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Existence Results and A Priori Bounds for Positive Solutions of Discrete Nonlinear Elliptic Equations

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1. Introduction

1.1. Motivation: Waveguide arrays

Telecommunication technology is concerned with the transport of data. One possibility to transmit data from one place to another is to send light through an optical fiber. In physics, light is a travelling electromagnetic wave and optical fibers which allow for coherent travelling waves are called *waveguides*. Since the worldwide data flow is more and more increasing it is common to use densely packed bundles of optical fibers. When these fibers are close to each other, then the light travelling through one waveguide can interact with light in neighbouring waveguides. In particular, the data signal can be perturbed or the energy loss over long distances may become so large that the arriving light is too weak to allow for reconstruction of the data.

We start the investigation of the underlying physical model in three dimensional space by considering a single straight waveguide W along the z-direction. We assume that the waveguide is infinitely long and surrounded by vacuum. In this situation the nonlinear Schrödinger equation

$$\mathrm{i}\,\frac{dE(z)}{dz} + \beta E(z) + \gamma |E(z)|^2 E(z) = 0, \quad z \in \mathbb{R},$$

is used to describe an approximation E of the amplitude of the electric field of a pulse travelling along the waveguide. Here, i denotes the imaginary unit whereas β is called the field propagation constant, a material parameter which depends on the waveguide. The nonlinear term $|E|^2 E$, is called Kerr-type nonlinearity and describes the self modulation, where the size of the real parameter γ measures the strength of this effect.

A quite simple arrangement which admits the interaction between pulses in different waveguides is a so-called waveguide array. This is a system of infinitely many (physically: a large number of), parallel optical fibers (see Figure 1.1). The distance between two neighbouring waveguides is fixed as the positive parameter h.

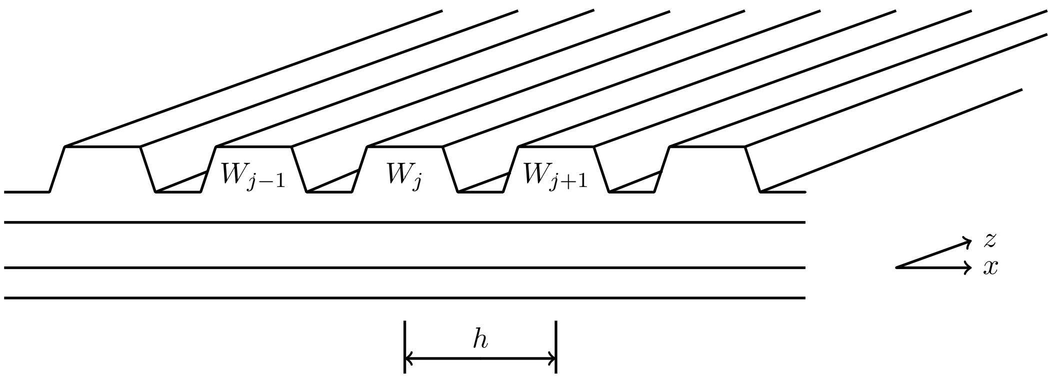


Figure 1.1: Sketch of a nonlinear array of coupled waveguides (cf. [8]).

We denote by E_j the amplitude of a pulse propagating along the j-th one-dimensional waveguide W_j, where j is an integer. Then, we obtain the following system of coupled nonlinear Schrödinger equations (see [8]):

$$\mathrm{i}\,\frac{dE_j}{dz} + \beta E_j + c(E_{j+1} + E_{j-1}) + \lambda |E_j|^2 E_j + \mu(|E_{j+1}|^2 + |E_{j-1}|^2)E_j = 0, \quad j \in \mathbb{Z}.$$

Again, z describes the position along the waveguide. Moreover, c is a positive coupling constant. Note that the corresponding coupling term $c(E_{j+1} + E_{j-1})$ only allows interaction with direct neighbours. Since pulses in waveguides further away have much less influence they are neglected. This assumption is called the nearest neighbour assumption. As the self-phase modulation dominates the non-linear interaction with the neighbours, the parameter λ is much larger than μ. Thus, it is reasonable to consider in the following only the case $\mu = 0$. This simplification leads to

$$\mathrm{i}\,\frac{dE_j}{dz} + \beta E_j + c(E_{j+1} + E_{j-1}) + \lambda |E_j|^2 E_j = 0, \quad j \in \mathbb{Z}.$$

We divide the equation by h^2 and rearrange it to see that

$$\frac{\mathrm{i}}{h^2}\frac{dE_j}{dz} + \frac{\beta + 2c}{h^2}E_j + \frac{c}{h^2}(E_{j+1} - 2E_j + E_{j-1}) + \frac{\lambda}{h^2}|E_j|^2 E_j = 0, \quad j \in \mathbb{Z}.$$

Then, we interpret E as a function $E(x,z)$ depending on $x \in h\mathbb{Z}$, $z \in \mathbb{R}$ and obtain

$$\begin{aligned}\frac{\mathrm{i}}{h^2}\frac{dE}{dz}(x,z) + \frac{\beta + 2c}{h^2}E(x,z) + \frac{c}{h^2}[E(x+1,z) - 2E(x,z) + E(x-1,z)]\\ + \frac{\lambda}{h^2}|E(x,z)|^2 E(x,z) = 0, \quad x \in h\mathbb{Z},\ z \in \mathbb{R}.\end{aligned}$$

Next, we look for monochromatic waves, i.e., we assume that the light travelling through the parallel waveguide has one single wavelength ω, which is a real parameter. This means that the function $E(x,z)$ is of the form $u(x)e^{i\omega z}$. Inserting this ansatz into the equation and dividing afterwards by the factor $e^{i\omega z}$, we eliminate the z-dependence and deduce

$$\frac{\beta + 2c - \omega}{h^2}u(x) + \frac{c}{h^2}[u(x+1) - 2u(x) + u(x-1)] + \frac{\lambda}{h^2}|u(x)|^2 u(x) = 0, \quad x \in h\mathbb{Z}.$$

Since the one-dimensional discrete Laplace operator $\Delta_h u(x)$ at the point x in $h\mathbb{Z}$ is given by $\frac{1}{h^2}[u(x+1) - 2u(x) + u(x-1)]$, we can rewrite the equation as the *discrete Schrödinger equation*

$$-c\Delta_h u + \frac{\omega - \beta - 2c}{h^2}u = \frac{\lambda}{h^2}|u|^2 u \quad \text{in } h\mathbb{Z}.$$

Arranging the waveguide array as 2d-stacks of parallel waveguides we arrive at the same equation in $h\mathbb{Z}^2$. If the wavelength ω is equal to the constant $\beta + 2c$, then the second summand vanishes. In this case, the rescaled function $v(x) := \sqrt{\frac{\lambda}{ch^2}}u(x)$ solves the *discrete Emden equation*

$$-\Delta_h v = v^3 \quad \text{in } h\mathbb{Z}^n$$

with $n = 1, 2$.

The finite difference method

The discrete Emden equation

$$-\Delta_h u = u^p$$

with exponent p in $(1, \infty)$ also arises in the field of numerical computations. Here, the finite difference method is used to approximate classical C^2-solutions of the continuous Emden equation

$$-\Delta u = u^p$$

by solutions of the discrete Emden equation. When the parameter h tends to zero and the corresponding solutions of the discrete Emden equation are uniformly bounded, then these solutions converge (in a certain sense) to a classical solution of the continuous Emden equation (see e.g. [30, Thm. 9.10]).

1.2. Outline

In this work we investigate positive solutions of discrete nonlinear elliptic equations. We use the discrete Emden equation

$$-\Delta_h u = u^p \tag{1}$$

as a prototype problem. Here, the exponent p is in the range $(1, \infty)$, and Δ_h denotes the discrete Laplacian (introduced in Definition 2.4) corresponding to the positive grid size h. We consider the discrete Emden equation (1) on subsets Ω_h of $\mathbb{R}^n_h := \{hz \colon z \in \mathbb{Z}^n\}$. If Ω_h is not the entire grid $\mathbb{R}^n_h$, we impose boundary conditions on the discrete boundary $\partial_h \Omega_h$. Here, we restrict ourselves to zero Dirichlet boundary conditions, i.e., we consider positive solutions $u \colon \overline{\Omega_h} \to [0, \infty)$ of

$$\begin{cases} -\Delta_h u = u^p & \text{in } \Omega_h, \\ \quad\; u = 0 & \text{on } \partial_h \Omega_h, \end{cases} \tag{2}$$

where the discrete closure $\overline{\Omega_h}$ is given by $\Omega_h \cup \partial_h \Omega_h$. Depending on the domain Ω_h and the exponent p, we are mainly interested in the following two questions:

(a) Do positive solutions $u = u_h$ exist or not?

(b) If solutions u_h exist, is the norm $\|u_h\|_{L^\infty(\Omega_h)}$ uniformly bounded with respect to h?

The importance of the second question, can be seen when we consider a sequence of grid sizes which tends to zero: If the corresponding solutions are uniformly bounded, then we can expect convergence to a classical C^2-solution (see e.g. [30, Thm. 9.10]). On the other hand, if the sequence of solutions is unbounded, then it either tends to a singular function or it does not converge and thus it contains spurious solutions, which do not correspond to continuous counterparts.

Next, we give a brief overview of the subsequent chapters. In Section 2, we explain our notation and collect some important results from the literature.

Afterwards, in Section 3, we prove the absence of positive solutions if the underlying domain Ω is a generalized orthant

$$\mathbb{R}^{n,k} := \{x \in \mathbb{R}^n \colon x_1, \ldots, x_k > 0\}$$

with $n \in \mathbb{N}$ and $k \in \{0, \ldots, n\}$ and if the exponent p is below some critical value p_*. The value p_* depends on Ω and is given by

$$p_* := \begin{cases} \dfrac{n+k}{n+k-2}, & \text{if } n+k-2 > 0, \\ +\infty, & \text{else.} \end{cases} \tag{3}$$

For such non-existence results the name *Liouville theorem* has been established in the literature and will be used in the sequel. Our main result reads as follows:

Theorem 1 (Discrete Liouville theorem for generalized orthants)
Let $h > 0$, $n \in \mathbb{N}$, $k \in \{0, \ldots, n\}$ and $1 < p < p_*$. Then, the only non-negative solution $u \colon \overline{\mathbb{R}_h^{n,k}} \to [0, \infty)$ of

$$-\Delta_h u \geq u^p \quad \text{in } \mathbb{R}_h^{n,k}$$

is the zero solution $u \equiv 0$.

Afterwards, we show that the exponent p_* is a critical exponent: On the one hand, if p is smaller than p_*, then there are no positive solution due to the Liouville theorem above. On the other hand, for p greater than p_*, we ensure the existence of positive solutions by the following theorem:

Theorem 2 (Existence of solutions)
Let $h > 0$, $n \geq 2$, $k \in \{0, \ldots, n\}$, where $k \neq 0$ if $n = 2$, and $p > p_*$. Then, there exists a positive solution $u \colon \overline{\mathbb{R}_h^{n,k}} \to (0, \infty)$ of

$$-\Delta_h u \geq u^p \quad \text{in } \mathbb{R}_h^{n,k}.$$

In dimension two, we can use techniques from the proof of the Liouville theorem above to obtain the subsequent variant of the discrete Liouville theorem for infinite cones:

Theorem 3 (Two-dimensional discrete Liouville theorem for cones)
For $m \in \{1, \dots, 8\}$ *let*

$$\Omega^m := \left\{(x_1, x_2)^T = (r\cos\varphi, r\sin\varphi)^T \in \mathbb{R}^2 \colon r > 0,\, \varphi \in \left(0, \frac{\pi}{4}m\right)\right\}.$$

Moreover, let $h > 0$ *and* $1 < p < p_* := \frac{m+2}{2}$. *Then, the only non-negative solution* $u \colon \overline{\Omega_h^m} \to [0, \infty)$ *of*

$$-\Delta_h u \geq u^p \quad \text{in } \Omega_h^m$$

is $u \equiv 0$.

In their celebrated paper [11], Gidas and Spruck used a scaling argument to obtain a priori bounds for positive C^2-solutions of the continuous Emden equation on bounded smooth domains. In order to get a contradiction, they employed rescalings of solutions violating the L^∞-bounds. After taking the rescaling limit, the contradiction is reached by two Liouville theorems for classical C^2-functions on the entire space $\mathbb{R}^n$ and the half space $\{x \in \mathbb{R}^n \colon x_n > 0\}$. In Section 4 we transfer this scaling approach to positive solutions $u \colon \overline{\Omega_h} \to [0, \infty)$ of the discrete problem (2), where $\Omega := (0, 1)^n$ and deduce the following result:

Theorem 4 (A priori bounds for cubes)
For dimensions $n \geq 2$ *let* $\Omega = (0, 1)^n$ *and* $1 < p < p_\star := \frac{n}{n-1}$. *Then, there exists a constant* $C > 0$ *such that for every grid size* $h = \frac{1}{\nu} > 0$ *with* $\nu \in \mathbb{N}$ *and every non-negative solution* $u_h \colon \overline{\Omega_h} \to [0, \infty)$ *of*

$$\begin{cases} -\Delta_h u = u^p & \text{in } \Omega_h, \\ \quad\;\; u = 0 & \text{on } \partial_h \Omega_h \end{cases}$$

the a priori estimate $\|u_h\|_{L^\infty(\Omega_h)} \leq C$ *holds.*

In the proof we assume for contradiction that there exists a sequence of grid sizes and corresponding unbounded solutions. Using the scaling approach, we obtain rescaled solutions on rescaled grids. Then, we show that the sequence of new grid sizes either converges to some positive constant (discrete limit) or to zero (continuous limit). In the discrete limit case, we construct a non-zero limit function, which contradicts the discrete Liouville theorem above. In the continuous limit case, we also obtain a non-zero limit function, which violates the corresponding Liouville theorem for C^2-functions ([5, Thm. 4.6]). The rescaling of the cube Ω can lead to different geometric limit configurations, i.e., $-\Delta_h u = u^p$ in $\mathbb{R}_h^{n,k}$ for $k \in \{0, \dots, n\}$, where k reflects that a cube has $n-k$–dimensional facets with $k = n$ (vertices), $k = n-1$ (edges), $k = 1$ (faces), $k = 0$ (inner point). For each k there is different Liouville exponent $p_*(k)$, which is given by (3) and is minimal for $k = n$. In this case $p_* = \frac{n}{n-1}$ is the exponent, which allows to deduce a contradiction by applying the Liouville theorem for all $k \in \{0, \dots, n\}$. Therefore, we can proof the a priori result for all p smaller than $\frac{n}{n-1}$.

Moreover, we generalize Theorem 4 to nonlinearities $f(u) = u^p\big(1 + \mathrm{o}(1)\big)$ with $\mathrm{o}(1) \to 0$ as $u \to \infty$. In dimension two, we also prove an a priori bound result for a right-angled isosceles triangle by means of the scaling approach:

Theorem 5 (A priori bounds for triangles)
Let $\Omega := \{x \in \mathbb{R}^2 \colon x_1, x_2 > 0,\ x_1 + x_2 < 1\}$ and $1 < p < p_\star := \frac{3}{2}$. Then, there exists a constant $C > 0$ such that for every grid size $h = \frac{1}{\nu} > 0$ with $\nu \in \mathbb{N}$ and every non-negative solution $u_h \colon \overline{\Omega_h} \to [0, \infty)$ of

$$\begin{cases} -\Delta_h u = u^p & \text{in } \Omega_h, \\ u = 0 & \text{on } \partial_h \Omega_h \end{cases}$$

the a priori estimate $\|u_h\|_{L^\infty(\Omega_h)} \le C$ holds.

In the proof the rescaling of the triangle leads to infinite cones. Thus, we use the discrete Liouville theorem for cones as well as the analogous Liouville theorem for C^2-functions from the literature to obtain a contradiction.

In Section 5 we return to the discrete Emden equation on the entire grid $\mathbb{R}^n_h$ and prove the following existence result:

Theorem 6 (Existence on entire space)
Let $h > 0$ and $n \ge 3$. Then, for every exponent $p > \frac{n+2}{n-2}$ there exists a positive solution $u \colon \mathbb{R}^n_h \to (0, \infty)$ of

$$-\Delta_h u = u^p \quad \text{in } \mathbb{R}^n_h.$$

This theorem is somehow complementary to the discrete Liouville theorem above: For dimensions $n \ge 3$ there is a positive solution of the discrete Emden equation on the entire grid $\mathbb{R}^n_h$ if the exponent p is larger than $\frac{n+2}{n-2}$. In contrast to that, the discrete Liouville theorem says that for p smaller than $\frac{n}{n-2}$ there are no positive solutions. It remains an open problem whether positive solutions exist if p is between $\frac{n}{n-2}$ and $\frac{n+2}{n-2}$.

We prove Theorem 6 with a concentration-compactness argument. If we consider the discrete Emden equation on an infinite strip, this method can be transferred. In this case we can apply the discrete Poincaré inequality instead of a Sobolev inequality to show the existence of a positive solution for all $p \in (1, \infty)$:

Theorem 7 (Existence on infinite strip)
Let $n \ge 2$ be the dimension, $S_h := \{x \in \mathbb{R}^n_h \colon 0 < x_n < 1\}$ an infinite strip *and $h = \frac{1}{\nu} > 0$ with $\nu \in \mathbb{N}$ and $\partial_h S_h = \{x \in \mathbb{R}^n_h \colon x_n = 0 \text{ or } x_n = 1\}$. Then, for every $p \in (1, \infty)$ there exists a positive solution of*

$$\begin{cases} -\Delta_h u = u^p & \text{in } S_h, \\ u = 0 & \text{on } \partial_h S_h. \end{cases}$$

2. Preliminaries and notation

Throughout this work we will use the following notations: For $n \in \mathbb{N}$ let $\{e_1, \dots, e_n\}$ be the standard basis in $\mathbb{R}^n$. For the *grid size* $h > 0$ we denote the *equidistant grid* by $\mathbb{R}^n_h := \{hz \colon z \in \mathbb{Z}^n\}$.

Definition 2.1 (Discrete interior, closure and boundary; admissibility)
For $A \subset \mathbb{R}^n$ and a grid size $h > 0$ we define

$$A_h := A \cap \mathbb{R}^n_h.$$

Let $\Omega \subset \mathbb{R}^n$ be open. Then, Ω_h is called discrete interior*. Furthermore, for $x \in \mathbb{R}^n_h$, the set of* discrete neighbours *is denoted by*

$$N_h(x) := \big\{x \pm he_i \colon i \in \{1, \dots, n\}\big\}.$$

With that, the discrete closure $\overline{\Omega_h}$ *and the* discrete boundary $\partial_h\Omega_h$ *are given by*

$$\overline{\Omega_h} := \Omega_h \cup \bigcup_{x \in \Omega_h} N_h(x) \quad \text{and} \quad \partial_h\Omega_h := \overline{\Omega_h} \setminus \Omega_h.$$

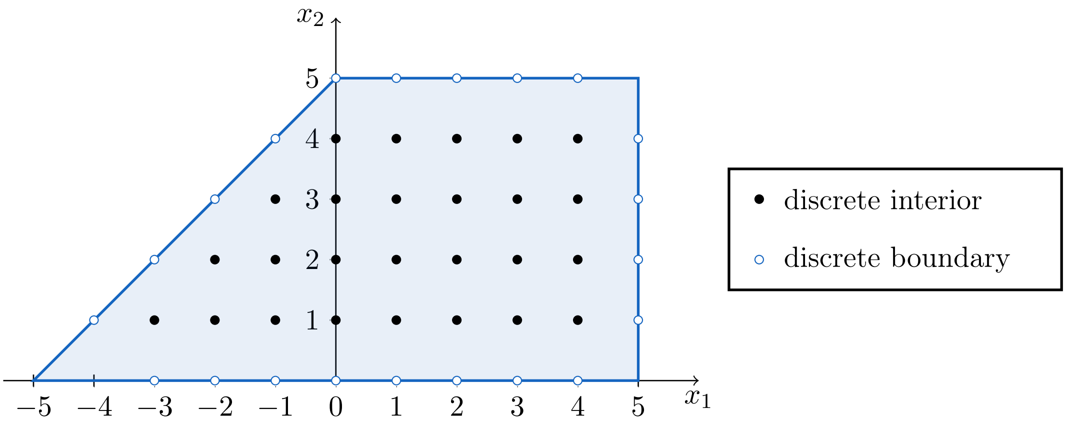


Figure 2.1: Discrete interior and boundary for a quadrangle Ω with $h = 1$.

We call h admissible *for Ω if $\partial_h\Omega_h \subset \partial\Omega$. In order to define function spaces corresponding to Ω_h, we specify parts of the boundary and set*

$$\partial_i^{\pm}\Omega_h := \{x \in \partial_h\Omega_h \colon \exists y \in \Omega_h \text{ with } x = y \pm he_i\}$$

for all $i \in \{1, \dots, n\}$. Furthermore, for a discrete set $B \subset \mathbb{R}^n_h$, we define the discrete closure *by*

$$\mathrm{cl}_h(B) = B \cup \bigcup_{x \in B} N_h(x). \tag{2.1}$$

Remark 2.2 (Admissibility)
There are open sets $\Omega \subset \mathbb{R}^n$ such that there is no admissible grid size $h > 0$, e.g. the two-dimensional rectangle $(0,1) \times (0,\sqrt{2})$. On the other hand, if Ω is a hypercube $\prod_{i=1}^n (a_i, b_i) \subset \mathbb{R}^n$ with $a_i, b_i \in \mathbb{Q}$ and $a_i < b_i$ for $i \in \{1,\dots,n\}$, then every $h > 0$, which divides all a_i and b_i (i.e., there are $y_i, z_i \in \mathbb{Z}$ such that $hy_i = a_i$ and $hz_i = b_i$), is admissible. Throughout this work, we only consider admissible grid sizes.

Definition 2.3 (Finite difference quotients)
Let $h > 0$ and $\Omega \subset \mathbb{R}^n$ be open. For a given function $u\colon \overline{\Omega_h} \to \mathbb{R}$ we define the forward and backward finite difference quotients *by*

$$D_i^+ u(x) \coloneqq \frac{u(x + he_i) - u(x)}{h} \quad \text{for all } x \in \Omega_h \cup \partial_i^- \Omega_h \quad \text{and}$$
$$D_i^- u(x) \coloneqq \frac{u(x) - u(x - he_i)}{h} \quad \text{for all } x \in \Omega_h \cup \partial_i^+ \Omega_h$$

for all $i \in \{1,\dots,n\}$.

Definition 2.4 (Discrete Laplacian)
Let $h > 0$, $\Omega \subset \mathbb{R}^n$ be open and $u\colon \overline{\Omega_h} \to \mathbb{R}$. For all $x \in \Omega_h$ the discrete Laplace operator *of u at x is given by*

$$\begin{aligned}\Delta_h u(x) &\coloneqq \sum_{i=1}^n D_i^- D_i^+ u(x) = \sum_{i=1}^n D_i^+ D_i^- u(x) \\ &= \frac{1}{h^2} \sum_{i=1}^n [u(x + he_i) - 2u(x) + u(x - he_i)].\end{aligned}$$

Definition 2.5 (Compact support)
Let $\Omega \subset \mathbb{R}^n$ be a domain and $h > 0$. A function $\varphi\colon \overline{\Omega_h} \to \mathbb{R}$ is said to have compact support if the set $\operatorname{supp}(\varphi) \coloneqq \{x \in \overline{\Omega_h}\colon \varphi(x) \neq 0\}$ is bounded and $\varphi = 0$ on $\partial_h \Omega_h$. We denote the set of all functions with compact support by

$$\mathcal{C}(\Omega_h) \coloneqq \{\varphi\colon \overline{\Omega_h} \to \mathbb{R}\colon \varphi \text{ has compact support}\}.$$

Definition 2.6 (Discrete harmonic, sub- and superharmonic functions)
Let $\Omega \subset \mathbb{R}^n$ be a domain and $h > 0$. A grid function $u\colon \overline{\Omega_h} \to \mathbb{R}$ is called discrete harmonic *or* subharmonic *or* superharmonic *in Ω_h if*

$$-\Delta_h u = 0 \text{ or } -\Delta_h u \leq 0 \text{ or accordingly } -\Delta_h u \geq 0 \quad \text{in } \Omega_h.$$

Notation 2.7 (Norms in $\mathbb{R}^n$)
For $q \in [1,\infty)$ and $x \in \mathbb{R}^n$ we use the following conventions:

$$|x|_\infty \coloneqq \max_{i \in \{1,\dots,n\}} |x_i| \qquad \text{and} \qquad |x|_q \coloneqq \left(\sum_{i=1}^n |x_i|^q\right)^{\frac{1}{q}}.$$

Instead of $|x|_2$ we write $|x|$.

2.1. Discrete function spaces

Let $h > 0$ and $\Omega \subset \mathbb{R}^n$ be a domain. For $1 \leq q < \infty$ and functions $u \colon \Omega_h \to \mathbb{R}$ we set

$$\|u\|_{L^q(\Omega_h)} \coloneqq \left(\sum_{x \in \Omega_h} |u(x)|^q h^n \right)^{\frac{1}{q}},$$
$$\|u\|_{L^\infty(\Omega_h)} \coloneqq \sup_{x \in \Omega_h} |u(x)|$$

and for $u \colon \overline{\Omega_h} \to \mathbb{R}$

$$\|\nabla_h^+ u\|_{L^q(\Omega_h)} \coloneqq \left(\sum_{i=1}^n \sum_{x \in \Omega_h \cup \partial_i^- \Omega_h} |D_i^+ u(x)|^q h^n \right)^{\frac{1}{q}},$$
$$\|u\|_{W^{1,q}(\Omega_h)} \coloneqq \left(\|\nabla_h^+ u\|_{L^q(\Omega_h)}^q + \|u\|_{L^q(\Omega_h)}^q \right)^{\frac{1}{q}}.$$

Corresponding to these norms we define the function spaces

$$\begin{aligned}
L^q(\Omega_h) &\coloneqq \{u \colon \Omega_h \to \mathbb{R} : \|u\|_{L^q(\Omega_h)} < \infty\}, \\
\dot{W}_0^{1,q}(\Omega_h) &\coloneqq \{u \colon \overline{\Omega_h} \to \mathbb{R} : \|\nabla_h^+ u\|_{L^q(\Omega_h)} < \infty \text{ and } u = 0 \text{ on } \partial_h \Omega_h\}, \\
W_0^{1,q}(\Omega_h) &\coloneqq \{u \colon \overline{\Omega_h} \to \mathbb{R} : \|u\|_{W^{1,q}(\Omega_h)} < \infty \text{ and } u = 0 \text{ on } \partial_h \Omega_h\}, \\
W^{1,q}(\Omega_h) &\coloneqq \{u \colon \overline{\Omega_h} \to \mathbb{R} : \|u\|_{W^{1,q}(\Omega_h)} < \infty\}.
\end{aligned}$$

In $\dot{W}_0^{1,q}(\mathbb{R}_h^n)$ we identify functions u_1 and u_2 which differ only by a constant. Then, all four spaces endowed with the corresponding norms are Banach spaces.

Further, for $1 < q < \infty$ we define the dual Hölder exponent by $q' \coloneqq \frac{q}{q-1}$. We also introduce the following shorthands:

$$\langle f, g \rangle_{\Omega_h} \coloneqq \sum_{x \in \Omega_h} f(x) g(x) h^n \quad \text{for } f \in L^q(\Omega_h), g \in L^{q'}(\Omega_h),$$
$$\langle\!\langle \nabla_h^+ u, \nabla_h^+ v \rangle\!\rangle_{\Omega_h} \coloneqq \sum_{i=1}^n \sum_{x \in \Omega_h \cup \partial_i^- \Omega_h} D_i^+ u(x) D_i^+ v(x) h^n \quad \text{for } u \in W^{1,q}(\Omega_h), v \in W^{1,q'}(\Omega_h).$$

If $q = 2$, then $q' = 2$ and $L^2(\Omega_h)$ equipped with the scalar product $\langle \cdot, \cdot \rangle_{\Omega_h}$ is a Hilbert space. Addionally, the spaces $H^1(\Omega_h) \coloneqq W^{1,2}(\Omega_h)$ with $\langle\!\langle \nabla_h^+ \cdot, \nabla_h^+ \cdot \rangle\!\rangle_{\Omega_h} + \langle \cdot, \cdot \rangle_{\Omega_h}$ and $\dot{H}_0^1(\Omega_h) \coloneqq \dot{W}_0^{1,2}(\Omega_h)$ endowed with $\langle\!\langle \nabla_h^+ \cdot, \nabla_h^+ \cdot \rangle\!\rangle_{\Omega_h}$ are Hilbert spaces as well.

2.2. Inequalities and embeddings for grid functions

For functions $u\colon \mathbb{R}^n \to \mathbb{R}$ the Poincaré, Sobolev and Hardy inequalities are well known ([1]). In the following sense they hold also for grid functions $u\colon \mathbb{R}^n_h \to \mathbb{R}$.

Theorem 2.8 (Discrete Poincaré inequality)
For $b > 0$ let $S := \{x \in \mathbb{R}^n : 0 < x_n < b\}$ be an infinite strip *and $h > 0$ be an admissible grid size. Moreover, let $q \in (1, \infty)$. Then, there exists a constant $C_{\mathrm{P}}(q, S) > 0$, which is independent of h, such that*

$$\|u\|_{L^q(S_h)} \leq C_{\mathrm{P}}(q, S)\|u\|_{\dot{W}_0^{1,q}(S_h)} \quad \text{for all } u \in \dot{W}_0^{1,q}(S_h).$$

Proof. For $n = 1$ and $q = 2$ the result follows, e.g., from [21, Lemma 3]. The proof can be generalized to all $q \in (1, \infty)$. Finally, using this one-dimensional result and summing over the other directions, we obtain the assertion. □

Theorem 2.9 (Discrete Sobolev inequality)
Let $h > 0$, $1 < q < n$ and $q^* := \frac{nq}{n-q}$ be the critical Sobolev exponent. Then, there exists a constant $C_{\mathrm{S}}(n, q) := 2q\frac{n-1}{n-q}n^{-\frac{1}{q}} > 0$, which is independent of the grid size h, with

$$\|u\|_{L^{q^*}(\mathbb{R}^n_h)} \leq C_{\mathrm{S}}(n, q)\|u\|_{\dot{W}_0^{1,q}(\mathbb{R}^n_h)} \quad \text{for all } u \in \dot{W}_0^{1,q}(\mathbb{R}^n_h).$$

Proof. For the special case $q = 2$ and $n \geq 3$ the result can be found in [24, Thm. 9]. For the general case $1 < q < n$ we can adapt the proof and the assertion follows. □

Theorem 2.10 (Discrete Hardy inequality)
Let $1 < q < n$ and $\gamma > 0$. For $x \in \mathbb{R}^n_h$, we write

$$|x|_h := \sqrt{|x|^2 + \gamma h^2}. \tag{2.2}$$

Then, there exists a constant $C_{\mathrm{H}}(\gamma, n, q) > 0$ such that

$$\sum_{x \in \mathbb{R}^n_h} \frac{|u(x)|^q}{|x|_h^q} h^q \leq C_{\mathrm{H}}^q(\gamma, n, q)\|u\|^q_{\dot{W}_0^{1,q}(\mathbb{R}^n_h)} \quad \text{for all } u \in \dot{W}_0^{1,q}(\mathbb{R}^n_h). \tag{2.3}$$

Proof. In [6] the so-called *vector field approach* is used to prove Hardy inequalities for functions in $C_{\mathrm{c}}^\infty(\mathbb{R}^n \setminus \{0\})$. This ansatz can be adapted to grid functions and we obtain (2.3) for all functions $u \in \mathcal{C}(\mathbb{R}^n_h)$. By density, the assertion follows. □

2.3. Sobolev spaces

In this section we briefly summarize the notation which we use for the well-know Sobolev spaces. Let $\Omega \subset \mathbb{R}^n$ be open. We denote

$$L^1_{\mathrm{loc}}(\Omega) := \{u\colon \Omega \to \mathbb{R} \text{ measurable and } \|u\|_{L^1(K)} < \infty \text{ for every compact set } K \subset \Omega\}.$$

Moreover, we define the *support* of a function $u\colon \Omega \to \mathbb{R}$ by

$$\operatorname{supp}(u) = \overline{\{x \in \Omega\colon u(x) \neq 0\}}$$

and use the convention

$$C_{\mathrm{c}}^{\infty}(\Omega) \coloneqq \big\{u \in C^{\infty}(\Omega)\colon \operatorname{supp}(u) \text{ is a compact subset of } \Omega\big\}.$$

Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and $i \in \{1, \dots, n\}$. If there exists a function $g \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega u \partial_i \varphi \, dx = -\int_\Omega g_i \varphi \, dx \quad \text{for all } \varphi \in C_{\mathrm{c}}^{\infty}(\Omega),$$

then we call $\partial_i u \coloneqq g_i$ the *weak derivative* of u in direction x_i. With that, for $1 < q < \infty$ and $u \in L^1_{\mathrm{loc}}(\Omega)$ we set

$$\|u\|_{L^q(\Omega)} \coloneqq \left(\int_\Omega |u(x)|^q \, dx\right)^{\frac{1}{q}} \quad \text{as well as} \quad \|u\|_{L^\infty(\Omega)} \coloneqq \sup_{x \in \Omega} |u(x)|$$

and

$$\begin{aligned}
\|\nabla u\|_{L^q(\Omega)} &\coloneqq \left(\int_\Omega \sum_{i=1}^n |\partial_i u(x)|^q \, dx\right)^{\frac{1}{q}} = \left(\int_\Omega |\nabla u(x)|_q^q \, dx\right)^{\frac{1}{q}}, \\
\|u\|_{W^{1,q}(\Omega)} &\coloneqq \left(\|\nabla u\|^q_{L^q(\Omega)} + \|u\|^q_{L^q(\Omega)}\right)^{\frac{1}{q}}.
\end{aligned}$$

Correspondingly, we define the Banach spaces

$$\begin{aligned}
L^q(\Omega) &\coloneqq \{u \in L^1_{\mathrm{loc}}(\Omega)\colon \|u\|_{L^q(\Omega)} < \infty\}, \\
W^{1,q}(\Omega) &\coloneqq \{u \in L^1_{\mathrm{loc}}(\Omega)\colon \|u\|_{W^{1,q}(\Omega)} < \infty\}, \\
W_0^{1,q}(\Omega) &\coloneqq \overline{C_{\mathrm{c}}^{\infty}(\Omega)}^{\|\cdot\|_{W^{1,q}(\Omega)}}, \\
\dot{W}_0^{1,q}(\Omega) &\coloneqq \overline{C_{\mathrm{c}}^{\infty}(\Omega)}^{\|\nabla \cdot\|_{L^q(\Omega)}}.
\end{aligned}$$

2.4. Frequently used results

In this section we collect some basic results for finite difference solutions of boundary value problems.

Definition 2.11 (Discretely connected set)
Let $h > 0$ and $\Omega \subset \mathbb{R}^n$. Then, Ω_h is called discretely connected *if for all $y, z \in \Omega_h$ there exist some $m \in \mathbb{N}$ and grid points $x_1, \dots, x_m \in \Omega_h$ such that $x_1 = y$, $x_m = z$ and $|x_i - x_{i+1}| = h$ for all $i \in \{1, \dots, m-1\}$.*

Lemma 2.12 (Discrete maximum principle)
Let $\Omega \subset \mathbb{R}^n$ and $h > 0$ such that Ω_h is discretely connected in the sense of Definition 2.11.

Moreover, let $u\colon \overline{\Omega_h} \to \mathbb{R}$ *with*

$$-\Delta_h u \leq 0 \quad \text{in } \Omega_h.$$

Then, the following is true:

(a) *If there exists some* $\hat{x} \in \Omega_h$ *with* $u(\hat{x}) = \max_{x \in \overline{\Omega_h}} u(x) \geq 0$*, then* $u \equiv u(\hat{x})$ *in* $\overline{\Omega_h}$*.*

(b) *If* Ω *is bounded and* $u \leq 0$ *on* $\partial_h \Omega_h$*, then* $u \leq 0$ *in* $\overline{\Omega_h}$*.*

Proof. The result can be found e.g. in [30, Lemma 5.15]. □

Definition 2.13 (Discrete Green's function)
Let $h > 0$ *and* $\Omega \subset \mathbb{R}^n$ *be a bounded domain. We introduce the function* $\delta_h\colon \mathbb{R}^n_h \to \mathbb{R}$ *by*

$$\delta_h(x) := \begin{cases} \dfrac{1}{h^n}, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

With that, for fixed $y \in \Omega_h$*, we define* $G(\cdot, y)\colon \overline{\Omega_h} \to \mathbb{R}$ *as the solution of the problem*

$$\begin{cases} -\Delta_{h,x} G(x,y) = \delta_h(x-y), & x \in \Omega_h, \\ \quad\quad G(x,y) = 0, & x \in \partial_h \Omega_h. \end{cases}$$

The resulting function $G\colon \overline{\Omega_h} \times \Omega_h \to \mathbb{R}$ *is called* discrete Green's function *(for zero Dirichlet boundary conditions).*

Theorem 2.14 (Solution formula for the discrete Poisson equation)
Let $h > 0$ *and* $\Omega \subset \mathbb{R}^n$ *be a bounded domain. Moreover, let* $f\colon \Omega_h \to \mathbb{R}$*. Then, there exists a unique solution* $u\colon \overline{\Omega_h} \to \mathbb{R}$ *of the* discrete Poisson equation

$$\begin{cases} -\Delta_h u = f & \text{in } \Omega_h, \\ \quad\quad u = 0 & \text{on } \partial_h \Omega_h. \end{cases}$$

This solution is given by the following representation formula*:*

$$u(x) = \sum_{y \in \Omega_h} G(x,y) f(y) h^n \quad \text{for all } x \in \overline{\Omega_h}.$$

Proof. The statements follow from Theorem 5.16 and Lemma 5.19 in [30]. □

Lemma 2.15 (Partial summation formula)
Let $\Omega \subset \mathbb{R}^n$ *with an admissible grid size* $h > 0$ *and* $u\colon \overline{\Omega_h} \to \mathbb{R}$*. Then, the following identity holds true:*

$$\langle\!\langle \nabla_h^+ u, \nabla_h^+ \varphi \rangle\!\rangle_{\Omega_h} = \langle -\Delta_h u, \varphi \rangle_{\Omega_h} \quad \text{for all } \varphi \in \mathcal{C}(\Omega_h).$$

Proof. If $\Omega = \mathbb{R}^n$, then the assertion follows directly from [24, Lemma 5]. So, let $\Omega \neq \mathbb{R}^n$ and $u\colon \overline{\Omega_h} \to \mathbb{R}$. Moreover, let $\varphi \in \mathcal{C}(\Omega_h)$ be an arbitrary test function. We extend u and φ by zero to grid functions defined on $\mathbb{R}^n_h$. Note that

$$D_i^+\varphi(x) = 0 \quad \text{for all } x \in \mathbb{R}^n_h \setminus (\Omega_h \cup \partial_i^-\Omega_h)$$

for all $i \in \{1, \dots, n\}$. Thus, we conclude

$$\langle\!\langle \nabla_h^+ u, \nabla_h^+ \varphi \rangle\!\rangle_{\Omega_h} = \langle\!\langle \nabla_h^+ u, \nabla_h^+ \varphi \rangle\!\rangle_{\mathbb{R}^n_h} = \langle -\Delta_h u, \varphi \rangle_{\mathbb{R}^n_h} = \langle -\Delta_h u, \varphi \rangle_{\Omega_h},$$

since the assertion holds true in the case $\Omega = \mathbb{R}^n$ by [24, Lemma 5]. □

3. Liouville theorems

If one considers any nonlinear boundary value problem in the field of partial differential equations, one natural question is whether solutions exist or not. In the case of non-existence of positive solutions the name Liouville theorem has been established in the literature and will be used in the sequel.

The Emden equation

$$-\Delta u = u^p$$

with exponents $p \in (1, \infty)$ is often used as a prototype problem. With a deeper understanding of this equation we can replace the Laplacian Δ by a more general differential operator in divergence form and allow other nonlinearities $f(u)$ instead of u^p.

