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ℓ^2-Invariants under Coarse Equivalence and Benjamini-Schramm Convergence

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Introduction

Perhaps the strongest link between the chapters of this thesis are ℓ^2-Betti numbers. ℓ^2-Betti numbers are topological invariants with values in the positive extended-real numbers $[0, \infty]$. Their history, or evolution, goes back to the year 1976, when Atiyah defined them for universal coverings of compact Riemannian manifolds by means of the heat kernel [Ati76]. This was followed by different definitions - a simplicial one was given by Dodziuk [Dod77] and, later, Lück [Lüc97, Lüc98] gave a homological one - all these definitions only apply to discrete groups acting on topological spaces. Using similar techniques as Cheeger and Gromov, who gave the first definition of ℓ^2-Betti numbers of an arbitrary discrete group [CG86], Gaboriau defined ℓ^2-Betti numbers of a *standard measure preserving equivalence relation* [Gab02]. This was succeeded by a generalization to *discrete measured groupoids*, due to Sauer [Sau05], which we will deal with in Chapter 4. Later on, Connes and Shlyakhtenko defined ℓ^2-Betti numbers of arbitrary finite von Neumann algebras [CS05]. Building on the homological approach of Lück, H. D. Petersen extended the notion of *von Neumann dimension* from finite to semi-finite von Neumann algebras and was thus able to define ℓ^2-Betti numbers of unimodular locally compact second countable groups [Pet13]. We will come back to this definition in Chapter 1.
In this thesis, I want to study different aspects and generalizations of ℓ^2-Betti numbers.

- Up to what extent do ℓ^2-Betti numbers depend on the coarse geometry of the group?
- How do ℓ^2-Betti numbers behave under Benjamini-Schramm convergence?
- Is there a way to define the ℓ^2-Betti numbers of groupoids, or spaces with an action of a groupoid, analogous to the definition of ℓ^2-Betti numbers of discrete groups?

The motivation for the last question is that such a definition may enables us to define ℓ^2-Betti numbers of objects for which it was not possible before. In the following we will address each of the three questions.

Coarse Geometry Invariance

The answer to the first question is that ℓ^2-Betti numbers are no quasi-isometry invariants and they do not coincide up to a constant factor. But, as we will see later, the vanishing of ℓ^2-Betti numbers is a coarse geometry invariant (this is a joint work with Roman Sauer [SS18]).
The insight that the vanishing of ℓ^2-Betti numbers provides a quasi-isometry invariant is due to Gromov [Gro93, Chapter 8], and positive results around this insight have a long history. Maybe the most important contribution is due to Pansu [Pan95] who introduced asymptotic ℓ^p-cohomology of discrete groups and proved its invariance under quasi-isometries.

If a group Γ is of type F_∞, then the ℓ^p-cohomology of Γ coincides with its asymptotic ℓ^p-cohomology [Pan95, Théorème 1]. The geometric explanation for the appearance of the type F_∞ condition is that the finite-dimensional skeleta of the universal covering of a classifying space of finite type are uniformly contractible. As an immediate consequence of Pansu's result, the vanishing of ℓ^2-Betti numbers is a quasi-isometry invariant among discrete groups of type F_∞. The same arguments work for totally disconnected groups admitting a topological model of finite type [Sau18].
Elek [Ele98] investigated the relation between ℓ^p-cohomology of discrete groups and Roe's coarse cohomology and proved similar results. Another independent treatment is due to Fan [Fan94]. Genton [Gen14] elaborated upon Pansu's methods in the case of metric measure spaces.
Oguni [Ogu10] generalised the quasi-isometry invariance of the vanishing of ℓ^2-Betti numbers from discrete groups of type F_∞ to discrete groups whose cohomology with coefficients in the group von Neumann algebra satisfies a certain technical condition. A similar technical condition appears in the proof of quasi-isometry invariance of Novikov-Shubin invariants of amenable groups [Sau06], and it is unclear how much this condition differs from the type F_∞ condition. Oguni's groupoid approach is inspired by [Gab02, Sau06] and quite different from the approaches by Elek, Fan, and Pansu. The coarse invariance of vanishing of ℓ^2-Betti numbers for discrete groups was shown by Mimura-Ozawa-Sako-Suzuki [MOSS15, Corollary 6.3]. Li [Li18] recently reproved this using groupoid techniques as a consequence of more general cohomological coarse invariance results.

Benjamini-Schramm Convergence

Now we turn to the second question. Benjamini and Schramm introduced the concept of random rooted graphs as probability measures on the space of connected rooted graphs [BS01]. This allowed them to define convergence - today known as *Benjamini-Schramm* convergence - of sequences of finite graphs and to study the corresponding limit random rooted graphs. It was realized by Aldous and Lyons [AL07] that Benjamini-Schramm limits of finite graphs share a useful mass-transport property, called *unimodularity*, which, when added to the definition of random rooted graphs, allows to extend several results from quasi-transitive graphs to this setting. These two insights provide the basis of a large number of subsequent developments.
The basic idea of Benjamini-Schramm convergence is not specific to graphs and was, in particular, generalized to simplicial complexes in the work of Elek [Ele10] and Bowen [Bow15], who further generalized it to metric measure spaces. In this thesis, I want to focus on Benjamini-Schramm convergence of simplicial complexes. Elek proved in [Ele10] that the limit $\lim_{n\to\infty} b_p(K_n)/|K_n^{(0)}|$ of the Betti numbers normalized by the number of vertices exists for a Benjamini-Schramm convergent sequence $(K_n)_n$ of finite simplicial complexes of globally bounded vertex degree.
I will give a definition of ℓ^2-Betti numbers of random rooted simplicial complexes - which coincides with $b_p(K)/|K^{(0)}|$ for a finite random rooted simplicial complex K - and show that

they are continuous under Benjamini-Schramm convergence on the space of *sofic* random rooted simplicial complexes [Sch19].
This result generalizes Lück's Approximation Theorem [Lüc94], which says in its classical version that for a free Γ-CW-complex X, with $\Gamma\backslash X$ of finite type,

$$\lim_{i\to\infty} \frac{b_p(\Gamma_i\backslash X;\mathbb{Q})}{[\Gamma:\Gamma_i]} = \beta_p^{(2)}(X,\Gamma)$$

holds for all finite index normal towers $(\Gamma_i)_{i\in\mathbb{N}}$. Farber [Far] extended this to so called Farber sequences, which weakens the assumption that $(\Gamma_i)_{i\in\mathbb{N}}$ is normal. Elek and Szabó [ES05] further extended this result to normal towers $(\Gamma_i)_{i\in\mathbb{N}}$ with Γ/Γ_i sofic for every $i\in\mathbb{N}$. In [ABB+17], Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault and Samet proved an approximation result for Benjamini-Schramm convergent sequences of orbifolds which arise from a uniformly discrete sequence of lattices in a semi-simple Lie group. Petersen, Sauer and Thom generalized the notion of Farber sequences to lattices in a totally disconnected second countable unimodular group G, which is also an instance of local weak convergence, and proved that the normalized Betti numbers of a uniformly discrete Farber sequence converge to the ℓ^2-Betti numbers of the group G [PST18]. Recently, Carderi, Gaboriau and de la Salle proved a theorem for graphed groupoids and ultra products of them, which for instance implies the convergence for Farber sequences [CGd18].
Now that we know that ℓ^2-Betti numbers are continuous under Benjamini-Schramm convergence, it is natural to ask if this also holds true for refinements of them. One classical refinement of ordinary Betti numbers are multiplicities of irreducible representations of a finite group which acts on the homology of a simplicial complex. In analogy to this, Kionke defined ℓ^2-multiplicities as a refinement of ℓ^2-Betti numbers of CW-complexes [Kio18].
In this thesis, I will give a definition of *ℓ^2-multiplicities* of random rooted simplicial complexes with an action of a finite group and prove that they are continuous with respect to an equivariant version of Benjamini-Schramm convergence for *sofic* random rooted simplicial G-complexes. This is a joint work with Steffen Kionke [KSB].

ℓ^2-Betti Numbers of Groupoids

Let us move on to the last of the three questions. Gaboriau defined ℓ^2-Betti numbers of an $\mathcal{R}$-simplicial complex Σ as the von Neumann dimension of the reduced homology of the chain complex $\int^{\oplus} C_n^{(2)}(\Sigma_x)d\mu(x)$ [Gab02]. If the simplicial complexes Σ_x are almost surely contractible, then the ℓ^2-Betti numbers are independent of Σ and are called the ℓ^2-Betti numbers of the equivalence relation $\mathcal{R}$.
For *discrete measured groupoids* there also exists a notion of ℓ^2-Betti numbers, due to Sauer [Sau05], which is purely homological and does not involve any simplicial complexes. Sauer showed that the two definitions coincide for orbit equivalence relations of discrete groups.
Later on, Takimoto adapted the ideas of Gaboriau and defined $\mathcal{G}$-simplicial complexes - for a discrete measured groupoid $\mathcal{G}$ - and their corresponding ℓ^2-Betti numbers [Tak15]. Furthermore, Takimoto proved that for contractible $\mathcal{G}$-simplicial complexes, the ℓ^2-Betti numbers

of the $\mathcal{G}$-simplicial complex coincides with the ℓ^2-Betti numbers, as defined by Sauer, of the groupoid.
Inspired by this, I will define *topological $\mathcal{G}$-spaces*, which are less restrictive than $\mathcal{G}$-simplicial complexes, and define a $\mathcal{G}$-equivariant singular homology on them. This enables us to define ℓ^2-Betti numbers of $\mathcal{G}$-spaces. Furthermore, to justify this definition, I will show that the ℓ^2-Betti numbers of the geometric realization of a $\mathcal{G}$-simplicial complex, considered as a topological $\mathcal{G}$-space, coincide with the ℓ^2-Betti numbers of the $\mathcal{G}$-simplicial complex as defined by [Tak15].

Before I will give a more detailed summary of the results, I would like to return to the three questions from the beginning of this introduction. We can give a positive answer to all three of them, atleast partially. It is clear that ℓ^2-Betti numbers can not be quasi-isometry invariants, but their vanishing is a coarse equivalence invariant. Before this work, it was only known for discrete groups of type F_∞, or discrete groups fulfilling another technical condition. Now, we know it for all unimodular locally compact second countable groups, and this are all groups for which ℓ^2-Betti numbers are defined. The question of the continuity of the ℓ^2-Betti numbers under Benjamini-Schramm convergence can also be answered positively for the space of sofic random rooted simplicial complexes. Before, it was only known for sequences of finite simplicial complexes, and furthermore, there was no notion of ℓ^2-Betti numbers for the limit objects. It would be nice to have a definition of ℓ^2-Betti numbers in a more general setting, in particular for probability measures on the space of rooted metric measure spaces or, atleast, on the space of rooted Riemannian manifolds as defined in [AB16]. I will give a definition of topological $\mathcal{G}$-spaces and ℓ^2-Betti numbers of them, which is a positive answer to the third question. This definition extends the scope of objects on which ℓ^2-Betti numbers are defined and it is compatible with previous definitions for groups, equivalence relations and discrete measured groupoids.

Statement of Results

This thesis is divided into two parts. The first part, consisting of the Chapters 1 to 4, represents original research. The second consists of the Appendices A to C and is mainly a summary of results from the literature that we use in the first part.
In Appendix A, I summarize the character theory of finite groups. Appendix B is dedicated to the theory of direct integral Hilbert spaces which is a major tool for the Chapters 2 and 3. The last appendix, Appendix C, is a collection of results from the dimension theory of modules over finite and semi-finite von Neumann algebras and also contains some results from homological algebra.

Coarse equivalence invariance

Chapter 1 is based on a joint work with Roman Sauer [SS18] and provides a proof of the following theorem:

Theorem (1.3.6). *Let G and H be unimodular locally compact second countable groups. If G and H are coarsely equivalent, then the nth ℓ^2-Betti number of G vanishes if and only if the nth ℓ^2-Betti number of H vanishes.*

To this end, we introduce the *coarse ℓ^2-cohomology* $HX^n_{(2)}(G)$ of a locally compact second countable group G. The idea is that we only consider (equivalence classes of) measurable functions $\alpha\colon G^{n+1} \to \mathbb{C}$ such that

$$\int_{G_R^{n+1}} |\alpha|^2 d\mu_{n+1} < \infty,$$

where μ_{n+1} is the $n+1$-fold product of a left-invariant Haar measure on G and $G_R^{n+1} \subset G^{n+1}$ is the tube consisting of tuples $(g_0, ..., g_n)$ such that $d(g_i, g_j) \leqslant R$ for all $0 \leqslant i, j \leqslant n$. Note that by a theorem from Struble every locally compact second countable group admits a left-invariant proper continuous metric (Theorem 1.1.2). Our definition of the coarse ℓ^2-cohomology is the continuous analogue of Elek's definition [Ele98, Definition1.3] in the discrete case, who gives credits to Roe [Roe93], and is very much related to Pansu's *asymptotic ℓ^2-cohomology* [Pan95].
In Theorem 1.3.5, we prove that coarsely equivalent locally compact second countable groups have isomorphic coarse ℓ^2-cohomology groups. This completes the proof of the main theorem (Theorem 1.3.6), since we verify in Theorem 1.3.3 that for unimodular locally compact second countable groups the coarse ℓ^2-cohomology is isomorphic to the continuous cohomology. I remark that coarse equivalence is a weaker notion than quasi-isometry, though for compactly generated groups, in particular for finitely generated discrete groups, they are equivalent.

Benjamini-Schramm Convergence

Most of the results of Chapter 2 already appeared in my article [Sch19]. Before I can describe the results, we briefly recall the definition of Benjamini-Schramm convergence. Every finite simplicial complex K gives rise to a random rooted simplicial complex, *i.e.* a *unimodular* probability measure on the space $\mathcal{SC}_*$ of isomorphism classes of locally finite connected rooted simplicial complexes, by picking a vertex uniformly at random as root. We say a sequence of finite simplicial complexes converges *Benjamini-Schramm* to a probability measure on $\mathcal{SC}_*$ if their associated random rooted simplicial complexes weakly converge to this probability measure. The Benjamini-Schramm limits of finite simplicial complexes are also called *sofic* random rooted simplicial complexes.
In order to define homology and ℓ^2-Betti numbers of random rooted simplicial complexes, we have to choose a representative for each isomorphism class in $\mathcal{SC}_*$. We can do this in a measurable way by Lemma 2.2.1. This enables us to define the *simplicial ℓ^2-chain complex*

$$C_*^{(2)}(\mathcal{SC}_*, \mu) = \int_{SC}^{\oplus} C_*^{(2)}([K,x]) d\mu([K,x])$$

of a random rooted simplicial complex μ (Definition 2.2.2) as a direct integral of the ℓ^2-chain complexes of the representatives of each isomorphism class $[K,x] \in \mathcal{SC}_*$. The Laplace operators on the fibres give rise to a Laplace operator on $C_*^{(2)}(\mathcal{SC}_*, \mu)$.

After we define a von Neumann algebra of bounded operators on $C_p^{(2)}(\mathcal{SC}_*, \mu)$ together with a trace, we can define the ℓ^2-Betti numbers $\beta_p^{(2)}(\mu)$ of a random rooted simplicial complex μ (Definition 2.2.8). The last section of Chapter 2 is dedicated to the proof of the following theorem:

Theorem (2.3.3). *Let $(\mu_n)_n$ be a sequence of sofic random rooted simplicial complexes with uniformly bounded vertex degree. If the sequence weakly converges to a random rooted simplicial complex μ, then the ℓ^2-Betti numbers of $(\mu_n)_n$ converge to the ℓ^2-Betti numbers of μ.*

From this theorem, together with Proposition 2.1.12, we recover the following generalization of Lück's approximation theorem of Elek and Szabó [ES05], which we already mentioned above:

Theorem. *Let K be a simplicial complex with a free and cocompact action of a discrete group Γ. If $(\Gamma_n)_n$ is a sequence of normal subgroups of Γ such that Γ/Γ_n is sofic, then*

$$\lim_{n\to\infty} \beta_p^{(2)}(K/\Gamma_n; \Gamma) = \beta_p^{(2)}(K; \Gamma).$$

Another corollary from Theorem 2.3.3 is a version of the Euler-Poincaré Formula:

Corollary (Euler-Poincaré Formula). *Let μ be a sofic random rooted simplicial complex of dimension n and with bounded degree. Then*

$$\sum_{p=0}^{n} (-1)^p \beta_p^{(2)}(\mu) = \sum_{p=0}^{n} (-1)^p \frac{\mathbb{E}_\mu(\deg_p)}{p+1},$$

where $\mathbb{E}_\mu(\deg_p)$ denotes the expected number of p-simplices containing the root.

ℓ^2-Multiplicities

Chapter 3 reflects the results of a joint work with Steffen Kionke [KSB], in which we generalize the concept of Benjamini-Schramm convergence to simplicial complexes with an action of a finite group G. Now, we do not consider simplicial complexes rooted at one vertex, but rooted at a G-orbit of vertices. If K is a finite simplicial complex and G acts on it, then G also acts on its homology group $H_p(K; \mathbb{C})$. This representation decomposes as a direct sum of irreducible representations of G and every irreducible representation $\sigma \in \mathrm{Irr}(G)$ occurs a finite number, say $m(\sigma, H_p(K, \mathbb{C}))$, of times in this decomposition. The number $m(\sigma, H_p(K, \mathbb{C}))$ is called the *multiplicity* of σ in $H_p(K, \mathbb{C})$. Drawing from this, and the concepts of Chapter 2, we define ℓ^2-multiplicities (Definition 3.2.6); these are non-negative real numbers $m_p^{(2)}(\sigma, \mu)$ for every random rooted simplicial G-complex μ and every irreducible representation $\sigma \in \mathrm{Irr}(G)$. In fact, if μ_K is the law of a finite G-complex, then

$$m_p^{(2)}(\sigma, \mu_K) = \frac{m(\sigma, H_p(K, \mathbb{C}))}{|K^{(0)}|};$$

see Example 3.2.8. Our main result is the following approximation theorem.

Theorem (3.3.2). *Let $(\mu_n)_n$ be a sequence of sofic random rooted simplicial G-complexes. If the sequence weakly converges to a random rooted simplicial G-complex μ_∞, then*

$$\lim_{n\to\infty} m_p^{(2)}(\sigma,\mu_n) = m_p^{(2)}(\sigma,\mu_\infty)$$

for every $p \in \mathbb{N}_0$ and every irreducible representation σ of G.

Along the way (in Section 3.1.1), we investigate induced G-complexes. Given a subgroup $H \leqslant G$ and a simplicial H-complex L, it is natural to construct a simplicial G-complex $K = G \times_H L$ *by inducing the action from H to G*. This can be promoted to an operation Ind_H^G which takes random rooted H-complexes to random rooted G-complexes. We provide a criterion to decide whether a sequence of finite G-complexes converges to an induced random rooted G-complex; see Proposition 3.1.8. This is relevant, since the ℓ^2-multiplicities of the induced random rooted complex $\mathrm{Ind}_H^G(\mu)$ can be computed from the ℓ^2-multiplicities of μ (see Theorem 3.3.3). As a special case, for $H = \{1\}$, we have the following result.

Corollary (3.3.4). *Let μ be a sofic random rooted simplicial complex and let G be a finite group. For all $\sigma \in \mathrm{Irr}(G)$ and $p \in \mathbb{N}_0$ the following identity holds:*

$$m_p^{(2)}(\sigma,\mathrm{Ind}_1^G(\mu)) = \frac{\dim_{\mathbb{C}}(\sigma)}{|G|}\beta_p^{(2)}(\mu).$$

Singular Groupoid Homology

In Chapter 4, I introduce *topological $\mathcal{G}$-spaces* (Definition 4.1.14) for a discrete measured groupoid $\mathcal{G}$ over a probability space X. A topological $\mathcal{G}$-space is a Borel fibred spaces Y over X, such that each fibre Y_x is a topological space, equipped with an action of $\mathcal{G}$. In comparison, $\mathcal{G}$-simplicial complexes are *discrete* $\mathcal{G}$-spaces, though their fibrewise geometric realizations form topological $\mathcal{G}$-spaces as we will see in Section 4.2.
Inspired by the singular homology of topological spaces, we define the *singular groupoid homology* of a topological $\mathcal{G}$-space in Section 4.3. The following is the main theorem of Chapter 4.

Theorem (4.4.1). *Let Σ be a simplicial $\mathcal{G}$-complex and $|\Sigma|$ its geometric realization as a topological $\mathcal{G}$-space. Then*

$$\beta_n^{(2)}(\Sigma,\mathcal{G}) = \beta_n^{(2)}(|\Sigma|,\mathcal{G})$$

for all $n \in \mathbb{N}_0$.

To this end, we show the homotopy invariance of the singular groupoid homology for a suitable notion of homotopy (see Corollary 4.3.9) and a corresponding version of the Excision Theorem (Proposition 4.3.10). Note that the ℓ^2-Betti numbers $\beta_n^{(2)}(\Sigma,\mathcal{G})$ of a simplicial $\mathcal{G}$-complex Σ and the ℓ^2-Betti numbers $\beta_n^{(2)}(|\Sigma|,\mathcal{G})$ of a topological $\mathcal{G}$-space are defined as the von Neumann dimension (Definition C.3) of the simplicial or singular groupoid homology groups, respectively, with respect to the groupoid von Neumann algebra $L\mathcal{G}$. In comparison to the classical Excision Theorem, our version is only a $L\mathcal{G}$-dimension isomorphism. That is

likewise the reason why we only have equal ℓ^2-Betti numbers and not isomorphic homology groups as in the classical case.
An immediate consequence of Theorem 4.4.1 and a result of Takimoto (Theorem 4.2.5) is the following corollary, where $\beta_n^{(2)}(\mathcal{G})$ denotes the ℓ^2-Betti number of the groupoid $\mathcal{G}$ as defined by Sauer [Sau05]:

Corollary (4.4.2)**.** *If Y is a contractible $\mathcal{G}$-space, then $\beta_n^{(2)}(Y, \mathcal{G}) = \beta_n^{(2)}(\mathcal{G})$ for every $n \in \mathbb{N}$.*

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Chapter 1.

Coarse Equivalence Invariance of ℓ^2-Betti Numbers of Locally Compact Groups

This chapter is a joint work with Roman Sauer [SS18]. We provide a proof that the vanishing of ℓ^2-Betti numbers of unimodular locally compact second countable groups is an invariance of coarse equivalence. To this end, we define coarse ℓ^2-cohomology for locally compact second countable groups and show that coarsely equivalent groups have isomorphic ℓ^2-cohomology groups. We complete the proof by showing that the coarse ℓ^2-cohomology is isomorphic to continuous cohomology for unimodular locally compact second countable groups.

We begin in Section 1.1 with a summary of results concerning the coarse geometry of locally compact groups and the existence of a semi-finite trace on the group von Neumann algebra. In Section 1.2 we review the necessary basics of ℓ^2-Betti numbers and continuous cohomology. We define coarse ℓ^2-cohomology for locally compact second countable groups and show that it is invariant under coarse equivalence in Section 1.3. Moreover, we prove that it is isomorphic to continuous cohomology if the group is unimodular and, by this, complete the proof of the main theorem of the chapter. In the end we shortly discuss what fails for non-unimodular groups.

1.1. Locally Compact Groups

In this section, we summarize some basic results for locally compact groups which we will need in the following. For more information about metric properties of topological groups consider [CdlH16], and [Ped18] for the regular representation and the group von Neumann algebra of unimodular locally compact groups.

A topological space is called *locally compact* if each of its points has a compact neighbourhood; it is called *second-countable* if its topology has a countable basis of open sets.

Definition 1.1.1. A *locally compact second countable group* is a group G which is a locally compact second countable topological space such that the map $(g, h) \mapsto gh^{-1}$ from $G \times G \to G$ is continuous, where $G \times G$ carries the product topology.

The following theorem of Struble is crucial:

Theorem 1.1.2. [Str74] *Every locally compact second countable group admits a left-invariant proper continuous metric, where* proper *means that balls are relatively compact.*

Now, that we know that there is always a metric, we introduce the notion of *coarse equivalences*. A map $f : (X, d_X) \to (Y, d_Y)$ between metric spaces is called *coarsely Lipschitz* if there is a non-decreasing function $a\colon [0, \infty) \to [0, \infty)$ such that

$$d_Y(f(x), f(x')) \leqslant a(d(x, x'))$$

for all $x, x' \in X$. We say that two such maps f, g are *close* to each other, denoted by $f \sim g$, if

$$\sup_{x \in X} d_Y(f(x), g(x)) < \infty.$$

Definition 1.1.3. A coarsely Lipschitz map $f\colon X \to Y$ is called a *coarse equivalence* if there is a coarsely Lipschitz map $g\colon Y \to X$ such that $f \circ g$ and $g \circ f$ are close to the identity. We say g is a *coarse inverse* of f.

In view of the following proposition, we can say that every locally compact second countable group has a well defined coarse geometry.

Proposition 1.1.4. [CdlH16, 4.A.6] *Let G be a locally compact second countable group and d_1 and d_2 are two left-invariant proper continuous metrics on G. The identity, viewed as a map* $\mathrm{Id}\colon (G, d_1) \to (G, d_2)$, *is a coarse equivalence.*

A locally compact second countable group does not only have a metric, but also a left- and a right-*Haar measure*, *i.e.* a left- and a right-translation invariant positive Radon measure on the Borel σ-algebra, respectively. These measures are uniquely determined, up to a constant factor, by their property of being translation invariant. A locally compact second countable group is called *unimodular* if its left- and right-Haar measures coincide.

Definition 1.1.5. Let G be a locally compact second countable group and μ a left-Haar measure on G. The *left-regular representation* $\lambda\colon G \to \mathcal{B}(L^2(G,\mu))$ of G is given by

$$\lambda(g)f(h) = f(g^{-1}h).$$

Usually, we denote $L^2(G,\mu)$ just by $L^2(G)$ if there are no different measures involved. The closure of $\lambda(L^1(G))$ in the weak-operator topology is the *group von Neumann algebra* of G and denoted by LG. The left-action of G on $L^2(G)$ commutes with the right-action of LG, therefore, we will consider $L^2(G)$ as a G-LG-module. Unimodularity is a necessary condition to define ℓ^2-Betti numbers of locally compact groups because of the following theorem:

Theorem 1.1.6. [Ped18, 7.2.7 and 7.2.8] *Let G be a locally compact second countable group. LG carries a canonical faithful, normal and semi-finite trace tr_μ if and only if G is unimodular. Moreover, the trace scales in the following way: $\operatorname{tr}_{c\mu} = \frac{1}{c}\operatorname{tr}_\mu$ for $c > 0$.*

If G is not unimodular, there exist only a faithful weight on G, which is not tracial. By Definition C.3 we have a dimension function $\dim_\mu$ for LG-modules, which is additive for short exact sequence (Proposition C.4). We added the Haar measure μ to the notation of the dimension $\dim_\mu$, since it scales in the same way as the trace, that is, $\dim_{c\mu} = \frac{1}{c}\dim_\mu$ for $c > 0$.

1.2. Continuous Cohomology and ℓ^2-Betti Numbers

Let X be a locally compact second countable space with Radon measure ν and let E be a Fréchet space. The space $C(X, E)$ of continuous functions from X to E becomes a Fréchet space when endowed with the topology of compact convergence. The semi-norms on $C(X, E)$ are then given by

$$p_{K,\|\cdot\|_q}(f) = \sup_{x\in K} \|f(x)\|_q$$

for $f \in C(X, E)$, where $K \subset X$ is compact and $\|\cdot\|_q$ is a semi-norm of E ([Gui80, D.1.3]). In a similar way we define the space $L^2_{loc}(X, E)$ of equivalence classes of measurable maps $f\colon X \to E$, up to ν-null sets, such that $\|f|_K\|_q$ is square-integrable for every compact subset $K \subset X$ and semi-norm $\|\cdot\|_q$ of E. So $L^2_{loc}(X, E)$ becomes again a Fréchet space with semi-norms given by

$$p_{K,\|\cdot\|_q}(f) = \left(\int_K \|f(x)\|_q^2 d\nu(x)\right)^{1/2},$$

for $f \in L^2_{loc}(X, E)$ ([Gui80, D.2.1]).

Let G be a unimodular locally compact second countable group. We call a Fréchet space E together with a continuous linear G-action (i.e $G \to E$, $g \mapsto gv$ is continuous for all $v \in E$) a *G-module*. A *homomorphism* of G-modules is a continuous linear G-equivariant map between G-modules. If E is a G-module and G acts continuously and ν-preserving on X, then $C(X, E)$ and $L^2_{loc}(X, E)$ become G-modules as well, via

$$(g \cdot f)(x) = gf(g^{-1}x),$$

for $g \in G$, $x \in X$ and $f \in C(X, E)$ or $f \in L^2_{loc}(X, E)$, respectively ([Bla79, Proposition 3.1.1]). Since the category of topological G-modules is not abelian, we have to restrict to relatively injective resolutions of E if you want to define the cohomology of G with coefficients in E independent of the choice of the resolution. We will not go into that because we only consider the following explicit cochain complexes of G-modules, which are relatively injective resolutions of E (cf. [Gui80, Chapter III, §1, Proposition 1.2 and 1.4]):

$$\begin{aligned} 0 &\xrightarrow{\epsilon} C(G, E) \xrightarrow{d^0} C(G^2, E) \xrightarrow{d^1} \dots \\ 0 &\xrightarrow{\epsilon} L^2_{loc}(G, E) \xrightarrow{d^0} L^2_{loc}(G^2, E) \xrightarrow{d^1} \dots, \end{aligned}$$

with $\epsilon(e)(g) = e$ and the usual homogeneous coboundary map

$$(d^{n-1}f)(g_0, ..., g_n) = \sum_{i=0}^{n} (-1)^i f(g_0, ..., \widehat{g}_i, ..., g_n),$$

for $g \in G$ and $f \in C(G^{n+1}, E)$, respectively $f \in L^2_{loc}(G^{n+1}, E)$. Note, here and in the following we use "$\widehat{\ }$" to mark an element which we remove from a tuple. The action of G on G^{n+1} is given by the diagonal action.

Definition 1.2.1. The *continuous cohomology of G* with coefficients in E is the cohomology

$$H^n(G,E) := H^n(C(G^{*+1},E)^G)$$

of the G-invariants of $C(G^{*+1},E)$. The *reduced continuous cohomology* $\underline{H}^n(G,E)$ of G is obtained by taking the quotient with the closure of $\operatorname{im} d^*|_{C(G^{*+1},E)^G}$.

We have the obvious inclusions

$$I^*\colon C(G^{*+1},E) \to L^2_{loc}(G^{*+1},E),$$

which form a cochain map of G-modules. We fix an arbitrary positive continuous function χ on G with compact support and integral 1. There is a cochain map $R^*\colon L^2_{loc}(G^{*+1},E) \to C(G^{*+1},E)$ of G-modules

$$(R^n f)(g_0,...,g_n) = \int_{G^{n+1}} f(h_0,...,h_n)\chi(g_0^{-1}h_0,...,g_n^{-1}h_n)d\mu(h_0,...,h_n)$$

such that $I^* \circ R^*$ and $R^* \circ I^*$ are homotopic, as cochain maps of G-modules, to the identity [Bla79, Proposition 4.8]. Therefore, we have the following useful fact:

Theorem 1.2.2. *The cochain map*

$$I^*\colon C(G^{*+1},E) \to L^2_{loc}(G^{*+1},E),$$

defined by the inclusions, induces isomorphisms in cohomology and reduced cohomology.

Now, we turn to the case where the coefficient module E is the (left)-regular representation $L^2(G)$ of G (Definition 1.1.5). Since the action of G and the action of LG on $L^2(G)$ commute, also the previously considered G-action on $C(G^{*+1},L^2(G))$ and $L^2_{loc}(G^{*+1},L^2(G))$ commutes with the LG-action induced from the right LG-action on $L^2(G)$. So the (reduced) continuous cohomology of G with coefficients in $L^2(G)$ is naturally a LG-module in the algebraic sense. Obviously, the cochain map I^* above is compatible with the LG-module structures. The groups $H^*(G,L^2(G))$ are called the *ℓ^2-cohomology* of G and similarly for the reduced cohomology. As mentioned after Theorem 1.1.6, the trace on LG induces a dimension function for LG-modules, therefore, we have the following definition:

Definition 1.2.3. The nth *ℓ^2-Betti number of* G is the LG-dimension of its nth reduced ℓ^2-cohomology group, *i.e.*

$$\beta_n^{(2)}(G) := \dim_\mu \underline{H}^n(G,L^2(G)) \in [0,\infty].$$

Remark 1.2.4. Equivalently, the nth ℓ^2-Betti number of G can be defined as the LG-dimension of the non-reduced cohomology $H^n(G,L^2(G))$ (see [KPV15, Theorem A]). For discrete groups this definition coincides with the one of Lück [Lüc98], this was shown in [PT11, Theorem 2.2].

The following useful lemma was observed in [Pet13, Proposition 3.8]. Since it is a direct consequence of Sauer's local criterion (Proposition C.8), we present the argument.

Lemma 1.2.5.
$$\beta_n^{(2)}(G) = 0 \Leftrightarrow \underline{H}^n(G, L^2(G)) = 0.$$

Proof. Let $\beta_n^{(2)}(G) = 0$ and let $f\colon G^{n+1} \to L^2(G))$ be a cocycle representing a cohomology class $[f] \in \underline{H}^n(G, L^2(G))$. By Proposition C.8 there is a increasing sequence of projections $p_j \in LG$ whose supremum is 1 and such that fp_j is a coboundary $d^{n-1}b_j$. Obviously, fp_j converges to f in the topology of $C(G^{n+1}, L^2(G))$, thus $[f] = 0$. □

1.3. Coarse Cohomology and Coarse Equivalence Invariance

Let G be a locally compact second countable group. We fix a proper continuous left-invariant metric d on G (see Theorem 1.1.2 for the existence). Let μ be a Haar measure on G and μ_n the n-fold product measure of μ on G^n.
For every $R > 0$ and $n \in \mathbb{N}$ we consider the closed subset

$$G_R^{n+1} := \{(g_0, ..., g_n) \in G^{n+1} \mid d(g_i, g_j) \leqslant R \text{ for all } 0 \leqslant i, j \leqslant n\}$$

and define a family of semi-norms on the space of measurable functions $\alpha\colon G^{n+1} \to \mathbb{C}$ by

$$\|\alpha\|_R^2 := \int_{G_R^{n+1}} |\alpha(g_0, ..., g_n)|^2 d\mu_{n+1}(g_0, ..., g_n) \in [0, \infty].$$

Let $CX^n_{(2)}(G)$ by the space of equivalence classes, up to μ_{n+1}-null sets, of measurable functions $\alpha\colon G^{n+1} \to \mathbb{C}$ such that $\|\alpha\|_R < \infty$ for every $R > 0$. The semi-norms $\|\cdot\|_R$ turn $CX^n_{(2)}(G)$ into a Fréchet space. We claim that the coboundary map

$$(d^{n-1}\alpha)(g_0, ..., g_n) = \sum_{i=0}^{n} (-1)^i \alpha(g_0, ..., \widehat{g}_i, ..., g_n)$$

yields a well-defined continuous homomorphism $CX^{n-1}_{(2)}(G) \to CX^n_{(2)}(G)$.

Proof.

$$\begin{aligned}
\|d^{n-1}\alpha\|_R^2 &= \int_{G_R^{n+1}} \left| \sum_{i=0}^{n} (-1)^i \alpha(g_0, ..., \widehat{g}_i, ..., g_n) \right|^2 d\mu_{n+1} \\
&\overset{(1)}{\leqslant} (n+1) \sum_{i=0}^{n} \int_{G_R^{n+1}} |\alpha(g_0, ..., \widehat{g}_i, ..., g_n)| d\mu_{n+1} \\
&= (n+1)^2 \int_{G_R^{n+1}} |\alpha(g_0, ..., g_{n-1})| d\mu_{n+1} \\
&\leqslant (n+1)^2 \mu(B_R(e)) \|\alpha\|_R^2
\end{aligned}$$

In (1) we used *Jensen's inequality* and in the last step we took into account that g_n does not appear any more in the integrand. So the measure of the acceptable domain for g_n factors in. The domain is precisely

$$\bigcap_{i=0}^{n-1} B_R(g_i),$$

where $B_R(g)$ denotes the closed ball of radius R around g, which is always contained in the ball $B_R(\tilde{g})$ around some $\tilde{g} \in G$. Since μ is left-translation invariant $\mu(B_R(\tilde{g})) = \mu(B_R(e))$, with e the neutral element of G. Further, since the metric is proper, closed balls are compact and hence $\mu(B_R(e)) < \infty$. $\square$

Thus, we obtain a cochain complex $CX^*_{(2)}(G)$ of Fréchet spaces.

Definition 1.3.1. We define the *coarse ℓ^2-cohomology* of G to be

$$HX^n_{(2)}(G) = H^n(CX^*_{(2)}(G)).$$

Similarly, we define the *reduced coarse ℓ^2-cohomology* $\underline{HX}^n_{(2)}(G)$ of G by taking the quotients by the closure of the differentials.

Remark 1.3.2. The previous definition is the continuous analogue of Elek's definition [Ele98, Definition 1.3] in the discrete case (Elek gives credits to Roe [Roe93]). It is very much related to Pansu's *asymptotic ℓ^2-cohomology* [Pan95], which was considered in the generality of metric measure spaces by Genton [Gen14]. The difference of our definition to the one in Genton [Gen14] is as follows: $CX^*_{(2)}(G)$ is an inverse limit of spaces $L^2(G_R^{*+1})$. Unlike us, Genton takes first the cohomology of $L^2(G_R^{*+1})$ and then the inverse limit. Under some uniform contractibility assumptions the two definitions coincide, but likely not in general.

