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1 Introduction

Partial differential equations are essential as theoretical models in many parts of science. However, often these equations do not allow for exact solutions and numerical approximations are necessary for the application of these models.
Among the many different approaches used in the numerical analysis of partial differential equations, we focus in this thesis on splitting methods (see [MQ02] for a general overview) which have been very successful in dealing with initial value problems of the form

$$\left.\begin{aligned} u'(t) &= (-Au)(t) + g(u(t)), \\ u(0) &= u_0, \end{aligned}\right\} \tag{1.1}$$

where $-A$ is an unbounded linear operator generating a strongly continuous semigroup and g is a nonlinear term so that the equations

$$u'(t) = \begin{cases} (-Au)(t), & (1.2a) \\ g(u(t)), & (1.2b) \end{cases}$$

with appropriate initial values can either be solved explicitly or allow further numerical approximations. Then, after choosing a time step size, we can approximate the solution by alternately following the solution of (1.2a) and (1.2b), starting with the initial value of the original equation and always using the most recent value of the approximation as initial value for the next step. If in the end, we followed all parts for the same total time, it seems reasonable to assume that we are not far off the exact solution at that time, given a sufficiently small size of the time steps. For better approximations, this idea can be iterated with appropriate decompositions of the time step. For a general overview of this so called time integration, see [Fao12].
Assuming that a unique solution to the original equation 1.1 exists, it is important to validate such an approach by proving the convergence of the numerical

approximation to the actual solution. Moreover, the speed of convergence as a function of the length h of the time step is of interest. In particular, one might ask whether the error is of order h^r. In practice, such time integration schemes have to be combined with spatial discretizations. We do not consider such questions, but note that the analysis does serve as a first step to treat full discretization. Different methods to approximate the exact solution are also popular, see [Hoc13] and the references therein.
In this thesis, we offer some analytical reflexions on the convergence of such splitting schemes which concentrate on the following topics.

Error estimates for initial values with low regularity

For example for the Schrödinger equation ($A = \mathrm{i}\Delta$) with a polynomial nonlinearity $g(u) = \pm\mathrm{i}|u|^2u$ on $L^2(\mathbb{R}^d)$, such error estimates (e.g. for orders $r = 2$ in case of the Strang splitting and $r = 1$ in case of the Lie splitting) are well established under high regularity assumptions on the initial value u_0 (e.g. $u_0 \in H^4(\mathbb{R}^d)$ for the Strang splitting or $u_0 \in H^2(\mathbb{R}^d)$ for the Lie splitting), see [ESS16] or [Lub08] and also [JMS17] for an equation involving damping and forcing.
However, the convergence of these methods for initial values of low regularity seems to be an open problem (as discussed in [ORS19], where a more intricate splitting is used to improve on this point). To obtain error estimates of order r with the present splitting method it is apparently necessary to assume that the initial value u_0 belongs to the fractional domain $D(A^r)$ of A (see [ESS16]), that is, in our example $u_0 \in H^{2r}(\mathbb{R}^d)$. However, further hurdles in the proof are Sobolev embeddings and Banach algebra properties of the Sobolev spaces $H^s(\mathbb{R}^d)$, which require $s > \frac{d}{2}$ in the L^2 setting. To get around this obstacle, we propose the following approaches.

- Employ different function space norms for the error estimate with more favourable Sobolev embeddings and Banach algebra properties. For parabolic problems , we use the spaces $L^p(\Omega)$ and $L^p(\Omega) \cap L^\infty(\Omega)$ for $1 < p \leqslant \infty$, where Ω is $\mathbb{R}^d$, $\mathbb{T}^d$ or a compact d-dimensional Riemannian manifold. The scale of modulation spaces $M^s_{p,q}$ has the advantage that the Schrödinger group acts on these spaces as bounded operators (in contrast to the L^p scale). For these norms, we can therefore obtain error estimates for arbitrarily low regularity for the nonlinear Schrödinger equation as well as the nonlinear heat equation.

- Use random initial values to improve the results. Given a $u_0 \in L^2(\mathbb{T}^d)$, we randomize its Fourier expansion $\sum_{n\in\mathbb{Z}} \hat{u}_0(n)\mathrm{e}^{2\pi i n}$ by for example introducing a sequence of independent standard Gaussian variables, that is

$$u_0^\omega = \sum_{n\in\mathbb{Z}} \hat{u}_0(n) g_n(\omega) \mathrm{e}^{2\pi i n}.$$

  Similar randomization exists for $u_0 \in L^2(\mathbb{R}^d)$, see also [BTT14]. This technique has been used very effectively in the theory of dispersive partial differential equations in order to construct Gibbs measures for the flow of solutions (starting with Bourgain in [Bou95]) or to find solutions for initial values in the subcritical domain (starting with Burq in [BTT13], see also the survey [BOP19] and the literature quoted therein). The point of this randomization is that the random variable u_0^ω does not only take values in $L^2(\mathbb{T}^d)$, but also in all $L^p(\mathbb{T}^d)$ for $2 \leqslant p \leqslant< \infty$. At least for parabolic problems, this allows us to give error estimates for random initial values in $H^s(\mathbb{T}^d)$ for all $s > 0$ with respect to the norm of $L^\infty(\mathbb{T}^d)$.

A unified framework for higher order splitting methods

The literature contains many papers that deal with specific nonlinear equations (for example Schrödinger equations on $\mathbb{R}^d$ and $\mathbb{T}^d$ (see [ESS16],[Lub08]), Harmonic oscillators (see [Gau11]) or heat and reaction diffusion equations (see [Fao09])) with various nonlinearities and gives estimates for the Lie, Strang and higher order splitting methods (at least up to order four).
In this thesis, we will isolate the common core of these arguments and present a unified framework that covers a large class of diverse situations. Special attention is given to the problem of initial values of low regularity which leads us to replace generous assumptions on Sobolev embeddings and Banach algebra assumptions with more delicate assumptions on the differentiability of the nonlinearity g. In the proofs, we avoid Lie derivatives and commutators and prefer iterative substitution of the solution in the variation of constants formula for the solution, which will lead to some technical complexity for higher order methods. These higher orders have been treated for the splitting of linear equations (see

[HO09], [EO14], [AHHK16]) and also the semilinear case (see [HO16], [Tha08]), but with stronger assumptions on either the operator A or the nonlinearity g.

Stochastic Schrödinger equation

A first result for the Lie splitting for the cubic stochastic Schrödinger equation with multiplicative noise was announced by Liu in [Liu13b]. Unfortunately, his proof contains a serious mistake (see Remark 5.9). In this thesis, we present an alternative proof which also extends the result announced by Liu in two ways: We also consider initial values in $H^s(\mathbb{R}^d)$ for $s \in (\frac{d}{2}, \frac{d}{2}+2]$ instead of $s > \frac{d}{2}+2$ as well as general skew adjoint operators as the linear part of the equation instead of the Schrödinger operator.

Structure of the thesis

The thesis consists of four parts and is organised in the following way.
In Chapter 2, as an introduction, we present an elementary proof for the error estimates of the Lie and Strang splitting which is inspired by the arguments in [Lub08] and [ESS16], but gives more information on initial values with low regularity and is written in such a way that it allows for considerable generalization (see Theorem 2.9).
In Chapter 3, we develop our general scheme for error estimates for splitting methods on general Banach spaces, semigroup generators $-A$ and differentiable nonlinearities g. We made an effort to also include higher order methods. It is originally motivated by [HO16].
In Chapter 4, we present our error estimates for initial values of low regularity in various scales of Banach spaces which are covered by our general approach in the chapter before. This includes $L^p(\Omega)$ and $L^p(\Omega) \cap L^\infty(\Omega)$ where Ω could be $\mathbb{R}^d$, $\mathbb{T}^d$ or a compact Riemannian manifold, uniform L^p spaces for $2 \leqslant p < \infty$ as well as modulation spaces $M^s_{p,s}$. We also introduce random initial values as a theoretical tool in the error analysis if splitting methods.
Chapter 5 is dedicated to the error analysis stochastic nonlinear Schrödinger equation.

2 Elementary proof for the cubic Schrödinger equation on $\mathbb{R}^d$

As a first specific example, we are taking a look at the Schrödinger equation with cubic nonlinearity, namely

$$\left.\begin{aligned} \mathrm{i}u'(t) &= (-\Delta u)(t) \pm |u(t)|^2 u(t), \\ u(x,0) &= u_0(x), \end{aligned}\right\} \tag{2.1}$$

with $-\Delta : H^2 \to H^0$ for some $d \in \mathbb{N}$, using the Bessel potential spaces

$$H^s := H^s(\mathbb{R}^d) := \{u \in L^2(\mathbb{R}^d) \mid (1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}u \in L^2\}$$

In fact, the proof also works for different operators and spaces, as long as they fulfil some properties (c.f. the end of the chapter). Moreover, the concept of the proofs is the same for nonlinearities of the form $|u|^{k-1}u$ for odd $k \in \mathbb{N}_{\geqslant 3}$, but we would have to work with more terms, which defeats the purpose of this section which is meant to be easy and comprehensible. A much more general case which encompasses this case will follow in Section 3.
According to [Kat95, Theorem 4.1], for $u_0 \in H^s$, there exists a $T > 0$ and a mild solution $u \in C([0,T], H^s)$ which is unique in a smaller space for $s \geqslant \max\{\frac{d}{2} - 1, 0\}$. It even is unique in $C([0,T], H^s)$ if $s \geqslant \frac{1}{6}$ (for $d = 1$), $s > \frac{d-1}{3}$ (for $d \in \{2,3\}$) and $s > \frac{d}{2} - 1$ (for $d \geqslant 4$). The former threshold is shown to be sharp in [Tho08, Theorem 1.3]. For all dimensions, we therefore have the result for $s > \frac{d}{2}$, which is all we need right now. The latter result also follows from the general result in [Paz92, Theorem 6.1.4], since H^s is an algebra for $s > \frac{d}{2}$ (see the proof of Lemma 2.2) and therefore the nonlinearity is locally Lipschitz continuous on H^s.
In order to define the splitting methods we want to use, we first need to split up

(3.1) into its linear and nonlinear part, respectively.

$$\mathrm{i}u'(t) = \begin{cases} (-\Delta u)(t), & (2.2a) \\ \pm|u(t)|^2 u(t), & (2.2b) \end{cases}$$

both having initial value $u(0) = u_0$. Equation (2.2a) has the solution $T(t)u_0 := \mathrm{e}^{\mathrm{i}t\Delta}u_0$ and equation (2.2b) has the solution $\mathrm{e}^{\mp\mathrm{i}t|u_0|^2}u_0$, both for all $u_0 \in L^2(\mathbb{R}^d)$ and $t \in \mathbb{R}$.

The principle behind the exponential splitting schemes is to approximate the exact solution u of (2.1) by alternately following the linear and nonlinear solutions of (2.2a) and (2.2b). For fixed $h > 0$, we therefore define the Lie splitting (for orders up to one) by

$$\begin{aligned} u^1_{\mathrm{Lie}} &:= S^h_{\mathrm{Lie}}(u_0) := T(h)\mathrm{e}^{\mp\mathrm{i}t|u_0|^2}u_0, \\ u^{k+1}_{\mathrm{Lie}} &:= S^h_{\mathrm{Lie}}(u^k_{\mathrm{Lie}}), \quad (k \in \mathbb{N}) \end{aligned} \tag{2.3}$$

as well as the Strang splitting (for orders up to two) by

$$\begin{aligned} u^1_{\mathrm{Strang}} &:= S^h_{\mathrm{Strang}}(u_0) := T(h/2)\mathrm{e}^{\mp\mathrm{i}t|T(h/2)u_0|^2}T(h/2)u_0, \\ u^{k+1}_{\mathrm{Strang}} &:= S^h_{\mathrm{Strang}}(u^k_{\mathrm{Strang}}). \quad (k \in \mathbb{N}) \end{aligned} \tag{2.4}$$

This is enough to formulate the result of this section. We need a couple of auxiliary results for the proof. Those are stated and proven after the Theorem and its proof for clarity's sake.

Theorem 2.1

Let $r > 0$ and $s > \frac{d}{2} - 2r$. If $u_0 \in H^{s+2r}$, then there exists a $T > 0$, a unique solution $u \in C([0,T], H^{s+2r})$ of (2.1) and a $h_0 \in (0,T]$ depending on d, s, T and $M := \sup_{0 \leqslant t \leqslant T} \|u(t)\|_{s+2r}$ such that, provided

a) *$r \in (0,1]$, we have*

$$\|u^N_{Lie} - u(Nh)\|_s \lesssim_{d,s,T,M} h^r$$

b) *$r \in (0,2]$, we have*

$$\|u^N_{Strang} - u(Nh)\|_s \lesssim_{d,s,T,M} h^r.$$

for all $0 \leqslant Nh \leqslant T$ and $h \leqslant h_0$.