Concerning classical positive solutions of the Emden equation on $\mathbb{R}^n$ with $n \geq 3$ the question of existence depends on the exponent $p \in (1, \infty)$ and was answered quite completely. A classical solution is a C^2-function $u \colon \mathbb{R}^n \to [0, \infty)$ which satisfies

$$-\Delta u = u^p \quad \text{in } \mathbb{R}^n. \tag{3.1}$$

For $p \in (1, \frac{n+2}{n-2})$ Gidas and Spruck ([12]) proved the following Liouville theorem: If $u \in C^2(\mathbb{R}^n)$ with $u \geq 0$ is a solution of (3.1), then $u \equiv 0$. For $p = \frac{n+2}{n-2}$ the instantons $u_\varepsilon \in C^\infty(\mathbb{R}^n)$,

$$u_\varepsilon(x) := \frac{\left[n(n-2)\varepsilon^2\right]^{\frac{n-2}{4}}}{\left[\varepsilon^2 + |x|^2\right]^{\frac{n-2}{2}}}, \quad x \in \mathbb{R}^n, \, \varepsilon > 0,$$

are up to translations the only positive solutions of (3.1) (e.g. [29, proof of Thm. III.2.1]). Finally, for $p \in (\frac{n+2}{n-2}, \infty)$ there are classical positive solutions of (3.1) due to a result of Joseph and Lundgren ([18, Thm. 1]). Therefore, $\frac{n+2}{n-2}$ is a threshold and is called the critical Liouville exponent for C^2-solutions of (3.1).

For weak solutions the critical exponent is the same: A weak solution of (3.1) is a function $u \in H^1_{\text{loc}}(\mathbb{R}^n)$, $u \geq 0$, such that

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla \varphi \, dx = \int_{\mathbb{R}^n} u^p \varphi \, dx \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n). \tag{3.2}$$

For $p \in (1, \frac{n+2}{n-2})$ we can show, by means of regularity theory, that every weak solution of (3.1) is in fact a classical C^2-solution of (3.1) (see e.g. Appendix B in [29]). Thus, we can apply the Liouville theorem for C^2-functions and conclude that $u \equiv 0$ is the unique solution of (3.2). Moreover, by definition, every classical solution is also a weak one. We showed above that for $p \in \left[\frac{n+2}{n-2}, \infty\right)$ there are classical positive solutions, which are therefore also weak solutions. In summary, the critical Liouville exponent for weak solutions is $\frac{n+2}{n-2}$.

Concerning very weak solutions there is a different critical exponent: A very weak solution of (3.1) is a function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $u \geq 0$, with

$$\int_{\mathbb{R}^n} u(-\Delta\varphi)dx = \int_{\mathbb{R}^n} u^p \varphi \, dx \quad \text{for all } \varphi \in C^2_c(\mathbb{R}^n). \tag{3.3}$$

For $p \in (1, \frac{n}{n-2})$ every very weak solution of (3.1) belongs to $W^{2,q}_{\mathrm{loc}}(\mathbb{R}^n)$ for all $q > 1$ according to e.g. [20, Thm. 2] and thus, by regularity theory, is a classical solution of (3.1). So, the Liouville theorem for C^2-functions by Gidas and Spruck applies and $u \equiv 0$ is the only solution of (3.3). For $p \in (\frac{n}{n-2}, \infty)$ the function $u_0 \in C^\infty(\mathbb{R}^n \setminus \{0\})$ defined by

$$u_0(x) := c_p |x|^{-\frac{2}{p-1}} \quad \text{with} \quad c_p^{p-1} = \frac{2}{p-1}\left(n - 2 - \frac{2}{p-1}\right)$$

solves $-\Delta u = u^p$ in $\mathbb{R}^n \setminus \{0\}$ and is a very weak solution of (3.1). Hence, the critical Liouville exponent for very weak solutions is $\frac{n}{n-2}$. The statements about very weak solutions can be found in [25].

All these existence and non-existence results are summarized in the chart below:

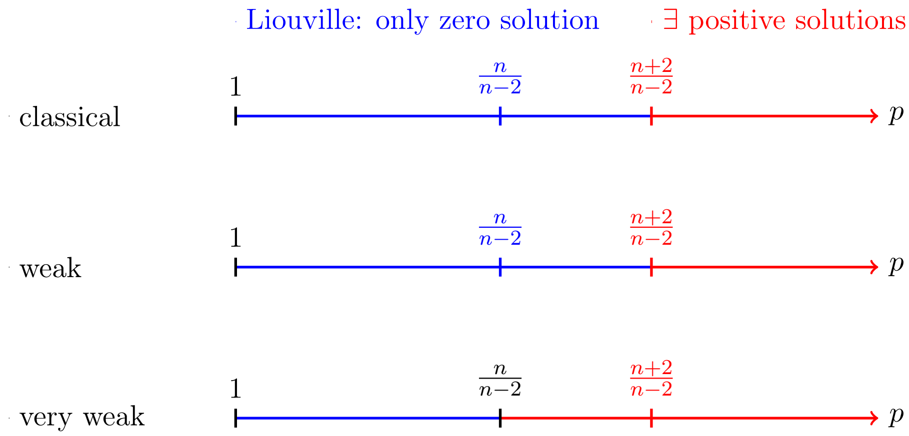


Figure 3.1: Overview of existence and non-existence results.

For finite differences, Anton Verbitzky showed in his dissertation ([30]) the following Liouville theorem for the discrete Emden equation

$$-\Delta_h u = u^p \quad \text{in } \mathbb{R}^n_h. \tag{3.4}$$

If $u \colon \mathbb{R}^n_h \to [0, \infty)$ is a solution of (3.4) with $p \in (1, \frac{n}{n-2})$, then $u \equiv 0$.

The next step is to analyse the Emden equation on a half space $H := \{x \in \mathbb{R}^n \colon x_n > 0\}$ with Dirichlet boundary conditions. For $p \in \left(1, \frac{n+2}{n-2}\right]$ Gidas and Spruck revealed in [11] that the only solution $u\colon \overline{H} \to [0,\infty)$, $u \in C^2(H) \cap C(\overline{H})$, of

$$\begin{cases} -\Delta u = u^p & \text{in } H, \\ u = 0 & \text{on } \partial H, \end{cases}$$

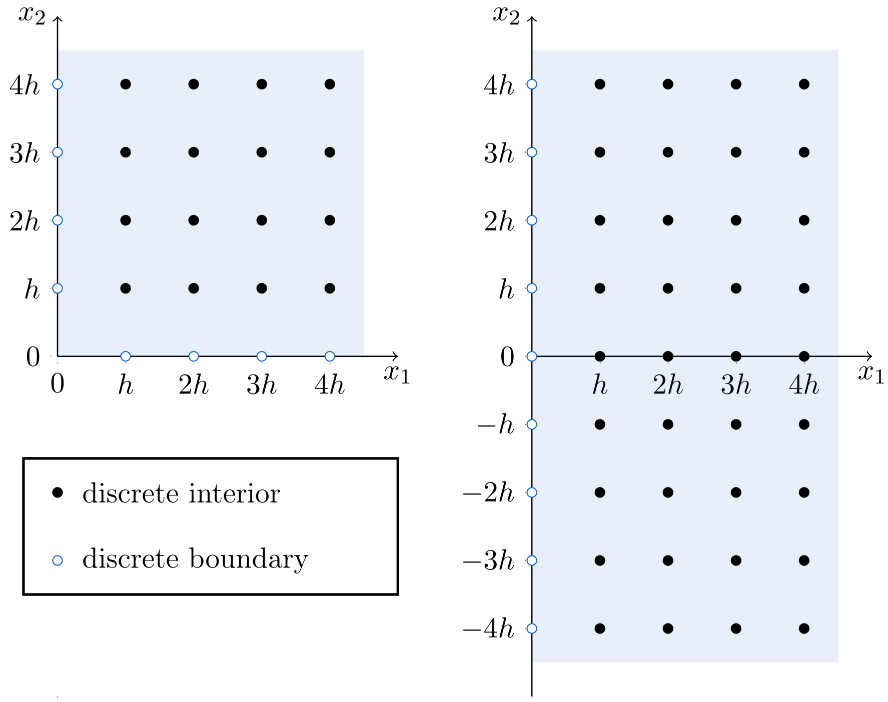
is the zero solution.

The goal of the subsequent section is to prove the discrete Liouville Theorem 3.4 for half spaces and more general unbounded domains, e.g. orthants. The method used is based on a *comparison argument* and the *asymptotic behaviour at infinity*. In the continuous case this approach goes back to Kondratiev, Liskevich and Moroz, especially Theorem 1.3. in [19].

3.1. Generalized orthant

Notation 3.1
For dimension $n \in \mathbb{N}$ and $k \in \{0, \dots, n\}$, we consider the generalized orthant

$$\mathbb{R}^{n,k} := \{x \in \mathbb{R}^n \colon x_1, \dots, x_k > 0\}.$$



(a) Orthant $(k = n)$ (b) Half space $(k = 1)$

Figure 3.2: Discretization of two generalized orthants with $n = 2$.

If $k = 0$ the special case $\mathbb{R}^{n,k} = \mathbb{R}^n$ occurs, if $k = 1$ we have a half space and if $k = n$ then $\mathbb{R}^{n,k}$ is an orthant. Moreover, for $h > 0$ the discrete analogon is defined by $\mathbb{R}^{n,k}_h = \mathbb{R}^{n,k} \cap \mathbb{R}^n_h$ and the corresponding discrete boundary $\partial_h \mathbb{R}^{n,k}_h$ as well as the discrete closure $\overline{\mathbb{R}^{n,k}_h}$ are given by Definition 2.1.

Remark 3.2
The discrete boundary $\partial_h \mathbb{R}^{n,k}_h$ is the empty set if $k = 0$. In this case we use the convention that a boundary condition on the empty set is an empty condition. Further, a product over the empty set is defined to be 1.

Notation 3.3
For $a > 0$ and $b \in \mathbb{R}$ we use the abbreviation

$$\frac{a}{b_+} := \begin{cases} \frac{a}{b}, & \text{if } b > 0, \\ +\infty, & \text{if } b \leq 0. \end{cases}$$

Theorem 3.4 (Discrete Liouville theorem for generalized orthants)
Let $h > 0$ be the grid size, $n \in \mathbb{N}$ be the dimension, $k \in \{0, \dots, n\}$ and $1 < p < \frac{n+k}{(n+k-2)_+}$. Then, the only non-negative solution $u \colon \overline{\mathbb{R}^{n,k}_h} \to [0, \infty)$ of

$$-\Delta_h u \geq u^p \quad \text{in } \mathbb{R}^{n,k}_h \tag{3.5}$$

is $u \equiv 0$.

Remark 3.5 (Scale invariance)
Let $h, h_* > 0$ be two grid sizes. Then, $\Psi \colon \overline{\mathbb{R}^{n,k}_{h_*}} \to \overline{\mathbb{R}^{n,k}_h}$, given by

$$\Psi(x) := \frac{h}{h_*} x,$$

is an isomorphism. Moreover, if $u \colon \overline{\mathbb{R}^{n,k}_h} \to [0, \infty)$ is a solution of (3.5) with respect to h, then $u \circ \Psi \colon \overline{\mathbb{R}^{n,k}_{h_*}} \to [0, \infty)$ is a solution of (3.5) with respect to h_*, where $u \circ \Psi$ denotes the composition of Ψ and u.

Let us mention the outline of this section: First, we proof Theorem 3.4 for dimensions $n \geq 3$. Later, the cases $n = 1, 2$ are treated separately. We want to point out that Theorem 3.4 for $n = 1$ is a consequence of Theorem 3.13, whereas the case $n = 2$ follows from Theorems 3.14 and 3.21.

The subsequent lemma goes back to the work of Bramble, Hubbard and Zlamal ([7]), especially the proof of Lemma 3.1 therein. There they prove neighbour estimates for $|x|_h = \sqrt{|x|^2 + \gamma h^2}$ with some fixed $\gamma > 0$, whereas we consider $|x|$ with a similar approach.

Lemma 3.6 (Neighbour estimates)
For all $\kappa \in (0,1)$, there exists a radius $R_\kappa := \max\left\{\frac{3}{\kappa^{-2}-1}, \frac{2}{1-\kappa^2}\right\} > 1$ such that for all $x \in \mathbb{R}^n_h$ with $|x| > R_\kappa$, $h \in (0,1]$ and $\xi \in \{x + \tau h e_i : \tau \in [-1,1]\}$ for $i \in \{1,\dots,n\}$ we have

$$\kappa^{-2}|x|^2 \geq |\xi|^2 \geq \kappa^2|x|^2.$$

Proof. Let $x \in \mathbb{R}^n_h$, $h \in (0,1]$, $\tau \in [-1,1]$, $\xi := x + \tau h e_i$, $\kappa \in (0,1)$ and $|x| > R_\kappa$. Using

$$|\xi|^2 = |x + \tau h e_i|^2 = |x|^2 + 2\tau h x_i + \tau^2 h^2$$

we obtain

$$|\xi|^2 - \kappa^2|x|^2 = \left(1-\kappa^2\right)|x|^2 + 2\tau h x_i + \tau^2 h^2 \geq \left(1-\kappa^2\right)|x|^2 - 2|x| \geq 0$$

since $|x| > R_\kappa \geq \frac{2}{1-\kappa^2}$, and likewise

$$\kappa^{-2}|x|^2 - |\xi|^2 = \left(\kappa^{-2}-1\right)|x|^2 - 2\tau h x_i - \tau^2 h^2 \geq \left(\kappa^{-2}-1\right)|x|^2 - 3|x| \geq 0$$

as $|x| > R_\kappa \geq \max\left\{\frac{3}{\kappa^{-2}-1}, 1\right\}$. □

The following auxiliary result is also based on the proof of Lemma 3.1 in [7]: For fixed $\beta < 0$ the authors estimate $-\Delta_h\left[|x|_h^\beta\right]$ for $x \in \mathbb{R}^n_h$. Instead of $x \mapsto |x|_h^\beta$, we investigate analogously the comparison function θ defined below.

Lemma 3.7 (Discrete subharmonic comparison function)
Let $h \in (0,1]$, $n \in \mathbb{N}$ with $n \geq 3$, $k \in \{0,\dots,n\}$ and $\theta \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ defined by

$$\theta(x) := \left(\prod_{j=1}^{k} x_j\right)|x|^\beta.$$

For every exponent $\beta < 2 - n - 2k$, the function θ is subharmonic in $\mathbb{R}^{n,k}$. Moreover, there exists a radius $R_\beta > 0$ such that

$$-\Delta_h\theta(x) \leq 0 \quad \text{for all } x \in \mathbb{R}^{n,k}_h \text{ with } |x|_\infty > R_\beta,$$

i.e., θ is discrete subharmonic in $\{x \in \mathbb{R}^{n,k}_h : |x|_\infty > R_\beta\}$. Further, $\theta = 0$ on $\partial_h\mathbb{R}^{n,k}_h \setminus \{0\}$.

Proof. For all $x \in \mathbb{R}^n \setminus \{0\}$, $i \in \{k+1,\dots,n\}$, direct computations show that

$$\frac{\partial}{\partial x_i}\theta(x) = \left(\prod_{j=1}^{k} x_j\right)\beta|x|^{\beta-2}x_i,$$

$$\frac{\partial^2}{\partial x_i^2}\theta(x) = \left(\prod_{j=1}^{k} x_j\right)\beta\left[(\beta-2)|x|^{\beta-4}x_i^2 + |x|^{\beta-2}\right],$$

$$\frac{\partial^3}{\partial x_i^3}\theta(x) = \left(\prod_{j=1}^{k} x_j\right)\beta(\beta-2)\left[(\beta-4)|x|^{\beta-6}x_i^3 + 3|x|^{\beta-4}x_i\right],$$

$$\begin{aligned}
\frac{\partial^4}{\partial x_i^4}\theta(x) &= \left(\prod_{j=1}^{k} x_j\right)\beta(\beta-2)\left[(\beta-4)(\beta-6)|x|^{\beta-8}x_i^4 + 6(\beta-4)|x|^{\beta-6}x_i^2 + 3|x|^{\beta-4}\right] \\
&= \left(\prod_{j=1}^{k} x_j\right)\beta(\beta-2)|x|^{\beta-4}\left[(\beta-4)(\beta-6)\frac{x_i^4}{|x|^4} + 6(\beta-4)\frac{x_i^2}{|x|^2} + 3\right] \\
&= \left(\prod_{j=1}^{k} x_j\right)\beta(\beta-2)|x|^{\beta-4}p\left(\frac{x_i^2}{|x|^2}\right)
\end{aligned}$$

with $p(s) \coloneqq (\beta-4)(\beta-6)s^2 + 6(\beta-4)s + 3$ for $s \in [0,1]$. Next, we consider the case $i \in \{1,\dots,k\}$ and introduce the abbreviation

$$J_i^k \coloneqq \big\{j \in \{1,\dots,k\} \colon j \neq i\big\}. \tag{3.6}$$

Then, for all $x \in \mathbb{R}^n \setminus \{0\}$, follows

$$\begin{aligned}
\frac{\partial}{\partial x_i}\theta(x) &= \left(\prod_{j\in J_i^k} x_j\right)\left[\beta|x|^{\beta-2}x_i^2 + |x|^{\beta}\right], \\
\frac{\partial^2}{\partial x_i^2}\theta(x) &= \left(\prod_{j\in J_i^k} x_j\right)\beta\left[(\beta-2)|x|^{\beta-4}x_i^3 + 3|x|^{\beta-2}x_i\right], \\
\frac{\partial^3}{\partial x_i^3}\theta(x) &= \left(\prod_{j\in J_i^k} x_j\right)\beta\left[(\beta-2)(\beta-4)|x|^{\beta-6}x_i^4 + 6(\beta-2)|x|^{\beta-4}x_i^2 + 3|x|^{\beta-2}\right], \\
\frac{\partial^4}{\partial x_i^4}\theta(x) &= \left(\prod_{j\in J_i^k} x_j\right)\beta(\beta-2)\left[(\beta-4)(\beta-6)|x|^{\beta-8}x_i^5 + 10(\beta-4)|x|^{\beta-6}x_i^3 + 15|x|^{\beta-4}x_i\right] \\
&= \left(\prod_{j=1}^{k} x_j\right)\beta(\beta-2)|x|^{\beta-4}\left[(\beta-4)(\beta-6)\frac{x_i^4}{|x|^4} + 10(\beta-4)\frac{x_i^2}{|x|^2} + 15\right] \\
&= \left(\prod_{j=1}^{k} x_j\right)\beta(\beta-2)|x|^{\beta-4}q\left(\frac{x_i^2}{|x|^2}\right)
\end{aligned} \tag{3.7}$$

with $q(s) \coloneqq (\beta-4)(\beta-6)s^2 + 10(\beta-4)s + 15$ for $s \in [0,1]$. Therefore,

$$\Delta\theta(x) = \sum_{i=1}^{k}\frac{\partial^2}{\partial x_i^2}\theta(x) + \sum_{i=k+1}^{n}\frac{\partial^2}{\partial x_i^2}\theta(x)$$

$$
\begin{aligned}
&= \sum_{i=1}^{k} \left(\prod_{j \in J_i^k} x_j \right) \beta \left[(\beta-2)|x|^{\beta-4} x_i^3 + 3|x|^{\beta-2} x_i \right] \\
&\quad + \sum_{i=k+1}^{n} \left(\prod_{j=1}^{k} x_j \right) \beta \left[(\beta-2)|x|^{\beta-4} x_i^2 + |x|^{\beta-2} \right] \\
&= \left(\prod_{j=1}^{k} x_j \right) \left[(3k+n-k)\beta |x|^{\beta-2} + \beta(\beta-2)|x|^{\beta-4} \sum_{i=1}^{n} x_i^2 \right] \\
&= \left(\prod_{j=1}^{k} x_j \right) \beta(2k+n+\beta-2)|x|^{\beta-2} > 0
\end{aligned}
$$

for all $x \in \mathbb{R}^{n,k}$ and $\beta < 2-n-2k < 0$, i.e., θ is subharmonic in $\mathbb{R}^{n,k}$.

In order to show that θ is discrete subharmonic for large $|x|_\infty$, we estimate the difference between $-\Delta\theta$ and $-\Delta_h\theta$, as it was done in [7] in a similar context: Let $x \in \mathbb{R}_h^{n,k}$ with $|x| \geq 2h$. By Taylor's theorem there exist $\xi^{(i)} \in \{x + \tau h e_i \colon \tau \in (0,1)\}$ and $\eta^{(i)} \in \{x - \tau h e_i \colon \tau \in (0,1)\}$, $i \in \{1, \dots, n\}$, with

$$
\begin{aligned}
-\Delta_h \theta(x) &= -\Delta\theta(x) - \frac{h^2}{24} \sum_{i=1}^{n} \left[\frac{\partial^4}{\partial x_i^4} \theta\big(\xi^{(i)}\big) + \frac{\partial^4}{\partial x_i^4} \theta\big(\eta^{(i)}\big) \right] \\
&= -\left(\prod_{j=1}^{k} x_j \right) \beta(2k+n+\beta-2)|x|^{\beta-2} - \frac{h^2}{24} \sum_{i=1}^{n} \left[\frac{\partial^4}{\partial x_i^4} \theta\big(\xi^{(i)}\big) + \frac{\partial^4}{\partial x_i^4} \theta\big(\eta^{(i)}\big) \right].
\end{aligned}
\tag{3.8}
$$

In the following we will verify that the right hand side is non-positive for all $x \in \mathbb{R}_h^{n,k}$ with $|x|_\infty \geq R$ for a sufficiently large radius R. Roughly speaking this works since the Laplacian is of order $\beta - 2$ and the forth derivatives are of order $\beta - 4$ with respect to $|x|$. Thus, the next step is to find a lower bound for $\frac{\partial^4}{\partial x_i^4}\theta(x)$ for all $x \in \overline{\mathbb{R}^{n,k}} \setminus \{0\}$. Elementary calculations reveal

$$
\begin{aligned}
\min_{s \in [0,1]} p(s) &= p\left(\frac{3}{6-\beta} \right) = \frac{-6(\beta-3)}{\beta-6} \quad \text{and} \\
\min_{s \in [0,1]} q(s) &= q\left(\frac{5}{6-\beta} \right) = \frac{-10(\beta-1)}{\beta-6}.
\end{aligned}
$$

As $0 \leq \frac{x_i^2}{|x|^2} \leq 1$ for all $x \in \mathbb{R}^{n,k}$ this yields for $i \in \{k+1, \dots, n\}$

$$
\frac{\partial^4}{\partial x_i^4} \theta(x) = \left(\prod_{j=1}^{k} x_j \right) \beta(\beta-2)|x|^{\beta-4} p\left(\frac{x_i^2}{|x|^2} \right) \geq \frac{-6\beta(\beta-2)(\beta-3)}{\beta-6} \left(\prod_{j=1}^{k} x_j \right) |x|^{\beta-4}
$$

and for $i \in \{1, \dots, k\}$

$$\frac{\partial^4}{\partial x_i^4}\theta(x) = \left(\prod_{j=1}^{k} x_j\right)\beta(\beta-2)|x|^{\beta-4} q\left(\frac{x_i^2}{|x|^2}\right) \geq \frac{-10\beta(\beta-1)(\beta-2)}{\beta-6}\left(\prod_{j=1}^{k} x_j\right)|x|^{\beta-4}.$$

In order to apply Lemma 3.6 for $\xi^{(i)}$ and $\eta^{(i)}$ we choose $\kappa \in (0,1)$ and set $R_\kappa := \max\left\{\frac{3}{\kappa^{-2}-1}, \frac{2}{1-\kappa^2}\right\}$. Hence, for $x \in \mathbb{R}_h^{n,k}$ with $|x| > R_\kappa$ and $i \in \{k+1, \dots, n\}$ we have

$$\begin{aligned}\frac{\partial^4}{\partial x_i^4}\theta\left(\xi^{(i)}\right) &\geq \frac{-6\beta(\beta-2)(\beta-3)}{\beta-6}\left(\prod_{j=1}^{k} \underbrace{\xi_j^{(i)}}_{=x_j}\right)\left|\xi^{(i)}\right|^{\beta-4} \\ &\geq \frac{-6\beta(\beta-2)(\beta-3)}{\beta-6}\left(\prod_{j=1}^{k} x_j\right)\kappa^{\beta-4}|x|^{\beta-4}\end{aligned}$$

and in the same manner

$$\frac{\partial^4}{\partial x_i^4}\theta\left(\eta^{(i)}\right) \geq \frac{-6\beta(\beta-2)(\beta-3)}{\beta-6}\left(\prod_{j=1}^{k} x_j\right)\kappa^{\beta-4}|x|^{\beta-4};$$

for $i \in \{1, \dots, k\}$ we use $\xi_i^{(i)} \leq 2x_i$ and (3.6) to obtain

$$\begin{aligned}\frac{\partial^4}{\partial x_i^4}\theta\left(\xi^{(i)}\right) &\geq \frac{-10\beta(\beta-1)(\beta-2)}{\beta-6}\left(\prod_{j\in J_i^k} \underbrace{\xi_j^{(i)}}_{=x_j}\right)\underbrace{\xi_i^{(i)}}_{\leq 2x_i}\left|\xi^{(i)}\right|^{\beta-4} \\ &\geq \frac{-20\beta(\beta-1)(\beta-2)}{\beta-6}\left(\prod_{j=1}^{k} x_j\right)\kappa^{\beta-4}|x|^{\beta-4}\end{aligned}$$

and $\eta_i^{(i)} \leq x_i$ yields

$$\begin{aligned}\frac{\partial^4}{\partial x_i^4}\theta\left(\eta^{(i)}\right) &\geq \frac{-10\beta(\beta-1)(\beta-2)}{\beta-6}\left(\prod_{j\in J_i^k} \underbrace{\eta_j^{(i)}}_{=x_j}\right)\underbrace{\eta_i^{(i)}}_{\leq x_i}\left|\eta^{(i)}\right|^{\beta-4} \\ &\geq \frac{-10\beta(\beta-1)(\beta-2)}{\beta-6}\left(\prod_{j=1}^{k} x_j\right)\kappa^{\beta-4}|x|^{\beta-4}.\end{aligned}$$

Thus, we can conclude with (3.8)

$$-\Delta_h\theta(x) = -\left(\prod_{j=1}^{k} x_j\right)\beta(2k+n+\beta-2)|x|^{\beta-2} - \frac{h^2}{24}\sum_{i=1}^{n}\left[\frac{\partial^4}{\partial x_i^4}\theta\left(\xi^{(i)}\right) + \frac{\partial^4}{\partial x_i^4}\theta\left(\eta^{(i)}\right)\right]$$

$$\leq \left(\prod_{j=1}^{k} x_j\right)|x|^{\beta-2}\Bigg[-\beta(2k+n+\beta-2)$$
$$-\frac{h^2}{24}\kappa^{\beta-4}|x|^{-2}\left(\frac{-12\beta(\beta-2)(\beta-3)}{\beta-6}(n-k)+\frac{-30\beta(\beta-1)(\beta-2)}{\beta-6}k\right)\Bigg]$$
$$\leq 0$$

for every $\beta < 2-n-2k$ and $x \in \mathbb{R}_h^{n,k}$ with $|x|_\infty > R_\beta$ for a sufficiently large radius $R_\beta > 0$.

Finally, by definition $\theta = 0$ on $\partial\mathbb{R}^{n,k} \setminus \{0\}$ and hence on $\partial_h\mathbb{R}_h^{n,k} \setminus \{0\}$. □

The lemma below is a variant of [30, Lemma 10.7]. Although the argumentation is similar to the original proof, we give all the details for the reader's convenience.

Lemma 3.8 (Reverse Hardy inequality)
There exists a sequence $(u_l)_{l\in\mathbb{N}} \subset \mathcal{C}(\mathbb{R}_h^n)$, $u_l \not\equiv 0$ with

$$\mathrm{cl}_h(\mathrm{supp}(u_l)) \subset \left\{x \in \mathbb{R}_h^{n,k} : |x|_\infty \geq lh,\, x_i \geq \frac{1}{3\sqrt{n}}|x| \ (i=1,\dots,k)\right\},$$

$l \in \mathbb{N}$, such that

$$\sum_{x\in\mathbb{R}_h^n}\sum_{i=1}^{n}|D_i^+u_l(x)|^2h^n \leq C \sum_{x\in\mathbb{R}_h^n\setminus\{0\}}\frac{|u_l(x)|^2}{|x|^2}h^n \tag{3.9}$$

for a constant $C > 0$ independent of h and l.

Proof. ⟨1⟩ *Reduction*
Using the scale invariance with respect to h we obtain

$$\frac{\sum_{x\in\mathbb{R}_h^n}\sum_{i=1}^n|D_i^+u_l(x)|^2h^n}{\sum_{x\in\mathbb{R}_h^n\setminus\{0\}}\frac{|u_l(x)|^2}{|x|^2}h^n} = \frac{\sum_{x\in\mathbb{R}_h^n}\sum_{i=1}^n|u_l(x+he_i)-u_l(x)|^2}{\sum_{z\in\mathbb{Z}^n\setminus\{0\}}\frac{|u_l(zh)|^2}{|z|^2}}.$$

Due to the norm equivalence in $\mathbb{R}^n$ it is therefore sufficient to show

$$\frac{\sum_{x\in\mathbb{R}_h^n}\sum_{i=1}^n|u_l(x+he_i)-u_l(x)|^2}{\sum_{z\in\mathbb{Z}^n\setminus\{0\}}\frac{|u_l(zh)|^2}{|z|_1^2}} \leq C. \tag{3.10}$$

⟨2⟩ *Construction of u_l*
For sets $A \subset \mathbb{R}^n$, the characteristic function $\chi_A\colon \mathbb{R}^n \to \mathbb{R}$ is given by

$$\chi_A(x) \coloneqq \begin{cases} 1, & x \in A, \\ 0, & \text{else}. \end{cases} \tag{3.11}$$

Defining also $e \coloneqq (1,\dots,1)^T \in \mathbb{R}^n$, we set

$$u_l \coloneqq \sum_{m=0}^{l} \frac{l-m}{l} \chi_{\{x\in\mathbb{R}_h^n\colon |x-2lhe|_1 = mh\}}.$$

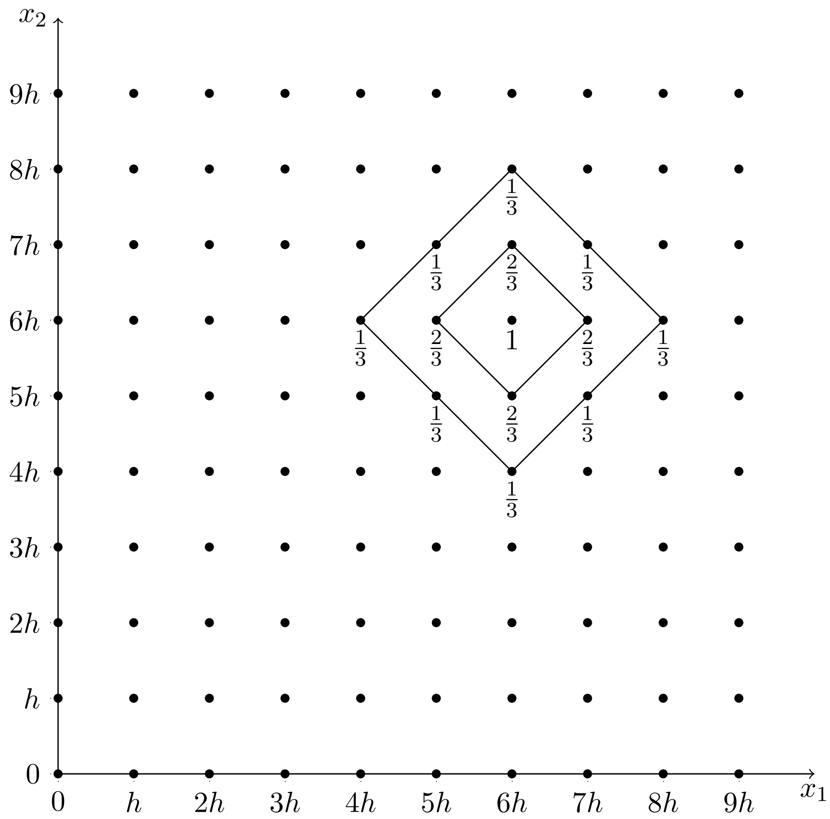


Figure 3.3: Illustration of u_3 in the case $n = 2$.

According to (2.1), the discrete closure of a discrete set $B \subset \mathbb{R}_h^n$ is given by

$$\mathrm{cl}_h(B) = B \cup \bigcup_{x\in B} N_h(x).$$

From the definition of u_l follows that $u_l \in \mathcal{C}(\mathbb{R}_h^n)$ with

$$\begin{aligned} \mathrm{cl}_h(\mathrm{supp}(u_l)) &= \{x \in \mathbb{R}_h^n\colon |x-2lhe|_1 \le lh\} \\ &\subset \left\{ x \in \mathbb{R}_h^{n,k}\colon |x|_\infty \ge lh,\ x_i \ge \frac{1}{3\sqrt{n}}|x|\ (i = 1,\dots,n) \right\}. \end{aligned}$$

For the last inclusion we used the following: $|x-2lhe|_1 \le lh$ implies $|x_i - 2lh| \le lh$

and hence $x_i \in [lh, 3lh]$ so that $|x|_\infty \geq lh$. From that we can conclude

$$x_i \geq lh = \frac{1}{3\sqrt{n}} 3lh\sqrt{n} = \frac{1}{3\sqrt{n}} \left(\sum_{i=1}^{n} (3lh)^2 \right)^{\frac{1}{2}} \geq \frac{1}{3\sqrt{n}} \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} = \frac{1}{3\sqrt{n}} |x|$$

for all $i \in \{1, \ldots, n\}$.

⟨3⟩ *Estimating the numerator in* (3.10)
First, we compute $|u_l(x + he_i) - u_l(x)|$ for $l \in \mathbb{N}$, $i \in \{1, \ldots, n\}$ and $x \in \mathbb{R}^n_h$. If x and $x + he_i$ are not contained in $\operatorname{supp}(u_l)$, then $|u_l(x + he_i) - u_l(x)| = 0$. Otherwise, due to the special construction of u_l (see also Figure 3.3), we deduce $|u_l(x + he_i) - u_l(x)| = \frac{1}{l}$. In summary, we obtain

$$|u_l(x + he_i) - u_l(x)|^2 = \begin{cases} 0, & \text{if } \{x, x + he_i\} \cap \operatorname{supp}(u_l) = \emptyset, \\ \dfrac{1}{l^2}, & \text{else,} \end{cases}$$

for all $l \in \mathbb{N}$, $i \in \{1, \ldots, n\}$, $x \in \mathbb{R}^n_h$. Next, we estimate the number of elements in

$$A_l := \bigcup_{i=1}^{n} \left\{ x \in \mathbb{R}^n_h \colon \{x, x + he_i\} \cap \operatorname{supp}(u_l) \neq \emptyset \right\}.$$

In view of

$$A_l \subset \operatorname{cl}_h(\operatorname{supp}(u_l)) = \{x \in \mathbb{R}^n_h \colon |x - 2lhe|_1 \leq lh\} \subset \{x \in \mathbb{R}^n_h \colon |x - 2lhe|_\infty \leq lh\}$$

we see that A_l contains at most $(2l + 1)^n$ elements. This yields

$$\sum_{x \in \mathbb{R}^n_h} \sum_{i=1}^{n} |u_l(x + he_i) - u_l(x)|^2 \leq (2l + 1)^n n \frac{1}{l^2} \leq 3^n n l^{n-2}.$$

⟨4⟩ *Estimating the denominator in* (3.10)
Since $\sum_{m=1}^{l} \left(1 - \frac{m}{l}\right)^2 \left(\frac{m}{l}\right)^{n-1} \frac{1}{l}$ is a Riemann sum for

$$\int_0^1 (1 - x)^2 x^{n-1} \, dx = \frac{2}{n(n+1)(n+2)} > 0,$$

there exists a constant $K(n) > 0$ with

$$\sum_{m=1}^{l} \left(1 - \frac{m}{l}\right)^2 \left(\frac{m}{l}\right)^{n-1} \frac{1}{l} \geq K(n) \qquad \text{for all } l \in \mathbb{N}. \tag{3.12}$$

Furthermore, there exists some constant $K'(n) > 0$ such that for all $l \in \mathbb{N}$ and

$m \in \{0, \dots, l\}$ we have

$$\begin{aligned}\sum_{z \in \mathbb{Z}^n \setminus \{0\}} \chi_{\{x \in \mathbb{R}^n_h \colon |x-2lhe|_1 = mh\}}(zh) &= \#\{x \in \mathbb{R}^n_h \colon |x - 2lhe|_1 = mh\} \\ &= \#\{y \in \mathbb{Z}^n \colon |y|_1 = m\} \\ &\geq K'(n) m^{n-1},\end{aligned} \tag{3.13}$$

where $\#$ denotes the counting measure. Moreover, from $|zh - 2lhe|_1 = mh$ we infer that $|z|_1 \leq m + 2l|e|_1 = m + 2ln$. Together with (3.12) and (3.13) this leads to

$$\begin{aligned}\sum_{z \in \mathbb{Z}^n \setminus \{0\}} \frac{|u_l(zh)|^2}{|z|_1^2} &= \sum_{z \in \mathbb{Z}^n \setminus \{0\}} \sum_{m=0}^{l} \frac{(l-m)^2}{l^2} \frac{\chi_{\{x \in \mathbb{R}^n_h \colon |x-2lhe|_1 = mh\}}(zh)}{|z|_1^2} \\ &\geq \sum_{z \in \mathbb{Z}^n \setminus \{0\}} \sum_{m=0}^{l} \frac{(l-m)^2}{l^2} \frac{\chi_{\{x \in \mathbb{R}^n_h \colon |x-2lhe|_1 = mh\}}(zh)}{(2nl+m)^2} \\ &\geq \sum_{m=0}^{l} \frac{1}{n^2} \frac{(l-m)^2}{l^2(2l+m)^2} \sum_{z \in \mathbb{Z}^n \setminus \{0\}} \chi_{\{x \in \mathbb{R}^n_h \colon |x-2lhe|_1 = mh\}}(zh) \\ &\geq \frac{K'(n)}{n^2} \frac{1}{l^2} \sum_{m=0}^{l} \frac{(l-m)^2}{(3l)^2} m^{n-1} \\ &= \frac{K'(n)}{9n^2} l^{n-2} \sum_{m=1}^{l} \left(1 - \frac{m}{l}\right)^2 \left(\frac{m}{l}\right)^{n-1} \frac{1}{l} \\ &\geq \frac{K(n)K'(n)}{9n^2} l^{n-2}.\end{aligned}$$

The estimates for numerator and denominator together prove (3.10) and thus, the reverse Hardy inequality (3.9) for u_l follows. □

Proof of Theorem 3.4 for $n \geq 3$. We prove the result by contradiction: *Suppose* there exists a solution $u \colon \mathbb{R}^{n,k}_h \to [0, \infty), u \not\equiv 0$ of (3.5). Due to the scale invariance of problem (3.5) (Remark 3.5), we may assume $h \in (0, 1]$.

⟨1⟩ *Positivity*
Assume $u(x_0) = \min_{x \in \mathbb{R}^{n,k}_h} u(x) = 0$ for some $x_0 \in \mathbb{R}^{n,k}_h$. Then, the discrete maximum principle (Lemma 2.12), applied to the function $-u$, directly yields $u = 0$ in $\mathbb{R}^{n,k}_h$, a contradiction. Thus, $u > 0$ in $\mathbb{R}^{n,k}_h$.

