Inverse Problems

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To cite this article: Andreas Kirsch 2019 *Inverse Problems* **35** 104004

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IOP Publishing
Inverse Problems

Inverse Problems **35** (2019) 104004 (34pp)
https://doi.org/10.1088/1361-6420/ab2e31

Scattering by a periodic tube in $\mathbb{R}^3$: part i. The limiting absorption principle*

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Received 21 November 2018, revised 19 April 2019
Accepted for publication 1 July 2019
Published 9 September 2019

Abstract

Scattering of time-harmonic waves from periodic structures at some fixed real-valued wave number becomes analytically difficult whenever there arise surface waves: These non-zero solutions to the homogeneous scattering problem physically correspond to modes propagating along the periodic structure and clearly imply non-uniqueness of any solution to the scattering problem. In this paper, we consider a medium, described by a refractive index which is periodic along the axis of an infinite cylinder in $\mathbb{R}^3$ and constant outside of the cylinder. We prove that there is a so-called limiting absorption solution to the associated scattering problem. By definition, such a solution is the limit of a sequence of unique solutions for artificial complex-valued wave numbers tending to the above-mentioned real-valued wave number. By the standard one-dimensional Floquet–Bloch transform and the introduction of the exterior Dirichlet–Neumann map we first reduce the scattering problem to a class of periodic problems in a bounded cell, depending on the wave number k and the Bloch parameter α. We use a functional analytic singular perturbation result to study this problem in a neighborhood of a singular pair (k, α). This abstract result allows us to derive explicitly a representation for the limiting absorption solution as a sum of a decaying part (along the axis of the cylinder) and a finite sum of propagating modes.

* Dedicated to the memory of my friend and colleague Armin Lechleiter.

Keywords: Helmholtz equation, limiting absorption principle, periodic wave guide

(Some figures may appear in colour only in the online journal)

1. Introduction

Periodic non-absorbing surface structures allow surface waves that propagate along the structure without decaying. These waves do physically arise at certain exceptional values of the Bloch parameter, and mathematically they are eigenfunctions of a certain periodic eigenvalue problem involving the periodic structure. The corresponding eigenvalue determines the surface wave's frequency, and the surface wave itself is the periodically extended eigenfunction.

Since the eigenfunction is a non-zero solution to a corresponding periodic scattering problem from the periodic structure, the latter scattering problem cannot be uniquely solvable at any of these eigenfrequencies. For this reason, such frequencies are often excluded from the analysis (see, e.g. [4]) by proper assumptions on the refractive index. In the past decade, however, the study of surface waves, also known as resonant scattering, has attracted a lot of attention. For an overview we suggest the interesting paper [24] by S. Shipman. From the mathematical point of view the formulation of a correct radiation condition is challenging which ensures both, uniqueness and the existence of surface waves.

In this paper we extend the paper [15] to the scattering problem in $\mathbb{R}^3$ by an infinite inhomogeneous cylinder involving a periodic (with respect to the axis of the cylinder) refractive index. We believe that the so-called limiting absorption principle is the natural approach for finding the physically correct solution of the scattering problem. By construction, this solution is, in a certain topology, limit of the unique solutions to a family of coercive problems with artificial complex-valued wave numbers.

This limiting absorption solution consists of two parts that we determine via the Floquet–Bloch transform: The first part belongs to H^1 in any cylinder of finite radius and the second part is made up of surface waves or *propagative modes*. This second part vanishes if no propagative mode exists. However, if such modes exist then the direction of propagation is determined through a finite-dimensional eigenvalue problem in the finite dimensional space of propagating modes. This paper seems to be a first instance of such a limiting absorption principle for a scattering problem by an infinite tube. There exist, however, several contributions concerning problems in $\mathbb{R}^3$ which are periodic with respect to two or all three variables, see, e.g. [24]. Our problems serves as a simple model how tubes in $\mathbb{R}^3$ are scattered by, e.g. point sources.

The structure of the spectrum, the limiting absorption principle, and the construction of radiation conditions for frequency scattering problems in free space, in closed waveguides, and in stratified media has a long history. We refer to [2, 3, 7–9, 13, 17, 22, 23, 25–27] for a few references. Further, in [10] (see also [14]) a limiting absorption principle for scattering in a closed waveguide has recently been shown that relies fundamentally on the Floquet–Bloch transform and has substantially motivated our first paper [15]. In [10], the authors decompose fields via the eigenfunctions of the generalized quasi-periodic Laplacian in the unit cell. This technique cannot be applied in our case—not even in the two-dimensional case—, as such decompositions cannot be directly transferred to structures that form open instead of closed waveguides. Our analysis is indeed rather different compared to the one in [10], and also compared to the independent study in [12].

Precisely, we consider the scattering of an incident field u^{inc} by an infinite cylinder $T_R := B_2(0,R) \times \mathbb{R} \subset \mathbb{R}^3$ where $B_2(0,R) \subset \mathbb{R}^2$ denotes the disc centered at the origin with radius $R > 0$. We assume that the index of refraction $n \in L^\infty(\mathbb{R}^3)$ is positive and 2π-periodic with respect to x_3 and equals to one for $x \notin T_R$ and construct a (weak) limiting absorption solution $u^s \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ such that the total field $u^t = u^{\mathrm{inc}} + u^s$ solves

$$\Delta u^t + \hat{k}^2 n\, u^t \;=\; 0 \quad \text{in } \mathbb{R}^3. \tag{1}$$

Here, $\hat{k} > 0$ denotes the wave number.

The limiting absorption principle leads to a special decomposition of the solution into a field $u^{(1)}$ which decays along the axis of the tube and a second field $u^{(2)}$ which consists of a finite combination of propagative modes. Outside of the cylinder T_R the field $u^{(1)}$ is a solution of the exterior boundary value problem $\Delta u^{(1)} + \hat{k}^2 u^{(1)} = -h$ for some $h \in L^2(\mathbb{R}^3 \setminus T_R)$ with $H^{1/2}$-boundary data on the boundary $\Gamma_R = \partial T_R$. This limiting absorption solution $u^{(1)}$ can be explicitly expressed by the (generalized) Fourier transform in terms of Hankel functions of the first kind.

Both parts, the decomposition of the field u into a decaying field $u^{(1)}$ along the axis and a propagating field $u^{(2)}$ and the particular form of $u^{(1)}$ outside of the cylinder allows the formulation of a radiation condition which we carry out in the second part of this paper. There, we will prove uniqueness under this radiation condition and also existence by a direct method; that is, without using the limiting absorption principle.

The methods we apply are all well-known and in principle simple enough to extend our analysis to more involved scattering problems in linear elasticity or electromagnetics. This, however, has to be done and is planed for the future.

To briefly comment on this paper's structure, the following section 2 discusses the scattering problem in more detail and transforms it into a family of periodic problems with the help of the Floquet–Bloch transform. We reduce the problems to a bounded cell by introducing the Dirichlet–Neumann operator for the exterior of the cylinder T_R. In section 3 we prove the limiting absorption principle and exhibit the particular form of $u^{(1)}$ in the exterior of T_R. Finally, in the appendix we prove several properties of Hankel functions with complex arguments and solve an exterior boundary value problem for the Helmholtz equation with the use of the Fourier transform.

2. Formulation of the scattering problem and the Floquet–Bloch transform

We begin by setting up some notations (see figure 1). Let $k \in \mathbb{C}$ with $\mathrm{Re}k > 0$ and $\mathrm{Im}k \geqslant 0$ be the wave number, $B_N(y,R) = \{x \in \mathbb{R}^N : |x-y| < R\}$ the ball in $\mathbb{R}^N$ with center y and radius $R > 0$, and $T_R = B_2(0,R) \times \mathbb{R} \subset \mathbb{R}^3$ the tube (or infinite cylinder) in x_3-direction with boundary $\Gamma_R := \partial T_R = \partial B_2(0,R) \times \mathbb{R}$. Furthermore, we define the finite cylinder by $C_R := B_2(0,R) \times (0,2\pi) \subset \mathbb{R}^3$ and $C_\infty := \mathbb{R}^2 \times (0,2\pi) \subset \mathbb{R}^3$ and the vertical part of the boundary by $\gamma_R := \Gamma_R \cap \overline{C_R}$. We consider in the following the case that a point source at some point $y \in \mathbb{R}^3$ is scattered by a tube $T_{R_0} \subset \mathbb{R}^3$ of radius R_0 which is filled by some medium with index of refraction $n \in L^\infty(\mathbb{R}^3)$ which is assumed to be 2π-periodic with respect to the variable x_3 and equals to one outside of T_{R_0}. The incident field u^{inc} is given by the fundamental solution Φ_k of the Helmholtz equation in $\mathbb{R}^3$; that is,

$$u^{\mathrm{inc}}(x) \;=\; \Phi_k(x;y) \;=\; \frac{\exp(ik|x-y|)}{4\pi\,|x-y|}, \quad x \neq y,$$

for some fixed $y \in \mathbb{R}^3$. The scattering problem is to determine the total field $u^t \in H^1_{loc}\big(\mathbb{R}^3 \setminus \{y\}\big)$ with

$$\Delta u^t + k^2 n\, u^t \;=\; 0 \text{ in } \mathbb{R}^3 \setminus \{y\}, \tag{2}$$

and such that the scattered field $u^s = u^t - u^{\text{inc}}$ is more regular than the incident field, in particular $u^s \in H^1_{loc}(\mathbb{R}^3)$. In other words, we wish to determine the Green's function of the differential operator $\Delta + k^2 n$. The solution is not uniquely determined by (2) because some kind of radiation condition for the scattered field is required. It is the purpose of this paper to derive a correct form of a radiation condition from the limiting absorption principle. First, we transform this problem into an inhomogeneous equation in $H^1_{loc}(\mathbb{R}^3)$ with a source term of bounded support. Indeed, choose $\epsilon > 0$ and a function $\chi \in C^\infty(\mathbb{R}^3)$ with $\chi(x) = 0$ for $|x| \leqslant \epsilon/2$ and $\chi(x) = 1$ for $|x| \geqslant \epsilon$ and set $u(x) = u^s(x) + \chi(x-y)\, u^{\text{inc}}(x)$. Then u coincides with u^s for $|x-y| \leqslant \epsilon/2$ and coincides with u^t for $|x-y| \geqslant \epsilon$. Setting $\chi_y(x) = \chi(x-y)$ we observe that u solves

$$\Delta u + k^2 n\, u \;=\; -f \quad \text{ in } \mathbb{R}^3 \tag{3}$$

where $f := -[k^2(1-n)(1-\chi_y) + \Delta\chi_y]\, u^{\text{inc}} - 2\nabla\chi_y \cdot \nabla u^{\text{inc}}$. We note that $f \in L^2(\mathbb{R}^3)$ has support in the ball $B_3(y,\epsilon) \subset \mathbb{R}^3$ and depends analytically on k. From now on we treat $f = f_k \in L^2(\mathbb{R}^3)$ as an arbitrary function with compact support in the disc $B_3(y,\epsilon) \subset \mathbb{R}^3$ such that $k \mapsto f_k \in L^2\big(B_3(y,\varepsilon)\big)$ is holomorphic in some (complex) neighborhood of some $\hat{k} \in \mathbb{R}_{>0}$. We enlarge the radius R_0 of the tube to include the support of the source function f. In the case of the scattering problem the scattered field is then given by $u^s = u - \chi_y\, u^{\text{inc}}$ and the total field by $u^t = u + (1-\chi_y)u^{\text{inc}}$. The solution of (3) is understood in the variational sense:

Definition 2.1. A function $u \in H^1_{loc}(\mathbb{R}^3)$ is called variational solution of (3) if

$$\int_{\mathbb{R}^3} \big[\nabla u \cdot \nabla \psi - k^2 n\, u\, \psi\big]\, \mathrm{d}x \;=\; \int_{T_{R_0}} f\, \psi\, \mathrm{d}x \tag{4}$$

for all $\psi \in H^1(\mathbb{R}^3)$ with compact support.

By choosing $\psi \in H^1(\mathbb{R}^3)$ in (4) with compact support in $\mathbb{R}^3 \setminus \overline{T_{R_0}}$ we note that u is a classical solution of the Helmholtz equation $\Delta u + k^2 u = 0$ for $x_1^2 + x_2^2 > R_0^2$. The solution u is therefore analytic in the exterior of the tube T_{R_0}.

In the following we will consider the source problem (3) for arbitrary functions $f \in L^2(\mathbb{R}^3)$ with compact support and make the following assumption on the data.

Assumption 2.2. *Let $k \in \mathbb{C}$ with $\mathrm{Re}k > 0$ and $\mathrm{Im}k \geqslant 0$ and let $n \in L^\infty(\mathbb{R}^3)$ which is assumed to be 2π-periodic with respect to the variable x_3 and equals to one outside of T_{R_0}. Furthermore, we assume that there exists $n_0 > 0$ with $n(x) \geqslant n_0$ for all $x \in \mathbb{R}^3$. Finally, we assume that $f \in L^2(\mathbb{R}^3)$ has compact support which is also contained in T_{R_0} and depends analytically on k in the neighborhood of some $\hat{k} > 0$.*

As mentioned above, a further condition is needed to assure uniqueness. It is one of the main goals of this paper to develop a proper radiation condition for real wave numbers. For wave numbers with positive imaginary part we simply require that $u \in H^1(\mathbb{R}^3)$.

Theorem 2.3. *Let assumption 2.2 hold. If $\mathrm{Im}k > 0$ then there exists a unique variational solution $u = u_k \in H^1(\mathbb{R}^3)$ of problem (4).*

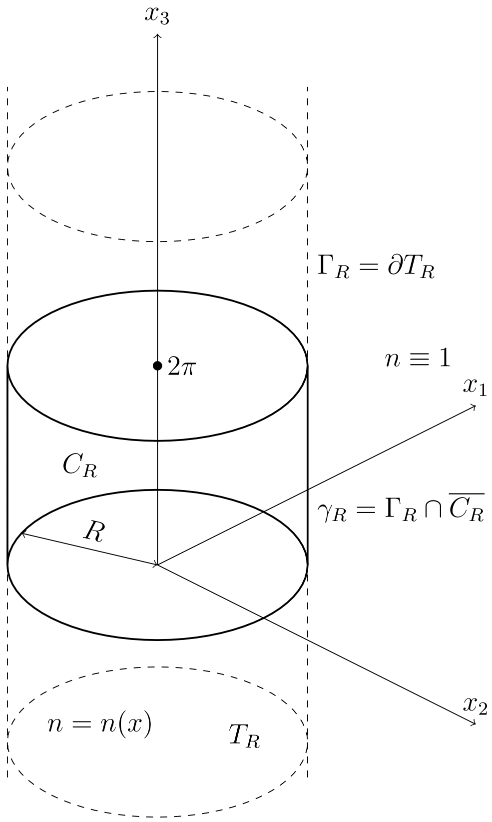

Figure 1. The geometry with its notations.

Proof. This is a standard application of the Lax–Milgram theorem. □

We will show in the first part of this paper that the solution $u = u_k$ converges (in some topology specified later) to some solution $u_{\hat{k}}$ of the Helmholtz equation $\Delta u_{\hat{k}} + \hat{k}^2 n\, u_{\hat{k}} = -f_{\hat{k}}$ in $\mathbb{R}^3$ when k tends to some real valued $\hat{k} > 0$; that is, the limiting absorption principle holds.

We use the (periodic) Floquet–Bloch transform to reformulate the problem as a family of 2π-periodic problems. Recall that the periodic Floquet–Bloch transform $F : L^2(\mathbb{R}) \to L^2\big((0, 2\pi) \times (-1/2, 1/2)\big)$ is defined by

$$(Ff)(t,\alpha) \;=\; \tilde{f}(t,\alpha) \;=\; \sum_{m\in\mathbb{Z}} f(t + 2\pi m)\, \mathrm{e}^{-\mathrm{i}\alpha(t+2\pi m)}, \quad t \in (0, 2\pi),\ \alpha \in [-1/2, 1/2].$$

This formula directly shows that for smooth functions f and fixed α the transformed function $t \mapsto (Ff)(t,\alpha) = \tilde{f}(t,\alpha)$ is 2π-periodic; while for fixed t the function $\alpha \mapsto (Ff)(t,\alpha) = \tilde{f}(t,\alpha)$ satisfies $\tilde{f}(t,\alpha+1) = \mathrm{e}^{\mathrm{i}t}\tilde{f}(t,\alpha)$. It is hence sufficient to consider $L^2\big((0, 2\pi) \times (-1/2, 1/2)\big)$ as image space of F.

The inverse transform is given by

$$(F^{-1}h)(t) \;=\; \int_{-1/2}^{1/2} h(t,\alpha)\, \mathrm{e}^{\mathrm{i}\alpha t} \mathrm{d}\alpha, \quad t \in \mathbb{R}, \tag{5}$$

where we extended $h(\cdot,\alpha)$ to a 2π-periodic function in $\mathbb{R}$.

In view of our scattering problem, we apply the Floquet–Bloch transforms to the variable x_3 and consider (x_1, x_2) as a parameter. We use the same symbol F for this extension. Then one can show that F is an isometry from $L^2(T_R)$ onto $L^2\big(C_R \times (-1/2, 1/2)\big)$; that is,

$$\|\tilde f\|^2_{L^2(C_R\times(-1/2,1/2))} \;=\; \int_{-1/2}^{1/2}\int_{C_R} |\tilde f(x,\alpha)|^2\,\mathrm{d}x\,\mathrm{d}\alpha \;=\; \int_{T_R} |f(x)|^2\,\mathrm{d}x \;=\; \|f\|^2_{L^2(T_R)}.$$

Further, the restriction of F to $H^1(T_R)$ is an isomorphism from $H^1(T_R)$ onto the space $L^2\big((-1/2,1/2), H^1_{\mathrm{per}}(C_R)\big)$ where $H^1_{\mathrm{per}}(C_R) = \big\{g \in H^1(C_R) : x_3 \mapsto g(x) \text{ is } 2\pi\text{-periodic}\big\}$ (see [20, section 6]).

We transform (3) using the Floquet–Bloch transform

$$\tilde f_\alpha(x) = \sum_{m\in\mathbb{Z}} f(x + 2\pi m e^{(3)})\,\mathrm{e}^{-\mathrm{i}\alpha(x_3+2\pi m)} \quad \text{and}$$
$$\tilde u_{k,\alpha}(x) = \sum_{m\in\mathbb{Z}} u(x + 2\pi m e^{(3)})\,\mathrm{e}^{-\mathrm{i}\alpha(x_3+2\pi m)} \quad \text{for } x \in \mathbb{R}^3 \text{ and } \alpha \in \mathbb{R},$$

where $e^{(3)} = (0,0,1)^\top$ denotes the third coordinate unit vector. We note that $\tilde f_\alpha$ depends analytically on α because the series reduces to a finite sum. We arrive at the problem to determine for every $\alpha \in [-1/2, 1/2]$ a solution $\tilde u_{k,\alpha} \in H^1_{\mathrm{per,loc}}(C_\infty)$ of

$$\Delta\tilde u_{k,\alpha} \;+\; 2\mathrm{i}\alpha\,\frac{\partial \tilde u_{k,\alpha}}{\partial x_3} \;+\; (k^2 n - \alpha^2)\,\tilde u_{k,\alpha} \;=\; -\tilde f_\alpha \quad \text{in } C_\infty. \tag{6}$$

Here, $H^1_{\mathrm{per,loc}}(C_\infty)$ is just the local space corresponding to $H^1_{\mathrm{per}}(C_\infty)$; that is, $H^1_{\mathrm{per,loc}}(C_\infty) = \big\{g : C_\infty \to \mathbb{C} : g|_{C_R} \in H^1_{\mathrm{per}}(C_R) \text{ for all } R > 0\big\}$. The trace space corresponding to $H^1_{\mathrm{per}}(C_R)$ with respect to the vertical boundary is denoted by $H^{1/2}_{\mathrm{per}}(\gamma_R)$. The spaces $H^{\pm 1/2}_{\mathrm{per}}(\gamma_R)$ are defined as the completion of $C^\infty_{\mathrm{per}}(\gamma_R)$ with respect to the norms

$$\|g\|^2_{H^{\pm1/2}(\gamma_R)} \;=\; R\sum_{\ell,m\in\mathbb{Z}} \left(1 + \frac{m^2}{R^2} + \ell^2\right)^{\pm 1/2} \big|g_{\ell,m}\big|^2 \tag{7}$$

where $g_{\ell,m} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} g(\varphi, x_3)\,\mathrm{e}^{-\mathrm{i}[m\varphi+\ell x_3]}\,\mathrm{d}\varphi\,\mathrm{d}x_3$, $\ell, m \in \mathbb{Z}$, are the Fourier coefficients of $g \in L^2(\gamma_R)$ (see [5]). Then $g(\varphi, x_3) = \frac{1}{2\pi}\sum_{\ell,m\in\mathbb{Z}} g_{\ell,m}\,\mathrm{e}^{\mathrm{i}[m\varphi+\ell x_3]}$.