Theorem 1.3.3. *Let G be a unimodular locally compact second countable group. For every $n \geqslant 0$, the nth continuous cohomology with coefficients in the left-regular representation $L^2(G)$ is isomorphic to the nth coarse ℓ^2-cohomology of G, and likewise for reduced cohomology.*

Proof. We have the obvious embedding

$$L^2_{loc}(G^{n+1}, L^2(G)) \subset L^2_{loc}(G^{n+1}, L^2_{loc}(G))$$

and an isomorphism [Gui80, D.2.2 (vii)]

$$L^2_{loc}(G^{n+1}, L^2_{loc}(G)) \cong L^2_{loc}(G^{n+1} \times G).$$

Thus, an element in $L^2_{loc}(G^{n+1}, L^2(G))$ is represented by a measurable complex function in $n+2$ variables. The G-invariant elements, *i.e.* $\alpha \in L^2_{loc}(G^{n+1}, L^2(G))$ with the property that

$$(g \cdot \alpha)(x_0, ..., x_n)(x) = \alpha(g^{-1}x_0, ..., g^{-1}x_n)(g^{-1}x) = \alpha(x_0, ..., x_n)(x),$$

are clearly represented by G-invariant elements of $L^2_{loc}(G^{n+2})$. We would like to turn such an invariant element $\alpha \in L^2_{loc}(G^{n+1}, L^2(G))^G$ into a complex valued function in $n+1$ variables by evaluating it at the neutral element, *i.e.* $\alpha(x_0, ..., x_n)(e)$, but this is in general not possible, since we are dealing with equivalence classes of measurable functions. Therefore, we define μ_{n+2}-almost everywhere

$$F^n\colon L^2_{loc}(G^{n+1}, L^2(G))^G \to L^2_{loc}(G^{n+2}),$$
$$F^n(\alpha)(x_0, ..., x_n, x) = \alpha(xx_0, ..., xx_n)(x).$$

The measurable function $F^n(\alpha)$ is invariant by translations in the $(n+2)$th variable because of the G-invariance of α. By Fubini's theorem, we may regard $F^n(\alpha)$ as a measurable function $E^n(\alpha)\colon G^{n+1} \to \mathbb{C}$ in the first $n+1$ variables and think of it as an evaluation at the neutral element e in the $(n+2)$th variable. An other way to define E^n is to pick an arbitrary non-empty relatively compact open set U (for example an open ball around the neutral element), thus $0 < \mu(U) < \infty$ holds, and set

$$E^n(\alpha)(x_0, ..., x_n) = \frac{1}{\mu(U)} \int_U \alpha(xx_0, ..., xx_n)(x) d\mu(x) \text{"="} \alpha(x_0, ..., x_n)(e).$$

We show that $\|E^n(\alpha)\|_R < \infty$ for every $R > 0$, thus $E(\alpha) \in CX^n_{(2)}(G)$. Let $B(R)$ denote the R-ball around $e \in G$. Since $\alpha \in L^2_{loc}(G^{n+1}, L^2(G))^G$, we have

$$\begin{aligned}\infty &> \int_{B(2R)^{n+1}} \int_G |\alpha(x_0, x_1, ..., x_n)(x)|^2 d\mu d\mu_{n+1} \\ &= \int_{B(2R)^{n+1}} \int_G |\alpha(x_0^{-1}x^{-1}x_0, x_0^{-1}x^{-1}x_1, ..., x_0^{-1}x^{-1}x_n)(x_0^{-1})|^2 d\mu d\mu_{n+1}\end{aligned}$$

We have to verify that the map

$$m \colon G^{n+2} \to G^{n+2}, \quad (x_0, ..., x_n, x) \mapsto (x_0^{-1}x^{-1}x_0, x_0^{-1}x^{-1}x_1, ..., x_0^{-1}x^{-1}x_n, x_0^{-1})$$

is measure preserving. Let $X, X_0, ..., X_n \subset G$ be measurable subsets:

$$\begin{aligned}&\mu_{n+2}(m(X_0 \times ... \times X_n \times X)) \\ &\qquad = \int_{G^{n+2}} \mathbb{1}_{x_0^{-1}X^{-1}x_0}(x) \mathbb{1}_{x_0^{-1}x^{-1}X_1}(x_1) ... \mathbb{1}_{x_0^{-1}x^{-1}X_n}(x_n) \mathbb{1}_{X_0^{-1}}(x_0) d\mu_{n+2} \\ &\qquad = \mu(X_1) \cdot ... \cdot \mu(X_n) \cdot \mu(X^{-1}) \cdot \mu(X_0^{-1}) \\ &\qquad = \mu_{n+2}(X_0 \times ... \times X_n \times X).\end{aligned}$$

Note that this requires unimodularity. Further, we have

$$m(G_R^{n+1} \times B(R)) \subset B(2R)^{n+1} \times G,$$

since for all $0 \leqslant j \leqslant n$

$$d(x_0^{-1}x^{-1}x_j, e) \leqslant d(x^{-1}x_j, x_j) + d(x_j, x_0) \leqslant 2R$$

holds. This implies the inequality below; the first equality follows from the fact that

$$(x_0, ..., x_n, x) \mapsto (xx_0, ..., xx_n, x)$$

is a measure preserving measurable automorphism of $G_R^{n+1} \times B(R)$:

$$\begin{aligned}&\int_{B(2R)^{n+1}} \int_G |\alpha(x_0^{-1}x^{-1}x_0, x_0^{-1}x^{-1}x_1, ..., x_0^{-1}x^{-1}x_n)(x_0^{-1})|^2 d\mu d\mu_{n+1} \\ &\qquad \geqslant \int_{G_R^{n+1}} \int_{B(R)} |\alpha(x_0, ..., x_n)(x)|^2 d\mu d\mu_{n+1} \\ &\qquad = \int_{G_R^{n+1}} \int_{B(R)} |\alpha(xx_0, ..., xx_n)(x)|^2 d\mu d\mu_{n+1} \\ &\qquad = \mu(B(R)) \|E^n(\alpha)\|_R.\end{aligned}$$

Hence, $\|E^n(\alpha)\|_R$ is finite for every $R > 0$. Further, this computation shows that E^n is continuous with respect to the Fréchet topologies. That E^* is a cochain map is obvious.
Given $\beta \in CX^n_{(2)}(G)$ we define

$$M^n(\beta)(g_0, ..., g_n)(g) = \beta(g^{-1}g_0, ..., g^{-1}g_n)$$

for μ_{n+2}-almost every $(g_0, ..., g_n, g)$. We show that $M^n(\beta)$ defines an element in $L^2_{loc}(G^{n+1}, L^2(G))^G$. We start with the G-invariance:

$$\begin{aligned}(h \cdot M^n(\beta))(g_0, ..., g_n)(g) &= M^n(\beta)(h^{-1}g_0, ..., h^{-1}g_n)(h^{-1}g)\\ &= \beta(g^{-1}g_0, ..., g^{-1}g_n) = M^n(\beta)(g_0, ..., g_n)(g)\end{aligned}$$

Next we have to show that $\|M^n(\beta)|_{B(R)^{n+1}}\|$ is square-integrable for every $R > 0$. This follows from the following computation, based on similar arguments as above:

$$\begin{aligned}\mu(B(R)) \int_{G^{n+1}_{2R}} |\beta(g_0, ..., g_n)|^2 d\mu_{n+1} &= \int_{G^{n+1}_{2R}} \int_{B(R)} |\beta(g_0, ..., g_n)|^2 d\mu_{n+1}\\ &\geqslant \int_{B(R)^{n+1}} \int_G |\beta(g^{-1}g_0, ..., g^{-1}g_n)|^2 d\mu d\mu_{n+1}\\ &= \int_{B(R)^{n+1}} \|M^n(\beta)\|^2 d\mu_{n+1}.\end{aligned}$$

It is clear that M^* is a chain map; continuity follows from the previous computation. An easy computation shows that M^* and E^* are mutually inverses:

$$\begin{aligned}E^n(M^n(\beta))(x_0, ..., x_n) &= M^n(\beta)(xx_0, ..., xx_n)(x) = \beta(x_0, ..., x_n);\\ M^n(E^n(\alpha))(g_0, ..., g_n)(g) &= E^n(\alpha)(g^{-1}g_0, ..., g^{-1}g_n) = \alpha(xg^{-1}g_0, ..., xg^{-1}g_n)(x)\\ &= \alpha(xg_0, ..., xg_n)(xg) = \alpha(g_0, ..., g_n)(g).\end{aligned}$$

Note that all the equalities only hold μ-almost everywhere. □

In order to compare the cohomology of coarsely equivalent locally compact groups we need the following lemma:

Lemma 1.3.4. *Coarsely equivalent locally compact second countable groups are measurably coarsely equivalent, i.e. if G and H are coarsely equivalent, then there are measurable coarsely Lipschitz maps $\varphi\colon G \to H$ and $\psi\colon H \to G$ such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are close to the identity.*

Proof. We choose left-invariant proper continuous metrics d_G and d_H on G and H, respectively. Let $\varphi\colon G \to H$ be a coarsely Lipschitz map with $d_H(\varphi(x), \varphi(x')) \leqslant a(d_G(x, x'))$. We fix a $t > 0$ and pick a countable measurable partition $\mathcal{U}$ of G whose element have diameter $\leqslant t$ and choose $x_U \in I$ for every $U \in \mathcal{U}$. In particular, we can construct such an cover by first covering G with open balls of diameter $\leqslant t$ and then pick a countable subcover $\{B_i\}_{i\in\mathbb{N}}$, which always exist by the Lindelöf property; finally, we define $U_i = B_i \backslash \bigcup U_{j<i}$ inductively.

By setting $\tilde{\varphi}(x) = \varphi(x_U)$ for $x \in U$, we obtain a coarsely Lipschitz map $\tilde{\varphi}\colon G \to H$ which satisfies

$$d_H(\tilde{\varphi}(x), \tilde{\varphi}(x')) = d_H(\varphi(x_U), \varphi(x_{U'})) \leqslant a(d_G(x_U, x_{U'})) \leqslant a(d_G(x, x') + 2t)$$

for $x \in U$ and $x' \in U'$ and which is close to φ with

$$d_H(\tilde{\varphi}(x), \varphi(x)) = a(d_G(x_U, x)) \leqslant a(t).$$

Analogously, we construct from a coarsely Lipschitz map $\psi\colon H \to G$, which is a coarse inverse of φ, a measurable coarsely Lipschitz map $\tilde{\psi}$. It is obvious that $\tilde{\psi}$ is a coarse inverse of $\tilde{\varphi}$, since

$$(\tilde{\psi} \circ \tilde{\varphi}) \sim (\psi \circ \tilde{\varphi}) \sim (\psi \circ \varphi) \sim \mathrm{Id}_G,$$

where $\varphi \sim \tilde{\varphi}$ means that φ and $\tilde{\varphi}$ are close to each other. □

Theorem 1.3.5. *Coarsely equivalent locally compact second countable groups have isomorphic reduced and non-reduced coarse ℓ^2-cohomology groups.*

Proof. Let G and H be locally compact second countable groups with left-Haar measures μ and ν, respectively. Let $\varphi\colon G \to H$ be a coarse equivalence with coarse inverse $\psi\colon H \to G$. Because of Lemma 1.3.4, we can further assume that φ and ψ are measurable. We define a map $\chi\colon G \times G \to \mathbb{R}$ by

$$\chi(x, y) = \frac{\mathbb{1}_{B_r(x)}(y)}{\mu(B_r(e))},$$

where we choose r such that $\mu(B_r(e)) \geqslant 1$. Thus, χ is a measurable function with $\chi(x, y) = \chi(y, x)$ and $\int_G \chi(x, y) d\mu(y) = 1$. We use the following notation:

$$\chi\colon G^{n+1} \times G^{n+1} \to \mathbb{R}, \quad \chi\big((x_0, \ldots, x_n), (y_0, \ldots, y_n)\big) = \chi(x_0, y_0) \cdot \ldots \cdot \chi(x_n, y_n).$$

Analogously, we define $\chi'\colon H^{n+1} \times H^{n+1} \to \mathbb{R}$ with some radius r'. We define the maps $\varphi^*\colon HX^*_{(2)}(H) \to HX^*_{(2)}(G)$ and $\psi^*\colon HX^*_{(2)}(G) \to HX^*_{(2)}(H)$ as follows:

$$(\varphi^n \alpha)(x_0, \ldots, x_n) = \int_{H^{n+1}} \alpha(y_0, \ldots, y_n) \chi'\big((\varphi(x_0), \ldots, \varphi(x_n)), (y_0, \ldots, y_n)\big) d\nu_{n+1}(y_0, \ldots, y_n),$$
$$(\psi^n \beta)(y_0, \ldots, y_n) = \int_{G^{n+1}} \beta(x_0, \ldots, x_n) \chi\big((\psi(y_0), \ldots, \psi(y_n)), (x_0, \ldots, x_n)\big) d\mu_{n+1}(x_0, \ldots, x_n),$$

where $\alpha \in HX^n_{(2)}(H)$ and $\beta \in HX^n_{(2)}(G)$, and where we used x_i and y_i to denote elements of G and H, respectively. The idea of averaging over a function like χ goes back to Pansu; it is necessary in our context, since the maps φ and ψ do not preserve measure classes in general.

First of all, we check that these are well-defined continuous cochain maps. Therefore, let $d_H(\varphi(x), \varphi(x')) \leqslant a(d_G(x, x'))$.

$$\begin{aligned}
\infty > \|\alpha\|^2_{a(R)+r'} &= \int_{H^{n+1}} |\alpha(y_0, ..., y_n)|^2 \mathbb{1}_{H^{n+1}_{a(R)+r'}}(y_0, ..., y_n) d\nu_{n+1} \\
&\geqslant \int_{H^{n+1}} |\alpha(y_0, ..., y_n)|^2 \int_{G^{n+1}_R} \chi'\big((\varphi(x_0), ..., \varphi(x_n)), (y_0, ..., y_n)\big) d\mu_{n+1} d\nu_{n+1} \\
&= \int_{G^{n+1}_R} \int_{H^{n+1}} |\alpha(y_0, ..., y_n)|^2 \chi'\big((\varphi(x_0), ..., \varphi(x_n)), (y_0, ..., y_n)\big) d\nu_{n+1} d\mu_{n+1} \\
&\geqslant \int_{G^{n+1}_R} \Big| \int_{H^{n+1}} \alpha(y_0, ..., y_n) \chi'\big((\varphi(x_0), ..., \varphi(x_n)), (y_0, ..., y_n)\big) d\nu_{n+1} \Big|^2 d\mu_{n+1} \\
&= \int_{G^{n+1}_R} |(\varphi^n \alpha)(x_0, ..., x_n)|^2 d\mu_{n+1} = \|\varphi^* \alpha\|^2_R.
\end{aligned}$$

It is a direct computation that $d^n \circ \alpha^n = \alpha^{n+1} \circ d^n$. It remains to show that there is a cochain homotopy $h\colon CX^*_{(2)}(H) \to CX^{*-1}_{(2)}(H)$ such that $\mathrm{Id} - \psi^* \varphi^* = hd + dh$. We define $h^{n+1}\colon CX^{n+1}_{(2)}(H) \to CX^n_{(2)}(H)$ by

$$\begin{aligned}
&(h^{n+1}_i \alpha)(y_0, ..., y_n) \\
&\qquad = \int_{H^{n+1}} \alpha(\tilde{y_0}, ..., \tilde{y_i}, y_i, ..., y_n) \chi'\big((y_0, ..., y_n), (\tilde{y_0}, ..., \tilde{y_n})\big) d\nu_{n+1}(\tilde{y})
\end{aligned}$$

and set

$$h^{n+1} = \sum_{i=0}^{n} (-1)^i h^{n+1}_i.$$

We show that h^* is well-defined and continuous.

$$\begin{aligned}
\|\alpha\|^2_{R+r'} &= \int_{H^{n+2}} |\alpha(y_0, ..., y_n, y_{n+1})|^2 \mathbb{1}_{H^{n+2}_{R+r'}}(y_0, ..., y_{n+1}) d\nu_{n+2} \\
&\geqslant \int_{H^{n+1}} \int_{H^{n+1}_R} |\alpha(\tilde{y_0}, ..., \tilde{y_n}, y_n)|^2 \chi'\big((y_0, ..., y_n), (\tilde{y_0}, ..., \tilde{y_n})\big) d\nu_{n+1}(y) d\nu_{n+1}(\tilde{y}) \\
&\geqslant \int_{H^{n+1}} \int_{H^{n+1}_R} |\alpha(\tilde{y_0}, ..., \tilde{y_i}, y_i, ..., y_n)|^2 \chi'\big((y_0, ..., y_n), (\tilde{y_0}, ..., \tilde{y_n})\big) d\nu_{n+1}(y) d\nu_{n+1}(\tilde{y}) \\
&\geqslant \int_{H^{n+1}_R} \Big| \int_{H^{n+1}} \alpha(\tilde{y_0}, ..., \tilde{y_i}, y_i, ..., y_n) \chi'\big((y_0, ..., y_n), (\tilde{y_0}, ..., \tilde{y_n})\big) d\nu_{n+1}(\tilde{y}) \Big|^2 d\nu_{n+1} \\
&= \|h^{n+1}_i \alpha\|_R
\end{aligned}$$

Now denote the ith term of the coboundary map by d^n_i, i.e.

$$(d^n_i \alpha)(y_0, ..., y_{n+1}) = \alpha(y_0, ..., \widehat{y_i}, ..., y_{n+1}).$$

It is straightforward to verify that we have the following relations:

$$\begin{aligned} h_n^{n+1} \circ d_{n+1}^n &= \psi^n \circ \varphi^n, \\ h_0^{n+1} \circ d_0^n &= \mathrm{Id}_{CX_{(2)}^n(H)}, \\ h_j^{n+1} \circ d_i^n &= d_i^{n-1} \circ h_{j-1}^n && \text{for } 1 \leqslant j \leqslant n \text{ and } i \leqslant j, \\ h_j^{n+1} \circ d_i^n &= d_{i-1}^{n-1} \circ h_j^n && \text{for } 1 \leqslant i \leqslant n \text{ and } i > j. \end{aligned}$$

Summa summarum, we get $h^{n+1}d^n + d^{n-1}h^n = \mathrm{Id}_{CX_{(2)}^n(H)} - \psi^n \circ \varphi^n$. The same construction applies to $\varphi^* \circ \psi^*$, which completes the proof. □

Theorem 1.3.6. *Let G and H be unimodular locally compact second countable groups. If G and H are coarsely equivalent, then the nth ℓ^2-Betti number of G vanishes if and only if the nth ℓ^2-Betti number of H vanishes.*

Proof. We have the following equivalences:

$$\begin{aligned} \beta_n^{(2)}(G) = 0 &\iff \underline{H}^n(G, L^2(G)) = 0 && \text{(Lemma 1.2.5)} \\ &\iff \underline{HX}_{(2)}^n(G) = 0 && \text{(Theorem 1.3.3)} \\ &\iff \underline{HX}_{(2)}^n(H) = 0 && \text{(Theorem 1.3.5).} \end{aligned}$$

Going the same steps backwards for the group H finishes the proof. □

Remark 1.3.7. Since the Borel subgroup $B < \mathrm{SL}_2(\mathbb{R})$ of upper triangular matrices is cocompact, the solvable Lie groups B and $\mathrm{SL}_2(\mathbb{R})$ are quasi-isometric. So B belongs to the class of amenable hyperbolic locally compact groups. For more details about this class consider [CCMT15].
The group B is not unimodular and thus ℓ^2-Betti numbers are not defined. Nevertheless, one may ask what exactly breaks down in the proof above which can be formulated to a large part without the notion of ℓ^2-Betti numbers. By a result of Delorme [Del77, Corollaire V.3], we have $\underline{H}^n(B, L^2(B)) = 0$. Since Theorem 1.3.5 does not require unimodularity, we have

$$\underline{HX}_{(2)}^1(B) \cong \underline{HX}_{(2)}^1(\mathrm{SL}_2(\mathbb{R})) \neq 0$$

because $\beta_1^{(2)}(\mathrm{SL}_2(\mathbb{R})) \neq 0$. So it is Theorem 1.3.3 that fails for the non-unimodular group B.

Chapter 2.

ℓ^2-Betti Numbers of Random Rooted Simplicial Complexes

Let $\mathcal{SC}_*$ be the space of *isomorphism classes of connected locally finite rooted simplicial complexes* (Definition 2.1.2). Every finite simplicial complex defines a *random rooted simplicial complex* (Definition 2.1.3), *i.e.* a probability measure on $\mathcal{SC}_*$, by choosing uniformly at random a vertex as root. A sequence $(K_n)_n$ of finite simplicial complexes converges *Benjamini-Schramm* (Definition 2.1.6) if their associated random rooted simplicial complexes weakly converge to a probability measure on $\mathcal{SC}_*$.

We will give a definition of ℓ^2-Betti numbers (Definition 2.2.8) for random rooted simplicial complexes. For a random rooted simplicial complex μ_K defined by a finite simplicial complex K, the ℓ^2-Betti numbers coincide with the ordinary Betti numbers $b_p(K)$ of K normalized by the number of vertices, *i.e.* (cf. Example 2.2.9)

$$\beta_p^{(2)}(\mu_K) = \frac{b_p(K)}{|K^{(0)}|}.$$

The main result of this chapter (Theorem 2.3.3) will be, that the ℓ^2-Betti numbers are continuous under weak convergence on the space of *sofic* random rooted simplicial complexes. Together with Proposition 2.1.12 this implies the sofic approximation theorem of Elek and Szabó [ES05].

This chapter is organized as follows. We define random rooted simplicial complexes (Definition 2.1.3) and give various examples in Section 2.1. We also give examples of Benjamini-Schramm convergent sequences of finite simplicial complexes. In particular, we show that every random rooted simplicial complex that arises from a simplicial complex with a cocompact action of a sofic group is a Benjamini-Schramm limit (Proposition 2.1.12). In Section 2.2, we will define ℓ^2-Betti numbers of random rooted simplicial complexes. To this end, we have to define the *simplicial ℓ^2-chain complex* of a random rooted simplicial complex (Definition 2.2.2), a von Neumann algebra and a trace function on it. We finish the section by considering the ℓ^2-Betti numbers of the examples given in Section 2.1. In order to prove our main result (Theorem 2.3.3) in Section 2.3, we show that for a weak convergence sequence of random rooted simplicial complexes the spectral measures of the associated Laplace operators weakly converge. We finish the chapter by a computation of the ℓ^2-Betti numbers of the random rooted simplicial complex defined by Sierpinski's Triangle (Example 2.3.9).

2.1. Random Rooted Simplicial Complexes

Simplicial complexes are one of the main objects we deal with in this work. There are several different definitions in the literature, therefore, and to introduce our notation, we will give a definition which, in particular, can be found in [tD08, 8.1].This definition is sometimes called an *abstract* simplicial complex, since it is a purely combinatorial object.

Definition 2.1.1. A *simplicial complex* (K, V) consists of a non-empty set V of *vertices* and a set K of finite subsets of V. A set $s \in K$ with $n+1$ elements is called an *n-simplex of K*. Further, a simplicial complex fulfils the following axioms:

- $\{v\} \in K$ for every $v \in V$;
- If $s \in K$, then every subset $t \subset s$ is also in K. We call $t \subset s$ a *face* of s.

Since V is contained in K as the single element sets, we usually denote a simplicial complex (K, V) only by K. We will denote the set of n-simplices of K by $K(n)$.
We say a simplicial complex is *n-dimensional* if it contains atleast one n-simplex but no $n+1$-simplices. The *degree* of a vertex $v \in V$ is the number of 1-simplices s with $v \in s$. A simplicial complex (K, V) is *locally finite* if $\deg v < \infty$ for all $v \in V$ and *uniform locally bounded*, or of *bounded degree*, if the *degree* $\deg(K) = \sup_{v \in V} \deg v$ of K is finite.
A *subcomplex* (L, W) of (K, V) consist of a subset $W \subset V$ of vertices and a subset of simplices $L \subset K$ such that (L, W) forms a simplicial complex. The n-skeleton $K^{(n)}$ of (K, V) is the subcomplex $(K^{(n)}, V)$, where $K^{(n)} = \{s \in K \mid \dim s \leqslant n\}$.
We have a metric on the vertices V of a simplicial complex K, called the *graph distance*, given by

$$d(v, w) := \inf\{r \in \mathbb{N} \mid \exists \{v_1, w_1\}, ..., \{v_r, w_r\} \in K(1) : v = v_1,\ w_i = v_{i+1},\ w_r = w\}.$$

A *rooted simplicial complex* is a triple (K, V, x) consisting of a simplicial complex (K, V) and a fixed vertex $x \in V$. Most of the time we will omit V and denote the rooted simplicial complex (K, V, x) just by (K, x). Two rooted simplicial complexes (K, x) and (L, y) are *isomorphic* if there is a simplicial isomorphism $\Phi\colon K \to L$ such that $\Phi(x) = y$.

Definition 2.1.2. We denote by $\mathcal{SC}_*$ the *space of isomorphism classes of locally finite connected rooted simplicial complexes* and write $[K, x]$ for the isomorphism class represented by (K, x).

Let $B_r(K, x)$ the rooted subcomplex of (K, x) spanned by all vertices of graph distance at most r from x. The following defines a metric on $\mathcal{SC}_*$:

$$d([K, x], [L, y]) := \inf_r \left\{ \frac{1}{2^r} \;\middle|\; B_r(K, x) \cong B_r(L, y) \right\}.$$

$\mathcal{SC}_*$ equipped with the topology induced by this metric is a totally disconnected Polish space, where the rooted isomorphism classes of finite simplicial complexes form a countable dense subset. For every finite rooted simplicial complex α and every $r \in \mathbb{N}$ we obtain an open set

$U_r(\alpha)$, called the *r-neighbourhood of* α, given by all isomorphism classes of rooted simplicial complexes $[K, x]$ such that $B_r(K, x) \cong B_r(\alpha)$, thus

$$U_r(\alpha) = \{[K, x] \in \mathcal{SC}_* \mid B_r(K, x) \cong B_r(\alpha)\}.$$

The sets $U_r(\alpha)$ are compact and open, and provide a basis for the topology. In a similar way, we define the *space* $\mathcal{SC}_{**}$ *of isomorphism classes of doubly rooted simplicial complexes*, where an element $[K, x, y]$ consists of a simplicial complex K and an ordered pair of vertices x and y. The metric on $\mathcal{SC}_{**}$ is given by:

$$d([K, x, y], [L, v, w]) := \max\left\{ d([K, x], [L, v])\, ,\, d([K, y], [L, w]) \right\}.$$

Definition 2.1.3. A *random rooted simplicial complex* is an *unimodular* probability measure μ on $\mathcal{SC}_*$, where unimodular means

$$\int_{\mathcal{SC}_*} \sum_{y \in K^{(0)}} f([K, x, y]) d\mu([K, x]) = \int_{\mathcal{SC}_*} \sum_{y \in K^{(0)}} f([K, y, x]) d\mu([K, x])$$

for all Borel functions $f\colon \mathcal{SC}_{**} \to \mathbb{R}_{\geqslant 0}$.

Remark 2.1.4. It was realized by Aldous and Lyons [AL07] that weak limits of probability measures arising from finite graphs share a useful *mass-transport property*, which they called unimodularity. If we consider $f([K, x, y])$ as the mass sent from x to y, then unimodularity says that the outgoing mass is equal the incoming mass of all vertices of the graph. There are two motivations for the name "unimodular". On the one hand, a graph has a unimodular automorphism group if and only if the uniform at random rooted graph is unimodular. On the other hand, unimodularity of an random rooted simplicial complex μ implies the equality of the following two measures on $\mathcal{SC}_{**}$ associated with μ, the left measure μ_L defined by

$$\int_{\mathcal{SC}_{**}} f d\mu_L := \int_{\mathcal{SC}_*} \sum_{y \in K^{(0)}} f([K, x, y]) d\mu([K, x])$$

and the right measure μ_R defined by

$$\int_{\mathcal{SC}_{**}} f d\mu_R := \int_{\mathcal{SC}_*} \sum_{y \in K^{(0)}} f([K, y, x]) d\mu([K, x]).$$

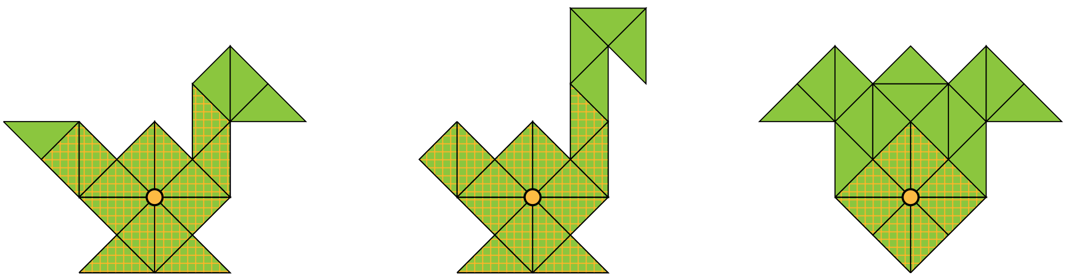


Figure 2.1. The first two rooted simplicial complexes are at distance $\frac{1}{4}$ to each other, where the third one has distance $\frac{1}{2}$ to the others.

A weak limit of random rooted simplicial complexes is again a random rooted simplicial complex. This can be deduced by observing that a probability measure μ on $\mathcal{SC}_*$ is unimodular if and only if for all $n \in \mathbb{N}$ and all Borel functions $f\colon \mathcal{SC}_{**} \to \mathbb{R}_{\geqslant 0}$ the equality

$$\int_{\mathcal{SC}_*} \sum_{\substack{y \in K^{(0)} \\ d(y,x) \leqslant n}} f([K,x,y]) d\mu([K,x]) = \int_{\mathcal{SC}_*} \sum_{\substack{y \in K^{(0)} \\ d(y,x) \leqslant n}} f([K,y,x]) d\mu([K,x])$$

holds, where the sum now runs only over the vertices with distance at most n to the root x. This follows immediately from the monotone convergence theorem. We can further restrict the assumptions from Borel functions to continuous functions by the monotone class theorem.

We define the *vertex degree* of an element $[K,x] \in \mathcal{SC}_*$ as $\deg([K,x]) = \sup\{\deg x \mid x \in K^{(0)}\}$ and the *degree* of a random rooted simplicial complex μ as the μ-essential supremum

$$\deg(\mu) := \operatorname{ess\,sup}_\mu \deg([K,x]).$$

In comparison, the *expected vertex degree* of a random rooted simplicial complex μ is

$$\mathbb{E}_\mu(\deg) = \int_{\mathcal{SC}_*} \deg(x) d\mu([K,x]).$$

Example 2.1.5. (1) Every finite simplicial complex K defines a random rooted simplicial complex

$$\mu_K = \sum_{x \in K^{(0)}} \frac{\delta_{[K_x,x]}}{|K^{(0)}|},$$

with K_x the connected component of $x \in K^{(0)}$ and $\delta_{[K_x,x]}$ the Dirac-measure of the point $[K_x,x]$. That μ_K is a probability measure is obvious, we only have to verify that it is unimodular. For the sake of simplicity let us assume that K is connected:

$$\begin{aligned}
\int_{\mathcal{SC}_*} \sum_{y \in L^{(0)}} f([L,x,y]) d\mu_K([L,x]) &= \sum_{x \in K^{(0)}} \frac{1}{|K^{(0)}|} \sum_{y \in K^{(0)}} f([K,x,y]) \\
&= \sum_{y \in K^{(0)}} \frac{1}{|K^{(0)}|} \sum_{x \in K^{(0)}} f([K,x,y]) \\
&= \int_{\mathcal{SC}_*} \sum_{x \in L^{(0)}} f([L,x,y]) d\mu_K([L,y]) \\
&= \int_{\mathcal{SC}_*} \sum_{y \in L^{(0)}} f([L,y,x]) d\mu_K([L,x])
\end{aligned}$$

Note that μ_K is unique in the sense that there is no other unimodular probability measure fully supported on the rooted isomorphism classes of K.

(2) There is a more general construction, which also applies to infinite complexes. Let K be a connected locally finite simplicial complex with unimodular *simplicial* automorphism group $\mathrm{Aut}(K)$, where simplicial means that we only consider *simplicial* automorphisms, *i.e.* the stabilizer of a simplex is equal the intersection of the stabilizers of its vertices. Further, let $\{x_1, x_2, ...\}$ be a complete orbit section in $K^{(0)}$. There exist an unique unimodular probability measure μ_K, fully supported on the rooted isomorphism classes of K, if

$$c := \sum_i \frac{1}{|\mathrm{Stab}(x_i)|} < \infty$$

holds, where $|\cdot|$ denotes the Haar measure on $\mathrm{Aut}(K)$. The random rooted simplicial complex is given by:

$$\mu_K := \frac{1}{c} \sum_i \frac{\delta_{[K,x_i]}}{|\mathrm{Stab}(x_i)|}.$$

In [AL07, Theorem 3.1], Aldous and Lyons give a proof for the graph case, which also applies to simplicial complexes. They proved even more: The measure μ_G, associated with a graph G, is unimodular if and only if the automorphism group $\mathrm{Aut}(G)$ of G is unimodular. An important special case of this is the following. Let K be connected locally finite simplicial complex with proper and cocompact action of a discrete group $\Gamma \trianglelefteq \mathrm{Aut}(K) =: G$. Let $\mathcal{F}$ be a fundamental domain for the Γ-action and F be a fundamental domain for the G-action in $K^{(0)}$. We remark that F is finite, since $\mathcal{F}$ is finite. Subsequently, we have the following formula for the random rooted simplicial complex:

$$\begin{aligned} \mu_K &:= \frac{1}{c} \sum_{x \in F} \frac{\delta_{[K,x]}}{|\mathrm{Stab}_G(x)|} = \frac{1}{c} \sum_{x \in F} \sum_{\substack{g \in G \\ gx \in \mathcal{F}}} \frac{\delta_{[K,gx]} |\mathrm{Stab}_{G/\Gamma}(\Gamma x)|}{|\mathrm{Stab}_G(x)||G/\Gamma|} \\ &= \frac{1}{c} \sum_{y \in \mathcal{F}} \frac{\delta_{[K,y]}}{|G/\Gamma||\mathrm{Stab}_\Gamma(y)|}. \end{aligned}$$

In a similar, way we obtain that

$$c = \sum_{y \in \mathcal{F}} \frac{1}{|G/\Gamma||\mathrm{Stab}_\Gamma(y)|},$$

hence, with $c(\Gamma) = \sum_{y \in \mathcal{F}} (|\mathrm{Stab}_\Gamma(y)|)^{-1}$, we deduce that

$$\mu_K := \frac{1}{c(\Gamma)} \sum_{y \in \mathcal{F}} \frac{\delta_{[K,y]}}{|\mathrm{Stab}_\Gamma(y)|}. \tag{2.1}$$

This shows that μ_K is independent of the group and the choice of the fundamental domain $\mathcal{F}$. Since this are important examples of random rooted simplicial complexes, we include a proof for the unimodularity of μ_K:

$$\begin{aligned}
&\int_{SC} \sum_{y \in (L)^{(0)}} f([L,x,y]) d\mu_K([L,x]) \\
=&\frac{1}{c(\Gamma)} \sum_{x \in \mathcal{F}^{(0)}} \frac{1}{\mathrm{Stab}_\Gamma(x)} \sum_{y \in K^{(0)}} f([K,x,y]) \\
=&\frac{1}{c(\Gamma)} \sum_{x \in \mathcal{F}^{(0)}} \frac{1}{\mathrm{Stab}_\Gamma(x)} \sum_{y \in \mathcal{F}^{(0)}} \sum_{\gamma \in \Gamma} \frac{1}{\mathrm{Stab}_\Gamma(y)} f([K,x,\gamma y]) \\
=&\frac{1}{c(\Gamma)} \sum_{x,y \in \mathcal{F}^{(0)}} \frac{1}{\mathrm{Stab}_\Gamma(x)} \frac{1}{\mathrm{Stab}_\Gamma(y)} \sum_{\gamma \in \Gamma} f([K,x\gamma^{-1},y]) \\
=&\cdots = \int_{SC} \sum_{y \in (L)^{(0)}} f([L,y,x]) d\mu_K([L,x]).
\end{aligned}$$

Note that if the Γ-action is free, μ_K has the following appearance:

$$\mu_K = \sum_{y \in \mathcal{F}^{(0)}} \frac{\delta_{[K,y]}}{|\mathcal{F}^{(0)}|}.$$

(3) Probability measure preserving equivalence relations $\mathcal{R} \subset X \times X$ on a standard Borel space (X,ν), or more generally discrete measured groupoids $\mathcal{G}$, also give rise to random rooted simplicial complexes. Let Σ be a simplicial $\mathcal{G}$-complex (see Definition 4.2.1), *i.e.* a measurable assignment $X \ni x \mapsto \Sigma_x$ of simplicial complexes together with an $\mathcal{G}$-action: For every $g \in \mathcal{G}$ with $s(g) = x$ and $t(g) = y$ there is a simplicial isomorphism $\psi_g \colon \Sigma_x \to \Sigma_y$ such that $\psi_g \circ \psi_h = \psi_{gh}$ and $\psi_{\mathrm{Id}_x} = \mathrm{Id}_{\Sigma_x}$. These complexes have been introduced by Gaboriau [Gab02] in the case of probability measure preserving equivalence relations and generalized my Takimoto to discrete measured groupoids [Tak15]. Let us assume that there is a Borel fundamental domain F for the action of $\mathcal{G}$ on the vertices with finite measure $c := \int_X |\pi^{-1}(\{x\}) \cap F| d\nu(x) < \infty$, then $(\Sigma, \mathcal{G}, \nu)$ defines a random rooted simplicial complex:

$$\mu_{\Sigma,\mathcal{G}} := \frac{1}{c} \int \sum_{f \in \pi^{-1}(\{x\})} \delta_{[\Sigma_x,f]} d\nu(x).$$

A special case of this is a field of graphs $x \mapsto \Phi_x$ defined by a graphing Φ of $\mathcal{R}$. In this situation, the fundamental domain is just the diagonal $(x,x) \in \mathcal{R}$ and the random rooted graph is given by

$$\mu_{\Phi,\mathcal{R}} := \int \delta_{[\Phi(x),x]} d\nu(x).$$

Random rooted graphs defined by graphings were also treated in [AL07]. For more information about measured equivalence relations see [FM77a, FM77b].

