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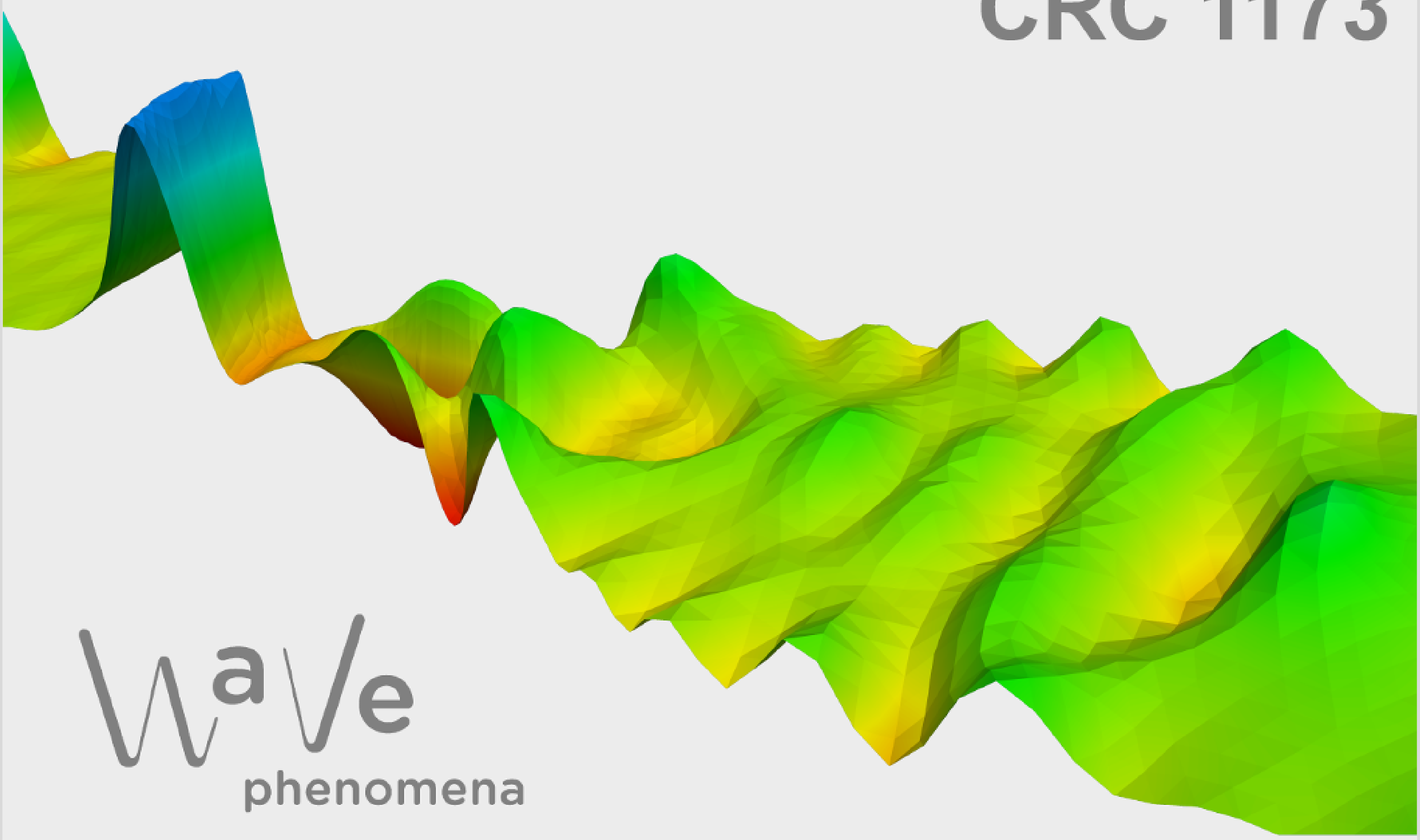

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ON THE GLOBAL WELL-POSEDNESS OF THE QUADRATIC NLS ON $L^2(\mathbb{R}) + H^1(\mathbb{T})$

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Abstract. We study the one dimensional nonlinear Schrödinger equation with power nonlinearity $|u|^{\alpha-1} u$ for $\alpha \in [2,5]$ and initial data $u_0 \in L^2(\mathbb{R}) + H^1(\mathbb{T})$. We show via Strichartz estimates that the Cauchy problem is locally well-posed. In the case of the quadratic nonlinearity ($\alpha = 2$) we obtain unconditional global well-posedness in the space $C(\mathbb{R}, L^2(\mathbb{R})+H^1(\mathbb{T}))$ via Gronwall's inequality.

1. Introduction and main results

We are interested in the Cauchy problem for the nonlinear Schrödinger equation (NLS) with power nonlinearity on the space $L^2(\mathbb{R}) + H^1(\mathbb{T})$, i.e.

$$\begin{cases} \mathrm{i}u_t(x,t) + \partial_x^2 u(x,t) \pm |u|^{\alpha-1} u = 0 & (x,t) \in \mathbb{R} \times \mathbb{R}, \\ u(\cdot, 0) = u_0, \end{cases} \tag{1}$$

where $u_0 = v_0 + w_0 \in L^2(\mathbb{R}) + H^1(\mathbb{T})$ and $\alpha \in [2,5]$. By $\mathbb{T}$ we denote the one-dimensional torus, i.e. $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, where we consider functions on $\mathbb{T}$ to be 2π-periodic functions on $\mathbb{R}$. Before we state our main results, let us mention that the NLS (1) is globally well-posed in $L^2(\mathbb{R})$ via Strichartz estimates and mass conservation (see [Tsu87]) and it is globally well-posed in $L^2(\mathbb{T})$ via the Fourier restriction norm method and mass conservation (see [Bou93a]). Motivation for the investigation of hybrid initial values $u_0 \in L^2(\mathbb{R}) + H^1(\mathbb{T})$ comes from high–speed optical fiber communications, where in a certain approximation the behavior of pulses in glass–fiber cables is described by a NLS equation. The NLS (1) with initial data in $H^s(\mathbb{R}) + H^s(\mathbb{T})$ was referred to in [CHKP18] as the tooth problem. A tooth, is, for example, w_0 restricted to one period. We think of the addition of v_0 to w_0 as eliminating finitely many of these teeth in the underlying periodic signal. A periodic signal is the simplest type of a non-decaying signal, encoding, for example, an infinite string of ones if there is exactly one tooth per period. However, such a purely periodic signal carries no information. One would like to be able to change it, at least locally. This leads necessarily to a hybrid formulation of the NLS where the signal is the sum of a periodic and a localized part. The localized part being able to remove one or more of the teeth in the underlying periodic signal. This way one can model, for example, a signal consisting of two infinite blocks of ones which are separated by a single zero, or even far more complicated patterns. In the optics literature the phenomenon of ghost pulses (see [MM99] and [ZM99]) occurs which in our terminology corresponds to the regrowth of missing teeth of the solution to the NLS (1).

The case of the cubic nonlinearity ($\alpha = 3$) and the initial data $u_0 \in H^s(\mathbb{R}) + H^s(\mathbb{T})$, where $s \geq 0$, was studied by the authors in [CHKP18], where the existence

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of weak solutions in the extended sense was established. Moreover, under some further assumptions, unconditional uniqueness was obtained. In this paper, due to the non-algebraic structure of the nonlinearity in (1) (for $\alpha \neq 3$) we have to use different methods. For the relation between the solutions of [CHKP18] and the solutions of Theorem 2 we refer to Remark 3.

To state the main results of this paper we need some preparation. Let $u = v + w \in C([0,T], L^2(\mathbb{R}) + H^1(\mathbb{T}))$ where w satisfies the periodic NLS on the torus with initial data w_0. The following is known about w.

Theorem 1. *(Cf.* [LRS88, Theorem 2.1]*). The Cauchy problem for the periodic NLS*

$$\begin{cases} \mathrm{i}w_t(x,t) + \partial_x^2 w(x,t) \pm |w|^{\alpha-1} w = 0 & (x,t) \in \mathbb{T} \times \mathbb{R}, \\ w(\cdot,0) = w_0. \end{cases} \tag{2}$$

is locally well-posed in $H^1(\mathbb{T})$ for $\alpha \geq 2$. That means that for any $w_0 \in H^1(\mathbb{T})$ there is a unique $w \in C([0,T], H^1(\mathbb{T}))$ satisfying (2) *in the mild sense. The guaranteed time of existence T depends only on $\|w\|_{H^1(\mathbb{T})}$.*

Then the local part v has to be a solution of the Cauchy problem for the *modified NLS*

$$\begin{cases} \mathrm{i}v_t(x,t) + \partial_x^2 v(x,t) \pm G_\alpha(w,v) = 0 & (x,t) \in \mathbb{R} \times \mathbb{R}, \\ v(\cdot,0) = v_0, \end{cases} \tag{3}$$

where

$$G_\alpha(w,v) := |v+w|^{\alpha-1}(v+w) - |w|^{\alpha-1} w. \tag{4}$$

The main results of the paper are the following two theorems.

Theorem 2 (Local well-posedness of the NLS (1))**.** *For $\alpha \in [2,5]$ the Cauchy problem* (3) *is locally well-posed in $C([0,T], L^2(\mathbb{R})) \cap L^{\frac{4(\alpha+1)}{\alpha-1}}([0,T], L^{\alpha+1}(\mathbb{R}))$ for any $v_0 \in L^2(\mathbb{R})$.*

Hence, the original Cauchy problem (1) *is locally well-posed.*

In the case $\alpha \in [2,5)$, the guaranteed time of existence T depends only on $\|v_0\|_2$ and $\|w_0\|_{H^1(\mathbb{T})}$, whereas, for $\alpha = 5$, T depends on the profile of v_0 and $\|w_0\|_{H^1(\mathbb{T})}$.

In the case $\alpha = 2$, the intersection above is not needed, i.e. one has unconditional uniqueness.

The method we employ for the proof of the Theorem 2 cannot be used to cover the range $\alpha \in (1,2)$. A more precise explanation is given in Remark 10.

Remark 3. *Notice that the weak solution in the extended sense $\tilde{u}$ constructed in* [CHKP18] *and the solution u from Theorem 2 coincide. This can be seen as follows: u is a weak solution in the extended sense, which follows by the definition, Plancherel's theorem and the dominated convergence theorem. Moreover, in the aforementioned paper it was observed that $\tilde{u}$ is unique among those solutions, which can be approximated by smooth solutions. This is true for u and hence $\tilde{u} = u$ follows.*

For $\alpha = 2$, we need the Cauchy problem for the periodic NLS (2) to be globally well-posed in $H^1(\mathbb{T})$. Although this is claimed to be well-known in the community, we could not find a suitable reference. Several people refer to [Bou93a] for this, however in [Bou93a, Propsition 5.73] he requires $\alpha \geq 3$ (in our notation). Moreover, in part ii) of the remark on page 152 in [Bou93a], Bourgain mentions that one could get existence of a solution for the quadratic nonlinearity using Schauder's fixed point theorem, but one would loose uniqueness. Hence, we provide a proof in the Appendix (Theorem 29). This global existence and uniqueness result on the torus, together with a close inspection of the mass $\int |v|^2 \, \mathrm{d}x$ are essential ingredients in our proof of global well-posedness of (1) on the "tooth space" $L^2(\mathbb{R}) + H^1(\mathbb{T})$.

Theorem 4 (Global well-posedness of the quadratic NLS)**.** *For $\alpha = 2$ and $v_0 \in L^2(\mathbb{R})$ the unique solution v of* (3) *from Theorem 2 extends globally and obeys the bound*

$$\|v(\cdot,t)\|_2 \leq \|v_0\|_2 \exp\left[\|w\|_{L_t^\infty L_x^\infty} t\right] \qquad \forall t \in [0,\infty). \tag{5}$$

Hence, the original Cauchy problem (1) *for $\alpha = 2$ is globally well-posed.*

Although the local well-posedness result from Theorem 2 covers the whole range $\alpha \in [2,5]$, the methods of the proof of Theorem 4 only work for $\alpha = 2$. A more precise explanation is given in Remark 15.

Of course, one can consider hybrid problems for other dispersive equations. Here we confine ourselves to a remark on the KdV.

Remark 5. *Observe that the tooth problem for the KdV reduces to a known setting. More precisely, consider real solutions of*

$$\begin{cases} u_t(x,t) + u_{xxx}(x,t) + u_x u = 0 & (x,t) \in \mathbb{R} \times \mathbb{R}, \\ & u(\cdot,0) = u_0 = v_0 + w_0 \in H^{s_1}(\mathbb{R}) + H^{s_2}(\mathbb{T}). \end{cases} \tag{6}$$

Let $u = v + w \in C([0,T], H^{s_1}(\mathbb{R}) + H^{s_2}(\mathbb{T}))$, where $s_2 \in \mathbb{N}$ and w is a global solution of the periodic KdV for the initial data w_0 (see [Bou93b, Theorem 5]*). Then v solves*

$$v_t + v_{xxx} + v_x v + (wv)_x = 0$$

with the initial data v_0, which is the KdV with the potential w. This problem has been studied in e.g. [ET16, Section 3.1] *using parabolic regularization. There it has been shown that v satisfies an exponential bound similar to* (5)*. Combining both results we obtain:*

> *For $s_1, s_2 \in \mathbb{N}$ satisfying $s_1 \geq 2$ and $s_2 \geq s_1 + 1$ the KdV tooth problem, i.e., the Cauchy problem* (6)*, is globally well-posed in $H^{s_1}(\mathbb{R}) + H^{s_2}(\mathbb{T})$.*

The paper is organized as follows: In Section 2 we state the required prerequisites for the proofs of the main theorems. In Section 3 we present the proof of Theorem 2 and in Section 4 we present the proof of Theorem 4. Finally, in the Appendix we justify why the quadratic periodic NLS (2) is globally well-posed in $H^1(\mathbb{T})$.

