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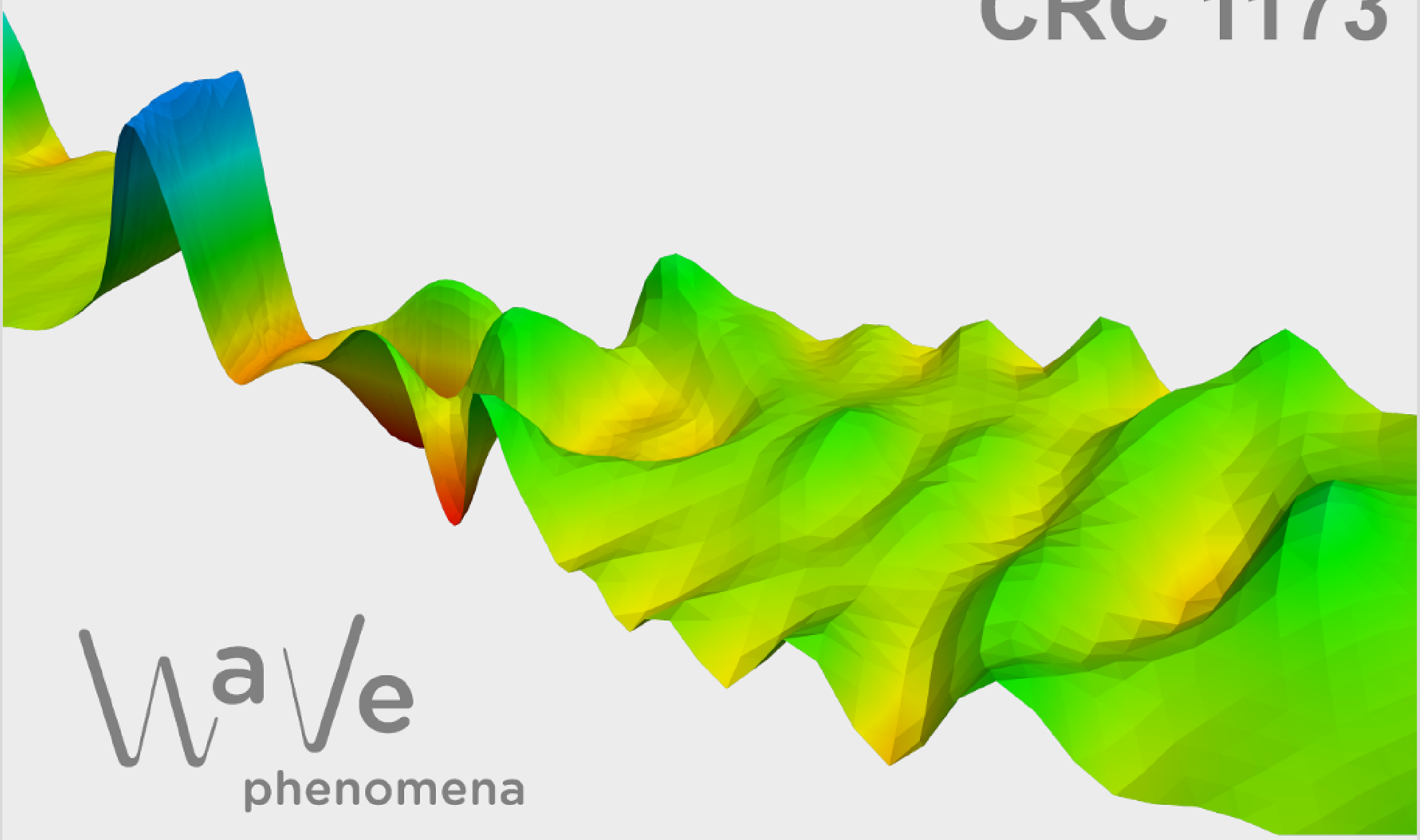

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Abstract

Solving the Klein-Gordon-Zakharov (KGZ) system in the high-plasma frequency regime $c \gg 1$ is numerically severely challenging due to the highly oscillatory nature or the problem. To allow reliable approximations classical numerical schemes require severe step size restrictions depending on the small parameter c^{-2}. This leads to large errors and huge computational costs. In the singular limit $c \to \infty$ the Zakharov system appears as the regular limit system for the KGZ system. It is the purpose of this paper to use this approximation in the construction of an effective numerical scheme for the KGZ system posed on the torus in the highly oscillatory regime $c \gg 1$. The idea is to filter out the highly oscillatory phases explicitly in the solution. This allows us to play back the numerical task to solving the non-oscillatory Zakharov limit system. The latter can be solved very efficiently without any step size restrictions. The numerical approximation error is then estimated by showing that solutions of the KGZ system in this singular limit can be approximated via the solutions of the Zakharov system and by proving error estimates for the numerical approximation of the Zakharov system. We close the paper with numerical experiments which show that this method is more effective than other methods in the high-plasma frequency regime $c \gg 1$.

