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Scattering by a periodic tube in $\mathbb{R}^3$: part ii. A radiation condition

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Abstract

This second part of a pair of papers complements the first part (see Kirsch 2018 (**35** 104004)) but can be read independently. Scattering of time-harmonic waves from periodic structures at some fixed real-valued wave number becomes analytically difficult whenever there arise surface waves: These non-zero solutions to the homogeneous scattering problem physically correspond to modes propagating along the periodic structure and clearly imply non-uniqueness of any solution to the scattering problem. As in the first part we consider a medium described by a refractive index which is periodic along the axis of an infinite cylinder in $\mathbb{R}^3$ and constant outside of the cylinder. We formulate a proper radiation condition which allows the existence of traveling modes (and is motivated by the limiting absorption principle proven in the first part) and prove uniqueness and existence.

Keywords: Helmholtz equation, periodic wave guide, radiation condition

(Some figures may appear in colour only in the online journal)

1. Introduction

This part continues the first part (see [2]) but can be read independently of it. While in the first part the limiting absorption principle for the scattering by a tube in $\mathbb{R}^3$ filled with a periodic refractive index was proven and the corresponding radiation condition was derived we now take a different point of view and assume the radiation condition as given. With this radiation condition, formulated for real and positive wave numbers $k > 0$, we prove uniqueness and existence of a solution. The limiting absorption principle is not used in this part but serves only as a motivation for the definition of the radiation condition. Indeed, from the purely mathematical point of view one can replace the radiation condition by several others which also (by essentially the same proof) yields uniqueness and existence of a solution, see remarks 3.4 and 4.6 below.

2. Formulation of the problem

We begin by setting up some notations (see figure 1). Let $k \in \mathbb{R}$ with $k > 0$ be the wave number which is kept fixed throughout the paper. Let $B_N(0, R) = \{x \in \mathbb{R}^N : |x| < R\}$ be the ball in $\mathbb{R}^N$ with center 0 and radius $R > 0$, and $T_R = B_2(0, R) \times \mathbb{R} \subset \mathbb{R}^3$ be the tube (or infinite cylinder) in x_3- direction. Furthermore, we define the finite cylinder by $C_R := B_2(0, R) \times (0, 2\pi) \subset \mathbb{R}^3$ and $C_\infty := \mathbb{R}^2 \times (0, 2\pi) \subset \mathbb{R}^3$. Furthermore, we assume that T_R is filled with some medium with index of refraction $n \in L^\infty(\mathbb{R}^3)$ which is assumed to be $2\pi-$periodic with respect to the variable x_3 and equals to one outside of T_{R_0} for some $R_0 > 0$. Finally, let $f \in L^2(\mathbb{R}^3)$ be given with support contained in T_{R_0}. The problem is to determine $u \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ with

$$\Delta u + k^2 n\, u \;=\; -f \quad \text{in } \mathbb{R}^3. \tag{1}$$

The solution is understood in the variational sense; that is, we search for $u \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \left[\nabla u \cdot \nabla\overline{\psi} - k^2 n\, u\, \overline{\psi}\right] \mathrm{d}x \;=\; \int_{T_{R_0}} f\, \overline{\psi}\, \mathrm{d}x \quad \text{for all } \psi \in H^1(\mathbb{R}^3) \text{ with compact support.}$$

Without a radiation condition the solution is not expected to be unique. In [2] we constructed the so called limiting absorption solution of the problem; that is, the limit of the solutions $u_\varepsilon \in H^1(\mathbb{R}^3)$ of the coercive problems $\Delta u_\varepsilon + (k + i\varepsilon)^2 n\, u_\varepsilon = f$ when $\varepsilon > 0$ tends to zero. The structure of the limiting absorption solution motivates the radiation condition below (definition 2.5). Its formulation needs some preparation.

Closely related to the source problem (1) is the family of quasi-periodic problems. Let $\alpha \in \mathbb{R}$. A function $u \in H^1(C_R)$ is called $\alpha-$quasi-periodic if $u(x_1, x_2, 2\pi) = \mathrm{e}^{\mathrm{i}\alpha 2\pi} u(x_1, x_2, 0)$ for all $(x_1, x_2) \in B_2(0, R)$ (in the sense of traces). It is obvious that one can restrict α to be in an interval of unit length and we take $\alpha \in [-1/2, 1/2]$. The subspace of $\alpha-$quasi-periodic functions is denoted by $H^1_\alpha(C_R)$, and the local space $H^1_{\alpha,\mathrm{loc}}(C_\infty)$ is defined by

$$H^1_{\alpha,\mathrm{loc}}(C_\infty) \;:=\; \left\{u \in H^1_{\mathrm{loc}}(C_\infty) : u|_{C_R} \in H^1_\alpha(C_R) \text{ for all } R > 0\right\}.$$

Therefore, the $\alpha-$quasi-periodic source problems are to determine $u_\alpha \in H^1_{\alpha,\mathrm{loc}}(C_\infty)$ such that

$$\Delta u_\alpha + k^2 n\, u_\alpha \;=\; -f_\alpha \quad \text{in } C_\infty \tag{2}$$

in the variational sense; that is,

$$\int_{C_\infty} \left[\nabla u_\alpha \cdot \nabla\overline{\psi} - k^2 n\, u_\alpha\, \overline{\psi}\right] \mathrm{d}x \;=\; \int_{C_R} f_\alpha\, \overline{\psi}\, \mathrm{d}x$$

for all $\psi \in H^1_\alpha(C_\infty)$ with $\psi = 0$ for $x_1^2 + x_2^2 \geqslant R^2$ for some $R > R_0$. Here $f_\alpha \in L^2(C_\infty)$ is some given function with compact support in $\overline{C_{R_0}}$. For the $\alpha-$quasi-periodic problem (2) a natural radiation condition is the extension of the classical Rayleigh expansion to our case; that is, the requirement that u_α has an expansion of the form

$$u_\alpha(r,\varphi,x_3) \;=\; \sum_{\ell,m\in\mathbb{Z}} a_{\ell,m}\,\frac{H_m^{(1)}\big(r\sqrt{k^2-(\ell+\alpha)^2}\big)}{H_m^{(1)}\big(R_1\sqrt{k^2-(\ell+\alpha)^2}\big)}\,\mathrm{e}^{\mathrm{i}[m\varphi+(\ell+\alpha)x_3]}, \quad r>R_1, \tag{3}$$

for some $R_1 > R_0$ and $a_{\ell,j} \in \mathbb{C}$. Here, $H_m^{(1)}(z)$ denote the Hankel functions of the first kind and order $m \in \mathbb{Z}$. The branch of the square root $\sqrt{z}$ for $z \in \mathbb{C}$ with $\mathrm{Im}z \geqslant 0$ is chosen such that $\mathrm{Re}z \geqslant 0$ and $\mathrm{Im}z \geqslant 0$. The series converges in $H^1(C_{R_2} \setminus C_{R_1})$ for every $R_2 > R_1$. This condition can equivalently be replaced by a one-dimensional radiation condition for the Fourier coefficients.

Lemma 2.1. *Let $u \in H^1_{\alpha,\mathrm{loc}}(C_\infty \setminus C_R)$ be a $\alpha-$quasi-periodic solution of the Helmholtz equation $\Delta u + k^2 u = 0$ in $C_\infty \setminus C_R$ for some $R > R_0$. Then the following conditions are equivalent:*

(a) u has a Rayleigh expansion of the form (3).

(b) All of the Fourier coefficients $u_{\ell,m}(r) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} u(r,\varphi,x_3)\mathrm{e}^{-\mathrm{i}[m\varphi+(\ell+\alpha)x_3]}\mathrm{d}x_3\,\mathrm{d}\varphi$ for $\ell, m \in \mathbb{Z}$, satisfy the one-dimensional radiation condition

$$\lim_{r\to\infty} \sqrt{r}\big[u'_{\ell,m}(r) - \mathrm{i}\,k_\ell\,u_{\ell,m}(r)\big] \;=\; 0 \tag{4}$$

where $k_\ell = \sqrt{k^2-(\ell+\alpha)^2}$.

Proof. It is obvious that (a) implies (b). Indeed, if u has a Rayleigh expansion of the form (3) then the Fourier coefficients are given by $u_{\ell,m}(r) = 2\pi\, a_{\ell,m}\frac{H_m^{(1)}\big(r\sqrt{k^2-(\ell+\alpha)^2}\big)}{H_m^{(1)}\big(R_1\sqrt{k^2-(\ell+\alpha)^2}\big)}$ which satisfy (4) by the asymptotic behaviour of the Hankel functions as r tends to infinity.

Let now $u_{\ell,m}(r)$ satisfy (4). The fact that u satisfies the Helmholtz equation implies that the Fourier coefficients satisfy Bessel's differential equation

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + k^2 - \frac{m^2}{r^2} - \ell^2\right] u_{\ell,m}(r) \;=\; 0, \quad r > R. \tag{5}$$

The general solution is given by $u_{\ell,m}(r) = c_{\ell,m}H_m^{(1)}(k_\ell r) + d_{\ell,m}H_m^{(2)}(k_\ell r)$ for some coefficients $c_{\ell,m}, d_{\ell,m}$. Condition (4) implies $d_{\ell,m} = 0$ which proves the assertion. □

But even with this Rayleigh expansion the solutions of (2) are not always unique. As the case of constant n explicitly shows there might exist parameters $\alpha \in [-1/2, 1/2]$ for which non-trivial quasi-periodic solutions of (2) for $f = 0$ exist. These parameters are called *exceptional values*. We define the set

$$A \;=\; \big\{\alpha \in [-1/2, 1/2] : \text{ there exists } \ell \in \mathbb{Z} \text{ with } |\alpha + \ell| = k\big\}$$

of cut-off values (note that A consists of one or two elements) and make the following assumption.

Assumption 2.2. *For every $\alpha \in A$ the only $\alpha-$quasi-periodic solution $u \in H^1_\alpha(C_\infty)$ of (2) for $f = 0$ which satisfies the Rayleigh expansion (3) has to be the trivial one. In other words, no $\alpha \in A$ is an exceptional value.*

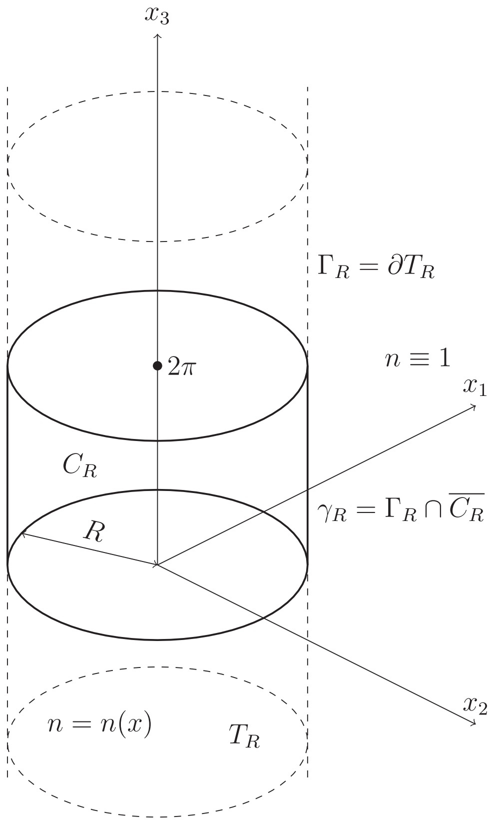

Figure 1. The geometry with its notations.

The following can be shown (see, e.g. lemma 2.9 of [2]).

Lemma 2.3. *Let assumption 2.2 hold. Then there exist only finitely many exceptional values $\alpha \in [-1/2, 1/2]$. Furthermore, if α is an exceptional value then also $-\alpha$. Therefore, the set of exceptional values can be described by $\{\hat{\alpha}_j : j \in J\}$ where $J \subset \mathbb{Z}$ is finite and symmetric with respect to the origin and $\hat{\alpha}_{-j} = -\hat{\alpha}_j$ for $j \in J$. The corresponding eigenspaces*

$$\hat{X}_j \;=\; \left\{\hat{\phi}_j \in H^1_{\hat{\alpha}_j}(C_\infty) : \Delta\hat{\phi}_j + k^2 n\hat{\phi}_j = 0 \text{ in } C_\infty,\ \hat{\phi}_j \text{ satisfies the Rayleigh expansion}\right\} \tag{6}$$

are finite dimensional. Furthermore, the expansion coefficients $a_{\ell,m}$ in (3) of any eigenfunction $\hat{\phi} \in X_j$ vanish for all $|\ell + \hat{\alpha}_j| \leqslant k$. This implies that every eigenfunction $\hat{\phi} \in \hat{X}_j$ is evanescent; that is, there exists $c > 0$ and $\sigma > 0$ with $|\hat{\phi}_j(x)| \leqslant c\,\mathrm{e}^{-\sigma|\tilde{x}|}$ for all $x \in C_\infty$ where $\tilde{x} = (x_1, x_2)$. We set $m_j = \dim\hat{X}_j$.

