

High-energy expansion of two-loop massive four-point diagrams

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ABSTRACT: We apply the method of regions to the massive two-loop integrals appearing in the Higgs pair production cross section at the next-to-leading order, in the high energy limit. For the non-planar integrals, a subtle problem arises because of the indefinite sign of the second Symanzik polynomial. We solve this problem by performing an analytic continuation of the Mandelstam variables such that the second Symanzik polynomial has a definite sign. Furthermore, we formulate the procedure of applying the method of regions systematically. As a by-product of the analytic continuation of the Mandelstam variables, we obtain crossing relations between integrals in a simple and systematic way. In our formulation, a concept of “template integral” is introduced, which represents and controls the contribution of each region. All of the template integrals needed in the computation of the Higgs pair production at the next-to-leading order are given explicitly. We also develop techniques to solve Mellin-Barnes integrals, and show them in detail. Although most of the calculation is shown for the concrete example of the Higgs pair production process, the application to other similar processes is straightforward, and we anticipate that our method can be useful also for other cases.

KEYWORDS: NLO Computations, QCD Phenomenology

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1 Introduction

Perturbative quantum field theory has been describing particle physics phenomena very well, yet improving the perturbative series, i.e., calculating Feynman integrals, has been always challenging. It becomes more and more important to include higher order corrections to theoretical predictions as particle physics experiments, especially at the Large Hadron Collider (LHC), become more and more precise over the years. This means that a deeper understanding of higher order corrections is required to obtain meaningful theoretical predictions.

One of the most interesting problems in this field is higher order corrections to multi-scale processes. The prime examples are the Higgs+jet, Higgs pair, and Higgs+ Z production cross sections at the LHC. These $2 \rightarrow 2$ processes involve two kinematical variables, such as the center of mass energy and the transverse momentum, and the masses of internal or external particles serve as additional scales. Here, the word “multi-scale” is used when there are more than three scales. The bottlenecks of the calculation of multi-scale processes are the integration-by-parts (IBP) reduction and the evaluation of the resulting master integrals. The subject of this paper is the second issue, namely, the evaluation of multi-scale Feynman integrals.

Efforts to solve multi-scale Feynman integrals have persisted over the years. One of the milestones is the analytic computation of all the planar master integrals contributing to the Higgs $\rightarrow 3$ partons process at two-loops [1]. Another milestone is the numeric evaluation of the Higgs+jet [2] and Higgs pair production [3, 4] cross sections at two-loop level using the program `SecDec` [5, 6]. An independent numerical evaluation of the Higgs pair production cross section is given recently [7]. It is worth mentioning some recent analytic calculations of three-scale four-point two-loop diagrams; the non-planar master integrals for μe scattering [8, 9], the planar double box integral relevant to top pair production [10] and the planar master integrals relevant to di-photon and di-jet production [11]. These works show that even three-scale problems are difficult to solve. Recently, some of the non-planar master integrals for these processes in the limit $m_H = 0$ have been solved [12], but there still remain unsolved master integrals.

It is a promising idea to reduce the number of scales entering integrals by expanding them in some small parameters. For a summary of this topic, see ref. [13]. In this direction, the large- m_t expansion of the Higgs+jet [14–20] and Higgs pair production [21–28] cross sections is very well investigated. However, it is not until recently that expansions in other parameters have been investigated. Concerning the Higgs+jet production cross section, the expansion in the small bottom quark mass $m_b \ll m_H$ [29, 30] and in the small top quark and Higgs masses $m_t > m_H$ [31] are performed. For the Higgs pair production cross section, the expansion in small m_t for the planar master integrals [32] and in small Higgs transverse momentum [33] are performed. The rest of the master integrals of Higgs pair production in the small- m_t expansion are obtained in ref. [34] together with the results of this paper. Many of the non-planar master integrals are the same as, or related to those of ref. [31] but we provide some new information needed for Higgs pair production. Furthermore, the method used in this paper — the method of regions — is completely different from the one in ref. [31] at all steps of the calculation, so it provides a complementary understanding of the massive non-planar integrals. We would like to emphasize that the method of regions is a generic and systematic procedure to expand integrals, and thus the calculations shown in this paper can be applied to other integrals in a straightforward way.

The concept of dividing the domain of integration variables into several regions and expanding the integrand according to hierarchies in each region was introduced by Beneke and Smirnov [35]. The method is now called “expansion by regions” or “strategy of regions”, and in this paper we call it the method of regions. A mathematical proof of the method of regions for a general integral is not yet known although many successful appli-

cations have been reported. In fact, the author of the most up-to-date textbook on this topic states in his book [36]

The strategy of expansion by regions still has the status of experimental mathematics.

In the cases of off-shell large-momentum expansion and large-mass expansion, a mathematical proof based on a graph-theoretical language is known and it is called as “expansion by subgraphs” [13, 37]. The procedure of expansion by subgraphs is implemented and can be performed in an automatic way [38, 39]. The large- m_t expansion mentioned above belongs to this category. A new proof of the method of regions was proposed by Jantzen [40] but its application is limited. The purpose of this paper is to show non-trivial examples where the method of regions works well, and our calculation shows the first application of the method to the high energy expansion of non-planar four-point integrals.

The remainder of the paper is organized as follows: in section 2, we briefly summarize the method of regions. In section 3 we introduce conventions, ideas, and techniques, which will be used in the following sections. In section 4, 5 and 6, we apply the method of regions to the one-loop box diagram, the two-loop planar massive diagrams, and the two-loop non-planar massive diagrams, respectively.

2 General idea of the method of regions

The procedure of the method of regions is the following [13, 35, 36, 40, 41]:

- Step 1: Assign a hierarchy to the dimensionful parameters.
- Step 2: Reveal the relevant scaling of the integration variable.
- Step 3: For each region, expand the integrand according to its scaling.
- Step 4: Integrate. Scaleless integrals such as $\int_0^\infty dx x^a$ are set to zero.
- Step 5: Sum over the contributions from all the relevant regions.

The Step 2 is the crucial part of the method of regions, and an algorithm to reveal such scalings for a general integral is established based on the analysis of the convex hull [42, 43]. One can use the algorithm, implemented in the `Mathematica` package `asy2.1.m` [43]. Although it is not proved that the algorithm works correctly for all the cases, no counterexample is known so far. Recently, a new idea to reveal relevant scalings is proposed based on the technique of power geometry, which is implemented in the `Mathematica` package `ASPIRE` [44]. In this paper we use `asy2.1.m`.

The practical bottleneck is Step 4, since the integration tends to be complicated even after the expansion if the original integral is very complicated. This is one of the reasons why testing the method of regions is difficult.

The method of regions was first applied to the momentum representation of the Feynman integrals, so the “regions” mean some domains of the loop momenta. Later, it was found that parametric representations such as the Feynman representation and the alpha

representation are more convenient to apply the method [45]. Recently, it was proposed in ref. [41] to use yet another parametric representation, Lee-Pomeransky representation [46], to apply the method of regions. For all the representations mentioned above, one has to follow Step 1 to 5 for practical calculation.

3 Notation and technical tools

3.1 Conventions

We distinguish the exact equal sign “=” and the equal sign under a certain analytic continuation. For this purpose, we introduce a sign “ $\stackrel{\text{AC}}{=}$ ” and use it as, e.g.,

$$\log(z + i0) \stackrel{\text{AC}}{=} \log(-z - i0) + i\pi, \tag{3.1}$$

where $i0$ represents an infinitesimal positive imaginary number. We interpret $\log(z)$ as the principal value of the complex logarithm whose imaginary part lies in the interval $(-\pi, \pi]$. Both the left-hand side and the right-hand side of eq. (3.1) are well-defined in the entire domain of z , but the equality is valid only in the upper half plane of z . This is how analytic continuation is performed, and that is why we add “AC” to the normal equal sign in eq. (3.1). The equality of a series expansion like

$$\frac{1}{1 - m/M} = \sum_{n=0}^{\infty} \left(\frac{m}{M}\right)^n \tag{3.2}$$

is in principle also regarded as an analytic continuation. However, when a hierarchy like $m \ll M$ is explicitly stated in the text, we use normal equal sign.

We use a simplified expression of the Landau \mathcal{O} notation for more than one variable as

$$X + \mathcal{O}((m_H^2)^{n_H}, (m_t^2)^{n_t}, \epsilon^n) \equiv X + \mathcal{O}((m_H^2)^{n_H}) + \mathcal{O}((m_t^2)^{n_t}) + \mathcal{O}(\epsilon^n). \tag{3.3}$$

The Euler-Mascheroni constant is denoted as γ_E .

We use the alpha representation to calculate Feynman integrals. The integration measure and the analytic regularization parameters are defined as

$$\int \mathfrak{D}^n \alpha^\delta \equiv \prod_{i=1}^n \left(\int_0^\infty \frac{d\alpha_i \alpha_i^{\delta_j}}{\Gamma(1 + \delta_j)} \right). \tag{3.4}$$

The analytic regularization parameters δ_j play one essential role and three secondary roles:

- (i). The essential role is to regularize the contribution of individual regions which are divergent if we naively expand in α_i . This means that individual contributions are regulator dependent, and the dependence on δ_j cancel after we sum all the contributions and take the limits $\delta_j \rightarrow 0$. In taking the limit, it is necessary to specify the order because some of them do not commute. We express the sequence of limits as

$$\lim_{\epsilon, \delta_n, \dots, \delta_2, \delta_1 \rightarrow 0} X \equiv \lim_{\epsilon \rightarrow 0} \lim_{\delta_n \rightarrow 0} \cdots \lim_{\delta_2 \rightarrow 0} \lim_{\delta_1 \rightarrow 0} X. \tag{3.5}$$

- (ii). We use δ_j to regularize the Mellin-Barnes integral. [See the text below eq. (A.2).]
- (iii). By shifting $\delta_j \rightarrow \delta_j + 1$, we can express polynomials of α_i in the integrand. For example, when $n = 2$,

$$\int \mathfrak{D}^2 \alpha^\delta (\alpha_1^2 + \alpha_1 \alpha_2) = \int \mathfrak{D}^2 \alpha^\delta \Big|_{\delta_1 \rightarrow \delta_1 + 2} + \int \mathfrak{D}^2 \alpha^\delta \Big|_{\delta_1 \rightarrow \delta_1 + 1, \delta_2 \rightarrow \delta_2 + 1} \quad (3.6)$$

This property is usually used to express the integrals with higher powers of propagators.

- (iv). We use the property of eq. (3.6) to express the higher order terms. [See the text below eq. (4.20).]

The sum of the variables will be expressed as

$$\alpha_{i_1 \dots i_n} \equiv \alpha_{i_1} + \dots + \alpha_{i_n}, \quad \delta_{i_1 \dots i_n \overline{i_{n+1}} \dots i_{n'}} \equiv \delta_{i_1} + \dots + \delta_{i_n} - \delta_{i_{n+1}} + \dots + \delta_{i_{n'}}. \quad (3.7)$$

The bar on an index indicates that the variable corresponding to the index is subtracted instead of added. Sometimes ϵ and δ_j are treated in the same way, and in those cases we express ϵ as δ_0 . For example, $\delta_{001\bar{2}} = 2\epsilon + \delta_1 - \delta_2$.

Also, we introduce the following compact notation for the product of Γ -functions

$$\Gamma[x_1, \dots, x_n] \equiv \prod_{i=1}^n \Gamma(x_i). \quad (3.8)$$

In Step 3 of section 2, we expand the integrand of Feynman integrals in terms of soft parameters. In order to control the expansion in a systematic way, we introduce an auxiliary soft-scaling parameter χ . For example, assume that we have four variables $\alpha_1, \dots, \alpha_4$ whose scalings are

$$\alpha_1 \sim m, \quad \alpha_2 \sim M, \quad \alpha_3 \sim m, \quad \alpha_4 \sim M, \quad (3.9)$$

where $m \sim \chi$ is a soft parameter and $M \sim 1$ is a hard parameter. In this case, we apply a substitution

$$m \rightarrow \chi m, \quad M \rightarrow M, \quad \alpha_1 \rightarrow \chi \alpha_1, \quad \alpha_2 \rightarrow \alpha_2, \quad \alpha_3 \rightarrow \chi \alpha_3, \quad \alpha_4 \rightarrow \alpha_4, \quad (3.10)$$

to the integrand and expand in χ . After that, we can set $\chi = 1$. In this paper we denote the scalings (3.9) as

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \stackrel{\chi}{\sim} (1, 0, 1, 0) \quad (3.11)$$

or simply $(1, 0, 1, 0)$.

3.2 Kinematics and high energy expansion

The assignment of the external momenta q_1, \dots, q_4 is illustrated in figure 1. We consider the 2 to 2 process but define all the external momenta as incoming. In addition to the usual Mandelstam variables s, t, u , (which we call physical Mandelstam variables), we introduce S, T, U as

$$S = -s = -(q_1 + q_2)^2, \quad T = -t = -(q_1 + q_3)^2, \quad U = -u = -(q_2 + q_3)^2. \quad (3.12)$$

In our calculation we assume that S, T , and U are positive and thus call them positive Mandelstam variables. Sometimes two of the three Mandelstam variables are sufficient to express four-point functions, and indeed for planar integrals we do not use U [See section 5]. However for non-planar integrals, all of S, T, U are required to make the second Symanzik polynomial positive [See subsection 6.1].

In this paper, we consider the master integrals of Higgs pair production at next-to-leading order, where the loops are induced by the top quark. Therefore there are two mass scales, the Higgs mass m_H and the top quark mass m_t . We consider the high energy limit where the following hierarchy is satisfied

$$m_H^2 < m_t^2 \ll S, T, U. \quad (3.13)$$

The high energy expansion in this case is two-fold. First, we treat m_H^2 as the soft parameter and m_t^2, S, T, U as the hard parameters. Afterwards, we treat m_t^2 as the soft parameter and S, T, U as the hard parameters.

In the first expansion, i.e. the m_H -expansion, m_H enters the integrals through the on-shell condition of the external momenta

$$q_1^2 = 0, \quad q_2^2 = 0, \quad q_3^2 = m_H^2, \quad q_4^2 = m_H^2. \quad (3.14)$$

In terms of χ , the scaling we impose here is

$$m_H^2 \sim \chi, \quad m_t^2 \sim 1, \quad S \sim 1, \quad T \sim 1, \quad U \sim 1, \quad (3.15)$$

and the resulting series expression of an integral I is expressed symbolically as¹

$$I(S, T, U, m_t^2, m_H^2) = \sum_{n_H} (m_H^2)^{n_H} c_{n_H}(S, T, U, m_t^2). \quad (3.16)$$

A note of caution should be made regarding the dependence on S, T, U . Since the physical Mandelstam variables satisfy the relation $s + t + u = 2m_H^2$, a similar relation should also hold for the positive Mandelstam variables. As a result, functions expressed in terms of S, T, U are not unique. However this is not a problem, because at the end of the calculation, we express the result in a unique way. [See the text below eq. (3.20).] We use the linear dependence of S, T, U to make the second Symanzik polynomial positive definite. [See subsection 6.1.] After the m_H -expansion, the external legs becomes massless legs.

¹In general, the coefficients c_{n_H} depend on $\log(m_H^2)$, but in our case not.

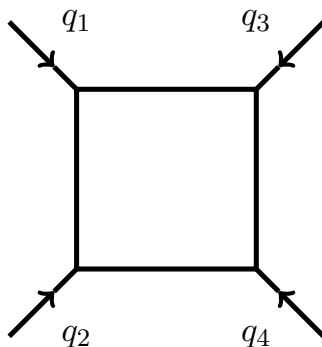


Figure 1. The convention of the external momenta.

In the second expansion, i.e. the m_t -expansion, the on-shell condition of the external momenta becomes

$$q_1^2 = 0, \quad q_2^2 = 0, \quad q_3^2 = 0, \quad q_4^2 = 0. \quad (3.17)$$

In terms of χ , the scaling we impose here is

$$m_t^2 \sim \chi, \quad S \sim 1, \quad T \sim 1, \quad U \sim 1, \quad (3.18)$$

and the resulting series for an integral I is expressed symbolically as

$$I(S, T, U, m_t^2, m_H^2) = \sum_{n_H} (m_H^2)^{n_H} \sum_{n_t} (m_t^2)^{n_t} c_{n_H, n_t}(S, T, U, \log(m_t^2)). \quad (3.19)$$

The result should be expressed in a way suitable for the evaluation with the physical kinematics. In order to achieve that, the analytic continuation

$$S \stackrel{\text{AC}}{=} e^{-i\pi+i0}(s+i0), \quad T \stackrel{\text{AC}}{=} sv, \quad U \stackrel{\text{AC}}{=} s(1-v), \quad (3.20)$$

is applied, where $i0$ represents an infinitesimal positive imaginary number, (note that the massless on-shell condition (3.17) is adopted), and $0 \leq v \leq 1$ in the physical kinematics. After expressing the result in terms of s and v , the expression is unique. The analytic continuation of T, U is trivial since their signs are consistent with those of the physical kinematics. The results are expressed in terms of harmonic polylogarithms (HPL) [47] and we introduce an abbreviation for HPL as

$$h_0 = H(0; v), \quad h_1 = H(1; v), \quad h_2 = H(2; v), \quad h_{2,1} = H(2, 1; v), \quad (3.21)$$

and so on. The argument of the HPL is always chosen to be v . For example,

$$H\left(3; -\frac{u}{s}\right) = H(3; 1-v) \stackrel{\text{AC}}{=} -h_{2,1} + h_2 h_1 - \frac{1}{2} h_1^2 h_0 - \frac{\pi^2}{6} h_1 + \zeta_3. \quad (3.22)$$

We use the *Mathematica* package `HPL.m` [48, 49] in dealing with HPL.

When diagrams of the same topology but with a different assignment of external momenta are considered, it is convenient to relate them by applying replacements of the

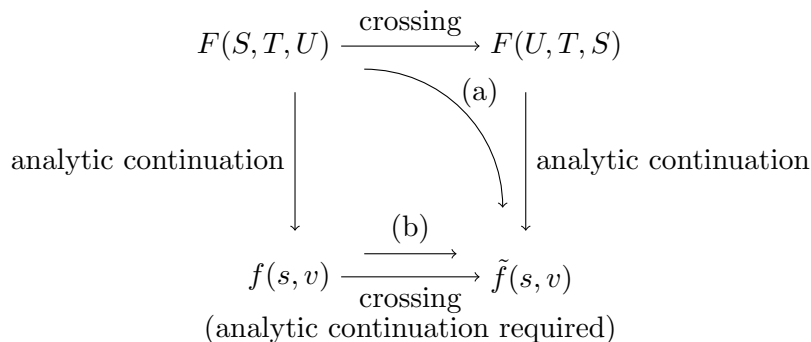


Figure 2. The commutative diagram of the crossing and the analytic continuation. Note that the crossing at upper level is a literal replacement of S and U whereas the crossing at bottom level changes the function in a nontrivial way. (a) Our approach. (b) Conventional approach.

external momenta, which means replacements of the Mandelstam variables. We call these replacements “crossing relations”. This subject is already well-established in the case of HPL [50], but we propose a simpler way to obtain the crossing relations. In figure 2, the commutative diagram of the crossing and analytic continuation is given. Usually an integral is given as the bottom-left expression, where the result is expressed in terms of the physical kinematic variables. On the other hand, we proceed the crossing in the upper expression, where the result is expressed in terms of the positive Mandelstam variables. The analytic continuation in the upper expression is the simple replacement of the positive Mandelstam variables, whereas the analytic continuation of the bottom expression requires the precise knowledge of the branch cuts. We take the approach (a) of figure 2 because it is easy to implement in programs, and crosscheck the result using approach (b). One can interpret the simplification by introducing the positive Mandelstam variables as the resolution of singularities by increasing the dimension, or in another words, we lose some information when we map the three-variable function F into the two-variable function f .

We would like to emphasize that the method to obtain the crossing relations explained above is a by-product of introducing the positive Mandelstam variables. The most important point in introducing the positive Mandelstam variables is that it makes the Symanzik polynomials positive and thus allows us to apply the method of regions safely. [See subsection 6.1.]

4 A first example: one-loop box diagram

We consider the massive one-loop Feynman integral family

$$J_{a_1, a_2, a_3, a_4} = \int \frac{d^d \ell}{i\pi^{d/2}} \frac{1}{(m_t^2 - \ell^2)^{a_1} (m_t^2 - (\ell + q_1)^2)^{a_2} (m_t^2 - (\ell + q_1 + q_2)^2)^{a_3} (m_t^2 - (\ell - q_3)^2)^{a_4}}, \tag{4.1}$$

where the infinitesimal negative imaginary part of each denominator is implicit. The external momenta q_i satisfy the on-shell conditions (3.14). We consider the box diagram shown

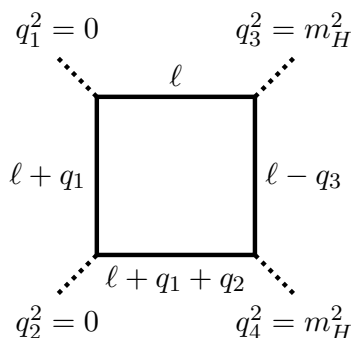


Figure 3. The one-loop box diagram considered in section 4.

in figure 3, $J_{1,1,1,1}$ and its alpha representation is

$$J_{1,1,1,1} \stackrel{\text{AC}}{=} I = \int \mathcal{D}^4 \alpha^\delta \mathcal{U}^{-d/2} e^{-\mathcal{F}/\mathcal{U}}, \tag{4.2}$$

where the first Symanzik polynomial \mathcal{U} and the second Symanzik polynomial \mathcal{F} are given by

$$\mathcal{U} = \alpha_{1234}, \quad \mathcal{F} = m_t^2 \alpha_{1234} \mathcal{U} + S \alpha_1 \alpha_3 + T \alpha_2 \alpha_4 - m_H^2 \alpha_{13} \alpha_4. \tag{4.3}$$

We make clear the analytic continuation in eq. (4.2) because the right hand side is regularized by δ_j whereas eq. (4.1) is explicitly δ_j -independent. Also, we assume $m_H^2 < 0$ in order to ensure the convergence of the integral and perform the analytic continuation of the result to $m_H^2 > 0$ at the end, which turns out to be trivial.