Analogously to theorem 2.3 the Lax–Milgram theorem yields:

Theorem 2.4. *Let assumption 2.2 hold. If* $\mathrm{Im}k > 0$ *then for every* $\alpha \in [-1/2, 1/2]$ *there exists a unique variational solution* $\tilde u_{k,\alpha} \in H^1_{\mathrm{per}}(C_\infty)$ *of (6).*

The solutions for $\mathrm{Im}k > 0$ necessarily satisfy a Rayleigh expansion. From now on we fix $R > R_0$.

Definition 2.5. Let $\mathrm{Im}k \geqslant 0$. A solution $\tilde u \in H^1_{\mathrm{per,loc}}(C_\infty \setminus C_R)$ of $\Delta\tilde u + 2\mathrm{i}\alpha\,\frac{\partial\tilde u}{\partial x_3} + (k^2 - \alpha^2)\tilde u = 0$ in $C_\infty \setminus C_R$ satisfies the Rayleigh expansion if there exists $R_1 > R$ and $a_{\ell,m} \in \mathbb{C}$ such that $\tilde u$ is given in cylindrical coordinates by

$$\tilde u(r,\varphi,x_3) \;=\; \frac{1}{2\pi}\sum_{\ell,m\in\mathbb{Z}} a_{\ell,m}\,\frac{H^{(1)}_m\big(r\sqrt{k^2-(\ell+\alpha)^2}\big)}{H^{(1)}_m\big(R_1\sqrt{k^2-(\ell+\alpha)^2}\big)}\,\mathrm{e}^{\mathrm{i}[m\varphi+\ell x_3]} \tag{8}$$

for $r \geqslant R_1$, $\varphi, x_3 \in (0, 2\pi)$. Here, $H_m^{(1)}(z)$ denote the Hankel functions of the first kind and order $m \in \mathbb{Z}$. The branch of the square root $\sqrt{z}$ for $z \in \mathbb{C}$ with $\mathrm{Im}z \geqslant 0$ is chosen such that $\mathrm{Re}z \geqslant 0$ and $\mathrm{Im}z \geqslant 0$. The series converges in $H^1(C_{R_2} \setminus C_{R_1})$ for every $R_2 > R_1$.

Remark. The series for $\tilde{u}(r, \varphi, x_3)$ and for $\partial\tilde{u}(r, \varphi, x_3)/\partial r$ converge uniformly for $r \geqslant R_1 + \delta$ for every $\delta > 0$. Indeed, we write (8) in the form

$$\tilde{u}(r, \varphi, x_3) \;=\; \frac{1}{2\pi} \sum_{\ell, m \in \mathbb{Z}} b_{\ell,m} \frac{H_m^{(1)}\big(r\sqrt{k^2 - (\ell+\alpha)^2}\big)}{H_m^{(1)}\big((R_1+\delta)\sqrt{k^2 - (\ell+\alpha)^2}\big)}\, \mathrm{e}^{\mathrm{i}[m\varphi + \ell x_3]}, \quad r \geqslant R_1 + \delta,$$

with $b_{\ell,m} = a_{\ell,m} \frac{H_m^{(1)}\left((R_1+\delta)\sqrt{k^2-(\ell+\alpha)^2}\right)}{H_m^{(1)}\left(R_1\sqrt{k^2-(\ell+\alpha)^2}\right)}$. Then $|\tilde{u}(r, \varphi, x_3)| \leqslant \frac{1}{2\pi} \sum_{\ell,m\in\mathbb{Z}} |b_{\ell,m}|$ because of the boundedness of $\frac{\left|H_m^{(1)}\left(r\sqrt{k^2-(\ell+\alpha)^2}\right)\right|}{\left|H_m^{(1)}\left((R_1+\delta)\sqrt{k^2-(\ell+\alpha)^2}\right)\right|}$ (see part (a) of lemma A.2). Furthermore, $b_{\ell,m}$ are the Fourier coefficients of $\tilde{u}(R_1 + \delta, \cdot, \cdot)$. Since this function is analytic we have convergence of $\sum_{\ell,m\in\mathbb{Z}} (1 + \ell^2 + m^2)^p |b_{\ell,m}|^2$ for all $p \in \mathbb{N}$. Therefore, also $\sum_{\ell,m\in\mathbb{Z}} |b_{\ell,m}|$ converges. The same argument holds for the derivative.

For $\mathrm{Im}k > 0$ the (unique solution) $\tilde{u}_{k,\alpha} \in H^1_{\mathrm{per}}(C_\infty)$ of (6) satisfies the Rayleigh expansion. Since $g = \tilde{u}_{k,\alpha}\big|_{\gamma_R} \in H^{1/2}_{\mathrm{per}}(\gamma_R)$ this follows from the first part of the following lemma.

Lemma 2.6. *Let assumption 2.2 hold and let $g \in H^{1/2}_{\mathrm{per}}(\gamma_R)$ with Fourier coefficients $g_{\ell,m} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} g(\varphi, x_3)\, \mathrm{e}^{-\mathrm{i}[m\varphi + \ell x_3]}\, \mathrm{d}\varphi\, \mathrm{d}x_3$, $\ell, m \in \mathbb{Z}$.*

(a) If $\mathrm{Im}k > 0$ then there exists a unique solution $\tilde{u} \in H^1_{\mathrm{per}}(C_\infty \setminus C_R)$ of the following Dirichlet boundary value problem

$$\Delta\tilde{u} \;+\; 2\mathrm{i}\alpha \frac{\partial\tilde{u}}{\partial x_3} \;+\; (k^2 - \alpha^2)\tilde{u} \;=\; 0 \text{ in } C_\infty \setminus C_R, \quad \tilde{u} = g \text{ on } \gamma_R. \tag{9}$$

The solution is given by

$$\tilde{u}(r, \varphi, x_3) \;=\; \frac{1}{2\pi} \sum_{\ell,m\in\mathbb{Z}} g_{\ell,m} \frac{H_m^{(1)}\big(r\sqrt{k^2 - (\ell+\alpha)^2}\big)}{H_m^{(1)}\big(R\sqrt{k^2 - (\ell+\alpha)^2}\big)}\, \mathrm{e}^{\mathrm{i}[m\varphi + \ell x_3]}, \tag{10}$$

for $r > R$ and $\varphi, x_3 \in (0, 2\pi)$. The solution $\tilde{u}$ and its derivative $\partial\tilde{u}/\partial r$ decay exponentially; that is, there exist $c > 0$ and $\sigma > 0$ such that $|\tilde{u}(r, \varphi, x_3)| \leqslant c \exp(-\sigma r)$ for all $r > R$ (and the same for the derivative).

(b) If $\mathrm{Im}k \geqslant 0$ then the series (10) converges in $H^1(C_{R_1} \setminus C_R)$ for every $R_1 > R$ to some $\tilde{u} \in H^1_{\mathrm{per,loc}}(C_\infty \setminus C_R)$. Furthermore, $\tilde{u}$ is the unique solution solution of (9) satisfying the Rayleigh expansion (8).

(c) For every $R_1 > R$ and $k \in \mathbb{C}$ with $\mathrm{Im}k \geqslant 0$ the operator $\tilde{S}_{k,\alpha} : H^{1/2}_{\mathrm{per}}(\gamma_R) \to H^1_{\mathrm{per}}(C_{R_1} \setminus C_R)$, given by $g \mapsto \tilde{u}|_{C_{R_1} \setminus C_R}$, is bounded.

Proof.

(a) Existence and uniqueness of a solution $\tilde{u} \in H^1_{\mathrm{per}}(C_\infty \setminus C_R)$ follows again by the Lax–Milgram theorem. The uniqueness and the proof of part (b) imply that $\tilde{u}$ has to be of the form (10). To show the exponential decay we set $k_\ell = \sqrt{k^2 - (\ell+\alpha)^2}$ and use part (a) of lemma A.2 which yields that, for any $R_1 > R$,

$$
\left|\tilde u(r,\varphi,x_3)\right| \leqslant \frac{1}{2\pi}\sum_{\ell,m\in\mathbb{Z}}\left|a_{\ell,m}\right|\left|\frac{H_m^{(1)}(rk_\ell)}{H_m^{(1)}(Rk_\ell)}\right| = \frac{1}{2\pi}\sum_{\ell,m\in\mathbb{Z}}\underbrace{\left|a_{\ell,m}\right|\left|\frac{H_m^{(1)}(R_1k_\ell)}{H_m^{(1)}(Rk_\ell)}\right|}_{=:\,|b_{\ell,m}|}\left|\frac{H_m^{(1)}(rk_\ell)}{H_m^{(1)}(R_1k_\ell)}\right|
$$

$$
\leqslant \frac{1}{2\pi}\sum_{\ell,m\in\mathbb{Z}}\left|b_{\ell,m}\right| \mathrm{e}^{-(\mathrm{Im}k_\ell)(r-R_1)} \leqslant \frac{1}{2\pi}\mathrm{e}^{-\sigma(r-R_1)}\sum_{\ell,m\in\mathbb{Z}}\left|b_{\ell,m}\right|, \quad r\geqslant R_1,
$$

where $\sigma = \min\{\mathrm{Im}k_\ell : \ell\in\mathbb{Z}\}$ which is positive because $\mathrm{Im}k>0$. This yields the estimate because $\sum_{\ell,m\in\mathbb{Z}}\left|b_{\ell,m}\right| < \infty$. The estimate for the derivative is obtained by the same argument.

(b), (c) Let $\mathrm{Im}k \geqslant 0$ and $\tilde u$ given by (10). We define

$$
\psi_{\ell,m}(r,\varphi) = \frac{H_m^{(1)}(r\,k_\ell)}{H_m^{(1)}(R\,k_\ell)}\,\mathrm{e}^{\mathrm{i}m\varphi}, \quad r\geqslant R,\ \ell,m\in\mathbb{Z},
$$

where again $k_\ell = \sqrt{k^2-(\ell+\alpha)^2}$, $\ell\in\mathbb{Z}$. Then

$$
\tilde u(r,\varphi,x_3) = \frac{1}{2\pi}\sum_{\ell,m\in\mathbb{Z}} g_{\ell,m}\,\psi_{\ell,m}(r,\varphi)\,\mathrm{e}^{\mathrm{i}\ell x_3}.
$$

By lemma A.4 of the appendix there exists $c>0$ such that

$$
\|\psi_{\ell,m}\|^2_{L^2(B_2(0,R_1)\setminus B_2(0,R))} \leqslant \frac{c}{\sqrt{1+\ell^2}} \quad\text{and}\quad \|\psi_{\ell,m}\|^2_{H^1_{\mathrm{per}}(B_2(0,R_1)\setminus B_2(0,R))} \leqslant c\sqrt{1+\ell^2+m^2}
$$

for all $\ell,m\in\mathbb{Z}$. Therefore, by the orthogonality of $\left\{\exp(im\varphi+\mathrm{i}\ell x_3) : \ell,m\in\mathbb{Z}\right\}$,

$$
\|\tilde u\|^2_{H^1_{\mathrm{per}}(C_{R_1}\setminus C_R)} = \frac{1}{2\pi}\sum_{\ell,m\in\mathbb{Z}}\left|g_{\ell,m}\right|^2\left[\|\psi_{\ell,m}\|^2_{H^1_{\mathrm{per}}(B_2(0,R_1)\setminus B_2(0,R))} + (\ell+\alpha)^2\|\psi_{\ell,m}\|^2_{L^2(B_2(0,R_1)\setminus B_2(0,R))}\right]
$$

$$
\leqslant c\sum_{\ell,m\in\mathbb{Z}}\left|g_{\ell,m}\right|^2\left[\sqrt{1+\ell^2+m^2} + \frac{(\ell+\alpha)^2}{\sqrt{1+\ell^2}}\right] \leqslant c'\,\|g\|^2_{H^{1/2}_{\mathrm{per}}(\gamma_R)}.
$$

This proves that the function given by (10) provides a solution of (9). The solution is also unique. Indeed, let $g=0$ and $\tilde u\in H^1_{\mathrm{per,loc}}(C_\infty\setminus C_R)$ be a solution satisfying the Rayleigh expansion (8). The Fourier coefficients $u_{\ell,m}(r) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\tilde u(r,\varphi,x_3)\,\mathrm{e}^{-\mathrm{i}m\varphi-\mathrm{i}\ell x_3}\mathrm{d}\varphi\,\mathrm{d}x_3$ satisfy Bessel's differential equation

$$
\frac{1}{r}\left(r\,u'_{\ell,m}(r)\right)' + \left(k^2-(\ell+\alpha)^2-\frac{m^2}{r^2}\right)u_{\ell,m}(r) = 0, \quad r>R,
$$

and the initial condition $u_{\ell,m}(R)=0$. By the Rayleigh expansion, $u_{\ell,m}$ is given by $u_{\ell,m}(r) = a_{\ell,m}\frac{H_m^{(1)}\left(r\sqrt{k^2-(\ell+\alpha)^2}\right)}{H_m^{(1)}\left(R_1\sqrt{k^2-(\ell+\alpha)^2}\right)}$ for $r>R_1$ and some $a_{\ell,m}\in\mathbb{C}$. By analyticity, $u_{\ell,m}(r)$ is given by this formula for all $r>R$. The initial condition yields $a_{\ell,m}=0$. □

Now we consider the source problem (6) also for the case of real wave numbers $k=\hat k>0$ and include the Rayleigh expansion.

Problem (P_α): Determine $\tilde u_{\hat k,\alpha}\in H^1_{\mathrm{per,loc}}(C_\infty)$ as a solution of (6) for $k=\hat k>0$ which satisfies also the Rayleigh expansion (8) for $k=\hat k>0$; that is,

$$\Delta \tilde{u}_{\hat{k},\alpha} \;+\; 2\mathrm{i}\alpha\,\frac{\tilde{u}_{\hat{k},\alpha}}{\partial x_3} \;+\; (\hat{k}^2 n-\alpha^2)\,\tilde{u}_{\hat{k},\alpha} \;=\; -\tilde{f}_\alpha \quad \text{in } \mathbb{R}^3, \qquad \text{and} \tag{11}$$

$$\tilde{u}_{\hat{k},\alpha}(r,\varphi,x_3) \;=\; \frac{1}{2\pi}\sum_{\ell,m\in\mathbb{Z}} a_{\ell,m}\,\frac{H_m^{(1)}\big(r\sqrt{\hat{k}^2-(\ell+\alpha)^2}\big)}{H_m^{(1)}\big(R_1\sqrt{\hat{k}^2-(\ell+\alpha)^2}\big)}\,\mathrm{e}^{\mathrm{i}[m\varphi+\ell x_3]}, \quad r\geqslant R_1, \tag{12}$$

for some $R_1 > R$ and $a_{\ell,m}\in\mathbb{C}$.

As the example of a constant index n shows for given $\hat{k}>0$ there might exist certain values of α such that the Problem (P_α) does not have a unique solution.

Definition 2.7. The values of $\alpha\in[-1/2,1/2]$ for which the homogeneous form of Problem (P_α); that is, (11), (12), with $\tilde{f}_\alpha=0$, admits non-trivial solutions are called *exceptional values* or *propagative wave numbers*. The corresponding periodic solutions $\tilde{\phi}\in H^1_{\mathrm{per,loc}}(C_\infty)$ of the homogeneous problem are called *propagating modes*.

We define the set

$$A \;=\; \big\{\alpha\in[-1/2,1/2]:\ \text{there exists } \ell\in\mathbb{Z} \text{ with } |\alpha+\ell|=\hat{k}\big\}$$

of cut-off values and note that A consists of one or two elements. We make the following assumption for the rest of the paper.

Assumption 2.8. *The cut-off values $\alpha\in A$ are not exceptional values; that is, for $\alpha\in A$ the only solution of (11) and (12) for $\tilde{f}_\alpha=0$ is the trivial one.*

The name 'propagating mode' is justified by the following.

Lemma 2.9. *Let assumption 2.8 hold and choose $R_1>R$.*

(a) If α is an exceptional value with corresponding propagating mode $\tilde{\phi}\in H^1_{\mathrm{per,loc}}(C_\infty)$ then $\tilde{\phi}$ is evanescent; that is, there exists $\sigma>0$ and $c>0$ with $|\tilde{\phi}(x)|\leqslant c\exp\big(-\sigma\sqrt{x_1^2+x_2^2}\big)$ for all $x\in C_\infty$. In particular, $\tilde{\phi}\in H^1_{\mathrm{per}}(C_\infty)$.

(b) If α is exceptional with propagating mode $\tilde{\phi}\in H^1_{\mathrm{per,loc}}(C_\infty)$ then $-\alpha$ is exceptional with propagating mode $\overline{\tilde{\phi}}$.

Proof.

(a) Let $\tilde{\phi}\in H^1_{\mathrm{per,loc}}(C_\infty)$ be a non-trivial solution of (P_α) with $\tilde{f}_\alpha=0$. Green's formula in C_{R_1} and the Rayleigh expansion (12) yields

$$\int_{C_{R_1}}\Big[|\nabla\tilde{\phi}|^2-2\mathrm{i}\alpha\frac{\partial\tilde{\phi}}{\partial x_3}\overline{\tilde{\phi}}+(\alpha^2-\hat{k}^2n)\,|\tilde{\phi}|^2\Big]\,\mathrm{d}x$$

$$=\int_{\gamma_{R_1}}\tilde{\phi}\,\frac{\partial\overline{\tilde{\phi}}}{\partial\nu}\,\mathrm{d}s \;=\; R_1\sum_{\ell,m\in\mathbb{Z}}|a_{\ell,m}|^2\,\frac{k_\ell\,H_m^{(1)\prime}(R_1k_\ell)}{H_m^{(1)}(R_1k_\ell)}$$

$$=R_1\sum_{m\in\mathbb{Z}}\sum_{|\ell+\alpha|<\hat{k}}|a_{\ell,m}|^2\,\frac{k_\ell\,H_m^{(1)\prime}(R_1k_\ell)}{H_m^{(1)}(R_1k_\ell)} \;+\; R_1\sum_{m\in\mathbb{Z}}\sum_{|\ell+\alpha|>\hat{k}}|a_{\ell,m}|^2\,\frac{\mathrm{i}|k_\ell|\,H_m^{(1)\prime}(\mathrm{i}R_1|k_\ell|)}{H_m^{(1)}(\mathrm{i}R_1|k_\ell|)}$$

$$=R_1\sum_{m\in\mathbb{Z}}\sum_{|\ell+\alpha|<\hat{k}}|a_{\ell,m}|^2\,\frac{k_\ell\,H_m^{(1)\prime}(R_1k_\ell)}{H_m^{(1)}(R_1k_\ell)} \;+\; R_1\sum_{m\in\mathbb{Z}}\sum_{|\ell+\alpha|>\hat{k}}|a_{\ell,m}|^2\,\frac{|k_\ell|\,K_m'(R_1|k_\ell|)}{K_m(R_1|k_\ell|)}$$

with $k_\ell = \sqrt{\hat{k}^2 - |\ell+\alpha|^2}$. The terms with $|\ell+\alpha| = \hat{k}$ do not appear because of assumption 2.8. Here, K_m are the modified Hankel functions which are related to the Hankel functions by $H_m^{(1)}(it) = \frac{2}{\pi} i^{-m-1} K_m(t)$ and are real valued (see [1]). The left hand side and the second series on the right hand side are real valued. Furthermore, for $s > 0$

$$\mathrm{Im} \frac{s\, H_m^{(1)\prime}(s)}{H_m^{(1)}(s)} \;=\; \frac{s\big[-J_m'(s)Y_m(s) + Y_m'(s)J_m(s)\big]}{\big|H_m^{(1)}(s)\big|^2} \;=\; \frac{2}{\pi \big|H_m^{(1)}(s)\big|^2} \;>\; 0$$

by the Wronskian relationship. This implies that $a_{\ell,m} = 0$ for all $(\ell, m) \in \mathbb{Z}^2$ with $|\ell+\alpha| < \hat{k}$. Finally, we use part (a) of lemma A.2 which yields as in the proof of lemma 2.6 that, for any $R_2 > R_1$,

$$\begin{aligned}
\big|\tilde{u}_{\hat{k},\alpha}(r,\varphi,x_3)\big| &\leqslant \frac{1}{2\pi} \sum_{m\in\mathbb{Z}} \sum_{|\ell+\alpha|>\hat{k}} |a_{\ell,m}| \left| \frac{H_m^{(1)}(rk_\ell)}{H_m^{(1)}(R_1 k_\ell)} \right| \\
&= \frac{1}{2\pi} \sum_{m\in\mathbb{Z}} \sum_{|\ell+\alpha|>\hat{k}} \underbrace{|a_{\ell,m}| \left| \frac{H_m^{(1)}(R_2 k_\ell)}{H_m^{(1)}(R_1 k_\ell)} \right|}_{=:\,|b_{\ell,m}|} \left| \frac{H_m^{(1)}(rk_\ell)}{H_m^{(1)}(R_2 k_\ell)} \right| \\
&\leqslant \frac{1}{2\pi} \sum_{m\in\mathbb{Z}} \sum_{|\ell+\alpha|>\hat{k}} \big|b_{\ell,m}\big|\, \mathrm{e}^{-|k_\ell|(r-R_2)} \\
&\leqslant \frac{1}{2\pi}\, \mathrm{e}^{-\sigma(r-R_2)} \sum_{m\in\mathbb{Z}} \sum_{|\ell+\alpha|>\hat{k}} \big|b_{\ell,m}\big|, \qquad r \geqslant R_2,
\end{aligned}$$

where $\sigma = \min\{|\ell+\alpha| - \hat{k} : |\ell+\alpha| > \hat{k},\ \ell \in \mathbb{Z}\}$. This proves part (a) because the series over $|b_{\ell,m}|$ converges.