2. Prerequisites

Let us fix the notation and state some results necessary for the proof of our main theorems. For the purpose of smoothing we will use the heat kernel $(\phi_\varepsilon)_{\varepsilon \geq 0}$. Recall, that $\phi_\varepsilon = \delta_0$, if $\varepsilon = 0$, and

$$\phi_\varepsilon(x) = \frac{1}{2\sqrt{\pi\varepsilon}} e^{-\frac{|x|^2}{4\varepsilon}} \qquad \forall x \in \mathbb{R},$$

if $\varepsilon > 0$. We shall denote the convolution (in the space variable x) by e.g. $u * \phi_\varepsilon$.

For $s \in \mathbb{R}$ and $\Omega \in \{\mathbb{R}, \mathbb{T}\}$ we shall denote by $H^s(\Omega)$ the Sobolev spaces on Ω. Also, we set $H^\infty(\Omega) := \cap_{s \in \mathbb{R}} H^s(\Omega)$.

A pair of exponents $(r,q) \in [2,\infty]^2$ is called *admissible* (in one dimension), if

$$\frac{2}{q} + \frac{1}{r} = \frac{1}{2}. \tag{7}$$

Let us denote by $q_{\mathrm{a}}(r)$ the solution of (7) for any $r \in [2,\infty]$. Another pair of exponents $(\rho,\gamma) \in [1,2]$ shall be called *dually admissible*, if $(\rho',\gamma') \in [2,\infty]$ is admissible, i.e. if

$$\frac{2}{\gamma} + \frac{1}{\rho} = \frac{5}{2}. \tag{8}$$

We denote by $\gamma_{\mathrm{a}}(\rho)$ the solution of (8) for any $\rho \in [1,2]$.

Proposition 6 (Strichartz estimates). *(Cf.* [KT98, Theorem 1.2]*) Let $(r, q_a(r))$ be admissible and $(\rho, \gamma_a(\rho))$ be dually admissible. Then there is a constant $C = C(r,\rho) > 0$ such that for any $T > 0$, any $v_0 \in L^2(\mathbb{R})$ and any $F \in L_a^{\gamma}(\rho)([0,T], L^{\rho}(\mathbb{R}))$ the* homogeneous *and* inhomogeneous *Strichartz estimate*

$$\left\| e^{\mathrm{i}t\partial_x^2} v_0 \right\|_{L^{q_a(r)}([0,T],L^r(\mathbb{R}))} \leq C \left\| v_0 \right\|_{L^2(\mathbb{R})}, \tag{9}$$

$$\left\| \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} F(\cdot,\tau) \right\|_{L^{q_a(r)}([0,T],L^r(\mathbb{R}))} \leq C \left\| F \right\|_{L^{\gamma_a(\rho)}([0,T],L^{\rho}(\mathbb{R}))}. \tag{10}$$

hold.

Lemma 7 (Gronwall, integral form). *(See* [Tao06, Theorem 1.10]*.) Let $A, T \geq 0$ and $u, B \in C([0,T], \mathbb{R}_0^+)$ be such that*

$$u(t) \leq A + \int_0^t B(s)u(s)\mathrm{d}s \qquad \forall t \in [0,T].$$

Then

$$u(t) \leq A \exp\left(\int_0^t B(s)\mathrm{d}s \right) \qquad \forall t \in [0,T].$$

Lemma 8 (Gronwall, differential form). *(See* [Tao06, Theorem 1.12]*.) Let $T > 0$, $u : [0,T] \to \mathbb{R}_0^+$ be absolutely continuous and $B \in C([0,T], \mathbb{R}_0^+)$ such that*

$$u'(t) \leq B(t)u(t) \qquad \text{for almost every } t \in [0,T].$$

Then

$$u(t) \leq u(0) \exp\left(\int_0^t B(s)\mathrm{d}s \right) \qquad \forall t \in [0,T].$$

Lemma 9. *(See* [CHKP18, Equation 18]*). Let $s \geq 0$. Then there is a constant $C = C(s) > 0$ such that for any $v \in H^s(\mathbb{R})$ and any $w \in H^{s+1}(\mathbb{T})$ one has $v \cdot w \in H^s(\mathbb{R})$ and*

$$\|vw\|_{H^s(\mathbb{R})} \leq C \|v\|_{H^s(\mathbb{R})} \|w\|_{H^{s+1}(\mathbb{T})}.$$

The above estimate is not optimal w.r.t. the assumed regularity. However, we do not need a stronger version and the proof is straight forward.

3. Proof of Theorem 2

Consider first the case $\alpha \in [2,5)$. Let us define the space

$$X := C([0,T], L^2(\mathbb{R})) \cap L^{q_{\mathrm{a}}(\alpha+1)}([0,T], L^{\alpha+1}(\mathbb{R}))$$

equipped with the norm

$$\|v\|_X := \|v\|_{L_t^\infty L_x^2} + \|v\|_{L_t^{q_{\mathrm{a}}(\alpha+1)} L^{\alpha+1}} \qquad \forall v \in X,$$

where T will be fixed later in the proof. The integral formulation of (3) reads as

$$v = e^{\mathrm{i}t\partial_x^2} v_0 \pm \mathrm{i} \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} G_\alpha(w,v)\mathrm{d}\tau =: \mathcal{T}(v). \tag{11}$$

By Banach's fixed-point theorem, it suffices to show that there are $R, T > 0$ such that $\mathcal{T}$ is a contractive self-mapping of

$$M(R,T) := \left\{ v \in X \,\middle|\, \|v\|_X \leq R \right\}.$$

Consider first the self-mapping property. For $r \in \{2, \alpha+1\}$ we have

$$\|\mathcal{T}v\|_{L_t^{q_a(r)} L_x^r} \leq \left\| e^{\mathrm{i}t\partial_x^2} v_0 \right\|_{L_t^{q_a(r)} L_x^r} + \left\| \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} G_\alpha(w,v) \mathrm{d}\tau \right\|_{L_t^{q_a(r)} L_x^r}.$$

By the homogeneous Strichartz estimate (9), we have

$$\left\| e^{\mathrm{i}t\partial_x^2} v_0 \right\|_{L_t^{q_a(r)} L_x^r} \lesssim \|v_0\|_2$$

for the first summand. This suggests the choice $R \approx \|v_0\|_2$. For the second summand, whose norm also needs to be comparable with R, we will split the integral term. We proceed with the estimates for the contraction property of $\mathcal{T}$, because the self-mapping property follows from them by setting $v = v_1$ and $v_2 = 0$. To that end, let us define $G_\alpha(w, v_1, v_2) := G_\alpha(w, v_1) - G_\alpha(w, v_2)$. Observe

$$\begin{aligned} G_\alpha(w, v_1, v_2) &= |w+v_1|^{\alpha-1}(w+v_1) - |w+v_2|^{\alpha-1}(w+v_2) \\ &= \left(|v_1+w|^{\alpha-1} - |v_2+w|^{\alpha-1}\right) v_1 \\ &\quad + \left(|v_2+w|^{\alpha-1} - |w|^{\alpha-1}\right)(v_1 - v_2) \\ &\quad + \left(|v_1+w|^{\alpha-1} - |v_2+w|^{\alpha-1}\right) w + |w|^{\alpha-1}(v_1 - v_2). \end{aligned} \tag{12}$$

Furthermore, one has

$$\left| |x|^{\alpha-1} - |y|^{\alpha-1} \right| \leq (\alpha-1) \max\left\{1, 2^{\alpha-2}\right\} \left(|x|^{\alpha-2} + |y|^{\alpha-2}\right) |x-y| \tag{13}$$

for any $x, y \in \mathbb{C}$. By the inhomogeneous Strichartz estimate, the above splitting of G_α and the size estimate (13), one obtains

$$\begin{aligned} &\left\| \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} G_\alpha(w, v_1, v_2) \mathrm{d}\tau \right\|_{L_t^{q_a(r)} L_x^r} \\ \lesssim\ & \left\| |v_1 - v_2|^{\alpha-1} v_1 \right\|_{L^{\gamma_a(\rho_1)}(L^{\rho_1})} + \left\| |v_2|^{\alpha-1}(v_1 - v_2) \right\|_{L^{\gamma_a(\rho_1)}(L^{\rho_1})} \\ &+ \left\| |v_1 - v_2|^{\alpha-1} w \right\|_{L^{\gamma_a(\rho_2)}(L^{\rho_2})} + \left\| |w|^{\alpha-1}(v_1 - v_2) \right\|_{L^{\gamma_a(\rho_3)}(L^{\rho_3})} \\ \lesssim\ & \left\| (|v_1|^{\alpha-2} + |v_2|^{\alpha-2}) |v_1 - v_2| v_1 \right\|_{L^{\gamma_a(\rho_1)}(L^{\rho_1})} + \left\| |v_2|^{\alpha-1}(v_1 - v_2) \right\|_{L^{\gamma_a(\rho_1)}(L^{\rho_1})} \\ &+ \left\| (|v_1|^{\alpha-2} + |v_2|^{\alpha-2}) |v_1 - v_2| w \right\|_{L^{\gamma_a(\rho_2)}(L^{\rho_2})} + \left\| |w|^{\alpha-1}(v_1 - v_2) \right\|_{L^{\gamma_a(\rho_3)}(L^{\rho_3})}, \end{aligned} \tag{14}$$

where the pairs $(\rho_i, \gamma_a(\rho_i))$ are dually admissible and are equal to the dual exponent of one plus the effective power of v of the corresponding term. Hence, $\rho_1 = \frac{\alpha+1}{\alpha}$, $\rho_2 = \frac{\alpha}{\alpha-1}$ and $\rho_3 = 2$. For the first summand, observe that

$$\begin{aligned} &\left\| (|v_1|^{\alpha-2} + |v_2|^{\alpha-2}) |v_1 - v_2| v_1 \right\|_{L^{\gamma_a(\rho_1)}(L^{\rho_1})} \\ \leq\ & T^{\frac{5-\alpha}{4}} \|v_1\|_{L^{q_a(\alpha+1)}(L^{\alpha+1})} \left(\|v_1\|^{\alpha-2}_{L^{q_a(\alpha+1)}(L^{\alpha+1})} + \|v_2\|^{\alpha-2}_{L^{q_a(\alpha+1)}(L^{\alpha+1})} \right) \\ & \cdot \|v_1 - v_2\|_{L^{q_a(\alpha+1)}(L^{\alpha+1})} \\ \leq\ & T^{\frac{5-\alpha}{4}} R^{\alpha-1} \|v_1 - v_2\|_X. \end{aligned}$$

The second summand is estimated in the same way. For the third summand we have

$$\begin{aligned}
&\left\| (|v_1|^{\alpha-2} + |v_2|^{\alpha-2}) \, |v_1 - v_2| \, w \right\|_{L^{\gamma_a(\rho_2)}(L^{\rho_2})} \\
\lesssim\ & \|w\|_{L^\infty([0,T],H^1(\mathbb{T}))} T^{\frac{6-\alpha}{4}} \left(\|v_1\|^{\alpha-2}_{L^{q_a(\alpha)}([0,T],L^\alpha)} + \|v_2\|^{\alpha-2}_{L^{q_a(\alpha)}([0,T],L^\alpha)} \right) \\
&\cdot \|v_1 - v_2\|_{L^{q_a(\alpha)}([0,T],L^\alpha)} \\
\lesssim\ & T^{\frac{6-\alpha}{4}} \|w\|_{L^\infty([0,T],H^1(\mathbb{T}))} R^{\alpha-2} \|v_1 - v_2\|_X .
\end{aligned}$$

The last term is estimated by

$$\left\| |w|^{\alpha-1} (v_1 - v_2) \right\|_{L^{\gamma_a(\rho_3)}(L^{\rho_3})} \lesssim \|w\|^{\alpha-1}_{L^\infty([0,T],H^1(\mathbb{T}))} T \|v_1 - v_2\|_X .$$

Choosing T small enough shows the contraction property of $\mathcal{T}$ and the proof, in the case $\alpha \in [2, 5)$, concludes.