4.1 Expansion in the Higgs mass

We first expand in m_H . By using `asy2.1.m`, we find that there is only one relevant scaling

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \stackrel{\times}{\sim} (0, 0, 0, 0). \tag{4.4}$$

The expansion corresponding to this scaling is actually just the Taylor expansion of the integrand in m_H^2 , and the original integral can be written as

$$I = \int \mathcal{D}^4 \alpha^\delta \mathcal{U}^{-d/2} \left[e^{-\mathcal{F}/\mathcal{U}} \Big|_{(m_H^2)=0} + (m_H^2) \frac{\partial(e^{-\mathcal{F}/\mathcal{U}})}{\partial(m_H^2)} \Big|_{(m_H^2)=0} + \dots \right]. \tag{4.5}$$

In particular, the leading term is identical to the box diagram with completely massless external legs.

The fact that the expansion in m_H^2 and the integration commute is reasonable because m_H in the denominator of the integrand (appearing through $q_3^2 = m_H^2$) is always accompanied by m_t which regulates the integral and the limit $m_H \rightarrow 0$ always exists.

In collider physics, it is common to use s and the transverse momentum p_T as the kinematic variables, where t and p_T are related as

$$p_T^2 = \frac{ut - m_H^4}{s}, \tag{4.6}$$

or equivalently²

$$t = -\frac{1}{2} \left[s - 2m_H^2 - \sqrt{s(s - 4m_H^2 - 4p_T^2)} \right]. \quad (4.7)$$

Since these relations are m_H -dependent, one should be careful when expanding in m_H . In order to clarify the point, let us consider two expressions

$$f_1(t, m_H^2) = f_2(p_T^2, m_H^2) \quad (4.8)$$

which are related by eq. (4.7). We would like to analyze the cross section for fixed p_T but not t , so let us consider the m_H -expansion of $f_2(p_T^2, m_H^2)$:

$$f_2(p_T^2, m_H^2) = f_2(p_T^2, 0) + (m_H^2) \left. \frac{\partial f_2}{\partial(m_H^2)} \right|_{m_H=0} + \frac{(m_H^2)^2}{2} \left. \frac{\partial^2 f_2}{\partial(m_H^2)^2} \right|_{m_H=0} + \mathcal{O}((m_H^2)^3). \quad (4.9)$$

On the other hand, the kinematic variable appearing in the Feynman integral is t and the natural representation is $f_1(t, m_H^2)$. Thus, we express the ingredients of eq. (4.9) in terms of $f_1(t, m_H^2)$ as

$$f_2(p_T^2, 0) = f_1(t_0, 0) \quad (4.10)$$

$$\left. \frac{\partial f_2}{\partial(m_H^2)} \right|_{m_H=0} = \left. \frac{\partial t}{\partial(m_H^2)} \right|_{m_H=0} \left. \frac{\partial f_1}{\partial t} \right|_{t=t_0, m_H=0} + \left. \frac{\partial f_1}{\partial(m_H^2)} \right|_{t=t_0, m_H=0} \quad (4.11)$$

$$\begin{aligned} \left. \frac{\partial^2 f_2}{\partial(m_H^2)^2} \right|_{m_H=0} &= \left. \frac{\partial^2 t}{\partial(m_H^2)^2} \right|_{m_H=0} \left. \frac{\partial f_1}{\partial t} \right|_{t=t_0, m_H=0} + \left[\left. \frac{\partial t}{\partial(m_H^2)} \right|_{m_H=0} \right]^2 \left. \frac{\partial^2 f_1}{\partial t^2} \right|_{t=t_0, m_H=0} \\ &+ 2 \left. \frac{\partial t}{\partial(m_H^2)} \right|_{m_H=0} \left. \frac{\partial^2 f_1}{\partial t \partial(m_H^2)} \right|_{t=t_0, m_H=0} + \left. \frac{\partial^2 f_1}{\partial(m_H^2)^2} \right|_{t=t_0, m_H=0} \end{aligned} \quad (4.12)$$

where $t_0 = t|_{m_H=0}$. Apparently, eq. (4.9) becomes complicated when eqs. (4.10), (4.11), (4.12) are substituted. However, taking into account the m_H -expansion of $f_1(t, 0)$

$$f_1(t, 0) = f_1(t_0, 0) + (m_H)^2 \left. \frac{\partial t}{\partial(m_H^2)} \right|_{m_H=0} \left. \frac{\partial f_1}{\partial t} \right|_{t=t_0, m_H=0} + \dots, \quad (4.13)$$

the expression becomes simpler

$$f_2(p_T^2, m_H^2) = f_1(t, 0) + (m_H^2) \left. \frac{\partial f_1}{\partial(m_H^2)} \right|_{m_H=0} + \frac{(m_H^2)^2}{2} \left. \frac{\partial^2 f_1}{\partial(m_H^2)^2} \right|_{m_H=0} + \mathcal{O}((m_H^2)^3), \quad (4.14)$$

where the m_H -dependent t (4.7) is used. In general, for a given order of m_H^2 , the difference between the strict m_H -expansion of $f_2(p_T^2, m_H^2)$ for fixed p_T and the m_H -expansion of $f_1(t, m_H^2)$ for the m_H -dependent t is higher order in m_H^2 .

²When we solve eq. (4.6) in t using the relation $s + t + u = 2m_H^2$, there are two solutions. The other solution has “+” in front of the square root in eq. (4.7), and it corresponds to u in this case. Note that the amplitude is symmetric under $t \leftrightarrow u$, so we could choose the sign the other way around. We choose the sign such that $t = -p_T^2 + \mathcal{O}(p_T^2/s, m_H^2/s)$.

The conclusions of this subsection are the following:

- (i). As the result of the m_H -expansion, the integrals reduce to integrals with massless external legs.
- (ii). The m_H -expansion for fixed p_T is obtained by the m_H -expansion with fixed t , keeping the m_H -dependence of t .

4.2 Expansion in the top quark mass

After the expansion in m_H , we have integrals which depend on m_t , S , and T . We now consider the expansion in m_t assuming the hierarchy (3.18). The integral of interest is

$$\int \mathcal{D}^4 \alpha^\delta \mathcal{U}^{-d/2} e^{-\mathcal{F}/\mathcal{U}}, \tag{4.15}$$

with

$$\mathcal{U} = \alpha_{1234}, \quad \mathcal{F} = m_t^2 \alpha_{1234} \mathcal{U} + S \alpha_1 \alpha_3 + T \alpha_2 \alpha_4. \tag{4.16}$$

Here we use the positive Mandelstam variables, S, T , to make all the terms in \mathcal{F} positive. Otherwise, hard terms could cancel and result in a soft term, which breaks the method of regions. The use of positive Mandelstam variables in eq. (4.16) is conceptually not new, since it corresponds to the integral in the u -channel where $s < 0, t < 0$ and $u = -s - t > 0$. The absence of a negative term in the u -channel is reasonable because there is no physical cut in those kinematics.

By using the package `asy2.1.m` [43], we reveal five relevant scalings:³

$$\underbrace{(0, 0, 0, 0)}_1, \quad \underbrace{(0, 0, 1, 1)}_2, \quad \underbrace{(0, 1, 1, 0)}_3, \quad \underbrace{(1, 0, 0, 1)}_4, \quad \underbrace{(1, 1, 0, 0)}_5. \tag{4.17}$$

The scalings of regions 2 to 5 reflect the symmetries of the integral, $\alpha_1 \leftrightarrow \alpha_3$ and $\alpha_2 \leftrightarrow \alpha_4$. Eq. (4.15) is thus expressed as the sum of the contributions from these five regions: $\sum_{i=1}^5 I^{(i)}$.

Region 1 (all-hard region). The region where all the alpha variables scale as χ^0 , i.e., $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \stackrel{\chi}{\sim} (0, 0, 0, 0)$ is special, and we call this region the ‘‘all-hard region’’. We can make several general statements about this region within our high-energy expansion:

- (a). Every integral has one all-hard region.
- (b). There is only one soft parameter in the all-hard region, which is m_t .
- (c). The contribution from the all-hard region can be expressed as the massless integral of the original topology. In particular, the leading order term is obtained by substituting $m_t = 0$ into the original integral.
- (d). The leading order contribution of the all-hard region is $\mathcal{O}(\chi^0)$.
- (e). The contribution from the all-hard region has no singularities in δ_j .

³More precisely, `asy2.1.m` reveals the scalings which lead to homogeneous and non-scaleless integrals.

Because of these properties, the contribution from the all-hard region can be calculated in two ways. The first is the procedure universal for any region, and the second is to use the momentum representation. We show them in order.

First, we show the universal procedure. By expanding eq. (4.15) in terms of χm_t^2 , we obtain the contribution of this region as

$$I^{(1)} = \int \mathcal{D}^4 \alpha^\delta \mathcal{U}^{(1)-d/2} e^{-\mathcal{F}^{(1)}/\mathcal{U}^{(1)}} \left[1 - \chi m_t^2 \alpha_{1234} + \frac{\chi^2}{2} (m_t^2 \alpha_{1234})^2 \right] + \mathcal{O}(\chi^3), \quad (4.18)$$

where $\mathcal{U}^{(1)} = \alpha_{1234}$ and $\mathcal{F}^{(1)} = S\alpha_1\alpha_3 + T\alpha_2\alpha_4$. As stated above, the leading order term is the massless box integral and we name it $\mathcal{T}_{\delta_1, \delta_2, \delta_3, \delta_4, \epsilon}^{(1)}$ for later use:

$$\mathcal{T}_{\delta_1, \delta_2, \delta_3, \delta_4, \epsilon}^{(1)} = \int \mathcal{D}^4 \alpha^\delta \mathcal{U}^{(1)-d/2} e^{-\mathcal{F}^{(1)}/\mathcal{U}^{(1)}} \quad (4.19)$$

$$= \int dz \frac{T^z \Gamma[-z, 1+z+\delta_2, 1+z+\delta_4, -1-z-\delta_{0124}, -1-z-\delta_{0234}, 2+z+\delta_{01234}]}{S^{2+z+\delta_{01234}} \Gamma[-\delta_{001234}, 1+\delta_1, 1+\delta_2, 1+\delta_3, 1+\delta_4]}. \quad (4.20)$$

The integrand of the higher order corrections has additional factors of α_i which can be expressed by some shifts of $\delta_j \rightarrow \delta_j + 1$. Indeed, eq. (4.18) can be expressed as

$$I^{(1)} = \mathcal{T}_{\delta_1, \delta_2, \delta_3, \delta_4, \epsilon}^{(1)} - \chi m_t^2 \left(\mathcal{P}_{1+\delta_1}^1 \mathcal{T}_{1+\delta_1, \delta_2, \delta_3, \delta_4, \epsilon}^{(1)} + \mathcal{P}_{1+\delta_2}^1 \mathcal{T}_{\delta_1, 1+\delta_2, \delta_3, \delta_4, \epsilon}^{(1)} + \mathcal{P}_{1+\delta_3}^1 \mathcal{T}_{\delta_1, \delta_2, 1+\delta_3, \delta_4, \epsilon}^{(1)} + \mathcal{P}_{1+\delta_4}^1 \mathcal{T}_{\delta_1, \delta_2, \delta_3, 1+\delta_4, \epsilon}^{(1)} \right) + \mathcal{O}(\chi^2), \quad (4.21)$$

where $\mathcal{P}_x^n = \Gamma(x+n)/\Gamma(x)$ is the Pochhammer symbol. We call $\mathcal{T}_{\delta_1, \delta_2, \delta_3, \delta_4, \epsilon}^{(1)}$ a “template integral” since all the higher order terms can be expressed in terms of $\mathcal{T}_{\delta_1, \delta_2, \delta_3, \delta_4, \epsilon}^{(1)}$ by shifting δ_j .

Since there is no singularity in δ_j , we can safely set $\delta_j = 0$ in eq. (4.21). Then, the leading order term is expressed as

$$\mathcal{T}_{0,0,0,0,\epsilon}^{(1)} = \int_{-1/2-i\infty}^{-1/2+i\infty} dz \frac{T^z \Gamma[-z, 1+z, 1+z, -1-\epsilon-z, -1-\epsilon-z, 2+\epsilon+z]}{S^{2+\epsilon+z} \Gamma(-2\epsilon)} \Bigg|_{\epsilon \simeq -1}, \quad (4.22)$$

where the technique explained in appendix A is used to set the integration contour to a straight line. The expression at $\epsilon \rightarrow 0$ is obtained by using the package MB.m [51], and the result is

$$\mathcal{T}_{0,0,0,0,\epsilon}^{(1)} = e^{i\pi\epsilon} \frac{e^{-\epsilon\gamma_E}}{s^{2+\epsilon\nu}} \left[-\frac{4}{\epsilon^2} + \frac{2h_0 + 2i\pi}{\epsilon} + \frac{4\pi^2}{3} + \mathcal{O}(\epsilon) \right]. \quad (4.23)$$

The higher order terms are given by $\mathcal{T}_{1,0,0,0,\epsilon}^{(1)}$, $\mathcal{T}_{0,1,0,0,\epsilon}^{(1)}$ etc, and can be calculated in a similar way.

The other procedure to calculate the contribution of the all-hard region is the following. We return to the momentum representation (4.1) and expand each propagator in m_t as

$$\frac{1}{m_t^2 - \ell^2} \rightarrow \sum_{n=0}^{\infty} \frac{(-m_t^2)^n}{(-\ell^2)^{n+1}}. \quad (4.24)$$

Then, the contribution can be expressed in terms of the integral family

$$J_{a_1, a_2, a_3, a_4}^{\text{massless}} = \int \frac{d^d \ell}{i\pi^{d/2}} \frac{1}{(-\ell^2)^{a_1} (-(\ell + q_1)^2)^{a_2} (-(\ell + q_1 + q_2)^2)^{a_3} (-(\ell - q_3)^2)^{a_4}} \quad (4.25)$$

as

$$I^{(1)} = J_{1,1,1,1}^{\text{massless}} - m_t^2 (J_{2,1,1,1}^{\text{massless}} + J_{1,2,1,1}^{\text{massless}} + J_{1,1,2,1}^{\text{massless}} + J_{1,1,1,2}^{\text{massless}}) + \mathcal{O}(m_t^4). \quad (4.26)$$

This corresponds to eq. (4.21). Applying the IBP-reduction, all the integrals appearing eq. (4.26) including higher order terms can be expressed by three master integrals

$$J_{1,0,1,0}^{\text{massless}}, J_{0,1,0,1}^{\text{massless}}, J_{1,1,1,1}^{\text{massless}}. \quad (4.27)$$

The IBP-reduction of massless integrals is computationally easy even at the two-loop level, and thus very useful. In the calculation of two-loop integrals, we adopt this approach to calculate the contribution from the all-hard region.

Regions 2, 3, 4, 5. The contribution of Region 2 is obtained by applying the second scaling of eq. (4.17) and expanding in $\chi m_t^2, \chi \alpha_3, \chi \alpha_4$,

$$I^{(2)} = \int \mathcal{D}^4 \alpha \mathcal{U}^{(2)-d/2} e^{-\mathcal{F}^{(2)}/\mathcal{U}^{(2)}} \left[1 - \chi \left(m_t^2 \alpha_{34} + \frac{d}{2} \frac{\alpha_{34}}{\mathcal{U}^{(2)}} + S \frac{\alpha_1 \alpha_3 \alpha_{34}}{(\mathcal{U}^{(2)})^2} + T \frac{\alpha_2 \alpha_4 \alpha_{34}}{(\mathcal{U}^{(2)})^2} \right) \right] + \mathcal{O}(\chi^2), \quad (4.28)$$

where $\mathcal{U}^{(2)} = \alpha_{12}$ and $\mathcal{F}^{(2)} = m_t^2 \alpha_{12} \mathcal{U}^{(2)} + S \alpha_1 \alpha_3 + T \alpha_2 \alpha_4$. The integration over $\alpha_1, \dots, \alpha_4$ can be performed using the relation (A.3) and a variant of it, and the template integral of this region is

$$\mathcal{T}_{\delta_1, \delta_2, \delta_3, \delta_4, \epsilon}^{(2)} = \frac{(m_t^2)^{-\epsilon - \delta_1 - \delta_2} \Gamma[\delta_1 - \delta_3, \delta_2 - \delta_4, \delta_1 + \delta_2 + \epsilon]}{S^{1+\delta_3} T^{1+\delta_4} \Gamma[\delta_{12} - \delta_{34}, 1 + \delta_1, 1 + \delta_2]}. \quad (4.29)$$

The higher order terms in eq. (4.28) which contain the inverse of $\mathcal{U}^{(2)}$ can be expressed by the shift $\epsilon \rightarrow \epsilon - 1$ in the template integral. Thus eq. (4.28) is written as

$$\begin{aligned} I^{(2)} = & \mathcal{T}_{\delta_1, \delta_2, \delta_3, \delta_4, \epsilon}^{(2)} + \chi m_t^2 \left(\mathcal{P}_{1+\delta_3}^1 \mathcal{T}_{\delta_1, \delta_2, \delta_3+1, \delta_4, \epsilon}^{(2)} + \mathcal{P}_{1+\delta_4}^1 \mathcal{T}_{\delta_1, \delta_2, \delta_3, \delta_4+1, \epsilon}^{(2)} \right) \\ & + \chi \frac{d}{2} \left(\mathcal{P}_{1+\delta_3}^1 \mathcal{T}_{\delta_1, \delta_2, \delta_3+1, \delta_4, \epsilon-1}^{(2)} + \mathcal{P}_{1+\delta_4}^1 \mathcal{T}_{\delta_1, \delta_2, \delta_3, \delta_4+1, \epsilon-1}^{(2)} \right) \\ & + \chi S \left(\mathcal{P}_{1+\delta_1}^1 \mathcal{P}_{1+\delta_3}^2 \mathcal{T}_{\delta_1+1, \delta_2, \delta_3+2, \delta_4, \epsilon-2}^{(2)} + \mathcal{P}_{1+\delta_1}^1 \mathcal{P}_{1+\delta_3}^1 \mathcal{P}_{1+\delta_4}^1 \mathcal{T}_{\delta_1+1, \delta_2, \delta_3+1, \delta_4+1, \epsilon-2}^{(2)} \right) \\ & + \chi T \left(\mathcal{P}_{1+\delta_2}^1 \mathcal{P}_{1+\delta_3}^1 \mathcal{P}_{1+\delta_4}^1 \mathcal{T}_{\delta_1, \delta_2+1, \delta_3+1, \delta_4+1, \epsilon-2}^{(2)} + \mathcal{P}_{1+\delta_2}^1 \mathcal{P}_{1+\delta_4}^2 \mathcal{T}_{\delta_1, \delta_2+1, \delta_3, \delta_4+2, \epsilon-2}^{(2)} \right) + \mathcal{O}(\chi^2). \end{aligned} \quad (4.30)$$

Recall that $\mathcal{P}_x^n = \Gamma(x+n)/\Gamma(x)$ is the Pochhammer symbol.

As mentioned in subsection 3.1, the result of the limits $\delta_j \rightarrow 0$ depends on the order in which we take them. For example, when we take the sequence of limits with ascending values of j , we obtain

$$\lim_{\epsilon, \delta_4, \delta_3, \delta_2, \delta_1 \rightarrow 0} \mathcal{T}_{\delta_1, \delta_2, \delta_3, \delta_4, \epsilon}^{(2)} = \frac{e^{-\epsilon \gamma_E}}{s^2 v (m_t^2)^\epsilon} \frac{1}{\epsilon} \left(\frac{1}{\delta_3} + \frac{1}{\delta_4} - \log s - h_0 + i\pi \right) + \mathcal{O}(\epsilon), \quad (4.31)$$

whereas with descending values of j , we instead obtain

$$\lim_{\epsilon, \delta_1, \delta_2, \delta_3, \delta_4 \rightarrow 0} \mathcal{T}_{\delta_1, \delta_2, \delta_3, \delta_4, \epsilon}^{(2)} = \frac{e^{-\epsilon\gamma_E}}{s^2 v(m_t^2)^\epsilon} \left[\frac{2}{\epsilon^2} - \frac{1}{\epsilon} \left(\frac{1}{\delta_1} + \frac{1}{\delta_2} - 2\log(m_t^2) \right) - \frac{\pi^2}{6} \right] + \mathcal{O}(\epsilon). \quad (4.32)$$

The order dependence is not problem, provided we use the same order throughout the calculation. The artifacts caused by the δ_j will cancel after we sum the contributions from all the relevant regions.

