symmetric)

$$\|S_k * f\|_{L^r(\mathbb{H}^N)} \leq C 2^{k/2} \|f\|_{L^2(\mathbb{H}^N)}. \tag{2.6}$$

(This is an improved version of (4.13) in [25].)

Interpolating now (2.6) and (2.4) we get

$$\|S_k * f\|_{L^q(\mathbb{H}^N)} \leq C 2^{k(1-\frac{1}{p}+M(1-\frac{2}{p}))} \|f\|_{L^p(\mathbb{H}^N)} \qquad q = \frac{rp}{2(p-1)}, r > 2.$$

So for any given $p \in (1,2), q > \frac{p}{p-1}$ we can choose $r > 2$ as in the previous line and take M sufficiently large to get

$$\|S_k * f\|_{L^q(\mathbb{H}^N)} \leq C2^{-k}\|f\|_{L^p(\mathbb{H}^N)}, \quad \text{provided } p \in (1,2),\ q > \frac{p}{p-1} > 2.$$

The dual version of this is

$$\|S_k * f\|_{L^q(\mathbb{H}^N)} \leq C2^{-k}\|f\|_{L^p(\mathbb{H}^N)}, \quad \text{provided } 2 < q < \frac{p}{p-1} < \infty$$

and interpolating both yields

$$\|S_k * f\|_{L^q(\mathbb{H}^N)} \leq C2^{-k}\|f\|_{L^p(\mathbb{H}^N)}, \quad \text{provided } p \in (1,2),\ q > 2.$$

The remaining case $p = 1, q > 2$ may again be obtained by interpolation with (2.4) and we are done. □

Having established locally uniform bounds and taking into account Theorem I.4.2 [28] we may define

$$\mathcal{R}_\lambda + i\mathcal{E}_\lambda := (L - \lambda^2 - i0)^{-1} := \lim_{\mu \to 0^+}(L - (\lambda_\mu + i\mu)^2)^{-1} \tag{2.7}$$

where $\lambda_\mu := \sqrt{\lambda^2 + \mu^2}$ as bounded linear operators on Lebegue spaces in $\mathbb{H}^N$. The main properties of these operators are summarized in the following result.

Corollary 2.1. *The operators $\mathcal{R}_\lambda, \mathcal{E}_\lambda : L^p(\mathbb{H}^N) \to L^q(\mathbb{H}^N)$ are linear and bounded provided that $1 \leq p < 2 < q$ and $\frac{1}{p} - \frac{1}{q} \leq \frac{2}{N}$ with $(p,q) \neq (1, \frac{N}{N-2}), (\frac{N}{2}, \infty)$. Moreover, we have the representation formula $\mathcal{R}_\lambda f = G * f$ for all $f \in C_0^\infty(\mathbb{H}^N)$ where the Green's function $G(t) := \lim_{\mu \to 0^+} \mathrm{Re}(G_{\lambda + i\mu}(t))$ satisfies*

$$\begin{aligned} |G(t)| &\leq C \max\{|t|^{2-N}, |\log(t)|\} \quad (|t| \leq 1), \\ |G(t)| &\leq Ce^{(1-N)t/2} \qquad\qquad\qquad (|t| \geq 1) \end{aligned} \tag{2.8}$$

For all $f \in L^p(\mathbb{H}^N)$ the function $\mathcal{R}_\lambda f$ is a strong solution of $L\phi - \lambda^2\phi = f$ in $\mathbb{H}^N$ and $\mathcal{E}_\lambda f$ solves $L\psi - \lambda^2\psi = 0$ in $\mathbb{H}^N$. Finally, we have the identities

$$\int_{\mathbb{H}^N} f(\mathcal{E}_\lambda g)\, dV = \frac{\pi}{2\lambda}\langle f, A_\lambda g\rangle, \qquad \int_{\mathbb{H}^N} f(\mathcal{R}_\lambda g)\, dV = p.v. \int_{\mathbb{R}} \frac{\langle f, A_s g\rangle}{s^2 - \lambda^2}\, ds \tag{2.9}$$

for all $f, g \in L^p(\mathbb{H}^N)$ with $1 \leq p < 2$ where $A_\lambda : L^p(\mathbb{H}^N) \to L^{p'}(\mathbb{H}^N)$ is given by

$$A_\lambda := \frac{d}{d\lambda}E_P(\lambda) = (\mathcal{F}_0(\lambda)^{(+)})^* \mathcal{F}_0(\lambda)^{(+)} = (\mathcal{F}_0(\lambda)^{(-)})^* \mathcal{F}_0(\lambda)^{(-)} \tag{2.10}$$

for the bounded linear operators $\mathcal{F}_0^{(\pm)}(\lambda) : L^p(\mathbb{H}^N) \to L^2(\mathbb{R}^{N-1})$ and $\mathcal{F}_0^{(\pm)}(\lambda)^* : L^2(\mathbb{R}^{N-1}) \to L^{p'}(\mathbb{H}^N)$ defined in (I.4.2),(I.4.10) [28].