Proof. **a)** We follow the standard concept called Lady Windermere's Fan. By artificially generating a telescoping sum, we see that

$$\|(S^h_{\mathrm{Lie}})^N(u_0) - u(Nh)\|_s \leqslant \sum_{k=0}^{N-1} \|(S^h_{\mathrm{Lie}})^{N-k}(u(kh)) - (S^h_{\mathrm{Lie}})^{N-(k+1)}(u((k+1)h))\|_s. \tag{2.5}$$

For the terms on the right hand side, we use the stability result from Proposition 2.3 **a)** $N-k-1$ times ($r_1 = s, r_2 = s+2(r-\varepsilon)$). It says that

$$S^h_{\mathrm{Lie}}(\psi) - S^h_{\mathrm{Lie}}(\varphi)_s \leqslant \mathrm{e}^{C(\widetilde{M},T)h}\|\psi - \varphi\|_s \tag{2.6}$$

as long as $\|\psi\|_{s+2(r-\varepsilon)}, \|\varphi\|_{s+2(r-\varepsilon)} \leqslant \widetilde{M}$ for some $\varepsilon \in (0,1]$ with $s+2(r-\varepsilon) > \frac{d}{2}$. Here, ψ and φ are given by $(S^h_{\mathrm{Lie}})^{N-j}(u(kh))$ for $k \in \{0,\dots,N\}$ and $j \in \{k+1,\dots,N\}$. The fact that their $H^{s+2(r-\varepsilon)}$ norms are uniformly bounded (by $\widetilde{M} = 2M$) for all k follows from Lemma 2.8. In this Lemma, the smallness of h_0 is needed. Hence, we obtain

$$\begin{aligned}\|(S^h_{\mathrm{Lie}})^{N-k}&(u(kh)) - (S^h_{\mathrm{Lie}})^{N-(k+1)}(u((k+1)h))\|_s \\ &\leqslant \mathrm{e}^{C(2M,T)h(N-k-1)}\|S^h_{\mathrm{Lie}}(u(kh)) - u((k+1)h)\|_s.\end{aligned}$$

Next, we use Proposition 2.5, which says that

$$\|S^h_{\mathrm{Lie}}(u(kh)) - u((k+1)h)\|_s \leqslant Ch^{1+r}, \tag{2.7}$$

where the constant just depends on the variables mentioned in the Theorem. This finally lets us go back to (2.5) in order to see that

$$\begin{aligned}\|(S^h_{\mathrm{Lie}})^N(u_0) - u(Nh)\|_s &\leqslant \sum_{k=0}^{N-1} \mathrm{e}^{C(2M,T)h(N-k-1)}\|S^h_{\mathrm{Lie}}(u(kh)) - u((k+1)h)\|_s \\ &\leqslant \sum_{k=0}^{N-1} \mathrm{e}^{C(2M,T)h(N-k-1)}Ch^{r+1} \leqslant Nh\mathrm{e}^{C(2M,T)hN}Ch^r \\ &\leqslant T\mathrm{e}^{C(2M,T)T}Ch^r.\end{aligned}$$

b) We replace S^h_{Lie} by S^h_{Strang} and use parts **b)** instead of **a)** in Propositions 2.3 and Lemma 2.8 as well as Proposition 2.7 instead of 2.5 to obtain the result in the exact same way as above.

■

Before we give the proof of (2.6) in Proposition 2.3 and of (2.7) in Propositions 2.5 and 2.7, we state multiplicative properties of the H^r norms.

Lemma 2.2

Let $0 \leqslant r_1 \leqslant r_2$ and $r_2 > \frac{d}{2}$. The following inclusions and estimates hold.

a) *If $f \in H^{r_1}$ and $g \in H^{r_2}$, then $fg \in H^{r_1}$ and*

$$\|fg\|_{r_1} \lesssim_{d,r_1,r_2} \|f\|_{r_1} \|g\|_{r_2}.$$

b) *If $f \in H^{\frac{r_1+r_2}{2}}$ and $g \in H^{\frac{r_1+r_2}{2}}$, then $fg \in H^{r_1}$ and*

$$\|fg\|_{r_1} \lesssim_{d,r_1,r_2} \|f\|_{\frac{r_1+r_2}{2}} \|g\|_{\frac{r_1+r_2}{2}}.$$

Proof. For $r_2 > \frac{d}{2}$, if $f \in H^{r_2}(\mathbb{R}^d)$, then $f \in L^\infty(\mathbb{R}^d)$ and by the Cauchy-Schwarz inequality

$$\|f\|_{L^\infty} \lesssim \|\mathcal{F}f\|_{L^1} \leqslant \|(1+|\cdot|^2)^{-\frac{s}{2}}\|_{L^2} \|f\|_{r_2}.$$

Hence, a trivial estimate gives

$$\|fg\|_{L^2} \leqslant \|f\|_{L^\infty} \|g\|_{L^2} \lesssim \|f\|_{r_2} \|g\|_{L^2}$$

for $f \in H^{r_2}$ and $g \in L^2$. If $f, g \in H^{r_2}(\mathbb{R}^d)$, then $fg \in H^{r_2}(\mathbb{R}^d)$ with $\|fg\|_{r_2} \lesssim \|f\|_{r_2} \|g\|_{r_2}$. To see this, we observe that

$$(1+|\xi|^2)^{\frac{r_2}{2}} \leqslant (1+3|\xi-\eta|^2+3|\eta|^2)^{\frac{r_2}{2}} \lesssim_{r_2} (1+|\xi-\eta|^2)^{\frac{r_2}{2}} + (1+|\eta|^2)^{\frac{r_2}{2}},$$

seen for natural numbers r_2 through estimating the mixed terms by the pure terms with highest order when multiplying (similar to the trick in the first estimate). Together with $\mathcal{F}(fg) = (\mathcal{F}f) * (\mathcal{F}g)$, this yields

$$(1+|\xi|^2)^{\frac{r_2}{2}} |\mathcal{F}(fg)(\xi)| \lesssim_{r_2} (|(1+|\cdot|^2)^{\frac{r_2}{2}} \mathcal{F}f| * |\mathcal{F}g|)(\xi) + (|\mathcal{F}f| * |(1+|\cdot|^2)^{\frac{r_2}{2}} \mathcal{F}g|)(\xi)$$

and therefore by Young's inequality and the same estimate as in the first estimate gives

$$\|fg\|_{r_2} \lesssim_s \|f\|_{r_2} \|\mathcal{F}g\|_{L^1} + \|\mathcal{F}f\|_{L^1} \|g\|_{r_2} \lesssim \|f\|_{r_2} \|g\|_{r_2}.$$

The two proven estimates mean that for fixed $f \in H^{r_2}$, the mapping $g \mapsto fg$ is linear and bounded on H^0 and H^s. Using complex interpolation between the two (see e.g. [BL76, Theorem 4.1.2], $\theta = \frac{r_1}{r_2}$) gives

$$\|fg\|_{r_1} \lesssim \|f\|_{r_2} \|g\|_{r_1}$$

for all $f \in H^{r_2}$ and $g \in H^{r_1}$, which is **a)**. Symmetry obviously also gives the mirrored version

$$\|fg\|_{r_1} \lesssim \|f\|_{r_1} \|g\|_{r_2}$$

for all $f \in H^{r_1}$ and $g \in H^{r_2}$. Interpreting $(f,g) \mapsto fg$ as a bilinear map, we can use multilinear interpolation (see [BL76, Theorem 4.4.1] for $\theta = \frac{1}{2}$) to obtain

$$\|fg\|_{r_1} \lesssim \|f\|_{\frac{r_1+r_2}{2}} \|g\|_{\frac{r_1+r_2}{2}}$$

for all $f \in H^{\frac{r_1+r_2}{2}}$ and $g \in H^{\frac{r_1+r_2}{2}}$ from those two estimates, giving us **b)** and hence ending the proof. ■

As mentioned, the first Proposition shows the stability of the splitting scheme used in (2.6) of the proof of Theorem 2.1.

Proposition 2.3

Let $0 \leqslant r_1 \leqslant r_2$ and $r_2 > \frac{d}{2}$. For $\psi, \varphi \in H^{r_2}$ with $\|\psi\|_{r_2}, \|\varphi\|_{r_2} \leqslant M$, then

a) $\|S^h_{Lie}(\psi) - S^h_{Lie}(\varphi)\|_{r_1} \leqslant \mathrm{e}^{C(M,T)h} \|\psi - \varphi\|_{r_1}$,

b) $\|S^h_{Strang}(\psi) - S^h_{Strang}(\varphi)\|_{r_1} \leqslant \mathrm{e}^{C(M,T)h} \|\psi - \varphi\|_{r_1}$,

Proof. **a)** We will repeatedly need the following estimate.

$$\begin{aligned} \|\mathrm{e}^{\mp \mathrm{i}h|\psi|^2}\|_{r_2} &\leqslant \sum_{j=0}^{\infty} \frac{h^j \| |\psi|^{2j} \|_{r_2}}{j!} \\ &\leqslant \sum_{j=0}^{\infty} \frac{h^j C^{2j-1} \|\psi\|_{r_2}^{2j}}{j!} = C^{-1} \mathrm{e}^{(C\|\psi\|_{r_2})^2 h} \leqslant C^{-1} \mathrm{e}^{(CM)^2 h} \end{aligned} \tag{2.8}$$

We define $\theta(t) = \mathrm{e}^{\mp \mathrm{i}t(|\varphi|^2-|\psi|^2)}$ and use the mean value Theorem as stated in [Car67, Theorem 3.3.2] to see that $\|1-\mathrm{e}^{\mp \mathrm{i}h(|\varphi|^2-|\psi|^2)}\|_{r_1} \leqslant h\sup_{0\leqslant t\leqslant h}\|\theta'(t)\|_{r_1}$. Because of $\theta'(t) = \mp\mathrm{i}(|\varphi|^2-|\psi|^2)\theta(t)$, we use $|\varphi|^2-|\psi|^2 = (\varphi-\psi)\overline{\varphi}+\psi(\overline{\varphi-\psi})$ to obtain that

$$\||\varphi|^2-|\psi|^2\|_{r_1} \leqslant C(\|\varphi-\psi\|_{r_1}\|\overline{\varphi}\|_{r_2}+\|\psi\|_{r_2}\|\overline{\varphi-\psi}\|_{r_1}) = C(\|\varphi\|_{r_2}+\|\psi\|_{r_2})\|\varphi-\psi\|_{r_1}$$

As in (2.8), this yields

$$\begin{aligned}
\|1-\mathrm{e}^{\mp \mathrm{i}h(|\varphi|^2-|\psi|^2)}\|_{r_1} &\leqslant h \sup_{0\leqslant t\leqslant h} \|(|\varphi|^2-|\psi|^2)\mathrm{e}^{\mp \mathrm{i}t(|\varphi|^2-|\psi|^2)}\|_{r_1} \\
&\leqslant Ch\||\varphi|^2-|\psi|^2\|_{r_1} \sup_{0\leqslant t\leqslant h} \|\mathrm{e}^{\mp \mathrm{i}t(|\varphi|^2-|\psi|^2)}\|_{r_2} \\
&\leqslant h(\|\varphi\|_{r_2}+\|\psi\|_{r_2})\|\varphi-\psi\|_{r_1} \sup_{0\leqslant t\leqslant h} \mathrm{e}^{C^2\||\varphi|^2-|\psi|^2\|_{r_2}t} \\
&\leqslant 2Mh\|\varphi-\psi\|_{r_1}\mathrm{e}^{C^2(\|\varphi\|_{r_2}^2+\|\psi\|_{r_2}^2)h} \\
&\leqslant 2Mh\mathrm{e}^{2(CM)^2T}\|\varphi-\psi\|_{r_1}
\end{aligned}$$

Since $T(h)$ is an isometry on H^{r_1}, we finally compute

$$\begin{aligned}
\|S^h_{\mathrm{Lie}}(\psi)-S^h_{\mathrm{Lie}}(\varphi)\|_{r_1} &= \|\mathrm{e}^{\mp \mathrm{i}h|\psi|^2}\psi-\mathrm{e}^{\mp \mathrm{i}h|\varphi|^2}\varphi\|_{r_1} \\
&\leqslant \|(1-\mathrm{e}^{\mp \mathrm{i}h(|\varphi|^2-|\psi|^2)})\psi\mathrm{e}^{\mp \mathrm{i}h|\psi|^2}\|_{r_1} + \|\mathrm{e}^{\mp \mathrm{i}h|\varphi|^2}(\psi-\varphi)\|_{r_1} \\
&\leqslant C^2\|1-\mathrm{e}^{\mp \mathrm{i}h(|\varphi|^2-|\psi|^2)}\|_{r_1}\|\psi\|_{r_2}\|\mathrm{e}^{\mp \mathrm{i}h|\psi|^2}\|_{r_2} \\
&\quad + C\|\mathrm{e}^{\mp \mathrm{i}h|\varphi|^2}\|_{r_2}\|\psi-\varphi\|_{r_1} \\
&\leqslant (1+2M^2C\mathrm{e}^{2(CM)^2T}h)\mathrm{e}^{(CM)^2h}\|\varphi-\psi\|_{r_1} \\
&\leqslant \mathrm{e}^{C(M,T)h}\|\varphi-\psi\|_{r_1}
\end{aligned}$$

b) We replace h by $h/2$ and φ,ψ by $T(h/2)\varphi, T(h/2)\psi$, both of which are still bounded by M in H^{r_2} since $T(h/2)$ defines an isometry there. The same exact computations as in **a)** yield

$$\|S^h_{\mathrm{Strang}}(\psi)-S^h_{\mathrm{Strang}}(\varphi)\|_{r_1} \leqslant \mathrm{e}^{C(M,T)h}\|T(h/2)\varphi-T(h/2)\psi\|_{r_1} = \mathrm{e}^{C(M,T)h}\|\varphi-\psi\|_{r_1}$$

■

To prove the local error estimate used in (2.6), we will need some Hölder estimates for semigroup orbits which we state in the following Lemma.