(4) The different models of random simplicial complexes, like the Erdös-Rényi model $G(n,p)$, the random d-complex $Y_d(n,p)$ of Linial and Meshulam [LM06], random flag complexes $X(n,p)$ [Kah09] or random geometric complexes (e.g. the random geometric graph of Penrose [Pen03]), give rise to unimodular measures on $\mathcal{SC}_*$ by applying the construction of Example 2.1.5.1. to a random sample.

Definition 2.1.6. We say a sequence $(K_n)_n$ of simplicial complexes is *Benjamini-Schramm convergent* if the weak limit $\lim_{n\to\infty}\mu_{K_n}$ of the associated random rooted simplicial complexes $(\mu_{K_n})_n$ exists. Moreover, we call a random rooted simplicial complex *sofic* if it is a weak limit of a Benjamini-Schramm convergent sequence of finite simplicial complexes.

Maybe the most prominent example of a Benjamini-Schramm convergent sequence is the following:

Example 2.1.7 (Towers of finite sheeted covering spaces). Let K be a simplicial complex of vertex degree bounded by some $D \in \mathbb{N}$ together with a simplicial, proper and cocompact action of a discrete group Γ. Suppose that Γ is residually finite and let $(N_n)_n$ be a descending sequence of finite index normal subgroups of Γ with $\bigcap_{n\in\mathbb{N}} N_n = \{1\}$, then the sequence $(K_n)_n := (K/N_n)_n$ converges Benjamini-Schramm to

$$\mu_K = \frac{1}{c(\Gamma)}\sum_{x\in\mathcal{F}^{(0)}}\frac{\delta_{[K,x]}}{|\operatorname{Stab}_\Gamma(x)|},$$

where $\mathcal{F}$ is a fundamental domain for the Γ-action. This can be deduced in the following way: We fix an $r>0$. The set S of elements $\gamma\in\Gamma$ such that

$$B_r(K,x)\cap\gamma B_r(K,x)\neq\varnothing \tag{2.2}$$

for some $x\in\mathcal{F}^{(0)}$ is finite, since K is locally finite. Therefore, $B_r(K,x)^{(0)}$ is finite and $\mathcal{F}^{(0)}$ is finite. We take an $n\in\mathbb{N}$ so large that $N_n\cap S=\{1\}$, hence we know if Equation (2.2) holds for an $\gamma\in N_n$ then $\gamma=1$. Hence, the quotient map $K\to K/N_n$ takes $B_r(K,x)$ injectively to $B_r(K/N_n, xN_n)$. We show that every simplex $s\in K_n$ has a unique lift in $B_r(K,x)$. Let $\tilde{s}$ be a lift of s with atleast one vertex in $B_r(K,x)$, thus $\tilde{s}$ lies completely in $B_{r+1}(K,x)$. Let y be an arbitrary vertex of $\tilde{s}$, then there must be an $\gamma\in N_n$ such that $d(\gamma y,x)\leqslant r$ in K. Therefore,

$$\gamma B_{r+1}(K,x)\cap B_{r+1}(K,x)\neq\varnothing$$

which yields that $\gamma=1$ and $\tilde{s}$ lies already in $B_r(K,x)$. This shows that for sufficiently large $n\in\mathbb{N}$ the r-ball $B_r(K,x)$ is isomorphic to the r-ball $B_r(K_n,xN_n)$, for all $x\in K^{(0)}$.
Now, choose n large enough such that $S\cap N_n=\{1\}$ and, additionally, that the action of N_n on K is free. Let $\mathcal{F}(N_n)$ be a fundamental domain for this action. Hence, we have for every $\alpha\in\mathcal{SC}_*$:

$$\begin{aligned}\mu_K(U_r(\alpha)) &= \frac{|\{x\in\mathcal{F}(N_n)^{(0)}\mid B_r(K,x)\cong\alpha\}|}{|\mathcal{F}(N_n)^{(0)}|} && \text{see Example 2.1.5.2.}\\ &= \frac{|\{x\in K_n^{(0)}\mid B_r(K_n,xN_n)\cong\alpha\}|}{|K_n^{(0)}|}\\ &= \mu_{K_n}(U_r(\alpha)).\end{aligned}$$

Example 2.1.8 (Sierpinski's triangle)**.** The following example is from a joint work with Kionke [KSB]. We describe a sequence $(T_n)_n$ of two-dimensional simplicial complexes which occur in the construction of the fractal Sierpinski triangle. It appeared to us that the example becomes clearer if we describe the geometric realizations of the T_n as subsets of $\mathbb{R}^2$ instead of working with the abstract simplicial complexes. Let $e_1 = (1, 0) \in \mathbb{R}^2$ and let $e_2 = \frac{1}{2}(1, \sqrt{3}) \in \mathbb{R}^2$. The points 0, e_1 and e_2 are the vertices of an equilateral triangle T_0 with sides of length 1; we consider T_0 to be a 2-simplex. We define inductively

$$T_{n+1} = T_n \cup (T_n + 2^n e_1) \cup (T_n + 2^n e_2).$$

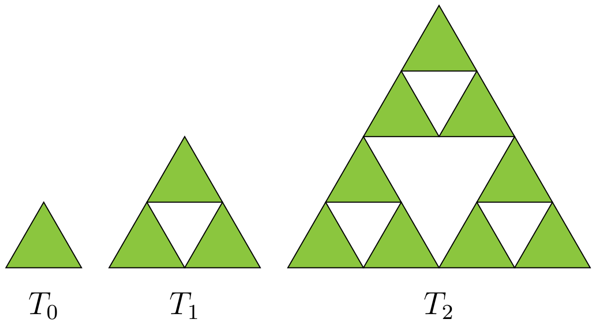


Figure 2.2.

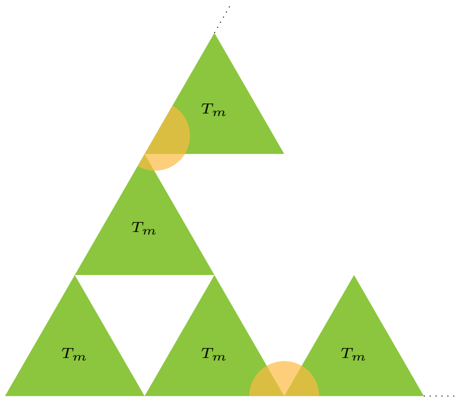
By induction it is easy to verify that T_n is a simplicial complex with $\frac{3^{n+1}+3}{2}$ vertices, 3^{n+1} edges and 3^n 2-simplices. The vertex degree of T_n is 4 for all $n \geqslant 1$. The three vertices of degree 2 will be called the *corners* of T_n. The distance between two corners of T_n is 2^n.

Claim: The sequence $(T_n)_n$ converges Benjamini-Schramm to a random rooted simplicial complex μ_τ.

Let $r > 0$ and let α be a finite rooted simplicial complex of radius at most r. Take m so large that $2^{m-1} > r$, then any r-ball in T_m contains at most one of the three corners of T_m. Let $N(k, \alpha)$ denote the number of vertices v in T_{m+k} such that the ball of radius r around v is isomorphic to α. We observe that

$$N(k + 1, \alpha) = 3N(k, \alpha) + c_\alpha,$$

for all $k \geqslant 0$, for some constant $c_\alpha \in \mathbb{Z}$. Indeed, r-balls around vertices of distance at least r from one of the corners in T_{m+k} occur exactly 3-times in T_{m+k+1}. In the small set of vertices which lie close to a corner of T_{m+k} we always see two copies of T_m being glued together at a corner. This shows that the effect of the operation does not depend on k; compare Figure 2.3.



T_{m+k}

Figure 2.3. Small balls touch at most two copies of T_m

Now it follows from a short calculation that $\left(|N(k,\alpha)| \middle/ |T^{(0)}_{m+k}|\right)_k$ is a Cauchy sequence. Since r and α were arbitrary, we conclude that the sequence $(T_n)_n$ converges in the sense of Benjamini-Schramm.

2.1.1. Sofic Approximation

In the following we will relate sofic groups to sofic random rooted simplicial complexes, but first, we will describe a procedure for assembling a collection of simplicial complexes together into a larger simplicial complex. This construction is inspired by graph of groups (cf. [Hat02, p.91]).

Definition 2.1.9. A *graph of simplicial complexes* $(G, \{K_v\}_v, \{\phi_e\}_e)$ consist of the following data:

(1) A locally finite, connected and simple (*i.e.* no loops or multiple edges) graph G;

(2) a finite simplicial complex K_v for every vertex $v \in G^{(0)}$ and

(3) a bijection $\phi_e \colon K^{(0)}_{v_1} \supset V_e \to U_e \subset K^{(0)}_{v_2}$, for every edge $e = \{v_1, v_2\}$ of G, between subsets V_e, U_e of the vertices of K_{v_1} and K_{v_2}, respectively; the ϕ_e are called *attaching maps*.

A graph of simplicial complexes determines a simplicial complex

$$K\big(G, \{K_v\}_v, \{\phi_e\}_e\big) := \bigsqcup_{v \in G^{(0)}} K_v / \sim,$$

where $(v_1, k) \sim (v_2, k')$ for $k \in K^{(0)}_{v_1}$ and $k' \in K^{(0)}_{v_2}$, if $\{v_1, v_2\} \in G^{(1)}$ and $\phi_{\{v_1,v_2\}}(k) = k'$. Note that by identifying vertices of K_{v_1} and K_{v_2} also the simplices spanned by these vertices get identified, even though the maps ϕ_e do not have to be bijections for the higher dimensional simplices. Consequently, it does not make a difference for the simplicial complex $K\big(G, \{K_v\}_v, \{\phi_e\}_e\big)$ if there is a simplex $\{x_0, \ldots, x_n\}$ in K_{v_1} and a simplex $\{\phi_e(x_0), \ldots, \phi_e(x_0)\}$

in K_{v_2} or the simplex is only contained in either K_{v_1} or K_{v_2} as long as all the vertices $x_0, \ldots, x_n$ are contained in the support of ϕ_e. On the other hand, it is also possible that a simplex of one of the K_v's vanishes if there is a cycle in the graph and two or more vertices of the simplex get identified by the attaching maps along the cycle.

For our purposes, the following is the most important example of a simplicial complex arising from a graph of simplicial complexes.

Example 2.1.10. Let Γ be a finitely generated discrete group with symmetric generating set S and let $G = \mathrm{Cay}(\Gamma; S)$ be the Cayley graph of Γ with respect to this generating set. Further, let K be a locally finite simplicial complex with a free, simplicial and cocompact action of Γ and denote by $\mathcal{F}$ a finite fundamental domain for this action. We can turn $\mathcal{F}$ into a simplicial complex by adding some lower dimensional simplices of K; let us denote this complex by $\overline{\mathcal{F}}$. To make things easier, we denote the vertices of G by corresponding elements $\gamma \in \Gamma$ of the group and set $K_\gamma = \overline{\mathcal{F}}$ for every vertex γ in G. Further, let $E_s = \overline{\mathcal{F}}^{(0)} \cap s\overline{\mathcal{F}}^{(0)}$ and $F_s = s^{-1}E_s \subset \overline{\mathcal{F}}^{(0)}$, both considered as subsets of $\overline{\mathcal{F}}$. We define a bijection

$$\phi_s \colon K_\gamma \supset E_s \to F_s \subset K_{s\gamma}, \quad x \mapsto s^{-1}x,$$

for every $s \in S$, and associate it with all edges labelled by s. In other words, we identify the vertices of the complex $\overline{\mathcal{F}}$ at the vertex γ of G with the corresponding, by s translated, vertices of $\overline{\mathcal{F}}$ at the vertex $s\gamma$ of G. Compare Figure 2.4. Obviously, we have that

$$K\big(\mathrm{Cay}(\Gamma; S), \{\overline{\mathcal{F}}\}_\gamma, \{\phi_s\}_s\big) = K,$$

since we attach the copies of the fundamental domain $\mathcal{F}$ according to the action of Γ.

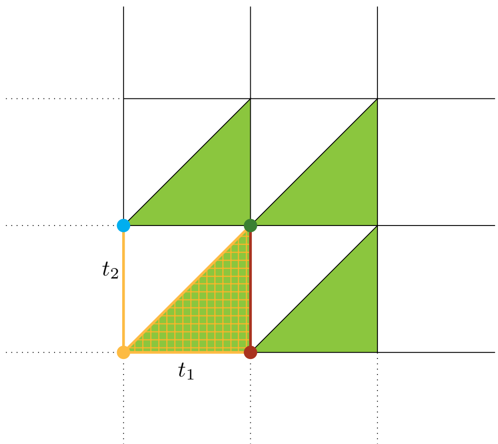


Figure 2.4. The group $\mathbb{Z}^2$ acts on this simplicial complex with fundamental domain given by the yellow coloured area; note there is only one vertex $\{\bullet\}$. To turn this into a simplicial complex we have to add the vertices $\{\bullet\}$, $\{\bullet\}$,$\{\bullet\}$ and the red edge on the right-hand side of the fundamental domain. We consider the grid as the Cayley graph of $\mathbb{Z}^2$ with edges labelled by t_1 and t_2 as indicated in the figure. The attaching map ϕ_{t_1}, which is associated with all edges labelled by t_1, is given by $\{\bullet\} \mapsto \{\bullet\}$ and $\{\bullet\} \mapsto \{\bullet\}$.

Figure 2.4. (Previous page.) We remark that, by identifying these vertices, also the edges $\{\bullet, \bullet\}$ and $\{\bullet, \bullet\}$ get identified. Further, note that there is no edge between $\{\bullet\}$ and $\{\bullet\}$ in the closed fundamental domain, but, since these vertices get identified by ϕ_{t_2} with $\{\bullet\}$ and $\{\bullet\}$, respectively, there is an edge between them in $K\big(\mathrm{Cay}(\mathbb{Z}^2, \{t_1, t_2\}), \{\overline{\mathcal{F}}\}_\gamma, \{\phi_{t_i}\}_i\big)$.

Definition 2.1.11. [Wei00, Definition 2.1] A finitely generated group Γ is called *sofic* if for some finite symmetric set of generators S, and any $\epsilon > 0$, and $r \in \mathbb{N}$, there is a finite directed graph G edge labelled by S, which has a finite subset of its vertices $V_0 \subset G^{(0)}$ satisfying:

(1) for each $v \in V_0$, the r-ball $B_r(G, v)$ is graph isomorphic as a labelled graph to the r-ball $B_r(\mathrm{Cay}(\Gamma, S), 1)$ in the Cayley graph of Γ and

(2) $|V_0| \geqslant (1 - \epsilon)|G^{(0)}|$.

Proposition 2.1.12. *Let Γ be a finitely generated sofic group. If Γ acts freely and cocompactly on a locally finite connected simplicial complex K, then there exist a sequence $(K_n)_n$ of finite connected simplicial complexes which converges Benjamini-Schramm to μ_K.*

Proof. Let $G = \mathrm{Cay}(\Gamma; S)$ be the Cayley graph of Γ for a symmetric generating set S and let $(G_n)_n$ be a sequence of S-labelled graphs such that there is a finite subset of vertices $V_n \subset G_n^{(0)}$ satisfying:

(1) for each $v \in V_n$ the n-ball $B_n(G_n, v)$ is graph isomorphic as a labelled graph to the n-ball $B_n(G, 1)$ and

(2) $|V_n| \geqslant (1 - \frac{1}{n})|G_n^{(0)}|$.

By Example 2.1.10, we know that K can be considered as the graph of simplicial complexes $K\big(G, \{\overline{\mathcal{F}}\}_\gamma, \{\phi_s\}_s\big)$ for some fundamental domain $\mathcal{F} \subset K$ of the Γ-action. We will show that the sequence

$$K_n := K\big(G_n, \{\overline{\mathcal{F}}\}_{v \in G_n^{(0)}}, \{\phi_s\}_s\big)$$

of simplicial complexes converges Benjamini-Schramm to $K = K\big(G, \{\overline{\mathcal{F}}\}_{\gamma\in\Gamma}, \{\phi_s\}_s\big)$, where $\big(G_n, \{\overline{\mathcal{F}}\}_{v \in G_n^{(0)}}, \{\phi_s\}_s\big)$ is the graph of simplicial complexes with the simplicial complex $\overline{\mathcal{F}}$ at every vertex and with the attaching maps $\{\phi_s\}_s$ given by the same maps as in K according to the labels of the edges.

Fix a radius $r > 0$ and choose $n \in \mathbb{N}$ so large that for at least $(1 - \epsilon/|\overline{\mathcal{F}}^{(0)}|)|G_n^{(0)}|$ many vertices in G_n the r-ball looks like the r-ball in G. Let us denote the set of these vertices by V_n. In the worst case, we only have one vertex in K_n for every vertex in V_n and $|\overline{\mathcal{F}}^{(0)}| - 1$ many vertices for every vertex in $G_n^{(0)} \backslash V_n$. Hence, since the attaching maps in K_n are the same as in K, we conclude that

$$|\mu_{K_n}(U_r(\alpha)) - \mu_K(U_r(\alpha))| < \epsilon$$

for every finite rooted simplicial complexes α. $\square$

2.2. ℓ^2-Betti Numbers of Random Rooted Simplicial Complexes

In order to define ℓ^2-Betti numbers, we introduce the ℓ^2-homology of random rooted simplicial complexes. Therefore, we will construct a chain complex for each probability measure on $\mathcal{SC}_*$. We begin by picking a representative for each isomorphism class $[K, x] \in \mathcal{SC}_*$ in a measurable way.

Let $\Delta(\mathbb{N}_0)$ be the simplicial complex consisting of all non-empty finite subsets of $\mathbb{N}_0$. Every subcomplex $\Lambda \subset \Delta(\mathbb{N}_0)$ can be encoded by an element $f_\Lambda \in \{0,1\}^{\Delta(\mathbb{N}_0)}$, such that $f_\Lambda(s) = 1$ if and only if the simplex s is contained in the subcomplex Λ. We endow $\{0,1\}^{\Delta(\mathbb{N}_0)}$ with the product topology, *i.e.* the topology generated by all cylinder sets. The subset $\mathrm{Sub}(\Delta(\mathbb{N}_0)) \subset \{0,1\}^{\Delta(\mathbb{N}_0)}$ which consists of elements encoding subcomplexes of $\Delta(\mathbb{N}_0)$ is closed.

Lemma 2.2.1. *There is a continuous map* $\Psi\colon \mathcal{SC}_* \to \mathrm{Sub}(\Delta(\mathbb{N}_0))$ *such that* $(\Psi([K,x]), 0)$ *is a representative for* $[K, x]$ *for all* $[K, x] \in \mathcal{SC}_*$, *i.e.*

$$[\Psi([K,x]), 0] = [K, x].$$

Proof. We enumerate the simplices of $\Delta(\mathbb{N}_0)$ in diagonal way,

$$(\{0\}, \{1\}, \{0,1\}, \{2\}, \{0,2\}, \{1,2\}, \{0,1,2\}, \{3\}, \ldots$$

and by this, identify the simplices of $\Delta(\mathbb{N}_0)$ with $\mathbb{N}_0$. We denote this map by $\Upsilon\colon \mathbb{N}_0 \to \Delta(\mathbb{N}_0)$. Subsequently, all simplices with vertices in $\{0, \ldots, n\}$ show up in the first 2^{n+1} entries of $\{0,1\}^{\Delta(\mathbb{N}_0)}$. The lexicographic order on $\{0,1\}^{\Delta(\mathbb{N}_0)}$, given by

$$f \prec g :\Leftrightarrow f(\Upsilon(i)) = g(\Upsilon(i)) \text{ for } 0 \leqslant i \leqslant k, \quad f(\Upsilon(k+1)) = 1 \text{ and } g(\Upsilon(k+1)) = 0,$$

defines an order on the subcomplexes of $\Delta(\mathbb{N}_0)$. Now, we define $\Psi([K,x])$ to be the smallest $\Lambda \in \mathrm{Sub}(\Delta(\mathbb{N}_0))$ with $(\Lambda, 0) \in [K, x]$. We have to remark two things: First, there is always a subcomplex Λ of $\Delta(\mathbb{N}_0)$ such that $(\Lambda, 0) \cong (K, x)$. We obtain such a Λ by enumerating the vertices of K in an arbitrary way with the only restriction that x is identified with 0. And second, for a rooted isomorphism class $[K, x]$ of a finite simplicial complex K it follows by the well-ordering principle that there is a unique smallest element in $\{0,1\}^{\Delta(\mathbb{N}_0)}$. We will show that for an infinite complexes $[K, x]$ there is also a minimal element. To this end, we claim that $\Psi([B_r(K,x)]) = B_r(\Psi([B_{r+t}(K,x)]))$ for all $r, t \geqslant 0$, hence the minimal element is given by

$$\Psi([K,x]) = \varinjlim \Psi([B_r(K,x), x]).$$

First, we verify that the set of vertices of $\Psi([B_r(K,x)])$ and also of $B_r(\Psi([B_{r+t}(K,x)]))$ is the interval $\{0, 1, \ldots, N_r\} \subset \mathbb{N}_0$, with $N_r = |B_r(K,x)^{(0)}|$. For $\Psi([B_r(K,x)])$ this is obvious by the definition of the order. For $B_r(\Psi([B_{r+t}(K,x)]))$ we show this by induction.

Suppose that $B_{r-1}(\Psi([B_{r+t}(K,x)]))^{(0)} = \{0, 1, \ldots, N_{r-1}\}$ and assume there exists an $k \in \{N_{r-1}+1, \ldots, N_r\}$ such that $k \notin B_r(\Psi([B_{r+t}(K,x)]))^{(0)}$. Therefore, there must be a vertex $l \in B_r(\Psi([B_{r+t}(K,x)]))^{(0)}$ with $l > N_r$. By assumption, the distance of k to the root

0 in $\Psi([B_{r+t}(K,x)])$ must be at least $r+1$, hence all simplices of the form $s \cup \{k\}$ with $s \subset \{0,1,...,N_{r-1}\}$ can not be contained in $\Psi([B_{r+t}(K,x)])$. But this is a contradiction to the minimality of $\Psi([B_{r+t}(K,x)])$, since by interchanging the roles of l and k we would get a smaller complex. After we verified that $\Psi([B_r(K,x)])$ and $B_r(\Psi([B_{r+t}(K,x)]))$ have the same set of vertices, it is a direct consequence of the definition of the order that

$$\Psi([B_r(K,x)]) = B_r(\Psi([B_{r+t}(K,x)])).$$

We check that Ψ is continuous. Let $\{k_0,k_1,...,k_n\} = \Upsilon(k) \in \Delta(\mathbb{N}_0)$ with $k_0 < k_1 < ... < k_n$ be the simplex which corresponds to the natural number $k \in \mathbb{N}_0$. From the previous part of the proof we can deduce that the simplex $\Upsilon(k) \in \Delta(\mathbb{N}_0)$ can have at most distance k_0 to the root in every minimal subcomplex of $\Delta(\mathbb{N}_0)$ which contains $\Upsilon(k)$. Let $[k]$ be the cylinder set defined by k. Since $\Upsilon(k) \in \Psi([K,x])$ if and only if it is in $\Psi(B_{k_0+1}(K,x))$, it follows that the preimage of $[k]$ is give by the countable union of k_0+1-neighbourhoods $U_{k_0+1}(\alpha)$ of finite rooted simplicial complexes α with $\Upsilon(k) \in \Psi(B_{k_0+1}(\alpha))$, *i.e.*

$$\Psi^{-1}([k]) = \bigcup_{\substack{\alpha\in\mathcal{SC}_* \text{ finite} \\ \Upsilon(k)\in\Psi(B_{k_0+1}(\alpha))}} U_{k_0+1}(\alpha).$$

This is a countable union of open sets, thus Ψ is continuous. □

For a simplicial complex K the complex Hilbert space of simplicial ℓ^2-p-chains $C_p^{(2)}(K)$ consists of formal sums $\sum_{s\in K(p)} c_s s$ with coefficients $c_s \in \mathbb{C}$ and such that $\sum_{s\in K(p)} |s|^2 < \infty$. The map Ψ, from the preceding lemma, gives rise to a field of complex Hilbert spaces (see Definition B.1) $[K,x] \to C_p^{(2)}(\Psi([K,x]))$ over $\mathcal{SC}_*$, for every $p \in \mathbb{N}_0$. In addition, every oriented p-simplex $s \in \Delta(\mathbb{N}_0)$ yields a *characteristic vector field* χ_s defined by

$$\chi_s([K,x]) = \begin{cases} s & \text{if } s \in \Psi([K,x]) \\ 0 & \text{otherwise.} \end{cases}$$

We observe that, since Ψ is continuous, the function

$$[K,x] \mapsto \langle \chi_s([K,x]), \chi_{s'}([K,x])\rangle$$

is continuous for all oriented p-simplices s and s'. Further, the collection of all characteristic vector fields $\{\chi_s \mid s \in \Delta(\mathbb{N}_0)(p)\}$ of oriented p-simplices $s \in \Delta(\mathbb{N}_0)$ form a total sequence for every complex Hilbert space $C_p^{(2)}(\Psi([K,x]))$. Therefore, by Proposition B.4, there exists a unique measurable field structure on $[K,x] \to C_p^{(2)}(\Psi([K,x])$ such that the characteristic vector fields are measurable and hence form a fundamental sequence. By Proposition B.3, a vector field $\sigma : [K,x] \to \sigma([K,x]) \in C_p^{(2)}(\Psi([K,x]))$ is measurable, with respect to this structure, if

$$[K,x] \mapsto \langle \sigma([K,x]), \chi_s([K,x])\rangle$$

is measurable for every oriented p-simplex s of $\Delta(\mathbb{N}_0)$. Now, let μ be a random rooted simplicial complex.

Definition 2.2.2. The Hilbert space of *simplicial ℓ^2-p-chains* of μ is the direct integral

$$C_p^{(2)}(\mathcal{SC}_*, \mu) := \int_{\mathcal{SC}_*}^{\oplus} C_p^{(2)}(\Psi([K,x]))d\mu([K,x]).$$

The elements of the direct integral (see Definition B.5) are (equivalence classes of) measurable vector fields σ with the property

$$\|\sigma\|^2 = \int_{\mathcal{SC}_*} \|\sigma([K,x])\|^2 d\mu([K,x]) < \infty,$$

where the vector fields that vanish almost every will be identified. The boundary operators

$$\partial_{p,[K,x]} \colon C_p^{(2)}(\Psi([K,x])) \to C_{p-1}^{(2)}(\Psi([K,x]))$$

and their adjoints

$$d^p_{[K,x]} \colon C_{p-1}^{(2)}(\Psi([K,x])) \to C_p^{(2)}(\Psi([K,x]))$$

define measurable fields of linear mappings $\partial_p \colon [K,x] \to \partial_{p,[K,x]}$ and $d^p \colon [K,x] \to d^p_{[K,x]}$.

By Proposition B.9, it is enough to check that

$$[K,x] \mapsto \langle \partial_{p,[K,x]}\chi_s([K,x]), \chi_{s'}([K,x])\rangle$$

is measurable for every p-simplex s and p-1-simplex s' in $\Delta(\mathbb{N}_0)$, but this is a direct consequence of the continuity of Ψ. Therefore, also the Laplace operators

$$\Delta_{p,[K,x]} = d^p_{[K,x]}\partial_{p,[K,x]} + \partial_{p+1,[K,x]}d^{p+1}_{[K,x]}$$

yield a measurable field of operators. By Proposition B.13, $\partial_{p,[K,x]}$ defines a closed linear mapping

$$\partial_p = \int_{\mathcal{SC}_*}^{\oplus} \partial_{p,[K,x]}d\mu \colon C_p^{(2)}(\mathcal{SC}_*, \mu) \to C_{p-1}^{(2)}(\mathcal{SC}_*, \mu),$$

and we obtain a chain complex

$$\cdots \to C_p^{(2)}(\mathcal{SC}_*, \mu) \xrightarrow{\partial_p} C_{p-1}^{(2)}(\mathcal{SC}_*, \mu) \to \cdots .$$

Definition 2.2.3. Let μ be a random rooted simplicial complex. We define the p-th *simplicial ℓ^2-homology of* μ as the Hilbert space

$$H_p^{(2)}(\mathcal{SC}_*, \mu) := \ker \partial_p / \overline{\operatorname{im} \partial_{p+1}}.$$

Later, we will usually consider the kernel of the Laplace operator, which is (atleast in the case when Δ_p is a bounded operator on $C_p^{(2)}(\mathcal{SC}_*, \mu)$) isomorphic to the homology by the Hodge isomorphism (cf. [Eck00, p. 192])

$$H_p^{(2)}(\mathcal{SC}_*, \mu) \cong \ker(\Delta_p) = \int_{\mathcal{SC}_*}^{\oplus} \ker \Delta^p_{[K,x]} d\mu.$$

This has the advantage that $\ker(\Delta_p)$ is a subspace of $C_p^{(2)}(\mathcal{SC}_*, \mu)$ and not a quotient space.

Proposition 2.2.4. *For every random rooted simplicial complex μ of degree bonded by D the boundary operator ∂_p, its adjoint d^p and the Laplace operator Δ_p are bounded operators for every $p \in \mathbb{N}$. Moreover, Δ_p is self-adjoint.*

Proof. To show that the operators are bounded, it is enough to proof that $\|\partial_p\| \leqslant R(D,p)$, for some constant $R(D,p)$ only depending on the dimension p and the degree D of μ, since $\|d^p\| = \|\partial_p^*\| = \|\partial_p\|$. Let us first consider a simplicial complex K with vertex degree bounded by D and let s be a p-simplex; $p \leqslant D$. Hence, every vertex v of s has already p neighbours. If s is contained in a $p+1$ simplex, then this simplex consist of the vertices of s plus one more vertex v_0. Therefore, v_0 is also a neighbour of all vertices in s and hence the number of $p+1$-simplices containing s is at most $D-p$. This gives us the following estimate of the norm of the boundary operator:

$$\|\partial_p(\sigma)\|^2 = \Big\|\sum_{i=0}^{p}(-1)^i\partial_p^i\sigma\Big\|^2 \leqslant (p+1)(D-p+1) =: R(D,p).$$

Now, since the essential supremum of the vertex degree is D, the essential supremum of $\|\partial_{p,[K,x]}\|$ is bounded by $R(D,p)$. By Proposition B.10, it follows that also

$$\Big\|\int^{\oplus}\partial_{p,[K,x]}d\mu([K,x])\Big\|^2 \leqslant R(D,p).$$

That Δ_p is self-adjoint, is a direct consequence of Proposition B.14, since for a simplicial complex of bounded degree the Laplace operator is self-adjoint. □

We would like to have a notion for the dimension of a subspace of $C_p^{(2)}(\mathcal{SC}_*,\mu)$, to this end we introduce a von Neumann algebra with a trace. Let $\mathcal{A}_p(\mu)$ be the von Neumann algebra of bounded decomposable operators $T = \int^{\oplus} T_{[K,x]}d\mu$ on $C_p^{(2)}(\mathcal{SC}_*,\mu)$ such that for almost all $[K,x] \in \mathcal{SC}_*$, all simplicial isomorphisms $\varphi\colon \Psi([K,x]) \to \Psi([L,y])$ and all $\sigma,\sigma' \in C_p^{(2)}(\mathcal{SC}_*,\mu)$ the identity

$$\langle T_{[K,x]}\sigma([K,x]),\sigma'([K,x])\rangle = \langle T_{[L,y]}\varphi_\sharp\sigma([K,x]),\varphi_\sharp\sigma'([K,x])\rangle$$

holds. It is clear that this condition is closed under the weak topology. Further, it implies that the operators are independent of the choice of the root, thus we will sometimes denote an operator by T_K instead of $T_{[K,x]}$. If ∂_p is a bounded operator for μ, then ∂_p, d^p and Δ_p are in $\mathcal{A}_p(\mu)$, since ∂_p and d^p commute with the chain map $\varphi_\sharp$ induced by an isomorphism $\varphi\colon \Psi([K,x]) \to \Psi([L,y])$. Now let

$$\mathcal{T}_p = \big\{\chi_s \in C_p^{(2)}(\mathcal{SC}_*,\mu) \mid s \in \Delta(\mathbb{N}_0)(p),\ 0 \in s\big\}$$

the characteristic vector fields of the oriented p-simplices of $\Delta(\mathbb{N}_0)$ which contain 0. We claim that

$$\operatorname{tr}_\mu(T) = \sum_{\tau\in\mathcal{T}_p}\frac{\langle T\tau,\tau\rangle}{p+1} \in [0,\infty]$$

defines a trace on $\mathcal{A}_p(\mu)$. Note that the formula does not depend on the orientation of τ. Let $T, S \in \mathcal{A}_p(\mu)$.

$$\sum_{\tau\in\mathcal{T}_p}\langle ST\tau,\tau\rangle = \sum_{\tau\in\mathcal{T}_p}\langle T\tau, S^*\tau\rangle = \int_{\mathcal{SC}_*}\sum_{\tau\in\mathcal{T}_p}\langle T_K\tau([K,x]), S_K^*\tau([K,x])\rangle d\mu([K,x])$$
$$= \int_{\mathcal{SC}_*}\sum_{s\in\Psi([K,x])(p)}\sum_{\tau\in\mathcal{T}_p}\langle T_K\tau([K,x]), s\rangle\langle s, S_K^*\tau([K,x])\rangle d\mu([K,x])$$
$$= \int_{\mathcal{SC}_*}\sum_{y\in\Psi([K,x])^{(0)}}\underbrace{\sum_{\substack{s\in\Psi([K,x])(p)\\ y\in s}}\sum_{\tau\in\mathcal{T}_p}\frac{\langle T_K\tau([K,x]), s\rangle\langle S_K s, \tau([K,x])\rangle}{p+1}}_{f([K,x,y])} d\mu([K,x]).$$

Now we use the unimodularity of μ (see Definition 2.1.3):

$$= \int_{\mathcal{SC}_*}\sum_{y\in K^{(0)}} f([K,y,x])d\mu([K,x])$$
$$= \int_{\mathcal{SC}_*}\sum_{y\in K^{(0)}}\sum_{\substack{s\in\Psi([K,x])(p)\\ y\in s}}\sum_{\tau\in\mathcal{T}_p}\frac{\langle T_K\tau([K,y]), s\rangle\langle S_K s, \tau([K,y])\rangle}{p+1}d\mu([K,x])$$
$$= \int_{\mathcal{SC}_*}\sum_{y\in K^{(0)}}\sum_{\substack{s\in\Psi([K,x])(p)\\ y\in s}}\sum_{\tau\in\mathcal{T}_p}\frac{\langle S_K s, \tau([K,y])\rangle\langle \tau([K,y]), T_K^* s\rangle}{p+1}d\mu([K,x])$$
$$= \int_{\mathcal{SC}_*}\sum_{\substack{s\in\Psi([K,x])(p)\\ 0\in s}}\langle S_K s, T_K^* s\rangle d\mu([K,x])$$
$$= \int_{\mathcal{SC}_*}\sum_{\tau\in\mathcal{T}}\langle S_K\tau([K,x]), T_K^*\tau([K,x])\rangle d\mu([K,x]) = \sum_{\tau\in\mathcal{T}_p}\langle TS\tau,\tau\rangle.$$

Proposition 2.2.5. *The trace* tr_μ *on* $\mathcal{A}_p(\mu)$ *is normal, faithful and semi-finite. If the expected number of* p*-simplices at the root is finite, then* tr *is a finite trace.*

Proof. Faithfulness follows from the fact that the operators in $\mathcal{A}_p(\mu)$ are independent from the choice of the root and from the unimodularity of μ. We check that tr_μ is semi-finite. Let $A \in \mathcal{A}_p(\mu)$ be positive. We define the operators

$$A_{n,[K,x]} := \frac{A_{[K,x]}}{\max\{\deg_p(x) - n, 1\}},$$

where we denote by $\deg_p(x)$ the number of p-simplices containing x. We will show that $(A_n)_n$ weakly converges to A, but first we verify that $\mathrm{tr}(A_n) < \infty$.

$$\begin{aligned}\mathrm{tr}_\mu(A_n) &= \int_{\mathcal{SC}_*} \sum_{\tau \in \mathcal{T}_p} \frac{\langle A\tau, \tau\rangle}{(p+1)\max\{\deg_p(x) - n, 1\}} d\mu([K,x]) \\ &\leqslant \int_{\mathcal{SC}_*} \frac{\|A\| \deg_p(x)}{(p+1)\max\{\deg_p(x) - n, 1\}} d\mu([K,x]) \\ &\leqslant \frac{\|A\|(n+1)}{p+1}.\end{aligned}$$

To prove the weak convergence, let $\sigma, \theta \in C_p^{(2)}(\mathcal{SC}_*, \mu)$.

$$\begin{aligned}\langle A_n\sigma, \theta\rangle &= \int_{\mathcal{SC}_*} \langle A_n\sigma, \theta\rangle d\mu = \int_{\mathcal{SC}_*} \frac{\langle A\sigma, \theta\rangle}{\max\{\deg_p(x) - n, 1\}} d\mu \\ &= \int_{\substack{[K,x]\in\mathcal{SC}_* \\ \deg_p(x)\leqslant n}} \langle A\sigma, \theta\rangle d\mu + \int_{\substack{[K,x]\in\mathcal{SC}_* \\ \deg_p(x)> n}} \frac{\langle A\sigma, \theta\rangle}{\deg_p(x) - n} d\mu \\ &\xrightarrow{n\to\infty} \int_{\mathcal{SC}_*} \langle A\sigma, \theta\rangle d\mu = \langle A\sigma, \theta\rangle.\end{aligned}$$

In the last step, we used the fact that the elements of $\mathcal{SC}_*$ are locally finite, hence the right summand tends to zero and the left one to $\int_{\mathcal{SC}_*} \langle A\sigma, \theta\rangle d\mu$. □

Definition 2.2.6. Let $\mathcal{K}\colon [K,x] \to \mathcal{K}([K,x])$ be a field of subspaces of $C_p^{(2)}(\mathcal{SC}_*, \mu)$ such that $\varphi_\sharp\mathcal{K}([K,x]) = \mathcal{K}([K,y])$ for every isomorphisms $\varphi\colon \Psi([K,x]) \to \Psi([K,y])$. Then the projection $P_\mathcal{K}\colon [K,x] \to P_{\mathcal{K}([K,x])}$ onto $\mathcal{K}$ is an element of $\mathcal{A}_p(\mu)$ and we define the *von Neumann dimension* of $\mathcal{K}$ as

$$\dim_\mu \mathcal{K} = \mathrm{tr}_\mu(P_\mathcal{K}).$$

We remark that, by the comment after Definition B.11,

$$\int_{\mathcal{SC}_*}^{\oplus} \mathcal{K}([K,x]) d\mu([K,x]) = P_\mathcal{K}(C_p^{(2)}(\mathcal{SC}_*, \mu)).$$

Example 2.2.7. We compute the dimension of $C_p^{(2)}(\mathcal{SC}_*, \mu)$:

$$\begin{aligned}\dim_\mu(C_p^{(2)}(\mathcal{SC}_*, \mu)) = \mathrm{tr}_\mu(\mathrm{Id}_{C_p^{(2)}(\mathcal{SC}_*,\mu)}) &= \sum_{\tau\in\mathcal{T}_p} \frac{\langle \mathrm{Id}\,\tau, \tau\rangle}{p+1} \\ &= \int_{\mathcal{SC}_*} \frac{|\{\tau \in \mathcal{T}_p \mid \tau([K,x]) \neq 0\}|}{p+1} d\mu([K,x]) \\ &= \int_{\mathcal{SC}_*} \frac{\deg_p(x)}{p+1} d\mu,\end{aligned}$$

where, again, we denoted by $\deg_p(x)$ the number of p-simplices containing the vertex x. If μ is the associated random rooted simplicial complex of a finite simplicial complex K, then the von Neumann dimension is equal

$$\frac{|K(p)|}{|K^{(0)}|} = \frac{\dim_{\mathbb{C}} C_p(K;\mathbb{C})}{|K^{(0)}|}.$$

Definition 2.2.8. Let μ be a random rooted simplicial complex. The pth *ℓ^2-Betti number of μ* is

$$\beta_p^{(2)}(\mu) := \dim_\mu \ker \Delta_p,$$

where Δ_p is the pth Laplace operator of the simplicial ℓ^2-chain complex $C_*^{(2)}(\mathcal{SC}_*,\mu)$. By the Hodge isomorphism, we can interpret this as the dimension of the pth homology group $H_p^{(2)}(\mathcal{SC}_*,\mu)$.