Proof. The asymptotics of G follows from Proposition 2.1. Since the boundedness of $\mathcal{R}_\lambda, \mathcal{E}_\lambda$ result from Theorem 2.2, we next prove (2.9) for $f, g \in C_0^\infty(\mathbb{H}^N)$. Let $h(s) := \langle f, A_s g\rangle$. Then we have

$$\begin{aligned} \int_{\mathbb{H}^N} f(L - \lambda^2 - i0)^{-1} g\, dV &= \lim_{\mu \to 0^+} \int_{\mathbb{H}^N} f(L - (\lambda_\mu + i\mu)^2)^{-1} g\, dV \\ &\overset{(2.3)}{=} \lim_{\mu \to 0^+} \Big\langle f, -\frac{1}{i\lambda_\mu - \mu} \int_0^\infty e^{(i\lambda_\mu - \mu)t} \cos(tP) g\, dt \Big\rangle \\ &= -\frac{1}{i\lambda} \lim_{\mu \to 0^+} \Big\langle f, \int_0^\infty e^{(i\lambda_\mu - \mu)t} \Big(\int_{\mathbb{R}} \cos(ts)\, dE_P(s) g\Big)\, dt \Big\rangle \\ &= -\frac{1}{i\lambda} \lim_{\mu \to 0^+} \int_{\mathbb{R}} \Big(\int_0^\infty e^{(i\lambda_\mu - \mu)t} \cos(ts)\, dt\Big)\, d\langle f, E_P(s) g\rangle \\ &= -\frac{1}{i\lambda} \lim_{\mu \to 0^+} \int_{\mathbb{R}} \frac{i(\lambda_\mu + i\mu)}{(\lambda_\mu + i\mu)^2 - s^2}\, d\langle f, E_P(s) g\rangle \end{aligned}$$

$$
\begin{aligned}
&= - \lim_{\mu\to 0^+} \int_{\mathbb{R}} \frac{1}{\lambda^2 - s^2 + 2i\mu\lambda_\mu}\, d\langle f, E_P(s) g\rangle \\
&= - \lim_{\mu\to 0^+} \int_{\mathbb{R}} \frac{h(s)}{\lambda^2 - s^2 + 2i\mu\lambda_\mu}\, ds \\
&= \frac{i\pi}{2\lambda} h(\lambda) - \lim_{\mu\to 0^+} \int_{\mathbb{R}} \frac{h(s)-h(\lambda)}{\lambda^2 - s^2 + 2i\mu\lambda_\mu}\, ds \\
&= \frac{i\pi}{2\lambda} h(\lambda) + \text{p.v.} \int_{\mathbb{R}} \frac{h(s)}{s^2-\lambda^2}\, ds.
\end{aligned}
$$

This computation and the definition (2.7) yield (2.9) by density of the test functions. Moreover, for all $f \in C_0^\infty(\mathbb{H}^N)$, we get from (2.9) and (I.4.3) [28] the identity

$$
\langle f, A_\lambda f\rangle = \frac{\lambda}{\pi i} \Big\langle \left((L-\lambda^2 - i0)^{-1} - (L-\lambda^2+i0)^{-1}\right) f, f \Big\rangle = \|\mathcal{F}_0^{(\pm)}(\lambda) f\|_{L^2(\mathbb{R}^{N-1})}^2
$$

for the operators $\mathcal{F}_0^{(\pm)}(\lambda)$ defined in (I.4.2) [28]. Hence, since A_λ is a symmetric operator, we deduce

$$
\langle f, A_\lambda g\rangle = \langle \mathcal{F}_0^{(\pm)}(\lambda) f, \mathcal{F}_0^{(\pm)}(\lambda) g\rangle_{L^2(\mathbb{R}^{N-1})}
$$

for all test functions f, g and (2.10) follows. Finally, since $\mathcal{E}_\lambda : L^p(\mathbb{H}^N) \to L^{p'}(\mathbb{H}^N)$ is bounded, the operators $\mathcal{F}_0^{(\pm)}(\lambda)^*$ and $\mathcal{F}_0^{(\pm)}(\lambda)$ are bounded as well. □