1 Introduction

We consider the Klein-Gordon-Zakharov (KGZ) system

$$c^{-2}\partial_t^2 u = \partial_x^2 u - c^2 u - uv, \qquad \partial_t^2 v = \partial_x^2 v + \partial_x^2(|u|^2), \tag{1}$$

with $u(x,t), v(x,t), t \in \mathbb{R}$ in the limit $c \to \infty$. For pratical implementation issues we consider the system (1) posed on the d-dimensional torus

$\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$. The subsequent approach in principle works for all space dimensions, i.e. $u(x,t), x \in \mathbb{T}^d$. However, for expository reasons we restrict ourselves to $d = 1$.

The KGZ system [14, Eq.(1.1)] is a model from plasma physics which is used to describe the interaction between so called Langmuir waves and ion sound waves in plasma. Here, $v(x,t)$ is proportional to the ion density fluctuation from a constant equilibrium density and $u(x,t)$ is proportional to the electric field.

We are interested in a robust numerical description of the KGZ system (1) for large values of c. Resolving the highly oscillatory behavior of the solutions in this regime is numerically very delicate, see, e.g., [1, 2, 4, 8, 11]. Severe time step restrictions need to be imposed, leading to high numerical costs. These can be avoided by passing to the regular limit system of the KGZ system (1) for $c \to \infty$. In this singular limit with the ansatz

$$\Psi_u(x,t) = \psi_u(x,t)e^{ic^2 t} + c.c., \qquad \Psi_v(x,t) = \psi_v(x,t) \tag{2}$$

for u, v the Zakharov system

$$2i\partial_t\psi_u = \partial_x^2\psi_u - \psi_u\psi_v, \qquad \partial_t^2\psi_v = \partial_x^2\psi_v + 2\partial_x^2(\psi_u^2) \tag{3}$$

can be derived. The numerical task to solve the KGZ equation for large c thus can be reduced to solving the corresponding non-oscillatory limit system (3). The latter can be carried out very efficiently without any additional step size restrictions. In the following we provide rigorous estimates between true solutions of the KGZ system (1) for large values of c and its numerical approximations obtained via the associated Zakharov system (3).

This *asymptotic* approach to handle highly oscillatory systems has attracted a lot of interest in the last years, cf. [2, 4, 8, 11]. In these highly oscillatory situations the approach via the regular limit system turned out to be more effective than other tools for highly oscillatory systems, such as Gautschi type approaches (see, e.g., [1]). In strong high plasma frequency limits $c \to \infty$ they are also far more effective than uniformly accurate oscillatory methods which have been recently invented for a number of Klein-Gordon type systems [4, 5, 8]. In particular, high order uniformly accurate oscillatory schemes are numerically very expensive such that the asymptotic approach of reducing the original complex system to the corresponding limit system is far more attractive from a computational point of view. Note that the Zakharov system (3) can be solved very efficiently with high order methods (in time and in space) without any step size restrictions (see, e.g., [3, 7, 12, 13]).

A sharp estimate on the difference between the exact solution u and the limit approximation $\psi_u(t,x)\mathrm{e}^{ic^2 t} + c.c.$ is essential for the global error bound of the effective numerical scheme

$$u^n_{\mathrm{num}} = (\psi_u)^n_{\mathrm{num}}\mathrm{e}^{ic^2 t_n} + c.c., \tag{4}$$

where $(\psi_u)^n_{\text{num}}$ denotes the numerical solution of the Zakharov system (3) at time $t_n = n\tau$. Such an estimate can be established with the triangle inequality. The full error can be reduced to the asymptotic error (KGZ to Zakharov) and the numerical error when solving the Zakharov system, i.e.,

$$\|u(t_n) - u_{\text{num}}(t_n)\| \le C \underbrace{\|u(t_n) - (\psi_u(t_n)\mathrm{e}^{ic^2 t_n} + c.c)\|}_{\text{asymptotic error}} + C \underbrace{\|\psi_u(t_n) - (\psi_u)^n_{\text{num}}\|}_{\text{numerical error (Zakharov)}},$$

see Section 3 below for the detailed error bounds.