We now choose a special basis in $\hat{X}_j$ which is justified by the limiting absorption principle (see part A, [2]). In every $\hat{X}_j$ we consider the m_j − dimensional self-adjoint eigenvalue problem to determine $\lambda \in \mathbb{R}$ and $\hat{\phi} \in \hat{X}_j$ with

$$-\mathrm{i}\int_{C_\infty} \frac{\partial\hat{\phi}}{\partial x_3}\,\overline{\psi}\,\mathrm{d}x \;=\; \lambda\, k\int_{C_\infty} n\,\hat{\phi}\,\overline{\psi}\,\mathrm{d}x \quad \text{for all } \psi \in \hat{X}_j.$$

We denote the eigenvalues and eigenfunctions by $\lambda_{\ell,j}$ and $\hat\phi_{\ell,j}$, respectively; that is,

$$-\mathrm{i}\int_{C_\infty}\frac{\partial\hat\phi_{\ell,j}}{\partial x_3}\,\overline{\psi}\,\mathrm{d}x \;=\; \lambda_{\ell,j}\,k\int_{C_\infty} n\,\hat\phi_{\ell,j}\,\overline{\psi}\,\mathrm{d}x \quad \text{for all } \psi\in\hat X_j \text{ and } \ell=1,\dots,m_j, \tag{7}$$

and every $j\in J$. We normalize the eigenfunctions $\left\{\hat\phi_{\ell,j}:\ell=1,\dots,m_j\right\}$ such that

$$2k\int_{C_\infty} n\,\hat\phi_{\ell,j}\,\overline{\hat\phi_{\ell',j}}\,\mathrm{d}x \;=\; \delta_{\ell,\ell'} \quad \text{for all } \ell,\ell'. \tag{8}$$

We make a second assumption and assume that the wave number k is regular in the following sense.

Definition 2.4. $k>0$ is called *regular*, if $\lambda_{\ell,j}\neq 0$ for all $\ell=1,\dots,m_j$ and $j\in J$ where $\lambda_{\ell,j}\in\mathbb{R}$, $\ell=1,\dots,m_j$, are the eigenvalues of the selfadjoint eigenvalue problem (7) in the finite dimensional space $\hat X_j$.

Then, for every $j\in J$ we can split the propagating modes $\hat\phi_{\ell,j}$ in those with $\lambda_{\ell,j}>0$ and those with $\lambda_{\ell,j}<0$. These describe the modes which travel upwards and downwards, respectively. The radiation condition, formulated below in definition 2.5, consists of two parts. The first part (see part (a) of definition 2.5) describes the behavior along the axis of the cylinder while the second part (part (b) of definition 2.5) describes the behavior orthogonal to the cylinder. The second part is formulated in terms of the Fourier transform $\mathcal{F}g:\mathbb{Z}\times\mathbb{R}\to\mathbb{C}$ of g with respect to cylindrical coordinates which is given by

$$(\mathcal{F}f)(m,\xi) \;=\; \frac{1}{2\pi}\int_{\mathbb{R}}\int_0^{2\pi} f(\varphi,y_3)\,\mathrm{e}^{-\mathrm{i}(m\varphi+\xi y_3)}\mathrm{d}\varphi\,\mathrm{d}y_3, \quad m\in\mathbb{Z},\ \xi\in\mathbb{R}.$$

Then $\mathcal{F}$ is well defined and bounded from $L^2(\Gamma_R)$ into

$$L^2(\mathbb{Z}\times\mathbb{R}) \;:=\; \left\{\hat g:\mathbb{Z}\times\mathbb{R}\to\mathbb{C}:\hat g(m,\cdot)\in L^2(\mathbb{R}) \text{ for all } m \text{ and } \sum_{m\in\mathbb{Z}}\int_{\mathbb{R}}\left|\hat g(m,\xi)\right|^2\mathrm{d}\xi<\infty\right\}.$$

The inverse transform is then

$$(\mathcal{F}^{-1}g)(\varphi,x_3) \;=\; \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}} g(m,\xi)\,\mathrm{e}^{\mathrm{i}(m\varphi+\xi x_3)}\,\mathrm{d}\xi.$$

Also, Parseval's identity holds in the form

$$\int_{\Gamma_R}\left|g(x)\right|^2\mathrm{d}s \;=\; R\int_{\mathbb{R}}\int_0^{2\pi}\left|g(\phi,x_3)\right|^2\mathrm{d}\phi\,\mathrm{d}x_3 \;=\; R\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}}\left|(\mathcal{F}g)(m,\xi)\right|^2\mathrm{d}\xi. \tag{9}$$

In the formulation of the radiation condition we separate the propagating modes which travel upwards or downwards. This separation is formulated by auxiliary functions $\psi^\pm\in C^\infty(\mathbb{R})$ with the properties

$$\left|\psi^+(t)-\sigma(t)\right| + \left|\frac{\mathrm{d}\psi^+(t)}{\mathrm{d}t}\right| + \left|\frac{\mathrm{d}^2\psi^+(t)}{\mathrm{d}t^2}\right| \leqslant \frac{c}{|t|}, \quad |t|\geqslant 1, \quad \psi^- \;=\; 1-\psi^+, \tag{10}$$

where $\sigma(t) = \frac{1}{2}\big(1 + \text{sign}t\big) = \begin{cases} 1, & t > 0, \\ 0, & t < 0. \end{cases}$ Here, the constant $c > 0$ is independent on t. In particular, $\psi^+(t)$ tends to zero as t tends to $-\infty$ while it tends to 1 as t tends to $+\infty$. The function ψ^- behaves analogously.

Definition 2.5 (Radiation condition). Let assumption 2.2 hold and let $k > 0$ be regular in the sense of definition 2.4 and let $\psi^\pm \in C^\infty(\mathbb{R})$ be given with the properties (10). The solution $u \in H^2_{\text{loc}}(\mathbb{R}^3)$ of (1) has a decomposition in the form $u = u^{(1)} + u^{(2)}$ where:

(a) $u^{(1)}|_{T_{\hat{R}}} \in H^1(T_{\hat{R}})$ for every $\hat{R} > R$, and $u^{(2)} \in W^{2,\infty}(\mathbb{R}^3)$ is given by

$$u^{(2)}(x) \;=\; \psi^+(x_3) \sum_{j\in J} \sum_{\lambda_{\ell,j}>0} a_{\ell j}\,\hat{\phi}_{\ell,j}(x) \;+\; \psi^-(x_3) \sum_{j\in J} \sum_{\lambda_{\ell,j}<0} a_{\ell j}\,\hat{\phi}_{\ell,j}(x), \quad x \in \mathbb{R}^3, \tag{11}$$

for some $a_{\ell j} \in \mathbb{C}$. Here, $\{\lambda_{\ell,j}, \hat{\phi}_{\ell j} : \ell = 1, \ldots, m_j\}$ are the eigenvalues and eigenfunctions, respectively, of the eigenvalue problem (7).

(b) The cylindrical Fourier transform $(\mathcal{F}u^{(1)})(r, m, \xi)$ of $u^{(1)}$ satisfies the one-dimensional radiation condition

$$\lim_{r\to\infty} \sqrt{r} \left[\frac{\partial}{\partial r}(\mathcal{F}u^{(1)})(r, m, \xi) - i\,k(\xi)\,(\mathcal{F}u^{(1)})(r, m, \xi) \right] \;=\; 0 \tag{12}$$

for all $m \in \mathbb{Z}$ and almost all $\xi \in \mathbb{R}$. Here, $k(\xi) = \sqrt{k^2 - \xi^2}$.

Remarks 2.6.

(a) From (11) we observe that for $x_3 \to \pm\infty$ the solution behaves as $\sum_{j\in J} u_j^\pm$ where $u_j^\pm$ are linear combinations of $\{\hat{\phi}_{\ell j} : \ell = 1, \ldots .m_j,\; \lambda_{\ell j} \gtrless 0\}$.

(b) Examples for functions ψ^+ with (10) are

$$\psi^+(t) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \mathrm{e}^{-s^2/2}\mathrm{d}s \quad \text{or} \quad \psi^+(t) \;=\; \frac{1}{2}\left[1 \;+\; \frac{2}{\pi}\int_0^t \frac{\sin s}{s}\,\mathrm{d}s\right], \quad t \in \mathbb{R}.$$

The asymptotic behaviour of $u^{(2)}$ is not changed by choosing different functions $\psi^\pm$ because for any functions $\psi_1^\pm$ and $\psi_2^\pm$ with (10) it holds that $\psi_1^\pm - \psi_2^\pm \in H^1(\mathbb{R})$. Therefore, the difference is subsumed in $u^{(1)}$.

(c) In part A we have shown that the limiting absorption solution satisfies this radiation condition. The coefficients $a_{\ell j}$ are explicitly given by

$$a_{\ell j} \;=\; \frac{2\pi \mathrm{i}}{|\lambda_{\ell,j}|} \int_{\mathbb{R}^3} f(x)\,\overline{\hat{\phi}_{\ell,j}(x)}\,\mathrm{d}x, \quad \ell = 1, \ldots, m_j,\; j \in J. \tag{13}$$

3. Uniqueness

If we interpret $\text{Im}\int_{\gamma_r} \overline{u}\,\frac{\partial u}{\partial \nu}\,\mathrm{d}s$ as an energy flow along the axis of the tube then the energies of the guided modes are constant and positive as the following lemma shows. In the case of a closed waveguide; that is, posing the boundary condition $u = 0$ or $\partial u/\partial r$ on ∂T_R, the following lemma implies almost directly uniqueness of the solution. We were not able to adjust the proof to the open waveguide problem but prove uniqueness in a different way (see below). The result of this lemma is, however, interesting in itself.

Lemma 3.1. *Let* $\gamma_r = \mathbb{R}^2 \times \{r\}$ *for* $r \in \mathbb{R}$ *and* $u^\pm = \sum_{j\in J} \sum_{\lambda_{\ell,j} \gtrless 0} a_{\ell,j}\, \hat{\phi}_{\ell,j}$ *for some* $a_{\ell,j} \in \mathbb{C}$. *Then, for every* $r \in \mathbb{R}$ *and* $\sigma \in \{+,-\}$,

$$\operatorname{Im} \int_{\gamma_r} \overline{u^\sigma}\, \frac{\partial u^\sigma}{\partial x_3}\, \mathrm{d}s \;=\; \frac{1}{4\pi} \sum_{j\in J} \sum_{\sigma\lambda_{\ell,j}>0} \lambda_{\ell,j}\, \left|a_{\ell,j}\right|^2.$$

Proof. Set $L_j^\pm = \{\ell : \lambda_{\ell,j} \gtrless 0\}$ and $u_j^\sigma = \sum_{\ell\in L_j^\sigma} a_{\ell,j}\, \hat{\phi}_{\ell,j}$ for $j \in J$. Then, for $j,j' \in J$ by Green's theorem in the region $C_{\infty,r} := \mathbb{R}^2 \times (r, r+2\pi)$,

$$\begin{aligned}
0 &= \int_{\partial C_{\infty,r}} \left(\overline{u_j^\sigma}\, \frac{\partial u_{j'}^\sigma}{\partial \nu} - u_{j'}^\sigma\, \frac{\partial \overline{u_j^\sigma}}{\partial \nu} \right) \mathrm{d}s \\
&= -\int_{\gamma_r} \left(\overline{u_j^\sigma}\, \frac{\partial u_{j'}^\sigma}{\partial x_3} - u_{j'}^\sigma\, \frac{\partial \overline{u_j^\sigma}}{\partial x_3} \right) \mathrm{d}s \;+\; \int_{\gamma_{r+2\pi}} \left(\overline{u_j^\sigma}\, \frac{\partial u_{j'}^\sigma}{\partial x_3} - u_{j'}^\sigma\, \frac{\partial \overline{u_j^\sigma}}{\partial x_3} \right) \mathrm{d}s \\
&= \left(\mathrm{e}^{\mathrm{i}(\hat{\alpha}_{j'} - \hat{\alpha}_j)2\pi} - 1 \right) \int_{\gamma_r} \left(\overline{u_j^\sigma}\, \frac{\partial u_{j'}^\sigma}{\partial x_3} - u_{j'}^\sigma\, \frac{\partial \overline{u_j^\sigma}}{\partial x_3} \right) \mathrm{d}s.
\end{aligned}$$