⟨2⟩ *Comparison argument*
From $1 < p < \frac{n+k}{n+k-2}$, we get $\delta := 2 - (p-1)(n+k-2) > 0$. By setting $\varepsilon := \frac{\delta}{2(p-1)}$ and $\beta := 2 - n - 2k - \varepsilon$, we infer

$$(k + \beta)(p - 1) + 2 > 0. \tag{3.14}$$

We define the comparison function $\theta\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by

$$\theta(x) := \left(\prod_{j=1}^{k} x_j\right)|x|^{\beta}.$$

Note that $x \mapsto \prod_{j=1}^{k} x_j$ is a positive, harmonic function in $\mathbb{R}^{n,k}$ which is zero on the boundary $\partial\mathbb{R}^{n,k}$. According to Lemma 3.7 there exists a radius $R_0 \in h\mathbb{N}$ such that

$$-\Delta_h\theta(x) \le 0 \quad \text{for all } x \in \mathbb{R}^{n,k}_h \text{ with } |x|_\infty > R_0.$$

As the set $\{x \in \mathbb{R}^{n,k}_h \colon |x|_\infty = R_0\}$ is finite, we can find a constant $C > 0$ with

$$C\theta(x) \le u(x) \quad \text{for all } x \in \mathbb{R}^{n,k}_h \text{ with } |x|_\infty = R_0.$$

Due to definition of β we have $\beta + k < 0$ and therefore $\theta(x)$ converges to 0 for $|x| \to \infty$. Hence, for all $w > 0$, there exists a radius $R_w \in h\mathbb{N}$, $R_w > R_0$, such that

$$C\theta(x) - w \le 0 \quad \text{for all } x \in \mathbb{R}^{n,k}_h \text{ with } |x|_\infty = R_w.$$

As u is non-negative, we infer that

$$C\theta(x) - w \le u(x) \quad \text{for all } x \in \mathbb{R}^{n,k}_h \text{ with } |x|_\infty = R_w.$$

We remark that $R_w \to +\infty$ as $w \to 0$. Now we can apply the discrete maximum principle (Lemma 2.12) to the function $x \mapsto C\theta(x) - w - u(x)$ defined on the set

$$\begin{aligned} M_w : &= \mathbb{R}^{n,k}_h \cap \{y \in \mathbb{R}^n_h \colon R_0 < |y|_\infty < R_w\} \\ &= \{y \in \mathbb{R}^n_h \colon y_1, \dots, y_k > 0,\, R_0 < |y|_\infty < R_w\}. \end{aligned}$$

With the considerations above we can easily conclude

$$\begin{cases} -\Delta_h[C\theta(x) - w - u(x)] \le 0 & \text{in } M_w, \\ \qquad C\theta(x) - w - u(x) \le 0 & \text{on } \partial_h M_w. \end{cases}$$

Thus, the maximum principle (Lemma 2.12) yields $C\theta(x) - w - u(x) \le 0$ in M_w. By taking the limit $w \to 0$, we get the comparison estimate

$$C\theta(x) \le u(x) \quad \text{for all } x \in \mathbb{R}^{n,k}_h \text{ with } |x|_\infty > R_0. \tag{3.15}$$

⟨3⟩ *Hardy-like inequality*

Due to (3.15), for all $x \in \mathbb{R}^{n,k}_h$ with $|x|_\infty > R_0$ and $x_i \ge \frac{1}{3\sqrt{n}}|x|$ $(i = 1, \dots, k)$, it follows

$$\begin{aligned}-\Delta_h u(x) &\geq u^{p-1}(x)u(x)\\ &\geq C^{p-1}\left(\prod_{j=1}^{k} x_j\right)^{p-1} |x|^{\beta(p-1)}\, u(x)\\ &= C^{p-1}\left(\prod_{j=1}^{k} \frac{x_j}{|x|}\right)^{p-1} |x|^{(k+\beta)(p-1)+2}\, |x|^{-2}\, u(x)\\ &\geq \left(\frac{C}{(3\sqrt{n})^k}\right)^{p-1} |x|^{(k+\beta)(p-1)+2}\, |x|^{-2}\, u(x).\end{aligned}$$

As the exponent $(k+\beta)(p-1)+2$ is strictly positive by (3.14), for every $K > 0$ there exists a radius $R_K > R_0$ such that

$$-\Delta_h u(x) \geq \frac{K}{|x|^2} u(x) \tag{3.16}$$

for all $x \in N_K := \left\{x \in \mathbb{R}_h^{n,k} \colon |x|_\infty \geq R_K,\, x_i \geq \frac{1}{3\sqrt{n}}|x|\ (i = 1, \ldots, k)\right\}$.

⟨4⟩ *Agmon principle*

Below, we use a discrete variant of the so-called Agmon positivity principle. A classical version thereof can be found in Agmon's work ([2, Thm. 3.1]) or in the book of Davies ([9, Thm. 1.5.12.]). The theory introduced by Agmon is based on preliminary studies of Allegretto ([4, Thm. 2]) and Piepenbrink ([26, Thm. 3.3]). Roughly speaking, the idea behind is the following: Inequality (3.16) says that u is a supersolution for the operator $-\Delta_h - \frac{K}{|x|^2}$ on N_K. This yields that the corresponding bilinear form is positive for all suitable functions $\psi\colon N_K \to \mathbb{R}$, i.e.,

$$\sum_{x\in N_K}\left[\sum_{i=1}^{n}(D_i^+\psi(x))^2 - \frac{K}{|x|^2}\psi^2(x)\right] \geq 0.$$

Next, we turn our attention to the details: For test functions $\psi \in \mathcal{C}(\mathbb{R}_h^n)$ with $\mathrm{cl}_h(\mathrm{supp}(\psi)) \subset N_K$, we multiply the inequality (3.16) by $\frac{\psi^2(x)}{u(x)} \geq 0$. Choosing test functions with such a support ensures that we can apply the partial summation formula from Lemma 2.15 and we obtain

$$\begin{aligned}0 &\leq \sum_{x\in N_K}\left[(-\Delta_h u(x))\frac{\psi^2(x)}{u(x)} - \frac{K}{|x|^2}u(x)\frac{\psi^2(x)}{u(x)}\right]\\ &= \sum_{x\in N_K}\left[-\sum_{i=1}^{n}(D_i^- D_i^+ u(x))\frac{\psi^2(x)}{u(x)} - \frac{K}{|x|^2}\psi^2(x)\right]\\ &= \sum_{x\in N_K}\left[\sum_{i=1}^{n} D_i^+ u(x) D_i^+\left(\frac{\psi^2(x)}{u(x)}\right) - \frac{K}{|x|^2}\psi^2(x)\right].\end{aligned} \tag{3.17}$$

Direct computations lead to

$$\begin{aligned}
D_i^+ u(x) D_i^+ \left(\frac{\psi^2(x)}{u(x)}\right) &= \frac{1}{h^2}(u(x+he_i)-u(x))\left(\frac{\psi^2(x+he_i)}{u(x+he_i)} - \frac{\psi^2(x)}{u(x)}\right) \\
&= \frac{1}{h^2}\left[\psi^2(x+he_i) - 2\psi(x+he_i)\psi(x) + \psi^2(x)\right] \\
&\quad - \frac{u(x+he_i)u(x)}{h^2}\left[\frac{\psi^2(x)}{u^2(x)} - 2\frac{\psi(x+he_i)\psi(x)}{u(x+hei)u(x)} + \frac{\psi^2(x+he_i)}{u^2(x+he_i)}\right] \\
&= \frac{1}{h^2}\left[\psi(x+he_i)-\psi(x)\right]^2 - \frac{u(x+he_i)u(x)}{h^2}\left[\frac{\psi}{u}(x+he_i) - \frac{\psi}{u}(x)\right]^2 \\
&= [D_i^+\psi(x)]^2 - u(x+he_i)u(x)\left[D_i^+\left(\frac{\psi}{u}(x)\right)\right]^2
\end{aligned}$$

for all $x \in N_K$ and all $i \in \{1,\ldots,n\}$. Together with (3.17), this implies

$$\begin{aligned}
&\sum_{x\in N_K}\left[\sum_{i=1}^n (D_i^+\psi(x))^2 - \frac{K}{|x|^2}\psi^2(x)\right] \\
&= \sum_{x\in N_K}\left[\sum_{i=1}^n \left(D_i^+ u(x) D_i^+\left(\frac{\psi^2(x)}{u(x)}\right) + u(x+he_i)u(x)\left[D_i^+\left(\frac{\psi}{u}(x)\right)\right]^2\right) - \frac{K}{|x|^2}\psi^2(x)\right] \\
&\geq \sum_{x\in N_K}\left[\sum_{i=1}^n D_i^+ u(x) D_i^+\left(\frac{\psi^2(x)}{u(x)}\right) - \frac{K}{|x|^2}\psi^2(x)\right] \geq 0.
\end{aligned} \tag{3.18}$$

⟨5⟩ *Contradiction to the reverse Hardy inequality*
Choosing K bigger than the constant C from Lemma 3.8 and $l \in \mathbb{N}$ so large that

$$\mathrm{cl}_h(\mathrm{supp}(u_l)) \subset N_K,$$

the estimates (3.18) and (3.9) yield

$$K\sum_{x\in N_K}\frac{u_l^2(x)}{|x|^2} \leq \sum_{x\in N_K}\sum_{i=1}^n |D_i^+ u_l(x)|^2 \leq C\sum_{x\in N_K}\frac{u_l^2(x)}{|x|^2},$$

which is a contradiction since $K > C$. □

Remark 3.9
In the case $\mathbb{R}_h^{n,k} = \mathbb{R}_h^n$ with $n \geq 3$, i.e., $k = 0$, the discrete Liouville Theorem 3.4 with "$\geq$" replaced by "$=$" in (3.5) was already proven by Anton Verbitsky in his dissertation ([30, Thm. 10.8.]) with a similar approach.

Theorem 3.10 (Existence of solutions for generalized orthants)
Let $h > 0$ be the grid size, $n \geq 2$ the dimension, $k \in \{0, \dots, n\}$, where $k \neq 0$ if $n = 2$, and $p > \frac{n+k}{n+k-2}$. Then, there exists a positive solution $u \colon \overline{\mathbb{R}_h^{n,k}} \to (0, \infty)$ of

$$-\Delta_h u \geq u^p \quad \text{in } \mathbb{R}_h^{n,k}. \tag{3.19}$$

Proof. We divide the proof into the two parts $k \in \{1, \dots, n\}$ and $k = 0$.

$\underline{k \in \{1, \dots, n\}}$: Due to the scale invariance (Remark 3.5) we may assume $h \in (0, 1]$. Furthermore, we reuse the comparison function $\theta \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ from Lemma 3.7, defined by

$$\theta(x) \coloneqq \left(\prod_{j=1}^{k} x_j \right) |x|^\beta,$$

where we choose $\beta \coloneqq \frac{-2}{p-1} - k < 0$. Thus, we see

$$(p-1)(\beta + k) + 2 = 0. \tag{3.20}$$

Moreover, the assumption $p > \frac{n+k}{n+k-2}$ yields

$$\beta > 2 - n - 2k. \tag{3.21}$$

Next, we recall some results from the proof of Lemma 3.7: By (3.8), for all $x \in \mathbb{R}_h^{n,k}$ with $|x| \geq 2h$ there exist $\xi^{(i)} \in \{x + \tau h e_i \colon \tau \in (0,1)\}$ and $\eta^{(i)} \in \{x - \tau h e_i \colon \tau \in (0,1)\}$, $i \in \{1, \dots, n\}$, such that

$$-\Delta_h \theta(x) = -\left(\prod_{j=1}^{k} x_j \right) \beta(2k + n + \beta - 2)|x|^{\beta-2} - \frac{h^2}{24} \sum_{i=1}^{n} \left[\frac{\partial^4}{\partial x_i^4} \theta\big(\xi^{(i)}\big) + \frac{\partial^4}{\partial x_i^4} \theta\big(\eta^{(i)}\big) \right]. \tag{3.22}$$

Further, for all $x \in \mathbb{R}^n \setminus \{0\}$, $i \in \{k+1, \dots, n\}$, we obtain

$$\frac{\partial^4}{\partial x_i^4} \theta(x) = \left(\prod_{j=1}^{k} x_j \right) \beta(\beta - 2)|x|^{\beta-4} q_1\left(\frac{x_i^2}{|x|^2} \right)$$

with $q_1(s) \coloneqq (\beta - 4)(\beta - 6)s^2 + 6(\beta - 4)s + 3$ for $s \in [0, 1]$. Similarly, for $i \in \{1, \dots, k\}$ we have

$$\frac{\partial^4}{\partial x_i^4} \theta(x) = \left(\prod_{j=1}^{k} x_j \right) \beta(\beta - 2)|x|^{\beta-4} q_2\left(\frac{x_i^2}{|x|^2} \right) \tag{3.23}$$

with $q_2(s) \coloneqq (\beta - 4)(\beta - 6)s^2 + 10(\beta - 4)s + 15$ for $s \in [0, 1]$. Note that $0 \leq \frac{x_i^2}{|x|^2} \leq 1$

for all $x \in \mathbb{R}_h^{n,k}$ and that the continuous functions q_1, q_2 attain a positive maximum on $[0,1]$, respectively. Hence, there exists a constant $C = C(\beta) > 0$ such that

$$\frac{\partial^4}{\partial x_i^4}\theta(x) \leq C\left(\prod_{j=1}^{k} x_j\right)|x|^{\beta-4} \tag{3.24}$$

for all $x \in \mathbb{R}_h^{n,k}$ and $i \in \{1,\ldots,n\}$. For $\kappa \in (0,1)$ fixed, let $R_\kappa > 0$ be the corresponding radius from Lemma 3.6, which allows the neighbour estimates for $|x| > R_\kappa$. From (3.24), we infer as in the proof of Lemma 3.7 that there exists some constant $C = C(\beta,\kappa) > 0$ with

$$\frac{\partial^4}{\partial x_i^4}\theta\big(\xi^{(i)}\big), \frac{\partial^4}{\partial x_i^4}\theta\big(\eta^{(i)}\big) \leq C\left(\prod_{j=1}^{k} x_j\right)|x|^{\beta-4}$$

for all $x \in \mathbb{R}^{n,k}$ with $|x| \geq R_\kappa$. We insert this in (3.22) and deduce

$$-\Delta_h\theta(x) \geq -\frac{1}{2}\left(\prod_{j=1}^{k} x_j\right)\beta(2k+n+\beta-2)|x|^{\beta-2} \tag{3.25}$$

for all $x \in \mathbb{R}^{n,k}$ with $|x| \geq R_0$ and a sufficiently large $R_0 > R_\kappa$, since $2k+n+\beta-2 > 0$ by (3.21) and $\beta < 0$. Below, we use the notation $c_\beta \coloneqq -\frac{1}{2}\beta(2k+n+\beta-2) > 0$. Next, we determine some $\tau > 0$ such that

$$-\Delta_h\left(\tau\theta(x)\right) \geq \left(\tau\theta(x)\right)^p \tag{3.26}$$

for large $|x|$. Due to (3.25), it is sufficient to show that

$$c_\beta\left(\prod_{j=1}^{k} x_j\right)|x|^{\beta-2} \geq \tau^{-1}\left(\tau\theta(x)\right)^p. \tag{3.27}$$

The last inequality is equivalent to

$$c_\beta \geq \tau^{p-1}|x|^{\beta(p-1)+2}\left(\prod_{j=1}^{k} x_j^{p-1}\right).$$

In view of $0 \leq x_i \leq |x|$ for all $x \in \mathbb{R}_h^{n,k}$ and $i \in \{1,\ldots,k\}$, we obtain the estimate

$$|x|^{\beta(p-1)+2}\left(\prod_{j=1}^{k} x_j^{p-1}\right) \leq |x|^{(\beta+k)(p-1)+2}.$$

Note that the exponent $(\beta+k)(p-1)+2$ is zero by (3.20). Therefore, the condition (3.27) is satisfied if $c_\beta \geq \tau^{p-1}$. In summary, for $\tau > 0$ such that $c_\beta \geq \tau^{p-1}$ the inequality

(3.26) holds true for all $x \in \mathbb{R}^{n,k}_h$ with $|x| \geq R_0$. Finally, let $e \in \mathbb{R}^n$ with

$$e_i := \begin{cases} 1, & i \leq k, \\ 0, & \text{else.} \end{cases}$$

Then, the function $u\colon \overline{\mathbb{R}^{n,k}_h} \to \mathbb{R}$, given by $u(x) := \tau\theta(x + R_0 e)$, solves (3.19).

<u>$k = 0$:</u> Note that $n > 2$ by assumption. In view of the scale invariance we may assume $h \in (0, 1]$. In the sequel, we consider the function $v\colon \mathbb{R}^n \to \mathbb{R}$, defined by

$$v(x) := |x|_h^\beta,$$

where we choose $\beta := \frac{-2}{p-1} < 0$ and recall that $|x|_h = \sqrt{|x|^2 + \gamma h^2}$ by (2.2). Here we fix some sufficiently large $\gamma = \gamma(\beta, n) > 0$ such that we can apply Lemma 6.2 in [30] and deduce the estimate

$$-\Delta_h v(x) \geq -\beta(\beta + n - 2)|x|_h^{\beta-2} \quad \text{for all } x \in \mathbb{R}^n_h \setminus \{0\}.$$

Note that the choice of β ensures $\beta - 2 = \beta p$. Using also the assumption $p > \frac{n}{n-2}$, we infer $\beta - 2 < \beta \frac{n}{n-2}$, which leads to $\beta > 2 - n$. Therefore, the constant $\tilde{c}_\beta := -\beta(\beta + n - 2)$ is positive and we conclude

$$-\Delta_h v(x) \geq \tilde{c}_\beta v^p(x) \quad \text{for all } x \in \mathbb{R}^n_h \setminus \{0\}. \tag{3.28}$$

In the next step we proof a similar estimate for $x = 0$: Direct computation shows that

$$\begin{aligned} -\Delta_h v(0) &= -\frac{1}{h^2} \sum_{i=1}^n \left[\sqrt{|he_i|^2 + \gamma h^2}^\beta - 2\sqrt{\gamma h^2}^\beta + \sqrt{|-he_i|^2 + \gamma h^2}^\beta \right] \\ &= h^{\beta-2} 2n \left[\gamma^{\frac{\beta}{2}} - (\gamma + 1)^{\frac{\beta}{2}} \right] > 0. \end{aligned}$$

Recalling that $\beta p = \beta - 2$, we see $v^p(0) = \sqrt{\gamma h^2}^{\beta p} = \gamma^{\frac{\beta-2}{2}} h^{\beta-2}$. Thus, with the shorthand $\hat{c}_\beta := 2n \dfrac{\gamma^{\frac{\beta}{2}} - (\gamma+1)^{\frac{\beta}{2}}}{\gamma^{\frac{\beta-2}{2}}}$ we deduce

$$-\Delta_h v(0) = h^{\beta-2} 2n \left[\gamma^{\frac{\beta}{2}} - (\gamma + 1)^{\frac{\beta}{2}} \right] = \hat{c}_\beta \gamma^{\frac{\beta-2}{2}} h^{\beta-2} = \hat{c}_\beta v^p(0).$$

Introducing $C_\beta := \min\{\tilde{c}_\beta, \hat{c}_\beta\} > 0$, we obtain by means of (3.28) that

$$-\Delta_h v \geq C_\beta v^p \quad \text{in } \mathbb{R}^n_h.$$

So, for all $\tau > 0$ with $\tau^{p-1} \leq C_\beta$, the function $u(x) = \tau v(x)$ solves (3.19) in $\mathbb{R}^n_h$. □

Remark 3.11 (Critical exponent)
For dimensions $n \geq 2$ and $k \in \{0, \dots, n\}$ (with $k \neq 0$ if $n = 2$) the exponent $p_* := \frac{n+k}{n+k-2}$ is a critical exponent for positive solutions $u\colon \overline{\mathbb{R}_h^{n,k}} \to (0, \infty)$ of

$$-\Delta_h u \geq u^p \quad \text{in } \mathbb{R}_h^{n,k}. \tag{3.29}$$

On the one hand, for $1 < p < p_*$ there is no positive solution due to Theorem 3.4. On the other hand, Theorem 3.10 provides the existence of a positive solution for all $p > p_*$. In the case $p = p_*$, it is still unclear whether positive solutions exist or not.

3.2. One-dimensional case

For the sake of completeness we consider the case $n = 1$. Using the techniques from Section 3.1, we can prove that the only non-negative solution $u\colon \mathbb{R}_h \to [0, \infty)$ of the one-dimensional discrete Emden equation

$$-\Delta_h u = u^p \quad \text{in } \mathbb{R}_h \tag{3.30}$$

is $u \equiv 0$ for all $p \in (1, \infty)$ and $h > 0$. Instead of giving all the details we focus on the following stronger result.

Proposition 3.12
In dimension one every discrete superharmonic function on $\mathbb{R}_h^n$, which is bounded from below, is constant.

Proof. Let $h > 0$. Suppose $v\colon \mathbb{R}_h \to \mathbb{R}$ be bounded from below and discrete superharmonic, i.e.,

$$-\Delta_h v(x) = -\frac{1}{h^2}\left[v(x+h) - 2v(x) + v(x-h)\right] \geq 0$$

for all $x \in \mathbb{R}_h$. Since problem (3.30) is scale invariant (Remark 3.5), it suffices to investigate the case $h = 1$. Then, $v\colon \mathbb{Z} \to \mathbb{R}$ satisfies

$$v(z+1) - v(z) \leq v(z) - v(z-1) \tag{3.31}$$

for all $z \in \mathbb{Z}$. *Assume* v is not constant. Thus, there exists some $z_0 \in \mathbb{Z}$ such that $\varepsilon := v(z_0) - v(z_0 - 1) \neq 0$. If $\varepsilon > 0$, applying inequality (3.31), leads iteratively to

$$\varepsilon = v(z_0) - v(z_0 - 1) \leq v(z_0 - k) - v(z_0 - k - 1)$$

for all $k \in \mathbb{N}_0$. Hence, for all $M \in \mathbb{N}$, we deduce

$$v(z_0) - v(z_0 - M) = \sum_{k=0}^{M-1} \left[v(z_0 - k) - v(z_0 - k - 1)\right] \geq M\varepsilon$$

and we infer

$$v(z_0 - M) \le v(z_0) - M\varepsilon \to -\infty$$

as $M \to \infty$. This contradicts the assertion that v is bounded from below. If $\varepsilon < 0$, similarly

$$v(z_0 + M) = v(z_0) + \sum_{k=1}^{M} [v(z_0 + k) - v(z_0 + k - 1)] \le v(z_0) + M\varepsilon \to -\infty$$

for $M \to \infty$ which yields a contradiction since v is bounded from below. □

Theorem 3.13
Let $h > 0$ and $p \in (1, \infty)$. Then, the only non-negative solution of

(a) the real line problem

$$-\Delta_h u \ge u^p \quad \text{in } \mathbb{R}_h,$$

(b) the half ray problem

$$\begin{cases} -\Delta_h u \ge u^p & \text{in } h\mathbb{N}, \\ u(0) \ge 0, \end{cases}$$

is $u \equiv 0$, respectively.

Proof. The first part is covered by Proposition 3.12. However, both problems can be treated as in the proof of Liouville Theorem 3.4 for $n \ge 3$. For exponents $\beta \in \left(\frac{-2}{p-1}, 0\right)$, the comparison function $\theta \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, defined by $\theta(x) := |x|^\beta$, is discrete subharmonic in $\{x \in \mathbb{R}_h \colon x \ge R\}$ if R is sufficiently large. Further, the crucial constraint $\beta(p-1)+2 > 0$ is valid. Thus, the *comparison argument* from the proof of Theorem 3.4 for $n \ge 3$ is applicable and the rest of the proof of Theorem 3.4 can be adapted. □

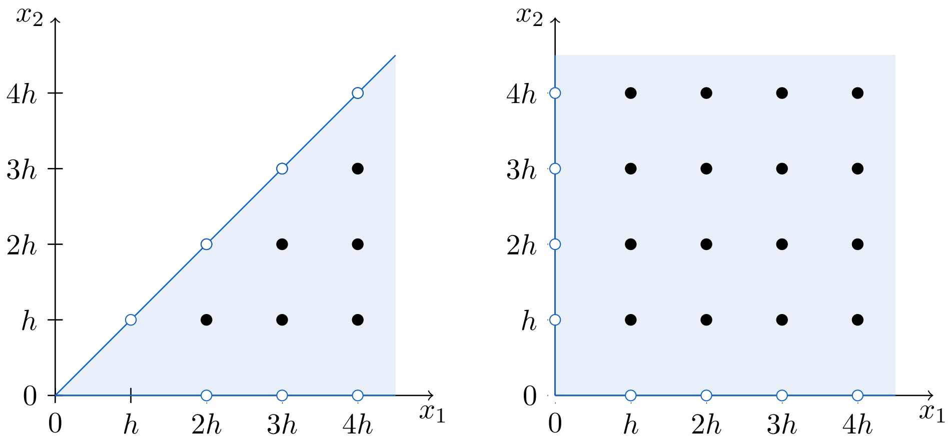
3.3. Two-dimensional case

3.3.1. Infinite cones

This section is devoted to examine the case $n = 2$ separately. Here it is possible to use spherical coordinates to find an appropriate comparison function. For $m \in \{1, \dots, 8\}$ we consider the *infinite cones*

$$\Omega^m := \left\{ (x_1, x_2)^T = (r\cos\varphi, r\sin\varphi)^T \in \mathbb{R}^2 \colon r > 0,\ \varphi \in \left(0, \frac{\pi}{4}m\right) \right\}. \tag{3.32}$$

The corresponding discrete boundary $\partial_h \Omega_h^m$, defined by Definition 2.1, satisfies by construction $\partial_h \Omega_h^m \subset \partial \Omega^m$.



(a) Cuspid corner ($m = 1$)

(b) Orthant ($m = 2$)

Figure 3.4: Discretization of two infinite cones.

Employing the spherical coordinates

$$(x_1, x_2) = (r \cos \varphi, r \sin \varphi)$$

we introduce the comparison function $v \colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ by

$$v(x) := r^{\beta + \frac{4}{m}} \sin\left(\frac{4}{m}\varphi\right) \tag{3.33}$$

with some $\beta < 0$. Note that $v(x) = |x|^\beta r^{\frac{4}{m}} \sin\left(\frac{4}{m}\varphi\right)$ and moreover, $x \mapsto r^{\frac{4}{m}} \sin\left(\frac{4}{m}\varphi\right)$ is a positive, harmonic function in Ω^m which vanishes on the boundary $\partial\Omega^m$. Thus, the ansatz for the comparison function is similar to the case of a generalized orthant.

The aim of this paragraph is the subsequent Liouville theorem which is proven afterwards with the help of some auxiliary lemmas.

Theorem 3.14 (Two-dimensional discrete Liouville theorem for cones)
Let $h > 0$ and $n = 2$. For $m \in \{1, \dots, 8\}$ let Ω^m be defined by (3.32) and $1 < p < p_\star := \frac{m+2}{2}$. Then, the only non-negative solution $u \colon \overline{\Omega_h^m} \to [0, \infty)$ of

$$-\Delta_h u \geq u^p \quad \textit{in } \Omega_h^m \tag{3.34}$$

is $u \equiv 0$.

Lemma 3.15 (Discrete subharmonic comparison function II)
Let $h \in (0, 1]$. For every exponent $\beta < -\frac{8}{m}$, the function v, defined by (3.33), is subharmonic in Ω^m and there exists a radius $R_\beta > 0$ such that

$$-\Delta_h v(x) \leq 0 \quad \textit{for all } x \in \Omega_h^m \textit{ with } |x| > R_\beta.$$

Further, $v = 0$ on $\partial_h \Omega_h^m \setminus \{0\}$.

Proof. Using the representation of the Laplacian in spherical coordinates reveals

$$\begin{aligned}\Delta v(x) &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right) v(r,\varphi)\\ &= r^{\beta+\frac{4}{m}-2}\sin\left(\frac{4}{m}\varphi\right)\left[\left(\beta+\frac{4}{m}\right)\left(\beta+\frac{4}{m}-1\right)+\beta+\frac{4}{m}-\left(\frac{4}{m}\right)^2\right]\\ &= r^{\beta+\frac{4}{m}-2}\sin\left(\frac{4}{m}\varphi\right)\beta\left(\beta+\frac{8}{m}\right) > 0\end{aligned}$$

for all $x = (r\cos(\varphi), r\sin(\varphi))^T \in \Omega^m$ as $\beta < -\frac{8}{m}$, i.e., v is subharmonic in Ω^m.

In the following we proceed as in the proof of Lemma 3.7 and show that v is discrete subharmonic for large $|x|$. Let $x \in \Omega_h^m$ with $|x| \geq 2h$. By Taylor's theorem there exist $\xi^{(i)} \in \{x + \tau h e_i : \tau \in (0,1)\}$ and $\eta^{(i)} \in \{x - \tau h e_i : \tau \in (0,1)\}$, $i \in \{1,2\}$, with

$$\begin{aligned}-\Delta_h v(x) &= -\Delta v(x) - \frac{h^2}{24}\sum_{i=1}^{2}\left[\frac{\partial^4}{\partial x_i^4}v\left(\xi^{(i)}\right) + \frac{\partial^4}{\partial x_i^4}v\left(\eta^{(i)}\right)\right]\\ &= -r^{\beta+\frac{4}{m}-2}\sin\left(\frac{4}{m}\varphi\right)\beta\left(\beta+\frac{8}{m}\right) - \frac{h^2}{24}\sum_{i=1}^{2}\left[\frac{\partial^4}{\partial x_i^4}v\left(\xi^{(i)}\right) + \frac{\partial^4}{\partial x_i^4}v\left(\eta^{(i)}\right)\right].\end{aligned} \tag{3.35}$$

Recall that $v(x) = r^{\beta+\frac{4}{m}}\sin\left(\frac{4}{m}\varphi\right)$ with $\beta + \frac{4}{m} < 0$. Thus, $v(x)$ is of order $\beta + \frac{4}{m}$ with respect to r. Due to

$$\begin{aligned}\frac{\partial}{\partial x_1} &= \cos(\varphi)\frac{\partial}{\partial r} - \frac{1}{r}\sin(\varphi)\frac{\partial}{\partial \varphi},\\ \frac{\partial}{\partial x_2} &= \sin(\varphi)\frac{\partial}{\partial r} + \frac{1}{r}\cos(\varphi)\frac{\partial}{\partial \varphi}\end{aligned}$$

every derivative in direction x_i decreases the order by 1 and therefore $\frac{\partial^4}{\partial x_i^4}v(x)$ is of order $\beta + \frac{4}{m} - 4$ for $i \in \{1,2\}$. In view of the neighbour estimates from Lemma 3.6 we infer that $\frac{\partial^4}{\partial x_i^4}v\left(\xi^{(i)}\right)$ and $\frac{\partial^4}{\partial x_i^4}v\left(\eta^{(i)}\right)$ are of order $\beta + \frac{4}{m} - 4$ for $i \in \{1,2\}$. Therefore, on the right hand side of (3.35) the first summand is negative and of order $\beta + \frac{4}{m} - 2$ whereas the second summand is of order $\beta + \frac{4}{m} - 4$ with respect to r. Hence, there exists a radius $R > 0$ such that

$$-\Delta_h v(x) \leq 0, \quad |x| > R.$$

Finally, by definition, we have $v = 0$ on $\partial\Omega^m \setminus \{0\}$ and hence on $\partial_h\Omega_h^m \setminus \{0\}$. □

The next ingredient needed is the reverse Hardy inequality. If $m \in \{2, \ldots, 8\}$, the proof of Lemma 3.8 can be adopted literally. If $m = 1$, then the supports of the functions u_l

from Lemma 3.8 are not contained in Ω^m. In this case it is sufficient to use translated versions of the functions u_l. For the sake of completeness, we prove a version of the reverse Hardy inequality which is applicable for all $m \in \{1, \dots, 8\}$.

Lemma 3.16 (Reverse Hardy inequality II)
Let $\varphi_1, \varphi_2 \in \left(0, \frac{\pi}{2}\right)$ with

$$\tan(\varphi_1) = \frac{1}{6} \quad \textit{and} \quad \tan(\varphi_2) = \frac{1}{2}. \tag{3.36}$$

Then, there exists a sequence $(u_l)_{l\in\mathbb{N}} \subset \mathcal{C}(\mathbb{R}^2_h)$, $u_l \not\equiv 0$ with

$$\mathrm{cl}_h(\mathrm{supp}(u_l)) \subset \left\{x \in \mathbb{R}^2_h \colon |x|_\infty \geq lh, \quad \varphi \in [\varphi_1, \varphi_2]\right\},$$

$l \in \mathbb{N}$, such that

$$\sum_{x\in\mathbb{R}^2_h} \sum_{i=1}^{2} |D_i^+ u_l(x)|^2 h^2 \leq C \sum_{x\in\mathbb{R}^2_h\setminus\{0\}} \frac{|u_l(x)|^2}{|x|^2} h^2 \tag{3.37}$$

with a constant $C > 0$ independent of h.

Proof. As in the proof of Lemma 3.8, for $l \in \mathbb{N}$ we define the functions $u_l \colon \mathbb{R}^2_h \to \mathbb{R}$ by

$$u_l \coloneqq \sum_{m=0}^{l} \frac{l-m}{l} \chi_{\{x\in\mathbb{R}^2_h \colon |x-2lhe|_1 = mh\}},$$

where $e \coloneqq (3,1)^T \in \mathbb{R}^2$ and $\chi_A \colon \mathbb{R}^n \to \mathbb{R}$ denotes the characteristic function corresponding to $A \subset \mathbb{R}^n$. Moreover, the discrete closure $\mathrm{cl}_h(B)$ of some set $B \subset \mathbb{R}^n_h$ is given by (2.1). Therefore, we obtain $u_l \in \mathcal{C}(\mathbb{R}^2_h)$ and

$$\begin{aligned} \mathrm{cl}_h(\mathrm{supp}(u_l)) &= \{x \in \mathbb{R}^2_h \colon |x - 2lhe|_1 \leq lh\} \\ &\subset \left\{x \in \mathbb{R}^2_h \colon |x|_\infty \geq lh, \, \varphi \in [\varphi_1, \varphi_2]\right\}. \end{aligned}$$

To see the last inclusion, we use the following: $|x - 2lhe|_1 \leq lh$ implies $|x_1 - 6lh| \leq lh$ as well as $|x_2 - 2lh| \leq lh$ and hence, $|x|_\infty \geq lh$. Further, for $x = (r\cos\varphi, r\sin\varphi)^T \in \mathbb{R}^2$ with $|x - 2lhe|_1 \leq lh$ the maximal angle φ_2 is achieved by $x = 2lhe + lhe_2 = lh(6,3)^T$, whereas the minimal angle φ_1 is attained for $x = 2lhe - lhe_2 = lh(6,1)^T$. Figure 3.5 illustrates the supports of the grid functions u_l.

Similar to the proof of Lemma 3.8, we can show that the functions u_l satisfy the reverse Hardy inequality (3.37) with a constant $C > 0$, which is independent of $l \in \mathbb{N}$. □

Proof of Theorem 3.14. The argumentation is analogous to the proof of Theorem 3.4 for $n \geq 3$. We suppose for contradiction that there exists a solution $u \colon \overline{\Omega^m_h} \to [0,\infty), u \not\equiv 0$ of (3.34). Due to the scale invariance of problem (3.34) (cf. Remark 3.5), we may assume $h \in (0, 1]$.

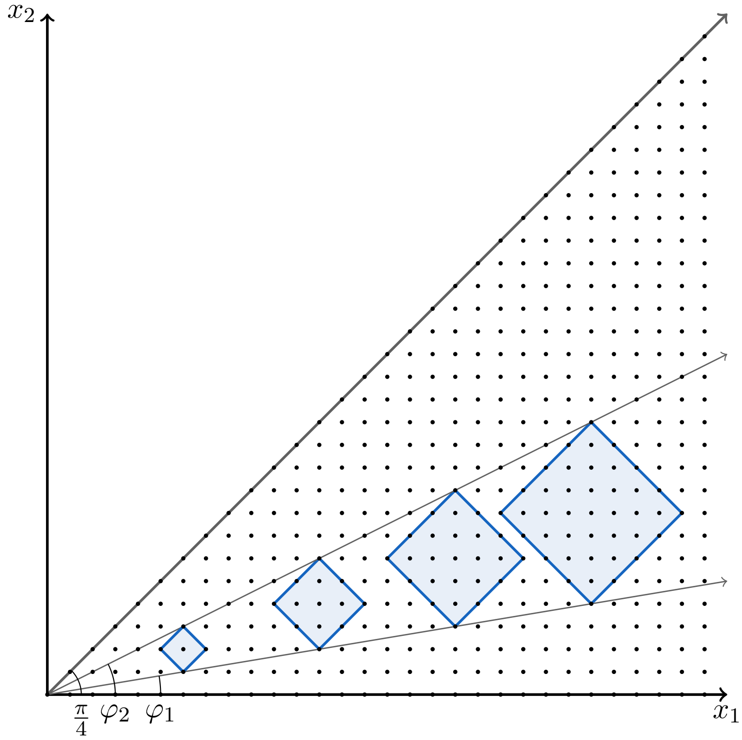


Figure 3.5: Support of u_l for $i = 1, \dots, 4$.

⟨1⟩ *Positivity*
Assume $u(x_0) = \min_{x \in \Omega_h^m} u(x) = 0$ for some $x_0 \in \Omega_h^m$. Then, the discrete maximum principle (Lemma 2.12) directly yields $u = 0$ in Ω_h^m, a contradiction. Thus, $u > 0$ in Ω_h^m.

⟨2⟩ *Comparison argument*
Since $1 < p < \frac{m+2}{2}$ we have $\delta := 2 - (p-1)\frac{4}{m} > 0$. Setting $\varepsilon := \frac{\delta}{2(p-1)}$ and $\beta := -\frac{8}{m} - \varepsilon$ implies

$$\left(\beta + \frac{4}{m}\right)(p-1) + 2 > 0. \tag{3.38}$$

Via spherical coordinates, we define the comparison function $v\colon \overline{\Omega^m} \setminus \{0\} \to \mathbb{R}$ by

$$v(x) := v(r, \varphi) := r^{\beta + \frac{4}{m}} \sin\left(\frac{4}{m}\varphi\right).$$

According to Lemma 3.15 there exists a radius $R_0 \in h\mathbb{N}$ such that

$$-\Delta_h v(x) \leq 0 \quad \text{for all } x \in \Omega_h^m \text{ with } |x|_\infty > R_0$$

and moreover $v = 0$ on $\partial_h\Omega_h^m \setminus \{0\}$. Since the set $\{x \in \Omega_h^m \colon |x|_\infty = R_0\}$ is finite, we can choose a constant $C > 0$ with $Cv(x) \leq u(x)$ for all $x \in \Omega_h^m$ with $|x|_\infty = R_0$. Applying the discrete maximum principle (Lemma 2.12), as in the proof of Theorem 3.4 for $n \geq 3$, yields the comparison estimate

$$Cv(x) \leq u(x) \tag{3.39}$$

for all $x \in \Omega_h^m$ with $|x|_\infty > R_0$.

⟨3⟩ *Hardy-like inequality*
Recall that $\varphi_1, \varphi_2 \in (0, \frac{\pi}{2})$ are given by (3.16). For all $x = (r\cos\varphi, r\sin\varphi) \in \Omega_h^m$ with $|x|_\infty > R_0$ and $\varphi \in [\varphi_1, \varphi_2]$ we deduce from (3.39)

$$\begin{aligned}
-\Delta_h u(x) &\geq u^{p-1}(x)u(x) \\
&\geq C^{p-1}v^{p-1}(x)u(x) \\
&= C^{p-1}|x|^{\left(\beta+\frac{4}{m}\right)(p-1)+2}\sin^{p-1}\left(\frac{4}{m}\varphi\right)|x|^{-2}u(x) \\
&\geq C^{p-1}\sin^{p-1}\left(\frac{4}{m}\varphi_1\right)|x|^{\left(\beta+\frac{4}{m}\right)(p-1)+2}|x|^{-2}u(x).
\end{aligned}$$

Since the exponent $\left(\beta + \frac{4}{m}\right)(p-1)+2$ is strictly positive by (3.38), for every $K > 0$ there exists a radius $R_K > R_0$ such that

$$-\Delta_h u(x) \geq \frac{K}{|x|^2}u(x) \tag{3.40}$$

for all $x \in N_K \coloneqq \{x \in \Omega_h^m \colon |x|_\infty \geq R_K,\ \varphi \in [\varphi_1, \varphi_2]\}$.

⟨4⟩ *Agmon principle*
Employing the discrete version of the Agmon principle (cf. proof of Theorem 3.4 for $n \geq 3$) we infer that for all test functions $\psi \colon \mathbb{R}_h^n \to \mathbb{R}$ with $\mathrm{cl}_h(\mathrm{supp}(\psi)) \subset N_K$:

$$\sum_{x\in N_K}\left[\sum_{i=1}^{2}(D_i^+\psi(x))^2 - \frac{K}{|x|^2}\psi^2(x)\right] \geq 0. \tag{3.41}$$

⟨5⟩ *Contradiction to the reverse Hardy inequality*
Choosing K bigger than the constant C from Lemma 3.16 and $l \in \mathbb{N}$ so large that

$$\mathrm{cl}_h(\mathrm{supp}(u_l)) \subset N_K,$$

the estimates (3.41) and (3.37) yield

$$K\sum_{x\in N_K}\frac{u_l^2(x)}{|x|^2} \leq \sum_{x\in N_K}\sum_{i=1}^{2}|D_i^+u_l(x)|^2 \leq C\sum_{x\in N_K}\frac{u_l^2(x)}{|x|^2}.$$

This is a contradiction since $K > C$. ∎

Remark 3.17 (Critical exponents for Liouville theorems and a priori bounds)
Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain with $\partial\Omega \setminus \{0\} \in C^\infty$ and $B_r(0) \cap \Omega = B_r(0) \cap \Omega^m$ for a number $m \in \{1, \dots, 8\}$ and a radius $r > 1$, i.e., Ω is a domain with one conical corner. In [22] McKenna and Reichel showed that every positive very weak solution u of

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.42}$$

is a priori bounded in $L^\infty(\Omega)$ provided that $1 < p < p^*$. In general p^* is given by

$$p^* := \frac{n + \gamma^*}{n + \gamma^* - 2} \quad \text{with} \quad \gamma^* := \frac{2-n}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + \tilde{\lambda}_1},$$

where $n \geq 2$ is the dimension and $\tilde{\lambda}_1$ is the first Dirichlet eigenvalue of the Laplace-Beltrami operator $-\Delta_{\mathbb{S}^{n-1}}$ on $\omega := \Omega \cap \mathbb{S}^{n-1}$. That is to say, the critical exponent p^* is determined by the so-called opening angle ω. Here $n = 2$ and $\tilde{\lambda}_1 = \left(\frac{4}{m}\right)^2$ and therefore it follows that $p^* = \frac{2+\sqrt{\tilde{\lambda}_1}}{\sqrt{\tilde{\lambda}_1}} = \frac{m+2}{2}$. So, in this case the critical exponent p^* for a priori bounds of very weak solutions from [22] and the Liouville exponent $p_\star$ for finite difference solutions from Theorem 3.14 coincide.