Due to the symmetries of the diagrams, the template integrals of the other regions can be expressed in terms of $\mathcal{T}_{\delta_1, \delta_2, \delta_3, \delta_4, \epsilon}^{(2)}$ with exchanged δ_j as

$$\mathcal{T}_{\delta_1, \delta_2, \delta_3, \delta_4, \epsilon}^{(3)} = \mathcal{T}_{\delta_1, \delta_4, \delta_3, \delta_2, \epsilon}^{(2)} \quad (4.33)$$

$$\mathcal{T}_{\delta_1, \delta_2, \delta_3, \delta_4, \epsilon}^{(4)} = \mathcal{T}_{\delta_3, \delta_2, \delta_1, \delta_4, \epsilon}^{(2)} \quad (4.34)$$

$$\mathcal{T}_{\delta_1, \delta_2, \delta_3, \delta_4, \epsilon}^{(5)} = \mathcal{T}_{\delta_3, \delta_4, \delta_1, \delta_2, \epsilon}^{(2)}, \quad (4.35)$$

and when we take the ascending order of limits, we obtain

$$\lim_{\epsilon, \delta_4, \delta_3, \delta_2, \delta_1 \rightarrow 0} \mathcal{T}_{\delta_1, \delta_2, \delta_3, \delta_4, \epsilon}^{(3)} = \frac{e^{-\epsilon\gamma_E}}{s^2 v(m_t^2)^\epsilon} \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left(\log(m_t^2) - \log s + i\pi + \frac{1}{\delta_3} - \frac{1}{\delta_4} \right) - \frac{\pi^2}{12} \right] + \mathcal{O}(\epsilon) \quad (4.36)$$

$$\lim_{\epsilon, \delta_4, \delta_3, \delta_2, \delta_1 \rightarrow 0} \mathcal{T}_{\delta_1, \delta_2, \delta_3, \delta_4, \epsilon}^{(4)} = \frac{e^{-\epsilon\gamma_E}}{s^2 v(m_t^2)^\epsilon} \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left(\log(m_t^2) - h_0 - \frac{1}{\delta_3} + \frac{1}{\delta_4} \right) - \frac{\pi^2}{12} \right] + \mathcal{O}(\epsilon) \quad (4.37)$$

$$\lim_{\epsilon, \delta_4, \delta_3, \delta_2, \delta_1 \rightarrow 0} \mathcal{T}_{\delta_1, \delta_2, \delta_3, \delta_4, \epsilon}^{(5)} = \frac{e^{-\epsilon\gamma_E}}{s^2 v(m_t^2)^\epsilon} \left[\frac{2}{\epsilon^2} + \frac{1}{\epsilon} \left(2\log(m_t^2) - \frac{1}{\delta_3} - \frac{1}{\delta_4} \right) - \frac{\pi^2}{6} \right] + \mathcal{O}(\epsilon). \quad (4.38)$$

Sum of all regions. Summing eqs. (4.23), (4.31), (4.36), (4.37), (4.38), we obtain the leading term of eq. (4.1):

$$\text{Eq. (4.1)} = e^{i\pi\epsilon} \frac{e^{-\epsilon\gamma_E}}{s^{2+\epsilon} v} \left\{ \pi^2 - 2 \left[\log\left(\frac{s}{m_t^2}\right) - i\pi \right] \left[\log\left(\frac{s}{m_t^2}\right) + h_0 \right] \right\} + \mathcal{O}(m_H^2, m_t^2, \epsilon). \quad (4.39)$$

As mentioned, the result is δ_j -independent. There are 24 possible ways to order $\delta_1, \delta_2, \delta_3, \delta_4$ in taking the limit, and we have confirmed that the result is the same for all of the orderings. Since the original integral is finite in the limit $\epsilon \rightarrow 0$, the poles of ϵ in the individual contributions from each region cancel.

4.3 Higher order terms in m_t

The method of regions can be used to obtain a series expansion up to arbitrary order in the soft parameters, m_H or m_t . For example, eqs. (4.21), (4.30) includes the contributions up to $\mathcal{O}(\chi)$. In general, the higher order terms can be expressed in terms of the template integrals. Therefore, in principle, it is straightforward to calculate higher order terms up to arbitrary order, once the template integrals have been obtained.

However, the number of integrals to calculate increases rapidly with the order of χ , and this makes it hard to calculate higher order terms. Especially in the two-loop case, some of the template integrals contain multi-dimensional Mellin-Barnes integrals, which

are not so easy to solve. Furthermore, the number of integrals increases more rapidly than in the case of the one-loop calculation. Therefore it is better to use another method to calculate the higher-order corrections.

The use of differential equations solves this issue [29, 31, 32, 34]. We use the differential equation with respect to m_t^2 to obtain higher order corrections in m_t^2 . Since we know that the integral has the form

$$\text{Eq. (4.15)} = \sum_{n_1} \sum_{n_2} c_{n_1, n_2}(S, T) (m_t^2)^{n_1} (\log m_t^2)^{n_2}, \quad (4.40)$$

the set of differential equations reduces to a set of linear relations of $c_{n_1, n_2}(S, T)$ which simplifies the problem a lot. In this sense, the leading order terms which we calculate in the previous subsections play the role of the boundary conditions of the differential equations.

4.4 Integrals with fewer lines

Once we have calculated the box integral, there are several shortcuts to calculate integrals with fewer lines such as the triangle integral and the self energy integral. Let us consider the s -channel triangle diagram, $J_{1,1,1,0}$, as an example.

The alpha representation of $J_{1,1,1,0}$ is obtained by setting $\alpha_4 \rightarrow 0$ in eq. (4.3), since the fourth propagator is absent in $J_{1,1,1,0}$. If we use `asy2.1.m` to reveal the relevant scalings for $J_{1,1,1,0}$, we obtain

$$(0, 0, 0), (0, 0, 1), (1, 0, 0), \quad (4.41)$$

however, we do not have to do that. We do not have to derive the template integrals for $J_{1,1,1,0}$ because they are derived from the template integrals of $J_{1,1,1,1}$.

Using the fact that the δ_j -dependence of the alpha representation is expressed by the replacement of $a_i \rightarrow 1 + \delta_j$, the triangle integral $J_{1,1,1,0}$ can be expressed by the limit $\delta_4 \rightarrow -1$. Therefore by taking the limit $\delta_4 \rightarrow -1$ to the template integral of $J_{1,1,1,1}$, one can obtain the template integral for $J_{1,1,1,0}$. For example,

$$\lim_{\delta_4 \rightarrow -1} \text{Eq. (4.29)} = \frac{(m_t^2)^{-\epsilon - \delta_1 - \delta_2}}{S^{1 + \delta_3}} \frac{\Gamma[\delta_1 - \delta_3, 1 + \delta_2, \delta_1 + \delta_2 + \epsilon]}{\Gamma[1 + \delta_{12} - \delta_3, 1 + \delta_1, 1 + \delta_2]}, \quad (4.42)$$

and this is the template integral for the region $(0, 0, 1)$. Sometimes the limit vanishes due to a suppression factor $1/\Gamma(1 + \delta_4)$. For example,

$$\lim_{\delta_4 \rightarrow -1} \text{Eq. (4.33)} = 0. \quad (4.43)$$

This fact is reasonable because the number of relevant regions for $J_{1,1,1,0}$ is smaller than for $J_{1,1,1,1}$. Two of the four soft template integrals are non-vanishing after taking the limit, and they are the two soft template integrals of $J_{1,1,1,0}$.



Figure 4. The two-loop massive planar diagrams defined in eq. (5.1) (left) and eq. (5.2) (right).

5 Two-loop planar diagrams

We consider the following Feynman integral families

$$J_{a_1, a_2, a_3, a_4, a_5, a_6, a_7}^{\text{PL1}} = \int \frac{d^d \ell_1}{i\pi^{d/2}} \frac{d^d \ell_2}{i\pi^{d/2}} \frac{1}{(-p_7^2)^{a_7}} \prod_{n=1}^6 \frac{1}{(m_t^2 - p_n^2)^{a_n}} \quad (5.1)$$

$$J_{a_1, a_2, a_3, a_4, a_5, a_6, a_7}^{\text{PL2}} = \int \frac{d^d \ell_1}{i\pi^{d/2}} \frac{d^d \ell_2}{i\pi^{d/2}} \prod_{n=1,2,3,7} \frac{1}{(m_t^2 - p_n^2)^{a_n}} \prod_{n=4}^6 \frac{1}{(-p_n^2)^{a_n}} \quad (5.2)$$

where the momenta of the lines are

$$\{p_1, p_2, p_3, p_4, p_5, p_6, p_7\} = \{\ell_1 + q_1, \ell_1, \ell_1 - q_2, \ell_2 - q_2, \ell_2 + q_1, \ell_2 + q_1, \ell_1 - \ell_2\}. \quad (5.3)$$

Recall that $q_{13} = q_1 + q_3$. We consider the integrals $I^{\text{PL1}} = J_{1,1,1,1,1,1,1}^{\text{PL1}}$ and $I^{\text{PL2}} = J_{1,1,1,1,1,1,1}^{\text{PL2}}$ whose diagrammatic representations are shown in figure 4.

The m_H -expansion can be performed in the same way as in subsection 4.1, and the alpha representations of the integrals after the m_H -expansion are given by

$$\mathcal{U}^{\text{PL1}} = \mathcal{U}^{\text{PL2}} = \alpha_{123}\alpha_{456} + \alpha_{123456}\alpha_7, \quad (5.4)$$

$$\mathcal{F}^{\text{PL1}} = m_t^2 \alpha_{123456} \mathcal{U}^{\text{PL1}} + S[\alpha_1(\alpha_4\alpha_{67} + \alpha_3\alpha_{4567}) + \alpha_6(\alpha_{23}\alpha_4 + \alpha_{34}\alpha_7)] + T\alpha_2\alpha_5\alpha_7, \quad (5.5)$$

$$\mathcal{F}^{\text{PL2}} = m_t^2 \alpha_{1237} \mathcal{U}^{\text{PL2}} + S[\alpha_1(\alpha_4\alpha_{67} + \alpha_3\alpha_{4567}) + \alpha_6(\alpha_{23}\alpha_4 + \alpha_{34}\alpha_7)] + T\alpha_2\alpha_5\alpha_7. \quad (5.6)$$

Conceptually there is no difference between the procedure of applying the method of regions to these integrals and the example discussed in section 4. In particular, the property of the \mathcal{F} -function that it is positive definite in the u -channel is the same [cf. the text below eq. (4.16)]. The only new ingredient is that now the template integrals are expressed by at most two-dimensional Mellin-Barnes integrals, which are not trivial to solve. However, their calculation is a subset of the calculation of the non-planar integrals, so we do not describe it here [cf. subsection 6.4]. Therefore we briefly summarize the important ingredients of the two-loop planar integrals in this section.

Double massive box diagram. By using the package `asy2.1.m`, we reveal thirteen relevant scalings:

$$\begin{aligned}
 & (0, 0, 0, 0, 0, 0, 0), (0, 0, 1, 0, 0, 1, 1), (0, 0, 1, 1, 0, 0, 1), (0, 0, 1, 1, 1, 0, 0), (0, 1, 1, 1, 0, 0, 0), \\
 & (1, 0, 0, 0, 0, 1, 1), (1, 0, 0, 0, 1, 1, 0), (1, 0, 0, 1, 0, 0, 1), (1, 1, 0, 0, 0, 1, 0), (0, 0, 1, 1, 1, 1, 1), \\
 & (1, 0, 0, 1, 1, 1, 1), (1, 1, 1, 0, 0, 1, 1), (1, 1, 1, 1, 0, 0, 1).
 \end{aligned} \tag{5.7}$$

The template integrals of these regions are summarized in appendix B, and can be found in the supplementary file attached to this paper. The result of this integral at the leading order in m_t , up to $\mathcal{O}(\epsilon)$, is given by

$$I^{\text{PL1}} = \sum_{n_1=0}^1 \sum_{n_2=0}^{n_1+4} \frac{d_{n_1, n_2}}{v} \epsilon^{n_1} \log^{n_2}(m_t) + \mathcal{O}(m_t, \epsilon^2), \tag{5.8}$$

where the coefficients d_{n_1, n_2} are given by

$$\begin{aligned}
 d_{0,4} &= 16, & d_{0,3} &= -\frac{64h_0}{3} + \frac{32i\pi}{3}, & d_{0,2} &= 8h_0^2 - 16i\pi h_0 - \frac{8\pi^2}{3}, \\
 d_{0,1} &= 8i\pi h_0^2 + \frac{16\pi^2 h_0}{3} - 8\zeta_3 + \frac{8i\pi^3}{3}, \\
 d_{0,0} &= 16h_0 h_3 - \frac{h_0^4}{3} - \frac{4}{3}i\pi h_0^3 - 4h_0^2 h_2 - 2\pi^2 h_0^2 - 8i\pi h_0 h_2 + 4h_0 \zeta_3 \\
 &\quad - \frac{4}{3}i\pi^3 h_0 + 16i\pi h_3 - 24h_4 - \frac{7\pi^4}{15}, \\
 d_{1,5} &= -\frac{128}{3}, & d_{1,4} &= \frac{152h_0}{3} - 56i\pi, & d_{1,3} &= -16h_0^2 + \frac{208i\pi h_0}{3} + \frac{160\pi^2}{9}, \\
 d_{1,2} &= \frac{4h_0^3}{3} + 4h_0^2 h_1 - 20i\pi h_0^2 + 8i\pi h_0 h_1 - 8h_0 h_2 - 28\pi^2 h_0 - 8i\pi h_2 + 8h_3 - 16\zeta_3 - \frac{4i\pi^3}{3}, \\
 d_{1,1} &= -\frac{4h_0^4}{3} - \frac{8h_0^3 h_1}{3} - 4i\pi h_0^3 - 4i\pi h_0^2 h_1 + \frac{28\pi^2 h_0^2}{3} - 8i\pi h_0 h_2 + 16h_0 h_3 + 24i\pi h_3 - 32h_4 \\
 &\quad - \frac{8}{3}\pi^2 h_0 h_1 + 16h_0 \zeta_3 - \frac{28}{3}i\pi^3 h_0 + \frac{8\pi^2 h_2}{3} + \frac{218\pi^4}{45}, \\
 d_{1,0} &= \frac{h_0^5}{2} + \frac{h_0^4 h_1}{2} + 6h_0^3 h_2 + 6h_0^2 h_1 h_2 + 4h_0^2 h_{21} - 20h_0^2 h_3 - 4h_0 h_2^2 - 12h_0 h_{22} \\
 &\quad + \frac{11}{6}i\pi h_0^4 + \frac{2}{3}i\pi h_0^3 h_1 + 18i\pi h_0^2 h_2 + 12i\pi h_0 h_1 h_2 - 24h_0 h_1 h_3 + 36h_1 h_4 - 4i\pi h_2^2 + 4h_2 h_3 \\
 &\quad + \frac{\pi^2 h_0^3}{9} + \frac{5}{3}\pi^2 h_0^2 h_1 - 6\pi^2 h_0 h_2 + 8i\pi h_0 h_{21} - 32i\pi h_0 h_3 - 24i\pi h_1 h_3 - 12i\pi h_{22} - 16i\pi h_4 \\
 &\quad - 4h_0^2 \zeta_3 + 5i\pi^3 h_0^2 + 2i\pi^3 h_0 h_1 - \frac{157\pi^4 h_0}{90} - \frac{2\pi^4 h_1}{5} + \frac{10}{3}i\pi^3 h_2 + \frac{26\pi^2 h_3}{3} + \frac{61i\pi^5}{90} \\
 &\quad + 24h_0 h_1 \zeta_3 + 24i\pi h_1 \zeta_3 - 56h_2 \zeta_3 + 4(4h_{23} + 7h_{32} + 21h_5 - 13\zeta_5) + 6\pi^2 \zeta_3.
 \end{aligned} \tag{5.9}$$

Single massive double box diagram. By using the package `asy2.1.m`, we reveal ten relevant scalings:

$$\begin{aligned}
 & (0, 0, 0, 0, 0, 0, 0), (0, 0, 1, 1, 1, 0, 0), (0, 1, 1, 1, 0, 0, 0), (1, 0, 0, 0, 1, 1, 0), \\
 & (1, 1, 0, 0, 0, 1, 0), (0, 0, 1, 1, 1, 1, 1), (1, 0, 0, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 0).
 \end{aligned} \tag{5.10}$$

The template integrals of these regions are summarized in appendix B, and can be found in the supplementary file attached to this paper. The result of this integral at the leading order in m_t , up to $\mathcal{O}(\epsilon)$, is

$$I^{\text{PL2}} = \sum_{n_1=-2}^1 \sum_{n_2=0}^{n_1+4} \frac{d_{n_1, n_2}}{v} \epsilon^{n_1} \log^{n_2}(m_t) + \mathcal{O}(m_t, \epsilon^2), \quad (5.11)$$

where the coefficients d_{n_1, n_2} are now given by

$$\begin{aligned} d_{-2,2} &= 8, & d_{-2,1} &= -4h_0 + 4i\pi, & d_{-2,0} &= -\pi^2 - 2i\pi h_0, \\ d_{-1,3} &= -\frac{32}{3}, & d_{-1,2} &= -4h_0 - 20i\pi, & d_{-1,1} &= 4h_0^2 + 4i\pi h_0 + \frac{20\pi^2}{3}, \\ d_{-1,0} &= \frac{h_0^3}{3} + h_0^2 h_1 + 3i\pi h_0^2 + 2i\pi h_0 h_1 - 2h_0 h_2 - \frac{4\pi^2 h_0}{3} - 2i\pi h_2 + 2h_3 - 14\zeta_3 + 2i\pi^3, \\ d_{0,4} &= 8, & d_{0,3} &= \frac{32h_0}{3} + \frac{80i\pi}{3}, & d_{0,2} &= -4h_0^2 + 8i\pi h_0 - \frac{70\pi^2}{3}, \\ d_{0,1} &= -4i\pi h_0^2 + 4\pi^2 h_0 + 20\zeta_3 - \frac{10i\pi^3}{3}, \\ d_{0,0} &= -\frac{5h_0^4}{6} - \frac{4h_0^3 h_1}{3} - \frac{h_0^2 h_1^2}{2} - 3h_0^2 h_2 + 2h_0 h_1 h_2 - 2h_0 h_{21} + 20h_0 h_3 + \frac{h_2^2}{2} - h_{22} \\ &\quad - \frac{10}{3}i\pi h_0^3 - 4i\pi h_0^2 h_1 - i\pi h_0 h_1^2 - 6i\pi h_0 h_2 + 2i\pi h_1 h_2 - 2h_1 h_3 - 2i\pi h_{21} + 20i\pi h_3 - 34h_4 \\ &\quad + \frac{11\pi^2 h_0^2}{6} + \frac{7}{3}\pi^2 h_0 h_1 + 14h_0 \zeta_3 - \frac{7}{3}i\pi^3 h_0 + 2h_1 \zeta_3 - \frac{1}{3}i\pi^3 h_1 - \frac{7\pi^2 h_2}{3} + 24i\pi \zeta_3 + \frac{259\pi^4}{180}, \\ d_{1,5} &= -\frac{64}{15}, & d_{1,4} &= -12h_0 - \frac{68i\pi}{3}, & d_{1,3} &= \frac{8h_0^2}{3} - \frac{56i\pi h_0}{3} + \frac{272\pi^2}{9}, \\ d_{1,2} &= \frac{2h_0^3}{3} + 2h_0^2 h_1 + 6i\pi h_0^2 + 4i\pi h_0 h_1 - 4h_0 h_2 + \frac{16\pi^2 h_0}{3} - 4i\pi h_2 + 4h_3 - \frac{220\zeta_3}{3} + \frac{40i\pi^3}{3}, \\ d_{1,1} &= -\frac{2h_0^4}{3} - \frac{4h_0^3 h_1}{3} - 2i\pi h_0^3 - 2i\pi h_0^2 h_1 - \frac{8\pi^2 h_0^2}{3} - 4i\pi h_0 h_2 + 8h_0 h_3 + 12i\pi h_3 - 16h_4 \\ &\quad - \frac{4}{3}\pi^2 h_0 h_1 + \frac{44h_0 \zeta_3}{3} - \frac{16}{3}i\pi^3 h_0 + \frac{4\pi^2 h_2}{3} - \frac{176i\pi \zeta_3}{3} + \frac{5\pi^4}{18}, \\ d_{1,0} &= \frac{43h_0^5}{60} + \frac{5h_0^4 h_1}{4} + \frac{2h_0^3 h_1^2}{3} + \frac{17h_0^3 h_2}{3} + \frac{h_0^2 h_1^3}{6} + 5h_0^2 h_1 h_2 - h_0 h_1^2 h_2 \\ &\quad + 5h_0^2 h_{21} + 2h_0 h_1 h_{21} - 3h_0 h_2^2 - \frac{h_1 h_2^2}{2} + h_1 h_{22} + \frac{h_2 h_{21}}{3} - \frac{h_{212}}{3} \\ &\quad - 22h_0^2 h_3 - 28h_0 h_1 h_3 - 2h_0(h_{211} + 7h_{22}) + h_1^2 h_3 + \frac{7h_2 h_3}{3} - h_{221} + \frac{53h_{23}}{3} \\ &\quad + \frac{13}{4}i\pi h_0^4 + \frac{13}{3}i\pi h_0^3 h_1 + 2i\pi h_0^2 h_1^2 + \frac{1}{3}i\pi h_0 h_1^3 + 46h_1 h_4 + 33h_{32} + 98h_5 \\ &\quad + 17i\pi h_0^2 h_2 + 10i\pi h_0 h_1 h_2 + 10i\pi h_0 h_{21} - i\pi h_1^2 h_2 + 2i\pi h_1 h_{21} - 3i\pi h_2^2 - 2i\pi(h_{211} + 7h_{22}) \\ &\quad - \frac{22}{9}\pi^2 h_0^3 - 3\pi^2 h_0^2 h_1 - \frac{7}{6}\pi^2 h_0 h_1^2 - 5\pi^2 h_0 h_2 - 40i\pi h_0 h_3 - 28i\pi h_1 h_3 - 8i\pi h_4 \\ &\quad + \frac{10}{3}i\pi^3 h_0^2 + \frac{4}{3}i\pi^3 h_0 h_1 + \frac{1}{6}i\pi^3 h_1^2 + \frac{7}{3}\pi^2 h_1 h_2 + 5i\pi^3 h_2 - \frac{7\pi^2 h_{21}}{3} + 16\pi^2 h_3 \\ &\quad - 9h_0^2 \zeta_3 + 26h_0 h_1 \zeta_3 - \frac{599\pi^4 h_0}{180} - h_1^2 \zeta_3 - \frac{161\pi^4 h_1}{180} - 64h_2 \zeta_3 - \frac{47i\pi^5}{90} \\ &\quad - \frac{32}{3}i\pi h_0 \zeta_3 + 26i\pi h_1 \zeta_3 + 29\pi^2 \zeta_3 - 121\zeta_5. \end{aligned} \quad (5.12)$$



Figure 5. The two-loop massive non-planar diagrams defined in eq. (6.1) (left) and eq. (6.3) (right).