Therefore, the last integral vanishes for $j \neq j'$. Thus we have

$$\begin{aligned}
2\operatorname{Im} \int_{\gamma_r} \overline{u^\sigma}\, \frac{\partial u^\sigma}{\partial x_3}\, \mathrm{d}s &= \int_{\gamma_r} \left[\overline{u^\sigma}\, \frac{\partial u^\sigma}{\partial x_3} - u^\sigma\, \frac{\partial \overline{u^\sigma}}{\partial x_3} \right] \mathrm{d}s \\
&= \sum_{j\in J} \int_{\gamma_r} \left[\overline{u_j^\sigma}\, \frac{\partial u_j^\sigma}{\partial x_3} - u_j^\sigma\, \frac{\partial \overline{u_j^\sigma}}{\partial x_3} \right] \mathrm{d}s \;=\; 2 \sum_{j\in J} \operatorname{Im} \int_{\gamma_r} \overline{u_j^\sigma}\, \frac{\partial u_j^\sigma}{\partial x_3}\, \mathrm{d}s \\
&= 2 \sum_{j\in J} \operatorname{Im} \int_{\gamma_0} \overline{u_j^\sigma}\, \frac{\partial u_j^\sigma}{\partial x_3}\, \mathrm{d}s
\end{aligned}$$

because $x_3 \mapsto \overline{u_j^\sigma}\, \frac{\partial u_j^\sigma}{\partial x_3}$ is 2π–periodic. Now we show that for every $j \in J$

$$\operatorname{Im} \int_{\gamma_0} \overline{u_j^\sigma}\, \frac{\partial u_j^\sigma}{\partial x_3}\, \mathrm{d}s \;=\; \frac{1}{2\pi} \operatorname{Im} \int_{C_\infty} \overline{u_j^\sigma}\, \frac{\partial u_j^\sigma}{\partial x_3}\, \mathrm{d}x.$$

Setting $v(x) = x_3\, u_j^\sigma(x)$ yields $\frac{\partial v(x)}{\partial x_3} = u_j^\sigma(x) + x_3\, \frac{\partial u_j^\sigma(x)}{\partial x_3}$ and $\Delta v + k^2 n v = 2 \frac{\partial u_j^\sigma}{\partial x_3}$. Therefore,

$$\begin{aligned}
2 \int_{C_\infty} \overline{u_j^\sigma}\, \frac{\partial u_j^\sigma}{\partial x_3}\, \mathrm{d}x &= \int_{C_\infty} \overline{u_j^\sigma}\, \left(\Delta v + k^2 n v \right) \mathrm{d}x \\
&= \int_{C_\infty} v \left(\Delta \overline{u_j^\sigma} + k^2 n \overline{u_j^\sigma} \right) \mathrm{d}x \;+\; \int_{\partial C_\infty} \left(\overline{u_j^\sigma}\, \frac{\partial v}{\partial \nu} - v\, \frac{\partial \overline{u_j^\sigma}}{\partial \nu} \right) \mathrm{d}s \\
&= -\int_{\gamma_0} |u_j^\sigma|^2 \mathrm{d}s + \int_{\gamma_{2\pi}} \left[\overline{u_j^\sigma} \left(u_j^\sigma + 2\pi\, \frac{\partial u_j^\sigma}{\partial x_3} \right) - 2\pi u_j^\sigma\, \frac{\partial \overline{u_j^\sigma}}{\partial x_3} \right] \mathrm{d}s \\
&= 2\pi \int_{\gamma_0} \left(\overline{u_j^\sigma}\, \frac{\partial u_j^\sigma}{\partial x_3} - u_j^\sigma\, \frac{\partial \overline{u_j^\sigma}}{\partial x_3} \right) \mathrm{d}s \;=\; 4\pi\, \mathrm{i} \operatorname{Im} \int_{\gamma_0} \overline{u_j^\sigma}\, \frac{\partial u_j^\sigma}{\partial x_3}\, \mathrm{d}s
\end{aligned}$$

which proves the equality. Furthermore,

$$\int_{C_\infty} \overline{u_j^\sigma}\, \frac{\partial u_j^\sigma}{\partial x_3}\, \mathrm{d}x = \sum_{\ell,\ell' \in L_j^\sigma} \overline{a_{\ell,j}}\, a_{\ell',j} \int_{C_\infty} \overline{\hat{\phi}_{\ell,j}}\, \frac{\partial \hat{\phi}_{\ell',j}}{\partial x_3}\, \mathrm{d}x$$

$$= \mathrm{i}k \sum_{\ell,\ell' \in L_j^\sigma} \overline{a_{\ell,j}}\, a_{\ell',j}\, \lambda_{\ell',j} \int_{C_\infty} n\, \overline{\hat{\phi}_{\ell,j}}\, \hat{\phi}_{\ell',j}\, \mathrm{d}x \;=\; \frac{\mathrm{i}}{2} \sum_{\ell \in L_j^\sigma} \lambda_{\ell,j}\, |a_{\ell,j}|^2$$

by the definiton of $\hat{\phi}_{\ell,j}$. Taking the imaginary part yields the assertion. □

The relationship between the original source problem (1) and the $\alpha-$quasi-periodic problems (2) is given by the Floquet–Bloch transform F which is defined as

$$(Ff)(t,\alpha) \;=\; \sum_{m\in\mathbb{Z}} f(t+2\pi m)\, \mathrm{e}^{-\mathrm{i}\alpha 2\pi m}, \quad t \in (0,2\pi),\ \alpha \in [-1/2, 1/2],$$

for $f \in C_0^\infty(\mathbb{R})$. From the definition we directly observe that for smooth functions f and fixed α the transformed function $t \mapsto (Ff)(t,\alpha)$ is $\alpha-$quasi-periodic while for fixed t the function $\alpha \mapsto (Ff)(t,\alpha)$ is $1-$ periodic. It is hence sufficient to consider $L^2\big((0,2\pi)\times(-1/2,1/2)\big)$ as image space of F. It is well known that F has an extension to a bounded operator from $L^2(\mathbb{R})$ onto $L^2\big((0,2\pi)\times(-1/2,1/2)\big)$ with inverse

$$(F^{-1}h)(t) \;=\; \int_{-1/2}^{1/2} h(t,\alpha)\, \mathrm{d}\alpha, \quad t \in \mathbb{R}, \tag{14}$$

where we extended $h(\cdot,\alpha)$ to a $\alpha-$quasiperiodic function in $\mathbb{R}$. Furthermore, the restriction of F to $H^1(\mathbb{R})$ is an isomorphism from $H^1(\mathbb{R})$ onto $L^2\big((-1/2,1/2), H^1_{qp}(0,2\pi)\big)$ where the latter space is defined as the completion of

$$\big\{v \in C^1\big([0,2\pi]\times[-1/2,1/2]\big) : v^{(j)}(2\pi,\alpha) = \mathrm{e}^{\mathrm{i}\alpha 2\pi} v^{(j)}(0,\alpha) \text{ for all } \alpha \text{ and } j = 0,1\big\}$$

with respect to the norm

$$\sqrt{\int_{-1/2}^{1/2} \|v(\cdot,\alpha)\|^2_{H^1(0,2\pi)}\, \mathrm{d}\alpha}.$$

Let

$$(\mathcal{F}_1 g)(t) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s)\, \mathrm{e}^{-\mathrm{i}st} \mathrm{d}s, \quad t \in \mathbb{R},$$

be the one dimensional Fourier transform which can be expressed by the Floquet–Bloch transform as

$$(\mathcal{F}_1 g)(\ell+\alpha) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} (Fg)(s,\alpha)\, \mathrm{e}^{-\mathrm{i}(\ell+\alpha)s}\, \mathrm{d}s \;=\; (Fg)_\ell(\alpha),\ \ell \in \mathbb{Z},\ \alpha \in (-1/2, 1/2], \tag{15}$$

where $(Fg)_\ell(\alpha)$ are the Fourier coefficients of $(Fg)(\cdot,\alpha)$, $\ell \in \mathbb{Z}$. Therefore,

$$(Fg)(t,\alpha) \;=\; \frac{1}{\sqrt{2\pi}} \sum_{\ell\in\mathbb{Z}} (\mathcal{F}_1 g)(\alpha+\ell)\, \mathrm{e}^{\mathrm{i}(\alpha+\ell)t}. \tag{16}$$

In the following we use the same symbol F also for the Floquet–Bloch transform with respect to the variable x_3 of functions on $\mathbb{R}^3$; that is,

$$(Ff)(x,\alpha) \;=\; \sum_{m\in\mathbb{Z}} f(x+2\pi m\hat{e}^{(3)})\,\mathrm{e}^{-\mathrm{i}\alpha 2\pi m}, \quad x\in C_\infty,\ \alpha\in[-1/2,1/2],$$

where $\hat{e}^{(3)} = (0,0,1)^\top$. Then the analogous of (15) and (16) are given by

$$(\mathcal{F}f)(r,m,\ell+\alpha) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}(Ff)(r,\varphi,x_3,\alpha)\,\mathrm{e}^{-\mathrm{i}(m\varphi+(\ell+\alpha)x_3)}\,\mathrm{d}x_3\,\mathrm{d}\varphi = (Ff)_{\ell,m}(r,\alpha), \tag{17}$$

$$(Ff)(x,\alpha) = \frac{1}{2\pi}\sum_{m,\ell\in\mathbb{Z}}(\mathcal{F}f)(r,m,\alpha+\ell)\,\mathrm{e}^{\mathrm{i}(m\varphi+(\alpha+\ell)x_3)} \tag{18}$$

for $r>0$, $m,\ell\in\mathbb{Z}$, and $\alpha\in(-1/2,1/2]$. Here, r,φ,x_3 are the cylindrical coordinates of $x\in\mathbb{R}^3$ and $(Ff)_{\ell,m}(r,\alpha)$ are the Fourier coefficients of $(Ff)(r,\cdot,\cdot,\alpha)$.

From part (a) of the radiation condition we observe that $u^{(1)}$ satisfies the differential equation $\Delta u^{(1)} + k^2 n u^{(1)} = -h - f$ where

$$\begin{aligned} h(x) &= \Delta u^{(2)}(x) + k^2 n(x)\,u^{(2)}(x) \\ &= \sum_{j\in J}\sum_{\sigma\in\{+,-\}}\sum_{\sigma\lambda_{\ell,j}>0} a_{\ell,j}\left[\hat{\phi}_{\ell,j}(x)\,\frac{\mathrm{d}^2\psi^\sigma(x_3)}{\mathrm{d}x_3^2} \;+\; 2\,\frac{\mathrm{d}\psi^\sigma(x_3)}{\mathrm{d}x_3}\,\frac{\partial\hat{\phi}_{\ell,j}(x)}{\partial x_3}\right] \\ &= \sum_{j\in J}\sum_{\ell=1}^{m_j}\mathrm{sign}(\lambda_{\ell,j})\,a_{\ell,j}\left[\hat{\phi}_{\ell,j}(x)\,\frac{\mathrm{d}^2\psi^+(x_3)}{\mathrm{d}x_3^2} \;+\; 2\,\frac{\mathrm{d}\psi^+(x_3)}{\mathrm{d}x_3}\,\frac{\partial\hat{\phi}_{\ell,j}(x)}{\partial x_3}\right] \end{aligned} \tag{19}$$

since $\psi^- = 1-\psi^+$. From the properties (10) we observe that h decays as $1/|x_3|$ as $|x_3|$ tends to infinity. Therefore, the Floquet–Bloch transform $(Fh)(x,\alpha)$ is well defined for all $\alpha\in\mathbb{R}$. The following lemma computes it for the terms in the sum.