Example 2.2.9. In the following, we will consider the ℓ^2-Betti numbers of the random rooted simplicial complexes defined in Example 2.1.5. We denote by $b_p(K)$ the pth ordinary Betti number of a simplicial complex K and by $\beta_p^{(2)}(K,\Gamma)$ the ℓ^2-Betti numbers of a Γ-simplicial complex with respect to the group von Neumann algebra of the discrete group Γ. The standard reference for ℓ^2-invariants is [Lüc02], we also recommend [Kam18]. Gaboriau gives a definition of ℓ^2-Betti numbers of $\mathcal{R}$-simplicial complexes which we denote by $\beta_p(\Sigma,\mathcal{R},\nu)$. This definition was generalized by Takimoto to $\mathcal{G}$-simplicial complexes [Tak15], where $\mathcal{G}$ is a discrete measured groupoid, we denote them by $\beta_p(\Sigma,\mathcal{G},\nu)$. If the $\mathcal{R}$-simplicial complex, respective $\mathcal{G}$-simplicial complex, is p-connected, then the definition is independent of the simplicial complex Σ and hence it is called the pth ℓ^2-Betti number $\beta_p(\mathcal{R},\nu)$ of the equivalence relation $\mathcal{R}$ [Gab02] or $\beta_p(\mathcal{G},\nu)$ of the groupoid $\mathcal{G}$ [Tak15], respectively. The later coincides with the original definition of ℓ^2-Betti numbers of discrete measured groupoids introduced by Sauer [Sau05].

(1) Given a finite simplicial complex L with associated random rooted simplicial complex μ_L and let P be the projection onto the kernel $\ker\Delta_p$ of the pth Laplace operator of $C_p^{(2)}(\mathcal{SC}_*,\mu_L)$:

$$\begin{aligned}
\beta_p^{(2)}(\mu_L) &= \sum_{\tau\in\mathcal{T}_p} \frac{\langle P\tau,\tau\rangle}{p+1} \\
&= \sum_{\substack{[K,x]\in\mathcal{SC}_*\\ K\cong L}} \mu_L([K,x]) \sum_{\substack{s\in\Psi([K,x])(p)\\ 0\in s}} \frac{\langle P_{[K,x]}s,s\rangle}{p+1} \\
&= \sum_{\substack{[K,x]\in\mathcal{SC}_*\\ K\cong L}} \frac{|\{y\in L^{(0)} \mid (L,y)\in[K,x]\}|}{|L^{(0)}|} \sum_{\substack{s\in\Psi([K,x])(p)\\ 0\in s}} \frac{\langle P_{[K,x]}s,s\rangle}{p+1} \\
&= \frac{1}{|L^{(0)}|} \sum_{y\in L^{(0)}} \sum_{\substack{s\in L(p)\\ y\in s}} \frac{\langle P_L s,s\rangle}{p+1}
\end{aligned}$$

$$= \frac{1}{|L^{(0)}|} \sum_{s \in L(p)} \langle P_L s, s \rangle = \frac{b_p(L)}{|L^{(0)}|} = \beta_p^{(2)}(L, \{e\})$$

From line three to four we used the fact that (L, y) and $(\Psi([K, x]), 0)$ are root isomorphic, so

$$\sum_{\substack{s \in \Psi([K,x])(p) \\ 0 \in s}} \frac{\langle Ps, s \rangle}{p+1} = \sum_{\substack{s \in L(p) \\ y \in s}} \frac{\langle Ps, s \rangle}{p+1}.$$

Further, note that if we sum over all p-simplices $s \in L(p)$ containing a fixed vertex y and then sum over all vertices $y \in L^{(0)}$, we hit every p-simplex exactly $p+1$ times. Therefore, we can cancel the denominator.

(2) Let L be an infinite connected locally finite simplicial complex with a free, simplicial and cocompact action of a discrete group Γ with finite fundamental domain F. The same arguments as in the previous example yield:

$$\begin{aligned}
\beta_p^{(2)}(\mu_L) &= \sum_{\tau \in \mathcal{T}_p} \frac{\langle P\tau, \tau \rangle}{p+1} \\
&= \sum_{\substack{[K,x] \in \mathcal{SC}_* \\ K \cong L}} \mu_L([K, x]) \sum_{\substack{s \in \Psi([K,x])(p) \\ 0 \in s}} \frac{\langle P_{[K,x]} s, s \rangle}{p+1} \\
&= \frac{1}{|F^{(0)}|} \sum_{y \in F^{(0)}} \sum_{\substack{s \in L(p) \\ y \in s}} \frac{\langle P_L s, s \rangle}{p+1} \\
&= \frac{1}{|F^{(0)}|} \sum_{s \in F(p)} \langle P_L s, s \rangle = \frac{\beta_p^{(2)}(L, \Gamma)}{|F^{(0)}|}.
\end{aligned}$$

We remark that for a finite index subgroup $\Gamma_0 < \Gamma$ the size of the fundamental domain scales by the same factor as the ℓ^2-Betti numbers do; therefore, the right side is independent of the group Γ as well.

(3) Given a probability measure preserving equivalence relation $\mathcal{R} \subset X \times X$ on a standard Borel space (X, ν). Let Σ be a $\mathcal{R}$-simplicial complex with fundamental domain $F = \bigsqcup_{j \in J} F_j$ of $\Sigma^{(0)}$ such that $\pi \colon F_j \to X$ is injective and $\sum_{j \in J} \pi(\nu(F_j)) < \infty$. We denote the vertex over x in F_j by $f_j(x)$. A fundamental domain for the ordered simplices of $\Sigma^{(p)}$ is given by

$$\{(f, v_1, ..., v_p) \in \Sigma^{(p)} \mid f \in F,\ v_i \in \Sigma^{(0)} \text{ for } 1 \leqslant i \leqslant p\}.$$

Among these ordered simplices we can choose a representative for each unoriented simplex, this gives us a fundamental domain $F' = \bigsqcup_{i \in I} F'_i$ for the action of $\mathcal{R} \times \mathfrak{S}_p$ on $\Sigma^{(p)}$ such that the projection $\pi \colon F'_i \to X$ is injective for each $i \in I$, where $\mathfrak{S}_p$ denotes the symmetric group on p elements. The characteristic functions of these sets define a total sequence $\{\sigma_i\}_{i \in I}$ for the direct integral $\int_X^{\oplus} C_p^{(2)}(\Sigma_x) d\nu$ and hence define a trace [Gab02, p.120]. Let $c := \int_X |\pi^{-1}(\{x\}) \cap F| d\nu(x) < \infty$. We compute the ℓ^2-Betti numbers:

$$
\begin{aligned}
&\beta_p^{(2)}(\mu_{\Sigma,\mathcal{R}}) \\
=&\sum_{\tau \in \mathcal{T}_p} \frac{\langle P_K \tau([K,x]), \tau([K,x])\rangle}{p+1} d\mu_{\Sigma,\mathcal{R}}([K,x]) \\
=&\int_X \frac{1}{c} \sum_{j\in J} \sum_{\tau \in \mathcal{T}_p} \chi_{\pi(F_j)}(x) \frac{\langle P_{\Sigma_x} \tau([\Sigma_x, f_j(x)]), \tau([\Sigma_x, f_j(x)])\rangle}{p+1} d\nu(x) \\
=&\int_X \frac{1}{c} \sum_{j\in J} \sum_{\substack{s \in \Sigma_x(p) \\ f_j(x)\in s}} \chi_{\pi(F_j)}(x) \frac{\langle P_{\Sigma_x} s, s\rangle}{p+1} d\nu(x) \\
=&\frac{1}{c} \int_X \sum_{i\in I} \chi_{\pi(F_i')}(x) \langle P_{\Sigma_x} \sigma_i(x), \sigma_i(x)\rangle d\nu(x) \\
=&\frac{\dim_{\mathcal{R}} \mathcal{H}_p^{(2)}(\Sigma)}{c} = \frac{\beta_p(\Sigma, \mathcal{R}, \nu)}{c}
\end{aligned}
$$

Again, we used the fact that summing over all p-simplices s containing a vertex $f_j(x)$ of the fundamental domain F and then summing over all vertices in F is equal to summing $p+1$ times over the elements in F', the fundamental domain of the unordered p-simplices.

2.3. Approximation of ℓ^2-Betti Numbers

In this section, we will show that the ℓ^2-Betti numbers of a random rooted simplicial complex μ can be, under some circumstances, approximated by a sequence $(\mu_n)_n$ of random rooted simplicial complexes that weakly converges to μ. To this end, we will apply the Spectral Theorem B.15 to the Laplace operator.

Remark 2.3.1. In order to apply the Spectral Theorem, the operator must be self-adjoint or, atleast, essentially self-adjoint, *i.e.* closable with a self-adjoint closure. The Laplace operator is always closable, but in general, when the vertex degree is not bounded, not essentially self-adjoint. We will summarize some results about the question, when the Laplace operator is essentially self-adjoint. In [Woj08], Wojciechowski proves that the 0th Laplace operator on a locally-finite graph is essentially self-adjoint. In [Bor, Proposition 2.2], Bordenave shows that the 0th Laplace operator for unimodular measures on the space of rooted locally finite graphs $\mathcal{G}_*$ is essentially self-adjoint. Anné and Torki-Hamza define in [ATH15] a property called χ-completeness for graphs, which implies that the 1st Laplace operator is essentially self-adjoint. This property was extended by Chebbi [Che18] to two-dimensional simplicial complexes, where it also implies essentially self-adjointness of the 1st and 2nd Laplace operator. Chebbi likewise gives an example of a two-dimensional simplicial complex with a non self-adjoint Laplace operator. In [LP16], Linial and Peled studied the spectral measures of random simplicial complexes of the type $Y(n, \frac{c}{n})$ and showed that they weakly converge to the spectral measure of a Poisson d-tree, which has a self-adjoint Laplace operator.

Even though the most things hold true in the unbounded case, as long as the Laplace operator is essentially self-adjoint, we will assume in the following that μ is a random rooted simplicial complex of bounded degree and therefore, the Laplacian Δ_p on $C_p^{(2)}(\mathcal{SC}_*, \mu)$ a bounded self-adjoint operator (see Proposition 2.2.4). Let E_{Δ_p} be the projection valued measure of Δ_p from the Spectral Theorem (Theorem B.15). Further, we define the *spectral measure of* Δ_p of a Borel set $B \subset \mathbb{R}$ to be

$$\nu_p(B) := \operatorname{tr}_\mu E_{\Delta_p}(B) \in [0, \infty).$$

By Proposition B.16, this is equivalent to

$$\nu_p(B) = \int_{\mathcal{SC}_*} \nu_{p,[K,x]}(B) d\mu([K, x]),$$

where

$$\nu_{p,[K,x]}(B) = \sum_{\substack{s \in K(p) \\ x \in s}} \frac{\langle E_{\Delta_{p,K}}(B)s, s\rangle}{p+1},$$

with $\Delta_{p,K}$ the pth Laplace operator of the simplicial chain complex of K and $E_{\Delta_{p,K}}$ the corresponding projection valued measure.

Remark 2.3.2. By the Spectral Theorem, ν_p satisfies

$$\operatorname{tr}_\mu(f(\Delta_p)) = \int_{\mathbb{R}} f(\lambda) d\nu_p(\lambda)$$

for all bounded Borel functions f on $\mathbb{R}$. Further, note that ν_p is not a probability measure, but it is finite as long as the expected number of p-simplices containing the root,

$$\mathbb{E}_\mu(\deg_p) = \int_{\mathcal{SC}_*} \deg_p(x) d\mu,$$

is finite (which is, in particular, satisfied if μ has bounded degree), since we have

$$\nu_p(\mathbb{R}) = \mathrm{tr}_\mu(E_p(\mathbb{R})) = \mathrm{tr}_\mu(\mathrm{Id}_p) = \mathbb{E}_\mu(\deg_p)/(p+1).$$

The spectral measure of the point $\{0\}$ is of special interest because

$$\nu_p(\{0\}) = \beta_p^{(2)}(\mu).$$

In the following, we will present a version of Lück's approximation theorem [Lüc94]. The proof adapts some ideas of [ATV11] to higher dimensions. This extends a result of Elek [Ele10], who proved that the limit $\lim_{n\to\infty} \frac{b_p(K_n)}{|K_n^{(0)}|}$ exists for a Benjamini-Schramm convergent sequence K_n of finite simplicial complexes of bounded degree.

Theorem 2.3.3. *Let $(\mu_n)_n$ be a sequence of sofic random rooted simplicial complexes with uniformly bounded vertex degree. If the sequence weakly converges to a random rooted simplicial complex μ, then the ℓ^2-Betti numbers of $(\mu_n)_n$ converge to the ℓ^2-Betti numbers of μ.*

Before we start with the proof, we recall the following lemmas. The first one is a consequence of the approximation theorem of Weierstraß and the second one is known as *Portmanteau theorem*:

Lemma 2.3.4. *Let μ be a Borel measure on $\mathbb{R}$ and let $(\mu_n)_n$ be a sequence of measures. Assume there is a compact set C which contains the support of μ_n for every $n \in \mathbb{N}$ and assume further, that*

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(t) d\mu_n(t) = \int_{\mathbb{R}} f(t) d\mu(t)$$

holds for all polynomials $f \in \mathbb{R}[x]$. Then $(\mu_n)_n$ weakly converges to μ.

Lemma 2.3.5. *Let $(\nu_n)_n$ be a sequence of finite Borel measures on $\mathbb{R}$ which weakly converges to ν. Then*

(1) $\liminf_{n\to\infty} \nu_n(U) \geqslant \nu(U)$ *for all open sets $U \subset \mathbb{R}$,*

(2) $\limsup_{n\to\infty} \nu_n(A) \leqslant \nu(A)$ *for all closed sets $A \subset \mathbb{R}$.*

The following lemma is the first step of the proof of Theorem 2.3.3:

Lemma 2.3.6. *Let $(\mu_n)_n$ be a sequence of random rooted simplicial complexes with $\deg(\mu_n) \leqslant D < \infty$ for all $n \in \mathbb{N}$ which weakly converges to μ_∞. Then the corresponding spectral measures $(\nu_p^n)_n$ of the pth Laplace operators Δ_p weakly converge to ν_p^∞, for every $p \in \mathbb{N}_0$.*

Proof. For the sake of simplicity, we denote the spectral measures ν_p^n and ν_p^∞ by ν_n and ν_∞, respectively. Since $\deg(\mu_n)$ is uniformly bounded, we know by Proposition 2.2.4 that $\|\Delta_p\|_{\mu_n} \leqslant R$ for all $n \in \mathbb{N}$ and hence the support of the spectral measures ν_p^n lies in $[-R, R]$ for all $n \in \mathbb{N}$. Lemma 2.3.4 implies that it is enough to check the identity

$$\lim_{n\to\infty} \int_{-R}^{R} f(\lambda) d\nu_n(\lambda) = \int_{-R}^{R} f(\lambda) d\nu_\infty(\lambda)$$

for polynomials $f \in \mathbb{R}[x]$. By linearity, we can further assume that $f = x^r$ for some $r \in \mathbb{N}$. Let us consider

$$\int_{-R}^{R} \lambda^r d\nu_n(\lambda) = \sum_{\tau\in\mathcal{T}_p} \int_{\mathcal{SC}_*} \frac{\langle(\Delta_p)^r\tau, \tau\rangle}{p+1} d\mu_n.$$

Let $s \in K(p)$ be a p-simplex with $x \in s$. The image $\Delta_p(s)$ of s is a linear combination of s and p-simplices which share a common $p-1$-face with s, hence $\Delta_p s$ lies in the 2-ball $B_2(K, x)$ centred at x. Therefore, $\langle(\Delta_p)^r s, s\rangle$ only depends on the $(r+1)$-neighbourhood of x. By the weak convergence of the sequence $(\mu_n)_n$, we know that

$$\lim_{n\to\infty} \mu_n(U_t(\alpha)) = \mu_\infty(U_t(\alpha))$$

for any finite rooted simplicial complex α and radius $t > 0$. Hence,

$$\lim_{n\to\infty} \sum_{\tau\in\mathcal{T}_p} \int_{U_{r+1}(\alpha)} \frac{\langle(\Delta_p)^r\tau, \tau\rangle}{p+1} d\mu_n = \sum_{\tau\in\mathcal{T}_p} \int_{U_{r+1}(\alpha)} \frac{\langle(\Delta_p)^r\tau, \tau\rangle}{p+1} d\mu_\infty$$

for all finite rooted simplicial complexes α. Finally, the claim follows from the fact that $\mathcal{SC}_*$ is a countable union of open sets of the form $U_{r+1}(\alpha)$. $\square$

Lemma 2.3.7. *For a sofic random rooted simplicial complex μ of degree bounded by some $D > 0$ there always exists a sequence of finite simplicial complexes $(K_n)_n$ with degree uniformly bounded by D and which converges Benjamini-Schramm to μ.*

Proof. Since μ is sofic, there exists a sequence $(L_n)_n$ of finite simplicial complexes converging Benjamini-Schramm to μ. We remove the edges of L_n which contain at least one vertex of degree greater than D, and, certainly, all higher dimensional simplices containing these edges, until the maximum degree is less or equal D. We denote this new sequence by $(K_n)_n$. We consider the difference of the associated random rooted simplicial complexes for open sets of the form $U_r(\alpha)$. Note that $K_n^{(0)} = L_n^{(0)}$. It follows that

$$\begin{aligned}
&\mu_{L_n}(U_r(\alpha)) - \mu_{K_n}(U_r(\alpha)) \\
\leqslant &\frac{|\{x \in L_n^{(0)} \mid B_r(L_n, x) \ncong B_r(K_n, x)\}|}{|L_n^{(0)}|} \\
= &\frac{|\{x \in L_n^{(0)} \mid \exists y \in B_r(L_n, x)^{(0)} \text{ with } \deg(y) > D\}|}{|L_n^{(0)}|} \\
&\xrightarrow{n\to\infty} 0,
\end{aligned}$$

since μ_{L_n} converges to μ and $\deg(\mu) \leqslant D$. Therefore, K_n converges Benjamini Schramm to μ, as well. $\square$

We are ready to proof Theorem 2.3.3:

Proof. Let ν_p^n and ν_p be the spectral measures of the sofic bounded degree random rooted simplicial complexes μ_n and μ, respectively. Considering Remark 2.3.2, we have to show that $\nu_p^n(\{0\})$ converges to $\nu_p(\{0\})$. By Lemma 2.3.6, we know that the spectral measures ν_p^n weakly converge to the spectral measure ν_p. First, let us assume that μ is the random rooted simplicial complex μ_K associated with a finite simplicial complex K with vertex degree at most D. The Laplace operator Δ_p of K is bounded $\|\Delta_p\| < \infty$ and can be viewed as a $d \times d$ matrix with integral coefficients, where $d = |K(p)|$, when we pick a basis by choosing an orientation for each p-simplex of K. Thus, the characteristic polynomial $q(x)$ of Δ_p is in $\mathbb{Z}[x]$ and of degree d. The product of the non-zero roots of $q(x)$, which are all in $[-\|\Delta_p\|, \|\Delta_p\|]$, is one of the coefficients of q, so it is in $\mathbb{Z}\backslash\{0\}$ and hence

$$1 \leqslant \Pi_{\lambda_i \neq 0}|\lambda_i|. \tag{2.3}$$

The spectral measure of any Borel set $S \subset \mathbb{R}$ has the following form, where we denote by v_i the normed eigenvector of the eigenvalue λ_i:

$$\begin{aligned}
\nu(S) &= \sum_{\tau \in \mathcal{T}_p} \int_{\mathcal{SC}_*} \frac{\langle E_{\Delta_p}(S)\tau, \tau\rangle}{p+1} d\mu = \sum_{s \in K(p)} \frac{\langle E_{\Delta_p,K}(S)s, s\rangle}{|K^{(0)}|} \\
&= \sum_{s \in K(p)} \sum_{\lambda_i \in S} \frac{\langle\langle s, v_i\rangle v_i, s\rangle}{|K^{(0)}|} = \sum_{\lambda_i \in S} \frac{\|v_i\|^2}{|K^{(0)}|} = \frac{|\{\lambda_i \in S\}|}{|K^{(0)}|}.
\end{aligned}$$

Thence, we can express the number of eigenvalues in the set S by

$$|\{\lambda_i \in S\}| = \nu(S)|K^{(0)}|. \tag{2.4}$$

Let $I_0 = (-\epsilon, \epsilon)\backslash\{0\}$ for some $0 < \epsilon < 1$. Equation (2.3) and (2.4) yields

$$1 \leqslant \Pi_{\lambda_i \neq 0}|\lambda_i| \leqslant \epsilon^{|K^{(0)}|\nu(I_0)} \|\Delta_p\|^d.$$

Therefore, we obtain

$$\begin{aligned}
0 &\leqslant -|K^{(0)}|\nu(I_0)\ln(1/\epsilon) + d\ln(\|\Delta_p\|) \\
\nu(I_0) &\leqslant \frac{d\ln(\|\Delta_p\|)}{|K^{(0)}|\ln(1/\epsilon)}.
\end{aligned}$$

The maximal number of p-simplices in a complex with vertex degree bounded by $D \in \mathbb{N}$ is $\binom{D}{p}\frac{|K^{(0)}|}{p+1}$; consequently, we can make the right side independent of the number $|K^{(0)}|$ of vertices:

$$\nu(I_0) \leqslant \frac{\ln(\|\Delta_p\|)\binom{D}{p}}{(p+1)\ln(1/\epsilon)}.$$

Note that $\|\Delta_p\|$ is also bounded by some number $R(D)$ only depending on the vertex degree D. Up to now, we only considered a random rooted simplicial complex associated to a finite simplicial complex, but we get the same inequality for any sofic random rooted simplicial complex μ of degree bounded by D, since by Lemma 2.3.7 we know that there is a sequence of finite simplicial complexes K_n, all of degree bounded by D, which converges Benjamini-Schramm to μ. We denote the associated spectral measures by ν, respectively ν_n, and apply Lemma 2.3.5:

$$\nu(I_0) \leqslant \liminf_{n\to\infty} \nu_n(I_0) \leqslant \frac{\ln(R(D))\binom{D}{p}}{(p+1)\ln(1/\epsilon)}. \tag{2.5}$$

After we have achieved Equation (2.5) for every sofic random rooted simplicial complex of bounded degree, we can finish the proof. First, we restate the assumptions of the Theorem. Let $(\nu_n)_n$ be a sequence of spectral measures associated with a sequence of sofic random rooted simplicial complexes $(\mu_n)_n$ of degree bounded by D which weakly converges to μ and let ν be the spectral measure of μ. We conclude:

$$\begin{aligned}\limsup_{n\to\infty} \nu_n(\{0\}) \leqslant \nu(\{0\}) \leqslant \nu((-\epsilon,\epsilon)) &\leqslant \liminf_{n\to\infty} \nu_n(-\epsilon,\epsilon) \\ &\leqslant \liminf_{n\to\infty} \nu_n(\{0\}) + \frac{\ln(R(D))\binom{D}{p}}{(p+1)\ln(1/\epsilon)}.\end{aligned}$$

Letting ϵ tend to zero, completes the proof. □

Corollary 2.3.8 (Euler-Poincaré Formula). *Let μ be a sofic random rooted simplicial complex of dimension n and of bounded degree. Then*

$$\sum_{p=0}^{n}(-1)^p\beta_p^{(2)}(\mu) = \sum_{p=0}^{n}(-1)^p\frac{\mathbb{E}_\mu(\deg_p)}{p+1},$$

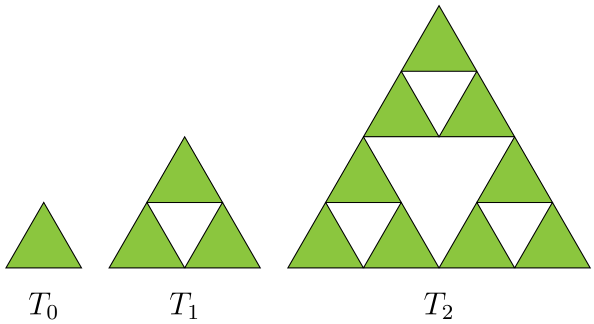
where $\mathbb{E}_\mu(\deg_p)$ denotes the expected number of p-simplices containing the root.

Proof. For a random rooted simplicial complex μ_K, associated with a finite simplicial complex K, we have, by Example 2.2.9 (1), that $\beta_p^{(2)}(\mu_K) = \frac{b_p(K)}{|K^{(0)}|}$ and therefore,

$$\begin{aligned}\sum_{p=0}^{n}(-1)^p\beta_p^{(2)}(\mu) = \sum_{p=0}^{n}(-1)^p\frac{b_p(K)}{|K^{(0)}|} = \frac{\chi(K)}{|K^{(0)}|} &= \sum_{p=0}^{n}(-1)^p\frac{|K(p)|}{|K^{(0)}|} \\ = \sum_{p=0}^{n}(-1)^p\int_{\mathcal{SC}_*}\frac{\deg_p(x)}{p+1}d\mu_K &= \sum_{p=0}^{n}(-1)^p\frac{\mathbb{E}_{\mu_K}(\deg_p)}{p+1},\end{aligned}$$

where we denote by $\chi(K)$ the ordinary Euler characteristic of K. If we now apply Theorem 2.3.3, we obtain the equality for sofic random rooted simplicial complexes of bounded degree. □

Example 2.3.9. Let $(T_n)_n$ be the sequence of two-dimension simplicial complexes which occurs in the construction of Sierpinski's triangle (see Example 2.1.8). As we have seen in Example 2.1.8, $(T_n)_n$ converges Benjamini-Schramm, or in other words, the sequence of associated random rooted simplicial complexes $(\mu_{T_n})_n$ weakly converges, to a random rooted simplicial complex μ_τ. We will use Theorem 2.3.3 to compute the first ℓ^2-Betti number of μ_τ.



First of all, observe that $b_1(T_0) = 0$, $b_1(T_1) = 1$ and in general, $b_1(T_n) = \frac{3^n-1}{2}$. Applying the equality in Example 2.2.9.(1) yields

$$\beta_1^{(2)}(\mu_{T_n}) = \frac{b_1(T_n)}{|T_n^{(0)}|} = \frac{3^n - 1}{3^{n+1} + 3}.$$

Therefore, by Theorem 2.3.3, we obtain as 1st ℓ^2-Betti number of μ_τ:

$$\beta_1^{(2)}(\mu_\tau) = \lim_{n\to\infty} \beta_1^{(2)}(\mu_{T_n}) = \lim_{n\to\infty} \frac{3^n - 1}{3^{n+1} + 3} = \frac{1}{3}.$$

Chapter 3.

ℓ^2-Multiplicities and Equivariant Benjamini-Schramm Convergence

The results of this chapter are a joint work with Steffen Kionke [KSB]. We define a variant of Benjamini-Schramm convergence for simplicial complexes with an action of a fixed finite group G which leads to the notion of random rooted simplicial G-complexes. For every random rooted simplicial G-complex we define a corresponding ℓ^2-homology and the ℓ^2-*multiplicity* of an irreducible representation of G in the homology (Definition 3.2.6). The ℓ^2-multiplicities generalize the ℓ^2-Betti numbers, defined in Chapter 2 (Definition 2.2.8). Further, we show that they are continuous on the space of sofic random rooted simplicial G-complexes. This result is a different approach to a theorem of Kionke [Kio18, Theorem 1.2], which says that for a tower of finite sheeted coverings $(X/\Gamma_n)_n$ of a Γ CW-complex X and every irreducible representation σ of G

$$\lim_{n\to\infty} \frac{m(\sigma, H_p(X/\Gamma_n, \mathbb{C}))}{[\Gamma : \Gamma_n]} = m_p^{(2)}(\sigma, X; \Gamma)$$

holds, where $m(\sigma, H_p(X/\Gamma_n, \mathbb{C}))$ are the ordinary multiplicities and $m_p^{(2)}(\sigma, X; \Gamma)$ the ℓ^2-multiplicities, defined in [Kio18], of a Γ-CW complex X. In addition, we study induction of random rooted simplicial complexes and discuss the effect on the ℓ^2-multiplicities. We will use a similar language and notation as in Chapter 2. Some proofs work in a similar way as in Chapter 2, in that case we will focus more on the differences, than on giving a detailed proof. If not stated otherwise, G will denote a finite group.

Structure of the chapter. We define random rooted simplicial G-complexes (Definition 3.1.2) in Section 3.1 and give a first example. Additionally, we introduce induction of random rooted simplicial complexes and finish the section with the prominent example of towers of finite sheeted covering spaces with action of a finite group. In Section 3.2, we define ℓ^2-multiplicities of random rooted simplicial complexes (Definition 3.2.6). To this end, we have to proceed as in Chapter 2 and define the simplicial ℓ^2-chain complex of a random rooted simplicial G-complex, a von Neumann algebra and a trace. Section 3.3 is dedicated to the proof of the continuity of ℓ^2-multiplicities (Theorem 3.3.2) and a reciprocity formula for induced random rooted simplicial complexes (Theorem 3.3.3). We end the chapter with Example 3.3.5, in which we compute the ℓ^2-multiplicities of Sierpinki's triangle with rotation.

3.1. Random Rooted Simplicial G-Complexes

A *rooted simplicial G-complex* is a pair (K, o) consisting of a simplicial complex K with G-action and a G-orbit o of vertices of K such that every connected component of K contains atleast one vertex of o. Actually, a rooted simplicial G-complex is not a pair but even a triple, but, as remarked in Chapter 2, we usually omit the set of vertices V of K.

Two rooted simplicial G-complexes (K, o) and (L, o') are *isomorphic* if there is a simplicial isomorphism $\Phi\colon K \to L$ such that $\Phi(o) = o'$ and $g \cdot \Phi(x) = \Phi(g \cdot x)$ for all $x \in K^{(0)}$ and $g \in G$. In this case, we write $(K, o) \overset{G}{\cong} (L, o')$ or, if there is no danger of confusion, we omit G and just write $(K, o) \cong (L, o')$.

Definition 3.1.1. We denote by $\mathcal{SC}_*(G)$ the space of *isomorphism classes of rooted simplicial G-complexes* and by $\mathcal{SC}_*^D(G)$ the subspace of isomorphism classes of vertex degree bounded by D.

To avoid technical issues, we will only consider simplicial complexes of degree bounded by some large constant $D \in \mathbb{N}$. Similarly as in Chapter 2, we define a metric on $\mathcal{SC}_*^D(G)$ by

$$d([K, o], [L, o']) = \inf_r \left\{ \frac{1}{2^r} \mid B_r(K, o) \overset{G}{\cong} B_r(L, o') \right\},$$

for $[K, o], [L, o'] \in \mathcal{SC}_*$, where $B_r(K, o)$ is the rooted subcomplex of K spanned by all vertices of K with distance at most r from a vertex in the orbit o. Note that subcomplexes of the form $B_r(K, o)$ are stable under the G-action. With the topology induced by this metric, $\mathcal{SC}_*^D(G)$ is a compact totally disconnected Polish space. We define the *r-neighbourhood* of a rooted simplicial G-complex α as the open subset $U_r(\alpha) \subset \mathcal{SC}_*^D(G)$ consisting of all rooted isomorphism classes $[K, o]$ such that $B_r(K, o) \overset{G}{\cong} B_r(\alpha)$. The $U_r(\alpha)$'s are compact and open, and provide a basis for the topology. In the same manner (cf. Chapter 2) we define the space $\mathcal{SC}_{**}^D(G)$ of *isomorphism classes of doubly rooted simplicial G-complexes*.

Definition 3.1.2. A *random rooted simplicial G-complex* is a *unimodular* probability measure μ on $\mathcal{SC}_*^D(G)$, where unimodular means

$$\int_{\mathcal{SC}_*^D(G)} \sum_{x \in K^{(0)}} f([K, o, Gx]) d\mu([K, o]) = \int_{\mathcal{SC}_*^D(G)} \sum_{x \in K^{(0)}} f([K, Gx, o]) d\mu([K, o])$$

for all Borel measurable functions $f\colon \mathcal{SC}_{**}^D(G) \to \mathbb{R}_{\geqslant 0}$.

Remark 3.1.3. A weak limit of random rooted simplicial G-complexes is again a random rooted simplicial G-complex, this follows precisely by the same arguments as in Remark 2.1.4.

Example 3.1.4. Let K be a finite simplicial G-complex of vertex degree bounded by D. There exists a unique random rooted simplicial G-complex, fully supported on the rooted isomorphism classes of K, given by

$$\mu_K^G := \sum_{x \in K^{(0)}} \frac{\delta_{[K, Gx]}}{|K^{(0)}|},$$

where $\delta_{[K,Gx]}$ denotes the Dirac-measure of the point $[K, Gx]$ in $\mathcal{SC}_*^D(G)$. That μ_K^G is a probability measure is obvious and unimodularity follows from the same computation as in Example 2.1.5 (1).

Definition 3.1.5. We say a sequence $(K_n)_n$ of finite simplicial G-complexes converges *Benjamini-Schramm* if the weak limit $\lim_{n\to\infty} \mu_{K_n}^G$ exists. Further, we call a random rooted simplicial G-complex *sofic* if it is the limit of a Benjamini-Schramm convergent sequence of finite simplicial G-complexes.

Before we can give more examples, we have to introduce induction of simplicial complexes.

3.1.1. Induction of Simplicial Complexes

Let G be a finite group and $H \leqslant G$ a subgroup. Given an H-set X, one can construct the induced G-set $G \times_H X$. The set

$$G \times_H X := G \times X/\sim$$

is obtained by forming the quotient of $G \times X$ under the equivalence relation

$$(g, x) = (gh, h^{-1} \cdot x)$$

for all $g \in G$, $x \in X$ and $h \in H$. The G-action is given by $g_1[g_2, x] = [g_1 g_2, x]$ for $g_1 \in G$ and $[g_2, x] \in G \times_H X$. In the same manner, a simplicial H-complex K gives rise to a simplicial G-complex $G \times_H K$ by induction. The vertices of $G \times_H K$ are the elements of $G \times_H K^{(0)}$ and the simplices are given by $s_g = \{[g, x] \mid x \in s\}$, for every simplex $s \in K$ and every $g \in G$. If we ignore the G-action, $G \times_H K$ is isomorphic to the disjoint union of $|G/H|$ copies of K. In particular, induction does not change the vertex degree.