6 Two-loop non-planar diagram

For the two-loop massive non-planar diagrams, we consider the following two Feynman integral families

$$J_{a_1, a_2, a_3, a_4, a_5, a_6, a_7}^{\text{NPL1}} = \int \frac{d^d \ell_1}{i\pi^{d/2}} \frac{d^d \ell_2}{i\pi^{d/2}} \prod_{n=1}^2 \frac{1}{(-p_n^2)^{a_n}} \prod_{n=3}^7 \frac{1}{(m_t^2 - p_n^2)^{a_n}} \quad (6.1)$$

where the momenta of the lines are given by

$$\{p_1, p_2, p_3, p_4, p_5, p_6, p_7\} = \{\ell_1 + q_{12}, \ell_1 - q_3, \ell_{12} + q_{2\bar{3}}, \ell_{12} + q_2, \ell_2 - q_1, \ell_2, \ell_2 + q_2\}, \quad (6.2)$$

and

$$J_{a_1, a_2, a_3, a_4, a_5, a_6, a_7}^{\text{NPL2}} = \int \frac{d^d \ell_1}{i\pi^{d/2}} \frac{d^d \ell_2}{i\pi^{d/2}} \prod_{n=1}^4 \frac{1}{(m_t^2 - p_n^2)^{a_n}} \prod_{n=5}^7 \frac{1}{(-p_n^2)^{a_n}} \quad (6.3)$$

where the momenta of the lines are

$$\{p_1, p_2, p_3, p_4, p_5, p_6, p_7\} = \{\ell_1, \ell_1 + q_3, \ell_{12} + q_{23}, \ell_{12} - q_1 \ell_2 - q_1, \ell_2, \ell_2 + q_2\}. \quad (6.4)$$

Recall that $q_{2\bar{3}} = q_2 - q_3$ and $q_{23} = q_2 + q_3$. We consider $I^{\text{NPL1}} = J_{1,1,1,1,1,1,1}^{\text{NPL1}}$ as an example in this section. The template integrals of $I^{\text{NPL2}} = J_{1,1,1,1,1,1,1}^{\text{NPL2}}$ is provided in appendix B. The Feynman diagrams corresponding I^{NPL1} and I^{NPL2} are illustrated in figure 5.

In subsection 4.1 we showed that the expansion in m_H can be obtained by the naive Taylor expansion of the integrand. This holds also in this case, and the m_H -expansion is straightforward. Therefore we again consider only integrals with massless external legs. The alpha representation of our non-planar integral is

$$I^{\text{NPL1}} \stackrel{\text{AC}}{=} \int \mathfrak{D}^7 \alpha^\delta \mathcal{U}^{-d/2} e^{-\mathcal{F}/\mathcal{U}}, \quad (6.5)$$

where

$$\mathcal{U} = \alpha_{12}\alpha_{34567} + \alpha_{34}\alpha_{567} \quad (6.6)$$

$$\mathcal{F} = m_t^2 \alpha_{34567} \mathcal{U} + S(\alpha_1 \alpha_7 \alpha_{45} + \alpha_2 \alpha_5 \alpha_{37} + \alpha_5 \alpha_7 \alpha_{34}) + T \alpha_1 \alpha_3 \alpha_6 + U \alpha_2 \alpha_4 \alpha_6. \quad (6.7)$$

6.1 Relevant scaling

A new feature appears in eq. (6.7): if we impose the relation $S + T + U = 0$, we obtain

$$\mathcal{F} \stackrel{\text{AC}}{=} m_t^2 \alpha_{34567} \mathcal{U} + S(\alpha_1 \alpha_7 \alpha_{45} + \alpha_2 \alpha_5 \alpha_{37} + \alpha_5 \alpha_7 \alpha_{34} - \alpha_2 \alpha_4 \alpha_6) + T(\alpha_1 \alpha_3 \alpha_6 - \alpha_2 \alpha_4 \alpha_6), \quad (6.8)$$

and the sign of \mathcal{F} becomes indefinite. In such a case, it is not guaranteed that the method to reveal the relevant scalings works properly [43]. An idea to solve this problem is to perform a proper change of variables and decompose the integration domain such that \mathcal{F} is positive-definite [43]. However in our case, this approach does not resolve the indefinite sign of \mathcal{F} since there is no simple change of variables to make \mathcal{F} positive-definite.

The solution to this problem is to keep S, T, U as independent variables. It is obvious that eq. (6.7) is positive definite in this case, and we can apply the method of regions, expand the integrand, and express the result in terms of Mellin-Barnes integrals in terms of the positive Mandelstam variables. The procedure to obtain expression (6.7) is the following: we first compute \mathcal{F} respecting the original definition of the Mandelstam variables (3.12). At this point, there are some redundant terms in \mathcal{F} such as $(S+T+U)\alpha_2\alpha_3\alpha_6$. We minimize the number of terms, under the condition that \mathcal{F} remains positive definite. The resulting \mathcal{F} is unique.

There are two commands in the package `asy2.1.m` to reveal the relevant regions. The first is `AlphaRepExpand`, which accepts a set of propagators and replacement rules as input. The other is `WilsonExpand`, which accepts the Symanzik polynomials as input. Here we must use `WilsonExpand` since the conventional routine used in `AlphaRepExpand` either eliminates U completely or keeps U completely, whereas we want to eliminate U partially, as explained above. There is an option `Preresolve` in `AlphaRepExpand` which makes it attempt some changes of variables to make \mathcal{F} positive definite, but in our cases this option did not solve the problem.

With this setup and the hierarchy (3.18), we obtain the following fourteen relevant scalings⁴

$$\begin{aligned} & (0, 0, 0, 0, 0, 0, 0), \left(0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, 1\right), \left(\frac{1}{2}, 0, 0, \frac{1}{2}, 1, \frac{1}{2}, 0\right), (0, 0, 0, 1, 1, 1, 0) \\ & (0, 0, 1, 0, 0, 1, 1), (0, 1, 0, 0, 0, 1, 1), (0, 1, 1, 0, 0, 0, 1), (1, 0, 0, 0, 1, 1, 0), (1, 0, 0, 1, 1, 0, 0), \\ & (1, 1, 0, 0, 0, 0, 1), (1, 1, 0, 0, 1, 0, 0), (1, 1, 0, 0, 1, 1, 1), (1, 1, 1, 1, 0, 0, 1), (1, 1, 1, 1, 1, 0, 0). \end{aligned} \quad (6.9)$$

In the case of the planar integrals [32], the scalings consist of only 0 and 1. Here, we additionally have a scaling $(0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2})$ which is particular to these non-planar integrals.

⁴In fact, it turns out that the correct scalings (6.9) can be obtained by using eq. (6.8) or by using `AlphaRepExpand`, provided suitable values for S and T are chosen (e.g. $S = 1, T = 1$), such that $U \neq 0$. This may be because there is no cancellation between two hard-scaling terms resulting in a soft-scaling term in eq. (6.8). However, this observation is made in hindsight, since in principle it is not guaranteed that the regions are found correctly. One could also use `ASPIRE` [44] instead of `asy2.1.m` and obtain the correct scalings, if one similarly chooses suitable values for S and T .

The contribution of the all-hard regions can be calculated by using the massless integral family and IBP-reduction [cf. the text below eq. (4.24)]. Therefore we do not need a template integral for Region 1, and we show the calculation of Region 1 separately. The template integrals for Regions 2 to 14 are calculated in the next subsection, and can be found in the supplementary file attached to this paper.

Region 1 (0, 0, 0, 0, 0, 0, 0) (all-hard region). As shown in subsection 4.2, the contribution from the all-hard region can be expressed in terms of massless integrals of the same topology. The massless integral that is relevant here has been calculated in ref. [52].

6.2 Template integrals for Regions 2 to 14

The template integral of each region is expressed as

$$\mathcal{T}_{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \epsilon}^{(j)} = \int \mathcal{D}^7 \alpha^\delta \left(\mathcal{U}^{(j)} \right)^{-d/2} e^{-\mathcal{F}^{(j)}/\mathcal{U}^{(j)}} \quad (6.10)$$

where $2 \leq j \leq 14$. For simplicity, we omit the subscripts of $\mathcal{T}_{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \epsilon}^{(j)}$ and write $\mathcal{T}^{(j)}$ when they are in the ordinary order.

Region 2 (0, $\frac{1}{2}$, $\frac{1}{2}$, 0, 0, $\frac{1}{2}$, 1), Region 3 ($\frac{1}{2}$, 0, 0, $\frac{1}{2}$, 1, $\frac{1}{2}$, 0). The Symanzik polynomials of Region 2 are given by

$$\mathcal{U}^{(2)} = \alpha_1 \alpha_{45} + \alpha_4 \alpha_5 \quad (6.11)$$

$$\mathcal{F}^{(2)} = m_t^2 \alpha_{45} \mathcal{U}^{(2)} + S(\alpha_2 \alpha_3 \alpha_5 + \alpha_1 \alpha_{45} \alpha_7 + \alpha_4 \alpha_5 \alpha_7) + T \alpha_1 \alpha_3 \alpha_6 + U \alpha_2 \alpha_4 \alpha_6. \quad (6.12)$$

The integration in α_7 can be performed using the relation (A.3). Then, we perform the following change of variables

$$\alpha_2 \rightarrow \beta_1 \beta_3 / \beta_2, \quad \alpha_3 \rightarrow \beta_1 \beta_2 / \beta_3, \quad \alpha_6 \rightarrow \beta_2 \beta_3 / \beta_1, \quad (6.13)$$

and the template integral becomes

$$\begin{aligned} \mathcal{T}^{(2)} &= 4S^{-1-\delta_7} \int_0^\infty d\alpha_1 d\alpha_4 d\alpha_5 d\beta_1 d\beta_2 d\beta_3 \alpha_1^{\delta_1} \alpha_4^{\delta_4} \alpha_5^{\delta_5} \beta_1^{\delta_{236}} \beta_2^{\delta_{236}} \beta_3^{\delta_{236}} \\ &\times \frac{(\alpha_1 \alpha_4 + \alpha_1 \alpha_5 + \alpha_4 \alpha_5)^{-d/2} e^{-m_t^2 \alpha_{45} - (S\beta_1^2 \alpha_5 + T\beta_2^2 \alpha_1 + U\beta_3^2 \alpha_4)/\mathcal{U}^{(2)}}}{\Gamma[\delta_1 + 1, \delta_2 + 1, \delta_3 + 1, \delta_4 + 1, \delta_5 + 1, \delta_6 + 1]}. \end{aligned} \quad (6.14)$$

The integration in $\beta_1, \beta_2, \beta_3$ is now straightforward. To integrate over the remaining variables, it is easiest to integrate in α_1 first using the relation (A.4) and then integrate in α_4, α_5 . Finally we obtain the template integral as

$$\begin{aligned} \mathcal{T}^{(2)} &= \frac{1}{2} (m_t^2)^{-(1+4\epsilon+\delta_{112344556})/2} S^{-(3+\delta_{23677})/2} T^{-(1+\delta_{236})/2} U^{-(1+\delta_{236})/2} \\ &\times \frac{\Gamma[\delta_{0\bar{1}2}, \frac{1+\delta_{236}}{2}, \frac{1+\delta_{236}}{2}, \frac{1+\delta_{236}}{2}, \frac{1+\delta_{11236}}{2}, \frac{1+\delta_{001123446}}{2}, \frac{1+\delta_{001123556}}{2}, \frac{1+\delta_{0000112344556}}{2}]}{\Gamma[\delta_1 + 1, \delta_2 + 1, \delta_3 + 1, \delta_4 + 1, \delta_5 + 1, \delta_6 + 1, \delta_{0011245} + 1, \frac{1-\delta_{00236}}{2}]}. \end{aligned} \quad (6.15)$$

The template integral of the Region 3 is obtained by the replacement $\alpha_1 \leftrightarrow \alpha_2, \alpha_3 \leftrightarrow \alpha_4, \alpha_5 \leftrightarrow \alpha_7, T \leftrightarrow U$ of $\mathcal{T}^{(2)}$

$$\mathcal{T}_{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \epsilon}^{(3)} = \mathcal{T}_{\delta_2, \delta_1, \delta_4, \delta_3, \delta_7, \delta_6, \delta_5, \epsilon}^{(2)} \Big|_{T \leftrightarrow U}. \quad (6.16)$$

Region 4 (0, 0, 0, 1, 1, 1, 0), Region 5 (0, 0, 1, 0, 0, 1, 1). The Symanzik polynomials of Region 4 are given by

$$\mathcal{U}^{(4)} = \alpha_{12}\alpha_{37} + \alpha_3\alpha_7 \quad (6.17)$$

$$\mathcal{F}^{(4)} = m_t^2\alpha_{37}\mathcal{U}^{(4)} + S(\alpha_3\alpha_5\alpha_7 + \alpha_1\alpha_{45}\alpha_7 + \alpha_2\alpha_5\alpha_{37}) + T\alpha_1\alpha_3\alpha_6. \quad (6.18)$$

The integrations in $\alpha_4, \alpha_5, \alpha_6$ can be done using the relation (A.3). Then the template integral is

$$\begin{aligned} \mathcal{T}^{(4)} &= \frac{1}{S^{2+\delta_{45}}T^{1+\delta_6}} \int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_7 \alpha_1^{-2+\delta_{14\bar{6}}} \alpha_2^{\delta_2} \alpha_3^{-1+\delta_{3\bar{6}}} \alpha_7^{-1+\delta_{47}} \\ &\times \frac{(\alpha_{12}\alpha_{37} + \alpha_3\alpha_7)^{3-d/2+\delta_{456}} (\alpha_{13}\alpha_7 + \alpha_2\alpha_{37})^{-1-\delta_5} e^{-m_t^2\alpha_{37}}}{\Gamma[\delta_1 + 1, \delta_2 + 1, \delta_3 + 1, \delta_7 + 1]}. \end{aligned} \quad (6.19)$$

We introduce a Mellin-Barnes integral to separate $\alpha_{12}\alpha_{37} + \alpha_3\alpha_7$ into two factors, $\alpha_{13}\alpha_7 + \alpha_2\alpha_{37}$ and $\alpha_1\alpha_3$, then the integration in α_1 and α_2 can be done using the relation (A.4). The remaining integration is also straightforward and we obtain

$$\mathcal{T}^{(4)} = \int dz_1 \frac{\Gamma[\delta_{\bar{0}\bar{1}\bar{2}}, \delta_{001237}, z_{\bar{1}} + \delta_{0267} + 1, z_{\bar{1}}, z_1 + \delta_{0123\bar{6}}, z_1 + \delta_{14\bar{6}} - 1, z_1 + \delta_{045\bar{6}} - 1]}{(m_t^2)^{\delta_{001237}} S^{2+\delta_{45}} T^{1+\delta_6} \Gamma[\delta_1 + 1, \delta_3 + 1, \delta_{\bar{0}\bar{4}\bar{5}\bar{6}} - 1, \delta_7 + 1, \delta_{0012237} + 1, z_1 + \delta_{\bar{0}\bar{4}\bar{6}}]}. \quad (6.20)$$

The template integral of Region 5 can be obtained in a similar manner and the result is

$$\mathcal{T}^{(5)} = \int dz_1 \frac{\Gamma[\delta_{\bar{0}\bar{1}\bar{2}}, \delta_{001245}, z_{\bar{1}} + \delta_{0156} + 1, z_{\bar{1}}, z_1 + \delta_{23\bar{6}} - 1, z_1 + \delta_{0124\bar{6}}, z_1 + \delta_{03\bar{6}\bar{7}} - 1]}{(m_t^2)^{\delta_{001245}} S^{2+\delta_{37}} U^{1+\delta_6} \Gamma[\delta_2 + 1, \delta_4 + 1, \delta_5 + 1, \delta_{0011245} + 1, \delta_{\bar{0}\bar{3}\bar{6}\bar{7}} - 1, z_1 + \delta_{\bar{0}\bar{3}\bar{6}}]}. \quad (6.21)$$

Region 6 (0, 1, 0, 0, 0, 1, 1), Region 8 (1, 0, 0, 0, 1, 1, 0). The Symanzik polynomials of Region 6 are given by

$$\mathcal{U}^{(6)} = \alpha_1\alpha_{345} + \alpha_{34}\alpha_5 \quad (6.22)$$

$$\mathcal{F}^{(6)} = m_t^2\alpha_{345}\mathcal{U}^{(6)} + S(\alpha_2\alpha_3\alpha_5 + \alpha_{34}\alpha_5\alpha_7 + \alpha_1\alpha_{45}\alpha_7) + T\alpha_1\alpha_3\alpha_6. \quad (6.23)$$

The integrations in $\alpha_2, \alpha_6, \alpha_7$ are straightforward using the relation (A.3) and the template integral is given by

$$\begin{aligned} \mathcal{T}^{(6)} &= \frac{1}{S^{2+\delta_{27}}T^{1+\delta_6}} \int_0^\infty d\alpha_1 d\alpha_3 d\alpha_4 d\alpha_5 \alpha_1^{-1+\delta_{1\bar{6}}} \alpha_3^{-2+\delta_{23\bar{6}}} \alpha_4^{\delta_4} \alpha_5^{-1+\delta_{25}} \\ &\times \frac{(\alpha_1\alpha_{345} + \alpha_{34}\alpha_5)^{3-d/2+\delta_{267}} (\alpha_{34}\alpha_5 + \alpha_1\alpha_{45})^{-1-\delta_7} e^{-m_t^2\alpha_{345}}}{\Gamma[\delta_1 + 1, \delta_3 + 1, \delta_4 + 1, \delta_5 + 1]}. \end{aligned} \quad (6.24)$$

This looks similar to eq. (6.19) but has one extra massive integral, and thus it is necessary to introduce two Mellin-Barnes integrals, giving

$$\begin{aligned} \mathcal{T}^{(6)} &= \int dz_1 dz_2 \frac{\Gamma[\delta_{015}, z_{\bar{1}}, z_1 + \delta_{1\bar{6}}, z_1 + \delta_{\bar{0}\bar{2}\bar{6}\bar{7}} - 1, z_2 + \delta_{0124} + 1]}{(m_t^2)^{\delta_{001345}} S^{2+\delta_{27}} T^{1+\delta_6} \Gamma[\delta_1 + 1, \delta_3 + 1, \delta_4 + 1, \delta_5 + 1]} \\ &\times \frac{\Gamma[z_{\bar{1}\bar{2}} + \delta_{0012456} + 1, z_{\bar{2}}, z_2 + \delta_{\bar{0}\bar{1}\bar{2}}, z_{12} + \delta_{23\bar{6}} - 1]}{\Gamma[\delta_{\bar{0}\bar{2}\bar{6}\bar{7}} - 1, z_1 + \delta_{\bar{0}\bar{2}\bar{6}}, z_2 + \delta_{0011245} + 1]}. \end{aligned} \quad (6.25)$$

The template integral of Region 8 can be obtained in a similar manner and the result is

$$\mathcal{T}^{(8)} = \int dz_1 dz_2 \frac{\Gamma[\delta_{\bar{0}\bar{1}\bar{2}}, z_{\bar{1}}, z_1 + \delta_{\bar{1}\bar{4}\bar{6}} - 1, z_1 + \delta_{\bar{0}\bar{1}\bar{5}\bar{6}} - 1, z_{\bar{1}\bar{2}} + \delta_{067}]}{(m_t^2)^{\delta_{002347}} S^{2+\delta_{15}} U^{1+\delta_6} \Gamma[\delta_2 + 1, \delta_3 + 1, \delta_4 + 1, \delta_{\bar{0}\bar{1}\bar{5}\bar{6}} - 1]} \times \frac{\Gamma[z_{\bar{2}}, z_2 + \delta_3 + 1, z_{12} + \delta_{2\bar{6}}, z_{12} + \delta_{0234\bar{6}}]}{\Gamma[\delta_7 + 1, z_1 + \delta_{\bar{0}\bar{1}\bar{6}}, z_{12} + \delta_{\bar{1}\bar{3}\bar{4}\bar{6}}]}. \quad (6.26)$$

Region 7 (0, 1, 1, 0, 0, 0, 1), Region 9 (1, 0, 0, 1, 1, 0, 0). The Symanzik polynomials of Region 7 are given by

$$\mathcal{U}^{(7)} = \alpha_1 \alpha_{456} + \alpha_4 \alpha_{56} \quad (6.27)$$

$$\mathcal{F}^{(7)} = m_t^2 \alpha_{456} \mathcal{U}^{(7)} + S(\alpha_4 \alpha_5 \alpha_7 + \alpha_1 \alpha_{45} \alpha_7) + T \alpha_1 \alpha_3 \alpha_6 + U \alpha_2 \alpha_4 \alpha_6. \quad (6.28)$$

The template integral is obtained in a similar way as that of Region 6 and the resulting two-dimensional integral is given by

$$\mathcal{T}^{(7)} = \int dz_1 dz_2 \frac{\Gamma[\delta_{014}, z_{\bar{1}} + \delta_{0012345} + 1, z_1 + \delta_{2\bar{3}\bar{6}} - 1, z_1 + \delta_{\bar{0}\bar{2}\bar{3}\bar{7}} - 1, z_{\bar{1}\bar{2}} + \delta_{\bar{1}\bar{3}}]}{(m_t^2)^{\delta_{001456}} S^{1+\delta_7} T^{1+\delta_3} U^{1+\delta_2} \Gamma[\delta_1 + 1, \delta_4 + 1, \delta_5 + 1, \delta_6 + 1]} \times \frac{\Gamma[z_{\bar{1}\bar{2}} + \delta_{0235} + 1, z_{\bar{2}}, z_2 + \delta_{1\bar{3}}, z_{12} + \delta_{\bar{0}\bar{2}\bar{3}}]}{\Gamma[\delta_{\bar{0}\bar{2}\bar{3}\bar{7}} - 1, z_1 + \delta_{\bar{0}\bar{2}\bar{3}}, z_{\bar{1}\bar{2}} + \delta_{0012345} + 1]}. \quad (6.29)$$