The functions $\mathcal{F}_0(\lambda)^* g$ with $g \in L^2(\mathbb{R}^{N-1})$ actually represent the totality of solutions of the homogeneous Helmholtz equation by Theorem I.4.3 [28]. They are the counterparts of Euclidean Hergoltz waves that are defined as the images of $L^2(S_\lambda)$-densities under the adjoint of the Fourier restriction operator $f \mapsto \hat{f}|_{S_\lambda}$, where $S_\lambda \subset \mathbb{R}^N$ denotes the sphere of radius λ. While hyperbolic Herglotz waves $\mathcal{F}_0(\lambda)^* g$ lie in $L^p(\mathbb{H}^N)$ for all $p > 2$, the optimal L^p decay rate of Euclidean Herglotz waves is given by the Stein-Tomas Theorem saying that $p \geq \frac{2(N+1)}{N-1}$. Better decay properties of the latter, namely pointwise decay like $|x|^{(1-N)/2}$ at infinity, can be obtained for densities of higher regularity via the method of stationary phase. Whether $L^p(\mathbb{R}^N)$ can be reached in the optimal range $p > \frac{2N}{N-1}$ only by assuming stronger integrability assumptions on the density, is a delicate question related to the Restriction Conjecture which is still unsolved for $N \geq 3$. So we see that hyperbolic Herglotz waves have much better integrability properties than their Euclidean counterparts. This will allow us to adopt a fixed point approach that is decidedly simpler than its Euclidean analogue [23, 31] where Helmholtz equations of the form (1.1) can only be discussed for a restricted set of exponents p.

3. Proof of Theorem 1.1

Given the results of the previous section, the proof is quite simple. Let $g \in L^2(\mathbb{R}^{N-1})$ be nontrivial. Then Corollary 2.1 implies $\psi_0 := \mathcal{F}_0^{(+)}(\lambda)^* g \in L^r(\mathbb{H}^N)$ for all $r > 2$ and $\psi(x,t) := e^{i\omega t}\psi_0(x)$ solves (1.3), but $\psi \notin L^p(\mathbb{R}, L^q(\mathbb{H}^N))$, since ψ is periodic in time. This proves the result. □

4. Proof of Theorem 1.2

In this section we prove Theorem 1.2 with the aid of the Contraction Mapping Principle and the estimates for the operators $\mathcal{F}_0(\lambda)^{(+)}, \mathcal{R}_\lambda$ from Corollary 2.1. As in [31] we use a smooth function $\chi \in C^\infty(\mathbb{R})$ such that $\chi(z) = z$ for $|z| \leq \frac{1}{2}$, $\chi(z) = 1$ for $|z| \geq 1$ and define, for any given $g \in L^2(\mathbb{R}^{N-1})$, the operator

$$
T_g(u) := \mathcal{F}_0^{(+)}(\lambda)^* g + \mathcal{R}_\lambda(f(\cdot, \chi(u))).
$$

As mentioned in the introduction we use the assumption $|f(x,z)| + |z||f_z(x,z)| \leq C|z|^{q-1}$. The operator T_g is well-defined as a map from $L^s(\mathbb{H}^N)$ to $L^s(\mathbb{H}^N)$ provided we choose s according to $\max\{2, \frac{N(q-2-\delta)}{2}, q-1-\delta\} < s < 2(q-1-\delta)$ for some $\delta \in (0, q-2)$. Indeed, under this assumption Corollary 2.1 applies and we obtain

$$\begin{aligned}
\|T_g(u)\|_{L^s(\mathbb{H}^N)} &\leq \|\mathcal{F}_0^{(+)}(\lambda)^* g\|_{L^s(\mathbb{H}^N)} + \|\mathcal{R}_\lambda(f(\cdot,\chi(u)))\|_{L^s(\mathbb{H}^N)} \\
&\leq C(\|g\|_{L^2(\mathbb{R}^{N-1})} + \|f(\cdot,\chi(u))\|_{L^{\frac{s}{q-1-\delta}}(\mathbb{H}^N)}) \\
&\leq C(\|g\|_{L^2(\mathbb{R}^{N-1})} + \||\chi(u)|^{q-1}\|_{L^{\frac{s}{q-1-\delta}}(\mathbb{H}^N)}) \\
&\leq C(\|g\|_{L^2(\mathbb{R}^{N-1})} + \||\chi(u)|^{q-1-\delta}\|_{L^{\frac{s}{q-1-\delta}}(\mathbb{H}^N)}) \\
&\leq C(\|g\|_{L^2(\mathbb{R}^{N-1})} + \|u\|_{L^s(\mathbb{H}^N)}^{q-1-\delta}).
\end{aligned}$$