Lemma 2.4

Let $\tilde{s} \geqslant 0$ and $\theta > 0$. For $t \geqslant 0$, $t_1 \in \mathbb{R}$ and $y \in H^{\tilde{s}+2\theta}$, the following estimates hold.

a) *For $\theta \in (0,1]$: $\|(T(t_1+t) - T(t_1))y\|_{\tilde{s}} \lesssim t^{\theta} \|y\|_{\tilde{s}+2\theta}$.*

b) *For $\theta \in (0,2]$: $\|(T(t_1+t) - 2T(t_1) + T(t_1-t))y\|_{\tilde{s}} \lesssim t^{\theta} \|y\|_{\tilde{s}+2\theta}$.*

Proof. Recall that if $A = \Delta$ on $H^{\tilde{s}}$, then $D(A^{\theta}) = H^{\tilde{s}+2\theta}$ for $\theta \geqslant 0$. By part **a)** of the Theorem on page 77 of [Tri95] ($p = \infty$, $m = 1$), combined with part **d)** of the Theorem on page 101 of [Tri95], we have

$$H^{\tilde{s}+2\theta} = D(A^{\theta}) \subseteq \{x \in H^{\tilde{s}} \mid \lim_{t\to 0+} t^{-\theta} \|(T(t) - I)x\|_{\tilde{s}} < \infty\} \qquad \forall \theta \in (0,1]$$

as well as

$$H^{\tilde{s}+2\theta} = D(A^{\theta}) \subseteq \{x \in H^{\tilde{s}} \mid \lim_{t\to 0+} t^{-\theta} \|(T(t) - I)^2 x\|_{\tilde{s}} < \infty\} \qquad \forall \theta \in (0,2].$$

a) For $\theta \in (0,1]$, the first inclusion gives

$$\|(T(t) - I)x\|_{\tilde{s}} \lesssim t^{\theta} \|x\|_{\tilde{s}+2\theta}$$

and with $x = T(t_1)y$

$$\|(T(t_1+t) - T(t_1))y\|_{\tilde{s}} \lesssim t^{\theta} \|T(t_1)y\|_{\tilde{s}+2\theta} \leqslant t^{\theta} \|y\|_{\tilde{s}+2\theta}.$$

b) For $\theta \in (0,2]$, the second inclusion gives

$$\|(T(2t) - 2T(t) + I)x\|_{\tilde{s}} \lesssim t^{\theta} \|x\|_{\tilde{s}+2\theta}$$

and with $x = T(t_1 - t)y$

$$\|(T(t_1+t) - 2T(t_1) + T(t_1-t))y\|_{\tilde{s}} \lesssim t^{\theta} \|T(t_1-t)y\|_{\tilde{s}+2\theta} \leqslant t^{\theta} \|y\|_{\tilde{s}+2\theta}.$$

■

For the local error, we first consider the Lie splitting.

Proposition 2.5

Let $r \in (0,1]$ and $s > \frac{d}{2} - 2r$. If $u_0 \in H^{s+2r}$, then there exists a $T > 0$, a unique solution $u \in C([0,T], H^{s+2r})$ of (2.1) such that with $M := \sup_{0 \leqslant t \leqslant T} \|u(t)\|_{s+2r}$ we have

$$\|S^h_{Lie}(u(kh)) - u((k+1)h)\|_s \lesssim_{d,s,T,M} h^{1+r}.$$

Proof. We start by finding a suitable representation of both the exact solution and its numerical approximation. We fix n and k and define

$$z(t) := T(t)u(kh), \qquad v(t) := \mp \mathrm{i} \int_0^t T(t-\tau)[\,|u(kh+\tau)|^2 u(kh+\tau)]\,\mathrm{d}\tau$$

for $t \in [0,h]$. For the exact solution, we obtain $u(kh+t) = z(t) + v(t)$ and

$$u((k+1)h) = z(h) \mp \mathrm{i} \int_0^h T(h-t)[\,|z(t)+v(t)|^2 (z(t)+v(t))]\,\mathrm{d}t$$

with

$$\begin{aligned} |z(t)+v(t)|^2(z(t)+v(t)) &= (\,|z(t)|^2 + z(t)\overline{v(t)} + \overline{z(t)}v(t) + |v(t)|^2)z(t) + |u(kh+t)|^2 v(t) \\ &= |z(t)|^2 z(t) + w(t) \end{aligned}$$

with

$$\begin{aligned} w(t) &= v(t)[\,|u(kh+t)|^2 + |z(t)|^2 + z(t)\overline{v(t)}] + \overline{v(t)}(z(t))^2 \\ &= v(t)[\,|u(kh+t)|^2 + z(t)\overline{u(kh+t)}] + \overline{v(t)}(z(t))^2. \end{aligned}$$

Then

$$u((k+1)h) = z(h) \underbrace{\mp \mathrm{i} \int_0^h T(h-t)[\,|z(t)|^2 z(t)]\,\mathrm{d}t}_{=:A_1} \underbrace{\mp \mathrm{i} \int_0^h T(h-t) w(t)\,\mathrm{d}t}_{=:I_1}.$$

To represent the numerical solution, we apply Taylor's Theorem to the function $a(t) = \mathrm{e}^{\mp \mathrm{i} t |u(kh)|^2}$ in zero so that

$$a(h) = 1 \mp \mathrm{i}h|u(kh)|^2 - \int_0^h (h-t)\,|u(kh)|^4 \mathrm{e}^{\mp \mathrm{i} t |u(kh)|^2}\,\mathrm{d}t$$

and therefore

$$S^h_{\mathrm{Lie}}(u(kh)) = T(h)a(h)u(kh) = z(h) \underbrace{\mp \mathrm{i} h T(h)[\,|u(kh)|^2 u(kh)]}_{=:A_2}$$
$$\underbrace{- T(h)\int_0^h (h-t)\,|u(kh)|^4 \mathrm{e}^{\mp \mathrm{i}t\,|u(kh)|^2} u(kh)\,\mathrm{d}t.}_{=:I_2}$$

This gives us

$$u((k+1)h) - S^h_{\mathrm{Lie}}(u(kh)) = (A_1 - A_2) + I_1 - I_2,$$

then

$$\|u((k+1)h) - S^h_{\mathrm{Lie}}(u(kh))\|_s = \|A_1 - A_2\|_s + \|I_1\|_s + \|I_2\|_s$$

and it remains to estimate the last three norms. We first notice that $T(t)$ is an isometry on H^{s+2r}, hence $\|z(t)\|_{s+2r} = \|u(kh)\|_{s+2r} \leqslant M$ for all $t \in [0,h]$. Since $s + 2r > \frac{d}{2}$, we see by Lemma 2.2 that

$$\begin{aligned}
\|v(t)\|_{s+2r} &\leqslant t \sup_{\tau\in[0,t]} \|T(t-\tau)[\,|u(kh+\tau)|^2 u(kh+\tau)]\|_{s+2r} \\
&= t \sup_{\tau\in[0,t]} \|\,|u(kh+\tau)|^2 u(kh+\tau)\|_{s+2r} \\
&\lesssim t \sup_{\tau\in[0,t]} \|u(kh+\tau)\|^3_{s+2r} \leqslant Mh
\end{aligned}$$

and

$$\begin{aligned}
\|w(t)\|_{s+2r} &\leqslant \|v(t)\,|u(kh+t)|^2\|_{s+2r} + \|v(t)z(t)\overline{u(kh+t)}\|_{s+2r} + \|\overline{v(t)}(z(t))^2\|_{s+2r} \\
&\lesssim \|v(t)\|_{s+2r}(\,\|u(kh+t)\|^2_{s+2r} + \|z(t)\|_{s+2r}\,\|u(kh+t)\|_{s+2r} + \|z(t)\|^2_{s+2r} \lesssim M^3 h
\end{aligned}$$

again for all $t \in [0,h]$. We now estimate

$$\|I_1\|_s \leqslant h \sup_{t\in[0,h]} \|T(h-t)w(t)\|_{s+2r} = h \sup_{t\in[0,h]} \|w(t)\|_{s+2r} \lesssim M^3 h^2 \leqslant M^3 \max\{1,T\} h^{1+r}$$

as well as

$$\|I_2\|_s = \Big\| \int_0^h (h-t)\,|u(kh)|^4 \mathrm{e}^{\mp \mathrm{i}t\,|u(kh)|^2} u(kh)\,\mathrm{d}t \Big\|_{s+2r}$$

$$\begin{aligned}&\lesssim h^2 \sup_{t\in[0,h]} \| |u(kh)|^4 \mathrm{e}^{\mp \mathrm{i}t|u(kh)|^2} u(kh)\|_{s+2r} \lesssim h^2 \sup_{t\in[0,h]} \|u(kh)\|_{s+2r}^5 \|\mathrm{e}^{\mp\mathrm{i}t|u(kh)|^2}\|_{s+2r}\\ &\lesssim M^5 \mathrm{e}^{C^2M^2T} \max\{1,T\} h^{1+r},\end{aligned}$$

by (2.8). Lastly, with $f(t) := T(h-t)[\,|z(t)|^2 z(t)]$, we obtain

$$\begin{aligned}\|A_1 - A_2\|_s &= \Big\| \int_0^h f(t)\,\mathrm{d}t - hf(0) \Big\|_s = \Big\| \int_0^h f(t) - f(0)\,\mathrm{d}t \Big\|_s \leqslant h \sup_{t\in[0,h]} \|f(t)-f(0)\|_s\\ &\leqslant h \sup_{t\in[0,h]} \Big(\|(T(h-t)-T(h))[\,|z(t)|^2 z(t)]\|_s + \|T(h)[\,|z(t)|^2(z(t)-z(0))]\|_s\\ &\qquad + \|T(h)[z(t)\overline{(z(t)-z(0))}z(0)]\|_s + \|T(h)[(z(t)-z(0))\,|z(0)|^2]\|_s \Big)\\ &\leqslant h \sup_{t\in[0,h]} \Big(t^r \|\,|z(t)|^2 z(t)\|_{s+2r} + \|\,|z(t)|^2(z(t)-z(0))\|_s\\ &\qquad + \|z(t)\overline{(z(t)-z(0))}z(0)\|_s + \|(z(t)-z(0))\,|z(0)|^2\|_s \Big)\\ &\leqslant h \sup_{t\in[0,h]} \Big(t^r \|z(t)\|_{s+2r}^3 + \|z(t)\|_{s+2r}^2 \|(T(t)-I)u(kh)\|_s + \|z(t)\|_{s+2r}\\ &\qquad \|z(0)\|_{s+2r} \|(T(t)-I)u(kh)\|_s + \|z(0)\|_{s+2r}^2 \|(T(t)-I)u(kh)\|_s \Big)\\ &\leqslant h \sup_{t\in[0,h]} t^r \Big(M^3 + 3M^2 \|u(kh)\|_{s+2r} \Big) \leqslant 4M^3 h^{1+r},\end{aligned}$$

where we used Lemma 2.4 **a)** at several points. Those three estimates together yield the desired result. ■

For the local error of the Strang splitting, we are going to need some estimates on cubic expressions of the linear solution which we collect in the following Lemma.

Lemma 2.6

Let $r \in (0,2]$ and $s > \frac{d}{2} - 2r$. For $x \in H^{s+2r}$ and $t \in [0,h]$ we define $z(t) := T(t)x$ and $y(t) := |z(h-t)|^2 z(h-t)$. It holds that for all $t \in [0,h]$ and $\|x\|_{s+2r} \leqslant M$,

a) $\|y(h/2)\|_{s+2r} \lesssim M^3$,

b) $\|y(h/2) - y(2t)\|_{s+r} \lesssim M^3 h^{r/2}$, *c)* $\|y(h/2+t) - 2y(h/2) + y(h/2-t)\|_s \lesssim M^3 h^r$,

Proof. **a)** It is easily seen that

$$\|y(h/2)\|_{s+2r} \lesssim \|z(h/2)\|_{s+2r}^3 \leqslant M^3$$

b) We see that

$$\begin{aligned} y(h/2) - y(2t) = {} & |z(h/2)|^2 (z(h/2) - z(h-2t))) \\ & + z(h/2)\overline{(z(h/2) - z(h-2t)))} z(h-2t)) \\ & + (z(h/2) - z(h-2t)))\,|z(h-2t))|^2 \end{aligned}$$

and hence by Lemmata 2.2 and 2.4 **a)** ($\tilde{s} = s + r, \theta = nicefracr2$),

$$\begin{aligned} \|y(h/2) - y(h-2t)\|_{s+r} \lesssim {} & \|z(h/2)\|_{s+2r}^2 \,\|(T(h/2) - I)x\|_{s+r} \\ & + \|z(h/2)\|_{s+2r}\,\|z(h-2t)\|_{s+2r}\,\|(T(h/2) - T(h-2t))x\|_{s+r} \\ & + \|z(h-2t)\|_{s+2r}^2\,\|(T(h/2) - T(h-2t))x\|_{s+r} \\ \lesssim {} & M^2\,\|u(kh)\|_{s+2r} h^{r/2} \leqslant M^3 h^{r/2} \end{aligned}$$

c) We compute that

$$\begin{aligned} y(h/2+t) - 2y(h/2) + y(h/2-t) = {} & [\,|z(h/2+t)|^2 - 2\,|z(h/2)|^2 + |z(h/2-t)|^2]z(h/2) \\ & + [\,|z(h/2+t)|^2 - |z(h/2)|^2][z(h/2+t) - z(h/2)] \\ & + [\,|z(h/2-t)|^2 - |z(h/2)|^2][z(h/2-t) - z(h/2)] \\ & + |z(h/2)|^2[z(h/2+t) - 2z(h/2) + z(h/2-t)] \\ = {} & w_1(t) z(h/2) \\ & + w_2(t)[z(h/2+t) - z(h/2)] \\ & + w_2(-t)[z(h/2-t) - z(h/2)] \\ & + |z(h/2)|^2[z(h/2+t) - 2z(h/2) + z(h/2-t)] \quad (2.9) \end{aligned}$$

with $w_1(t) = |z(h/2+t)|^2 - 2\,|z(h/2)|^2 + |z(h/2-t)|^2$ and $w_2(t) = |z(h/2+t)|^2 - |z(h/2)|^2$ This is easily checked by simplifying the right hand side. A similar calculation yields

$$w_1(t) = [z(h/2+t) - 2z(h/2) + z(h/2-t)]\overline{z(h/2)} + |z(h/2+t) - z(h/2)|^2 + |z(h/2-t) - z(h/2)|^2 + z(h/2)[\overline{z(h/2+t) - 2z(h/2) + z(h/2-t)}].$$