The template integral of Region 9 can also be obtained in a similar manner and the result is given by

$$\mathcal{T}^{(9)} = \int dz_1 dz_2 \frac{\Gamma[\delta_{023}, z_{\bar{1}} + \delta_{0012347} + 1, z_1 + \delta_{\bar{0}\bar{1}\bar{4}\bar{5}} - 1, z_1 + \delta_{\bar{1}\bar{4}\bar{6}} - 1, z_{\bar{1}\bar{2}} + \delta_{\bar{2}\bar{4}}]}{(m_t^2)^{\delta_{002367}} S^{1+\delta_5} T^{1+\delta_1} U^{1+\delta_4} \Gamma[\delta_2 + 1, \delta_3 + 1, \delta_{\bar{0}\bar{1}\bar{4}\bar{5}} - 1, \delta_6 + 1]} \times \frac{\Gamma[z_{\bar{1}\bar{2}} + \delta_{0147} + 1, z_{\bar{2}}, z_2 + \delta_{2\bar{4}}, z_{12} + \delta_{\bar{0}\bar{1}\bar{4}}]}{\Gamma[\delta_7 + 1, z_1 + \delta_{\bar{0}\bar{1}\bar{4}}, z_{\bar{1}\bar{2}} + \delta_{0012347} + 1]}. \quad (6.30)$$

Region 10 (1, 1, 0, 0, 0, 0, 1), Region 11 (1, 1, 0, 0, 1, 0, 0). The Symanzik polynomials of Region 10 are given by

$$\mathcal{U}^{(10)} = \alpha_{34} \alpha_{56} \quad (6.31)$$

$$\mathcal{F}^{(10)} = m_t^2 \alpha_{3456} \mathcal{U}^{(10)} + S(\alpha_2 \alpha_3 \alpha_5 + \alpha_{34} \alpha_5 \alpha_7) + T \alpha_1 \alpha_3 \alpha_6 + U \alpha_2 \alpha_4 \alpha_6. \quad (6.32)$$

The template integral is obtained in a similar way as that of Region 5 and the resulting one-dimensional integral is given by

$$\mathcal{T}^{(10)} = \int dz_1 \frac{\Gamma[\delta_{034}, \delta_{056}, z_{\bar{1}} + \delta_{\bar{1}\bar{2}\bar{3}} - 1, z_{\bar{1}} + \delta_{\bar{2}\bar{5}\bar{7}} - 1, z_{\bar{1}}, z_1 + \delta_2 + 1, z_1 + \delta_4 + 1, z_1 + \delta_{\bar{1}\bar{6}}]}{(m_t^2)^{\delta_{003456}} S^{2+z_1+\delta_{27}} T^{1+\delta_1} U^{z_{\bar{1}}} \Gamma[\delta_2 + 1, \delta_3 + 1, \delta_4 + 1, \delta_{\bar{1}\bar{2}\bar{3}\bar{4}}, \delta_5 + 1, \delta_6 + 1, \delta_{\bar{1}\bar{2}\bar{5}\bar{6}\bar{7}} - 1]}. \quad (6.33)$$

The template integral of Region 11 can be obtained in a similar manner and the result is given by

$$\mathcal{T}^{(11)} = \int dz_1 \frac{\Gamma[\delta_{034}, \delta_{067}, z_{\bar{1}} + \delta_{\bar{1}\bar{2}\bar{4}} - 1, z_{\bar{1}} + \delta_{\bar{1}\bar{5}\bar{7}} - 1, z_{\bar{1}}, z_1 + \delta_1 + 1, z_1 + \delta_3 + 1, z_1 + \delta_{2\bar{6}}]}{(m_t^2)^{\delta_{003467}} S^{2+z_1+\delta_{15}} T^{z_{\bar{1}}} U^{1+\delta_2} \Gamma[\delta_1 + 1, \delta_3 + 1, \delta_4 + 1, \delta_{\bar{1}\bar{2}\bar{3}\bar{4}}, \delta_6 + 1, \delta_7 + 1, \delta_{\bar{1}\bar{2}\bar{5}\bar{6}\bar{7}} - 1]}. \quad (6.34)$$

Region 12 (1, 1, 0, 0, 1, 1, 1). The Symanzik polynomials of Region 12 are given by

$$\mathcal{U}^{(12)} = \alpha_{34}\alpha_{12567} \quad (6.35)$$

$$\mathcal{F}^{(12)} = m_t^2 \alpha_{34} \mathcal{U}^{(12)} + S(\alpha_2\alpha_3\alpha_5 + \alpha_1\alpha_4\alpha_7 + \alpha_{34}\alpha_5\alpha_7) + T\alpha_1\alpha_3\alpha_6 + U\alpha_2\alpha_4\alpha_6. \quad (6.36)$$

We introduce Mellin-Barnes integrals four times, and the template integral is given by

$$\begin{aligned} \mathcal{T}^{(12)} = & \int dz_1 dz_2 dz_3 dz_4 \frac{\Gamma[\delta_{034}, z_{\bar{1}}, z_{\bar{2}}, z_{12} + \delta_6 + 1, z_{\bar{1}\bar{2}\bar{3}} + \delta_{\bar{0}\bar{1}\bar{2}\bar{5}\bar{6}} - 2, z_{\bar{3}}, z_{13} + \delta_3 + 1, z_{23} + \delta_2 + 1]}{(m_t^2)^{\delta_{034}} S^{3+z_{12}+\delta_{012567}} T^{z_{\bar{1}}} U^{z_{\bar{2}}} \Gamma[\delta_1 + 1, \delta_2 + 1, \delta_3 + 1, \delta_4 + 1]} \\ & \times \frac{\Gamma[z_{\bar{1}\bar{3}\bar{4}} + \delta_{\bar{0}\bar{1}\bar{2}\bar{4}\bar{5}\bar{6}\bar{7}} - 2, z_{\bar{2}\bar{3}\bar{4}} + \delta_{\bar{0}\bar{2}\bar{5}\bar{6}\bar{7}} - 2, z_{\bar{4}}, z_{34} + \delta_5 + 1, z_{1234} + \delta_{012567} + 3]}{\Gamma[\delta_5 + 1, \delta_6 + 1, \delta_{\bar{0}\bar{0}\bar{1}\bar{2}\bar{5}\bar{6}\bar{7}} - 1, \delta_7 + 1, z_{\bar{4}} + \delta_{\bar{0}\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}} - 1]}. \end{aligned} \quad (6.37)$$

Region 13 (1, 1, 1, 1, 0, 0, 1), Region 14 (1, 1, 1, 1, 1, 0, 0). The Symanzik polynomials of Region 13 are given by

$$\mathcal{U}^{(13)} = \alpha_{1234}\alpha_{56} \quad (6.38)$$

$$\mathcal{F}^{(13)} = m_t^2 \alpha_{56} \mathcal{U}^{(13)} + S(\alpha_{134}\alpha_5\alpha_7 + \alpha_2\alpha_{37}\alpha_5) + T\alpha_1\alpha_3\alpha_6 + U\alpha_2\alpha_4\alpha_6. \quad (6.39)$$

It is necessary to introduce the Mellin-Barnes integral twice in order to separate the terms proportional to S, T , or U , respectively, and we obtain

$$\begin{aligned} \mathcal{T}^{(13)} = & \int dz_1 dz_2 \frac{\Gamma[\delta_{056}, z_{\bar{1}} + \delta_{\bar{0}\bar{1}\bar{2}\bar{3}\bar{4}\bar{6}} - 1, z_{\bar{1}}, z_1 + \delta_{57}, z_{\bar{2}} + \delta_{\bar{0}\bar{1}\bar{3}\bar{4}} - 1, z_{\bar{1}\bar{2}} + \delta_{\bar{0}\bar{1}\bar{2}\bar{3}} - 1, z_{\bar{2}}]}{(m_t^2)^{\delta_{056}} S^{1+z_{\bar{1}}+\delta_7} T^{z_{\bar{2}}} U^{2+z_{12}+\delta_{01234}} \Gamma[\delta_1 + 1, \delta_2 + 1, \delta_3 + 1, \delta_{\bar{0}\bar{0}\bar{1}\bar{2}\bar{3}\bar{4}}]} \\ & \times \frac{\Gamma[z_2 + \delta_1 + 1, z_{12} + \delta_3 + 1, z_{12} + \delta_{01234} + 2]}{\Gamma[\delta_4 + 1, \delta_5 + 1, \delta_6 + 1, \delta_{\bar{0}\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}} - 1]}. \end{aligned} \quad (6.40)$$

The template integral of Region 14 can be obtained in a similar manner and the result is

$$\begin{aligned} \mathcal{T}^{(14)} = & \int dz_1 dz_2 \frac{\Gamma[\delta_{067}, z_{\bar{1}} + \delta_{\bar{0}\bar{1}\bar{2}\bar{3}\bar{4}\bar{6}} - 1, z_{\bar{1}}, z_1 + \delta_{57}, z_{\bar{2}} + \delta_{\bar{0}\bar{2}\bar{3}\bar{4}} - 1, z_{\bar{1}\bar{2}} + \delta_{\bar{0}\bar{1}\bar{2}\bar{4}} - 1, z_{\bar{2}}]}{(m_t^2)^{\delta_{067}} S^{1+z_{\bar{1}}+\delta_5} T^{2+z_{12}+\delta_{01234}} U^{z_{\bar{2}}} \Gamma[\delta_1 + 1, \delta_2 + 1, \delta_3 + 1, \delta_{\bar{0}\bar{0}\bar{1}\bar{2}\bar{3}\bar{4}}]} \\ & \times \frac{\Gamma[z_2 + \delta_2 + 1, z_{12} + \delta_4 + 1, z_{12} + \delta_{01234} + 2]}{\Gamma[\delta_4 + 1, \delta_6 + 1, \delta_7 + 1, \delta_{\bar{0}\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}} - 1]}. \end{aligned} \quad (6.41)$$

In the limit where all the regularization parameters go to zero, eq. (6.40) becomes

$$\begin{aligned} \mathcal{T}^{(13)} = & -2\epsilon \int dz_1 dz_2 \frac{S^{-1+z_1} T^{z_2}}{U^{2+z_1+z_2}} \Gamma[z_{\bar{1}} - 1, z_{\bar{1}}, z_1, z_{\bar{2}} - 1, z_{\bar{2}}, z_2 + 1, z_{\bar{1}\bar{2}} - 1, z_{12} + 1, z_{12} + 2] \\ & + (\text{one-dimensional Mellin-Barnes integrals}) + \mathcal{O}(\epsilon^2), \end{aligned} \quad (6.42)$$

which means that the two-dimensional integral does not contribute at ϵ^0 -order. Thus, the representation (6.40) is most useful when the required order is ϵ^0 .

There is another representation of $\mathcal{T}^{(13)}$ which is more suitable if we require calculation beyond order ϵ^0 :

$$\begin{aligned} \mathcal{T}^{(13)} = & \int dz_1 \frac{\Gamma[\delta_{\bar{0}}, \delta_{\bar{0}\bar{1}\bar{2}}, \delta_{012} + 1, \delta_{056}, \delta_{57}, z_{\bar{1}} + \delta_{\bar{0}\bar{1}} - 1, z_{\bar{1}}, z_1 + 1, z_1 + \delta_{\bar{0}\bar{1}\bar{2}\bar{6}}]}{(m_t^2)^{\delta_{056}} S^{2+z_1+\delta_7} T^{z_{\bar{1}}} U^{1+\delta_{012}} \Gamma[\delta_{\bar{0}\bar{0}\bar{1}\bar{2}}, \delta_2 + 1, \delta_5 + 1, \delta_6 + 1, z_{\bar{1}} + \delta_{\bar{0}}, z_1 + \delta_{\bar{0}\bar{1}\bar{2}\bar{5}\bar{6}\bar{7}}]} \\ & - \int dz_1 dz_2 \frac{\Gamma[\delta_{\bar{0}}, \delta_{\bar{0}\bar{1}\bar{2}}, \delta_{056}, z_{\bar{1}} + \delta_{\bar{0}\bar{1}\bar{2}\bar{5}\bar{7}} - 1, z_{\bar{1}}, z_1 + \delta_1 + 1, z_1 + \delta_{012} + 1]}{(m_t^2)^{\delta_{056}} S^{2+z_{12}+\delta_{0127}} T^{1+z_{12}} \Gamma[\delta_1 + 1, \delta_{\bar{0}\bar{0}\bar{1}\bar{2}}, \delta_2 + 1, \delta_5 + 1]} \\ & \times \frac{\Gamma[z_{12} + \delta_6, z_{\bar{2}}, z_2 + 1, z_{12} + \delta_{\bar{0}\bar{1}}]}{\Gamma[\delta_6 + 1, z_{\bar{2}} + \delta_{\bar{0}\bar{1}\bar{2}\bar{5}\bar{6}\bar{7}} - 1, z_2 + \delta_{\bar{0}} + 1]}. \end{aligned} \quad (6.43)$$

This representation consists of two integrals with at most two kinematic parameters, and their calculation is simpler. It has some disadvantages, however; it holds only when δ_3 and δ_4 are non-negative integers, and eq. (6.43) is shown in the special case that $\delta_3 = 0, \delta_4 = 0$, since that is the typical situation (here we may set these values because $\mathcal{T}^{(13)}$ is regular in δ_3 and δ_4). Another disadvantage is that each integral produces singularities in ϵ which cancel in their sum, however as a by-product, higher-order derivatives of Γ -functions contribute to the ϵ^0 order.

6.3 Analytic continuation

As mentioned in the text below eq. (A.1), the Mellin-Barnes integrals in the template integrals are assumed to be regularized by choosing suitable values of δ_j . However, the quantity we need is the one where $\delta_j \rightarrow 0$ for all j . Therefore we need to analytically continue the Mellin-Barnes integrals in terms of δ_j . We describe the procedure of the analytic continuation showing the case of Region 7, which is one of the most involved cases, as a concrete example.⁵ We take the limit of the ascending order of δ_j ,

$$\lim_{\epsilon, \delta_7, \delta_6, \delta_5, \delta_4, \delta_3, \delta_2, \delta_1 \rightarrow 0} \mathcal{T}^{(7)}. \tag{6.44}$$

As mentioned below eq. (A.1), in general the integral contours of z_1, \dots, z_4 are assumed to be straight lines parallel to the imaginary axis [cf. figure 6 (b)]. For Region 7, we have only z_1, z_2 as integration variables. We choose $\text{Re}(z_1) = -1/5, \text{Re}(z_2) = -1/3$. Then, we may choose

$$\delta_1 = 0, \delta_2 = 0, \delta_3 = -\frac{13}{30}, \delta_4 = \frac{1}{300}, \delta_5 = \frac{1}{500}, \delta_6 = \frac{17881}{15540}, \delta_7 = -\frac{64003}{62160}, \epsilon = -\frac{2507}{10360} \tag{6.45}$$

to regularize the template integral $\mathcal{T}^{(7)}$.⁶ We try to set as many parameters as possible to zero in ascending j order in eq. (6.45). The limit of $\delta_{1,2} \rightarrow 0$ in eq. (6.44) is now trivial.

The analytic continuation of δ_3 from $-13/30$ to 0 makes the first rightmost left poles of $\Gamma(z_1 + \delta_{36} - 1), \Gamma(z_2 - \delta_3),$ and $\Gamma(z_{12} - \delta_{03})$ into right poles, which must be compensated by adding their residues. This analytic continuation procedure is automatized in the `Mathematica` package `MB.m` [51]. After the analytic continuation in terms of δ_3 , the integral depends on $\delta_4, \dots, \delta_7$ and ϵ , and we repeat the same procedure for δ_4 , then, δ_5 and so on. In this way, we obtain a combination of integrals for which the arguments of the Γ -functions in the integrand contain only z_1 and z_2 such as

$$\int dz_1 dz_2 \frac{\Gamma[1 - z_1, -1 + z_1, -1 + z_1, -z_{12}, -z_2, z_2, z_{12}]}{\Gamma(z_1)}. \tag{6.46}$$

The methods to solve these integrals are explained in the next subsection.

⁵In terms of the dimension of the Mellin-Barnes integrals, the most involved is Region 12. However, the analytic continuation turns out to be rather simple in this region.

⁶These values can be changed, provided they do not cross the integration contours.

6.4 Solving the Mellin-Barnes integrals

The usual idea to solve the Mellin-Barnes integral is to apply the first and the second Barnes lemma and variants of them. The `Mathematica` package `barnesroutines.m` [53] performs this procedure in an automatic way, and solves some of the Mellin-Barnes integrals we encounter. Unfortunately, not all of them are solved by this package, and we describe here how to treat such cases. The essential points are mentioned in ref. [32], and we fit or extend them to our integrals.

Three- and four-dimensional Mellin-Barnes integrals. The template integral of the Region 12 (6.37) is expressed as a four-dimensional Mellin-Barnes integral. Thus, the contribution from Region 12 contain a four-dimensional integral of the form

$$\int dz_1 dz_2 dz_3 dz_4 \left(\frac{T}{S}\right)^{z_1} \left(\frac{U}{S}\right)^{z_2} \frac{\Gamma[-z_4, 1 + z_{34}, -2 - z_{134}, -2 - z_{234}, 3 + z_{1234}]}{\Gamma(-1 - z_4)} \times \Gamma[-z_1, -z_2, -z_3, 1 + z_{12}, 1 + z_{13}, 1 + z_{23}, -2 - z_{123}]. \quad (6.47)$$

We use the relation

$$\begin{aligned} & \frac{\Gamma[-z_4, 1 + z_{34}, -2 - z_{134}, -2 - z_{234}, 3 + z_{1234}]}{\Gamma(-1 - z_4)} \\ &= \Gamma[-2 - z_{134}, -1 - z_{234}, 1 + z_{34}, 3 + z_{1234}] \\ &+ \frac{\Gamma[2 + z_{23}, -2 - z_{134}, -2 - z_{234}, 1 + z_{34}, 3 + z_{1234}]}{\Gamma(1 + z_{23})} \end{aligned} \quad (6.48)$$

to reduce the number of the Γ -functions whose argument contains z_4 from 6 to 4. Now one can apply the first Barnes lemma to solve the z_4 -integral. The resulting three-dimensional integral can be easily reduced to a sum of two-dimensional integrals since the z_3 -integral can be solved by the variants of the first and second Barnes lemmas. Thus, we have two-dimensional integrals, and the way to solve them will be explained below.

Note that the reduction in eq. (6.48) can be done only after the limits $\delta_j \rightarrow 0$ and $\epsilon \rightarrow 0$ since the Γ -functions of the denominator and the numerator have different dependence on ϵ , thus the cancellation does not occur before the limits have been taken.

The content of this subsection is not formulated in an algorithmic way and has been done manually.

Two-dimensional Mellin-Barnes integrals. In the cases of integrals with no argument such as

$$\int dz_1 dz_2 \Gamma[-z_1, -1 + z_1, -z_2, z_2, 1 - z_{12}, -1 + z_{12}] \psi(z_1) \psi(1 - z_{12}), \quad (6.49)$$

or integrals with a single argument of the form

$$\int dz_1 dz_2 X^{z_2} \Gamma[-z_1, -1 + z_1, -z_2, z_2, 1 - z_{12}, -1 + z_{12}] \psi(z_1) \psi(1 - z_{12}), \quad (6.50)$$

we first reduce them to a one-dimensional integral using the generalized Barnes lemma [32]

$$\int_C \frac{dz}{2\pi i} \frac{\Gamma[a_1 - z, a_2 - z, a_3 + z, a_4 + z, a_5 + z]}{\Gamma(-a_6 + z)} = \frac{\Gamma[a_{13}, a_{23}, a_{14}, a_{24}, a_{15}, a_{25}]}{\Gamma[a_{1235}, a_{1245}, -a_{56}]} {}_3F_2 \left(\begin{matrix} a_{15}, a_{25}, a_{123456} \\ a_{1235}, a_{1245} \end{matrix}; 1 \right), \quad (6.51)$$

where ${}_3F_2$ is the generalized hypergeometric function [54, 55]. A useful corollary of eq. (6.51) is presented in appendix C. The resulting one-dimensional integrals can be solved by the method below.

Integrals with two different arguments are difficult to solve. However in our case, such integrals only appear at higher orders in ϵ so we do not need to consider them.

One-dimensional Mellin-Barnes integrals. For integrals with no argument such as⁷

$$\int dz_1 \Gamma[-z_1, -1 + z_1, 1 - z_1, -1 + z_1] \psi(z_1) \psi(-2 - z_1), \quad (6.52)$$

we evaluate them numerically using `Mathematica` and apply the PSLQ algorithm [56, 57] to fit them to a basis of constants which consists of all possible products of

$$\{1, \gamma_E, \pi^2, \zeta_3, \zeta_5\} \quad (6.53)$$

up to a transcendental weight of five. The results turn out to only require constants to weight four. Typically, 50–70 digits of the numerical result are sufficient to obtain the correct answer, which we verify with 100 more digits.

The integrals with one argument typically have the form

$$\int dz_1 X^{z_1} \Gamma[-z_1, -1 + z_1, 1 - z_1, -1 + z_1], \quad X = \frac{X_1}{X_2}, \quad X_1, X_2 \in \{S, T, U\}. \quad (6.54)$$

Various combinations of X_1, X_2 appear since the template integrals contain them. We obtain the series expansion of eq. (6.54) by taking the residue of the left poles or the right poles. By adjusting which poles we consider, we can choose the series in terms of either $T/S, U/S$, or T/U :⁸

$$\int dz_1 X^{z_1} \Gamma[-z_1, -1 + z_1, 1 - z_1, -1 + z_1] \stackrel{\text{AC}}{=} \sum_{n_1=0}^4 \sum_{n_2=0}^{\infty} (\log X)^{n_1} X^{n_2}. \quad (6.55)$$

⁷After the analytic continuation described in subsection 6.3, we may have an expression where some of the poles merge. The following procedure can be applied also in these cases.

