Geometric Cycles in Locally Symmetric Spaces Covered by Products of the Hyperbolic Plane

Zur Erlangung des akademischen Grades eines

DOKTORS DER NATURWISSENSCHAFTEN

von der KIT-Fakultät für Mathematik des
Karlsruher Instituts für Technologie (KIT)
genehmigte

DISSERTATION

von

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Tag der mündlichen Prüfung: 24. Juli 2019

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Contents

Chapter 1

Introduction

In this thesis, we study the homology of locally symmetric spaces. A *locally symmetric space* is a Riemannian manifold M in which for any point $p \in M$, the geodesic reflection in p is a local isometry of M. These manifolds appear naturally in a wide range of mathematical areas, for instance in topology as examples of aspherical manifolds or in geometry as moduli spaces of lattices.

It is in general hard to compute the homology of a locally symmetric space, and even if one can do so, then the geometric meaning of the homology classes is often lost during the computation. We choose a more geometric approach going back to Millson [Mil76], in which one studies the images of fundamental classes of certain totally geodesic submanifolds in the homology of a locally symmetric space. These totally geodesic submanifolds are obtained from subspaces of the symmetric space that is the universal covering space of the locally symmetric space and are called *geometric cycles*. In order to conclude that the fundamental class of a geometric cycle is nontrivial in the homology of the locally symmetric space, one finds another geometric cycle which intersects the first one in such a way that their intersection product is nonzero. Often the two geometric cycles are of complementary dimensions and intersect transversally, in which case it suffices to show that their intersection numbers are all of the same sign.

This technique has been successfully applied to find nontrivial homology classes in the homology of some families of compact locally symmetric spaces by Millson and Raghunathan in [Mil76; MR81] and in the homology of the locally symmetric space $\Gamma\backslash \mathrm{SL}_3(\mathbb{R})/\mathrm{SO}(3)$ for a torsion-free lattice Γ commensurable with $\mathrm{SL}_3(\mathbb{Z})$ by Lee and Schwermer in [LS86]. The geometric cycles considered in these articles are so-called *special cycles*, that is, they arise from the fixed point sets of rational involutions of the symmetric space that universally covers the locally symmetric space. When the fixed point sets of two such rational involutions intersect orthogonally, and some compatibility conditions are satisfied, then it is possible to control the intersection numbers of the corresponding special cycles in a finite covering space of the locally symmetric space.

A different approach to find geometric cycles that give rise to nontrivial homology classes was developed by Avramidi and Nguyen-Phan in [AN15] for the locally symmetric space $\Gamma\backslash \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$, where $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$ is a torsion-free subgroup of finite index. In that article, the authors study geometric cycles that come from maximal flat subspaces of the symmetric space $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$ and complementary subspaces isomorphic to $\mathrm{SL}_{n-1}(\mathbb{R})\backslash\mathrm{SO}(n-1)\times\mathbb{R}$. These subspaces are in general not fixed point sets of rational involutions and they do not need to intersect orthogonally. Nevertheless, it is possible to control the intersection numbers in some finite covering space of the locally symmetric space. This method gives more flexibility in choosing the geometric cycles compared to the method of special cycles and makes it possible to find a large number of linearly independent homology classes coming from geometric cycles.

It is known that any nonpositively curved locally symmetric space M of finite volume has a compact flat totally geodesic immersed submanifold of dimension equal to the rank of M, which is the maximal dimension of a flat totally geodesic immersed submanifold of M. Furthermore, Pettet and Souto have shown in [PS14, Theorem 1.2] that these submanifolds are *non-peripheral*, that is, they cannot be homotoped outside of every compact subset of M. This raises the question of whether or not these submanifolds contribute to the homology of M. In [AN15], it has been shown that for $M=\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$, they give rise to nontrivial $(n-1)$-dimensional homology classes in the free part of the homology of finite covering spaces of M. Note that here $n-1$ is the rank of M. But there exist also counterexamples: Any locally symmetric space of finite volume covered by quaternionic hyperbolic n-space has rank one, but vanishing first Betti number, because lattices in $\mathrm{Sp}(n,1)$ for $n\geq 2$ have Property (T), and so one can only expect torsion homology classes in this case.

We give in this thesis an answer to the above question for all locally symmetric spaces of finite volume that are covered by a product of the real hyperbolic plane $\mathbb{H}^2$. An interesting example of such a locally symmetric space is a so-called *Hilbert modular surface*, which is constructed as follows: Consider the real quadratic number field $F=\mathbb{Q}(\sqrt{d})$ associated to a square-free integer $d>0$. There are exactly two distinct field embeddings $\sigma_1,\sigma_2\colon F\hookrightarrow\mathbb{R}$, determined by $\sigma_1(\sqrt{d})=\sqrt{d}$ and $\sigma_2(\sqrt{d})=-\sqrt{d}$, respectively. Let $\mathcal{O}_F$ be the ring of integers of F. Then the group $\mathrm{SL}_2(\mathcal{O}_F)$ acts properly discontinuously on the product $\mathbb{H}^2\times\mathbb{H}^2$ by

$$g\cdot(z_1,z_2):=(\sigma_1(g)\cdot z_1,\sigma_2(g)\cdot z_2),$$

where $\sigma_i(g)\cdot z_i$ is the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}^2$ by fractional linear transformations. Any torsion-free subgroup of finite index $\Gamma\subset\mathrm{SL}_2(\mathcal{O}_F)$ acts freely and properly discontinuously on $\mathbb{H}^2\times\mathbb{H}^2$, and the quotient space $\Gamma\backslash(\mathbb{H}^2\times\mathbb{H}^2)$ is a locally symmetric space of finite volume which is not compact and not finitely covered by a product of hyperbolic surfaces. This example shows that the lattices that we consider are in general not commensurable to $\mathrm{SL}_n(\mathbb{Z})$ and they do not have to be cocompact, though there are also cocompact ones due to a theorem of Borel [Bor63]. Hence our work is different from [Mil76; MR81; LS86] and [AN15].

We prove the following theorem, which shows that the fundamental classes of compact flat totally geodesic submanifolds of maximal dimension contribute significantly to the homology of these locally symmetric spaces:

Theorem 1.1. *Let M be a locally symmetric space of finite volume covered by $(\mathbb{H}^2)^r$. Then for any $n \in \mathbb{N}$, there exists a connected finite covering $M' \to M$ and compact oriented flat totally geodesic r-dimensional submanifolds $A_1, \ldots, A_n \subset M'$ such that the images of the fundamental classes $[A_1], \ldots, [A_n]$ in $H_r(M'; \mathbb{R})$ are linearly independent.*

In particular, it follows that the r-th Betti number of a locally symmetric space of finite volume that is covered by $(\mathbb{H}^2)^r$ can be made arbitrarily large by going to a finite covering space of the locally symmetric space.

For the proof of this theorem, it is convenient to decompose a locally symmetric space into a product whenever this is possible. Therefore, we define:

Definition 1.2. A locally symmetric space M is said to be *reducible* if it is finitely covered by a product $M_1 \times M_2$ of two locally symmetric spaces M_1 and M_2 of positive dimensions. Otherwise, M is said to be *irreducible*.

Using induction on $\dim(M)$, one sees that for every locally symmetric space M, there exist irreducible locally symmetric spaces $M_1, \ldots, M_k$ and a finite covering $M_1 \times \ldots \times M_k \to M$. An application of the Künneth theorem for homology now shows that it suffices to prove Theorem 1.1 for all irreducible locally symmetric spaces (see Proposition 6.29 in Chapter 6).

Every irreducible locally symmetric space of finite volume that is covered by $(\mathbb{H}^2)^r$ is also finitely covered by a quotient $\Gamma \backslash (\mathbb{H}^2)^r$ for an irreducible lattice $\Gamma \subset \mathrm{SL}_2(\mathbb{R})^r$. By Margulis' arithmeticity theorem, the lattice Γ is *arithmetically defined* whenever $r \geq 2$. Roughly speaking, this means that Γ is commensurable with a group obtained from the integer points of an algebraic group over $\mathbb{Q}$ (see Definition 4.47 for a precise definition). Examples of arithmetically defined lattices are $\mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{SL}_2(\mathbb{R})$ and groups coming from the integer points of algebraic groups over number fields, such as the image of the group $\mathrm{SL}_2(\mathcal{O}_F)$ in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ with $F = \mathbb{Q}(\sqrt{d})$ in the case of a Hilbert modular surface. It is possible to give a uniform description of all arithmetically defined lattices in $\mathrm{SL}_2(\mathbb{R})^r$ up to commensurability using quaternion algebras. We give a proof of this characterization and use it to prove Theorem 1.1 for all locally symmetric spaces that are covered by a product with at least two factors of $\mathbb{H}^2$. The locally symmetric spaces that are covered by a single factor of $\mathbb{H}^2$ are hyperbolic surfaces and we give a separate and more geometric proof for them.

This thesis is structured as follows: In Chapter 2, we study homology classes of closed geodesics in the homology of hyperbolic surfaces and prove Theorem 1.1 for the case $r = 1$. In Chapter 3, we introduce general symmetric spaces and describe locally symmetric spaces as quotient spaces of symmetric spaces by discrete groups of isometries. In Chapter 4, we discuss algebraic groups and their arithmetic subgroups and state Margulis' arithmeticity theorem. In Chapter 5, we introduce quaternion algebras and prove that every arithmetically defined lattice in $\mathrm{SL}_2(\mathbb{R})^r$ is commensurable with a subgroup derived from a quaternion algebra. Finally, we use this description in Chapter 6, where we apply the method of geometric cycles to complete the proof of Theorem 1.1.

Acknowledgments

First of all, I want to thank my advisor Enrico Leuzinger for his support and guidance during my time as a PhD student and for always having the right advice at the right time. I am very grateful to the DFG Research Training Group 2229 and the Karlsruhe Institute of Technology for giving me the opportunity to pursue mathematics by providing financial support and a research environment for me while I was working on this project. Further, I want to thank Roman Sauer for helpful discussions and for acting as a second referee for this thesis. I also want to thank Stefan Kühnlein for reading an early manuscript of this work and giving important feedback, and Grigori Avramidi, Holger Kammeyer, Aurel Page, Benjamin Waßermann and Mike Miller for answering questions and helpful discussions. The treatment of algebraic groups and arithmetic groups in this thesis largely benefited by the presentation given by Steffen Kionke in his RTG lecture in the winter term 2018/19.

Finally, I want to thank my parents Rolf and Inge for invaluably supporting me throughout my life, and Miriam, for being there for me.

Chapter 2

Closed Geodesics in Hyperbolic Surfaces

In this chapter, we prove Theorem 1.1 for hyperbolic surfaces (the case $r = 1$). A *hyperbolic surface* is a complete 2-dimensional Riemannian manifold of constant sectional curvature -1. The universal covering space of such a surface is the hyperbolic plane $\mathbb{H}^2$ and so these surfaces are the simplest examples of locally symmetric spaces covered by a product of $\mathbb{H}^2$. We study closed geodesics in a hyperbolic surface because their images are the compact flat totally geodesic one-dimensional submanifolds, where by flat we mean that their curvature tensor vanishes. The topology of a hyperbolic surface is not as complicated as the topology of a general locally symmetric space, and so we do not yet need the general theory of symmetric spaces which we will develop later in Chapter 3.

It is known that every connected orientable hyperbolic surface of finite area is homeomorphic to a connected compact orientable surface of genus g minus a finite number m of points (see [Rat06, Theorem 9.8.2]). We denote this topological surface by $\Sigma_{g,m}$. The removed points can be thought of as punctures of the surface, near which the distances in the hyperbolic metric become arbitrarily small (see Figure 1).

Definition 2.1. Let M be a connected orientable hyperbolic surface of finite area. A closed subset of M is called a *neighborhood of a puncture* if it is homeomorphic to a punctured disc. A *closed curve* in M is a continuous map $S^1 \to M$, and a closed curve is called *simple* if it is injective.

Observe that if a closed curve in M is freely homotopic into a neighborhood of a puncture, then it can be homotoped to be arbitrarily small because the

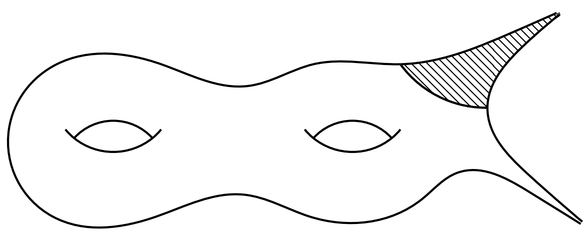


Figure 1: A hyperbolic surface and a neighborhood of a puncture.

distances in M become smaller and smaller when approaching a puncture. For any other closed curve in M, we can find a unique closed geodesic in its free homotopy classes by [FM12, Propositions 1.3 and 1.6]:

Lemma 2.2. *Let M be a connected orientable hyperbolic surface of finite area and let $\gamma\colon S^1 \to M$ be a closed curve which is not freely homotopic into a neighborhood of a puncture. Then there exists a unique closed geodesic $\widetilde{\gamma}\colon S^1 \to M$ which is freely homotopic to γ. If γ is simple, then $\widetilde{\gamma}$ is simple.*

Using this lemma, we can now prove the first statement about the homology classes of closed geodesics in the homology of a hyperbolic surface.

Proposition 2.3. *In every connected orientable hyperbolic surface M of finite area and genus g, there exist simple closed geodesics $\gamma_1, \dots, \gamma_g\colon S^1 \to M$ such that the homology classes $[\gamma_1], \dots, [\gamma_g] \in H_1(M;\mathbb{R})$ are linearly independent.*

Proof. We know that M is homeomorphic to the topological surface $\Sigma_{g,m}$ for some m. By [JS87, p. 262], the fundamental group of M has a presentation of the form

$$\pi_1(M) = \left\langle a_1, b_1, \dots, a_g, b_g, r_1, \dots, r_m \,\middle|\, \prod_{i=1}^{g} a_i^{-1} b_i^{-1} a_i b_i \prod_{j=1}^{m} r_j \right\rangle, \tag{2.1}$$

where a_i, b_i and r_j are path homotopy classes of simple closed curves as depicted in Figure 2. The first homology group $H_1(M;\mathbb{Z})$ is the abelianization of $\pi_1(M)$, and so the curves $a_1, \dots, a_g$ in the above presentation determine linearly independent real homology classes $[a_1], \dots, [a_g] \in H_1(M;\mathbb{R})$. Moreover, because the curves a_i cannot be freely homotoped into a neighborhood of a puncture, there exists by Lemma 2.2 in the free homotopy class of each a_i a simple closed geodesic $\gamma_i\colon S^1 \to M$. Freely homotopic curves are homologous, and so the geodesics γ_i determine the same real homology classes as the curves a_i. Thus, the homology classes $[\gamma_1], \dots, [\gamma_g] \in H_1(M;\mathbb{R})$ are linearly independent. □

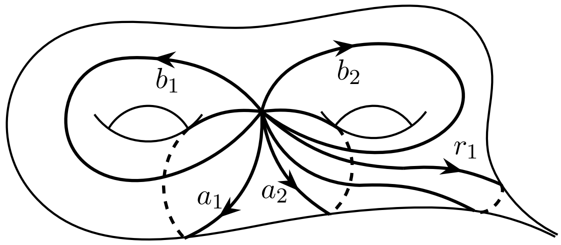


Figure 2: Generators of the fundamental group of a hyperbolic surface.

We have now found a number of linearly independent homology classes coming from closed geodesics in the homology of a hyperbolic surface that is proportional to its genus. It remains to show that the genus can be made arbitrarily large by going to a finite covering space of the surface. This is what we do next.

Definition 2.4. The surface $\Sigma_{g,m}$ is said to be of *hyperbolic type* if it admits the structure of a hyperbolic surface of finite area.

Note that $\Sigma_{g,m}$ is of hyperbolic type if and only if its Euler characteristic is negative, that is, if $\chi(\Sigma_{g,m}) = 2 - 2g - m < 0$.

Lemma 2.5. *Every surface $\Sigma_{g,m}$ of hyperbolic type is finitely covered by a surface $\Sigma_{g',m'}$ of hyperbolic type and strictly larger genus $g' > g$.*

Proof. First, assume that $g > 1$. Then we have $2g - 1 > 1$, and by identifying every point of the surface $\Sigma_{2g-1,0}$ with its image under the point reflection in the center of the middle hole, as shown in Figure 3, we obtain a double covering $\Sigma_{2g-1,0} \to \Sigma_{g,0}$. Next, we remove m points from the base space, and so we get a double covering $\Sigma_{2g-1,2m} \to \Sigma_{g,m}$ of the given surface $\Sigma_{g,m}$. The genus of the so obtained covering space is $2g - 1 > 2g - g = g$, hence we are done.

Assume now that $g = 1$. Then we must have $m > 0$ because $\Sigma_{g,m}$ is of hyperbolic type. We now use the four-fold covering $\Sigma_{2,2} \to \Sigma_{1,1}$ constructed in [Sch06, Example 2.1]. After removing $m - 1$ points from the base space, we get a four-fold covering $\Sigma_{2,4m-2} \to \Sigma_{1,m}$ with a covering space of genus $2 > 1 = g$.

Finally, assume that $g = 0$. Then $m > 2$, and so the fundamental group $\pi_1(\Sigma_{g,m})$ is free by (2.1). In particular, $\pi_1(\Sigma_{g,m})$ has a normal subgroup of some index $d \geq -2/\chi(\Sigma_{g,m})$. So there exists a d-fold connected covering $M' \to \Sigma_{g,m}$. With the lifted metric, M' is an orientable hyperbolic surface of finite area and hence is homeomorphic to $\Sigma_{g',m'}$ for some g' and m'. If $g' > 0$, then we are done. Otherwise, if $g' = 0$, then since the Euler characteristic is multiplicative under finite coverings and $\chi(\Sigma_{g,m}) < 0$, we have

$$2 - m' = \chi(\Sigma_{g',m'}) = d \cdot \chi(\Sigma_{g,m}) \leq -\frac{2}{\chi(\Sigma_{g,m})} \cdot \chi(\Sigma_{g,m}) = -2.$$

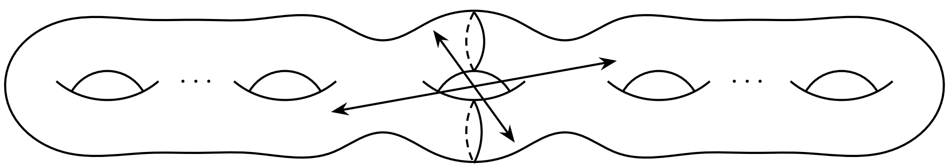


Figure 3: Point reflection symmetry of the surface $\Sigma_{2g-1,0}$ for $g > 1$.

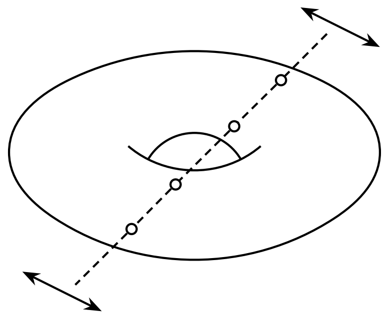


Figure 4: Line reflection symmetry of the four-times punctured torus $\Sigma_{1,4}$.

So $m' \geq 4$. Now we use the two-fold covering $\Sigma_{1,4} \to \Sigma_{0,4}$ that one obtains by identifying every point of $\Sigma_{1,4}$ with its images under the reflection at a line that goes through all the four puncture points as depicted in Figure 4 (see also [Ful95, Section 20e]). After removing $m' - 4$ points from the base space, we get a covering $\Sigma_{1,4+2(m'-4)} \to \Sigma_{g',m'}$. By composing this covering with the covering $\Sigma_{g',m'} \to \Sigma_{g,m}$ from above, we obtain a covering space of positive genus of the given surface $\Sigma_{g,m}$. □

By combining the results obtained so far, we can now prove the following theorem, which is a restatement of Theorem 1.1 for hyperbolic surfaces:

Theorem 2.6. *Let M be a connected hyperbolic surface of finite area. Then for every $n \in \mathbb{N}$, there exists a connected finite covering $M' \to M$ and simple closed geodesics $\gamma_1, \ldots, \gamma_n \colon S^1 \to M'$ such that $[\gamma_1], \ldots, [\gamma_n] \in H_1(M'; \mathbb{R})$ are linearly independent.*

Proof. If M is not orientable, then we can pass to its orientation covering. So assume that M is orientable. Then M is homeomorphic to $\Sigma_{g,m}$ for some g and m. By repeatedly applying Lemma 2.5, we obtain a connected finite covering $M' \to M$ so that M' is with the lifted metric a hyperbolic surface of finite area and genus at least n. Now Proposition 2.3 applied to M' yields simple closed geodesics $\gamma_1, \ldots, \gamma_n$ in M' as required. □

Chapter 3

Symmetric Spaces

In this chapter, we introduce the general theory of symmetric spaces and locally symmetric spaces. We present here only the parts of the theory that we need for this thesis. For a detailed exposition to the topic, we refer the reader to the textbooks [Hel78] and [Ebe96], and the article [Ji05].

Definition 3.1. Let M be a connected Riemannian manifold. The *geodesic reflection* in a point $p \in M$ is the local diffeomorphism

$$s_p := \exp_{M,p} \circ (-\operatorname{id}_{T_pM}) \circ {\exp_{M,p}}^{-1},$$

where $\exp_{M,p}$ is the Riemannian exponential map of M at p. We say that M is a *locally symmetric space* if for each point $p \in M$, the map s_p is a local isometry, and M is called a *(globally) symmetric space* if for each point $p \in M$, the map s_p can be extended to a global isometry of M.

One can show that a symmetric space is complete and has a transitive isometry group. Examples of symmetric spaces are the simply connected Riemannian manifolds of constant sectional curvature $\mathbb{R}^n$, S^n and $\mathbb{H}^n$ for $n > 1$. Every symmetric space is locally symmetric, but the converse is not true. Indeed, the hyperbolic surfaces of finite area that we have seen in Chapter 2 are locally symmetric but not globally symmetric, because they each have a finite and thus non-transitive isometry group. By [Hel78, Theorem IV.5.6], there is the following connection between locally and globally symmetric spaces:

Theorem 3.2. *Every simply connected complete locally symmetric space is a globally symmetric space.*

In particular, it follows that the universal covering space of a complete locally symmetric space is globally symmetric. We will focus first on symmetric spaces and will see later how to describe locally symmetric spaces.

Recall that we have defined in Definition 1.2 the notion of a reducible locally symmetric space. A simply connected symmetric space is reducible if and only if it is a nontrivial product of symmetric spaces of positive dimensions. Such a decomposition is unique in the following sense by [KN63, Theorem IV.6.2]:

Theorem 3.3 (de Rham Decomposition Theorem). *Every simply connected symmetric space M is isometric to a product*

$$M_0 \times M_1 \times \ldots \times M_k,$$

where M_0 is isometric to $\mathbb{R}^m$ for some $m \in \mathbb{N}_0$, and all M_i for $i > 0$ are irreducible symmetric spaces of positive dimension not isometric to $\mathbb{R}$. This decomposition is unique up to isometry and the order of the factors.

We call the manifolds M_i in the above decomposition the *de Rham factors of* M. One often considers the following two types of symmetric spaces:

Definition 3.4. A simply connected symmetric space is said to be of the *compact type* or the *noncompact type* if it has no nontrivial Euclidean de Rham factor and all other de Rham factors are compact or noncompact, respectively.

We are primarily interested in symmetric spaces of the noncompact type. By [Ji05, p. 65], they can all be constructed from semisimple Lie groups as follows:

Proposition 3.5. *Let G be a semisimple Lie group with finite center and finitely many connected components. Then the following holds:*

(i) G has a maximal compact subgroup, and any two maximal compact subgroups of G are conjugate.