Lemma 3.2. *Set $\varphi(x_3) = \frac{\mathrm{d}\psi^+(x_3)}{\mathrm{d}x_3}$ for abbreviation. Then*

$$F\left(\hat{\phi}_{\ell,j}\frac{\mathrm{d}\varphi}{\mathrm{d}x_3} + 2\,\varphi\,\frac{\partial\hat{\phi}_{\ell,j}}{\partial x_3}\right)(x,\alpha) = \big(\Delta + k^2 n(x)\big)\big[\hat{\phi}_{\ell,j}(x)\,\rho(x_3,\alpha-\hat{\alpha}_j)\big] + \delta_{\alpha-\hat{\alpha}_j}\frac{1}{\pi}\,\frac{\partial\hat{\phi}_{\ell,j}(x)}{\partial x_3}$$

for all $\alpha\in(-1/2,1/2]$ and almost all $x\in\mathbb{R}^3$. Here, $\delta_\beta = 1$ for $\beta\in\mathbb{Z}$ and $\delta_\beta = 0$ for $\beta\notin\mathbb{Z}$ and ρ is given by

$$\rho(x_3,\beta) \;=\; \begin{cases} \dfrac{1}{\sqrt{2\pi}}\displaystyle\sum_{\ell\in\mathbb{Z}}\frac{(\mathcal{F}_1\varphi)(\ell+\beta)}{\mathrm{i}(\ell+\beta)}\,\mathrm{e}^{\mathrm{i}(\ell+\beta)x_3}, & \beta\notin\mathbb{Z}, \\ \dfrac{1}{\sqrt{2\pi}}\displaystyle\sum_{\ell\neq 0}\frac{(\mathcal{F}_1\varphi)(\ell)}{\mathrm{i}\ell}\,\mathrm{e}^{\mathrm{i}\ell x_3}, & \beta\in\mathbb{Z}. \end{cases}$$

Proof. Using that $\hat{\phi}_{\ell,j}$ is $\hat{\alpha}_j$–quasi-periodic, we observe that

$$F\big(\hat{\phi}_{\ell,j}\,\varphi\big)(x,\alpha) \;=\; \hat{\phi}_{\ell,j}(x)\sum_{m\in\mathbb{Z}}\varphi(x_3+2\pi m)\,\mathrm{e}^{2\pi m(\hat{\alpha}_j-\alpha)i} \;=\; \hat{\phi}_{\ell,j}(x)\,(F\varphi)(x_3,\alpha-\hat{\alpha}_j)$$

and analogously for $\frac{\partial\hat{\phi}_{\ell,j}}{\partial x_3}$ replacing $\hat{\phi}_{\ell,j}$. We compute the Floquet–Bloch transform of φ. From (16) we conclude that

$$(F\varphi)(x_3,\beta) \;=\; \frac{1}{\sqrt{2\pi}}\sum_{\ell\in\mathbb{Z}}(\mathcal{F}_1\varphi)(\ell+\beta)\,\mathrm{e}^{\mathrm{i}(\ell+\beta)x_3} \;=\; \frac{\partial\rho(x_3,\beta)}{\partial x_3} \;+\; \frac{(\mathcal{F}_1\varphi)(0)}{\sqrt{2\pi}}\,\delta_\beta \tag{20}$$

for $x_3,\beta\in\mathbb{R}$. Analogously, by (16),

$$(F\varphi')(x_3,\beta) \;=\; \frac{1}{\sqrt{2\pi}}\sum_{\ell\in\mathbb{Z}}(cF_1\varphi')(\ell+\beta)\,\mathrm{e}^{\mathrm{i}(\ell+\beta)x_3} \;=\; \frac{\partial^2\rho(x_3,\beta)}{\partial x_3^2}.$$

Using $(\mathcal{F}_1\varphi)(0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(t)\,\mathrm{d}t=\frac{1}{\sqrt{2\pi}}$ this yields

$$\begin{aligned}F\left(\hat\phi_{\ell,j}\frac{\mathrm{d}\varphi}{\mathrm{d}x_3}+2\,\varphi\frac{\partial\hat\phi_{\ell,j}}{\partial x_3}\right)(x,\alpha) &= \hat\phi_{\ell,j}(x)\,\frac{\partial^2\rho(x_3,\alpha-\hat\alpha_j)}{\partial x_3^2}+2\,\frac{\partial\rho(x_3,\alpha-\hat\alpha_j)}{\partial x_3}\,\frac{\partial\hat\phi_{\ell,j}(x)}{\partial x_3}\\ &\quad+\frac{1}{\pi}\,\frac{\partial\hat\phi_{\ell,j}(x)}{\partial x_3}\,\delta_{\alpha-\hat\alpha_j}\\ &= \left(\Delta+k^2n\right)\left[\hat\phi_{\ell,j}(x)\,\rho(x_3,\alpha-\hat\alpha_j)\right]+\frac{1}{\pi}\,\frac{\partial\hat\phi_{\ell,j}(x)}{\partial x_3}\,\delta_{\alpha-\hat\alpha_j}\end{aligned}$$

and ends the proof. □

Now we are able to show uniqueness under the radiation condition of definition 2.5.

Theorem 3.3. *Let assumption 2.2 hold and let $k>0$ be regular in the sense of definition 2.4. Then there exist at most one solution $u\in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ of the source problem (1) satisfying the radiation condition of definition 2.5.*

Proof. Let $u\in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ be a solution of the source problem (1) corresponding to $f=0$ which satisfies the radiation condition. We recall that $u^{(1)}$ satisfies the differential equation $\Delta u^{(1)}+k^2nu^{(1)}=-h$ where h is given by (19). Taking the Floquet–Bloch transform and using the previous lemma yields

$$\Delta(Fu^{(1)})(x,\alpha)+k^2n(x)(Fu^{(1)})(x,\alpha) \;=\; -(Fh)(x,\alpha) \;=\; -\Delta w(x,\alpha)-k^2n(x)w(x,\alpha)$$

for almost all α (in particular $\alpha\notin\left\{\hat\alpha_j : j\in J\right\}+\mathbb{Z}$) where

$$w(x,\alpha) \;=\; \sum_{j\in J}\sum_{\ell=1}^{m_j}\mathrm{sign}(\lambda_{\ell,j})\,a_{\ell,j}\,\hat\phi_{\ell,j}(x)\,\rho(x_3,\alpha-\hat\alpha_j),\qquad x\in C_\infty, \tag{21}$$

and ρ from lemma 3.2. We note that $w(\cdot,\alpha)$ is α−quasi-periodic. Now we set $v(x,\alpha)=(Fu^{(1)})(x,\alpha)+w(x,\alpha)$ for $x\in C_\infty$ and almost all $\alpha\in\mathbb{R}$. Then we observe that $v(\cdot,\alpha)$ is α−quasi-periodic and $\Delta v(\cdot,\alpha)+k^2n\,v(\cdot,\alpha)=0$ in C_∞ for almost all $\alpha\in\mathbb{R}$. Next we show that $v(\cdot,\alpha)$ satisfies a Rayleigh expansion for $x\notin C_R$. By lemma 2.1 it is sufficient to show that the Fourier coefficients of $v(\cdot,\alpha)$ satisfy the one dimensional radiation condition. This is clear for the Fourier coefficients of w because of the exponential decay of $\hat\phi_{\ell,j}(x)$ as r tends to infinity. The Fourier coefficients $\hat u_{\ell,m}(r,\alpha)$ of $(Fu^{(1)})(\cdot,\alpha)$ are given by (18); that is,

$$\hat u_{\ell,m}(r,\alpha) \;=\; \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}(Fu^{(1)})(r,\varphi,x_3,\alpha)\,\mathrm{e}^{-\mathrm{i}(m\varphi+(\ell+\alpha)x_3)}\,\mathrm{d}\varphi\,\mathrm{d}x_3 \;=\; (\mathcal{F}u^{(1)})(r,m,\ell+\alpha),$$

and this satisfies the radiation condition of definition 2.5, part (b), by assumption. By lemma 2.1 this is equivalent to the Rayleigh expansion. This holds for almost all $\alpha \in [-1/2, 1/2]$. The trivial uniqueness result for the α−quasi-periodic scattering problem at non-exceptional wave numbers implies that $v(\cdot, \alpha)$ vanishes in C_∞ for almost all α. Thus, $(Fu^{(1)})(\cdot, \alpha) = -w(\cdot, \alpha)$ in C_∞ for almost all $\alpha \in (-1/2, 1/2]$. Now fix any $j_0 \in J$ and choose a small open interval I such that $\hat{\alpha}_{j_0} \in I$ and $\hat{\alpha}_j \notin I$ for $j \neq j_0$. Then, for almost all $\alpha \in I$,

$$\begin{aligned}
&(Fu^{(1)})(x, \alpha) + \sum_{j \neq j_0} \sum_{\ell=1}^{m_j} \operatorname{sign}(\lambda_{\ell,j})\, a_{\ell,j}\, \hat{\phi}_{\ell,j}(x)\, \rho(x_3, \alpha - \hat{\alpha}_j) \\
&= -\left[\sum_{\ell=1}^{m_{j_0}} \operatorname{sign}(\lambda_{\ell,j_0})\, a_{\ell,j_0}\, \hat{\phi}_{\ell,j_0}(x)\right] \rho(x_3, \alpha - \hat{\alpha}_{j_0}) \\
&= -\left[\sum_{\ell=1}^{m_{j_0}} \operatorname{sign}(\lambda_{\ell,j_0})\, a_{\ell,j_0}\, \hat{\phi}_{\ell,j_0}(x)\right] \left[\frac{1}{\sqrt{2\pi}} \frac{\varphi(\alpha - \hat{\alpha}_{j_0})}{i(\alpha - \hat{\alpha}_{j_0})}\, \mathrm{e}^{\mathrm{i}(\alpha - \hat{\alpha}_{j_0})x_3} \right. \\
&\qquad \left. + \frac{1}{\sqrt{2\pi}} \sum_{\ell \neq 0} \frac{\varphi(\ell + \alpha - \hat{\alpha}_{j_0})}{i(\ell + \alpha - \hat{\alpha}_{j_0})}\, \mathrm{e}^{\mathrm{i}(\ell + \alpha - \hat{\alpha}_{j_0})x_3}\right].
\end{aligned}$$

This equation has the form

$$g(x, \alpha) = \frac{1}{\sqrt{2\pi}} \left[\sum_{\ell=1}^{m_{j_0}} \operatorname{sign}(\lambda_{\ell,j_0})\, a_{\ell,j_0}\, \hat{\phi}_{\ell,j_0}(x)\right] \frac{\varphi(\alpha - \hat{\alpha}_{j_0})}{\mathrm{i}(\alpha - \hat{\alpha}_{j_0})}\, \mathrm{e}^{\mathrm{i}(\alpha - \hat{\alpha}_{j_0})x_3}$$

for some g which is in $L^2\big(C_{\hat{R}} \times I\big)$ for every $\hat{R} > R$. Therefore,

$$\|g(\cdot, \alpha)\|^2_{L^2(C_{\hat{R}})} = \frac{1}{2\pi\,(\alpha - \hat{\alpha}_{j_0})^2} \left|\varphi(\alpha - \hat{\alpha}_{j_0})\right|^2 \left\|\sum_{\ell=1}^{m_{j_0}} \operatorname{sign}(\lambda_{\ell,j_0})\, a_{\ell,j_0}\, \hat{\phi}_{\ell,j_0}\right\|^2_{L^2(C_{\hat{R}})}.$$

The left-hand side is integrable over I in contrast to the right hand side unless the sum vanishes identically in $C_{\hat{R}}$. From the linear independence of $\big\{\hat{\phi}_{\ell,j_0} : \ell = 1, \ldots, m_{j_0}\big\}$ we conclude that all of the coefficients a_{ℓ,j_0} vanish. This holds for all j_0; that is, $u^{(2)}$ vanishes identically. It remains to show that $u^{(1)}$ vanishes. Then $u = u^{(1)}$ solves $\Delta u + k^2 n u = 0$ in $\mathbb{R}^3$ and $u \in H^1(T_{\hat{R}})$ for all $\hat{R} > R$. Taking the Floquet–Bloch transform yields that Fu satisfies the Rayleigh expansion and $\Delta(Fu) + k^2 n(Fu) = 0$ in C_∞ for almost all $\alpha \in (-1/2, 1/2)$. Since the set of exceptional is finite by lemma 2.3 we conclude that Fu vanishes for almost all α and thus also $u = 0$ almost everywhere. □

Remark 3.4. In the radiation condition the signs of $\lambda_{\ell,j}$ determine whether the corresponding propagating mode travels to $x_3 \to +\infty$ or to $x_3 \to -\infty$. From the proof we note that this particular decomposition $\{1, \ldots, m_j\} = \{\ell : \lambda_{\ell,j} > 0\} \cup \{\ell : \lambda_{\ell,j} < 0\}$—which is justified by the limiting absorption principle—is not necessary. Any prescribed decomposition of $\{1, \ldots, m_j\}$ into disjoint sets $L_j^{(1)}$ and $L_j^{(2)}$ would also provide uniqueness.