⁸Below eq. (3.2), it was stated that we use the normal equal sign for series representations when the hierarchy is obvious. However, here we have to introduce additional assumptions for $X = T/S, U/S, T/U$, since hierarchies between the positive Mandelstam variables have not been fixed up to this point. Thus we use the sign “ $\stackrel{\text{AC}}{=}$ ” in eq. (6.55), indicating that a certain analytic continuation should be performed in order to ensure $X \ll 1$.

Now we apply analytic continuation and obtain

$$X = T/S : \quad \log X \stackrel{\text{AC}}{=} h_0 - \log s + i\pi, \quad X \stackrel{\text{AC}}{=} -v \quad (6.56)$$

$$X = U/S : \quad \log X \stackrel{\text{AC}}{=} -h_1 - \log s + i\pi, \quad X \stackrel{\text{AC}}{=} -(1-v) \quad (6.57)$$

$$X = T/U : \quad \log X \stackrel{\text{AC}}{=} h_0 + h_1, \quad X \stackrel{\text{AC}}{=} \frac{v}{1-v}. \quad (6.58)$$

Recall that $h_1 = -\log(1-v)$. We fit the series with HPL and express the result in terms of h_N . In the case of (6.56), the series in v is directly fit to h_N . In the case of (6.57), we first fit the series to HPL with the argument of $(1-v)$ and then express them in terms of h_N . In the case of (6.58), we first fit the series to HPL with the argument of $v/(1-v)$ and then express them in terms of h_N . Taking into account that $0 \leq v \leq 1$ and the branch cut of $h_{N>0}$ lies on the real axis of $v > 1$, we never cross the branch cut in the above procedure. The information of the branch cut is encoded in the analytic continuation of $\log X$.

We already cover all of the combinations of $X = X_1/X_2$, so the calculation of the crossed diagrams can be done with the same procedure. For our sample integral (6.1), there are about 50 one-dimensional Mellin-Barnes integrals which are treated in this way.

6.5 Combining the results

Summing the contributions from all the relevant regions, we obtain for our sample integral

$$I = \frac{i\pi^3 e^{-2\epsilon\gamma_E}}{m_t s^{\frac{5}{2}} \sqrt{v} \sqrt{1-v}} \left(\frac{1}{\epsilon} - 2 \log(m_t^2) - 10 \log 2 \right) + \frac{e^{2i\pi\epsilon} e^{-2\epsilon\gamma_E}}{s^{3+2\epsilon}} \sum_{i_1=-1}^0 \sum_{i_2=0}^{4+i_1} \frac{d_{i_1, i_2}}{v(1-v)} \epsilon^{i_1} \log^{i_2}(m_t) + \mathcal{O}(m_t, \epsilon) \quad (6.59)$$

where

$$\begin{aligned} d_{-1,3} &= -\frac{4}{3}, & d_{-1,2} &= h_0(4v+2) + h_1(4v-6) + 6i\pi, \\ d_{-1,1} &= -h_0^2 + 2h_0h_1 + 8h_0v + 2i\pi h_0(2v-1) - h_1^2 + 8h_1(v-1) + 2i\pi h_1(2v-1) - \frac{10\pi^2}{3} + 8i\pi, \\ d_{-1,0} &= -\frac{1}{2}i\pi h_0^2 + 4h_0h_1 + i\pi h_0h_1 - \frac{1}{2}i\pi h_1^2 - 8i\pi + 4i\pi h_1v \\ &\quad + \frac{1}{6}h_0^3(1-2v) + \frac{1}{3}\pi^2 h_0(5-8v) + h_1(8-8v) + \pi^2 h_1 \left(1 - \frac{8v}{3}\right) - \frac{4i\pi^3}{3} \\ &\quad + h_0^2 h_1 \left(\frac{1}{2} - v\right) + h_0 h_1^2 \left(\frac{1}{2} - v\right) + 4i\pi h_0(v-1) - 8h_0v + \frac{1}{6}h_1^3(1-2v), \\ d_{0,4} &= -\frac{10}{3}, & d_{0,3} &= h_0(4-8v) + h_1(4-8v) - \frac{20i\pi}{3}, \\ d_{0,2} &= -h_0^2 + 6h_0h_1 - 2i\pi h_0(6v+1) - h_1^2 - 2i\pi h_1(6v-7) + \frac{47\pi^2}{3}, \\ d_{0,1} &= -i\pi h_0^2 + 16h_0h_1 - 16i\pi h_0 + 16i\pi h_1 - 32i\pi + \pi^2 h_0 \left(2v - \frac{7}{3}\right) \\ &\quad + \frac{1}{3}h_0^3(-2v-1) - 2i\pi h_0h_1 - i\pi h_1^2 - i\pi^3 + 16\pi^2 + \pi^2 h_1 \left(2v + \frac{1}{3}\right) \\ &\quad + h_0^2 h_1(-2v-1) + h_0 h_1^2(3-2v) - 32h_0v + h_1^3 \left(1 - \frac{2v}{3}\right) - 32h_1(v-1) + 6\zeta_3, \end{aligned}$$

$$\begin{aligned}
 d_{0,0} = & 8h_0h_1^2 - 32h_0h_1 - 12i\pi h_1h_2 + h_1h_3(8-16v) + h_4(60-48v) - 16\pi^2 + \frac{22}{3}\pi^2h_2(1-2v) \\
 & + h_0^2h_1(8-8v) + h_0^2h_2(10-8v) + \frac{11}{12}\pi^2h_1^2(1-4v) + h_0h_1^3\left(\frac{1}{2}-2v\right) + \frac{1}{8}h_1^4(5-4v) \\
 & + h_0^2h_1^2\left(-v-\frac{1}{4}\right) + 16h_0h_2(v-2) - \frac{2}{3}\pi^2h_1(v-13) - 16i\pi h_0(v-2) - \frac{8}{3}h_1^3(v-1) \\
 & - \frac{8h_0^3v}{3} - 8i\pi h_0^2(v-1) + 8i\pi h_1^2v + 8i\pi h_0h_2(v+1) - \frac{2}{3}\pi^2h_0(v+12) - 16i\pi h_1(v+1) \\
 & - \frac{1}{6}i\pi h_0^3(2v-11) + \frac{1}{6}i\pi^3h_1(2v-7) + \pi^4\left(\frac{151v}{90} - \frac{19}{9}\right) + h_0^3h_1\left(2v-\frac{3}{2}\right) - 8i\pi h_0h_1(2v-1) \\
 & - \frac{1}{2}i\pi h_0^2h_1(2v+1) - \frac{1}{6}i\pi h_1^3(2v+9) - 4i\pi h_3(2v+5) + \frac{11}{12}\pi^2h_0^2(4v-3) + 8h_0h_3(4v-5) \\
 & + \frac{1}{8}h_0^4(4v+1) + \frac{1}{2}i\pi h_0h_1^2(6v-1) - \frac{2}{3}i\pi^3(4v+1) + \frac{1}{6}i\pi^3h_0(2v+17) - 16h_3(v-2) \\
 & + h_0h_1h_2(16v-8) + \frac{1}{6}\pi^2h_0h_1(44v-31) + h_2^2(14v-7) + 16i\pi h_2(2v-1) \\
 & + h_0(h_{21}(20-40v) + 64v) + 4h_1(4h_{21}v + h_{21} + 16(v-1)) - 4i\pi(h_{21}(2v-7) - 16) \\
 & + h_0(-30v-3)\zeta_3 + h_1(21-30v)\zeta_3 - 2(8h_{21}(v+1) + 6h_{211}(4v+1) + 7h_{22}(2v-1)) \\
 & + 16v\zeta_3 + i\pi(8v-61)\zeta_3. \tag{6.60}
 \end{aligned}$$

This result is consistent with ref. [31] after a proper analytic continuation.

One remarkable feature of the result is that it contains terms proportional to $1/m_t$. Higher m_t -order correction also contain odd-power terms. These odd-power terms come from Region 2 and 3:

$$\begin{aligned}
 & \lim_{\epsilon, \delta_7, \delta_6, \delta_5, \delta_4, \delta_3, \delta_2, \delta_1 \rightarrow 0} \left(\mathcal{T}^{(2)} + \mathcal{T}^{(3)} \right) \times e^{2\epsilon\gamma_E} \\
 & = \frac{i\pi^3}{m_t s^{\frac{5}{2}} \sqrt{v} \sqrt{1-v}} \left(\frac{1}{\epsilon} - 2 \log(m_t^2) - 10 \log 2 \right) + m_t^0 \Delta(1/\delta_j) + \mathcal{O}(\epsilon, m_t), \tag{6.61}
 \end{aligned}$$

where $\Delta(1/\delta_j)$ has poles in δ_j , and these poles are cancelled by the contributions from the other regions.

6.6 Other master integrals

Seven-line integrals. After the IBP-reduction described in ref. [34], we find that there are four more diagrams which have seven internal lines. We consider $J_{2,1,1,1,1,1,1}^{\text{NPL1}}$, $J_{1,1,2,1,1,1,1}^{\text{NPL1}}$, $J_{1,1,1,2,1,1,1}^{\text{NPL1}}$, $J_{1,1,1,1,1,2,1}^{\text{NPL1}}$ which are used to calculate the Higgs pair production cross section. (For the detail, see ref. [34].) These integrals can be expressed by the proper shifts of δ_j : $\delta_1 \rightarrow \delta_1 + 1$ for $J_{2,1,1,1,1,1,1}^{\text{NPL1}}$, for example.

Six-line integrals. We can use the method described in subsection 4.4 to compute the integral with fewer lines. For example, $J_{1,1,1,1,1,1,0}^{\text{NPL1}}$ is obtained by shifting $\delta_7 \rightarrow \delta_7 - 1$ and repeat the same procedure described above. Note that the analytic continuation of δ_j may change due to the shift. For example, the template integral of Region 12 (6.37) can be regularized with $\delta_{i>0} = 0$, whereas $J_{1,1,1,1,1,1,0}^{\text{NPL1}}$ requires a non-zero δ_7 to regularize that template integral.

7 Summary

Asymptotic expansion is useful to extract information from multi-scale Feynman integrals, which are difficult to solve exactly. The method of regions plays an essential role in this extraction. The crucial part of the method of regions is to reveal the relevant regions correctly, and a naive application of the conventional method fails in the case of non-planar integrals for which the second Symanzik polynomial does not have a definite sign. We solve this problem by performing an analytic continuation of the Mandelstam variables such that the second Symanzik polynomial is positive definite.

We show the applicability of the method of regions by the explicit calculation of the master integrals of the Higgs pair production cross section at two-loop order, in the high energy limit. It is straightforward to extend our calculation to other four-point two-loop integral which satisfy $q_i^2 \ll m^2 \ll S, T, U$ where q_i are the external momenta and m is the mass of the internal lines. We anticipate that our idea to make the second Symanzik polynomial positive definite works in more general cases.

In addition to solving the issue of the sign of the second Symanzik polynomial, we formulate the procedure of the calculation in a systematic way. The contribution from each region is expressed in terms of Mellin-Barnes integrals, and a way to solve them is presented. The procedure presented here to solve the Mellin-Barnes integrals beyond the Barnes lemmas is not applicable to the general case, although it is sufficient to solve our master integrals completely. The automatization of this part is a future project.

As a by-product of introducing the positive Mandelstam variables, it becomes easier to obtain the crossed integrals, since the crossing of the positive Mandelstam variables does not cross any branch cut.

We compute the first few terms of the series in m_t , and the higher order terms can be obtained by the use of the m_t -differential equations. In this sense, our results can be considered as the boundary conditions of the differential equations with respect to m_t .

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A Mellin-Barnes integrals

Here we explain elementary aspects of the Mellin-Barnes integrals. The basic tool is the identity

$$\frac{1}{(A+B)^\lambda} = \int_C \frac{dz}{2\pi i} \frac{B^z}{A^{\lambda+z}} \frac{\Gamma[-z, \lambda+z]}{\Gamma(\lambda)}, \quad (\text{A.1})$$

where the integration contour C satisfies the following three properties:

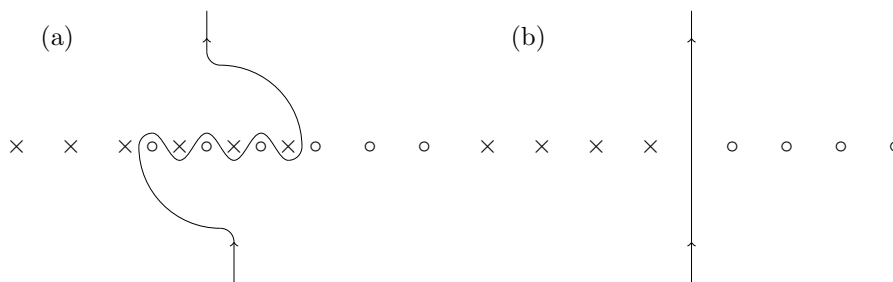


Figure 6. A circle (o) represents the right pole and a cross (x) represents the left pole. (a) An example of the contour in eq. (A.1) with $\lambda = -5/2$. (b) An example of the contour in eq. (A.2) with $\lambda = 2, c_0 = 1$.

- o C runs from $-i\infty$ to $+i\infty$.
- o The poles of $\Gamma(-z)$ lie on the right side of C . (We call them right poles.)
- o The poles of $\Gamma(\lambda + z)$ lie on the left side of C . (We call them left poles.)

The separation of the left poles and right poles is possible if $\lambda \notin \{0, -1, -2, -3, \dots\}$. For example, when $\lambda = -7/2$, some of the left poles lie on the right side of the left-most right pole, but there exists a contour which separates the left poles and the right poles in the proper manner. [See figure 6(a).] In our calculation, λ is a combination of ϵ and δ_j and integration variables of other Mellin-Barnes integrals, and A, B are combinations of the positive Mandelstam variables and the alpha parameters. In particular, $A > 0$ and $B > 0$ are always satisfied. We choose δ_j to ensure the condition $\lambda \notin \{0, -1, -2, -3, \dots\}$.⁹ When $\text{Re}(\lambda) > 0$, it is possible to set C as a straight line along the imaginary axis [See figure 6(b).]

$$\frac{1}{(A+B)^\lambda} = \int_{-c_0-i\infty}^{-c_0+i\infty} \frac{dz}{2\pi i} \frac{B^z}{A^{\lambda+z}} \frac{\Gamma[-z, \lambda+z]}{\Gamma(\lambda)} \quad 0 < c_0 < \text{Re}(\lambda). \quad (\text{A.2})$$

The integrand converges rapidly to zero at $\text{Im}(z) = \pm\infty$ rapidly so that we can shift the end points of the integration.

It is sometimes difficult to fix the contour C immediately when the Mellin-Barnes representation is introduced because additional convergence conditions concerning the α_i -integration should be taken into account. For example, the relations

$$\int_0^\infty d\alpha \alpha^a e^{-A\alpha} \stackrel{\text{AC}}{=} \Gamma(1+a) A^{-1-a}, \quad (\text{A.3})$$

$$\int_0^\infty d\alpha \alpha^a (A+B\alpha)^b \stackrel{\text{AC}}{=} A^{1+a+b} B^{-1-a} \frac{\Gamma[1+a, -1-a-b]}{\Gamma(-b)}, \quad (\text{A.4})$$

require $\text{Re}(a) > -1$ and $\text{Re}(a+b) < -1$, and if a or b contains the Mellin-Barnes integration variable z , C should be chosen accordingly. This kind of convergence condition appears

⁹If some of the left poles and right poles merge for any choice of δ_j , it is necessary to compensate the contributions of the merged poles by adding or subtracting the residue of the poles. However, it turns out that the merger of poles does not happen in our cases.

every time an integration is performed, and it can happen that an analytic continuation of the regularization parameters is necessary between two successive integrations. Therefore it is necessary to deform C smoothly in each step of the integration assuming a flexible contour like figure 6 (a) in order to ensure convergence. Note that after the integration, the analytic continuation of Γ -functions in the left hand sides of eqs. (A.3), (A.4) is possible.

After the integration with respect to all α_i is solved, we deform C to a straight line by tuning δ_j to separate the right poles and the left poles as in figure 6 (b). We obtain an expression which is valid for certain values of δ_j . In order to take the limit $\delta_j \rightarrow 0$, it is necessary to apply a final analytic continuation of δ_j from their initial values into a neighborhood around 0. This time, we allow poles to cross C . For this procedure, we use the package `MB.m` [51].

B Template integrals for the two-loop master integrals of Higgs pair production

The results for the template integrals listed here can be found in the supplementary file attached to this paper, in a computer-readable format.

Template integrals of NPL2. The template integrals of I^{NPL2} are

$$\mathcal{T}^{(2)} = \frac{1}{2} (m_t^2)^{-(1+4\epsilon+\delta_{112344556})/2} S^{-(3+\delta_{23677})/2} T^{-(1+\delta_{236})/2} U^{-(1+\delta_{236})/2} \times \frac{\Gamma[\delta_{056}, \frac{1+\delta_{236}}{2}, \frac{1+\delta_{236}}{2}, \frac{1+\delta_{236}}{2}, \frac{1+\delta_{23556}}{2}, \frac{1+\delta_{001123556}}{2}, \frac{1+\delta_{002344556}}{2}, \frac{1+\delta_{0000112344556}}{2}]}{\Gamma[\delta_1+1, \delta_2+1, \delta_3+1, \delta_4+1, \delta_5+1, \delta_6+1, \delta_{0014556}+1, \frac{1-\delta_{00236}}{2}]} \quad (\text{B.1})$$

$$\mathcal{T}^{(3)} = \frac{1}{2} (m_t^2)^{-(1+4\epsilon+\delta_{122334677})/2} S^{-(3+\delta_{14556})/2} T^{-(1+\delta_{146})/2} U^{-(1+\delta_{146})/2} \times \frac{\Gamma[\delta_{067}, \frac{1+\delta_{146}}{2}, \frac{1+\delta_{146}}{2}, \frac{1+\delta_{146}}{2}, \frac{1+\delta_{14677}}{2}, \frac{1+\delta_{001224677}}{2}, \frac{1+\delta_{001334677}}{2}, \frac{1+\delta_{0000122334677}}{2}]}{\Gamma[\delta_1+1, \delta_2+1, \delta_3+1, \delta_4+1, \delta_6+1, \delta_7+1, \delta_{0023677}+1, \frac{1-\delta_{00146}}{2}]} \quad (\text{B.2})$$

$$\mathcal{T}^{(4)} = \int dz_1 \frac{(m_t^2)^{-\delta_{001234}} \Gamma[\delta_{012}, \delta_{034}, z_{\bar{1}} + \delta_{356} - 1, z_{\bar{1}} + \delta_{167} - 1, z_{\bar{1}}, z_1 + \delta_{25}, z_1 + \delta_6 + 1, z_1 + \delta_{47}]}{S^{2+\delta_{57}} T^{1+z_1+\delta_6} U^{z_{\bar{1}}} \Gamma[\delta_1+1, \delta_2+1, \delta_3+1, \delta_4+1, \delta_6+1, \delta_{12567} - 1, \delta_{34567} - 1]} \quad (\text{B.3})$$

$$\mathcal{T}^{(5)} = \int dz_1 \frac{\Gamma[\delta_{037}, \delta_{001237}, \delta_{47}, z_{\bar{1}} + \delta_{0267} + 1, z_{\bar{1}}, z_1 + \delta_{146} - 1, z_1 + \delta_{0456} - 1, z_1 + \delta_{067}]}{(m_t^2)^{\delta_{001237}} S^{2+\delta_{45}} T^{1+\delta_6} \Gamma[\delta_1+1, \delta_2+1, \delta_3+1, \delta_{0456} - 1, \delta_7+1, \delta_{00123477}, z_1 + \delta_{046}]} \quad (\text{B.4})$$

$$\mathcal{T}^{(6)} = \int dz_1 \frac{\Gamma[\delta_{35}, \delta_{045}, \delta_{001245}, z_{\bar{1}} + \delta_{0156} + 1, z_{\bar{1}}, z_1 + \delta_{236} - 1, z_1 + \delta_{056}, z_1 + \delta_{0367} - 1]}{(m_t^2)^{\delta_{001245}} S^{2+\delta_{37}} U^{1+\delta_6} \Gamma[\delta_1+1, \delta_2+1, \delta_4+1, \delta_5+1, \delta_{00123455}, \delta_{0367} - 1, z_1 + \delta_{036}]} \quad (\text{B.5})$$

$$\mathcal{T}^{(7)} = \int dz_1 \frac{\Gamma[\delta_{015}, \delta_{25}, \delta_{001345}, z_{\bar{1}} + \delta_{0456} + 1, z_{\bar{1}}, z_1 + \delta_{236} - 1, z_1 + \delta_{056}, z_1 + \delta_{0267} - 1]}{(m_t^2)^{\delta_{001345}} S^{2+\delta_{27}} T^{1+\delta_6} \Gamma[\delta_1+1, \delta_3+1, \delta_4+1, \delta_5+1, \delta_{00123455}, \delta_{0267} - 1, z_1 + \delta_{026}]} \quad (\text{B.6})$$

$$\mathcal{T}^{(8)} = \frac{\Gamma[\delta_{056}, \delta_{236} - 1, \delta_{01356}, \delta_{001456}, \delta_{02456}, \delta_{067}]}{(m_t^2)^{\delta_{001456}} S^{1+\delta_7} T^{1+\delta_3} U^{1+\delta_2} \Gamma[\delta_1+1, \delta_4+1, \delta_{06} + 1, \delta_6+1, \delta_{0012345566}, \delta_{0237} - 1]} \quad (\text{B.7})$$

$$\mathcal{T}^{(9)} = \int dz_1 \frac{\Gamma[\delta_{\bar{1}7}, \delta_{027}, \delta_{002347}, z_{\bar{1}} + \delta_{0367} + 1, z_{\bar{1}}, z_1 + \delta_{\bar{1}46} - 1, z_1 + \delta_{\bar{0}156} - 1, z_1 + \delta_{\bar{0}67}]}{(m_t^2)^{\delta_{002347}} S^{2+\delta_{15}} U^{1+\delta_6} \Gamma[\delta_2 + 1, \delta_3 + 1, \delta_4 + 1, \delta_{\bar{0}156} - 1, \delta_7 + 1, \delta_{00123477}, z_1 + \delta_{\bar{0}16}]} \quad (\text{B.8})$$

$$\mathcal{T}^{(10)} = \frac{\Gamma[\delta_{\bar{0}56}, \delta_{\bar{1}46} - 1, \delta_{\bar{0}67}, \delta_{\bar{0}1367}, \delta_{002367}, \delta_{02467}]}{(m_t^2)^{\delta_{002367}} S^{1+\delta_5} T^{1+\delta_1} U^{1+\delta_4} \Gamma[\delta_2 + 1, \delta_3 + 1, \delta_{\bar{0}145} - 1, \delta_{\bar{0}6} + 1, \delta_6 + 1, \delta_{0012346677}]} \quad (\text{B.9})$$

$$\mathcal{T}^{(11)} = \int dz_1 dz_2 dz_3 dz_4 \frac{\Gamma[\delta_{012}, z_{\bar{1}}, z_{\bar{2}}, z_{12} + \delta_6 + 1, z_{\bar{3}}, z_{13} + \delta_2 + 1, z_{23} + \delta_3 + 1, z_{\bar{1}23} + \delta_{\bar{0}3456} - 2]}{(m_t^2)^{\delta_{012}} S^{3+z_{12}+\delta_{034567}} T^{z_2} U^{z_1} \Gamma[\delta_1 + 1, \delta_2 + 1, \delta_3 + 1, \delta_4 + 1]} \\ \times \frac{\Gamma[z_{\bar{1}34} + \delta_{\bar{0}134567} - 2, z_{\bar{4}}, z_{34} + \delta_5 + 1, z_{1234} + \delta_{034567} + 3, z_{\bar{2}34} + \delta_{\bar{0}3567} - 2]}{\Gamma[\delta_5 + 1, \delta_6 + 1, \delta_{\bar{0}034567} - 1, \delta_7 + 1, z_{\bar{4}} + \delta_{\bar{0}1234567} - 1]} \quad (\text{B.10})$$

$$\mathcal{T}^{(12)} = \int dz_1 dz_2 dz_3 dz_4 \frac{\Gamma[\delta_{034}, z_{\bar{1}}, z_{\bar{2}}, z_{12} + \delta_6 + 1, z_{\bar{3}}, z_{13} + \delta_3 + 1, z_{23} + \delta_2 + 1, z_{\bar{1}23} + \delta_{\bar{0}1256} - 2]}{(m_t^2)^{\delta_{034}} S^{3+z_{12}+\delta_{012567}} T^{z_1} U^{z_2} \Gamma[\delta_1 + 1, \delta_2 + 1, \delta_3 + 1, \delta_4 + 1]} \\ \times \frac{\Gamma[z_{\bar{1}34} + \delta_{\bar{0}124567} - 2, z_{\bar{4}}, z_{34} + \delta_5 + 1, z_{1234} + \delta_{012567} + 3, z_{\bar{2}34} + \delta_{\bar{0}2567} - 2]}{\Gamma[\delta_5 + 1, \delta_6 + 1, \delta_{\bar{0}012567} - 1, \delta_7 + 1, z_{\bar{4}} + \delta_{\bar{0}1234567} - 1]} \quad (\text{B.11})$$