(ii) If $K \subset G$ is a maximal compact subgroup, then G/K is simply connected and has a G-invariant Riemannian metric with which it is a symmetric space of the noncompact type and which is unique up to constant scaling on its de Rham factors.

Definition 3.6. We write X_G for the symmetric space G/K specified in Proposition 3.5 and call it the *symmetric space associated to* G.

Remark 3.7. For a semisimple Lie group G as in Proposition 3.5 and a maximal compact subgroup $K \subset G$, we have by [HT94, Lemma 3.10] that $K^0 = K \cap G^0$ is a maximal compact subgroup of G^0 and $G = G^0 K$. So the inclusion $G^0 \hookrightarrow G$ induces a diffeomorphism $X_{G^0} \xrightarrow{\cong} X_G$. Therefore, we can assume that G is connected when considering the symmetric space associated to G. Similarly, one can assume that G has no compact factors.

3.1 Discrete Subgroups of Semisimple Lie Groups

The following proposition gives us a description of locally symmetric spaces as quotient spaces of symmetric spaces by discrete groups of isometries:

Proposition 3.8. *Let G be a connected semisimple Lie group with finite center and without compact factors. Then the following holds:*

(i) For every torsion-free discrete subgroup $\Gamma \subset G$, the quotient space $\Gamma\backslash X_G$ is a complete locally symmetric space with universal covering space X_G.

(ii) Conversely, if M is a locally symmetric space with universal covering space X_G, then M is complete and there exists a torsion-free discrete subgroup $\Gamma \subset G$ and a finite covering $\Gamma\backslash X_G \to M$.

Proof. First, let $\Gamma \subset G$ be a torsion-free discrete subgroup. Then Γ acts freely and properly discontinuously on X_G. So the quotient $\Gamma\backslash X_G$ is a manifold by the quotient manifold theorem and the projection $X_G \to \Gamma\backslash X_G$ is a universal covering map. It follows that $\Gamma\backslash X_G$ is a complete locally symmetric space.

Let now M be a locally symmetric space that is universally covered by X_G. Then by covering theory, M is isometric to $\Delta\backslash X_G$ for some discrete subgroup $\Delta \subset \mathrm{Isom}(X_G)$, and Δ is torsion-free because M is a manifold. Since X_G is a homogeneous space, the group $\mathrm{Isom}(X_G)$ has only finitely many connected components by [Qui06, Corollary 2.2], and so $\Delta' := \Delta \cap \mathrm{Isom}(X_G)^0$ is a subgroup of finite index in Δ. This gives us a finite covering

$$\Delta'\backslash X_G \to \Delta\backslash X_G.$$

Let $\rho\colon G \to \mathrm{Isom}(X_G)^0$ be the map given by left translation. Then ρ is surjective by [Ebe96, p. 70] because X_G is a symmetric space of the noncompact type. Moreover, we have $\ker(\rho) \subset K$, and so $\ker(\rho)$ is a compact normal subgroup of G and hence must be finite because G has no compact factors. This shows that ρ is a finite covering map. So the preimage $\Gamma := \rho^{-1}(\Delta')$ is a torsion-free discrete subgroup of G. The corresponding quotient space $\Gamma\backslash X_G$ is isometric to $\Delta'\backslash X_G$, and so is indeed a finite covering space of M. □

Definition 3.9. A *lattice* in a Lie group G is a discrete subgroup $\Gamma \subset G$ such that the quotient $\Gamma\backslash G$ has a finite G-invariant Haar measure. It is called *cocompact* if $\Gamma\backslash G$ is compact.

Remark 3.10. The locally symmetric space $\Gamma\backslash X_G$ in Proposition 3.8 has finite volume if and only if the discrete subgroup $\Gamma \subset G$ is a lattice, and $\Gamma\backslash X_G$ is compact if and only if Γ is cocompact (see [Mor15, p. 15 and Exercise 1.3.6]).

3.2 Flat Subspaces

We now study the flat subspaces of a symmetric space and discuss an algebraic description of these subspaces. We use the following terminology:

Definition 3.11. Let M be a Riemannian manifold. A submanifold $S \subset M$ is *totally geodesic* if any geodesic in S with its induced metric is a geodesic in M.

Definition 3.12. Let M be a symmetric space. A *flat in M* is a connected complete totally geodesic submanifold of M whose curvature tensor vanishes. A *maximal flat in M* is a flat of maximal dimension among all flats in M.

Remark 3.13. Let X_G be the symmetric space of noncompact type associated to a semisimple Lie group G. Then the group G acts transitively on the set of all maximal flats in X_G by [Hel78, Theorem V.6.2].

The maximal flats of a symmetric space encode important information about the symmetric space in the form of the following invariant:

Definition 3.14. The *rank* of a symmetric space M is the dimension of a maximal flat in M. We define the *rank* of a complete locally symmetric space to be the rank of its universal covering space.

Next, we give an algebraic description of the maximal flats in a symmetric space of the noncompact type. We will need the following notions for this:

Definition 3.15. A matrix $g \in \mathrm{GL}_n(\mathbb{R})$ is called *semisimple* if it is diagonalizable over $\mathbb{C}$, and g is called *unipotent* if $(g - I_n)^k = 0$ for some $k \in \mathbb{N}$. A semisimple matrix $g \in \mathrm{GL}_n(\mathbb{R})$ is called *hyperbolic* if all its eigenvalues are real and positive, and g is called *elliptic* if all its eigenvalues have absolute norm 1.

Using the Jordan decomposition and the polar decomposition, every matrix in $\mathrm{GL}_n(\mathbb{R})$ can be written uniquely as a product of a unipotent, a hyperbolic and an elliptic matrix such that they all commute with each other. We say that a Lie group G is *linear* if it admits an embedding $G \hookrightarrow \mathrm{GL}_n(\mathbb{R})$ for some $n \in \mathbb{N}$. By [Hel78, p. 431], we have the following decomposition:

Proposition 3.16 (Real Jordan Decomposition). *Let G be a connected linear semisimple Lie group. Then every $g \in G$ can be written uniquely as*

$$g = g_u g_h g_e,$$

such that the images of $g_u, g_h, g_e \in G$ in some (and therefore any) embedding $G \hookrightarrow \mathrm{GL}_n(\mathbb{R})$ are unipotent, hyperbolic and elliptic, respectively, and they all commute with each other.

We will say that an element in a connected linear semisimple Lie group G is *semisimple*, *unipotent*, *hyperbolic* or *elliptic* if its image in an embedding $G \hookrightarrow \mathrm{GL}_n(\mathbb{R})$ has the respective property. This is well-defined by Proposition 3.16.

Definition 3.17. Let G be a connected linear semisimple Lie group. An element $g \in G$ is called *polar regular* if for all $g' \in G$, we have $\dim C_G(g_h) \leq \dim C_G(g'_h)$, where $C_G(g_h)$ and $C_G(g'_h)$ are the centralizers of g_h and g'_h in G, respectively.

By [Mos73, Lemma 5.2], we have the following relationship between the polar regular elements of a semisimple Lie group and the maximal flats in the associated symmetric space. Here, we use the following notation: If G is a group acting on a symmetric space M and $A \subset M$ is a flat, then we write $G_A := \{g \in G : g \cdot A = A\}$ for the stabilizer subgroup of the flat A in G.

Proposition 3.18 (Mostow). *Let G be a connected linear semisimple Lie group and let $g \in G$ be polar regular. Then g is semisimple, and there exists a unique maximal flat $A \subset X_G$ such that $g \cdot A = A$. Moreover, $C_G(g)$ is a subgroup of G_A and acts transitively on A.*

The next lemma shows that the condition of being a polar regular element behaves well with respect to direct products of Lie groups:

Lemma 3.19. *Let $G = G_1 \times \ldots \times G_n$ be a direct product of connected linear semisimple Lie groups. Then an element of G is polar regular if and only if its projections to every direct factor G_i of G is polar regular.*

Proof. We denote by $\pi_i \colon G \to G_i$ the i-th projection map. Note that G is linear and semisimple, and for each $g \in G$, we have $g_h = (\pi_1(g)_h, \ldots, \pi_n(g)_h)$. Thus,

$$C_G(g_h) = C_G(\pi_1(g)_h, \ldots, \pi_n(g)_h) = C_{G_1}(\pi_1(g)_h) \times \ldots \times C_{G_n}(\pi_n(g)_h).$$

The statement now follows because $\dim C_G(g_h) = \sum_{i=1}^n \dim C_{G_i}(\pi_i(g)_h)$ is minimal if and only if the dimension of each $C_{G_i}(\pi_i(g)_h)$ is minimal. □

One is often interested in flats in a symmetric space whose projections to a given locally symmetric space are compact. Therefore, we define:

Definition 3.20. Let X_G be the symmetric space associated to a semisimple Lie group G and let $\Gamma \subset G$ be a lattice. A flat $A \subset X_G$ is called Γ-*compact* if the quotient space $\Gamma_A \backslash A$ is compact, where $\Gamma_A = \{\gamma \in \Gamma : \gamma \cdot A = A\}$.

Remark 3.21. If $A \subset X_G$ is a Γ-compact flat, then the image of A in $\Gamma \backslash X_G$ is compact because the projection $A \to \Gamma \backslash X_G$ factors through $\Gamma_A \backslash A$. Moreover, a Γ-compact flat A is also Γ'-compact for every subgroup of finite index $\Gamma' \subset \Gamma$, because in this situation $\Gamma'_A \backslash A$ is a finite covering space of $\Gamma_A \backslash A$.

It is known from [Mos73, Lemma 8.3'] that the Γ-compact maximal flats are dense in the space of all maximal flats in a symmetric space:

Theorem 3.22 (Density of Γ-Compact Maximal Flats). *Let G be a connected linear semisimple Lie group, let $\Gamma \subset G$ be a lattice and let $A \subset X_G$ be a maximal flat. Then for every open neighborhood of the identity $U \subset G$, there exists some $u \in U$ such that $u \cdot A$ is a Γ-compact maximal flat in X_G which is stabilized by a polar regular element of Γ.*

3.3 Boundary at Infinity

We now discuss the concept of the boundary at infinity of a symmetric space of the noncompact type. This is useful for computing the intersection of subspaces and leads to a compactification of the symmetric space.

Definition 3.23. Let M be a symmetric space of the noncompact type. Two unit speed geodesics $\gamma_1, \gamma_2 \colon \mathbb{R} \to M$ are called *asymptotic* if there exists a constant $C > 0$ such that for all $t > 0$, we have $d(\gamma_1(t), \gamma_2(t)) \leq C$. The set of all equivalence classes of asymptotic unit speed geodesics in M is called the *boundary at infinity of M* and is denoted by $\partial_\infty M$.

For a symmetric space M of the noncompact type, there is a natural topology on the disjoint union $\overline{M} := M \sqcup \partial_\infty M$ that is called the *cone topology.* This topology is defined by requiring that the induced topology on M is the original topology of M, and that for any point $x \in \partial_\infty M$, the set of truncated cones originating from a point in M towards x is a neighborhood basis for x. The space $\overline{M}$ is compact with the cone topology and is called the *geodesic compactification of M*. We refer the reader to [Ebe96, pp. 28–30] for a precise definition of the cone topology and a proof of the following proposition:

Proposition 3.24. *Let M be an n-dimensional symmetric space of the noncompact type. Then the following holds:*

(i) The geodesic compactification $\overline{M}$ is homeomorphic to a closed n-ball, and $\partial_\infty M$ is homeomorphic to an $(n-1)$-sphere.

(ii) The action of the isometry group of M on M extends to a continuous action of the same group on $\overline{M}$ by $\phi \cdot [\gamma] := [\phi \circ \gamma]$ for all $[\gamma] \in \partial_\infty M$.

Example 3.25. In the Poincaré disc model of the hyperbolic plane $\mathbb{H}^2$, the boundary at infinity can be identified with the boundary circle of the disk.

In the next section, we will see how the boundary at infinity can help to understand the intersections of subspaces in the example of the hyperbolic plane.

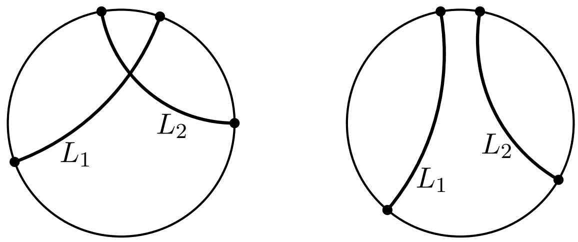


Figure 5: Two pairs of geodesic lines in $\mathbb{H}^2$ and their endpoints in $\partial_\infty \mathbb{H}^2$.

3.4 Geometry of the Hyperbolic Plane

We now take a closer look at the hyperbolic plane $\mathbb{H}^2$. This is the symmetric space associated to the semisimple Lie group $\mathrm{SL}_2(\mathbb{R})$. It is a symmetric space of the noncompact type and has rank 1. Hence, the maximal flat subspaces in $\mathbb{H}^2$ are the images of maximal geodesics.

Definition 3.26. A *geodesic line* in $\mathbb{H}^2$ is the image of a maximal geodesic in $\mathbb{H}^2$. If L is a geodesic line, then we call the two equivalence classes of asymptotic unit speed geodesics parametrizing L the *endpoints of L in $\partial_\infty \mathbb{H}^2$*.

Definition 3.27. Let M be a smooth manifold and let S_1 and S_2 be submanifolds of M. We say that S_1 and S_2 *intersect transversally* if for each $p \in S_1 \cap S_2$, the natural map $T_pS_1 \oplus T_pS_2 \to T_pM$ is surjective.

The following lemma shows that the intersection of two generic geodesic lines in $\mathbb{H}^2$ is transverse and stable under small perturbations:

Lemma 3.28 (Perturbation Lemma). *Let L_1 and L_2 be two geodesic lines in $\mathbb{H}^2$ whose endpoints in $\partial_\infty \mathbb{H}^2$ are pairwise distinct. Then the following holds:*

- *(i) L_1 and L_2 are either disjoint or intersect transversally in a single point.*
- *(ii) There exists an open neighborhood of the identity $U \subset \mathrm{SL}_2(\mathbb{R})$ such that for every $u, v \in U$, the endpoints of $u \cdot L_1$ and $v \cdot L_2$ in $\partial_\infty \mathbb{H}^2$ are pairwise distinct, and $u \cdot L_1$ and $v \cdot L_2$ intersect if and only if L_1 and L_2 intersect.*

Proof. Note that two geodesic lines in $\mathbb{H}^2$ intersect if and only if their endpoints in $\partial_\infty \mathbb{H}^2$ are linked (see Figure 5), in which case their intersection is necessarily transverse if the endpoints are distinct. We now denote the four endpoints of L_1 and L_2 by $v_1, v_2, v_3, v_4 \in \partial_\infty \mathbb{H}^2$. By Proposition 3.24, the geodesic compactification $\overline{\mathbb{H}^2}$ is Hausdorff, and so we can find pairwise disjoint open neighborhoods $V_i \subset \overline{\mathbb{H}^2}$ of the points v_i. The group $\mathrm{SL}_2(\mathbb{R})$ acts continuously on $\overline{\mathbb{H}^2}$, hence the maps $\phi_i\colon \mathrm{SL}_2(\mathbb{R}) \to \overline{\mathbb{H}^2}$ given by $\phi_i(g) := g \cdot v_i$ are continuous, and

$$U := \bigcap_{i=1}^{4} \phi_i^{-1}(V_i)$$

is an open neighborhood of the identity in $\mathrm{SL}_2(\mathbb{R})$. Then for every $u, v \in U$, the endpoints of $u \cdot L_1$ and $v \cdot L_2$ in $\partial_\infty \mathbb{H}^2$ are linked if and only if whose of L_1 and L_2 are linked, and so the statement of the lemma follows. □

Next, we give an algebraic description of geodesic lines in $\mathbb{H}^2$. It is convenient for this to use the Poincaré half-plane model for the hyperbolic plane

$$\mathbb{H}^2 = \{x + iy \in \mathbb{C} : x, y \in \mathbb{R} \text{ and } y > 0\},$$

where the Riemannian metric is $\mathrm{d}s^2 = y^{-2}(\mathrm{d}x^2 + \mathrm{d}y^2)$. The action of the group $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}^2$ is given by fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az + b}{cz + d}.$$

We now characterize the polar regular elements in $\mathrm{SL}_2(\mathbb{R})$ by their eigenvalues:

Lemma 3.29. *An element of* $\mathrm{SL}_2(\mathbb{R})$ *is polar regular if and only if it has two distinct real eigenvalues.*

Proof. Let $g \in \mathrm{SL}_2(\mathbb{R})$ and consider its hyperbolic part $g_h \in \mathrm{SL}_2(\mathbb{R})$. If $g_h = I_2$, then the dimension of $C_{\mathrm{SL}_2(\mathbb{R})}(g_h) = \mathrm{SL}_2(\mathbb{R})$ is three. Otherwise, g_h has two distinct positive eigenvalues λ and λ^{-1}. Then g_h is diagonalizable over $\mathbb{R}$, and so we have

$$g_h = T \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} T^{-1}$$

for some matrix $T \in \mathrm{GL}_2(\mathbb{R})$. We conclude that the centralizer of g_h is

$$C_{\mathrm{SL}_2(\mathbb{R})}(g_h) = T C_{\mathrm{SL}_2(\mathbb{R})}\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\right) T^{-1} = T \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^\times \right\} T^{-1}$$

and has dimension one. This shows that g is polar regular if and only if $g_h \neq I_2$.

In particular, any matrix in $\mathrm{SL}_2(\mathbb{R})$ with two distinct real eigenvalues is polar regular. Conversely, a polar regular element $g \in \mathrm{SL}_2(\mathbb{R})$ is semisimple by Proposition 3.18. Its eigenvalues are of the form λ and λ^{-1} for some $\lambda \in \mathbb{C}$ and satisfy $\lambda + \lambda^{-1} = \mathrm{tr}(g) \in \mathbb{R}$ and $|\lambda| \neq 1$, because of $g_h \neq I_2$. From this, we conclude $\lambda \in \mathbb{R}$, and so g has two distinct real eigenvalues. □

For our computations later in Chapter 6, it is useful to extend the action of the group $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}^2$ to an action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{H}^2$ as follows:

Definition 3.30. We define the action of the group $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{H}^2$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \begin{cases} (az+b)(cz+d)^{-1}, & \text{if } ad-bc>0, \\ \overline{(az+b)(cz+d)^{-1}}, & \text{otherwise.} \end{cases}$$

Geometrically, this means that matrices with negative determinant act on $\mathbb{H}^2$ by a composition of a Möbius transformation with a reflection on the imaginary axis. In particular, the action of a matrix $g \in \mathrm{GL}_2(\mathbb{R})$ on $\mathbb{H}^2$ is isometric, and the action of g is orientation-preserving if and only if $\det(g) > 0$. Similar as in Proposition 3.18, we have the following algebraic description of geodesic lines:

Proposition 3.31. *Let $g \in \mathrm{GL}_2(\mathbb{R})$ be a matrix with two distinct real eigenvalues. Then there exists a unique geodesic line $L \subset \mathbb{H}^2$ with $g \cdot L = L$. Moreover, the centralizer $C_{\mathrm{GL}_2(\mathbb{R})}(g)$ is a subgroup of index two in $\mathrm{GL}_2(\mathbb{R})_L$ and acts by orientation-preserving isometries on L.*

Proof. Any matrix in $\mathrm{GL}_2(\mathbb{R})$ with two distinct real eigenvalues can be conjugated to a diagonal matrix, so it suffices to prove the statement for a diagonal matrix. Let $g \in \mathrm{GL}_2(\mathbb{R})$ be a diagonal matrix with two distinct real eigenvalues. Then g acts either as $z \mapsto \lambda z$ for some $\lambda > 0$ with $\lambda \neq 1$, or as $z \mapsto \lambda \overline{z}$ for some $\lambda < 0$ on $\mathbb{H}^2$. So g stabilizes the unique geodesic line $L := i\mathbb{R}_{>0} \subset \mathbb{H}^2$, and the centralizer of g in $\mathrm{GL}_2(\mathbb{R})$ consists of diagonal matrices, which act by orientation-preserving isometries on L. We claim that the stabilizer subgroup of L in $\mathrm{GL}_2(\mathbb{R})$ is

$$\mathrm{GL}_2(\mathbb{R})_L = C_{\mathrm{GL}_2(\mathbb{R})}(g) \sqcup C_{\mathrm{GL}_2(\mathbb{R})}(g) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.1}$$

One can check that the matrices in the right hand side of (3.1) stabilize L. For the other direction, consider a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})_L$$

in the stabilizer subgroup of L. Then for all $t > 0$, we have

$$0 = \mathrm{Re}\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot it\Big) = \mathrm{Re}\Big(\frac{ait+b}{cit+d}\Big) = \frac{tcd+bd}{c^2t^2+d^2}.$$

Letting $t \to 0$, we see that $bd = 0$. Moreover, if we let $t = 1$, then we get $cd = 0$. The condition $ad - bc \neq 0$ then shows that either $b = c = 0$ or $a = d = 0$, and so the matrix is in the right hand side of (3.1). $\square$

Chapter 4

Arithmetic Groups

In this chapter, we discuss arithmetic groups and give the necessary background in the theory of algebraic groups. We take the approach to define algebraic groups as certain group valued functors, which allows us to view them as special cases of affine group schemes of finite type over a commutative ring. This will also be convenient later when we work with orders in quaternion algebras in Chapters 5 and 6. For details on this viewpoint, we refer the reader to the books [Wat79] and [Mil17]. A more classical treatment of algebraic groups as algebraic varieties from the viewpoint of algebraic geometry can be found in [Hum75] and [Bor91].

4.1 Affine Group Schemes

Throughout this section, we assume that R is a commutative ring. We denote by Alg_R the category of commutative R-algebras and by Set and Grp the categories of sets and groups, respectively. A functor $\mathbf{F}\colon \mathrm{Alg}_R \to \mathrm{Set}$ is *representable* if there exists a commutative R-algebra A so that $\mathbf{F}$ is naturally isomorphic to the functor

$$\mathbf{h}^A\colon \mathrm{Alg}_R \to \mathrm{Set}, \quad \mathbf{h}^A(B) = \hom_{\mathrm{Alg}_R}(A, B).$$

If A and B are commutative R-algebras, then every natural transformation $\Phi\colon \mathbf{h}^A \to \mathbf{h}^B$ is of the form $\Phi(f) = f \circ g$ for a unique R-algebra homomorphism $g\colon B \to A$ by the Yoneda lemma (see for example [Mac98, p. 61]). In particular, if a functor $\mathbf{F}\colon \mathrm{Alg}_R \to \mathrm{Set}$ is representable by a commutative R-algebra, then this algebra is uniquely determined by $\mathbf{F}$ up to an isomorphism.

Definition 4.1. An *affine group scheme over R* is a functor $\mathbf{G}\colon \mathrm{Alg}_R \to \mathrm{Grp}$ whose composition with the forgetful functor $\mathrm{Grp} \to \mathrm{Set}$ is representable by a commutative R-algebra, which we then denote by $\mathcal{O}(\mathbf{G})$ and call the *coordinate ring of* $\mathbf{G}$. We say that $\mathbf{G}$ is of *finite type* if $\mathcal{O}(\mathbf{G})$ is finitely generated.