(b) This is clear from the definition. □

Next, we reduce the problem to a boundary problem in the bounded cell C_R using the Dirichlet–Neumann operator.

Definition 2.10. Let $\mathrm{Re}k > 0$ and $\mathrm{Im}k \geqslant 0$. The periodic Dirichlet–Neumann operator $\tilde{\Lambda}_{k,\alpha} : H^{1/2}_{\mathrm{per}}(\gamma_R) \to H^{-1/2}_{\mathrm{per}}(\gamma_R)$ is defined by

$$(\tilde{\Lambda}_{k,\alpha} g)(\varphi, x_3) \;=\; \frac{1}{2\pi} \sum_{\ell,m\in\mathbb{Z}} g_{\ell,m} \frac{k_\ell\, H_m^{(1)\prime}(k_\ell R)}{H_m^{(1)}(k_\ell R)}\, \mathrm{e}^{\mathrm{i}m\varphi + \mathrm{i}\ell x_3} \quad \text{for } \varphi, x_3 \in (0, 2\pi), \tag{13}$$

where again $k_\ell = \sqrt{k^2 - (\ell+\alpha)^2}$ for $\ell \in \mathbb{Z}$. Here we cut the complex plane along the negative imaginary axis so that the square root is holomorphic in $\mathbb{C} \setminus (\mathrm{i}\mathbb{R}_{\leqslant 0})$. Furthermore,

$$g_{\ell,m} \;=\; \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} g(\varphi, x_3)\, \mathrm{e}^{-\mathrm{i}[m\varphi + \ell x_3]}\, \mathrm{d}\varphi\, \mathrm{d}x_3, \qquad \ell, m \in \mathbb{Z},$$

are the Fourier coefficients of $g \in H^{1/2}_{\mathrm{per}}(\gamma_R)$. The operator is well defined and bounded by part (b) of lemma A.2.

We can even extend this operator $\tilde{\Lambda}_{k,\alpha}$ to $(k,\alpha)\in\mathbb{C}\times\mathbb{C}$ in a neighborhood of real values $(\hat{k},\hat{\alpha})\in\mathbb{R}_{>0}\times[-1/2,1/2]\subset\mathbb{R}\times\mathbb{R}$. This and other properties are shown in the following theorem.

Theorem 2.11.

(a) The operators $\tilde{\Lambda}_{k,\alpha}$ are well defined and bounded from $H^{1/2}_{\mathrm{per}}(\gamma_R)$ into $H^{-1/2}_{\mathrm{per}}(\gamma_R)$ for all $k\in\mathbb{C}$ and $\alpha\in[-1/2,1/2]$ with $\mathrm{Re}k>0$ and $\mathrm{Im}k\geqslant 0$.

(b) Let $\hat{k}>0$ and $\hat{\alpha}\in[-1/2,1/2]$ such that $|\ell+\hat{\alpha}|\neq\hat{k}$ for all $\ell\in\mathbb{Z}$. Choose $\delta>0$ such

$$P=\left\{(k,\alpha)\in\mathbb{C}\times\mathbb{C}:|\alpha-\hat{\alpha}|+|k-\hat{k}|<\delta\right\}$$

satisfies $|\ell+\alpha|\neq k$ and $k^2-(\ell+\alpha)^2\notin i\mathbb{R}_{\leqslant 0}$ for all $(k,\alpha)\in P$ and $\ell\in\mathbb{Z}$. Then the mapping $(k,\alpha)\mapsto\tilde{\Lambda}_{k,\alpha}$ is well defined and strongly holomorphic from P into $\mathcal{L}\big(H^{1/2}_{\mathrm{per}}(\gamma_R),H^{-1/2}_{\mathrm{per}}(\gamma_R)\big)$.

(c) $\tilde{\Lambda}_{k,\alpha}-\tilde{\Lambda}_{i,0}$ is compact from $H^{1/2}_{\mathrm{per}}(\gamma_R)$ into $H^{-1/2}_{\mathrm{per}}(\gamma_R)$. Here, $\tilde{\Lambda}_{i,0}$ denotes the Dirichlet–Neumann operator for $k=i$ and $\alpha=0$.

(d) $\tilde{\Lambda}_{i,0}$ is self-adjoint and negative; that is, $\langle\tilde{\Lambda}_{i,0}g,g\rangle<0$ for all $g\in H^{1/2}_{\mathrm{per}}(\gamma_R)$ with $g\neq 0$. Here, $\langle\cdot,\cdot\rangle$ denotes the dual form; that is, the extension of the L^2-inner product to $H^{-1/2}_{\mathrm{per}}(\gamma_R)\times H^{1/2}_{\mathrm{per}}(\gamma_R)$.

Proof.

(a) This follows directly from the estimate of part (b) of lemma A.2 of the appendix.

(b) Boundedness of $\tilde{\Lambda}_{k,\alpha}$ for every $(k,\alpha)\in P$ follows again by the estimate of lemma A.2 of the appendix. For the analyticity of the mapping $(k,\alpha)\mapsto\tilde{\Lambda}_{k,\alpha}$ it is sufficient to show that this mapping is weakly holomorphic; that is, the mapping $(k,\alpha)\mapsto\big\langle\tilde{\Lambda}_{k,\alpha}g,h\big\rangle$ is holomorphic in P for every $g,h\in H^{1/2}_{\mathrm{per}}(\gamma_R)$. The fact that every weakly holomorphic function is strongly holomorphic is shown, e.g. in Chapter 8 of [6] for operator valued functions of one complex variable. Since the proof uses only Cauchy's integral formula - which is valid also in $\mathbb{C}^2$ - this property holds also for functions of two complex variables. The series

$$\langle\tilde{\Lambda}_{k,\alpha}g,h\rangle \;=\; R\sum_{\ell,m\in\mathbb{Z}} g_{\ell,m}\,\overline{h_{\ell,m}}\,\frac{k_\ell\,H^{(1)\prime}_m(k_\ell R)}{H^{(1)}_m(k_\ell R)}$$

converges uniformly by lemma A.2 which proves analyticity of this function.

(c) Let $\tilde{\Lambda}_{k,\alpha}g=\partial u/\partial r|_{\gamma_R}$ and $\tilde{\Lambda}_{i,0}g=\partial v/\partial r|_{\gamma_R}$ where u and v solve

$$\Delta u\;+\;2i\alpha\,\frac{\partial u}{\partial x_3}\;+\;(k^2-\alpha^2)\,u\;=\;0,\quad \Delta v\;-v\;=\;0\quad\text{in } C_\infty\setminus C_R$$

and $u=v=g$ on γ_R and the Rayleigh expansions. The difference $u-v$ satisfies

$$\Delta(u-v)\;+\;2i\alpha\,\frac{\partial(u-v)}{\partial x_3}\;+\;(k^2-\alpha^2)\,(u-v)\;=\;-2i\alpha\,\frac{\partial v}{\partial x_3}\;-\;(k^2-\alpha^2+1)\,v$$

for $r>R$ and $u-v$ vanishes on γ_R. Now we choose $\phi\in C^\infty(\mathbb{R}^3)$ which is equal to one in the cylinder $\{x\in\mathbb{R}^3:x_1^2+x_2^2\leqslant\hat{R}^2,\ -1<x_3<2\pi+1\}$ and vanishes outside of the larger cylinder $\{x\in\mathbb{R}^3:x_1^2+x_2^2\leqslant(\hat{R}+1)^2,\ -2<x_3<2\pi+2\}$ for some $\hat{R}>R$.

We define the shell $S := \{x \in \mathbb{R}^3 : R^2 < x_1^2 + x_2^2 < (\hat{R}+1)^2,\ -2 < x_3 < 2\pi + 2\}$. The product $w = \phi(u - v)$ satisfies

$$\Delta w \;+\; 2\mathrm{i}\alpha\,\frac{\partial w}{\partial x_3} \;+\; (k^2 - \alpha^2)\,w \;=\; -h \quad \text{in } S$$

and vanishes on the boundary of S. Here,

$$h \;=\; \left[2\mathrm{i}\alpha\,\frac{\partial v}{\partial x_3} + (k^2 - \alpha^2 + 1)\,v\right]\phi - 2\,\nabla\phi\cdot\nabla(u - v) - (u - v)\,\Delta\phi - 2\mathrm{i}\alpha\,(u - v)\,\frac{\partial \phi}{\partial x_3}.$$

We notice that $h \in L^2(S)$ and the mapping $g \mapsto h$ is bounded from $H^{1/2}_{\mathrm{per}}(\gamma_R)$ into $L^2(S)$ (by the continuous dependence of u and v on g). Standard regularity results for elliptic partial differential equations imply continuity of $h \to w$ from $L^2(S)$ into $H^2(S)$. This shows continuity of $g \mapsto w$ from $H^{1/2}_{\mathrm{per}}(\gamma_R)$ into $H^2(S)$ and thus compactness as a mapping into $H^1(S)$. The trace theorem yields compactness of $g \mapsto \partial w/\partial r = \partial(u - v)/\partial r = (\tilde{\Lambda}_{k,\alpha} - \tilde{\Lambda}_{i,0})g$.

(d) Let again $\tilde{\Lambda}_{i,0} g_j = \partial v_j/\partial r|_{\gamma_R}$ for $j = 1, 2$ where $v_j \in H^1_{\mathrm{per}}(C_\infty \setminus C_R)$ solves $\Delta v_j - v_j = 0$ in $C_\infty \setminus C_R$ and $v_j = g_j$ on γ_R. Then, for any $R_1 > R$,

$$\langle \tilde{\Lambda}_{i,0} g_1, g_2\rangle \;=\; \int_{\gamma_R} \frac{\partial v_1}{\partial r}\,\overline{v_2}\,\mathrm{d}s \;=\; \int_{r=R_1} \frac{\partial v_1}{\partial r}\,\overline{v_2}\,\mathrm{d}s \;-\; \int_{C_{R_1}\setminus C_R} \nabla v_1\cdot\nabla\overline{v_2} + v_1\overline{v_2}\,\mathrm{d}x$$

which converges to the hermitean form $-\int_{C_\infty\setminus C_R}\left[\nabla v_1\cdot\nabla\overline{v_2} + v_1\overline{v_2}\right]\mathrm{d}x$ as $R_1 \to \infty$ because v_j and $\partial v_j/\partial r$ decay exponentially by part (a) of lemma 2.6. This shows that $\tilde{\Lambda}_{i,0}$ is selfadjoint. Furthermore, for $g = g_1 = g_2$ we observe that $\tilde{\Lambda}_{i,0}$ is negative because $\langle \tilde{\Lambda}_{i,0} g, g\rangle = 0$ holds only for $v = 0$ which implies $g = 0$.

We recall the bounded cell $C_R = B_2(0, R) \times (0, 2\pi)$ and formulate the source problem for $\tilde{u}_{k,\alpha} \in H^1_{\mathrm{per}}(C_R)$ as the variational equation

$$\int_{C_R}\left[\nabla\tilde{u}_{k,\alpha}\cdot\nabla\overline{\psi} + 2\mathrm{i}\alpha\,\tilde{u}_{k,\alpha}\,\frac{\partial\overline{\psi}}{\partial x_3} + (\alpha^2 - k^2 n)\,\tilde{u}_{k,\alpha}\,\overline{\psi}\right]\mathrm{d}x \;-\; \langle \tilde{\Lambda}_{k,\alpha}\tilde{u}_{k,\alpha}, \psi\rangle \;=\; \int_{C_R} \tilde{f}_\alpha\,\overline{\psi}\,\mathrm{d}x \qquad (14)$$

for all $\psi \in H^1_{\mathrm{per}}(C_R)$.

The proof of the following lemma is simple and left to the reader.

Lemma 2.12. *Let $k \in \mathbb{C}$ with $\mathrm{Re}k > 0$ and $\mathrm{Im}k \geqslant 0$.*

(a) If $\tilde{u}_{k,\alpha} \in H^1_{\mathrm{per,loc}}(C^\infty)$ is a solution of the scattering problem (6) and Rayleigh expansion (8) then the restriction $\tilde{u}_{k,\alpha}|_{C_R} \in H^1_{\mathrm{per}}(C_R)$ solves (14).

(b) If $\tilde{u}_{k,\alpha} \in H^1_{\mathrm{per}}(C_R)$ is a solution of (14) then the extension

$$\tilde{u}_{k,\alpha}(x) \;=\; \begin{cases} \tilde{u}_{k,\alpha}(x), & x \in C_R, \\ \tilde{S}_{k,\alpha}\big(\tilde{u}_{k,\alpha}|_{\Gamma_R}\big)(x), & x \in C_\infty \setminus C_R, \end{cases}$$

with the operator $\tilde{S}_{k,\alpha}$ introduced in lemma 2.6 is the solution of the scattering problem (6) and Rayleigh expansion (8).

We show that the equation (14) is of Fredholm type. Indeed, we first decompose $\tilde{\Lambda}_{k,\alpha}$ into $\tilde{\Lambda}_{k,\alpha} = \tilde{\Lambda}_{i,0} + [\tilde{\Lambda}_{k,\alpha} - \tilde{\Lambda}_{i,0}]$. From theorem 2.11 we observe that

$$(v,\psi)_* \;:=\; \int_{C_R}\left[\nabla v\cdot\nabla\overline{\psi}+v\,\overline{\psi}\right]\mathrm{d}x \;-\; \langle\tilde{\Lambda}_{i,0}v,\psi\rangle,\quad v,\psi\in H^1_{\mathrm{per}}(C_R), \tag{15}$$

defines an inner product in $H^1_{\mathrm{per}}(C_R)$ which is equivalent to the ordinary norm in $H^1_{\mathrm{per}}(C_R)$. Therefore, (14) is equivalent to

$$(\tilde{u}_{k,\alpha},\psi)_* \;-\; a_{k,\alpha}(\tilde{u}_{k,\alpha},\psi) \;=\; \int_{C_R}\tilde{f}_\alpha\,\overline{\psi}\,\mathrm{d}x,\quad \psi\in H^1_{\mathrm{per}}(C_R),$$

where

$$a_{k,\alpha}(v,\psi) := -\int_{C_R}\left[\mathrm{i}\alpha\left(v\,\frac{\partial\overline{\psi}}{\partial x_3}-\overline{\psi}\,\frac{\partial v}{\partial x_3}\right)+(\alpha^2-k^2n-1)\,v\,\overline{\psi}\right]\mathrm{d}x \\ -\;\left\langle[\tilde{\Lambda}_{i,0}-\tilde{\Lambda}_{k,\alpha}]\,v,\psi\right\rangle,\quad v,\psi\in H^1_{\mathrm{per}}(C_R). \tag{16}$$

We note that the source term $\tilde{f}_\alpha$ in (14) can also depend on k (as in the original scattering problem (3)), and we write $\tilde{f}_{k,\alpha}$ from now on.

Let again $k\in\mathbb{C}$ with $\mathrm{Re}k>0$ and $\mathrm{Im}k\geqslant 0$. By the theorem of Riesz, the compact imbedding of $H^1_{\mathrm{per}}(C_R)$ in $L^2(C_R)$, and the compactness of $\Lambda_{i,0}-\Lambda_{k,\alpha}$ there exists a compact operator $K_{k,\alpha}$ from $H^1_{\mathrm{per}}(C_R)$ into itself with $a_{k,\alpha}(u,\psi)=(K_{k,\alpha}u,\psi)_*$ for all $u,\psi\in H^1_{\mathrm{per}}(C_R)$. Furthermore, there exists $r_{k,\alpha}\in H^1_{\mathrm{per}}(C_R)$ with $\int_{C_R}\tilde{f}_{k,\alpha}\,\overline{\psi}\,\mathrm{d}x=(r_{k,\alpha},\psi)_*$ for all $\psi\in H^1_{\mathrm{per}}(C_R)$. Then we can rewrite the variational equation (14) as an operator equation in the form

$$\tilde{u}_{k,\alpha} \;-\; K_{k,\alpha}\tilde{u}_{k,\alpha} \;=\; r_{k,\alpha}\quad\text{in } H^1_{\mathrm{per}}(C_R). \tag{17}$$

The operator equation is well defined for all $k\in\mathbb{C}$ with $\mathrm{Re}k>0$ and $\mathrm{Im}k\geqslant 0$. For $\mathrm{Im}k>0$ we have uniqueness and existence by theorem 2.4.

For real values $k=\hat{k}>0$, however, we expect non-uniqueness at certain values of α that we called exceptional values (see definition 2.7) . In other words, we expect that for some $\alpha\in[-1/2,1/2]$ there is an eigenvalue $\lambda=1$ of the non-selfadjoint operator $K_{\hat{k},\alpha}$. We note that by lemma 2.12 the corresponding eigenfunctions are exactly the propagating modes of definition 2.7.

Lemma 2.13. *Let $\hat{k}>0$ and assume that assumptions 2.2 and 2.8 hold. Then there exist at most finitely many exceptional values $\{\hat{\alpha}_j : j\in J\}\subset[-1/2,1/2]$ for some finite index set $J\subset\mathbb{Z}$. By part (b) of lemma 2.9 we can assume that J is symmetric with respect to the origin and $\hat{\alpha}_{-j}=-\hat{\alpha}_j$ for all $j\in J$.*

Proof. Assume on the contrary that there exists an (infinite) sequence $(\hat{\alpha}_j)_j$ in $[-1/2,1/2]$ and a sequence $(w_j)_j$ in $H^1_{\mathrm{per}}(C_R)$ of corresponding normalized functions such that $(I-K_{\hat{k},\hat{\alpha}_j})w_j=0$ for all j. Let again $A=\{\alpha\in[-1/2,1/2] : |\ell+\alpha|=\hat{k}$ for some $\ell\in\mathbb{Z}\}$. We can assume that the sequence belongs to one of the at most three intervals of $[-1/2,1/2]\setminus A$, say to $\mathcal{I}=[-1/2,\tau)$ where $|\ell+\tau|=\hat{k}$ for some $\ell\in\mathbb{Z}$. By theorem 2.11 there exists an open set U such that $\mathcal{I}\subset U$ and the mapping $\alpha\mapsto K_{\hat{k},\alpha}$ is analytic from U into $\mathcal{L}(H^1_{0,per}(C_R))$. From [11, theorem 5.1] it follows that the equation $(I-K_{\hat{k},\alpha})w=0$ has the same number of linearly independent solutions at every parameter $\alpha\in\mathcal{I}$ except for finitely many. Since for the infinite sequence $\hat{\alpha}_j$ this number is at least one, it has to be at least one for all $\alpha\in\mathcal{I}$ except for finitely many. From the continuity of $\alpha\mapsto\tilde{K}_{\hat{k},\alpha}$ and the injectivity of $\tilde{K}_{\hat{k},\tau}$ by assumption 2.8 the operators $\tilde{K}_{\hat{k},\alpha}$ have to be injective for all α in a neighborhood of τ. This is a contradiction. The other cases of $\mathcal{I}$ are treated in the same way. □

3. The limiting absorption principle

In this section we fix an arbitrary wave number $\hat{k} \in \mathbb{R}_{>0}$ and investigate the operator equation (17) in a neighborhood of the exceptional values $\hat{\alpha}_j$ for $j \in J$. (Of course, such exceptional values do not need to exist for every $\hat{k} > 0$.) The following lemma is obvious by the Fredholm property of the operator $K_{k,\alpha}$ and the definition of an exceptional value.

Lemma 3.1. *For any fixed $\delta > 0$ the solutions $\tilde{u}_{k,\alpha} \in H^1_{\text{per}}(C_R)$ of (17) for $\operatorname{Im} k > 0$ converge to $\tilde{u}_{\hat{k},\alpha}$ in $H^1_{\text{per}}(C_R)$ as $k \to \hat{k}$ uniformly with respect to $\{\alpha \in [-1/2, 1/2] : |\alpha - \hat{\alpha}_j| \geqslant \delta \ \forall j \in J\}$.*

It remains to study the convergence of $\tilde{u}_{k,\alpha}$ in neighborhoods of the exceptional values $\hat{\alpha}_j$. To this end, we formulate the following result from abstract functional analysis.