Template integrals of PL1. The template integrals of $I^{\text{PL1}} = J_{1,1,1,1,1,1}^{\text{PL1}}$ defined in eq. (5.1) are

$$\mathcal{T}^{(2)} = \int dz_1 \frac{\Gamma[\delta_{012}, \delta_{045}, z_{\bar{1}} + \delta_{137} - 1, z_{\bar{1}} + \delta_{467} - 1, z_{\bar{1}}, z_1 + \delta_2 + 1, z_1 + \delta_5 + 1, z_1 + \delta_7 + 1]}{(m_t^2)^{\delta_{001245}} S^{3+z_1+\delta_{367}} T^{z_1} \Gamma[\delta_1 + 1, \delta_2 + 1, \delta_4 + 1, \delta_5 + 1, \delta_{1237}, \delta_{4567}, \delta_7 + 1]} \quad (\text{B.12})$$

$$\mathcal{T}^{(3)} = \frac{\Gamma[\delta_{012}, \delta_{13}, \delta_{46}, \delta_{056}, \delta_{27}, \delta_{57}]}{(m_t^2)^{\delta_{001256}} S^{2+\delta_{34}} T^{1+\delta_7} \Gamma[\delta_1 + 1, \delta_2 + 1, \delta_5 + 1, \delta_6 + 1, \delta_{1237}, \delta_{4567}]} \quad (\text{B.13})$$

$$\mathcal{T}^{(4)} = \int dz_1 dz_2 \frac{\Gamma[\delta_{067}, z_{\bar{1}}, z_1 + \delta_{\bar{0}345} - 1, z_1 + \delta_{57}, z_{\bar{2}} + \delta_{017}, z_{\bar{1}2} + \delta_{001567}, z_{\bar{2}}, z_2 + \delta_{\bar{0}7} + 1]}{(m_t^2)^{\delta_{001267}} S^{2+\delta_{34}} T^{1+\delta_5} \Gamma[\delta_1 + 1, \delta_2 + 1, \delta_{\bar{0}345} - 1, \delta_6 + 1, \delta_7 + 1, z_1 + \delta_{\bar{0}35}]} \\ \times \frac{\Gamma[z_{12} + \delta_{25}, z_{12} + \delta_{\bar{0}35}]}{\Gamma[z_{\bar{2}} + \delta_{001677}, z_{12} + \delta_{\bar{0}5} + 1]} \quad (\text{B.14})$$

$$\mathcal{T}^{(5)} = \int dz_1 dz_2 \frac{\Gamma[\delta_{017}, z_{\bar{1}}, z_1 + \delta_{\bar{0}234} - 1, z_1 + \delta_{27}, z_{\bar{2}} + \delta_{067}, z_{\bar{1}2} + \delta_{001267}, z_{\bar{2}}, z_2 + \delta_{\bar{0}7} + 1]}{(m_t^2)^{\delta_{001567}} S^{2+\delta_{34}} T^{1+\delta_2} \Gamma[\delta_1 + 1, \delta_{\bar{0}234} - 1, \delta_5 + 1, \delta_6 + 1, \delta_7 + 1, z_1 + \delta_{\bar{0}24}]} \\ \times \frac{\Gamma[z_{12} + \delta_{\bar{0}24}, z_{12} + \delta_{\bar{2}5}]}{\Gamma[z_{\bar{2}} + \delta_{001677}, z_{12} + \delta_{\bar{0}2} + 1]} \quad (\text{B.15})$$

$$\mathcal{T}^{(6)} = \frac{\Gamma[\delta_{13}, \delta_{023}, \delta_{045}, \delta_{46}, \delta_{27}, \delta_{57}]}{(m_t^2)^{\delta_{002345}} S^{2+\delta_{16}} T^{1+\delta_7} \Gamma[\delta_2 + 1, \delta_3 + 1, \delta_4 + 1, \delta_5 + 1, \delta_{1237}, \delta_{4567}]} \quad (\text{B.16})$$

$$\mathcal{T}^{(7)} = \int dz_1 dz_2 \frac{\Gamma[\delta_{047}, z_{\bar{1}}, z_1 + \delta_{\bar{0}156} - 1, z_1 + \delta_{57}, z_{\bar{2}} + \delta_{037}, z_{\bar{1}2} + \delta_{003457}, z_{\bar{2}}, z_2 + \delta_{\bar{0}7} + 1]}{(m_t^2)^{\delta_{002347}} S^{2+\delta_{16}} T^{1+\delta_5} \Gamma[\delta_2 + 1, \delta_3 + 1, \delta_4 + 1, \delta_{\bar{0}156} - 1, \delta_7 + 1, z_1 + \delta_{\bar{0}15}]} \\ \times \frac{\Gamma[z_{12} + \delta_{\bar{0}15}, z_{12} + \delta_{\bar{2}5}]}{\Gamma[z_{\bar{2}} + \delta_{003477}, z_{12} + \delta_{\bar{0}5} + 1]} \quad (\text{B.17})$$

$$\mathcal{T}^{(8)} = \int dz_1 \frac{\Gamma[\delta_{023}, \delta_{056}, z_{\bar{1}} + \delta_{137} - 1, z_{\bar{1}} + \delta_{467} - 1, z_{\bar{1}}, z_1 + \delta_2 + 1, z_1 + \delta_5 + 1, z_1 + \delta_7 + 1]}{(m_t^2)^{\delta_{002356}} S^{3+z_1+\delta_{147}} T^{z_1} \Gamma[\delta_2 + 1, \delta_3 + 1, \delta_5 + 1, \delta_6 + 1, \delta_{1237}, \delta_{4567}, \delta_7 + 1]} \quad (\text{B.18})$$

$$\mathcal{T}^{(9)} = \int dz_1 dz_2 \frac{\Gamma[\delta_{037}, z_{\bar{1}}, z_1 + \delta_{\bar{0}126} - 1, z_1 + \delta_{27}, z_{\bar{2}} + \delta_{047}, z_{\bar{1}2} + \delta_{002347}, z_{\bar{2}}, z_2 + \delta_{\bar{0}7} + 1]}{(m_t^2)^{\delta_{003457}} S^{2+\delta_{16}} T^{1+\delta_2} \Gamma[\delta_3 + 1, \delta_4 + 1, \delta_5 + 1, \delta_{\bar{0}126} - 1, \delta_7 + 1, z_1 + \delta_{\bar{0}26}]} \\ \times \frac{\Gamma[z_{12} + \delta_{25}, z_{12} + \delta_{\bar{0}26}]}{\Gamma[z_{\bar{2}} + \delta_{003477}, z_{12} + \delta_{\bar{0}2} + 1]} \quad (\text{B.19})$$

$$\begin{aligned} \mathcal{T}^{(10)} = & \int dz_1 dz_2 \frac{\Gamma[\delta_{012}, z_{\bar{1}}, z_1 + \delta_2 + 1, z_1 + \delta_5 + 1, z_{\bar{1}} + \delta_{\bar{0}\bar{5}\bar{6}\bar{7}} - 1, z_{\bar{2}}, z_2 + \delta_{1\bar{3}}, z_{12} + \delta_7 + 1]}{(m_t^2)^{\delta_{012}} S^{3+z_1+\delta_{034567}} T^{z_{\bar{1}}} \Gamma[\delta_1 + 1, \delta_2 + 1, \delta_4 + 1, \delta_5 + 1, \delta_6 + 1, \delta_{\bar{0}\bar{0}\bar{4}\bar{5}\bar{6}\bar{7}}]} \\ & \times \frac{\Gamma[z_{12} + \delta_{04567} + 2, z_{\bar{1}\bar{2}} + \delta_{\bar{0}\bar{4}\bar{5}\bar{7}} - 1]}{\Gamma[\delta_7 + 1, z_{12} + \delta_{1\bar{2}\bar{3}} + 1]} \end{aligned} \quad (\text{B.20})$$

$$\begin{aligned} \mathcal{T}^{(11)} = & \int dz_1 dz_2 \frac{\Gamma[\delta_{023}, z_{\bar{1}}, z_1 + \delta_2 + 1, z_1 + \delta_5 + 1, z_{\bar{1}} + \delta_{\bar{0}\bar{4}\bar{5}\bar{7}} - 1, z_{\bar{2}}, z_2 + \delta_{\bar{1}\bar{3}}, z_{12} + \delta_7 + 1]}{(m_t^2)^{\delta_{023}} S^{3+z_1+\delta_{014567}} T^{z_{\bar{1}}} \Gamma[\delta_2 + 1, \delta_3 + 1, \delta_4 + 1, \delta_5 + 1, \delta_6 + 1, \delta_{\bar{0}\bar{0}\bar{4}\bar{5}\bar{6}\bar{7}}]} \\ & \times \frac{\Gamma[z_{12} + \delta_{04567} + 2, z_{\bar{1}\bar{2}} + \delta_{\bar{0}\bar{5}\bar{6}\bar{7}} - 1]}{\Gamma[\delta_7 + 1, z_{12} + \delta_{\bar{1}\bar{2}\bar{3}} + 1]} \end{aligned} \quad (\text{B.21})$$

$$\begin{aligned} \mathcal{T}^{(12)} = & \int dz_1 dz_2 \frac{\Gamma[\delta_{045}, z_{\bar{1}}, z_1 + \delta_2 + 1, z_1 + \delta_5 + 1, z_{\bar{1}} + \delta_{\bar{0}\bar{2}\bar{3}\bar{7}} - 1, z_{\bar{2}}, z_2 + \delta_{\bar{4}\bar{6}}, z_{12} + \delta_7 + 1]}{(m_t^2)^{\delta_{045}} S^{3+z_1+\delta_{012367}} T^{z_{\bar{1}}} \Gamma[\delta_1 + 1, \delta_2 + 1, \delta_3 + 1, \delta_4 + 1, \delta_5 + 1, \delta_{\bar{0}\bar{0}\bar{1}\bar{2}\bar{3}\bar{7}}]} \\ & \times \frac{\Gamma[z_{12} + \delta_{01237} + 2, z_{\bar{1}\bar{2}} + \delta_{\bar{0}\bar{1}\bar{2}\bar{7}} - 1]}{\Gamma[\delta_7 + 1, z_{12} + \delta_{\bar{4}\bar{5}\bar{6}} + 1]} \end{aligned} \quad (\text{B.22})$$

$$\begin{aligned} \mathcal{T}^{(13)} = & \int dz_1 dz_2 \frac{\Gamma[\delta_{056}, z_{\bar{1}}, z_1 + \delta_2 + 1, z_1 + \delta_5 + 1, z_{\bar{1}} + \delta_{\bar{0}\bar{1}\bar{2}\bar{7}} - 1, z_{\bar{2}}, z_2 + \delta_{\bar{4}\bar{6}}, z_{12} + \delta_7 + 1]}{(m_t^2)^{\delta_{056}} S^{3+z_1+\delta_{012347}} T^{z_{\bar{1}}} \Gamma[\delta_1 + 1, \delta_2 + 1, \delta_3 + 1, \delta_5 + 1, \delta_6 + 1, \delta_{\bar{0}\bar{0}\bar{1}\bar{2}\bar{3}\bar{7}}]} \\ & \times \frac{\Gamma[z_{12} + \delta_{01237} + 2, z_{\bar{1}\bar{2}} + \delta_{\bar{0}\bar{2}\bar{3}\bar{7}} - 1]}{\Gamma[\delta_7 + 1, z_{12} + \delta_{\bar{4}\bar{5}\bar{6}} + 1]}. \end{aligned} \quad (\text{B.23})$$

Template integrals of PL2. The template integrals of $I^{\text{PL2}} = J_{1,1,1,1,1,1,1}^{\text{PL2}}$ defined in eq. (5.2) are

$$\begin{aligned} \mathcal{T}^{(2)} = & \int dz_1 dz_2 \frac{\Gamma[\delta_{067}, z_{\bar{1}} + \delta_{0156} + 1, z_{\bar{1}}, z_1 + \delta_{\bar{0}\bar{3}\bar{4}\bar{5}} - 1, z_1 + \delta_{\bar{0}\bar{5}\bar{6}}, z_{\bar{1}\bar{2}} + \delta_{001567}, z_{\bar{2}}, z_2 + \delta_6 + 1]}{(m_t^2)^{\delta_{001267}} S^{2+\delta_{34}} T^{1+\delta_5} \Gamma[\delta_1 + 1, \delta_2 + 1, \delta_{\bar{0}\bar{3}\bar{4}\bar{5}} - 1, \delta_6 + 1, \delta_7 + 1, z_{\bar{1}} + \delta_{0015667} + 1]} \\ & \times \frac{\Gamma[z_{12} + \delta_{2\bar{5}}, z_{12} + \delta_{\bar{0}\bar{3}\bar{5}}]}{\Gamma[z_1 + \delta_{\bar{0}\bar{3}\bar{5}}, z_{12} + \delta_{\bar{0}\bar{5}} + 1]} \end{aligned} \quad (\text{B.24})$$

$$\mathcal{T}^{(3)} = \int dz_1 \frac{\Gamma[\delta_{\bar{0}\bar{4}\bar{5}}, \delta_{\bar{0}\bar{5}\bar{6}}, \delta_{001567}, z_{\bar{1}} + \delta_{0126} + 1, z_{\bar{1}}, z_1 + \delta_{\bar{0}\bar{2}\bar{3}\bar{4}} - 1, z_1 + \delta_{\bar{2}\bar{5}}, z_1 + \delta_{02567}]}{(m_t^2)^{\delta_{001567}} S^{2+\delta_{34}} T^{1+\delta_2} \Gamma[\delta_1 + 1, \delta_{\bar{0}\bar{2}\bar{3}\bar{4}} - 1, \delta_{\bar{0}\bar{5}} + 1, \delta_5 + 1, \delta_7 + 1, \delta_{0015667} + 1, z_1 + \delta_{\bar{0}\bar{2}\bar{4}}]} \quad (\text{B.25})$$

$$\begin{aligned} \mathcal{T}^{(4)} = & \int dz_1 dz_2 \frac{\Gamma[\delta_{047}, z_{\bar{1}} + \delta_{0345} + 1, z_{\bar{1}}, z_1 + \delta_{\bar{0}\bar{4}\bar{5}}, z_1 + \delta_{\bar{0}\bar{1}\bar{5}\bar{6}} - 1, z_{\bar{1}\bar{2}} + \delta_{003457}, z_{\bar{2}}, z_2 + \delta_4 + 1]}{(m_t^2)^{\delta_{002347}} S^{2+\delta_{16}} T^{1+\delta_5} \Gamma[\delta_2 + 1, \delta_3 + 1, \delta_4 + 1, \delta_{\bar{0}\bar{1}\bar{5}\bar{6}} - 1, \delta_7 + 1, z_{\bar{1}} + \delta_{0034457} + 1]} \\ & \times \frac{\Gamma[z_{12} + \delta_{\bar{0}\bar{1}\bar{5}}, z_{12} + \delta_{\bar{2}\bar{5}}]}{\Gamma[z_1 + \delta_{\bar{0}\bar{1}\bar{5}}, z_{12} + \delta_{\bar{0}\bar{5}} + 1]} \end{aligned} \quad (\text{B.26})$$

$$\mathcal{T}^{(5)} = \int dz_1 \frac{\Gamma[\delta_{\bar{0}\bar{4}\bar{5}}, \delta_{\bar{0}\bar{5}\bar{6}}, \delta_{003457}, z_{\bar{1}} + \delta_{0234} + 1, z_{\bar{1}}, z_1 + \delta_{\bar{2}\bar{5}}, z_1 + \delta_{\bar{0}\bar{1}\bar{2}\bar{6}} - 1, z_1 + \delta_{02457}]}{(m_t^2)^{\delta_{003457}} S^{2+\delta_{16}} T^{1+\delta_2} \Gamma[\delta_3 + 1, \delta_{\bar{0}\bar{5}} + 1, \delta_5 + 1, \delta_{\bar{0}\bar{1}\bar{2}\bar{6}} - 1, \delta_7 + 1, \delta_{0034457} + 1, z_1 + \delta_{\bar{0}\bar{2}\bar{6}}]} \quad (\text{B.27})$$

$$\begin{aligned} \mathcal{T}^{(6)} = & \int dz_1 dz_2 \frac{\Gamma[\delta_{012}, z_{\bar{1}}, z_1 + \delta_2 + 1, z_1 + \delta_5 + 1, z_{\bar{1}} + \delta_{\bar{0}\bar{5}\bar{6}\bar{7}} - 1, z_{\bar{1}\bar{2}} + \delta_{\bar{0}\bar{1}\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}} - 2, z_{\bar{2}}, z_2 + \delta_6 + 1]}{(m_t^2)^{\delta_{012}} S^{3+z_1+\delta_{034567}} T^{z_{\bar{1}}} \Gamma[\delta_1 + 1, \delta_2 + 1, \delta_4 + 1, \delta_5 + 1, \delta_6 + 1, \delta_{\bar{0}\bar{0}\bar{4}\bar{5}\bar{6}\bar{7}}]} \\ & \times \frac{\Gamma[z_{12} + \delta_{04567} + 2, z_{\bar{2}} + \delta_{\bar{0}\bar{4}\bar{5}\bar{6}} - 1]}{\Gamma[\delta_7 + 1, z_{\bar{2}} + \delta_{\bar{0}\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}} - 1]} \end{aligned} \quad (\text{B.28})$$

$$\begin{aligned} \mathcal{T}^{(7)} = & \int dz_1 dz_2 \frac{\Gamma[\delta_{023}, z_{\bar{1}}, z_1 + \delta_2 + 1, z_1 + \delta_5 + 1, z_{\bar{1}} + \delta_{\bar{0}\bar{4}\bar{5}\bar{7}} - 1, z_{\bar{1}\bar{2}} + \delta_{\bar{0}\bar{1}\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}} - 2, z_{\bar{2}}, z_2 + \delta_4 + 1]}{(m_t^2)^{\delta_{023}} S^{3+z_1+\delta_{014567}} T^{z_{\bar{1}}} \Gamma[\delta_2 + 1, \delta_3 + 1, \delta_4 + 1, \delta_5 + 1, \delta_6 + 1, \delta_{\bar{0}\bar{0}\bar{4}\bar{5}\bar{6}\bar{7}}]} \\ & \times \frac{\Gamma[z_{12} + \delta_{04567} + 2, z_{\bar{2}} + \delta_{\bar{0}\bar{4}\bar{5}\bar{6}} - 1]}{\Gamma[\delta_7 + 1, z_{\bar{2}} + \delta_{\bar{0}\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}} - 1]} \end{aligned} \quad (\text{B.29})$$

$$\begin{aligned} \mathcal{T}^{(8)} = & \int dz_1 \frac{\Gamma[\delta_{07} - 1, z_{\bar{1}}, z_1 + \delta_2 + 1, z_1 + \delta_5 + 1, z_1 + \delta_{0123456} + 4, z_{\bar{1}} + \delta_{\bar{0}\bar{2}\bar{3}\bar{4}\bar{5}} - 2, z_{\bar{1}} + \delta_{\bar{0}\bar{1}\bar{2}\bar{5}\bar{6}} - 2]}{(m_t^2)^{-1+\delta_{07}} S^{4+z_1+\delta_{0123456}} T^{z_{\bar{1}}} \Gamma[\delta_2 + 1, \delta_{34} + 2, \delta_5 + 1, \delta_{\bar{0}\bar{0}\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}\bar{6}} - 2, \delta_{16} + 2, \delta_7 + 1]}. \end{aligned} \quad (\text{B.30})$$

C A corollary of the generalized Barnes lemma

The generalized Barnes lemma (6.51)

$$\int_C \frac{dz}{2\pi i} \frac{\Gamma[a_1-z, a_2-z, b_1+z, b_2+z, b_3+z]}{\Gamma(c+z)} = \frac{\Gamma[a_1+b_1, a_2+b_1, a_1+b_2, a_2+b_2, a_1+b_3, a_2+b_3]}{\Gamma[a_{12}+b_{13}, a_{12}+b_{23}, -b_3+c]} {}_3F_2 \left(\begin{matrix} a_1+b_3, a_2+b_3, a_{12}+b_{123}-c \\ a_{12}+b_{13}, a_{12}+b_{23} \end{matrix}; 1 \right), \quad (C.1)$$

contains the generalized hypergeometric function ${}_3F_2$. We try to reduce ${}_3F_2$ to a product of Γ -functions on a case-by-case basis, using the relations given in refs. [54, 55]. In this appendix we present a formula which is, to the knowledge of the author, not published. It yields a useful corollary of the generalized Barnes lemma (6.51).