Lemma 3.1.6. *The function* $\operatorname{Ind}_H^G\colon \mathcal{SC}_*^D(H) \to \mathcal{SC}_*^D(G)$ *which maps* $[K, o]$ *to* $\big[G \times_H K, G[1, o]\big]$ *is continuous. In particular, the push-forward of measures with* Ind_H^G *is weakly-continuous.*

Proof. We observe that the ball of radius r in $G \times_H K$ around $G[1, o]$ is isomorphic to $G \times_H B_r(K, o)$. This implies that

$$d\big(\operatorname{Ind}_H^G([K, o]), \operatorname{Ind}_H^G([L, o'])\big) \leqslant d([K, o], [L, o'])$$

and proves the assertion. □

Lemma 3.1.7. *Let* μ *be a random rooted simplicial* H*-complex. Then the* push-forward measure $\operatorname{Ind}_H^G(\mu)$ *is a random rooted simplicial* G*-complex.*

Proof. The push-forward preserves the total mass, hence $\mathrm{Ind}_H^G(\mu)$ is a probability measure as well. It remains to verify the unimodularity of $\mathrm{Ind}_H^G(\mu)$. Recall the change-of-variable formula [Bou67, §6]

$$\int_{\mathcal{SC}_*^D(G)} f(\alpha)d\,\mathrm{Ind}_H^G(\mu)(\alpha) = \int_{\mathcal{SC}_*^D(H)} f(\mathrm{Ind}_H^G(\beta))d\mu(\beta).$$

We obtain for all Borel maps $f\colon \mathcal{SC}_{**}(G) \to \mathbb{R}_{\geqslant 0}$:

$$\begin{aligned}
&\int_{\mathcal{SC}_*(G)} \sum_{x\in L^{(0)}} f([L,o,Gx])d\,\mathrm{Ind}_H^G(\mu)([L,o])\\
&\overset{(1)}{=} \int_{\mathcal{SC}_*(H)} \sum_{x\in G\times_H K^{(0)}} f([G\times_H K, o, Gx])d\mu([K,o])\\
&= \int_{\mathcal{SC}_*(H)} |G/H| \sum_{y\in K^{(0)}} f([G\times_H K, o, G[1,y]])d\mu([K,o])\\
&\overset{(2)}{=} \int_{\mathcal{SC}_*(H)} |G/H| \sum_{y\in K^{(0)}} f([G\times_H K, G[1,y], o])d\mu([K,o])\\
&= \int_{\mathcal{SC}_*(H)} \sum_{x\in G\times_H K^{(0)}} f([G\times_H K, Gx, o])d\mu([K,o])\\
&\overset{(3)}{=} \int_{\mathcal{SC}_*(G)} \sum_{x\in L^{(0)}} f([L, Gx, o])d\,\mathrm{Ind}_H^G(\mu)([L,o]),
\end{aligned}$$

where we used the change-of-variable formula in steps (1) and (3), and the unimodularity of μ in step (2). $\square$

The following criterion is useful to show that a sequence of finite simplicial G-complexes converges Benjamini-Schramm to an induced random rooted simplicial G-complex.

Proposition 3.1.8. *Let $(K_n)_n$ be a sequence of finite simplicial G-complexes with vertex degree bounded by D and $H \leqslant G$ be a subgroup. Assume that the sequence $(K_n)_n$, considered as simplicial H-complexes, converges Benjamini-Schramm to a random rooted simplicial H-complex μ_∞. Then $(K_n)_n$ converges Benjamini-Schramm as simplicial G-complexes to $\mathrm{Ind}_H^G(\mu_\infty)$ on $\mathcal{SC}_*^D(G)$ if and only if*

$$\lim_{n\to\infty} \frac{|E(K_n,g,C)|}{|K^{(0)}|} = 0, \tag{3.1}$$

for all $g \in G\backslash H$ and all $C > 0$, where $E(K,g,C) = \{x \in K^{(0)} \mid d(x,gx) \leqslant C\}$.

Proof. Assume that Equation (3.1) holds for all $g \in G\backslash H$ and all $C > 0$. Define $E(K,C) = \bigcup_{g\in G\backslash H} E(K,g,C)$ and let $r > 0$ be given. We want to verify that for all $x \in K_n^{(0)}\backslash E(K_n, 2r+1)$ the ball of radius r around Gx in K_n is isomorphic to $G\times_H B_r(K_n,Hx)$. This is the case if

$$B_r(K_n,Hx) \cap gB_r(K_n,Hx) = \varnothing$$

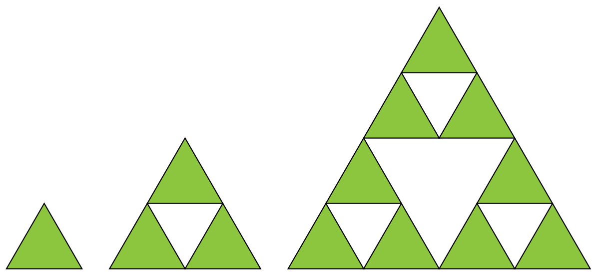
for all $g \in G \backslash H$ and further, if there is no edge between these two sets. Suppose that there is an element in the intersection or an edge between $B_r(K_n, Hx)$ and $gB_r(K_n, Hx)$. In both cases, we can find $h, h' \in H$ such that $d(hx, gh'x) \leqslant 2r+1$. However, this implies $x \in E(K_n, 2r+1)$, since $d(x, h^{-1}gh'x) \leqslant 2r+1$ and $h^{-1}gh' \notin H$.

Let α be a finite rooted simplicial G-complex of radius at most r. Let $\epsilon > 0$ and take n sufficiently large such that
$$\frac{|E(K_n, 2r+1)|}{|K_n^{(0)}|} < \epsilon.$$
The inverse image $V = (\mathrm{Ind}_H^G)^{-1}(U_r(\alpha)) \subseteq \mathcal{SC}_*^D(H)$ is a finite (possibly empty) union of sets of the form $U_r(\alpha')$; thus it is open and compact. The weak convergence $\mu_{K_n}^H \xrightarrow{w} \mu_\infty$ shows that for all sufficiently large n the inequality $|\mu_\infty(V) - \mu_{K_n}^H(V)| < \epsilon$ holds. Moreover, by the observation above,
$$\left|\mu_{K_n}^G(U_r(\alpha)) - \mu_{K_n}^H(V)\right| \leqslant \frac{|E(K_n, 2r+1)|}{|K_n^{(0)}|} < \epsilon.$$
We deduce the convergence $\mu_{K_n}^G \xrightarrow{w} \mathrm{Ind}_H^G(\mu_\infty)$.

Conversely, suppose that the sequence $(\mu_{K_n}^G)_n$ converges to $\mathrm{Ind}_H^G(\mu_\infty)$. Let $g \in G \backslash H$, $C > 0$ and $x \in E(K_n, g, C)$. The ball $B_C(K_n, Hx)$ contains a path from x to gx and hence it can not be isomorphic to a simplicial complex induced from an H-complex, since in that case $B_C(K_n, Hx)$ and $gB_C(K_n, Hx)$ would not be path-connected. By assumption the limit $\lim_{n\to\infty} \mu_{K_n}^G$ is supported on induced complexes and thus Equation (3.1) is satisfied.

□

Example 3.1.9 (Sierpinski's triangle with rotation).



T_0 T_1 T_2

We have seen in Example 2.1.8 that the sequence $(T_n)_n$ Benjamini-Schramm converges to a random rooted simplicial complex $\mu_\mathcal{T}$. Now we introduce an action of the finite cyclic group $G = \langle \rho \rangle$ of order 3. We let ρ act by rotation of $2\pi/3$ around the barycentre $c_n = 2^{n-1}(1, \sqrt{3}^{-1})$ of T_n. All vertices of T_n have Euclidean distance at least $\frac{2^{n-2}}{\sqrt{3}}$ from the barycentre, thus every vertex is moved by an Euclidean distance of at least 2^{n-2} under the non-trivial rotations ρ and

ρ^2, which in particular also holds for the path distance in T_n. Proposition 3.1.8 implies that the sequence $(T_n)_n$, considered as simplicial G-complexes, converges Benjamini-Schramm to the induced random rooted simplicial G-complex $\mathrm{Ind}_1^G(\mu_\tau)$. Roughly speaking, the sequence converges to three copies of Sierpinski's triangle which are permuted cyclically by G.

Next we will discuss the prominent example of towers of finite sheeted covering spaces which we have already treated in Example 2.1.7; this time with an additional action of a finite group G.

3.1.2. Towers of finite sheeted covering spaces with action of a finite group

We recall the situation of Example 2.1.7. Let K be a simplicial complex of vertex degree bounded by some $D \in \mathbb{N}$ with a simplicial, proper and cocompact action of a discrete group Γ. Suppose that Γ is residually finite and let $(N_n)_n$ be a descending sequence of finite index normal subgroups of Γ with $\bigcap_{n\in\mathbb{N}} N_n = \{1\}$. We have seen in Example 2.1.7 that the sequence $(K_n)_n := (K/N_n)_n$ Benjamini-Schramm converges to the the random rooted simplicial complex

$$\mu_K = \frac{1}{c(\Gamma)} \sum_{x\in\mathcal{F}} \frac{\delta_{[K,x]}}{|\operatorname{Stab}_\Gamma(x)|},$$

where $\mathcal{F}$ is a finite fundamental domain for the Γ-action on $K^{(0)}$ and $c(\Gamma) = \sum_{x\in\mathcal{F}} \frac{1}{|\operatorname{Stab}_\Gamma(x)|}$. Let $G \leqslant \Gamma$ be a finite subgroup. For every normal subgroup N_n the quotient simplicial complex K_n carries a G-action. In the following, we will describe the limit taking the G-action into account.
We say that an element $\gamma \in \Gamma$ is *FC* if it has a finite conjugacy class, *i.e.* $|\Gamma : C_\Gamma(\gamma)| < \infty$, where $C_\Gamma(\gamma)$ is the centralizer of γ. Consider the subgroup $H = \{g \in G \mid g \text{ is FC in } \Gamma\} \leqslant G$ of FC-elements which lie in G and let $\Gamma_0 \leqslant_{f.i.} \Gamma$ be a finite index subgroup which satisfies

$$\Gamma_0 \subseteq \bigcap_{h\in H} C_\Gamma(h).$$

Lemma 3.1.10. *If $\mathcal{F}_0 \subseteq K^{(0)}$ is a fundamental domain for the action of Γ_0 on $K^{(0)}$, then the measure*

$$\mu_K^H = \frac{1}{c(\Gamma_0)} \sum_{x\in\mathcal{F}_0} \frac{\delta_{[K,Hx]}}{|\operatorname{Stab}_{\Gamma_0}(x)|}$$

on $\mathcal{SC}_*(H)$ is unimodular and does not depend on the choices of Γ_0 and $\mathcal{F}_0$.

Proof. Observe that any element $\gamma \in \Gamma_0$ commutes with all $h \in H$ and thus defines an isomorphism between (K, Hx) and $(K, H\gamma x)$ as simplicial H-complexes. This shows that the measure is independent of the fundamental domain.

In order to verify that μ_K^H does not depend on Γ_0, it is sufficient to show that we can replace Γ_0 by some finite index normal subgroup $\Gamma_1 \trianglelefteq_{f.i.} \Gamma_0$. Let $\mathcal{F}_1$ be a fundamental domain for Γ_1. We obtain

$$\begin{aligned}
\sum_{x\in\mathcal{F}_1} \frac{\delta_{[K,Hx]}}{|\mathrm{Stab}_{\Gamma_1}(x)|} &= \sum_{y\in\mathcal{F}_0} |\frac{1}{\mathrm{Stab}_{\Gamma_0}(y)|} \sum_{\substack{\gamma\in\Gamma_0\\ \gamma y\in\mathcal{F}_1}} \frac{\delta_{[K,H\gamma y]}}{|\mathrm{Stab}_{\Gamma_1}(\gamma y)|}\\
&= \sum_{y\in\mathcal{F}_0} \frac{\delta_{[K,Hy]}}{|\mathrm{Stab}_{\Gamma_0}(y)|} \sum_{\substack{\gamma\in\Gamma_0\\ \gamma y\in\mathcal{F}_1}} \frac{1}{|\mathrm{Stab}_{\Gamma_1}(y)|}\\
&= |\Gamma_0 : \Gamma_1| \sum_{y\in\mathcal{F}_0} \frac{\delta_{[K,Hy]}}{|\mathrm{Stab}_{\Gamma_0}(y)|}
\end{aligned}$$

and, from a similar calculation, that $c(\Gamma_1) = |\Gamma_0 : \Gamma_1| c(\Gamma_0)$.

It remains to show that μ_K^H is unimodular. Let $f\colon \mathcal{SC}_{**}(H) \to \mathbb{R}_{\geqslant 0}$ be a measurable function. Unimodularity follows from a short calculation:

$$\begin{aligned}
&\int_{\mathcal{SC}_* H} \sum_{y\in L^{(0)}} f([L,o,Hy]) d\mu_K^H([L,o])\\
&= \frac{1}{c(\Gamma_0)} \sum_{x\in\mathcal{F}_0} \frac{1}{|\mathrm{Stab}_{\Gamma_0}(x)|} \sum_{y\in K^{(0)}} f([K,Hx,Hy])\\
&= \frac{1}{c(\Gamma_0)} \sum_{x\in\mathcal{F}_0} \frac{1}{|\mathrm{Stab}_{\Gamma_0}(x)|} \sum_{y\in\mathcal{F}_0} \sum_{\gamma\in\Gamma_0} \frac{1}{|\mathrm{Stab}_{\Gamma_0}(y)|} f([K,Hx,H\gamma y])\\
&= \frac{1}{c(\Gamma_0)} \sum_{x,y\in\mathcal{F}_0} \frac{1}{|\mathrm{Stab}_{\Gamma_0}(x)| \cdot |\mathrm{Stab}_{\Gamma_0}(y)|} \sum_{\gamma\in\Gamma_0} f([K,H\gamma^{-1}x,Hy])\\
&= \cdots = \int_{\mathcal{SC}_* H} \sum_{x\in L^{(0)}} f([L,Hx,o]) d\mu_K^H([L,o]). \qquad \square
\end{aligned}$$

Proposition 3.1.11. *Let $G \leqslant \Gamma$ be a finite subgroup and let $H \leqslant G$ be the subgroup of FC-elements for Γ. If $(N_n)_n$ is a descending chain of finite index normal subgroups in Γ with $\bigcap_{n\in\mathbb{N}} N_n = \{1\}$, then the sequence of simplicial G-complexes $(K/N_n)_n$ converges to the random rooted simplicial complex $\mu_K^G := \mathrm{Ind}_H^G(\mu_K^H)$.*

Proof. The proof consists of two steps. First we show that $(K/N_n)_n$ converges as a sequence of H-complexes to μ_K^H (Claims 1 and 2) and in the second step (Claim 3) we apply Proposition 3.1.8.

Claim 1: Let $r > 0$ be fixed. For all sufficiently large n and all $x \in K^{(0)}$, the r-ball $B_r(K,Hx)$ in K and the r-ball $B_r(K/N_n, HN_nx)$ in K/N_n are isomorphic as H-complexes.

Let $\Gamma_0 \leqslant \Gamma$ be as above and let $\mathcal{F}_0$ be a fundamental domain for Γ_0 acting on $K^{(0)}$. The action is proper, the sets $\mathcal{F}_0$ and H are finite and the vertex degree of K is bounded, hence the set S of elements $\gamma \in \Gamma$ such that

$$B_{r+1}(K, Hx_0) \cap \gamma B_{r+1}(K, Hx_0) \neq \varnothing \tag{3.2}$$

for some $x_0 \in \mathcal{F}_0$ is finite.
Take $n \in \mathbb{N}$ so large that $S \cap N_n = \{1\}$. Then for all $x_0 \in K^{(0)}$ and $\gamma \in N_n$ Property (3.2) implies that $\gamma = 1$. Indeed, find $\gamma_0 \in \Gamma_0$ with $\gamma_0 x_0 \in \mathcal{F}_0$ then multiplication with γ_0 yields $B_{r+1}(K, H\gamma_0 x_0) \cap \gamma_0\gamma\gamma_0^{-1} B_{r+1}(K, H\gamma_0 x_0) \neq \varnothing$. We deduce $\gamma_0\gamma\gamma_0^{-1} = \gamma = 1$. In particular, the quotient map takes the vertices of the r-ball $B_r(K, Hx)$ injectively to the r-ball $B_r(K/N_n, HN_n x)$. We have to verify that every simplex in $B_r(K/N_n, HN_n x)$ lifts to a unique simplex in $B_r(K, Hx)$. Let s be a simplex in $B_r(K/N_n, HN_n x)$ and let $\tilde{s}$ be a lift in K such that at least one vertex lies in $B_r(K, Hx)$. As a consequence $\tilde{s}$ is a simplex in $B_{r+1}(K, Hx)$. Let y be any vertex of $\tilde{s}$. There is an element $k \in N_n$ such that $d(ky, Hx) \leqslant r$. This means that $B_{r+1}(K, Hx) \cap kB_{r+1}(K, Hx) \neq \varnothing$ and shows that $k = 1$. In particular, the simplex $\tilde{s}$ lives in $B_r(K, Hx)$.

Claim 2: The sequence $(\mu^H_{K/N_n})_n$ converges to μ^H_K.

Let $r > 0$ and let α be a finite rooted simplicial H-complex. Let $n \in \mathbb{N}$ sufficiently large such that N_n acts freely on K and so that Claim 1 applies. In addition, we may take $\Gamma_0 \leqslant N_n$; the action of Γ_0 is also free. Now, every point in K/N_n is covered by exactly $|N_n : \Gamma_0|$ points in $\mathcal{F}_0$ and we deduce

$$\begin{aligned}
\mu^H_K(U_r(\alpha)) &= \frac{|\{x \in \mathcal{F}_0 \mid B_r(K, Hx) \overset{H}{\cong} \alpha\}|}{|\mathcal{F}_0|} \\
&= \frac{|\{x \in K^{(0)}/N_n \mid B_r(K, HN_n x) \overset{H}{\cong} \alpha\}|}{|K^{(0)}/N_n|} = \mu^H_{K/N_n}(U_r(\alpha)).
\end{aligned}$$

Claim 3: Equation (3.1) holds for all $C > 0$ and all $g \in G\backslash H$.

Let $i \in \mathbb{N}$ be chosen so that N_i acts freely on K and let $\mathcal{F}$ be a fundamental domain for the action of N_i on $K^{(0)}$. Let $Z \subseteq \Gamma$ be the finite set of elements $\gamma \in \Gamma$ such that $d(\gamma x, x) \leqslant C$ for some $x \in \mathcal{F}$. For $n \geqslant i$ the vertices of K/N_n correspond bijectively to $N_i/N_n \times \mathcal{F}$.
Take $x \in K^{(0)}$ and write $\bar{x} = N_n x \in K^{(0)}/N_n$. Suppose that $\bar{x} \in E(K/N_n, g, C)$; *i.e.* there is $\gamma_n \in N_n$ with $d(gx, \gamma_n x) \leqslant C$. There is a unique $x_0 \in \mathcal{F}$ and an element $\gamma_i \in N_i$ satisfying $x = \gamma_i x_0$. This shows that $d(\gamma_i^{-1}\gamma_n^{-1} g\gamma_i x_0, x_0) \leqslant C$ and so $\gamma_i^{-1} g\gamma_i \in ZN_n$. How many elements has the finite set $e_n(g, Z) = \{k \in N_i/N_n \mid k^{-1}gk \in ZN_n/N_n\}$? Clearly, its cardinality is bounded above by $|Z| \cdot |C_{N_i/N_n}(gN_n)|$. The element $g \in G$ has an infinite conjugacy class in Γ. Let Y be a set of left coset representatives of $C_\Gamma(g)$. For every finite subset $S \subset Y$, the elements in $\{sgs^{-1} \mid s \in S\}$ are distinct modulo N_n for sufficiently large n. Hence, $|\Gamma/N_n : C_{\Gamma/N_n}(gN_n)| \geqslant |S|$ and therefore,

$$\lim_{n\to\infty} \frac{|C_{\Gamma/N_n}(gN_n)|}{|\Gamma : N_n|} = 0.$$

We deduce that

$$\lim_{n\to\infty} \frac{|E(K/N_n, g, C)|}{|K^{(0)}/N_n|} \leqslant \lim_{n\to\infty} \frac{|e_n(g, Z)|}{|K^{(0)}/N_n|} \leqslant \lim_{n\to\infty} \frac{|Z| \cdot |C_{\Gamma/N_n}(gN_n)|}{|N_i : N_n||K^{(0)}/N_i|} = 0.$$

This proves the last claim and Proposition 3.1.8 completes the proof. □

3.2. ℓ^2-Multiplicities of Random Rooted Simplicial Complexes

Again, let G be a finite group. In order to define the ℓ^2-multiplicities of a random rooted simplicial G-complex, we have to proceed as in Chapter 2. We begin by picking a representative for each isomorphism class $[K, o] \in \mathcal{SC}_*^D(G)$ of rooted simplicial G-complexes in a measurable way. To this end, let

$$\mathbb{N}_G := \bigsqcup_{X \in \mathrm{Orb}(G)} \mathbb{N}_0 \times X,$$

where $\mathrm{Orb}(G)$ denotes the finite set of isomorphism classes of transitive G-sets and further, let $\Delta^D(\mathbb{N}_G)$ be the simplicial complex consisting of all non-empty subset of $\mathbb{N}_G$ with at most $D+1$ elements. The action of G on $\Delta^D(\mathbb{N}_G)$ is defined by the second coordinate. We encode every subcomplex $\Lambda \subset \Delta^D(\mathbb{N}_G)$ by an element $f_\Lambda \in \{0,1\}^{\Delta^D(\mathbb{N}_G)}$ such that $f_\Lambda(s) = 1$ if and only if s is contained in Λ and endow $\{0,1\}^{\Delta^D(\mathbb{N}_G)}$ with the product topology, *i.e.* the topology generated by all cylinder sets. The subset $\mathrm{Sub}(\Delta^D(\mathbb{N}_G)) \subset \{0,1\}^{\Delta^D(\mathbb{N}_G)}$ which consists of elements encoding G-invariant subcomplexes of $\Delta^D(\mathbb{N}_G)$ is closed.

Lemma 3.2.1. *There is a continuous map $\Psi \colon \mathcal{SC}_*^D(G) \to \mathrm{Sub}(\Delta^D(\mathbb{N}_G))$ such that*

$$\big[\Psi([K,o]), \{0\} \times X)\big] = [K,o],$$

for all $[K,o] \in \mathcal{SC}_*^D(G)$ and such that the elements of $\{0\} \times X$ are the only vertices contained in $\Psi([K,o])$ with first coordinate 0, where X is the isomorphism class of the orbit o.

Proof. We enumerate the set $\mathbb{N}_G$ of vertices in the following way: First, we enumerate the set of isomorphism classes of G-sets $X_1, \ldots, X_k \in \mathrm{Orb}(G)$, then we choose an order on the elements of each $X_i = \{x_{i1}, \ldots, x_{im_i}\}$ and finally we enumerate the elements of $\mathbb{N}_G$ diagonally

$$(0, x_{11}), \ldots, (0, x_{1m_1}), (0, x_{21}), \ldots, (0, x_{km_k}), (1, x_{11}), \ldots.$$

Now we can proceed as in Lemma 2.2.1 and enumerate the simplices of $\Delta^D(\mathbb{N}_G)$ in a diagonal way which gives us a map $\Upsilon \colon \mathbb{N} \to \Delta^D(\mathbb{N}_G)$. Hence, we have an order on $\mathrm{Sub}(\Delta^D(\mathbb{N}_G))$ given by the lexicographic order on $\{0,1\}^{\mathbb{N}}$:

$$f \prec g :\Leftrightarrow f(\Upsilon(i)) = g(\Upsilon(i)) \text{ for } 1 \leqslant i \leqslant k,\ f(\Upsilon(k+1)) = 1 \text{ and } g(\Upsilon(k+1)) = 0.$$

We define Ψ to map an isomorphism class $[K,o]$ to the minimal subcomplex Λ of $\Delta^D(\mathbb{N}_G)$ such that $(\Lambda, \{0\} \times X_i)) \in [K,o]$, where X_i is the isomorphism class of o, and such that the vertices in $\{0\} \times X_i$ are the only vertices of Λ with first coordinate equals zero. The second assumption is for some technical reasons which will appear later. The proof, that there exist a minimal subcomplex and that Ψ is continuous, is precisely the same as in Lemma 2.2.1, since the set of subcomplexes of $\Delta^D(\mathbb{N}_G)$ with vertices in

$$\bigsqcup_{j \neq i} \mathbb{N} \times X_j \sqcup \mathbb{N}_0 \times X_i$$

is closed. □

We follow Section 2.2 and refer to the details there.

Definition 3.2.2. Let μ be a random rooted simplicial G-complex. The *pth simplicial ℓ^2-chain module* of μ is

$$C_p^{(2)}(\mathcal{SC}_*^D(G), \mu) := \int_{\mathcal{SC}_*^D(G)}^{\oplus} C_p^{(2)}(\Psi([K,o])d\mu([K,o]).$$

The boundary operator

$$\partial_p = \int_{\mathcal{SC}_*^D(G)}^{\oplus} \partial_{p,[K,o]} d\mu \colon C_p^{(2)}(\mathcal{SC}_*^D(G), \mu) \to C_{p-1}^{(2)}(\mathcal{SC}_*^D(G), \mu)$$

and its adjoint

$$d^p = \int_{\mathcal{SC}_*^D(G)}^{\oplus} d^p_{[K,o]} d\mu \colon C_{p-1}^{(2)}(\mathcal{SC}_*^D(G), \mu) \to C_p^{(2)}(\mathcal{SC}_*^D(G), \mu)$$

commute with the induced unitary G-action on $C_p^{(2)}(\mathcal{SC}_*^D(G), \mu)$, since they commute fibrewise and G preserves fibres. Therefore, we have for every random rooted simplicial G-complex μ a chain complex $C_*^{(2)}(\mathcal{SC}_*^D(G), \mu)$ and a Laplace operator Δ_* which commutes with the G-action. Note that by Proposition 2.2.4 the boundary operator ∂_p, its adjoint d^p and the Laplace operator Δ_p are bounded operators.

Definition 3.2.3. We define the *pth simplicial ℓ^2-homology of a random rooted simplicial G-complex* μ as the Hilbert space

$$H_p^{(2)}(\mathcal{SC}_*^D(G), \mu) := \ker \Delta_p,$$

equipped with the natural unitary action of G.

The bounded decomposable operators

$$T = \int^{\oplus} T_{[K,o]} d\mu \colon C_p^{(2)}(\mathcal{SC}_*^D(G), \mu) \to C_p^{(2)}(\mathcal{SC}_*^D(G), \mu)$$

with the property that for almost all $[K,o] \in \mathcal{SC}_*^D(G)$, all simplicial G-equivariant isomorphisms $\varphi\colon \Psi([K,o]) \to \Psi([L,o'])$ and all $\sigma, \sigma' \in C_p^{(2)}(\mathcal{SC}_*^D(G), \mu)$ the identity

$$\langle T_{[K,o]}\sigma([K,o]), \sigma'([K,o])\rangle = \langle T_{[L,o']}\varphi_\sharp\sigma([K,o]), \varphi_\sharp\sigma'([K,o])\rangle$$

holds form a von Neumann algebra $\mathcal{A}_p(\mu)$. Of course, the operators defined by elements of G are contained in $\mathcal{A}_p(\mu)$, since we only consider isomorphisms which commute with the G-action. Moreover, Δ_p is in $\mathcal{A}_p(\mu)$ because ∂_* and d^* commute with the chain map $\varphi_\sharp$ induced by an isomorphism φ. Let

$$\mathcal{T}_p(x) := \{\chi_s \in C_p^{(2)}(\mathcal{SC}_*^D(G), \mu) \mid s \in \Delta^D(\mathbb{N}_G)(p),\ (0,x) \in s\}$$

be the set of characteristic vector fields χ_s of p-simplices s which contain the vertex $(0, x)$ for $x \in X$ and $X \in \mathrm{Orb}(G)$. For $T \in \mathcal{A}_p(\mu)$, we define

$$\mathrm{tr}_\mu(T) := \sum_{X \in \mathrm{Orb}(G)} \sum_{x \in X} \sum_{\tau \in \mathcal{T}_p(x)} \frac{\langle T\tau, \tau \rangle}{|X|(p+1)} \in [0, \infty).$$

The formula does not depend on the chosen orientation of τ. As in the nonequivariant case (before Proposition 2.2.5), one can verify that $\mathrm{tr}_\mu(ST) = \mathrm{tr}_\mu(TS)$ and hence we obtain a normal, faithful and finite trace on $\mathcal{A}_p(\mu)$ (cf. Proposition 2.2.5).

Definition 3.2.4. Let $\mathcal{K} \colon [K, o] \to \mathcal{K}([K, o])$ be a G-invariant subspace of $C_p^{(2)}(\mathcal{SC}_*^D(G), \mu)$ such that $\varphi_\sharp \mathcal{K}([K, o]) = \mathcal{K}([K, o'])$ for every isomorphism $\varphi \colon \Psi([K, o]) \to \Psi([K, o'])$. Then the projection $P_\mathcal{K} \colon [K, o] \to P_{\mathcal{K}([K,o])}$ onto $\mathcal{K}$ is an element of $\mathcal{A}_p(\mu)$ and we define the *von Neumann dimension of* $\mathcal{K}$ as

$$\dim_\mu \mathcal{K} := \mathrm{tr}_\mu(P_\mathcal{K}).$$

Example 3.2.5. Let L be a finite simplicial G-complex and μ_L^G the associated random rooted simplicial G-complex (cf. Example 3.1.4). Let $\mathcal{K}$ be a field of G-invariant subspaces of $C_n^{(2)}(\mathcal{SC}_*^D(G), \mu_L^G)$ as in Definition 3.2.4. Given an orbit $o \subseteq L^{(0)}$ and an isomorphism $\eta \colon L \to \Psi([L, o])$ we define

$$\mathcal{K}(L) = \eta^{-1}(\mathcal{K}([L, o])) \subseteq C_n^{(2)}(L);$$

this subspace does not depend on o and η. We compute the dimension of $\mathcal{K}$:

$$\begin{aligned}
\dim_{\mu_L^G}(\mathcal{K}) = \mathrm{tr}_{\mu_L^G}(P_\mathcal{K}) &= \sum_{X \in \mathrm{Orb}(G)} \sum_{x \in X} \sum_{\tau \in \mathcal{T}_p(x)} \frac{\langle P_\mathcal{K}\tau, \tau \rangle}{|X|(p+1)} \\
&= \sum_{y \in L^{(0)}} \sum_{x \in Gy} \sum_{\substack{s \in L(p) \\ x \in s}} \frac{\langle P_{\mathcal{K}(L)} s, s \rangle}{|L^{(0)}||Gy|(p+1)} \\
&= \sum_{y \in L^{(0)}} \sum_{\substack{s \in L(p) \\ y \in s}} \frac{\langle P_{\mathcal{K}(L)} s, s \rangle}{|L^{(0)}|(p+1)} = \frac{\dim_\mathbb{C} \mathcal{K}(L)}{|L^{(0)}|}
\end{aligned}$$

Definition 3.2.6. Let μ be a random rooted simplicial G-complex and σ be an irreducible representation of G. The ℓ^2*-multiplicity* of σ in the pth homology of μ is

$$m_p^{(2)}(\sigma, \mu) := \frac{1}{\chi_\sigma(1)} \dim_\mu \sigma(H_p^{(2)}(\mathcal{SC}_*^D(G), \mu)),$$

where $\sigma(H_p^{(2)}(\mathcal{SC}_*^D(G), \mu))$ denotes the direct sum of the irreducible representations of $H_p^{(2)}(\mathcal{SC}_*^D(G), \mu)$ isomorphic to σ.

In addition, we define the pth σ-Laplace operator

$$\Delta_{p,\sigma} := (\mathrm{Id} - P_\sigma) + \Delta_p,$$

where Δ_p is the Laplace operator on $C_p^{(2)}(\mathcal{SC}_*^D(G), \mu)$ and

$$P_\sigma = \frac{\chi_\sigma(1)}{|G|} \sum_{g \in G} \overline{\chi_\sigma}(g) g \in \mathbb{C}[G]$$

is the central idempotent defined in Theorem A.5. Note that $\sigma(H_p^{(2)}(\mathcal{SC}_*^D(G), \mu))$ is independent of the chosen decomposition of $H_p^{(2)}(\mathcal{SC}_*^D(G), \mu)$ into irreducible representations by Theorem A.5.

Remark 3.2.7. If $G = \{1\}$ is the trivial group and σ the unique irreducible representation of G, *i.e.* the trivial 1-dimensional representation, then $m_p^{(2)}(\sigma, \mu) = \beta_p^{(2)}(\mu)$ is simply the pth ℓ^2-Betti number of μ defined in Definition 2.2.8.

Example 3.2.8. Let K be a finite simplicial G-complex and μ_K^G the associated random rooted simplicial G-complex (cf. Example 3.1.4). If (σ, V) is an irreducible representation of G, then $m_p^{(2)}(\sigma, \mu_K^G)$ is the ordinary multiplicity of the representation σ in $H_p(K, \mathbb{C})$ divided by the number of vertices of K:

$$\begin{aligned} m^{(2)}(\sigma, \mu_K^G) &= \frac{1}{\chi_\sigma(1)} \dim_\mu \sigma(H_p^{(2)}(\mathcal{SC}_*^D(G), \mu_K^G)) \\ &= \frac{1}{\chi_\sigma(1)|K^{(0)}|} \sum_{s \in K(p)} \langle P_{\sigma(H_p(K,\mathbb{C}))} s, s\rangle \\ &= \frac{\dim_\mathbb{C}(\sigma(H_p(K,\mathbb{C}))}{\dim_\mathbb{C} V \cdot |K^{(0)}|} = \frac{m(\sigma, H_p(K,\mathbb{C}))}{|K^{(0)}|}. \end{aligned}$$

Lemma 3.2.9. *The operator $\Delta_{p,\sigma}$ is positive self-adjoint and its operator norm is bounded above by a constant $R(p, D)$ only depending on p and D. Moreover, the kernel of $\Delta_{p,\sigma}$ is $\sigma(H_p^{(2)}(\mathcal{SC}_*^D(G), \mu))$.*

Proof. Since Δ_p and $\mathrm{Id} - P_\sigma$ are positive self-adjoint and commute, $\Delta_{p,\sigma}$ also inherits these properties. By Proposition 2.2.4 we know that $\|\Delta_p\|$ is bounded above by a constant only depending on p and D and $\mathrm{Id} - P_\sigma$ is a projection, hence $\|\Delta_{p,\sigma}\| \leqslant \|\Delta_p\| + 1$.
Observe that a vector v lies in $\ker \Delta_{p,\sigma}$ if and only if $\langle \Delta_{p,\sigma} v, \Delta_{p,\sigma} v\rangle = 0$. Further, note that

$$\langle \Delta_{p,\sigma} v, \Delta_{p,\sigma} v\rangle = \|(\mathrm{Id} - P_\sigma)v\| + \|\Delta_p v\| + 2\langle \Delta_p(\mathrm{Id} - P_\sigma)v, v\rangle,$$

since Δ_p and $\mathrm{Id} - P_\sigma$ are self-adjoint and commute. All three summands are non-negative because $\Delta_p(\mathrm{Id} - P_\sigma)$ is positive. Hence,

$$\ker \Delta_{p,\sigma} = \ker \Delta_p \cap \ker(\mathrm{Id} - P_\sigma) = \ker \Delta_p \cap \operatorname{im} P_\sigma = \sigma(H_p^{(2)}(\mathcal{SC}_*^D(G), \mu)).$$

□

3.3. Approximation of ℓ^2-Multiplicities

In this section, we want to prove a statement analogous to Theorem 2.3.3 for ℓ^2-multiplicities; the strategy will be the same. Let σ be an irreducible representation of G and let $E_{\Delta_{p,\sigma}}$ denote the projection valued measure of $\Delta_{p,\sigma}$ we obtain from the Spectral Theorem (Theorem B.15). Further, we define the *spectral measure* $\nu_{p,\sigma}$ of $\Delta_{p,\sigma}$ to be

$$\nu_{p,\sigma} \colon \mathrm{Bor}(\mathbb{R}) \to [0, \infty), \quad \nu_{p,\sigma}(B) = \mathrm{tr}_\mu \, E_{\Delta_{p,\sigma}}(B).$$

Lemma 3.3.1. *Let $(\mu_n)_n$ be a sequence of random rooted simplicial G-complexes which converges weakly to μ_∞ and let $(\nu^n_{p,\sigma})_n$ and $\nu^\infty_{p,\sigma}$ the corresponding spectral measures of the pth σ-Laplace operators $\Delta_{p,\sigma}$ on their respective chain complexes $C^{(2)}_p(\mathcal{SC}^D_*(G), \mu_n)$ and $C^{(2)}_p(\mathcal{SC}^D_*(G), \mu_\infty)$. Then $(\nu^n_{p,\sigma})_n$ converges weakly to $\nu^\infty_{p,\sigma}$.*

Proof. For the sake of simplicity, we denote $\nu^n_{p,\sigma}$ by ν^n and $\nu^\infty_{p,\sigma}$ by ν^∞. By the same arguments as in Lemma 2.3.6, it is enough to show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \lambda^r d\nu^n(\lambda) = \int_{\mathbb{R}} \lambda^r d\nu^\infty(\lambda)$$

for some $r \in \mathbb{N}$. We have

$$\begin{aligned}\int_{\mathbb{R}} \lambda^r d\nu^n(\lambda) &= \mathrm{tr}_{\mu_n}(\Delta^r_{p,\sigma}) \\ &= \sum_{X \in \mathrm{Orb}(G)} \frac{1}{|X|(p+1)} \sum_{x\in X} \sum_{\tau \in \mathcal{T}_p(x)} \int_{\mathcal{SC}^D_*(G)} \langle \Delta^r_{p,\sigma}\tau([K,o]), \tau([K,o])\rangle d\mu_n([K,o])\end{aligned}$$

Let us consider $\Delta^r_{p,\sigma}$ and observe that

$$\begin{aligned}\Delta^r_{p,\sigma} = ((\mathrm{Id} - P_\sigma) + \Delta_p))^r &= \sum_{k=0}^{r} \binom{r}{k} (\mathrm{Id} - P_\sigma)^{r-k} \Delta_p^k \\ &= \Delta^r_p + \sum_{k=0}^{r-1} \binom{r}{k} \Delta^k_p (\mathrm{Id} - P_\sigma).\end{aligned}$$

Let $s \in \Delta^D(\mathbb{N}_G)(p)$ with $(0, x) \in s$ for some $x \in X$ and suppose that $s \in \Psi([K, o])$ for a rooted isomorphism class $[K, o]$ with $o \cong X$. Then $(\mathrm{Id} - P_\sigma)(s)$ lies in the 1-ball around the orbit $\{0\} \times X$, since P_σ is a linear combination of elements $g \in G$ which only act on the second coordinate, hence $(0, x)$ stays in $\{0\} \times X$. Further, $\Delta^k_p(s)$ lies in the $k+1$-ball around $(0, x)$ (cf. proof of Lemma 2.3.6). Therefore, $\langle \Delta^r_{p,\sigma} s, s\rangle$ only depends on the $r+1$-neighbourhood of the orbit $\{0\} \times X$. By the weak convergence of the random rooted simplicial G-complexes μ_n, we know that $\mu_n(U_{r+1}(\alpha))$ converges to $\mu_\infty(U_{r+1}(\alpha))$ for all finite rooted simplicial G-complexes α, hence

$$\lim_{n\to\infty} \int_{U_{r+1}(\alpha)} \langle \Delta^r_{p,\sigma}\tau, \tau\rangle d\mu_n = \int_{U_{r+1}(\alpha)} \langle \Delta^r_{p,\sigma}\tau, \tau\rangle d\mu_\infty.$$

Now the claim follows from the fact that $\mathcal{SC}^D_*(G)$ is a finite union of open sets of the form $U_{r+1}(\alpha)$. □

Theorem 3.3.2. *Let $(\mu_n)_n$ be a sequence of sofic random rooted simplicial G-complexes. If the sequence weakly converges to a random rooted simplicial G-complex μ_∞, then*

$$\lim_{n\to\infty} m_p^{(2)}(\sigma,\mu_n) = m_p^{(2)}(\sigma,\mu_\infty)$$

for every $p \in \mathbb{N}$ and every irreducible representation σ of G.