So we conclude that T_g is a selfmap on any given sufficiently small ball in $L^s(\mathbb{H}^N)$ provided $g \in L^2(\mathbb{R}^{N-1})$ is chosen small enough. Similarly, using $|f_z(x,z)| \leq C|z|^{q-2}$ for $|z| \leq 1$ we obtain that T_g is a contraction on small balls. Hence, by the Contraction Mapping Principle, for every given small enough $g \in L^2(\mathbb{R}^{N-1})$ the operator T_g has a unique fixed point in a small ball and thus a solution $u \in L^s(\mathbb{H}^N)$ of $Lu - \lambda^2 u = f(x,\chi(u))$. Elliptic L^p-estimates imply $u \in L^\infty(\mathbb{H}^N) \cap L^s(\mathbb{H}^N)$ and using the mapping properties of $\mathcal{F}_0^{(+)}(\lambda)^*, \mathcal{R}_\lambda$ from Corollary 2.1 iteratively, we actually find $u \in L^r(\mathbb{H}^N)$ for all $r \in (2,\infty]$. Applying global L^p-estimates from from Theorem A [40] we find $u \in W^{2,r}(\mathbb{H}^N)$ for all $r \in (2,\infty)$. Moreover, choosing g sufficiently small, we may choose the ball and hence the $L^s(\mathbb{H}^N)$-norm of u so small that $\|u\|_{L^\infty(\mathbb{H}^N)} \leq C\|u\|_{L^s(\mathbb{H}^N)} \leq \frac{1}{2}$ holds. But then we have $\chi(u) = u$ and u is the solution of the nonlinear Helmholtz equation (1.1) we were looking for. We finally mention that different g yield different solutions since $\mathcal{F}_0^{(+)}(\lambda)^*$ is injective, see Corollary I.4.6 [28]. $\square$

5. Proof of Theorem 1.3

In this section we prove the existence of nontrivial solutions to

$$Lu - \lambda^2 u = \Gamma |u|^{p-2} u \quad \text{in } \mathbb{H}^N$$

under the assumptions of Theorem 1.3 using dual variational methods. In the context of the nonlinear Helmholtz equation, this approach was introduced by Evequoz and Weth in order to treat the corresponding equation in $\mathbb{R}^N$ for $N \geq 3$ [20] or $N = 2$ [18]. We show that many of their ideas carry over to the Euclidean setting. It actually turns out that the main difficulty in their approach, namely the verification of the "Nonvanishing Property", is much simpler due to a variant of the Stein-Kunze phenomenon, as we will show later. Given that the fundamental ideas are all present in the literature, we keep the presentation short. Following [20] the dual variational method works as follows. Using that Γ is nonnegative, we may set $v := \Gamma^{1/p'}|u|^{p-2}u \in L^{p'}(\mathbb{H}^N)$ so that the task is to find nontrivial solutions of (1.1) by solving the integral equation

$$\Gamma^{1/p}\mathcal{R}_\lambda(\Gamma^{1/p}v) = |v|^{p'-2}v. \tag{5.1}$$

Notice that the mapping properties of $\mathcal{R}_\lambda$ from Corollary 2.1 ensure that this equation makes sense for $v \in L^{p'}(\mathbb{H}^N)$ as long as $2 < p < 2^*$. Since $\mathcal{R}_\lambda$ is symmetric, solutions of (5.1) are critical points of the functional $J \in C^1(L^{p'}(\mathbb{H}^N), \mathbb{R})$ defined by

$$J(v) := \frac{1}{p'}\int_{\mathbb{H}^N} |v|^{p'}\,dV - \frac{1}{2}\int_{\mathbb{H}^N} \Gamma^{1/p} v \mathcal{R}_\lambda(\Gamma^{1/p}v)\,dV. \tag{5.2}$$

In the proof of our statements (i),(ii),(iii), we will apply Critical Point Theory to prove the existence of one respectively infinitely many nontrivial critical points of J. For the same reasons as in the previous section, these solutions are actually strong solutions and belong to $W^{2,r}(\mathbb{H}^N)$ for all $r \in (2,\infty)$.

Proof of (i): We only prove the claim for constant Γ. Note that the proof in this special case requires the verification of the "Nonvanishing property" from [20] so that the claim in the general case $\Gamma \geq \Gamma_0 = \lim_{x\to\infty}\Gamma(x) > 0$ follows precisely as in Theorem 4.3 [19]. So from now on we assume w.l.o.g. $\Gamma \equiv 1$ so that $J(u) = J(u\circ\tau)$ for all hyperbolic tranlations τ.