Theorem 3.2. *Let H be a (complex) Hilbert space, $I_\varepsilon = (-\varepsilon_0, \varepsilon_0) \subset \mathbb{R}$ and $I_\alpha = (-\alpha_0, \alpha_0) \subset \mathbb{R}$ open intervals containing 0. Let $K(\varepsilon, \alpha) : H \to H$ and $f(\varepsilon, \alpha) \in H$, $(\varepsilon, \alpha) \in I_\varepsilon \times I_\alpha$, be families of compact operators and elements, respectively, such that $(\varepsilon, \alpha) \mapsto K(\varepsilon, \alpha)$ is twice continuously differentiable on $I_\varepsilon \times I_\alpha$ and $(\varepsilon, \alpha) \mapsto f(\varepsilon, \alpha)$ is Lipschitz continuous on $I_\varepsilon \times I_\alpha$. Set $L(\varepsilon, \alpha) = I - K(\varepsilon, \alpha)$ and assume the following:*

- *(a) The null space $\mathcal{N} := \mathcal{N}\big(L(0,0)\big)$ is not trivial and the Riesz number of $L(0,0)$ is one; that is, the algebraic and geometric multiplicities of the eigenvalue 1 of $K(0,0)$ coincide; that is, $\mathcal{N}\big(L(0,0)^2\big) = \mathcal{N}\big(L(0,0)\big)$. Let $P : H \to \mathcal{N} \subset H$ be the projection operator onto $\mathcal{N}$ corresponding to the direct decomposition $H = \mathcal{N} \oplus \mathcal{R}\big(L(0,0)\big)$,*
- *(b) $L(\varepsilon, \alpha)$ is one-to-one; that is, also onto, for all $(\varepsilon, \alpha) \in I_\varepsilon \times I_\alpha$, $(\varepsilon, \alpha) \neq (0,0)$,*
- *(c) $A := \frac{1}{i} P \frac{\partial}{\partial \varepsilon} K(0,0)|_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$ is selfadjoint and positive definite and $B := P \frac{\partial}{\partial \alpha} K(0,0)|_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$ is selfadjoint and one-to-one.*

Let $u(\varepsilon, \alpha) \in H$ be the unique solution of $L(\varepsilon, \alpha) u(\varepsilon, \alpha) = f(\varepsilon, \alpha)$ for all $(\varepsilon, \alpha) \in I_\varepsilon \times I_\alpha$, $\varepsilon > 0$. Then there exists $\varepsilon_1 \in (0, \varepsilon_0)$ and $\delta \in (0, \alpha_0)$ such that $u(\varepsilon, \alpha)$ has the form

$$u(\varepsilon, \alpha) \; = \; u^{(1)}(\varepsilon, \alpha) \; - \; \sum_{\ell=1}^{m} \frac{f_\ell}{\mathrm{i}\varepsilon - \lambda_\ell \, \alpha} \, \phi_\ell \quad \text{for } (\varepsilon, \alpha) \in (0, \varepsilon_1) \times (-\delta, \delta).$$

Here, $\|u^{(1)}(\varepsilon, \alpha)\|_H$ is uniformly bounded with respect to (ε, α), and $\{\lambda_\ell, \phi_\ell : \ell = 1, \ldots, m\}$ is an orthonormal eigensystem of the following generalized eigenvalue problem in the finite dimensional space $\mathcal{N}$ (where $m = \dim \mathcal{N}$):

$$-B\phi_\ell \; = \; \lambda_\ell A \phi_\ell \quad \text{in } \mathcal{N} \quad \text{with normalization} \quad \big(A\phi_\ell, \phi_{\ell'}\big)_H \; = \; \delta_{\ell,\ell'} \tag{18}$$

for $\ell, \ell' = 1, \ldots, m$. Finally, $f_\ell = \big(Pf(0,0), \phi_\ell\big)_H$ are the expansion coefficients of $A^{-1}Pf(0,0)$ with respect to the inner product $(A\cdot, \cdot)_H$.

For the proof we refer to [16].

We want to apply this theorem to the equation (17) and set $K(\varepsilon, \alpha) = K_{\hat{k}+\mathrm{i}\varepsilon, \hat{\alpha}_j + \alpha}$ and $f(\varepsilon, \alpha) = r_{\hat{k}+\mathrm{i}\varepsilon, \hat{\alpha}_j + \alpha}$ where $\hat{k} > 0$ is fixed and $\hat{\alpha}_j$, $j \in J$, is one of the exceptional values. In the following we assume always that assumptions 2.2 and 2.8 are satisfied. We have to show the assumptions of the previous theorem. Let $\tilde{X}_j = \mathcal{N}(I - K_{\hat{k}, \hat{\alpha}_j})$ denote the kernel of $I - K_{\hat{k}, \hat{\alpha}_j}$. By lemma 2.12 it is given by

$$\tilde X_j = \left\{ \tilde\phi_j|_{C_R} : \tilde\phi_j \in H^1_{\rm per}(C_\infty) \text{ satisfies } \begin{array}{l} \Delta\tilde\phi_j + 2{\rm i}\hat\alpha_j\,\partial\tilde\phi_j/\partial x_3 + (\hat k^2 n - \hat\alpha_j^2)\tilde\phi_j = 0 \text{ in } C_\infty, \\ \tilde\phi_j \text{ is evanescent} \end{array} \right\}$$

and recall that $\tilde X_j \subset H^1_{\rm per}(C_R)$ is finite dimensional. The space $H^1_{\rm per}(C_R)$ is again equipped with the inner product $(\cdot,\cdot)_*$ from (15). We set $m_j = \dim \tilde X_j$.

Lemma 3.3. *Fix $j \in J$.*

(a) The decomposition $H^1_{\rm per}(C_R) = \tilde X_j \oplus \mathcal{R}(I - K_{\hat k,\hat\alpha_j})$ is orthogonal with respect to $(\cdot,\cdot)_*$. Therefore, the projection operator $P_j : H^1_{\rm per}(C_R) \to \tilde X_j$ is the orthogonal projection.
(b) The Riesz number of $I - K_{\hat k,\hat\alpha_j}$ is one.
(c) $\frac{1}{i}\, P_j \frac{\partial}{\partial\varepsilon} K(0,0)\big|_{\tilde X_j} = P_j \frac{\partial}{\partial k} K_{\hat k,\hat\alpha_j}\Big|_{\tilde X_j} : \tilde X_j \to \tilde X_j$ is selfadjoint and positive definite and $P_j \frac{\partial}{\partial\alpha} K_{\hat k,\hat\alpha_j}\Big|_{\tilde X_j} : \tilde X_j \to \tilde X_j$ is selfadjoint.
(d) The eigenvalue problem (18) takes the form

$$\int_{C_\infty} \left[-{\rm i}\frac{\partial\tilde\phi_{\ell,j}}{\partial x_3} + \hat\alpha_j\,\tilde\phi_{\ell,j} \right] \overline{\psi}\,{\rm d}x \;=\; \lambda_{\ell,j}\,\hat k \int_{C_\infty} n\,\tilde\phi_{\ell,j}\,\overline{\psi}\,{\rm d}x \quad \text{for all } \psi \in \tilde X_j \tag{19}$$

and $\ell = 1,\dots,m_j$. The normalization takes the form

$$2\hat k \int_{C_\infty} n\,\tilde\phi_{\ell,j}\,\overline{\tilde\phi_{\ell',j}}\,{\rm d}x \;=\; \delta_{\ell,\ell'}.$$

Here, $\tilde\phi_{\ell,j} \in \tilde X_j$ is identified with its extension in C_∞.

Proof. Since $j \in J$ is fixed we drop j from the notation.

(a) First we note that for $v \in \tilde X$ and $\psi \in H^1_{\rm per}(C_R)$ the dual form takes the form (see proof of lemma 2.9)

$$\langle \Lambda_{\hat k,\hat\alpha} v,\psi\rangle = R \sum_{|\ell+\hat\alpha|>\hat k}\sum_{m\in\mathbb{Z}} v_{\ell,m}\,\overline{\psi_{\ell,m}}\,\frac{k_\ell H^{(1)\prime}_m(k_\ell R)}{H^{(1)}_m(k_\ell R)} \;=\; R \sum_{|\ell+\hat\alpha|>\hat k}\sum_{m\in\mathbb{Z}} v_{\ell,m}\,\overline{\psi_{\ell,m}}\,\frac{|k_\ell| K_m'(|k_\ell| R)}{K_m(|k_\ell| R)}$$
$$= \langle v, \Lambda_{\hat k,\hat\alpha}\psi\rangle$$

because the ratio is real valued. Again, K_m are the modified Bessel functions and $k_\ell = k_\ell(\hat k,\hat\alpha) = \sqrt{\hat k^2 - (\ell+\hat\alpha)^2}$. Therefore, for $v \in \tilde X$ and $\psi \in H^1_{\rm per}(C_R)$ we have by partial integration

$$a_{\hat k,\hat\alpha}(v,\psi) = -\int_{C_R} \left[2{\rm i}\hat\alpha\, v\,\frac{\partial\overline{\psi}}{\partial x_3} + (\hat\alpha^2 - \hat k^2 n - 1)\, v\,\overline{\psi} \right] {\rm d}x \;-\; \langle \Lambda_{\hat k,\hat\alpha} v,\psi\rangle$$
$$= -\int_{C_R} \left[-2{\rm i}\hat\alpha\,\overline{\psi}\,\frac{\partial v}{\partial x_3} + (\hat\alpha^2 - \hat k^2 n - 1)\, v\,\overline{\psi} \right] {\rm d}x \;-\; \langle v, \Lambda_{\hat k,\hat\alpha}\psi\rangle \;=\; \overline{a_{\hat k,\hat\alpha}(\psi,v)}.$$

From this we conclude for $v \in \mathcal{N}(I - K_{\hat k,\hat\alpha})$ and $\psi \in H^1_{\rm per}(C_R)$ that

$$\big((I - K_{\hat k,\hat\alpha})\psi, v\big)_* \;=\; (\psi,v)_* - a_{\hat k,\hat\alpha}(\psi,v) \;=\; \overline{(v,\psi)_* - a_{\hat k,\hat\alpha}(v,\psi)} \;=\; \overline{\big((I - K_{\hat k,\hat\alpha})v,\psi\big)_*} \;=\; 0.$$

(b) Let $u \in \mathcal{N}\big((I-K_{\hat{k},\hat{\alpha}})^2\big)$ and set $v=(I-K_{\hat{k},\hat{\alpha}})u$. Then $v \in \mathcal{N}(I-K_{\hat{k},\hat{\alpha}})$. Therefore

$$\begin{aligned}\|v\|_*^2 = \big(v,(I-K_{\hat{k},\hat{\alpha}})u\big)_* \;&=\; \overline{\big((I-K_{\hat{k},\hat{\alpha}})u,v\big)_*} \;=\; \overline{a_{\hat{k},\hat{\alpha}}(u,v)}\\ = a_{\hat{k},\hat{\alpha}}(v,u) \;&=\; \big((I-K_{\hat{k},\hat{\alpha}})v,u\big)_* \;=\; 0\end{aligned}$$

because $(I-K_{\hat{k},\hat{\alpha}})v=0$. This implies $v=0$; that is, $u \in \mathcal{N}(I-K_{\hat{k},\hat{\alpha}})$.

(c) We note that $a_{\hat{k},\hat{\alpha}}$ has the following form on $\tilde{X}\times\tilde{X}$

$$a_{\hat{k},\hat{\alpha}}(v,\psi) \;=\; -\int_{C_\infty}\left[2\mathrm{i}\hat{\alpha}\,v\,\frac{\partial\overline{\psi}}{\partial x_3}+(\hat{\alpha}^2-\hat{k}^2n-1)\,v\,\overline{\psi}\right]\mathrm{d}x,\quad v,\psi\in\tilde{X},$$

because v and ψ decay exponentially as $x_1^2+x_2^2$ tends to infinity. On the right hand side $v,\psi \in \tilde{X}$ are again identified with their extensions in C_∞. Therefore,

$$\frac{\partial}{\partial k}a_{\hat{k},\hat{\alpha}}(v,\psi) = 2\hat{k}\int_{C_\infty} n\,v\,\overline{\psi}\,\mathrm{d}x,\quad v,\psi\in\tilde{X},$$

$$\frac{\partial}{\partial \alpha}a_{\hat{k},\hat{\alpha}}(v,\psi) = -2\int_{C_\infty}\left[\mathrm{i}\,v\,\frac{\partial\overline{\psi}}{\partial x_3}+\hat{\alpha}\,v\,\overline{\psi}\right]\mathrm{d}x,\quad v,\psi\in\tilde{X}.$$

This proves part (c) because $\tilde{X}$ is finite dimensional.

(d) This is obvious of the forms of $\frac{\partial}{\partial k}a_{\hat{k},\hat{\alpha}}$ and $\frac{\partial}{\partial \alpha}a_{\hat{k},\hat{\alpha}}$ of part (c). □

Therefore, all of the assumptions of theorem 3.2 are satisfied if $\lambda_{\ell,j}\neq 0$ for all $\ell=1,\ldots,m_j$ and $j\in J$.

For a more convenient formulation we translate the spaces $\tilde{X}_j$ into the (isomorphic) spaces $\hat{X}_j$ of $\hat{\alpha}_j$-quasi-periodic solutions of the homogeneous Helmholtz equation; that is, we replace the periodic function $\tilde{\phi}_j$ by $\hat{\phi}_j(x)=\mathrm{e}^{\mathrm{i}\hat{\alpha}_j x_3}\tilde{\phi}(x)$. Then $\tilde{\phi}_j\in\tilde{X}_j$ if, and only if, $\hat{\phi}_j\in\hat{X}_j$ where

$$\hat{X}_j \;=\; \big\{\hat{\phi}_j|_{C_R} : \hat{\phi}_j \in H^1_{\hat{\alpha}_j}(C_\infty) \text{ satisfies } \Delta\hat{\phi}_j+\hat{k}^2n\hat{\phi}_j=0 \text{ in } C_\infty,\ \hat{\phi}_j \text{ is evanescent}\big\}. \tag{20}$$

Here, $H^1_{\hat{\alpha}_j}(C_\infty)$ denotes the space of $\hat{\alpha}_j$-quasi-periodic functions (wrt x_3); that is, the subspace of $H^1(C_\infty)$ consisting of functions $\hat{\phi}$ such that $\hat{\phi}(x_1,x_2,2\pi)=\mathrm{e}^{\mathrm{i}\hat{\alpha}_j 2\pi}\hat{\phi}(x_1,x_2,0)$ for all x_1,x_2. The eigenvalue problem (19) is equivalent to

$$-\mathrm{i}\int_{C_\infty}\frac{\partial\hat{\phi}_{\ell,j}}{\partial x_3}\,\overline{\psi}\,\mathrm{d}x \;=\; \lambda_{\ell,j}\,\hat{k}\int_{C_\infty} n\,\hat{\phi}_{\ell,j}\,\overline{\psi}\,\mathrm{d}x\quad\text{for all } \psi\in\hat{X}_j \text{ and } \ell=1,\ldots,m_j. \tag{21}$$

Again, $\hat{\phi}_{\ell,j}$ are normalized by

$$2\hat{k}\int_{C_\infty} n\,\hat{\phi}_{\ell,j}\,\overline{\hat{\phi}_{\ell',j}}\,\mathrm{d}x \;=\; \delta_{\ell,\ell'}.$$

Therefore, all of the assumptions of theorem 3.2 are satisfied if $\hat{k}>0$ is regular in the following sense.

Definition 3.4. $\hat{k}>0$ is called *regular*, if $\lambda_{\ell,j}\neq 0$ for all $\ell=1,\ldots,m_j$ and $j\in J$ where $\lambda_{\ell,j}\in\mathbb{R}$, $\ell=1,\ldots,m_j$, are the eigenvalues of the selfadjoint eigenvalue problem (21) in the finite dimensional space $\hat{X}_j$.

We fix $j\in J$ and consider α in a neighborhood of an exceptional point $\hat{\alpha}_j$. Application of theorem 3.2 to the equation (17) for $k=\hat{k}+\mathrm{i}\varepsilon$; that is, writing $\tilde{u}_{\varepsilon,\alpha}$, $K_{\varepsilon,\alpha}$, and $r_{\varepsilon,\alpha}$ for $\tilde{u}_{\hat{k}+\mathrm{i}\varepsilon,\alpha}$, $K_{\hat{k}+\mathrm{i}\varepsilon,\alpha}$, and $r_{\hat{k}+\mathrm{i}\varepsilon,\alpha}$, respectively,

$$\tilde{u}_{\varepsilon,\alpha} \;-\; K_{\varepsilon,\alpha}\tilde{u}_{\varepsilon,\alpha} \;=\; r_{\varepsilon,\alpha} \quad \text{in } H^1_{\text{per}}(C_R),$$

yields the decomposition

$$\tilde{u}_{\varepsilon,\alpha} \;=\; \tilde{u}^{(1)}_{\varepsilon,\alpha} \;+\; \tilde{u}^{(2)}_{\varepsilon,\alpha} \tag{22}$$

of $\tilde{u}_{\varepsilon,\alpha}$ for α in a neighborhood $\{\alpha : |\alpha-\hat{\alpha}_j|<\delta\}$ of an exceptional point $\hat{\alpha}_j$ where $\tilde{u}^{(1)}_{\varepsilon,\alpha} \in H^1_{\text{per}}(C_R)$ is bounded uniformly in (ε,α) and

$$\tilde{u}^{(2)}_{\varepsilon,\alpha} \;=\; -\sum_{\ell=1}^{m_j} \frac{f_{\ell,j}}{\mathrm{i}\varepsilon - \lambda_{\ell,j}(\alpha-\hat{\alpha}_j)}\,\tilde{\phi}_{\ell,j} \quad \text{for } (\varepsilon,\alpha)\in(0,\varepsilon_1)\times(\hat{\alpha}_j-\delta,\hat{\alpha}_j+\delta).$$

Here, $\lambda_{\ell,j}$ denote the eigenvalues of (19) with corresponding eigenfunctions $\tilde{\phi}_{\ell,j}\in\tilde{X}_j$. Furthermore, $f_{\ell,j}=\big(P_j r_{0,\hat{\alpha}_j},\tilde{\phi}_{\ell,j}\big)_* = \big(r_{0,\hat{\alpha}_j},\tilde{\phi}_{\ell,j}\big)_* = \int_{C_R}\tilde{f}_{\hat{k},\hat{\alpha}_j}\,\tilde{\phi}_{\ell,j}\,\mathrm{d}x$ by the definition of $r_{\hat{k},\hat{\alpha}_j}$ and part (a) of lemma 3.3. We note that $\tilde{u}^{(2)}_{\varepsilon,\alpha}\in\tilde{X}_j$ has an extension to all of $\mathbb{R}^3$.

We set $\tilde{u}^{(2)}_{\varepsilon,\alpha}=0$ and $\tilde{u}^{(1)}_{\varepsilon,\alpha}=\tilde{u}_{\varepsilon,\alpha}$ for $\alpha\in\mathcal{I}_\delta := \{\alpha\in[-1/2,1/2] : |\alpha-\hat{\alpha}_j|\geqslant\delta \ \forall j\in J\}$. Then we have the decomposition $\tilde{u}_{\varepsilon,\alpha}=\tilde{u}^{(1)}_{\varepsilon,\alpha}+\tilde{u}^{(2)}_{\varepsilon,\alpha}$ for all $\alpha\in[-1/2,1/2]$ and $\varepsilon\in(0,\varepsilon_1)$.