Example 4.2. The functor $\mathbf{GL}_n\colon A \mapsto \mathrm{GL}_n(A)$ is an affine group scheme of finite type over R. In fact, for every commutative R-algebra A, the map

$$\mathrm{GL}_n(A) \to \hom_{\mathrm{Alg}_R}(R[X_{11}, X_{12}, \dots, X_{nn}, T]/(\det(X_{ij})_{ij}T - 1), A)$$

which sends a matrix $g = (g_{ij})_{ij} \in \mathrm{GL}_n(A)$ to the homomorphism induced by $X_{ij} \mapsto g_{ij}$ and $T \mapsto \det(g)^{-1}$ is a natural bijection.

Remark 4.3. Every affine group scheme of finite type can be considered as a group valued functor defined by polynomial equations. To see this, let $\mathbf{G}$ be an affine group scheme of finite type over R. Since $\mathcal{O}(\mathbf{G})$ is finitely generated by assumption, there exists a surjection $\pi\colon R[X_1, \dots, X_n] \twoheadrightarrow \mathcal{O}(\mathbf{G})$. Let $I := \ker(\pi)$. Then for every commutative R-algebra A, we have a natural inclusion

$$\mathbf{G}(A) \xrightarrow{\cong} \mathbf{h}^{\mathcal{O}(\mathbf{G})}(A) \xhookrightarrow{\pi^*} \hom_{\mathrm{Alg}_R}(R[X_1, \dots, X_n], A) \xrightarrow{\cong} A^n$$

which identifies $\mathbf{G}(A)$ with the set $V_{A^n}(I) := \{x \in A^n : f(x) = 0 \text{ for all } f \in I\}$.

Definition 4.4. An *R-homomorphism* $\mathbf{G} \to \mathbf{H}$ of affine group schemes over R is a natural transformation of functors $\mathrm{Alg}_R \to \mathrm{Grp}$.

The next lemma gives us a functorial way to topologize for affine group schemes $\mathbf{G}$ of finite type the groups $\mathbf{G}(A)$ for topological R-algebras A.

Lemma 4.5. *Let $\mathbf{G}$ be an affine group scheme of finite type over R. Then:*

(i) For every topological commutative R-algebra A, there is a unique weakest topology on the group $\mathbf{G}(A)$ so that the maps $\mathbf{G}(A) \hookrightarrow A^n$ in Remark 4.3 are continuous for every realization of $\mathbf{G}(A)$ as a vanishing set of polynomials.

(ii) This topology is functorial in the sense that for every R-homomorphism $\mathbf{G} \to \mathbf{H}$, the induced maps $\mathbf{G}(A) \to \mathbf{H}(A)$ are continuous.

Proof. Consider an R-homomorphism $\Phi\colon \mathbf{G} \to \mathbf{H}$ of affine group schemes of finite type and choose identifications $\mathbf{G}(A) \xrightarrow{\cong} V_{A^n}(I)$ and $\mathbf{H}(A) \xrightarrow{\cong} V_{A^m}(J)$ as in Remark 4.3 for ideals $I \subset R[X_1, \dots, X_n]$ and $J \subset R[Y_1, \dots, Y_m]$. Then Φ determines a unique map f making the diagram

$$\begin{array}{ccc} \mathbf{G}(A) & \longrightarrow & \mathbf{H}(A) \\ \downarrow \cong & & \downarrow \cong \\ V_{A^n}(I) & \overset{f}{\dashrightarrow} & V_{A^m}(J) \end{array}$$

commute, and f is given by polynomials, hence is continuous. It follows that also the map $\mathbf{G}(A) \to \mathbf{H}(A)$ induced by Φ is continuous when we give $\mathbf{G}(A)$ and $\mathbf{H}(A)$ the topologies induced by the above identifications. In particular, for $\Phi = \mathrm{id}_{\mathbf{G}}$, this argument shows that the topology that we have put on $\mathbf{G}(A)$ is independent of the realization of $\mathbf{G}(A)$ as a vanishing set of polynomials. $\square$

Definition 4.6. The *comorphism* of an R-homomorphism $\Phi\colon \mathbf{G} \to \mathbf{H}$ is the unique R-algebra homomorphism $\Phi^*\colon \mathcal{O}(\mathbf{H}) \to \mathcal{O}(\mathbf{G})$ that makes the diagram

$$\begin{array}{ccc} \mathbf{G} & \xrightarrow{\Phi} & \mathbf{H} \\ \downarrow{\cong} & & \downarrow{\cong} \\ \mathbf{h}^{\mathcal{O}(\mathbf{G})} & \xrightarrow{f \mapsto f \circ \Phi^*} & \mathbf{h}^{\mathcal{O}(\mathbf{H})} \end{array}$$

commute. We call Φ a *closed R-embedding* if Φ^* is surjective, and we denote a closed R-embedding by $\mathbf{G} \hookrightarrow \mathbf{H}$.

Lemma 4.7. *Let $\Phi\colon \mathbf{G} \hookrightarrow \mathbf{H}$ be a closed R-embedding of affine group schemes of finite type over R. Then for every topological commutative R-algebra A, the map $\mathbf{G}(A) \to \mathbf{H}(A)$ induced by Φ is a closed embedding.*

Proof. Choose a surjection $\pi\colon R[X_1, \dots, X_n] \twoheadrightarrow \mathcal{O}(\mathbf{H})$. By assumption, the comorphism $\Phi^*\colon \mathcal{O}(\mathbf{H}) \twoheadrightarrow \mathcal{O}(\mathbf{G})$ is surjective, and so the composition $\Phi^* \circ \pi$ is also surjective. Let $J := \ker(\pi)$ and $I := \ker(\Phi^* \circ \pi)$. Then we have $J \subset I$, and for a topological R-algebra A, we obtain a commutative diagram

$$\begin{array}{ccc} \mathbf{G}(A) & \longrightarrow & \mathbf{H}(A) \\ \downarrow{\cong} & & \downarrow{\cong} \\ V_{A^n}(I) & \hookrightarrow & V_{A^n}(J), \end{array}$$

where $V_{A^n}(I) \hookrightarrow V_{A^n}(J)$ is the inclusion map, which is a closed embedding. Hence the map $\mathbf{G}(A) \to \mathbf{H}(A)$ induced by Φ is also a closed embedding. □

Definition 4.8. Let $\sigma\colon R \hookrightarrow S$ be an injective ring homomorphism and let $\mathbf{G}$ be an affine group scheme over R. Then the *extension of scalars of* $\mathbf{G}$ *along* σ is

$$\mathbf{G}_\sigma\colon \mathrm{Alg}_S \to \mathrm{Grp}, \quad \mathbf{G}_\sigma(A) := \mathbf{G}(\mathrm{res}_\sigma(A)),$$

where $\mathrm{res}_\sigma(A)$ is A considered as R-algebra with multiplication $\lambda \cdot a := \sigma(\lambda) \cdot a$. If the map σ is clear from the context, then we write $\mathbf{G}_S := \mathbf{G}_\sigma$.

It follows from the change of rings adjunction that $\mathbf{G}_\sigma$ is an affine group scheme over S with coordinate ring $\mathcal{O}(\mathbf{G}_\sigma) = \mathcal{O}(\mathbf{G}) \otimes_R S$.

4.2 Algebraic Groups

We now specialize to the case where the commutative ground ring is a field. Throughout this section, we assume that K is a field of characteristic zero.

Definition 4.9. An *(affine) algebraic group over* K or a K*-group* is an affine group scheme of finite type over K. A K*-subgroup* of an algebraic group $\mathbf{G}$ over K is an algebraic group $\mathbf{H}$ over K so that $\mathbf{H}(A) \subset \mathbf{G}(A)$ is a subgroup for all A and the inclusion $\mathbf{H} \to \mathbf{G}$ is a closed K-embedding.

We remark that every K-homomorphism $\mathbf{H} \to \mathbf{G}$ of algebraic groups with trivial kernel is automatically a closed K-embedding by [Wat79, p. 115]. All algebraic groups that we will encounter in this thesis are affine, and so we will usually just speak of algebraic groups instead of affine algebraic groups.

Every algebraic group can be embedded into a group of matrices by the following proposition (see [Wat79, p. 25] for a proof):

Proposition 4.10. *For every algebraic group* $\mathbf{G}$ *over* K*, there exists a closed* K*-embedding* $\mathbf{G} \hookrightarrow \mathbf{GL}_n$ *for some* $n \in \mathbb{N}$*.*

In particular, it follows from Proposition 4.10 and the closed subgroup theorem that for any algebraic group $\mathbf{G}$ over $\mathbb{R}$, the group $\mathbf{G}(\mathbb{R})$ is a real Lie group. By a theorem of Whitney [Whi57], the group $\mathbf{G}(\mathbb{R})$ has only finitely many connected components.

The classical approach is to consider an algebraic group as an algebraic variety with a group structure such that the group multiplication and inversion operations are regular maps. With our definition of an algebraic group as a representable functor $\mathbf{G}\colon \mathrm{Alg}_K \to \mathrm{Grp}$, this object can be obtained from $\mathbf{G}$ as follows: Let $\overline{K}$ be an algebraic closure of K, and identify the group $\mathbf{G}(\overline{K})$ with the vanishing set $V_{\overline{K}^n}(I)$ for some ideal $I \subset K[X_1, \dots, X_n]$ as in Remark 4.3. This set is an affine variety when equipped with the subspace topology induced by the $\overline{K}$-Zariski topology on $\overline{K}^n$, and the group operations are regular maps.

Definition 4.11. The *underlying affine variety* $|\mathbf{G}|$ of an algebraic group $\mathbf{G}$ over K is the group $\mathbf{G}(\overline{K})$ with the structure of an affine variety obtained by realizing it as a vanishing set of a system of polynomials as above.

It is important to note that the topology on $|\mathbf{G}|$ is different from the topology on $\mathbf{G}(\overline{K})$ given by Lemma 4.5. We can now define the following properties of an algebraic group:

Definition 4.12. An algebraic group $\mathbf{G}$ is *connected* or *finite* if its underlying affine variety $|\mathbf{G}|$ is connected or finite, respectively.

Example 4.13. The algebraic groups $\mathbf{GL}_n$ and $\mathbf{SL}_n$ are connected for any $n \in \mathbb{N}$. For $\mathbf{GL}_n$, this can be seen from the fact that $|\mathbf{GL}_n|$ is a principal open set in an affine space. The connectedness of $\mathbf{SL}_n$ is proven in [Hum75, pp. 55–56].

Remark 4.14. If $\mathbf{G}$ is a connected algebraic group over K, then the group $\mathbf{G}(K)$ is not necessarily connected. For example, $\mathbf{GL}_1$ is connected as an algebraic group, but the group $\mathbf{GL}_1(\mathbb{R}) \cong \mathbb{R}^\times$ has two connected components.

Definition 4.15. A K*-epimorphism* is a K-homomorphism $\mathbf{G} \to \mathbf{H}$ of algebraic groups whose induced map $|\mathbf{G}| \to |\mathbf{H}|$ on the underlying affine varieties is surjective. We denote a K-epimorphism by $\mathbf{G} \twoheadrightarrow \mathbf{H}$.

In the theory of algebraic groups, there is a counterpart to the scalar extension for affine group schemes as defined in Definition 4.8. Given an algebraic group $\mathbf{G}$ over a finite field extension L/K, it is possible to construct an algebraic group over K whose group of K-points is isomorphic to $\mathbf{G}(L)$ as follows:

Definition 4.16. Let L/K be a finite field extension and let $\mathbf{G}$ be an algebraic group over L. The *restriction of scalars of* $\mathbf{G}$ *from* L *to* K is

$$\operatorname{Res}_{L/K}\mathbf{G}\colon \operatorname{Alg}_K \to \operatorname{Grp}, \quad (\operatorname{Res}_{L/K}\mathbf{G})(A) := \mathbf{G}(A \otimes_K L).$$

The functor $\operatorname{Res}_{L/K}\mathbf{G}$ is an algebraic group over K by [Mil17, p. 57], and the map $K \otimes_K L \xrightarrow{\cong} L$, $\lambda \otimes x \mapsto \lambda \cdot x$ induces an isomorphism of topological groups

$$(\operatorname{Res}_{L/K}\mathbf{G})(K) \xrightarrow{\cong} \mathbf{G}(L).$$

An important special case is that of a number field $F/\mathbb{Q}$. In this case, the algebraic group obtained from restriction of scalars has the following properties (see [Mil17, pp. 58–59]):

Proposition 4.17. *Let* $\mathbf{G}$ *be an algebraic group over a number field* F*. Let* $\sigma_1, \dots, \sigma_{r_1}\colon F \hookrightarrow \mathbb{R}$ *be the real embeddings of* F *and* $\sigma_{r_1+1}, \dots, \sigma_{r_1+r_2}\colon F \hookrightarrow \mathbb{C}$ *be representatives of the complex embeddings of* F *modulo complex conjugation. Then we have an* $\mathbb{R}$*-isomorphism*

$$(\operatorname{Res}_{F/\mathbb{Q}}\mathbf{G})_{\mathbb{R}} \xrightarrow{\cong} \prod_{i=1}^{r_1} \mathbf{G}_{\sigma_i} \times \prod_{j=1}^{r_2} \operatorname{Res}_{\mathbb{C}/\mathbb{R}}(\mathbf{G}_{\sigma_{r_1+j}})$$

induced by the natural ring isomorphisms $A \otimes_{\mathbb{Q}} F \xrightarrow{\cong} \prod_{i=1}^{r_1} A \times \prod_{j=1}^{r_2} A \otimes_{\mathbb{R}} \mathbb{C}$ *for* $\mathbb{R}$*-algebras* A*.*

Corollary 4.18. *Let* $\mathbf{G}$ *be an algebraic group over a number field* F*. Then we have*

$$(\operatorname{Res}_{F/\mathbb{Q}}\mathbf{G})_{\mathbb{C}} \cong \prod_{\sigma\colon F \hookrightarrow \mathbb{C}} \mathbf{G}_\sigma,$$

where σ *runs over all (real and complex) embeddings of* F *into* $\mathbb{C}$*.*

We now discuss some special types of algebraic groups. Since we always assume that K is of characteristic zero, we can use the following definition:

Definition 4.19. A connected algebraic group $\mathbf{G}$ over K is called *semisimple* if every connected commutative normal K-subgroup of $\mathbf{G}$ is trivial. It is called *almost K-simple* if it is noncommutative and every proper normal K-subgroup of $\mathbf{G}$ is finite.

For example, $\mathbf{SL}_n$ is almost K-simple for each $n > 1$. Every almost K-simple algebraic group is semisimple. By [Mil17, Theorem 21.51], semisimple algebraic groups decompose into almost K-simple algebraic groups as follows:

Lemma 4.20. *Every connected semisimple algebraic group $\mathbf{G}$ over K has only finitely many almost K-simple normal K-subgroups $\mathbf{G}_1, \ldots, \mathbf{G}_n$, and the multiplication map $\mathbf{G}_1 \times \ldots \times \mathbf{G}_n \to \mathbf{G}$ is a K-epimorphism with finite kernel.*

The algebraic groups $\mathbf{G}_i$ in the above lemma are called the *almost K-simple factors of* $\mathbf{G}$.

Remark 4.21. The property of being a semisimple algebraic group is invariant under scalar extension to an algebraic closure. In fact, a connected algebraic group $\mathbf{G}$ over K is semisimple if and only if $\mathbf{G}_{\overline{K}}$ is semisimple, where $\overline{K}$ is an algebraic closure of K, by [Mil17, Proposition 19.3]. However, the property of being an almost K-simple algebraic group does depend more strongly on the field K as the next example shows.

Example 4.22. The algebraic group $\mathbf{G} := \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbf{SL}_2$ is almost $\mathbb{R}$-simple since $\mathrm{SL}_2(\mathbb{C})$ is a simple Lie group, but $\mathbf{G}_{\mathbb{C}} \cong \mathbf{SL}_2 \times \mathbf{SL}_2$ is not almost $\mathbb{C}$-simple.

The algebraic groups of the following type play a fundamental role in the theory of semisimple algebraic groups:

Definition 4.23. An algebraic group $\mathbf{T}$ over K is called a *torus* if it becomes isomorphic to a direct product of copies of $\mathbf{GL}_1$ over an algebraic closure $\overline{K}$ of K, that is, if

$$\mathbf{T}_{\overline{K}} \cong \mathbf{GL}_1 \times \ldots \times \mathbf{GL}_1.$$

The number of factors of $\mathbf{GL}_1$ is called the *dimension of* $\mathbf{T}$. A torus $\mathbf{T}$ is called *K-split* if $\mathbf{T}$ itself is K-isomorphic to a direct product of copies of $\mathbf{GL}_1$ over K.

Definition 4.24. The *K-rank* of a semisimple algebraic group $\mathbf{G}$ over K is the maximal dimension of a K-split torus which is a K-subgroup of $\mathbf{G}$. We denote the K-rank of $\mathbf{G}$ by $\operatorname{rank}_K(\mathbf{G})$, and we say that $\mathbf{G}$ is *K-anisotropic* if $\operatorname{rank}_K(\mathbf{G}) = 0$ and *K-isotropic* if $\operatorname{rank}_K(\mathbf{G}) > 0$.

Example 4.25. For each $n > 1$, the K-subgroup of $\mathbf{SL}_n$ defined by

$$\mathbf{D}_n(A) := \{\operatorname{diag}(x_1, \ldots, x_{n-1}, 1/(x_1 \cdots x_{n-1})) : x_1, \ldots, x_{n-1} \in A^\times\}$$

is a K-split torus of dimension $n-1$. One can show that $\operatorname{rank}_K(\mathbf{SL}_n) = n-1$.

Remark 4.26. Let $\mathbf{G}$ be a connected semisimple algebraic group over $\mathbb{R}$. Then the $\mathbb{R}$-rank of $\mathbf{G}$ agrees with the rank of the symmetric space associated to the semisimple Lie group $\mathbf{G}(\mathbb{R})$ as defined in Definition 3.14 (see [Mar91, IX.7.10]).

Definition 4.27. A connected algebraic group $\mathbf{G}$ over K is called *reductive* if every connected commutative normal K-subgroup of $\mathbf{G}$ is a torus.

It follows from the definitions that every semisimple algebraic group is reductive. The property of being a reductive algebraic group is also invariant under scalar extension to an algebraic closure (see [Mil17, Proposition 19.12]).

Example 4.28. For every $n \in \mathbb{N}$, the algebraic group $\mathbf{GL}_n$ is reductive, but not semisimple because the image of the diagonal embedding $\mathbf{GL}_1 \hookrightarrow \mathbf{GL}_n$ is a nontrivial torus and a normal K-subgroup. Moreover, every torus is reductive.

4.3 Integral Forms and Arithmetic Subgroups

We now introduce the notion of an arithmetic subgroup of an algebraic group over a number field. These subgroups arise from the following objects:

Definition 4.29. Let $\mathbf{G}$ be an algebraic group over a number field F. An *integral form of* $\mathbf{G}$ is an affine group scheme $\mathbf{G}_0$ of finite type over the ring of integers $\mathcal{O}_F$ of F together with an F-isomorphism $(\mathbf{G}_0)_F \xrightarrow{\cong} \mathbf{G}$.

Definition 4.30. We say that two subgroups Γ_1 and Γ_2 of a group G are *commensurable* if their intersection $\Gamma_1 \cap \Gamma_2$ is of finite index in both Γ_1 and Γ_2.

Lemma 4.31. *Let* $\mathbf{G}$ *be an algebraic group over a number field* F*. Then* $\mathbf{G}$ *has an integral form, and for any two integral forms* $\mathbf{G}_0$ *and* $\mathbf{G}_0'$ *of* $\mathbf{G}$*, the images of the groups* $\mathbf{G}_0(\mathcal{O}_F)$ *and* $\mathbf{G}_0'(\mathcal{O}_F)$ *in* $\mathbf{G}(F)$ *are commensurable.*

Proof. We first prove the existence of an integral form. Since F is Noetherian, we can identify the coordinate ring of $\mathbf{G}$ with a quotient ring $\mathcal{O}(\mathbf{G}) = F[X_1, \ldots, X_n]/(f_1, \ldots, f_k)$ for some $n \in \mathbb{N}$ and polynomials $f_1, \ldots, f_k \in F[X_1, \ldots, X_n]$. After normalizing if necessary, we may assume that the coefficients of the f_i all lie in $\mathcal{O}_F$. Any affine group scheme over $\mathcal{O}_F$ which represents the $\mathcal{O}_F$-algebra $\mathcal{O}_F[X_1, \ldots, X_n]/(f_1, \ldots, f_k)$ is an integral form of $\mathbf{G}$.

For the second part, let $\mathbf{G}_0$ and $\mathbf{G}_0'$ be two integral forms of $\mathbf{G}$. Then we have an F-isomorphism

$$(\mathbf{G}_0)_F \xrightarrow{\cong} \mathbf{G} \xrightarrow{\cong} (\mathbf{G}_0')_F.$$

In [PR94, Proposition 4.1], it is proven that the image of $\mathbf{G}_0(\mathcal{O}_F)$ under any such F-isomorphism is commensurable with $\mathbf{G}_0'(\mathcal{O}_F)$. □

Definition 4.32. Let $\mathbf{G}$ be an algebraic group over a number field F. An *arithmetic subgroup of* $\mathbf{G}$ is a subgroup $\Gamma \subset \mathbf{G}(F)$ which is commensurable with the image of $\mathbf{G}_0(\mathcal{O}_F)$ in $\mathbf{G}(F)$ for some integral form $\mathbf{G}_0$ of $\mathbf{G}$.

Remark 4.33. Every arithmetic subgroup can be considered as an arithmetic subgroup of an algebraic group over $\mathbb{Q}$. In fact, if $\Gamma \subset \mathbf{G}(F)$ is an arithmetic subgroup of an F-group $\mathbf{G}$, then the image of Γ under the map

$$\mathbf{G}(F) \xrightarrow{\cong} (\operatorname{Res}_{F/\mathbb{Q}} \mathbf{G})(\mathbb{Q})$$

is an arithmetic subgroup of the $\mathbb{Q}$-group $\operatorname{Res}_{F/\mathbb{Q}} \mathbf{G}$ by [PR94, pp. 50–51].

In 1962, Borel and Harish-Chandra [BH62] proved that arithmetic subgroups of semisimple algebraic groups are lattices. This is a fundamental result in the theory of arithmetic groups and gives us many examples of lattices.

Theorem 4.34 (Borel, Harish-Chandra). *Every arithmetic subgroup of a semisimple algebraic group* $\mathbf{G}$ *over* $\mathbb{Q}$ *is a lattice in* $\mathbf{G}(\mathbb{R})$.

4.4 Adeles and Congruence Subgroups

We now introduce the ring of finite adeles of a number field and discuss the closely related notion of a congruence subgroup of an arithmetic subgroup. Let F be a number field with ring of integers $\mathcal{O}_F$. Then for every nonzero prime ideal $\mathfrak{p} \subset \mathcal{O}_F$, we have a $\mathfrak{p}$-adic absolute value $v_\mathfrak{p}$ on F. The completion of F with respect to $v_\mathfrak{p}$ is a locally compact topological field $F_\mathfrak{p}$, and the closure of $\mathcal{O}_F$ in $F_\mathfrak{p}$ is a compact open subring $\mathcal{O}_\mathfrak{p} \subset F_\mathfrak{p}$. In order to be able to look at all these completions at once, we introduce the following construction:

Definition 4.35. Let $(G_i)_{i\in I}$ be a family of locally compact topological groups and let there be given for each $i \in I$ an open compact subgroup $K_i \subset G_i$. Then the *restricted direct product of* $(G_i)_{i\in I}$ *with respect to* $(K_i)_{i\in I}$ is the group

$$\prod_{i\in I}(G_i, K_i) := \left\{(g_i)_{i\in I} \in \prod_{i\in I} G_i : g_i \in K_i \text{ for all but finitely many } i \in I\right\}$$

with the topology in which a basis of open neighborhoods of the identity is given by all sets of the form $\prod_{i\in I} U_i$, where $U_i \subset G_i$ is an open neighborhood of the identity for each $i \in I$ and $U_i = K_i$ for all but finitely many $i \in I$.