As in Lemma 4.2 (i) [20] one finds that J has the mountain pass geometry and that there is a Palais-Smale sequence (v_n) in $L^{p'}(\mathbb{H}^N)$ at its Mountain Pass level $c > 0$ given by

$$c = \inf_{\gamma\in P}\max_{t\in[0,1]} J(\gamma(t)) > 0. \tag{5.3}$$

In other words,

$$\begin{aligned} |v_n|^{p'-2}v_n - G * v_n \to 0 \quad \text{in } L^p(\mathbb{H}^N),\\ \frac{1}{p'}\int_{\mathbb{H}^N} |v_n|^{p'}\,dV - \frac{1}{2}\int_{\mathbb{H}^N} v_n(G*v_n)\,dV \to c. \end{aligned} \tag{5.4}$$

From this one infers that (v_n) is bounded in $L^{p'}(\mathbb{H}^N)$ and

$$\left(\frac{1}{p'}-\frac{1}{2}\right)\int_{\mathbb{H}^N} v_n(G*v_n)\,dV \to c. \tag{5.5}$$

Next we show that after some hyperbolic translations (v_n) converges to some non-trivial critical point $v \in L^{p'}(\mathbb{H}^N)$ of J.

The Stein-Kunze estimate from Lemma 4.1 [6] yields

$$\begin{aligned} &\left(\frac{1}{p'}-\frac{1}{2}\right)\left|\int_{\mathbb{H}^N} v_n\left[(1_{[\frac{1}{R},R]^c}G)*v_n\right]dV\right| \\ &\leq \|v_n\|_{L^{p'}(\mathbb{H}^N)}\|(1_{[\frac{1}{R},R]^c}G)*v_n\|_{L^p(\mathbb{H}^N)} \\ &\leq C\|v_n\|^2_{L^{p'}(\mathbb{H}^N)}\left(\int_0^{1/R}+\int_R^\infty (\sinh r)^{N-1}(1+r)e^{-(N-1)r/2}|G(r)|^{p/2}\,dr\right)^{2/p} \\ &\leq \frac{c}{2} \end{aligned}$$

for all $n\in\mathbb{N}$ provided R is large enough. Here we used the boundedness of (v_n) in $L^{p'}(\mathbb{H}^N)$, the asymptotics of G from (2.8) and $2<p<2^*$. So we have

$$\left(\frac{1}{p'}-\frac{1}{2}\right)\liminf_{n\in\mathbb{N}}\int_{\mathbb{H}^N} v_n(1_{[\frac{1}{R},R]}G*v_n)\,dV \geq \frac{c}{2}.$$

Next let $(Q_l)_{l\in\mathbb{N}}$ be a family of disjoint geodesic balls of radius R the centers of which have the geodesic distance $R/2$ and that cover the whole $\mathbb{H}^N$. Denoting by $2Q_l$ the ball with the same center but doubled radius we get for almost all n

$$\begin{aligned} \frac{c}{4} &\leq \left(\frac{1}{p'}-\frac{1}{2}\right)\int_{\mathbb{H}^N} v_n\left[(1_{[\frac{1}{R},R]}G)*v_n\right]dV \\ &\leq C\sum_{l=1}^\infty \int_{Q_l}\left(\int_{1/R<d(x,y)<R} |G(d(x,y))||v_n(x)||v_n(y)|\,dV(y)\right)dV(x) \\ &\leq C\max_{1/R\leq d(x,y)\leq R}|G(d(x,y))|\sum_{l=1}^\infty\int_{Q_l}\left(\int_{2Q_l}|v_n(x)||v_n(y)|\,dV(y)\right)dV(x) \end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{l=1}^{\infty}\left(\int_{2Q_l}|v_n(x)|\,dV(x)\right)^2\\
&\leq C\sum_{l=1}^{\infty}\left(\int_{2Q_l}|v_n(x)|^{p'}\,dV(x)\right)^{2/p'}\\
&\leq C\left(\sup_{m\in\mathbb{N}}\int_{2Q_m}|v_n(x)|^{p'}\,dV(x)\right)^{2/p'-1}\cdot\sum_{l=1}^{\infty}\int_{2Q_l}|v_n(x)|^{p'}\,dV(x)\\
&\leq C\left(\sup_{m\in\mathbb{N}}\int_{2Q_m}|v_n(x)|^{p'}\,dV(x)\right)^{2/p'-1}\cdot\|v_n\|_{L^{p'}(\mathbb{H}^N)}\\
&\leq C\left(\sup_{m\in\mathbb{N}}\int_{2Q_m}|v_n(x)|^{p'}\,dV(x)\right)^{2/p'-1}.
\end{aligned}
$$

Here we used that the balls $2Q_m$ cover $\mathbb{H}^N$ only a finite number of times because $\mathbb{H}^N$ has bounded geometry. The latter estimate implies that there are centers $x_n\in\mathbb{H}^N$ such that

$$
\liminf_{n\to\infty}\int_{B_{2R}(x_n)}|v_n|^{p'}\,dV>0. \tag{5.6}
$$

Denoting by τ_{x_n} the hyperbolic translation with $\tau_{x_n}0=x_n$, we obtain that $w_n(x):=v_n(\tau_{x_n}x)$ is another Palais-Smale sequence of J. Given that it is bounded as well, it converges weakly to some $w\in L^{p'}(\mathbb{H}^N)$. Combining the first line of (5.4) with local L^p-estimates, we infer that w_n is bounded in $W^{2,p}(B_{4R}(0))$ and hences converges in $L^{p'}(B_{2R}(0))$ to its weak limit w so that (5.6) implies $w\neq 0$. Moreover, one checks that w is a critical point of J at the mountain pass level and the proof is finished. □