Using Lemmata 2.2 and 2.4 (**a**) with $\tilde{s} = s + r, \theta = r/2$, **b**) with $\tilde{s} = s, \theta = r$), we see that

$$\begin{aligned}\|w_1\|_s &\lesssim \|z(h/2+t) - 2z(h/2) + z(h/2-t)\|_s \|z(h/2)\|_{s+2r} + \|z(h/2+t) - z(h/2)\|_{s+r}^2 \\ &\quad + \|z(h/2-t) - z(h/2)\|_{s+r}^2 + \|z(h/2)\|_{s+2r} \|z(h/2+t) - 2z(h/2) + z(h/2-t)\|_s \\ &\lesssim M^2 h^r.\end{aligned}$$

Moreover, we observe that with

$$w_2(t) = z(h/2+t)[\overline{z(h/2+t) - z(h/2)}] + [z(h/2+t) - z(h/2)]\overline{z(h/2)},$$

it holds that

$$\|w_2(t)\|_{s+r} \lesssim \|z(h/2+t)\|_{s+2r} \|\overline{z(h/2+t) - z(h/2)}\|_{s+r} + \|z(h/2+t) - z(h/2)\|_{s+r} \|z(h/2)\|_{s+2r} \lesssim M^2 t^{r/2},$$

again using Lemmata 2.2 and 2.4 **a**) ($\tilde{s} = s + r, \theta = nicefracr2$). The analogous estimate holds for $w_2(-t)$, hence we return to (2.9) and end up with

$$\begin{aligned}\|y(h/2+t) - 2y(h/2) + y(h/2-t)\|_s &\lesssim \|w_1(t)\|_s \|z(h/2)\|_{s+2r} \\ &\quad + \|w_2(t)\|_{s+r} \|z(h/2+t) - z(h/2)\|_{s+r} \\ &\quad + \|w_2(-t)\|_{s+r} \|z(h/2-t) - z(h/2)\|_{s+r} \\ &\quad + \|z(h/2)\|_{s+2r}^2 \|z(h/2+t) - 2z(h/2) + z(h/2-t)\|_s \\ &\lesssim M^3 h^r.\end{aligned}$$

■

Proposition 2.7

Let $r \in (0,2]$ and $s > \frac{d}{2} - 2r$. If $u_0 \in H^{s+2r}$, then there exists a $T > 0$, a unique solution $u \in C([0,T], H^{s+2r})$ of (2.1) such that with $M := \sup_{0 \leqslant t \leqslant T} \|u(t)\|_{s+2r}$ we have

$$\|S^h_{Strang}(u(kh)) - u((k+1)h)\|_s \lesssim_{d,s,T,M} h^{1+r}.$$

for all $0 < Nh \leqslant T$ and $k \in \{0, \dots, N-1\}$.

Proof. We again start by finding a representation of both the exact solution and its numerical approximation, still using the notation $z(t)$ and $v(t)$ from Proposition 2.5. For the exact solution, we obtain

$$\begin{aligned} u((k+1)h) &= T(h)u(kh) \mp \mathrm{i} \int_0^h T(h-t)[\,|u(kh+t)|^2 u(kh+t)]\,\mathrm{d}t \\ &= z(h) \mp \mathrm{i} \int_0^h T(h-t)[\,|z(t)+v(t)|^2 (z(t)+v(t))]\,\mathrm{d}t \end{aligned}$$

and

$$|z(t)+v(t)|^2(z(t)+v(t)) = |z(t)|^2 z(t) + 2\,|z(t)|^2 v(t) + (z(t))^2 \overline{v(t)} + w_1(t)$$

with

$$w_1(t) = |v(t)|^2(2z(t)+v(t)) + (v(t))^2\overline{z(t)} = |v(t)|^2(z(t)+u(kh+t)) + (v(t))^2\overline{z(t)}$$

that

$$\begin{aligned} u((k+1)h) = z(h) &\underbrace{\mp \mathrm{i} \int_0^h T(h-t)[\,|z(t)|^2 z(t)]\,\mathrm{d}t}_{=:A_1} \\ &\underbrace{\mp \mathrm{i} \int_0^h T(h-t)[2\,|z(t)|^2 v(t) + (z(t))^2\overline{v(t)}]\,\mathrm{d}t}_{I} \underbrace{\mp \mathrm{i} \int_0^h T(h-t) w_1(t)\,\mathrm{d}t}_{=:I_1}. \end{aligned}$$

In I we substitute $v(t)$ by its above definition to obtain

$$\left.\begin{aligned} I \quad &= -2\int_0^h T(h-t)[\,|z(t)|^2 \int_0^t T(t-s)[\,|z(s)|^2 z(s)]\,\mathrm{d}s]\,\mathrm{d}t \\ &\quad + \int_0^h T(h-t)[(z(t))^2 \overline{\int_0^t T(t-s)[\,|z(s)|^2 z(s)]\,\mathrm{d}s}]\,\mathrm{d}t \end{aligned}\right\} =: B_1$$

$$\left.\begin{aligned} &-2\int_0^h T(h-t)[\,|z(t)|^2 \int_0^t T(t-s) w_2(s)\,\mathrm{d}s]\,\mathrm{d}t \\ &+ \int_0^h T(h-t)[(z(t))^2 \overline{\int_0^t T(t-s) w_2(s)\,\mathrm{d}s}]\,\mathrm{d}t \end{aligned}\right\} =: I_2$$

with

$$\begin{aligned} w_2(s) &= 2|z(s)|^2 v(s) + (z(s))^2\overline{v(s)} + 2z(s)\,|v(s)|^2 + \overline{z(s)}(z(s))^2 + |v(s)|^2 v(s) \\ &= (|u(s)|^2 + |z(s)|^2)v(s) + z(s)u(s)\overline{v(s)}. \end{aligned}$$

For the numerical solution, we use Taylor's Theorem with one order more than before on the function $a(t) = \mathrm{e}^{\mp \mathrm{i}t\,|z(h/2)|^2}$ in zero to see that

$$a(h) = 1 \mp \mathrm{i}h\,|z(h/2)|^2 - \frac{h^2}{2}\,|z(h/2)|^4 \pm \frac{\mathrm{i}}{2}\int_0^h (h-t)^2\,|z(h/2)|^6 \mathrm{e}^{\mp \mathrm{i}t\,|z(h/2)|^2}\,\mathrm{d}t$$

and therefore

$$\begin{aligned} S^h_{\mathrm{Strang}}(u(kh)) = T(h/2)a(h)z(h/2) = z(h) &\underbrace{\mp \mathrm{i}hT(h/2)[\,|z(h/2)|^2 z(h/2)]}_{=:A_2} \\ &\underbrace{-\frac{h^2}{2}T(h/2)[\,|z(h/2)|^4 z(h/2)]}_{=:B_2} \\ &\underbrace{\pm\frac{\mathrm{i}}{2}T(h/2)\int_0^h (h-t)^2\,|z(h/2)|^6 \mathrm{e}^{\mp \mathrm{i}t\,|z(h/2)|^2} z(h/2)\,\mathrm{d}t}_{=:I_3}. \end{aligned}$$

This gives us

$$u((k+1)h) - S^h_{\mathrm{Lie}}(u(kh)) = (A_1 - A_2) + (B_1 - B_2) + I_1 + I_2 - I_3,$$

hence

$$\|u((k+1)h) - S^h_{\text{Strang}}(u(kh))\|_s \leqslant \|A_1 - A_2\|_s + \|B_1 - B_2\|_s + \|I_1\|_s + \|I_2\|_s + \|I_3\|_s.$$

and it remains to estimate the last five norms. We recall from Proposition 2.5 that

$$\|v(t)\|_{s+2r} \lesssim Mh, \quad \|z(t)\|_{s+2r} \leqslant M.$$

and therefore

$$\begin{aligned}\|w_1(t)\|_{s+2r} &\leqslant \| |v(t)|^2 z(t)\|_{s+2r} + \| |v(t)|^2 u(kh+t)\|_{s+2r} + \|(v(t))^2 \overline{z(t)}\|_{s+2r} \\ &\lesssim \|v(t)\|^2_{s+2r}(2\|z(t)\|_{s+2r} + \|u(kh+t)\|_{s+2r}) \lesssim M^3 h^2\end{aligned}$$

as well as

$$\begin{aligned}\|w_2(s)\| &\leqslant \|(|u(s)|^2 + |z(s)|^2)v(s) + z(s)u(s)\overline{v(s)}\|_{s+2r} \\ &\lesssim \|v(s)\|_{s+2r}(\|u(s)\|^2_{s+2r} + \|z(s)\|^2_{s+2r} + \|z(s)\|^2_{s+2r}\|u(s)\|^2_{s+2r}) \\ &\lesssim M^3 h\end{aligned}$$

for $s, t \in [0, h]$. Those estimates yield

$$\begin{aligned}\|I_1\|_s &\leqslant h \sup_{t\in[0,h]} \|T(h-t)w_1(t)\|_{s+2r} \\ &= h \sup_{t\in[0,h]} \|w_1(t)\|_{s+2r} \lesssim M^3 h^3 \leqslant M^3 \max\{1, T^2\} h^{1+r},\end{aligned}$$

followed by

$$\begin{aligned}\|I_2\|_s &\leqslant 2h \sup_{t\in[0,h]} \left\| T(h-t)[\,|z(t)|^2 \int_0^t T(t-s)w_2(s)\,\mathrm{d}s] \right\|_{s+2r} \\ &\quad + h \sup_{t\in[0,h]} \left\| T(h-t)[(z(t))^2 \overline{\int_0^t T(t-s)w_2(s)\,\mathrm{d}s} \right\|_{s+2r} \\ &\leqslant 2h \sup_{t\in[0,h]} \left\| |z(t)|^2 \int_0^t T(t-s)w_2(s)\,\mathrm{d}s \right\|_{s+2r} \\ &\quad + h \sup_{t\in[0,h]} \left\| (z(t))^2 \overline{\int_0^t T(t-s)w_2(s)\,\mathrm{d}s} \right\|_{s+2r}\end{aligned}$$

$$
\begin{aligned}
&\lesssim h \sup_{t\in[0,h]} \|z(t)\|_{s+2r}^2 \left\| \int_0^t T(t-s)w_2(s)\,\mathrm{d}s \right\|_{s+2r} \\
&\leqslant M^2h^2 \sup_{t\in[0,h]} \sup_{s\in[0,t]} \|T(t-s)w_2(s)\|_{s+2r} \\
&= M^2h^2 \sup_{s\in[0,t]} \|w_2(s)\|_{s+2r} \lesssim M^5h^3 \leqslant M^5 \max 1, T^2 h^{1+r}.
\end{aligned}
$$

Next, we see that

$$
\begin{aligned}
\|I_3\|_s &= \frac{1}{2} \left\| \int_0^h (h-t)^2 |z(h/2)|^6 \mathrm{e}^{\mp \mathrm{i} t |z(h/2)|^2} z(h/2)\,\mathrm{d}t \right\|_{s+2r} \\
&\leqslant \frac{1}{2} h^3 \sup_{t\in[0,h]} \| |u(kh)|^6 \mathrm{e}^{\mp \mathrm{i} t |u(kh)|^2} z(h/2)\|_{s+2r} \\
&\lesssim h^3 \sup_{t\in[0,h]} \|u(kh)\|_{s+2r}^6 \|z(h/2)\|_{s+2r} \|\mathrm{e}^{\mp \mathrm{i} t |u(kh)|^2}\|_{s+2r} \\
&\lesssim M^7 \mathrm{e}^{C^2M^2T} \max\{1, T^2\} h^{1+r}
\end{aligned}
$$

by (2.8). Moving on, we again work with $f(t) := T(h-t)[\,|z(t)|^2 z(t)]$ to see that

$$
A_1 - A_2 = \int_0^h f(t)\,\mathrm{d}t - hf(h/2) = \underbrace{\int_0^{\frac{h}{2}} f(t)\,\mathrm{d}t}_{\overset{s=\frac{h}{2}-t}{=} \int_0^{\frac{h}{2}} f(h/2-s)\,\mathrm{d}s} + \underbrace{\int_{\frac{h}{2}}^h f(t)\,\mathrm{d}t}_{\overset{s=t-\frac{h}{2}}{=} \int_0^{\frac{h}{2}} f(h/2+s)\,\mathrm{d}s} - 2\int_0^{\frac{h}{2}} f(h/2)
$$

$$
= \int_0^{\frac{h}{2}} f(h/2-t) - 2f(h/2) + f(h/2+t)\,\mathrm{d}s.
$$

Abbreviating our terms by defining $y(t) := |z(h-t)|^2 z(h-t)$, we therefore obtain