Remark 4.36. The topology on the restricted direct product $\prod_{i\in I}(G_i, K_i)$ is different from the subspace topology induced by the direct product $\prod_{i\in I} G_i$.

In the following, we denote for a number field F by $P(F)$ the set of all nonzero prime ideals in the ring of integers $\mathcal{O}_F$. For each $\mathfrak{p} \in P(F)$, we denote as above by $F_\mathfrak{p}$ the completion of F with respect to the $\mathfrak{p}$-adic absolute value and by $\mathcal{O}_\mathfrak{p}$ the closure of $\mathcal{O}_F$ in $F_\mathfrak{p}$.

Definition 4.37. Let F be a number field. The *ring of finite adeles of* F is the restricted direct product

$$\mathbb{A}_{f,F} := \prod_{\mathfrak{p}\in P(F)} (F_\mathfrak{p}, \mathcal{O}_\mathfrak{p}),$$

and the *ring of integral finite adeles of* F is the open subring

$$\mathbb{O}_{f,F} := \prod_{\mathfrak{p}\in P(F)} \mathcal{O}_\mathfrak{p}.$$

Remark 4.38. The ring $\mathbb{A}_{f,F}$ is a commutative topological F-algebra, where F is considered as a subring of $\mathbb{A}_{f,F}$ by the diagonal embedding $F \hookrightarrow \mathbb{A}_{f,F}$. Similarly, $\mathcal{O}_F$ is a subring of $\mathbb{O}_{f,F}$ by the diagonal embedding $\mathcal{O}_F \hookrightarrow \mathbb{O}_{f,F}$, and $\mathbb{O}_{f,F}$ is a commutative topological $\mathcal{O}_F$-algebra.

The next proposition describes for an algebraic group over a number field the topology of its group of points with values in the finite adele ring.

Proposition 4.39. *Let* $\mathbf{G}$ *be an algebraic group over a number field* F*. Then for every integral form* $\mathbf{G}_0$ *of* $\mathbf{G}$*, the group* $\mathbf{G}_0(\mathbb{O}_{f,F})$ *is an open compact subgroup of* $\mathbf{G}(\mathbb{A}_{f,F})$ *and a basis of open neighborhoods of the identity in both groups is given by the sets*

$$\mathbf{G}_0(\mathbb{O}_{f,F})(\mathfrak{a}) := \ker\Big(\mathbf{G}_0(\mathbb{O}_{f,F}) \to \mathbf{G}_0(\mathbb{O}_{f,F}/\mathfrak{a}\mathbb{O}_{f,F})\Big)$$

for all nonzero ideals $\mathfrak{a} \subset \mathcal{O}_F$*, where* $\mathfrak{a}\mathbb{O}_{f,F}$ *is the ideal in* $\mathbb{O}_{f,F}$ *generated by* $\mathfrak{a}$.

Proof. By [PR94, pp. 108, 243–244], we know that for each $\mathfrak{p} \in P(F)$, the group $\mathbf{G}(F_\mathfrak{p})$ is locally compact, $\mathbf{G}_0(\mathcal{O}_\mathfrak{p})$ is an open compact subgroup of $\mathbf{G}(F_\mathfrak{p})$ and the projections $\mathbb{A}_{f,F} \to F_{\mathfrak{p}}$ induce an isomorphism of topological groups

$$\mathbf{G}(\mathbb{A}_{f,F}) \xrightarrow{\cong} \prod_{\mathfrak{p}\in P(F)} (\mathbf{G}(F_{\mathfrak{p}}), \mathbf{G}_0(\mathcal{O}_{\mathfrak{p}})).$$

Moreover, a basis of open neighborhoods of the identity in $\mathbf{G}(F_{\mathfrak{p}})$ is given by the sets

$$\mathbf{G}_0(\mathcal{O}_{\mathfrak{p}})(\mathfrak{p}^n) := \ker\Big(\mathbf{G}_0(\mathcal{O}_{\mathfrak{p}}) \to \mathbf{G}_0(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^n\mathcal{O}_{\mathfrak{p}})\Big)$$

for all $n \in \mathbb{N}$, where $\mathfrak{p}^n\mathcal{O}_{\mathfrak{p}}$ is the ideal in $\mathcal{O}_{\mathfrak{p}}$ generated by $\mathfrak{p}^n$. Hence by the definition of the restricted direct product topology, a basis of open neighborhoods of the identity in $\prod_{\mathfrak{p}\in P(F)}(\mathbf{G}(F_{\mathfrak{p}}), \mathbf{G}_0(\mathcal{O}_{\mathfrak{p}}))$ is given by the sets of the form

$$\prod_{i=1}^{k} \mathbf{G}_0(\mathcal{O}_{\mathfrak{p}_i})(\mathfrak{p}_i^{e_i}) \times \prod_{\mathfrak{p}\notin\{\mathfrak{p}_1,\ldots,\mathfrak{p}_k\}} \mathbf{G}_0(\mathcal{O}_{\mathfrak{p}}),$$

where $\mathfrak{p}_1, \ldots, \mathfrak{p}_k \subset \mathcal{O}_F$ are nonzero prime ideals and $e_1, \ldots, e_k \in \mathbb{N}$. These sets are precisely the images of the sets $\mathbf{G}_0(\mathbb{O}_{f,F})(\mathfrak{a})$ under the above isomorphism for nonzero ideals $\mathfrak{a} \subset \mathcal{O}_F$, where $\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_k^{e_k}$. This completes the proof. □

The following lemma describes the behavior of the ring of finite adeles with respect to finite field extensions (see [PR94, pp. 15, 50–51] for a proof):

Lemma 4.40. *Let E/F be a finite extension of number fields. Then there exists a natural ring isomorphism $\mathbb{A}_{f,F} \otimes_F E \xrightarrow{\cong} \mathbb{A}_{f,E}$, and for every algebraic group $\mathbf{G}$ over E, this isomorphism induces an isomorphism of topological groups*

$$(\operatorname{Res}_{E/F} \mathbf{G})(\mathbb{A}_{f,F}) \xrightarrow{\cong} \mathbf{G}(\mathbb{A}_{f,E}).$$

We now come to the definition of congruence subgroups, which provides us with an important source of subgroups of finite index in an arithmetic subgroup.

Definition 4.41. Let $\mathbf{G}_0$ be an affine group scheme of finite type over the ring of integers $\mathcal{O}_F$ of a number field F. Let $\Gamma \subset \mathbf{G}_0(\mathcal{O}_F)$ be a subgroup and let $\mathfrak{a} \subset \mathcal{O}_F$ be a nonzero ideal. Then the *principal congruence subgroup of* Γ *of level* $\mathfrak{a}$ is

$$\Gamma(\mathfrak{a}) := \Gamma \cap \mathbf{G}_0(\mathbb{O}_{f,F})(\mathfrak{a}) = \ker(\Gamma \to \mathbf{G}_0(\mathcal{O}_F/\mathfrak{a}\mathcal{O}_F)).$$

A subgroup of Γ is called a *congruence subgroup of* Γ if it contains a principal congruence subgroup of Γ.

We will sometimes also use the notation $\Gamma(\mathfrak{a}) := \Gamma \cap \mathbf{G}_0(\mathbb{O}_{f,F})(\mathfrak{a})$ for subgroups $\Gamma \subset \mathbf{G}_0(\mathbb{O}_{f,F})$, but we only call it a congruence subgroup when $\Gamma \subset \mathbf{G}_0(\mathcal{O}_F)$.

Remark 4.42. Every congruence subgroup of Γ is a subgroup of finite index in Γ since the quotient $\mathcal{O}_F/\mathfrak{a}\mathcal{O}_F$ is finite for any nonzero ideal $\mathfrak{a} \subset \mathcal{O}_F$. However, in some affine group schemes, there exist subgroups of finite index that are not congruence subgroups. Examples of such subgroups in $\mathbf{SL}_2(\mathbb{Z})$ have already been known to Fricke and Klein in the 19th century (see [Rag04, p. 299]).

Definition 4.43. An affine group scheme $\mathbf{G}_0$ of finite type over the ring of integers $\mathcal{O}_F$ of a number field F is said to have the *congruence subgroup property* if every subgroup of finite index in $\mathbf{G}_0(\mathcal{O}_F)$ is a congruence subgroup of $\mathbf{G}_0(\mathcal{O}_F)$.

As noted above, $\mathbf{SL}_2$ does not have this property. In 1951, Chevalley proved that $\mathbf{GL}_1$ has the congruence subgroup property (see [Che51]):

Theorem 4.44 (Chevalley). *Let F be a number field with ring of integers $\mathcal{O}_F$. Then every subgroup of finite index in $\mathcal{O}_F^\times$ contains a principal congruence subgroup $\mathcal{O}_F^\times(\mathfrak{a})$ for some nonzero ideal $\mathfrak{a} \subset \mathcal{O}_F$.*

4.5 Margulis' Arithmeticity Theorem

In this section, we discuss the question of whether or not a given lattice in a Lie group can be defined by an arithmetic construction, and we state Margulis' arithmeticity theorem. It is convenient and sufficient for us to consider in this discussion only lattices that are irreducible in the following sense:

Definition 4.45. Let G be a connected semisimple Lie group without compact factors. A lattice $\Gamma \subset G$ is called *reducible* if there exist two connected normal subgroups G_1 and G_2 of G with $G_1G_2 = G$ such that the intersection $G_1 \cap G_2$ is discrete and $\Gamma/((\Gamma \cap G_1)(\Gamma \cap G_2))$ is finite. Otherwise, Γ is called *irreducible*.

Note that every subgroup of a Lie group that is commensurable with a lattice is again a lattice (see for example [Mor15, p. 47]). The next lemma from [Zim84, p. 114] gives us another way to obtain new lattices from a given lattice:

Lemma 4.46. *Let $\varphi\colon G \twoheadrightarrow H$ be a surjective homomorphism of Lie groups such that $\ker(\varphi)$ is compact. Then for any lattice $\Gamma \subset G$, the image $\varphi(\Gamma)$ is a lattice in H.*

The following definition of an arithmetically defined lattice now includes all lattices that are obtained in one of these ways from an arithmetic subgroup.

Definition 4.47. Let $\mathbf{G}$ be a connected semisimple algebraic group over $\mathbb{R}$ without $\mathbb{R}$-anisotropic almost $\mathbb{R}$-simple factors. An irreducible lattice $\Delta \subset$

$\mathbf{G}(\mathbb{R})^0$ is said to be *arithmetically defined* if there exists a connected almost $\mathbb{Q}$-simple $\mathbb{Q}$-group $\mathbf{H}$ and an $\mathbb{R}$-epimorphism

$$\Phi\colon \mathbf{H}_{\mathbb{R}} \twoheadrightarrow \mathbf{G}$$

such that $(\ker \Phi)(\mathbb{R})$ is compact and Δ is commensurable with $\Phi(\mathbf{H}(\Gamma))$ for an arithmetic subgroup $\Gamma \subset \mathbf{H}(\mathbb{Q})$.

Example 4.48. There exist many non-arithmetically defined lattices in the group $\mathbf{SL}_2(\mathbb{R})$. This follows from a counting argument: It is known that there are uncountably many non-isometric hyperbolic surfaces of any genus $g \geq 2$ (see [Mor15, Corollary 15.3.4]), but there are only countably many non-isomorphic arithmetically defined lattices in $\mathbf{SL}_2(\mathbb{R})$ (see also [Ji08, p. 63]).

In 1984, Margulis proved in his remarkable arithmeticity theorem that every irreducible lattice in a semisimple Lie group of rank at least two is arithmetically defined. From [Mar91, Theorem IX.1.11], we have the following version:

Theorem 4.49 (Margulis' Arithmeticity Theorem). *Let* $\mathbf{G}$ *be a connected semisimple* $\mathbb{R}$*-group without* $\mathbb{R}$*-anisotropic almost* $\mathbb{R}$*-simple factors and with* $\operatorname{rank}_{\mathbb{R}}(\mathbf{G}) > 1$*. Then every irreducible lattice in* $\mathbf{G}(\mathbb{R})^0$ *is arithmetically defined.*

Chapter 5

Unit Groups in Quaternion Algebras

In this chapter, we introduce quaternion algebras and study groups of units in these algebras, which give us interesting examples of algebraic groups. We discuss the concept of orders in quaternion algebras over number fields and see how they give rise to arithmetically defined lattices in the Lie group $\mathrm{SL}_2(\mathbb{R})^r$. In the last section of this chapter, we prove that conversely every arithmetically defined lattice in $\mathrm{SL}_2(\mathbb{R})^r$ can be described up to commensurability in this way by a quaternion algebra.

5.1 Quaternion Algebras

Throughout this section, we assume that K is a field of characteristic zero.

Definition 5.1. An algebra D over K is a *quaternion algebra* if there exist $a, b \in K^\times$ and a vector space basis $\{1, i, j, k\}$ for D such that

$$i^2 = a, \quad j^2 = b, \quad k = ij = -ji.$$

Such a basis is called a *quaternionic basis for D*. We denote the quaternion algebra determined by the above relations by $(a, b)_K$.

Note that a quaternion algebra D with quaternionic basis $\{1, i, j, k\}$ is generated as an algebra over K by i and j and that its center is $Z(D) = K$.

Remark 5.2. The constants $a, b \in K^\times$ in Definition 5.1 are not unique for a quaternion algebra. By [MR03, p. 78], for all $a, b, c \in K^\times$, we have isomorphisms

$$(b, a)_K \cong (a, b)_K \cong (c^2 a, c^2 b)_K.$$

Example 5.3. The matrix algebra $M_2(K)$ is a quaternion algebra over K. In fact, the assignment

$$i \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

defines an isomorphism $(1, 1)_K \xrightarrow{\cong} M_2(K)$ of K-algebras.

Definition 5.4. A quaternion algebra over K is said to be *split* if it is isomorphic to the matrix algebra $M_2(K)$.

Example 5.5. In 1843, Hamilton discovered the quaternion algebra $\mathcal{H}(\mathbb{R}) := (-1,-1)_{\mathbb{R}}$ and proved that it is a division algebra, which means that every nonzero element is invertible (see [Ros88, p. 385]). In particular, it is not split.

The algebras $M_2(\mathbb{R})$ and $\mathcal{H}(\mathbb{R})$ are the only real quaternion algebras up to isomorphism. We have the following classification by [MR03, Theorem 2.5.1]:

Theorem 5.6. *For quaternion algebras over* $\mathbb{R}$*, we have:*

$$(a,b)_{\mathbb{R}} \cong \begin{cases} M_2(\mathbb{R}), & \text{if } a > 0 \text{ or } b > 0, \\ \mathcal{H}(\mathbb{R}), & \text{otherwise.} \end{cases}$$

Every quaternion algebra can be embedded into a matrix algebra over some field extension of the ground field. From [Kat92, p. 114], we have:

Proposition 5.7. *Let $D = (a,b)_K$ be a quaternion algebra and let $K(\sqrt{a})$ be the field obtained by adjoining a square root of a to K. Then the map*

$$D \to M_2(K(\sqrt{a})), \quad x + yi + zj + wk \mapsto \begin{pmatrix} x + y\sqrt{a} & z + w\sqrt{a} \\ b(z - w\sqrt{a}) & x - y\sqrt{a} \end{pmatrix}$$

is an injective K-algebra homomorphism, and an isomorphism if $K(\sqrt{a}) = K$.

In particular, it follows from Proposition 5.7 that a quaternion algebra over an algebraically closed field is split because any such field is quadratically closed.

Definition 5.8. Let D be a quaternion algebra over K. A *splitting field for D* is a field extension L/K such that $D \otimes_K L \cong M_2(L)$.

We have the following characterization of quadratic splitting fields for quaternion algebras from [GS06, Proposition 1.2.3]:

Proposition 5.9 (Splitting Criterion for Quadratic Extensions). *Let D be quaternion algebra over K and let $a \in K^\times$ be such that $K(\sqrt{a})/K$ is a quadratic field extension. Then $K(\sqrt{a})$ is a splitting field for D if and only if D is isomorphic to $(a,b)_K$ for some $b \in K^\times$.*

Definition 5.10. Let D be a quaternion algebra over K with quaternionic basis $\{1, i, j, k\}$. Then we define the subspace of *pure quaternions D_0 of D* to be the K-span of $\{i, j, k\}$.

One can show that a nonzero element $x \in D$ belongs to D_0 if and only if $x \notin K$ and $x^2 \in K$ (see [MR03, Lemma 2.1.4]). Hence the subspace $D_0 \subset D$ is independent of the choice of the quaternionic basis for D, and so we obtain a canonical decomposition $D = K \oplus D_0$.

Definition 5.11. Let D be a quaternion algebra over K. The *conjugate* of an element $x = \lambda + x_0 \in D$, with $\lambda \in K$ and $x_0 \in D_0$, is $\overline{x} := \lambda - x_0$, and the *reduced norm of* x is $N(x) := x\overline{x} = \lambda^2 - x_0^2 \in K$.

Remark 5.12. The reduced norm is a multiplicative map $N\colon D \to K$, that is, we have $N(xy) = N(x)N(y)$ for all $x, y \in D$. It follows that $x \in D$ is invertible if and only if $N(x) \neq 0$, in which case its inverse is given by $x^{-1} = N(x)^{-1}\overline{x}$.

We write $D^1 := \{x \in D : N(x) = 1\}$ for the set of all elements of reduced norm one in a quaternion algebra D, which is a subgroup of $D^\times$ by Remark 5.12.

Next, we associate an algebraic group to a quaternion algebra:

Definition 5.13. The *general linear group* $\mathbf{GL}_D$ over a quaternion algebra D over K is the functor

$$\mathbf{GL}_D\colon \mathrm{Alg}_K \to \mathrm{Grp}, \quad \mathbf{GL}_D(A) := (D \otimes_K A)^\times.$$

In order to show that $\mathbf{GL}_D$ is an algebraic group over K, we extend the definition of the reduced norm as follows: Let A be a commutative K-algebra. Then we have $D \otimes_K A = A \oplus A_0$, where $A_0 := D_0 \otimes_K A$. For $x = \lambda + x_0 \in D \otimes_K A$ with $\lambda \in A$ and $x_0 \in A_0$, we define $\overline{x} := \lambda - x_0$ and $N(x) := x\overline{x} = \lambda^2 - x_0^2 \in A$. As in Remark 5.12, we see that an element $x \in D \otimes_K A$ is invertible if and only if $N(x) \in A^\times$. Now every quaternionic basis $\{1, i, j, k\}$ for D is also a basis for $D \otimes_K A$ as a free A-module, and with respect to this basis, we have

$$N(x + yi + zj + wk) = x^2 - ay^2 - bz^2 + abw^2 \tag{5.1}$$

for all $x, y, z, w \in A$. We can now show the following (compare Example 4.2):

Proposition 5.14. *The functor $\mathbf{GL}_D$ is an algebraic group over K for every quaternion algebra D over K.*

Proof. Let $\{1, i, j, k\}$ be a quaternionic basis for D. Since an element of $D \otimes_K A$ is invertible if and only if its reduced norm is invertible, we have by (5.1) a natural bijection

$$(D \otimes_K A)^\times \xrightarrow{\cong} \hom_{\mathrm{Alg}_K}(K[X, Y, Z, W, T]/((X^2 - aY^2 - bZ^2 + abW)T - 1), A)$$

that sends an element $g = x + yi + zj + wk \in (D \otimes_K A)^\times$ to the homomorphism induced by $X \mapsto x$, $Y \mapsto y$, $Z \mapsto z$, $W \mapsto w$ and $T \mapsto N(g)^{-1}$. $\square$

Definition 5.15. The *special linear group* $\mathbf{SL}_D$ over a quaternion algebra D over K is the functor

$$\mathbf{SL}_D\colon \mathrm{Alg}_K \to \mathrm{Grp}, \quad \mathbf{SL}_D(A) := \{x \in D \otimes_K A : N(x) = 1\}.$$

Since the reduced norm defines a K-homomorphism $N\colon \mathbf{GL}_D \to \mathbf{GL}_1$ whose kernel is $\mathbf{SL}_D$, it follows that $\mathbf{SL}_D$ is also an algebraic group over K.

The next theorem shows that any two homomorphisms between fixed quaternion algebras differ only by conjugation with a unit (see [MR03, Theorem 2.9.8]):

Theorem 5.16 (Skolem-Noether). *Let A and B be quaternion algebras over K. Then for every two K-algebra homomorphisms $\phi, \psi\colon A \to B$, there exists some $b \in B^\times$ such that $\phi(a) = b\psi(a)b^{-1}$ for all $a \in A$.*

We finish this section by proving a lemma about the intersection of the centralizers of two elements in a quaternion algebra, which we will use in Chapter 6.

Lemma 5.17. *Let D be a quaternion algebra over K. If $x, y \in D$ do not commute with each other, then $C_D(x) \cap C_D(y) = K$, where $C_D(x)$ and $C_D(y)$ are the centralizers of x and y in D, respectively.*

Proof. Suppose to the contrary that there exists some $z \in D$ with $z \notin K$ that satisfies $xz = zx$ and $yz = zy$. Because $xy \neq yx$, we have $x, y \notin K$, and so we see that $\{1, z, x, y\}$ is a linearly independent subset with four distinct elements of the linear subspace $C_D(z) \subset D$. Since $\dim_K(D) = 4$, it follows that $C_D(z) = D$, and so $z \in Z(D) = K$. This contradicts our assumption $z \notin K$. □

5.2 Quaternion Algebras over Number Fields

In this section, we study quaternion algebras over number fields. The situation here is not as simple as over $\mathbb{C}$ or $\mathbb{R}$. Instead, we will see that a quaternion algebra over a number field F is determined up to an isomorphism by its behavior at the completions of F with respect to the different absolute values on F.

Definition 5.18. A *place* of a number field F is an equivalence class of nontrivial absolute values on F, where two nontrivial absolute values on F are *equivalent* if they induce equivalent metrics on F.

We write F_v for the completion of a number field F with respect to the metric induced by a place v of F. If the completion F_v is a nonarchimedean field, then v is called a *finite place* and is represented by a $\mathfrak{p}$-adic absolute value on F

for some nonzero prime ideal $\mathfrak{p} \subset \mathcal{O}_F$ in the ring of integers. Otherwise, if F_v is archimedean, then either $F_v \cong \mathbb{R}$ or $F_v \cong \mathbb{C}$, and v is called a *real place* or *complex place*, respectively. In any number field F, there is one real place for each real embedding $F \hookrightarrow \mathbb{R}$ and one complex place for each equivalence class of complex embeddings $F \hookrightarrow \mathbb{C}$ modulo complex conjugation.

Definition 5.19. Let F be a number field and let v be a place of F. We say that D is *split at v* if $D \otimes_F F_v$ is split. Otherwise, D is said to be *ramified at v*. The set of all places of F at which D is ramified is denoted by $\mathrm{Ram}(D)$.