Proof of (ii): Using the formula (2.9) we may verify the assumptions of the Symmetric Mountain Pass Theorem as in Lemma 3.2 [32]. Indeed, for every $m\in\mathbb{N}$, we can choose radially symmetric functions $\psi_1,\dots,\psi_m\in C_0^\infty(\mathbb{R}_+)$ with mutually disjoint supports contained in the exterior of the ball of radius λ. Then $\{z_1,\dots,z_m\}$ is a linearly independent set if we set

$$
z_j:=\max\{\Gamma^{-1/p},\delta\}\cdot\psi_j(P)h
$$

for some fixed $h\in C_0^\infty(\mathbb{H}^N)$ and sufficiently small $\delta>0$. Indeed, due to (2.9) these functions are even mutually orthogonal and we have $J(tz_j)\to-\infty$ as $t\to\infty$ because of

$$
\begin{aligned}
J(tz_j)&=\frac{t^{p'}}{p'}\|z_j\|_{L^{p'}(\mathbb{H}^N)}^{p'}-\frac{t^2}{2}\int_{\mathbb{H}^N}\Gamma^{1/p}z_j\mathcal{R}_\lambda(\Gamma^{1/p}z_j)\,dV\\
&\leq\frac{t^{p'}}{p'}\|z_j\|_{L^{p'}(\mathbb{H}^N)}^{p'}-\frac{t^2}{4}\int_{\mathbb{H}^N}\psi_j(P)h\mathcal{R}_\lambda(\psi_j(P)h)\,dV\\
&\overset{(2.9)}{=}\frac{t^{p'}}{p'}\|z_j\|_{L^{p'}(\mathbb{H}^N)}^{p'}-\frac{t^2}{4}\text{p.v.}\int_{\mathbb{R}}\frac{\langle\psi_j(P)h,\frac{d}{ds}E_P(s)(\psi_j(P)h)\rangle}{s^2-\lambda^2}\,ds\\
&=\frac{t^{p'}}{p'}\|z_j\|_{L^{p'}(\mathbb{H}^N)}^{p'}-\frac{t^2}{4}\text{p.v.}\int_{\mathbb{R}}\frac{|\psi_j(s)|^2}{s^2-\lambda^2}\langle h,\frac{d}{ds}E_P(s)h\rangle\,ds\\
&=\frac{t^{p'}}{p'}\|z_j\|_{L^{p'}(\mathbb{H}^N)}^{p'}-\frac{t^2}{4}\int_{\mathrm{supp}(\psi_j)}\frac{\psi_j(s)^2}{s^2-\lambda^2}\|\mathcal{F}_0^{(+)}(s)h\|_{L^2(\mathbb{R}^{N-1})}^2\,ds,
\end{aligned}
$$

which tends to $-\infty$ if h is chosen suitably. From $\Gamma(x)\to 0$ as $x\to\infty$ one deduces as in Lemma 5.2 [20] that the Palais-Smale condition holds so that the existence of an unbounded sequence of solutions follows from the Symmetric Mountain Pass Theorem, see Theorem 6.5 [38] for the Euclidean version. □

Proof of (iii): We show that under the assumptions of part (iii) the functional J restricted to $L^{p'}_{rad}(\mathbb{H}^N)$ satisfies again the assumptions of the Symmetric Mountain Pass Theorem. First, the functions $z_1,\ldots,z_m$ from above are radial if h is radial, so it remains to verify the Palais-Smale condition. A Palais-Smale sequence (v_n) in $L^{p'}_{rad}(\mathbb{H}^N)$ is bounded and hence without loss of generality weakly convergent to some $v\in L^{p'}_{rad}(\mathbb{H}^N)$. Since $L^{p'}_{rad}(\mathbb{H}^N)$ is a uniformly convex Banach space, the convergence $v_n\to v$ in $L^{p'}_{rad}(\mathbb{H}^N)$ is proved once we show $\|v_n\|_{L^{p'}(\mathbb{H}^N)}\to\|v\|_{L^{p'}(\mathbb{H}^N)}$, see Proposition 3.32 [14]. In view of (5.4) this holds once we show that $\psi_n:=G*v_n$ has a convergent subsequence in $L^p_{rad}(\mathbb{H}^N)$. This is checked as follows. Corollary 2.1 implies that $\psi_n:=G*v_n=\mathcal{R}_\lambda v_n$ is bounded in $L^q_{rad}(\mathbb{H}^N)$ for all $2<q<\frac{Np'}{(N-2p')_+}$ because v_n is bounded in $L^{p'}_{rad}(\mathbb{H}^N)$. Then $L\psi_n+k^2\psi_n=(k^2+\lambda)\psi_n+v_n\in L^q_{rad}(\mathbb{H}^N)+L^{p'}_{rad}(\mathbb{H}^N)$ and the L^p-estimates from Theorem A [40] show that (ψ_n) is bounded in $W^{2,q}_{rad}(\mathbb{H}^N)+W^{2,p'}_{rad}(\mathbb{H}^N)$. By Theorem 2 in [24] (see also Theorem 3.1 [12] for a more elementary proof of a related result), this space imbeds compactly into $L^p_{rad}(\mathbb{H}^N)$ if q is chosen smaller than but sufficiently close to $p\in(2,2^*)$. So (ψ_n) has a convergent subsequence in $L^p_{rad}(\mathbb{H}^N)$ and J restricted to $L^{p'}_{rad}(\mathbb{H}^N)$ satisfies the Palais-Smale condition. As above, we obtain an unbounded sequence of critical points of J, which finishes the proof. □