4. Existence

In the first part [2] we have shown existence indirectly by the limiting absorption principle. It is the aim to present a direct proof of existence which is solely based on the radiation condition.

The radiation condition suggests that we search for the solution u of $\Delta u + k^2 nu = -f$ in the form $u = u^{(1)} + u^{(2)}$ where $u^{(1)} \in H^1(T_{\hat{R}})$ for every $\hat{R} > R$ and $u^{(2)} \in W^{2,\infty}(\mathbb{R}^3)$ is given by

$$u^{(2)}(x) \;=\; \psi^+(x_3)\sum_{j\in J}\sum_{\lambda_{\ell,j}>0} a_{\ell,j}\,\hat{\phi}_{\ell,j}(x) \;+\; \psi^-(x_3)\sum_{j\in J}\sum_{\lambda_{\ell,j}<0} a_{\ell,j}\,\hat{\phi}_{\ell,j}(x), \quad x\in\mathbb{R}^3, \tag{22}$$

for $a_{\ell,j} \in \mathbb{C}$ given by (13). Here, $\{\lambda_{\ell,j}, \hat{\phi}_{\ell,j} : \ell = 1,\dots,m_j\}$ are the eigenvalues and eigenfunctions, respectively, of the eigenvalue problem (7) for every $j \in J$. Furthermore, we choose explicitly $\psi^\pm \in C^\infty(\mathbb{R})$ to be $\psi^+(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^t \mathrm{e}^{-s^2/2}\mathrm{d}s$, $t \in \mathbb{R}$, and $\psi^- = 1 - \psi^+$. We set again $h = \Delta u^{(2)} + k^2 nu^{(2)}$ in $\mathbb{R}^3$. Then $u^{(1)}$ has to solve $\Delta u^{(1)} + k^2 nu^{(1)} = -f - h$ in $\mathbb{R}^3$. Furthermore, since f and also h are in $L^2(\mathbb{R}^3)$ (for h this follows from the form (19) and the decay of $\mathrm{d}\psi^+/\mathrm{d}t$ and $\mathrm{d}^2\psi^+/\mathrm{d}t^2$) we can take the Floquet–Bloch transforms. Because of the exponential decay of f and h as $|x_3| \to \infty$ we note that Ff and Fh are continuous with respect to α. Therefore, it is the aim to solve

$$\Delta u_\alpha \;+\; k^2 nu_\alpha \;=\; -(Ff)(\cdot,\alpha) \;-\; (Fh)(\cdot,\alpha) \quad \text{in } C_\infty \tag{23}$$

for every $\alpha \in [-1/2, 1/2]$. Assume for the moment that there exists a solution $u_\alpha \in H^1_{\alpha,\mathrm{loc}}(C_\infty)$ of (23) for every $\alpha \in [-1/2, 1/2]$ which satisfies also the radiation condition (4) such that $\alpha \mapsto \|u_\alpha\|_{H^1(C_{\hat{R}})}$ is continuous for every $\hat{R} > R$. Then $u^{(1)} = \int_{-1/2}^{1/2} u_\alpha\,\mathrm{d}\alpha$ belongs to $H^1(T_{\hat{R}})$ for every $\hat{R} > R$ and $u = u^{(1)} + u^{(2)}$ satisfies $\Delta u + k^2 nu = -f$ in $\mathbb{R}^3$ and the radiation condition (12) by (17). Therefore, we have to study (23) and (4) with respect to solvability and continuous dependence on α.

In the first part we reduce the problem (23) and (4) to an operator equation of the form

$$\tilde{u}_\alpha \;-\; K_\alpha \tilde{u}_\alpha \;=\; r_\alpha \quad \text{in } H^1_{\mathrm{per}}(C_R) \tag{24}$$

with a compact operator K_α and right hand side $r_\alpha \in H^1_{\mathrm{per}}(C_R)$ which depend continuously on α (see lemma 4.2 below). Here, $H^1_{\mathrm{per}}(C_R)$ denotes the subspace of $H^1(C_R)$ consisting of 2π–periodic (wrt x_3) functions. The reduction to this equation on the bounded domain C_R is not quite standard because the part $(Fh)(\cdot,\alpha)$ in (23) does not vanish outside of any $C_{\hat{R}}$—in contrast to $(Ff)(\cdot,\alpha)$ which vanishes outside of C_R.

The equation (24) is singular in the sense of Colton and Kress (section 1.4 of [1]) because it is uniquely solvable for all α which are not exceptional. For $\alpha \in \{\hat{\alpha}_j : j \in J\}$, however, the kernel of $I - K_\alpha$ is not trivial. We will apply a theorem from [1] (see theorem A.1 of the appendix) which proves that the mapping $\alpha \mapsto \tilde{u}_\alpha$ can be continuously extended to the whole interval $[-1/2, 1/2]$. Therefore, the inverse Floquet–Bloch transform yields that $u^{(1)} \in H^1(T_{\hat{R}})$ for any $\hat{R} > R$ and provides the solution $u = u^{(1)} + u^{(2)}$.

For the reduction of (23) to an equation of the type (24) we need to investigate the α–quasi-periodic problem (2) for right hand sides f_α which do not vanish for large values of r but decay sufficiently large as $r \to \infty$. We recall that in our case $f_\alpha = (Ff)(\cdot,\alpha) + (Fh)(\cdot,\alpha)$. We define the weighted space $(C_\infty) = \{f \in L^2(C_\infty) :$

$\tilde{f}_\sigma \in L^2(C_\infty)\big\}$ where $\tilde{f}_\sigma(x) = (1+x_1^2+x_2^2)^{\sigma/2} f(x),\ x \in C_\infty$. This space is equipped with the canonical norm $\|f\|_{L^2_\sigma(C_\infty)} = \|\tilde{f}_\sigma\|_{L^2(C_\infty)}$. The spaces $L^2_\sigma(C_\infty \setminus C_R)$ for $R>0$ are defined analogously.

For given $\sigma > 1$ and $f_\alpha \in L^2_\sigma(C_\infty)$ we consider the problem to determine $u_\alpha \in H^1_{\alpha,\mathrm{loc}}(C_\infty)$ with

$$\Delta u_\alpha + k^2 n u_\alpha \;=\; -f_\alpha \text{ in } C_\infty,, \tag{25}$$

satisfying the family of one-dimensional radiation conditions (4) for the Fourier coefficients of u_α. Again, later we will set $f_\alpha = (Ff)(\cdot,\alpha) + (Fh)(\cdot,\alpha)$.

To reduce this problem (25) and (4) to a boundary value problem on the bounded tube C_R we consider first the analog of problem (35) of Part A and solve the boundary value problem in the exterior of C_R explicitly.

Theorem 4.1. *Let $\alpha \in [-1/2, 1/2]$ and $f_\alpha \in L^2_\sigma(C_\infty \setminus C_R)$ for some $\sigma > 1$ and $g_\alpha \in H^{1/2}_\alpha(\gamma_R)$. The function*

$$v(r,\varphi,x_3) \;=\; \frac{1}{2\pi}\sum_{\ell,m\in\mathbb{Z}}\left[\int_R^\infty G_\alpha(r,\rho;m,\ell) f_{\ell,m}(\rho)\,\rho\,\mathrm{d}\rho \;+\; \frac{H_m^{(1)}(k_\ell r)}{H_m^{(1)}(k_\ell R)}\, g_{\ell,m}\right] \mathrm{e}^{\mathrm{i}m\varphi+\mathrm{i}(\ell+\alpha)x_3} \tag{26}$$

for $r>R$, $\varphi\in[0,2\pi]$, and $x_3 \in (0,2\pi)$, is the unique solution $v \in H^1_{\alpha,\mathrm{loc}}(C_\infty\setminus C_R)$ of the $\alpha-$quasi-periodic boundary value problem

$$\Delta v + k^2 v = -f_\alpha \text{ in } C_\infty\setminus C_R, \quad v = g_\alpha \text{ on } \gamma_R, \tag{27}$$

satisfying the one-dimensional radiation condition (4) for all $\ell, m \in \mathbb{Z}$. Here, $f_{\ell,m}(\rho)$ and $g_{\ell,m}$ are the Fourier coefficients of $f_\alpha(\rho,\cdot,\cdot)$ and g_α, respectively, and G_α is given by (see (38) of part A)

$$G_\alpha(r,\rho;m,\ell) \;=\; \frac{\mathrm{i}\pi}{2}\left[H_m^{(1)}(k_\ell r_+)J_m(k_\ell r_-) \;-\; \frac{H_m^{(1)}(k_\ell \rho)}{H_m^{(1)}(k_\ell R)} H_m^{(1)}(k_\ell r) J_m(k_\ell R)\right], \tag{28}$$

for $r,\rho \geqslant R$ and $m,\ell\in\mathbb{Z}$ where $r_+ = \max\{r,\rho\}$ and $r_- = \min\{r,\rho\}$. Set $A = \big\{\alpha \in [-1/2,1/2] : \text{there exists } \ell\in\mathbb{Z} \text{ with } |\alpha+\ell| = k\big\}$. Then the mapping $\alpha\mapsto v$ is continuous on $[-1/2,1/2]$ and continuously differentiable on $[-1/2,1/2]\setminus A$ as a mapping into $H^1_\alpha(C_{\hat{R}}\setminus C_R)$ for every $\hat{R} > R$.

The proof uses the same arguments as the proof of theorem 4.1 of part A and is omitted.

For $f_\alpha = 0$ this theorem provides the Dirichlet-to-Neumann map $\Lambda_\alpha : H^{1/2}_\alpha(\gamma_R) \to H^{-1/2}_\alpha(\gamma_R)$ by

$$(\Lambda_\alpha g_\alpha)(\varphi,x_3) \;=\; \frac{1}{2\pi}\sum_{\ell,m\in\mathbb{Z}} g_{\ell,m}\frac{k_\ell\, H_m^{(1)\prime}(Rk_\ell)}{H_m^{(1)}(Rk_\ell)}\, \mathrm{e}^{\mathrm{i}[m\varphi+(\ell+\alpha)x_3]}, \tag{29}$$

where $g_{\ell,m} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} g(R,\varphi,x_3)\,\mathrm{e}^{-\mathrm{i}[m\varphi+(\ell+\alpha)x_3]}\mathrm{d}\varphi\,\mathrm{d}x_3$ are again the Fourier coefficients of g_α. Let, furthermore, $v_\alpha \in H^1_{\alpha,\mathrm{loc}}(C_\infty\setminus C_R)$ solve (27) and (4) with $g_\alpha = 0$ on γ_R. Existence of v_α is again assured by theorem 4.1. Then (25) and (4) is equivalent to the following $\alpha-$quasi-periodic boundary value problem (see lemma 4.2 below):

$$\Delta u_\alpha + k^2 n u_\alpha = -f_\alpha \text{ in } C_R, \quad \frac{\partial u_\alpha}{\partial \nu}\Big|_- = \Lambda_\alpha u_\alpha + \frac{\partial v_\alpha}{\partial \nu}\Big|_+ \text{ on } \gamma_R.$$