We need to investigate convergence of the inverse Floquet–Bloch transformations

$$\begin{aligned}
u_\varepsilon(x) &= \int_{-1/2}^{1/2}\tilde{u}^{(1)}_{\varepsilon,\alpha}(x)\,\mathrm{e}^{\mathrm{i}\alpha x_3}\mathrm{d}\alpha \;+\; \sum_{j\in J}\int_{\hat{\alpha}_j-\delta}^{\hat{\alpha}_j+\delta}\tilde{u}^{(2)}_{\varepsilon,\alpha}(x)\,\mathrm{e}^{\mathrm{i}\alpha x_3}\mathrm{d}\alpha\\
&= \int_{-1/2}^{1/2}\tilde{u}^{(1)}_{\varepsilon,\alpha}(x)\,\mathrm{e}^{\mathrm{i}\alpha x_3}\mathrm{d}\alpha \;-\; \sum_{j\in J}\sum_{\ell=1}^{m_j} f_{\ell,j}\,\tilde{\phi}_{\ell,j}(x)\int_{\hat{\alpha}_j-\delta}^{\hat{\alpha}_j+\delta}\frac{1}{\mathrm{i}\varepsilon-\lambda_{\ell,j}\,(\alpha-\hat{\alpha}_j)}\mathrm{e}^{\mathrm{i}\alpha x_3}\mathrm{d}\alpha\\
&= \int_{-1/2}^{1/2}\tilde{u}^{(1)}_{\varepsilon,\alpha}(x)\,\mathrm{e}^{\mathrm{i}\alpha x_3}\mathrm{d}\alpha \;-\; \sum_{j\in J}\sum_{\ell=1}^{m_j} f_{\ell,j}\,\hat{\phi}_{\ell,j}(x)\int_{-\delta}^{\delta}\frac{1}{\mathrm{i}\varepsilon-\lambda_{\ell,j}\,\alpha}\mathrm{e}^{\mathrm{i}\alpha x_3}\mathrm{d}\alpha\\
&= \int_{-1/2}^{1/2}\tilde{u}^{(1)}_{\varepsilon,\alpha}(x)\,\mathrm{e}^{\mathrm{i}\alpha x_3}\mathrm{d}\alpha \;+\; \sum_{j\in J}\sum_{\ell=1}^{m_j} f_{\ell,j}\,\hat{\phi}_{\ell,j}(x)\int_{\delta<|\alpha|<1/2}\frac{1}{\mathrm{i}\varepsilon-\lambda_{\ell,j}\,\alpha}\,\mathrm{e}^{\mathrm{i}\alpha x_3}\mathrm{d}\alpha\\
&\quad -\sum_{j\in J}\sum_{\ell=1}^{m_j} f_{\ell,j}\,\hat{\phi}_{\ell,j}(x)\int_{-1/2}^{1/2}\frac{1}{\mathrm{i}\varepsilon-\lambda_{\ell,j}\,\alpha}\,\mathrm{e}^{\mathrm{i}\alpha x_3}\mathrm{d}\alpha\\
&= u^{(1)}_\varepsilon(x) \;+\; u^{(2)}_\varepsilon(x)
\end{aligned}$$

where

$$u^{(2)}_\varepsilon(x) \;=\; -\sum_{j\in J}\sum_{\ell=1}^{m_j} f_{\ell,j}\,\hat{\phi}_{\ell,j}(x)\int_{-1/2}^{1/2}\frac{1}{\mathrm{i}\varepsilon-\lambda_{\ell,j}\alpha}\mathrm{e}^{\mathrm{i}\alpha x_3}\mathrm{d}\alpha \tag{23}$$

and $u^{(1)}_\varepsilon = u_\varepsilon - u^{(2)}_\varepsilon$. Note that we switched from $\tilde{\phi}_{\ell,j}(x)$ to $\hat{\phi}_{\ell,j}(x)=\mathrm{e}^{\mathrm{i}\hat{\alpha}_j x_3}\tilde{\phi}_{\ell,j}(x)$.

The functions $\tilde{u}^{(1)}_{\varepsilon,\alpha}$ converge to $\tilde{u}^{(1)}_{0,\alpha}$ in $H^1_{\text{per}}(C_R)$ as ε tends to zero for every $\alpha\notin\{\hat{\alpha}_j : j\in J\}$ because $I-K_{\hat{k},\alpha}$ is an isomorphism for all such α. Furthermore, $\|\tilde{u}^{(1)}_{\varepsilon,\alpha}-\tilde{u}^{(1)}_{0,\alpha}\|_{H^1_{\text{per}}(C_R)}$ is uniformly bounded for $\alpha\in\mathcal{I}_\delta$ and also for $\alpha\in\bigcup_{j\in J}(\hat{\alpha}_j-\delta,\hat{\alpha}_j+\delta)$ by theorem 3.2. Therefore, Lebesgue's theorem on dominated convergence yields $\int_{-1/2}^{1/2}\|\tilde{u}^{(1)}_{\varepsilon,\alpha}-\tilde{u}^{(1)}_{0,\alpha}\|^2_{H^1_{\text{per}}(C_R)}\mathrm{d}\alpha\to 0$ as $\varepsilon\to 0$. The boundedness of the inverse Floquet–Bloch transform yields convergence $u^{(1)}_\varepsilon$ to $u^{(1)}_0$ in $H^1(T_R)$ as $\varepsilon\to 0$. Therefore, it remains to study the convergence of $u^{(2)}_\varepsilon$.

Lemma 3.5. *Let $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Then*

$$\lim_{\varepsilon\to 0}\int_{-1/2}^{1/2}\frac{1}{\mathrm{i}\varepsilon-\lambda\,\alpha}\,\mathrm{e}^{\mathrm{i}\alpha x_3}\mathrm{d}\alpha \;=\; -\frac{\mathrm{i}\pi}{|\lambda|}\left[1+\mathrm{sign}(\lambda)\,\frac{2}{\pi}\int_0^{x_3/2}\frac{\sin t}{t}\,\mathrm{d}t\right] \tag{24}$$

uniformly with respect to $|x_3| \leqslant a$ for every $a > 0$. Also, the derivative of the integral with respect to x_3 converges uniformly for $|x_3| \leqslant a$ for every $a > 0$.

Proof. We compute

$$\begin{aligned}\int_{-1/2}^{1/2}\frac{1}{\mathrm{i}\varepsilon-\lambda\,\alpha}\,\mathrm{e}^{\mathrm{i}\alpha x_3}\mathrm{d}\alpha &= \int_{-1/2}^{1/2}\frac{-\mathrm{i}\varepsilon-\lambda\alpha}{\varepsilon^2+\lambda^2\alpha^2}\,\mathrm{e}^{\mathrm{i}\alpha x_3}\mathrm{d}\alpha\\ &= -\mathrm{i}\varepsilon\int_{-1/2}^{1/2}\frac{\cos(\alpha x_3)}{\varepsilon^2+\lambda^2\alpha^2}\,\mathrm{d}\alpha \;-\; \mathrm{i}\lambda\int_{-1/2}^{1/2}\frac{\alpha\sin(\alpha x_3)}{\varepsilon^2+\lambda^2\alpha^2}\,\mathrm{d}\alpha.\end{aligned}$$

In the first integral we substitute $\alpha = t\varepsilon/|\lambda|$ and in the second integral $t = \alpha x_3$. This yields

$$\int_{-1/2}^{1/2}\frac{1}{\mathrm{i}\varepsilon-\lambda\,\alpha}\,\mathrm{e}^{\mathrm{i}\alpha x_3}\mathrm{d}\alpha \;=\; \frac{-\mathrm{i}}{|\lambda|}\int_{-|\lambda|/(2\varepsilon)}^{|\lambda|/(2\varepsilon)}\frac{\cos(t\varepsilon x_3/|\lambda|)}{1+t^2}\,\mathrm{d}t \;-\; \mathrm{i}\lambda\int_{-x_3/2}^{x_3/2}\frac{t\sin t}{x_3^2\varepsilon^2+\lambda^2t^2}\,\mathrm{d}t.$$

For $\varepsilon \to 0$ the expression on the right converges to

$$-\frac{\mathrm{i}}{|\lambda|}\int_{-\infty}^{\infty}\frac{1}{1+t^2}\,\mathrm{d}t \;-\; \frac{2\mathrm{i}}{\lambda}\int_0^{x_3/2}\frac{\sin t}{t}\,\mathrm{d}t \;=\; -\frac{\mathrm{i}\pi}{|\lambda|}\left[1+\mathrm{sign}(\lambda)\,\frac{2}{\pi}\int_0^{x_3/2}\frac{\sin t}{t}\,\mathrm{d}t\right]$$

uniformly with respect to $|x_3| \leqslant a$, for arbitrary $a > 0$. The derivative of the integral with respect to x_3 converges uniformly for $|x_3| \leqslant a$ for every $a > 0$ as well. □

Remark 3.6. As $\lim_{a\to\infty}\int_0^a \sin(t)/t\,\mathrm{d}t = \pi/2$ we observe that $\psi^\pm \in C^\infty(\mathbb{R})$, defined by

$$\psi^\pm(x_3) \;=\; \frac{1}{2}\left[1 \;\pm\; \frac{2}{\pi}\int_0^{x_3/2}\frac{\sin t}{t}\,\mathrm{d}t\right], \quad x_3 \in \mathbb{R}, \tag{25}$$

tends to 1 as $x_3 \to \pm\infty$ while it converges to 0 for $x_3 \to \mp\infty$. Thus, as ε tends to zero, $u_\varepsilon^{(2)}$ from (23) converges to

$$u_0^{(2)}(x) \;=\; \psi^+(x_3)\sum_{j\in J}u_j^+(x) \;+\; \psi^-(x_3)\sum_{j\in J}u_j^-(x), \quad x \in T_R, \tag{26}$$

where

$$u_j^\pm(x) \;=\; 2\pi\mathrm{i}\sum_{\lambda_{\ell,j}\gtrless 0}\frac{f_{\ell,j}}{|\lambda_{\ell,j}|}\,\hat{\phi}_{\ell,j}(x), \quad x \in \mathbb{R}^3. \tag{27}$$

This separates $u_0^{(2)}$ into groups of modes propagating to the left and the right.

Setting $a_{\ell,j} = 2\pi\mathrm{i}\,\frac{f_{\ell,j}}{|\lambda_{\ell,j}|}$ in (27) yields the following main result.

Theorem 3.7 (The limiting absorption principle). *Let assumptions 2.2 and 2.8 hold and let $\hat{k} > 0$ be regular in the sense of definition 3.4. Then the solution $u_{\hat{k}+\mathrm{i}\varepsilon}$ of (3) for $k = \hat{k} + \mathrm{i}\varepsilon$ has a decomposition in the form $u_{\hat{k}+\mathrm{i}\varepsilon} = u_\varepsilon^{(1)} + u_\varepsilon^{(2)}$ where $u_\varepsilon^{(1)} \in H^1_{loc}(\mathbb{R}^3)$ and $u_\varepsilon^{(2)} \in W^{2,\infty}(\mathbb{R}^3)$ is given by (23). Furthermore, for every $R > R_0$ we have that $u_\varepsilon^{(1)} \in H^1(T_R)$ converges in $H^1(T_R)$ to some $u^{(1)} \in H^1(T_R)$ and $u_\varepsilon^{(2)} \in W^{2,\infty}(\mathbb{R}^3)$ converges for every $a > 0$ in $W^{2,\infty}\big(\mathbb{R}^2 \times (-a, a)\big)$ to $u^{(2)} \in W^{2,\infty}(\mathbb{R}^3)$ which has the form*

$$u^{(2)}(x) \;=\; \psi^+(x_3) \sum_{j\in J} \sum_{\lambda_{\ell,j}>0} a_{\ell,j}\, \hat{\phi}_{\ell,j}(x) \;+\; \psi^-(x_3) \sum_{j\in J} \sum_{\lambda_{\ell,j}<0} a_{\ell,j}\, \hat{\phi}_{\ell,j}(x), \quad x \in \mathbb{R}^3, \tag{28}$$

for $a_{\ell,j} \in \mathbb{C}$ given by

$$a_{\ell,j} \;=\; \frac{2\pi\mathrm{i}}{|\lambda_{\ell,j}|} \int_{C_\infty} (Ff)(x, \hat{\alpha}_j)\, \overline{\hat{\phi}_{\ell,j}(x)}\, \mathrm{d}x \;=\; \frac{2\pi\mathrm{i}}{|\lambda_{\ell,j}|} \int_{\mathbb{R}^3} f(x)\, \overline{\hat{\phi}_{\ell,j}(x)}\, \mathrm{d}x, \tag{29}$$

$\ell = 1, \ldots, m_j$, $j \in J$. Here, the functions $\psi^\pm$ are defined in (25), and $\hat{\phi}_{\ell,j}$ are the $\hat{\alpha}_j$-quasi-periodic solutions of $\Delta\hat{\phi}_{\ell,j} + k^2 n \hat{\phi}_{\ell,j} = 0$ in $\mathbb{R}^3$, given by the eigenvalue problem (21). The function $u = u^{(1)} + u^{(2)} \in H^1_{loc}(\mathbb{R}^3)$ is a solution of the source problem $\Delta u + \hat{k}^2 n u = -f$ in $\mathbb{R}^3$.

Proof. Only the second equality in (29) has to be shown. But this follows directly from

$$\begin{aligned}\int_{C_\infty} (Ff)(x, \hat{\alpha}_j)\, \overline{\hat{\phi}_{\ell,j}(x)}\, \mathrm{d}x &= \sum_{m\in\mathbb{Z}} \int_{C_\infty} f(x + 2\pi m e^{(3)})\, \mathrm{e}^{-2\pi\mathrm{i} m\hat{\alpha}_j}\, \overline{\hat{\phi}_{\ell,j}(x)}\, \mathrm{d}x \\ &= \sum_{m\in\mathbb{Z}} \int_{C_\infty} f(x + 2\pi m e^{(3)})\, \overline{\hat{\phi}_{\ell,j}(x + 2\pi m e^{(3)})}\, \mathrm{d}x \;=\; \int_{\mathbb{R}^3} f(x)\, \overline{\hat{\phi}_{\ell,j}(x)}\, \mathrm{d}x.\end{aligned}$$

□

This result holds for any $R > R_0$. Therefore, the solution $u = u^{(1)} + u^{(2)}$ is defined in all of $\mathbb{R}^3$ and a solution of the differential equation (3) for $k = \hat{k}$.

Remarks 3.8.

(a) We note that we can replace the functions $\psi^\pm$ by any functions with $\psi^+(x_3) = 1 + \mathcal{O}(1/x_3)$ as $x_3 \to \infty$ and $\psi^+(x_3) = \mathcal{O}(1/|x_3|)$ as $x_3 \to -\infty$ and $\frac{\mathrm{d}}{\mathrm{d}x_3}\psi^+(x_3) = \mathcal{O}(1/|x_3|)$ as $x_3 \to \pm\infty$ (and ψ^- analogously) because the difference of this choice of $\psi^\pm$ and the one of (25) differ only by a H^1-function. In particular, one can choose ψ^+ such that $\psi^+(x_3) = 0$ for $x_3 \leqslant -\tau$ and $\psi^+(x_3) = 1$ for $x_3 \geqslant \tau$ for some $\tau > 0$ (and ψ^- analogously) or

$$\psi^+(x_3) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_3} \mathrm{e}^{-t^2/2} \mathrm{d}t,\ x_3 \in \mathbb{R}, \quad \psi^- \;=\; 1 \;-\; \psi^+.$$

(b) The representation (28) (with (29)) corresponds to the asymptotic formulas for closed waveguides T_R; that is, with boundary condition $u = 0$ on ∂T_R see, e.g. theorem 7 in [10].

(c) From part (b) of lemma 2.9 and the eigenvalue problem [21) we note that we can assume that $\hat{\phi}_{\ell,-j} = \overline{\hat{\phi}_{\ell,j}}$ for all ℓ and $j \in J$ and thus $\lambda_{\ell,-j} = -\lambda_{\ell,j}$. Therefore, there exist as many propagating modes propagating upwards as ones propagating downwards.

(d) In the special case that $\hat{X}_j$ is one-dimensional with basis $\{\hat{\phi}_j\}$ such that $2\hat{k}\int_{C_\infty} n|\hat{\phi}_j|^2 \mathrm{d}x = 1$ the wave number is regular if $\lambda_j = -2\mathrm{i}\int_{C_\infty} \frac{\partial\hat{\phi}_j}{\partial x_3}\overline{\hat{\phi}_j}\mathrm{d}x \neq 0$. In particular $\frac{\lambda_j}{2} = \mathrm{Im}\int_{C_\infty} \frac{\partial\hat{\phi}_j}{\partial x_3}\overline{\hat{\phi}_j}\mathrm{d}x$ corresponds to the group velocity of this eigenfunction and measures its energy flux in vertical direction: In the the case $\lambda_j > 0$ the energy of the wave is traveling upwards, in the case $\lambda_j < 0$ the energy of the wave is traveling downwards (see [18]).

As a corollary we apply this result to the special case of the scattering of a point source $u^{\mathrm{inc}}(x) = \Phi_k(x,y)$ by the periodic waveguide. In this case the total field $u^t = u^{\mathrm{inc}} + u^s$ is the Green's function of the differential operator $\Delta + k^2 n$ and is given by $u^t = u + (1-\chi_y)u^{\mathrm{inc}}$ where $\chi_y(x) = \chi(x-y)$ and $\chi \in C^\infty(\mathbb{R}^3)$ is such that $\chi(x) = 0$ for $|x| \leqslant \epsilon/2$ and $\chi(x) = 1$ for $|x| \geqslant \epsilon$ for some $\epsilon > 0$ and u solves (3) with $f := (\Delta + k^2 n)\left[(1-\chi_y)\, u^{\mathrm{inc}}\right]$. In this case we can compute the coefficients $a_{\ell,j}$ explicitly.

Corollary 3.9. *Let assumptions 2.2 and 2.8 hold and let $\hat{k} > 0$ be regular in the sense of definition 3.4. Then the Green's function $G(x,y)$ has the form*

$$G(x,y) = \Phi_k(x,y) + G^{(1)}(x,y) + 2\pi\mathrm{i}\,\psi^+(x_3)\sum_{j\in J}\sum_{\lambda_{\ell,j}>0}\frac{1}{|\lambda_{\ell,j}|}\hat{\phi}_{\ell,j}(x)\,\overline{\hat{\phi}_{\ell,j}(y)}$$
$$+ 2\pi\mathrm{i}\,\psi^-(x_3)\sum_{j\in J}\sum_{\lambda_{\ell,j}<0}\frac{1}{|\lambda_{\ell,j}|}\hat{\phi}_{\ell,j}(x)\,\overline{\hat{\phi}_{\ell,j}(y)}, \qquad x,y\in\mathbb{R}^3,\ x\neq y, \tag{30}$$

where $\psi^\pm$ are as in theorem 3.7 and $G^{(1)}(\cdot,y) \in H^1(T_R)$ for all $R > 0$ and $y \in \mathbb{R}^3$.

Proof. We have to compute $a_{\ell,j}$ from (29) for the special form $f := (\Delta + k^2 n)\left[(1-\chi_y)\,\Phi_k(\cdot,y)\right]$. By Green's theorem we have for any $\delta \in (0,\varepsilon/2)$ (note that f has compact support and $\chi_y(x) = \chi(x-y) = 0$ for $|x-y| = \delta$)

$$\begin{aligned}
\int_{|x-y|>\delta} f(x)\,\overline{\hat{\phi}_{\ell,j}(x)}\,\mathrm{d}x &= \int_{|x-y|>\delta}(\Delta_x + k^2 n)\left[(1-\chi_y)\,\Phi_k(\cdot,y)\right]\overline{\hat{\phi}_{\ell,j}}\,\mathrm{d}x \\
&= -\int_{|x-y|=\delta}\left[\frac{\partial\Phi_k(x,y)}{\partial\nu(x)}\,\overline{\hat{\phi}_{\ell,j}(x)} - \frac{\partial\overline{\hat{\phi}_{\ell,j}(x)}}{\partial\nu}\,\Phi_k(x,y)\right]\mathrm{d}s(x) \\
&= \int_{|x-y|<\delta}\Phi_k(x,y)\left[\Delta\overline{\hat{\phi}_{\ell,j}}(x) + k^2\,\overline{\hat{\phi}_{\ell,j}}(x)\right]\mathrm{d}x + \overline{\hat{\phi}_{\ell,j}(y)} \\
&= k^2\int_{|x-y|<\delta}\Phi_k(x,y)\left(1-n(x)\right)\overline{\hat{\phi}_{\ell,j}}(x)\,\mathrm{d}x + \overline{\hat{\phi}_{\ell,j}(y)}
\end{aligned}$$

and this converges to $\overline{\hat{\phi}_{\ell,j}(y)}$ as δ tends to zero. Therefore, $a_{\ell,j} = \frac{2\pi\mathrm{i}}{|\lambda_{\ell,j}|}\,\overline{\hat{\phi}_{\ell,j}(y)}$. The assertion follows from the decomposition

$$G(x,y) = u(x) + \left(1-\chi(x-y)\right)\Phi_k(x,y) = \Phi_k(x,y) + \underbrace{\left[u^{(1)}(x) - \chi(x-y)\,\Phi_k(x,y)\right]}_{=:\,G^{(1)}(x,y)} + u^{(2)}(x)$$

and the form of $u^{(2)}$. □

4. The radiation condition

It is obvious that $u^{(1)}$ from theorem 3.7 is a solution of the following boundary value problem

$$\Delta u^{(1)} + \hat{k}^2 u^{(1)} = -h_0 \text{ in } \mathbb{R}^3 \setminus T_R, \quad u^{(1)} = g_0 \text{ on } \Gamma_R, \tag{31}$$

with $h_0 := \Delta u^{(2)} + \hat{k}^2 u^{(2)}$ in $\mathbb{R}^3 \setminus T_R$ and $g_0 := u^{(1)} = u - u^{(2)}$ on Γ_R. The solution of this boundary value problem is not unique without a condition away from the waveguide. We will derive such a radiation condition below. First we note that $h_0 \in L^2(\mathbb{R}^3 \setminus T_R)$. Indeed, from (26) we observe that for $x \notin T_R$ the right hand side h_0 is given by

$$\begin{aligned} h_0(x) &= \Delta u^{(2)}(x) + \hat{k}^2 u^{(2)}(x) \\ &= \sum_{j\in J}\left[2\,\frac{\mathrm{d}}{\mathrm{d}x_3}\psi^+(x_3)\,\frac{\partial u_j^+(x)}{\partial x_3} + u_j^+(x)\,\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}\psi^+(x_3)\right] \\ &\quad + \sum_{j\in J}\left[2\,\frac{\mathrm{d}}{\mathrm{d}x_3}\psi^-(x_3)\,\frac{\partial u_j^-(x)}{\partial x_3} + u_j^-(x)\,\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}\psi^-(x_3)\right] \\ &= \frac{1}{2\pi}\sum_{j\in J}\left[\frac{\sin(x_3/2)}{x_3/2}\left(\frac{\partial u_j^+(x)}{\partial x_3} - \frac{\partial u_j^-(x)}{\partial x_3}\right) + \frac{\mathrm{d}}{\mathrm{d}x_3}\,\frac{\sin(x_3/2)}{x_3/2}\,\big(u_j^+(x) - u_j^-(x)\big)\right], \end{aligned} \tag{32}$$

where $u_j^\pm$ are linear combinations of the evanescent modes $\hat{\phi}_{\ell,j}$, $\ell = 1,\dots,m_j$, see (27). From this we observe that not only $h_0 \in L^2(\mathbb{R}^3 \setminus T_R)$ but even $h_0 \in L^2_\sigma(\mathbb{R}^3 \setminus T_R)$ for any $\sigma \geqslant 0$ where $L^2_\sigma(\mathbb{R}^3 \setminus T_R) = \{h \in L^2(\mathbb{R}^3 \setminus T_R) : \tilde{h}_\sigma \in L^2(\mathbb{R}^3 \setminus T_R)\}$ and $\tilde{h}_\sigma(x) = (1 + x_1^2 + x_2^2)^{\sigma/2} h(x)$, $x \notin T_R$. This space is equipped with the canonical norm $\|h\|_{L^2_\sigma(\mathbb{R}^3\setminus T_R)} = \|\tilde{h}_\sigma\|_{L^2(\mathbb{R}^3\setminus T_R)}$.