6. Proof of Theorem 1.4

For the proof we have to analyze the unique solution of the ODE initial value problem

$$-u''-\frac{f'(r)}{f(r)}u'-V(r)u=\Gamma(r)|u|^{p-2}u,\qquad u(0)=\gamma,\ u'(0)=0 \tag{6.1}$$

where γ will be assumed to be positive without loss of generality. The first step is to find suitable bounds for u,u',u''. To this end we introduce the positive function

$$Z(r):=\frac12u'(r)^2+\frac12V(r)u(r)^2+\frac1p\Gamma(r)|u(r)|^p. \tag{6.2}$$

From $f'\geq 0$ and (H2),(H3) we get that there is an integrable function m such that

$$\begin{aligned}Z'(r)&\overset{(6.2)}{=}u'(r)\big(u''(r)+V(r)u(r)+\Gamma(r)|u(r)|^{p-2}u(r)\big)\\&\quad+\frac12V'(r)u(r)^2+\frac1p\Gamma'(r)|u(r)|^p\\&\overset{(6.1)}{=}-\frac{f'(r)}{f(r)}|u'(r)|^2+\frac12V'(r)u(r)^2+\frac1p\Gamma'(r)|u(r)|^p\\&\leq m(r)Z(r)\end{aligned} \tag{6.3}$$

and thus

$$Z(r)\leq Z(0)\exp\left(\int_0^\infty m(s)\,ds\right)\qquad\text{for all } r>0. \tag{6.4}$$

Since Γ is nonnegative and V is positive we deduce that u,u',u'' exist globally.

Next we show that u has an unbounded sequence of zeros. Indeed, the function $v(r) := f(r)^{1/2}u(r)$ satisfies

$$\begin{aligned} &v''(r) + c(r)v(r) = 0, \qquad \text{where} \\ &c(r) := \Gamma(r)|u(r)|^{p-2} + V(r) - \frac{f''(r)}{2f(r)} + \frac{f'(r)^2}{4f(r)^2} \qquad (6.5) \\ &\quad = \Gamma(r)|u(r)|^{p-2} + V(r) - \frac{1}{2}\log(f)''(r) - \frac{1}{4}(\log(f)'(r))^2 \end{aligned}$$

From $\Gamma \geq 0$, (H1), (H2) we deduce

$$\liminf_{r\to\infty} c(r) \geq V_\infty - \frac{\kappa^2}{4} > 0.$$

Hence, the function c is uniformly positive near infinity so that Sturm's oscillation theorem implies that v and hence u has an unbounded sequence of zeros.

Next we prove the estimates (1.4). To this end we define

$$\psi(r) := \frac{1}{2}v'(r)^2 + f(r)\left(\frac{1}{2}\tilde{V}(r)u(r)^2 + \frac{1}{p}\Gamma(r)|u(r)|^p\right), \qquad (6.6)$$