The variational form of this boundary value problem is to find $u_\alpha \in H^1_\alpha(C_R)$ with

$$\int_{C_R} \left[\nabla u_\alpha \cdot \overline{\nabla \psi} - k^2 n\, u_\alpha\, \overline{\psi}\right] \mathrm{d}x \;-\; \int_{\gamma_R} (\Lambda_\alpha u_\alpha)\, \overline{\psi}\, \mathrm{d}s \;=\; \int_{C_R} f_\alpha\, \overline{\psi}\, \mathrm{d}x \;+\; \int_{\gamma_R} \frac{\partial v_\alpha}{\partial \nu}\, \overline{\psi}\, \mathrm{d}s \tag{30}$$

for all $\psi \in H^1_\alpha(C_R)$. Later we will study the dependence on α. Therefore, it is convenient to eliminate the dependence of the solution space on α by replacing the $\alpha-$quasi-periodic function u_α by $\tilde{u}_\alpha(x) = \exp(-\mathrm{i}\alpha x_3) u_\alpha(x)$ and, analogously for f_α, v_α, and the test functions ψ. We indicate the periodic functions by using the tilde sign on top of the symbol. Therefore, we search for $\tilde{u}_\alpha \in H^1_{\text{per}}(C_R)$ with

$$\int_{C_R} \left[\nabla \tilde{u}_\alpha \cdot \overline{\nabla \psi} - 2\mathrm{i}\alpha\, \overline{\psi}\, \frac{\partial \tilde{u}_\alpha}{\partial x_3} + (\alpha^2 - k^2 n)\, \tilde{u}_\alpha\, \overline{\psi}\right] \mathrm{d}x \;-\; \int_{\gamma_R} (\tilde{\Lambda}_\alpha \tilde{u}_\alpha)\, \overline{\psi}\, \mathrm{d}s$$
$$= \int_{C_R} \tilde{f}_\alpha\, \overline{\psi}\, \mathrm{d}x \;+\; \int_{\gamma_R} \left[\frac{\partial \tilde{v}_\alpha}{\partial \nu} + \mathrm{i}\alpha \tilde{v}_\alpha\right] \overline{\psi}\, \mathrm{d}s \quad \text{for all } \psi \in H^1_{\text{per}}(C_R).$$

Here,

$$(\tilde{\Lambda}_\alpha g)(\varphi, x_3) \;=\; \mathrm{e}^{-\mathrm{i}\alpha x_3}\, \Lambda_\alpha(\mathrm{e}^{\mathrm{i}\alpha x_3} g)(\varphi, x_3) \;=\; \frac{1}{2\pi} \sum_{\ell, m \in \mathbb{Z}} g_{\ell,m} \frac{k_\ell\, H_m^{(1)\prime}(Rk_\ell)}{H_m^{(1)}(Rk_\ell)}\, \mathrm{e}^{\mathrm{i}[m\varphi + \ell x_3]},$$

denotes the corresponding periodic Dirichlet–Neumann operator which is bounded from $H^{1/2}_{\text{per}}(\gamma_R)$ into $H^{-1/2}_{\text{per}}(\gamma_R)$. Recall that $k_\ell = \sqrt{k^2 - (\ell + \alpha)^2}$ and $g_{\ell,m}$ denote the Fourier coefficients of the periodic function g. As in part A we write this variational equation as $(\tilde{u}_\alpha, \psi)_* - a_\alpha(\tilde{u}_\alpha, \psi) = \ell_\alpha(\psi)$ where

$$(u, \psi)_* \;=\; \int_{C_R} \left[\nabla u \cdot \nabla \overline{\psi} + u\, \overline{\psi}\right] \mathrm{d}x \;-\; \int_{\gamma_R} (\tilde{\Lambda}_{0,i} u)\, \overline{\psi}\, \mathrm{d}s, \quad u, \psi \in H^1_{\text{per}}(C_R), \tag{31}$$

defines an inner product in $H^1_{\text{per}}(C_R)$ which is equivalent to the ordinary inner product and

$$a_\alpha(u, \psi) := -\int_{C_R} \left[\mathrm{i}\alpha \left(u\, \frac{\partial \overline{\psi}}{\partial x_3} - \overline{\psi}\, \frac{\partial u}{\partial x_3}\right) + (\alpha^2 - k^2 n - 1)\, u\, \overline{\psi}\right] \mathrm{d}x$$
$$- \int_{\gamma_R} \left[\tilde{\Lambda}_{0,i} u - \tilde{\Lambda}_\alpha u\right] \overline{\psi}\, \mathrm{d}s, \quad u, \psi \in H^1_{\text{per}}(C_R), \tag{32}$$

$$\ell_\alpha(\psi) = \int_{C_R} \tilde{f}_\alpha\, \overline{\psi}\, \mathrm{d}x \;+\; \int_{\gamma_R} \left[\frac{\partial \tilde{v}_\alpha}{\partial \nu} + \mathrm{i}\alpha \tilde{v}_\alpha\right] \overline{\psi}\, \mathrm{d}s, \quad \psi \in H^1_{\text{per}}(C_R). \tag{33}$$

Here, $\tilde{\Lambda}_{0,i}$ denotes the operator $\tilde{\Lambda}_\alpha$ for $\alpha = 0$ and $k = i$. Recall that $\tilde{v}_\alpha(x) = \mathrm{e}^{-\mathrm{i}\alpha x_3} v_\alpha(x)$ where $v_\alpha \in H^1_{\alpha,\text{loc}}(C_\infty \setminus C_R)$ solves (27), (4) with $g_\alpha = 0$ on γ_R.

Furthermore, by the representation theorem of Riesz there exists a unique operator K_α from $H^1_{\text{per}}(C_R)$ into itself and $r_\alpha \in H^1_{\text{per}}(C_R)$ such that $\left(K_\alpha u, \psi\right)_* = a_\alpha(u, \psi)$ and $\left(r_\alpha, \psi\right)_* = \ell_\alpha(\psi)$ for all $u, \psi \in H^1_{\text{per}}(C_R)$. Therefore, (25) and (4) is equivalent to the equation (24). Indeed, as shown as for lemma 2.12 of part A (i.e. [2]) we have the following result.

Lemma 4.2.

(a) Let $u_\alpha \in H^1_{\alpha,\mathrm{loc}}(C_\infty)$ *satisfy (25) and (4). Then the restriction* $\tilde u_\alpha(x) := \exp(-\mathrm{i}\alpha x_3)u_\alpha(x)$, $x \in C_R$, *is in* $H^1_{\mathrm{per}}(C_R)$ *and satisfies the operator equation*

$$\tilde u_\alpha \;-\; K_\alpha \tilde u_\alpha \;=\; r_\alpha \quad \text{in } H^1_{\mathrm{per}}(C_R), \tag{34}$$

where $K_\alpha : H^1_{\mathrm{per}}(C_R) \to H^1_{\mathrm{per}}(C_R)$ *and* $r_\alpha \in H^1_{\mathrm{per}}(C_R)$ *are defined by* $\big(K_\alpha u,\psi\big)_* = a_\alpha(u,\psi)$ *and* $\big(r_\alpha,\psi\big)_* = \ell_\alpha(\psi)$ *for all* $u,\psi \in H^1_{\mathrm{per}}(C_R)$ *and* $a_\alpha(u,\psi)$ *and* ℓ_α *are defined in (32) and (33), respectively*

(b) Let $\tilde u_\alpha \in H^1_{\mathrm{per}}(C_R)$ *satisfy (34). Then the function* $u_\alpha(x) = \begin{cases} \exp(\mathrm{i}\alpha x_3)\tilde u_\alpha(x) & \text{in } C_R, \\ v_\alpha(x) + u_{\mathrm{ext}}(x) & \text{in } C_\infty \setminus C_R, \end{cases}$ *solves (25) and (4) where*

$$u_{\mathrm{ext}}(r,\varphi,x_3) \;=\; \frac{1}{2\pi}\sum_{\ell,m\in\mathbb{Z}} u_{\ell,m}\frac{H^{(1)}_m(rk_\ell)}{H^{(1)}_m(Rk_\ell)}\,\mathrm{e}^{\mathrm{i}[m\varphi+(\ell+\alpha)x_3]}, \qquad r > R.$$

We note that $I - K_\alpha$ is one-to-one (and thus also onto) if, and only if, α is not an exceptional value. Properties of K_α and r_α are collected in the following lemma.

Lemma 4.3. *Let assumption 2.2 hold and let again* $A = \big\{\alpha \in [-1/2, 1/2] : \text{there exists } \ell \in \mathbb{Z} \text{ with } |\alpha+\ell| = k\big\}$. *The mappings* $\alpha \mapsto r_\alpha$ *and* $\alpha \mapsto K_\alpha$ *are continuous on* $[-1/2,1/2]$ *and continuously differentiable on* $[-1/2,1/2]\setminus A$ *as mappings into* $H^1_{\mathrm{per}}(C_R)$ *and* $\mathcal{L}\big(H^1_{\mathrm{per}}(C_R)\big)$, *respectively. Furthermore,* $I - K_\alpha$ *has Riesz number one for every exceptional value* α; *that is,* $\mathcal{N}\big((I-K_\alpha)^2\big) = \mathcal{N}(I-K_\alpha)$.

Proof. The smoothness with respect to α follows from the definitions of r_α and K_α, see also theorem 4.1. To show the last statement let $\tilde u \in \mathcal{N}\big((I-K_\alpha)^2\big)$ and set $\tilde w = (I-K_\alpha)\tilde u$. Then $\tilde w - K_\alpha \tilde w = 0$; that is, $(\tilde w,\psi)_* - a_\alpha(\tilde w,\psi) = 0$ for all $\psi \in H^1_{\mathrm{per}}(C_R)$. Therefore, the extension of the corresponding $w(x) = \mathrm{e}^{\mathrm{i}\alpha x_3}\tilde w(x)$ into $\mathbb{R}^3$ is an evanescent $\alpha-$quasi-periodic solution of $\Delta w + k^2 n w = 0$ in C_∞. Since $\frac{k_\ell H^{(1)\prime}_m(Rk_\ell)}{H^{(1)}_m(Rk_\ell)}$ is real valued for $|\ell+\alpha| \geqslant k$ we observe that $a_\alpha(\tilde w,\psi) = \overline{a_\alpha(\psi,\tilde w)}$ for all $\psi \in H^1_{\mathrm{per}}(C_R)$. We rewrite the equations $\tilde w = (I-K_\alpha)\tilde u$ and $\tilde w - K_\alpha\tilde w = 0$ again as $(\tilde u,\psi_1)_* - a_\alpha(\tilde u,\psi_1) = (\tilde w,\psi_1)_*$ and $(\tilde w,\psi_2)_* - a_\alpha(\tilde w,\psi_2) = 0$, respectively, for all $\psi_1,\psi_2 \in H^1_{\mathrm{per}}(C_R)$. Taking $\psi_1 = \tilde w$ and $\psi_2 = \tilde u$ yields

$$\|\tilde w\|_*^2 \;=\; (\tilde u,\tilde w)_* - a_\alpha(\tilde u,\tilde w) \;=\; \overline{(\tilde w,\tilde u)_* - a_\alpha(\tilde w,\tilde u)} \;=\; 0$$

by using $a_\alpha(\tilde w,\psi) = \overline{a_\alpha(\psi,\tilde w)}$ for all ψ. Therefore, $\tilde w = (I-K_\alpha)\tilde u = 0$; that is, $\tilde u \in \mathcal{N}(I-K_\alpha)$. □

The next theorem proves a Fredholm property of the quasi-periodic source problem (25) and (4).

Theorem 4.4. *Let assumption 2.2 hold and* $\alpha \in [-1/2,1/2]$ *and* $f_\alpha \in L^2_\sigma(C_\infty)$ *for some* $\sigma > 1$. *There exists* $u_\alpha \in H^1_{\alpha,\mathrm{loc}}(C_\infty)$ *with (25) and (4) if, and only if,*

$$\int_{C_\infty} f_\alpha\,\overline{\phi}\,\mathrm{d}x \;=\; 0 \quad \text{for all } \phi \in H^1_\alpha(C_\infty) \quad \begin{array}{l}\text{with } \Delta\phi + k^2 n\phi = 0 \text{ in } C_\infty \\ \text{satisfying the Rayleigh expansion.}\end{array} \tag{35}$$

Proof. Let first $u_\alpha \in H^1_{\alpha,\mathrm{loc}}(C_\infty)$ satisfy (25) and (4). We write u and f instead of u_α and f_α, respectively, for brevity. If α is not an exceptional value then condition (35) is trivially satisfied because there are no such nontrivial ϕ. Therefore, let α be an exceptional value and $\phi \in H^1_\alpha(C_\infty)$ be a solution of $\Delta\phi + k^2 n\phi = 0$ in C_∞ which satisfies the Rayleigh expansion. By lemma 2.3 ϕ is evanescent. Let $\phi_{\ell,m}(r)$ and $u_{\ell,m}(r)$, $\ell, m \in \mathbb{Z}$, be the Fourier coefficients of ϕ and u, respectively. We split the integral $\int_{C_\infty} f\,\overline{\phi}\,\mathrm{d}x$ into the sum $\int_{C_\infty} f\,\overline{\phi}\,\mathrm{d}x = \int_{C_R} f\,\overline{\phi}\,\mathrm{d}x + \int_{C_\infty\setminus C_R} f\,\overline{\phi}\,\mathrm{d}x$. For the first integral we use first Green's second theorem and then Parseval's identity for the boundary term which yields

$$\begin{aligned}\int_{C_R} f\,\overline{\phi}\,\mathrm{d}x = -\int_{C_R} (\Delta u + k^2 nu)\,\overline{\phi}\,\mathrm{d}x \;&=\; -\int_{\gamma_R}\left[\overline{\phi}\,\frac{\partial u}{\partial r} - u\,\frac{\partial\overline{\phi}}{\partial r}\right]\mathrm{d}s \\ &= -R\sum_{\ell,m\in\mathbb{Z}}\left[\overline{\phi_{\ell,m}(R)}\,u'_{\ell,m}(R) - u_{\ell,m}(R)\,\overline{\phi'_{\ell,m}(R)}\right].\end{aligned} \tag{36}$$