For the remaining critical case $\alpha = 5$, consider the complete metric space

$$M(R,T) := \left\{ v \in X \;\middle|\; \left\| v - e^{\mathrm{i}t\partial_x^2} v_0 \right\|_{L_t^\infty L_x^2} + \|v\|_{L_t^6 L_x^6} \le R \right\}.$$

It is again to show that $\mathcal{T}$ is a contractive self-mapping of $M(R,T)$ for some $R, T > 0$. Candidates for R and T are determined from the first term of (12), corresponding to the effective power $|v|^5$, exactly as in the treatment of the usual mass critical NLS (see e.g. [LP15, Theorem 5.3]). Subsequently, the remaining terms corresponding to the effective powers $|v|^4$ and $|v|^1$ are treated via the Strichartz estimates as in the case $\alpha \in [2, 5)$ enforcing a possibly smaller choice of T. We omit the details. ☐

Remark 10. *Observe, that for $\alpha \in (1, 2)$, the proof would proceed unchanged up to the inequality* (14)*. However, the term with the norm index ρ_2 needs to be controlled by a space-time norm with the space index in the interval* $[2, \infty]$*. This is not possible under the above assumption, as $(\alpha - 1)\rho_2 \in (1, 2)$.*

4. Proof of Theorem 4

The proof of Theorem 4 will be done by looking at the mass $\frac{1}{2} \|v(t)\|_2^2$ of the solution. In order to make this rigorous we have to work with solutions which are differentiable in time. We will get time regularity from regularity in space. Hence we replace G_2 in (3) by its smooth version G^ε. We obtain

$$\begin{cases} \mathrm{i}v_t(x,t) + \partial_x^2 v(x,t) \pm G^\varepsilon(w,v) = 0 & (x,t) \in \mathbb{R} \times \mathbb{R}, \\ v(\cdot,0) = v_0, \end{cases} \tag{15}$$

where

$$G^\varepsilon(w,v) := [|v + w| * \phi_\varepsilon](v + w) - [|w| * \phi_\varepsilon]w. \tag{16}$$

Theorem 11 (Local well-posedness of the smoothened modified NLS)**.** *Let $\varepsilon \ge 0$. Then there is a constant $C > 0$ such that for any $v_0 \in L^2$ and any $w \in C(\mathbb{R}, L_x^\infty)$ the Cauchy problem* (15) *has a unique solution in $C([0,T], L^2(\mathbb{R}))$, provided*

$$T \le C \min \left\{ \|v_0\|_2^{-\frac{4}{3}}, \|w\|^{-1}_{L_t^\infty, L_x^\infty} \right\}. \tag{17}$$

Observe, that the time T above does not depend on ε.

Proof. Consider the integral formulation of (15), i.e.

$$v = e^{\mathrm{i}t\partial_x^2} v_0 \pm \mathrm{i} \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} G^\varepsilon(w,v) \mathrm{d}\tau =: \mathcal{T}^\varepsilon(v) \tag{18}$$

and notice that

$$G^\varepsilon(w,v) = \underbrace{\left([|v+w|-|w|]*\phi_\varepsilon\right)v}_{=:G_1^\varepsilon(w,v)} + \underbrace{\left([|v+w|-|w|]*\phi_\varepsilon\right)w + [|w|*\phi_\varepsilon]v}_{=:G_2^\varepsilon(w,v)}.$$

By Banach's fixed-point theorem, it suffices to show that there are $R, T > 0$ such that $\mathcal{T}^\varepsilon$ is a contractive self-mapping of

$$M(R,T) := \left\{ v \in C([0,T], L^2(\mathbb{R})) \,\middle|\, \|v\| \le R \right\}.$$

Consider first the self-mapping property. We have

$$\|\mathcal{T}^\varepsilon v\|_{L_t^\infty L_x^2} \le \left\| e^{\mathrm{i}t\partial_x^2} v_0 \right\|_{L_t^\infty L_x^2} + \left\| \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} G^\varepsilon(w,v)\mathrm{d}\tau \right\|_{L_t^\infty L_x^2}.$$

Since the operator $e^{it\partial_x^2}$ is an isometry on L^2 we have

$$\left\| e^{\mathrm{i}t\partial_x^2} v_0 \right\|_{L_t^\infty L_x^2} = \|v_0\|_2$$

for the first summand. This suggests the choice $R \approx \|v_0\|_2$. For the second summand, whose norm needs to also be comparable with R, we split the integral term and obtain

$$\begin{aligned} &\left\| \int_0^T e^{\mathrm{i}(t-\tau)\partial_x^2} G^\varepsilon(w,v)\mathrm{d}\tau \right\|_{L_t^\infty L_x^2} \\ \le\ & \left\| \int_0^T e^{\mathrm{i}(t-\tau)\partial_x^2} G_1^\varepsilon(w,v)\mathrm{d}\tau \right\|_{L_t^\infty L_x^2} + \left\| \int_0^T e^{\mathrm{i}(t-\tau)\partial_x^2} G_2^\varepsilon(w,v)\mathrm{d}\tau \right\|_{L_t^\infty L_x^2}. \end{aligned}$$

Now, both summands are treated via the inhomogeneous Strichartz estimate as in the proof of Theorem 2. More precisely, one has

$$\begin{aligned} \left\| \int_0^T e^{\mathrm{i}(t-\tau)\partial_x^2} G_1^\varepsilon(w,v)\mathrm{d}\tau \right\|_{L_t^\infty L_x^2} &\lesssim \|([|v+w|-|w|]*\phi_\epsilon)v)\|_{L^{\gamma}(L^\rho)} \\ &\le \left\| \|[|v+w|-|w|]*\phi_\epsilon\|_{L_x^{2\rho}} \|v\|_{L_x^{2\rho}} \right\|_{L_t^\gamma} \\ &\le \left\| \|v\|_{L_x^{2\rho}}^2 \right\|_{L_t^\gamma} = \|v\|_{L^{2\gamma}(L^{2\rho})}^2. \end{aligned}$$

Above, we used the Cauchy-Schwartz inequality to arrive at the second line and Young's inequality (if $\varepsilon \ne 0$) and a size estimate to pass to the last line (all in the space variable).

As we want to arrive at the norm in $C([0,T], L^2(\mathbb{R}))$, we put $2\rho = 2$, i.e. $\rho = 1$. Then, from the admissibility condition (7) for (ρ', γ'), one obtains $\gamma = \frac{4}{3}$. As $2\gamma = \frac{8}{3} < \infty = q_{\mathrm{a}}(2)$, one can raise the time exponent to ∞ by Hölder's inequality for the time variable, i.e.

$$\|v\|_{L^{2\gamma}(L^{2\rho})}^2 \le T^{\frac{3}{4}} \|v\|_{L^\infty(L^2)}^2 \le T^{\frac{3}{4}} R^2 \overset{!}{\lesssim} R. \tag{19}$$

This inequality holds under the condition

$$T \lesssim_1 \|v_0\|_2^{-\frac{4}{3}},$$

which is satisfied by (17).

For G_2^ε we similarly obtain

$$\begin{aligned}\left\| \int_0^T e^{\mathrm{i}(t-\tau)\partial_x^2} G_2^\varepsilon(w,v)\mathrm{d}\tau \right\|_{L_t^\infty L_x^2} &\lesssim \|([|v+w|-|w|]*\phi_\varepsilon])w\|_{L^{\tilde\gamma}(L^{\tilde\rho})} + \|[|w|*\phi_\varepsilon]v\|_{L^{\tilde\gamma}(L^{\tilde\rho})} \\ &\leq \|w\|_{L^\infty(L^\infty)} \|[|v+w|-|w|]*\phi_\varepsilon]\|_{L^{\tilde\gamma}(L^{\tilde\rho})} \\ &\quad + \|[|w|*\phi_\varepsilon]\|_{L^\infty(L^\infty)} \|v\|_{L^{\tilde\gamma}(L^{\tilde\rho})} \\ &\lesssim \|w\|_{L^\infty(L^\infty)} \|v\|_{L^{\tilde\gamma}(L^{\tilde\rho})},\end{aligned}$$

where we employed Young's inequality and a size estimate to obtain the last line. In contrast to the G_1-case, we choose $\tilde\rho = 2$ to arrive at the norm in $C([0,T], L^2(\mathbb{R}))$. Then, by the admissibility condition (7), $\tilde\gamma = 1 < \infty = q_{\mathrm{a}}(2)$. Hence, by exploiting again the Hölder's inequality for the time variable, we get

$$\begin{aligned}\|w\|_{L^\infty(L^\infty)} \|v\|_{L^{\tilde\gamma}(L^{\tilde\rho})} &= \|w\|_{L^\infty(L^\infty)} \|v\|_{L^1(L^2)} \\ &\leq \|w\|_{L^\infty(L^\infty)} T \|v\|_{L^\infty(L^2)} \\ &\leq \|w\|_{L^\infty(L^\infty)} RT \\ &\overset{!}{\lesssim}_1 R.\end{aligned}$$

From this we obtain the additional condition

$$T \lesssim \|w\|_{L^\infty(L^\infty)}^{-1},$$

which is also satisfied by (17).

For the contraction property, consider the splitting

$$\begin{aligned}G^\varepsilon(w,v_1,v_2) &:= G^\varepsilon(w,v_1) - G^\varepsilon(w,v_2) \\ &= [|v_1+w|*\phi_\varepsilon](v_1+w) - [|v_2+w|*\phi_\varepsilon](v_2+w) \\ &= \underbrace{([|v_1+w|-|w|]*\phi_\varepsilon)(v_1-v_2) + ([|v_1+w|-|v_2+w|]*\phi_\varepsilon)v_2}_{=:G_1^\varepsilon(w,v_1,v_2)} \\ &\quad + \underbrace{([|v_1+w|-|v_2+w|]*\phi_\varepsilon)w + [|w|*\phi_\varepsilon](v_1-v_2)}_{=:G_2^\varepsilon(w,v_1,v_2)}.\end{aligned}$$

Arguments similar to those used in the proof of the self-mapping property shown above yield the contraction property of $\mathcal{T}^\varepsilon$, possibly requiring an even smaller implicit constant in (17). □

Lemma 12 (Convergence of the solutions for vanishing smoothing)**.** *Fix $v_0 \in L^2$ and $w \in C(\mathbb{R}, C(\mathbb{T}))$, and denote by $v^\varepsilon \in C([0,T], L^2(\mathbb{R}))$ for all $\varepsilon \geq 0$ the unique solution of the Cauchy problem* (15) *from Theorem 11. Then,*

$$\left\|v^\varepsilon - v^0\right\|_{L_t^\infty L_x^2} \xrightarrow{\varepsilon\to 0+} 0.$$

Proof. Recall, that by construction v^ε and v^0 are fixed points of $\mathcal{T}^\varepsilon$ and $\mathcal{T}^0$ respectively and hence

$$\begin{aligned}\left\|v^\varepsilon - v^0\right\|_{L_t^\infty L_x^2} &\leq \left\| \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} \left(G^\varepsilon(w,v^\varepsilon) - G^0(w,v^0)\right) \mathrm{d}\tau \right\|_{L_t^\infty L_x^2} \\ &\leq \left\| \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} \left(G^\varepsilon(w,v^\varepsilon) - G^\varepsilon(w,v^0)\right) \mathrm{d}\tau \right\|_{L_t^\infty L_x^2} \\ &\quad + \left\| \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} \left(G^\varepsilon(w,v^0) - G^0(w,v^0)\right) \mathrm{d}\tau \right\|_{L_t^\infty L_x^2}.\end{aligned}$$

Due to the fact that $\mathcal{T}^\varepsilon$ is contractive, the first summand is controlled by

$$\left\| \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} \left(G^\varepsilon(w,v^\varepsilon) - G^\varepsilon(w,v^0)\right) \mathrm{d}\tau \right\|_{L_t^\infty L_x^2} \leq C \left\| v^0 - v^\varepsilon \right\|_{L_t^\infty L_x^2},$$

where $C<1$ is the contraction constant. Thus, it suffices to show that the second summand converges to zero. To that end we first gather terms with the same effective powers of v^0 and w, i.e.

$$\begin{aligned}
&\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} \left(G^\varepsilon(w,v^0) - G^0(w,v^0)\right) \mathrm{d}\tau \\
&= \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} \big(\left[\left|w+v^0\right| * \phi_\varepsilon\right] (v^0+w) - [|w| * \phi_\varepsilon]\, w \\
&\quad - \left|w+v^0\right| (v^0+w) + |w|\, w \big) \mathrm{d}\tau \\
&= \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} \left(\left[\left(\left|w+v^0\right| - |w|\right) * \phi_\varepsilon - \left(\left|w+v^0\right| - |w|\right)\right] v^0 \right) \mathrm{d}\tau \qquad (20)\\
&\quad + \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} \big(\left[\left(\left|w+v^0\right| - |w|\right) * \phi_\varepsilon - \left(\left|w+v^0\right| - |w|\right)\right] w \qquad (21)\\
&\quad + \left(|w| * \phi_\varepsilon - |w|\right) v^0 \big) \mathrm{d}\tau.
\end{aligned}$$

The first summand corresponding to $\left|v^0\right|^2$ is treated in the same way as the G_1^ε-term in the proof of Theorem 11, i.e. via a Strichartz estimate and Hölder's inequality. We arrive at

$$\begin{aligned}
&\left\| \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} \left(\left[\left(\left|w+v^0\right| - |w|\right) * \phi_\varepsilon - \left(\left|w+v^0\right| - |w|\right)\right] v^0 \right) \mathrm{d}\tau \right\|_{L_t^\infty L_x^2} \\
\leq\ & \left\| \left(\left|w+v^0\right| - |w|\right) * \phi_\varepsilon - \left(\left|w+v^0\right| - |w|\right) \right\|_{L_t^{\frac{4}{3}} L_x^2} \cdot \left\| v^0 \right\|_{L_t^\infty L_x^2}.
\end{aligned}$$

It suffices to show that the fist factor above goes to zero, as ε goes to zero. For almost every $t \in [0,T]$ we have that $\left(\left|w+v^0\right| - |w|\right) \in L^2$, which implies, due to the fact that $(\phi_\varepsilon)_{\varepsilon>0}$ is an approximation to the identity, that

$$\left\| \left(\left|w+v^0\right| - |w|\right) * \phi_\varepsilon - \left(\left|w+v^0\right| - |w|\right) \right\|_{L_x^2} \xrightarrow{\varepsilon \to 0+} 0.$$

Furthermore, by Young's inequality,

$$\left\| \left(\left|w+v^0\right| - |w|\right) * \phi_\varepsilon - \left(\left|w+v^0\right| - |w|\right) \right\|_{L_x^2}^{\frac{4}{3}} \lesssim \left\| v^0 \right\|_{L_x^2}^{\frac{4}{3}}$$

for every $\varepsilon>0$ and almost every $t \in [0,T]$. Also,

$$\int_0^T \left\| v^0(\cdot,\tau) \right\|_{L_x^2}^{\frac{4}{3}} \mathrm{d}\tau = \left\| v^0 \right\|_{L_t^{\frac{4}{3}} L_x^2}^{\frac{4}{3}} \lesssim_T \left\| v^0 \right\|_{L_t^\infty L_x^2}^{\frac{4}{3}}$$

and hence the claim follows by the dominated convergence theorem.