$$
\begin{aligned}
\|A_1 - A_2\|_s &\leqslant \frac{h}{2} \sup_{t\in[0,h/2]} \|T(h/2+t)y(h/2+t) - 2T(h/2)y(h/2) + T(h/2-t)y(h/2-t)\|_s \\
&\leqslant \frac{h}{2} \sup_{t\in[0,h/2]} \big(\|(T(h/2+t) - 2T(h/2) + T(h/2-t))y(h/2)\|_s \\
&\quad + \|(T(h/2+t) - T(h/2))[y(h/2+t) - y(h/2)]\|_s \\
&\quad + \|(T(h/2-t) - T(h/2))[y(h/2-t) - y(h/2)]\|_s \\
&\quad + \|T(h/2)[y(h/2+t) - 2y(h/2) + y(h/2-t)]\|_s \big) \\
&\lesssim h \sup_{t\in[0,h/2]} \Big(h^r \|y(h/2)\|_{s+2r} + h^{r/2} \|y(h/2+t) - y(h/2)\|_{s+r}
\end{aligned}
$$

$$
\begin{aligned}
&+ h^{r/2}\|y(h/2-t)-y(h/2)\|_{s+r} + \|y(h/2+t)-2y(h/2)+y(h/2-t)\|_s\Big) \\
&\lesssim M^3 h^{1+r}
\end{aligned}
$$

by Lemmata 2.4 and 2.6. Finally, we move on to the last term which is again easier to handle. Defining

$$
F(t,s) = -2T(h-t)[\,|z(t)|^2 T(t-s)[\,|z(s)|^2 z(s)]] + T(h-t)[(z(t))^2 \overline{T(t-s)[\,|z(s)|^2 z(s)]}],
$$

we see that

$$
\begin{aligned}
\frac{h^2}{2}F(h/2,h/2) &= -h^2 T(h/2)[\,|z(h/2)|^2\,|z(h/2)|^2 z(h/2)] + \frac{h^2}{2}T(h/2)[(z(h/2))^2\,|z(h/2)|^2\overline{z(h/2)}] \\
&= -\frac{h^2}{2}T(h/2)[\,|z(h/2)|^4 z(h/2)] = B_2
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\|B_1-B_2\|_s &= \Big\|\int_0^h\int_0^t F(t,s)\,\mathrm{d}s\,\mathrm{d}t - \frac{h^2}{2}F(h/2,h/2)\Big\|_s \\
&= \Big\|\int_0^h\int_0^t F(h/2,h/2)-F(t,s)\,\mathrm{d}s\,\mathrm{d}t\Big\|_s \\
&\leqslant \frac{h^2}{2}\sup_{0\leqslant s\leqslant t\leqslant h}\|F(h/2,h/2)-F(t,s)\|_s \\
&= \frac{h^2}{2}\sup_{0\leqslant s\leqslant t\leqslant h}\|F_1(t,s)-2F_2(t,s)\|_s. \qquad (2.10)
\end{aligned}
$$

with

$$
\begin{aligned}
F_1(t,s) &= T(h/2)[\,|z(h/2)|^4 z(h/2)] - T(h-t)[(z(t))^2\overline{T(t-s)[\,|z(s)|^2 z(s)]}] \\
F_2(t,s) &= T(h/2)[\,|z(h/2)|^4 z(h/2)] - T(h-t)[\,|z(t)|^2 T(t-s)[\,|z(s)|^2 z(s)]]
\end{aligned}
$$

With

$$
\begin{aligned}
F_2(t,s) = (T(h/2)&-T(h-t)[\,|z(h/2)|^4 z(h/2)] \\
&+ T(h-t)[(z(h/2)-z(t))\,|z(h/2)|^4] \\
&+ T(h-t)[z(t)(\overline{z(h/2)-z(t)})\,|z(h/2)|^2 z(h/2)] \\
&+ T(h-t)[\,|z(t)|^2(I-T(t-s))[\,|z(h/2)|^2 z(h/2)]] \\
&+ T(h-t)[\,|z(t)|^2 T(t-s)[y(h/2)-y(h-s)],
\end{aligned}
$$

We can use Lemmata 2.2, 2.4 and 2.6 to obtain

$$\begin{aligned}
\|F_2(t,s)\|_s &\leqslant \|(T(h/2) - T(h-t)[\,|z(h/2)|^4 z(h/2)]\|_{s+r} \\
&\quad + \|T(h-t)[(z(h/2) - z(t))\,|z(h/2)|^4]\|_{s+r} \\
&\quad + \|T(h-t)[z(t)\overline{(z(h/2) - z(t))}\,|z(h/2)|^2 z(h/2)]\|_{s+r} \\
&\quad + \|T(h-t)[\,|z(t)|^2 (I - T(t-s))[\,|z(h/2)|^2 z(h/2)]]\|_{s+r} \\
&\quad + \|T(h-t)[\,|z(t)|^2 T(t-s)[y(h/2) - y(h-s)]]\|_{s+r} \\
&\lesssim h^{r/2}\,\|\,|z(h/2)|^4 z(h/2)\|_{s+2r} \\
&\quad + \|z(h/2) - z(t)\|_{s+r}\,\|z(h/2)\|_{s+2r}^4 \\
&\quad + \|z(t)\|_{s+2r}\,\|z(h/2) - z(t)\|_{s+r}\,\|z(h/2)\|_{s+r}^3 \\
&\quad + \|z(t)\|_{s+2r}^2\,\|(I - T(t-s))[\,|z(h/2)|^2 z(h/2)]\|_{s+r} \\
&\quad + \|z(t)\|_{s+2r}^2\,\|y(h/2) - y(h-s)\|_{s+r} \\
&\lesssim h^{r/2}\,\|z(h/2)\|_{s+2r}^5 \\
&\quad + \|z(h/2) - z(t)\|_{s+r}\,\|z(h/2)\|_{s+2r}^4 \\
&\quad + \|z(t)\|_{s+2r}^2\,\|z(h/2) - z(t)\|_{s+r}\,\|z(h/2)\|_{s+2r}^3 \\
&\quad + h^{r/2}\,\|z(t)\|_{s+2r}^2\,\|z(h/2)\|_{s+r}^3 \\
&\quad + \|z(t)\|_{s+2r}^2\,\|y(h/2) - y(h-s)\|_{s+r} \\
&\lesssim M^5 h^{r/2}
\end{aligned}$$

for $t,s \in [0,h]$. The estimate of F_1 works analogously. Plugging this into (2.10) gives

$$\|B_1 - B_2\|_s \lesssim M^5 h^{2+r/2} = M^5 h^{1-r/2} h^{1+r} \leqslant M^5 \max\{1,T\} h^{1+r},$$

which is the last part we need to conclude the proof. ■

The last result we need shows a uniform bound on the H^{s+2r} norm of all terms we want to use Proposition 2.3 on in the proof of Theorem 2.1.

Lemma 2.8

Let $\|u(t)\|_{s+2r} \leqslant M$ for all $t \in [0,T]$. Choose $\varepsilon \in (0,r]$ such that $s+2(r-\varepsilon) > \frac{d}{2}$ and put $h_0 = \min\{\left(\frac{M}{T e^{C(2M,T)T} C_{loc}}\right)^{\frac{1}{\varepsilon}}, T\}$. $C(M,T)$ is the the constant from Proposition 2.3, C_{loc} is the constant from Proposition 2.5 with s replaced by $s+2(r-\varepsilon)$. Then, for $h \in (0,h_0]$, $Nh \leqslant T$, $k \in \{0,\dots,N\}$ and $j \in \{1,\dots,k\}$, we have

a) $\|(S^h_{Lie})^{N-j}(u(kh))\|_{s+2(r-\varepsilon)} \leqslant 2M$

b) $\|(S^h_{Strang})^{N-j}(u(kh))\|_{s+2(r-\varepsilon)} \leqslant 2M$

Proof. **a)** We show a stronger result by induction over N, namely

$$\|(S^h_{\mathrm{Lie}})^{N-j}(u(kh)) - u((N-j+k)h)\|_{s+2(r-\varepsilon)} \leqslant T\mathrm{e}^{C(2M,T)T} C_{\mathrm{loc}} h^{\varepsilon}. \tag{2.11}$$

for $Nh \leqslant T$ and all $k \in \{0,\dots,N\}$, $j \in \{k,\dots,N\}$. Indeed, by the triangle inequality and $h \leqslant h_0$, our bound on u in H^{s+2r} and hence in $H^{s+2(r-\varepsilon)}$ and (2.11), we get

$$\|(S^h_{\mathrm{Lie}})^{N-j}(u(kh))\|_{s+2(r-\varepsilon)} \leqslant 2M. \tag{2.12}$$

We start with $N = 0$, for which the difference in (2.12) is 0 and hence the estimate is trivial. Assume (2.11) for some $N \in \mathbb{N}_0$ with $(N+1)h \leqslant T$ and all $k \in \{0,\dots,N\}$, $j \in \{1,\dots,k\}$. For $k = N+1$, we get $j = N+1$ and the estimate is once again trivial. For $k \in \{0,\dots,N\}$ and $j \in \{k+1,\dots N+1\}$, the resulting term is already covered by the induction assumption. Let $k \in \{0,\dots,N\}$ and $j = k$. We compute that

$$\begin{aligned}
&\|(S^h_{\mathrm{Lie}})^{N+1-k}(u(kh)) - u((N+1)h)\|_{s+2(r-\varepsilon)} \\
&\quad \leqslant \sum_{l=0}^{N-k} \|(S^h_{\mathrm{Lie}})^{N+1-k-l}(u((k+l)h)) - (S^h_{\mathrm{Lie}})^{N-k-l}(u((k+l+1)h))\|_{s+2(r-\varepsilon)}
\end{aligned}$$

Now, we can use the stability property from Proposition 2.3 **a)** $N-k-l$ times. M can be taken to be $2M$ by our induction assumption (see (2.12)). This yields

$$\begin{aligned}
&\|(S^h_{\mathrm{Lie}})^{N+1-k}(u(kh)) - u((N+1)h)\|_{s+2(r-\varepsilon)} \\
&\qquad \leqslant \sum_{l=0}^{N-k} \mathrm{e}^{C(2M,T)h(N-k-l)} \|u((k+l)h))) - S^h_{\mathrm{Lie}}(u((k+l+1)h))\|_{s+2(r-\varepsilon)}.
\end{aligned}$$

Now, we use a version of Proposition 2.5 with H^s replaced by $H^{s+2(r-\varepsilon)}$ and r replaced by ε, that is,

$$\|S^h_{\mathrm{Lie}}(u(kh)) - u((k+1)h)\|_{s+2(r-\varepsilon)} \leqslant C_{\mathrm{loc}} h^{1+\varepsilon},$$

the proof working exactly the same way. Therefore, we finally obtain

$$\begin{aligned}\|(S_{\text{Lie}}^h)^{N+1-k}(u(kh)) - u((N+1)h)\|_{s+2(r-\varepsilon)} &\leqslant \sum_{l=0}^{N-k} \mathrm{e}^{C(2M,T)T} C_{\text{loc}} h^{1+\varepsilon} \\ &\leqslant (N+1)\mathrm{e}^{C(2M,T)T} C_{\text{loc}} h^{1+\varepsilon} \leqslant T\mathrm{e}^{C(2M,T)T} C_{\text{loc}} h^{\varepsilon},\end{aligned}$$

which is (2.11) for N replaced by $N+1$ and $j = k$. This concludes the induction as well as the proof by (2.12).

b) The proof is identical, using part **b)** instead of **a)** in Proposition 2.3 and Proposition 2.7 instead of 2.5.

■

If we examine precisely which properties of Δ on $L^2(\mathbb{R}^d)$ we used in the above proofs, we realize that with the same argument, we actually proved a much more general result: We can replace Δ on $L^2(\mathbb{R}^d)$ by a general semigroup generator $-A$ on $L^p(\Omega)$ if we replace the Sobolev scale H^{2s} by the scale of fractional domain spaces $D(A^s)$. More precisely, we have the following assumption.

Assumption

(i) *$-A$ generates a bounded semigroup on a space $X = L^p(\Omega, \Sigma, \mu)$, $1 \leqslant p \leqslant \infty$, on a σ-finite measure space (Ω, Σ, μ).*

(ii) *Let $Y_s = D(A^s)$ equipped with the graph norm of A^s and assume that for some $s_\infty > 0$ and $s > s_\infty$, we have*

- *continuous embeddings $Y_s \hookrightarrow L^\infty(\Omega, \Sigma, \mu)$*
- *Y_s is a Banach algebra with respect to the pointwise multiplication, that is for $y_1, y_2 \in Y_s$,*

$$\|y_1 y_2\|_s \leqslant C_s \|y_1\|_s \|y_2\|_s$$

Consider now the equation

$$\left.\begin{aligned} u'(t) &= (-Au)(t) \pm \mathrm{i}|u(t)|^2 u(t), \\ u(x,0) &= u_0(x), \end{aligned}\right\} \tag{2.13}$$

Then the arguments of this section yield the result on the convergence speed r of the Lie and Strang splitting applied to (2.13).

Theorem 2.9

Let $r > 0$ and $s > \frac{d}{2} - 2r$. If $u_0 \in Y_{s+r}$ then there exists a $T > 0$, a unique solution $u \in C([0,T], Y_{s+r})$ of (2.13) and a $h_0 \in (0,T]$ depending on d, s, T and $M := \sup_{0 \leqslant t \leqslant T} \|u(t)\|_{s+r}$ such that, provided

a) *$r \in (0,1]$, we have*

$$\|u_{Lie}^N - u(Nh)\|_s \lesssim_{d,s,T,M} h^r$$

b) *$r \in (0,2]$, we have*

$$\|u_{Strang}^N - u(Nh)\|_s \lesssim_{d,s,T,M} h^r.$$

for all $0 \leqslant Nh \leqslant T$ and $h \leqslant h_0$.

For a more general statement, see the Chapter 3, more precisely Section 3.3. For concrete examples, see Section 4.