Proof. Since all the μ_n are sofic, it is sufficient to prove the theorem under the assumption that all the μ_n are random rooted simplicial G-complexes associated with finite simplicial G-complexes K_n (cf. Theorem 2.3.3). Let σ be an irreducible representation and let $(\nu_n)_n$ and ν_∞ the corresponding spectral measures of the pth σ-Laplace operator $\Delta_{p,\sigma}$. We have to show that $\nu_n(\{0\})$ converges to $\nu_\infty(\{0\})$. First, observe that by Proposition A.8 and Theorem A.11 there exist a finite Galois extension E of $\mathbb{Q}$ such that all characters of G take values in $\mathcal{O}_E$. We pick a basis of $C_p^{(2)}(K_n)$ by choosing an orientation for every p-simplex of K_n. Then $\Delta_{p,\sigma}$ can be realized on $C_p^{(2)}(K_n)$ as a $d \times d$-matrix A_σ with coefficients in $\frac{1}{|G|}\mathcal{O}_E$, where d is the numbers of p-simplices in K_n. The product $c_{p,\sigma}$ of the non-zero eigenvalues of A_σ is a coefficient in the characteristic polynomial and hence it is in $\frac{1}{|G|^d}\mathcal{O}_E$. Further, since the norm $\|\Delta_{p,\sigma}\|$ is bounded by a constant $R(p,D)$, only depending on p and the vertex degree D, we get an upper bound $|c_{p,\sigma}| \leqslant R(p,D)^d$. Consider the action of the Galois group $\mathrm{Gal}(E/\mathbb{Q})$ on the irreducible representations of G. Let $\eta \in \mathrm{Gal}(E/\mathbb{Q})$, then $\eta A_\sigma = A_{\eta(\sigma)}$ and hence $\eta c_{p,\sigma} = c_{p,\eta(\sigma)}$. Look at the element

$$c = |G|^{d[E:\mathbb{Q}]} \prod_{\eta\in\mathrm{Gal}(E/\mathbb{Q})} \eta c_{p,\sigma}.$$

We observe that c must be in $\mathcal{O}_E$, since $\eta c_{p,\sigma} \in \frac{1}{|G|^d}\mathcal{O}_E$. Further, c is fixed by the elements of $\mathrm{Gal}(E/\mathbb{Q})$ and therefore, c must be in $\mathbb{Q}$. We conclude that $c \in \mathbb{Z}\backslash\{0\}$ because $\mathbb{Q} \cap \mathcal{O}_E = \mathbb{Z}$. Now, let $I_0 = (0,\epsilon)$ for some $0 < \epsilon < 1$; we proceed as in Theorem 2.3.3:

$$\begin{aligned}
&1 \leqslant |c| \leqslant |c_{p,\sigma}||G|^{d[E:\mathbb{Q}]}R(p,D)^{d([E:\mathbb{Q}]-1)}\\
\Longrightarrow\ & |G|^{-d[E:\mathbb{Q}]}R(p,D)^{-d([E:\mathbb{Q}]-1)} \leqslant |c_{p,\sigma}| \leqslant \epsilon^{|K_n^{(0)}|\nu_n(I_0)}R(p,D)^d\\
\Longrightarrow\ & \nu_n(I_0) \leqslant \frac{d[E:\mathbb{Q}]\ln(|G|) + d[E:\mathbb{Q}]\ln(R(p,D))}{|K_n^{(0)}|\ln(1/\epsilon)}.
\end{aligned}$$

The number d of p-simplices in K_n is bounded by $\binom{D}{p}\frac{|K_n^{(0)}|}{p+1}$, hence we can make the inequality independent of K_n:

$$\nu_n(I_0) \leqslant \frac{\binom{D}{p}[E:\mathbb{Q}]\ln(|G|) + \binom{D}{p}[E:\mathbb{Q}]\ln(R(p,D))}{(p+1)\ln(1/\epsilon)}.$$

We finish the proof by applying two times Lemma 2.3.5 and letting ϵ tend to zero:

$$\begin{aligned}
\limsup_{n\to\infty} \nu_n(\{0\}) &\leqslant \nu(\{0\}) \leqslant \nu((0,\epsilon)) \leqslant \liminf_{n\to\infty} \nu_n(0,\epsilon)\\
&\leqslant \liminf_{n\to\infty} \nu_n(\{0\}) + \frac{\binom{D}{p}[E:\mathbb{Q}](\ln(|G|) + \ln(R(p,D)))}{(p+1)\ln(1/\epsilon)}.
\end{aligned}$$

□

Let G be a finite group and H a subgroup. Let (σ, V) be a finite dimensional complex representation of H. We have seen in the paragraph after Theorem A.7 that the multiplicity of an irreducible representation ρ of G in the induced representation $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ can be computed with the following formula:

$$m(\rho, \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V) = \sum_{\theta \in \mathrm{Irr}(H)} m(\theta, \rho|_H) m(\theta, V).$$

The following theorem provides an analogous reciprocity formula for induced sofic random rooted simplicial G-complexes.

Theorem 3.3.3. *Let μ be a sofic random rooted simplicial H-complex, then $\mathrm{Ind}_H^G(\mu)$ is sofic and moreover,*

$$m_p^{(2)}(\sigma, \mathrm{Ind}_H^G(\mu)) = \frac{|H|}{|G|} \sum_{\theta \in \mathrm{Irr}(H)} m(\theta, \sigma) m_p^{(2)}(\theta, \mu),$$

for every $\sigma \in \mathrm{Irr}(G)$ and $p \in \mathbb{N}_0$.

Proof. Since μ is sofic, we can find a sequence of finite simplicial complexes $(K_n)_n$ such that the associated random rooted simplicial complexes $(\mu_n)_n$ weakly converge to μ. Since induction is continuous (Lemma 3.1.6), the sequence $(\mathrm{Ind}_H^G(\mu_n))_n$ weakly converges to $\mathrm{Ind}_H^G(\mu)$. This shows that $\mathrm{Ind}_H^G(\mu)$ is sofic, since $\mathrm{Ind}_H^G(\mu_n)$ is the random rooted simplicial complex associated with the finite simplicial complex $G \times_H K_n$. The complex $G \times_H K_n$ consists of $|G : H|$ disjoint copies of K_n which are permuted by the action of G. Therefore,

$$H_p(G \times_H K_n, \mathbb{C}) \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} H_p(K_n, \mathbb{C}).$$

Applying Frobenius reciprocity (Theorem A.7), we obtain that

$$m\big(\sigma, \mathbb{C}[G] \otimes_{\mathbb{C}[H]} H_p(K_n, \mathbb{C})\big) = \sum_{\theta \in \mathrm{Irr}(H)} m(\theta, \sigma|_H) m(\theta, H_p(K_n, \mathbb{C})).$$

Now, using the relation of multiplicities and ℓ^2-multiplicities from Example 3.2.8 and applying Theorem 3.3.2 completes the proof. ☐

Corollary 3.3.4. *Let μ be a sofic random rooted simplicial complex and G be a finite group. For all $(\sigma, V) \in \mathrm{Irr}(G)$ the following identity holds:*

$$m_p^{(2)}(\sigma, \mathrm{Ind}_1^G(\mu)) = \frac{\dim_{\mathbb{C}} V}{|G|} \beta_p^{(2)}(\mu).$$

Proof. Just apply Theorem 3.3.3 to $H = \{1\}$. ☐

Example 3.3.5 (Sierpinski's triangle with rotation)**.** One last time, we come back to our running example, Sierpinski's triangle (Example 3.1.9). We want to compute the ℓ^2-multiplicities of the Benjamini-Schramm limit of the sequence T_n with rotation action of the cyclic group $G = \langle \rho \rangle$ of order 3. We have seen that the limit is a induced random rooted simplicial G-complex, $\operatorname{Ind}_1^G(\mu_\tau)$. By Corollary 3.3.4, we can compute the ℓ^2-multiplicities $m_p^{(2)}(\sigma, \operatorname{Ind}_1^G(\mu_\tau))$ of an irreducible representation $\sigma \in \operatorname{Irr}(G)$ by computing the ℓ^2-Betti numbers of μ_τ. We already know that $\beta_1^{(2)}(\mu_\tau) = \frac{1}{3}$ (cf. Example 2.3.9). Hence, we obtain that

$$m_1^{(2)}(\sigma, \operatorname{Ind}_1^G(\mu_\tau) = \frac{\dim_{\mathbb{C}} V}{|G|} \beta_1^{(2)}(\mu_\tau) = \frac{1}{9},$$

for every irreducible representation $\sigma \in \operatorname{Irr}(G)$, where we used the fact the all irreducible representations of G are 1-dimensional.

Chapter 4.

ℓ^2-Betti Numbers of Groupoids and Fibred Spaces

Gaboriau defined ℓ^2-Betti numbers $\beta_*^{(2)}(\Sigma, \mathcal{R})$ of an $\mathcal{R}$-simplicial complex Σ [Gab02], where $\mathcal{R}$ is a probability measure preserving equivalence relation on some probability measure space X. If Σ is contractible, then $\beta_*^{(2)}(\Sigma, \mathcal{R})$ is the ℓ^2-Betti number of the equivalence relation $\mathcal{R}$. Sauer defined ℓ^2-Betti numbers of a discrete measured groupoid $\mathcal{G}$ by a homological approach [Sau05] and proved that they coincide with the definition of Gaboriau if $\mathcal{G}$ is the orbit equivalence relation of a discrete group acting on a probability measure space X. Takimoto adapted the ideas of Gaboriau to define ℓ^2-Betti numbers of $\mathcal{G}$-simplicial complexes for a discrete measured groupoid $\mathcal{G}$ [Tak15]. Further, he showed that if the $\mathcal{G}$-simplicial complex Σ is contractible, the ℓ^2-Betti numbers of Σ coincide with the ℓ^2-Betti numbers of $\mathcal{G}$ defined by Sauer.

In this chapter, we will give a definition for *topological* $\mathcal{G}$-spaces (Definition 4.1.14), which are in some sense more flexible than $\mathcal{G}$-simplicial complexes. In order to define ℓ^2-Betti numbers of topological $\mathcal{G}$-spaces (Definition 4.3.4), we have to define the *singular groupoid homology* (Definition 4.3.3) of a $\mathcal{G}$-space. Our main result (Theorem 4.4.1) will be the equivalence of the ℓ^2-Betti numbers

$$\beta_n^{(2)}(\Sigma, \mathcal{G}) = \beta_n^{(2)}(|\Sigma|, \mathcal{G})$$

of a $\mathcal{G}$-simplicial complex Σ and its geometric realization $|\Sigma|$ regarded as topological $\mathcal{G}$-space.

The chapter is organized as follows. In Section 4.1, we define discrete measured groupoids and summarize some facts about the groupoid von Neumann algebra $L\mathcal{G}$. Further, we will give the definition of (*topological*) $\mathcal{G}$-spaces. We recall the theory of simplicial $\mathcal{G}$-complexes and their ℓ^2-Betti numbers in Section 4.2. In Section 4.3, we introduce the $\mathcal{G}$-singular groupoid homology of a $\mathcal{G}$-space and show that it is $\mathcal{G}$-homotopy invariant. Additionally, we prove a version of the Excision Theorem. Section 4.4 is dedicated to the proof of the main theorem, which adapts the structure of the proof from Hatcher [Hat02] of the equivalence of singular and simplicial homology in the classical theory, though we have to deal with a bunch of new problems.

4.1. Discrete Measured Groupoids

First, we will review some definitions and facts about discrete measures groupoids; for more details consider [Con79, Ram82, Sau02, Sau05]. Afterwards, in Section 4.1.1, we will define (topological) $\mathcal{G}$-spaces.

Definition 4.1.1. A *groupoid* is a small category in which every morphism is invertible. The set of *objects* X can be considered as a subset of the morphisms $\mathcal{G}$ by identifying every $x \in X$ with the identity morphism $\mathrm{Id}_x \in \mathcal{G}$; therefore, we identify the groupoid with the set of its morphisms $\mathcal{G}$. A groupoid fulfils the following properties:

- There is a *source map* $s\colon \mathcal{G} \to X$, $s(f\colon x \mapsto y) = x$
- and a *target map* $t\colon \mathcal{G} \to X$, $t(f\colon x \mapsto y) = y$.
- The *composition* $\circ\colon \mathcal{G}^{(2)} \to \mathcal{G}$ is associative, where $\mathcal{G}^{(2)} := \{g, h \in \mathcal{G} \mid s(g) = t(h)\}$. Usually we denote the composition by gh instead of $g \circ h$.
- For every g in $\mathcal{G}$ there is an *inverse* $g^{-1} \in \mathcal{G}$.

Definition 4.1.2. A *standard Borel space* is a space X together with a σ-algebra $\mathcal{S}$ such that $(X, \mathcal{S})$ is isomorphic to some Polish space Y with its Borel σ-algebra or, equivalently, there is a Polish topology $\mathcal{T}$ on X such that $\mathcal{S}$ is equal to the Borel σ-algebra generated by $\mathcal{T}$.

Definition 4.1.3. We call a groupoid $\mathcal{G}$ with the structure of a standard Borel space a *discrete measurable groupoid*, if

- the maps $i\colon \mathcal{G} \ni g \mapsto g^{-1} \in \mathcal{G}$, $\circ\colon \mathcal{G}^{(2)} \ni (g,h) \to gh \in G$ and $s, t\colon \mathcal{G} \to X$ are all measurable and
- $s^{-1}(\{x\})$ and $t^{-1}(\{x\})$ are countable for every $x \in X$.

Let μ be a probability measure on the space of objects X. The maps $x \mapsto \sharp\{s^{-1}(\{x\}) \cap A\}$ and $x \mapsto \sharp\{t^{-1}(\{x\}) \cap A\}$ are measurable for every measurable subset $A \subset \mathcal{G}$ and define σ-finite measures μ_s and μ_t on $\mathcal{G}$ by

$$\mu_s(A) = \int_X \sharp\{s^{-1}(\{x\}) \cap A\} d\mu$$

and analogously for t. If $\mu_s = \mu_t$ holds, we call μ an *invariant measure*.

Definition 4.1.4. A discrete measurable groupoid $\mathcal{G}$ together with an invariant probability measure μ is called a *discrete measured groupoid*. We denote the measure induced by μ on $\mathcal{G}$ by $\mu_{\mathcal{G}}$.

The condition that μ is invariant is equivalent to say that $\mu(s(E)) = \mu(t(E))$ for every *one-sheeted* set E, *i.e.* $s|_E$ and $t|_E$ are injective. The one-sheeted sets have the following useful property, which is essentially a theorem of Lusin-Novikov (Theorem 4.1.13):

Lemma 4.1.5. *Every discrete measurable groupoid can be decomposed into a countable disjoint union of one-sheeted sets.*

As usual, we denote by $L^\infty(\mathcal{G}, \mu_{\mathcal{G}})$ the complex-valued, measurable and $\mu_{\mathcal{G}}$-essentially bounded functions on $\mathcal{G}$, where we identify functions which coincide $\mu_{\mathcal{G}}$-almost everywhere. We call $\operatorname{supp}(\phi) := \{g \in \mathcal{G} \mid \phi(g) \neq 0\}$ the support of $\phi \in L^\infty(\mathcal{G}, \mu_{\mathcal{G}})$ and define

$$S(\phi)(x) := \sharp\{\operatorname{supp}(\phi) \cap s^{-1}(\{x\})\} \text{ and}$$
$$T(\phi)(x) := \sharp\{\operatorname{supp}(\phi) \cap t^{-1}(\{x\})\} \in \mathbb{N} \cup \infty.$$

Definition 4.1.6. The *groupoid ring* $\mathbb{C}\mathcal{G}$ of $\mathcal{G}$ is

$$\mathbb{C}\mathcal{G} := \{\phi \in L^\infty(\mathcal{G}, \mu_{\mathcal{G}}) \mid S(\phi), T(\phi) \in L^\infty(X)\}.$$

$\mathbb{C}\mathcal{G}$ is a ring with involution, where the addition is the pointwise addition, the multiplication is given by

$$(\phi\psi)(g) = \sum_{\substack{g_1, g_2 \in \mathcal{G} \\ g_1 g_2 = g}} \phi(g_1)\psi(g_2)$$

for $\phi, \psi \in \mathbb{C}\mathcal{G}$ and $g \in \mathcal{G}$, and the involution is defined by $\phi^*(g) = \overline{\phi(g^{-1})}$. Further, $\mathbb{C}\mathcal{G}$ contains $L^\infty(X)$ as a subring.

Lemma 4.1.7. [Sau05, Lemma 3.3] *Every element ϕ of $\mathbb{C}\mathcal{G}$ can be written as finite sum $\sum_{i=1}^n f_i \chi_{E_i}$ with $f_i \in L^\infty(X)$ and χ_{E_i} the characteristic function of a one-sheeted set E_i.*

Let $L^2(\mathcal{G}) := L^2(\mathcal{G}, \mu_{\mathcal{G}})$ be the Hilbert space associated with a discrete measured groupoid $\mathcal{G}$. The mapping $L_\phi\colon \mathbb{C}\mathcal{G} \to \mathbb{C}\mathcal{G},\ \psi \mapsto \phi\psi$, for $\phi \in \mathbb{C}\mathcal{G}$, extends by continuity to a bounded operator on $L^2(\mathcal{G}, \mu_{\mathcal{G}})$; we denote the bounded operators on $L^2(\mathcal{G})$ by $\mathcal{B}(L^2(\mathcal{G}))$ and call the action $\mathbb{C}\mathcal{G} \curvearrowright L^2(\mathcal{G}, \mu_{\mathcal{G}})$ the *left regular representation of* $\mathcal{G}$.

Definition 4.1.8. The *(left) von Neumann algebra $L\mathcal{G}$* of a discrete measured groupoid $\mathcal{G}$ is the weak closure of the operators $\{L_\phi \mid \phi \in \mathbb{C}\mathcal{G}\}$ in $\mathcal{B}(L^2(\mathcal{G}))$. By the von Neumann Bicommutant Theorem, this is equivalent to say that $L\mathcal{G} = \mathbb{C}\mathcal{G}''$.

We summarize some properties of $\mathbb{C}\mathcal{G}$ and $L\mathcal{G}$:

Lemma 4.1.9. [Sau05, Lemma 4.5, Lemma 4.8]

- *The groupoid ring $\mathbb{C}\mathcal{G}$ is flat over $L^\infty(X)$;*
- *$\mathbb{C}\mathcal{G}$ is a dimension-compatible $L^\infty(X)$-$L^\infty(X)$-bimodule;*
- *Since $L^\infty(X) \subset L\mathcal{G}$ is an inclusion of von Neumann algebras, $L\mathcal{G}$ is a dimension-compatible $L\mathcal{G}$-$L^\infty(X)$-bimodule.*

I did not found the following Lemma in the literature, therefore, I included a proof for the slightly more general Proposition C.13 in the appendix.

Lemma 4.1.10. *If A and B are $\mathbb{C}\mathcal{G}$-modules and $\varphi\colon A \to B$ is a $L^\infty(X)$-dimension isomorphic $\mathbb{C}\mathcal{G}$-module map, then*

$$\dim_{L\mathcal{G}} L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} A = \dim_{L\mathcal{G}} L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} B.$$

4.1.1. $\mathcal{G}$-Spaces

Definition 4.1.11. Let X be a standard Borel space and μ a probability measure on X. Further, let Y be a Borel space. We say Y is *fibred over* X if there is a Borel surjection $\pi\colon Y \to X$. For $x \in X$, we call the inverse image $\pi^{-1}(\{x\})$ the *fibre over* x and denote it by Y_x.

The *fibre product* of two fibred spaces Y_1 and Y_2 over X, with surjections π_1 and π_2, is defined by

$$Y_1 * Y_2 := \{(y_1, y_2) \in Y_1 \times Y_2 \mid \pi_1(y_1) = \pi_2(y_2)\}.$$

This is again a fibred space over X. Similarly, we define the fibre product $\ast_{n\in I} Y_n$ for a countable set I.

In the following, we will distinguish between two types of fibred spaces. We say Y is a *discrete fibred space* if Y is a standard Borel space and each fibre Y_x is countable and we call Y a *topological fibred space* if each fibre Y_x is a topological space. Note that we do not assume Y to be standard for topological fibred spaces. Further, we say a fibred space Y is *contractible* if μ-almost every fibre Y_x is contractible.

A *fibred map* φ from Y_1 to Y_2 is a Borel map such that $\pi_1 = \pi_2 \circ \varphi$. If Y_1, Y_2 are topological fibred spaces, we call a Borel fibred map *topological* if $f_x\colon Y_{1,x} \to Y_{2,x}$ is continuous for all $x \in X$ and if it is a homeomorphism for all fibres, we call it a *$\mathcal{G}$-homeomorphism*.

Remark 4.1.12. In the case that Y is discrete, the probability measure μ on X induces a measure μ_Y on Y via $\mu_Y(U) = \int_X \sharp\{\pi^{-1}(\{x\}) \cap U\} d\mu$. Further, by the following theorem, every discrete fibred space has a countable partition into Borel sets E_n such that $\pi|_{E_n}$ is injective and hence, there is an injection from Y to $X \times \mathbb{N}$.

Theorem 4.1.13 (Lusin-Novikov). *Let X, Y be standard Borel spaces and $E \subset X \times Y$ be Borel. Further, let $\pi\colon X \times Y \to X$ be the projection onto X. If every section $\pi^{-1}(x) \cap E$ is countable, then there is a countable partition $E = \bigcup_n E_n$, where E_n is Borel and $\pi|_{E_n}$ is injective for every n.*

Definition 4.1.14. Let $\mathcal{G}$ be a measurable groupoid considered as a fibred space over its objects X with surjection $s\colon \mathcal{G} \to X$. A *$\mathcal{G}$-space* is a fibred space $\pi\colon Y \to X$ together with a Borel map $(g, y) \to gy$ from $\mathcal{G} * Y$ to Y, called the *$\mathcal{G}$-action*, such that for all $g, h \in \mathcal{G}$ and $y \in Y$

- $g(hy) = (gh)y$,
- $\pi(gy) = r(g)$ and
- $\pi(y)y = y$.

Again, if Y is a discrete or topological fibred space we call it a *discrete or topological $\mathcal{G}$*-space, respectively. Every $\mathcal{G}$-invariant Borel subset Z of a $\mathcal{G}$-space Y is a $\mathcal{G}$-space as well. For topological $\mathcal{G}$-spaces we say Z is a *closed or open $\mathcal{G}$-subspace* if Z is a $\mathcal{G}$-invariant Borel subset and $Z_x \subset Y_x$ is a closed or open subspace for all $x \in X$, respectively. For $\mathcal{G}$-spaces Y_1 and Y_2 the fibre product is again a $\mathcal{G}$-space with $\mathcal{G}$-action given by $g(y_1, y_2) = (gy_1, gy_2)$. Furthermore,

we call a fibred map $\varphi\colon Y_1 \to Y_2$ a *$\mathcal{G}$-map* if it is $\mathcal{G}$-equivariant, *i.e.* $\varphi(gy) = g\varphi(y)$ for all $y \in Y$ and $g \in \mathcal{G}$.
A measurable subset $D \subset Y$ of a discrete $\mathcal{G}$-space is called *fundamental domain* for the $\mathcal{G}$-action if $t\colon \mathcal{G} * D \to X$ and $\pi\colon Y \to X$ are isomorphic as $\mathcal{G}$-fibred spaces. If $gy = y$ implies $g = \pi(y)$ for μ_Y-almost all $y \in Y$, we call the $\mathcal{G}$-action *essentially free*; but, since we can restrict an essentially free action to a free one, we will usually assume that the action is free.

We cite the following lemma, since it is essential for the proof of the immediately succeeding one.

Lemma 4.1.15. [Kec95, Corollary 15.2] *Let X and Y be standard Borel spaces and $f\colon X \to Y$ be Borel. If $A \subset X$ is Borel and $f|_A$ is injective, then $f(A)$ is Borel and f is a Borel isomorphism of A and $f(A)$.*

Lemma 4.1.16. *Let $\pi\colon Y \to X$ be a discrete $\mathcal{G}$-space with free action and fundamental domain D. Then there exists an injective $\mathcal{G}$-map from Y to $\bigsqcup_{i\in I} \mathcal{G} = \mathcal{G} \times I$, where the surjection $\mathcal{G} \times I \to X$ is given by $(g, i) \mapsto t(g)$ and the $\mathcal{G}$-action is given by $g_1(g_2, i) = (g_1 g_2, i)$.*

Proof. Since $\pi^{-1}(\{x\}) \cap D$ is countable, there exists, by Theorem 4.1.13, a countable partition $D = \bigsqcup_{i\in I} D_i$ such that $\pi|_{D_i}$ is injective. Hence we have $Y \cong \mathcal{G} * D = \bigsqcup_{i\in I} \mathcal{G} * D_i$. The map $\pi|_{D_i}\colon D_i \to \pi(D_i) =: X_i$ is a Borel isomorphism, by Lemma 4.1.15, and we have a Borel injection $\mathcal{G} * X_i \to Y$ given by $(g, x) \mapsto g\pi|_{D_i}^{-1}(x)$ with image $\mathcal{G}D_i$. Therefore, $\mathcal{G}D_i$ is Borel and, again by Lemma 4.1.15, $f_i\colon \mathcal{G}D_i \to \mathcal{G}X_i;\ gy \mapsto g$ is a Borel isomorphism. Thus, the injection we are looking for is given by $f\colon Y \to \mathcal{G} \times I$, with $f|_{\mathcal{G}D_i} := f_i$. $\square$

4.2. Simplicial Groupoid Homology

We summarize the theory of simplicial $\mathcal{G}$-complexes of Takimoto [Tak15] which generalizes the theory of probability measure preserving equivalence relations of Gaboriau [Gab02] to groupoids. Let $\mathcal{G}$ be a discrete measured groupoid with object space X and probability measure μ on X. Recall that we denote by $\mathfrak{S}_n$ the symmetric group on n elements.

Definition 4.2.1. A *simplicial $\mathcal{G}$-complex* is a sequence $\Sigma = (\Sigma^{(n)})_{n\geqslant 0}$ of discrete $\mathcal{G}$-spaces, where we assume the action to be free and admitting a fundamental domain, such that each $\Sigma^{(n)}$ is a $\mathcal{G}$-invariant measurable subset of the $n+1$ times fibre product of $\Sigma^{(0)}$ with the restriction of the $\mathcal{G}$-action of the fibre product to $\Sigma^{(n)}$ and such that

- if $(v_0, .., v_n) \in \Sigma^{(n)}$, then $(v_{\varsigma(0)}, ..., v_{\varsigma(n)}) \in \Sigma^{(n)}$ for any permutation $\varsigma \in \mathfrak{S}_{n+1}$,
- if $(v_0, .., v_n) \in \Sigma^{(n)}$, then $v_i \neq v_j$ for $i \neq j$ and
- if $(v_0, ..., v_n) \in \Sigma^{(n)}$, then $(v_1, ..., \widehat{v}_j, ..., v_n) \in \Sigma^{(n-1)}$, for every $0 \leqslant j \leqslant n$.

Remark 4.2.2. The fibres $\Sigma_x = (\Sigma_x^{(n)})_{n\geqslant 0}$ are usual simplicial complexes and the geometric realization $|\Sigma|$ of Σ is given by the disjoint union of the fibrewise geometric realizations $|\Sigma_x|$. By Lemma 4.1.16, there is an injection of $\mathcal{G}$-spaces $\Sigma^{(0)} \to \mathcal{G} \times I$, given by $(gv) \mapsto (g, i)$ for $v \in F_i$, where $F = \bigsqcup_{i\in I} F_i$ is a fundamental domain of $\Sigma^{(0)}$. Further, by Lemma 4.1.5, we have an injection of $\mathcal{G}$ into $X \times \mathbb{N}$, given by $g \mapsto (x, n)$ for $g \in E_n$ and $x = t(g)$, where E_n is a partition of $\mathcal{G}$ into one-sheeted sets. Thus, there is an injection of $\Sigma^{(0)}$ into $X \times \mathbb{N} \times I$ which extends to an embedding $\Sigma \to X \times \Delta(\mathbb{N} \times I)$, where $\Delta(\mathbb{N} \times I)$ is the simplicial complex consisting of all finite subsets of $\mathbb{N} \times I$. The embedding is given by

$$(v_0, ..., v_m) \mapsto \big((x, n_0, i_0), (x, n_1, i_1), ..., (x, n_m, i_m)\big).$$

Such an embedding yields a Borel injection $\phi\colon |\Sigma| \to X \times |\Delta(\mathbb{N} \times I)|$, where we define the (standard) Borel structure on $|\Sigma|$ to be the restriction of the product Borel structure on $X \times |\Delta(N \times I)|$. By Lemma 4.1.15, $\phi(|\Sigma|) \subset X \times |\Delta(\mathbb{N} \times I)|$ is a Borel subset and the Borel structure is independent of the choice of the embedding of $\Sigma^{(0)}$ into $X \times \mathbb{N} \times I$ (see [Sau09, Lemma 2.13]). Hence, $|\Sigma|$ is a topological $\mathcal{G}$-space in the sense of Definition 4.1.14. The $\mathcal{G}$-action is given by the linear extension of the action of $\mathcal{G}$ on the vertices, *i.e.*

$$gp = \sum_{i=0}^{n} t_i g v_i$$

for a point $p = \sum_{i=0}^{n} t_i v_i$ in the geometric realization of the simplex $(v_0, ..., v_n)$ and $g \in \mathcal{G}$.

We say a simplicial $\mathcal{G}$-complex is *uniformly locally bounded (ULB)* if $\Sigma^{(0)}$ has a fundamental domain F such that

$$\int \sharp\, \{\, \pi^{-1}(x) \cap F \,\}\, d\mu < \infty$$

and there exists an integer N such that $\sharp\, \{\, s \in \Sigma_x \mid v \in s \,\} \leqslant N$ holds for all $v \in \Sigma^{(0)}$ and almost all $x \in X$. A $\mathcal{G}$-subcomplex of Σ is a simplicial $\mathcal{G}$-complex Ξ such that each $\Xi^{(n)}$

is a $\mathcal{G}$-subspace of $\Sigma^{(n)}$. An *ULB exhaustion* $(\Xi_i)_{i\geqslant 1}$ of Σ is a sequence of $\mathcal{G}$-subcomplexes such that each Ξ_i is ULB and such that $(\Xi_{i,x}^{(n)})_{i\geqslant 1}$ is an increasing sequence of $\Sigma_x^{(n)}$ satisfying $\varinjlim \Xi_{i,x}^{(n)} = \Sigma_x^{(n)}$ for almost all $x \in X$.

We denote by $C_n^b(\Sigma)$ the space of (equivalence classes of) measurable functions $f \in L^\infty(\Sigma^{(n)})$ such that

- $\sharp\{\pi_{\Sigma^{(n)}}^{-1}(\{x\}) \cap \operatorname{supp}(f)\} \in L^\infty(X)$ and
- $f(\varsigma s) = \operatorname{sgn}(\varsigma) f(s)$, for all $\varsigma \in \mathfrak{S}_{n+1}$;

and by $C_n^{(2)}(\Sigma)$ the (equivalence classes of) measurable functions $f\colon \Sigma^{(n)} \to \mathbb{C}$ such that

- $\int_X \sum_{s\in\Sigma_x^{(n)}} |f(s)|^2 d\mu < \infty$ and
- $f(\varsigma s) = \operatorname{sgn}(\varsigma) f(s)$, for all $\varsigma \in \mathfrak{S}_{n+1}$.

In other words, $C_n^{(2)}(\Sigma)$ is the direct integral Hilbert space $\int_X^\oplus C_n^{(2)}(\Sigma_x) d\mu$ of square-summable simplicial chains (see Definition B.5).

$C_n^b(\Sigma)$ and $C_n^{(2)}(\Sigma)$ have a natural $\mathbb{C}\mathcal{G}$-module structure given by

$$(\omega f)(s) := \sum_{g\in r^{-1}(\{\pi(s)\})} \omega(g) f(g^{-1}s)$$

for $\omega \in \mathbb{C}\mathcal{G}$ and $s \in \Sigma^{(n)}$. Furthermore, $C_n^{(2)}(\Sigma)$ is a Hilbert $L\mathcal{G}$-module. The boundary operators $\partial_{n,x}$ of the simplicial chain complexes of the fibres $\Sigma_x^{(n)}$ define a $\mathbb{C}\mathcal{G}$-module map $\partial_n\colon C_n^b(\Sigma) \to C_{n-1}^b(\Sigma)$ by $(\partial_n f)(s) = \partial_{n,x} f_x(s)$ for $s \in \Sigma_x^{(n-1)}$. Additionally, if Σ is ULB, we get a bounded $L\mathcal{G}$-module map $\partial_n^{(2)}\colon C_n^{(2)}(\Sigma) \to C_n^{(2)}(\Sigma)$, which turns $C_*^{(2)}(\Sigma)$ into a Hilbert $L\mathcal{G}$-chain complex.

We recall Takimoto's definition of ℓ^2-Betti numbers of a simplicial $\mathcal{G}$-complex [Tak15].

Definition 4.2.3. For an ULB simplicial $\mathcal{G}$-complex Σ the *nth reduced ℓ^2-Homology* is

$$\overline{H}_n^{(2)}(\Sigma,\mathcal{G}) := \ker \partial_n^{(2)} / \overline{\operatorname{im} \partial_{n+1}^{(2)}}$$

and the nth *ℓ^2-Betti number of* Σ is

$$\beta_n^{(2)}(\Sigma,\mathcal{G}) := \dim_{L\mathcal{G}} \overline{H}_n^{(2)}(\Sigma,\mathcal{G}).$$

For an arbitrary simplicial $\mathcal{G}$-complex Σ we have to pick an ULB exhaustion $(\Sigma_i)_{i\geqslant 1}$, where for $i \leqslant j$ we have bounded $L\mathcal{G}$-chain maps $J_n^{i,j}\colon C_n^{(2)}(\Sigma_i) \to C_n^{(2)}(\Sigma_j)$ induced by the inclusions $\Sigma_i \subset \Sigma_j$. We denote by $\overline{H}_n^{(2)}(J_*^{i,j})\colon \overline{H}_n^{(2)}(\Sigma_i,\mathcal{G}) \to \overline{H}_n^{(2)}(\Sigma_j,\mathcal{G})$ the map induced by $J_*^{i,j}$ and define the *nth ℓ^2-Betti number of* Σ as

$$\beta_n^{(2)}(\Sigma,(\Sigma_i)_{i\geqslant 1},\mathcal{G}) := \lim_{i\geqslant 1}\lim_{j\geqslant i} \dim_{L\mathcal{G}} \overline{\operatorname{im}(\overline{H}_n^{(2)}(J_*^{i,j}))}.$$

We can omit $(\Sigma_i)_{i\geqslant 1}$ and just write $\beta_n^{(2)}(\Sigma,\mathcal{G})$ because of the following proposition:

Proposition 4.2.4. [Tak15, Proposition 3.7] *For any simplicial $\mathcal{G}$-complex Σ and any ULB exhaustion $(\Sigma_i)_{i\geqslant 1}$ of Σ we have*

$$\begin{aligned}\beta_n^{(2)}(\Sigma, (\Sigma_i)_{i\geqslant 1}, \mathcal{G}) &= \dim_{L\mathcal{G}} H_n(L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} \varinjlim C_*^b(\Sigma_i)) \\ &= \dim_{L\mathcal{G}} H_n(L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} C_*^b(\Sigma)).\end{aligned}$$

Takimoto justifies this definition of ℓ^2-Betti numbers of discrete measured groupoids by the next theorem. Recall that Sauer [Sau05] defined the nth ℓ^2-Betti number of a discrete measured groupoid $\mathcal{G}$ by $\beta_n^{(2)}(\mathcal{G}) := \dim_{L\mathcal{G}} \operatorname{Tor}_n^{\mathbb{C}\mathcal{G}}(L\mathcal{G}, L^\infty(X))$.