The second summand (Equation (21)), corresponding to $\left|v^0 w\right|$, is treated like the G_2^ε-term and we arrive at

$$\begin{aligned}
&\left\| \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} \left(\left[\left(\left|w+v^0\right| - |w|\right) * \phi_\varepsilon - \left(\left|w+v^0\right| - |w|\right)\right] w + \left(|w| * \phi_\varepsilon - |w|\right) v^0 \right) \mathrm{d}\tau \right\|_{L_t^\infty L_x^2} \\
\leq\ & \left\| \left(\left|w+v^0\right| - |w|\right) * \phi_\varepsilon - \left(\left|w+v^0\right| - |w|\right) \right\|_{L_t^1 L_x^2} \cdot \|w\|_{L_t^\infty L_x^\infty} + \left\| v^0 \right\|_{L_t^\infty L_x^2} \left\| |w| * \phi_\varepsilon - |w| \right\|_{L_t^1 L_x^\infty}.
\end{aligned}$$

Observe, that $|w|$ is uniformly continuous in the x-variable on the whole of $\mathbb{R}$. Hence, as for (20), the fact that $(\phi_\varepsilon)_{\varepsilon>0}$ is an approximation to the identity implies the convergence to zero of (21). □

Lemma 13 (Smooth solutions for smooth initial data)**.** *(Cf.* [Tao06, Proposition 3.11]*.) Let $\varepsilon > 0$, $w \in C([0,T], H^\infty(\mathbb{T}))$ and $v_0 \in \mathcal{S}$ and let v denote the unique solution of* (15)*. Then $v \in C^1([0,T], H^\infty(\mathbb{R}))$ and for any $s > \frac{1}{2}$ one has*

$$\|v\|_{L_t^\infty H_x^s} \leq C \|v_0\|_{H^s} \exp\left(\|v\|_{L_t^1 L_x^\infty} + T\|w\|_{C(H^{s+1}(\mathbb{T}))}\right) \tag{22}$$

for some $C = C(\varepsilon, s) > 0$.

Proof. We begin by showing that $v \in C([0,T], H^s(\mathbb{R}))$ for any $s \in \mathbb{N}$. To that end, we will show that the operator $\mathcal{T}^\varepsilon$ from Theorem 11 is a contractive mapping in $M(R, T') \subseteq H^s$, for a possibly smaller $T' \leq T$. We only show the self-mapping property. To that end, observe that

$$\begin{aligned} \|\mathcal{T}^\varepsilon v\|_{H^s} &\leq \left\|e^{\mathrm{i}t\partial_x^2} v_0\right\|_{H^s} + \left\|\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} G^\varepsilon(w,v)\mathrm{d}\tau\right\|_{H^s} \\ &\leq \|v_0\|_{H^s} + \int_0^t \|G^\varepsilon(w,v)\|_{H^s}\,\mathrm{d}\tau. \end{aligned}$$

The first summand fixes $R \approx \|v_0\|_{H^s}$. For the integrand in the second summand we have (the variable τ is omitted in the notation)

$$\begin{aligned} &\|G^\varepsilon(w,v)\|_{H^s} \\ \leq\ &\underbrace{\|([|w+v|-|w|]*\phi_\varepsilon)\,v\|_{H^s}}_{=:I} + \underbrace{\|(|w|*\phi_\varepsilon)\,v\|_{H^s}}_{=:II} + \underbrace{\|([|w+v|-|w|]*\phi_\varepsilon)\,w\|_{H^s}}_{=:III}. \end{aligned} \tag{23}$$

As $H^s(\mathbb{R})$ is an algebra with respect to the point-wise multiplication, the first summand is estimated against

$$\|([|w+v|-|w|]*\phi_\varepsilon)\,v\|_{H^s} \lesssim \|[|w+v|-|w|]*\phi_\varepsilon\|_{H^s}\,\|v\|_{H^s}.$$

The first product above is further estimated via the characterization of the H^s norm in terms of derivatives and Young's inequality as

$$\begin{aligned} \|[|w+v|-|w|]*\phi_\varepsilon\|_{H^s} &\approx \sum_{|\alpha|\leq s} \|([|w+v|-|w|])*[D^\alpha\phi_\varepsilon]\|_2 \\ &\leq \left(\sum_{|\alpha|\leq s}\|D^\alpha\phi_\varepsilon\|_1\right)\|v\|_2. \end{aligned} \tag{24}$$

Further estimating $\|v\|_2 \leq \|v\|_{H^s} \leq R$ and recalling the integral concludes the discussion of this term. The second summand (II) is treated via Lemma 9

$$\|(|w|*\phi_\varepsilon)\,v\|_{H^s} \lesssim_s \||w|*\phi_\varepsilon\|_{H^{s+1}(\mathbb{T})}\,\|v\|_{H^s}.$$

We again estimate $\|v\|_{H^s} \leq R$ and observe for the other factor that

$$\begin{aligned} \||w|*\phi_\varepsilon\|_{H^{s+1}(\mathbb{T})} &\approx \sum_{|\alpha|\leq(s+1)} \||w|*[D^\alpha\phi_\varepsilon]\|_{L^2(\mathbb{T})} \\ &\leq \|w\|_\infty \sum_{|\alpha|\leq(s+1)}\|D^\alpha\phi_\varepsilon\|_{L^1(\mathbb{R})} \\ &\lesssim_{\varepsilon,s} \|w\|_{H^{s+1}(\mathbb{T})}. \end{aligned}$$

The last summand (III) is estimated via

$$\|([|w+v|-|w|]*\phi_\varepsilon)\,w\|_{H^s} \lesssim_{\varepsilon,s} \|v\|_{H^s}\,\|w\|_{H^{s+1}(\mathbb{T})}.$$

The proof of the above requires no new techniques and is omitted. All in all this shows the local well-posedness of (15) in $C([0,T'], H^s)$, where the guaranteed time of existence is

$$T' \approx_{\varepsilon,s} \left\{\|w\|_{H^{s+1}(\mathbb{T})}^{-1}, \|v_0\|_{H^s(\mathbb{R})}^{-1}\right\}.$$

To prove the estimate (22), we will employ Lemma 7 (Gronwall's inequality). To that end, let T' be now the maximal time of existence of the solution $v \in C([0,T'), H^s)$. Observe that

$$\|v(\cdot,t)\|_{H^s} = \|(\mathcal{T}^\varepsilon v)(\cdot,t)\|_{H^s} \le \|v_0\|_{H^s} + \int_0^t \|G^\varepsilon(w,v)(\cdot,\tau)\|_{H^s}\,\mathrm{d}\tau \qquad \forall t \in [0,T').$$

The integrand above is estimated as in inequality (23). The first term (I), however, needs retreatment, as it is quadratic in $\|v\|_{H^s}$. The algebra property of $H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$ implies

$$I \le \|([|w+v| - |w|] * \phi_\varepsilon)\|_{H^s} \|v\|_\infty + \|([|w+v| - |w|] * \phi_\varepsilon)\|_\infty \|v\|_{H^s}.$$

We estimate the first factor in the first summand by (24). For the first factor of the second summand we have

$$\|([|w+v| - |w|] * \phi_\varepsilon)\|_\infty \le \|[|w+v| - |w|]\|_\infty \|\phi_\varepsilon\|_1 \le \|v\|_\infty$$

by Young's inequality. Reinserting the estimates for the terms (II) and (III) yields

$$\|v(\cdot,t)\|_{H^s} \lesssim_{s,\varepsilon} \|v_0\|_{H^s} + \int_0^t \Big(\|v(\cdot,\tau)\|_\infty + \|w(\cdot,\tau)\|_{H^{s+1}(\mathbb{T})}\Big) \|v(\cdot,\tau)\|_{H^s}\,\mathrm{d}\tau.$$

Gronwall's inequality now implies

$$\begin{aligned} \|v(\cdot,t)\|_{H^s} \quad &\lesssim_{\varepsilon,s} \quad \|v_0\|_{H^s} \exp\left(\int_0^t \Big(\|v(\cdot,\tau)\|_\infty + \|w(\cdot,\tau)\|_{H^{s+1}(\mathbb{T})}\Big)\,\mathrm{d}\tau\right) \\ &\le \quad \|v_0\|_{H^s} \exp\Big(\|v\|_{L^1_t L^\infty_x} + T' \|w\|_{C(H^{s+1}(\mathbb{T}))}\Big) \qquad \forall t \in [0,T'). \end{aligned}$$

Thus we see that a blowup cannot occur for any $T' < T$ and so $T' = T$.

This indeed shows that $v \in C([0,T], H^s)$. As $v_0 \in \mathcal{S}$ and $w \in C([0,T], H^\infty(\mathbb{T}))$ are smooth, a classical result from the semi-group theory (see [Paz92, Theorem 4.2.4]) implies that $v \in C^1([0,T], H^s)$. Because $s \in \mathbb{N}$ was arbitrary, the proof is complete. □

Proposition 14. *The unique solution v of* (15) *from Theorem 11 satisfies*

$$\|v(\cdot,t)\|_2 \le \|v_0\|_2 \exp\Big[\|w\|_{L^\infty_t L^\infty_x}\, t\Big] \qquad \forall t \in [0,T]. \tag{25}$$

Proof. Let $w^n \in C([0,T], H^\infty(\mathbb{T}))$ be functions with the property

$$\|w^n - w\|_{C([0,T],H^1(\mathbb{T}))} \xrightarrow{n\to\infty} 0$$

and let $v_n \xrightarrow{n\to\infty} v_0$ in the L^2-norm where $v_n \in \mathcal{S}$ for all $n \in \mathbb{N}$. Moreover, let $v^{\varepsilon,n} \in C^1([0,T], L^2))$ be the solution of (15) with initial data v_n and nonlinearity $G^\varepsilon(w^n, v^{\varepsilon,n})$ (the smoothness of $v^{\varepsilon,n}$ follows from Lemma 13). We have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v^{\varepsilon,n}(\cdot,t)\|_2^2 \quad &= \quad \mathrm{Re}\,\langle \dot{v}^{\varepsilon,n}(\cdot,t), v^{\varepsilon,n}(\cdot,t)\rangle = \mathrm{Re}\,\big\langle \mathrm{i}\partial_x^2 v^{\varepsilon,n} \pm \mathrm{i}G^\varepsilon(w^n, v^{\varepsilon,n}), v^{\varepsilon,n}\big\rangle \\ &= \quad \underbrace{-\,\mathrm{Re}\,\mathrm{i}\,\langle \nabla v^{\varepsilon,n}, \nabla v^{\varepsilon,n}\rangle}_{=0} \\ &\quad\ \pm \mathrm{Re}\,\mathrm{i}\,\langle (|v^{\varepsilon,n} + w^n| * \phi_\varepsilon)(v^{\varepsilon,n} + w^n) - (|w^n| * \phi_\varepsilon) w^n, v^{\varepsilon,n}\rangle \\ &= \quad \pm \underbrace{\mathrm{Re}\,\mathrm{i}\,\langle (|v^{\varepsilon,n} + w^n| * \phi_\varepsilon) v^{\varepsilon,n}, v^{\varepsilon,n}\rangle}_{=0} \\ &\quad\ \pm \mathrm{Re}\,\mathrm{i}\,\langle ([|v^{\varepsilon,n} + w^n| - |w^n|] * \phi_\varepsilon) w^n, v^{\varepsilon,n}\rangle \end{aligned} \tag{26}$$

and hence

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v^{\varepsilon,n}(\cdot,t)\|_2^2 &\leq |\langle([|v^{\varepsilon,n}+w^n|-|w^n|]*\phi_\varepsilon)w^n, v^{\varepsilon,n}\rangle| \\ &\leq \|([|v^{\varepsilon,n}+w^n|-|w^n|]*\phi_\varepsilon)w^n\|_{L_x^2}\|v^{\varepsilon,n}\|_{L_x^2} \\ &\leq \|w^n\|_{L_t^\infty L_x^\infty}\|v^{\varepsilon,n}\|_{L_x^2}^2 \qquad (27)\end{aligned}$$

for all $t\in[0,T]$. Above, we obtained the first estimate by the Cauchy-Schwartz inequality and the second one by Hölder's inequality, Young's inequality and the size estimate. By the differential form of the Gronwall's inequality from Lemma 8, we obtain

$$\|v^{\varepsilon,n}(\cdot,t)\|_2\leq\|v_n\|_2\exp\left[\|w^n\|_{L_t^\infty L_x^\infty}t\right]\qquad\forall t\in[0,T].$$