3 Splitting methods for general semilinear evolution equations

3.1 The equation

We are interested in numerical approximations of solutions to equations of the form

$$\left.\begin{aligned} u'(t) &= (-Au)(t) + g(u(t)), \\ u(0) &= u_0. \end{aligned}\right\} \tag{3.1}$$

Our assumptions on the operator A and the nonlinearity g are as follows, where $r > 0$ will denote our desired order of convergence.

Assumption 3.1

Let $(Y, \|\cdot\|_Y)$ be a Banach space and $A : D(A) \subseteq Y \to Y$ a linear operator such that $-A$ generates by $T(t) := \mathrm{e}^{-tA} : Y \to Y$

- $r \leqslant 2$*: A* C_0 *semigroup for* $t \geqslant 0$
- $r > 2$*: A* C_0 *group for* $t \in \mathbb{R}$ *or an analytic semigroup for* t *in a sector* $\Sigma_\varphi := \{z \in \mathbb{C} \mid |\arg(z)| \leqslant \varphi\}$ *for some* $\varphi \in [0, \frac{\pi}{2})$.

Remark

A (semi)group generator fulfilling Assumption 3.1 has fractional powers A^s with domain $X_s := D(A^s)$ for all $s \geqslant 0$ and the graph norm $\|\cdot\|_s$. Each (semi)group has the growth bound

$$\|T(t)\|_{X_s \to X_s} \leqslant C\mathrm{e}^{\omega|t|}$$

for some $C, \omega \geqslant 0$ and all $s \geqslant 0$ as well as all t in question. By using the norm $|||\cdot|||_s := \sup_t \mathrm{e}^{-\omega|t|}\|T(t)\cdot\|_s$, we see that the two are equivalent since

$$\|x\|_s \leqslant |||x|||_s := \sup_t \mathrm{e}^{-\omega|t|} \|T(t)x\|_s \leqslant C\|x\|_s$$

for $x \in X_s$ and that $|||T(t_0)|||_{X_s \to X_s} \leqslant \mathrm{e}^{\omega|t_0|}$ since

$$\begin{aligned} |||T(t_0)x|||_s &= \sup_t \mathrm{e}^{-\omega|t|} \|T(t+t_0)x\|_s = \sup_t \mathrm{e}^{-\omega|t|} \|T(t+t_0)x\|_s \\ &\leqslant \mathrm{e}^{\omega|t_0|} \sup_t \mathrm{e}^{-\omega|t+t_0|} \|T(t+t_0)x\|_s \leqslant \mathrm{e}^{\omega|t_0|} |||T(t_0)x|||_s. \end{aligned}$$

To simplify the following computations, we assume that $\omega = 0$. It is easily checked that $w > 0$ does not pose a problem, whereas $C > 1$ creates a problem in the proof of the main Theorem 3.4 since by repeatedly applying the stability result from Proposition 3.6, we obtain a factor C^n where $n \to \infty$ as our numerical step size approaches zero.

Our conditions on the nonlinearity g are motivated by the paper [HO16] by Hansen and Ostermann which considers nonlinearities $g : X_s \to X_s$ for $s = 0, \dots, r$ with $r \in \mathbb{N}$ assuming that

- g is locally Lipschitz on X_r (∗)
- g is k times Fréchet differentiable on X_{r-k} for $k = 1, \dots, r$ (∗∗)

In order to accommodate more nonlinearities g, even polynomials, and also in order to consider fractional convergence orders $r > 0$, we found it necessary to weaken these assumptions in several directions. To this end, we write $r = n-1+\theta$ with $n \in \mathbb{N}$ and $\theta \in (0,1]$.

Assumption 3.2

Let $g : D_g \to Y$ be a nonlinearity with a domain dense in Y and the following properties.

(i) $X_r \subseteq D_g$ and g is locally Lipschitz on X_r

(ii) For $b = 1, \dots, n-1$ and $b = \theta, \theta+1, \dots, n-2+\theta, r$, we require that

- *For some $s(b) \in [r-b, r]$, we have $X_{s(b)} \subseteq D_g$ and g is $\lceil b \rceil$ times Fréchet differentiable on $X_{s(b)}$*
- *For every $M > 0$ there is a constant $C(M)$ so that for all $k \in \{1, \dots, \lceil b \rceil\}$ and $a_i \in \{0, \dots, n-1, \theta, 1+\theta, \dots, n-2+\theta, r\}$ for $i = 1, \dots, k$ with*

*$\sum_{i=1}^{k} a_i = b$, we require that for $x \in X_r$, the multilinear map $g^{(k)}(x) : (X_{s(b)})^k \to X_{s(b)}$ determines (see part **b)** of the Remark below) a continuous multilinear map*

$$\widetilde{g^{(k)}}(x) : X_{r-a_1} \times \cdots \times X_{r-a_k} \to X_{r-b}$$

with $\|\widetilde{g^{(k)}}(x)\| \leqslant C(M)$ for all $x \in X_r$ with $\|x\|_r \leqslant M$.

(iii) For some $s \in [0, r)$, g is Lipschitz continuous on bounded subsets of

- *X_s with respect to the norm of $X_0 = Y$*
- *X_r with respect to the norm of X_s*

Under this assumption, for $u_0 \in X_r$, there exists a $T > 0$ and a unique (mild) solution $u \in C([0,T], X_r)$ of (3.1) in case of a semigroup and $C([-T,T], X_r)$ in case of a group. Since the (semi)group generated by A also operates on X_r, this follows from [Paz92, Theorem 6.1.4]. The size of T for a fixed function g only depends on $\|u_0\|_{X_r}$, as can be seen from the proof. If g is continuously differentiable on $D(A^{r-1})$, then we even obtain a classical solution.

Remark

a) *If one chooses $D_g = Y$ and $s(b) = r - b$ in Assumption 3.2, one recovers the assumption by Hansen and Ostermann mentioned above.*
Indeed, for $D_g = Y$, (i) reduces to $(*)$. Since $a_i \leqslant b$, we have $X_{r-a_i} \subseteq X_{r-b}$ and (ii) follows by restricting $(**)$, assuming the $\|g^{(k)}(x)\|$ is bounded for x in bounded subsets of X_r. (iii) with $s = 0$ follows from $(*)$ if $\|g'(x)\|_Y$ is bounded for x in bounded subsets of Y. One might also choose larger values for s in order to get (two) different requirements on the boundedness of $g'(x)$. The assumptions on the boundedness on bounded sets seem to be missing in [HO16].

b) *For general choices of a_i in (ii) it may happen that (upon rearrangement by symmetry)*

$$r - a_1, \ldots, r - a_{i_0} < s(b) \leqslant r - a_{i_0+1}, \ldots, r - a_k.$$

Since $X_{r-b}, X_{r-a_1}, \ldots, X_{r-a_{i_0}} \supseteq X_{s(b)} \supseteq X_{r-a_{i_0+1}}, \ldots, X_{r-a_k}$, we obtain using these continuous inclusions for domain and range spaces a continuous *map identical to* $g^{(k)}(x)$ *from* $(X_{s(b)})^{i_0} \times X_{r-a_{i_0+1}} \times \ldots \times X_{r-a_k}$ *to* X_{r-b} *and* *assume now that this map can be extended to the continuous map*

$$\widetilde{g^{(k)}}(x) : X_{r-a_1} \times \cdots \times X_{r-a_k} \to X_{r-b}$$

required in (ii). For notational reasons, we will identify $\widetilde{g^{(k)}}(x)$ *and* $g^{(k)}(x)$*.*

c) *Since Assumption 3.2 is very general and quite technical, we try to make if comprehensible by restating it for small order* r*. Continuing with the notation* $X_s = D(A^s)$*, note that* $X_0 = Y$*. For details on the definition of* $\widetilde{g'}(x)$ *and* $\widetilde{g''}(x)$*, check* ***b)*** *above.*

- *For* $0 < r \leqslant 1$ *(which means* $n = 1$*,* $\theta = r$*), we require* g *to be once Fréchet differentiable on* $X_{s(1)}$ *for some* $s(1) \in [0, r]$*. For* $x \in X_r$ *with* $\|x\|_r \leqslant M$*, we need to extend* $g'(x) : X_{s(1)} \to X_{s(1)}$ *to a map*

$$\widetilde{g'}(x) : X_0 \to X_0 \quad \textit{with} \quad \|g'(x)\|_{X_0 \to X_0} \leqslant C(M).$$

for some $C(M) > 0$*. Additionally, we need an* $s \in [0, r)$ *with*

$$\|g(u) - g(v)\|_0 \leqslant C(M_s)\,\|u - v\|_0$$

for $\|u\|_s, \|v\|_s \leqslant M_s$ *and*

$$\|g(u) - g(v)\|_s \leqslant C(M)\,\|u - v\|_s$$

as well as

$$\|g(u) - g(v)\|_r \leqslant C(M)\,\|u - v\|_r$$

for $\|u\|_r, \|v\|_r \leqslant M$*.*
To illustrate how these 'weaker' forms of differentiability and Lipschitz continuity work, we consider the example $g(u) = |u|^2 u$ *,* $Y = X_0 = L^p(\mathbb{R}^d)$ *and hence* $X_s = H^s_p(\mathbb{R}^d)$ *for the Bessel potential spaces in* $L^p(\mathbb{R}^d)$ *with regularity* $s \geqslant 0$*. Choosing* $r > \frac{d}{p}$*,* X_r *is a function algebra (see [RS96][4.6.4]) and hence* g *is real Fréchet differentiable on* X_r *with*

$$g'(x)[v] = 2\,|x|^2 v + x^2 \overline{v},$$

see Lemma 4.1 ***b)*** *and the Remark below it. Although this map is not the Fréchet derivative of g on any X_s with $s \leqslant \frac{d}{p}$ because the difference quotient does not converge, it still makes sense as a map from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for $x \in X_r$, simply because of the Sobolev embedding $X_r \hookrightarrow L^\infty(\mathbb{R}^d)$ (see [RS96][2.4.4]), which gives the multiplication estimate*

$$\|fg\|_0 \lesssim \|f\|_0 \|g\|_r \qquad \forall f \in X_0, g \in X_r.$$

This estimate corresponds to the extendability requirement of $g'(x)$ since for $x, v \in X_r$ with $\|x\|_r \leqslant M$ for some $M \geqslant 0$, we have

$$\|g'(x)[v]\|_0 \lesssim \|x\|_r^2 \|v\|_0 \leqslant M^2 \|v\|_0,$$

letting us extend $g'(x)$ to the required map $\widetilde{g'}(x)$. The three estimates on Lipschitz continuity follow by the two multiplication estimates when choosing $\frac{d}{p} < s < r$, interpolating between the two estimates to obtain the multiplication estimate

$$\|fg\|_s \lesssim \|f\|_s \|g\|_r \qquad \forall f \in X_s, g \in X_r$$

for the second estimate. For details, see Lemma 4.1.

- *For $1 < r \leqslant 2$ (hence $r = 1 + \theta = n - 1 + \theta$ for $n = 2$, $\theta = r - 1$), we require g to be once Fréchet differentiable on $X_{s(1)}$ for some $s(1) \in [1, r]$. For $x \in X_r$ with $\|x\|_r \leqslant M$, we need to be able to define*

  $$\widetilde{g'}(x) : X_1 \to X_1 \quad \text{with} \quad \|g'(x)\|_{X_1 \to X_1} \leqslant C(M)$$

  as well as, since $X_{s(1)} \subseteq X_\theta$, a continuous extension

  $$\widetilde{g'}(x) : X_\theta \to X_\theta \quad \text{with} \quad \|\widetilde{g'}(x)\|_{X_\theta \to X_\theta} \leqslant C(M),$$

  as in Remark ***b)*** *above. Moreover, we require g to be twice Fréchet differentiable on $X_{s(2)}$ for some $s(2) \in [0, r]$. For $x \in X_r$ with $\|x\|_r \leqslant M$, we need to be able to extend $g''(x) : X_{s(2)} \times X_{s(2)} \to X_{s(2)}$ to am* *map*

$$\widetilde{g''}(x) : X_0 \times X_r \to X_0 \quad \textit{with} \quad \|\widetilde{g''}(x)\|_{X_0 \times X_r \to X_0} \leqslant C(M)$$

as well as

$$\widetilde{g''}(x) : X_\theta \times X_1 \to X_0 \quad \textit{with} \quad \|g''(x)\|_{X_\theta \times X_1 \to X_0} \leqslant C(M).$$

*as in Remark **b)** above. Finally, we once again need an* $s \in [0, r)$ *with*

$$\|g(u) - g(v)\|_0 \leqslant C(M_s)\|u - v\|_0$$

for $\|u\|_s, \|v\|_s \leqslant M_s$ *and*

$$\|g(u) - g(v)\|_s \leqslant C(M)\|u - v\|_s$$

as well as

$$\|g(u) - g(v)\|_r \leqslant C(M)\|u - v\|_r$$

for $\|u\|_r, \|v\|_r \leqslant M$.