Regarding cones Ω^m with $m \in \{1, \dots, 7\}$, the exponent $p_\star = \frac{m+2}{2}$ is critical for positive solutions of

$$-\Delta_h u \geq u^p \quad \text{in } \Omega_h^m.$$

If $1 < p < p_\star$, there is no positive solution in view of Theorem 3.14 and the following theorem ensures the existence of a positive solution for all $p > p_\star$.

Theorem 3.18 (Existence of solutions for cones)
Let $h > 0$ and $n = 2$. For $m \in \{1, \dots, 7\}$ let Ω^m be defined by (3.32) and $p > p_\star = \frac{m+2}{2}$. Then, there exists a positive solution $u \colon \overline{\Omega_h^m} \to (0, \infty)$ of

$$-\Delta_h u \geq u^p \quad \text{in } \Omega_h^m. \tag{3.43}$$

Proof. The argumentation is analogous to the proof of Theorem 3.10. Due to the scale invariance of the problem we may assume $h \in (0, 1]$. Further, we consider the comparison function $v \colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ from Lemma 3.15, given by

$$v(x) := r^{\beta + \frac{4}{m}} \sin\left(\frac{4}{m}\varphi\right),$$

with $\beta := \frac{-2}{p-1} - \frac{4}{m} < 0$.

Hence, we have

$$(p-1)\left(\beta + \frac{4}{m}\right) + 2 = 0 \tag{3.44}$$

and the assumption $p > \frac{m+2}{2}$ yields

$$\beta > -\frac{8}{m}. \tag{3.45}$$

Next, let $x \in \Omega_h^m$ with $|x| \geq 2h$. Due to (3.35), there exist $\xi^{(i)} \in \{x + \tau h e_i \colon \tau \in (0,1)\}$ and $\eta^{(i)} \in \{x - \tau h e_i \colon \tau \in (0,1)\}$, $i \in \{1,2\}$, with

$$-\Delta_h v(x) = -r^{\beta+\frac{4}{m}-2} \sin\left(\frac{4}{m}\varphi\right) \beta \left(\beta + \frac{8}{m}\right) - \frac{h^2}{24} \sum_{i=1}^{2} \left[\frac{\partial^4}{\partial x_i^4} v\left(\xi^{(i)}\right) + \frac{\partial^4}{\partial x_i^4} v\left(\eta^{(i)}\right)\right].$$

Since $\beta > -\frac{8}{m}$ by (3.45), we can choose some sufficiently large radius $R_0 > 0$ such that

$$-\Delta_h v(x) \geq -\frac{1}{2} r^{\beta+\frac{4}{m}-2} \sin\left(\frac{4}{m}\varphi\right) \beta \left(\beta + \frac{8}{m}\right) \tag{3.46}$$

for all $x \in \Omega_h^m$ with $|x|_\infty \geq R_0$ (cf. proof of Thm. 3.10). Additionally, let $\tau > 0$ such that $\tau^{p-1} \leq c_\beta := -\frac{1}{2}\beta\left(\beta + \frac{8}{m}\right)$. Thus, with (3.46) and (3.44) we infer

$$-\Delta_h\left(\tau v(x)\right) \geq c_\beta\, \tau\, r^{\beta+\frac{4}{m}-2} \sin\left(\frac{4}{m}\varphi\right) \geq \tau^p\, r^{p\left(\beta+\frac{4}{m}\right)} \sin^p\left(\frac{4}{m}\varphi\right) = \left(\tau v(x)\right)^p$$

for all $x \in \Omega_h^m$ with $|x|_\infty \geq R_0$.

Finally, let $e \in \mathbb{R}^2$ be defined as follows:

$$e := \begin{cases} (2,1)^T, & \text{if } m \in \{1,2\}, \\ (0,1)^T, & \text{if } m \in \{3,4\}, \\ (-3,1)^T, & \text{if } m \in \{5,6,7\}. \end{cases}$$

Then, the function $u \colon \overline{\Omega_h^m} \to \mathbb{R}$, given by $u(x) := \tau v(x + R_0 e)$, solves (3.43) in Ω_h^m. □

3.3.2. Entire space problem in dimension two

In the previous section the case $\Omega = \mathbb{R}^2$ was not included and this gap will be closed below. The subsequent two results for the entire space go back to Hans Heilbronn ([15]) and an unpublished manuscript of Michael Plum ([27]), respectively: We prove that all discrete harmonic, bounded functions are constant in all dimensions $n \in \mathbb{N}$. Thereby, it can be verified that all discrete superharmonic functions, which are bounded from below, are constant in the two-dimensional case.

Theorem 3.19 (Liouville theorem for discrete harmonic functions)
Let $h > 0$ and $n \in \mathbb{N}$ be arbitrary. Then, every bounded, discrete harmonic function on $\mathbb{R}^n_h$ is constant.

This result was first published by Hans Heilbronn ([15, Thm. 5]). For the reader's convenience we give an alternative proof which is based on the explanations of Michael Plum ([27]).

Proof. In view of the scale invariance (Remark 3.5) we may assume $h = 1$ and use the notation $\Delta_{\mathbb{Z}^n}$ instead of Δ_h. For $n = 1$ the result is a direct consequence of Proposition 3.12. So, let $n \geq 2$ and $u\colon \mathbb{Z}^n \to \mathbb{R}$ be bounded and discrete harmonic.

⟨1⟩ *Reduction*
First, for all $z \in \mathbb{Z}$ we show that

$$m(z) \coloneqq \inf\{u(z,y)\colon y \in \mathbb{Z}^{n-1}\} = \inf\{u(x)\colon x \in \mathbb{Z}^n\} \eqqcolon M. \tag{3.47}$$

Let $z \in \mathbb{Z}$ and $y \in \mathbb{Z}^{n-1}$. Then,

$$\begin{aligned} 0 = \Delta_{\mathbb{Z}^n} u(z,y) &= \sum_{i=1}^{n} [u((z,y)+e_i) - 2u(z,y) + u((z,y)-e_i)] \\ &= u(z+1,y) + u(z-1,y) - 2nu(z,y) + \sum_{i=2}^{n} [u((z,y)+e_i) + u((z,y)-e_i)] \\ &\geq m(z+1) + m(z-1) - 2nu(z,y) + 2(n-1)m(z), \end{aligned}$$

which leads to

$$2nu(z,y) \geq m(z+1) + m(z-1) + 2(n-1)m(z).$$

Taking the infimum over all $y \in \mathbb{Z}^{n-1}$ yields

$$-\Delta_{\mathbb{Z}} m(z) = -m(z+1) + 2m(z) - m(z-1) \geq 0$$

for all $z \in \mathbb{Z}$. Thus, $m\colon \mathbb{Z} \to \mathbb{R}$ is bounded and discrete superharmonic. Applying Proposition 3.12 ensures that m is constant. Since the discrete hyperplanes $\{(z,y)\colon y \in \mathbb{Z}^{n-1}\}$ ($z \in \mathbb{Z}$) cover $\mathbb{Z}^n$, the constant has to be M and (3.47) is proven. Analogously, this result can be shown for hyperplanes with fixed jth component ($j = 2, \ldots, n$), i.e.,

$$\inf\{u(\xi,z,y)\colon \xi \in \mathbb{Z}^{j-1}, y \in \mathbb{Z}^{n-j}\} = \inf\{u(x)\colon x \in \mathbb{Z}^n\}$$

for all $z \in \mathbb{Z}$.

⟨2⟩ *Symmetry*
The next auxiliary statement is that u is symmetric with respect to all hyperplanes

$\{a\} \times \mathbb{Z}^{n-1}$, $a \in \mathbb{Z}$. To check this, let $a \in \mathbb{Z}$ and $v\colon \mathbb{Z}^n \to \mathbb{R}$ be given by

$$v(z, y) \coloneqq u(a + z, y) - u(a - z, y) \quad (z \in \mathbb{Z},\, y \in \mathbb{Z}^{n-1}).$$

With u also $\pm v$ is discrete harmonic and bounded. Applying (3.47) to $\pm v$ entails

$$\inf\{\pm v(0, y) : y \in \mathbb{Z}^{n-1}\} = \inf\{\pm v(x)\colon x \in \mathbb{Z}^n\}.$$

As $v(0, y) = 0$ for all $y \in \mathbb{Z}^{n-1}$, it follows $\inf \pm v = 0$ and hence, $v \equiv 0$. This is just the desired symmetry property for u. The symmetry of u with respect to the hyperplanes $\mathbb{Z}^{j-1} \times \{a\} \times \mathbb{Z}^{n-j}$ with $j = 2, \ldots, n$ can be obtained mutatis mutandis.

⟨3⟩ *Conclusion*
In view of the symmetry with respect to the hyperplanes $\mathbb{Z}^{i-1} \times \{x_i + 1\} \times \mathbb{Z}^{n-i}$ for all $x = (x_1, \ldots, x_n)^T$ and $i \in \{1, \ldots, n\}$, the function u is 2 periodic in all coordinate directions e_i, i.e.,

$$u(x) = u(x + 2e_i)$$

for all $x \in \mathbb{Z}^n$ and $i \in \{1, \ldots, n\}$. Therefore, u attains only finitely many values and there exists some $\hat{x} \in \mathbb{Z}^n$ with $\hat{x}_i \in \{0, 1\}$ and

$$u(\hat{x}) = \min_{x \in \mathbb{Z}^n} u(x) = M.$$

From

$$0 = \Delta_{\mathbb{Z}^n} u(\hat{x}) = -2nM + \sum_{i=1}^{n} [u(\hat{x} + e_i) + u(\hat{x} - e_i)]$$

we infer $u(\hat{x} \pm e_i) = M$ for all $i \in \{1, \ldots, n\}$. Inductively, this implies $u \equiv u(\hat{x})$ on $\mathbb{Z}^n$. □

With the Liouville theorem in the discrete harmonic setting, we are now in the position to prove it in the discrete superharmonic, two-dimensional case.

Theorem 3.20 (Liouville theorem for superharmonic two-dimensional case)
Let $h > 0$ be fixed. Then, every discrete superharmonic function on $\mathbb{R}^2_h$, which is bounded from below, is constant.

Proof. This proof is based on ideas of Michael Plum ([27]). Due to the scale invariance (Remark 3.5) of the problem we assume $h = 1$.

⟨1⟩ First, we show the result under the additional *assumption* boundedness from above. So, let $u\colon \mathbb{Z}^2 \to \mathbb{R}$ be bounded and discrete superharmonic. The idea is to verify that u is in fact discrete harmonic and therefore, by Theorem 3.19, constant. We define the operator Θ by

$$\Theta u(x, y) = u(x + 1, y) + u(x - 1, y) + u(x, y + 1) + u(x, y - 1) \quad \text{for } x, y \in \mathbb{Z}.$$

Then, the superharmonicity just reads $4u \geq \Theta u$. Moreover, we introduce the sequences of grid functions (u_k), (d_k) and (w_k) by

$$\begin{aligned} u_0 &:= u, \qquad u_{k+1} := \frac{1}{4}\Theta u_k, \\ d_k &:= -\Delta_{\mathbb{Z}^2} u_k, \\ w_k(x,y) &:= \begin{cases} \dfrac{1}{4^k}\dbinom{k}{\frac{1}{2}(k+x+y)}\dbinom{k}{\frac{1}{2}(k+x-y)}, & \text{if } |x|+|y| \leq k,\ k+x+y \text{ even}, \\ 0, & \text{else}, \end{cases} \end{aligned}$$

for all $k \in \mathbb{N}_0$ and $x, y \in \mathbb{Z}$, where the binomial coefficient is given by

$$\binom{k}{x} = \begin{cases} \dfrac{k!}{(k-x)!\,x!}, & \text{if } k \geq x \geq 0, \\ 0, & \text{else.} \end{cases}$$

With these notations the following relations which are established in the appendix (Lemma A.2) hold:

(a) $4(u_k - u_{k+1}) = d_k$,

(b) $d_{k+1} = \frac{1}{4}\Theta d_k$,

(c) $w_{k+1} = \frac{1}{4}\Theta w_k$,

(d) $d_k(x,y) = \sum\limits_{\mu,\nu\in\mathbb{Z}} w_k(\mu,\nu)[-\Delta_{\mathbb{Z}^2} u](x+\mu, y+\nu)$,

(e) $\sum\limits_{k=0}^{\infty} w_k(0,0) = +\infty$.

From the definition of u_k we inductively deduce $u_k \geq \inf u$ for all $k \in \mathbb{N}$. With (a), (d) and the non-positivity of $-\Delta_{\mathbb{Z}^2} u$, w_k we obtain

$$\begin{aligned} 4(u(x,y) - \inf u) \geq 4(u(x,y) - u_{j+1}(x,y)) &= \sum_{k=0}^{j} 4(u_k(x,y) - u_{k+1}(x,y)) \\ &= \sum_{k=0}^{j} d_k(x,y) = \sum_{k=0}^{j}\left[\sum_{\mu,\nu\in\mathbb{Z}} w_k(\mu,\nu)[-\Delta_{\mathbb{Z}^2} u](x+\mu,y+\nu)\right] \\ &\geq \sum_{k=0}^{j} w_k(0,0)[-\Delta_{\mathbb{Z}^2} u](x,y) \end{aligned}$$

for every $(x,y) \in \mathbb{Z}^2$. Due to (e), this gives $-\Delta_{\mathbb{Z}^2} u \leq 0$. Thus, u is indeed discrete harmonic on $\mathbb{Z}^2$ and hence constant by Theorem 3.19.

⟨2⟩ Next, let $u\colon \mathbb{Z}^2 \to \mathbb{R}$ be discrete superharmonic and only bounded from below.

Defining $a := u(0,0) + 1$, we introduce $v\colon \mathbb{Z}^2 \to \mathbb{R}$ by

$$v(x,y) := \min\{u(x,y), a\} \quad \text{for all } x, y \in \mathbb{Z}.$$

The superharmonicity of u ensures for all $x, y \in \mathbb{Z}$

$$4u(x,y) \geq \Theta u(x,y) \geq \Theta v(x,y).$$

By definition, we have

$$4a \geq \Theta v(x,y)$$

and this yields immediately

$$4v(x,y) \geq \Theta v(x,y).$$

So, v is discrete superharmonic. Furthermore, v is bounded from above and from below. According to the first part v is constant. Hence,

$$v(x,y) = \min\{u(x,y), a\} = \min\{u(0,0), a\} = \min\{a-1, a\} = a - 1$$

which leads to $u(x,y) = a - 1 = u(0,0)$ for all $x, y \in \mathbb{Z}$, i.e., u is constant. □

After this excursus about discrete superharmonic functions, we return to the investigation of the discrete Emden equation:

Theorem 3.21 (Discrete Liouville theorem)
Let $h > 0$ be the grid size, $n = 2$ the dimension and $1 < p < \infty$. Then, the only non-negative solution $u\colon \mathbb{R}^2_h \to [0,\infty)$ of

$$-\Delta_h u \geq u^p \quad \text{in } \mathbb{R}^2_h \tag{3.48}$$

is $u \equiv 0$.

Proof. The result follows directly from Theorem 3.20. □

Remark 3.22 (Alternative proof of Theorem 3.21)
We can also prove Theorem 3.21 by applying the comparison argument *used in the proof of Theorem 3.4 for $n \geq 3$. For some fixed $\beta \in \left(\frac{-2}{p-1}, 0\right)$, the comparison function*

$$\theta\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R},\ \theta(x) := |x|^\beta$$

is discrete subharmonic in $\{x \in \mathbb{R}^2_h\colon |x|_\infty \geq R\}$ if R is sufficiently large (cf. Lemma 3.7). Moreover, β satisfies the crucial constraint $\beta(p-1) + 2 > 0$. The rest of the proof of Theorem 3.4 for $n \geq 3$ can be transferred mutatis mutandis.

Remark 3.23 (Applicability of the comparison approach)
We investigate for which $p \in (1,\infty)$ our comparison approach is applicable on $\mathbb{R}^n_h$, depending on the dimension $n \in \mathbb{N}$. Thereto, we need the comparison function $\theta\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$,

$\theta(x) := |x|^\beta$ to be discrete subharmonic and therefore the exponent β has to satisfy

$$\beta < 0 \quad \text{and} \quad \beta(p-1)+2 > 0. \tag{3.49}$$

Case $n = 1, 2$: For every $p \in (1, \infty)$ we can fix any $\beta \in \left(\frac{-2}{p-1}, 0\right)$. Then, (3.49) holds true and the comparison argument can be applied.

Case $n \geq 3$: Due to Lemma 3.7 we have to ensure additionally that

$$\beta < 2 - n. \tag{3.50}$$

For $p \geq \frac{n}{n-2}$ the conditions (3.49) and (3.50) contradict each other. If $p < \frac{n}{n-2}$, we can find some β (see proof of Theorem 3.4) such that (3.49) and (3.50) are satisfied. Hence, for $n \geq 3$, the comparison approach is applicable if and only if $p < \frac{n}{n-2}$.

3.4. More complex geometries

With our comparison argument it is possible to obtain finite difference Liouville theorems for unbounded domains with more complex structures. Exemplarily, we consider for dimensions $n \geq 3$, $k \in \{1, \ldots, n-2\}$ and $m \in \{1, \ldots, 8\}$ the unbounded domains

$$\begin{aligned} A^{k,m} :=& \mathbb{R}_+^k \times \mathbb{R}^{n-2-k} \times \Omega^m \\ =& \left\{ (x_1, \ldots, x_n)^T \in \mathbb{R}^n \colon x_1, \ldots, x_k > 0,\ (x_{n-1}, x_n) = (\rho\cos\varphi, \rho\sin\varphi) \text{ with} \right. \\ & \left. \rho > 0,\ \varphi \in \left(0, \frac{\pi}{4} m\right) \right\}. \end{aligned}$$

By construction, we have $\partial_h A_h^{k,m} \subset \partial A^{k,m}$. Using spherical coordinates

$$(x_{n-1}, x_n) = (\rho\cos\varphi, \rho\sin\varphi)$$

we define the comparison function $\theta \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by

$$\theta(x) := |x|^\beta \left(\prod_{j=1}^k x_j \right) \rho^{\frac{4}{m}} \sin\left(\frac{4}{m}\varphi\right). \tag{3.51}$$

Note that $x \mapsto \left(\prod_{j=1}^k x_j\right) \rho^{\frac{4}{m}} \sin\left(\frac{4}{m}\varphi\right)$ is a positive, harmonic function on $A^{k,m}$ which is zero on the boundary $\partial A^{k,m}$.

The further proceeding is analogous to Section 3.1: First, the Liouville theorem is stated. Afterwards, we show that the chosen comparison function is discrete subharmonic. Then, a suitable version of the reverse Hardy inequality is given. Finally, we prove the Liouville theorem by means of these auxiliary statements.

Theorem 3.24 (Discrete Liouville theorem for more complex geometries)
Let $h > 0$ and $n \geq 3$. For $k \in \{1, \dots, n-2\}$ and $m \in \{1, \dots, 8\}$ let

$$A^{k,m} := \mathbb{R}_+^k \times \mathbb{R}^{n-2-k} \times \Omega^m$$

and $1 < p < \frac{n+k+\frac{4}{m}}{n+k+\frac{4}{m}-2}$. Then, the only non-negative solution $u\colon \overline{A_h^{k,m}} \to [0,\infty)$ of

$$-\Delta_h u \geq u^p \quad \text{in } A_h^{k,m} \tag{3.52}$$

is $u \equiv 0$.

Lemma 3.25 (Discrete subharmonic comparison function III)
Let $h \in (0,1]$. For every exponent $\beta < 2 - n - 2k - \frac{8}{m}$, the function θ, defined by (3.51), is subharmonic in $A^{k,m}$ and there exists a radius $R = R_\beta > 0$ such that

$$-\Delta_h \theta(x) \leq 0$$

for all $x \in A_h^{k,m}$ with $|x|_\infty > R_\beta$. Moreover, we have $\theta = 0$ on $\partial_h A_h^{k,m} \setminus \{0\}$.

Proof. For all $x \in \mathbb{R}^n \setminus \{0\}$, $i \in \{1, \dots, k\}$, we recall that

$$\frac{\partial^2}{\partial x_i^2}\left(|x|^\beta \prod_{j=1}^k x_j\right) = \left(\prod_{j=1}^k x_j\right)\beta\left[(\beta-2)|x|^{\beta-4}x_i^2 + 3|x|^{\beta-2}\right] \tag{3.53}$$

by (3.7). For $i \in \{k+1, \dots, n-2\}$ it follows similarly

$$\frac{\partial^2}{\partial x_i^2}|x|^\beta = \beta\left[(\beta-2)|x|^{\beta-4}x_i^2 + |x|^{\beta-2}\right]. \tag{3.54}$$

Employing the spherical coordinates $(x_{n-1}, x_n) = (\rho\cos\varphi, \rho\sin\varphi)$ we obtain

$$\sum_{i=n-1}^n \frac{\partial^2}{\partial x_i^2}\left(|x|^\beta \rho^{\frac{4}{m}} \sin\left(\frac{4}{m}\varphi\right)\right) = \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial}{\partial\varphi^2}\right)\left(|x|^\beta \rho^{\frac{4}{m}} \sin\left(\frac{4}{m}\varphi\right)\right).$$

Moreover, we have

$$\frac{\partial}{\partial\rho}\left(|x|^\beta \rho^{\frac{4}{m}}\right) = \beta|x|^{\beta-2}\rho^{\frac{4}{m}+1} + \frac{4}{m}|x|^\beta \rho^{\frac{4}{m}-1}$$

as well as

$$\frac{\partial^2}{\partial\rho^2}\left(|x|^\beta \rho^{\frac{4}{m}}\right) = \beta(\beta-2)|x|^{\beta-4}\rho^{\frac{4}{m}+2} + \beta\left(\frac{8}{m}+1\right)|x|^{\beta-2}\rho^{\frac{4}{m}} + \frac{4}{m}\left(\frac{4}{m}-1\right)|x|^\beta \rho^{\frac{4}{m}-2}.$$

Using the last two identities, we compute

$$
\begin{aligned}
&\sum_{i=n-1}^{n} \frac{\partial^2}{\partial x_i^2}\left(|x|^\beta \rho^{\frac{4}{m}} \sin\left(\frac{4}{m}\varphi\right)\right) \\
&= \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial}{\partial \varphi^2}\right)\left(|x|^\beta \rho^{\frac{4}{m}} \sin\left(\frac{4}{m}\varphi\right)\right) \\
&= \Bigg[\beta(\beta-2)|x|^{\beta-4}\rho^{\frac{4}{m}+2} + \beta\left(\frac{8}{m}+1\right)|x|^{\beta-2}\rho^{\frac{4}{m}} + \frac{4}{m}\left(\frac{4}{m}-1\right)|x|^\beta \rho^{\frac{4}{m}-2} \qquad (3.55) \\
&\quad + \beta|x|^{\beta-2}\rho^{\frac{4}{m}} + \frac{4}{m}|x|^\beta \rho^{\frac{4}{m}-2} - \left(\frac{4}{m}\right)^2 |x|^\beta \rho^{\frac{4}{m}-2}\Bigg] \sin\left(\frac{4}{m}\varphi\right) \\
&= \left[\beta\left(\frac{8}{m}+2\right)|x|^{\beta-2} + \beta(\beta-2)|x|^{\beta-4}\rho^2\right]\rho^{\frac{4}{m}} \sin\left(\frac{4}{m}\varphi\right).
\end{aligned}
$$

By means of (3.53), (3.54) and (3.55), we conclude

$$
\begin{aligned}
\Delta\theta(x) &= \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\theta(x) \\
&= \sum_{i=1}^{k} \frac{\partial^2}{\partial x_i^2}\left(|x|^\beta \prod_{j=1}^{k} x_j\right)\rho^{\frac{4}{m}} \sin\left(\frac{4}{m}\varphi\right) \\
&\quad + \sum_{i=k+1}^{n-2} \frac{\partial^2}{\partial x_i^2}\left(|x|^\beta\right)\left(\prod_{j=1}^{k} x_j\right)\rho^{\frac{4}{m}} \sin\left(\frac{4}{m}\varphi\right) \\
&\quad + \sum_{i=n-1}^{n} \frac{\partial^2}{\partial x_i^2}\left(|x|^\beta \rho^{\frac{4}{m}} \sin\left(\frac{4}{m}\varphi\right)\right)\left(\prod_{j=1}^{k} x_j\right) \\
&= \sum_{i=1}^{k}\left(\prod_{j=1}^{k} x_j\right)\left[3\beta|x|^{\beta-2} + \beta(\beta-2)|x|^{\beta-4}x_i^2\right]\rho^{\frac{4}{m}} \sin\left(\frac{4}{m}\varphi\right) \\
&\quad + \sum_{i=k+1}^{n-2}\left[\beta|x|^{\beta-2} + \beta(\beta-2)|x|^{\beta-4}x_i^2\right]\left(\prod_{j=1}^{k} x_j\right)\rho^{\frac{4}{m}} \sin\left(\frac{4}{m}\varphi\right) \\
&\quad + \left[\beta\left(\frac{8}{m}+2\right)|x|^{\beta-2} + \beta(\beta-2)|x|^{\beta-4}\rho^2\right]\rho^{\frac{4}{m}} \sin\left(\frac{4}{m}\varphi\right)\left(\prod_{j=1}^{k} x_j\right) \\
&= \left(\prod_{j=1}^{k} x_j\right)\rho^{\frac{4}{m}} \sin\left(\frac{4}{m}\varphi\right)\beta\left[\left(2k+n+\frac{8}{m}\right)|x|^{\beta-2} + (\beta-2)|x|^{\beta-4}\sum_{i=1}^{n} x_i^2\right] \\
&= \left(\prod_{j=1}^{k} x_j\right)\rho^{\frac{4}{m}} \sin\left(\frac{4}{m}\varphi\right)\beta\left[2k+n+\beta-2+\frac{8}{m}\right]|x|^{\beta-2} > 0
\end{aligned}
$$

for all $x \in A^{k,m}$ and $\beta < 2-n-2k-\frac{8}{m} < 0$, i.e., θ is subharmonic in $A^{k,m}$.

In the following we proceed as in the proofs of Lemma 3.7 and Lemma 3.15 and show that θ is discrete subharmonic if $|x|_\infty$ is large enough. Let $x \in A_h^{k,m}$ with $|x| \geq 2h$. By Taylor's theorem there exist $\xi^{(i)} \in \{x+\tau h e_i \colon \tau \in (0,1)\}$ and $\eta^{(i)} \in \{x-\tau h e_i \colon \tau \in (0,1)\}$, $i \in \{1,\dots,n\}$, with

$$\begin{aligned}
-\Delta_h\theta(x) =& -\Delta\theta(x) - \frac{h^2}{24}\sum_{i=1}^n \left[\frac{\partial^4}{\partial x_i^4}\theta\left(\xi^{(i)}\right) + \frac{\partial^4}{\partial x_i^4}\theta\left(\eta^{(i)}\right)\right] \\
=& -\left(\prod_{j=1}^k x_j\right)\rho^{\frac{4}{m}}\sin\left(\frac{4}{m}\varphi\right)\beta\left[2k+n+\beta-2+\frac{8}{m}\right]|x|^{\beta-2} \\
& - \frac{h^2}{24}\sum_{i=1}^n \left[\frac{\partial^4}{\partial x_i^4}\theta\left(\xi^{(i)}\right) + \frac{\partial^4}{\partial x_i^4}\theta\left(\eta^{(i)}\right)\right] \\
\leq& 0,
\end{aligned}$$

provided $|x| = r > R$ for a sufficiently large radius $R > 0$ (cf. proof of Lemma 3.15). Finally, by definition we get $\theta = 0$ on $\partial A^{k,m} \setminus \{0\}$ and hence on $\partial_h A_h^{k,m} \setminus \{0\}$. □

Lemma 3.26 (Reverse Hardy inequality III)
Let $\varphi_1, \varphi_2 \in \left(0, \frac{\pi}{2}\right)$ *with*

$$\tan(\varphi_1) = \frac{1}{6} \quad \textit{and} \quad \tan(\varphi_2) = \frac{1}{2}. \tag{3.56}$$

Then, there exists a sequence $(u_l)_{l\in\mathbb{N}} \subset \mathcal{C}(\mathbb{R}_h^n)$, $u_l \not\equiv 0$ *with*

$$\mathrm{cl}_h(\mathrm{supp}(u_l)) \subset \left\{x \in \mathbb{R}_h^{n,k} \colon |x|_\infty \geq lh,\, x_i \geq \frac{1}{7\sqrt{n}}|x| \ (i=1,\dots,n),\, \varphi \in [\varphi_1,\varphi_2]\right\},$$

$l \in \mathbb{N}$, *such that*

$$\sum_{x\in\mathbb{R}_h^n}\sum_{i=1}^n |D_i^+ u_l(x)|^2 h^n \leq C \sum_{x\in\mathbb{R}_h^n\setminus\{0\}} \frac{|u_l(x)|^2}{|x|^2} h^n \tag{3.57}$$

with a constant $C > 0$ *independent of* h *and* l.

Proof. The functions u_l are constructed similarly to the proof of Lemma 3.8: Setting $e \coloneqq (1,\dots,1,3,1)^T \in \mathbb{R}^n$, we define

$$u_l \coloneqq \sum_{m=0}^l \frac{l-m}{l}\chi_{\{x\in\mathbb{R}_h^n \colon |x-2lhe|_1 = mh\}} \quad \text{for all } l \in \mathbb{N},$$

where $\chi_{\mathcal{A}} \colon \mathbb{R}^n \to \mathbb{R}$ denotes the characteristic function corresponding to $\mathcal{A} \subset \mathbb{R}^n$. Recall that the discrete closure $\mathrm{cl}_h(B)$ of some discrete set $B \subset \mathbb{R}_h^n$ is given by (2.1). From the definition of u_l we obtain $u_l \in \mathcal{C}(\mathbb{R}^n_h)$ with

$$\begin{aligned}\mathrm{cl}_h(\mathrm{supp}(u_l)) &= \{x \in \mathbb{R}^n_h \colon |x - 2lhe|_1 \le lh\} \\ &\subset \left\{ x \in \mathbb{R}^n_h \colon |x|_\infty \ge lh,\, x_i \ge \frac{1}{7\sqrt{n}}|x| \,(i = 1, \dots, n),\, \varphi \in [\varphi_1, \varphi_2] \right\},\end{aligned}$$

where the last inclusion can be justified in the following way: $|x - 2lhe|_1 \le lh$ implies $|x_{n-1} - 6lh| \le lh$ as well as $|x_i - 2lh| \le lh$ for all $i \in \{1, \dots, n-2, n\}$. Therefore, $x_{n-1} \in [5lh, 7lh]$, $x_i \in [lh, 3lh]$ for all $i \in \{1, \dots, n-2, n\}$ and hence $|x|_\infty \ge lh$. This yields

$$x_i \ge lh = \frac{1}{7\sqrt{n}} 7lh\sqrt{n} = \frac{1}{7\sqrt{n}} \left(\sum_{i=1}^{n} (7lh)^2 \right)^{\frac{1}{2}} \ge \frac{1}{7\sqrt{n}} \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} = \frac{1}{7\sqrt{n}} |x|$$

for all $i \in \{1, \dots, n\}$. Furthermore, for $x = (x_1, \dots, x_{n-2}, \rho\cos\varphi, \rho\sin\varphi) \in \mathbb{R}^n$ with $|x - 2lhe|_1 \le lh$ the maximal angle φ_2 is achieved by $x = 2lhe + lhe_n = lh(2, \dots, 2, 6, 3)^T$, whereas the minimal angle φ_1 is attained for $x = 2lhe - lhe_n = lh(2, \dots, 2, 6, 1)^T$ (cf. two-dimensional case, especially Figure 3.5).

As in the proof the Lemma 3.8, we can show that the functions u_l satisfy (3.57) with a constant $C > 0$ independent of h and l. □

Proof of Theorem 3.24. Once more, we follow the lines of the proofs of Theorem 3.4 for $n \ge 3$ as well as Theorem 3.14 and suppose for contradiction that there exists a non-negative solution $u \colon \overline{A^{k,m}_h} \to [0, \infty), u \not\equiv 0$ of (3.52).

⟨1⟩ *Positivity*
By the same arguments as in the proof of Theorem 3.4 for $n \ge 3$, we may assume $h \in (0, 1]$ and $u > 0$ in $A^{k,m}_h$.

⟨2⟩ *Comparison argument*
Since $1 < p < \frac{n+k+\frac{4}{m}}{n+k+\frac{4}{m}-2}$ we have $\delta := 2 - (p-1)\left(n + k + \frac{4}{m} - 2\right) > 0$. Setting $\varepsilon := \frac{\delta}{2(p-1)}$ and $\beta := 2 - n - 2k - \frac{8}{m} - \varepsilon < 0$ yields

$$\left(k + \beta + \frac{4}{m} \right)(p-1) + 2 > 0. \tag{3.58}$$

By means of the spherical coordinates $(x_{n-1}, x_n) = (\rho\cos\varphi, \rho\sin\varphi)$, we define the comparison function $\theta \colon \overline{A^{k,m}} \setminus \{0\} \to \mathbb{R}$ by

$$\theta(x) := |x|^\beta \left(\prod_{j=1}^{k} x_j \right) \rho^{\frac{4}{m}} \sin\left(\frac{4}{m}\varphi \right).$$

In view of Lemma 3.25 there exists a radius $R_0 \in h\mathbb{N}$ such that

$$-\Delta_h \theta(x) \le 0$$

for all $x \in A_h^{k,m}$ with $|x|_\infty > R_0$ and $\theta = 0$ on $\partial_h A_h^{k,m} \setminus \{0\}$. Since the set $\{x \in A_h^{k,m} \colon |x|_\infty = R_0\}$ is finite, we can choose a constant $C > 0$ with $C\theta(x) \le u(x)$ for all $x \in A_h^{k,m}$ with $|x|_\infty = R_0$. Applying the discrete maximum principle, as in the proof of Theorem 3.4 for $n \ge 3$, entails the comparison estimate

$$C\theta(x) \le u(x) \tag{3.59}$$

for all $x \in A_h^{k,m}$ with $|x|_\infty > R_0$.

⟨3⟩ *Hardy-like inequality*
Let $\varphi_1, \varphi_2 \in (0, \frac{\pi}{2})$ be given by (3.56). For all $x = (x_1, \ldots, x_{n-2}, \rho\cos\varphi, \rho\sin\varphi)^T \in A_h^{k,m}$ with $|x|_\infty > R_0$, $x_i \ge \frac{1}{7\sqrt{n}}|x|$ for all $i \in \{1, \ldots, n\}$ and $\varphi \in [\varphi_1, \varphi_2]$ we deduce from (3.59)

$$\begin{aligned}
&-\Delta_h u(x) \\
&\quad\ge u^{p-1}(x)u(x) \\
&\quad\ge C^{p-1}\theta^{p-1}(x)u(x) \\
&\quad= C^{p-1}|x|^{\left(k+\beta+\frac{4}{m}\right)(p-1)+2}\left(\prod_{j=1}^{k}\frac{x_j}{|x|}\right)^{p-1}\sin^{p-1}\left(\frac{4}{m}\varphi\right)\left(\frac{\rho}{|x|}\right)^{\frac{4}{m}(p-1)}|x|^{-2}u(x) \\
&\quad\ge \left(\frac{C}{(7\sqrt{n})^{k+\frac{4}{m}}}\right)^{p-1}\sin^{p-1}\left(\frac{4}{m}\varphi_1\right)|x|^{\left(k+\beta+\frac{4}{m}\right)(p-1)+2}|x|^{-2}u(x).
\end{aligned}$$

Since the exponent $\left(k + \beta + \frac{4}{m}\right)(p-1) + 2$ is strictly positive by (3.58), for every $K > 0$ there exists a radius $R_K > R_0$ such that

$$-\Delta_h u(x) \ge \frac{K}{|x|^2}u(x) \tag{3.60}$$

for all $x \in N_K$ with

$$N_K := \left\{x \in A_h^{k,m} \colon |x|_\infty \ge R_K,\, x_i \ge \frac{1}{7\sqrt{n}}|x|\,(i = 1, \ldots, n),\, \varphi \in [\varphi_1, \varphi_2]\right\}.$$

⟨4⟩ *Agmon principle*
Employing the discrete version of the Agmon principle (cf. proof of Theorem 3.4 for $n \ge 3$), we conclude that for all test functions $\psi \colon \mathbb{R}_h^n \to \mathbb{R}$ with $\mathrm{cl}_h(\mathrm{supp}(\psi)) \subset N_K$:

$$\sum_{x \in N_K}\left[\sum_{i=1}^{n}(D_i^+\psi(x))^2 - \frac{K}{|x|^2}\psi^2(x)\right] \ge 0. \tag{3.61}$$

⟨5⟩ *Contradiction to the reverse Hardy inequality*
Choosing K bigger than the constant C from Lemma 3.26 and $l \in \mathbb{N}$ so large that

$$\mathrm{cl}_h(\mathrm{supp}(u_l)) \subset N_K,$$

the estimates (3.61) and (3.57) yield

$$K \sum_{x \in N_K} \frac{u_l^2(x)}{|x|^2} \leq \sum_{x \in N_K} \sum_{i=1}^{n} |D_i^+ u_l(x)|^2 \leq C \sum_{x \in N_K} \frac{u_l^2(x)}{|x|^2},$$

which is a contradiction since $K > C$. □

Finally, we complete this chapter with the following existence theorem, which ensures that the exponent $\frac{n+k+\frac{4}{m}}{n+k+\frac{4}{m}-2}$ from the Liouville Theorem 3.24 is a critical exponent.

Theorem 3.27 (Existence of solutions)
Let $h > 0$ and $n \geq 3$. For $k \in \{1, \dots, n-2\}$ and $m \in \{1, \dots, 7\}$ let

$$A^{k,m} = \mathbb{R}_+^k \times \mathbb{R}^{n-2-k} \times \Omega^m$$

and $p > \frac{n+k+\frac{4}{m}}{n+k+\frac{4}{m}-2}$. Then, there exists a positive solution $u \colon \overline{A_h^{k,m}} \to (0, \infty)$ of

$$-\Delta_h u \geq u^p \quad \text{in } A_h^{k,m}.$$

Proof. We show the result similar to the proofs of Theorem 3.10 and 3.18 by considering the comparison function $\theta \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, given by

$$\theta(x) = |x|^\beta \left(\prod_{j=1}^{k} x_j \right) \rho^{\frac{4}{m}} \sin\left(\frac{4}{m} \varphi \right),$$

with exponent $\beta \coloneqq \frac{-2}{p-1} - \frac{4}{m} - k < 0$. □

4. A priori bounds

It is a quite natural question to ask whether all solutions of a nonlinear elliptic boundary value problem are bounded. In their celebrated paper, [11], Gidas and Spruck considered non-negative solutions of

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with $\partial\Omega \in C^1$ and $n \geq 2$. If $1 < p < \frac{n+2}{(n-2)_+}$, then every non-negative solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of (4.1) is a priori bounded, i.e., there exists some uniform constant $C > 0$ (depending on p and Ω but not on u) such that

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

Since we use a similar approach, we give a short outline of their proof: The result is proven by contradiction. Therefore, they assume there exists a sequence of solutions $(u_k)_{k\in\mathbb{N}} \subset C^2(\Omega) \cap C(\overline{\Omega})$ with $\|u_k\|_{L^\infty(\Omega)} \to \infty$ for $k \to \infty$. After some rescaling, they extract a non-negative, nontrivial limit function $v \in C^2(A) \cap C(\overline{A})$ which solves

$$\begin{cases} -\Delta u = u^p & \text{in } A, \\ u = 0 & \text{on } \partial A, \end{cases}$$

where A is either the entire space $\mathbb{R}^n$ or a half space. As $1 < p < \frac{n+2}{(n-2)_+}$ this leads in both cases to a contradiction to the nonlinear Liouville theorem. So, below the minimum of the Liouville exponents on the whole space and on the half space (which is given by $\frac{n+2}{(n-2)_+}$), the scaling argument of Gidas and Spruck gives rise to a priori bounds on a bounded C^1-domain.