In the limit $n\to\infty$, the right-hand side above converges to the right-hand side of (25). It remains to show

$$\|v^{\varepsilon,n}-v^\varepsilon\|_{L^\infty L^2}\xrightarrow{n\to\infty}0, \qquad (28)$$

because then the left-hand side converges to $\|v^\varepsilon\|_{L_t^\infty L_x^2}$ in the limit $n\to\infty$. But by Lemma 12,

$$\|v^\varepsilon\|_{L_t^\infty L_x^2}\xrightarrow{\varepsilon\to 0}\left\|v^0\right\|_{L_t^\infty L_x^2}.$$

To prove (28), observe that the linear evolution poses no problems and hence it suffices to control the integral term

$$\left\|\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2}\left[G^\varepsilon(w,v^\varepsilon)-G^\varepsilon(w^n,v^{\varepsilon,n})\right]\mathrm{d}\tau\right\|_{L^\infty L^2}.$$

To that end, we will split the difference of the nonlinear terms according to their effective power up to one exception. We begin by observing that

$$\begin{aligned}&G^\varepsilon(w,v^\varepsilon)-G^\varepsilon(w^n,v^{\varepsilon,n})\\ =\;&(|w+v^\varepsilon|*\phi_\varepsilon)v^\varepsilon-(|w^n+v^{\varepsilon,n}|*\phi_\varepsilon)v^{\varepsilon,n}\\ &+([|w+v^\varepsilon|-|w|]*\phi_\varepsilon)w-([|w^n+v^{\varepsilon,n}|-|w^n|]*\phi_\varepsilon)w^n\end{aligned}$$

and gather the first and the second summand, as well as the third and the last summand. In the first sum we have

$$\begin{aligned}&(|w+v^\varepsilon|*\phi_\varepsilon)v^\varepsilon-(|w^n+v^{\varepsilon,n}|*\phi_\varepsilon)v^{\varepsilon,n}\\ =\;&\underbrace{(|w+v^\varepsilon|*\phi_\varepsilon)v^\varepsilon-(|w+v^\varepsilon|*\phi_\varepsilon)v^{\varepsilon,n}}_{=:I}\\ &+\underbrace{(|w+v^\varepsilon|*\phi_\varepsilon)v^{\varepsilon,n}-(|w^n+v^{\varepsilon,n}|*\phi_\varepsilon)v^{\varepsilon,n}}_{=:II},\end{aligned}$$

whereas for the second sum

$$\begin{aligned}&([|w+v^\varepsilon|-|w|]*\phi_\varepsilon)w-([|w^n+v^{\varepsilon,n}|-|w^n|]*\phi_\varepsilon)w^n\\ =\;&\underbrace{([|w^n+v^\varepsilon|-|w^n|]*\phi_\varepsilon)w^n-([|w^n+v^{\varepsilon,n}|-|w^n|]*\phi_\varepsilon)w^n}_{=:III}\\ &+\underbrace{([|w+v^\varepsilon|-|w|]*\phi_\varepsilon)w-([|w^n+v^\varepsilon|-|w^n|]*\phi_\varepsilon)w^n}_{=:IV}\end{aligned}$$

holds. We now complete the splitting of $G^\varepsilon(w,v^{\varepsilon,n})-G^\varepsilon(w^n,v^{\varepsilon,n})$ into terms of the same effective powers. We have

$$\begin{aligned}
I &= (|w+v^\varepsilon|*\phi_\varepsilon)(v^\varepsilon-v^{\varepsilon,n})\\
&= ([|w+v^\varepsilon|-|w|]*\phi_\varepsilon)(v^\varepsilon-v^{\varepsilon,n})+(|w|*\phi_\varepsilon)(v^\varepsilon-v^{\varepsilon,n}),\\
II &= ([|w+v^\varepsilon|-|w^n+v^{\varepsilon,n}|]*\phi_\varepsilon)v^{\varepsilon,n}\\
&= ([|w+v^\varepsilon|-|w+v^{\varepsilon,n}|]*\phi_\varepsilon)v^{\varepsilon,n}+([|w+v^{\varepsilon,n}|-|w^n+v^{\varepsilon,n}|]*\phi_\varepsilon)v^{\varepsilon,n},\\
III &= ([|w^n+v^\varepsilon|-|w^n+v^{\varepsilon,n}|]*\phi_\varepsilon)w^n \text{ and}\\
IV &= ([|w+v^\varepsilon|-|w|]*\phi_\varepsilon)w-([|w+v^\varepsilon|-|w|]*\phi_\varepsilon)w^n\\
&\quad -([|w^n+v^\varepsilon|-|w^n|]*\phi_\varepsilon)w^n+([|w+v^\varepsilon|-|w|]*\phi_\varepsilon)w^n\\
&= ([|w+v^\varepsilon|-|w|]*\phi_\varepsilon)(w-w^n)\\
&\quad +([|w+v^\varepsilon|-|w|-|w^n+v^\varepsilon|+|w^n|]*\phi_\varepsilon)w^n,
\end{aligned}$$

from which the effective powers are obvious, and put

$$\begin{aligned}
\tilde G_1^\varepsilon(w,w^n,v^\varepsilon,v^{\varepsilon,n}) &:= ([|w+v^\varepsilon|-|w|]*\phi_\varepsilon)(v^\varepsilon-v^{\varepsilon,n})\\
&\quad +([|w+v^\varepsilon|-|w+v^{\varepsilon,n}|]*\phi_\varepsilon)v^{\varepsilon,n},\\
\tilde G_2^\varepsilon(w,w^n,v^\varepsilon,v^{\varepsilon,n}) &:= (|w|*\phi_\varepsilon)(v^\varepsilon-v^{\varepsilon,n})+([|w^n+v^\varepsilon|-|w^n+v^{\varepsilon,n}|]*\phi_\varepsilon)w^n\\
&\quad +([|w+v^{\varepsilon,n}|-|w^n+v^{\varepsilon,n}|]*\phi_\varepsilon)v^{\varepsilon,n}\\
&\quad +([|w+v^\varepsilon|-|w|]*\phi_\varepsilon)(w-w^n)\\
&\quad +([|w+v^\varepsilon|-|w|-|w^n+v^\varepsilon|+|w^n|]*\phi_\varepsilon)w^n.
\end{aligned}$$

Now, by the triangle inequality and the inhomogeneous Strichartz estimate, one has

$$\begin{aligned}
&\left\|\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2}\left[G^\varepsilon(w,v^{\varepsilon,n})-G^\varepsilon(w^n,v^{\varepsilon,n})\right]\mathrm{d}\tau\right\|_{L^\infty L^2}\\
\le\ &\left\|\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2}\tilde G_1^\varepsilon(w,w^n,v^\varepsilon,v^{\varepsilon,n})\mathrm{d}\tau\right\|_{L^\infty L^2}\\
&+\left\|\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2}\tilde G_2^\varepsilon(w,w^n,v^\varepsilon,v^{\varepsilon,n})\mathrm{d}\tau\right\|_{L^\infty L^2}\\
\lesssim\ &\left\|\tilde G_1^\varepsilon(w,w^n,v^\varepsilon,v^{\varepsilon,n})\right\|_{L_t^{\frac43}L_x^1}+\left\|\tilde G_2^\varepsilon(w,w^n,v^\varepsilon,v^{\varepsilon,n})\right\|_{L_t^1L_x^2}.
\end{aligned}$$

We begin by estimating the first summand above. In fact, we have

$$\begin{aligned}
&\|([|w+v^\varepsilon|-|w|]*\phi_\varepsilon)(v^\varepsilon-v^{\varepsilon,n})\|_{L_t^{\frac43}L_x^1}\\
\le\ &\left\|t\mapsto\|[|w+v^\varepsilon|-|w|]*\phi_\varepsilon\|_{L_x^2}\|v^\varepsilon-v^{\varepsilon,n}\|_{L_x^2}\right\|_{\frac43}\\
\le\ &\left\|t\mapsto\|v^\varepsilon\|_{L_x^2}\|v^\varepsilon-v^{\varepsilon,n}\|_{L_x^2}\right\|_{\frac43}\\
\le\ &T^{\frac34}\|v^\varepsilon\|_{L_t^\infty L_x^2}\|v^\varepsilon-v^{\varepsilon,n}\|_{L_t^\infty L_x^2}.
\end{aligned}$$

by the Cauchy-Schwarz, Young's and the inverse triangle inequalities for the space variable and Hölder's inequality for the time variable. Choosing T sufficiently small shows that

$$\|([|w+v^\varepsilon|-|w|]*\phi_\varepsilon)(v^\varepsilon-v^{\varepsilon,n})\|_{L_t^{\frac43}L_x^1}\le\frac15\|v^\varepsilon-v^{\varepsilon,n}\|_{L_t^\infty L_x^2}.$$

For the second term in the definition of $\tilde G_1^\varepsilon$ the same techniques are applied to yield the bound

$$\|([|w+v^\varepsilon|-|w+v^{\varepsilon,n}|]*\phi_\varepsilon)v^{\varepsilon,n}\|_{L_t^{\frac43}L_x^1}\le T^{\frac34}\|v^{\varepsilon,n}\|_{L_t^\infty L_x^2}\|v^\varepsilon-v^{\varepsilon,n}\|_{L_t^\infty L_x^2}.$$

By the proof of Theorem 11, one has

$$\|v^{\varepsilon,n}\|_{L_t^\infty L_x^2} \lesssim \|v_n\|_2 \approx \|v_0\|_2 \tag{29}$$

and thus choosing T sufficiently small, again yields

$$\|([|w+v^\varepsilon| - |w+v^{\varepsilon,n}|] * \phi_\varepsilon)v^{\varepsilon,n}\|_{L_t^{\frac{4}{3}} L_x^1} \leq \frac{1}{5} \|v^\varepsilon - v^{\varepsilon,n}\|_{L_t^\infty L_x^2}.$$

The first term in the definition of $\tilde{G}_2^\varepsilon$ is treated similarly to the above. The same is true for the second term, where we additionally observe that

$$\sup_{n\in\mathbb{N}} \|w^n\|_{C([0,T],H^1(\mathbb{T}))} < \infty. \tag{30}$$

For the third term, we have

$$\begin{aligned} & \|([|w+v^{\varepsilon,n}| - |w^n+v^{\varepsilon,n}|] * \phi_\varepsilon)v^{\varepsilon,n}\|_{L_t^1 L_x^2} \\ \leq\ & \|[|w+v^{\varepsilon,n}| - |w^n+v^{\varepsilon,n}|] * \phi_\varepsilon\|_{L_t^\infty L_x^\infty} \|v^{\varepsilon,n}\|_{L_t^\infty L_x^2} \\ \leq\ & \|w-w^n\|_{L_t^\infty L_x^\infty} \|v^{\varepsilon,n}\|_{L_t^\infty L_x^2} \\ \lesssim\ & \|w-w^n\|_{L_t^\infty H_x^1(\mathbb{T})} \xrightarrow{n\to\infty} 0, \end{aligned}$$

where the Cauchy-Schwarz inequality was used for the first estimate, the embedding $L_t^\infty \hookrightarrow L_t^1$, Young's inequality and the inverse triangle inequality for the second estimate and the embedding $C([0,T],H^1(\mathbb{T})) \hookrightarrow L_t^\infty L_x^\infty$ together with (29) for the last estimate. By the same techniques, one obtains the convergence of the fourth term to zero.

Finally, for the last term in the definition of $\tilde{G}_2^\varepsilon$, one has

$$\begin{aligned} & \|([|w+v^\varepsilon| - |w| - |w^n+v^\varepsilon| + |w^n|] * \phi_\varepsilon)w^n\|_{L_t^1 L_x^2} \\ \leq\ & \||w+v^\varepsilon| - |w| - |w^n+v^\varepsilon| + |w^n|\|_{L_t^1 L_x^2} \|w^n\|_{L^\infty H_x^1(\mathbb{T})} \\ \lesssim\ & \||w+v^\varepsilon| - |w| - |w^n+v^\varepsilon| + |w^n|\|_{L_t^1 L_x^2}, \end{aligned}$$

where Hölder's inequality, the embedding $C([0,T],H^1(\mathbb{T})) \hookrightarrow L_t^\infty L_x^\infty$ and Young's inequality were used for the first estimate and (30) for the second estimate. Observe that by the inverse triangle inequality, the bound

$$||w+v^\varepsilon| - |w| - |w^n+v^\varepsilon| + |w^n|| \leq 2\min\{|w-w^n|, |v^\varepsilon|\} \leq 2|v^\varepsilon|$$

holds pointwise (in t and x). This implies that

$$|w+v^\varepsilon| - |w| - |w^n+v^\varepsilon| + |w^n| \xrightarrow{n\to\infty} 0$$

and hence, by the theorem of dominated convergence for the space variable,

$$g_n(t) := \||w+v^\varepsilon| - |w| - |w^n+v^\varepsilon| + |w^n|\|_{L_x^2} \xrightarrow{n\to\infty} 0 \qquad \forall t \in [0,T].$$

Moreover, for all $t \in [0,T]$, we have $g_n(t) \leq 2\|v^\varepsilon(\cdot,t)\|_2$ and $\|v^\varepsilon\|_{L_t^1 L_x^2} \lesssim \|v^\varepsilon\|_{L_t^\infty L_x^2} < \infty$. Hence, reapplying the theorem of dominated convergence for the time variable yields

$$\|([|w+v^\varepsilon| - |w| - |w^n+v^\varepsilon| + |w^n|] * \phi_\varepsilon)w^n\|_{L_t^1 L_x^2} \xrightarrow{n\to\infty} 0$$

as claimed. □

Notice that (25) together with the local well-posedness of v from Theorem 11 imply that v is globally well-posed, i.e. Theorem 4 is proved.