If v is a place of F and $\sigma\colon F \hookrightarrow F_v$ the embedding of F into the completion, then we also say that D is *split at σ* or *ramified at σ* if D is split at v or ramified at v, respectively. By [MR03, Theorem 7.3.6], we have the following theorem:

Theorem 5.20 (Classification of Quaternion Algebras). *Let F be a number field. Then the following holds:*

- *(i) For every quaternion algebra D over F, the set $\mathrm{Ram}(D)$ consists of a finite and even number of non-complex places of F.*
- *(ii) Conversely, for every set S consisting of a finite and even number of non-complex places of F, there exists a quaternion algebra D over F, uniquely determined up to an isomorphism, with $\mathrm{Ram}(D) = S$.*

In particular, it follows that a quaternion algebra over a number field F is split if and only if it is split at all places of F, because of $\mathrm{Ram}(M_2(F)) = \emptyset$.

We now study splitting fields for quaternion algebras over number fields.

Definition 5.21. Let E/F be a finite extension of number fields. A place w of E is said to *lie above* a place v of F if there is an absolute value on E representing w that extends an absolute value on F representing v.

Remark 5.22. Let E/F be a quadratic extension of number fields. Then E/F is a Galois extension, and so by [Neu99, Proposition II.9.1], we have that for each place v of F, the group $\mathrm{Aut}(E/F)$ of F-linear field automorphisms of E acts transitively on the set of all places of E above v. In particular, the completions of E at the places above v are all isomorphic to each other.

By [MR03, Theorem 7.3.3 and its proof], we have the following criterion to decide if a quadratic extension is a splitting field for a quaternion algebra:

Proposition 5.23 (Splitting Criterion for Number Fields). *Let D be a quaternion algebra over a number field F. Then a quadratic field extension E/F is a splitting field for D if and only if for each $v \in \mathrm{Ram}(D)$, the completion of E at a place above v is a quadratic field extension of the completion F_v.*

The next theorem allows us to prescribe the completions of a quadratic extension of a number field at finitely many places (see [Roq05, p. 29]):

Theorem 5.24 (Grunwald-Wang Theorem for Quadratic Extensions). *Let F be a number field. Suppose that S is a finite set of non-complex places of F and let there be given for each $v \in S$ a trivial or quadratic field extension E_v/F_v. Then there exists a quadratic field extension E/F so that for every $v \in S$, the completions of E at places above v are isomorphic to the field E_v.*

We can now prove a proposition that we will need later in Chapter 6.

Proposition 5.25. *Let D be a quaternion algebra over a number field F. Then D is isomorphic to $(a,b)_F$ for some $a,b \in \mathcal{O}_F$ in the ring of integers of F such that $\sigma(a) > 0$ for all real embeddings $\sigma\colon F \hookrightarrow \mathbb{R}$ at which D is split.*

Proof. Let $S := \mathrm{Ram}(D) \cup S_\infty$, where S_∞ is the set of all real places of F at which D is split. For each $v \in \mathrm{Ram}(D)$, we let E_v/F_v be a quadratic extension. Note that such an extension exists because F_v is either isomorphic to $\mathbb{R}$ or to a finite extension of $\mathbb{Q}_p$ for some prime number p. For each place $v \in S_\infty$, we set $E_v := F_v \cong \mathbb{R}$. By Theorem 5.24, there exists a quadratic field extension E/F whose completions at places above v are isomorphic to E_v for each $v \in S$.

Since E/F is a quadratic extension, we can write $E = F(\sqrt{a})$ for some $a \in F^\times$ with $a \notin (F^\times)^2$. Now let $\sigma\colon F \hookrightarrow \mathbb{R}$ be a real embedding at which D is split. Then σ corresponds to a place $v \in S_\infty$. We can extend σ to an embedding

$$\widetilde{\sigma}\colon E \hookrightarrow \mathbb{C}, \quad \widetilde{\sigma}(x + y\sqrt{a}) := \sigma(x) + \sigma(y)\sqrt{\sigma(a)}$$

for $x, y \in F$. The image of this embedding must lie in $\mathbb{R}$, because we have chosen E so that its completion with respect to places of E above v is isomorphic to $F_v \cong \mathbb{R}$. Thus, we must have $\sigma(a) > 0$.

By Proposition 5.23, we know that E is a splitting field for D, and so by Proposition 5.9 there exists some $b \in F^\times$ with $D \cong (a,b)_F$. Finally, since $(a,b)_F \cong (c^2a, c^2b)_F$ for all $c \in F^\times$, we can also achieve that $a, b \in \mathcal{O}_F$. □

5.3 Orders in Quaternion Algebras

We now study orders in quaternion algebras over number fields. These are the analogs for quaternion algebras of the ring of integers in a number field.

Definition 5.26. Let D be a quaternion algebra over a number field F. An *order in D* is a subring $\Lambda \subset D$ which is a finitely generated $\mathcal{O}_F$-submodule such that the map $\Lambda \otimes_{\mathcal{O}_F} F \to D$, $x \otimes \lambda \mapsto \lambda x$ is an isomorphism.

Example 5.27. The ring $M_2(\mathcal{O}_F)$ is an order in $M_2(F)$. Moreover, for the quaternion algebra $D = (a,b)_F$ with $a, b \in \mathcal{O}_F$, the $\mathcal{O}_F$-span of the quaternionic basis $\{1, i, j, k\}$ for D is an order in D.

Any order Λ contains the ring of integers $\mathcal{O}_F$ (see [MR03, Lemma 2.2.7]), and so Λ is closed under the conjugation of D. We write $\Lambda^1 := \Lambda \cap D^1$ for its group of units of reduced norm one. By [JR16, Lemmas 4.6.6 and 4.6.9], we have:

Lemma 5.28. *Let Λ_1 and Λ_2 be two orders. Then $\Lambda_1 \cap \Lambda_2$ is again an order and $(\Lambda_1 \cap \Lambda_2)^\times$ is of finite index in both $\Lambda_1^\times$ and $\Lambda_2^\times$. In particular, the group $\Lambda_1^\times$ is commensurable with $\Lambda_2^\times$, and the group Λ_1^1 is commensurable with Λ_2^1.*

Definition 5.29. The *general linear group* $\mathbf{GL}_\Lambda$ over an order Λ is the functor

$$\mathbf{GL}_\Lambda \colon \mathrm{Alg}_{\mathcal{O}_F} \to \mathrm{Grp}, \quad \mathbf{GL}_\Lambda(A) := (\Lambda \otimes_{\mathcal{O}_F} A)^\times.$$

Next, we would like to show that $\mathbf{GL}_\Lambda$ is an affine group scheme over $\mathcal{O}_F$. For this, it is convenient to assume that the order comes from a quaternionic basis as in Example 5.27. We say that an order $\Lambda \subset D$ is a *standard order* if Λ is the $\mathcal{O}_F$-span of a quaternionic basis $\{1, i, j, k\}$ for D with $i^2 \in \mathcal{O}_F$ and $j^2 \in \mathcal{O}_F$. If Λ is a standard order, then for every commutative $\mathcal{O}_F$-algebra A, we have that $\{1, i, j, k\}$ is a basis for $\Lambda \otimes_{\mathcal{O}_F} A$ as a free A-module and so we can use the expression from (5.1) to define the reduced norm

$$N \colon \Lambda \otimes_{\mathcal{O}_F} A \to A, \quad N(x + yi + wj + wk) := x^2 - ay^2 - bz^2 + abw^2 \tag{5.2}$$

for all $x, y, z, w \in A$. Note that an element of $\Lambda \otimes_{\mathcal{O}_F} A$ is invertible if and only if its reduced norm is invertible in A.

Proposition 5.30. *Let $\Lambda \subset D$ be a standard order. Then the functor $\mathbf{GL}_\Lambda$ is an affine group scheme of finite type over $\mathcal{O}_F$ and an integral form of $\mathbf{GL}_D$.*

Proof. As in the proof of Proposition 5.14, it follows from (5.2) that $\mathbf{GL}_\Lambda$ is represented by a finitely generated $\mathcal{O}_F$-algebra. For any commutative F-algebra A, we have a natural ring isomorphism

$$\Lambda \otimes_{\mathcal{O}_F} A \xrightarrow{\cong} (\Lambda \otimes_{\mathcal{O}_F} F) \otimes_F A \xrightarrow{\cong} D \otimes_F A.$$

These isomorphisms induce an F-isomorphism $(\mathbf{GL}_\Lambda)_F \xrightarrow{\cong} \mathbf{GL}_D$, and so $\mathbf{GL}_\Lambda$ is an integral form of $\mathbf{GL}_D$. □

Definition 5.31. The *special linear group* $\mathbf{SL}_\Lambda$ over a standard order Λ is

$$\mathbf{SL}_\Lambda \colon \mathrm{Alg}_{\mathcal{O}_F} \to \mathrm{Grp}, \quad \mathbf{SL}_\Lambda(A) := \{x \in \Lambda \otimes_{\mathcal{O}_F} A : N(x) = 1\},$$

where $N\colon \Lambda \otimes_{\mathcal{O}_F} A \to A$ is the map defined in (5.2).

The functor $\mathbf{SL}_\Lambda$ is also an affine group scheme of finite type over $\mathcal{O}_F$ and an integral form of the algebraic group $\mathbf{SL}_D$. We conclude that an order gives us arithmetic subgroups as follows:

Proposition 5.32. *Let Λ be an order in a quaternion algebra D. Then the groups $\Lambda^\times$ and Λ^1 are arithmetic subgroups of $\mathbf{GL}_D$ and $\mathbf{SL}_D$, respectively.*

Proof. By Lemma 5.28, the groups $\Lambda^\times$ and Λ^1 are commensurable with the respective unit groups of a standard order. The statement now follows because the general linear group and special linear group over a standard order are integral forms of the algebraic groups $\mathbf{GL}_D$ and $\mathbf{SL}_D$, respectively. □

5.4 Subgroups Derived from Quaternion Algebras

Let $r \in \mathbb{N}$. We now study lattices in the Lie group $\mathrm{SL}_2(\mathbb{R})^r$ that are defined by quaternion algebras. Let D be a quaternion algebra over a totally real number field F such that D is split at exactly r distinct real embeddings $\sigma_1, \dots, \sigma_r\colon F \hookrightarrow \mathbb{R}$. Then for each $1 \leq i \leq r$, there exists an $\mathbb{R}$-algebra isomorphism

$$\tau_i\colon D \otimes_F \mathrm{res}_{\sigma_i}(\mathbb{R}) \xrightarrow{\cong} M_2(\mathbb{R}).$$

We call τ_i a *splitting map for D at σ_i* and the collection $(\tau_1, \dots, \tau_r)$ a *family of splitting maps for D*. These maps are not unique, but by Theorem 5.16, any two splitting maps for an embedding differ only by conjugation with a matrix in $\mathrm{GL}_2(\mathbb{R})$.

Definition 5.33. A subgroup $\Delta \subset \mathrm{SL}_2(\mathbb{R})^r$ is said to be *derived from a quaternion algebra* if there exists a quaternion algebra D over a totally real number field F which is split at exactly r distinct real embeddings of F, a family of splitting maps $(\tau_1, \dots, \tau_r)$ for D and an order $\Lambda \subset D$ such that

$$\Delta = \{(\tau_1(x), \dots, \tau_r(x)) : x \in \Lambda^1\}.$$

Proposition 5.34. *Every subgroup of $\mathrm{SL}_2(\mathbb{R})^r$ that is derived from a quaternion algebra is an arithmetically defined lattice in $(\mathbf{SL}_2)^r(\mathbb{R})$.*

Proof. Let $\Delta \subset \mathrm{SL}_2(\mathbb{R})^r$ be a subgroup derived from a quaternion algebra D as in Definition 5.33. It is shown in [Vig80, Theorem IV.1.1] that Δ is an irreducible lattice in $\mathrm{SL}_2(\mathbb{R})^r$. We prove that Δ is arithmetically defined. To see this, we consider the connected almost $\mathbb{Q}$-simple $\mathbb{Q}$-group $\mathbf{H} := \mathrm{Res}_{F/\mathbb{Q}}\, \mathbf{SL}_D$. Let $\sigma_1, \dots, \sigma_d\colon F \hookrightarrow \mathbb{R}$ be the distinct real embeddings of F, ordered in such a way that D is split at the first r embeddings and ramified at the remaining embeddings. By Proposition 4.17, we have an $\mathbb{R}$-isomorphism

$$\mathbf{H}_{\mathbb{R}} \cong \prod_{i=1}^{d} (\mathbf{SL}_D)_{\sigma_i},$$

and or each $1 \le i \le r$, we have $(\mathbf{SL}_D)_{\sigma_i} \cong \mathbf{SL}_2$. So by projecting to the first r direct factors of $\mathbf{H}_{\mathbb{R}}$, the splitting maps $\tau_1, \ldots, \tau_r$ define an $\mathbb{R}$-epimorphism

$$\Phi\colon \mathbf{H}_{\mathbb{R}} \twoheadrightarrow (\mathbf{SL}_2)^r$$

such that $\Phi(\Lambda^1) = \Delta$ and $(\ker \Phi)(\mathbb{R}) \cong (\mathcal{H}(\mathbb{R})^1)^{d-r}$ is compact. This shows that Δ is arithmetically defined. $\square$

In the next section, we will prove the converse direction of Proposition 5.34. We finish this section by taking a closer look at the role of the choice of the order and the splitting maps for a subgroup derived from a quaternion algebra.

Definition 5.35. Two subgroups Δ_1 and Δ_2 of $\mathrm{SL}_2(\mathbb{R})^r$ are *commensurable in the wide sense* if Δ_1 is commensurable with $g\Delta_2 g^{-1}$ for some $g \in \mathrm{GL}_2(\mathbb{R})^r$.

Proposition 5.36. *Any two subgroups of* $\mathrm{SL}_2(\mathbb{R})^r$ *that are derived from the same quaternion algebra are commensurable in the wide sense.*

Proof. We know from Lemma 5.28 that the groups of units of reduced norm one in any two orders in a quaternion algebra are commensurable. By Theorem 5.16, any two splitting maps for an embedding differ only by conjugation with a matrix in $\mathrm{GL}_2(\mathbb{R})$, and so the statement of the proposition follows. $\square$

5.5 Classification of Arithmetically Defined Subgroups

In this section, we prove that every arithmetically defined lattice in $(\mathbf{SL}_2)^r(\mathbb{R})$ is commensurable with a subgroup derived from a quaternion algebra. We will need some results on the classification of semisimple algebraic groups for this, which we now discuss. More details on this classification can be found in [Tit66]. Throughout this section, we assume that K is a field of characteristic zero.

Definition 5.37. A connected semisimple algebraic group $\mathbf{G}$ over K is called *simply connected* if for any connected semisimple group $\mathbf{G}'$ over K, every K-epimorphism $\mathbf{G}' \twoheadrightarrow \mathbf{G}$ with finite kernel is a K-isomorphism.

If the algebraic group that we start with is not simply connected, we can use the following proposition from [Mar91, Proposition I.1.4.11]:

Proposition 5.38. *For every connected semisimple algebraic group* $\mathbf{G}$ *over* K*, there exists a simply connected algebraic group* $\widetilde{\mathbf{G}}$ *over* K *and a* K*-epimorphism* $\widetilde{\mathbf{G}} \twoheadrightarrow \mathbf{G}$ *with finite kernel. The algebraic group* $\widetilde{\mathbf{G}}$ *is uniquely determined by this property up to an isomorphism.*

By Lemma 4.20, every simply connected algebraic group over K is a direct product of its almost K-simple factors. We now turn to the almost K-simple algebraic groups. Recall from Example 4.22 that an almost K-simple algebraic group does not have to stay almost simple over field extensions of K.

Definition 5.39. An algebraic group $\mathbf{G}$ over K is called *absolutely almost simple* if, for an algebraic closure $\overline{K}$ of K, the group $\mathbf{G}_{\overline{K}}$ is almost $\overline{K}$-simple.

The next proposition from [Mar91, I.1.7] shows that every simply connected almost simple algebraic group is the restriction of scalars of an absolutely almost simple algebraic group.

Proposition 5.40. *For every simply connected almost* K*-simple algebraic group* $\mathbf{G}$ *over* K*, there exists a finite field extension* L/K *and a simply connected absolutely almost simple* L*-group* $\mathbf{H}$ *such that* $\mathrm{Res}_{L/K}\,\mathbf{H}$ *is* K*-isomorphic to* $\mathbf{G}$*.*

We now give a complete list of the simply connected almost simple algebraic groups over an algebraically closed field up to isomorphism. A proof of the following theorem and the definitions of the corresponding algebraic groups can be found in [Mil17, Chapters 23 and 24] and [Tit66, pp. 33–38].

Theorem 5.41 (Classification Theorem). *Let* K *be an algebraically closed field of characteristic zero. Then every simply connected almost* K*-simple algebraic group over* K *is isomorphic to exactly one on the following list:*

(i) The special linear groups $\mathbf{SL}_{n+1}$ *for* $n \geq 1$*.*

(ii) The special orthogonal groups $\mathbf{SO}_{2n+1}$ *for* $n \geq 2$*.*

(iii) The symplectic groups $\mathbf{Sp}_{2n}$ *for* $n \geq 3$*.*

(iv) The special orthogonal groups $\mathbf{SO}_{2n}$ *for* $n \geq 4$*.*

(v) The five exceptional algebraic groups of type E_6, E_7, E_8, F_4 *and* G_2*.*

In order to understand algebraic groups over non-algebraically closed fields, it is useful to introduce the concept of a K-form of an algebraic group over K:

Definition 5.42. Let $\mathbf{G}$ and $\mathbf{H}$ be two algebraic groups over K. We say that $\mathbf{G}$ is a *K-form of* $\mathbf{H}$ if there exists a field extension L/K such that $\mathbf{G}_L \cong \mathbf{H}_L$.

By [Mil17, p. 421], the F-forms of the algebraic group $\mathbf{SL}_2$ over a number field F can be described using quaternion algebras as follows:

Theorem 5.43 (Classification of Forms of $\mathbf{SL}_2$). *Let F be a number field. Every F-form of $\mathbf{SL}_2$ is isomorphic to $\mathbf{SL}_D$ for a quaternion algebra D over F.*

We can now prove that every arithmetically defined lattice in $(\mathbf{SL}_2)^r(\mathbb{R})$ is coming from a quaternion algebra. Our proof is based on [Moc98, pp. 6–7] and uses the concept of the Lie algebra $\mathrm{Lie}(\mathbf{G})$ of an algebraic group $\mathbf{G}$. We refer the reader to [Mil17, Chapter 10] for the definition and elementary properties of Lie algebras of algebraic groups.

Theorem 5.44. *A subgroup of $(\mathbf{SL}_2)^r(\mathbb{R})$ is an arithmetically defined lattice if and only if it is commensurable with a subgroup derived from a quaternion algebra.*

Proof. We have already seen in Proposition 5.34 that subgroups derived from quaternion algebras are arithmetically defined lattices. Let now $\Delta \subset (\mathbf{SL}_2)^r(\mathbb{R})$ be an arithmetically defined lattice. Then by definition, there exists a connected almost $\mathbb{Q}$-simple $\mathbb{Q}$-group $\mathbf{H}$ and an $\mathbb{R}$-epimorphism

$$\Phi\colon \mathbf{H}_{\mathbb{R}} \twoheadrightarrow (\mathbf{SL}_2)^r$$

such that $(\ker \Phi)(\mathbb{R})$ is compact and Δ is commensurable with $\Phi(\Gamma)$ for an arithmetic subgroup $\Gamma \subset \mathbf{H}(\mathbb{Q})$. By [Mar91, Remark IX.1.6 (i)], we may assume that $\mathbf{H}$ is simply connected. So Proposition 5.40 yields a number field F and a simply connected absolutely almost simple F-group $\mathbf{F}$ such that $\mathbf{H} = \mathrm{Res}_{F/\mathbb{Q}} \mathbf{F}$. We now show that $\mathbf{F}$ is an F-form of $\mathbf{SL}_2$. By Corollary 4.18, we have

$$(\mathrm{Res}_{F/\mathbb{Q}} \mathbf{F})_{\mathbb{C}} \cong \prod_{\sigma\colon F \hookrightarrow \mathbb{C}} \mathbf{F}_{\sigma},$$

and so the Lie algebra of this algebraic group is the direct sum of the Lie algebras of the direct factors $\mathbf{F}_\sigma$. Because $\mathbf{F}$ is absolutely almost simple, the algebraic groups $\mathbf{F}_\sigma$ are almost $\mathbb{C}$-simple and so their Lie algebras are simple. Now observe that the $\mathbb{R}$-epimorphism Φ induces a surjection of Lie algebras (see [Hum75, p. 44])

$$\mathrm{Lie}\big((\mathrm{Res}_{F/\mathbb{Q}} \mathbf{F})_{\mathbb{C}}\big) = \bigoplus_{\sigma\colon F \hookrightarrow \mathbb{C}} \mathrm{Lie}(\mathbf{F}_\sigma) \twoheadrightarrow \mathfrak{sl}_2(\mathbb{C})^r.$$

It follows that at least one of the simple Lie algebras $\mathrm{Lie}(\mathbf{F}_\sigma)$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Then $\mathbf{F}_\sigma$ is isomorphic to $\mathbf{SL}_2$ by Theorem 5.41 because it is simply connected and absolutely almost simple, and so $\mathbf{F}$ is an F-form of $\mathbf{SL}_2$.

By Theorem 5.43, we can assume that $\mathbf{F} = \mathbf{SL}_D$ for some quaternion algebra D over F. Next, we show that the number field F is totally real. Assume to the contrary that $\sigma\colon F \hookrightarrow \mathbb{C}$ is a non-real embedding. Then $(\mathbf{SL}_D)_\sigma \cong \mathbf{SL}_2$, and so $\mathbf{H}_\mathbb{R}(\mathbb{R}) = (\operatorname{Res}_{F/\mathbb{Q}} \mathbf{SL}_D)(\mathbb{R})$ contains by Proposition 4.17 a direct factor isomorphic to

$$(\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbf{SL}_2)(\mathbb{R}) \cong \mathrm{SL}_2(\mathbb{C}),$$

where here $\mathrm{SL}_2(\mathbb{C})$ is considered as a real Lie group. If Φ were trivial on this direct factor, then $(\ker \Phi)(\mathbb{R})$ would contain a subgroup isomorphic to $\mathrm{SL}_2(\mathbb{C})$, and so $(\ker \Phi)(\mathbb{R})$ would be noncompact. Hence Φ induces, after projecting to a direct factor in its image, a nontrivial Lie group homomorphism $\varphi\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{R})$. The kernel of φ is a closed normal subgroup of $\mathrm{SL}_2(\mathbb{C})$ and hence is discrete because $\mathrm{SL}_2(\mathbb{C})$ is a simple complex Lie group. Thus, φ is an immersion, in contradiction to the fact that $\mathrm{SL}_2(\mathbb{C})$ has real dimension six and $\mathrm{SL}_2(\mathbb{R})$ has real dimension three. So F must be totally real.