The statement in [11] allows also for more general uniformly elliptic differential operators and nonlinearities $f(x,u) = g_1(x)u^p + g_2(x,u)$ with g_1 in L^∞ and $|g_2(x,u)| = \mathrm{o}(u^p)$ as $u \to \infty$. The idea to prove a priori bounds by means of a scaling ansatz and corresponding Liouville theorems has been adopted many times, e.g. in Reichel and Weth ([28, Thm. 1]) for higher order differential operators or in Hirsch ([17, Thm. 4.3]) for cylindrically symmetric solutions of the curl-curl problem. In the context of finite differences Verbitzky employed this approach in [30, Thm. 10.9] to show a priori bounds for a discrete Schrödinger equation on the entire grid $\mathbb{R}^n_h$. In this case only the Liouville theorems on $\mathbb{R}^n$ and $\mathbb{R}^n_h$ were needed and boundary issues were not involved. For finite difference solutions on a hypercube McKenna, Reichel and Verbitzky used in [24] a comparison argument which was based on the knowledge of the first eigenfunction and gave explicit a priori bounds.

In the following we transfer the scaling approach of Gidas and Spruck from C^2-functions to grid functions. The advantage of our method is that it can be applied to other domains than hypercubes. Exemplarily, we prove later a priori bounds for right-angled isosceles 2d-triangles. In order to get a better understanding of the new method, we first analyse the approach on hypercubes and turn later to more general domains.

4.1. Hypercubes

Notation 4.1
Let $\Omega := \prod_{i=1}^{n}(a_i, b_i) \subset \mathbb{R}^n$ *be a bounded hypercube for dimensions* $n \geq 2$ *and* $a_i < b_i$ *for* $i \in \{1, \ldots, n\}$. *Further, we denote by* $p_\star := \frac{n}{n-1}$ *the discrete Liouville exponent for orthants from Chapter 3. For admissible grid sizes* $h > 0$ *and exponents* $p \in (1, p_\star)$ *we consider positive solutions of the discrete Emden equation*

$$\begin{cases} -\Delta_h u = u^p & \text{in } \Omega_h, \\ \quad u = 0 & \text{on } \partial_h \Omega_h. \end{cases} \tag{4.2$_h$}$$

Remark 4.2 (Liouville exponents for orthants)
According to Theorem 3.4 the Liouville exponent for discrete generalized orthants

$$\mathbb{R}_h^{n,k} = \{x \in \mathbb{R}_h^n \colon x_1, \ldots, x_k > 0\}$$

is $\frac{n+k}{(n+k-2)_+}$ *for all* $n \in \mathbb{N}$, $k \in \{0, \ldots, n\}$ *and grid sizes* $h > 0$. *In the special case of orthants we have* $n \geq 2$ *as well as* $k = n$ *and therefore the Liouville exponent is* $\frac{n}{n-1}$.

Theorem 4.3 (A priori bounds for hypercubes)
Let $1 < p < p_\star = \frac{n}{n-1}$ *and* $\Omega = \prod_{i=1}^{n}(a_i, b_i) \subset \mathbb{R}^n$. *Then, there exists a constant* $C > 0$ *such that for every admissible grid size* $h > 0$ *and every solution* $u_h \colon \overline{\Omega_h} \to [0, \infty)$ *of* $(4.2)_h$ *the a priori estimate* $\|u_h\|_{L^\infty(\Omega_h)} \leq C$ *holds.*

This is our main results in the section and it is proven by contradiction with the aid of a scaling argument inspired by Gidas and Spruck. The idea is to construct appropriate limit functions which violate corresponding Liouville theorems: In the discrete limit case we can use Theorem 3.4 and in the continuous one we will employ the following corresponding Liouville theorem for classical solutions which is a collection of several known results from the literature.

Theorem 4.4 (Liouville theorem on generalized orthants for C^2-functions)
Let $n \in \mathbb{N}$, $n \geq 2$, $k \in \{0, \ldots, n\}$ *and* $1 < p < \frac{n+k}{(n+k-2)_+}$. *Then, the only non-negative solution* $u \in C^2(\mathbb{R}^{n,k}) \cap C(\overline{\mathbb{R}^{n,k}})$ *of*

$$\begin{cases} -\Delta u = u^p & \text{in } \mathbb{R}^{n,k}, \\ \quad u = 0 & \text{on } \partial \mathbb{R}^{n,k} \end{cases}$$

is $u \equiv 0$.

Proof. In the case of $k=0$, this is just the classical Liouville theorem for the entire space $\mathbb{R}^n$ of Gidas and Spruck if $n \geq 3$ or a variant of it which can be found in the work of Wei and Xu ([31]) if $n=2$.

If $k \in \{1, \ldots, n\}$ the result is based on [5, Thm. 4.6]. For the reader's convenience we illustrate how to apply this theorem such that it yields the desired statement: For $x \in \mathbb{R}^n$ let $(r, \theta) \in [0, \infty) \times \mathbb{S}^{n-1}$ be the spherical coordinates of x abbreviated by $x = (r, \theta)$. Thereby, for $\omega \subset \mathbb{S}^{n-1}$ we define the *infinite cone*

$$\mathcal{C}_\omega := \{x = (r, \theta) \colon r > 0,\, \theta \in \omega\}$$

and consider non-negative solutions $u \in C^2(\mathcal{C}_\omega) \cap C(\overline{\mathcal{C}_\omega})$ of

$$\begin{cases} -\Delta u = u^p & \text{in } \mathcal{C}_\omega, \\ \quad u = 0 & \text{on } \partial \mathcal{C}_\omega. \end{cases} \tag{4.3}$$

Moreover, let $(\tilde{\lambda}_1, \tilde{\psi}_1)$ be the first Dirichlet eigenpair of the Laplace-Beltrami operator $-\Delta_{\mathbb{S}^{n-1}}$ on ω and denote the two roots of the equation $\gamma(\gamma + n - 2) - \tilde{\lambda}_1 = 0$ by

$$\gamma^{\pm} := \frac{2-n}{2} \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + \tilde{\lambda}_1}.$$

With that, Theorem 4.6. in [5] says that for $1 < p < p_{\mathrm{BT}}$ the Emden equation (4.3) has only the zero solution, where

$$p_{\mathrm{BT}} := \frac{\gamma^- - 2}{\gamma^-} = \frac{n + \gamma^+}{n + \gamma^+ - 2}$$

is the so-called *generalized Brezis-Turner exponent* (see [23]). It remains to verify that $\gamma^+ = k$ if the cone $\mathcal{C}_\omega$ is a generalized orthant $\mathbb{R}^{n,k}$ with $k \in \{1, \ldots, n\}$. Indeed, in this case

$$\omega = \mathbb{R}^{n,k} \cap \mathbb{S}^{n-1} = \{x \in \mathbb{S}^{n-1} \colon x_1, \ldots, x_k > 0\}$$

and according to Lemma A.3 the principle eigenfunction of $-\Delta_{\mathbb{S}^{n-1}}$ on ω is given by $\tilde{\psi}_1(x) = \prod_{i=1}^k x_i$ with corresponding eigenvalue $\tilde{\lambda}_1 = k(k+n-2)$. Thus,

$$\begin{aligned} \gamma^+ &= \frac{2-n}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + \tilde{\lambda}_1} \\ &= \frac{2-n}{2} + \frac{1}{2}\sqrt{(n-2)^2 + 4k(k+n-2)} \\ &= \frac{2-n}{2} + \frac{1}{2}\sqrt{(n+2k-2)^2} = k \end{aligned}$$

and the assertion is valid. □

Definition 4.5 (Tensor product interpolation)
Let $\Omega \subset \mathbb{R}^n$ be a hypercube and $h > 0$ be an admissible grid size. We define the discrete sets

$$\begin{aligned} \hat{\Omega}_h &:= \overline{\Omega} \cap \mathbb{R}^n_h, \\ \hat{\partial}_h \hat{\Omega}_h &:= \partial\Omega \cap \mathbb{R}^n_h \quad \text{as well as} \\ \hat{\partial}_i^{\pm} \hat{\Omega}_h &:= \{x \in \hat{\partial}_h \hat{\Omega}_h : x \pm h e_i \notin \hat{\Omega}_h\} \quad \text{for all } i \in \{1, \dots, n\}. \end{aligned}$$

For $u \colon \hat{\Omega}_h \to \mathbb{R}$ and $q \in (1, \infty)$, we assign

$$\|u\|_{\dot{W}_0^{1,q}(\hat{\Omega}_h)} := \left(\sum_{i=1}^{n} \sum_{x \in \hat{\Omega}_h \setminus \hat{\partial}_i^+ \hat{\Omega}_h} |D_i^+ u(x)|^q h^n \right)^{\frac{1}{q}}.$$

Furthermore, we denote by $\hat{u} \colon \overline{\Omega} \to \mathbb{R}$ the corresponding tensor product interpolant from [30, Def. 8.12]. If $v \colon \overline{\Omega_h} \to \mathbb{R}$, then $\hat{v} \colon \overline{\Omega} \to \mathbb{R}$ denotes the tensor product interpolant corresponding to the by zero extended function $v \colon \hat{\Omega}_h \to \mathbb{R}$.

Lemma 4.6 (Norm estimates for interpolants)
Let $\Omega \subset \mathbb{R}^n$ be a hypercube, $h > 0$ be an admissible grid size. Moreover, let $u \colon \hat{\Omega}_h \to \mathbb{R}$ and $\hat{u} \colon \overline{\Omega} \to \mathbb{R}$ be the corresponding tensor product interpolant.

(a) There exists a constant $C = C(n) > 0$ such that

$$\|\hat{u}\|_{L^\infty(\Omega)} \leq C \|u\|_{L^\infty(\hat{\Omega}_h)}.$$

(b) For all $q \in (1, \infty)$, there exists a constant $C = C(n, q) > 0$ such that

$$\|\nabla \hat{u}\|_{L^q(\Omega)} \leq C \|u\|_{\dot{W}_0^{1,q}(\hat{\Omega}_h)}.$$

Proof. The first estimate follows from [30, Cor. 8.8], whereas the second estimate can be proven as in the proof of [30, Thm. 8.13]. □

Lemma 4.7 (Special estimate for interpolants)
Let $q \in (1, \infty)$, $\Omega \subset \mathbb{R}^n$ and $h > 0$ be an admissible grid size. Moreover, let $u \colon \overline{\Omega_h} \to \mathbb{R}$ with $u = 0$ on $\partial_h \Omega_h$. Then, there exists a constant $C = C(n, q) > 0$ such that for every hypercube $A \subset \Omega$ (with h is admissible for A) the following estimate holds true

$$\|\hat{u}\|_{W^{1,q}(A)} \leq C |A|^{\frac{1}{q}} \left(\|u\|_{L^\infty(\Omega_h)} + \sum_{i=1}^{n} \left\|D_i^+ u\right\|^q_{L^\infty(\partial_i^- \Omega_h \cup \Omega_h)} \right)^{\frac{1}{q}},$$

where we extend the function u by zero to a function $u \colon \hat{A}_h \to \mathbb{R}$ and denote by $\hat{u} \colon \overline{A} \to \mathbb{R}$ the corresponding interpolant.

Proof. As $u = 0$ on $\partial_h \Omega_h$ we see that

$$\begin{aligned} \|u\|_{L^\infty(\hat{A}_h)} &\leq \|u\|_{L^\infty(\Omega_h)}\,, \\ \left\|D_i^+ u\right\|_{L^\infty(\hat{A}_h \setminus \hat{\partial}_i^+ \hat{A}_h)} &\leq \left\|D_i^+ u\right\|_{L^\infty(\partial_i^- \Omega_h \cup \Omega_h)} \quad \text{for all } i \in \{1, \dots, n\}. \end{aligned} \tag{4.4}$$

In the following $C > 0$ denotes a positive constant, which depends only on n, q and can vary from line to line. From Lemma 4.6 and (4.4) we deduce

$$\begin{aligned} \|\nabla \hat{u}\|_{L^q(A)} &\leq C \|u\|_{\dot{W}_0^{1,q}(\hat{A}_h)} \\ &= C \left(\sum_{i=1}^n \sum_{x \in \hat{A}_h \setminus \hat{\partial}_i^+ \hat{A}_h} |D_i^+ u(x)|^q h^n \right)^{\frac{1}{q}} \\ &\leq C |A|^{\frac{1}{q}} \left(\sum_{i=1}^n \left\|D_i^+ u\right\|^q_{L^\infty(\partial_i^- \Omega_h \cup \Omega_h)} \right)^{\frac{1}{q}} \end{aligned}$$

as well as

$$\|\hat{u}\|_{L^q(A)} = \left(\int_A |\hat{u}|^q \, dx \right)^{\frac{1}{q}} \leq |A|^{\frac{1}{q}} \|\hat{u}\|_{L^\infty(A)} \leq C |A|^{\frac{1}{q}} \|u\|_{L^\infty(\hat{A}_h)} \leq C |A|^{\frac{1}{q}} \|u\|_{L^\infty(\Omega_h)}.$$

Combining the last two estimate we conclude

$$\begin{aligned} \|\hat{u}\|_{W^{1,q}(A)} &= \left(\|\nabla \hat{u}\|^q_{L^q(A)} + \|\hat{u}\|^q_{L^q(A)} \right)^{\frac{1}{q}} \\ &\leq C |A|^{\frac{1}{q}} \left(\|u\|_{L^\infty(\Omega_h)} + \sum_{i=1}^n \left\|D_i^+ u\right\|^q_{L^\infty(\partial_i^- \Omega_h \cup \Omega_h)} \right)^{\frac{1}{q}}. \end{aligned}$$

□

Proof of Theorem 4.3. The result is shown by a contradiction argument. Hence, *assume* there exists a sequence of admissible grid sizes $(h_l)_{l \in \mathbb{N}} \subset (0, \infty)$ and corresponding solutions $u_l \coloneqq u_{h_l} \colon \overline{\Omega_{h_l}} \to [0, \infty)$ of $(4.2)_{h_l}$ such that

$$\|u_l\|_{L^\infty(\Omega_{h_l})} \to \infty \quad (l \to \infty). \tag{4.5}$$

This means that there exist points $P_l \in \Omega_{h_l}$ with

$$M_l \coloneqq \max_{x \in \Omega_{h_l}} u_l(x) = u_l(P_l) \to \infty \quad (l \to \infty).$$

In the sequel, we suppose without loss of generality $M_l > 0$ for all $l \in \mathbb{N}$. As $\overline{\Omega} \subset \mathbb{R}^n$ is compact, we can assume that $P_l \to P \in \overline{\Omega}$ for $l \to \infty$.

⟨1⟩ *Boundedness*
Let $h > 0$ be admissible and $u_h\colon \overline{\Omega_h} \to [0,\infty)$ be a non-negative solution of $(4.2)_h$. From

$$u_h^p(x) = -\Delta_h u_h(x) = -\frac{1}{h^2}\sum_{i=1}^{n}(u_h(x+he_i) - 2u_h(x) + u_h(x-he_i)) \le \frac{2n}{h^2}u_h(x)$$

we infer that

$$u_h^{p-1}(x) \le \frac{2n}{h^2} \quad \text{for all } x \in \Omega_h.$$

This yields the uniform boundedness of the non-negative solution u_h if the grid size $h > 0$ is fixed and moreover

$$hu_h^{\frac{p-1}{2}}(x) \le \sqrt{2n} \quad \text{for all } x \in \Omega_h. \tag{4.6}$$

Combining (4.5) and (4.6), we see that $h_l \to 0$ as $l \to \infty$.

⟨2⟩ *Scaling*
We introduce the scaling parameter $\lambda_l := M_l^{\frac{1-p}{2}}$ for $l \in \mathbb{N}$. Note that $\lambda_l \to 0$ as $l \to \infty$. Further, the rescaled function $v_l\colon \overline{D_{\tau_l}^l} \to [0,\infty)$ is given by

$$v_l(x) := \frac{1}{M_l}u_l(\lambda_l x + P_l),$$

where the new grid size $\tau_l := \frac{h_l}{\lambda_l} = M_l^{\frac{p-1}{2}}h_l$ and the domain $D^l := \frac{1}{\lambda_l}(\Omega - P_l)$ is chosen such that $\lambda_l x + P_l \in \Omega_{h_l} = \Omega \cap \mathbb{R}^n_{h_l}$ if and only if $x \in D^l_{\tau_l} = D^l \cap \mathbb{R}^n_{\tau_l}$. This entails

$$\|v_l\|_{L^\infty(D^l_{\tau_l})} = v_l(0) = 1.$$

Since $x \in \partial_{\tau_l}D^l_{\tau_l}$ is equivalent to $\lambda_l x + P_l \in \partial_{h_l}\Omega_{h_l}$ we obtain

$$\|v_l\|_{L^\infty(\partial_{\tau_l}D^l_{\tau_l})} = 0.$$

Moreover, for all $x \in D^l_{\tau_l}$ we have the equality

$$\begin{aligned}
-\Delta_{\tau_l}v_l(x) &= \frac{-1}{\tau_l^2}\sum_{i=1}^{n}(v_l(x+\tau_l e_i) - 2v_l(x) + v_l(x-\tau_l e_i)) \\
&= \frac{-1}{M_l^p h_l^2}\sum_{i=1}^{n}(u_l(\lambda_l(x+\tau_l e_i)+P_l) - 2u_l(\lambda_l x + P_l) + u_l(\lambda_l(x-\tau_l e_i)+P_l)) \\
&= \frac{-1}{M_l^p}\Delta_{h_l}u_l(\lambda_l x + P_l) = \frac{1}{M_l^p}u_l^p(\lambda_l x + P_l) = v_l^p(x).
\end{aligned} \tag{4.7}$$

⟨3⟩ *Alternatives for* $(\tau_l)_{l\in\mathbb{N}}$
From the definition of M_l and (4.6) we deduce

$$\tau_l = M_l^{\frac{p-1}{2}} h_l = \left(\max_{x\in\Omega_{h_l}} u_l(x)\right)^{\frac{p-1}{2}} h_l \leq \sqrt{2n}.$$

Hence, $(\tau_l)_{l\in\mathbb{N}} \subset (0,\infty)$ is a bounded sequence and the following two alternatives can occur: Either $\tau_l \to \tau = 0$ or, up to a subsequence, $\tau_l \to \tau > 0$ for $l \to \infty$.

From now on, we separately discuss the two possibilities $P \in \Omega$ and $P \in \partial\Omega$. In each of the two cases we consider the alternatives $\tau > 0$ (discrete limit) and $\tau = 0$ (continuous limit) separately.

Case 1: $P \in \Omega$. In this situation we can deduce a contradiction with the aid of the two Liouville theorems on $\mathbb{R}^n_h$ and $\mathbb{R}^n$:

⟨4⟩ *Domain convergence*
Defining $d := \frac{1}{2}\operatorname{dist}(P,\partial\Omega) = \frac{1}{2}\min\{\|y-P\|_1 \colon y \in \partial\Omega\}$, we ensure that the ball $B_{2d}(P) = \{\|y-P\|_1 < 2d \colon y \in \mathbb{R}^n\}$ is a subset of Ω. Since $P_l \to P$ we have

$$B_d(P_l) \subset \Omega$$

and thus

$$B_{\frac{d}{\lambda_l}}(0) \cap \mathbb{R}^n_{\tau_l} \subset D^l_{\tau_l} \tag{4.8}$$

for sufficiently large $l \in \mathbb{N}$. This is an important feature since $\frac{d}{\lambda_l} \to \infty$ as $l \to \infty$.

⟨5⟩ *Discrete limit*
Below, we analyse the case $\tau_l \to \tau > 0$ for a subsequence which is again denoted by (τ_l). We construct the limit function $v_\tau \colon \mathbb{R}^n_\tau \to [0,1]$ as follows. First, note that

$$v_\tau(0) := \lim_{l\to\infty} v_l(0) = 1.$$

Next, we choose $l_0 \in \mathbb{N}$ such that $v_l(\tau_l e_1)$ is well-defined for all $l \geq l_0$. This is possible in view of (4.8). According to the Bolzano-Weierstraß theorem, the sequence $(v_l(\tau_l e_1))_{l\geq l_0} \subset (0,1]$ contains a convergent subsequence $(v_{l_m}(\tau_{l_m} e_1))_{m\in\mathbb{N}}$ with a limit in $[0,1]$ and we assign

$$v_\tau(\tau e_1) := \lim_{m\to\infty} v_{l_m}(\tau_{l_m} e_1).$$

Analogously, we extract a convergent subsequence from $(v_{l_m}(\tau_{l_m} e_2))_{m\geq m_0}$ and call the limit $v_\tau(\tau e_2)$, where again the starting index $m_0 \in \mathbb{N}$ is chosen so that $v_{l_m}(\tau_{l_m} e_2)$ is well-defined for all $m \geq m_0$. Extracting iteratively more and more subsequences and using a diagonal sequence, we define $v_\tau(x)$ for all $x \in \mathbb{R}^n_\tau$. Taking the limit for the renamed diagonal sequence in (4.7) and employing (4.8) yields

$$-\Delta_\tau v_\tau(x) = v_\tau^p(x)$$

for all $x \in \mathbb{R}^n_\tau$. Keeping in mind that $v_\tau(0) = 1$, this is contradictory to the discrete Liouville Theorem 3.4.

⟨6⟩ *Continuous limit*
In the sequel, we regard the case $\tau_l \to 0$. Let $(R_l)_{l\in\mathbb{N}} \subset (0,\infty)$ be a non-decreasing sequence of radii with $R_l \to \infty$ for $l \to \infty$. Further, we define $R_l^{(m)} := \left\lceil \frac{R_l}{\tau_m} \right\rceil \tau_m$ for $l, m \in \mathbb{N}$, i.e., $R_l^{(m)} \geq R_l$ and $\left(-R_l^{(m)}, R_l^{(m)}\right)^n$ is admissible for τ_m. Due to (4.8) we have

$$\left[-2R_1^{(m)}, 2R_1^{(m)}\right]^n_{\tau_m} \subset B_{\frac{d}{\lambda_m}}(0) \cap \mathbb{R}^n_{\tau_m} \subset D^m_{\tau_m},$$

for all sufficiently large $m \in \mathbb{N}$. Thus, we see that $\|v_m\|_{L^\infty} = 1$ and $\|\Delta_{\tau_m} v_m\|_{L^\infty} = 1$ on $\left[-2R_1^{(m)}, 2R_1^{(m)}\right]^n_{\tau_m}$ and therefore Theorem 5.31 from [30] yields

$$\|D_i^+ v_m\|_{L^\infty([-R_1^{(m)}, R_1^{(m)}]^n_{\tau_m})} \leq \tilde{C} \tag{4.9}$$

for a fixed constant $\tilde{C} > 0$, every $i \in \{1, \ldots, n\}$ and all sufficiently large $m \in \mathbb{N}$. According to Definition 4.5, there exists an interpolant $\hat{v}_m \in C([-R_1^{(m)}, R_1^{(m)}]^n)$ associated with $v_m \colon [-R_1^{(m)}, R_1^{(m)}]^n_{\tau_m} \to [0,1]$ for all large enough $m \in \mathbb{N}$ (cf. (4.8)). Next, we fix some $q > n$. In view of (4.9), $\|v_l\|_{L^\infty(D^l_{\tau_l})} = 1$ and $v_l = 0$ on $\partial_{\tau_l} D^l_{\tau_l}$, Lemma 4.7 ensures

$$\|\hat{v}_m\|_{W^{1,q}([-R_1^{(m)}, R_1^{(m)}]^n)} \leq C$$

for a constant $C > 0$ and all large enough $m \in \mathbb{N}$. As $q > n$ we can use the compact embedding $W^{1,q}([-R_1, R_1]^n) \hookrightarrow C^{0,\alpha}([-R_1, R_1]^n)$ for some $\alpha \in (0, 1 - \frac{n}{q})$ (see [1, Thm. 6.3]) and extract from $\left(\hat{v}_m|_{[-R_1,R_1]^n}\right)_{m\in\mathbb{N}}$ a uniformly convergent subsequence with limit function $v \in C^{0,\alpha}([-R_1, R_1]^n) \cap W^{1,q}([-R_1, R_1]^n)$, $v \geq 0$ and $v(0) = 1$. In the next step, we use the resulting subsequence of $(v_m)_{m\in\mathbb{N}}$ as a starting point to repeat this argument on $[-R_2, R_2]^n$. Using a diagonal sequence, we obtain a limit function $v \in C^{0,\alpha}_{\text{loc}}(\mathbb{R}^n) \cap W^{1,q}_{\text{loc}}(\mathbb{R}^n)$ with $v(0) = 1 = \|v\|_{L^\infty(\mathbb{R}^n)}$. From the proofs of Lemma 9.8 and 9.9 in [30] we infer that the limit function v satisfies

$$\int_{\mathbb{R}^n} v(-\Delta\psi)\, dx = \int_{\mathbb{R}^n} v^p \psi\, dx$$

for all $\psi \in C^\infty_c(\mathbb{R}^n)$. Thus, regularity theory (see Lemma A.7) guarantees that

$v \in C^2(\mathbb{R}^n)$ and solves

$$-\Delta v = v^p \quad \text{in } \mathbb{R}^n.$$

By virtue of $v(0) = 1$ this contradicts Theorem 4.4.

<u>Case 2:</u> $P \in \partial\Omega$.

⟨4′⟩ *Bounded discrete gradients*
Recall that $\|v_l\|_{L^\infty(\overline{D^l_{\tau_l}})} = 1$ and $\|\Delta_{\tau_l} v_l\|_{L^\infty(D^l_{\tau_l})} = 1$ for all $l \in \mathbb{N}$. Employing the discrete Schwarz reflection principle, which is carried out in Proposition A.5, and applying subsequently Theorem 5.31 from [30] yields a uniform constant $C > 0$ with

$$\left\|D_i^+ v_l\right\|_{L^\infty(\partial_i^- D^l_{\tau_l} \cup D^l_{\tau_l})} \leq C \tag{4.10}$$

for all $l \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$.

⟨5′⟩ *Distance to the boundary*
For $l \in \mathbb{N}$ let $\delta_l \coloneqq \min\left\{\|y\|_1 \colon y \in \partial_{\tau_l} D^l_{\tau_l}\right\} \in \tau_l \mathbb{N}$ be the distance between 0, which is the maximizer of the functions v_l, and the discrete boundary $\partial_{\tau_l} D^l_{\tau_l}$. Then, there exist an integer $J = J(l) \in \mathbb{N}$ and a minimizer $y_J \in \partial_{\tau_l} D^l_{\tau_l}$ so that

$$\delta_l = \min\left\{\|y\|_1 \colon y \in \partial_{\tau_l} D^l_{\tau_l}\right\} = \|y_J\|_1 = J\tau_l.$$

Using the definition $y_0 \coloneqq 0 \in \mathbb{R}^n$, we can choose $y_j \in D^l_{\tau_l}$, $j = 1, \ldots, J-1$, such that

$$|y_{j+1} - y_j| = \tau_l$$

and together with (4.10) it follows that

$$\frac{1}{\tau_l} |v_l(y_{j+1}) - v_l(y_j)| \leq C$$

for all $j = 0, \ldots, J-1$. This finally leads to

$$\begin{aligned} 1 &= v_l(0) - v_l(y_J) \\ &= v_l(y_0) - v_l(y_J) \\ &= \tau_l \sum_{j=0}^{J-1} \frac{1}{\tau_l} (v_l(y_j) - v_l(y_{j+1})) \\ &\leq \tau_l J C \\ &= C\delta_l \end{aligned}$$

and therefore,

$$\delta_l \geq \frac{1}{C} > 0 \quad \text{for all } l \in \mathbb{N}. \tag{4.11}$$

Thus, these two alternatives may occur: Either $\delta_l \to \infty$ or there exists a convergent subsequence, which is again denoted by $(\delta_l)_{l\in\mathbb{N}}$, such that $\delta_l \to \delta > 0$ for $l \to \infty$.

$\langle 6' \rangle$ *Drifting away*
First, the case $\delta_l \to \infty$ is investigated: The definition of δ_l assures

$$B_{\delta_l}(0) \cap \mathbb{R}^n_{\tau_l} = \left\{ y \in \mathbb{R}^n_{\tau_l} \colon \|y\|_1 < \delta_l \right\} \subset D^l_{\tau_l} \tag{4.12}$$

for all $l \in \mathbb{N}$. Using (4.12) instead of (4.8), the steps $\langle 5 \rangle$ and $\langle 6 \rangle$ can be transferred literally and we obtain a contradiction to the Liouville theorems on $\mathbb{R}^n_\tau$ and $\mathbb{R}^n$, respectively. In a manner of speaking, the discrete as well as the continuous limit function does not *see* the boundary.

$\langle 7' \rangle$ *Staying near the boundary - half space case*
The case $\delta_l \to \delta > 0$ can be much more delicate, especially when $P \in \partial\Omega$ is a vertex. Before we come to that we analyse the situation when the boundary $\partial\Omega$ coincides, in a neighbourhood of P, with a hyperplane. Up to a rotation and a translation we may assume that $P = 0$ and there exists a radius $\varrho > 0$ such that

$$\begin{aligned} &B_\varrho(P) \cap \{x_n > 0\} \subset \Omega, \\ &B_\varrho(P) \cap \{x_n = 0\} \subset \partial\Omega \text{ and} \\ &B_\varrho(P) \cap \{x_n < 0\} \subset \mathbb{R}^n \setminus \Omega. \end{aligned} \tag{4.13}$$

Employing the notation $x = (x', x_n) \in \mathbb{R}^n$ with $x' := (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ we introduce the modified scaled functions $w_l \colon \overline{\mathcal{D}^l_{\tau_l}} \to [0, \infty)$ given by

$$w_l(x) := \frac{1}{M_l} u_l(\lambda_l x + (P'_l, 0)) = v_l\left(x - \frac{1}{\lambda_l}(0', P_{l,n})\right)$$

with domain $\mathcal{D}^l := \frac{1}{\lambda_l}(\Omega - (P'_l, 0)) = D^l + \frac{1}{\lambda_l}(0', P_{l,n})$. Thus, according to (4.10) there is a constant $C > 0$ so that

$$\left\| D^+_i w_l \right\|_{L^\infty(\partial^-_i \mathcal{D}^l_{\tau_l} \cup \mathcal{D}^l_{\tau_l})} \leq C \tag{4.14}$$

uniformly for all $l \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$. Other important properties of the functions w_l are

$$\begin{aligned} -\Delta_{\tau_l} w_l &= w_l^p \quad \text{in } \mathcal{D}^l_{\tau_l}, \\ \|w_l\|_{L^\infty(\partial_{\tau_l} \mathcal{D}^l_{\tau_l})} &= 0, \\ \|w_l\|_{L^\infty(\mathcal{D}^l_{\tau_l})} &= w_l(0', \lambda_l^{-1} P_{l,n}) = \frac{1}{M_l} u_l(P_l) = 1 \end{aligned} \tag{4.15}$$

for all $l \in \mathbb{N}$. Furthermore, since

$$\begin{aligned}\lambda_l^{-1} P_{l,n} &= \lambda_l^{-1} \min\{\|z - P_l\|_1 : z \in \partial_{h_l}\Omega_{h_l}\}\\ &= \min\left\{\left\|\frac{z - P_l}{\lambda_l}\right\|_1 : z \in \partial_{h_l}\Omega_{h_l}\right\}\\ &= \min\{\|y\|_1 : \lambda_l y + P_l \in \partial_{h_l}\Omega_{h_l}\}\\ &= \min\left\{\|y\|_1 : y \in \partial_{\tau_l} D^l_{\tau_l}\right\}\\ &= \delta_l\end{aligned} \tag{4.16}$$

for large enough $l \in \mathbb{N}$, we see that

$$(0', \lambda_l^{-1} P_{l,n}) = (0', \delta_l) \to (0', \delta) \qquad (l \to \infty). \tag{4.17}$$

As $B_\varrho(0) \cap \{x_n > 0\} \subset \Omega$ by (4.13) we have

$$\begin{aligned}\mathcal{D}^l &= \frac{1}{\lambda_l}[\Omega - (P_l', 0)]\\ &\supset \frac{1}{\lambda_l}[(B_\varrho(0) \cap \{x_n > 0\}) - (P_l', 0)]\\ &= \frac{1}{\lambda_l} B_\varrho\left(-P_l', 0\right) \cap \{x_n > 0\}.\end{aligned} \tag{4.18}$$

Employing $\lambda_l \to 0$ and $P_l' \to 0'$ directly leads to

$$\mathcal{D}^l \to \{x_n > 0\} =: H \quad \text{for } l \to \infty. \tag{4.19}$$

The shorthand $\mathcal{D}^l \to H$ means that for all $x \in H$ there exists some $l_0(x) \in \mathbb{N}$ such that $x \in \mathcal{D}^l$ for all $l \geq l_0(x)$. In the same manner we obtain

$$\partial\mathcal{D}^l \to \{x_n = 0\} = \partial H \tag{4.20}$$

for $l \to \infty$. In the case of $P \in \Omega$ the correlation (4.8) has been important. Below, (4.19) and (4.20) are used instead. Again the distinction between $\tau_l \to \tau > 0$ and $\tau_l \to 0$ is appropriate.

First, the case of a discrete limit $\tau_l \to \tau > 0$ is considered. Then (4.19) and (4.20) entail

$$\begin{aligned}\mathcal{D}^l_{\tau_l} &\to \{x \in \mathbb{R}^n_\tau : x_n > 0\} = H_\tau,\\ \partial_{\tau_l}\mathcal{D}^l_{\tau_l} &\to \{x \in \mathbb{R}^n_\tau : x_n = 0\} = \partial_\tau H_\tau\end{aligned}$$

for $l \to \infty$. In analogy to ⟨5⟩ we generate a limit function $w_\tau : \overline{H_\tau} \to [0, 1]$ with

$$\begin{aligned}w_\tau(0', \delta) &:= 1 = \lim_{l\to\infty} w_{\tau_l}(0', \delta_l),\\ w_\tau &= 0 \quad \text{on } \partial_\tau H_\tau.\end{aligned} \tag{4.21}$$

Moreover, taking the limit $l \to \infty$ in (4.15) reveals

$$-\Delta_\tau w_\tau(x) = w_\tau^p(x) \quad \text{for all } x \in H_\tau.$$

In view of (4.21) this contradicts the discrete Liouville Theorem 3.4.

In the situation $\tau_l \to 0$ our approach is similar to ⟨6⟩. Let $(R_l)_{l\in\mathbb{N}} \subset (0,\infty)$ be an increasing sequence with $R_l \to \infty$ for $l \to \infty$. Assigning $R_l^{(m)} := \left\lceil \frac{R_l}{\tau_m} \right\rceil \tau_m$ for $l, m \in \mathbb{N}$ yields $R_l^{(m)} \geq R_l$ and the sets $A^{l,m} := \left(-R_l^{(m)}, R_l^{(m)}\right)^{n-1} \times \left(0, R_l^{(m)}\right)$ are admissible for τ_m. Moreover, due to Definition 4.5, there exists an interpolant $\hat{w}_m \in C(\overline{A^{1,m}}, [0,1])$ corresponding to w_m for $m \in \mathbb{N}$ large enough (cf. (4.18)). Since $w_m = 0$ on $\partial_{\tau_m} \mathcal{D}_{\tau_m}^m$ it is guaranteed by Lemma 8.11 in [30] that

$$\hat{w}_m = 0 \quad \text{on} \left[-R_1^{(m)}, R_1^{(m)}\right]^{n-1} \times \{0\}. \tag{4.22}$$

Again, we fix some $q > n$. Due to (4.14) and (4.15), Lemma 4.7 yields a constant $C > 0$ such that

$$\|\hat{w}_m\|_{W^{1,q}(A^{1,m})} \leq C$$

for all sufficiently large $m \in \mathbb{N}$. Since $R_1^{(m)} \geq R_1$, we can restrict the interpolant $\hat{w}_m$ to the set $A_1 := (-R_1, R_1)^{n-1} \times (0, R_1)$ for all $m \in \mathbb{N}$ large enough. As $q > n$ we can apply the compact embedding $W^{1,q}(A_1) \hookrightarrow C^{0,\alpha}(\overline{A_1})$ for some $\alpha \in (0, 1 - \frac{n}{q})$ (see [1, Thm. 6.3]) and extract from $\left(\hat{w}_m|_{\overline{A_1}}\right)_{m\in\mathbb{N}}$ a uniformly convergent subsequence with limit $w \in C^{0,\alpha}(\overline{A_1}) \cap W^{1,q}(A_1)$. As in ⟨6⟩, we obtain a limit function $w \in C^{0,\alpha}_{\text{loc}}(\overline{H}) \cap W^{1,q}_{\text{loc}}(H)$. In view of (4.15), (4.17) and (4.22) we see that

$$w(0', \delta) = 1 = \|w\|_{L^\infty(H)} \quad \text{and}$$
$$w = 0 \quad \text{on } \partial H.$$

Repeating the argumentation in the proofs of Lemmas 9.8 and 9.9 in [30] reveals that the limit function w satisfies

$$\int_H w(-\Delta\psi)\, dx = \int_H w^p \psi\, dx$$

for all $\psi \in C_c^\infty(H)$. Therefore, classical regularity theory (Lemma A.7) ensures that $w \in C^2(H) \cap C(\overline{H})$ with

$$\begin{cases} -\Delta w = w^p & \text{in } H, \\ w = 0 & \text{on } \partial H. \end{cases}$$

Since $w(0', \delta) = 1$ and $p \in \left(1, \frac{n}{n-1}\right)$, this contradicts the Liouville Theorem 4.4.

⟨8′⟩ *Staying near the boundary - generalized orthant case*
We still investigate the situation $\delta_l \to \delta > 0$ and $P \in \partial\Omega$. This time no special case is considered as in the previous section and we only know that the domain Ω coincides in a neighbourhood of P with a generalized orthant. Up to an isometric transformation and a translation we may assume that $P = 0$ and there exists a radius $\varrho > 0$ such that

$$\begin{aligned} B_\varrho(P) \cap \mathbb{R}^{n,k} &\subset \Omega, \\ B_\varrho(P) \cap \partial\mathbb{R}^{n,k} &\subset \partial\Omega \text{ and} \\ B_\varrho(P) \setminus \mathbb{R}^{n,k} &\subset \mathbb{R}^n \setminus \Omega, \end{aligned} \tag{4.23}$$

where $\mathbb{R}^{n,k} = \{x \in \mathbb{R}^n : x_1, \ldots, x_k > 0\}$ for some $k \in \{1, \ldots, n\}$. In Section ⟨7′⟩ the relation $\lambda_l^{-1} P_{l,n} = \delta_l$ has been important. Inspired by that, we define

$$\delta_{l,j} \coloneqq \lambda_l^{-1} P_{l,j}$$

for $j \in \{1, \ldots, k\}$ and see similar to (4.16) that

$$\begin{aligned} \delta_{l,j} &= \lambda_l^{-1} P_{l,j} \\ &\geq \lambda_l^{-1} \operatorname{dist}(P_l, \partial_{h_l}\Omega_{h_l}) \\ &= \lambda_l^{-1} \min\{\|z - P_l\|_1 : z \in \partial_{h_l}\Omega_{h_l}\} = \delta_l \end{aligned}$$

for sufficiently large $l \in \mathbb{N}$ due to (4.23). Fixing any $j \in \{1, \ldots, k\}$ and recalling that $\delta_l \to \delta > 0$ as $l \to \infty$, there are two alternatives for $(\delta_{l,j})_{l\in\mathbb{N}}$: Either $\delta_{l,j} \to \infty$ or $\delta_{l,j} \to \delta_{\infty,j} \geq \delta > 0$ up to a renamed subsequence. With that, we assign

$$\kappa \coloneqq \#\big\{j \in \{1, \ldots, k\} : (\delta_{l,j})_{l\in\mathbb{N}} \text{ converges to } \delta_{\infty,j}\big\},$$

where # denotes the counting measure. The case $\kappa = 0$ entails $\delta_l \to \infty$ and was already treated in subsection "⟨6′⟩ *Drifting away*". Thus, we may assume $\kappa \geq 1$. In order to simplify the notation we make once more use of an isometric transformation such that without loss of generality the sequences $(\delta_{l,j})_{l\in\mathbb{N}}$ converge to $\delta_{\infty,j}$ for $j \in \{1, \ldots, \kappa\}$ and accordingly to ∞ for $j \in \{\kappa + 1, \ldots, k\}$.