Remark 15. *Observe, that in the case $\alpha > 2$, the proof would proceed roughly unchanged up to Equation* (26)*. However, the differential inequality* (27) *would then be replaced by*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v^{\varepsilon,n}(\cdot,t)\|_2^2 \lesssim \|w^n\|_{L_t^\infty L_x^\infty}^{\alpha-1} \|v^{\varepsilon,n}\|_{L_x^2}^2 + \|w^n\|_{L_t^\infty L_x^\infty} \|v^{\varepsilon,n}\|_{L_x^\alpha}^\alpha$$

and this bound does not suffice to exclude a blow-up of the L^2-norm.

Appendix A. Quadratic NLS on the torus

To prove global existence of solutions to the Cauchy problem of the quadratic nonlinear Schrödinger equation on $\mathbb{T}$ (that is (2) with $\alpha = 2$), we will employ the mass and energy conservation laws. The justification of conservation laws requires solutions which are differentiable in time. This time regularity will be obtained from the regularity in space. To that end we will smoothen out the rough quadratic nonlinearity, in such a way that the solutions of the resulting equation still admit suitable conservation laws. The regularization is slightly different from the one used in the proof of Theorem 4. Let us mention that the ideas presented here are borrowed from [GV79] where the same problem was studied on $\mathbb{R}^d$, using a contraction argument and conservation laws. Since our setting is based on the torus, we have to work with Bourgain spaces. For the convenience of the reader, we present some of the arguments in detail.

Observe, that if w is a sufficiently nice 2π-periodic function and $\varepsilon > 0$, then

$$\begin{aligned}(w * \phi_\varepsilon)(x) &= \int_{-\infty}^{\infty} w(y)\phi_\varepsilon(x-y)\mathrm{d}y = \sum_{n\in\mathbb{Z}} \int_{(2n-1)\pi}^{(2n+1)\pi} w(y)\phi_\varepsilon(x-y)\mathrm{d}y \\ &= \int_{-\pi}^{\pi} w(y) \sum_{n\in\mathbb{Z}} \phi_\varepsilon(x-y-2n\pi)\mathrm{d}y.\end{aligned}$$

Hence, the convolution of w with ϕ_ε on $\mathbb{R}$ corresponds to the convolution of w with the periodization of ϕ_ε on $\mathbb{T}$. For the rest of the paper we will slightly abuse the notation and denote this periodization also by ϕ_ε. In the same spirit we will use from now on $*$ to denote the convolution on $\mathbb{T}$.

The smooth version of (2) for $\alpha = 2$ reads as

$$\begin{cases} \mathrm{i}w_t(x,t) + \partial_x^2 w(x,t) \pm (|w * \phi_\varepsilon|\,(w * \phi_\varepsilon)) * \phi_\varepsilon = 0 & (x,t) \in \mathbb{T}\times\mathbb{R}, \\ & w(\cdot,0) = w_0 * \phi_\varepsilon \end{cases} \tag{31}$$

and the corresponding Duhamel's formula is (cf. [GV79, Equations (2.14), (2.13), (2.11) and (1.15)])

$$w(\cdot,t) = e^{\mathrm{i}t\partial_x^2}(w_0 * \phi_\varepsilon) \pm \mathrm{i} \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} \left[(|w * \phi_\varepsilon|\,(w * \phi_\varepsilon)) * \phi_\varepsilon(\cdot,\tau)\right] \mathrm{d}\tau. \tag{32}$$

We denote the *Fourier transform* by $\mathcal{F}$ and the inverse Fourier transform by $\mathcal{F}^{(-1)}$, where we use the symmetric choice of constants and write also

$$\begin{aligned}\hat{f}(\xi) &:= (\mathcal{F}f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-\mathrm{i}\xi\cdot x} f(x)\mathrm{d}x, \\ \check{g}(x) &:= \left(\mathcal{F}^{(-1)}g\right)(x) = \frac{1}{\sqrt{2\pi}} \sum_{\xi\in\mathbb{Z}} e^{\mathrm{i}\xi\cdot x} g(\xi).\end{aligned}$$

One has $\mathcal{F}(f * g) = \sqrt{2\pi}\hat{f}\hat{g}$. Furthermore, let $\langle\xi\rangle := \sqrt{1+|\xi|^2}$ for any $\xi \in \mathbb{R}$ and $J^s w := \mathcal{F}^{(-1)}\langle\cdot\rangle^s \mathcal{F}w$ for any $w \in (C^\infty(\mathbb{T}))'$.

A.1. Prerequisites.

In this section, we present some technical results from the literature, needed for treatment of the quadratic nonlinearity.

Lemma 16. *Let $p \in [1,\infty]$ and $\varepsilon \geq 0$. Then for any $w \in L^p(\mathbb{T})$ one has*

$$\|w * \phi_\varepsilon\|_{L^p(\mathbb{T})} \leq \|w\|_{L^p(\mathbb{T})}.$$

Lemma 17. *Let $s \in \mathbb{R}$ and $w \in H^s(\mathbb{T})$. Then*

$$\|w * \phi_\varepsilon\|_{H^s(\mathbb{T})} \leq \|w\|_{H^s(\mathbb{T})} \qquad \text{and} \qquad \|w * \phi_\varepsilon\|_{\dot{H}^s(\mathbb{T})} \leq \|w\|_{\dot{H}^s(\mathbb{T})} \qquad \forall \varepsilon \geq 0.$$

Furthermore, if $\varepsilon > 0$, then

$$\|w * \phi_\varepsilon\|_{H^s(\mathbb{T})} \lesssim_{\varepsilon,s} \|w\|_{L^2(\mathbb{T})}.$$

Lemma 18 (à la Banach-Alaoglu)**.** *(Cf.* [Bre11, Theorem 3.16]*.) Let $w^n \xrightarrow{n\to\infty} w$ in $L^2(\mathbb{T})$ and $\sup_{n\in\mathbb{N}} \|w^n\|_{H^1(\mathbb{T})} < \infty$. Then $w \in H^1(\mathbb{T})$,*

$$\|w\|_{H^1(\mathbb{T})} \leq \liminf_{n\to\infty} \|w^n\|_{H^1(\mathbb{T})}, \qquad \|w\|_{\dot{H}^1(\mathbb{T})} \leq \liminf_{n\to\infty} \|w^n\|_{\dot{H}^1(\mathbb{T})}, \tag{33}$$

and $w^n \rightharpoonup w$ in $H^1(\mathbb{T})$, i.e. for any $u \in H^1(\mathbb{T})$ one has

$$\lim_{n\to\infty} \langle w^n, u\rangle_{H^1(\mathbb{T})} = \lim_{n\to\infty} \sum_{k\in\mathbb{Z}} \langle k\rangle^2 \overline{\hat{w}_k^n} \hat{u}_k = \langle w, u\rangle_{H^1(\mathbb{T})}. \tag{34}$$

If additionally $\|w^n\|_{H^1} \xrightarrow{n\to\infty} \|w\|_{H^1}$, then $w^n \xrightarrow{n\to\infty} w$ in $H^1(\mathbb{T})$.

In the following we are going to use the $X^{s,b}$ spaces on the torus where $s, b \in \mathbb{R}$. They are defined via the norm (see equation (3.49) in [ET16])

$$\|w\|_{X^{s,b}} = \|\langle k\rangle^s \langle \tau + k^2\rangle^b \hat{w}(\tau, k)\|_{L^2_\tau l^2_k}. \tag{35}$$

Lemma 19 ($X^{0,\frac{3}{8}} \hookrightarrow L^4(\mathbb{T}\times\mathbb{R})$)**.** *(See* [Tao06, Proposition 2.13]*.) We have*

$$\|w\|_{L^4(\mathbb{T}\times\mathbb{R})} \lesssim \|w\|_{X^{0,\frac{3}{8}}}$$

for any $w \in \mathcal{S}(\mathbb{R}, C^\infty(\mathbb{T}))$.

Lemma 20 ($X^{s,b}_\delta \hookrightarrow C(H^s)$)**.** *(Cf.* [ET16, Lemma 3.9]*.) Let $b > \frac{1}{2}$ and $s \in \mathbb{R}$. Then*

$$\|w\|_{C([0,\delta],H^s(\mathbb{T}))} \lesssim \|w\|_{X^{s,b}_\delta}.$$

Lemma 21 (Linear Schrödinger evolution in $X^{s,b}_\delta$)**.** *(Cf.* [ET16, Lemma 3.10]*.) Let $b, s \in \mathbb{R}$, $\delta \in (0, 1]$ and η a smooth cut-off in time. Then*

$$\left\|\eta(t) e^{\mathrm{i}t\partial_x^2} w_0\right\|_{X^{s,b}_\delta} \lesssim \|w_0\|_{H^s(\mathbb{T})} \qquad \forall w_0 \in H^s(\mathbb{T}).$$

Lemma 22 (Treating the integral term in $X^{s,b}_\delta$)**.** *(Cf.* [ET16, Lemma 3.12]*.) Let $b \in \left(\frac{1}{2}, 1\right]$, $s \in \mathbb{R}$ and $\delta \leq 1$. Set $b' := b - 1$. Then*

$$\left\|\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} F(\tau) \mathrm{d}\tau\right\|_{X^{s,b}_\delta} \lesssim_b \|F\|_{X^{s,b'}_\delta} \qquad \forall F \in X^{s,b'}_\delta.$$

Lemma 23 (Changing b in $X^{s,b}_\delta$)**.** *(Cf.* [ET16, Lemma 3.11]*.) Let $b, b' \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ with $b' < b$, $s \in \mathbb{R}$ and $\delta \in (0, 1]$. Then*

$$\|w\|_{X^{s,b'}_\delta} \lesssim \delta^{b-b'} \|w\|_{X^{s,b}_\delta} \qquad \forall w.$$

The next proposition appears in [ET16] for the case of the cubic nonlinearity and $\varepsilon = 0$. Since we need the corresponding result for the quadratic nonlinearity, which is more complicated than the cubic nonlinearity, which has an algebraic structure, we present the proof, too.

Proposition 24 (Control of the nonlinearity in $X^{s,b}_\delta$)**.** *(Cf.* [ET16, Proposition 3.26]*.) Let $s \geq 0$ and $\varepsilon > 0$ or $\varepsilon = s = 0$. Then, for all w_1, w_2 we have*

$$\begin{aligned}&\|(|w_1 * \phi_\varepsilon| (w_1 * \phi_\varepsilon)) * \phi_\varepsilon - (|w_2 * \phi_\varepsilon| (w_2 * \phi_\varepsilon)) * \phi_\varepsilon\|_{X^{s,-\frac{3}{8}}_\delta} \\ \lesssim_{\varepsilon,s} &\left(\|w_1\|_{X^{0,\frac{3}{8}}_\delta} + \|w_2\|_{X^{0,\frac{3}{8}}_\delta}\right)\left(\|w_1 - w_2\|_{X^{0,\frac{3}{8}}_\delta}\right).\end{aligned}$$

Proof. Fix w_1, w_2. Then, by Plancherel theorem and duality in $L^2(\mathbb{R}\times\mathbb{T})$, one has

$$\begin{aligned}
&\left\|(|w_1*\phi_\varepsilon|(w_1*\phi_\varepsilon))*\phi_\varepsilon-(|w_2*\phi_\varepsilon|(w_2*\phi_\varepsilon))*\phi_\varepsilon\right\|_{X_\delta^{s,-\frac{3}{8}}}\\
=&\sup_{\|w\|_{X_\delta^{-s,\frac{3}{8}}}=1}\left|\langle(|w_1*\phi_\varepsilon|(w_1*\phi_\varepsilon)-|w_2*\phi_\varepsilon|(w_2*\phi_\varepsilon))*\phi_\varepsilon,w\rangle_{L^2(\mathbb{R}\times\mathbb{T})}\right|.
\end{aligned}$$