3.2 The splitting method

In order to define the general exponential splitting method we want to use, we first need to split up (3.1) into its linear and nonlinear part, respectively.

$$u'(t) = \begin{cases} (-Au)(t), & (3.2a) \\ g(u(t)), & (3.2b) \end{cases}$$

both having initial value $u(0) = u_0$. Equation (3.2a) has the solution $T(t)u_0 := \mathrm{e}^{-tA}u_0$ for all $u_0 \in Y$ and t depending on the kind of (semi)group A generates. Again from [Paz92, Theorem 6.1.4], setting $A = 0$, we obtain for $u_0 \in X_r$ a $T > 0$ and a (mild) solution $\psi_{u_0} \in C([-T, T], X_r)$ of (3.2b), where T once again only depends on $\|u_0\|_{X_r}$.
The principle behind the splitting process is to approximate the exact solution u of (3.1) by alternately following the linear and nonlinear solutions of (3.2a)

and (3.2b). For fixed $h > 0$ and $q \in \mathbb{N}$, we therefore define

$$\begin{aligned} S_1^h(u_0) &:= T(\alpha_1 h)u_0, \\ S_{i+1}^h(u_0) &:= T(\alpha_{i+1} h)\psi_{S_i^h(u_0)}(\beta_i h), \qquad i \in \{1,\dots,q-1\}, \\ S^h(u_0) &:= S_q^h(u_0). \end{aligned} \tag{3.3}$$

Our assumption on the coefficients α_i and β_i depend on our order of convergence $r > 0$ as well as the kind of (semi)group generated by $-A$. We first state possible values for orders up to four. The exact requirements for arbitrary orders will be given in Assumption 3.5 after Theorem 3.4. Any reader who is not interested in higher orders or the exact assumption on the coefficients which can be checked to fulfil the general Assumption 3.5 which we give below, as will be shown in Remark 3.12.

Remark 3.3

Possible splitting schemes for $r \leqslant 4$:

- $r \leqslant 1$: $q = 2$, $\alpha_1 = 0$, $\alpha_2 = 1$, $\beta_1 = 1$ *(Lie Splitting),*
- $r \leqslant 2$: $q = 2$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{2}$, $\beta_1 = 1$ *(Strang Splitting),*
- $r \leqslant 3$: $q = 3$, $\alpha_1 = \frac{1}{4} + \mathrm{i}\frac{\sqrt{3}}{12}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{1}{4} - \mathrm{i}\frac{\sqrt{3}}{12}$, $\beta_1 = \frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{6}$, $\beta_2 = \frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{6}$, *if A generates a fitting analytic semigroup and ψ admits complex times (otherwise, see $r \leqslant 4$),*
- $r \leqslant 4$: $q = 6$, $\alpha_1 = \alpha_6 = \frac{1}{8-2\cdot 4^{\frac{1}{3}}}$, $\alpha_2 = \alpha_5 = \frac{1}{4-4^{\frac{1}{3}}}$, $\alpha_3 = \alpha_4 = \frac{1-4^{\frac{1}{3}}}{8-2\cdot 4^{\frac{1}{3}}}$, $\beta_1 = \beta_2 = \beta_4 = \beta_5 = \frac{1}{4-4^{\frac{1}{3}}}$, $\beta_3 = -\frac{4^{\frac{1}{3}}}{4-4^{\frac{1}{3}}}$, *if A generates a C_0 group, or* $q = 5$, $\alpha_1 = \alpha_5 = \frac{1}{10} - \mathrm{i}\frac{1}{30}$, $\alpha_2 = \alpha_4 = \frac{4}{15} + \mathrm{i}\frac{2}{15}$, $\alpha_3 = \frac{4}{15} - \mathrm{i}\frac{1}{5}$, $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \frac{1}{4}$, *if A generates a fitting analytic semigroup.*

3.3 The result

Now that we have given all assumptions on our equation and splitting scheme, we are able to state the main result of this chapter, the global error estimate in Y on $[0, T]$. We mention here once that all constants depend on the choice of the operator A, the nonlinearity g as well as the variables q, α_i, β_i of the splitting scheme implicitly.

Theorem 3.4

Let $r > 0$ and $u_0 \in X_r$. Let assumptions 3.1, 3.2 and 3.5 (see Remark 3.3) hold and assume the mild solution $u \in C([0,T], X_r)$ of (3.1) fulfils $\|u(t)\|_{X_r} \leqslant R$ for all $t \in [0,T]$ and some $R \geqslant 0$. Then, we conclude that there exists an $h_0 \in (0,T]$ such that

$$\|(S^h)^N(u_0) - u(Nh)\|_Y \leqslant Ch^r$$

for all $h \in (0, h_0]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T$, where C only depends on R and T.

We give the proof directly at this point, citing three results which will be stated and shown over the course of this chapter.

Proof. We follow the standard concept called Lady Windermere's fan. By artificially generating a telescoping sum, we see that

$$\|(S^h)^N(u_0) - u(Nh)\|_Y \leqslant \sum_{k=0}^{N-1} \|(S^h)^{N-k}(u(kh)) - (S^h)^{N-(k+1)}(u((k+1)h))\|_Y \quad (3.4)$$

For the terms on the right hand side, we use the stability result from Proposition 3.6 **a)** $N-k-1$ times. It says that

$$\|S^h(w_1) - S^h(w_2)\|_Y \leqslant \mathrm{e}^{C(2R_s)h}\|w_1 - w_2\|_Y$$

as long as $\|w_1\|_{X_s}, \|w_2\|_{X_s} \leqslant R_s$ for s as in Assumption 3.2. Here, w_1 and w_2 are given by $(S^h)^{N-j}(u(kh))$ for $k \in \{0, \dots, N\}$ and $j \in \{k+1, \dots, N\}$. The fact that their Y norm is uniformly bounded (by $R_s = 2R$) for all k and j follows from Lemma 3.14. Hence, we obtain

$$\begin{aligned}\|(S^h)^{N-k}(u(kh)) &- (S^h)^{N-(k+1)}(u((k+1)h))\|_Y \\ &\leqslant \mathrm{e}^{C(4R)|\beta|h(N-k-1)}\|S^h(u(kh)) - u((k+1)h)\|_Y,\end{aligned}$$

where $|\beta| = \sum_{i=1}^{q-1}|\beta_i|$. Next, we use Proposition 3.13, which says that

$$\|S^h(u_0) - u(h)\|_Y \leqslant \widetilde{C}(R)h^{1+r}$$

as long as $\|u_0\|_{X_r} \leqslant R$. This is the result that actually requires Assumption 3.5 and Assumption 3.2 and gives us most of the work. Since we assume $\|u(t)\|_{X_r} \leqslant R$ for all $t \in [0,T]$, we can just replace u_0 by $u(kh)$ and end up with

$$\|S^h(u(kh)) - u((k+1)h)\|_Y \leqslant Ch^{r+1},$$

which finally lets us go back to (3.4) to see that

$$\begin{aligned}\|(S^h)^N(u_0) - u(Nh)\|_Y &\leqslant \sum_{k=0}^{N-1} \mathrm{e}^{C(4R)|\beta|h(N-k-1)} \|S^h(u(kh)) - u((k+1)h)\|_Y \\ &\leqslant \sum_{k=0}^{N-1} \mathrm{e}^{C(4R)|\beta|h(N-k-1)} Ch^{r+1} \leqslant Nh\mathrm{e}^{C(4R)|\beta|hN} Ch^r \leqslant T\mathrm{e}^{C(4R)|\beta|T} Ch^r.\end{aligned}$$

■

We now state the general assumption on the splitting scheme, which requires some technical definitions. A reader only interested in smaller orders and the well known schemes from Remark 3.3 can skip these technicalities. When comparing the exact solution to its numerical approximation at hand, we are going to use Taylor's Theorem on the function g and its derivatives. These functions are going to appear in composite expressions such as for example

$$g'''(x_1)\Big[g''(x_{2,1})\Big[g(x_{3,1}), g(x_{3,2})\Big], g'(x_{2,2})\Big[g'(x_{3,3})[g(x_{4,1})]\Big], g(x_{2,3})\Big],$$

with additional (semi)groups in front of every derivative of g which we omit here. We will denote the orders of those derivatives by $k_{j,r} \in \mathbb{N}_0$ ($j, r \in \mathbb{N}$), where j stands for the level in the composition and r is used for numbering them within one of those levels. In the above example, this means $k_{1,1} = 3, k_{2,1} = 2, k_{2,2} = 1, k_{2,3} = 0, k_{3,1} = 0, k_{3,2} = 0, k_{3,3} = 1, k_{4,1} = 0$. We now define

$$K_{-1} := k_{0,1} := 1, \quad K_j := \sum_{r=1}^{K_{j-1}} k_{j,r}, \quad S_m := \sum_{l=0}^{m} K_l, \qquad K_{j+1}^{(p)} = \sum_{s=1}^{\sum_{r=1}^{p} k_{j,r}} k_{j+1,s},$$

with $j, m \in \mathbb{N}_0$ and $p \in \{0, \ldots, K_{j-1}\}$. Notice that $p = 0$ renders the sum in the upper bound empty, hence $K_{j+1}^{(0)} = 0$. The K_j are obviously the sum of all orders on one level (in the example, $K_1 = 3, K_2 = 3, K_3 = 1, K_4 = 0$, giving us the number of arguments we have on the next level. The $K_{j+1}^{(p)}$ are auxiliary numbers which add the orders of derivatives on level $j+1$ occurring within the first p arguments on level j. In the example, this is only non-trivial on the third level, where $K_3^{(1)} = 0, K_3^{(2)} = K_3^{(3)} = 1 = K_3$, since the only (first) derivative of g appears in the second variable. S_m is the sum of all orders up to level m (in the example, $S_1 = 3, S_2 = 6, S_3 = 7, S_4 = 0$), important to separate the easy remainder terms from the main terms.

Moreover, repeated use of the variation of constants formula will leave us with two types of integration sets depending on the $k_{j,r}$ from above, namely

$$N_m^{(k_{j,r})} := \{0 \leqslant t_{1,1} \leqslant 1,\ \ 0 \leqslant t_{j,s} \leqslant t_{j-1,r} \text{ for } s \in \{K_{j-1}^{(r-1)}+1,\dots,K_{j-1}^{(r)}\} \text{ and } j = 2,\dots,m\}$$

for the representation of the exact solution as well as

$$\begin{aligned} M_m^{(k_{j,r}),(i_{j,s})} := \{&0 \leqslant t_{1,1} \leqslant 1,\ \ 0 \leqslant t_{j,s} \leqslant t_{j-1,r}\ (i_{j,s} = i_{j-1,r}),\ \ 0 \leqslant t_{j,s} \leqslant 1\ (i_{j,s} < i_{j-1,r}),\\ &j = 1,\dots,m,\ \ r = 1,\dots,K_{j-2},\ \ s = K_{j-1}^{(r-1)}+1,\dots,K_{j-1}^{(r)}\}. \end{aligned}$$

for the numerical approximation. Finally, we abbreviate the sum of the first parameters α_i from the linear part of the numerical approximation (see 3.3), that is $c_i := \sum_{l=1}^{j} \alpha_i$, since this is the only way they are going to appear in later calculations. The assumption now looks as follows.

Assumption 3.5

Let $r = n-1+\theta$ with $n \in \mathbb{N}$ and $\theta \in (0,1]$. Let all α_i be either nonnegative, real or lie in a sector $\Sigma_{\varphi'}$ for a $\varphi' \in [0,\varphi)$ depending on whether A generates a C_0 semigroup, a C_0 group or a analytic semigroup. Moreover, let all β_i be real, since the nonlinear flow might not be defined for complex times. If it is, one might also choose β_i to be complex.

For all $k_{j,r} \in \mathbb{N}_0$ for $j \in \{1,\dots,n\}$, $r \in \{1,\dots,K_{j-1}\}$ with $S_{n-1} \leqslant n$, we require that

$$\sum_{i_{1,1}=1}^{q-1} \sum_{\substack{i_{j,s}=1;\, j=2,\dots,n \\ r=1\dots,K_{j-2};\, s=K_{j-1}^{(r-1)}+1,\dots,K_{j-1}^{(r)}}}^{i_{j-1,r}} |M_n^{(i_{j,s})}| \left(\prod_{j=1}^{n} \prod_{s=1}^{K_{j-1}} \beta_{i_{j,s}} \right) P((c_{i_{j,s}})) = \int_{N_n^{(k_{j,r})}} P((t_{j,r}))\, \mathrm{d}t_{n,K_{n-1}} \cdots \mathrm{d}t_{1,1}$$

for all monomials $P \in \{(t_{j,r})^\gamma \mid \gamma \in \mathbb{N}_0^{S_{n-1}}, |\gamma| \leqslant n - S_{n-1}\}$ *in multi index notation.*

We are not able to prove the general existence of a q and (α_j), (β_j) which fulfil the equations in Assumption 3.5, and can also not reduce the number of equations for general $n \in \mathbb{N}$. But we suspect that the set of equations can be drastically reduced down to $(k_{j,r}) = (k_{1,1}, \dots, k_{N,1}) = (1, \dots, 1)$ for all $N \in \{1, \dots, n\}$ and the pure monomials in $t_{1,1}$, namely $P((t_{j,s})) = t_{1,1}^M$ for all $M \in \{0, \dots, n-1\}$. This definitely works for $n \leqslant 4$, as will be shown in Remark 3.12.

3.4 The stability

We start off with the stability result since it is the easier part of the proof and we are going to need part of it in subsequent computations.

Proposition 3.6

Let $w_1, w_2 \in X_r$ *with* $\|w_1\|_{X_s}, \|w_2\|_{X_s} \leqslant R_s$ *(in **a)**) or* $\|w_1\|_{X_r}, \|w_2\|_{X_r} \leqslant R$ *(in **b)**) and* $h \in \left(0, \frac{\log(2)}{C(2R)|\beta|}\right)$*, where* $|\beta| = \sum_{i=1}^{q-1} |\beta_i|$*. Then the following is true:*

a) $\|S^h(w_1) - S^h(w_2)\|_Y \leqslant \mathrm{e}^{C(2R_s)|\beta|h} \|w_1 - w_2\|_Y$.

b) $\|S^h(w_1) - S^h(w_2)\|_{X_s} \leqslant \mathrm{e}^{C(2R)|\beta|h} \|w_1 - w_2\|_{X_s}$.

c) $\|S_i^h(w_1)\|_{X_r} \leqslant 2R$ *for* $i \in \{1, \dots, q\}$.