Establishing the notation $x = (x', x'') \in \mathbb{R}^n$ with $x' \coloneqq (x_1, \ldots, x_\kappa) \in \mathbb{R}^\kappa$, $x'' \coloneqq (x_{\kappa+1}, \ldots, x_n) \in \mathbb{R}^{n-\kappa}$ we introduce the modified scaled functions $\omega_l : \overline{\mathfrak{D}^l_{\tau_l}} \to [0, \infty)$ given by

$$\omega_l(x) \coloneqq \frac{1}{M_l} u_l(\lambda_l x + (0', P_l'')) = v_l\left(x - \frac{1}{\lambda_l}(P_l', 0'')\right)$$

with the domain $\mathfrak{D}^l \coloneqq \frac{1}{\lambda_l}(\Omega - (0', P_l'')) = D^l + \frac{1}{\lambda_l}(P_l', 0'')$. Hence, due to (4.10) there is a constant $C > 0$ so that

$$\left\|D_i^+ \omega_l\right\|_{L^\infty(\partial_i^- \mathfrak{D}^l_{\tau_l} \cup \mathfrak{D}^l_{\tau_l})} \leq C \tag{4.24}$$

uniformly for all $l \in \mathbb{N}$ and $i \in \{1, \dots, n\}$. Moreover,

$$\begin{aligned} -\Delta_{\tau_l}\omega_l &= \omega_l^p \quad \text{in } \mathfrak{D}^l_{\tau_l}, \\ \|\omega_l\|_{L^\infty(\partial_{\tau_l}\mathfrak{D}^l_{\tau_l})} &= 0, \\ \|\omega_l\|_{L^\infty(\mathfrak{D}^l_{\tau_l})} &= \omega_l(\lambda_l^{-1}P_l', 0'') = \frac{1}{M_l}u_l(P_l) = 1 \end{aligned} \tag{4.25}$$

for all $l \in \mathbb{N}$ and

$$(\lambda_l^{-1}P_l', 0'') = (\delta_{l,1}, \dots, \delta_{l,\kappa}, 0'') \to (\delta_{\infty,1}, \dots, \delta_{\infty,\kappa}, 0'') \in \mathbb{R}^{n,\kappa} \tag{4.26}$$

as $l \to \infty$. According to (4.23) there exists a radius $\varrho > 0$ with $B_\varrho(0) \cap \mathbb{R}^{n,k} \subset \Omega$. Since $P_l \to 0$ we can therefore fix a constant $R > 0$ such that

$$\begin{aligned} B^l :=& (0,R)^\kappa \times (0, 2P_{l,\kappa+1}) \times \dots \times (0, 2P_{l,k}) \times (-R,R)^{n-k} \\ =& (0,R)^\kappa \times \prod_{j=\kappa+1}^{k} (0, 2P_{l,j}) \times \prod_{i=k+1}^{n} (-R,R) \\ \subset& B_\varrho(0) \cap \mathbb{R}^{n,k} \subset \Omega \end{aligned}$$

for sufficiently large $l \in \mathbb{N}$ and thus

$$\begin{aligned} \mathfrak{D}^l &= \frac{1}{\lambda_l}[\Omega - (0', P_l'')] \\ &\supset \frac{1}{\lambda_l}[B^l - (0', P_l'')] \\ &= \frac{1}{\lambda_l}\left[\left((0,R)^\kappa \times \prod_{j=\kappa+1}^{k}(0, 2P_{l,j}) \times \prod_{i=k+1}^{n}(-R,R)\right) - (0', P_{l,\kappa+1}, \dots, P_{l,n})\right] \\ &= \frac{1}{\lambda_l}\left[(0,R)^\kappa \times \prod_{j=\kappa+1}^{k}(-P_{l,j}, P_{l,j}) \times \prod_{i=k+1}^{n}(-R - P_{l,j}, R - P_{l,j})\right]. \end{aligned} \tag{4.27}$$

Due to $\lambda_l \to 0$, $P_l'' \to 0''$ and $\delta_{l,j} = \lambda_l^{-1}P_{l,j} \to \infty$ for $j \in \{\kappa+1, \dots, k\}$ this reveals

$$\mathfrak{D}^l \to \mathbb{R}^{n,\kappa} \tag{4.28}$$

and correspondingly

$$\partial\mathfrak{D}^l \to \partial\mathbb{R}^{n,\kappa} \tag{4.29}$$

as $l \to \infty$. In the case of $P \in \Omega$ the relation (4.8) was pivotal. In the sequel (4.28) and (4.29) are used as a substitute. Once more the distinction between $\tau_l \to \tau > 0$ and $\tau_l \to 0$ is convenient.

To begin with, the case of a discrete limit $\tau_l \to \tau > 0$ is considered. The relations (4.28) and (4.29) lead to

$$\begin{aligned}\mathfrak{D}^l_{\tau_l} &\to \{x \in \mathbb{R}^n_\tau \colon x_1, \dots, x_\kappa > 0\} = \mathbb{R}^{n,\kappa}_\tau,\\ \partial_{\tau_l}\mathfrak{D}^l_{\tau_l} &\to \partial_\tau \mathbb{R}^{n,\kappa}_\tau\end{aligned}$$

as $l \to \infty$. Similar to $\langle 5\rangle$ we construct a limit function $\omega_\tau \colon \overline{\mathbb{R}^{n,\kappa}_\tau} \to [0,1]$ with

$$\begin{aligned}\omega_\tau(\delta_{\infty,1}, \dots, \delta_{\infty,\kappa}, 0'') &:= 1 = \lim_{l\to\infty} \omega_{\tau_l}(\delta_{l,1}, \dots, \delta_{l,\kappa}, 0''),\\ \omega_\tau(\tau z) &:= 0 = \lim_{l\to\infty} \omega_{\tau_l}(\tau_l z) \quad \text{for all } z \in \partial\mathbb{R}^{n,\kappa} \cap \mathbb{Z}^n.\end{aligned} \tag{4.30}$$

As in $\langle 5\rangle$, passing to the limit $l \to \infty$ for the renamed diagonal sequence in (4.25) results in

$$-\Delta_\tau \omega_\tau(x) = \omega^p_\tau(x) \quad \text{for all } x \in \mathbb{R}^{n,\kappa}_\tau.$$

In view of (4.30) this contradicts the discrete Liouville Theorem 3.4.

In the case of $\tau_l \to 0$ the argumentation is like in $\langle 6\rangle$. Let $(R_l)_{l\in\mathbb{N}} \subset (0,\infty)$ be a increasing sequence with $R_l \to \infty$ for $l \to \infty$. Denoting $R^{(m)}_l := \left\lceil \frac{R_l}{\tau_m} \right\rceil \tau_m$ for $l, m \in \mathbb{N}$ yields $R^{(m)}_l \geq R_l$ and the sets $\mathcal{A}^{l,m} := \left(0, R^{(m)}_l\right)^\kappa \times \left(-R^{(m)}_l, R^{(m)}_l\right)^{n-\kappa}$ are admissible for τ_m. According to Definition 4.5, there exists a tensor product interpolant $\hat{\omega}_m \in C(\overline{\mathcal{A}^{1,m}}, [0,1])$ corresponding to ω_m for $m \in \mathbb{N}$ sufficiently large (cf. (4.27)). In view of [30, Lemma 8.11] the boundary condition $\omega_m = 0$ on $\overline{\mathcal{A}^{1,m}_{\tau_m}} \cap \partial_{\tau_m}\mathbb{R}^{n,\kappa}_{\tau_m} \subset \partial_{\tau_m}\mathfrak{D}^m_{\tau_m}$ ensures

$$\hat{\omega}_m = 0 \quad \text{on } \overline{\mathcal{A}^{1,m}} \cap \partial\mathbb{R}^{n,\kappa}. \tag{4.31}$$

We fix some $q > n$. By means of (4.24) and (4.25), Lemma 4.7 guarantees that

$$\|\hat{\omega}_m\|_{W^{1,q}(\mathcal{A}^{1,m})} \leq C$$

for a constant $C > 0$ and $m \in \mathbb{N}$ sufficiently large. Since $R^{(m)}_1 \geq R_1$ we can restrict $\hat{\omega}_m$ to $\mathcal{A}_1 := (0, R_1)^\kappa \times (-R_1, R_1)^{n-\kappa}$. As $q > n$ we can apply the compact embedding $W^{1,q}(\mathcal{A}_1) \hookrightarrow C^{0,\alpha}(\overline{\mathcal{A}_1})$ for some $\alpha \in (0, 1 - \frac{n}{q})$ ([1, Thm. 6.3]) and extract from $\left(\hat{\omega}_m|_{\overline{\mathcal{A}_1}}\right)_{m\in\mathbb{N}}$ a uniformly convergent subsequence with limit function $\omega \in C^{0,\alpha}(\overline{\mathcal{A}_1}) \cap W^{1,q}(\mathcal{A}_1)$. As in $\langle 6\rangle$ we construct a limit function $\omega \in C^{0,\alpha}_{\text{loc}}(\overline{\mathbb{R}^{n,\kappa}}) \cap W^{1,q}_{\text{loc}}(\mathbb{R}^{n,\kappa})$. In view of (4.25), (4.26) and (4.31) we see that

$$\begin{aligned}\omega(\delta_1, \dots, \delta_\kappa, 0'') = 1 = \|\omega\|_{L^\infty(\mathbb{R}^{n,\kappa})} \quad &\text{and}\\ \omega = 0 \quad \text{on } \partial\mathbb{R}^{n,\kappa}.&\end{aligned}$$

Similar to ⟨6⟩ we conclude that $\omega \in C^2(\mathbb{R}^{n,\kappa}) \cap C(\overline{\mathbb{R}^{n,\kappa}})$ and ω solves

$$\begin{cases} -\Delta\omega = \omega^p & \text{in } \mathbb{R}^{n,\kappa}, \\ \omega = 0 & \text{on } \partial\mathbb{R}^{n,\kappa}. \end{cases}$$

Since $\omega(\delta_1, \dots, \delta_\kappa, 0'') = 1$ and $p \in \left(1, \frac{n}{n-1}\right)$, a contradiction is reached by the Liouville Theorem 4.4. □

In the sequel, we still consider a bounded hypercube $\Omega = \prod_{i=1}^n (a_i, b_i) \subset \mathbb{R}^n$ with $n \geq 2$. Further, recall that the discrete Liouville exponent for orthants is given by $p_\star = \frac{n}{n-1}$. In the previous proof we used the scaling argument for nonlinearities u^p with $p < p_\star$. This scaling approach can be generalized to nonlinearities $f(u)$ with a continuous function $f\colon [0,\infty) \to [0,\infty)$ such that

(A1) $\lim_{y\to\infty} \frac{f(y)}{y^p} = \kappa > 0$ for some $p \in (1, p_\star)$ and

(A2) $f(0) = 0$.

Condition (A1) ensures that limit equation is the Emden equation, as before. Furthermore, condition (A2) allows the application of the Schwarz reflexion principle.

Theorem 4.8 (A priori bounds for hypercubes)
Let $\Omega = \prod_{i=1}^n (a_i, b_i) \subset \mathbb{R}^n$ and $f\colon [0,\infty) \to [0,\infty)$ be a continuous function satisfying (A1) and (A2). Then, there exists a constant $C > 0$ such that for every admissible grid size $h > 0$ and every solution $u_h\colon \overline{\Omega_h} \to [0,\infty)$ of

$$\begin{cases} -\Delta_h u = f(u) & \text{in } \Omega_h, \\ u = 0 & \text{on } \partial_h\Omega_h \end{cases} \tag{4.32$_h$}$$

the a priori estimate $\|u_h\|_{L^\infty(\Omega_h)} \leq C$ holds.

Proof. The following argumentation is similar to the proof of Theorem 4.3. Thus, we only explain the new ideas. Without loss of generality we may assume $\kappa = 1$. Otherwise, for any solution $u_h\colon \overline{\Omega_h} \to [0,\infty)$ of $(4.32)_h$, we consider the corresponding rescaled function $w_h\colon \overline{\Omega_h} \to [0,\infty)$, given by $w_h(x) := \kappa^{\frac{1}{p-1}} u(x)$. Then, w_h solves

$$\begin{cases} -\Delta_h w_h = \tilde{f}(w_h) & \text{in } \Omega_h, \\ w_h = 0 & \text{on } \partial_h\Omega_h \end{cases}$$

with rescaled nonlinearity $\tilde{f}(y) := \kappa^{\frac{1}{p-1}} f\left(\kappa^{\frac{-1}{p-1}} y\right)$ and $\lim_{y\to\infty} \frac{\tilde{f}(y)}{y^p} = 1$.

Assume for contradiction that there exists a sequence of grid sizes $(h_l)_{l\in\mathbb{N}} \subset (0,\infty)$, corresponding solutions $u_l := u_{h_l}\colon \overline{\Omega_{h_l}} \to [0,\infty)$ of $(4.32)_{h_l}$ and points $P_l \in \Omega_{h_l}$ such that

$$M_l := \max_{x\in\Omega_{h_l}} u_l(x) = u_l(P_l) \to \infty \quad (l \to \infty). \tag{4.33}$$

As $\overline{\Omega} \subset \mathbb{R}^n$ is compact, we may assume that $P_l \to P \in \overline{\Omega}$ for $l \to \infty$.

⟨1⟩ *Boundedness*
Let $h > 0$ and $u_h : \overline{\Omega_h} \to [0,\infty)$ be a non-negative solution of $(4.32)_h$ with $u_h \not\equiv 0$. The discrete maximum principle (Lemma 2.12) ensures $u_h > 0$ in Ω_h. Since

$$f\big(u_h(x)\big) = -\Delta_h u_h(x) = -\frac{1}{h^2}\sum_{i=1}^{n}(u_h(x+he_i) - 2u_h(x) + u_h(x-he_i)) \le \frac{2n}{h^2}u_h(x)$$

we deduce

$$h\left(\frac{f\big(u_h(x)\big)}{u_h(x)}\right)^{\frac{1}{2}} \le \sqrt{2n} \quad \text{for all } x \in \Omega_h. \tag{4.34}$$

Combining (A1), (4.33) and (4.34), we infer for large $l \in \mathbb{N}$

$$h_l u_l^{\frac{p-1}{2}}(P_l) \le 2h_l\left(\frac{f\big(u_l(P_l)\big)}{u_l(P_l)}\right)^{\frac{1}{2}} \le 2\sqrt{2n}. \tag{4.35}$$

In view of (4.33), we infer $h_l \to 0$ as $l \to \infty$.

⟨2⟩ *Scaling*
For $l \in \mathbb{N}$, we introduce the scaling parameter $\lambda_l := M_l^{\frac{1-p}{2}}$ and the rescaled function $v_l : \overline{D^l_{\tau_l}} \to [0,\infty)$ given by

$$v_l(x) := \frac{1}{M_l}u_l(\lambda_l x + P_l),$$

with new grid size $\tau_l := \frac{h_l}{\lambda_l} = M_l^{\frac{p-1}{2}}h_l$ and new domain $D^l := \frac{1}{\lambda_l}(\Omega - P_l)$. Thus, we have

$$\|v_l\|_{L^\infty(D^l_{\tau_l})} = v_l(0) = 1 \quad \text{and} \quad \|v_l\|_{L^\infty(\partial_{\tau_l}D^l_{\tau_l})} = 0. \tag{4.36}$$

Additionally, for all $x \in D^l_{\tau_l}$ we obtain the identity

$$-\Delta_{\tau_l}v_l(x) = \frac{-1}{M_l^p}\Delta_{h_l}u_l(\lambda_l x + P_l) = \frac{1}{M_l^p}f\big(u_l(\lambda_l x + P_l)\big) = \frac{1}{M_l^p}f\big(M_l v_l(x)\big). \tag{4.37}$$

⟨3⟩ *Alternatives for* $(\tau_l)_{l\in\mathbb{N}}$
For sufficiently large $l \in \mathbb{N}$, estimate (4.35) leads to

$$\tau_l = h_l M_l^{\frac{p-1}{2}} = h_l u_l^{\frac{p-1}{2}}(P_l) \le 2\sqrt{2n}.$$

Hence, $(\tau_l)_{l\in\mathbb{N}} \subset (0,\infty)$ is a bounded sequence and we may assume $\tau_l \to \tau$ as $l \to \infty$ with some limit $\tau \in [0,\infty)$.

Below, we separately discuss the two possibilities $P \in \Omega$ and $P \in \partial\Omega$. In both cases we consider the alternatives $\tau > 0$ (discrete limit) and $\tau = 0$ (continuous limit).

<u>Case 1:</u> $P \in \Omega$. In this situation we can deduce a contradiction with the aid of the two Liouville theorems on $\mathbb{R}^n_h$ and $\mathbb{R}^n$:

⟨4⟩ *Discrete limit*

Firstly, we investigate the case $\tau_l \to \tau > 0$. As in the proof of Theorem 4.3, we use a renamed diagonal sequence to construct a limit function $v_\tau \colon \mathbb{R}^n_\tau \to [0,1]$ with

$$v_\tau(\tau z) = \lim_{l\to\infty} v_l(\tau_l z) \quad \text{for all } z \in \mathbb{Z}^n.$$

Moreover, for all $z \in \mathbb{Z}^n$ with $M_l v_l(\tau_l z) \to \infty$ as $l \to \infty$, we infer from assumption (A1) that

$$\lim_{l\to\infty} \frac{1}{M_l^p} f\big(M_l v_l(\tau_l z)\big) = \lim_{l\to\infty} \frac{f\big(M_l v_l(\tau_l z)\big)}{\big(M_l v_l(\tau_l z)\big)^p} v_l^p(\tau_l z) = v_\tau^p(\tau z).$$

On the other hand, if $(M_l v_l(\tau_l z))_{l\in\mathbb{N}}$ is bounded, then $v_\tau(\tau z) = \lim_{l\to\infty} v_l(\tau_l z) = 0$ since $M_l \to \infty$ as $l \to \infty$. Thus, we deduce

$$\lim_{l\to\infty} \frac{1}{M_l^p} f\big(M_l v_l(\tau_l z)\big) = 0 = v_\tau^p(\tau z).$$

In summary, we have

$$\lim_{l\to\infty} \frac{1}{M_l^p} f\big(M_l v_l(\tau_l z)\big) = v_\tau^p(\tau z) \quad \text{for all } z \in \mathbb{Z}^n. \tag{4.38}$$

Due to (4.38), taking the limit in (4.37) yields

$$-\Delta_\tau v_\tau(x) = v_\tau^p(x) \quad \text{for all } x \in \mathbb{R}^n_\tau.$$

As $v_\tau(0) = \lim_{l\to\infty} v_l(0) = 1$, this contradicts the discrete Liouville Theorem 3.4.

⟨5⟩ *Continuous limit*

In the following, we consider the case $\tau_l \to 0$. Recall that $\|v_l\|_{L^\infty(\overline{D^l_{\tau_l}})} = 1$ for all $l \in \mathbb{N}$ by (4.36). Furthermore, we aim to show that there exists some constant $c > 0$ such that

$$\|\Delta_{\tau_l} v_l\|_{L^\infty(D^l_{\tau_l})} \le c \quad \text{for all } l \in \mathbb{N}. \tag{4.39}$$

Note that identity (4.37) leads to

$$\|\Delta_{\tau_l} v_l\|_{L^\infty(D^l_{\tau_l})} = \left\| \frac{1}{M_l^p} f\big(M_l v_l(\cdot)\big) \right\|_{L^\infty(D^l_{\tau_l})}.$$

In view of (A1), there is some $y_0 \in (0,\infty)$ such that

$$\left| \frac{f(y)}{y^p} - 1 \right| < \frac{1}{2} \quad \text{for all } y \geq y_0.$$

Hence, if $M_l v_l(x) \geq y_0$, we deduce

$$\left| \frac{1}{M_l^p} f\big(M_l v_l(x)\big) \right| = \left| \frac{f\big(M_l v_l(x)\big)}{(M_l v_l(x))^p} v_l^p(x) \right| \leq \frac{3}{2} \left| v_l^p(x) \right| \leq \frac{3}{2}.$$

Otherwise, if $M_l v_l(x) < y_0$, we estimate

$$\left| \frac{1}{M_l^p} f\big(M_l v_l(x)\big) \right| \leq \frac{1}{\min_l M_l^p} \max_{y \in [0,y_0]} |f(y)|.$$

Altogether, (4.39) holds true. Corresponding to the grid function $v_l \colon \overline{D^l_{\tau_l}} \to [0,\infty)$, there exists the tensor product interpolant $\hat{v}_l \in C(\overline{D^l}, [0,1])$ from Definition 4.5. Using (4.36) and (4.39), we construct a limit function $v \in C^{0,\alpha}_{\text{loc}}(\mathbb{R}^n) \cap W^{1,q}_{\text{loc}}(\mathbb{R}^n)$ with $v(0) = 1 = \|v\|_{L^\infty(\mathbb{R}^n)}$ and

$$\hat{v}_l \to v \quad (l \to \infty) \tag{4.40}$$

uniformly on compact subsets of $\mathbb{R}^n$, similar to $\langle 6 \rangle$ in the proof of Theorem 4.3. For all $\psi \in C^\infty_c(\mathbb{R}^n)$, the proof of Lemma 9.8 in [30] shows that

$$\sum_{x \in D^l_{\tau_l}} v_l(x)(-\Delta_{\tau_l}\psi(x))\tau_l^n \to \int_{\mathbb{R}^n} v\,(-\Delta\psi)\,dx \tag{4.41}$$

as $l \to \infty$. Next, we verify that

$$\frac{1}{M_l^p} f\big(M_l \hat{v}_l\big) \to v^p$$

uniformly on compact sets as $l \to \infty$. To this end, we define $g \colon [0,\infty) \to \mathbb{R}$ by $g(y) \coloneqq f(y) - y^p$. Due to (A1), $g \in \mathrm{o}(y^p)$ as $y \to \infty$, and according to Lemma A.9 we have

$$\frac{g(yt)}{y^p} \to 0 \quad (y \to \infty)$$

uniformly for all $t \in [0,1]$.

Together with (4.40), this reveals

$$\frac{f\big(M_l \hat{v}_l(x)\big)}{M_l^p} = \hat{v}_l^p(x) + \frac{g\big(M_l \hat{v}_l(x)\big)}{M_l^p} \to v^p(x)$$

uniformly on compact sets as $l \to \infty$. Thus, we obtain for every $\psi \in C_c^\infty(\mathbb{R}^n)$

$$\begin{aligned}
&\sum_{x \in D_{\tau_l}^l} \frac{1}{M_l^p} f\big(M_l v_l(x)\big)\psi(x)\tau_l^n \\
&= \sum_{x \in D_{\tau_l}^l} \left[\frac{1}{M_l^p} f\big(M_l \hat{v}_l(x)\big) - v^p(x)\right]\psi(x)\tau_l^n + \sum_{x \in D_{\tau_l}^l} v^p(x)\psi(x)\tau_l^n \\
&\to \int_{\mathbb{R}^n} v^p \psi \, dx \qquad (l \to \infty).
\end{aligned} \tag{4.42}$$

Moreover, identity (4.37) and partial summation (Lemma 2.15) yield for sufficiently large $l \in \mathbb{N}$ that

$$\sum_{x \in D_{\tau_l}^l} v_l(x)(-\Delta_{\tau_l}\psi(x))\tau_l^n = \sum_{x \in D_{\tau_l}^l} \frac{1}{M_l^p} f\big(M_l v_l(x)\big)\psi(x)\tau_l^n.$$

Taking the limit $l \to \infty$, we see by means of (4.41) and (4.42) that the limit function v satisfies

$$\int_{\mathbb{R}^n} v\,(-\Delta\psi)\,dx = \int_{\mathbb{R}^n} v^p \psi \, dx \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}^n).$$

Hence, regularity theory (see Lemma A.7) ensures that $v \in C^2(\mathbb{R}^n)$ and solves

$$-\Delta v = v^p \quad \text{in } \mathbb{R}^n.$$

Since $v(0) = 1$, this contradicts Theorem 4.4.

Case 2: $P \in \partial\Omega$.

$\langle 4' \rangle$ *Bounded discrete gradients*
Recall that $\|v_l\|_{L^\infty(\overline{D_{\tau_l}^l})} = 1$ for all $l \in \mathbb{N}$ by (4.36). Using the same arguments as in the case $P \in \Omega$ ((4.39) is still valid here), we obtain some constant $c > 0$ such that

$$\|\Delta_{\tau_l} v_l\|_{L^\infty(D_{\tau_l}^l)} \le c \quad \text{for all } l \in \mathbb{N}.$$

Hence, using the discrete Schwarz reflection principle (cf. Proposition A.5) and applying subsequently Theorem 5.31 from [30] yields a uniform constant $C > 0$

with

$$\left\| D_i^+ v_l \right\|_{L^\infty(\partial_i^- D_{\tau_l}^l \cup D_{\tau_l}^l)} \leq C \tag{4.43}$$

for all $l \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$.

The subsections "$\langle 5' \rangle$ *Distance to the boundary*" and "$\langle 6' \rangle$ *Drifting away*" can be transferred literally from the proof of Theorem 4.3.

$\langle 7' \rangle$ *Staying near the boundary - generalized orthant case*
The argumentation is similar to subsection $\langle 8' \rangle$ in the proof of Theorem 4.3. We only point out the two significant changes: In case of a discrete limit, we proceed as in $\langle 4 \rangle$ in the proof of Theorem 4.3 and use assumption (A1) to ensure that the limit equation is the discrete Emden equation. If the limit is continuous, we argue like in $\langle 5 \rangle$ in the proof of Theorem 4.3 to show by means of (A1) that the limit equation is the continuous Emden equation. The rest of $\langle 8' \rangle$ in the proof of Theorem 4.3 can be transferred mutatis mutandis. □

4.2. More general convex domains

The statement of Theorem 4.3 was already proven by McKenna, Reichel and Verbitzky in [24]. They used a comparison argument which requires knowledge of the principle Dirichlet eigenfunction of $-\Delta_h$ on Ω_h. On the one hand, if $\Omega = \prod_{i=1}^n (a_i, b_i) \subset \mathbb{R}^n$ is a hypercube the corresponding principle eigenfunction is known and the comparison approach gives explicit a priori bounds as a function of Ω. On the other hand, it is not clear how to compute the first eigenfunction if Ω is not a hypercube.

The advantage of our scaling method is that it can be applied for a class of convex, admissible domains and not only for hypercubes. As a prototype domain we consider right-angled isosceles triangles and give the details in the subsequent theorem. Before, we have a closer look at the proof of Theorem 4.3 and point out the crucial steps of the scaling approach we applied:

(a) It has to be guaranteed that after scaling the maximizer does not tend to the boundary of the rescaled domain, i.e., $(\delta_l)_{l \in \mathbb{N}}$ has to be bounded away from zero as it was done in section $\langle 5' \rangle$. For hypercubes we could achieve this by uniformly bounding the discrete gradients with respect to the L^∞-norm by means of the discrete Schwarz reflection principle. Thus, this strategy is possible for all domains Ω which allow the application of the reflection principle.

At this point, we suppose that the scaling approach can be employed for more general convex domains. The key idea is to universalize Theorem 7.9 from [30] in the following way:

Conjecture 4.9 (Discrete L^p-estimates)
Let $\Omega \subset \mathbb{R}^n$ be a bounded, convex domain with admissible grid size $h > 0$ and

$q \in (1, \infty)$. Further, let $g\colon \Omega_h \to \mathbb{R}$ be given and $u\colon \overline{\Omega_h} \to \mathbb{R}$ be the solution of

$$\begin{cases} -\Delta_h u = g & \text{in } \Omega_h, \\ \quad\;\; u = 0 & \text{on } \partial_h \Omega_h. \end{cases}$$

Then, there exists a constant $C > 0$ which depends only on n, q and $|\Omega|$ such that

$$\|u\|_{\dot{W}_0^{1,q}(\Omega_h)} \leq C \|g\|_{L^q(\Omega_h)}.$$

If this were true, then this result for $q > n$, multiplication with a suitable cut-off function together with Lemma A.8 would ensure that $(\delta_l)_{l \in \mathbb{N}}$ stays away from zero without using the Schwarz reflection principle.

Conjecture 4.9 is a discrete analogue of a special case of Theorem 1 in Alkhutov and Kondratiev ([3]) and was verified by Verbitzky in [30] for hypercubes. Turning to more general convex domains most of the proofs in the discrete case can be transferred mutatis mutandis but the crucial step, the estimate for the *obstacle function* in [30, Lem. 6.6], remains an open problem.

(b) The construction of the interpolants demands evaluations of functions at each vertex of every reference box (cf. Chapter 8 in [30]). For hypercubes everything works fine but e.g. for right-angled isosceles triangles there are missing values along boundary segments which are diagonal to the coordinate axes. If so, we prescribe the missing values by odd reflexion.

(c) The contradiction is always achieved by means of classical or discrete Liouville theorems. Therefore, we have to make sure that for all limit domains the corresponding Liouville statements are available. For hypercubes these limit domains are the generalized orthants and thus, Theorem 3.4 can be applied if the limit is discrete and accordingly Theorem 4.4 if the limit is continuous.

Exemplarily, for a more general bounded, convex domain which permits our scaling argument for a priori bounds we consider in the sequel a right-angled isosceles 2d-triangle.

Theorem 4.10 (A priori bounds for triangles)
Let $b > 0$, $\Omega := \{x \in \mathbb{R}^2\colon x_1, x_2 > 0,\ x_1 + x_2 < b\}$ and $1 < p < p_\bullet := \frac{3}{2}$. Then, there exists a constant $C > 0$ such that for every admissible grid size $h > 0$ and every non-negative solution $u_h\colon \overline{\Omega_h} \to [0, \infty)$ of

$$\begin{cases} -\Delta_h u = u^p & \text{in } \Omega_h, \\ \quad\;\; u = 0 & \text{on } \partial_h \Omega_h \end{cases} \tag{4.44$_h$}$$

the a priori estimate $\|u_h\|_{L^\infty(\Omega_h)} \leq C$ holds.

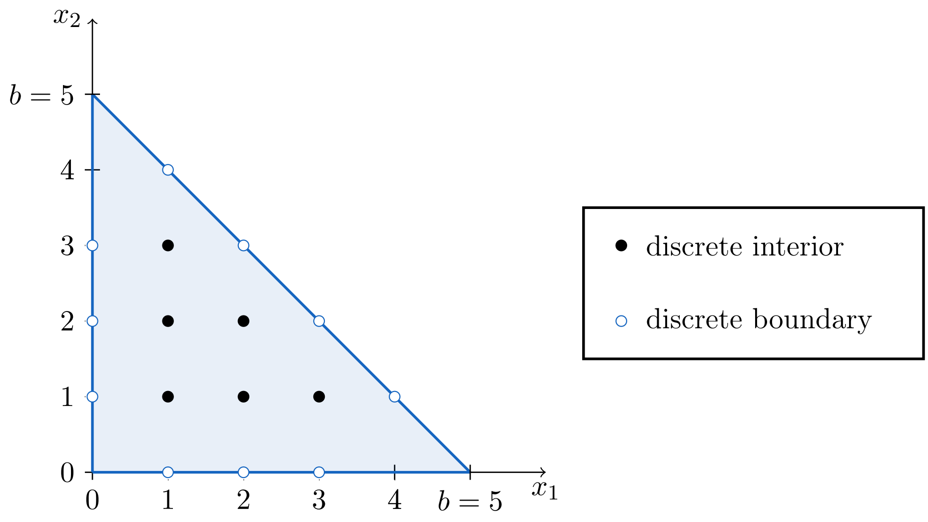


Figure 4.1: Discrete interior and boundary for triangle $\Omega = \{x_1, x_2 > 0,\ x_1 + x_2 < b\}$ with $b = 5$ and $h = 1$.

Remark 4.11 (Liouville theorems and exponents for cones)

(a) According to Theorem 3.14 the discrete Liouville exponent for cones

$$\Omega^m := \left\{(x_1, x_2)^T = (r\cos\varphi, r\sin\varphi)^T \in \mathbb{R}^2 \colon r > 0, \varphi \in \left(0, \frac{\pi}{4}m\right)\right\}$$

is $\frac{m+2}{2}$ for all $m \in \{1, \ldots, 8\}$ and grid sizes $h > 0$. Further, in the case of a triangle $\Omega = \{x \in \mathbb{R}^2 \colon x_1, x_2 > 0,\ x_1 + x_2 < b\}$ the smallest internal angle is $\frac{\pi}{4}$ and thus, the smallest Liouville exponent is attained for $m = 1$ and equals $\frac{3}{2}$.

(b) For all $m \in \{1, \ldots, 8\}$ as well as $1 < p < \frac{m+2}{2}$ the only non-negative solution $u \in C^2(\Omega^m) \cap C(\overline{\Omega^m})$ of

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega^m, \\ u = 0 & \text{on } \partial\Omega^m \end{cases}$$

is the zero solution.

Proof of (b). This Liouville-type result follows as in the proof of Theorem 4.4: The first eigenfunction of $-\Delta_{\mathbb{S}^1}$ on $\omega := \Omega^m \cap \mathbb{S}^1$ is given by $\tilde{\psi} \colon \varphi \mapsto \sin\left(\frac{4}{m}\varphi\right)$ and $\tilde{\lambda}_1 = \left(\frac{4}{m}\right)^2$ is the corresponding eigenvalue. This directly yields $p_{\mathrm{BT}} = \frac{m+2}{2}$. □

Proof of Theorem 4.10. The argumentation is similar to the proof of Theorem 4.3 for hypercubes. Thus, we only recall the main steps and notation. New aspects are illuminated in detail whereas parts which do not change substantially are only mentioned.

Again, we *assume* for contradiction that there exists a sequence of admissible grid sizes

$(h_l)_{l\in\mathbb{N}} \subset (0,\infty)$ and solutions $u_l := u_{h_l} : \overline{\Omega_{h_l}} \to [0,\infty)$ of $(4.44)_{h_l}$ such that

$$\|u_l\|_{L^\infty(\Omega_{h_l})} \to \infty \quad (l \to \infty). \tag{4.45}$$

Further, there exist points $P_l \in \Omega_{h_l}$ with

$$M_l := \max_{x\in\Omega_{h_l}} u_l(x) = u_l(P_l) \to \infty \quad (l \to \infty)$$

and after passing to a subsequence we may assume that $P_l \to P \in \overline{\Omega}$ for $l \to \infty$ thanks to the compactness of $\overline{\Omega}$.

⟨1⟩ *Boundedness*
For $h > 0$ and $u_h : \overline{\Omega_h} \to [0,\infty)$ a solution of $(4.44)_h$, we have the bound

$$hu_h^{\frac{p-1}{2}}(x) \le \sqrt{2n} = 2 \quad \text{for all } x \in \Omega_h. \tag{4.46}$$

Thus, together with (4.45) we see that $h_l \to 0$ for $l \to \infty$.

⟨2⟩ *Scaling*
For $l \in \mathbb{N}$, we introduce the scaling parameter $\lambda_l := M_l^{\frac{1-p}{2}}$. Then, the function $v_l : \overline{D^l_{\tau_l}} \to [0,\infty)$ is given by

$$v_l(x) := \frac{1}{M_l} u_l(\lambda_l x + P_l)$$

with the new grid size $\tau_l := \frac{h_l}{\lambda_l} = M_l^{\frac{p-1}{2}} h_l$ and domain $D^l := \frac{1}{\lambda_l}(\Omega - P_l)$. This entails

$$\begin{aligned} \|v_l\|_{L^\infty(D^l_{\tau_l})} &= v_l(0) = 1, \\ \|v_l\|_{L^\infty(\partial_{\tau_l} D^l_{\tau_l})} &= 0, \\ -\Delta_{\tau_l} v_l(x) &= v_l^p(x) \quad \text{for all } x \in D^l_{\tau_l}. \end{aligned} \tag{4.47}$$

⟨3⟩ *Alternatives for* $(\tau_l)_{l\in\mathbb{N}}$
Since $\tau_l \le 2$ by (4.46), the following two alternatives can occur: Either $\tau_l \to 0$ or, up to a subsequence, $\tau_l \to \tau > 0$ for $l \to \infty$.

From now on, we separately discuss the two possibilities $P \in \Omega$ and $P \in \partial\Omega$.

Case 1: $P \in \Omega$. Then we can find a contradiction with the aid of the two Liouville theorems on $\mathbb{R}^2_h$ and $\mathbb{R}^2$. The details can be transferred almost literally from the proof of Theorem 4.3 for $n \ge 3$.

Case 2: $P \in \partial\Omega$.

⟨4′⟩ *Bounded discrete gradients*
Recall that $\|v_l\|_{L^\infty(\overline{D^l_{\tau_l}})} = 1$ and $\|\Delta_{\tau_l} v_l\|_{L^\infty(D^l_{\tau_l})} = 1$ for all $l \in \mathbb{N}$. Applying first

the discrete Schwarz reflection principle as carried out in Propositions A.6 and A.5 and subsequently Theorem 5.31 from [30] yields a uniform constant $C > 0$ with

$$\left\| D_i^+ v_l \right\|_{L^\infty(\partial_i^- D_{\tau_l}^l \cup D_{\tau_l}^l)} \leq C \tag{4.48}$$

for all $l \in \mathbb{N}$ and $i \in \{1, 2\}$.

⟨5′⟩ *Distance to the boundary*
For all $l \in \mathbb{N}$, (4.48) leads to

$$\delta_l := \min \left\{ \|y\|_1 : y \in \partial_{\tau_l} D_{\tau_l}^l \right\} \geq \frac{1}{C} > 0.$$

Thus, these two alternatives may occur as $l \to \infty$: Either $\delta_l \to \infty$ or, up to a subsequence, $\delta_l \to \delta > 0$.

⟨6′⟩ *Drifting away*
Once again, the situation $\delta_l \to \infty$ can be treated as the inner point case $P \in \Omega$.

⟨7′⟩ *Staying near the boundary*
Next, we analyse the situation $\delta_l \to \delta > 0$ when $P \in \partial\Omega$. There are four possibilities:

(i) $P = (0, 0)^T$: Here Ω coincides in a neighbourhood of P with an orthant. This setting was already discussed in the proof of Theorem 4.3 (Subitem ⟨8′⟩).

(ii) $P \in \{0\} \times (0, b)$ or $P \in (0, b) \times \{0\}$: That is just the local half space case also analysed in Theorem 4.3 (Subitem ⟨7′⟩).

(iii) $P = (b, 0)^T$ or $P = (0, b)^T$: This time, Ω coincides locally around P with an infinite cone. Thus, up to an isometric transformation and a translation we may assume that $P = (0, 0)^T \in \mathbb{R}^2$ and $\Omega = \{x \in \mathbb{R}^2 : 0 < x_2 < x_1 < b\}$.

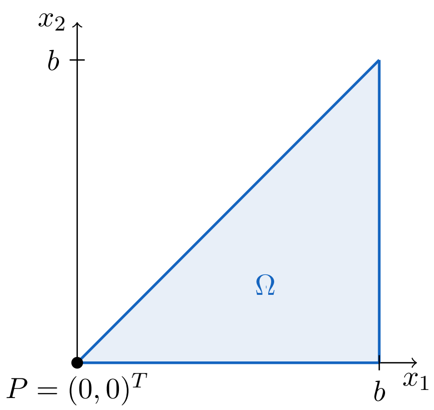


Figure 4.2: Situation after the transformation.

So, given the cone $\mathcal{S}$ with

$$\begin{aligned}\mathcal{S} :=& \{x \in \mathbb{R}^2 \colon 0 < x_2 < x_1\} \\ =& \left\{(x_1, x_2)^T = (r\cos\varphi, r\sin\varphi)^T \in \mathbb{R}^2 \colon r > 0,\ \varphi \in \left(0, \frac{\pi}{4}\right)\right\},\end{aligned}$$

there exists a radius $\varrho > 0$ such that

$$\begin{aligned} B_\varrho(P) \cap \mathcal{S} &\subset \Omega, \\ B_\varrho(P) \cap \partial\mathcal{S} &\subset \partial\Omega \text{ and} \\ B_\varrho(P) \setminus \mathcal{S} &\subset \mathbb{R}^n \setminus \Omega. \end{aligned} \tag{4.49}$$

Analogously to the local orthant case, we define

$$\delta_{l,1} := \lambda_l^{-1} P_{l,2} \quad \text{and} \quad \delta_{l,2} := \lambda_l^{-1}(P_{l,1} - P_{l,2})$$

in such a way that the relations

$$\begin{aligned} \delta_{l,1} =& \lambda_l^{-1} \operatorname{dist}(P_l, \partial_{h_l}\Omega_{h_l} \cap \{x_2 = 0\}) \quad \text{and} \\ \delta_{l,2} =& \lambda_l^{-1} \operatorname{dist}(P_l, \partial_{h_l}\Omega_{h_l} \cap \{x_1 = x_2\}) \end{aligned} \tag{4.50}$$

hold (see Figure 4.3).