Fix any $w\in X_\delta^{-s,-\frac{3}{8}}$ with $\|w\|_{X_\delta^{-s,-\frac{3}{8}}}=1$. Then

$$\begin{aligned}
&\left|\langle(|w_1*\phi_\varepsilon|(w_1*\phi_\varepsilon)-|w_2*\phi_\varepsilon|(w_2*\phi_\varepsilon))*\phi_\varepsilon,w\rangle_{L^2(\mathbb{R}\times\mathbb{T})}\right|\\
=&\left|\left\langle J^s\left[(|w_1*\phi_\varepsilon|(w_1*\phi_\varepsilon)-|w_2*\phi_\varepsilon|(w_2*\phi_\varepsilon))*\phi_\varepsilon\right],J^{-s}w\right\rangle_{L^2(\mathbb{R}\times\mathbb{T})}\right|\\
\le&\left\|(|w_1*\phi_\varepsilon|(w_1*\phi_\varepsilon)-|w_2*\phi_\varepsilon|(w_2*\phi_\varepsilon))*(J^s\phi_\varepsilon)\right\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})}\left\|J^{-s}w\right\|_{L^4(\mathbb{R}\times\mathbb{T})}\\
\lesssim_{\varepsilon,s}&\left\||w_1*\phi_\varepsilon|(w_1*\phi_\varepsilon)-|w_2*\phi_\varepsilon|(w_2*\phi_\varepsilon)\right\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})}\underbrace{\left\|J^{-s}w\right\|_{X_\delta^{0,\frac{3}{8}}}}_{=\|w\|_{X_\delta^{-s,-\frac{3}{8}}}=1}\\
\le&\left\||w_1*\phi_\varepsilon|(w_1*\phi_\varepsilon)-|w_2*\phi_\varepsilon|(w_2*\phi_\varepsilon)\right\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})}\\
\le&\left\||w_1*\phi_\varepsilon|((w_1-w_2)*\phi_\varepsilon)\right\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})}\\
&+\left\|(|w_1*\phi_\varepsilon|-|w_2*\phi_\varepsilon|)(w_2*\phi_\varepsilon)\right\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})},
\end{aligned}$$

where, for the first estimate, we used Hölder's inequality and Young's inequality and Lemma 19 for the second. For the sake of brevity, we will only show the estimate for the second summand above. The first one is treated with the same techniques and is less difficult to handle. By the Hölder's and Young's inequalities, one has

$$\begin{aligned}
&\left\|(|w_1*\phi_\varepsilon|-|w_2*\phi_\varepsilon|)(w_2*\phi_\varepsilon)\right\|_{L^{\frac{4}{3}}(\mathbb{R}\times\mathbb{T})}\\
\le&\left\||w_1*\phi_\varepsilon|-|w_2*\phi_\varepsilon|\right\|_{L^4(\mathbb{R}\times\mathbb{T})}\|w_2*\phi_\varepsilon\|_{L^2(\mathbb{R}\times\mathbb{T})}\\
\le&\|w_1-w_2\|_{L^4(\mathbb{R}\times\mathbb{T})}\|w_2\|_{L^2(\mathbb{R}\times\mathbb{T})}\\
\le&\|w_1-w_2\|_{X_\delta^{0,\frac{3}{8}}}\|w_2\|_{X_\delta^{0,\frac{3}{8}}},
\end{aligned}$$

where we used Lemma 19 in the last step. This finishes the proof. ☐

A.2. **Results.** First, we consider local wellposedness:

Theorem 25. *(Cf.* [ET16, Theorem 3.27] *for the cubic NLS.) Let $\varepsilon>0$ and $s\ge 0$ or $\varepsilon=s=0$. Then the (smoothened) quadratic NLS* (31) *is* locally *well-posed in $H^s(\mathbb{T})$.*

Proof. It suffices to show that the right-hand side of (32) defines a contractive self-mapping $\mathcal{T}:M(R,\delta)\to M(R,\delta)$ for some $R,\delta>0$, where

$$M(R,\delta):=\left\{w\in Y\middle|\ \|w\|_Y\le R\right\}$$

and Y is a suitable subspace of $C([0,\delta],H^s(\mathbb{T}))$.

We consider the case $s\ge 1$ first. Put $Y=C([0,\delta],H^s(\mathbb{T}))$. Due to $e^{\mathrm{i}t\partial_x^2}$ being an isometry on $H^s(\mathbb{T})$ and Lemma 17 for any $t\in\mathbb{R}$, we have

$$\begin{aligned}
\|\mathcal{T}w\|_Y\le&\left\|e^{\mathrm{i}t\partial_x^2}(w_0*\phi_\varepsilon)\right\|_{H^s(\mathbb{T})}+\left\|\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2}\left[(|w*\phi_\varepsilon|(w*\phi_\varepsilon))*\phi_\varepsilon\right]\mathrm{d}\tau\right\|_Y\\
\le&\|w_0\|_{H^s(\mathbb{T})}+\delta\left\|(|w*\phi_\varepsilon|(w*\phi_\varepsilon))*\phi_\varepsilon\right\|_Y.
\end{aligned}$$

This suggests the choice $R \approx \|w_0\|_{H^s}$. Fix any $\tau \in [0,\delta]$. Then, due to Lemma 17 and the embedding $H^s(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$, we have that

$$\begin{aligned} & \|((|w*\phi_\varepsilon|\,(w*\phi_\varepsilon))*\phi_\varepsilon)(\cdot,\tau)\|_{H^s(\mathbb{T})} \\ \lesssim_{\varepsilon,s} \quad & \|(|w*\phi_\varepsilon|\,(w*\phi_\varepsilon))(\cdot,\tau)\|_{L^2(\mathbb{T})} \\ \leq \quad & \|(w*\phi_\varepsilon)(\cdot,\tau)\|_{L^\infty(\mathbb{T})} \|(w*\phi_\varepsilon)(\cdot,\tau)\|_{L^2(\mathbb{T})} \\ \lesssim \quad & R^2. \end{aligned}$$

By the above, the condition $\|\mathcal{T}w\|_Y \leq R$ is satisfied, if $\delta \lesssim_{\varepsilon,s} \frac{1}{R}$. The contraction property of $\mathcal{T}$ is shown in the same way, possibly requiring a smaller implicit constant in the last inequality.

In the case $s \in [0,1)$ and $\varepsilon > 0$, consider any $b \in \left(\frac{1}{2}, \frac{5}{8}\right)$ and put $Y = X_\delta^{s,b}$ (by Lemma 20 one indeed has $Y \hookrightarrow C([0,\delta], H^s(\mathbb{T}))$. Then, by the triangle inequality and Lemmata 21 and 22 we have

$$\begin{aligned} & \|\mathcal{T}w\|_{X_\delta^{s,b}} \\ \leq \quad & \left\| e^{\mathrm{i}t\partial_x^2}(w_0*\phi_\varepsilon)\right\|_{X_\delta^{s,b}} + \left\| \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2}\left[(|w*\phi_\varepsilon|\,(w*\phi_\varepsilon))*\phi_\varepsilon\right]\mathrm{d}\tau \right\|_{X_\delta^{s,b}} \\ \lesssim \quad & \|w_0\|_{H^s(\mathbb{T})} + \|(|w*\phi_\varepsilon|\,(w*\phi_\varepsilon))*\phi_\varepsilon\|_{X_\delta^{s,b-1}}. \end{aligned}$$

This estimate suggests $R \approx \|w_0\|_{H^s(\mathbb{T})}$. For the second summand, apply Lemma 23 and Proposition 24 (with $w=0$) to obtain the upper bound

$$\begin{aligned} \|(|w*\phi_\varepsilon|\,(w*\phi_\varepsilon))*\phi_\varepsilon\|_{X_\delta^{s,b-1}} \quad & \lesssim \quad \delta^{1-b-\frac{3}{8}} \|(|w*\phi_\varepsilon|\,(w*\phi_\varepsilon))*\phi_\varepsilon\|_{X_\delta^{s,-\frac{3}{8}}} \\ & \lesssim \quad \delta^{1-b-\frac{3}{8}} \|w\|_{X_\delta^{0,\frac{3}{8}}} \|w\|_{X_\delta^{s,\frac{3}{8}}} \\ & \leq \quad \delta^{1-b-\frac{3}{8}} \|w\|^2_{X_\delta^{s,\frac{3}{8}}} \leq \delta^{1-b-\frac{3}{8}} R^2. \end{aligned} \tag{36}$$

As the exponent of δ is positive, we can choose δ small enough to make $\mathcal{T}$ a self-mapping of $M(R,\delta)$. The fact that $\mathcal{T}$ is contractive is proven similarly, possibly requiring a smaller δ.

The remaining case $\varepsilon = s = 0$ is treated exactly as the last case. □

In order to prove the conservation laws, we need to be able to approximate by smooth solutions. As in the case of the modified NLS (15) (Lemma 13) one proves the following

Lemma 26 (Smooth solutions for smooth initial data)**.** *(Cf.* [Tao06, Proposition 3.11]*.) Let $\varepsilon > 0$, and $w_0 \in L^2(\mathbb{T})$ and let w denote the unique solution of* (32)*. Then $w \in C([0,\delta], H^\infty(\mathbb{R}))$ and for any $s > \frac{1}{2}$ one has*

$$\|w\|_{L_t^\infty H_x^s} \leq C \|w_0\|_{H^s} e^{C\|w\|_{L_t^1 L_x^\infty}} \tag{37}$$

for some $C = C(\varepsilon,s) > 0$.

Theorem 27. *Let $\varepsilon > 0$ and $s \in [1,\infty)$. Then the smoothened NLS* (31) *is* globally *well-posed in $H^s(\mathbb{T})$.*

Proof. Local well-posedness has already been shown in Theorem 25 and it remains to show that the solution w exists globally. By the blow-up alternative, it suffices to see that $\|w(\cdot,t)\|_{H^s(\mathbb{T})}$ cannot explode. Moreover, by Lemma 26 and the fact that for any $s' > \frac{1}{2}$ one has

$$\|w\|_{L_t^1 L_x^\infty} \lesssim_{\delta,s'} \|w\|_{C([0,\delta],H^{s'}(\mathbb{T}))}, \tag{38}$$

it suffices to consider $s = 1$. By the same Lemma, one has that $w \in C([0,\delta], H^\infty(\mathbb{T}))$ and in particular, $w \in C^1([0,\delta], H^1(\mathbb{T}))$. Hence, the energy conservation (cf. [GV79, Equations (3.14) and (1.18)])

$$E_\varepsilon(w(\cdot,t)) := \int_{\mathbb{T}} \frac{1}{2} |\nabla w(x,t)|^2 \mp \frac{1}{3} |(w * \phi_\varepsilon)(x,t)|^3 \, \mathrm{d}x = E_\varepsilon(w_0 * \phi_\varepsilon) \tag{39}$$

is applicable to w. But

$$\|w(\cdot,t)\|^2_{H^1(\mathbb{T})} = \|w_0\|_2^2 + 2E_\varepsilon(w_0 * \phi_\varepsilon) \pm \frac{2}{3} \|w(\cdot,t)\|_3^3 \tag{40}$$

and so $\|w(\cdot,t)\|_{\dot{H}^1(\mathbb{T})}$ is controlled by $E_\varepsilon(w_0 * \phi_\varepsilon)$ in the defocusing case. In the focusing case we can assume w.l.o.g. that $\|w(\cdot,t)\|^2_{\dot{H}^1(\mathbb{T})}$ is an unbounded function of t, (otherwise, there is nothing to show) and say that $\|w(\cdot,t)\|^2_{\dot{H}^1(\mathbb{T})}$ is large. Then, by the Gagliardo-Nirenberg inequality from [Bre11, Equation (42)], we have

$$\|w(\cdot,t)\|_3^3 \lesssim \|w(\cdot,t)\|_2^{\frac{5}{2}} \|w(\cdot,t)\|^{2-\frac{3}{2}}_{H^1(\mathbb{T})} \le \frac{1}{2} \|w(\cdot,t)\|^2_{H^1(\mathbb{T})}, \tag{41}$$

where above we additionally used the mass conservation

$$\|w(\cdot,t)\|_{L^2(\mathbb{T})} = \|w(\cdot,0)\|_{L^2(\mathbb{T})}.$$

Hence, inserting (41) into (40) and rearranging the inequality shows that $\|w(\cdot,t)\|^2_{H^1(\mathbb{T})}$ is bounded, in contradiction to the assumption. This completes the proof. ☐

Theorem 28. *(Cf.* [ET16, Theorem 3.28] *for the cubic NLS.) The Cauchy problem for the quadratic periodic NLS (*(2) *with* $\alpha = 2$*) is* globally *well-posed in* $L^2(\mathbb{T})$ *and the solution* u *enjoys mass conservation* $\|u(\cdot,t)\|_{L^2(\mathbb{T})} = \|u_0\|_{L^2(\mathbb{T})}$.

Proof. Local well-posedness has already been shown in Theorem 25. Let w denote this local solution. By the blow-up alternative, it suffices to show mass conservation. To that end, let us denote by w^ε the global solution of (31) for $\varepsilon > 0$ from Theorem 27. Observe that for any $b \in \left(\frac{1}{2}, \frac{5}{8}\right)$ one has

$$\begin{aligned}
&\|w^\varepsilon - w\|_{X^{0,b}_\delta} && (42)\\
\le\ & \left\| e^{\mathrm{i}t\partial_x^2}(w_0 * \phi_\varepsilon - w_0)\right\|_{X^{0,b}_\delta} \\
&+ \left\| \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} \left[(|w^\varepsilon * \phi_\varepsilon| (w^\varepsilon * \phi_\varepsilon)) * \phi_\varepsilon - |w|\, w \right] \mathrm{d}\tau \right\|_{X^{0,b}_\delta} && (43)\\
\lesssim\ & \|w_0 * \phi_\varepsilon - w_0\|_{L^2(\mathbb{T})} + \|(|w^\varepsilon * \phi_\varepsilon| (w^\varepsilon * \phi_\varepsilon)) * \phi_\varepsilon - |w|\, w\|_{X^{0,b-1}_\delta} \\
\lesssim\ & \|w_0 * \phi_\varepsilon - w_0\|_{L^2(\mathbb{T})} + \delta^{1-b} \|(|w^\varepsilon * \phi_\varepsilon| (w^\varepsilon * \phi_\varepsilon)) * \phi_\varepsilon - |w|\, w\|_{X^{0,0}_\delta},
\end{aligned}$$

where we used the fact that w and w^ε solve the corresponding fixed-point equations and Lemmata 21, 22 and 23.