The plan of the paper is as follows. In Section 2 we provide bounds for the error made by the Zakharov approximation which show that the Zakharov system (3) allows us to make correct predictions about the dynamics of the KGZ system (1) for large values of c, in particular we explain why existing error estimates for the problem posed on the real line transfer to the problem posed on the torus. In Section 3 we present a numerical scheme which allows us an effective simulation of the dynamics of the Zakharov system (3) and give error bounds for this numerical approximation. After that we bring together the estimates from Section 2 and Section 3 and present the error bound for this effective numerical simulation of the KGZ system via the Zakharov system (3) for large c. In Section 4 we close the paper with some numerical illustrations showing the strengh of the method in this highly oscillatory regime.

2 From the KGZ system to the Zakharov system

It is the purpose of this section to provide error estimates for the Zakharov approximation of the KGZ system posed on a one-dimensional torus for large values of c. This approximation question has been addressed in a number of papers. However, the results [6, 9, 16] all have been established for the KGZ system posed in $\mathbb{R}^d$. As the example of another singular limit of the KGZ system, namely the Klein-Gordon approximation, shows, such transfers can be wrong. In [10] it has been shown that for the problem posed on the torus a modified Klein-Gordon equation replaces the Klein-Gordon equation as regular limit system in this other singular limit. As we will see below also for the Zakharov approximation of the KGZ system such a transfer is non-trivial.

In [6] with the 'harmonic' ansatz

$$u(x,t) = \psi_u(x,t)e^{ic^2 t}, \qquad v(x,t) = \psi_v(x,t) \tag{5}$$

a Zakharov system with slightly different coefficients has been derived and convergence results for $c \to \infty$ have been established. Here we will concentrate on the 'real' case of the introduction.

We start with an approximation result in the spaces of 2π-spatially periodic analytic functions

$$H^{\infty}_{\mu,s,per} = \{u \in L^2_{\rm per}(\mathbb{T}) : e^{\mu|\cdot|}(1+|\cdot|^2)^{\frac{s}{2}}\widehat{u}(\cdot) \in \ell^2(\mathbb{Z})\},$$

equipped with the norm

$$\|u\|_{H^{\infty}_{\mu,s,per}} = \left(\sum_{k\in\mathbb{Z}} |\widehat{u}(k)|^2 e^{2\mu|k|}(1+|k|^2)^s\right)^{\frac{1}{2}},$$

where $\mu \geq 0$ and $s \geq 0$. Functions $u \in H^{\infty}_{\mu,0,per}$ can be extended to functions that are analytic on the strip $\{z \in \mathbb{C}\colon |\Im(z)| < \mu\}$, cf. [15]. The KGZ system is then solved in a space

$$\mathcal{X}_{\mu_A,s} = (H^{\infty}_{\mu_A,s+1,per} \times H^{\infty}_{\mu_A,s,per} \times H^{\infty}_{\mu_A,s,per} \times H^{\infty}_{\mu_A,s-1,per}).$$

We transfer [16] to the 2π-spatially periodic situation and obtain

Theorem 2.1. *Fix $\beta \in (0,2]$, $\mu_A > 0$, $s \geq 1$. Let $(\psi_u,\psi_v) \in C([0,T_0], H^{\infty}_{\mu_A,s+5,per} \times H^{\infty}_{\mu_A,s+4,per})$ be a solution of the Zakharov system (3). Then there exist $c_0 > 0$, $C_1 > 0$, $C_2 > 0$ and $T_1 \in (0,T_0]$ such that for all $c \geq c_0$ and all initial conditions $(u,\partial_t u, v, \partial_t v)(\cdot,0)$ of the KGZ system (1) satisfying*

$$\|(u-\Psi_u, \varepsilon^2\partial_t(u-\Psi_u), v-\Psi_v, \partial_t(v-\Psi_v))(x,0)\|_{\mathcal{X}_{\mu_A,s}} \leq C_1 c^{-\beta},$$

there are solutions (u,v) of the KGZ system (1) with

$$\sup_{t\in[0,T_1]} \|(u-\Psi_u, \varepsilon^2\partial_t(u-\Psi_u), v-\Psi_v, \partial_t(v-\Psi_v))(x,t)\|_{\mathcal{X}_{0,s}} \leq C_2 c^{-\beta}.$$

Remark 2.2. *The theorem is also true if the scaling ε^{β} of the error is replaced by a scaling function which decays to zero as $o(1)$ for $\varepsilon \to 0$.*