In order to formulate a correct radiation condition for $u^{(1)}$ normal to the axis of the cylinder we need the cylindrical Fourier transform $\mathcal{F}g : \mathbb{Z}\times\mathbb{R} \to \mathbb{C}$ of g which is given by

$$(\mathcal{F}g)(m,\xi) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_0^{2\pi} g(\varphi, y_3)\,\mathrm{e}^{-\mathrm{i}(m\varphi+\xi y_3)}\mathrm{d}\varphi\,\mathrm{d}y_3, \quad m\in\mathbb{Z},\ \xi\in\mathbb{R}.$$

Here, $\varphi \in (0, 2\pi)$ and $y_3 \in \mathbb{R}$ are the cylindrical coordinates of $y \in \Gamma_R$. Then $\mathcal{F}$ is well defined and bounded from $L^2(\Gamma_R)$ into

$$L^2(\mathbb{Z}\times\mathbb{R}) := \left\{\hat{g} : \mathbb{Z}\times\mathbb{R}\to\mathbb{C} : \hat{g}(m,\cdot)\in L^2(\mathbb{R}) \text{ for all } m \text{ and } \sum_{m\in\mathbb{Z}}\int_{\mathbb{R}}\big|\hat{g}(m,\xi)\big|^2\mathrm{d}\xi < \infty\right\}.$$

The inverse transform is then

$$g(\varphi, x_3) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}}(\mathcal{F}g)(m,\xi)\,\mathrm{e}^{\mathrm{i}(m\varphi+\xi x_3)}\,\mathrm{d}\xi.$$

Also, Parseval's identity holds in the form

$$\int_{\Gamma_R}\big|g(x)\big|^2\,\mathrm{d}s = R\int_{\mathbb{R}}\int_0^{2\pi}\big|g(\phi, x_3)\big|^2\mathrm{d}\phi\,\mathrm{d}x_3 = R\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}}\big|(\mathcal{F}g)(m,\xi)\big|^2\mathrm{d}\xi. \tag{33}$$

The one-dimensional Fourier tranform $\mathcal{F}_1$ is related to the Floquet–Bloch transform F by

$$(\mathcal{F}_1 f)(\xi+\ell) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(s)\,\mathrm{e}^{-\mathrm{i}s(\xi+\ell)}\,\mathrm{d}s \;=\; \frac{1}{\sqrt{2\pi}}\sum_{m\in\mathbb{Z}}\int_{2\pi m}^{2\pi(m+1)} f(s)\,\mathrm{e}^{-\mathrm{i}s(\xi+\ell)}\,\mathrm{d}s$$
$$= \frac{1}{\sqrt{2\pi}}\sum_{m\in\mathbb{Z}}\int_0^{2\pi} f(s+2\pi m)\,\mathrm{e}^{-\mathrm{i}(s+2\pi m)(\xi+\ell)}\,\mathrm{d}s \;=\; \frac{1}{\sqrt{2\pi}}\int_0^{2\pi}(Ff)(s,\xi)\,\mathrm{e}^{-\mathrm{i}\ell s}\,\mathrm{d}s$$

for $\ell\in\mathbb{Z}$ and $\xi\in(-1/2,1/2]$. This translates to the cylindrical Fourier transform $\mathcal{F}$ as

$$(\mathcal{F}f)(r,m,\xi+\ell) \;=\; \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}(Ff)(x,\xi)\,\mathrm{e}^{-\mathrm{i}(m\varphi+\ell x_3)}\,\mathrm{d}x_3\,\mathrm{d}\varphi \tag{34}$$

for $r>0$, $\ell,m\in\mathbb{Z}$, $\xi\in(-1/2,1/2]$ where r,φ,x_3 are the cylindrical coordinates of $x\in\mathbb{R}^3$. Analogously, the inverse of $\mathcal{F}$ is expressed as

$$(\mathcal{F}^{-1}g)(x) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}} g(r,m,\xi)\,\mathrm{e}^{\mathrm{i}(m\varphi+\xi x_3)}\,\mathrm{d}\xi \;=\; \frac{1}{2\pi}\sum_{m,\ell\in\mathbb{Z}}\int_{\ell-1/2}^{\ell+1/2} g(r,m,\xi)\,\mathrm{e}^{\mathrm{i}(m\varphi+\xi x_3)}\,\mathrm{d}\xi$$
$$= \frac{1}{2\pi}\int_{-1/2}^{1/2}\sum_{m,\ell\in\mathbb{Z}} g(r,m,\xi+\ell)\,\mathrm{e}^{\mathrm{i}(m\varphi+(\xi+\ell)x_3)}\,\mathrm{d}\xi$$

where r,φ,x_3 are the cylindrical coordinates of $x\in\mathbb{R}^3$. Therefore, by (5),

$$(F\mathcal{F}^{-1}g)(x,\alpha) \;=\; \frac{1}{2\pi}\sum_{m,\ell\in\mathbb{Z}} g(r,m,\alpha+\ell)\,\mathrm{e}^{\mathrm{i}(m\varphi+\ell x_3)}. \tag{35}$$

The Fourier transform is, in particular, useful to define the trace space $H^{1/2}(\Gamma_R)$ by

$$H^{1/2}(\Gamma_R) \;=\; \left\{ g\in L^2(\Gamma_R) : \sum_{m\in\mathbb{Z}}\int_{\mathbb{R}}\big(1+(m/R)^2+\xi^2\big)^{1/2}|(\mathcal{F}g)(m,\xi)|^2\mathrm{d}\xi<\infty\right\}.$$

Then the trace theorem holds for $u\in H^1(T_R)$ or $u\in H^1(T_{\hat{R}}\setminus T_R)$ for $\hat{R}>R$ (see [5]).

We will now study the following exterior boundary value problem for the Helmholtz equation in the exterior of the cylinder T_R

$$\Delta v + k^2 v = -h \text{ in } \mathbb{R}^3\setminus T_R, \quad v = g \text{ on } \Gamma_R, \tag{36}$$

such that the Fourier transform satisfies the family of one-dimensional radiation conditions

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial}{\partial r}(\mathcal{F}v)(r,m,\xi) - \mathrm{i}\,k(\xi)\,(\mathcal{F}v)(r,m,\xi)\right) \;=\; 0 \tag{37}$$

for all $m\in\mathbb{Z}$ and almost all $\xi\in\mathbb{R}$ where $k(\xi)=\sqrt{k^2-\xi^2}$.

Theorem 4.1. *Let $h\in L^2_\sigma(\mathbb{R}^3\setminus T_R)$ for some $\sigma>1$ and $g\in H^{1/2}(\Gamma_R)$ and $k\in\mathbb{C}$ with* $\mathrm{Re}k>0$ *and* $\mathrm{Im}k\geqslant 0$. *The function*

$$v(r,\theta,x_3) = \frac{1}{2\pi}\int_{\mathbb{R}}\sum_{m\in\mathbb{Z}}\left[\int_R^\infty G(r,\rho;m,\xi)\,(\mathcal{F}h)(\rho,m,\xi)\,\rho\,\mathrm{d}\rho + \frac{H_m^{(1)}(k(\xi)r)}{H_m^{(1)}(k(\xi)R)}\,(\mathcal{F}g)(m,\xi)\right]\mathrm{e}^{\mathrm{i}m\theta+\mathrm{i}\xi x_3}\mathrm{d}\xi \tag{38}$$

for $r > R$, $\theta \in [0, 2\pi]$, *and* $x_3 \in \mathbb{R}$, *is the unique solution of the boundary value problem (36), (37). Again,* $k(\xi) = \sqrt{k^2 - \xi^2}$, *and G is given by*

$$G(r,\rho;m,\xi) \;=\; \frac{\mathrm{i}\pi}{2}\left[H_m^{(1)}(k(\xi)r_+)J_m(k(\xi)r_-) \;-\; \frac{H_m^{(1)}(k(\xi)\rho)}{H_m^{(1)}(k(\xi)R)}H_m^{(1)}(k(\xi)r)J_m(k(\xi)R)\right], \tag{39}$$

for $r, \rho \geqslant R$, $m \in \mathbb{Z}$, *and* $\xi \in \mathbb{R}$ *where* $r_+ = \max\{r, \rho\}$ *and* $r_- = \min\{r, \rho\}$. *Furthermore, the solution depends continuously on h, g, and k, in the sense that for any* $\hat{R} > R$ *and* $\sigma > 1$ *the mapping* $(h, g, k) \mapsto v|_{T_{\hat{R}}\setminus T_R}$ *is continuous from* $L^2_\sigma(\mathbb{R}^3 \setminus T_R) \times H^{1/2}(\Gamma_R) \times \mathbb{C}^+$ *to* $H^1(T_{\hat{R}} \setminus T_R)$. *Here* $\mathbb{C}^+ = \{k \in \mathbb{C} : \mathrm{Re}k > 0,\ \mathrm{Im}k \geqslant 0\}$.

Proof. In this—and only this—proof we abbreviate the (cylindrical) Fourier transform by writing $\hat{f}$ instead of $\mathcal{F}f$. Let $k_0 > 0$ be arbitrary, but fixed, and $k \in \mathbb{C}$ with $\mathrm{Re}k > 0$ and $\mathrm{Im}k \geqslant 0$ and $|k| \leqslant k_0$. The proof is divided into five parts.

(a) We show that v is well defined and depends continuously on h and g. The function v consists of two parts; i.e. $v = v_1 + v_2$ with Fourier transforms

$$\hat{v}_1(r,m,\xi) = \int_R^\infty G(r,\rho;m,\xi)\,\hat{h}(\rho,m,\xi)\,\rho\,\mathrm{d}\rho, \quad \hat{v}_2(r,m,\xi) = \frac{H_m^{(1)}(k(\xi)r)}{H_m^{(1)}(k(\xi)R)}\,\hat{g}(m,\xi).$$

First we discuss $\hat{v}_1$. We have $\frac{\partial}{\partial r}\hat{v}_1(r,m,\xi) = \int_R^\infty \frac{\partial}{\partial r}G(r,\rho;m,\xi)\,\hat{h}(\rho,m,\xi)\,\rho\,\mathrm{d}\rho$ by the continuity of G. From part (d) of lemma A.2 we conclude that

$$\begin{aligned}
&\left(1 + \frac{m^2}{r^2} + \xi^2\right)\left|\hat{v}_1(r,m,\xi)\right|^2 + \left|\frac{\partial \hat{v}_1(r,m,\xi)}{\partial r}\right|^2 \\
&\leqslant c\left[\int_R^\infty \left|\hat{h}(\rho,m,\xi)\right|\rho\,\mathrm{d}\rho\right]^2 \;=\; c\left[\int_R^\infty \frac{1}{\rho^{\sigma-1/2}}\left|\hat{h}(\rho,m,\xi)\right|\rho^{\sigma+1/2}\,\mathrm{d}\rho\right]^2 \\
&\leqslant c\int_R^\infty \frac{\mathrm{d}\rho}{\rho^{2\sigma-1}}\int_R^\infty \left|\hat{h}(\rho,m,\xi)\,\rho^\sigma\right|^2\rho\,\mathrm{d}\rho \;=\; \frac{c}{2(\sigma-1)\,R^{2(\sigma-1)}}\int_R^\infty \left|\hat{h}(\rho,m,\xi)\,\rho^\sigma\right|^2\rho\,\mathrm{d}\rho
\end{aligned} \tag{40}$$

and thus by Parseval's identity

$$\begin{aligned}
\|v_1\|^2_{H^1(T_{\hat{R}}\setminus T_R)} &= \int_{\mathbb{R}}\sum_{m\in\mathbb{Z}}\left[\int_R^{\hat{R}}\left(1 + \frac{m^2}{r^2} + \xi^2\right)\left|\hat{v}_1(r,m,\xi)\right|^2 r\,\mathrm{d}r \;+\; \int_R^{\hat{R}}\left|\frac{\partial \hat{v}_1(r,m,\xi)}{\partial r}\right|^2 r\,\mathrm{d}r\right]\mathrm{d}\xi \\
&\leqslant\; c\int_{\mathbb{R}}\sum_{m\in\mathbb{Z}}\int_R^\infty \left|\hat{h}(\rho,m,\xi)\,\rho^\sigma\right|^2\rho\,\mathrm{d}\rho\,\mathrm{d}\xi \;\leqslant\; c\int_{\mathbb{R}^3\setminus T_R}\left|\tilde{h}_\sigma(x)\right|^2\mathrm{d}x \;=\; c\,\|h\|^2_{L^2_\sigma(\mathbb{R}^3\setminus T_R)}
\end{aligned}$$

where c depends only on k_0, R, and $\hat{R}$. This shows the estimate $\|v_1\|_{H^1(T_{\hat{R}}\setminus T_R)} \leqslant c\,\|h\|_{L^2_\sigma(\mathbb{R}^3\setminus T_R)}$.

Now we discuss $\hat{v}_2$ as in [5]. Part (a) of lemma A.2 yields $\left|\hat{v}_2(r,m,\xi)\right| \leqslant \left|\hat{g}(m,\xi)\right|$. Furthermore,

$$\xi^2 \int_R^{\hat{R}} \left|\hat{v}_2(r,m,\xi)\right|^2 r\,\mathrm{d}r \leqslant \xi^2 \int_R^{\hat{R}} \mathrm{e}^{-2\mathrm{Im}k(\xi)(r-R)}\, r\,\mathrm{d}r\, \left|\hat{g}(m,\xi)\right|^2$$

$$\leqslant \left|\hat{g}(m,\xi)\right|^2 \begin{cases} \xi^2 \hat{R} \int_R^{\hat{R}} \mathrm{e}^{-|\xi|(r-R)}\,\mathrm{d}r, & |\xi| \geqslant 2|k|, \\ 4|k|^2 \hat{R}^2/2, & |\xi| \leqslant 2|k|, \end{cases}$$

$$\leqslant c\sqrt{1+\xi^2}\left|\hat{g}(m,\xi)\right|^2$$

where we used the elementary estimate $\mathrm{Im}k(\xi) \geqslant |\xi|/2$ for $|\xi| \geqslant 2|k|$ (see lemma A.5). Furthermore, with lemma A.4,

$$\int_R^{\hat{R}} \left[\left(1+\frac{m^2}{r^2}\right)\left|\hat{v}_2(r,m,\xi)\right|^2 + \left|\frac{\partial \hat{v}_2(r,m,\xi)}{\partial r}\right|^2\right] r\,\mathrm{d}r = \frac{1}{2\pi}\left\|\psi_{\xi,m}\right\|^2_{H^1(B_2(0,\hat{R})\setminus B_2(0,R))}\left|\hat{g}(m,\xi)\right|^2$$

$$\leqslant c\sqrt{1+\xi^2+m^2}\left|\hat{g}(m,\xi)\right|^2.$$

Altogether we have shown that

$$\int_R^{\hat{R}} \left[\left(1+\xi^2+\frac{m^2}{r^2}\right)\left|\hat{v}_2(r,m,\xi)\right|^2 + \left|\frac{\partial \hat{v}_2(r,m,\xi)}{\partial r}\right|^2\right] r\,\mathrm{d}r \;\leqslant\; c\sqrt{1+\xi^2+m^2}\left|\hat{g}(m,\xi)\right|^2. \quad (41)$$

Taking the inverse Fourier transform proves that $v_2 \in H^1(T_{\hat{R}} \setminus T_R)$ and $\|v_2\|_{H^1(T_{\hat{R}}\setminus T_R)} \leqslant c\|g\|_{H^{1/2}(\Gamma_R)}$ where c depends only on k_0, R, and $\hat{R}$.

(b) Now we show that v from (38) satisfies the differential equation. Let $h \in C^\infty(\mathbb{R}^3 \setminus \overline{T_R})$ and $g \in C^\infty(\Gamma_R)$ have compact supports. Then $\hat{h} = \mathcal{F}h$ and $\hat{g} = \mathcal{F}g$ are smooth with respect to ρ and ξ and $\hat{h}$ vanishes for large values of ρ. The partial sections

$$w^N(r,\theta,x_3) = \frac{1}{2\pi}\int_{-N}^{N} \sum_{|m|\leqslant N} \left[\int_R^\infty G(r,\rho;m,\xi)\,\hat{h}(\rho,m,\xi)\,\rho\,\mathrm{d}\rho + \frac{H_m^{(1)}(k(\xi)r)}{H_m^{(1)}(k(\xi)R)}\,\hat{g}(m,\xi)\right]\mathrm{e}^{\mathrm{i}m\theta+\mathrm{i}\xi x_3}\mathrm{d}\xi$$

satisfy $\Delta w^N + k^2 w^N = -h^N$ in $\mathbb{R}^3 \setminus \overline{T_R}$ and $w^N = g^N$ on Γ_R where

$$h^N(r,\theta,x_3) = \frac{1}{2\pi}\int_{-N}^{N}\sum_{|m|\leqslant N} \hat{h}(r,m,\xi)\,\mathrm{e}^{\mathrm{i}m\theta+\mathrm{i}\xi x_3}\mathrm{d}\xi \quad \text{and}$$

$$g^N(\theta,x_3) = \frac{1}{2\pi}\int_{-N}^{N}\sum_{|m|\leqslant N} \hat{g}(r,m,\xi)\,\mathrm{e}^{\mathrm{i}m\theta+\mathrm{i}\xi x_3}\mathrm{d}\xi$$

as it is shown by using (A.7). Let $\psi \in C^\infty(\mathbb{R}^3 \setminus \overline{T_R})$ have compact support in some $T_{\hat{R}} \setminus \overline{T_R}$. Then

$$\int_{T_{\hat{R}}\setminus\overline{T_R}} \left[\nabla w^N \cdot \nabla\psi - k^2 w^N \psi\right]\mathrm{d}x \;=\; \int_{T_{\hat{R}}\setminus\overline{T_R}} h^N \psi\,\mathrm{d}x.$$

Both sides converge as N tends to infinity which shows that

$$\int_{T_{\hat{R}}\setminus\overline{T_R}} \left[\nabla w \cdot \nabla\psi - k^2 w\,\psi\right]\mathrm{d}x \;=\; \int_{T_{\hat{R}}\setminus\overline{T_R}} h\,\psi\,\mathrm{d}x;$$

that is, w satisfies $\Delta w + k^2 w = -h$ in $\mathbb{R}^3 \setminus \overline{T_R}$. For arbitrary $h \in L^2_\sigma(\mathbb{R}^3 \setminus \overline{T_R})$ and $g \in H^{1/2}(\Gamma_R)$ we approximate h and g by smooth functions with compact support and repeat the argument. This ends the second part of the proof.

(c) Now we study the dependence of the solution on k. Here we use Lebesgue's theorem on dominated convergence. Concerning v_1 estimate (40) provides an integrable bound uniformly with respect to k. Furthermore, the integrand (left hand side of (40)) is continuous with respect k. Lebesgue's theorem, applied to the integration with respect to r and ξ and to the summation with respect to m yields the continuous dependence of v_1 on k. The same arguments applies to v_2 using estimate (41). This ends the proof of part (c).