For the first summand, observe that

$$\|w_0 * \phi_\varepsilon - w_0\|_{L^2(\mathbb{T})} = \left\| \left(\langle k \rangle^s \hat{w}_0(k) (\sqrt{2\pi} \hat{\phi}_\varepsilon(k) - 1) \right)_k \right\|_{l^2(\mathbb{Z})}$$

and the right-hand side above converges to 0 as $\varepsilon \to 0+$ by the dominated convergence theorem and the definition of ϕ_ε.

For the second summand, note that $X^{0,0}_\delta = L^2([0,\delta] \times \mathbb{T})$ and hence

$$\begin{aligned}
&\|(|w^\varepsilon * \phi_\varepsilon| (w^\varepsilon * \phi_\varepsilon)) * \phi_\varepsilon - |w|\, w\|_{L^2([0,\delta]\times\mathbb{T})} \\
\le\ & \|(|w|\, w) * \phi_\varepsilon - |w|\, w\|_{L^2([0,\delta]\times\mathbb{T})} \\
&+ \|(|w^\varepsilon * \phi_\varepsilon| (w^\varepsilon * \phi_\varepsilon) - |w|\, w) * \phi_\varepsilon\|_{L^2([0,\delta]\times\mathbb{T})}.
\end{aligned}$$

The first summand above goes to zero due to $(\phi_\varepsilon)_\varepsilon$ being an approximation to the identity on $L^2(\mathbb{Z})$. The other summand is further estimated by

$$\begin{aligned}
& \|(|w^\varepsilon * \phi_\varepsilon|\,(w^\varepsilon * \phi_\varepsilon) - |w|\,w) * \phi_\varepsilon\|_{L^2([0,\delta]\times\mathbb{T})} \\
\le\ & \||w^\varepsilon * \phi_\varepsilon|\,(w^\varepsilon * \phi_\varepsilon) - |w|\,w\|_{L^2([0,\delta]\times\mathbb{T})} \\
\le\ & \||w^\varepsilon * \phi_\varepsilon|\,(w^\varepsilon * \phi_\varepsilon - w)\|_{L^2([0,\delta]\times\mathbb{T})} + \|(|w^\varepsilon * \phi_\varepsilon| - |w|)\,w\|_{L^2([0,\delta]\times\mathbb{T})} \\
\le\ & \||w^\varepsilon * \phi_\varepsilon|\,(w * \phi_\varepsilon - w)\|_{L^2([0,\delta]\times\mathbb{T})} + \||w^\varepsilon * \phi_\varepsilon|\,((w^\varepsilon - w) * \phi_\varepsilon)\|_{L^2([0,\delta]\times\mathbb{T})} \\
& + \|(|w^\varepsilon * \phi_\varepsilon| - |w * \phi_\varepsilon|)\,w\|_{L^2([0,\delta]\times\mathbb{T})} + \|(|w * \phi_\varepsilon| - |w|)\,w\|_{L^2([0,\delta]\times\mathbb{T})}
\end{aligned}$$

and we now need to treat the four summands above. For the first one we have

$$\||w^\varepsilon * \phi_\varepsilon|\,(w * \phi_\varepsilon - w)\|_{L^2([0,\delta]\times\mathbb{T})} \le \|w^\varepsilon * \phi_\varepsilon\|_{L^4([0,\delta]\times\mathbb{T})}\,\|w * \phi_\varepsilon - w\|_{L^4([0,\delta]\times\mathbb{T})}$$

and the first factor above is bounded by

$$\|w^\varepsilon * \phi_\varepsilon\|_{L^4([0,\delta]\times\mathbb{T})} \lesssim \|w^\varepsilon\|_{X_\delta^{0,\frac{3}{8}}} \lesssim \|w_0 * \phi_\varepsilon\|_{L^2(\mathbb{T})} \le \|w_0\|_{L^2}$$

by Lemma 19 and the construction of w^ε. The second factor goes to zero due to $(\phi_\varepsilon)_\varepsilon$ being an approximation to identity.

For the third summand, observe that

$$\begin{aligned}
& \|(|w^\varepsilon * \phi_\varepsilon| - |w * \phi_\varepsilon|)\,w\|_{L^2([0,\delta]\times\mathbb{T})} \\
\le\ & \||w^\varepsilon * \phi_\varepsilon| - |w * \phi_\varepsilon|\|_{L^4([0,\delta]\times\mathbb{T})}\,\|w\|_{L^4([0,\delta]\times\mathbb{T})} \\
\le\ & \|(w^\varepsilon - w) * \phi_\varepsilon\|_{L^4([0,\delta]\times\mathbb{T})}\,\|w\|_{L^4([0,\delta]\times\mathbb{T})} \\
\lesssim\ & \|w^\varepsilon - w\|_{X_\delta^{0,b}}\,\|w\|_{X_\delta^{0,b}}\,.
\end{aligned}$$

Recall that in front of this term is $\delta^{1-b} \ll 1$ and hence we can just move it to the left-hand side of (42). The remaining two terms are treated with the same techniques.

The estimates above and Lemma 20 show that

$$\begin{aligned}
\|w\|_{C([0,T],L^2(\mathbb{T}))} &\le \limsup_{\varepsilon\to 0+} \left[\|w^\varepsilon - w\|_{X_\delta^{0,b}} + \|w^\varepsilon\|_{C([0,T],L^2(\mathbb{T}))}\right] \\
&\le \limsup_{\varepsilon\to 0+} \left[\|w_0 * \phi_\varepsilon\|_{L^2(\mathbb{T})}\right] = \|w_0\|_{L^2(\mathbb{T})}
\end{aligned}$$

and hence the solution w indeed enjoys mass conservation. This finishes the proof. □

In addition to mass conservation, we also have conservation of the energy.

Theorem 29. *(Cf.* [GV79, Theorem 3.1] *and* [LRS88, Theorem 2.1]*.) The Cauchy problem for the quadratic periodic NLS ((2) with $\alpha = 2$) is* globally *well-posed in $H^1(\mathbb{T})$ and the solution u enjoys energy conservation $E(u(\cdot,t)) = E(u_0)$.*

Remark 30. *In* [LRS88] *it is claimed that the quadratic NLS is globally well-posed on the torus. They refer to* [GV79]*, where it is done on the real line. While our proof of Theorem 29 borrows some ideas from* [GV79]*, we believe that in order to be able to do the torus case, one needs the result of Bourgain* [Bou93a]*, in particular, the Bourgain spaces, which appeared 5 years after* [LRS88].

Proof. Let $w_0 \in H^1(\mathbb{T}) \subseteq L^2(\mathbb{T})$. By Theorem 28, the quadratic periodic NLS has the unique global solution $w \in C_{\mathrm{b}}(\mathbb{R}, L^2(\mathbb{T}))$. It remains to show that $w \in C_{\mathrm{b}}(\mathbb{R}, H^1(\mathbb{T}))$. To show that for any $t \in \mathbb{R}$ one has $w(\cdot,t) \in H^1(\mathbb{T})$ we first prove that

$$\sup_{\varepsilon>0} \|w^\varepsilon\|_{C(\mathbb{R},H^1(\mathbb{T}))} < \infty. \tag{44}$$

By the calculations similar to those in the proof of Theorem 28, it suffices to prove the corresponding bound for the energy $E_\varepsilon(w^\varepsilon(\cdot,t))$.

To that end let w^ε be the unique global solution of the modified NLS (31) for $\varepsilon > 0$ from Theorem 27. The energy conservation from Equation (39) implies

$$E_\varepsilon(w^\varepsilon(\cdot,t)) = E_\varepsilon(w_0 * \phi_\varepsilon) = \frac{1}{2}\|w_0 * \phi_\varepsilon\|^2_{\dot{H}^1(\mathbb{T})} \mp \frac{1}{3}\|w_0 * \phi_\varepsilon\|^3_{L^3(\mathbb{T})}.$$

Observe that by Lemma 17 the first summand above satisfies

$$\|w_0 * \phi_\varepsilon\|^2_{\dot{H}^1(\mathbb{T})} \le \|w_0\|^2_{\dot{H}^1(\mathbb{T})}.$$

If the sign of the second summand is negative (focusing case), there is nothing left to do. If the sign is positive (defocusing case), one has

$$\|w_0 * \phi_\varepsilon\|_3^3 \le \|w_0\|_3^3 \le \|w_0\|_{L^\infty(\mathbb{T})}\|w_0\|^2_{L^2(\mathbb{T})} \le \|w_0\|^3_{H^1(\mathbb{T})}$$

by Lemma 16. Therefore, the bound (44) holds.

Assume for now that $t \in [0,\delta]$, where δ is the guaranteed time of existence of w in $L^2(\mathbb{T})$. From the proof of Theorem 28, one has that

$$\lim_{\varepsilon\to 0+}\|w^\varepsilon - w\|_{C([0,T],L^2(\mathbb{T}))} = 0. \tag{45}$$

Hence, from Equations (44) and (45) and Lemma 18 it follows that

$$\|w(\cdot,t)\|_{H^1(\mathbb{T})} \le \liminf_{\varepsilon\to 0+}\|w^\varepsilon(\cdot,t)\|_{H^1(\mathbb{T})} < \infty.$$

Observe, that by the above we have

$$\begin{aligned}
& \|w^\varepsilon(\cdot,t) * \phi_\varepsilon - w\|^3_{L^3(\mathbb{T})} \\
\lesssim\ & \|(w^\varepsilon(\cdot,t) - w(\cdot,t)) * \phi_\varepsilon\|^3_{L^3(\mathbb{T})} + \|(w(\cdot,t) * \phi_\varepsilon - w(\cdot,t)\|^3_{L^3(\mathbb{T})} \\
\le\ & (\|w^\varepsilon(\cdot,t)\|_{L^\infty} + \|w(\cdot,t)\|_{L^\infty})\|w^\varepsilon(\cdot,t) - w(\cdot,t)\|^2_{L^2(\mathbb{T})} \\
& + \|(w(\cdot,t) * \phi_\varepsilon - w(\cdot,t)\|^3_{L^3(\mathbb{T})} \xrightarrow{\varepsilon\to 0+} 0
\end{aligned}$$

and hence

$$E_0(w(\cdot,t)) \le \liminf_{\varepsilon\to 0+} E_\varepsilon(w^\varepsilon(\cdot,t)) \le E_0(w_0).$$

Interchanging 0 and t shows the reverse inequality and proves the energy conservation $E_0(w_0) = E_0(w(\cdot,t))$.

Reiterating the argument proves that $w \in L^\infty(\mathbb{R}, H^1(\mathbb{T}))$. It remains to show that $w \in C(\mathbb{R}, H^1(\mathbb{T}))$. To that end, observe that $t \mapsto w(\cdot,t)$ is weakly continuous in $L^2(\mathbb{T})$. But, by the above, $\sup_{t\in\mathbb{R}}\|w(\cdot,t)\|_{H^1(\mathbb{T})} < \infty$ and hence $t \mapsto w(\cdot,t)$ is weakly continuous in $H^1(\mathbb{T})$. By the observation

$$\|w(\cdot,t) - w(\cdot,s)\|^2_{H^1(\mathbb{T})} = \|w(\cdot,t)\|^2_{H^1(\mathbb{T})} + \|w(\cdot,s)\|^2_{H^1(\mathbb{T})} - 2\,\mathrm{Re}\,\langle w(\cdot,t), w(\cdot,s)\rangle_{H^1(\mathbb{T})},$$

it is enough to show that $t \mapsto \|w(\cdot,t)\|_{H^1(\mathbb{T})}$ is continuous. (See [Bre11, Proposition 3.32] for this result in a more general setting.)

To that end, observe that by the mass and energy conservation we have

$$\begin{aligned}
\|w(\cdot,t)\|^2_{H^1(\mathbb{T})} &= 2E(w(\cdot,t)) \pm \frac{2}{3}\|w(\cdot,t)\|^3_{L^3(\mathbb{T})} + \|w(\cdot,t)\|^2_{L^2(\mathbb{T})} \\
&= 2E_0(w_0) \pm \frac{2}{3}\|w(\cdot,t)\|^3_{L^3(\mathbb{T})} + \|w_0\|^2_{L^2(\mathbb{T})}.
\end{aligned}$$

Moreover, for any $t, s \in \mathbb{R}$ we have

$$\begin{aligned}
&\left| \|w(\cdot,t)\|_{L^3(\mathbb{T})}^3 - \|w(\cdot,s)\|_{L^3(\mathbb{T})}^3 \right| \\
\leq\ & \int_{\mathbb{T}} |w(x,t) - w(x,s)| \left(|w(x,t)|^2 + |w(x,t)|\,|w(x,s)| + |w(x,s)|^2 \right) \mathrm{d}x \\
\lesssim\ & 3\,\|w\|_{L^\infty(\mathbb{R},H^1(\mathbb{T}))}^2 \|w(\cdot,t) - w(\cdot,s)\|_{L^2(\mathbb{T})} .
\end{aligned}$$

The fact that $w \in C_{\mathrm{b}}(\mathbb{R}, L^2(\mathbb{T}))$ concludes the argument. □

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