Remark 2.3. *The Sobolev version of this theorem for the KGZ system posed on $\mathbb{R}^d$ can be found in [9]. More precisely, the scaling of the Zakharov limit of the KGZ system is given in [9, (2.6)]. Rewriting the statements of [9] in the above form and transferring them from $x \in \mathbb{R}^d$ to $x \in \mathbb{T}^d$ yields the following theorem:*

Theorem 2.4. *Theorem 2.1 remains true also in case $\mu_A = 0$, i.e. if $H^{\infty}_{0,s} = H^s$. Moreover we have $T_1 = T_0$.*

Remark 2.5. *Error estimates for the Zakharov approximation are nontrivial due to some c in front of the nonlinear terms if the KGZ system is written as first order system. The problem has been solved in [9] and [16] by completely different approaches. In [9] a detailed analysis of the bilinear terms through an averaging approach has been used, whereas in [16] a Cauchy-Kowalevsjaya like approach has been chosen. Although the appraoch of [9] gives stronger results, the approach of [16] is conceptually more simple and more robust, in the sense that it applies for other systems, too, without a detailed analysis of the underlying system.*

Remark 2.6. *In order to prove Theorem 2.1 and Theorem 2.4 we have to prove that the formal error made by the Zakharov approximation is sufficiently small. The largest terms which do not cancel are* $\psi_u(x,t)^2 e^{2ic^2t}$ *and* $\overline{\psi_u}(x,t)^2 e^{-2ic^2t}$ *in the v-equation. There are two possibilities to prove that the influence of these terms on the dynamics on the given* $\mathcal{O}(1)$ *time interval, is less or equal order* $\mathcal{O}(\varepsilon^\beta)$*. These are averaging methods in the variation of constant formula or adding higher order terms to the approximation. In the following we explain for the second approach why the transfer from* $x \in \mathbb{R}$ *to* $x \in \mathbb{T}$ *is in general a non-trivial question. However, for the Zakharov approximation of the KGZ system the transfer is possible.*

In order to make the residual terms to be of order $\mathcal{O}(\varepsilon^2)$ *the Zakharov ansatz has to be extended to*

$$\begin{aligned}\Psi_u(x,t) &= \psi_u(x,t)e^{ic^2t} + c.c.,\\ \Psi_v(x,t) &= \psi_v(x,t) + c^{-4}\psi_{v,+}(x,t)e^{2ic^2t} + c^{-4}\psi_{v,-}(x,t)e^{-2ic^2t},\end{aligned}$$

with $-\psi_{v,+} = \partial_x^2(\psi_u^2)$ *and* $-\psi_{v,-} = \partial_x^2(\overline{\psi_u}^2)$*. It is an easy exercise to show that the all terms down to order* $\mathcal{O}(1)$ *vanish. The remaining terms are*

$$\begin{aligned}Res_u &= -c^{-2}(\partial_t^2\psi_u)e^{ic^2t} + c.c.\\ &\quad -(\psi_u e^{ic^2t} + c.c.)(c^{-4}\psi_{v,+}e^{2ic^2t} + c^{-4}\psi_{v,-}e^{-2ic^2t}),\\ Res_v &= -c^{-4}(\partial_t^2\psi_{v,+})e^{2ic^2t} - c^{-4}(\partial_t^2\psi_{v,-})e^{-2ic^2t}\\ &\quad -2ic^{-2}(\partial_t\psi_{v,+})e^{2ic^2t} + 2ic^{-2}(\partial_t\psi_{v,-})e^{-2ic^2t}\\ &\quad +\partial_x^2(c^{-4}\psi_{v,+}(x,t)e^{2ic^2t} + c^{-4}\psi_{v,-}(x,t)e^{-2ic^2t}).\end{aligned}$$

Writing the v equation as first order system makes it necessary to estimate $\partial_x^{-1}Res_v$*, cf. [16]. It can be bounded if* Res_v *can be written as a derivative or alternatively for* $x \in \mathbb{R}^d$ *with* L^p*-*L^q *estimates. In* Res_v *all terms have a* ∂_x *in front, except of the pure time derivatives* $\partial_t^2\psi_{v,\pm}$ *and* $\partial_t\psi_{v,\pm}$*. Using*$-\psi_{v,+} = \partial_x^2(\psi_u^2)$ *and* $-\psi_{v,-} = \partial_x^2(\overline{\psi_u}^2)$ *they can be written as* $\partial_x^2\partial_t^2(\psi_u^2)$ *resp.* $\partial_x^2\partial_t(\psi_u^2)$ *such that* $\partial_x^{-1}Res_v$ *can be estimated. See [10] for an example about what happens for* $x \in \mathbb{T}$ *when the residual terms cannot be written as a derivative.*