Similarly, one sees that D must be split at exactly r real embeddings of F. Let now $\sigma_1, \dots, \sigma_d\colon F \hookrightarrow \mathbb{R}$ be the real embeddings of F ordered in such a way that D is split at the first r embeddings, and choose a family of splitting maps $(\tau_1, \dots, \tau_r)$ for D. Then by Proposition 4.17, we have

$$\mathbf{H}_\mathbb{R}(\mathbb{R}) \cong \mathrm{SL}_2(\mathbb{R})^r \times (\mathcal{H}(\mathbb{R})^1)^{d-r},$$

and so, as in the proof of Proposition 5.34, the splitting maps $\tau_1, \dots, \tau_r$ define an $\mathbb{R}$-epimorphism

$$\Psi\colon \mathbf{H}_\mathbb{R} \twoheadrightarrow (\mathbf{SL}_2)^r$$

so that $(\ker \Psi)(\mathbb{R})$ is compact. Then Φ and Ψ are both trivial on the maximal compact subgroup of $\mathbf{H}_\mathbb{R}(\mathbb{R})$, and the restrictions of Φ and Ψ to the noncompact factor of $\mathbf{H}_\mathbb{R}(\mathbb{R})$ are both automorphisms of $\mathrm{SL}_2(\mathbb{R})^r$. Every such automorphism is of the form

$$(g_1, \dots, g_r) \mapsto (\varphi_1(g_{\pi(1)}), \dots, \varphi_r(g_{\pi(r)})),$$

where $\varphi_1, \dots, \varphi_r$ are automorphisms of $\mathrm{SL}_2(\mathbb{R})$ and π is a permutation of $\{1, \dots, r\}$ (see [Kra11, p. 2631]). By [Die80, p. 18], any two automorphisms of $\mathrm{SL}_2(\mathbb{R})$ differ by conjugation with a matrix in $\mathrm{GL}_2(\mathbb{R})$. So after possibly rearranging the embeddings $\sigma_1, \dots, \sigma_r$ and conjugating the splitting maps $\tau_1, \dots, \tau_r$ with matrices in $\mathrm{GL}_2(\mathbb{R})$, we have that Φ and Ψ agree on $\mathbf{H}_\mathbb{R}(\mathbb{R})$. If we choose an order $\Lambda \subset D$, then Λ^1 is commensurable with Γ by Proposition 5.32. So Δ is commensurable with a subgroup derived from a quaternion algebra. □

Chapter 6

Construction of Flat Homology Classes

In this chapter, we discuss geometric cycles and intersection numbers and complete the proof of Theorem 1.1 by constructing families of linearly independent homology classes coming from compact flat totally geodesic r-dimensional submanifolds in the homology of a locally symmetric space covered by $(\mathbb{H}^2)^r$. This construction is based on the method developed by Avramidi and Nguyen-Phan in [AN15] and works for every locally symmetric space that is a quotient of $(\mathbb{H}^2)^r$ by an arithmetically defined lattice in $(\mathbf{SL}_2)^r(\mathbb{R})$.

6.1 Geometric Cycles

In this section, we describe a method to obtain totally geodesic submanifolds in a locally symmetric space which is given as a quotient of a symmetric space by an arithmetic subgroup. We first see how to obtain totally geodesic submanifolds in a symmetric space. For this, we need the following statements from [HN12, Theorems 14.1.3 and 14.3.11] and [HT94, Lemma 3.10]:

Theorem 6.1 (Maximal Compact Subgroups of Lie Groups). *Let H be a Lie group with finitely many connected components. Then we have:*

(i) Every compact subgroup of H is contained in a maximal compact subgroup of H, and any two maximal compact subgroups of H are conjugate.

(ii) For any maximal compact subgroup $K \subset H$, the quotient H/K is diffeomorphic to a Euclidean space.

For semisimple Lie groups, we have already seen most of these statements in Proposition 3.5. In analogy to the situation there, we use the following notation:

Definition 6.2. We write $X_H := H/K$ for the quotient of a Lie group H with finitely many connected components by a maximal compact subgroup $K \subset H$.

Assume now that G is a semisimple Lie group with finitely many connected components and finite center, and consider the symmetric space X_G. Let $H \subset G$ be a closed subgroup with finitely many connected components. By Theorem 6.1, there exists a maximal compact subgroup $K_H \subset H$, and K_H is contained in some maximal compact subgroup $K \subset G$. We must have $K_H = K \cap H$, because $K \cap H$ is a compact subgroup of H containing K_H. The inclusion $H \hookrightarrow G$ induces a closed embedding

$$j_H \colon X_H \hookrightarrow X_G$$

of the quotients $X_H = H/K_H$ and $X_G = G/K$ whose image is a totally geodesic submanifold of X_G (see [Sch10, p. 213] and [Wal08, p. 5]). This embedding does in general not descend to an embedding of a totally geodesic submanifold into the locally symmetric space $\Gamma\backslash X_G$ for a torsion-free discrete subgroup $\Gamma \subset G$. However, the following theorem from [Sch10, Theorem D] states that this can always be achieved in the arithmetic setting by replacing Γ with a subgroup of finite index:

Theorem 6.3. *Let* $\mathbf{G}$ *be a connected semisimple algebraic group over* $\mathbb{Q}$ *and let* $\mathbf{H}$ *be a connected reductive* $\mathbb{Q}$*-subgroup of* $\mathbf{G}$*. Let* $G := \mathbf{G}(\mathbb{R})$ *and* $H := \mathbf{H}(\mathbb{R})$*. Then any arithmetic subgroup* $\Gamma \subset \mathbf{G}(\mathbb{Q})$ *has a torsion-free subgroup of finite index* $\Gamma_0 \subset \Gamma$ *such that for every subgroup of finite index* $\Gamma' \subset \Gamma_0$*, the map*

$$j_{H|\Gamma'} \colon (\Gamma' \cap H)\backslash X_H \to \Gamma'\backslash X_G$$

induced by a map j_H *as above is a closed embedding and its image is an orientable totally geodesic submanifold of the locally symmetric space* $\Gamma'\backslash X_G$*.*

Definition 6.4. If the map $j_{H|\Gamma'}$ in Theorem 6.3 is a closed embedding, then we call its image a *geometric cycle* in the locally symmetric space $\Gamma'\backslash X_G$.

An effective strategy to show that the fundamental class of a geometric cycle is nontrivial in the homology of the locally symmetric space is to find another geometric cycle such that their intersection product is nontrivial.

6.2 Intersection Numbers and de Rham Cohomology

In order to show that homology classes of submanifolds are linearly independent, we will use the concept of the intersection numbers of transversally intersecting submanifolds. Recall that two submanifolds $S_1, S_2 \subset M$ intersect transversally if for each $p \in S_1 \cap S_2$, the natural map $T_pS_1 \oplus T_pS_2 \to T_pM$ is surjective.

Definition 6.5. Let M be a smooth oriented manifold and let S_1 and S_2 be oriented submanifolds of M such that S_1 and S_2 intersect transversally and $\dim(S_1) + \dim(S_2) = \dim(M)$. Then for each $p \in S_1 \cap S_2$, the *intersection*

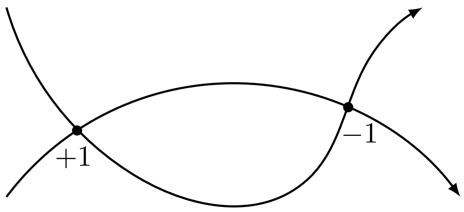


Figure 6: Intersection numbers of two curves in the plane.

number of S_1 *and* S_2 *in* p is

$$I_p(S_1, S_2) := \begin{cases} +1, & \text{if } T_pS_1 \oplus T_pS_2 \xrightarrow{\cong} T_pM \text{ is orientation-preserving,} \\ -1, & \text{otherwise.} \end{cases}$$

Let M be a smooth oriented n-manifold. We write $H^k_{\mathrm{dR}}(M)$ for the k-th de Rham cohomology group of M and $H^k_{\mathrm{dR,c}}(M)$ for the k-th de Rham cohomology group with compact support of M. By [GHV72, p. 197], we have:

Theorem 6.6 (Poincaré Duality for de Rham Cohomology). *Let* M *be a smooth oriented* n*-manifold. Then for any* $0 \leq k \leq n$*, the bilinear map*

$$H^k_{\mathrm{dR,c}}(M) \times H^{n-k}_{\mathrm{dR}}(M) \to \mathbb{R}, \quad (\omega, \tau) \mapsto \int_M \omega \wedge \tau$$

has the property that the induced map $H^{n-k}_{\mathrm{dR}}(M) \to \left(H^k_{\mathrm{dR,c}}(M)\right)^*$ *is a linear isomorphism.*

Since integration of differential k-forms with compact support on M over a closed oriented k-submanifold of M defines a linear form on $H^k_{\mathrm{dR,c}}(M)$, we can now assign by Theorem 6.6 a cohomology class to a submanifold as follows:

Definition 6.7. Let M be a smooth oriented n-manifold and let $S \subset M$ be a closed oriented k-submanifold. Then the *closed Poincaré dual of* S is the unique cohomology class $\eta_S \in H^{n-k}_{\mathrm{dR}}(M)$ such that for all $\omega \in H^k_{\mathrm{dR,c}}(M)$, we have

$$\int_M \omega \wedge \eta_S = \int_S i^*\omega,$$

where $i\colon S \hookrightarrow M$ is the inclusion map.

Next, we want to assign to a submanifold another cohomology class that lives in the cohomology group with compact support. We therefore consider now the map $H^k_{\mathrm{dR,c}}(M) \to (H^{n-k}_{\mathrm{dR}}(M))^*$ induced by the pairing in Theorem 6.6. Note that this map does not have to be an isomorphism if $H^k_{\mathrm{dR}}(M)$ and $H^k_{\mathrm{dR,c}}(M)$ are not finite-dimensional (see [BT82, Remark I.5.7]). In order to guarantee this property, we impose the following condition on M:

Definition 6.8. A smooth manifold is said to be of *finite type* if it is diffeomorphic to the interior of a compact smooth manifold with boundary.

Example 6.9. Let $\mathbf{G}$ be a connected semisimple $\mathbb{R}$-group and let $\Gamma \subset \mathbf{G}(\mathbb{R})^0$ be a torsion-free arithmetically defined lattice. Then the locally symmetric space $\Gamma\backslash X_{\mathbf{G}(\mathbb{R})}$ is a smooth manifold of finite type (see [Rag68]).

If M is of finite type and diffeomorphic to the interior of a compact manifold with boundary $\overline{M}$, then the inclusion $M \hookrightarrow \overline{M}$ is a homotopy equivalence. So in this case, the cohomology groups $H^k_{\mathrm{dR}}(M)$ and $H^k_{\mathrm{dR,c}}(M)$ are finite-dimensional for any k by Theorem 6.6. Then the pairing from Theorem 6.6 also induces (after swapping k and $n-k$) a linear isomorphism

$$H^{n-k}_{\mathrm{dR,c}}(M) \to \left(H^k_{\mathrm{dR}}(M)\right)^*.$$

Integration of differential k-forms on M over a compact oriented k-submanifold of M defines a linear form on $H^k_{\mathrm{dR}}(M)$, and so we can assign to such a submanifold a cohomology class as follows:

Definition 6.10. Let M be a smooth oriented n-manifold of finite type and let $S \subset M$ be a compact oriented k-submanifold. Then the *compact Poincaré dual of* S is the unique cohomology class $\eta'_S \in H^{n-k}_{\mathrm{dR,c}}(M)$ such that for all $\omega \in H^k_{\mathrm{dR}}(M)$, we have

$$\int_M \omega \wedge \eta'_S = \int_S i^*\omega,$$

where $i\colon S \hookrightarrow M$ is the inclusion map.

Remark 6.11. The closed and compact Poincaré duals of a compact oriented k-submanifold $S \subset M$ are related by the natural map $H^{n-k}_{\mathrm{dR,c}}(M) \to H^{n-k}_{\mathrm{dR}}(M)$, which sends η'_S to η_S (see [BT82, p. 51]). Moreover, the class η'_S is the image of the fundamental class $[S] \in H_k(S;\mathbb{R})$ of S under the composition

$$H_k(S;\mathbb{R}) \to H_k(M;\mathbb{R}) \xrightarrow{\cong} H^{n-k}_c(M;\mathbb{R}) \xrightarrow{\cong} H^{n-k}_{\mathrm{dR,c}}(M),$$

where the second map is the Poincaré duality isomorphism and the third map is the de Rham isomorphism for cohomology with compact support.

By [BT82, p. 69], the Poincaré dual of a transverse intersection of submanifolds of complementary dimensions is related to the intersection numbers as follows:

Theorem 6.12. *Let M be a smooth oriented manifold and let S_1 and S_2 be two closed oriented submanifolds of M intersecting transversally with* $\dim(S_1) + \dim(S_2) = \dim(M)$. *Then $S_1 \cap S_2$ is a discrete submanifold of M and for the orientation on it induced by the intersection numbers of S_1 and S_2, we have*

$$\eta_{S_1} \wedge \eta_{S_2} = \eta_{S_1 \cap S_2}.$$

6.3 Building a Configuration of Flats

We now construct a configuration of maximal flats in $(\mathbb{H}^2)^r$ that we will later project to a quotient of $(\mathbb{H}^2)^r$ by an arithmetically defined lattice in $(\mathbf{SL}_2)^r(\mathbb{R})$. Recall from Theorem 5.44 that any such lattice is commensurable to a subgroup derived from a quaternion algebra.

Throughout this section, we fix the following notation and assumptions: Let F be a totally real number field whose real embeddings are $\sigma_1, \dots, \sigma_d \colon F \hookrightarrow \mathbb{R}$. We assume that $F \subset \mathbb{R}$ and σ_1 is the identity embedding. Let $D = (a, b)_F$ be a quaternion algebra over F which is split at the first r embeddings of F and ramified at the remaining embeddings. We assume that $a, b \in \mathcal{O}_F$ and $\sigma_i(a) > 0$ for all $i \leq r$ (see Proposition 5.25). Let Λ be the standard order spanned by the quaternionic basis for D and let $\Gamma \subset \Lambda^1$ be a torsion-free subgroup of finite index. Let $(\tau_1, \dots, \tau_r)$ be a family of splitting maps for D so that τ_1 is a splitting map with $\tau_1(D) \subset M_2(F(\sqrt{a}))$ as constructed from Proposition 5.7. We define an action of $D^\times$ on $(\mathbb{H}^2)^r$ by

$$x \cdot (z_1, \dots, z_r) := (\tau_1(x) \cdot z_1, \dots, \tau_r(x) \cdot z_r),$$

where $\tau_i(x) \cdot z_i$ is the action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{H}^2$ defined in Definition 3.30.

Under these assumptions, the group Γ acts freely and properly discontinuously on $(\mathbb{H}^2)^r$ and the quotient space $\Gamma \backslash (\mathbb{H}^2)^r$ is an irreducible locally symmetric space of finite volume. We will see later that every quotient of $(\mathbb{H}^2)^r$ by an arithmetically defined lattice is finitely covered by a quotient space of this form.

Recall from Section 3.2 that if G is a group acting on a symmetric space M and $A \subset M$ is a flat, then we write $G_A := \{g \in G : g \cdot A = A\}$.

Proposition 6.13. *Let $x \in D^\times$ and assume that $\tau_1(x), \dots, \tau_r(x) \in \mathrm{GL}_2(\mathbb{R})$ each have two distinct real eigenvalues. Then there exists a unique maximal flat $A \subset (\mathbb{H}^2)^r$ such that $x \cdot A = A$. Moreover, the centralizer $C_{D^\times}(x)$ is a subgroup of finite index in $(D^\times)_A$ and acts by orientation-preserving isometries on A.*

Proof. By Proposition 3.31, each $\tau_i(x)$ stabilizes a unique geodesic line $L_i \subset \mathbb{H}^2$ and $C_{\mathrm{GL}_2(\mathbb{R})}(\tau_i(x))$ acts by orientation-preserving isometries on L_i. Every maximal flat in $(\mathbb{H}^2)^r$ is a product of geodesic lines, and so $A := L_1 \times \ldots \times L_r$ is the unique maximal flat in $(\mathbb{H}^2)^r$ with $x \cdot A = A$. Since $\tau_i(C_{D^\times}(x)) \subset C_{\mathrm{GL}_2(\mathbb{R})}(\tau_i(x))$, we see that $C_{D^\times}(x)$ stabilizes the flat A and acts on it by orientation-preserving isometries. The map τ_1 induces a group homomorphism $(D^\times)_A \to \mathrm{GL}_2(\mathbb{R})_{L_1}$ and $C_{D^\times}(\alpha)$ is the preimage under this homomorphism of $C_{\mathrm{GL}_2(\mathbb{R})}(\tau_1(\alpha))$. Since the latter group is by Proposition 3.31 a subgroup of finite index in $\mathrm{GL}_2(\mathbb{R})_{L_1}$, it follows that $C_{D^\times}(\alpha)$ is a subgroup of finite index in $(D^\times)_A$. □

Proposition 6.14. *For any $n \in \mathbb{N}$, there exist maximal flats $A_1, \ldots, A_n \subset (\mathbb{H}^2)^r$ and $B_1, \ldots, B_n \subset (\mathbb{H}^2)^r$ so that for all $1 \leq i \leq n$ and $1 \leq j \leq n$, we have:*

(i) A_i and B_j are disjoint if $i > j$, and they intersect transversally in a single point if $i \leq j$.

(ii) A_i is Γ-compact.

(iii) A_i is stabilized by an element $\alpha_i \in D^1$ so that $\tau_1(\alpha_i), \ldots, \tau_r(\alpha_i)$ each have two distinct real eigenvalues.

(iv) B_j is stabilized by an element $\beta_j \in D^\times$ so that $\tau_1(\beta_j), \ldots, \tau_r(\beta_j)$ each have two distinct real eigenvalues and $\tau_1(\beta_j)$ is diagonalizable over $F(\sqrt{a})$.

Proof. We start by constructing maximal flats that satisfy the first condition. For this, let $L_1, \ldots, L_n$ and $M_1, \ldots, M_n$ be geodesic lines in $\mathbb{H}^2$ such that L_i and M_j intersect if and only if $i \leq j$ and such that all their endpoints in $\partial_\infty \mathbb{H}^2$ are pairwise distinct. See Figure 7 for an example in the case $n = 3$. Now let $A_i := L_i \times \cdots \times L_i \subset (\mathbb{H}^2)^r$ and $B_j := M_j \times \cdots \times M_j \subset (\mathbb{H}^2)^r$ for each i and j.

Since for each i and j, the endpoints of L_i and M_j in $\partial_\infty \mathbb{H}^2$ are pairwise distinct, we have by Lemma 3.28 that L_i and M_j are either disjoint or intersect transversally in a single point. Moreover, there is an open neighborhood of the identity $U_{ij} \subset \mathrm{SL}_2(\mathbb{R})$ such that the same is true for all $u, v \in U_{ij}$ for the geodesic lines $u \cdot L_i$ and $v \cdot M_j$. It follows that $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ are maximal flats in $(\mathbb{H}^2)^r$ that satisfy the first condition, and we can move them by elements of the open neighborhood of the identity

$$U := \bigcap_{i=1}^{n} \bigcap_{j=1}^{n} (U_{ij})^r \subset \mathrm{SL}_2(\mathbb{R})^r$$

without invalidating the first condition. By the density of Γ-compact maximal flats from Theorem 3.22, there exist $u_1, \ldots, u_n \in U$ such that for each i, the flat $u_i \cdot A_i$ is Γ-compact and stabilized by an element $\alpha_i \in D^1$ so that $(\tau_1(\alpha_i), \ldots, \tau_r(\alpha_i)) \in \mathrm{SL}_2(\mathbb{R})^r$ is polar regular. We now replace each A_i with $u_i \cdot A_i$ to achieve that A_i is Γ-compact. By Lemma 3.19, the matrices

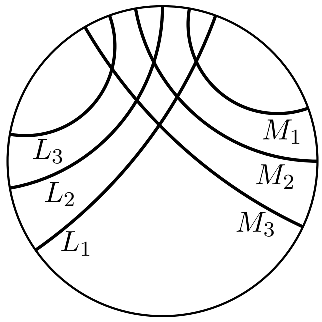


Figure 7: A pattern of geodesics lines in $\mathbb{H}^2$.

$\tau_1(\alpha_i), \ldots, \tau_r(\alpha_i) \in \mathrm{SL}_2(\mathbb{R})$ are polar regular, and so they each have two distinct real eigenvalues by Lemma 3.29. Now the first three conditions are satisfied.

In order to satisfy the last condition, we choose some $x_0 \in D^\times$ with $(x_0)^2 = a$ and $x_0 \notin F$. Then for each $k \in \{1, \ldots, r\}$, we have $(\tau_k(x_0))^2 = \sigma_k(a) I_2$ and $\tau_k(x_0) \notin \mathbb{R} \cdot I_2$. Recall that $\sigma_k(a) > 0$ by assumption. So the minimal polynomial of $\tau_k(x_0)$ over $\mathbb{R}$ is

$$(X + \sqrt{\sigma_k(a)})(X - \sqrt{\sigma_k(a)}).$$

Hence, each $\tau_k(x_0)$ has two distinct real eigenvalues and $\tau_1(x_0)$ is diagonalizable over $F(\sqrt{a})$. By Proposition 6.13, there exists a unique maximal flat $B_0 \subset (\mathbb{H}^2)^r$ that is stabilized by x_0. The group $\mathrm{SL}_2(\mathbb{R})^r$ acts transitively on the set of all maximal flats in $(\mathbb{H}^2)^r$, and so for each $j \in \{1, \ldots, n\}$, we can find some $T_j \in \mathrm{SL}_2(\mathbb{R})^r$ with $B_j = T_j \cdot B_0$. By the real approximation theorem from [Mil17, Theorem 25.70], the image of D^1 in $\mathrm{SL}_2(\mathbb{R})^r$ under the splitting maps is dense. So because the subsets $UT_j \subset \mathrm{SL}_2(\mathbb{R})^r$ are open, there exist $x_j \in D^1$ and $v_j \in U$ with

$$(\tau_1(x_j), \cdots, \tau_r(x_j)) = v_j T_j.$$

Next, from $v_j \cdot B_j = v_j T_j \cdot (T_j^{-1} \cdot B_j) = x_j \cdot B_0$, we conclude that

$$x_j x_0 x_j^{-1} \cdot (v_j \cdot B_j) = x_j x_0 \cdot B_0 = x_j \cdot B_0 = v_j \cdot B_j.$$

So the maximal flat $v_j \cdot B_j \subset (\mathbb{H}^2)^r$ is stabilized by the conjugate $x_j x_0 x_j^{-1} \in D^\times$ of x_0. We now replace each B_j by $v_j \cdot B_j$ and all conditions are satisfied. □

6.4 Controlling the Intersections

Throughout this section, we keep the notations and assumptions from the previous section and now study the images of the flats which we have constructed in Proposition 6.14 in finite covering spaces of $\Gamma\backslash(\mathbb{H}^2)^r$. Our goal to find a finite covering space of $\Gamma\backslash(\mathbb{H}^2)^r$ in which the images of these flats are totally geodesic submanifolds and such that we can control their intersections.

We assume that A and B are maximal flats in $(\mathbb{H}^2)^r$ that are either disjoint or intersect transversally in a single point. Moreover, we assume that A is Γ-compact, and that A and B are stabilized by elements $\alpha \in D^1$ and $\beta \in D^\times$, respectively, such that $\tau_1(\alpha), \ldots, \tau_r(\alpha)$ and $\tau_1(\beta), \ldots, \tau_r(\beta)$ each have two distinct real eigenvalues and $\tau_1(\beta)$ is diagonalizable over $F(\sqrt{a})$.