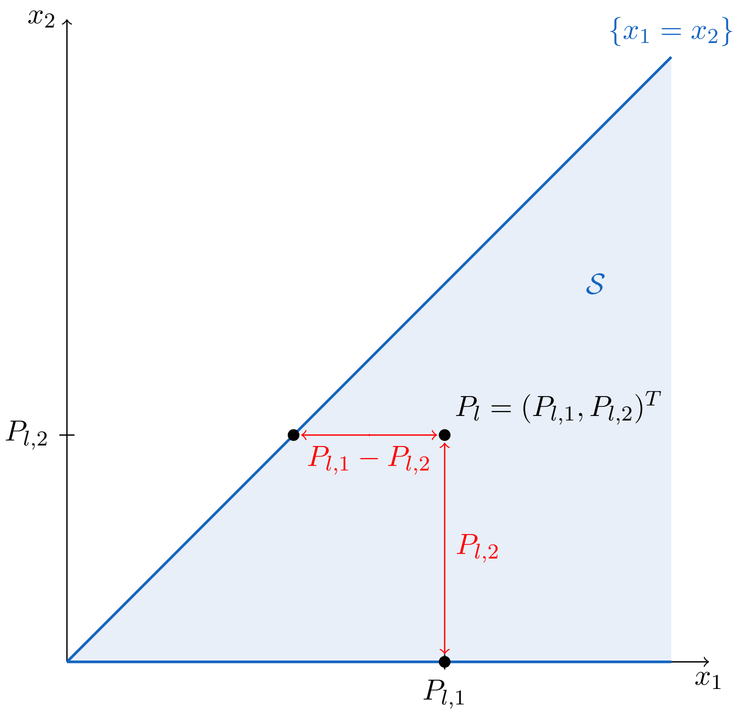


Figure 4.3: Illustration of the distance between P_l and the boundary $\partial\mathcal{S}$.

Consequently, we have

$$\delta_{l,j} \geq \lambda_l^{-1} \operatorname{dist}(P_l, \partial_{h_l}\Omega_{h_l}) = \lambda_l^{-1} \min\{\|z - P_l\|_1 : z \in \partial_{h_l}\Omega_{h_l}\} = \delta_l,$$

similar to (4.16). Since $\delta_l \to \delta > 0$ as $l \to \infty$, for both sequences $(\delta_{l,j})_{l\in\mathbb{N}}$ there are two alternatives: Either $\delta_{l,j} \to \infty$ or $\delta_{l,j} \to \delta_{\infty,j} \geq \delta > 0$ up to a subsequence. Generally speaking, the *limit domain* $\mathcal{L}$ *sees* the boundary $\{x_2 = 0\}$ if and only if $(\delta_{l,1})_{l\in\mathbb{N}}$ is bounded and respectively $\{x_1 = x_2\}$ is *seen* if and only if $(\delta_{l,2})_{l\in\mathbb{N}}$ is bounded. More precisely, the following four alternatives can occur. Note that the contradiction is always achieved with the aid of the Liouville theorems on $\mathcal{L}$ respectively $\mathcal{L}_h$ (cf. Theorem 4.3).

1.) If the sequences $(\delta_{l,1})_{l\in\mathbb{N}}$ and $(\delta_{l,2})_{l\in\mathbb{N}}$ are both unbounded, then $\delta_l \to \infty$ in view of (4.50). Here the limit domain is $\mathbb{R}^2$ and this has already been considered in "$\langle 6' \rangle$ *Drifting away*".

2.) In the case that $(\delta_{l,1})_{l\in\mathbb{N}}$ is bounded and $(\delta_{l,2})_{l\in\mathbb{N}}$ is unbounded, the limit domain $\mathcal{L}$ is the halfspace $\{x_2 > 0\}$ (cf. $\langle 7' \rangle$ in the proof of Theorem 4.3).

3.) The situation $(\delta_{l,1})_{l\in\mathbb{N}}$ is unbounded and $(\delta_{l,2})_{l\in\mathbb{N}}$ is bounded can be treated mutatis mutandis and here the limit domain is the halfspace $\{x_1 < x_2\}$. For specifics we refer to Subitem "(iv) $P \in \{x \in \mathbb{R}^2 : x_1, x_2 > 0,\, x_2 = b - x_1\}$" which is carried out later.

4.) Now let $(\delta_{l,1})_{l\in\mathbb{N}}$ and $(\delta_{l,2})_{l\in\mathbb{N}}$ be both bounded. For this alternative we explain the details in the sequel. Initially, we introduce the modified scaled functions $\omega_l : \overline{\mathfrak{D}^l_{\tau_l}} \to [0, \infty)$ given by

$$\omega_l(x) := \frac{1}{M_l} u_l(\lambda_l x) = v_l\left(x - \frac{1}{\lambda_l} P_l\right)$$

with the domain

$$\mathfrak{D}^l := \frac{1}{\lambda_l}\Omega = D^l + \frac{1}{\lambda_l} P_l = \left\{x \in \mathbb{R}^2 : 0 < x_2 < x_1 < \frac{b}{\lambda_l}\right\}. \tag{4.51}$$

Hence, due to (4.48) there is a constant $C > 0$ so that

$$\left\|D_i^+ \omega_l\right\|_{L^\infty(\partial_i^- \mathfrak{D}^l_{\tau_l} \cup \mathfrak{D}^l_{\tau_l})} \leq C \tag{4.52}$$

uniformly for all $l \in \mathbb{N}$ and $i \in \{1, 2\}$. Moreover,

$$\begin{aligned} -\Delta_{\tau_l}\omega_l &= \omega_l^p \quad \text{in } \mathfrak{D}^l_{\tau_l}, \\ \|\omega_l\|_{L^\infty(\partial_{\tau_l}\mathfrak{D}^l_{\tau_l})} &= 0, \\ \|\omega_l\|_{L^\infty(\mathfrak{D}^l_{\tau_l})} &= \omega_l(\lambda_l^{-1} P_l) = \frac{1}{M_l} u_l(P_l) = 1 \end{aligned} \tag{4.53}$$

for all $l \in \mathbb{N}$ and

$$\lambda_l^{-1} P_l = (\delta_{l,1} + \delta_{l,2}, \delta_{l,1}) \to (\delta_{\infty,1} + \delta_{\infty,2}, \delta_{\infty,1}) \in \mathcal{S} \tag{4.54}$$

as $l \to \infty$. Further, from (4.51) and $\lambda_l \to 0$ it follows that

$$\mathfrak{D}^l \to \mathcal{S} \tag{4.55}$$

as well as

$$\partial \mathfrak{D}^l \to \partial \mathcal{S} \tag{4.56}$$

for $l \to \infty$. Therefore, the *limit domain* $\mathcal{L}$ is the cone $\mathcal{S}$. Once more, the distinction between $\tau_l \to \tau > 0$ and $\tau_l \to 0$ is convenient.

To begin with, the discrete limit case $\tau_l \to \tau > 0$ is considered. The coherencies (4.55) and (4.56) lead to

$$\begin{aligned} \mathfrak{D}^l_{\tau_l} &\to \mathcal{S}_\tau, \\ \partial_{\tau_l} \mathfrak{D}^l_{\tau_l} &\to \partial_\tau \mathcal{S}_\tau \end{aligned}$$

for $l \to \infty$. As in the proof of Theorem 4.3 subsection $\langle 6 \rangle$, we construct a discrete limit function $\omega_\tau \colon \overline{\mathcal{S}_\tau} \to [0,1]$ with

$$\begin{aligned} \omega_\tau(\delta_{\infty,1} + \delta_{\infty,2}, \delta_{\infty,1}) &:= 1 = \lim_{l \to \infty} \omega_{\tau_l}(\delta_{l,1} + \delta_{l,2}, \delta_{l,1}), \\ \omega_\tau(\tau z) &:= 0 = \lim_{l \to \infty} \omega_{\tau_l}(\tau_l z) \quad \text{for all } z \in \partial \mathcal{S} \cap \mathbb{Z}^2. \end{aligned} \tag{4.57}$$

Moreover, taking the limit $l \to \infty$ in (4.53) entails

$$-\Delta_\tau \omega_\tau(x) = \omega_\tau^p(x) \quad \text{for all } x \in \mathcal{S}_\tau.$$

In light of (4.57) this contradicts the discrete Liouville Theorem 3.14.

Next, the continuous limit case $\tau_l \to 0$ is investigated. Let $(R_l)_{l \in \mathbb{N}} \subset (0, \infty)$ a non-decreasing sequence with $R_l \to \infty$ for $l \to \infty$. Assigning $R_l^{(m)} := \left\lceil \frac{R_l}{\tau_m} \right\rceil \tau_m$ for $l, m \in \mathbb{N}$ yields $R_l^{(m)} \geq R_l$ and the sets $\mathcal{A}^{l,m} := \left(0, R_l^{(m)}\right)^2$ are admissible for τ_m. Due to (4.51), we have $\mathfrak{D}^m = \left\{0 < x_2 < x_1 < \frac{b}{\lambda_m}\right\}$. Thus, we can extend ω_m by the discrete Schwarz reflection principle (cf. Prop. A.6) to a function $\omega_m \colon \hat{\mathcal{M}}^m_{\tau_m} \to [0,1]$ with $\mathcal{M}^m := \left(0, \frac{b}{\lambda_m}\right)^2$ and $\omega_m = 0$ on $\hat{\partial}_{\tau_m} \hat{\mathcal{M}}^m_{\tau_m}$. Therefore, for m large enough, $\omega_m \colon \hat{\mathcal{A}}^{1,m}_{\tau_m} \to [0,1]$ is well-defined and in view of Definition 4.5 there exists the corresponding tensor product interpolant $\hat{\omega}_m \in C(\overline{\mathcal{A}^{1,m}}, [0,1])$. According to [30, Lemma 8.11] the discrete boundary condition $\omega_m = 0$ on $\hat{\mathcal{A}}^{1,m}_{\tau_m} \cap \partial\mathbb{R}^{2,2}$ ensures

$$\hat{\omega}_m = 0 \quad \text{on } \overline{\mathcal{A}^{1,m}} \cap \partial\mathbb{R}^{2,2}.$$

Since ω_m is extended by odd reflection and $\omega_m = 0$ on $\hat{\mathcal{A}}^{1,m}_{\tau_m} \cap \{x_1 = x_2\}$, the interpolant inherits the non-negativity $\hat{\omega}_m \geq 0$ in $\mathcal{A}^{1,m} \cap \mathcal{S}$ and the boundary values

$$\hat{\omega}_m = 0 \quad \text{on } \overline{\mathcal{A}^{1,m}} \cap \{x_1 = x_2\}$$

and we can summarize the boundary conditions of interest by

$$\hat{\omega}_m = 0 \quad \text{on } \overline{\mathcal{A}^{1,m}} \cap \partial\mathcal{S}. \tag{4.58}$$

Let $q > n$ be fixed. Using (4.52) and (4.53), we obtain with Lemma 4.7 that

$$\|\hat{\omega}_m\|_{W^{1,q}(\mathcal{A}^{1,m})} \leq C$$

for a constant $C > 0$ and $m \in \mathbb{N}$ sufficiently large. Since $R_1^{(m)} \geq R_1$ we may restrict $\hat{\omega}_m$ to $\mathcal{A}_1 := (0, R_1)^2$ for large enough $m \in \mathbb{N}$. As $q > n$ we can apply the compact embedding $W^{1,q}(\mathcal{A}_1) \hookrightarrow C^{0,\alpha}(\overline{\mathcal{A}_1})$ for some $\alpha \in (0, 1 - \frac{n}{q})$ (see [1, Thm. 6.3]) and extract from $\left(\hat{\omega}_m|_{\overline{\mathcal{A}_1}}\right)_{m\in\mathbb{N}}$ a uniformly convergent subsequence with limit $\omega \in C^{0,\alpha}(\overline{\mathcal{A}_1}) \cap W^{1,q}(\mathcal{A}_1)$. Similar to the proof of Theorem 4.3 subsection $\langle 6 \rangle$, we obtain a limit function $\omega \in C^{0,\alpha}_{\text{loc}}(\overline{\mathbb{R}^{2,2}}) \cap W^{1,q}_{\text{loc}}(\mathbb{R}^{2,2})$, which can then be restricted on $\overline{\mathcal{S}} \subset \overline{\mathbb{R}^{2,2}} = \{x_1, x_2 \geq 0\}$. In view of (4.53), (4.54) and (4.58) we deduce that

$$\begin{aligned} \omega(\delta_1 + \delta_2, \delta_1) = 1 = \|\omega\|_{L^\infty(\mathcal{S})} \quad &\text{and} \\ \omega = 0 \quad \text{on } \partial\mathcal{S}.& \end{aligned}$$

As in the proof of Theorem 4.3 subitem $\langle 7' \rangle$, we see that $\omega \in C^2(\mathcal{S}) \cap C(\overline{\mathcal{S}})$ and ω solves

$$\begin{cases} -\Delta\omega = \omega^p & \text{in } \mathcal{S}, \\ \omega = 0 & \text{on } \partial\mathcal{S}. \end{cases}$$

Since $\omega(\delta_1 + \delta_2, \delta_1) = 1$ and $p \in \left(1, \frac{3}{2}\right)$ this contradicts Remark 4.11.

(iv) $P \in \{x \in \mathbb{R}^2 \colon x_1, x_2 > 0,\, x_2 = b - x_1\}$. This alternative can be handled as the previous one. We only want to point out two crucial steps:

1.) The discrete limit leads to the Liouville theorem on the half space $\{x \in \mathbb{R}^2_h \colon 0 < x_2 < x_1\}$ which can be verified similar to Theorem 3.14.

2.) In the continuous limit case it is suitable to apply the discrete Schwarz reflection with respect to the straight line $\{x_1 = x_2\}$ (cf. Prop. A.6) before introducing the tensor product interpolants. □

Finally, we point out that we can extend the a priori result for triangles of the form $\Omega := \{x \in \mathbb{R}^2 : x_1, x_2 > 0,\ x_1 + x_2 < b\}$ by replacing the nonlinearity u^p with the more general $f(u)$. In the previous proof we applied the scaling argument for nonlinearities u^p with $1 < p < p_\bullet := \frac{3}{2}$, where $p_\bullet$ is the smallest Liouville exponent which occurs in the limit (cf. Remark 4.11). For the general nonlinearity $f(u)$ we consider as in Section 4.1 continuous functions $f\colon [0,\infty) \to [0,\infty)$ such that

(A1) $\lim\limits_{y\to\infty} \frac{f(y)}{y^p} = \kappa > 0$ for some $p \in (1, p_\bullet)$ and

(A2) $f(0) = 0$.

Combining the methods in the proofs of Theorem 4.10 and 4.8, we obtain the following result:

Theorem 4.12 (A priori bounds for triangles with general nonlinearity)
Let $b > 0$, $\Omega = \{x \in \mathbb{R}^2 : x_1, x_2 > 0,\ x_1 + x_2 < b\}$. Further, let $f\colon [0,\infty) \to [0,\infty)$ be a continuous function satisfying (A1) and (A2). Then, there exists a constant $C > 0$ such that for every admissible grid size $h > 0$ and every non-negative solution $u_h\colon \overline{\Omega_h} \to [0,\infty)$ of

$$\begin{cases} -\Delta_h u = f(u) & \text{in } \Omega_h, \\ u = 0 & \text{on } \partial_h \Omega_h \end{cases}$$

the a priori estimate $\|u_h\|_{L^\infty(\Omega_h)} \le C$ holds.

5. Existence of solutions

In Section 3 about Liouville theorems we showed the non-existence of positive solutions of the finite difference Emden equation for exponents $p \in (1, p_\star)$. The critical value $p_\star$ depends on the unbounded domains we considered, especially on the dimension n. In contrast to these negative results, we give two positive results in this paragraph and prove the existence of positive solutions on the entire space as well as on an infinite strip for exponents p larger than some threshold value $p^\star$.

In both cases, the proof is based on a concentrated compactness argument and an embedding $\dot{H}_0^1 \hookrightarrow L^q$ is used. The procedure requires the condition $p+1 > q$. On the entire grid the application of the Sobolev embedding with $q = 2^* = \frac{2n}{n-2}$ leads to $p > \frac{n+2}{n-2}$, whereas on a infinite strip $q = 2$ can be chosen in view of Poincaré inequality and all $p > 1$ are allowed.

5.1. Entire space

Theorem 5.1 (Existence on entire space)
Let $h > 0$ be a fixed grid size and $n \geq 3$ be the dimension. Then, for every $p > \frac{n+2}{n-2}$ there exists a positive solution $u\colon \mathbb{R}_h^n \to (0, \infty)$, $u \in \dot{H}_0^1(\mathbb{R}_h^n)$, of

$$-\Delta_h u = u^p \quad \text{in } \mathbb{R}_h^n. \tag{5.1}$$

Before proving the theorem, two lemmas are required. Both statements are verified similar to their classical analogue.

Lemma 5.2 (Lions' lemma for finite differences)
Let $h > 0$ be a fixed grid size and $n \in \mathbb{N}$ the dimension. For $q \in [1, \infty)$ let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in $L^q(\mathbb{R}_h^n)$, i.e., there exists a constant $C > 0$ such that

$$\sum_{x\in\mathbb{R}_h^n} |u_k(x)|^q h^n < C \quad \text{for all } k \in \mathbb{N}. \tag{5.2}$$

Further, assume

$$\sup_{y\in\mathbb{R}_h^n} \sum_{x\in B_R^h(y)} |u_k(x)|^q h^n \to 0 \quad (k \to \infty) \tag{5.3}$$

for some $R > 0$, where $B_R^h(y) \coloneqq \{x \in \mathbb{R}_h^n \colon \|x - y\|_\infty < R\}$ denotes a discrete ball *with radius R. Then,*

$$u_k \to 0 \quad \text{in } L^s(\mathbb{R}_h^n) \quad (k \to \infty)$$

for all $q < s \leq \infty$.

Proof. Let $u\colon \mathbb{R}^n_h \to \mathbb{R}$. Since the grid size $h>0$ is fixed, we have for $q<s<\infty$

$$\|u\|_{L^s(\mathbb{R}^n_h)} = \left(\sum_{x\in\mathbb{R}^n_h} |u(x)|^s h^n\right)^{\frac{1}{s}} \le \left(\sum_{x\in\mathbb{R}^n_h} |u(x)|^q h^n\right)^{\frac{1}{q}} h^{\frac{n}{s}-\frac{n}{q}} = h^{\frac{n}{s}-\frac{n}{q}} \|u\|_{L^q(\mathbb{R}^n_h)}.$$

For every $\emptyset \ne \Omega_h \subset \mathbb{R}^n_h$, we therefore obtain

$$\|u\|^s_{L^s(\Omega_h)} \le h^{n-\frac{ns}{q}} \|u\|^s_{L^q(\Omega_h)}. \tag{5.4}$$

Next, we choose for some fixed $R>0$ a sequence of midpoints $(y_l)_{l\in\mathbb{N}} \subset \mathbb{R}^n_h$ such that $\bigcup_{l\in\mathbb{N}} B^h_R(y_l)$ is a disjoint covering of $\mathbb{R}^n_h$, i.e., every $z\in\mathbb{R}^n_h$ is contained in exactly one ball $B^h_R(y_l)$. Using (5.4), $q<s<\infty$, (5.3) as well as (5.2) we conclude

$$\begin{aligned}
\|u_k\|^s_{L^s(\mathbb{R}^n_h)} &= \sum_{x\in\mathbb{R}^n_h} |u_k(x)|^s h^n \\
&= \sum_{l=1}^{\infty} \sum_{x\in B^h_R(y_l)} |u_k(x)|^s h^n \\
&\le \sum_{l=1}^{\infty} h^{n-\frac{ns}{q}} \left(\sum_{x\in B^h_R(y_l)} |u_k(x)|^q h^n\right)^{\frac{s}{q}} \\
&= h^{n-\frac{ns}{q}} \sum_{l=1}^{\infty} \left[\left(\sum_{x\in B^h_R(y_l)} |u_k(x)|^q h^n\right)^{\frac{s}{q}-1} \left(\sum_{x\in B^h_R(y_l)} |u_k(x)|^q h^n\right)\right] \\
&\le h^{n-\frac{ns}{q}} \left(\sup_{y\in\mathbb{R}^n_h} \sum_{x\in B^h_R(y)} |u_k(x)|^q h^n\right)^{\frac{s}{q}-1} \sum_{l=1}^{\infty} \left(\sum_{x\in B^h_R(y_l)} |u_k(x)|^q h^n\right) \\
&= h^{n-\frac{ns}{q}} \underbrace{\left(\sup_{y\in\mathbb{R}^n_h} \sum_{x\in B^h_R(y)} |u_k(x)|^q h^n\right)^{\frac{s}{q}-1}}_{\to 0} \underbrace{\sum_{x\in\mathbb{R}^n_h} |u_k(x)|^q h^n}_{<C} \to 0
\end{aligned}$$

as $k\to\infty$. An immediate consequence is

$$\|u_k\|_{L^\infty(\mathbb{R}^n_h)} \le \left(\sum_{x\in\mathbb{R}^n_h} |u_k(x)|^s h^n\right)^{\frac{1}{s}} h^{-\frac{n}{s}} = h^{-\frac{n}{s}} \|u_k\|_{L^s(\mathbb{R}^n_h)} \to 0 \qquad (k\to\infty).$$

□

Lemma 5.3 (Discrete Brezis-Lieb lemma)
Let $r\in[1,\infty)$ and $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in $L^r(\mathbb{R}^n_h)$ with $u_k\to u$ pointwise in $\mathbb{R}^n_h$. Then

$$\|u\|^r_{L^r(\mathbb{R}^n_h)} = \lim_{k\to\infty} \left(\|u_k\|^r_{L^r(\mathbb{R}^n_h)} - \|u_k-u\|^r_{L^r(\mathbb{R}^n_h)}\right).$$

Proof. The result is proven as its classical analogue, Lemma 1.32. in [32]: According to Fatou's lemma, we have

$$\|u\|^r_{L^r(\mathbb{R}^n_h)} \leq \liminf_{k\in\mathbb{N}} \|u_k\|^r_{L^r(\mathbb{R}^n_h)} < \infty$$

and thus $u \in L^r(\mathbb{R}^n_h)$. Now let $\varepsilon > 0$ and define $f_k^{(\varepsilon)} \colon \mathbb{R}^n_h \to \mathbb{R}$ by

$$f_k^{(\varepsilon)} := \big(\big||u_k|^r - |u_k - u|^r - |u|^r\big| - \varepsilon|u_k - u|^r\big)_+ ,$$

where $t_+ = \max\{t, 0\}$ denotes the positive part for all $t \in \mathbb{R}$. Due to Lemma A.1 there exists a constant $C = C(\varepsilon) \geq 1$ with

$$||a + b|^r - |a|^r| \leq \varepsilon|a|^r + C|b|^r$$

for all $a, b \in \mathbb{R}$. Using also the monotonicity of the function $t \mapsto t_+$, we estimate

$$\begin{aligned} f_k^{(\varepsilon)} &= \big(\big||u_k|^r - |u_k - u|^r - |u|^r\big| - \varepsilon|u_k - u|^r\big)_+ \\ &\leq \big(\big||u_k|^r - |u_k - u|^r\big| + |u|^r - \varepsilon|u_k - u|^r\big)_+ \\ &\leq (\varepsilon|u_k - u|^r + C|u|^r + |u|^r - \varepsilon|u_k - u|^r)_+ \\ &= (C + 1)|u|^r. \end{aligned}$$

Since $f_k^{(\varepsilon)} \to 0$ pointwise in $\mathbb{R}^n_h$ for $k \to \infty$, Lebesgue's dominated convergence theorem guarantees $\|f_k^{(\varepsilon)}\|_{L^1(\mathbb{R}^n_h)} \to 0$ as $k \to \infty$. Moreover, from the definition of $f_k^{(\varepsilon)}$ we infer

$$\big||u_k|^r - |u_k - u|^r - |u|^r\big| \leq f_k^{(\varepsilon)} + \varepsilon|u_k - u|^r$$

and obtain

$$\limsup_{k\in\mathbb{N}} \sum_{\mathbb{R}^n_h} \big||u_k|^r - |u_k - u|^r - |u|^r\big| h^n \leq \varepsilon \sup_{k\in\mathbb{N}} \sum_{\mathbb{R}^n_h} |u_k - u|^r h^n.$$

As $\sup_{k\in\mathbb{N}} \sum_{\mathbb{R}^n_h} |u_k - u|^r h^n \leq \big(\|u\|_{L^r(\mathbb{R}^n_h)} + \sup_{k\in\mathbb{N}} \|u_k\|_{L^r(\mathbb{R}^n_h)}\big)^r < \infty$ by assumption, taking the limit $\varepsilon \to 0$ ensures

$$\sum_{\mathbb{R}^n_h} \big||u_k|^r - |u_k - u|^r - |u|^r\big| h^n \to 0 \quad (k \to \infty),$$

which leads to

$$\sum_{\mathbb{R}^n_h} (|u_k|^r - |u_k - u|^r) h^n \to \sum_{\mathbb{R}^n_h} |u|^r h^n \quad (k \to \infty).$$

This is exactly the claim. $\square$

Proof of Theorem 5.1. We minimize the functional $J\colon \dot{H}_0^1(\mathbb{R}_h^n) \to \mathbb{R}$,

$$J[u] \coloneqq \|u\|^2_{\dot{H}_0^1(\mathbb{R}_h^n)} = \|\nabla_h^+ u\|^2_{L^2(\mathbb{R}_h^n)} = \sum_{i=1}^{n} \sum_{x\in\mathbb{R}_h^n} |D_i^+ u(x)|^2 h^n,$$

over the set

$$\mathcal{M} \coloneqq \{u \in \dot{H}_0^1(\mathbb{R}_h^n)\colon L[u] = 1\}$$

with $L[u] \coloneqq \|u\|^{p+1}_{L^{p+1}(\mathbb{R}_h^n)}$ and define $d \coloneqq \inf_{u\in\mathcal{M}} J[u] \geq 0$.

⟨1⟩ *Existence of a minimizer*
Let $(u_k)_{k\in\mathbb{N}}$ be a minimizing sequence of J in the set $\mathcal{M}$. The Sobolev embedding $\dot{H}_0^1(\mathbb{R}_h^n) \hookrightarrow L^{\frac{2n}{n-2}}(\mathbb{R}_h^n)$ yields that $\left(\|u_k\|_{L^{\frac{2n}{n-2}}(\mathbb{R}_h^n)}\right)_{k\in\mathbb{N}}$ is bounded. The next step is the application of Lions' Lemma 5.2. *Assume*

$$\sup_{y\in\mathbb{R}_h^n} \sum_{B_R^h(y)} |u_k|^{\frac{2n}{n-2}} h^n \to 0 \quad (k \to 0)$$

for a radius $R > 0$ and discrete balls $B_R^h(y) \coloneqq \{x \in \mathbb{R}_h^n\colon \|x-y\|_\infty < R\}$. Employing Lions' lemma with $q = \frac{2n}{n-2}$ entails

$$u_k \to 0 \quad \text{in } L^s(\mathbb{R}_h^n)$$

for all $s \in \left(\frac{2n}{n-2}, \infty\right)$. The choice $s \coloneqq p+1 > \frac{2n}{n-2}$ leads to a contradiction as $u_k \in \mathcal{M}$ for all $k \in \mathbb{N}$. Thus, there exists a radius $R > 0$ and a sequence of midpoints $(y_k)_{k\in\mathbb{N}} \subset \mathbb{R}_h^n$ with

$$\sum_{B_R^h(y_k)} |u_k|^{\frac{2n}{n-2}} h^n \geq \delta > 0.$$

Shifting back by y_k we define $v_k(x) \coloneqq u_k(x + y_k)$. The sequence $(v_k)_{k\in\mathbb{N}}$ has the following properties:

$$\|v_k\|_{L^{p+1}(\mathbb{R}_h^n)} = \|u_k\|_{L^{p+1}(\mathbb{R}_h^n)} \quad \text{and} \quad \|v_k\|_{\dot{H}_0^1(\mathbb{R}_h^n)} = \|u_k\|_{\dot{H}_0^1(\mathbb{R}_h^n)}.$$

Going to a subsequence we may assume that there exists a limit function $v\colon \mathbb{R}_h^n \to \mathbb{R}$ with

$$\begin{aligned} v_k &\rightharpoonup v \quad \text{in } \dot{H}_0^1(\mathbb{R}_h^n),\\ v_k &\rightharpoonup v \quad \text{in } L^{p+1}(\mathbb{R}_h^n) \text{ and}\\ v_k &\to v \quad \text{pointwise in } \mathbb{R}_h^n. \end{aligned}$$

The function v is not the zero function since by pointwise convergence

$$\sum_{B_R^h(0)} |v|^{\frac{2n}{n-2}} h^n = \lim_{k\to\infty} \sum_{B_R^h(0)} |v_k|^{\frac{2n}{n-2}} h^n = \lim_{k\to\infty} \sum_{B_R^h(y_k)} |u_k|^{\frac{2n}{n-2}} h^n \geq \delta > 0.$$

From the weakly lower semicontinuity of the norms $\|\cdot\|_{\dot{H}_0^1(\mathbb{R}_h^n)}$ and $\|\cdot\|_{L^{p+1}(\mathbb{R}_h^n)}$, we further deduce

$$\begin{aligned}
\|v\|^2_{\dot{H}_0^1(\mathbb{R}_h^n)} &\leq \liminf_{k\in\mathbb{N}} \|v_k\|^2_{\dot{H}_0^1(\mathbb{R}_h^n)} = \liminf_{k\in\mathbb{N}} \|u_k\|^2_{\dot{H}_0^1(\mathbb{R}_h^n)} = \liminf_{k\in\mathbb{N}} J[u_k] = d,\\
\|v\|_{L^{p+1}(\mathbb{R}_h^n)} &\leq \liminf_{k\in\mathbb{N}} \|v_k\|_{L^{p+1}(\mathbb{R}_h^n)} = \liminf_{k\in\mathbb{N}} \|u_k\|_{L^{p+1}(\mathbb{R}_h^n)} = 1.
\end{aligned}$$

Consequently, we obtain $d > 0$ and $\|v\|_{L^{p+1}(\mathbb{R}_h^n)} \in (0,1]$. In order to show that v is the desired minimizer, it remains to verify that $v \in \mathcal{M}$, i.e., $\|v\|_{L^{p+1}(\mathbb{R}_h^n)} = 1$. This is done by means of the Brezis-Lieb Lemma 5.3 which yields

$$\|v\|^{p+1}_{L^{p+1}(\mathbb{R}_h^n)} = \lim_{k\to\infty} \left(1 - \|v_k - v\|^{p+1}_{L^{p+1}(\mathbb{R}_h^n)}\right) \tag{5.5}$$

as $\|v_k\|^{p+1}_{L^{p+1}(\mathbb{R}_h^n)} = 1$ for all $k \in \mathbb{N}$. Moreover,

$$d = \inf_{u\in\mathcal{M}} \|u\|^2_{\dot{H}_0^1(\mathbb{R}_h^n)} = \inf_{0\neq w\in \dot{H}_0^1(\mathbb{R}_h^n)} \frac{\|w\|^2_{\dot{H}_0^1(\mathbb{R}_h^n)}}{\|w\|^2_{L^{p+1}(\mathbb{R}_h^n)}}$$

entails

$$d\|w\|^2_{L^{p+1}(\mathbb{R}_h^n)} \leq \|w\|^2_{\dot{H}_0^1(\mathbb{R}_h^n)} \tag{5.6}$$

for all $w \in \dot{H}_0^1(\mathbb{R}_h^n)$. Furthermore, for all $\lambda \in (0,1)$ and $\beta \in (0,1)$, we have

$$(1-\lambda)^\beta + \lambda^\beta > 1. \tag{5.7}$$

Employing the weak convergence $v_k \rightharpoonup v$ in $\dot{H}_0^1(\mathbb{R}_h^n)$ together with (5.6), (5.5) and (5.7), we conclude

$$\begin{aligned}
d &= \lim_{k\to\infty} \|v_k\|^2_{\dot{H}_0^1(\mathbb{R}_h^n)}\\
&= \lim_{k\to\infty} \left(\|v_k - v\|^2_{\dot{H}_0^1(\mathbb{R}_h^n)} - \|v\|^2_{\dot{H}_0^1(\mathbb{R}_h^n)} + 2\langle\!\langle \nabla_h^+ v, \nabla_h^+ v_k\rangle\!\rangle_{\mathbb{R}_h^n}\right)\\
&= \lim_{k\to\infty} \|v_k - v\|^2_{\dot{H}_0^1(\mathbb{R}_h^n)} + \|v\|^2_{\dot{H}_0^1(\mathbb{R}_h^n)}\\
&\geq d\left(\lim_{k\to\infty} \|v_k - v\|^2_{L^{p+1}(\mathbb{R}_h^n)} + \|v\|^2_{L^{p+1}(\mathbb{R}_h^n)}\right)\\
&= d\left(\left(1 - \|v\|^{p+1}_{L^{p+1}(\mathbb{R}_h^n)}\right)^{\frac{2}{p+1}} + \|v\|^{(p+1)\cdot\frac{2}{p+1}}_{L^{p+1}(\mathbb{R}_h^n)}\right) > d,
\end{aligned}$$

provided $\|v\|_{L^{p+1}(\mathbb{R}^n_h)} \in (0,1)$. This would be a contradiction. So, $\|v\|_{L^{p+1}(\mathbb{R}^n_h)} = 1$ and therefore $v \in \mathcal{M}$ is a minimizer of the functional J over the set $\mathcal{M}$.

⟨2⟩ *The minimizer $v \in \mathcal{M}$ yields a weak solution of $-\Delta_h w = |w|^{p-1}w$*
Above we verified that $v \in \mathcal{M} = \{u \in \dot{H}^1_0(\mathbb{R}^n_h) \colon L[u] = 1\}$ satisfies $J[v] = d = \inf_{u\in\mathcal{M}} J[u]$. Hence, there exists a pair of Lagrange multipliers $(\alpha,\beta) \in \mathbb{R}^2 \setminus \{(0,0)\}$ with

$$\alpha J'[v] = \beta L'[v]. \tag{5.8}$$

Here J' and L' denote the Fréchet derivative of J and L, respectively. They are given by

$$J'[w](\varphi) = 2\langle\!\langle \nabla_h^+ w, \nabla_h^+ \varphi\rangle\!\rangle_{\mathbb{R}^n_h} \quad \text{and} \quad L'[w](\varphi) = (p+1)\langle |w|^{p-1}w, \varphi\rangle_{\mathbb{R}^n_h}$$

for all $w, \varphi \in \dot{H}^1_0(\mathbb{R}^n_h)$. Evaluating (5.8) at v ensures

$$2\alpha \|v\|^2_{\dot{H}^1_0(\mathbb{R}^n_h)} = \beta(p+1)\|v\|^{p+1}_{L^{p+1}(\mathbb{R}^n_h)}.$$

Due to $\|v\|^2_{\dot{H}^1_0(\mathbb{R}^n_h)} = J[v] = d > 0$ and $\|v\|_{L^{p+1}(\mathbb{R}^n_h)} = 1$ we deduce $\alpha, \beta \neq 0$ and $\frac{\beta}{\alpha} = \frac{2d}{p+1}$. Thus, we infer $J'[v] = \frac{2d}{p+1} L'[v]$, i.e.,

$$\langle\!\langle \nabla_h^+ v, \nabla_h^+ \varphi\rangle\!\rangle_{\mathbb{R}^n_h} = d\langle |v|^{p-1}v, \varphi\rangle_{\mathbb{R}^n_h}$$

for all $\varphi \in \dot{H}^1_0(\mathbb{R}^n_h)$. In other words, v is a weak solution of

$$-\Delta_h v = d|v|^{p-1}v \quad \text{in } \mathbb{R}^n_h.$$

Hence, the rescaled function $w := d^{\frac{1}{p-1}} v \in \dot{H}^1_0(\mathbb{R}^n_h)$ is a weak solution of

$$-\Delta_h w = |w|^{p-1}w \quad \text{in } \mathbb{R}^n_h.$$

⟨3⟩ *Pointwise solution*
The last statement means that w satisfies

$$\langle\!\langle \nabla_h^+ w, \nabla_h^+ \varphi\rangle\!\rangle_{\mathbb{R}^n_h} = \langle |w|^{p-1}w, \varphi\rangle_{\mathbb{R}^n_h}$$

for all $\varphi \in \dot{H}^1_0(\mathbb{R}^n_h)$, in particular for all $\varphi \in \mathcal{C}(\mathbb{R}^n_h)$. So, partial summation (Lemma 2.15) leads to

$$\langle -\Delta_h w, \varphi\rangle_{\mathbb{R}^n_h} = \langle |w|^{p-1}w, \varphi\rangle_{\mathbb{R}^n_h}$$

for all $\varphi \in \mathcal{C}(\mathbb{R}^n_h)$. Choosing the indicator function $\chi_z \in \mathcal{C}(\mathbb{R}^n_h)$ as test function φ, we obtain

$$-\Delta_h w(z) = |w(z)|^{p-1} w(z) \quad \text{for all } z \in \mathbb{R}^n_h.$$

⟨4⟩ *Positivity*
Without loss of generality we may assume $u_k \geq 0$ for all $k \in \mathbb{N}$ since for every minimizing sequence $(u_k)_{k\in\mathbb{N}}$ the sequence $(|u_k|)_{k\in\mathbb{N}}$ is also minimizing. Then, by construction, we deduce $v \geq 0$ in $\mathbb{R}^n_h$. Consequently $w = d^{\frac{1}{p-1}} v \geq 0$ solves pointwise

$$-\Delta_h w = w^p \geq 0 \quad \text{in } \mathbb{R}^n_h.$$

Finally, the discrete maximum principle (Lemma 2.12) yields $w > 0$ in $\mathbb{R}^n_h$. □

Let us collect our results for the discrete Emden equation $-\Delta_h u = u^p$ in $\mathbb{R}^n_h$: The Liouville Theorem 3.4 yields that for $1 < p < p_\star = \frac{n}{n-2}$ there is only the zero solution, whereas Theoreom 5.1 guarantees the existence of positive solutions for $p > p^\star = \frac{n+2}{n-2}$. For $p \in [p_\star, p^\star]$, it remains an open question whether positive solutions do exist or not. Thereby, the overview for classical and very weak solutions (Figure 3.1) can be extended to the finite difference solutions:

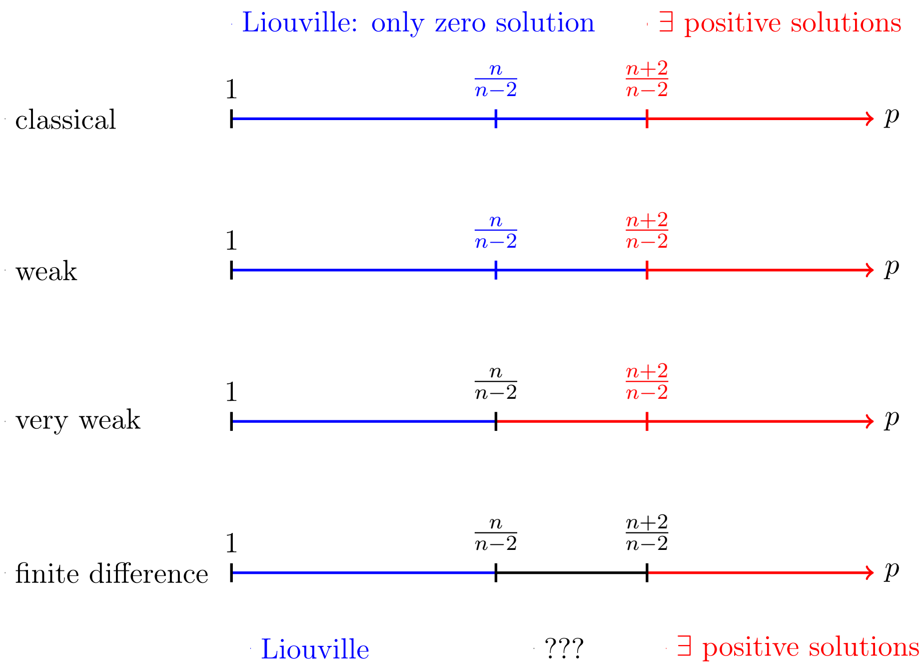


Figure 5.1: Extended overview of existence and non-existence results.