Remark 2.7. *In [17] the Zakharov approximation has been justified for the original Euler-Maxwell system. Hence the procedure of approximating the Euler-Maxwell system in the singular limit with the regular limit system is in principle possible for the original Euler-Maxwell system, too.*

3 Error bounds for the numerical scheme

The asymptotic approximation result given in Theorem 2.4 allows us to develop an efficient numerical scheme for the KGZ system (1) in the Zakharov limit. The idea is thereby the following: Let

$$\begin{aligned}&(\psi_u^n)_{\text{num}}, \quad (\psi_v^n)_{\text{num}}, \quad (\psi_{v'}^n)_{\text{num}} \\ &(\psi_u^0)_{\text{num}} = \frac{1}{2}(u(\cdot,0) + \frac{1}{ic^2}\partial_t u(\cdot,0)), \quad (\psi_v^0)_{\text{num}} = v(\cdot,0), \quad (\psi_{v'}^0)_{\text{num}} = \partial_t v(\cdot,0)\end{aligned} \tag{6}$$

denote the numerical solution at time $t = t_n$ of the Zakahrov system (3) obtained, e.g., with the trigonometric integrator proposed in [12]. Then we choose, motivated by (5), the scheme defined through

$$u_{\text{num}}^n = (\psi_u^n)_{\text{num}} \, \mathrm{e}^{ic^2 t_n} + \overline{(\psi_u^n)_{\text{num}}} \, \mathrm{e}^{-ic^2 t_n}, \quad v_{\text{num}}^n = (\psi_v^n)_{\text{num}} \tag{7}$$

as a numerical approximation to the exact solution $(u(t_n), v(t_n), v'(t_n))$ of the KGZ system (1). The choice of initial values in (6) is thereby motivated as follows: From (2) we find for $t = 0$ that

$$u = \psi_u + \overline{\psi_u}, \qquad \partial_t u = ic^2\psi_u - ic^2\overline{\psi_u} + \mathcal{O}(1)$$

which implies

$$\psi_u = \frac{1}{2}(u + \frac{1}{ic^2}\partial_t u)$$

when the terms indicated with $\mathcal{O}(1)$ are ignored.

Thanks to Theorem 2.4, which allows us to control the asymptotic error, the scheme (7) allows for the following global error estimate.

Theorem 3.1. *Fix $s \geq 1$, and let $(\psi_u, \psi_v) \in C([0,T_0], H^{s+5}_{per} \times H^{s+4}_{per})$ be a solution of the Zakharov system* (3). *Assume that the numerical scheme* (6) *approximates the solution of the Zakharov system* (3) *with order p in H^s, i.e., there exist $C, \tau_0 > 0$ such that for all $\tau \leq \tau_0$ and $t_n \leq T$*

$$\|(\psi_u^n)_{num} - \psi_{u(t_n)}\|_{s+1} + \|(\psi_v^n)_{num} - \psi_{v(t_n)}\|_s + \|(\psi_{v'}^n)_{num} - \psi_{v'(t_n)}\|_{s-1} \leq C\tau^p. \tag{8}$$

Then the scheme (7) *converges to the solution (u, v) of the KGZ system* (1) *in the limit $c \to \infty$, $\tau \to 0$. In detail, there exist $C, c_0, \tau_0 > 0$ such that for all $c > c_0$ and $\tau < \tau_0$ we have*

$$\|u_{num}^n - u(t_n)\|_{s+1} + \|v_{num}^n - v(t_n)\|_s \leq C\left(\tau^p + c^{-2}\right).$$

Proof. The proof follows by Theorem 2.4 together with the triangle inequality. Note that

$$\begin{aligned}\|u(t_n) - u_{\text{num}}^n\|_{s+1} &\leq \|u(t_n) - \Psi_u(t_n) + \Psi_u(t_n) - u_{\text{num}}^n\|_{s+1} \\ &\leq \|u(t_n) - \Psi_u(t_n)\|_{s+1} + C\|\psi_{u(t_n)} - (\psi_u^n)_{\text{num}}\|_{s+1}.\end{aligned} \tag{9}$$

Thanks to Theorem 2.4 we can bound the first term by Cc^{-2}. The second term in (9) is bounded by τ^p by assumption (8). This yields the assertion for the error in u. The other terms can be bounded in a similar way. □

The second-order trigonometric integrator [12, Eq. (3.9)] developed for the Zakharov system (3) together with the ansatz (7) allow us to obtain a second-order asymptotic and time convergent scheme for the KGZ system (1).