For the second integral we use Parseval's identity directly which yields

$$\int_{C_\infty\setminus C_R} f\,\overline{\phi}\,\mathrm{d}x \;=\; \sum_{\ell,m\in\mathbb{Z}}\int_R^\infty f_{\ell,m}(r)\,\overline{\phi_{\ell,m}(r)}\,r\,\mathrm{d}r. \tag{37}$$

We compute the one dimensional integrals. First we note that $u_{\ell,m}(r)$ and $\phi_{\ell,m}(r)$ satisfy the ordinary differential equations of Bessel type

$$\begin{aligned}&\frac{1}{r}\left(r\,u'_{\ell,m}(r)\right)' \;+\; \left(k_\ell^2 - \frac{m^2}{r^2}\right)u_{\ell,m}(r) = -f_{\ell,m}(r), \quad r > R,\\ &\frac{1}{r}\left(r\,\overline{\phi'_{\ell,m}(r)}\right)' \;+\; \left(k_\ell^2 - \frac{m^2}{r^2}\right)\overline{\phi_{\ell,m}(r)} = 0, \quad r > R,\end{aligned}$$

where again $k_\ell = \sqrt{k^2 - (\ell+\alpha)^2}$. Multiplying the first equation by $r\overline{\phi_{\ell,m}(r)}$ and the second by $ru_{\ell,m}(r)$, integrating from R to some $\hat{R} > R$, and using partial integration yields

$$-\int_R^{\hat{R}} f_{\ell,m}(r)\,\overline{\phi_{\ell,m}(r)}\,r\,\mathrm{d}r \;=\; \left[r\,\overline{\phi_{\ell,m}(r)}\,u'_{\ell,m}(r) - r\,\overline{\phi'_{\ell,m}(r)}\,u_{\ell,m}(r)\right]\Big|_R^{\hat{R}}.$$

Now we let $\hat{R}$ tend to infinity and use the boundedness of $\sqrt{r}|u_{\ell,m}(r)|$ and $\sqrt{r}|u'_{\ell,m}(r)|$ (see lemma A.2 of the appendix) and the fact that $\phi_{\ell,m}(r)$ and $\phi'_{\ell,m}(r)$ tend to zero exponentially when r tends to infinity. This yields

$$\int_R^\infty f_{\ell,m}(r)\,\overline{\phi_{\ell,m}(r)}\,r\,\mathrm{d}r \;=\; R\left[\overline{\phi_{\ell,m}(R)}\,u'_{\ell,m}(R) - \overline{\phi'_{\ell,m}(R)}\,u_{\ell,m}(R)\right]. \tag{38}$$

Substituting this into (37) and combining it with (36) yields $\int_{C_\infty} f\,\overline{\phi}\,\mathrm{d}x = 0$.

For the reverse part we assume that $\int_{C_\infty} f_\alpha\,\overline{\phi}\,\mathrm{d}x = 0$ for all $\alpha-$quasi-periodic solutions of $\Delta\phi + k^2 n\phi = 0$ in C_∞ which satisfy the Rayleigh expansion. By lemma 4.2 we have to show existence of $u_\alpha \in H^1_\alpha(C_R)$ which solves the variational equation (30). Recall the definition of $v_\alpha \in H^1_{\alpha,\mathrm{loc}}(C_\infty \setminus C_R)$ as a solution of (27) and (4) for boundary data $g_\alpha = 0$. Using Fredholm's alternative for the equivalent form (34) it is straight forward to show that (30) is solvable in $H^1_\alpha(C_R)$ if, and only if,

$$\int_{C_R} f_\alpha\,\overline{\phi}\,\mathrm{d}x \;+\; \int_{\gamma_R} \frac{\partial v_\alpha}{\partial\nu}\,\overline{\phi}\,\mathrm{d}s \;=\; 0 \quad \text{for all } \phi\in H^1_\alpha(C_R) \text{ with} \tag{39}$$

$$\int_{C_R}\big[\nabla\psi\cdot\overline{\nabla\phi}-k^2n\,\psi\,\overline{\phi}\big]\,\mathrm{d}x \;-\; \int_{\gamma_R}(\Lambda_\alpha\psi)\,\overline{\phi}\,\mathrm{d}s \;=\; 0 \quad \text{for all } \psi\in H^1_\alpha(C_R). \tag{40}$$

If α is not an exceptional value then the only solution ϕ of (40) is the trivial one and, therefore, (39) is trivially satisfied. If α is an exceptional value then we set $\psi=\phi$ in (40) and use (29) to show again that ϕ has an extension to an evanescent solution of $\Delta\phi+k^2n\phi=0$ in C_∞. Green's theorem yields

$$\begin{aligned}\int_{C_R} f_\alpha\,\overline{\phi}\,\mathrm{d}x + \int_{\gamma_R}\frac{\partial v_\alpha}{\partial\nu}\,\overline{\phi}\,\mathrm{d}s &= \int_{C_R} f_\alpha\,\overline{\phi}\,\mathrm{d}x + \int_{\gamma_R}\left[\frac{\partial v_\alpha}{\partial\nu}\,\overline{\phi}-\frac{\partial\overline{\phi}}{\partial\nu}\,v_\alpha\right]\mathrm{d}s\\ &= \int_{C_R} f_\alpha\,\overline{\phi}\,\mathrm{d}x - \int_{C_\infty\setminus C_R}\big[\overline{\phi}\Delta v_\alpha - v_\alpha\Delta\overline{\phi}\big]\,\mathrm{d}x \;=\; \int_{C_\infty} f_\alpha\,\overline{\phi}\,\mathrm{d}x \;=\; 0.\end{aligned}$$

This ends the proof. □

Now we are able to prove existence of a solution to $\Delta u+k^2nu=-f$ satisfying the radiation condition of definition 2.5. From the considerations at the beginning of this section we have to study solvability of the equations (23) and (4).

Theorem 4.5. *Let assumption 2.2 hold and let $k>0$ be regular in the sense of definition 2.4. Then, for every $f\in L^2(\mathbb{R}^3)$ with compact support there exists a unique solution $u\in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ of (1); that is,*

$$\Delta u + k^2 n\,u \;=\; -f \quad \text{in } \mathbb{R}^3,$$

satisfying the radiation condition of definition 2.5. The coefficients are given by

$$a_{\ell,j} \;=\; \frac{2\pi\mathrm{i}}{|\lambda_{\ell,j}|}\int_{\mathbb{R}^3} f(x)\,\overline{\hat{\phi}_{\ell,j}(x)}\,\mathrm{d}x, \quad \ell=1,\dots,m_j,\ j\in J. \tag{41}$$

Proof. We define $a_{\ell,j}$ by (41) and consider equation (23) with the radiation condition (4) where the right hand side $Fh+Ff$ is now given. By theorem 4.4 we have to show that $(Fh)(\cdot,\alpha)+(Ff)(\cdot,\alpha)$ is orthogonal to all radiating α−quasi-periodic solutions ϕ_α of the homogeneous equation $\Delta\phi_\alpha+k^2n\phi_\alpha=0$ in C_∞. For $\alpha\notin\{\hat{\alpha}_j : j\in J\}$ this is obvious. Let now $\alpha=\hat{\alpha}_j$ for some $j\in J$. From the form (19) of h and lemma 3.2 we conclude that

$$\begin{aligned}(Fh)(x,\hat{\alpha}_j) &= \sum_{j'\in J}\sum_{\ell=1}^{m_{j'}}\operatorname{sign}(\lambda_{\ell,j'})\,a_{\ell,j'}\,\big(\Delta+k^2n(x)\big)\big[\hat{\phi}_{\ell,j'}(x)\rho(x_3,\hat{\alpha}_j-\hat{\alpha}_{j'})\big]\\ &\quad+\sum_{\ell=1}^{m_j}\operatorname{sign}(\lambda_{\ell,j})\,a_{\ell,j}\,\frac{1}{\pi}\,\frac{\partial\hat{\phi}_{\ell,j}(x)}{\partial x_3}\\ &= \Delta w(x,\hat{\alpha}_j)+k^2n(x)w(x,\hat{\alpha}_j) \;+\; \sum_{\ell=1}^{m_j}\operatorname{sign}(\lambda_{\ell,j})\,a_{\ell,j}\,\frac{1}{\pi}\,\frac{\partial\hat{\phi}_{\ell,j}(x)}{\partial x_3}\end{aligned}$$

with w from (21) for $\alpha=\hat\alpha_j$. Then, for any $\ell\in\{1,\dots,m_j\}$,

$$\begin{aligned}
&\int_{C_\infty}\big[(Ff)(\cdot,\hat\alpha_j)+(Fh)(\cdot,\hat\alpha_j)\big]\,\overline{\hat\phi_{\ell,j}}\,\mathrm{d}x\\
&=\int_{C_\infty}\big[(Ff)(\cdot,\hat\alpha_j)+(\Delta+k^2n)w(\cdot,\hat\alpha_j)\big]\,\overline{\hat\phi_{\ell,j}}\,\mathrm{d}x\;+\;\sum_{\ell'=1}^{m_j}\mathrm{sign}(\lambda_{\ell',j})\,a_{\ell',j}\,\frac{1}{\pi}\int_{C_\infty}\frac{\partial\hat\phi_{\ell',j}}{\partial x_3}\,\overline{\hat\phi_{\ell,j}}\,\mathrm{d}x\\
&=\int_{C_\infty}(Ff)(\cdot,\hat\alpha_j)\,\overline{\hat\phi_{\ell,j}}\,\mathrm{d}x\;+\;\mathrm{sign}(\lambda_{\ell,j})\,a_{\ell,j}\,\frac{1}{2\pi}\,i\,\lambda_{\ell,j}\\
&=\sum_{m\in\mathbb{Z}}\int_{C_\infty}f\big(x+2\pi me^{(3)}\big)\,\overline{\hat\phi_{\ell,j}\big(x+2\pi me^{(3)}\big)}\,\mathrm{d}x\;+\;\mathrm{sign}(\lambda_{\ell,j})\,a_{\ell,j}\,\frac{1}{2\pi}\,\mathrm{i}\,\lambda_{\ell,j}\\
&=\int_{\mathbb{R}^3}f(y)\,\overline{\hat\phi_{\ell,j}(y)}\,\mathrm{d}y\;+\;\mathrm{sign}(\lambda_{\ell,j})\,a_{\ell,j}\,\frac{1}{2\pi}\,\mathrm{i}\,\lambda_{\ell,j}\;=\;0
\end{aligned}$$

by the definition of $a_{\ell,j}$. Here we have used Green's second theorem in C_∞ and the normalization of the eigenfunctions $\hat\phi_{\ell,j}$ (see (7) and (8)). Therefore, the equation (23) has a radiating solution for all α.