Theorem 4.2.5. [Tak15, Theorem 3.6] *If Σ is a contractible simplicial $\mathcal{G}$-complex, then*

$$\beta_n^{(2)}(\Sigma, \mathcal{G}) = \beta_n^{(2)}(\mathcal{G})$$

for every $n \in \mathbb{N}_0$.

This is all that we need to know about the theory of simplicial $\mathcal{G}$-complexes. In the next section, we will introduce the singular groupoid homology of a topological $\mathcal{G}$-space.

4.3. Singular Groupoid Homology

In this section, $\mathcal{G}$ will always be a discrete measured groupoid (Definition 4.1.4) with objects X and probability measure μ on X.

Definition 4.3.1. Let $\pi_Y : Y \to X$ be a topological $\mathcal{G}$-space. A *$\mathcal{G}$-singular n-simplex* is a topological fibred map $\sigma\colon X \times \Delta^n \to Y$, where Δ^n is the standard n-simplex, *i.e.* $\pi_Y(\sigma(x,p)) = x$ and $\sigma|_{\{x\}\times\Delta^n}$ is continuous for all $x \in X$.

Definition 4.3.2. Let $S_n^{\mathcal{G}}$ be the set of all $\mathcal{G}$-singular n-simplices. We define the $L^\infty(X)$-module of *$\mathcal{G}$-singular n-chains of Y* as

$$C_n^{sing}(Y,\mathcal{G}) := \Big\{ c = \sum_{\sigma\in S_n^{\mathcal{G}}} c_\sigma \sigma \mid c_\sigma \in L^\infty(X) \Big\} / \sim,$$

where two chains are equivalent $c \sim c'$ if $c|_{\{x\}\times\Delta^n} = c'|_{\{x\}\times\Delta^n}$ for almost all $x \in X$. To relax the notation, we will usually denote the space of $\mathcal{G}$-singular n-chains just by $C_n(Y,\mathcal{G})$.

The $\mathcal{G}$-action on Y induces a $\mathcal{G}$-action on $C_n(Y,\mathcal{G})$ in the following way:

$$(g\cdot c)(x,p) = \sum_{\sigma\in S_n^{\mathcal{G}}} c_\sigma(g^{-1}x) g\sigma(g^{-1}x,p),$$

for $g \in \mathcal{G}$, $x = t(g)$ and $c \in C_n(Y,\mathcal{G})$. We will show that this defines a natural $\mathbb{C}\mathcal{G}$-module structure on $C_n(Y,\mathcal{G})$. By Lemma 4.1.7, we know that we can write $\omega \in \mathbb{C}\mathcal{G}$ as a finite sum $\sum_{i=1}^k \omega_i \chi_{E_i}$, where $\omega_i \in L^\infty(X)$ and the E_i's are Borel subsets of $\mathcal{G}$ such that $s|_{E_i}$ and $t|_{E_i}$ are injective. We assume that $\omega = \omega_E \chi_E$ and $c = c_\sigma \sigma \in C_n(Y,\mathcal{G})$, then

$$(\omega c)(x,p) = \omega_E(x) c_\sigma(s(t|_E^{-1}(x))) t|_E^{-1}(x) \sigma(s(t|_E^{-1}(x)), p),$$

which is again a $\mathcal{G}$-singular n-simplex

$$\sigma'(x,p) = t|_E^{-1}(x)\sigma(s(t|_E^{-1}(x)),p)$$

together with the coefficient

$$\omega_E(x) c_\sigma(s(t|_E^{-1}(x))) \in L^\infty(X).$$

We define boundary operators $\partial_n\colon C_n(Y,\mathcal{G}) \to C_{n-1}(Y,\mathcal{G})$ by restricting the $\mathcal{G}$-singular n-simplices to the $(n-1)$-faces Δ_i^{n-1} of Δ^n:

$$\partial_n \sigma = \sum_{i=0}^n (-1)^i \sigma|_{A\times\Delta_i^{n-1}},$$

where we consider $\sigma\colon X \times \Delta_i^{n-1} \to Y$ as a map from $X \times \Delta^{n-1}$ to Y. It is clear that this defines a $\mathbb{C}\mathcal{G}$-module map and that $\partial_n \circ \partial_{n-1} = 0$. Hence, we have a chain complex $C_*(Y,\mathcal{G})$ of $\mathbb{C}\mathcal{G}$-modules:

$$\dots \xrightarrow{\partial_{n+1}} C_n(Y,\mathcal{G}) \xrightarrow{\partial_n} C_{n-1}(Y,\mathcal{G}) \xrightarrow{\partial_{n-1}} \dots$$

Definition 4.3.3. We define the nth *$\mathcal{G}$-singular homology* of a topological $\mathcal{G}$-space Y as

$$H_n^{sing}(Y, L\mathcal{G}) := H_n(L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} C_*^{sing}(Y, \mathcal{G})).$$

Definition 4.3.4. The *nth ℓ^2-Betti number* of a topological $\mathcal{G}$-space Y is the von Neumann dimension of its nth $\mathcal{G}$-singular homology

$$\beta_n^{(2)}(Y, \mathcal{G}) := \dim_{L\mathcal{G}} H_n^{sing}(Y, L\mathcal{G}).$$

Remark 4.3.5. Analogously to the ordinary singular chains of a topological space, we define relative $\mathcal{G}$-singular chains by

$$C_*^{sing}(Y, Z, \mathcal{G}) := C_*^{sing}(Y, \mathcal{G}) / C_*^{sing}(Z, \mathcal{G})$$

for a pair (Y, Z) of $\mathcal{G}$-spaces, where Z is a $\mathcal{G}$-subspace of Y.

4.3.1. $\mathcal{G}$-Homotopy Invariance

The proof of the homotopy invariance and the proof of the Excision Theorem 4.3.10 in the next subsection are adapted from [Hat02] to our setting.

Definition 4.3.6. Let $f, g\colon Y_1 \to Y_2$ be topological $\mathcal{G}$-maps between $\mathcal{G}$-spaces Y_1 and Y_2. We say f and g are *$\mathcal{G}$-homotopic*, and denote it by $f \simeq g$, if there is a topological $\mathcal{G}$-map

$$F\colon Y_1 * (X \times [0,1]) \to Y_2,$$

such that $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$ almost everywhere, where $Y_1 * (X \times [0,1])$ is the fibre product of Y_1 and the constant $\mathcal{G}$-space $X \times [0,1]$. For pairs of $\mathcal{G}$-spaces (Y_1, Z_1) and (Y_2, Z_2) we further have to assume that $F(Z_1 * (X \times [0,1])) \subset Z_2$. Consequently, we call a topological $\mathcal{G}$-map $f\colon Y_1 \to Y_2$ a *$\mathcal{G}$-homotopy equivalence* if there is a topological $\mathcal{G}$-map $g\colon Y_2 \to Y_1$ such that $f \circ g \simeq \mathrm{Id}_{Y_2}$ and $g \circ f \simeq \mathrm{Id}_{Y_1}$.

Proposition 4.3.7. *If two $\mathcal{G}$-maps $f, g\colon Y_1 \to Y_2$ are $\mathcal{G}$-homotopic, then they induce chain homotopic maps*

$$f_\sharp \simeq g_\sharp\colon C_n^{sing}(Y_1, \mathcal{G}) \to C_n^{sing}(Y_2, \mathcal{G}),$$

for every $n \in \mathbb{N}_0$. In particular, they induce the same homomorphism in the homology

$$f_* = g_*\colon H_n^{sing}(Y_1, L\mathcal{G}) \to H_n^{sing}(Y_2, L\mathcal{G}).$$

Proof. First, we remark that the induced maps between chain complexes

$$f_\sharp, g_\sharp \colon C_n(Y_1, \mathcal{G}) \to C_n(Y_2, \mathcal{G})$$

are given by composing each $\mathcal{G}$-singular simplex $\sigma \colon X \times \Delta^n \to Y_1$ with f or g, respectively. Let us denote the homotopy by $F \colon Y_1 * (X \times [0,1])$ with $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$. The idea is to consider the prism $\Delta^n \times [0,1]$ as if it were triangulated with vertices $\{ v_0, ..., v_n \}$ of $\Delta^n \times \{ 0 \}$ and $\{ w_0, ..., w_n \}$ of $\Delta^n \times \{ 1 \}$. We define the prism operator $P \colon C_n(Y_1, \mathcal{G}) \to C_{n+1}(Y_2, \mathcal{G})$ as follows:

$$P(\sigma) = \sum_{i=1}^{n} (-1)^i F \circ (\sigma * \mathrm{Id}_{X \times [0,1]})|_{[v_0, \ldots, v_i, w_i, \ldots, w_n]}.$$

The composition

$$F \circ (\sigma * \mathrm{Id}_{X \times [0,1]}) \colon (X \times \Delta^n) * (X \times [0,1]) \to Y_1 * (X \times [0,1]) \to Y_2$$

is again measurable and fibrewise continuous. A straightforward computation (cf. [Hat02, Theorem 2.10]) shows that

$$\partial P + P \partial = g_\sharp - f_\sharp,$$

which finishes the proof. $\square$

Definition 4.3.8. A $\mathcal{G}$*-deformation retract* of a $\mathcal{G}$-space Y is a $\mathcal{G}$-subspace Z such that there is a $\mathcal{G}$-homotopy from Id_Y to a *retraction* $r \colon Y \to Y$, *i.e.* a topological $\mathcal{G}$-map with $r|_Z = \mathrm{Id}_Z$.

Corollary 4.3.9. *If $f \colon Y \to Z$ is a $\mathcal{G}$-homotopy equivalence, in particular, if Y deformation retracts onto Z, then f induces an isomorphism*

$$f_* \colon H_n^{sing}(Y, L\mathcal{G}) \to H_n^{sing}(Z, L\mathcal{G}),$$

for every $n \in \mathbb{N}_0$.

4.3.2. Excision

Proposition 4.3.10 (Excision Theorem). *Let Y be a topological $\mathcal{G}$-space and $A, B \subset Y$ be $\mathcal{G}$-subspaces such that for almost all $x \in X$ the interiors of A_x and B_x cover Y_x. Then the inclusion $(B, A \cap B) \hookrightarrow (Y, A)$ induces an $L\mathcal{G}$-dimension isomorphism*

$$\dim_{L\mathcal{G}} H_n^{sing}(B, A \cap B, L\mathcal{G}) = \dim_{L\mathcal{G}} H_n^{sing}(Y, A, L\mathcal{G}),$$

for every $n \in \mathbb{N}_0$.

In order to proof Proposition 4.3.10, we need a tool which allows us to compute the dimension of the homology by considering only "small" simplices. Let $\mathcal{U} = \{ U_j \}_j$ be a countable collection of $\mathcal{G}$-subspaces of Y such that the interiors of $\mathcal{U}_x = \{ U_{j,x} \}_j$ form a cover of Y_x for almost all $x \in X$. We define $C_n^{\mathcal{U}}(Y, \mathcal{G})$ to be the module of $\mathcal{G}$-singular chains such that the image of each $\mathcal{G}$-singular simplex is contained in an element of $\mathcal{U}$.

Proposition 4.3.11. *The inclusions $i_n\colon C_n^{\mathcal{U}}(Y,\mathcal{G}) \hookrightarrow C_n(Y,\mathcal{G})$ induces $L\mathcal{G}$-dimension isomorphisms in homology*
$$\dim_{L\mathcal{G}} H_n(L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} C_*^{\mathcal{U}}(Y,\mathcal{G})) = \dim_{L\mathcal{G}} H_n^{sing}(Y, L\mathcal{G}),$$
for every $n \in \mathbb{N}_0$.

Proof. We need a construction called barycentric subdivision. The points of a standard n-simplex $[v_0, ..., v_n]$, with vertices $\{ v_0, ..., v_n \}$, can be described in barycentric coordinates as linear combinations $\sum_{i=0}^n t_i v_i$ with $t_i \in [0,1]$. The barycentre of $[v_0, ..., v_n]$ is the point $b = \sum_{i=0}^n \frac{1}{n+1} v_i$. We define the barycentric subdivision of an n-simplex as its decomposition in n-simplices $[b, w_0, ..., w_{n-1}]$, where $[w_0, ..., w_{n-1}]$ is an $(n-1)$-simplex in the barycentric subdivision of a face $[v_0, ..., \hat{v}_i, ..., v_n]$. The induction starts with the barycentric subdivision of $[v_0]$ which is just $[v_0]$.

In the following, we will need the fact that the diameter of a simplex in the barycentric subdivision of $[v_0, ..., v_n]$ is $\frac{n}{n+1}$ times the diameter of $[v_0, ..., v_n]$. The diameter of a simplex is the maximal distance between any two of its points. The first observation we make is that the diameter is equal to the distance between any two of its vertices, since for two points v and $\sum_i t_i v_i$ we have the inequality
$$|v - \sum_i t_i v_i| \leqslant \sum t_i |v - v_i| \leqslant \sum_i t_i \max |v - v_i| = \max |v - v_i|,$$
where for vertices of $[v_0, ..., v_n]$ equality prevails. Therefore, we only have to verify that the distance between any two vertices w_i, w_j in an n-simplex in the barycentric subdivision of $[v_0, ..., v_n]$ is at most $\frac{n}{n+1}$ times the diameter of $[v_0, ..., v_n]$. If neither w_i nor w_j is the barycentre of $[v_0, ..., v_n]$, then they both lie in a face of $[v_0, ..., v_n]$ and we are done by induction. So let us assume that one of them is b and the other one a vertex v_i of $[v_0, ..., v_n]$. Hence, we have
$$|b - v_i| = |\sum_{j=0}^n \frac{1}{n+1} v_j - v_i| \leqslant \sum_{j=0}^n \frac{1}{n+1} |v_j - v_i| = \frac{n}{n+1} \operatorname{diameter}([v_0, ..., v_n]),$$
since $|v_j - v_i|$ is equal the diameter of $[v_0, ..., v_n]$ for $i \neq j$ and 0 otherwise.

If we have n points $\{ v_0, ..., v_n \}$ in a vector space, we can add any other point b to the simplex and build a cone $b \cdot [v_0, ..., v_n] = [b, v_0, ..., v_n]$. Let us denote the barycentre of a simplex τ by b_τ. We define the *barycentric subdivision* inductively by $S\tau = b_\tau \cdot S(\partial\tau)$ of τ, where $S[\varnothing] = [\varnothing]$. Note that $\partial S\tau = S\partial\tau$. Let $\sigma\colon X \times \Delta^n \to Y$ be a $\mathcal{G}$-singular simplex; we define $S\sigma = \sigma|_{X \times S\Delta^n}$ regarded as a signed sum of restrictions of σ to the n-simplices in the barycentric subdivision of Δ^n. It is clear that $S\sigma$ is again measurable and fibrewise continuous. Another point of view is to consider $X \times S\Delta^n$ as $\mathcal{G}$-singular n-chain of the constant $\mathcal{G}$-space $X \times \Delta^n$. That is, as an element of $C^{sing}(X \times \Delta^n, \mathcal{G})$, and then define $S\sigma = \sigma_\sharp(X \times S\Delta^n)$, where $\sigma_\sharp$ is the induced chain map. The operator $S\colon C_n^{sing}(Y,\mathcal{G}) \to C_n^{sing}(Y,\mathcal{G})$ is a chain map, since
$$\begin{aligned}\partial S\sigma &= \partial(\sigma_\sharp(X \times S\Delta^n)) = \sigma_\sharp(\partial(X \times S\Delta^n)) \\ &= \sigma_\sharp((X \times \partial S\Delta^n)) = \sigma_\sharp((X \times S\partial\Delta^n)) \\ &= S(\sigma_\sharp((X \times \partial\Delta^n)) = S\partial\sigma.\end{aligned}$$

Next, we define a chain homotopy $T\colon C_n^{sing}(Y,\mathcal{G}) \to C_{n+1}^{sing}(Y,\mathcal{G})$ between S and Id. For a simplex τ let $T\tau = b_\tau(\tau - T\partial\tau)$ and $T[\varnothing] = 0$. This is the decomposition of $\Delta^n \times [0,1]$ into $n+1$ simplices, which we obtained by joining the simplices of $\Delta^n \times \{0\} \cup \partial\Delta^n \times [0,1]$ to the barycentre of $\Delta^n \times \{1\}$ and then projecting it down to Δ^n. Note that $\partial T + T\partial = \mathrm{Id} - S$. We regard $X \times T\Delta^n$ as an element of $C_{n+1}(X \times \Delta^n)$ and define $T\sigma = \sigma_\sharp(X \times T\Delta^n)$ for $\sigma \in C_n(Y,\mathcal{G})$. Since $T\Delta^n$ is a sum of singular $n+1$-simplices and $X \times T\Delta^n$ is constant in the first coordinate, $T\sigma$ is measurable and fibrewise continuous. We check that $T\colon C_n(Y,\mathcal{G}) \to C_{n+1}(Y,\mathcal{G})$ is a chain homotopy:

$$\begin{aligned}\partial T\sigma &= \partial\sigma_\sharp(X \times T\Delta^n)\\ &= \sigma_\sharp(X \times \partial T\Delta^n)\\ &= \sigma_\sharp((X \times \Delta^n) - (X \times S\Delta^n) - (X \times T\partial\Delta^n))\\ &= \sigma - S\sigma - \sigma_\sharp(X \times T\partial\Delta^n) = \sigma - S\sigma - T\partial\sigma.\end{aligned}$$

Later, we will have to iterate the subdivision to make the simplices smaller and smaller. Therefore, a chain homotopy between the m-fold iteration S^m and Id is given by $D_m = \sum_{i=0}^{m-1} TS^i$:

$$\begin{aligned}\partial D_m + D_m\partial &= \sum_{i=0}^{m-1} \partial TS^i + \sum_{i=0}^{m-1} TS^i\partial = \sum_{i=0}^{m-1} (\partial T + T\partial)S^i\\ &= \sum_{i=0}^{m-1} (\mathrm{Id} - S)S^i = \mathrm{Id} - S^m.\end{aligned}$$

Until now, every step worked more or less as in the classical proof, but now we have to be careful. For each $\mathcal{G}$-singular n-simplex $\sigma\colon X \times \Delta^n \to Y$ and almost all $x \in X$ there exists an $m_\sigma(x)$ such that each $\mathcal{G}$-singular n-simplex in the subdivision $(S^{m_\sigma(x)}\sigma)(\{x\} \times \Delta^n)$ lies in some $U_{j,x}$ for $U_j \in \mathcal{U}$, since we can choose $m_\sigma(x)$ such that the diameter of the simplices in $S^{m_\sigma(x)}\Delta^n$ will be less than the Lebesgue number of the cover of Δ^n given by $\sigma^{-1}|_{\{x\}\times\Delta^n}(\operatorname{int}\mathcal{U}_x)$. Nevertheless, we can not assume, unlike in the situation of ordinary singular homology of a topological space, that we find an m_σ such that $S^{m_\sigma}\sigma \in C_n^{\mathcal{U}}(Y,\mathcal{G})$, since the diameter of a simplex can grow bigger and bigger when we move in the fibres. To solve this problem, we define the sets

$$X_m(\sigma) = \left\{ x \in X \;\middle|\; \operatorname{im}(\sigma) \subset \bigcup_{i=1}^m U_i,\ m_\sigma(x) \leqslant m \right\}.$$

This is an increasing sequence with $\mu(X \backslash \bigcup_{m=1}^\infty X_m(\sigma)) = 0$. For each n-simplex $\tilde{\Delta}$ in the iterated subdivision $S^m\Delta^n$ we set $\hat{O}_i = \sigma|^{-1}_{X_m(\sigma)\times\tilde{\Delta}}(U_i)$ and define

$$O_i = \hat{O}_i \backslash \bigcup_{j<i} O_j.$$

The $\hat{O}_i$ are preimages of measurable sets under a measurable map and therefore measurable as well. Hence, we can write $(S^m\sigma)\chi_{X_m(\sigma)}$ as a signed linear combination of elements

$$\sigma|_{X_m(\sigma)\times\tilde{\Delta}} = \sum_{i=1}^m \sigma|_{O_i\times\tilde{\Delta}}.$$

By definition, $(S^m\sigma)\chi_{X_{m(\sigma)}} \in C_n^{\mathcal{U}}(Y,\mathcal{G})$ and thus the chains, which lie in $C_n^{\mathcal{U}}(Y,\mathcal{G})$ after a finite iteration of subdivisions, are $L^\infty(X)$-dense in $C_n(Y,\mathcal{G})$. Let use denote them by $C_n^{small}(Y,\mathcal{G})$. We conclude that the inclusion

$$i : C_n^{small}(Y,\mathcal{G}) \hookrightarrow C_n(Y,\mathcal{G})$$

is a $\dim_{L^\infty(X)}$-isomorphic $\mathbb{C}\mathcal{G}$-module map and by Lemma 4.1.10 we have

$$\dim_{L\mathcal{G}} H_n(L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} C_*^{small}(Y,\mathcal{G})) = \dim_{L\mathcal{G}} H_n^{sing}(Y, L\mathcal{G}). \tag{4.1}$$

We can define a chain homotopy $D\colon C_n^{small}(Y,\mathcal{G}) \to C_{n+1}^{small}(Y,\mathcal{G})$ for the small chains by setting $D\sigma = D_{m_\sigma}\sigma$, where m_σ is the smallest $m \in \mathbb{N}$ such that $S^m\sigma \in C_n^{\mathcal{U}}(Y,\mathcal{G})$. Let $\rho\colon C_n^{small}(Y,\mathcal{G}) \to C_n^{small}(Y,\mathcal{G})$ be the chain map given by

$$\rho(\sigma) = \sigma - \partial D\sigma - D\partial\sigma.$$

We have to show that ρ takes $C_n^{small}(Y,\mathcal{G})$ into $C_n^{\mathcal{U}}(Y,\mathcal{G})$.

$$\begin{aligned}\rho(\sigma) &= \sigma - \partial D\sigma - D\partial\sigma \quad \text{using } D\partial\sigma + \partial D\sigma = S^{m_\sigma}\sigma - \sigma \\ &= S^{m_\sigma}\sigma + D_{m_\sigma}(\partial\sigma) - D_{m_{\partial\sigma}}(\partial\sigma).\end{aligned}$$

The first term, S^{m_σ}, is clearly an element of $C_n^{\mathcal{U}}(Y,\mathcal{G})$. Further, $D_{m_\sigma}(\partial\sigma) - D_{m_{\partial\sigma}}(\partial\sigma)$ consists of terms $TS^i(\partial\sigma)$, with $i \geqslant m_{\partial\sigma}$, which all lie in $C_n^{\mathcal{U}}(Y,\mathcal{G})$, since $m_{\partial\sigma} \leqslant m_\sigma$. We remark that for the inclusion $i\colon C_n^{\mathcal{U}}(Y,\mathcal{G}) \to C_n^{small}(Y,\mathcal{G})$, we have $\rho \circ i = \mathrm{Id}$ and $D\partial + \partial D = \mathrm{Id} - i \circ \rho$. Hence, we have shown that

$$H_n(L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} C_n^{small}(Y,\mathcal{G})) = H_n(L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} C_n^{\mathcal{U}}(Y,\mathcal{G}))$$

which together with Equation (4.1) finishes the proof. □

Proof of Proposition 4.3.10. We set $\mathcal{U} = \{A, B\}$. The maps ρ and i from the preceding proposition take chains in A to chains in A and hence induce quotient maps such that the inclusion

$$C_n^{\mathcal{U}}(Y,\mathcal{G})/C_n^{\mathcal{U}}(A,\mathcal{G}) \hookrightarrow C_n^{small}(Y,\mathcal{G})/C_n^{small}(A,\mathcal{G})$$

is a chain homotopy equivalence. The superscripts for the chains in A are superfluous in a certain way, since for $\mathcal{U} = \{A, B\}$ all chains in A are small and in $\mathcal{U}$. Therefore, with the same argument as in Proposition 4.3.11, $C_n^{small}(Y,\mathcal{G})/C_n^{small}(A,\mathcal{G})$ is $L^\infty(X)$-dense in $C_n(Y,A,\mathcal{G})$. Furthermore, the map

$$C_n(B,\mathcal{G})/C_n(A \cap B,\mathcal{G}) \hookrightarrow C_n^{\mathcal{U}}(Y,\mathcal{G})/C_n^{\mathcal{U}}(A,\mathcal{G})$$

induced by the inclusion is an isomorphism, since both are generated by the $\mathcal{G}$-singular simplices $\sigma\colon X \times \Delta^n \to B$ with $\sigma|_{\{x\}\times\Delta^n} \subset B_x \backslash A_x$ for almost all $x \in X$. So we summarize:

$$\begin{aligned}\dim_{L\mathcal{G}} H_n^{sing}(Y,A,L\mathcal{G}) &= \dim_{L\mathcal{G}} H_n\left(L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} C_*^{small}(Y,\mathcal{G}))/C_*(A,\mathcal{G})\right) \\ &= \dim_{L\mathcal{G}} H_n\left(L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} C_*^{\mathcal{U}}(Y,\mathcal{G})/C_*(A,\mathcal{G})\right) \\ &= \dim_{L\mathcal{G}} H_n\left(L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} C_*(B,\mathcal{G})/C_*(A \cap B,\mathcal{G})\right) \\ &= \dim_{L\mathcal{G}} H_n^{sing}(B, A \cap B, L\mathcal{G}).\end{aligned}$$

□

Remark 4.3.12. We extract from the proofs of the preceding two propositions that $C_n^{\mathcal{U}}(Y,\mathcal{G})$ and $C_n^{small}(Y,\mathcal{G})$ are chain homotopy equivalent and $C_n(B, A\cap B,\mathcal{G})$ and $C_n^{\mathcal{U}}(Y,A,\mathcal{G})$ are isomorphic. Hence, we have that $C_n^{small}(Y,A,\mathcal{G})$ and $C_n(B,A\cap B,\mathcal{G})$ are chain homotopy equivalent.

Before we will present the last lemma of this subsection, we have to introduce the following notation: For a pair of topological fibred spaces Y and Z with $Z\subset Y$ and $\mathcal{B}$ the σ-algebra of Y, we define the quotient Y/Z to be $\bigsqcup_{x\in X} Y_x/Z_x$ with the quotient σ-algebra

$$\mathcal{F} = \big\{F\subset Y/Z \mid \exists B = \bigsqcup_{x\in X} B_x \in \mathcal{B} \text{ such that } F = \bigsqcup_{x\in X} B_x/Z_x\big\}.$$

This is again a topological fibred space over X. If Y is a $\mathcal{G}$-space and Z a $\mathcal{G}$-invariant subspace, then the quotient is again a $\mathcal{G}$-space.

Lemma 4.3.13. *Let (Y,Z) be a pair of $\mathcal{G}$-spaces such that there is a $\mathcal{G}$-space $Z\subset U\subset Y$ with U_x open for almost all $x\in X$ and such that Z is a $\mathcal{G}$-deformation retract of U. Then the quotient map $q:(Y,Z)\to(Y/Z,Z/Z)$ induces $L\mathcal{G}$-dimension isomorphisms*

$$q_*\colon H_n^{sing}(Y,Z,L\mathcal{G})\to H_n^{sing}(Y/Z,Z/Z,L\mathcal{G}),$$

for every $n\in\mathbb{N}_0$.

Proof. We consider the commutative diagram

$$\begin{tikzcd}
H_n^{sing}(Y,Z,L\mathcal{G}) \arrow[r] \arrow[d, "q_*"] & H_n^{sing}(Y,U,L\mathcal{G}) \arrow[d, "q_*"] & H_n^{sing}(Y\backslash Z,U\backslash Z,L\mathcal{G}) \arrow[l] \arrow[d, "q_*"] \\
H_n^{sing}(Y/Z,Z/Z,L\mathcal{G}) \arrow[r] & H_n^{sing}(Y/Z,U/Z,L\mathcal{G}) & H_n^{sing}(Y/Z\backslash Z/Z,U/Z\backslash Z/Z,L\mathcal{G}). \arrow[l]
\end{tikzcd}$$

The upper left arrow is an isomorphism, since the homology module $H_n^{sing}(U,Z,L\mathcal{G})$ is isomorphic to $H_n^{sing}(Z,Z,L\mathcal{G})=0$ in the long exact sequence of (Y,U,Z) by Corollary 4.3.9. The same is true for the lower left arrow, since U/Z deformation retracts onto Z/Z. The other two horizontal maps are $L\mathcal{G}$-dimension isomorphisms, which follows from the Excision Theorem 4.3.10. The map q restricted to $Y\backslash Z$ is a $\mathcal{G}$-homeomorphism, hence the right-hand vertical map q_* is an isomorphism. □

Remark 4.3.14. Considering Remark 4.3.12 and the previous proof, we have a chain homotopy equivalence

$$C_n^{small}(Y,Z,\mathcal{G})\simeq C_n^{small}(Y/Z,Z/Z,\mathcal{G}).$$

4.4. Equivalence of Simplicial and Singular Homology

This section is dedicated to the proof of the main theorem of this chapter:

Theorem 4.4.1. *Let Σ be a simplicial $\mathcal{G}$-complex and $|\Sigma|$ its geometric realization as a topological $\mathcal{G}$-space. Then*

$$\beta_n^{(2)}(\Sigma, \mathcal{G}) = \beta_n^{(2)}(|\Sigma|, \mathcal{G})$$

for every $n \in \mathbb{N}_0$.

Proof. Note that by Proposition 4.2.4 it is enough to prove that

$$\dim_{L\mathcal{G}} H_n^{sing}(|\Sigma|, L\mathcal{G}) = \dim_{L\mathcal{G}} H_n(L\mathcal{G} \otimes C_*^b(\Sigma)).$$

The proof is divided into three parts. First, we consider the situation where Σ is the constant discrete $\mathcal{G}$-space $X \times [v_0, \ldots, v_k]$, in the second part, we assume Σ to be an ULB simplicial $\mathcal{G}$-complex and in the last part, we prove the statement for arbitrary simplicial $\mathcal{G}$-complexes.

1. Let $\Sigma^{(0)}$ be the constant discrete $\mathcal{G}$-space $X \times \{ v_0, \ldots, v_k \}$ and

   $$\Sigma = (\Sigma^{(j)})_{0 \leqslant j \leqslant k} = X \times \left\{ (v_{\varsigma(0)}, \ldots, v_{\varsigma(j)}) \mid \varsigma \in \mathfrak{S}_{k+1} \right\},$$

   where $\mathfrak{S}_{k+1}$ denotes the symmetric group on $k+1$ elements. The $\mathcal{G}$-action is given by

   $$g(x, (v_0, \ldots, v_j)) = (gx, (gv_0, \ldots, gv_j)) = (t(g), (v_{\varsigma(0)}, \ldots, v_{\varsigma(j)}))$$

   for some permutation ς depending on g. The geometric realization of $\Sigma^{(k)}$ is just

   $$|\Sigma| = X \times \Delta^k$$

   and the $\mathcal{G}$-action is given by permuting the barycentric coordinates of Δ^k according to the action of $\mathcal{G}$ on $(v_0, \ldots, v_k)$. Moreover, the geometric realization of the simplicial $\mathcal{G}$-subcomplex $\Sigma^{(k-1)}$ is given by

   $$|\Sigma^{(k-1)}| = X \times \partial\Delta^k.$$

   A map from $C_k^b(\Sigma)$ to $C_k(|\Sigma|, \mathcal{G})$ is given by

   $$\begin{aligned}[f \colon \Sigma \to \mathbb{C}] \longmapsto [|f| &\colon X \times \Delta^k \to X \times \Delta^k] \\ |f|(x,p) &= f(\pi^{-1}(x)) \operatorname{Id}_{X \times \Delta^k}(x,p).\end{aligned}$$

   $f(\pi^{-1}(x))$ is well defined, since f commutes with permutations of the vertices and there is only one simplex, up to permutations, in $\pi^{-1}(x)$. Further note that the nth relative homology of the pair $(\Sigma^{(k)}, \Sigma^{(k-1)})$ is

   $$H_n\left(L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} C_*^b(\Sigma^{(k)}, \Sigma^{(k-1)})\right) = L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} C_k^b(\Sigma)$$

for $n = k$ and zero otherwise, since for $n < k$ we have $C_n^b(\Sigma^{(k)}) = C_n^b(\Sigma^{(k-1)})$ and $C_n^b(\Sigma^{(j)}) = 0$ for $n > k$. Moreover,

$$C_k^b(\Sigma) = \left\{ f : \Sigma^{(k)} \to \mathbb{C} \;\middle|\; f \in L^\infty(\Sigma^{(k)}),\ f(\varsigma) = \operatorname{sgn}(\varsigma) f \right\} \cong L^\infty(X)$$

and hence,

$$H_k\left(L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} C_*^b(\Sigma, \Sigma^{(k-1)})\right) \cong L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} L^\infty(X).$$

We will show by induction that also

$$\dim_{L\mathcal{G}} H_k^{sing}(|\Sigma^{(k)}|, |\Sigma^{(k-1)}|, L\mathcal{G}) = \dim_{L\mathcal{G}} L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} L^\infty(X), \tag{4.2}$$

and zero otherwise. For $k = 0$ we have

$$\begin{aligned} C_0(X \times \Delta^0, \mathcal{G}) &= \left\{ \sum_{j=1}^{k} c_j \operatorname{Id}_{X \times \Delta^0} \;\middle|\; c_j \in L^\infty(X) \right\} \Big/ \sim \\ &= \left\{ c \operatorname{Id}_{X \times \Delta^0} \mid c \in L^\infty(X) \right\} \\ &\cong L^\infty(X). \end{aligned}$$

Further, since $\partial \Delta^0 = \varnothing$ and

$$\begin{aligned} \operatorname{Id} \otimes \partial_1 \colon L\mathcal{G} \times_{\mathbb{C}\mathcal{G}} C_1(X \times \Delta^0, \mathcal{G}) &\to L\mathcal{G} \times_{\mathbb{C}\mathcal{G}} C_0(X \times \Delta^0, \mathcal{G}) \\ (f, \sigma) &\mapsto (f, \operatorname{Id}_{X \times \Delta^0} - \operatorname{Id}_{X \times \Delta^0}) = 0, \end{aligned}$$

it follows that

$$\begin{aligned} H_0^{sing}(X \times \Delta^0, X \times \partial\Delta^0, L\mathcal{G}) &= L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} C_0(X \times \Delta^0, X \times \partial\Delta^0, \mathcal{G}) \\ &= L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} C_0(X \times \Delta^0, \mathcal{G}) \\ &\cong L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} L^\infty(X). \end{aligned}$$

Let $\Lambda^k \subset \Delta^k$ be the union of all but one of the $(k-1)$-dimensional faces of Δ^k. We get from the long exact sequence of the triple $(X \times \Delta^k, X \times \partial\Delta^k, X \times \Lambda^k)$ that

$$H_n^{sing}(X \times \Delta^k, X \times \partial\Delta^k, L\mathcal{G}) \cong H_{n-1}^{sing}(X \times \partial\Delta^k, X \times \Lambda^k, L\mathcal{G}), \tag{4.3}$$

since $X \times \Delta^k$ deformation retracts onto $X \times \Lambda^k$. For $n = k$, the isomorphism maps a cycle $\sigma \in C_k(X \times \Delta^k)$ to $\partial\sigma$. The pairs $(X \times \partial\Delta^k, X \times \Lambda^k)$ and $(X \times \Delta^{k-1}, X \times \partial\Delta^{k-1})$ fulfil the assumptions of Lemma 4.3.13 and furthermore, we have a $\mathcal{G}$-homeomorphism

$$X \times \Delta^{k-1}/\partial\Delta^{k-1} \approx X \times \partial\Delta^k/\Lambda^k$$

given by the inclusion $\Delta^{k-1} \hookrightarrow \partial\Delta^k$ as the face not contained in Λ^k, hence we have an $L\mathcal{G}$-dimension isomorphism

$$H_{n-1}^{sing}(X \times \partial\Delta^k, X \times \Lambda^k, L\mathcal{G}) \cong_{L\mathcal{G}} H_{n-1}^{sing}(X \times \Delta^{k-1}, X \times \partial\Delta^{k-1}, L\mathcal{G}). \tag{4.4}$$

Before we finish the proof of Equation (4.2) by induction on k, note that $(f, \mathrm{Id}_{X\times\Delta^k})$ is a cycle in $L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} C_k(X \times \Delta^k, X \times \partial\Delta^k, \mathcal{G})$, since

$$(\mathrm{Id} \otimes \partial_k)(f, \mathrm{Id}_{X\times\Delta^k}) = (f, \mathrm{Id}_{X\times\partial\Delta^k}) \in L\mathcal{G} \otimes C_k(X \times \partial\Delta^k, \mathcal{G}).$$

The isomorphism in Equation (4.3) maps $\mathrm{Id}_{X\times\Delta^k}$ to $\mathrm{Id}_{X\times\partial\Delta^k}$ and the $L\mathcal{G}$-dimension isomorphism in Equation (4.4) is given by

$$\mathrm{Id}_{X\times\partial\Delta^k} \xrightarrow{q} \mathrm{Id}_{X\times\partial\Delta^k/\Lambda^k} \xleftarrow{i} \mathrm{Id}_{X\times\Delta^{k-1}/\partial\Delta^{k-1}} \xleftarrow{q} \mathrm{Id}_{X\times\Delta^{k-1}},$$

therefore, $\mathrm{Id}_{X\times\Delta^k}$ is mapped to $\pm\,\mathrm{Id}_{X\times\Delta^{k-1}}$, which generates a dimension isomorphic submodule of $H_{k-1}^{sing}(X \times \Delta^{k-1}, X \times \partial\Delta^{k-1}, L\mathcal{G})$, hence Equation (4.2) follows.