Corollary 3.2 (A second-order scheme)**.** *Fix $s \geq 1$, and let $(\psi_u, \psi_v) \in C([0,T_0], H^{s+5}_{per} \times H^{s+4}_{per})$ be a solution of the Zakharov system* (3)*. Then the scheme [12, Eq. (3.9)] converges to the solution (u,v) of the KGZ system* (1) *in the limit $c \to \infty$, $\tau \to 0$. In detail, there exist $C, c_0, \tau_0 > 0$ such that for all $c > c_0$ and $\tau < \tau_0$ we have*

$$\|u^n_{num} - u(t_n)\|_{s+1} + \|v^n_{num} - v(t_n)\|_s \leq C\left(\tau^2 + c^{-2}\right).$$

4 Some numerical illustrations

In this section we compare various numerical schemes for the solution of the KGZ system (1). Our numerical experiments confirm that in the high plasma frequency regime $c \gg 1$ the ansatz (4), based on the Zakharov limit approximation, is more efficient than directly solving the KGZ system (1) with a uniformly accurate scheme such as [5]. Furthermore, the numerical experiments underline the second-order convergence rate (in time and in c^{-2}) established in Corollary 3.2.

For the practical implementation we consider $x \in \mathbb{T} = [0, 2\pi]$ and a finite time interval, i.e., $t \in [0,1]$. For the spatial approximation we use a standard Fourier pseudospectral method with $M = 256$ Fourier modes (i.e., $\Delta x = 0.0245$) and choose the initial values

$$\begin{aligned} u(x,0) &= \frac{\sin(2x)\cos(4x)}{2-\cos(x)\sin(2x)}, \quad \partial_t u(x,0) = c^2(-\sin(2x)\cos(x)), \\ v(x,0) &= \frac{\sin(x)\cos(2x)}{2-\sin(2x)^2}, \quad \partial_t v(x,0) = \frac{\sin(x)}{2-\cos(2x)^2}. \end{aligned} \tag{10}$$

Efficiency

In Figure 1 and Figure 2 we compare the error versus the computational time of different numerical methods for the KGZ system (1). The work-precision plots show the efficiency of the different methods for different values of c.

More precisely, we compare the following schemes:

- The first- and second-order schemes (4) based on the asymptotic approximation result given in Theorem 2.4. Thereby we use the trigonometric integration method [12] for the numerical solution of the Zakharov system. This allows for a global error of order (cf. Corollary 3.2)
$$c^{-2} + \tau^p \quad \text{with } p = 1, 2.$$

- The uniformly accurate methods for the KGZ system (1) developed in [5] which allow for a global error of order

$$\tau^2 \quad \text{with } p = 1, 2.$$

- A Gautschi method which was developed for the KGZ system (1) in [2]. The latter allows for a global error of order

$$c^2\tau^2.$$

We plot the corresponding error against the computation time (in seconds) of the corresponding numerical method. The reference solution is computed via the uniformly accurate method by [5] with a very small step size $\tau_{ref} = 1.19 \cdot 10^{-7}$.
In the numerical experiments we observe the following: Although the uniformly accurate methods allow uniform convergence (i.e., error bounds independently of c) the limit integrators are faster for very large values of c.

Asymptotic consistency plot

In this section we numerically underline that the Zakharov system (3) approximates the KGZ (1) with rate $\mathcal{O}(c^{-2})$ for sufficiently smooth solutions (cf. Theorem 2.4). To solve the Zakharov and KGZ system we use the uniformly accurate scheme [5] and limit integrator [12], respectively.

In Figure 3 and Figure 4 we use the smooth initial data (10) and initial data in H^2, respectively. To test the convergence rate in c^{-2} both simulations are carried out with the constant small time step size $\tau = 1.53 \cdot 10^{-5}$ in order to not see the time discretization error in the plots. In Figure 4 the initial values in H^2 are computed by choosing uniformly distributed random numbers in the interval $[0, 1]$ for the real and imaginary part of the N Fourier coefficients, respectively. These coefficients are then divided by $(1 + |k|)^{2+1/2}$ for $k = -\frac{N}{2}, ..., \frac{N}{2} - 1$ and finally transformed back with the discrete Fourier transform to get the desired discrete initial data in physical space. The reference solution is computed via the uniformly accurate method [5] with $\tau_{ref} = 1.19 \cdot 10^{-7}$.