Proposition 6.15. *There exists a subgroup of finite index* $\Gamma_{\mathrm{cent}} \subset \Gamma$ *such that every element of* Γ_{cent} *which stabilizes* A *commutes with* α*, and every element of* Γ_{cent} *which stabilizes* B *commutes with* β*.*

Proof. By Proposition 3.31, we know that the centralizer $C_{D^\times}(\alpha)$ is a subgroup of finite index in $(D^\times)_A$. Hence, there exist $y_1, \ldots, y_m \in (D^\times)_A$ with

$$(D^\times)_A = C_{D^\times}(\alpha) \sqcup C_{D^\times}(\alpha)y_1 \sqcup \ldots \sqcup C_{D^\times}(\alpha)y_m.$$

Consider now the algebraic group $\mathbf{GL}_D$ over F from Definition 5.13 and the ring of finite adeles $\mathbb{A}_{f,F}$ from Definition 4.37. Each $C_{\mathbf{GL}_D(\mathbb{A}_{f,F})}(\alpha)y_i \subset \mathbf{GL}_D(\mathbb{A}_{f,F})$ is a closed subset that does not contain the identity. So by using the basis of open neighborhoods of the identity in $\mathbf{GL}_D(\mathbb{A}_{f,F})$ from Proposition 4.39 induced by the integral form $\mathbf{GL}_\Lambda$, we find a nonzero ideal $\mathfrak{a} \subset \mathcal{O}_F$ such that

$$\mathbf{GL}_\Lambda(\mathbb{O}_{f,F})(\mathfrak{a}) \cap C_{\mathbf{GL}_D(\mathbb{A}_{f,F})}(\alpha)y_i = \emptyset$$

for all $i \in \{1, \ldots, m\}$. So the principal congruence subgroup $\Gamma(\mathfrak{a}) \subset \Gamma$ satisfies

$$\Gamma(\mathfrak{a})_A \subset (D^\times)_A \cap \mathbf{GL}_\Lambda(\mathbb{O}_{f,F})(\mathfrak{a}) \subset C_{D^\times}(\alpha).$$

Similarly, we find a nonzero ideal $\mathfrak{b} \subset \mathcal{O}_F$ with $\Gamma(\mathfrak{b})_B \subset C_{D^\times}(\beta)$. The subgroup $\Gamma_{\mathrm{cent}} := \Gamma(\mathfrak{a}) \cap \Gamma(\mathfrak{b})$ is of finite index in Γ, and so the proof is complete. □

We can now show that the images of A and B in some finite covering space of the locally symmetric space $\Gamma\backslash(\mathbb{H}^2)^r$ are totally geodesic submanifolds:

Proposition 6.16. *There exists a subgroup of finite index* $\Gamma_{\mathrm{emb}} \subset \Gamma_{\mathrm{cent}}$ *so that for every subgroup of finite index* $\Gamma' \subset \Gamma_{\mathrm{emb}}$*, the natural maps*

$$\Gamma'_A\backslash A \to \Gamma'\backslash(\mathbb{H}^2)^r \quad \textit{and} \quad \Gamma'_B\backslash B \to \Gamma'\backslash(\mathbb{H}^2)^r$$

are closed embeddings and their images are orientable flat totally geodesic r*-dimensional submanifolds of the locally symmetric space* $\Gamma'\backslash(\mathbb{H}^2)^r$*.*

Proof. Note that the algebraic group $\mathbf{SL}_D$ from Definition 5.15 is connected and semisimple. Since $\alpha \in \mathbf{SL}_D(F)$, there exists by [Mil17, pp. 33–35] a unique smallest F-subgroup $C_{\mathbf{SL}_D}(\alpha)$ of $\mathbf{SL}_D$ such that for all fields K with $F \subset K$, we have

$$(C_{\mathbf{SL}_D}(\alpha))(K) = C_{\mathbf{SL}_D(K)}(\alpha).$$

Since $\tau_1(\alpha)$ is a diagonalizable matrix, we see that $C_{\mathbf{SL}_D}(\alpha)$ becomes isomorphic to $\mathbf{GL}_1$ over an algebraic closure of F, hence it is connected and reductive.

We now have that $\mathbf{G} := \mathrm{Res}_{F/\mathbb{Q}}(\mathbf{SL}_D)$ is a connected semisimple $\mathbb{Q}$-group and $\mathbf{H} := \mathrm{Res}_{F/\mathbb{Q}}(C_{\mathbf{SL}_D}(\alpha))$ is a connected reductive $\mathbb{Q}$-subgroup of $\mathbf{G}$. For each $i \in \{r+1, \ldots, d\}$, we choose an isomorphism $\rho_i \colon D \otimes_F \mathrm{res}_{\sigma_i}(\mathbb{R}) \xrightarrow{\cong} \mathcal{H}(\mathbb{R})$, where $\mathcal{H}(\mathbb{R})$ is the division algebra from Example 5.5. Then the splitting maps $\tau_1, \ldots, \tau_r$ and the maps $\rho_{r+1}, \ldots, \rho_d$ induce by Proposition 4.17 an isomorphism

$$G := \mathbf{G}(\mathbb{R}) \xrightarrow{\cong} \mathrm{SL}_2(\mathbb{R})^r \times (\mathcal{H}(\mathbb{R})^1)^{d-r}$$

which maps the group $H := \mathbf{H}(\mathbb{R})$ to $\prod_{i=1}^r C_{\mathrm{SL}_2(\mathbb{R})}(\tau_i(\alpha)) \times \prod_{i=r+1}^d C_{\mathcal{H}(\mathbb{R})^1}(\rho_i(\alpha))$. Let $K \subset G$ be the preimage of $\mathrm{SO}(2)^r \times (\mathcal{H}(\mathbb{R})^1)^{d-r}$ under this isomorphism. Then K is a maximal compact subgroup of G and $K_H := K \cap H$ is a maximal compact subgroup of H. Consider now the quotient spaces $X_G := G/K$ and $X_H := H/K_H$ and the embedding $j_H \colon X_H \hookrightarrow X_G$ induced by the inclusion $H \hookrightarrow G$. Fix a point $x_0 \in A$. Then the diffeomorphism

$$X_G \xrightarrow{\cong} (\mathbb{H}^2)^r, \quad gK \mapsto g \cdot x_0$$

maps $j_H(X_H)$ onto the flat A. Note that $\mathbf{H}(\mathbb{Q}) = C_{D^1}(\alpha)$. So by Propositions 6.13 and 6.15, we have $\Gamma'_A = \Gamma' \cap \mathbf{H}(\mathbb{Q})$ for every subgroup of finite index $\Gamma' \subset \Gamma_{\mathrm{cent}}$. Hence, by Theorem 6.3, there exists a subgroup of finite index $\Gamma_0 \subset \Gamma_{\mathrm{cent}}$ such that for all subgroups of finite index $\Gamma' \subset \Gamma_0$, the map $\Gamma'_A \backslash A \to \Gamma' \backslash (\mathbb{H}^2)^r$ is a closed embedding with the required properties.

Similarly, we obtain a subgroup of finite index $\Gamma_1 \subset \Gamma_{\mathrm{cent}}$ such that for every subgroup of finite index $\Gamma' \subset \Gamma_1$, the map $\Gamma'_B \backslash B \to \Gamma' \backslash (\mathbb{H}^2)^r$ is a closed embedding with the required properties. So we set $\Gamma_{\mathrm{emb}} := \Gamma_0 \cap \Gamma_1$ and the proof is complete. □

Lemma 6.17. *Let $\Gamma' \subset \Gamma_{\mathrm{emb}}$ be a subgroup of finite index. Then $\Gamma' A$ is a disjoint union of copies of A, that is, for any $\gamma \in \Gamma'$, we have $\gamma A = A$ or $\gamma A \cap A = \emptyset$. Similarly, $\Gamma' B$ is a disjoint union of copies of B.*

Proof. Let $\gamma \in \Gamma'$ and assume that $\gamma A \cap A \neq \emptyset$. Then there exist $x_1, x_2 \in A$ with $x_2 = \gamma x_1$, and so we have $\Gamma' x_1 = \Gamma' x_2$. Since the map $\Gamma'_A \backslash A \to \Gamma' \backslash (\mathbb{H}^2)^r$ is injective by Proposition 6.16, it follows that $\Gamma'_A x_1 = \Gamma'_A x_2$. So there exists some

$\delta \in \Gamma'_A$ with $x_1 = \delta x_2$. Hence we have $\delta\gamma x_1 = x_1$, and because Γ' is torsion-free, this implies that $\gamma = \delta^{-1}$. So we have $\gamma \in \Gamma'_A$, or, in other words, $\gamma A = A$. The statement for $\Gamma' B$ can be proven in an analogous way. □

Remark 6.18. For any subgroup of finite index $\Gamma' \subset \Gamma_{\mathrm{emb}}$, the images of the flats A and B in $\Gamma'\backslash(\mathbb{H}^2)^r$ can be oriented as follows: We choose orientations A^+ on A and B^+ on B and define Γ'-invariant orientations on $\Gamma' A$ and $\Gamma' B$ by $(\gamma A)^+ := \gamma A^+$ and $(\gamma B)^+ := \gamma B^+$ for any $\gamma \in \Gamma'$. By Proposition 6.16, the maps $\Gamma' A \to \Gamma'_A\backslash A$ and $\Gamma' B \to \Gamma'_B\backslash B$ are covering maps and their images are diffeomorphic to the images of A and B in $\Gamma'\backslash(\mathbb{H}^2)^r$, respectively. Since $\Gamma_{\mathrm{emb}} \subset \Gamma_{\mathrm{cent}}$, we have by Propositions 6.13 and 6.15 that Γ'_A and Γ'_B act by orientation-preserving isometries on A and B, respectively, and so we get induced orientations on the images of A and B in $\Gamma'\backslash(\mathbb{H}^2)^r$.

Our next task is to find a finite covering space of the locally symmetric space $\Gamma\backslash(\mathbb{H}^2)^r$ in which we can control the intersection of the images of A and B.

Lemma 6.19. *Let $\Gamma' \subset \Gamma_{\mathrm{emb}}$ be a subgroup of finite index. Then for any $\gamma_1, \gamma_2 \in \Gamma'$, we have $\Gamma'_B\gamma_1 A = \Gamma'_B\gamma_2 A$ if and only if there exists some $\delta \in \Gamma'_B$ with $\gamma_1 A = \delta\gamma_2 A$.*

Proof. If there exists some $\delta \in \Gamma'_B$ with $\gamma_1 A = \delta\gamma_2 A$, then we also have $\Gamma'_B\gamma_1 A = \Gamma'_B\gamma_2 A$. Conversely, if $\Gamma'_B\gamma_1 A = \Gamma'_B\gamma_2 A$, then $\gamma_1 A \cap \delta\gamma_2 A \neq \emptyset$ for some $\delta \in \Gamma'_B$, and so $\gamma_1 A = \delta\gamma_2 A$ by Lemma 6.17. □

Proposition 6.20. *For every subgroup of finite index $\Gamma' \subset \Gamma_{\mathrm{emb}}$, we have*

$$\#\{\Gamma'_B\gamma A : \gamma \in \Gamma' \text{ with } \gamma A \cap B \neq \emptyset\} < \infty.$$

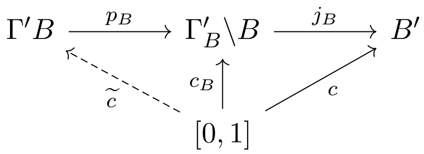
Proof. Let $\pi\colon (\mathbb{H}^2)^r \to \Gamma'\backslash(\mathbb{H}^2)^r$ be the projection map. We write $A' := \pi(A)$ and $B' := \pi(B)$. Since A' and B' are closed totally geodesic submanifolds of $\Gamma'\backslash(\mathbb{H}^2)^r$ and A' is compact, it follows that $A' \cap B'$ is a compact manifold. In particular, $A' \cap B'$ has only finitely many path components. Thus, it suffices to show that for all $\gamma_0, \gamma_1 \in \Gamma'$ for which there is a continuous path in $A' \cap B'$ connecting a point in $\pi(\gamma_0 A \cap B)$ to a point in $\pi(\gamma_1 A \cap B)$, we have

$$\Gamma'_B\gamma_0 A = \Gamma'_B\gamma_1 A.$$

Let $c\colon [0,1] \to A' \cap B'$ be such a path and choose preimages $x_i \in \gamma_i A \cap B$ with $\pi(x_i) = c(i)$ for $i \in \{0,1\}$. Since $j_B\colon \Gamma'_B\backslash B \to B'$ is a diffeomorphism, c induces a path

$$c_B := j_B^{-1} \circ c\colon [0,1] \to \Gamma'_B\backslash B.$$

The map $p_B\colon \Gamma' B \to \Gamma'_B\backslash B$, $\gamma\cdot b \mapsto \Gamma'_B b$ is well-defined by Proposition 6.16 and is a covering map. By the lifting property of p_B and the fact that $p_B(x_0) = c_B(0)$, there exists a path $\widetilde{c}\colon [0,1] \to \Gamma' B$ with $\widetilde{c}(0) = x_0$ such that the diagram



commutes. From $c([0,1]) \subset A'$, we deduce that $\widetilde{c}([0,1]) \subset \Gamma' A$. But $\Gamma' A$ is a disjoint union of copies of A by Lemma 6.17, and so $\widetilde{c}(0) = x_0 \in \gamma_0 A$ implies that the image of $\widetilde{c}$ must be fully contained in $\gamma_0 A$. In particular, $\widetilde{c}(1) \in \gamma_0 A$.

On the other hand, using $\pi(\widetilde{c}(1)) = c(1) = \pi(x_1)$ and the injectivity of j_B, we see that

$$\Gamma'_B\widetilde{c}(1) = p_B(\widetilde{c}(1)) = p_B(x_1) = \Gamma'_B x_1.$$

Because of $x_1 \in \gamma_1 A$, this shows that $\widetilde{c}(1) \in \delta\gamma_1 A$ for some $\delta \in \Gamma'_B$. In conclusion, we have $\widetilde{c}(1) \in \gamma_0 A \cap \delta\gamma_1 A$, and so Lemma 6.17 implies that $\delta\gamma_1 A = \gamma_0 A$. Hence, by Lemma 6.19, we have

$$\Gamma'_B\gamma_0 A = \Gamma'_B\gamma_1 A. \qquad \square$$

Corollary 6.21. *Let $\Gamma' \subset \Gamma_{\mathrm{emb}}$ be a subgroup of finite index. Then there exist $\gamma_1, \dots, \gamma_m \in \Gamma'$ such that the intersection of the images of A and B in $\Gamma'\backslash(\mathbb{H}^2)^r$ is the image of the projection map*

$$\bigcup_{i=1}^{m} \gamma_i A \cap B \to \Gamma'\backslash(\mathbb{H}^2)^r.$$

Proof. By Proposition 6.20, there exist $\gamma_1, \dots, \gamma_m \in \Gamma'$ with

$$\{\Gamma'_B\gamma A : \gamma \in \Gamma' \text{ with } \gamma A \cap B \neq \emptyset\} = \{\Gamma'_B\gamma_1 A, \dots, \Gamma'_B\gamma_m A\}. \tag{6.1}$$

Let $\pi\colon (\mathbb{H}^2)^r \to \Gamma'\backslash(\mathbb{H}^2)^r$ be the projection map. For each $i \in \{1, \dots, m\}$, we have $\pi(\gamma_i A \cap B) \subset \pi(A) \cap \pi(B)$. Conversely, let $z \in \pi(A) \cap \pi(B)$. Then there is some $x \in \Gamma' A \cap B$ with $z = \Gamma' x$. Let $\gamma \in \Gamma'$ with $x \in \gamma A$. By (6.1), we have $\Gamma'_B\gamma A = \Gamma'_B\gamma_i A$ for some $i \in \{1, \dots, m\}$. So by Lemma 6.19, there exists some $\delta \in \Gamma'_B$ with $\gamma A = \delta\gamma_i A$. Hence, there is some $y \in A$ with $x = \delta\gamma_i y$. From $\gamma_i y = \delta^{-1} x \in B$ and $\Gamma' x = \Gamma'\delta\gamma_i y = \Gamma'\gamma_i y$, we deduce $z = \Gamma' x \in \pi(\gamma_i A \cap B)$. $\square$

The next two lemmas will be used in the proof of Proposition 6.25.

Lemma 6.22. *For the intersection of the centralizers of α and β in $D \otimes_F \mathbb{A}_{f,F}$, we have*

$$C_{D\otimes_F \mathbb{A}_{f,F}}(\beta) \cap C_{D\otimes_F \mathbb{A}_{f,F}}(\alpha) = \mathbb{A}_{f,F}.$$

Proof. Note that α and β do not commute with each other, because otherwise Proposition 6.13 would imply $A = B$ in contradiction to our assumptions on A and B. So by Lemma 5.17, we have $C_D(\beta) \cap C_D(\alpha) = F$. The desired equation now follows by tensoring this equation with $\mathbb{A}_{f,F}$ over F, because tensor products commute with centralizers. □

Definition 6.23. Let R be a ring and let $k \in \mathbb{N}$. The *group of k-th roots of unity in R* is the group $\mu_k(R) := \{\nu \in R : \nu^k = 1\}$.

To simplify the notation, we will from now on write $C_G(g) := \{h \in G : gh = hg\}$ for a group G whenever multiplication with g is defined (even if $g \notin G$).

Lemma 6.24. *The group $\mu_2(\mathbb{O}_{f,F})$ of second roots of unity in $\mathbb{O}_{f,F}$ acts transitively by $\nu \cdot (u, v) := (\nu^{-1}u, \nu v)$ on the fibers of the multiplication map*

$$C_{\mathbf{SL}_\Lambda(\mathbb{O}_{f,F})}(\beta) \times C_{\mathbf{SL}_\Lambda(\mathbb{O}_{f,F})}(\alpha) \to \mathbf{SL}_\Lambda(\mathbb{O}_{f,F}), \quad (u, v) \mapsto uv.$$

Proof. Let $(u_1, v_1), (u_2, v_2) \in C_{\mathbf{SL}_\Lambda(\mathbb{O}_{f,F})}(\beta) \times C_{\mathbf{SL}_\Lambda(\mathbb{O}_{f,F})}(\alpha)$ with $u_1v_1 = u_2v_2$. Then $\nu := u_2^{-1}u_1 = v_2v_1^{-1}$ commutes with both α and β, and so ν is a scalar in $\mathbb{O}_{f,F}$ by Lemma 6.22. Since $N(\nu) = 1$ and $N(\nu) = \nu^2$, we have $\nu \in \mu_2(\mathbb{O}_{f,F})$. Now $\nu^{-1}u_1 = u_1\nu^{-1} = u_2$ and $\nu v_1 = v_2$, hence we have $\nu \cdot (u_1, v_1) = (u_2, v_2)$. □

Recall from Proposition 6.13 that the centralizers $C_{D^\times}(\alpha)$ and $C_{D^\times}(\beta)$ act by orientation-preserving isometries on the flats A and B, respectively. The next two propositions will, combined with Corollary 6.21, allow us to control the intersection of the images of A and B in a finite covering space of $\Gamma\backslash(\mathbb{H}^2)^r$.

Proposition 6.25. *For every $\gamma \in \Lambda^1$ that is in the closure of $C_{\Lambda^1}(\beta)C_{\Lambda^1}(\alpha)$ in $\mathbf{SL}_\Lambda(\mathbb{O}_{f,F})$, there exist $x \in C_{D^\times}(\beta)$ and $y \in C_{D^\times}(\alpha)$ such that $\gamma = xy$ and such that x acts by orientation-preserving isometries on $(\mathbb{H}^2)^r$.*

Proof. **Step 1:** We first find $x' \in C_{\mathbf{SL}_\Lambda(\mathbb{O}_{f,F})}(\beta)$ and $y' \in C_{\mathbf{SL}_\Lambda(\mathbb{O}_{f,F})}(\alpha)$ such that $\gamma = x'y'$. This is possible because $C_{\mathbf{SL}_\Lambda(\mathbb{O}_{f,F})}(\beta)$ and $C_{\mathbf{SL}_\Lambda(\mathbb{O}_{f,F})}(\alpha)$ are both closed in $\mathbf{SL}_\Lambda(\mathbb{O}_{f,F})$, so their product is also closed and contains the set $C_{\Lambda^1}(\beta)C_{\Lambda^1}(\alpha)$, hence also the closure point γ of this set.

Step 2: Next, we find some $c \in \mathbb{A}_{f,F}$ with $cx' \in D^\times$. To achieve this, we observe that x' is a solution in $D \otimes_F \mathbb{A}_{f,F}$ of the homogeneous system of linear equations

$$x'\alpha = (\gamma\alpha\gamma^{-1})x',$$
$$x'\beta = \beta x'.$$

The coefficients of this system are in F. Let $\mathcal{B}$ be an F-basis for the space of solutions of this system in D. Then the solution space in $D \otimes_F \mathbb{A}_{f,F}$ is the $\mathbb{A}_{f,F}$-span of $\mathcal{B}$. In particular, $\mathcal{B} \neq \emptyset$. Moreover, the function $x \mapsto N(x)$ on the solution space in D can be expressed in coordinates with respect to $\mathcal{B}$ by some multivariate polynomial $P \in F[X_1, \dots, X_m]$. Because of $N(x') \neq 0$, we have $P \neq 0$. So since F is an infinite field, there exists a solution with nonzero reduced norm in D, that is, there exists an element $x \in D^\times$ satisfying

$$\begin{aligned} x\alpha &= (\gamma\alpha\gamma^{-1})x, \\ x\beta &= \beta x. \end{aligned}$$

The element $x^{-1}x' \in D \otimes_F \mathbb{A}_{f,F}$ commutes with β. It also commutes with α, because from the above two linear systems of equations, we deduce that

$$x^{-1}x'\alpha = x^{-1}(\gamma\alpha\gamma^{-1})x' = x^{-1}(x\alpha x^{-1})x' = \alpha x^{-1}x'.$$

So by Lemma 6.22, we have $x = cx' \in D^\times$ for some $c \in \mathbb{A}_{f,F}$ as required. Let $y := c^{-1}y'$. Then $y = c^{-1}(x')^{-1}\gamma = x^{-1}\gamma \in D^\times$, and so we now have $\gamma = xy$ with $x \in C_{D^\times}(\beta)$ and $y \in C_{D^\times}(\alpha)$. It remains to show that x acts by orientation-preserving isometries on $(\mathbb{H}^2)^r$, which we do in the next two steps.

Step 3: Next, we show that $C_{\Lambda^1}(\beta)x' \cap \mu_2(\mathbb{O}_{f,F})U \neq \emptyset$ for every open neighborhood of the identity $U \subset \mathbf{SL}_\Lambda(\mathbb{O}_{f,F})$. Assume to the contrary that $U \subset \mathbf{SL}_\Lambda(\mathbb{O}_{f,F})$ is an open neighborhood of the identity with $C_{\Lambda^1}(\beta)x' \cap \mu_2(\mathbb{O}_{f,F})U = \emptyset$. Then we have

$$\left(C_{\Lambda^1}(\beta)x' \times y'C_{\Lambda^1}(\alpha)\right) \cap \left(\mu_2(\mathbb{O}_{f,F})U \times C_{\mathbf{SL}_\Lambda(\mathbb{O}_{f,F})}(\alpha)\right) = \emptyset.$$

The set $\mu_2(\mathbb{O}_{f,F})U \times C_{\mathbf{SL}_\Lambda(\mathbb{O}_{f,F})}(\alpha)$ is invariant under the action of $\mu_2(\mathbb{O}_{f,F})$ defined in Lemma 6.24 and $\mu_2(\mathbb{O}_{f,F})$ acts transitively on the fibers of the multiplication map by this lemma. Hence, for the images under this map, we obtain

$$C_{\Lambda^1}(\beta)\gamma C_{\Lambda^1}(\alpha) \cap \mu_2(\mathbb{O}_{f,F})UC_{\mathbf{SL}_\Lambda(\mathbb{O}_{f,F})}(\alpha) = \emptyset. \tag{6.2}$$

Since U is an open neighborhood of the identity in $\mathbf{SL}_\Lambda(\mathbb{O}_{f,F})$, there exists by Proposition 4.39 a nonzero ideal $\mathfrak{a} \subset \mathcal{O}_F$ with $\mathbf{SL}_\Lambda(\mathbb{O}_{f,F})(\mathfrak{a}) \subset U$. Hence we have $\Lambda^1(\mathfrak{a}) \subset U$, and so (6.2) implies that

$$C_{\Lambda^1}(\beta)\gamma C_{\Lambda^1}(\alpha) \cap \Lambda^1(\mathfrak{a}) = \emptyset. \tag{6.3}$$

On the other hand, γ is in the closure of $C_{\Lambda^1}(\beta)C_{\Lambda^1}(\alpha)$ and so we have $C_{\Lambda^1}(\beta)C_{\Lambda^1}(\alpha) \cap \Lambda^1(\mathfrak{a})\gamma \neq \emptyset$. Hence, there exist $\gamma_\beta \in C_{\Lambda^1}(\beta)$, $\gamma_\alpha \in C_{\Lambda^1}(\alpha)$

and $\gamma_0 \in \Lambda^1(\mathfrak{a})$ with $\gamma_\beta\gamma_\alpha = \gamma_0\gamma$. Since $\Lambda^1(\mathfrak{a})$ is normal in Λ^1, we obtain

$$\gamma_\beta^{-1}\gamma_0^{-1}\gamma_\beta = \gamma_\beta^{-1}\gamma\gamma_\alpha^{-1} \in C_{\Lambda^1}(\beta)\gamma C_{\Lambda^1}(\alpha) \cap \Lambda^1(\mathfrak{a})$$

in contradiction to (6.3). We are now finished with this step.