Therefore, we have shown that the operator equation (34) is solvable in $H^1_{\mathrm{per}}(C_R)$ for all $\alpha\in[-1/2,1/2]$. The smoothness properties of $\alpha\mapsto K_\alpha$ and $\alpha\mapsto r_\alpha$ are shown in lemma 4.3. In the same lemma it has been shown that the Riesz number of $I-K_{\hat\alpha_j}$ is one. Therefore, all of the assumptions of theorem A.1 of the appendix has been shown except of the injectivity of the projected operator $PK'_{\hat\alpha_j}\big|_{\mathcal N}:\mathcal N\to\mathcal N$ where $\mathcal N=\mathcal N(I-K_{\hat\alpha_j})$ and $P:H^1_{\mathrm{per}}(C_R)\to\mathcal N$ is the projection with respect to the direct sum $H^1_{\mathrm{per}}(C_R)=\mathcal N\oplus\mathcal R(I-K_{\hat\alpha_j})$. We fix $\hat\alpha_j$ for some $j\in J$ and let α be in a small neighborhood of $\hat\alpha_j$. Let $u,\psi\in\mathcal N\big(I-K_{\hat\alpha_j}\big)$. We recall that

$$\begin{aligned}
(K_\alpha u,\psi)_*\;=\;a_\alpha(u,\psi)&=-\int_{C_R}\left[\mathrm{i}\alpha\left(u\,\frac{\partial\overline{\psi}}{\partial x_3}-\overline{\psi}\,\frac{\partial u}{\partial x_3}\right)+(\alpha^2-k^2n-1)\,u\,\overline{\psi}\right]\mathrm{d}x\\
&\quad-\int_{\gamma_R}\big[\tilde\Lambda_{0,i}u-\tilde\Lambda_\alpha u\big]\,\overline{\psi}\,\mathrm{d}s,\qquad u,\psi\in H^1_{\mathrm{per}}(C_R)
\end{aligned}$$

and

$$\int_{\gamma_R}\overline{\psi}\,\tilde\Lambda_\alpha u\;\mathrm{d}s\;=\;R\sum_{|\ell+\hat\alpha|>k}\sum_{m\in\mathbb{Z}}\overline{\psi_{\ell,m}}\,u_{\ell,m}\,\frac{k_\ell H^{(1)\prime}_m(k_\ell(\alpha)R)}{H^{(1)}_m k_\ell(\alpha)R)}$$

where $k_\ell(\alpha)=\sqrt{k^2-|\ell+\alpha|^2}$ and $\psi_{\ell,m}$ and $u_{\ell,m}$ are the Fourier coefficients of the periodic functions $\psi|_{\gamma_R}$ and $u|_{\gamma_R}$, respectively. We extend the α−quasi-periodic functions $\mathrm{e}^{\mathrm{i}\alpha x_3}u(x)$ and $\mathrm{e}^{\mathrm{i}\alpha x_3}\psi(x)$ as in lemma 4.2 into all of C_∞. Then, by Green's formula,

$$\begin{aligned}
\int_{\gamma_R}\overline{\psi}\,\tilde\Lambda_\alpha u\;\mathrm{d}s&=\int_{\gamma_R}\overline{\big(\psi(x)\mathrm{e}^{\mathrm{i}\alpha x_3}\big)}\,\frac{\partial}{\partial r}\big(u(x)\mathrm{e}^{\mathrm{i}\alpha x_3}\big)\,\mathrm{d}s(x)\\
&=-\int_{C_\infty\setminus C_R}\big(\nabla u+\mathrm{i}\alpha u\,\hat e^{(3)}\big)\cdot\overline{\big(\nabla\psi+\mathrm{i}\alpha\psi\,\hat e^{(3)}\big)}-k^2u\,\overline{\psi}\,\mathrm{d}x\\
&=-\int_{C_\infty\setminus C_R}\nabla u\cdot\nabla\overline{\psi}+\mathrm{i}\alpha\left(u\,\frac{\partial\overline{\psi}}{\partial x_3}-\overline{\psi}\,\frac{\partial u}{\partial x_3}\right)+(\alpha^2-k^2)\,u\,\overline{\psi}\,\mathrm{d}x.
\end{aligned}$$

Therefore,

$$(K_\alpha u,\psi)_* = -\int_{C_\infty}\left[\mathrm{i}\alpha\left(u\,\frac{\partial\overline{\psi}}{\partial x_3} - \overline{\psi}\,\frac{\partial u}{\partial x_3}\right) + (\alpha^2 - k^2 n)\,u\,\overline{\psi}\right]\mathrm{d}x \\ - \int_{C_\infty\setminus C_R}\nabla u\cdot\nabla\overline{\psi}\,\mathrm{d}x + \int_{C_R} u\,\overline{\psi}\,\mathrm{d}x - \int_{\gamma_R}\overline{\psi}\,\tilde{\Lambda}_{0,i}u\,\mathrm{d}s.$$

Differentiating this with respect to α yields for $\alpha = \hat{\alpha}_j$

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}(K_{\hat{\alpha}_j}u,\psi)_* = 2i\int_{C_\infty}\left(\frac{\partial u}{\partial x_3} + i\hat{\alpha}_j u\right)\overline{\psi}\,\mathrm{d}x = \int_{C_\infty}\overline{\left(\psi(x)\mathrm{e}^{\mathrm{i}\hat{\alpha}_j x_3}\right)}\,\frac{\partial}{\partial x_3}\left(u(x)\mathrm{e}^{\mathrm{i}\hat{\alpha}_j x_3}\right)\mathrm{d}x.$$

From this we observe that $P\frac{\mathrm{d}}{\mathrm{d}\alpha}K_{\hat{\alpha}_j}u$ vanishes for some nontrivial $u \in \mathcal{N}\big(I - K_{\hat{\alpha}_j}\big)$ if, and only if, for some $\ell \in \{1,\dots,m_j\}$ the eigenvalue $\lambda_{\ell,j}$ of (7) vanishes. Therefore, the regularity of k (definition 2.4) implies injectivity of $P\frac{\mathrm{d}}{\mathrm{d}\alpha}K_{\hat{\alpha}_j}$ on $\mathcal{N}\big(I - K_{\hat{\alpha}_j}\big)$. Application of theorem A.1 yields the continuous extension of of $\alpha \mapsto u_\alpha$ into $\hat{\alpha}_j$. For this extension the mapping $\alpha \mapsto \int_{-1/2}^{1/2} u_\alpha\,\mathrm{d}\alpha$ is continuous which proves the assertion. □

Remark 4.6. Again, as already mentioned in remark 3.4, the proof of existence does not use the particular form of the decomposition $\{1,\dots,m_j\} = \{\ell : \lambda_{\ell,j} > 0\} \cup \{\ell : \lambda_{\ell,j} < 0\}$. Any prescribed decomposition of $\{1,\dots,m_j\}$ into disjoint sets $L_j^{(1)}$ and $L_j^{(2)}$ will also provide existence.

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Appendix

The following result is a special case of a slightly more general result of Colton and Kress (see section 1.4 in [1]).

Theorem A.1. *Let X be a Banach space, $I \subset \mathbb{R}$ an open interval and $r_\alpha \in X$ and $K_\alpha : X \to X$ for $\alpha \in I$ families of linear and compact operators such that $\alpha \mapsto r_\alpha$ and $\alpha \mapsto K_\alpha$ are continuously differentiable from a neighborhood of some $\hat{\alpha} \in I$ into X and $\mathcal{L}(X,X)$, respectively.*

Let $I - K_\alpha$ be bijective for $\alpha \neq \hat{\alpha}$ but $\mathcal{N}\big(I - K_{\hat{\alpha}}\big) \neq \{0\}$. Let the Riesz number of $I - K(\hat{\alpha})$ be one; that is, $\mathcal{N}\big((I - K_{\hat{\alpha}})^2\big) = \mathcal{N}\big(I - K_{\hat{\alpha}}\big)$ and $P : X \to \mathcal{N} := \mathcal{N}\big(I - K_{\hat{\alpha}}\big)$ be the projection operator onto the null space with respect to the direct sum $X = \mathcal{N} \oplus \mathcal{R}\big(I - K_{\hat{\alpha}}\big)$. Assume, furthermore, that $P\frac{\mathrm{d}}{\mathrm{d}\alpha}K_{\hat{\alpha}}\big|_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$ is one-to-one and $r_{\hat{\alpha}} \in \mathcal{R}\big(I - K_{\hat{\alpha}}\big)$.

Then the unique solution $u_\alpha \in X$ of $\big(I - K_\alpha\big)u_\alpha = r_\alpha$ for $\alpha \neq \hat{\alpha}$ converges to a solution $u(\hat{\alpha})$ of $\big(I - K_{\hat{\alpha}}\big)u_{\hat{\alpha}} = r_{\hat{\alpha}}$. In other words, the mapping $\alpha \mapsto u_\alpha$ from $I \setminus \{\hat{\alpha}\}$ into X has a continuous extension into $\hat{\alpha}$.

The following lemma estimates the growth of the Fourier coefficients of a function u satisfying the radiation condition (4).

Lemma A.2. *For some* $R>0$ *let* $f\in L^2(R,\infty)$ *such that also* $\int_R^\infty |f(s)|\sqrt{s}\,\mathrm{d}s<\infty$. *Let* $t\in\mathbb{C}\setminus\{0\}$ *with* $\mathrm{Re}t\geqslant 0$ *and* $\mathrm{Im}t\geqslant 0$ *and* $w\in H^1_{\mathrm{loc}}(R,\infty)$ *a solution of*

$$\frac{1}{r}\big(r\,w'(r)\big)'\;+\;\left(t^2-\frac{m^2}{r^2}\right)w(r)\;=\;-f(r),\quad r>R, \tag{A.1}$$

satisfying the radiation condition

$$\lim_{r\to\infty}\sqrt{r}\big[w'(r)-\mathrm{i}\,t\,w(r)\big]\;=\;0.$$

Then there exists $c>0$ *such that* $\sqrt{r}\,\big[|w(r)|+|w'(r)|\big]\leqslant c$ *for all* $r\geqslant R$.

Proof. Set $\varepsilon_r=r\big|w'(r)-itw(r)\big|^2=r\big[|tw(r)|^2+|w'(r)|^2\big]-2r\mathrm{Im}\big[\overline{tw(r)}w'(r)\big]$. Then $\varepsilon(r)$ tends to zero and is thus bounded on (R,∞) by some constant $c_1>0$. This implies the estimate

$$r\big[|t\,w(r)|^2+|w'(r)|^2\big]\;\leqslant\;c_1\;+\;2r\mathrm{Im}\big[\bar t\,\overline{w(r)}\,w'(r)\big]. \tag{A.2}$$

Multiplying (A.1) for variable s instead of r by $\overline{sw(s)}$ and integrating from R to some $r>R$ yields

$$\int_R^r\big(s\,w'(s)\big)'\,\overline{w(s)}\,\mathrm{d}s\;+\;\int_R^r\left(t^2-\frac{m^2}{s^2}\right)s\,|w(s)|^2\mathrm{d}s\;=\;-\int_R^r s\,f(s)\,\overline{w(s)}\,\mathrm{d}s.$$

We use partial integration of the first term, multiply by $\bar t$ and take the imaginary part. This yields

$$\begin{aligned}-\mathrm{Im}\left[\bar t\int_R^r s\,f(s)\,\overline{w(s)}\,\mathrm{d}s\right]&=r\,\mathrm{Im}\big[\bar t\,\overline{w(r)}\,w'(r)\big]-R\,\mathrm{Im}\big[\bar t\,\overline{w(R)}\,w'(R)\big]\\&\quad+\;\mathrm{Im}t\int_R^r s\,|w'(s)|^2\,\mathrm{d}s\;+\;\mathrm{Im}t\int_R^r\left(|t|^2+\frac{m^2}{s^2}\right)s\,|w(s)|^2\mathrm{d}s\\&\geqslant r\,\mathrm{Im}\big[\bar t\,\overline{w(r)}\,w'(r)\big]\;-\;c_2\end{aligned}$$

with $c_2=R\,\mathrm{Im}\big[\bar t\,\overline{w(R)}\,w'(R)\big]$. Therefore,

$$r\,\mathrm{Im}\big[\bar t\,\overline{w(r)}\,w'(r)\big]\;\leqslant\;c_2\;+\;|t|\max_{R\leqslant s\leqslant r}\big[\sqrt{s}\,|w(s)|\big]\int_R^\infty|f(s)|\sqrt{s}\,\mathrm{d}s\;\leqslant\;c_2\;+\;c_3\,\psi(r)$$

where $c_3=\int_R^\infty|f(s)|\sqrt{s}\,\mathrm{d}s$ and $\psi(r)=|t|\max_{R\leqslant s\leqslant r}\big[\sqrt{s}\,|w(s)|\big]$. Substituting this estimate for $R\leqslant r\leqslant\rho$ into (A.2) and taking the supremum for $r\in[R,\rho]$ yields

$$\psi(\rho)^2\;\leqslant c_1+2c_2\;+\;2c_3\,\psi(\rho);\ \text{that is,}\ \big[\psi(\rho)-c_3\big]^2\;\leqslant\;c_1+2c_2+c_3^2$$

which shows boundedness $\psi(\rho)\leqslant c_3+\sqrt{c_1+2c_2+c_3^2}$. Boundedness of $\sqrt{r}|w'(r)|$ follows now from (A.2). □

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References

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