For H^2 data we numerically observe an order reduction down to order $c^{-1/2}$ in the asymptotic error of the Zakharov approximation of the KGZ system which underlines the necessity of sufficiently smooth solutions for the validity of the asymptotic approximation (cf. Theorem 2.4).

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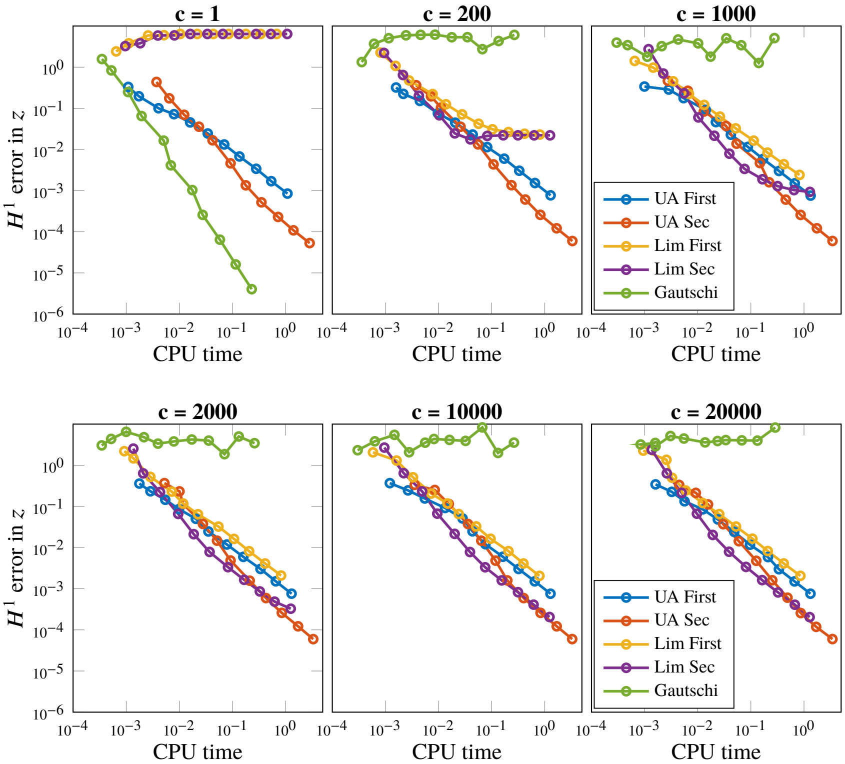

Figure 1: Efficiency plot of z for different values of c. The blue and red lines correspond to the first- and second-order uniformly accurate method of [5]. In yellow and purple we plot the first- and second-order limit integrator based on (4) using the trigonometric integrator [12] for the numerical solution of the Zakharov limit system. The green line corresponds to the Gautschi method [2].