2. Next, we assume that Σ is an ULB simplicial $\mathcal{G}$-complex. We can decompose $\Sigma^{(k)}$ into a countable disjoint union of Borel sections $\Sigma_i^{(k)}$. Further, we can assume that the $\Sigma_i^{(k)}$'s are $\mathcal{G}$-invariant, since we can construct them by decomposing an exact fundamental domain $F = \bigsqcup_i F_i$ of $\Sigma^{(k)}$ and then set $\Sigma_i^{(k)} = \mathcal{G}F_i$. We consider the map

$$\Phi\colon \bigsqcup_i \left(X_i \times \Delta^k, X_i \times \partial\Delta^k\right) \to (|\Sigma^{(k)}|, |\Sigma^{(k-1)}|), \tag{4.5}$$

with $X_i = \pi(\Sigma_i^{(k)})$, formed by the characteristic maps of the k-simplices $X_i \times \Delta^k \to \Sigma_i^{(k)}$, which we regard as $\mathcal{G}$-singular simplices. For almost all $x \in X$ we have a homeomorphism

$$\bigsqcup_i \{\, x \,\} \times \Delta^k / \bigsqcup_i \{\, x \,\} \times \partial\Delta^k \approx |\Sigma_x^{(k)}| / |\Sigma_x^{(k-1)}|$$

induced by Φ, and hence a $\mathcal{G}$-homeomorphism

$$\bigsqcup_i X_i \times \Delta^k / \bigsqcup_i X_i \times \partial\Delta^k \approx |\Sigma^{(k)}| / |\Sigma^{(k-1)}|.$$

These pairs of $\mathcal{G}$-spaces fulfil the assumptions of Lemma 4.3.13, hence

$$\dim_{L\mathcal{G}} H_n^{sing}\left(|\Sigma^{(k)}|, |\Sigma^{(k-1)}|, L\mathcal{G}\right) = \dim_{L\mathcal{G}} H_n^{sing}\left(\bigsqcup_i X_i \times \Delta^k, \bigsqcup_i X_i \times \partial\Delta^k, L\mathcal{G}\right).$$

Thus, with part 1 of the proof, we know that there is a dimension isomorphism

$$\dim_{L\mathcal{G}} H_n^{sing}\left(|\Sigma^{(k)}|, |\Sigma^{(k-1)}|, L\mathcal{G}\right) = \dim_{L\mathcal{G}} H_n\left(L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} C_*(\Sigma^{(k)}, \Sigma^{(k-1)})\right).$$

Let us consider the following commutative diagram, where we write $H_n(\Sigma)$ for $H_n(L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} C_*^b(\Sigma))$ and $H_n(|\Sigma|)$ for $H_n^{sing}(|\Sigma|, L\mathcal{G})$) to simplify the notation.

$$\begin{tikzcd}
H_{n+1}(\Sigma^{(k)}, \Sigma^{(k-1)}) \arrow[r] \arrow[d] & H_n(\Sigma^{(k-1)}) \arrow[r] \arrow[d] & H_n(\Sigma^{(k)}) \arrow[r] \arrow[d] & H_n(\Sigma^{(k)}, \Sigma^{(k-1)}) \arrow[r] \arrow[d] & H_{n-1}(\Sigma^{(k-1)}) \arrow[d] \\
H_{n+1}(|\Sigma^{(k)}|, |\Sigma^{(k-1)}|) \arrow[r] & H_n(|\Sigma^{(k-1)})| \arrow[r] & H_n(|\Sigma^{(k)}|) \arrow[r] & H_n(|\Sigma^{(k)}|, |\Sigma^{(k-1)}|) \arrow[r] & H_{n-1}(|\Sigma^{(k-1)}|)
\end{tikzcd}$$

We already know that the first and fourth vertical map are dimension isomorphisms; by induction on k we can further assume that the second and fifth vertical map are also dimension isomorphisms. Hence, together with the Five-Lemma C.7, the claim for an ULB simplicial $\mathcal{G}$-complex follows.

3. Finally, let Σ be an arbitrary simplicial $\mathcal{G}$-complex with ULB exhaustion $\{ \Sigma_i \}_i$. By Proposition 4.2.4 we know that

$$\begin{aligned}\dim_{L\mathcal{G}} H_n^{(2)}(\Sigma) &= \dim_{L\mathcal{G}} H_n(L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} C_*^b(\Sigma)) \\ &= \dim_{L\mathcal{G}} H_n(L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} \varinjlim C_*^b(\Sigma_i)).\end{aligned}$$

Since the image of an $\mathcal{G}$-singular simplex $\sigma\colon X \times \Delta^n \to \Sigma$ is compact in each fibre, $\varinjlim C_n(|\Sigma_i|, \mathcal{G})$ is $L^\infty(X)$-dense in $C_n(|\Sigma|, \mathcal{G})$ and hence

$$\dim_{L\mathcal{G}} H_n^{sing}(|\Sigma|, L\mathcal{G}) = \dim_{L\mathcal{G}} H_n \left(L\mathcal{G} \otimes_{L\mathcal{G}} \varinjlim C_*(|\Sigma_i|, \mathcal{G}) \right).$$

Further, for $i < j$ and $\Sigma_i \subset \Sigma_j$ the following diagram commutes

$$\begin{tikzcd} H_n(L\mathcal{G} \otimes C_*^b(\Sigma_i)) \arrow[r] \arrow[d] & H_n(L\mathcal{G} \otimes C_*^b(\Sigma_j)) \arrow[d] \\ H_n(L\mathcal{G} \otimes C_*(|\Sigma_i|, \mathcal{G})) \arrow[r] & H_n(L\mathcal{G} \otimes C_*(|\Sigma_j|, \mathcal{G})), \end{tikzcd}$$

where the vertical maps are dimension isomorphisms by part 2 of the proof. Together with the colimit property this implies that

$$\dim_{L\mathcal{G}} H_n(L\mathcal{G} \otimes_{L\mathcal{G}} \varinjlim C_*(|\Sigma_i|, \mathcal{G})) = \dim_{L\mathcal{G}} H_n(L\mathcal{G} \otimes_{\mathbb{C}\mathcal{G}} \varinjlim C_*^b(\Sigma_i)). \qquad \square$$

As a direct consequence of Theorem 4.2.5 and this theorem, we obtain the following corollary:

Corollary 4.4.2. *If Y is a contractible topological $\mathcal{G}$-space, then*

$$\beta_n^{(2)}(Y, \mathcal{G}) = \beta_n^{(2)}(\mathcal{G})$$

for every $n \in \mathbb{N}_0$.

Appendix

The following appendix is a summary of basic definitions and results and fixes notation to be used throughout. We shortly recall the character theory of finite groups in Appendix A. In Appendix B, we discuss direct integral Hilbert spaces. To this end, we introduce measurable fields of Hilbert spaces and decomposable linear operators. We finish the section with a version of the Spectral Theorem for direct integrals. Direct integrals are necessary for the theory we develop in Chapters 2 and 3, but also interesting in connection with Chapter 4. The last appendix is dedicated to dimension functions on modules over von Neumann algebras; Appendix C also contains some homological algebra results, which are mainly used in Chapters 1 and 4.

A. Character Theory of Finite Groups

The theory of characters of finite groups is only needed for Chapter 3. Nevertheless, I have decided to present it here in the appendix, since one or the other readers are already familiar with it. This is only a summary of results we need; this should not be considered as an introduction to the topic; therefore, I refer the reader to the main sources [Ser77] and [Isa76].

Let G be a finite group and V a finite dimensional complex vector space. A representation (ρ, V), $\rho\colon G \to \mathrm{GL}(V)$, is called *irreducible* if V is not 0 and (ρ, V) is not the direct sum of two representations. Usually, we will not mention the underlying vector space V and simply speak of the representation ρ. The following proposition justifies the restriction to irreducible representations:

Proposition A.1. [Ser77, §2 Theorem 2] *Every representation is a direct sum of irreducible representations.*

We call the complex valued function

$$\chi_\rho\colon G \to \mathbb{C}, \quad g \mapsto \mathrm{tr}(\rho(g))$$

the *character* of the representation ρ, where $\mathrm{tr}(A) = \sum_{i=1}^n a_{ii}$ for $A \in \mathrm{GL}(V)$ and $\dim V = n$. We summarize some properties of characters:

Proposition A.2. *Let χ be the character of a representation (ρ, V) with $\dim V = n$. Then*

(1) $\chi(1) = n$;

(2) $\chi(g^{-1}) = \overline{\chi(g)}$ *for all* $g \in G$;

(3) $\chi(hgh^{-1}) = \chi(g)$ *for all* $h, g \in G$;

(4) *if ρ is the direct sum of two representations $\rho_1 \oplus \rho_2$, then $\chi = \chi_{\rho_1} + \chi_{\rho_2}$.*

We define a *scalar product* on the complex valued functions on G by

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\psi}(g),$$

for $\phi, \psi \in \mathbb{C}[G]$.

Proposition A.3. [Ser77, §2 Theorem 3]

(1) *If χ is the character of an irreducible representation, then $\langle \chi, \chi \rangle = 1$.*

(2) *If χ and χ' are the character of two non-isomorphic irreducible representations, then $\langle \chi, \chi' \rangle = 0$.*

Theorem A.4. [Ser77, §2 Theorem 4, Corollary 1 and 2] *Let ρ be a representation of G and χ its character. Suppose ρ decomposes into a direct sum of irreducible representations:*

$$\rho = \rho_1 \oplus \cdots \oplus \rho_k.$$

Then, if σ is an irreducible representation with character φ, the number of ρ_i's isomorphic to σ is equal to $\langle \varphi, \chi \rangle$; thus independent of the chosen decomposition. Further, it follows that two representations with the same character are isomorphic.

Let $\{\chi_1, ..., \chi_k\}$ denote the set of all irreducible character of G, *i.e.* the characters of a set $\mathrm{Irr}(G) = \{\sigma_1, ..., \sigma_k\}$ of representatives of the irreducible representations of G. The first part of the following theorem is a direct consequence of the previous one.

Theorem A.5. [Ser77, §2 Theorem 8] *Let ρ be a linear representation of G with decomposition into irreducible representations given by $\rho = \rho_1 \oplus \ldots \rho_l$. Then*

(1) *the decomposition $\rho = \bigoplus_{i=1}^k \sigma_i(\rho)$, with $\sigma_i(\rho)$ the direct sum of all irreducible representations ρ_j which are isomorphic to σ_i, is unique, i.e. it does not depend on the chosen decomposition of ρ into irreducible representations $\rho = \rho_1 \oplus \ldots \rho_l$;*

(2) *the projection P_i from ρ onto $\sigma_i(\rho)$ is given by:*

$$P_i = \frac{\chi_i(1)}{|G|} \sum_{g \in G} \overline{\chi_i}(g) g \in \mathbb{C}[G].$$

The $\sigma_i(\rho)$ in Theorem A.5 are direct sums of irreducible representations isomorphic to σ_i, *i.e.* $\sigma_i(\rho) \cong \sigma_i^{m(\sigma_i, \rho)}$. We call the number $m(\sigma, \rho)$ the *multiplicity* of σ in ρ. Note that by Theorem A.4 $m(\sigma_i, \rho) = \langle \chi_i, \chi_\rho \rangle$ if χ_ρ is the character of ρ.

We recall some properties of representations and characters which are induced by a representation of a subgroup $H \leqslant G$. Let (σ, V) be a representation of H. We define the *induced representation* by $\mathrm{Ind}_H^G(\sigma) := \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$.

Proposition A.6. [Ser77, §7, Proposition 20] *Let (σ, V) be a representation of H and χ_σ its character, then*

$$\mathrm{Ind}_H^G(\chi_\sigma) \colon G \to \mathbb{C}, \quad g \mapsto \frac{1}{|H|} \sum_{\substack{k \in G \\ k^{-1} g k \in H}} \chi_\sigma(k^{-1} g k)$$

is the character of $\mathrm{Ind}_H^G(\sigma)$.

Theorem A.7 (Frobenius reciprocity). [Ser77, §7, Theorem 13] *Let H be a subgroup of G and σ a representation of H with character χ_σ and ρ a representation of G with character χ_ρ. If $\rho|_H$ is the restriction of χ_ρ to elements in H, then*

$$\langle \chi_\sigma, \chi_{\rho|_H} \rangle_H = \langle \mathrm{Ind}_H^G \chi_\sigma, \rho \rangle_G,$$

where $\langle \cdot, \cdot \rangle_H$ or $\langle \cdot, \cdot \rangle_G$ denotes the scalar product on the characters of H or G, respectively.

Note that by Proposition A.2(4) and Theorem A.5 we can write every character χ_σ of a representation of H as a linear combination:

$$\chi_\sigma = \sum_{\theta \in \mathrm{Irr}(H)} \langle \chi_\sigma, \chi_\theta \rangle \chi_\theta$$

of characters χ_θ of irreducible representations $\theta \in \mathrm{Irr}(H)$. Together with the Frobenius reciprocity we obtain that

$$m(\rho, \mathrm{Ind}_H^G(\sigma)) = \sum_{\theta \in \mathrm{Irr}(H)} m(\theta, \rho|_H) m(\theta, \sigma),$$

for every representation ρ of G.

Last, we want to give some results concerning the values of characters:

Proposition A.8. [Ser77, §6 Proposition 15] *Let χ be the character of a representation σ of a finite group G. Then $\chi(g)$ is an algebraic integer for every $g \in G$.*

Let $E \subset \mathbb{C}$ denote a field of characteristic 0.

Definition A.9. We say a representation σ of a finite group G can be *realized* over a field E, if there exists a linear representation (ρ, V), where V is an E-vector space such that σ is isomorphic to

$$\rho_{\mathbb{C}} \colon G \to \mathrm{GL}(V) \to \mathrm{GL}(\mathbb{C} \otimes_E V).$$

Proposition A.10. [Ser77, §12 Proposition 33] *Let σ be a representation of a finite group G with character χ, then σ is realizable over a field E if and only if $\chi(g) \in E$ for all $g \in G$.*

Theorem A.11 (Brauer). [Ser77, §12 Theorem 24] *Let G be a finite group and m be the least common multiple of the orders of the elements of G. Let χ be the character of a representation σ of G, then $\chi(g) \in \mathbb{Q}(\xi_m)$ for all $g \in G$, where ξ_m denotes the mth root of unity.*

B. Direct Integrals

The definition of direct integrals goes back to John von Neumann. Though, for this summary I mainly used the book [Dix81] of Dixmier. The theory of direct integrals is requisite for Chapters 2 and 3. In Chapter 4 direct integrals appear, but more than an alternative view; they are not necessary for the understanding of the results or the proofs of them. The theory for unbounded operators can be found in [Nus64] and [DNSZ15].

Let X be a Borel space and μ a positive measure on X. If it is not necessary, we omit μ and just say *measurable* instead of μ-measurable.

Definition B.1. We call a mapping $x \to H(x)$ on X, with $H(x)$ a Hilbert space for every $x \in X$, a *field of Hilbert spaces over* X. An element v of $\prod_{x\in X} H(x)$ can be regarded as a mapping $v\colon x \mapsto v(x)$, such that $v(x) \in H(x)$, and is called a *vector field over* X.

Definition B.2. A *measurable field of Hilbert spaces* is a field of Hilbert spaces $x \to H(x)$ together with a linear subspace $\mathcal{M}$ of $\prod_{x\in X} H(x)$, called the *measurable* vector fields, with the following properties:

- For every $v \in \mathcal{M}$, $x \to \|v(x)\|$ is measurable;
- If for $w \in \prod_{x\in X} H(x)$ the function $x \mapsto \langle v(x), w(x)\rangle$ is measurable for every $v \in \mathcal{M}$, then $w \in \mathcal{M}$;
- There exists a sequence $(b_1, b_2, \ldots.)$ of element of $\mathcal{M}$ such that for every $x \in X$, $b_n(x)$ is a total sequence in $H(x)$; this means, every element of $H(x)$ can be written as a countable linear combination of the $b_n(x)$'s. We call such a sequence a *fundamental sequence* of measurable vector fields.

Proposition B.3. *Let $(b_1, b_2, \ldots)$ be a fundamental sequence of measurable fields. A vector field v is measurable if and only if $x \mapsto \langle v(x), b_i(x)\rangle$ is measurable for every i.*

Proposition B.4. *Let $x \to H(x)$ be a field of Hilbert spaces and $(v_1, v_2, \ldots)$ a sequence of vector fields possessing the following properties:*

- *$x \mapsto \langle v_i(x), v_j(x)\rangle$ is measurable for every i and j;*
- *$(v_1(x), v_2(x), \ldots)$ is a total sequence in $H(x)$ for every $x \in X$.*

Then there exist a unique measurable field structure on $x \to H(x)$ such that the v_i's are measurable.

We say a vector field v is *square-integrable* if it is measurable and

$$\int_X \|v(x)\|^2 d\mu(x) < \infty.$$

The square-integrable vector fields form a complex vector space $\mathcal{K}$ with inner product

$$\langle v, w\rangle = \int_X \langle v(x), w(x)\rangle d\mu(x).$$

Note that the vector fields v which vanish almost everywhere are just those with

$$\|v\|^2 = \int_X \|v(x)\|^2 d\mu(x) = 0.$$

Definition B.5. Let $x \to H(x)$ be a measurable field of Hilbert spaces with $\mathcal{M}$ the measurable vector fields and $\mathcal{K}$ the square integrable ones. The *direct integral* of $x \to H(x)$ with respect to $\mathcal{M}$ is

$$\int^{\oplus,\mathcal{M}} H(x)d\mu(x) := \mathcal{K}/\sim,$$

where $\sim$ is given by identifying the vector fields which vanish almost everywhere.

The direct integral depends on the measurable structure $\mathcal{M}$, but if it is clear from the context what the measurable vector fields are, especially in regard of Proposition B.4, we omit $\mathcal{M}$ in the notation.

Proposition B.6. *The direct integral*

$$\int^{\oplus} H(x)d\mu(x)$$

is a Hilbert space. If X is a standard Borel space, then it is separable.

The next proposition gives us a criterion to decide whether a field of subspaces is measurable or not.

Proposition B.7. *Let $x \to H(x)$ a measurable field of Hilbert spaces, $K(x)$ a closed linear subspace of $H(x)$ for every $x \in X$ and $P(x)$ the projection onto $K(x)$. Let $\mathcal{M}_K$ be the set of measurable vector fields v such that $v(x) \in K(x)$ for every $x \in X$. The following conditions are equivalent:*

- *The field of Hilbert spaces $x \to K(x)$ together with $\mathcal{M}_K$ is a measurable field;*
- *There exist a sequence $(v_1, v_2, ..)$ of measurable vector fields in $x \to H(x)$ such that for every $x \in X$, $(v_1(x), v_2(x), ...)$ is a total sequence for $K(x)$;*
- *For any measurable vector field v in $x \to H(x)$ the field $x \mapsto P(x)v(x)$ is measurable.*

Now let $x \to H(x)$ and $x \to H'(x)$ be measurable fields of Hilbert spaces over X and $T(x) \in L(H(x), H'(x))$ a (possibly unbounded) linear mapping from $H(x)$ into $H'(x)$ for every $x \in X$. We call $x \to T(x)$ a *field of linear mappings*. If H and H' coincide we speak of a field of *operators*. First, we consider bounded linear mappings.

Definition B.8. A field of continuous linear mappings $x \to T(x)$ is called *measurable* if for every measurable vector field $x \to v(x) \in H(x)$ the vector field $x \to T(x)v(x) \in H'(x)$ is measurable.

Proposition B.9. *Let $(b_1, b_2, \dots)$ and $(b'_1, b'_2, \dots)$ be fundamental sequences for $x \to H(x)$ and $x \to H'(x)$, respectively. A field of continuous linear mappings $x \to T(x)$ is measurable if and only if*

$$x \mapsto \langle T(x) b_i(x), b'_j \rangle$$

is measurable for every i and j.

A measurable field $x \to T(x)$ of linear mappings is called *essentially bounded* if the essentially supremum of $x \to T(x)$ is finite. Recall, the *essentially supremum* of a measurable function $f \colon X \to H$ is given by

$$\operatorname{ess\,sup}_X \|f\| = \inf \big\{ D \geqslant 0 \mid \mu(\{x \in X \mid \|f\| > D\}) = 0 \big\},$$

for a measure space (X, μ) and H a Banach space.

Proposition B.10. *Let $x \to T(x) \in L(H(x), H'(x))$ a measurable and essentially bounded field of linear mappings with $\operatorname{ess\,sup}_X \|T(x)\| = D < \infty$. Then $x \to T(x)$ defines a continuous linear mapping*

$$\widetilde{T} \colon \int^{\oplus} H d\mu \to \int^{\oplus} H' d\mu$$

with $\|\widetilde{T}\| = D$. Further, if there is an other field of linear mappings $x \to T'(x)$ such that $\widetilde{T} = \widetilde{T}'$, then $T(x) = T'(x)$ for almost all $x \in X$.

Definition B.11. A continuous linear mapping $T \colon \int^{\oplus} H d\mu \to \int^{\oplus} H' d\mu$ is called *decomposable* if it can be expressed by a measurable and essentially bounded field of linear mappings $x \to T(x)$. In that case, we also write

$$T = \int^{\oplus} T(x) d\mu(x).$$

In regard of Proposition B.7, let $x \to P(x)$ be a measurable field of projections onto the linear subspaces $x \to K(x)$ and

$$P = \int^{\oplus} P(x) d\mu(x),$$

then the direct integral of $x \to K(x)$ and the image of P coincide, *i.e.*

$$\int^{\oplus} K(x) d\mu(x) = P\Big(\int^{\oplus} H(x) d\mu(x) \Big).$$

We also want to deal with unbounded mappings. For an linear mapping $T \colon H \to H'$ let us denote the projection from $H \times H'$ onto the closure of its graph $\overline{\mathcal{G}(T)}$ by P_T.

Definition B.12. A field of linear mappings $x \to T(x)$ is called *measurable* if the field of bounded operators $x \to P_T$ is measurable.

For continuous linear mappings this definition coincide with the previous one. Also a field of unbounded mappings $x \to T(x)$ gives rise to an operator on the direct integral:

Proposition B.13. *Let $x \to T(x)$ be a field of measurable closed linear mappings $H(x) \to H'(x)$ with domains $\mathcal{D}(T(x))$. Let $\mathcal{D}$ be the set of square-integrable vector fields $x \to v(x)$ such that $v(x) \in \mathcal{D}(T(x))$, for all $x \in X$, and such that the vector field $x \to T(x)v(x)$ is square-integrable. Then $x \to T(x)$ defines a closed linear mapping*

$$T = \int^{\oplus} T(x)d\mu(x)$$

from $\int^{\oplus} H(x)d\mu(x)$ to $\int^{\oplus} H'(x)d\mu(x)$ with domain $\mathcal{D}$.

In regards of this proposition, we can extend Definition B.11 to the unbounded case and say a linear mapping

$$T\colon \int^{\oplus} H(x)d\mu(x) \to \int^{\oplus} H'(x)d\mu(x)$$

is *decomposable* if it can be expresses as $T = \int^{\oplus} T(x)d\mu(x)$. A decomposable operator inherits many of the properties of its field of operators. We only state some of them, which we will need later:

Proposition B.14. *For an decomposable operator $T = \int^{\oplus} T(x)d\mu(x)$ its adjoint T^* exist if and only if $T(x)^*$ exist for almost all $x \in X$. In that case, we have*

$$T^* = \int^{\oplus} T(x)^* d\mu(x).$$

Moreover, T is self-adjoint if and only if $T(x)$ is self-adjoint for almost all $x \in X$.

There is also a "decomposable" version of the spectral theorem. We first state the usual version to introduce the notation:

Theorem B.15 (Spectral Theorem)**.** *For every self-adjoint operator T on a Hilbert space H there exist a unique projection valued measure $E_T\colon \operatorname{Bor}(\mathbb{R}) \to \mathcal{P}(H)$ such that for all bounded Borel functions f on $\mathbb{R}$*

$$f(T) = \int_{\mathbb{R}} f(\lambda)dE_T(\lambda).$$

Proposition B.16. [DNSZ15, Proposition 4.2] *Let $T = \int^{\oplus} T(x)d\mu(x)$ be a decomposable self-adjoint operator on the direct integral Hilbert space $\int^{\oplus} H(x)d\mu(x)$. For every Borel subset $B \subset \mathbb{R}$ we have*

$$E_T(B) = \int^{\oplus} E_{T(x)}(B)d\mu(x).$$

Moreover, for every Borel measurable function $f\colon \mathbb{R} \to \mathbb{R}$, the following holds

$$f(T) = \int^{\oplus} f(T(x))d\mu(x).$$

C. Dimension Theory

We will give the definition and summarize some properties of the dimension function for modules over (semi)-finite von Neumann algebras. The semi-finite case is only needed in Chapter 1, therefore, we formulate only the lemmas and propositions needed their in the generality of semi-finite von Neumann algebras. When we say $\mathcal{A}$ is a semi-finite or finite von Neumann algebra with trace $\mathrm{tr}_{\mathcal{A}}$, we mean a semi-finite, σ-finite von Neumann algebra $\mathcal{A}$ and a faithful, normal, semi-finite trace $\mathrm{tr}_{\mathcal{A}}$ on $\mathcal{A}$, respectively, a finite von Neumann algebra $\mathcal{A}$ and a faithful, normal, finite trace $\mathrm{tr}_{\mathcal{A}}$ on $\mathcal{A}$. Most of the results concerning the finite case can be found in [Lüc94] and the semi-finite case in [Pet13]. I only included a proof of the very last proposition, since I did not found it in the literature; though, I do not claim originality.

Definition C.1. Let $\mathcal{A}$ be a semi-finite von Neumann algebra with trace $\mathrm{tr}_{\mathcal{A}}$ and M be an $\mathcal{A}$-module. We say M is $\mathrm{tr}_{\mathcal{A}}$*-finitely generated*, if there is a short exact sequence of $\mathcal{A}$-modules

$$0 \to N \to \mathcal{A}^n p \to M \to 0,$$

where p is a projection in $M_n(\mathcal{A})$ with finite trace $\sum_{i=1}^n \mathrm{tr}_{\mathcal{A}}(p_{ii})$.

Definition C.2. For a $\mathrm{tr}_{\mathcal{A}}$-finitely generated projective $\mathcal{A}$-module $M \cong \mathcal{A}^n p$, where $p \in M_n(\mathcal{A})$ is a projection, the *von Neumann dimension* is

$$\dim_{\mathcal{A}} M = \sum_{i=1}^{n} \mathrm{tr}_{\mathcal{A}}(p_{ii}).$$

Note that the von Neumann dimension is independent of the choice of the projection p. This definition can be extended to arbitrary $\mathcal{A}$-modules.

Definition C.3. The *von Neumann dimension* of an arbitrary $\mathcal{A}$-module M is

$$\dim_{\mathcal{A}}(M) := \sup\{\dim_{\mathcal{A}} N \mid N \leqslant M \ \ \mathrm{tr}_{\mathcal{A}}\text{-finitely generated projective submodule}\}.$$

Proposition C.4. *For a semi-finite von Neumann algebra $\mathcal{A}$ with trace $\mathrm{tr}_{\mathcal{A}}$ the dimension function $\dim_{\mathcal{A}}$ satisfies the following properties:*

- *A $\mathrm{tr}_{\mathcal{A}}$-finitely generated projective $\mathcal{A}$-module P is trivial if and only if $\dim_{\mathcal{A}} P = 0$.*
- Additivity
  For any short exact sequence of $\mathcal{A}$-modules

  $$0 \to M \to N \to Q \to 0$$

  the von Neumann dimension satisfies $\dim_{\mathcal{A}} N = \dim_{\mathcal{A}} M + \dim_{\mathcal{A}} Q$.
- Cofinality
  Let $\{M_i \mid i \in I\}$ be a cofinal system of submodules of $M = \bigcup_{i \in I} M_i$. Then

  $$\dim_{\mathcal{A}} M = \sup\{\dim_{\mathcal{A}} M_i \mid i \in I\}.$$

Definition C.5. Let M and N be $\mathcal{A}$-modules. An $\mathcal{A}$-homomorphism $f\colon M \to N$ is called *$\mathcal{A}$-dimension isomorphism*, or *$\dim_{\mathcal{A}}$-isomorphism*, if $\dim_{\mathcal{A}}(\ker f) = \dim_{\mathcal{A}}(\operatorname{coker} f) = 0$.

Remark C.6. The $\mathcal{A}$-modules M with $\dim_{\mathcal{A}} M = 0$ are a *Serre subcategory* $\mathcal{A}\text{-Mod}_0$ of $\mathcal{A}\text{-Mod}$, *i.e.* $\mathcal{A}\text{-Mod}_0$ is closed under subobjects, quotients and extensions which follows from Proposition C.4 (cf. [ST10, §4.2]). Therefore, we can form the *localized category* $\mathcal{A}\text{-Mod}/\mathcal{A}\text{-Mod}_0 = \mathcal{A}\text{-Mod}_{loc}$, which has the same objects as $\mathcal{A}$-Mod, is abelian and there is an exact functor $q\colon \mathcal{A}\text{-Mod} \to \mathcal{A}\text{-Mod}_{loc}$ such that:

- $q(f)$ is an isomorphism if f is an $\mathcal{A}$-dimension isomorphism;
- For any other abelian category $\mathcal{C}$ and exact functor $F\colon \mathcal{A}\text{-Mod} \to \mathcal{C}$, such that $F(f)$ is an isomorphism for all $\dim_{\mathcal{A}}$-isomorphisms f, there is (up to natural isomorphisms) an unique functor $\overline{F}\colon \mathcal{A}\text{-Mod}_{loc} \to \mathcal{C}$ such that the following diagram commutes up to natural equivalences:

$$\begin{tikzcd} \mathcal{A}\text{-Mod} \arrow[d, "q"'] \arrow[dr, "F"] & \\ \mathcal{A}\text{-Mod}_{loc} \arrow[r, "\overline{F}"] & \mathcal{C}. \end{tikzcd}$$

This is a powerful tool, for instances we get the following version of the Five lemma:

Lemma C.7 (Five Lemma)**.** *Given the following commutative diagram of $\mathcal{A}$-modules with exact rows and $\mathcal{A}$-dimension isomorphisms a, b, d and e:*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "a"] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & D \arrow[r] \arrow[d, "d"] & E \arrow[d, "e"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'. \end{tikzcd}$$

Then c is also an $\mathcal{A}$-dimension isomorphism.

Proof. This follows directly from Remark C.6 and the Five lemma for abelian categories. □

The next proposition is due to Sauer [Sau05, Theorem 2.4] in the finite case and was extended by Petersen [Pet13, B.27] to semi-finite von Neumann algebras.

Proposition C.8 (Sauer's local criterion)**.** *An $\mathcal{A}$-module M satisfies $\dim_{\mathcal{A}} M = 0$ if and only if for every $m \in M$ there is an increasing sequence of projections $p_n \in \mathcal{A}$ such that $p_n m = 0$ for every $n \in \mathbb{N}$ and $\sup p_n = 1$.*

Definition C.9. Let $\mathcal{A}$ be a finite von Neumann algebra. For any $\mathcal{A}$-module M we define the *rank function* $\operatorname{rk}_{\mathcal{A}}\colon M \to [0, 1]$ by

$$\operatorname{rk}_{\mathcal{A}}(m) := \inf\{\operatorname{tr}_{\mathcal{A}}(p) \mid p \in \operatorname{Proj}(\mathcal{A}),\ p^{\perp} m = 0\}.$$

The rank function induces a pseudo-metric $d_{\mathcal{A}}$, called the *rank metric*, on M by

$$d_{\mathcal{A}}(m,n) := \operatorname{rk}(m-n).$$

The following corollary is a direct consequence of Sauer's local criterion and enables us to use the rank metric to decide whether a submodule has the same von Neumann dimension as the ambient module or not.

Corollary C.10. *A submodule $N \leqslant M$ of an $\mathcal{A}$-module is dense with respect to the rank metric if and only if $\dim_{\mathcal{A}} M/N = 0$ and $\dim_{\mathcal{A}} M = \dim_{\mathcal{A}} N$. Therefore, the inclusion of an $d_{\mathcal{A}}$-dense submodule is an $\mathcal{A}$-dimension isomorphism.*

Proof. Suppose N is $d_{\mathcal{A}}$-dense in M, hence for every $m \in M$ and $\epsilon > 0$ there is a $n \in N$ and a $p_\epsilon \in \operatorname{Proj}(\mathcal{A})$ such that $\operatorname{tr}_{\mathcal{A}}(p) < \epsilon$ and $p_\epsilon^\perp(m-n) = 0$. This is equivalent to say that $p_\epsilon^\perp(m+N) = 0 \in M/N$, which shows together with the local criterion that $\dim_{\mathcal{A}} M/N = 0$. □

Definition C.11. An $\mathcal{A}$-$\mathcal{B}$-bimodule M is called *dimension compatible* if

$$\dim_{\mathcal{A}} M \otimes_{\mathcal{B}} N = 0$$

for every $\mathcal{B}$-module N with $\dim_{\mathcal{B}} N = 0$.

Lemma C.12. [Sau05, Lemma 4.7] *Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of finite von Neumann algebras. Then $\mathcal{A}$ is a dimension compatible $\mathcal{A}$-$\mathcal{B}$-bimodule.*

Since I did not found a reference for the following proposition, I will present the argument.

Proposition C.13. *Let $\mathcal{A} \subset R \subset \mathcal{B}$ be an inclusion of rings, where $\mathcal{A}$ and $\mathcal{B}$ are finite von Neumann algebras. Further, let $\varphi\colon M \to N$ be an $\mathcal{A}$-dimension isomorphism. Then*

$$\operatorname{Id}_{\mathcal{B}} \otimes \varphi\colon \mathcal{B} \otimes_R M \to \mathcal{B} \otimes_R N$$

is a $\mathcal{B}$-dimension isomorphism.

Proof. Since φ is an $\mathcal{A}$-dimension isomorphism, we know that

$$\dim_{\mathcal{A}} \ker \varphi = \dim_{\mathcal{A}} \operatorname{coker} \varphi = 0.$$

Hence, with Lemma C.12, it follows that

$$\dim_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \ker \varphi) = \dim_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \operatorname{coker} \varphi) = 0$$

and by the additivity of the dimension function (see Proposition C.4) that

$$\dim_{\mathcal{B}}(\mathcal{B} \otimes_R \ker \varphi) = \dim_{\mathcal{B}}(\mathcal{B} \otimes_R \operatorname{coker} \varphi) \leqslant \dim_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \operatorname{coker} \varphi) = 0.$$

Further, we have for any free R-resolution $P_\bullet$ of $\mathcal{B}$ that $\dim_{\mathcal{B}}(P_\bullet \otimes_R \operatorname{coker}\varphi) = 0$ and hence

$$\dim_{\mathcal{B}} \operatorname{Tor}_\bullet^R(\mathcal{B}, \operatorname{coker}\varphi) = \dim_{\mathcal{B}} H_\bullet(P_\bullet \otimes_R \operatorname{coker}\varphi) = 0;$$

again, this follows from the additivity and cofinality of the dimension function. We consider the short exact sequences:

$$\begin{aligned} &0 \to \ker\varphi \to M \to \operatorname{im}\varphi \to 0, \\ &0 \to \operatorname{im}\varphi \to N \to \operatorname{coker}\varphi \to 0; \end{aligned}$$

and obtain the following long exact sequences:

$$\begin{aligned} &\cdots \to \operatorname{Tor}_1^R(\mathcal{B}, \operatorname{im}\varphi) \to \mathcal{B} \otimes_R \ker\varphi \to \mathcal{B} \otimes_R M \to \mathcal{B} \otimes_R \operatorname{im}\varphi \to 0, \\ &\cdots \to \operatorname{Tor}_1^R(\mathcal{B}, \operatorname{coker}\varphi) \to \mathcal{B} \otimes_R \operatorname{im}\varphi \to \mathcal{B} \otimes_R N \to \mathcal{B} \otimes_R \operatorname{coker}\varphi \to 0. \end{aligned}$$

We already know that $\dim_{\mathcal{B}}(\mathcal{B} \otimes_R \ker\varphi) = 0$, $\dim_{\mathcal{B}} \operatorname{Tor}_1^R(\mathcal{B}, \operatorname{coker}\varphi) = 0$ and $\dim_{\mathcal{B}} \mathcal{B} \otimes_R \operatorname{coker}\varphi = 0$, hence

$$\begin{aligned} \mathcal{B} \otimes_R M &\to \mathcal{B} \otimes_R \operatorname{im}\varphi \\ \mathcal{B} \otimes_R \operatorname{im}\varphi &\to \mathcal{B} \otimes_R N \end{aligned}$$

are $\mathcal{B}$-dimension isomorphisms and, therefore, their composition. $\square$

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