5.2. Infinite Strip

Notation 5.4
For dimensions $n \geq 2$ we consider the infinite strip $S := \{x \in \mathbb{R}^n : 0 < x_n < b\}$ *with width $b > 0$. Let the grid size $h > 0$ be admissible, i.e., there exists $j \in \mathbb{N}$ such that $j \geq 2$ and $jh = b$. Moreover, the discrete analogon is defined by $S_h := S \cap \mathbb{R}^n_h$, the corresponding discrete boundary is $\partial_h S := \partial S \cap \mathbb{R}^n_h$ and the discrete closure is given by $\overline{S_h} := S_h \cup \partial_h S_h = \overline{S} \cap \mathbb{R}^n_h$. For exponents $p \in (1, \infty)$ we consider the discrete Emden equation*

$$\begin{cases} -\Delta_h u = u^p & \text{in } S_h, \\ u = 0 & \text{on } \partial_h S_h. \end{cases} \tag{5.9}$$

Theorem 5.5 (Existence on infinite strip)
Let $n \geq 2$ be the dimension, $S = \{x \in \mathbb{R}^n : 0 < x_n < b\}$ a infinite strip *and $h > 0$ a fixed admissible grid size. Then, for every $p \in (1, \infty)$ there exist two in S_h positive solutions of* (5.9)*: One solution $u_1 \in \dot{H}^1_0(S_h)$ has* finite energy*, the other one u_2 has* infinite energy *and does not belong to $L^q(S_h)$ for all $q \in [1, \infty)$.*

Proof. Solution with infinite energy
Let $v : [0, b]_h \to [0, \infty)$ be a in $(0, b)_h$ positive solution of

$$\begin{cases} -\dfrac{1}{h^2}[v(x+h) - 2v(x) + v(x-h)] = v^p(x), & x \in (0, b)_h, \\ v(x) = 0, & x = 0 \text{ or } x = b. \end{cases}$$

Indeed, we construct v by minimizing the functional $F : \dot{H}^1_0((0, b)_h) \to \mathbb{R}$, $F(w) := \|w\|^2_{\dot{H}^1_0((0,b)_h)}$, on the set $\left\{w \in \dot{H}^1_0((0, b)_h) : \|w\|_{L^{p+1}((0,b)_h)} = 1\right\}$ similarly to the proof of Theorem 5.1. Then $u_2 : \overline{S_h} \to [0, \infty)$ defined by $u_2(x) := v(x_n)$ solves (5.9) and $u_2 \notin L^q(S_h)$ for all $q \in [1, \infty)$.

Solution with finite energy
The approach is analogous to the proof of Theorem 5.1. The main difference is that the Sobolev embedding $\dot{H}^1_0(\mathbb{R}^n_h) \hookrightarrow L^{\frac{2n}{n-2}}(\mathbb{R}^n_h)$ is replaced by Poincaré's inequality. For the reader's convenience the most important steps are executed below:

In the following we identify functions in $\dot{H}^1_0(S_h)$ and $L^q(S_h)$ with their zero extensions in $\dot{H}^1_0(\mathbb{R}^n_h)$ and $L^q(\mathbb{R}^n_h)$, respectively. Note that

$$\dot{H}^1_0(S_h) = \left\{w : \overline{S_h} \to \mathbb{R} : \|w\|_{\dot{H}^1_0(S_h)} < \infty,\ w = 0 \text{ on } \partial_h S_h\right\},$$

where

$$\|w\|_{\dot{H}^1_0(S_h)} = \left(\sum_{i=1}^{n-1} \sum_{x \in S_h} |D_i^+ w(x)|^2 h^n + \sum_{x \in \partial_n^- S_h \cup S_h} |D_n^+ w(x)|^2 h^n\right)^{\frac{1}{2}}.$$

We minimize the functional $J\colon \dot{H}_0^1(S_h) \to \mathbb{R}$,

$$J[u] := \|u\|^2_{\dot{H}_0^1(S_h)} = \sum_{i=1}^{n} \sum_{x\in\mathbb{R}_h^n} |D_i^+ u(x)|^2 h^n = \|u\|^2_{\dot{H}_0^1(\mathbb{R}_h^n)},$$

over the set

$$\mathcal{M} := \{u \in \dot{H}_0^1(S_h)\colon L[u] = 1\}$$

with $L[u] := \|u\|^{p+1}_{L^{p+1}(S_h)} = \sum_{x\in S_h} |u(x)|^{p+1} h^n$ and define $d := \inf_{u\in\mathcal{M}} J[u] > 0$.

⟨1⟩ *Existence of a minimizer*
Let $(u_k)_{k\in\mathbb{N}}$ be a minimizing sequence of J in the set $\mathcal{M} \subset \dot{H}_0^1(S_h)$. The Poincaré inequality ensures the embedding $\dot{H}_0^1(S_h) \hookrightarrow L^2(S_h)$ and thus, $\big(\|u_k\|_{L^2(S_h)}\big)_{k\in\mathbb{N}}$ is bounded in $\mathbb{R}$. Next, we *assume*

$$\sup_{y\in S_h} |u_k(y)| \to 0 \quad (k \to 0).$$

Employing the Lions' Lemma 5.2 for a radius $R \in (0, h)$ and $q = 2$ results in

$$u_k \to 0 \quad \text{in } L^s(S_h)$$

for all $s \in (2, \infty)$. Choosing $s := p + 1 > 2$ leads to a contradiction as $u_k \in \mathcal{M}$ for all $k \in \mathbb{N}$. Hence, there exists a sequence of points $(y_k)_{k\in\mathbb{N}} \subset S_h$ with

$$|u_k(y_k)| \geq \delta > 0.$$

Shifting back by $\hat{y}_k := (y_{k,1}, \dots, y_{k,n-1}, 0)$ we define $v_k(x) := u_k(x+\hat{y}_k)$. Immediate consequences are

$$\|v_k\|_{L^{p+1}(S_h)} = \|u_k\|_{L^{p+1}(S_h)} \quad \text{and} \quad \|v_k\|_{\dot{H}_0^1(S_h)} = \|u_k\|_{\dot{H}_0^1(S_h)}.$$

Up to a subsequence we may assume that there exists a limit function $v\colon \overline{S_h} \to \mathbb{R}$ such that

$$\begin{aligned} v_k &\rightharpoonup v \quad \text{in } \dot{H}_0^1(S_h),\\ v_k &\rightharpoonup v \quad \text{in } L^{p+1}(S_h) \text{ and}\\ v_k &\to v \quad \text{pointwise in } S_h. \end{aligned}$$

Considering the finite set $A_h := \{x \in S_h\colon x_1, \dots, x_{n-1} = 0\}$ we note that

$$\sum_{x\in A_h} |v(x)| = \lim_{k\to\infty} \sum_{x\in A_h} |v_k(x)| = \lim_{k\to\infty} \sum_{x\in A_h} |u_k(x + \hat{y}_k)| \geq \limsup_{k\in\mathbb{N}} |u_k(y_k)| \geq \delta > 0$$

and conclude that $v \not\equiv 0$. Due to the weakly lower semicontinuity of norms, we obtain

$$\|v\|^2_{\dot{H}^1_0(S_h)} \leq \liminf_{k\in\mathbb{N}} \|v_k\|^2_{\dot{H}^1_0(S_h)} = \liminf_{k\in\mathbb{N}} \|u_k\|^2_{\dot{H}^1_0(S_h)} = \liminf_{k\in\mathbb{N}} J[u_k] = d,$$
$$\|v\|_{L^{p+1}(S_h)} \leq \liminf_{k\in\mathbb{N}} \|v_k\|_{L^{p+1}(S_h)} = \liminf_{k\in\mathbb{N}} \|u_k\|_{L^{p+1}(S_h)} = 1.$$

The rest of the proof is akin to the proof of Theorem 5.1. □

Remark 5.6 (Infinite energy solutions on entire space)
For dimensions $n \geq 4$ and $p \geq \frac{n+1}{n-3}$ there are also positive solutions with infinite energy of the entire space problem (5.1), where $\frac{n+1}{n-3}$ is larger than the critical exponent $\frac{n+2}{n-2}$ from Theorem 5.1. The construction works as in the proof of Theorem 5.5: In this case, we can apply Theorem 5.1 with $n-1 \geq 3$ and $p \geq \frac{(n-1)+2}{(n-1)-2}$, which yields a positive solution v on $\mathbb{R}^{n-1}_h$. By setting $u_2(x) \coloneqq v(x_1, \ldots, x_{n-1})$ for $x \in \mathbb{R}^n_h$, we get a solution u_2 on $\mathbb{R}^n_h$ with infinite energy.

With the methods executed in the proof of Theorem 5.5 it is possible to obtain the subsequent result for more general strips with minor adjustments since the Poincaré inequality and the concentrated compactness argument can be employed in this situation.

Theorem 5.7 (Existence on general infinite strip)
Let $n \geq 2$ be the dimension, $S \coloneqq (a_1, b_1) \times \ldots \times (a_k, b_k) \times \mathbb{R}^{n-k}$ an infinite strip *for some $k \in \{1, \ldots, n-1\}$ and let $h > 0$ be an admissible grid size for S. Then, for every $p \in (1, \infty)$ there exist two in S_h positive solutions of*

$$\begin{cases} -\Delta_h u = u^p & \text{in } S_h, \\ u = 0 & \text{on } \partial_h S_h. \end{cases}$$

One solution $u_1 \in \dot{H}^1_0(S_h)$ has finite energy*, the other one u_2 has* infinite energy *and does not belong to $L^q(S_h)$ for all $q \in [1, \infty)$.*

A. Appendix

Lemma A.1
Let $r \in [1, \infty)$. For every $\varepsilon > 0$ there exists a constant $C = C(\varepsilon) \geq 1$ such that

$$||a+b|^r - |a|^r| \leq \varepsilon |a|^r + C|b|^r$$

for all $a, b \in \mathbb{R}$.

Proof. Let $\varepsilon > 0$ and $s \in \mathbb{R}$. By continuity there exists some $\delta > 0$ such that $|s| < \delta$ implies $||1+s|^r - 1| \leq \varepsilon$. On the other hand, we can fix some $C = C(\varepsilon) \geq 1$ so that $|s| \geq \delta$ leads to $\frac{||1+s|^r - 1|}{|s|^r} \leq C$. This results in

$$||1+s|^r - 1| \leq \varepsilon + C|s|^r$$

for all $s \in \mathbb{R}$. For $a \neq 0$, the choice $s := \frac{b}{a}$ and multiplication with $|a|^r$ yields

$$||a+b|^r - |a|^r| \leq \varepsilon |a|^r + C|b|^r$$

for all $a \neq 0$, $b \in \mathbb{R}$. If $a = 0$, the inequality is obviously satisfied as $C \geq 1$. □

Below, some auxiliary statements for the proof of Theorem 3.20 are recaped and verified.

Lemma A.2
Let $u \colon \mathbb{Z}^2 \to \mathbb{R}$ be bounded and discrete superharmonic. The operator Θ is given by

$$\Theta u(x,y) = u(x+1,y) + u(x-1,y) + u(x,y+1) + u(x,y-1)$$

for $x, y \in \mathbb{Z}$ and the sequences of grid functions (u_k), (d_k) and (w_k) are defined by

$$\begin{aligned}
&u_0 := u, \qquad u_{k+1} := \frac{1}{4}\Theta u_k, \\
&d_k := -\Delta_{\mathbb{Z}^2} u_k, \\
&w_k(x,y) := \begin{cases} \frac{1}{4^k}\binom{k}{\frac{1}{2}(k+x+y)}\binom{k}{\frac{1}{2}(k+x-y)}, & \text{if } |x|+|y| \leq k,\ k+x+y \text{ even}, \\ 0, & \text{else}, \end{cases}
\end{aligned}$$

for all $k \in \mathbb{N}_0$ and $x, y \in \mathbb{Z}$, , where the binomial coefficient is given by

$$\binom{k}{x} = \begin{cases} \dfrac{k!}{(k-x)!\,x!}, & \text{if } k \geq x \geq 0, \\ 0, & \text{else}. \end{cases}$$

Then, the following assertions hold:

(a) $4(u_k - u_{k+1}) = d_k$,

(b) $d_{k+1} = \frac{1}{4}\Theta d_k$,

(c) $w_{k+1} = \frac{1}{4}\Theta w_k$,

(d) $d_k(x, y) = \sum_{\mu,\nu\in\mathbb{Z}} w_k(\mu, \nu)[-\Delta_{\mathbb{Z}^2} u](x + \mu, y + \nu)$ *(Note that only finitely many summands are different from* 0 *as* w_k *has compact support.),*

(e) $\sum_{k=0}^{\infty} w_k(0, 0) = +\infty$.

Proof. (a) The definitions of u_k and d_k directly show

$$4(u_k - u_{k+1}) = 4u_k - \Theta u_k = -\Delta_{\mathbb{Z}^2} u_k = d_k.$$

(b) By means of the definitions of d_k and u_k as well as (a), we see

$$\begin{aligned} d_{k+1} &= -\Delta_{\mathbb{Z}^2} u_{k+1} = 4u_{k+1} - \Theta u_{k+1} = \Theta u_k - \Theta u_{k+1} \\ &= \Theta(u_k - u_{k+1}) = \frac{1}{4}\Theta d_k. \end{aligned}$$

(c) Let $k \in \mathbb{N}_0$ and $x, y \in \mathbb{Z}$ fixed. If $|x| + |y| > k + 1$, then $|x + 1| + |y| > k$, $|x - 1| + |y| > k$, $|x| + |y + 1| > k$ and $|x| + |y - 1| > k$. Thus, $w_{k+1}(x, y) = w_k(x + 1, y) = w_k(x - 1, y) = w_k(x, y + 1) = w_k(x, y - 1) = 0$ by construction and the promised equality is valid.

If $k + 1 + x + y$ is odd, then $k + (x + 1) + y$, $k + (x - 1) + y$, $k + x + (y + 1)$ and $k+x+(y-1)$ are odd as well and hence, $w_{k+1}(x, y) = w_k(x+1, y) = w_k(x-1, y) = w_k(x, y + 1) = w_k(x, y - 1) = 0$.

Now, let $|x| + |y| \leq k + 1$ and $k + 1 + x + y$ be even. With the shorthands $c_1 := \frac{1}{2}(k+1+x+y)$, $c_2 := \frac{1}{2}(k+1+x-y) \in \{0, \ldots, k+1\}$ and $\binom{k}{-1} = \binom{k}{k+1} = 0$ we obtain

$$\begin{aligned} w_k(x + 1, y) &= \frac{1}{4^k}\binom{k}{c_1}\binom{k}{c_2}, & w_k(x - 1, y) &= \frac{1}{4^k}\binom{k}{c_1 - 1}\binom{k}{c_2 - 1}, \\ w_k(x, y + 1) &= \frac{1}{4^k}\binom{k}{c_1}\binom{k}{c_2 - 1}, & w_k(x, y - 1) &= \frac{1}{4^k}\binom{k}{c_1 - 1}\binom{k}{c_2} \end{aligned}$$

and thus

$$\begin{aligned} \frac{1}{4}\Theta w_k(x, y) &= \frac{1}{4^{k+1}}\left[\binom{k}{c_1} + \binom{k}{c_1 - 1}\right]\left[\binom{k}{c_2} + \binom{k}{c_2 - 1}\right] \\ &= \frac{1}{4^{k+1}}\binom{k+1}{c_1}\binom{k+1}{c_2} = w_{k+1}(x, y). \end{aligned}$$

(d) As $d_0 = -\Delta_{\mathbb{Z}^2} u$ and

$$w_0(x,y) \coloneqq \begin{cases} 1, & \text{if } x = y = 0, \\ 0, & \text{else,} \end{cases}$$

the assertion holds true for $k = 0$. The induction step $k \curvearrowright k+1$ is shown by means of (b), the induction hypothesis and (c):

$$\begin{aligned}
4d_{k+1}(x,y) &= \Theta d_k(x,y) \\
&= d_k(x+1,y) + d_k(x-1,y) + d_k(x,y+1) + d_k(x,y-1) \\
&= \sum_{\mu,\nu\in\mathbb{Z}} w_k(\mu,\nu)[-\Delta_{\mathbb{Z}^2} u](x+1+\mu, y+\nu) \\
&\quad + \sum_{\mu,\nu\in\mathbb{Z}} w_k(\mu,\nu)[-\Delta_{\mathbb{Z}^2} u](x-1+\mu, y+\nu) \\
&\quad + \sum_{\mu,\nu\in\mathbb{Z}} w_k(\mu,\nu)[-\Delta_{\mathbb{Z}^2} u](x+\mu, y+1+\nu) \\
&\quad + \sum_{\mu,\nu\in\mathbb{Z}} w_k(\mu,\nu)[-\Delta_{\mathbb{Z}^2} u](x+\mu, y-1+\nu) \\
&= \sum_{\mu,\nu\in\mathbb{Z}} w_k(\mu-1,\nu)[-\Delta_{\mathbb{Z}^2} u](x+\mu, y+\nu) \\
&\quad + \sum_{\mu,\nu\in\mathbb{Z}} w_k(\mu+1,\nu)[-\Delta_{\mathbb{Z}^2} u](x+\mu, y+\nu) \\
&\quad + \sum_{\mu,\nu\in\mathbb{Z}} w_k(\mu,\nu-1)[-\Delta_{\mathbb{Z}^2} u](x+\mu, y+\nu) \\
&\quad + \sum_{\mu,\nu\in\mathbb{Z}} w_k(\mu,\nu+1)[-\Delta_{\mathbb{Z}^2} u](x+\mu, y+\nu) \\
&= \sum_{\mu,\nu\in\mathbb{Z}} \Theta w_k(\mu,\nu)[-\Delta_{\mathbb{Z}^2} u](x+\mu, y+\nu) \\
&= 4 \sum_{\mu,\nu\in\mathbb{Z}} w_{k+1}(\mu,\nu)[-\Delta_{\mathbb{Z}^2} u](x+\mu, y+\nu).
\end{aligned}$$

(e) By construction

$$\sum_{k=0}^{M} w_k(0,0) = \sum_{\substack{k=0 \\ k \text{ even}}}^{M} \frac{1}{4^k} \binom{k}{\frac{k}{2}}^2 = \sum_{k=0}^{\lfloor \frac{M}{2} \rfloor} \frac{1}{16^k} \binom{2k}{k}^2$$

and the divergence $\sum_{k=0}^{\lfloor \frac{M}{2} \rfloor} \frac{1}{16^k} \binom{2k}{k}^2 \to +\infty$ for $M \to \infty$ follows in view of

$$\frac{\frac{1}{16^k}\binom{2k}{k}^2}{\frac{1}{16^{k-1}}\binom{2(k-1)}{k-1}^2} = \frac{1}{16}\left[\frac{2k(2k-1)}{k^2}\right]^2 = \left(1 - \frac{1}{2k}\right)^2 = 1 - \frac{1}{k} + \frac{1}{4k^2} \geq 1 - \frac{1}{k}$$

with Raabe's test (see e.g. [16, Section 33.10]). □

Lemma A.3 (Spherical harmonics)
For $n \geq 2$, $k \in \{1, \ldots, n\}$, let

$$\omega := \mathbb{R}^{n,k} \cap \mathbb{S}^{n-1} = \{x \in \mathbb{S}^{n-1} : x_1, \ldots, x_k > 0\}.$$

Then, the first eigenvalue-eigenfunction-pair $(\tilde{\lambda}_1, \tilde{\psi}_1)$ for

$$\begin{cases} -\Delta_{\mathbb{S}^{n-1}} \tilde{\psi} = \tilde{\lambda}\tilde{\psi} & \text{in } \omega, \\ \tilde{\psi} = 0 & \text{on } \partial\omega \end{cases} \tag{A.1}$$

is given by $\tilde{\psi}_1(x) = \prod_{i=1}^k x_i$ and $\tilde{\lambda}_1 = k(k+n-2)$.

Proof. The function $f \colon \mathbb{R}^n \to \mathbb{R}$, $f(x) := \prod_{i=1}^k x_i$ is harmonic and a homogeneous polynomial of degree k, i.e.,

$$f(tx) = t^k f(x)$$

for all $x \in \mathbb{R}^n$ and $t > 0$. Therefore, Proposition 1.8 in Gallier's article ([10]) yields that the restriction $f|_{\mathbb{S}^{n-1}} =: \tilde{\psi}_\star$ solves

$$-\Delta_{\mathbb{S}^{n-1}} \tilde{\psi} = \tilde{\lambda}\tilde{\psi} \quad \text{in } \mathbb{S}^{n-1}$$

with eigenvalue $\tilde{\lambda}_\star := k(k+n-2)$. By construction we have $\tilde{\psi}_\star = 0$ on $\partial\omega$ and hence, $\tilde{\psi}_\star$ is an eigenfunction for (A.1) with corresponding eigenvalue $\tilde{\lambda}_\star$. The positivity of $\tilde{\psi}_\star$ in ω shows that $\tilde{\lambda}_\star$ is indeed the first eigenvalue ([5]). □

Definition A.4 (Odd reflected function)
Let $\Omega := \prod_{i=1}^n (a_i, b_i) \subset \mathbb{R}^n$ be a hypercube and $h > 0$ be admissible. For $x \in \mathbb{R}^n$ and fixed $i \in \{1, \ldots, n\}$ we denote by $x^- := x + 2(a_i - x_i)e_i$ and $x^+ := x + 2(b_i - x_i)e_i$ the image points of x reflected with respect to the hyperplanes $\{x_i = a_i\}$ and $\{x_i = b_i\}$, respectively. Thereby, the corresponding reflected domains *are given by*

$$\Omega^{i-} := \{x \in \mathbb{R}^n : x^- \in \Omega\},$$
$$\Omega^{i+} := \{x \in \mathbb{R}^n : x^+ \in \Omega\}$$

as well as $M := \left(\overline{\Omega} \cup \overline{\Omega^{i+}} \cup \overline{\Omega^{i-}}\right)^{\circ}$. *Moreover, for a given function* $u\colon \overline{\Omega_h} \to \mathbb{R}$, *we define the* odd reflected *function* $v\colon \overline{M_h} \to \mathbb{R}$ *by*

$$v(x) := \begin{cases} u(x), & x \in \overline{\Omega_h}, \\ -u(x^-), & x \in \overline{\Omega_h^{i-}}, \\ -u(x^+), & x \in \overline{\Omega_h^{i+}}, \\ 0, & \textit{if } x_i = a_i \textit{ or } x_i = b_i. \end{cases}$$

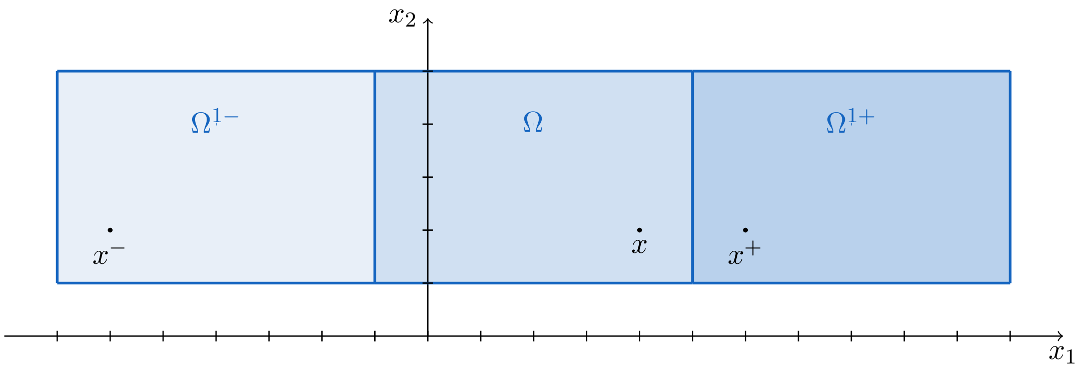


Figure A.1: Reflected domains for the rectangle $\Omega = (-1, 5) \times (1, 5)$.

Proposition A.5 (Discrete Schwarz reflection principle for hypercubes)
Let $p \in (1, \infty)$, $\Omega := \prod_{i=1}^n (a_i, b_i) \subset \mathbb{R}^n$ *be a hypercube and correspondingly* $h > 0$ *be an admissible grid size. Moreover, let* $u\colon \overline{\Omega_h} \to \mathbb{R}$ *be a solution of*

$$\begin{cases} -\Delta_h u = |u|^{p-1} u & \textit{in } \Omega_h, \\ u = 0 & \textit{on } \partial_h \Omega_h. \end{cases} \tag{A.2}$$

Then, for fixed $i \in \{1, \dots, n\}$, *the* odd reflected *function* $v\colon \overline{M_h} \to \mathbb{R}$ *given by Definition A.4 solves*

$$\begin{cases} -\Delta_h v = |v|^{p-1} v & \textit{in } M_h, \\ v = 0 & \textit{on } \partial_h M_h \end{cases}$$

and $\|v\|_{L^\infty(\overline{M_h})} = \|u\|_{L^\infty(\overline{\Omega_h})}$.

Proof. The assertion is verified by direct computation. First, for $x \in \Omega_h$ we have

$$-\Delta_h v(x) = -\Delta_h u(x) = |u(x)|^{p-1} u(x) = |v(x)|^{p-1} v(x).$$

Moreover, for $x \in \Omega_h^{i-}$ we obtain

$$-\Delta_h v(x) = \Delta_h u(x^-) = -|u(x^-)|^{p-1} u(x^-) = |v(x)|^{p-1} v(x).$$

A similar calculation can be done for $x \in \Omega_h^{i+}$. Finally, let $x \in M_h$ with $x_i = a_i$. Then, from $0 = v(x) = v(x \pm he_j)$ for all $j \neq i$ together with $(x - he_i)^- = x + he_i$ we deduce

$$-\Delta_h v(x) = -\frac{1}{h^2}\sum_{j=1}^{n}[v(x+he_j) - 2v(x) + v(x-he_j)] = -\frac{1}{h^2}[v(x+he_i) + v(x-he_i)]$$
$$= -\frac{1}{h^2}[u(x+he_i) - u(x+he_i)] = 0 = |v(x)|^{p-1}v(x).$$

If $x \in M_h$ with $x_i = b_i$, the argumentation is analogous. Additionally, by construction we see that $\|v\|_{L^\infty(\overline{M_h})} = \|u\|_{L^\infty(\overline{\Omega_h})}$. □

Proposition A.6 (Discrete Schwarz reflection principle for triangles)
Let $p \in (1,\infty)$, $b > 0$, $\Omega := \{x \in \mathbb{R}^2 : x_1, x_2 > 0,\ x_1 + x_2 < b\}$ a triangle and $h > 0$ an admissible grid size. Further, let $u\colon \overline{\Omega_h} \to \mathbb{R}$ be a solution of

$$\begin{cases} -\Delta_h u = |u|^{p-1}u & \text{in } \Omega_h, \\ u = 0 & \text{on } \partial_h\Omega_h. \end{cases} \tag{A.3}$$

Moreover, for $x \in \mathbb{R}^2$ we define the reflected element *with respect to the straight line $\{x \in \mathbb{R}^2 : x_2 = b - x_1\}$ by $x^* := (b - x_2, b - x_1)^T$ and the* reflected domain

$$\Omega^* := \{x \in \mathbb{R}^2 : x^* \in \Omega\}$$

as well as $M := \left(\overline{\Omega} \cup \overline{\Omega^*}\right)^\circ = (0,b)^2$.

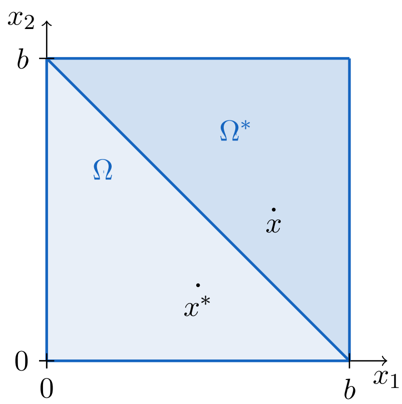


Figure A.2: Schwarz reflexion principle for a triangle $\Omega = \{x_1, x_2 > 0,\ x_1 + x_2 < b\}$.

Then, the odd reflected *function $v\colon \overline{M_h} \to \mathbb{R}$ given by*

$$v(x) := \begin{cases} u(x), & x \in \overline{\Omega_h}, \\ -u(x^*), & x \in \overline{\Omega_h^*}, \\ 0, & x \in \partial_h M_h \end{cases}$$

solves

$$\begin{cases} -\Delta_h v = |v|^{p-1} v & \text{in } M_h, \\ v = 0 & \text{on } \partial_h M_h \end{cases}$$

and $\|v\|_{L^\infty(\overline{M_h})} = \|u\|_{L^\infty(\overline{\Omega_h})}$.

Proof. For $x \in \Omega_h$ we have

$$-\Delta_h v(x) = -\Delta_h u(x) = |u(x)|^{p-1} u(x) = |v(x)|^{p-1} v(x).$$

Besides, for $x \in \Omega_h^*$ we obtain

$$-\Delta_h v(x) = \Delta_h u(x^*) = -|u(x^*)|^{p-1} u(x^*) = |v(x)|^{p-1} v(x).$$

Finally, let $x \in M_h$ with $x_2 = b - x_1$. Then, recalling $h > 0$,

$$(x + he_1)^* = (x_1 + h, x_2)^* = (b - x_2, b - x_1 - h) = (x_1, x_2 - h) = x - he_2 \in \overline{\Omega_h} \quad \text{and}$$
$$(x + he_2)^* = (x_1, x_2 + h)^* = (b - x_2 - h, b - x_1) = (x_1 - h, x_2) = x - he_1 \in \overline{\Omega_h}.$$

Therefore, also using $v(x) = 0$, we conclude

$$\begin{aligned} -\Delta_h v(x) &= -\frac{1}{h^2} \sum_{j=1}^{2} [v(x + he_j) - 2v(x) + v(x - he_j)] \\ &= -\frac{1}{h^2} [v(x + he_1) + v(x - he_1) + v(x + he_2) + v(x - he_2)] \\ &= -\frac{1}{h^2} [-u(x - he_2) + u(x - he_1) - u(x - he_1) + u(x - he_2)] \\ &= 0 = |v(x)|^{p-1} v(x). \end{aligned}$$

By construction, the boundary condition is fulfilled and $\|v\|_{L^\infty(\overline{M_h})} = \|u\|_{L^\infty(\overline{\Omega_h})}$ holds true. □

Lemma A.7 (Regularity in the interior)
Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a (possibly unbounded) domain. Moreover, let $v \in C^{0,\alpha}_{\mathrm{loc}}(\overline{\Omega}) \cap W^{1,q}_{\mathrm{loc}}(\Omega)$ with $q > n$ and $\alpha \in (0, 1 - \frac{n}{q})$ be a non-negative solution of

$$\int_\Omega v(-\Delta\psi)\, dx = \int_\Omega v^p \psi\, dx \quad \text{for all } \psi \in C^\infty_{\mathrm{c}}(\Omega).$$

Then, $v \in C^2(\Omega)$ and v solves

$$-\Delta v = v^p \quad \text{in } \Omega.$$

Proof. Let $B \subset \Omega$ an open ball. We consider the problem

$$\begin{cases} -\Delta w = v^p & \text{in } B, \\ \quad w = v & \text{on } \partial B. \end{cases} \tag{A.4}$$

Due to $v \in C^{0,\alpha}(\overline{B}, [0,\infty))$ we deduce $v^p \in C^{0,\alpha}(B) \cap L^\infty(B)$ and $v|_{\partial B} \in C(\partial B)$. Hence, according to classical existence and elliptic regularity theory (e.g. [13, Thm. 4.3]) there exists a unique solution $w \in C^2(\Omega) \cap C(\overline{\Omega})$ of (A.4). Therefore,

$$\int_B w(-\Delta\psi)\,dx = \int_B (-\Delta w)\psi\,dx = \int_B v^p\psi\,dx = \int_B v(-\Delta\psi)\,dx$$

and integration by parts yields

$$0 = \int_B (w-v)(-\Delta\psi)\,dx = \int_B \nabla(w-v)\cdot\nabla\psi\,dx$$

for all $\psi \in C_c^\infty(B)$. In view of $q > n \geq 2$ we have $w - v \in W^{1,2}(B)$ as well as $w - v = 0$ on ∂B. Thus, $w - v \in W_0^{1,2}(B)$ and since $C_c^\infty(B)$ is dense in $W_0^{1,2}(B)$, we may choose the special test function $\psi = w - v$ and obtain $w = v$. As $B \subset \Omega$ was arbitrary, we conclude $v \in C^2(\Omega)$ with $-\Delta v = v^p$ in Ω. □

Lemma A.8 (Pointwise estimates from gradient bounds)
Let $Q = \prod_{i=1}^n [a_i, b_i]$ be a closed cube in $\mathbb{R}^n$ with dimension $n \geq 2$ and edge length $r := b_i - a_i$ for all $i = 1, \ldots, n$. Further let $q \in (n, \infty)$ and $\beta := 1 - \frac{n}{q} \in (0,1)$.

(a) *There exists a constant $C > 0$ independent of Q such that*

$$|f(x) - f(y)| \leq C \| |\nabla f|_q \|_{L^q(Q)} |x - y|_\infty^\beta$$

for all $x, y \in \partial Q$ with $|x - y|_\infty = r$ and all $f \in W^{1,q}(Q)$.

(b) *Additionally, let $h > 0$ and $a_i, b_i \in \mathbb{R}_h$ ($i = 1, \ldots, n$), i.e., Q is admissible for h. Then,*

$$|f(x) - f(y)| \leq C \|f\|_{\dot{W}_0^{1,q}(\hat{Q}_h)} |x - y|_\infty^\beta = C \left(\sum_{i=1}^n \sum_{z \in \hat{Q}_h \setminus \hat{\partial}_i^+ \hat{Q}_h} |D_i^+ f(z)|^q h^n \right)^{\frac{1}{q}} |x - y|_\infty^\beta$$

for all $f \colon \hat{Q}_h \to \mathbb{R}$ and $x, y \in \hat{\partial}_h \hat{Q}_h$ with $|x - y|_\infty = r$, where the constant $C > 0$ depends only on n and p.

Proof. (a) The following argumentation goes back to the proof of Lemma 4.28 in [1]. Let $f \in C^1(Q)$ and $x_0 \in Q$. We define $M := \frac{1}{|Q|}\int_Q f\,dx = r^{-n}\int_Q f\,dx$. By means of the transformation $y = t(x - x_0)$ and Hölder's inequality we compute

$$|f(x_0) - M| = \left| r^{-n} \int_Q (f(x_0) - f(x))\,dx \right|$$

$$
\begin{aligned}
&= r^{-n} \left| \int_Q \int_1^0 \frac{d}{dt} f(x_0 + t(x - x_0))\, dt\, dx \right| \\
&\le r^{-n} \int_Q \int_0^1 |\nabla f(x_0 + t(x - x_0)) \cdot (x - x_0)|\, dt\, dx \\
&\le r^{1-n} \int_0^1 \int_Q |\nabla f(x_0 + t(x - x_0))|_1 \, dx\, dt \\
&= r^{1-n} \int_0^1 \int_{t(Q-x_0)} |\nabla f(x_0 + y)|_1\, t^{-n}\, dy\, dt \\
&\le r^{1-n} \int_0^1 \left(\int_{t(Q-x_0)} |\nabla f(x_0 + y)|_1^q \, dy \right)^{\frac{1}{q}} \left(\int_{t(Q-x_0)} 1\, dy \right)^{\frac{1}{q'}} t^{-n}\, dt \\
&\le r^{1+\frac{n}{q'}-n} \int_0^1 \left(\int_{tQ+(1-t)x_0} |\nabla f(z)|_1^q \, dz \right)^{\frac{1}{q}} t^{\frac{n}{q'}-n}\, dt \\
&\le c r^{1+\frac{n}{q'}-n} \left(\int_Q |\nabla f(z)|_q^q \, dz \right)^{\frac{1}{q}} \int_0^1 t^{\frac{n}{q'}-n}\, dt \\
&= C r^{\beta} \| |\nabla f|_q \|_{L^q(Q)}
\end{aligned}
$$

with constants $c, C > 0$ which depend only on n and q. The crucial constraint $\frac{n}{q'} - n > -1$ is equivalent to the assumption $q > n$ whereas $tQ + (1-t)x_0 \subset Q$ for all $t \in [0, 1]$ is due to the convexity of the cube Q. Now, let $x, y \in \partial Q$ with $|x - y|_\infty = r$. By means of the estimate above we conclude

$$
\begin{aligned}
|f(x) - f(y)| &\le |f(x) - M| + |f(y) - M| \\
&\le 2C r^{\beta} \| |\nabla f|_q \|_{L^q(Q)} \\
&= 2C \| |\nabla f|_q \|_{L^q(Q)} |x - y|_\infty^{\beta}.
\end{aligned}
$$

As $C_c^\infty(Q) = \{f|_Q \colon f \in C_c^\infty(\mathbb{R}^n)\}$ is dense in $W^{1,q}(Q)$ due to [14, Thm. 1.4.2.1] we obtain the estimate also for all $f \in W^{1,q}(Q)$.

(b) By means of interpolation theory, the second statement is a direct consequence of the first one: Let $f \colon \hat{Q}_h \to \mathbb{R}$. According to the proof of [30, Theorem 8.12.] there exists a corresponding tensor product interpolant $\hat{f} \colon \overline{Q} \to \mathbb{R}$ such that

(i) $\hat{f} = f$ on $\hat{Q}_h$ and

(ii) there is a constant $C(n, q) > 0$ such that

$$
\|\hat{f}\|_{\dot{W}_0^{1,q}(Q)} \le C(n, q) \|f\|_{\dot{W}_0^{1,q}(\hat{Q}_h)}.
$$

Note that in the statement of Theorem 8.12. a boundary condition $f = 0$ on $\hat{\partial}_h \hat{Q}_h$ is required which is not needed in the proof. Employing the interpolant $\hat{f}$ directly yields

$$\begin{aligned}|f(x) - f(y)| &= |\hat{f}(x) - \hat{f}(y)| \\ &\le C\||\nabla \hat{f}|_q\|_{L^q(Q)}|x-y|_\infty^\beta \\ &= C\|\hat{f}\|_{\dot{W}_0^{1,q}(Q)}|x-y|_\infty^\beta \\ &\le \tilde{C}\|f\|_{\dot{W}_0^{1,q}(\hat{Q}_h)}|x-y|_\infty^\beta \\ &= \tilde{C}\left(\sum_{i=1}^n \sum_{z\in\hat{Q}_h\setminus\hat{\partial}_i^+\hat{Q}_h} |D_i^+ f(z)|^q h^n\right)^{\frac{1}{q}} |x-y|_\infty^\beta\end{aligned}$$

for all $x, y \in \hat{\partial}_h \hat{Q}_h$ with $|x-y|_\infty = r$ and constants $C, \tilde{C} > 0$ depending only on q and n. □

Lemma A.9
Let $g\colon [0,\infty) \to \mathbb{R}$ be continuous with $g \in o(y^p)$ as $y \to \infty$. Then,

$$\frac{g(yt)}{y^p} \to 0 \qquad (y \to \infty)$$

uniformly for all $t \in [0,1]$.

Proof. Let $\varepsilon > 0$. Since $g \in \mathrm{o}(y^p)$ as $y \to \infty$, there exists some $y_0 > 0$ such that

$$\left|\frac{g(y)}{y^p}\right| \le \varepsilon \quad \text{for all } y \ge y_0.$$

In view of the continuity of g, we may assign

$$\mathcal{B} \coloneqq \max_{y\in[0,y_0]} |g(y)|.$$

Next, we fix some $t \in [0,1]$. Then, for all $y \ge \max\left\{y_0, \left(\frac{\mathcal{B}}{\varepsilon}\right)^{\frac{1}{p}}\right\}$ we deduce

$$\left|\frac{g(yt)}{y^p}\right| \le \frac{\mathcal{B}}{y^p} \le \varepsilon \quad \text{if } yt \le y_0,$$

as well as

$$\left|\frac{g(yt)}{y^p}\right| = \left|\frac{g(yt)}{y^p t^p}\right| t^p \le \varepsilon t^p \le \varepsilon \quad \text{if } yt > y_0.$$

□

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