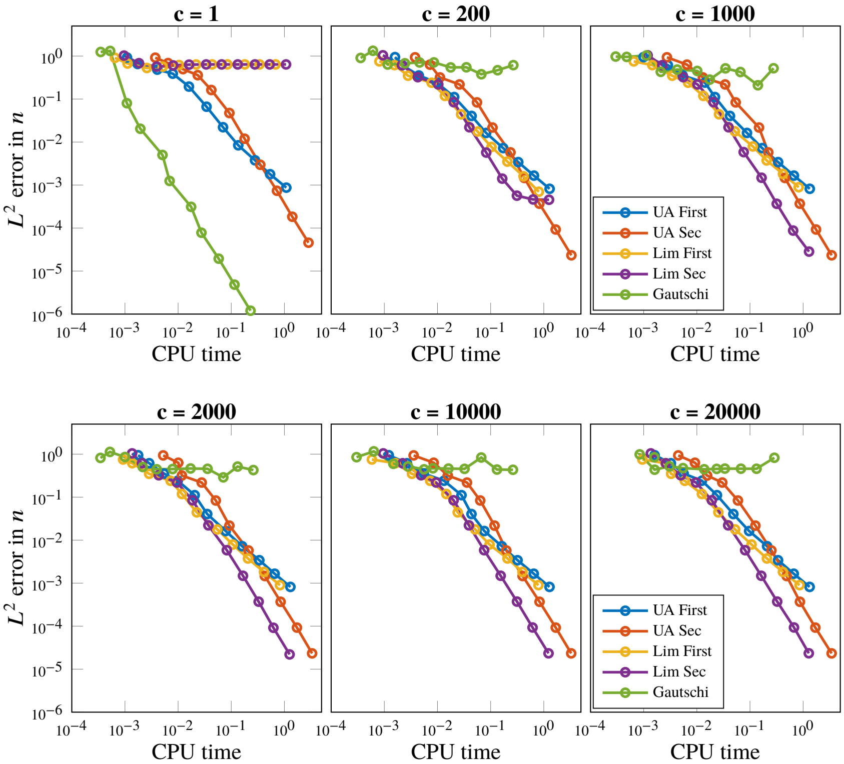

Figure 2: Efficiency plot of n for different values of c. The blue and red lines correspond to the first- and second-order uniformly accurate method of [5]. In yellow and purple we plot the first- and second-order limit integrator based on (4) using the trigonometric integrator [12] for the numerical solution of the Zakharov limit system. The green line corresponds to the Gautschi method [2].

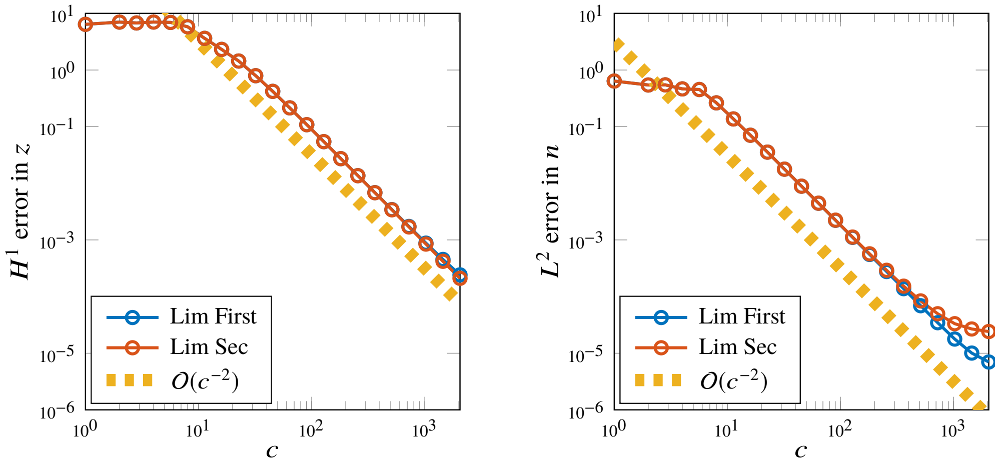

Figure 3: Asymptotic consistency plot with initial data (10). The blue and red line correspond to the first- and second-order limit integrator for the Zaharov system of [12]. Yellow: reference line of order c^{-2}.

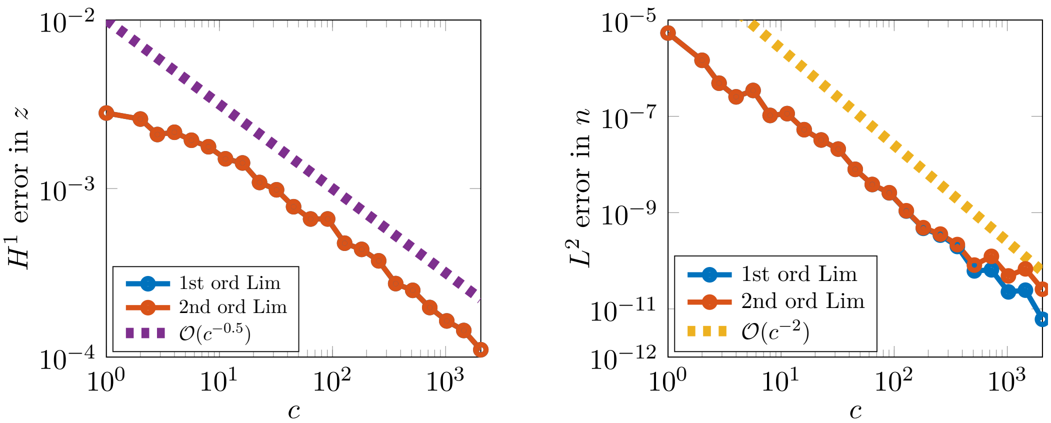

Figure 4: Asymptotic consistency plot with initial data in H^2. The blue and red line correspond to the first- and second-order limit integrator for the Zakharov system of [12]. Reference line of order $c^{-1/2}$ and c^{-2} in purple (left) and yellow (right), respectively.

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