Step 4: Finally, we show that $x = cx'$ acts by orientation-preserving isometries on $(\mathbb{H}^2)^r$. Assume to the contrary that this is not the case. Then there must exist some $i \in \{1, \ldots, r\}$ with $\det(\tau_i(x)) = \sigma_i(N(x)) = \sigma_i(c^2) < 0$. Let $E := F(\sqrt{a})$. We can extend σ_i to a real embedding $\widetilde{\sigma_i}\colon E \hookrightarrow \mathbb{R}$ as in the proof of Proposition 5.25, because by assumption we have $\sigma_i(a) > 0$. Recall that $\tau_1(D^\times) \subset M_2(E)$, and so τ_1 induces an F-algebra homomorphism $D \to M_2(E)$ and hence also an F-homomorphism $\mathbf{GL}_D \to \operatorname{Res}_{E/F}\mathbf{GL}_2$. By Lemma 4.40, we have $(\operatorname{Res}_{E/F}\mathbf{GL}_2)(\mathbb{A}_{f,F}) \cong \mathbf{GL}_2(\mathbb{A}_{f,E})$, and so we get a continuous group homomorphism

$$\Phi\colon \mathbf{GL}_D(\mathbb{A}_{f,F}) \to \mathbf{GL}_2(\mathbb{A}_{f,E})$$

that extends τ_1 on $D^\times$. The matrix $\Phi(\beta) = \tau_1(\beta) \in \mathrm{GL}_2(E)$ is by assumption diagonalizable over E with two distinct eigenvalues. So there exists a one-dimensional subspace $L \subset E^2$ which is invariant under $\tau_1(\beta)$. The corresponding eigenspace in $(\mathbb{A}_{f,E})^2$ of $\tau_1(\beta)$ is $\mathbb{A}_{f,E}\cdot L$, and so every matrix in $M_2(\mathbb{A}_{f,E})$ that commutes with $\tau_1(\beta)$ stabilizes $\mathbb{A}_{f,E}\cdot L$. Let now $v \in L$ be a nonzero vector and let $\ell \in \{1, 2\}$ be such that the ℓ-th coordinate of v is $v_\ell \neq 0$. Consider the map

$$s\colon C_{\mathbf{GL}_D(\mathbb{A}_{f,F})}(\beta) \to \mathbf{GL}_1(\mathbb{A}_{f,E}), \quad g \mapsto \left(\frac{(\Phi(g)v)_\ell}{v_\ell}\right)^2. \tag{6.4}$$

Note that s is multiplicative and so its image is contained in $\mathbf{GL}_1(\mathbb{A}_{f,E}) = \mathbb{A}_{f,E}^\times$. Moreover, s is continuous because Φ is continuous and $\mathbb{A}_{f,E}$ is a topological E-algebra. We have $s(C_{D^\times}(\beta)) \subset (E^\times)^2$, and so by writing $x' = c^{-1}cx'$, we see that

$$s(\mu_2(\mathbb{O}_{f,F})C_{\Lambda^1}(\beta)x') \subset (E^\times)^2c^{-2}s(cx')$$

is contained in $E^\times$ and has only negative images under $\widetilde{\sigma_i}$ because of $\widetilde{\sigma_i}(c^2) < 0$. Note that $V_+ := \{v \in \mathcal{O}_E^\times : \widetilde{\sigma_i}(v) > 0\}$ is a subgroup of finite index in $\mathcal{O}_E^\times$. So by Theorem 4.44, there exists a nonzero ideal $\mathfrak{a} \subset \mathcal{O}_E$ with $\mathcal{O}_E^\times(\mathfrak{a}) \subset V_+$. Hence $\mathbb{O}_{f,E}^\times(\mathfrak{a}) \cap E^\times \subset \mathcal{O}_E^\times(\mathfrak{a}) \subset V_+$, and so we obtain

$$\mu_2(\mathbb{O}_{f,F})C_{\Lambda^1}(\beta)x' \cap s^{-1}(\mathbb{O}_{f,E}^\times(\mathfrak{a})) = \emptyset.$$

Since s is continuous, the preimage $s^{-1}(\mathbb{O}_{f,E}^\times(\mathfrak{a}))$ is an open neighborhood of the identity in $C_{\mathbf{GL}_D(\mathbb{A}_{f,F})}(\beta)$, and so there exists a nonzero ideal $\mathfrak{b} \subset \mathcal{O}_F$

with $C_{\mathbf{GL}_\Lambda(\mathbb{O}_{f,F})}(\beta)(\mathfrak{b}) \subset s^{-1}(\mathbb{O}_{f,E}^\times(\mathfrak{a}))$. Let $U := \mathbf{SL}_\Lambda(\mathbb{O}_{f,F})(\mathfrak{b})$. Then we have $\mu_2(\mathbb{O}_{f,F})C_{\Lambda^1}(\beta)x' \cap U = \emptyset$, or equivalently,

$$C_{\Lambda^1}(\beta)x' \cap \mu_2(\mathbb{O}_{f,F})U = \emptyset.$$

This contradicts the result from the previous step. So x must act by orientation-preserving isometries on $(\mathbb{H}^2)^r$ and the proof is complete. □

Proposition 6.26. *There exists a subgroup of finite index $\Gamma_{\mathrm{prod}} \subset \Gamma_{\mathrm{emb}}$ such that every $\gamma \in \Gamma_{\mathrm{prod}}$ with $\gamma A \cap B \neq \emptyset$ can be written as $\gamma = xy$ with $x \in C_{D^\times}(\beta)$ and $y \in C_{D^\times}(\alpha)$ so that x acts by orientation-preserving isometries on $(\mathbb{H}^2)^r$.*

Proof. By Proposition 6.20, there exist $\gamma_1, \ldots, \gamma_m \in \Gamma_{\mathrm{emb}}$ with

$$\{(\Gamma_{\mathrm{emb}})_B \gamma A : \gamma \in \Gamma_{\mathrm{emb}},\, \gamma A \cap B \neq \emptyset\} = \{(\Gamma_{\mathrm{emb}})_B \gamma_1 A, \ldots, (\Gamma_{\mathrm{emb}})_B \gamma_m A\}.$$

We now choose for every $i \in \{1, \ldots, m\}$ a subgroup $\Gamma_{(i)} \subset \Gamma_{\mathrm{emb}}$ as follows: If γ_i is in the closure of $C_{\Lambda^1}(\beta)C_{\Lambda^1}(\alpha)$ in $\mathbf{SL}_\Lambda(\mathbb{O}_{f,F})$, then we set $\Gamma_{(i)} := \Gamma_{\mathrm{emb}}$. Otherwise, there exists by Proposition 4.39 a nonzero ideal $\mathfrak{a}_i \subset \mathcal{O}_F$ with $C_{\Lambda^1}(\beta)C_{\Lambda^1}(\alpha) \cap \mathbf{SL}_\Lambda(\mathbb{O}_{f,F})(\mathfrak{a}_i)\gamma_i = \emptyset$ and we set $\Gamma_{(i)} := \Gamma_{\mathrm{emb}}(\mathfrak{a}_i)$. Consider

$$\Gamma_{\mathrm{prod}} := \Gamma_{(1)} \cap \ldots \cap \Gamma_{(m)}.$$

Let $\gamma \in \Gamma_{\mathrm{prod}}$ with $\gamma A \cap B \neq \emptyset$. Then we have $(\Gamma_{\mathrm{emb}})_B \gamma A = (\Gamma_{\mathrm{emb}})_B \gamma_i A$ for some $i \in \{1, \ldots, m\}$. So by Lemma 6.19, there exists some $\delta \in (\Gamma_{\mathrm{emb}})_B$ with $\gamma A = \delta \gamma_i A$. Hence we have $\gamma^{-1}\delta\gamma_i A = A$. Let $\tau := \gamma^{-1}\delta\gamma_i$. Then $\tau \in (\Gamma_{\mathrm{emb}})_A$ and

$$\delta^{-1}\tau = (\delta^{-1}\gamma^{-1}\delta)\gamma_i. \tag{6.5}$$

Since Γ_{prod} is normal in Γ_{emb}, we have $\delta^{-1}\gamma^{-1}\delta \in \Gamma_{\mathrm{prod}}$. Moreover, we have $\delta^{-1} \in (\Gamma_{\mathrm{emb}})_B \subset C_{\Lambda^1}(\beta)$ and $\tau \in (\Gamma_{\mathrm{emb}})_A \subset C_{\Lambda^1}(\alpha)$. Hence, (6.5) shows that

$$C_{\Lambda^1}(\beta)C_{\Lambda^1}(\alpha) \cap \Gamma_{\mathrm{prod}}\gamma_i \neq \emptyset.$$

Now because of $\Gamma_{\mathrm{prod}} \subset \Gamma_{(i)}$, we have that γ_i must be in the closure of $C_{\Lambda^1}(\beta)C_{\Lambda^1}(\alpha)$ in $\mathbf{SL}_\Lambda(\mathbb{O}_{f,F})$. So by Proposition 6.25, we can write $\gamma_i = x_i y_i$ with $x_i \in C_{D^\times}(\beta)$ and $y_i \in C_{D^\times}(\alpha)$ such that x_i acts by orientation-preserving isometries on $(\mathbb{H}^2)^r$. Observe that

$$\gamma = \delta\gamma_i\tau^{-1} = \delta x_i y_i \tau^{-1}.$$

So we have $\gamma = xy$ for $x := \delta x_i \in C_{D^\times}(\beta)$ and $y := y_i\tau^{-1} \in C_{D^\times}(\alpha)$, and x acts by orientation-preserving isometries on $(\mathbb{H}^2)^r$ because of $N(\delta) = 1$. □

We can now show the following result about the intersection of the images of A and B in a finite covering space of the locally symmetric space $\Gamma\backslash(\mathbb{H}^2)^r$:

Proposition 6.27. *Let $\Gamma' \subset \Gamma_{\text{prod}}$ be a subgroup of finite index. Then the images of A and B in $\Gamma'\backslash(\mathbb{H}^2)^r$ intersect if and only if A and B intersect in $(\mathbb{H}^2)^r$, in which case their intersection is transverse and consists of finitely many points all of which have the same intersection number.*

Proof. Let A' and B' be the images of the flats A and B in $\Gamma'\backslash(\mathbb{H}^2)^r$, respectively. By Corollary 6.21, there exist $\gamma_1, \ldots, \gamma_m \in \Gamma'$ such the projection map induces a surjection

$$\bigcup_{i=1}^{m} \gamma_i A \cap B \twoheadrightarrow A' \cap B'.$$

Because of $\Gamma' \subset \Gamma_{\text{prod}}$, we can apply Proposition 6.26 and write each γ_i as $\gamma_i = x_i y_i$ for some $x_i \in C_{D^\times}(\beta)$ and $y_i \in C_{D^\times}(\alpha)$ such that x_i acts by orientation-preserving isometries on $(\mathbb{H}^2)^r$. Now choose orientations A^+ on A and B^+ on B, and let A' and B' carry the induced orientations as described in Remark 6.18. By Proposition 6.13, we have $x_i \cdot B^+ = B^+$ and $y_i \cdot A^+ = A^+$, and so we obtain

$$\gamma_i \cdot A^+ \cap B^+ = x_i \cdot A^+ \cap B^+ = x_i \cdot (A^+ \cap x_i^{-1} \cdot B^+) = x_i \cdot (A^+ \cap B^+).$$

This shows that A' and B' intersect if and only if A and B intersect. Furthermore, we see that in this case the intersection of A' and B' is transverse and the intersection number is the same in each point of intersection because for each $i \in \{1, \ldots, m\}$, the action of x_i maps the intersection $A^+ \cap B^+$ to the intersection $\gamma_i A^+ \cap B^+$ while preserving the orientation of the symmetric space $(\mathbb{H}^2)^r$. □

6.5 Finishing the Proof of the Main Theorem

In this section, we put together what we have proven so far to finish the proof of Theorem 1.1. It is convenient for us to introduce the following notion:

Definition 6.28. We say that a complete locally symmetric space M of rank r satisfies the *flat homology condition* if for any $n \in \mathbb{N}$, there exists a connected covering $M' \to M$ and compact oriented flat totally geodesic r-dimensional submanifolds $A_1, \ldots, A_n \subset M'$ such that the images of $[A_1], \ldots, [A_n]$ in $H_r(M'; \mathbb{R})$ are linearly independent.

Now Theorem 1.1 can be rephrased to be the claim that every locally symmetric space of finite volume covered by $(\mathbb{H}^2)^r$ satisfies the flat homology condition. We have already proven this for $r = 1$ in Theorem 2.6. The next proposition shows that it suffices to check the flat homology condition for irreducible locally symmetric spaces:

Proposition 6.29. *Let M and N be two complete locally symmetric spaces that satisfy the flat homology condition. Then the product $M \times N$ also satisfies the flat homology condition.*

Proof. We denote the ranks of M and N by r and s, respectively. By assumption, there exist for any $n \in \mathbb{N}$ connected finite coverings $M' \to M$ and $N' \to N$, and we have compact oriented flat totally geodesic r-dimensional submanifolds $A_1, \dots, A_n \subset M'$ and compact oriented flat totally geodesic s-dimensional submanifolds $B_1, \dots, B_n \subset N'$ so that the images of their fundamental classes are linearly independent in $H_r(M'; \mathbb{R})$ and $H_s(N'; \mathbb{R})$, respectively. Now the product $M' \times N'$ is a finite covering space of $M \times N$, and $A_1 \times B_1, \dots, A_n \times B_n$ are compact oriented flat totally geodesic $(r+s)$-dimensional submanifolds of $M' \times N'$. By the Künneth theorem for homology (see [Hat02, Corollary 3B.7]), the homological cross product induces an injective linear map

$$H_r(M'; \mathbb{R}) \otimes_{\mathbb{R}} H_s(N'; \mathbb{R}) \hookrightarrow H_{r+s}(M' \times N'; \mathbb{R}), \quad a \otimes b \mapsto a \times b. \tag{6.6}$$

Let $\iota_{A_i} \colon A_i \hookrightarrow M'$ and $\iota_{B_i} \colon B_i \hookrightarrow N'$ denote the inclusion maps. Since $(\iota_{A_1})_*[A_1], \dots, (\iota_{A_n})_*[A_n]$ and $(\iota_{B_1})_*[B_1], \dots, (\iota_{B_n})_*[B_n]$ are both linearly independent, these sets can be extended to basis for $H_r(M'; \mathbb{R})$ and $H_s(N'; \mathbb{R})$, respectively. It follows that the tensor products

$$(\iota_{A_i})_*[A_i] \otimes (\iota_{B_i})_*[B_i] \quad \text{for } i \in \{1, \dots, n\}$$

in $H_r(M'; \mathbb{R}) \otimes H_s(N'; \mathbb{R})$ are part of a basis and so they are linearly independent. Using the naturality of the homological cross product, we obtain

$$(\iota_{A_i \times B_i})_*[A_i \times B_i] = (\iota_{A_i})_*[A_i] \times (\iota_{B_i})_*[B_i],$$

where $\iota_{A_i \times B_i} \colon A_i \times B_i \hookrightarrow M' \times N'$ is the inclusion map. So by the injectivity of the map in (6.6), the images of $[A_1 \times B_1], \dots, [A_n \times B_n]$ in $H_{r+s}(M' \times N'; \mathbb{R})$ are linearly independent. Hence $M \times N$ satisfies the flat homology condition. □

Using the results from the previous two sections, we can now show the following theorem:

Theorem 6.30. *Every locally symmetric space that is a quotient of $(\mathbb{H}^2)^r$ by an arithmetically defined lattice in $(\mathbf{SL}_2)^r(\mathbb{R})$ satisfies the flat homology condition.*

Proof. Let $\Delta \subset (\mathbf{SL}_2)^r(\mathbb{R})$ be an arithmetically defined lattice and consider the quotient $M := \Delta \backslash (\mathbb{H}^2)^r$. By Theorem 5.44, we know that Δ is commensurable with a subgroup derived from a quaternion algebra D over a totally real number field F. Our goal is to get into the situation of the assumptions in Section 6.3, so that we can apply the results proven there.

Therefore, we now denote by $\sigma_1, \dots, \sigma_d \colon F \hookrightarrow \mathbb{R}$ the real embeddings of F, ordered in such a way that D is split exactly at the first r real embeddings. We assume that $F \subset \mathbb{R}$ and σ_1 is the identity embedding. By Proposition 5.25, we can write $D = (a, b)_F$ for some $a, b \in \mathcal{O}_F$ with $\sigma_i(a) > 0$ for all $i \leq r$. We also choose a family of splitting maps $(\tau_1, \dots, \tau_r)$ as assumed in Section 6.3. Now by Proposition 5.36, we know that any two subgroup derived from D are commensurable in the wide sense and so the corresponding quotients of $(\mathbb{H}^2)^r$ have a common finite covering space. Hence, without loss of generality, we can assume that we are in the situation of Section 6.3 and that Δ is the subgroup

$$\Delta = \big\{(\tau_1(x), \dots, \tau_r(x)) \colon x \in \Gamma\big\} \subset \mathrm{SL}_2(\mathbb{R})^r$$

for some torsion-free subgroup of finite index $\Gamma \subset \Lambda^1$, where Λ is the standard order spanned by the quaternionic basis for D. Thus, we have $M = \Gamma\backslash(\mathbb{H}^2)^r$.

Let now $n \in \mathbb{N}$. By Proposition 6.14, there exists a configuration of maximal flats $A_1, \dots, A_n \subset (\mathbb{H}^2)^r$ and $B_1, \dots, B_n \subset (\mathbb{H}^2)^r$ so that Proposition 6.27 can be applied for every $i, j \in \{1, \dots, n\}$ to the flats $A = A_i$ and $B = B_j$. It follows that there is a subgroup of finite index $\Gamma' \subset \Gamma$ such that the images of the flats $A_1, \dots, A_n$ and $B_1, \dots, B_n$ in $M' := \Gamma'\backslash(\mathbb{H}^2)^r$ are closed orientable flat totally geodesic r-dimensional submanifolds $A'_1, \dots, A'_n \subset M'$ and $B'_1, \dots, B'_n \subset M'$, and each A'_i is compact. We choose orientations on them as in Remark 6.18. By Theorem 6.12, we have

$$\int_{M'} \eta'_{A'_i} \wedge \eta_{B'_j} = \int_{M'} \eta_{A'_i} \wedge \eta_{B'_j} = \int_{M'} \eta_{A'_i \cap B'_j} = \sum_{p \in A'_i \cap B'_j} I_p(A'_i, B'_j).$$

Furthermore, by what we know about the intersections of A'_i and B'_j from Proposition 6.27, this sum is nonzero if and only if $A_i \cap B_j \neq \emptyset$, which is the case if and only if $i \leq j$. It follows that the matrix

$$\left(\int_{M'} \eta'_{A'_i} \wedge \eta_{B'_j}\right)_{ij} \in M_n(\mathbb{R})$$

is upper triangular with nonzero entries on the diagonal, hence is in $\mathrm{GL}_n(\mathbb{R})$. Since the map $H^r_{\mathrm{dR,c}}(M') \times H^r_{\mathrm{dR}}(M') \to \mathbb{R}$, $(\omega, \tau) \mapsto \int_{M'} \omega \wedge \tau$ is bilinear, it follows that the classes $\eta'_{A'_1}, \dots, \eta'_{A'_n} \in H^r_{\mathrm{dR},c}(M')$ are linearly independent. By Remark 6.11, they are mapped by the Poincaré duality isomorphism to the images of the fundamental classes $[A'_1], \ldots, [A'_n]$ in $H_r(M'; \mathbb{R})$, and so these homology classes are also linearly independent. Thus M satisfies the flat homology condition. □

By combining Theorem 6.30 with the results from Chapter 2 and Margulis' arithmeticity theorem, we can now finish the proof of our main theorem:

Theorem 1.1. *Let M be a locally symmetric space of finite volume covered by $(\mathbb{H}^2)^r$. Then for any $n \in \mathbb{N}$, there exists a connected finite covering $M' \to M$ and compact oriented flat totally geodesic r-dimensional submanifolds $A_1, \ldots, A_n \subset M'$ such that the images of the fundamental classes $[A_1], \ldots, [A_n]$ in $H_r(M'; \mathbb{R})$ are linearly independent.*

Proof. If $r = 1$, then M is a hyperbolic surface of finite area and we have already shown the claim in Theorem 2.6. So assume that $r \geq 2$. By repeatedly applying Proposition 6.29, we can reduce the proof to the case where M is irreducible. So assume now that M is irreducible. By Proposition 3.8, there exists a torsion-free lattice $\Delta \subset \mathrm{SL}_2(\mathbb{R})^r$ such that M is finitely covered by the quotient $\Delta\backslash(\mathbb{H}^2)^r$, and so $\Delta\backslash(\mathbb{H}^2)^r$ is also an irreducible locally symmetric space. We now show that Δ is an irreducible lattice: If Δ were reducible, then we would have a decomposition $\mathrm{SL}_2(\mathbb{R})^r = \mathrm{SL}_2(\mathbb{R})^{r_1} \times \mathrm{SL}_2(\mathbb{R})^{r_2}$ with $r_1 + r_2 = r$ and $r_1, r_2 > 0$ such that the quotient $\Delta/(\Delta_1\Delta_2)$ is finite, where Δ_i for $i \in \{1, 2\}$ is the intersection of Δ with the image of the i-th factor in the above decomposition of $\mathrm{SL}_2(\mathbb{R})^r$. Then the map

$$\Delta_1\backslash(\mathbb{H}^2)^{r_1} \times \Delta_2\backslash(\mathbb{H}^2)^{r_2} \to \Delta\backslash(\mathbb{H}^2)^r, \quad (\Delta_1 x_1, \Delta_2 x_2) \mapsto \Delta(x_1, x_2)$$

would be a finite covering with $\dim(\Delta_i\backslash(\mathbb{H}^2)^{r_i}) = 2r_i > 0$, in contradiction to the irreducibility of $\Delta\backslash(\mathbb{H}^2)^r$. So Δ is an irreducible lattice. Since $r \geq 2$, Margulis' arithmeticity theorem (see Theorem 4.49) implies that Δ is arithmetically defined. So the claim follows by Theorem 6.30. □

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