(d) Next we show that $\hat{v} = \mathcal{F}v$ satisfies (37). For $\hat{v}_2$ this follows from $H_m^{(1)\prime}(z) - \mathrm{i}\,H_m^{(1)}(z) = \mathcal{O}(1/|z|^{3/2})$ as $|z| \to \infty$, see [1]. Furthermore,

$$\hat{v}_1(r,m,\xi) = H_m^{(1)}(k(\xi)r) \int_R^r J_m(k(\xi)\rho)\,\hat{h}(\rho,m,\xi)\,\rho\,\mathrm{d}\rho + J_m(k(\xi)r) \int_r^\infty H_m^{(1)}(k(\xi)\rho)\,\hat{h}(\rho,m,\xi)\,\rho\,\mathrm{d}\rho$$

and thus

$$\begin{aligned}
&\frac{\partial}{\partial r}\hat{v}_1(r,m,\xi) - \mathrm{i}\,k(\xi)\,\hat{v}_1(r,m,\xi) \\
&= k(\xi)\,\left[H_m^{(1)\prime}(k(\xi)r) - \mathrm{i}\,H_m^{(1)}(k(\xi)r)\right] \int_R^r J_m(k(\xi)\rho)\,\hat{h}(\rho,m,\xi)\,\rho\,\mathrm{d}\rho \\
&+\; k(\xi)\,\left[J_m'(k(\xi)r) - \mathrm{i}\,J_m(k(\xi)r)\right] \int_r^\infty H_m^{(1)}(k(\xi)\rho)\,\hat{h}(\rho,m,\xi)\,\rho\,\mathrm{d}\rho.
\end{aligned}$$

Now we use again that $H_m^{(1)\prime}(z) - \mathrm{i}\,H_m^{(1)}(z) = \mathcal{O}(1/|z|^{3/2})$ and also $J_m'(z), J_m(z) = \mathcal{O}(1/|z|^{1/2})$ as $|z|$ tends to infinity. This ends the proof of this part because also $\int_r^\infty H_m^{(1)}(k(\xi)\rho)\,\hat{h}(\rho,m,\xi)\,\rho\,\mathrm{d}\rho \to 0$ as $r \to \infty$.

(e) Finally, we show uniqueness under this radiation condition. Let $h = 0$ and $g = 0$ and $v \in H^1(\mathbb{R}^3 \setminus T_R)$ be a solution of (36) satisfying (37). We take the Fourier transform $\hat{v}(r) = (\mathcal{F}v)(r,m,\xi)$ of v. Then $\hat{v}$ solves the Bessel differential equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial \hat{v}(r,m,\xi)}{\partial r}\right) \;+\; \left(k^2 - \xi^2 - \frac{m^2}{r^2}\right)\hat{v}(r,m,\xi) \;=\; 0, \qquad r > R,$$

and $\hat{v}(R,m,\xi) = 0$ for all parameters $m \in \mathbb{Z}$ and almost all $\xi \in \mathbb{R}$. The general solution is given by $\hat{v}(r) = a\,H_m^{(1)}(k(\xi)r) + b\,H_m^{(2)}(k(\xi)r)$ for $r > R$ and some $a, b \in \mathbb{C}$. The boundary condition and the one-dimensional radiation condition $\hat{v}'(r) - \mathrm{i}k(\xi)\hat{v}(r) = o(1/\sqrt{r})$ yields $(\mathcal{F}v)(r,m,\xi) = \hat{v}(r) = 0$ for all $r > R$, $m \in \mathbb{Z}$ and almost all $\xi \in \mathbb{R}$. This yields $v = 0$ and ends the proof. □

Theorem 4.2. *Let $u = u^{(1)} + u^{(2)}$ be the solution of the source problem $\Delta u + \hat{k}^2 n u = -f$ in $\mathbb{R}^3$ derived by the limiting absorption principle of theorem 3.7. Then the Fourier transform $(\mathcal{F}u^{(1)})(r,m,\xi)$ of $u^{(1)}$ satisfies the one-dimensional radiation condition (37) for all $m \in \mathbb{Z}$ and almost all $\xi \in \mathbb{R}$.*

Proof. First we consider the case $k = \hat{k} + \mathrm{i}\varepsilon$ for $\varepsilon > 0$. The unique solution $u_\varepsilon^{(1)} \in H^1(\mathbb{R}^3 \setminus T_R)$ of (36) for $h = h_\varepsilon = \Delta u_\varepsilon^{(2)} + k^2 u_\varepsilon^{(2)}$ in $\mathbb{R}^3 \setminus T_R$ and $g = g_\varepsilon = u_\varepsilon^{(1)}$ on γ_R satisfies the radiation condition (37) because they decay exponentially as $r \to \infty$. Therefore, by the previous theorem it suffices to show convergence of h_ε and g_ε to $h_0 = \Delta u^{(2)} + \hat{k}^2 u^{(2)}$ and $g_0 := u^{(1)} = u - u^{(2)}$, respectively, in $L^2_\sigma(\mathbb{R}^3 \setminus T_R)$ and $H^{1/2}(\Gamma_R)$, respectively. For g_ε this is clear by the trace theorem and the convergence of $u_\varepsilon^{(1)}$ to $u^{(1)}$ in $H^1(T_R)$ (theorem 3.7). For h_ε and h_0 we compute, using (23) and (26), (27),

$$
\begin{aligned}
h_\varepsilon(x) &= \Delta u_\varepsilon^{(2)}(x) + (\hat{k}+\mathrm{i}\varepsilon)^2 u_\varepsilon^{(2)}(x) \\
&= \sum_{j\in J}\sum_{\ell=1}^{m_j} f_{\ell,j}\Bigg[\left((\hat{k}+\mathrm{i}\varepsilon)^2-\hat{k}^2\right)\hat{\phi}_{\ell,j}(x)\int_{-1/2}^{1/2}\frac{1}{\mathrm{i}\varepsilon-\lambda_{\ell,j}\alpha}\mathrm{e}^{\mathrm{i}\alpha x_3}\mathrm{d}\alpha \\
&\quad + 2\mathrm{i}\,\frac{\partial\hat{\phi}_{\ell,j}(x)}{\partial x_3}\int_{-1/2}^{1/2}\frac{\alpha}{\mathrm{i}\varepsilon-\lambda_{\ell,j}\alpha}\mathrm{e}^{\mathrm{i}\alpha x_3}\mathrm{d}\alpha - \hat{\phi}_{\ell,j}(x)\int_{-1/2}^{1/2}\frac{\alpha^2}{\mathrm{i}\varepsilon-\lambda_{\ell,j}\alpha}\mathrm{e}^{\mathrm{i}\alpha x_3}\mathrm{d}\alpha\Bigg],\ x\notin T_R, \\
h_0(x) &= \Delta u^{(2)}(x) + \hat{k}^2 u^{(2)}(x) \\
&= -\mathrm{i}\sum_{j\in J}\sum_{\ell=1}^{m_j}\frac{f_{\ell,j}}{\lambda_{\ell,j}}\left[4\,\frac{\partial\hat{\phi}_{\ell,j}(x)}{\partial x_3}\,\frac{\sin(x_3/2)}{x_3} + 2\,\hat{\phi}_{\ell,j}(x)\,\frac{\mathrm{d}}{\mathrm{d}x_3}\,\frac{\sin(x_3/2)}{x_3}\right] \\
&= -\sum_{j\in J}\sum_{\ell=1}^{m_j}\frac{f_{\ell,j}}{\lambda_{\ell,j}}\left[2\mathrm{i}\,\frac{\partial\hat{\phi}_{\ell,j}(x)}{\partial x_3}\int_{-1/2}^{1/2}\mathrm{e}^{\mathrm{i}\alpha x_3}\mathrm{d}\alpha - \hat{\phi}_{\ell,j}(x)\int_{-1/2}^{1/2}\alpha\,\mathrm{e}^{\mathrm{i}\alpha x_3}\mathrm{d}\alpha\right],\ x\notin T_R.
\end{aligned}
$$

For the proof that h_ε converges to h_0 we observe that we have to estimate three terms corresponding to the three integrals of h_ε. For the first we set $\varphi_\varepsilon^{(0)}(x_3) = \left((\hat{k}+\mathrm{i}\varepsilon)^2-\hat{k}^2\right)\int_{-1/2}^{1/2}\frac{1}{\mathrm{i}\varepsilon-\lambda\alpha}\mathrm{e}^{\mathrm{i}\alpha x_3}\mathrm{d}\alpha$ for the moment (where $\lambda\in\mathbb{R}$ is one of the $\lambda_{\ell,j}$). Then, with Parseval's identity,

$$
\|\varphi_\varepsilon^{(0)}\|_{L^2(\mathbb{R})}^2 \leqslant c\,\varepsilon^2\int_{-1/2}^{1/2}\frac{\mathrm{d}\alpha}{|\mathrm{i}\varepsilon-\lambda\alpha|^2} = c\,\varepsilon^2\int_{-1/2}^{1/2}\frac{\mathrm{d}\alpha}{\varepsilon^2+\lambda^2\alpha^2} = \frac{c\,\varepsilon}{|\lambda|}\int_{-|\lambda|/(2\varepsilon)}^{|\lambda|/(2\varepsilon)}\frac{\mathrm{d}t}{1+t^2} \leqslant \frac{c\pi\,\varepsilon}{|\lambda|}.
$$

Analogously, with $\varphi_\varepsilon^{(j)}(x_3) = \int_{-1/2}^{1/2}\frac{\alpha^j}{\mathrm{i}\varepsilon-\lambda\alpha}\mathrm{e}^{\mathrm{i}\alpha x_3}\mathrm{d}\alpha$ for $j=1,2$ and $\varepsilon\geqslant 0$ we compute $\varphi_\varepsilon^{(j)}(x_3)-\varphi_0^{(j)}(x_3) = \frac{\mathrm{i}\varepsilon}{\lambda}\int_{-1/2}^{1/2}\frac{\alpha^{j-1}}{\mathrm{i}\varepsilon-\lambda\alpha}\mathrm{e}^{\mathrm{i}\alpha x_3}\mathrm{d}\alpha$ and thus

$$
\|\varphi_\varepsilon^{(j)}-\varphi_0^{(j)}\|_{L^2(\mathbb{R})}^2 \leqslant \frac{\varepsilon^2}{\lambda^2}\int_{-1/2}^{1/2}\frac{\alpha^{2j-2}}{\varepsilon^2+\lambda^2\alpha^2}\,\mathrm{d}\alpha \leqslant \begin{cases}\pi\varepsilon/|\lambda|^3, & j=1,\\ \varepsilon^2/|\lambda|^4, & j=2.\end{cases}
$$

Furthermore, we have an estimate of the form $\left|\hat{\phi}_{\ell,j}(x)\right| + \left|\partial\hat{\phi}_{\ell,j}(x)/\partial x_3\right| \leqslant c\,\mathrm{e}^{-\mu r}$ for $r\geqslant R$ where $r=\sqrt{x_1^2+x_2^2}$ because $\hat{\phi}_{\ell,j}$ is evanescent. This shows that there exists $c>0$ with

$$
\begin{aligned}
\int_{\mathbb{R}^3\setminus T_R}\left|h_\varepsilon(x)-h_0(x)\right|^2(x_1^2+x_2^2)^p\,\mathrm{d}x &= \int_R^\infty\int_0^{2\pi}\int_{\mathbb{R}}\left|h_\varepsilon(r,\theta,x_3)-h_0(r,\theta,x_3)\right|^2 r^{2p+1}\,\mathrm{d}x_3\,\mathrm{d}\theta\,\mathrm{d}r \\
&\leqslant c\,\varepsilon\int_R^\infty\mathrm{e}^{-2\mu r}r^{2p+1}\,\mathrm{d}r
\end{aligned}
$$

for every $p\geqslant 0$ and $\varepsilon>0$. Therefore, h_ε converges to h_0 in $L_\sigma^2(\mathbb{R}^3\setminus T_R)$ for every $\sigma\geqslant 0$. □

Therefore, we have shown that the limiting absorption solution satisfies the following radiation condition.

Definition 4.3 (Radiation condition). Let assumptions 2.2 and 2.8 hold and let $k>0$ be regular in the sense of definition 3.4. Let $\psi^\pm\in C^\infty(\mathbb{R})$ be any functions with $\psi^\pm(t)=1+\mathcal{O}(1/|t|)$ as $t\to\pm\infty$ and $\psi^\pm(t)=\mathcal{O}(1/|t|)$ as $t\to\mp\infty$ and $\frac{\mathrm{d}}{\mathrm{d}t}\psi^\pm(t)=\mathcal{O}(1/|t|)$ as $|t|\to\infty$. The solution $u\in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ of (3) has a decomposition in the form $u=u^{(1)}+u^{(2)}$ where:

(a) $u^{(1)}|_{T_{\hat R}} \in H^1(T_{\hat R})$ for every $\hat R > R$, and $u^{(2)} \in W^{2,\infty}(\mathbb{R}^3)$ is given by

$$u^{(2)}(x) \;=\; \psi^+(x_3)\sum_{j\in J}\sum_{\lambda_{\ell,j}>0} a_{\ell,j}\,\hat\phi_{\ell,j}(x) \;+\; \psi^-(x_3)\sum_{j\in J}\sum_{\lambda_{\ell,j}<0} a_{\ell,j}\,\hat\phi_{\ell,j}(x), \quad x\in\mathbb{R}^3,$$

for some $a_{\ell,j}\in\mathbb{C}$.

(b) The cylindrical Fourier transform $(\mathcal{F}u^{(1)})(r,m,\xi)$ of $u^{(1)}$ satisfies the one-dimensional radiation condition (37); that is,

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial}{\partial r}(\mathcal{F}u^{(1)})(r,m,\xi) - \mathrm{i}\,k(\xi)\,(\mathcal{F}u^{(1)})(r,m,\xi)\right) \;=\; 0$$

for all $m\in\mathbb{Z}$ and almost all $\xi\in\mathbb{R}$. Here again, $k(\xi) = \sqrt{k^2-\xi^2}$.

We note that the coefficients $a_{\ell,j}$ are given by (29). Again, the representation of (a) corresponds to the well known radiation condition for closed waveguides as in definition 5 of [10]. Part (b) is needed for open waveguides to describe the behavior normal to the axis of the cylinder.

We will show in Part (B) that this radiation condition assures uniqueness and existence directly; that is, without using the limiting absorption property.

Acknowledgments

The author gratefully acknowledge the financial support by Deutsche Forschungsgemeinschaft (DFG) through CRC 1173.

Appendix

We collect some properties of Hankel functions.

Lemma A.1. *Let* $\mathbb{C}^+ := \{z\in\mathbb{C} : \mathrm{Re}z > 0,\ \mathrm{Im}z > 0\}$. *Then*

$$\left|\frac{H_m^{(1)}(\rho z)}{H_n^{(1)}(rz)}\right| \;\leqslant\; \mathrm{e}^{-\mathrm{Im}z(\rho-r)}$$

for all $\rho\geqslant r>0$, $m,n\in\mathbb{Z}$ *with* $|m|\leqslant|n|$, *and* $z\in\overline{\mathbb{C}^+}$.

Proof. It suffices to consider the case $0\leqslant m\leqslant n$. We use the representation of the modified Hankel functions

$$K_m(w) \;=\; \int_0^\infty \mathrm{e}^{-w\cosh(s)}\cosh(ms)\,\mathrm{d}s, \quad \mathrm{Re}w>0,\ m\in\mathbb{N}_0, \tag{A.1}$$

(see [21], 6.22, formula (5)) and Nicholson's formula

$$\left|H_m^{(1)}(a)\right|^2 \;=\; \frac{8}{\pi^2}\int_0^\infty K_0\big(2a\sinh(s)\big)\cosh(2ms)\,\mathrm{d}s, \quad a>0,\ m\in\mathbb{N}_0. \tag{A.2}$$

([21], 13.73, formula (1)).

We define for fixed $0 \leqslant m \leqslant n$ and $\rho \geqslant r > 0$ the function f by

$$f(z) \;=\; \frac{H_m^{(1)}(\rho z)}{H_n^{(1)}(rz)}\, \mathrm{e}^{\mathrm{i}z(r-\rho)}, \quad z \in \mathbb{C}^+.$$

Then f is holomorphic in $\mathbb{C}^+$ and continuous in $\overline{\mathbb{C}^+}$, in particular at the origin. The latter is seen by the asymptoticis of the Hankel functions for small arguments, see, e.g. [1]. Now we show boundedness of f on $\partial\mathbb{C}^+$.

(a) Let $z = t > 0$. Then

$$\big|f(z)\big| \;=\; \left|\frac{H_m^{(1)}(\rho t)}{H_n^{(1)}(rt)}\right| \;\leqslant\; 1$$

by the monotonicity properties of (A.1) and (A.2).

(b) Let $z = \mathrm{i}t$ with $t > 0$. Then, because $H_m^{(1)}(it) = \frac{2}{\pi} i^{-m-1} K_m(t)$ (see, e.g. [1])

$$\big|f(z)\big| \;=\; \left|\frac{K_m(\rho t)}{K_n(rt)}\,\mathrm{e}^{-t(r-\rho)}\right| \;=\; \frac{\int_0^\infty \mathrm{e}^{-\rho t(\cosh(s)-1)} \cosh(ms)\,\mathrm{d}s}{\int_0^\infty \mathrm{e}^{-rt(\cosh(s)-1)} \cosh(ns)\,\mathrm{d}s} \;\leqslant\; 1$$

again by the monotonicity properties of (A.1). Therefore, $\big|f(z)\big| \leqslant 1$ on $\partial\mathbb{C}^+$. Now we consider $f(z)$ for large values of $|z|$. We have (see again, e.g. [19], section 5.11)

$$H_m^{(1)}(z) \;=\; \sqrt{\frac{2}{\pi z}}\, \mathrm{e}^{\mathrm{i}(z - m\pi/2 - \pi/4)} \big[1 + \mathcal{O}\big(1/|z|\big)\big], \quad |z| \to \infty,$$

uniformly with respect to $z \in \overline{\mathbb{C}^+}$. This yields

$$f(z) \;=\; \sqrt{\frac{r}{\rho}}\, \mathrm{e}^{\mathrm{i}z(\rho-r)}\, \mathrm{e}^{\mathrm{i}z(r-\rho)} \big[1 + \mathcal{O}\big(1/|z|\big)\big] \;=\; \sqrt{\frac{r}{\rho}}\, \big[1 + \mathcal{O}\big(1/|z|\big)\big], \quad |z| \to \infty,$$

and thus $\lim_{|z|\to\infty} \big|f(z)\big| = \sqrt{\frac{r}{\rho}} \leqslant 1$. The maximum principle for holomorphic functions yields the assertion. □

Lemma A.2. *Let* $k(\xi) = \sqrt{k^2 - \xi^2}$ *for* $\xi \in \mathbb{R}$ *and* $k \in \mathbb{C}$ *with* $\mathrm{Re}k > 0$ *and* $\mathrm{Im}k \geqslant 0$. *Here, the branch of the square root is taken such that the square root is holomorphic in* $\mathbb{C} \setminus \mathrm{i}\mathbb{R}_{\leqslant 0}$. *Then*

(a)

$$\left|\frac{H_m^{(1)}\big(k(\xi)\rho\big)}{H_n^{(1)}\big(k(\xi)r\big)}\right| \;\leqslant\; \mathrm{e}^{-\mathrm{Im}k(\xi)(\rho-r)}$$

for all $m, n \in \mathbb{Z}$ *with* $|m| \leqslant |n|$, $\xi \in \mathbb{R}$, $\rho \geqslant r > 0$ *and* $k \in \mathbb{C}$ *with* $\mathrm{Re}k > 0$ *and* $\mathrm{Im}k \geqslant 0$.

(b) For all $k_0 > 0$ *there exists* $c > 0$ *such that*

$$\left|\frac{k(\xi) H_m^{(1)\prime}\big(k(\xi)\rho\big)}{H_n^{(1)}\big(k(\xi)r\big)}\right| \;\leqslant\; c\,\big[\big|k(\xi)\big| + |m|\big]\, \mathrm{e}^{-\mathrm{Im}k(\xi)(\rho-r)}$$

for all $m, n \in \mathbb{Z}$ *with* $|m| \leqslant |n|$, $\xi \in \mathbb{R}$, $\rho \geqslant r > 0$, *and* $k \in \mathbb{C}$ *with* $|k| \leqslant k_0$ *and* $\mathrm{Re}k > 0$ *and* $\mathrm{Im}k \geqslant 0$.

(c) *For all* $k_0 > 0$ *and* $0 < R < \hat{R}$ *there exists* $c > 0$ *with*

$$\left|J_m\big(k(\xi)r\big)H_m^{(1)}\big(k(\xi)r\big)\right| \;\leqslant\; \frac{c}{\sqrt{1+m^2+\xi^2}}$$

for all $m \in \mathbb{Z} \setminus \{0\}$, $\xi \in \mathbb{R}$, $R \leqslant r \leqslant \hat{R}$, *and* $k \in \mathbb{C}$ *with* $|k| \leqslant k_0$ *and* $\mathrm{Re}k > 0$ *and* $\mathrm{Im}k \geqslant 0$.