Consider the case where

$$a_1 = n_1 - b_1, a_2 = n_2 - b_2, \quad n_1, n_2 \in \mathbb{N}, \quad (C.2)$$

where \mathbb{N} is the set of positive integers. Note that $n_1 \neq 0, n_2 \neq 0$ because otherwise the left pole and the right pole merge.

Substituting eq. (C.2) into eq. (C.1), the arguments of ${}_3F_2$ become

$${}_3F_2 \left(\begin{matrix} -b_1 + b_3 + n_1, -b_2 + b_3 + n_2, b_3 - c + n_1 + n_2 \\ -b_1 + b_3 + n_1 + n_2, -b_2 + b_3 + n_1 + n_2 \end{matrix}; 1 \right). \quad (C.3)$$

and here we express this type of ${}_3F_2$ in terms of the Γ -function only. To this end, consider

$$\begin{aligned} \mathcal{I} &= {}_3F_2 \left(\begin{matrix} x_1, x_2, x_3 \\ x_1 + n_1, x_2 + n_2 \end{matrix}; 1 \right) \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\Gamma[x_1+m, x_2+m, x_3+m, x_1+n_1, x_2+n_2]}{\Gamma[x_1+m+n_1, x_2+m+n_2, x_3, x_1, x_2]} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \left(\prod_{i_1=0}^{n_1-1} \frac{1}{x_1+m+i_1} \right) \left(\prod_{i_2=0}^{n_2-1} \frac{1}{x_2+m+i_2} \right) \frac{\Gamma[x_3+m, x_1+n_1, x_2+n_2]}{\Gamma[x_3, x_1, x_2]}, \quad (C.4) \end{aligned}$$

where x_1, x_2, x_3 can contain n_1, n_2 . The products are resolved by the partial fraction decomposition

$$\prod_{n=0}^j \frac{1}{x+n} = \sum_{n=0}^j \frac{(-1)^n}{n!(j-n)!} \frac{1}{x+n} \quad (C.5)$$

and we obtain

$$\mathcal{I} = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{i_1=0}^{n_1-1} \sum_{i_2=0}^{n_2-1} C_{i_1, i_2} \frac{1}{(x_1+m+i_1)(x_2+m+i_2)} \frac{\Gamma[x_3+m, x_1+n_1, x_2+n_2]}{\Gamma[x_3, x_1, x_2]} \quad (C.6)$$

$$C_{i_1, i_2} = \frac{(-1)^{i_1+i_2}}{i_1! i_2! (n_1-i_1-1)! (n_2-i_2-1)!}. \quad (C.7)$$

We can apply the partial fraction decomposition further and obtain

$$\mathcal{I} = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{i_1=0}^{n_1-1} \sum_{i_2=0}^{n_2-1} C_{i_1, i_2} \frac{\frac{1}{x_1+m+i_1} - \frac{1}{x_2+m+i_2}}{x_2 - x_1 + i_2 - i_1} \frac{\Gamma[x_3 + m, x_1 + n_1, x_2 + n_2]}{\Gamma[x_3, x_1, x_2]}. \quad (\text{C.8})$$

Here the equation in the case of $x_1 - x_2 \notin \mathbb{Z}$ is shown, but the case of $x_1 - x_2 \in \mathbb{Z}$ is similar. It is also possible to set $x_2 \rightarrow x_1 + n, n \in \mathbb{Z}$ at the end. The infinite sum of m in eq. (C.8) is now possible, and we obtain

$$\mathcal{I} = \sum_{i_1=0}^{n_1-1} \sum_{i_2=0}^{n_2-1} \frac{C_{i_1, i_2} \Gamma[x_1 + n_1, x_2 + n_2]}{(x_2 - x_1 + i_2 - i_1) \Gamma[x_1, x_2]} \left\{ \frac{\Gamma[x_1 + i_1, 1 - x_3]}{\Gamma[1 + x_1 + i_1 - x_3]} - \frac{\Gamma[x_2 + i_2, 1 - x_3]}{\Gamma[1 + x_2 + i_2 - x_3]} \right\}. \quad (\text{C.9})$$

Substituting the replacement $x_1 \rightarrow -b_2 + b_3 + n_2, x_2 \rightarrow -b_1 + b_3 + n_1, x_3 \rightarrow b_3 - c + n_1 + n_2$ into eq. (C.9), we obtain

$$\begin{aligned} & \int_C \frac{dz}{2\pi i} \frac{\Gamma[n_1 - b_1 - z, n_2 - b_2 - z, b_1 + z, b_2 + z, b_3 + z]}{\Gamma(c + z)} \\ &= \sum_{i_1=0}^{n_1-1} \sum_{i_2=0}^{n_2-1} \frac{(-1)^{i_1+i_2}}{b_1 - b_2 - n_1 + n_2 + i_1 - i_2} \left\{ \frac{\Gamma[-b_1 + b_3 + n_1 + i_2]}{\Gamma[1 - b_1 + c - n_2 + i_2]} - \frac{\Gamma[-b_2 + b_3 + n_2 + i_1]}{\Gamma[1 - b_2 + c - n_1 + i_1]} \right\} \\ & \quad \times \frac{\Gamma[n_1, n_2, -b_1 + b_2 + n_1, b_1 - b_2 + n_2, 1 - b_3 + c - n_1 - n_2]}{\Gamma[i_1 + 1, i_2 + 1, n_1 - i_1, n_2 - i_2, -b_3 + c]}. \end{aligned} \quad (\text{C.10})$$

In this expression, there are only finite sums, so they are evaluated in a straightforward way. Among the six variables in eq. (C.10), two of them (n_1, n_2) should be positive integers, but the other four (b_1, b_2, b_3, c) can take any value, provided the left poles and right poles do not merge.

Once the integral is expressed in terms of the Γ -function, we can easily compute derivatives of, or analytically continue the result.

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References

- [1] R. Bonciani, V. Del Duca, H. Frellesvig, J.M. Henn, F. Moriello and V.A. Smirnov, *Two-loop planar master integrals for Higgs \rightarrow 3 partons with full heavy-quark mass dependence*, *JHEP* **12** (2016) 096 [[arXiv:1609.06685](https://arxiv.org/abs/1609.06685)] [[INSPIRE](https://inspirehep.net/literature/1609066)].
- [2] S.P. Jones, M. Kerner and G. Luisoni, *Next-to-Leading-Order QCD Corrections to Higgs Boson Plus Jet Production with Full Top-Quark Mass Dependence*, *Phys. Rev. Lett.* **120** (2018) 162001 [[arXiv:1802.00349](https://arxiv.org/abs/1802.00349)] [[INSPIRE](https://inspirehep.net/literature/1802001)].
- [3] S. Borowka et al., *Higgs Boson Pair Production in Gluon Fusion at Next-to-Leading Order with Full Top-Quark Mass Dependence*, *Phys. Rev. Lett.* **117** (2016) 012001 [*Erratum ibid.* **117** (2016) 079901] [[arXiv:1604.06447](https://arxiv.org/abs/1604.06447)] [[INSPIRE](https://inspirehep.net/literature/1604064)].

- [4] S. Borowka et al., *Full top quark mass dependence in Higgs boson pair production at NLO*, *JHEP* **10** (2016) 107 [[arXiv:1608.04798](#)] [[INSPIRE](#)].
- [5] S. Borowka, G. Heinrich, S.P. Jones, M. Kerner, J. Schlenk and T. Zirke, *SecDec-3.0: numerical evaluation of multi-scale integrals beyond one loop*, *Comput. Phys. Commun.* **196** (2015) 470 [[arXiv:1502.06595](#)] [[INSPIRE](#)].
- [6] S. Borowka et al., *pySecDec: a toolbox for the numerical evaluation of multi-scale integrals*, *Comput. Phys. Commun.* **222** (2018) 313 [[arXiv:1703.09692](#)] [[INSPIRE](#)].
- [7] J. Baglio, F. Campanario, S. Glaus, M. Mühlleitner, M. Spira and J. Streicher, *Gluon fusion into Higgs pairs at NLO QCD and the top mass scheme*, [arXiv:1811.05692](#) [[INSPIRE](#)].
- [8] P. Mastrolia, M. Passera, A. Primo and U. Schubert, *Master integrals for the NNLO virtual corrections to μe scattering in QED: the planar graphs*, *JHEP* **11** (2017) 198 [[arXiv:1709.07435](#)] [[INSPIRE](#)].
- [9] S. Di Vita, S. Laporta, P. Mastrolia, A. Primo and U. Schubert, *Master integrals for the NNLO virtual corrections to μe scattering in QED: the non-planar graphs*, *JHEP* **09** (2018) 016 [[arXiv:1806.08241](#)] [[INSPIRE](#)].
- [10] L. Adams, E. Chaubey and S. Weinzierl, *Analytic results for the planar double box integral relevant to top-pair production with a closed top loop*, *JHEP* **10** (2018) 206 [[arXiv:1806.04981](#)] [[INSPIRE](#)].
- [11] M. Becchetti and R. Bonciani, *Two-Loop Master Integrals for the Planar QCD Massive Corrections to Di-photon and Di-jet Hadro-production*, *JHEP* **01** (2018) 048 [[arXiv:1712.02537](#)] [[INSPIRE](#)].
- [12] X. Xu and L.L. Yang, *Towards a new approximation for pair-production and associated-production of the Higgs boson*, *JHEP* **01** (2019) 211 [[arXiv:1810.12002](#)] [[INSPIRE](#)].
- [13] V.A. Smirnov, *Applied Asymptotic Expansions in Momenta and Masses*, *Springer Tracts Mod. Phys.* **177** (2002) 1.
- [14] R. Boughezal, F. Caola, K. Melnikov, F. Petriello and M. Schulze, *Higgs boson production in association with a jet at next-to-next-to-leading order in perturbative QCD*, *JHEP* **06** (2013) 072 [[arXiv:1302.6216](#)] [[INSPIRE](#)].
- [15] X. Chen, T. Gehrmann, E.W.N. Glover and M. Jaquier, *Precise QCD predictions for the production of Higgs + jet final states*, *Phys. Lett. B* **740** (2015) 147 [[arXiv:1408.5325](#)] [[INSPIRE](#)].
- [16] R. Boughezal, F. Caola, K. Melnikov, F. Petriello and M. Schulze, *Higgs boson production in association with a jet at next-to-next-to-leading order*, *Phys. Rev. Lett.* **115** (2015) 082003 [[arXiv:1504.07922](#)] [[INSPIRE](#)].
- [17] R. Boughezal, C. Focke, W. Giele, X. Liu and F. Petriello, *Higgs boson production in association with a jet at NNLO using jetiness subtraction*, *Phys. Lett. B* **748** (2015) 5 [[arXiv:1505.03893](#)] [[INSPIRE](#)].
- [18] F. Caola, K. Melnikov and M. Schulze, *Fiducial cross sections for Higgs boson production in association with a jet at next-to-next-to-leading order in QCD*, *Phys. Rev. D* **92** (2015) 074032 [[arXiv:1508.02684](#)] [[INSPIRE](#)].
- [19] X. Chen, J. Cruz-Martinez, T. Gehrmann, E.W.N. Glover and M. Jaquier, *NNLO QCD corrections to Higgs boson production at large transverse momentum*, *JHEP* **10** (2016) 066 [[arXiv:1607.08817](#)] [[INSPIRE](#)].

- [20] T. Neumann and C. Williams, *The Higgs boson at high p_T* , *Phys. Rev. D* **95** (2017) 014004 [[arXiv:1609.00367](#)] [[INSPIRE](#)].
- [21] D. de Florian et al., *Differential Higgs Boson Pair Production at Next-to-Next-to-Leading Order in QCD*, *JHEP* **09** (2016) 151 [[arXiv:1606.09519](#)] [[INSPIRE](#)].
- [22] D. de Florian and J. Mazzitelli, *Two-loop virtual corrections to Higgs pair production*, *Phys. Lett. B* **724** (2013) 306 [[arXiv:1305.5206](#)] [[INSPIRE](#)].
- [23] J. Grigo, J. Hoff, K. Melnikov and M. Steinhauser, *On the Higgs boson pair production at the LHC*, *Nucl. Phys. B* **875** (2013) 1 [[arXiv:1305.7340](#)] [[INSPIRE](#)].
- [24] D. de Florian and J. Mazzitelli, *Higgs Boson Pair Production at Next-to-Next-to-Leading Order in QCD*, *Phys. Rev. Lett.* **111** (2013) 201801 [[arXiv:1309.6594](#)] [[INSPIRE](#)].
- [25] F. Maltoni, E. Vryonidou and M. Zaro, *Top-quark mass effects in double and triple Higgs production in gluon-gluon fusion at NLO*, *JHEP* **11** (2014) 079 [[arXiv:1408.6542](#)] [[INSPIRE](#)].
- [26] J. Grigo, K. Melnikov and M. Steinhauser, *Virtual corrections to Higgs boson pair production in the large top quark mass limit*, *Nucl. Phys. B* **888** (2014) 17 [[arXiv:1408.2422](#)] [[INSPIRE](#)].
- [27] J. Grigo, J. Hoff and M. Steinhauser, *Higgs boson pair production: top quark mass effects at NLO and NNLO*, *Nucl. Phys. B* **900** (2015) 412 [[arXiv:1508.00909](#)] [[INSPIRE](#)].
- [28] G. Degrossi, P.P. Giardino and R. Gröber, *On the two-loop virtual QCD corrections to Higgs boson pair production in the Standard Model*, *Eur. Phys. J. C* **76** (2016) 411 [[arXiv:1603.00385](#)] [[INSPIRE](#)].
- [29] K. Melnikov, L. Tancredi and C. Wever, *Two-loop $gg \rightarrow Hg$ amplitude mediated by a nearly massless quark*, *JHEP* **11** (2016) 104 [[arXiv:1610.03747](#)] [[INSPIRE](#)].
- [30] K. Melnikov, L. Tancredi and C. Wever, *Two-loop amplitudes for $qg \rightarrow Hq$ and $q\bar{q} \rightarrow Hg$ mediated by a nearly massless quark*, *Phys. Rev. D* **95** (2017) 054012 [[arXiv:1702.00426](#)] [[INSPIRE](#)].
- [31] K. Kudashkin, K. Melnikov and C. Wever, *Two-loop amplitudes for processes $gg \rightarrow Hg, qg \rightarrow Hq$ and $q\bar{q} \rightarrow Hg$ at large Higgs transverse momentum*, *JHEP* **02** (2018) 135 [[arXiv:1712.06549](#)] [[INSPIRE](#)].
- [32] J. Davies, G. Mishima, M. Steinhauser and D. Wellmann, *Double-Higgs boson production in the high-energy limit: planar master integrals*, *JHEP* **03** (2018) 048 [[arXiv:1801.09696](#)] [[INSPIRE](#)].
- [33] R. Bonciani, G. Degrossi, P.P. Giardino and R. Gröber, *Analytical Method for Next-to-Leading-Order QCD Corrections to Double-Higgs Production*, *Phys. Rev. Lett.* **121** (2018) 162003 [[arXiv:1806.11564](#)] [[INSPIRE](#)].
- [34] J. Davies, G. Mishima, M. Steinhauser and D. Wellmann, *Double Higgs boson production at NLO in the high-energy limit: complete analytic results*, *JHEP* **01** (2019) 176 [[arXiv:1811.05489](#)] [[INSPIRE](#)].
- [35] M. Beneke and V.A. Smirnov, *Asymptotic expansion of Feynman integrals near threshold*, *Nucl. Phys. B* **522** (1998) 321 [[hep-ph/9711391](#)] [[INSPIRE](#)].
- [36] V.A. Smirnov, *Analytic Tools for Feynman Integrals*, *Springer Tracts Mod. Phys.* **250** (2012) 1.
- [37] V.A. Smirnov, *Asymptotic expansions in limits of large momenta and masses*, *Commun. Math. Phys.* **134** (1990) 109 [[INSPIRE](#)].
- [38] R. Harlander, T. Seidensticker and M. Steinhauser, *Corrections of $O(\alpha_s)$ to the decay of the Z boson into bottom quarks*, *Phys. Lett. B* **426** (1998) 125 [[hep-ph/9712228](#)] [[INSPIRE](#)].

- [39] T. Seidensticker, *Automatic application of successive asymptotic expansions of Feynman diagrams*, in *6th International Workshop on New Computing Techniques in Physics Research: Software Engineering, Artificial Intelligence Neural Nets, Genetic Algorithms, Symbolic Algebra, Automatic Calculation (AIHENP 99)*, Heraklion, Crete, Greece, April 12–16, 1999 [[hep-ph/9905298](#)] [[INSPIRE](#)].
- [40] B. Jantzen, *Foundation and generalization of the expansion by regions*, *JHEP* **12** (2011) 076 [[arXiv:1111.2589](#)] [[INSPIRE](#)].
- [41] T.Y. Semenova, A.V. Smirnov and V.A. Smirnov, *On the status of expansion by regions*, [arXiv:1809.04325](#) [[INSPIRE](#)].
- [42] A. Pak and A. Smirnov, *Geometric approach to asymptotic expansion of Feynman integrals*, *Eur. Phys. J. C* **71** (2011) 1626 [[arXiv:1011.4863](#)] [[INSPIRE](#)].
- [43] B. Jantzen, A.V. Smirnov and V.A. Smirnov, *Expansion by regions: revealing potential and Glauber regions automatically*, *Eur. Phys. J. C* **72** (2012) 2139 [[arXiv:1206.0546](#)] [[INSPIRE](#)].
- [44] B. Ananthanarayan, A. Pal, S. Ramanan and R. Sarkar, *Unveiling Regions in multi-scale Feynman Integrals using Singularities and Power Geometry*, *Eur. Phys. J. C* **79** (2019) 57 [[arXiv:1810.06270](#)] [[INSPIRE](#)].
- [45] V.A. Smirnov, *Problems of the strategy of regions*, *Phys. Lett. B* **465** (1999) 226 [[hep-ph/9907471](#)] [[INSPIRE](#)].
- [46] R.N. Lee and A.A. Pomeransky, *Critical points and number of master integrals*, *JHEP* **11** (2013) 165 [[arXiv:1308.6676](#)] [[INSPIRE](#)].
- [47] E. Remiddi and J.A.M. Vermaseren, *Harmonic polylogarithms*, *Int. J. Mod. Phys. A* **15** (2000) 725 [[hep-ph/9905237](#)] [[INSPIRE](#)].
- [48] D. Maître, *HPL, a mathematica implementation of the harmonic polylogarithms*, *Comput. Phys. Commun.* **174** (2006) 222 [[hep-ph/0507152](#)] [[INSPIRE](#)].
- [49] D. Maître, *Extension of HPL to complex arguments*, *Comput. Phys. Commun.* **183** (2012) 846 [[hep-ph/0703052](#)] [[INSPIRE](#)].
- [50] C. Anastasiou, T. Gehrmann, C. Oleari, E. Remiddi and J.B. Tausk, *The Tensor reduction and master integrals of the two loop massless crossed box with lightlike legs*, *Nucl. Phys. B* **580** (2000) 577 [[hep-ph/0003261](#)] [[INSPIRE](#)].
- [51] M. Czakon, *Automatized analytic continuation of Mellin-Barnes integrals*, *Comput. Phys. Commun.* **175** (2006) 559 [[hep-ph/0511200](#)] [[INSPIRE](#)].
- [52] J.B. Tausk, *Nonplanar massless two loop Feynman diagrams with four on-shell legs*, *Phys. Lett. B* **469** (1999) 225 [[hep-ph/9909506](#)] [[INSPIRE](#)].
- [53] D. Kosower, <https://mbtools.hepforge.org>.
- [54] L.J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press (1966).
- [55] A.P. Prudnikov, Y.A. Brychkov and O.I. Marichev, *Integrals and series, volume 3*, Gordon and Breach Science Publishers (1990).
- [56] H.R.P. Ferguson and D.H. Bailey, *A Polynomial Time, Numerically Stable Integer Relation Algorithm*, RNR Technical Report, RNR-91-032 (1992).
- [57] H.R.P. Ferguson, D.H. Bailey and S. Arno, *Analysis of PSLQ, an Integer Relation Finding Algorithm*, NAS Technical Report, NAS-96-005 (1996).