where $\tilde{V}(r) := V(r) - \frac{\kappa^2}{4}$. Differentiation yields

$$\begin{aligned} \psi' &= v'v'' + f'\left(\frac{1}{2}\tilde{V}u^2 + \frac{1}{p}\Gamma|u|^p\right) \\ &\quad + f\left(\frac{1}{2}\tilde{V}'u^2 + \frac{1}{p}\Gamma'|u|^p + (\tilde{V}u + \Gamma|u|^{p-2}u)u'\right) \\ &\overset{(6.5)}{=} -cvv' + fu'(\tilde{V}u + \Gamma|u|^{p-2}u) \\ &\quad + f'\left(\frac{1}{2}\tilde{V}u^2 + \frac{1}{p}\Gamma|u|^p\right) + f\left(\frac{1}{2}V'u^2 + \frac{1}{p}\Gamma'|u|^p\right) \\ &= -cvv' + \left(f^{1/2}v' - \frac{1}{2}f'f^{-1/2}v\right)(\tilde{V}u + \Gamma|u|^{p-2}u) \\ &\quad + f'\left(\frac{1}{2}\tilde{V}u^2 + \frac{1}{p}\Gamma|u|^p\right) + f\left(\frac{1}{2}V'u^2 + \frac{1}{p}\Gamma'|u|^p\right) \\ &= \left(-c + \tilde{V} + \Gamma|u|^{p-2}\right)vv' + (\frac{1}{p} - \frac{1}{2})f'\Gamma|u|^p + f\left(\frac{1}{2}V'u^2 + \frac{1}{p}\Gamma'|u|^p\right) \\ &= \left(\frac{f''}{2f} - \frac{(f')^2}{4f^2} - \frac{\kappa^2}{4}\right)vv' \\ &\quad + \left((\frac{1}{p} - \frac{1}{2})\frac{f'}{f}\Gamma|u|^{p-2} + \frac{1}{2}V' + \frac{1}{p}\Gamma'|u|^{p-2}\right)v^2. \end{aligned}$$

To prove the upper bounds in (1.4) we prove an upper bound for ψ as follows. From the previous identity we get on the interval $[R,\infty)$ for large R

$$\begin{aligned} \psi' &\leq \left|\frac{f''}{2f} - \frac{(f')^2}{4f^2} - \frac{\kappa^2}{4}\right|\frac{v^2 + (v')^2}{2} + \left(\frac{|V'|}{\min_{[R,\infty)}\tilde{V}} + m\right)\left(\frac{1}{2}\tilde{V} + \frac{1}{p}\Gamma|u|^{p-2}\right)v^2 \\ &\leq \frac{1}{\min\{1, \min_{[R,\infty)}\tilde{V}\}}\left(\left|\frac{f''}{2f} - \frac{(f')^2}{4f^2} - \frac{\kappa^2}{4}\right| + |V'| + m\right)\psi. \end{aligned}$$

Here we used $p > 2, f' \geq 0, \Gamma \geq 0$ and $\min_{[R,\infty)}\tilde{V} > 0$ for sufficiently large R by (H2). Since the prefactor is integrable over $[R,\infty)$ and ψ is positive on this interval, we infer from Gronwall's inequality that

$$\psi(r) \leq C\psi(R) \qquad \text{for all } r \geq R$$

where C is independent of γ. Combining this inequality, the simple inequality

$$\psi(R) \leq 2\left(f(R) + \frac{f'(R)^2}{4f(R)\min_{\mathbb{R}} V}\right) Z(R)$$

and (6.4) we get for some $A > 0$ independent of γ

$$Z(r) + |\psi(r)| \leq AZ(0) = A(V(0)\gamma^2 + \Gamma(0)|\gamma|^p) \qquad \text{for all } r \geq 0.$$

This proves the upper estimate in (1.4). This upper estimate may be used in the proof of the lower estimate. Indeed, $|u(r)| \leq C_\gamma(1+f(r))^{-1/2}$ implies

$$\begin{aligned}\psi' &\geq -\left|\frac{f''}{2f} - \frac{(f')^2}{4f^2} - \frac{\kappa^2}{4}\right| \frac{v^2+(v')^2}{2} - \left(\|\Gamma\|_\infty C_\gamma^{p-2} f' f^{-p/2} + \frac{|V'|}{\min_{[R,\infty)} \tilde{V}} + m\right)\psi \\ &\geq -\frac{1}{\min\{1, \min_{[R,\infty)} \tilde{V}\}}\left(\left|\frac{f''}{2f} - \frac{(f')^2}{4f^2} - \frac{\kappa^2}{4}\right| + \|\Gamma\|_\infty C_\gamma^{p-2} f' f^{-p/2} + |V'| + m\right)\psi.\end{aligned}$$

Again, the prefactor is integrable over $[R,\infty)$ and we obtain

$$\psi(r) \geq c\psi(R) \qquad \text{for all } r \geq R$$

where c only depends on the L^1-norm of this prefactor. Since u oscillates, we may choose R such that additionally $u(R)u'(R) = 0$ holds. For such R one has the simple inequality $\psi(R) \geq f(R)Z(R)$ so that the differential inequality for Z' finally yields a positive number $B > 0$ independent of γ such that

$$\psi(r) \geq BZ(0) = B(V(0)\gamma^2 + \Gamma(0)|\gamma|^p) \qquad \text{for all } r \geq 0.$$

This finishes the proof of (1.4). □

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