(d) *Define*

$$G(r,\rho;m,\xi) \;=\; \frac{\mathrm{i}\pi}{2}\left[H_m^{(1)}(k(\xi)r_+)J_m(k(\xi)r_-) \;-\; \frac{H_m^{(1)}(k(\xi)\rho)}{H_m^{(1)}(k(\xi)R)}H_m^{(1)}(k(\xi)r)J_m(k(\xi)R)\right]$$

for $r, \rho \geqslant R > 0$ *where* $r_+ = \max\{r,\rho\}$ *and* $r_- = \min\{r,\rho\}$. *Then, for every* $k_0 > 0$ *and* $\hat{R} > R > 0$ *there exists* $c > 0$ *such that*

$$\left(1+\frac{m^2}{r^2}+\xi^2\right)\left|G(r,\rho;m,\xi)\right|^2 \;+\; \left|\frac{\partial}{\partial r}G(r,\rho;m,\xi)\right|^2 \;\leqslant\; c^2$$

for all $r, \rho \geqslant R$ *with* $r_- \leqslant \hat{R}$, $m \in \mathbb{Z}$, $\xi \in \mathbb{R}$, *and* $k \in \mathbb{C}$ *with* $|k| \leqslant k_0$ *and* $\mathrm{Re}k > 0$ *and* $\mathrm{Im}k \geqslant 0$. *Furthermore,* $(r,\rho,\xi) \mapsto G(r,\rho;m,\xi)$ *is continuous on* $[R,\infty) \times [R,\infty) \times \mathbb{R}$ *for every* $m \in \mathbb{Z}$.

Proof.

(a) The estimate follows directly from lemma A.1 (set $z = k(\xi)$ and note that $k(\xi) \in \overline{\mathbb{C}^+}$).

(b) Again, we assume that $n \geqslant m \geqslant 0$. The recursion formula $zH_m^{(1)\prime}(z) = zH_{m-1}^{(1)}(z) - mH_m^{(1)}(z)$ and the estimate of (a) yields for $m \geqslant 1$

$$\begin{aligned}\left|\frac{k(\xi)H_m^{(1)\prime}\big(k(\xi)\rho\big)}{H_n^{(1)}\big(k(\xi)r\big)}\right| &\leqslant \left|k(\xi)\,\frac{H_{m-1}^{(1)}\big(k(\xi)\rho\big)}{H_n^{(1)}\big(k(\xi)r\big)}\right| + \left|\frac{m}{\rho}\,\frac{H_m^{(1)}\big(k(\xi)\rho\big)}{H_n^{(1)}\big(k(\xi)r\big)}\right| \\ &\leqslant \left[\left|k(\xi)\right| \;+\; \frac{|m|}{R}\right]\mathrm{e}^{-\mathrm{Im}k(\xi)(\rho-r)}.\end{aligned}$$

For $m = 0$ we have that $H_0^{(1)\prime} = -H_1^{(1)}$. Therefore, the same estimate holds for $n \geqslant 1$. If $n = m = 0$ then

$$\left|\frac{k(\xi)\,H_0^{(1)\prime}\big(k(\xi)\rho\big)}{H_0^{(1)}\big(k(\xi)r\big)}\right| \;=\; \left|k(\xi)\right|\left|\frac{H_1^{(1)}\big(k(\xi)\rho\big)}{H_0^{(1)}\big(k(\xi)\rho\big)}\right|\left|\frac{H_0^{(1)}\big(k(\xi)\rho\big)}{H_0^{(1)}\big(k(\xi)r\big)}\right|.$$

The last factor can be estimated by $\mathrm{e}^{-\mathrm{Im}k(\xi)(\rho-r)}$. The second factor is bounded because $\lim_{|z|\to\infty}\left|\frac{H_1^{(1)}(z)}{H_0^{(1)}(z)}\right| = 1$, and $\lim_{|z|\to 0}\left|\frac{H_1^{(1)}(z)}{H_0^{(1)}(z)}\right| = 0$ uniformly for $z \in \mathbb{C}^+$.

(c) By [1], formula 9.6.4, we have $\frac{\pi}{2}\left|H_m^{(1)}\big(k(\xi)r\big)\right| = \frac{\pi}{2}\left|H_m^{(1)}\big(ir\sqrt{\xi^2-k^2}\big)\right| = \left|K_m(mz_r)\right|$ and $\left|J_m\big((k(\xi)r\big)\right| = \left|I_m(mz_r)\right|$ with $z_r = \frac{r}{m}\sqrt{\xi^2-k^2}$. Since $\arg\sqrt{\xi^2-k^2} \in (-\pi,\pi/2]$ we can use formulas 9.7.8 and 9.7.8 of [1] and have the asymptotic forms

$$\frac{\pi}{2}\left|H_m^{(1)}\big(k(\xi)r\big)\right| = \sqrt{\frac{\pi}{2m}}\left|\frac{\mathrm{e}^{-m\eta_r}}{(1+z_r^2)^{1/4}}\right|\left|1+\rho_1(z_r,m)\right|,$$
$$\left|J_m\big(k(\xi)r\big)\right| = \sqrt{\frac{1}{2m\pi}}\left|\frac{\mathrm{e}^{m\eta_r}}{(1+z_r^2)^{1/4}}\right|\left|1+\rho_2(z_r,m)\right|$$

where

$$\eta_r = (1+z_r)^{1/2} + \ln\frac{z_r}{1+(1+z_r)^{1/2}} \quad \text{and}$$
$$\left|\rho_j(z_r,m)\right| \leqslant \frac{c_1}{m\sqrt{|1+z_r^2|}} = \frac{c_1}{\sqrt{|m^2+r^2(\xi^2-k^2)|}},\ j=1,2,$$

for some $c_1>0$ which is independent of r, ξ, and m. Substituting the form of z_r we arrive at

$$\left|H_m^{(1)}\big(k(\xi)r\big)\,J_m\big(k(\xi)r\big)\right| = \frac{1}{m\pi\,|1+z_r^2|^{1/2}}\left|1+\rho_1(z_r,m)\right|\left|1+\rho_2(z_r,m)\right|$$
$$= \frac{1}{\pi\sqrt{\left|m^2+r^2\big(\xi^2-k^2\big)\right|}}\left|1+\rho_3(r,\xi,m)\right|$$

where $\left|\rho_3(r,\xi,m)\right| \leqslant \frac{c_2}{\sqrt{|m^2+r^2(\xi^2-k^2)|}}$. We choose M and $\Xi \geqslant 2k_0$ such that $\left|\rho_j(\xi,m)\right| \leqslant \frac{1}{2}$ for $|\xi|\geqslant\Xi$ and $|m|\geqslant M$ and $j=1,2,3$. Then also $\xi^2-k_0^2 \geqslant \frac{3}{4}\xi^2$ for $|\xi|\geqslant\Xi$ and $|m|\geqslant M$.

We distinguish between four cases:

Case 1: $|\xi|\geqslant\Xi$ and $|m|\geqslant M$. From the above asymptotic form we conclude that

$$\left|H_m^{(1)}\big(k(\xi)r\big)\,J_m\big(k(\xi)r\big)\right| \leqslant \frac{c}{\sqrt{m^2+r^2(\xi^2-k_0^2)}} \leqslant \frac{2c}{\sqrt{m^2+\frac{3}{4}R^2\,\xi^2}} \leqslant \frac{c_1}{\sqrt{1+m^2+\xi^2}}.$$

Case 2: $|\xi|\geqslant\Xi$ and $|m|\leqslant M$. Now we use the asymptotic formulas for the Hankel functions for large arguments (see [1], formulas ?)

$$H_m^{(1)}(z) = \sqrt{\frac{2}{\pi z}}\,\mathrm{e}^{\mathrm{i}(z-m\pi/2-\pi/4)}\big[1+\mathcal{O}(1/|z|)\big], \tag{A.3}$$

$$J_m(z) = \sqrt{\frac{1}{\pi z}}\left[\mathrm{e}^{\mathrm{i}(z-m\pi/2-\pi/4)}+\mathrm{e}^{-\mathrm{i}(z-m\pi/2-\pi/4)}\right]\big[1+\mathcal{O}(1/|z|)\big], \tag{A.4}$$

as $|z|$ tends to infinity uniformly with respect to $|m|\leqslant M$ and thus

$$\left|J_m\big(k(\xi)r\big)H_m^{(1)}\big(k(\xi)r\big)\right| \leqslant \frac{c}{r|t(\xi)|}\left[\mathrm{e}^{-2r|k(\xi)|}+1\right] \leqslant \frac{c}{R\,|t(\xi)|}.$$

This proves the estimate because $|k(\xi)| \geqslant \sqrt{\xi^2-k_0^2} \geqslant \frac{\sqrt{3}}{2}\,\xi \geqslant c_1\sqrt{1+\xi^2+m^2}$ for all $|\xi|\geqslant\Xi$ and $|m|\leqslant M$. We note that this holds also for $m=0$!

Case 3: $|\xi|\leqslant\Xi$ and $|m|\geqslant M$. Now we use the asymptotics for large order:

$$H_m^{(1)}(z) = \sqrt{\frac{2}{\pi\mathrm{i}}}\,(m-1)!\left(\frac{2}{z}\right)^m\big[1+\mathcal{O}(1/m)\big], \tag{A.5}$$

$$J_m(z) = \frac{1}{m!}\left(\frac{z}{2}\right)^m \left[1 + \mathcal{O}(1/m)\right], \tag{A.6}$$

for $m \to \infty$ uniformly with respect to $|z| \leqslant R\sqrt{\Xi^2 + k_0^2}$. This yields

$$\left|J_m\big(k(\xi)r\big)H_m^{(1)}\big(k(\xi)r\big)\right| \leqslant \frac{c}{m} \leqslant \frac{c_1}{\sqrt{1+m^2+\xi^2}}$$

for $m \geqslant M$ uniformly with respect to $|\xi| \leqslant \Xi$ and $R \leqslant r \leqslant \hat{R}$.

Case 4: $|\xi| \leqslant \Xi$ and $0 < |m| \leqslant M$. We have to estimate finitely many terms. Each of them is continuous with respect to ξ and k, also in cases where $\xi^2 = k^2$. This proves part (c).

(d) For $m \neq 0$ this follows easily from (a), (b), and (c). Indeed, by (a) and (c) we have

$$\left|G(r,\rho;m,\xi)\right| \leqslant \frac{\pi}{2}\left|H_m^{(1)}(k(\xi)r_-)J_m(k(\xi)r_-)\right| + \frac{\pi}{2}\left|H_m^{(1)}(k(\xi)R)J_m(k(\xi)R)\right| \leqslant \frac{c}{\sqrt{1+(m/r)^2+\xi^2}}.$$

Furthermore, if $r \leqslant \rho$ we have by the same arguments

$$\begin{aligned}
\left|\frac{\partial}{\partial r}G(r,\rho;m,\xi)\right| &\leqslant \frac{\pi}{2}|k(\xi)|\left|J_m'(k(\xi)r)H_m^{(1)}(k(\xi)r)\right| + \frac{\pi}{2}|k(\xi)|\left|H_m^{(1)\prime}(k(\xi)r)J_m(k(\xi)R)\right| \\
&\leqslant \frac{\pi}{2}|k(\xi)|\left|J_m(k(\xi)r)H_m^{(1)\prime}(k(\xi)r)\right| + \frac{1}{r} + \frac{\pi}{2}|k(\xi)|\left|H_m^{(1)\prime}(k(\xi)r)J_m(k(\xi)R)\right| \\
&\leqslant \frac{1}{R} + \frac{\pi}{2}\left|\frac{k(\xi)H_m^{(1)\prime}(k(\xi)r)}{H_m^{(1)}(k(\xi)r)}\right|\Bigg(\left|H_m^{(1)}(k(\xi)r)J_m(k(\xi)r)\right| + \\
&\quad + \left|\frac{H_m^{(1)}(k(\xi)r)}{H_m^{(1)}(k(\xi)R)}\right|\left|H_m^{(1)}(k(\xi)R)J_m(k(\xi)R)\right|\Bigg) \\
&\leqslant \frac{1}{R} + c\left[\left|k(\xi)\right| + \frac{|m|}{R}\right]\frac{1}{\sqrt{1+m^2+\xi^2}} \leqslant c_1.
\end{aligned}$$

Here we used the Wronskian $zJ_m'(z)H_m^{(1)}(z) = zJ_m(z)H_m^{(1)\prime}(z) - \frac{2\mathrm{i}}{\pi}$ and the estimates of parts (a) and (b) of lemma A.2. For $r > \rho$ we argue analogously. If $m = 0$ and $\xi \geqslant \Xi$ we can argue analogously (see remark in the proof of part (c) at the end of case 2). Finally, in the case $m = 0$ and $\xi \leqslant \Xi$ we use the asymptotics of the Bessel and Hankel functions for small arguments and find that

$$\lim_{t\to 0}\left|H_0^{(1)}(tr_+)J_0(tr_-) - \frac{H_0^{(1)}(t\rho)}{H_0^{(1)}(tR)}H_0^{(1)}(tr)J_0(k(\xi)R)\right| = \frac{2}{\pi}\ln\frac{r_+}{r} \leqslant \frac{2}{\pi}\ln\frac{\hat{R}}{R}.$$

For the derivative we argue analogously using $\lim_{t\to 0}\left[tH_0^{(1)\prime}(t)\right] = \frac{2\mathrm{i}}{\pi}$. □

Remark A.3. We note that G is essentially a Green's function of the ordinary differential operator

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{m^2}{r^2} + k^2 - \xi^2.$$

More precisely, for any $h \in L^2(R,\infty)$ with compact support in $[R,\infty)$,

$$v(r) \;=\; \int_R^\infty G(r,\rho,m,\xi)\,h(\rho)\,\rho\,\mathrm{d}\rho, \qquad r \geqslant R,$$

satisfies the boundary value problem

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\,\frac{\mathrm{d}v(r)}{\mathrm{d}r}\right) + \left(k^2-\xi^2-\frac{m^2}{r^2}\right)v(r) = -h(r),\ r>R, \quad v(R)=0. \tag{A.7}$$

This is seen by a direct calculation using the Bessel differential equation and the Wronskian of the Bessel functions in the form $H_m^{(1)\prime}(z)J_m(z) - H_m^{(1)}(z)J_m'(z) = \frac{2\mathrm{i}}{\pi z}$.

Lemma A.4. *Let $k \in \mathbb{C}$ with $\mathrm{Re}k > 0$ and $\mathrm{Im}k \geqslant 0$. For $\xi \in \mathbb{R}$ and $m \in \mathbb{Z}$ define*

$$\psi_{\xi,m}(r,\varphi) \;=\; \frac{H_m^{(1)}\big(r\,k(\xi)\big)}{H_m^{(1)}\big(R\,k(\xi)\big)}\,\mathrm{e}^{\mathrm{i}m\varphi}, \qquad r \geqslant R,\ \ell,m \in \mathbb{Z},$$

where $k(\xi) = \sqrt{k^2-\xi^2}$, $\xi \in \mathbb{R}$. Then, for all $k_0 > 0$ and $\hat{R} > R$ there exists $c > 0$ such that

$$\|\psi_{\xi,m}\|^2_{L^2(B_2(0,\hat{R})\setminus B_2(0,R))} \;\leqslant\; \frac{c}{\sqrt{1+\xi^2}} \quad \text{and} \quad \|\psi_{\xi,m}\|^2_{H^1(B_2(0,\hat{R})\setminus B_2(0,R))} \;\leqslant\; c\,\sqrt{1+\xi^2+m^2}$$

for all $\xi \in \mathbb{R}$ and $m \in \mathbb{Z}$ and $k \in \mathbb{C}$ with $|k| \leqslant k_0$ and $\mathrm{Re}k > 0$ and $\mathrm{Im}k \geqslant 0$.

Proof. We follow closely the proof in [5]. We set $A_{R,\hat{R}} = B_2(0,\hat{R}) \setminus B_2(0,R)$ and use the elementary estimate $\mathrm{Im}k(\xi) = \mathrm{Im}\sqrt{k^2-\xi^2} \geqslant \frac{\sqrt{3}}{2\sqrt{2}}\,|\xi| \geqslant \frac{1}{2}\,|\xi|$ for $|\xi| \geqslant 2|k|$ (see lemma A.5). First we have

$$\|\psi_{\xi,m}\|^2_{L^2(A_{R,\hat{R}})} \;=\; 2\pi\int_R^{\hat{R}} \left|\frac{H_m^{(1)}\big(r\,k(\xi)\big)}{H_m^{(1)}\big(R\,k(\xi)\big)}\right|^2 r\,\mathrm{d}r \;\leqslant\; 2\pi\int_R^{\hat{R}} \mathrm{e}^{-2\mathrm{Im}k(\xi)(r-R)}\,r\,\mathrm{d}r$$

by part (a) of the previous lemma. If $|\xi| \leqslant 2|k|$ then we use $\mathrm{e}^{-2\mathrm{Im}k(\xi)(r-R)} \leqslant 1$ for $r \geqslant R$, and $\|\psi_{\xi,m}\|_{L^2(A_{R,\hat{R}})}$ is bounded uniformly with respect to ξ and m. If $|\xi| \geqslant 2|k|$ then, by the above elementary estimate

$$\|\psi_{\xi,m}\|^2_{L^2(A_{R,\hat{R}})} \;\leqslant\; 2\pi\int_R^{\hat{R}} \mathrm{e}^{-|\xi|(r-R)}\,r\,\mathrm{d}r \;\leqslant\; \frac{2\pi}{|\xi|}\left(R+\frac{1}{2k_0}\right).$$

This proves the assertion for $\|\psi_{\xi,m}\|_{L^2(B_2(0,\hat{R})\setminus B_2(0,R))}$. For the gradient we use Green's formula and have

$$\|\nabla\psi_{\xi,m}\|^2_{L^2(A(R,\hat{R}))} = \int_{A(R,\hat{R})} |\nabla\psi_{\xi,m}|^2\,\mathrm{d}x$$

$$= -\int_{A(R,\hat{R})} \overline{\psi_{\xi,m}}\,\Delta\psi_{\xi,m}\,\mathrm{d}x + \int_{r=\hat{R}} \overline{\psi_{\xi,m}}\,\frac{\partial\psi_{\xi,m}}{\partial r}\,\mathrm{d}s - \int_{r=R} \overline{\psi_{\xi,m}}\,\frac{\partial\psi_{\xi,m}}{\partial r}\,\mathrm{d}s$$

$$\leqslant |k(\xi)|^2\,\|\psi_{\xi,m}\|^2_{L^2(A(R,\hat{R}))}$$

$$+ 2\pi\,|k(\xi)|\left[\hat{R}\left|\frac{\overline{H_m^{(1)}(\hat{R}\,k(\xi))}\,H_m^{(1)\prime}(\hat{R}\,k(\xi))}{|H_m^{(1)}(\hat{R}\,k(\xi))|^2}\right| + R\left|\frac{\overline{H_m^{(1)}(R\,k(\xi))}\,H_m^{(1)\prime}(R\,k(\xi))}{|H_m^{(1)}(R\,k(\xi))|^2}\right|\right]$$

$$\leqslant c_1\,\frac{|k(\xi)|^2}{\sqrt{1+\xi^2}} + c_2\,\big[|k(\xi)| + |m|\big] \leqslant c\,\sqrt{1+\xi^2+m^2}$$

for all $\xi \in \mathbb{R}$ and $m \in \mathbb{Z}$. Here we used (a) and (b) of the previous lemma. This ends the proof. □

Lemma A.5. *For $k \in \mathbb{C}$ with $\mathrm{Re}k > 0$ and $\mathrm{Im}k \geqslant 0$ we have*

$$\mathrm{Im}\sqrt{k^2-\xi^2} \geqslant \frac{1}{2}\,|\xi| \quad \text{for all } |\xi| \geqslant 2|k|.$$

Proof. We have $k^2-\xi^2 = |k^2-\xi^2|\,\exp(\mathrm{i}\alpha)$ with some $\alpha \in [0,\pi]$. Taking the real part yields $|k^2-\xi^2|\cos\alpha = (\mathrm{Re}k)^2 - (\mathrm{Im}k)^2 - \xi^2 \leqslant |k|^2 - \xi^2 < 0$. Therefore, $\alpha \in [\pi/2,\pi]$. Then

$$\mathrm{Im}\sqrt{k^2-\xi^2} = \sqrt{|k^2-\xi^2|}\sin\frac{\alpha}{2} \geqslant \sqrt{|k^2-\xi^2|}\sin\frac{\pi}{4} \geqslant \frac{1}{\sqrt{2}}\sqrt{\xi^2-|k|^2}$$

$$\geqslant \frac{1}{\sqrt{2}}\sqrt{\frac{3}{4}}\,|\xi| \geqslant \frac{1}{2}\,|\xi| \quad \text{for } |\xi| \geqslant 2|k|.$$ □

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