Here, C is the constant from Assumption 3.2.

Proof. We will use combinations $(W, X_W, R_W) \in \{(Y, X_s, R_s), (X_s, X_r, R), (X_r, X_r, R)\}$ in this proof, all of which are justified by Assumption 3.2. First of all, we know that for any $v \in X_r$, ψ_v is the mild solution of (3.2b) and hence

$$\psi_v(t) = v + \int_0^t g(\psi_v(s)) \, \mathrm{d}s$$

for all t for which the solution exists. This means that for any $v_1, v_2 \in X_r$ with $\|v_1\|_{X_W}, \|v_2\|_{X_W} \leqslant \widetilde{R}$, we obtain

$$\|\psi_{v_1}(t) - \psi_{v_2}(t)\|_W \leqslant \|v_1 - v_2\|_W + \int_0^t \|g(\psi_{v_1}(s)) - g(\psi_{v_2}(s))\|_W \, \mathrm{d}s. \tag{3.5}$$

By Assumption 3.2, we obtain

$$\|g(v_1) - g(v_2)\|_W \leqslant C(\widetilde{R})\|v_1 - v_2\|_W,$$

which turns (3.5) into

$$\|\psi_{v_1}(t) - \psi_{v_2}(t)\|_W \leqslant \|v_1 - v_2\|_W + \int_0^t C(\widetilde{R})\|\psi_{v_1}(s) - \psi_{v_2}(s)\|_W \, \mathrm{d}s.$$

By Gronwall's inequality (see [Wal00, §29 VI.]), we conclude that

$$\|\psi_{v_1}(t) - \psi_{v_2}(t)\|_W \leqslant \mathrm{e}^{C(\widetilde{R})t}\|v_1 - v_2\|_W. \tag{3.6}$$

We now prove the statement inductively by showing that for every $i \in \{1,\dots,q\}$, we have

$$\|S_i^h(w_1) - S_i^h(w_2)\|_W \leqslant \mathrm{e}^{C(2R_W)\sum_{j=1}^{i-1}|\beta_j|h}\|w_1 - w_2\|_W. \tag{3.7}$$

The induction start ($i = 1$) is trivial since

$$\begin{aligned}\|S_1^h(w_1) - S_1^h(w_2)\|_W &= \|T(\alpha_1 h)w_1 - T(\alpha_1 h)w_2\|_W \\ &= \|T(\alpha_1 h)[w_1 - w_2]\|_W \leqslant \|w_1 - w_2\|_W,\end{aligned}$$

which is (3.7) because the sum in the exponential is empty. Assuming (3.7) for some $i \in \{1,\dots,q-1\}$, we see that

$$\begin{aligned}\|S_{i+1}^h(w_1) - S_{i+1}^h(w_2)\|_W &= \|T(\alpha_{i+1}h)[\psi_{S_i^h(w_1)}(\beta_i h) - \psi_{S_i^h(w_2)}(\beta_i h)]\|_W \\ &\leqslant \|\psi_{S_i^h(w_1)}(\beta_i h) - \psi_{S_i^h(w_2)}(\beta_i h)\|_W.\end{aligned}$$

Here, we use (3.6) with $v_1 = S_i^h(w_1)$ and $v_2 = S_i^h(w_2)$, meaning $\widetilde{R} = \mathrm{e}^{C(2R_W)\sum_{j=1}^{i-1}|\beta_j|h} R_W$ by (3.7) for either $w_1 = 0$ or $w_2 = 0$ plus the assumption $\|w_1\|_{X_W}, \|w_2\|_{X_W} \leqslant R_W$, to obtain

$$\|S_{i+1}^h(w_1) - S_{i+1}^h(w_2)\|_W \leqslant \mathrm{e}^{C(\mathrm{e}^{C(2R_W)\sum_{j=1}^{i-1}|\beta_j|h}R_W)|\beta_i|h}\|S_i^h(w_1) - S_i^h(w_2)\|_W.$$

By assumption, $\mathrm{e}^{C(2R_W)\sum_{j=1}^{i-1}|\beta_j|h} \leqslant 2$. Combining this with the induction assumption yields

$$\begin{aligned}\|S_{i+1}^h(w_1) - S_{i+1}^h(w_2)\|_W &\leqslant \mathrm{e}^{C(2R_W)|\beta_i|h}\mathrm{e}^{C(2R_W)\sum_{j=1}^{i-1}|\beta_j|h}\|w_1 - w_2\|_W \\ &= \mathrm{e}^{C(2R_W)\sum_{j=1}^{i}|\beta_j|h}\|w_1 - w_2\|_W,\end{aligned}$$

which is (3.7) for $i+1$ instead of i. This ends the induction.
The results now follow from (3.7): Part **a)** by using $(W, X_W, R_W) = (Y, X_s, R_s)$ and $i = q-1$, part **b)** by using $(W, X_W, R_W) = (X_s, X_r, R)$ and part **c)** by using $(W, X_W, R_W) = (X_r, X_r, R)$ and $w_2 = 0$ as well as the upper bound on h. ■

3.5 The local error

We start off by developing the exact solution at time h in terms of orders of h. We fix $h > 0$ and begin by noticing that the mild solution u of (3.1) fulfils the equation

$$u(h) = \underbrace{T(h)u_0}_{=:u_l(h)} + \underbrace{\int_0^h T(h-t)g(u(t))\,\mathrm{d}t}_{=:u_{nl}(h)} \tag{3.8}$$

in X_r (and also on all X_s where g is at least once differentiable by Assumption 3.2) by the variation of constants formula. Replacing $u(t)$ in the integral by the same formula, using Taylor's Theorem and iterating this process, we get the following representation.

Proposition 3.7

If g is n-times Fréchet differentiable on some X_s, then

$$u(h) = u_l(h) + \sum_{\substack{k_{j,r}=0 \\ j=1,\dots,n-1;r=1,\dots,K_{j-1} \\ S_{n-1}\leqslant n}}^{n-S_{j-1}} \frac{h^{S_{n-1}}}{\prod_{j=1}^{n-1}\prod_{r=1}^{K_{j-1}} k_{j,r}!} \int_{N_n^{(k_{j,r})}} F_{(k_{j,r})}((t_{j,r}h))\, \mathrm{d}t_{n,K_{n-1}} \cdots \mathrm{d}t_{1,1}$$

$$+ \sum_{m=1}^{n} \sum_{\substack{k_{j,r}=0 \\ j=1,\dots,m;r=1,\dots,K_{j-1} \\ S_{m-1}\leqslant n; S_m>n}}^{n-S_{j-1}} \frac{h^{S_{m-1}}}{\prod_{j=1}^{m}\prod_{r=1}^{K_{j-1}} k_{j,r}!} \int_{N_{m+1}^{(k_{j,r})}} \widetilde{F}_{(k_{j,r}),m}\Big[\Big(\int_0^1 g^{(k_{m,r})}(A_{k_{m,s}}^{n-S_{m-1}}(t_{m,s}))$$

$$\Big[\Big(T((t_{m,r}-t_{m+1,s})h)g(u(t_{m+1,s}h))\Big)_{s=K_m^{(r-1)}+1}^{K_m^{(r)}}\Big]\,\mathrm{d}\xi\Big)_{r=1}^{K_{m-1}}\Big]((t_{j,r}h))$$

$$\mathrm{d}t_{m+1,K_m} \cdots \mathrm{d}t_{1,1},$$

with

$$A_k^n(t) = \begin{cases} u_l(t) & ,k \in \{1,\dots,n-1\}, \\ u_l(t) + \xi u_{nl}(t) & ,k = n, \end{cases} \tag{3.9}$$

where the integrands $\widetilde{F}_{(k_{j,r}),m} := \widetilde{F}_{(k_{j,r})_{\substack{j=1,\dots,m-1 \\ r=1,\dots,K_{j-1}}}}$ *are defined inductively by*

$$\widetilde{F}_{(k_{j,r}),1}[v](t_{1,1}) := T((1-t_{1,1})h)v \qquad \forall v \in X_s,$$

$$\widetilde{F}_{(k_{j,r}),m+1}\Big[(v_s)_{s=1}^{K_m}\Big]((t_{j,r})) := \widetilde{F}_{(k_{j,r}),m}\Big[\Big(g^{(k_{m,r})}(u_l(t_{m,r}h))$$

$$[(T((t_{m,r}-t_{m+1,s})h)v_s)_{s=K_m^{(r-1)}+1}^{K_m^{(r)}}\Big)_{r=1}^{K_{m-1}}\Big]((t_{j,r}))$$

for all $(v_s)_{s=1}^{K_m} \in (X_s)^{K_m}$ *with* $(t_{j,r}) = (t_{j,r})_{\substack{j=1,\dots,m+1 \\ r=1,\dots,K_{j-1}}}$ *and* $m \in \mathbb{N}$. *The functions* $F_{(k_{j,r})} := F_{(k_{j,r})_{\substack{j=1,\dots,n-1 \\ r=1,\dots,K_{j-1}}}}$ *are now just defined by*

$$F_{(k_{j,r})}((t_{j,r})) = \widetilde{F}_{(k_{j,r}),n}\Big[\Big(g(u_l(t_{n,r}h)\Big)_{r=1}^{K_{n-1}}\Big]((t_{j,r}h)).$$

Proof. We show the result inductively, meaning we want to prove

$$u(h) = u_l(h) + \sum_{\substack{k_{j,r}=0 \\ j=1,\dots,p;r=1,\dots,K_{j-1} \\ S_p \leqslant n}}^{n-S_{j-1}} \frac{1}{\prod_{j=1}^{p} \prod_{r=1}^{K_{j-1}} k_{j,r}!} \int_{N_{p+1,h}^{(k_{j,r})}} \widetilde{F}_{(k_{j,r}),p}\Big[\Big(g^{(k_{p,r})}(u_l(t_{p,r}))$$

$$\Big[\Big(T(t_{p,r} - t_{p+1,s})g(u(t_{p+1,s}))\Big)_{s=K_p^{(r-1)}+1}^{K_p^{(r)}}\Big]\Big)_{r=1}^{K_{p-1}}\Big]((t_{j,r}))\, \mathrm{d}t_{p+1,K_p} \cdots \mathrm{d}t_{1,1}$$

$$+ \sum_{m=1}^{p} \sum_{\substack{k_{j,r}=0 \\ j=1,\dots,m;r=1,\dots,K_{j-1} \\ S_{m-1} \leqslant n; S_m > n}}^{n+1-S_{j-1}} \frac{1}{\prod_{j=1}^{m} \prod_{r=1}^{K_{j-1}} k_{j,r}!} \int_{N_{m+1,h}^{(k_{j,r})}} \widetilde{F}_{(k_{j,r}),m}\Big[\Big(\int_0^1 g^{(k_{m,r})}(A_{k_{m,s}}^{n-S_{m-1}}(t_{m,s}))$$

$$\Big[\Big(T(t_{m,r} - t_{m+1,s})g(u(t_{m+1,s}))\Big)_{s=K_m^{(r-1)}+1}^{K_m^{(r)}}\Big]\, \mathrm{d}\xi\Big)_{r=1}^{K_{m-1}}\Big]((t_{j,r}))\, \mathrm{d}t_{m+1,K_m} \cdots \mathrm{d}t_{1,1}, \tag{3.10}$$

for $p \in \{1, \dots, n\}$, where

$$N_{m,h}^{(k_{j,r})} := \{0 \leqslant t_{1,1} \leqslant h,\ \ 0 \leqslant t_{j,s} \leqslant t_{j-1,r} \text{ for } s \in \{K_{j-1}^{(r-1)} + 1, \dots, K_{j-1}^{(r)}\} \text{ and } j = 2, \dots, m\}. \tag{3.11}$$

For $p = n$, this is our desired formula. Comparing the two, we see that this is the case since on the one hand, we obtain the integration set $N_m^{(k_{j,r})}$ from $N_{m,h}^{(k_{j,r})}$ through the simple substitution $(t_{j,s}) \mapsto \left(\frac{t_{j,s}}{h}\right)$, giving us a power of h for every t and a factor h in front of every t as well. On the other hand, we have $k_{n,1} = \dots = k_{n,K_{n-1}} = 0$, which means we do not need to sum over those indices, $S_n = S_{n-1}$ and the $t_{n+1,r}$ do not occur. To see that this is in fact true, notice that for $k_{1,1} = 0$, the sums over $k_{2,j}$ are empty, so that they only occur for $k_{1,1} \geqslant 1$. This in turn means $S_1 = 1 + k_{1,1} \geqslant 2$, so that the upper bound for those sums is at most $n - 2$. This pattern continues, hence after iterating this process for a total of n times, the newest indices $k_{n,j}$ all need to be zero.

Starting off with $p = 1$, we use Taylor's Theorem for a function $f : \mathbb{R} \to X_s$. If f is n-times differentiable, then

$$f(1) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} + \frac{1}{n!} \int_0^1 f^{(n)}(\xi)\, \mathrm{d}\xi. \tag{3.12}$$

We set $f(s) := g(u_l(t) + su_{nl}(t))$ and compute

$$f^{(k)}(s) = g^{(k)}(u_l(t) + su_{nl}(t))\underbrace{[u_{nl}(t), \dots, u_{nl}(t)]}_{k \text{ times}},$$

therefore

$$\begin{aligned} g(u(t)) = f(1) = &\sum_{k=0}^{n-1} \frac{g^{(k)}(u_l(t))[u_{nl}(t), \dots, u_{nl}(t)]}{k!} \\ &+ \frac{1}{n!} \int_0^1 g^{(n)}(u_l(t) + \xi u_{nl}(t))[u_{nl}(t), \dots, u_{nl}(t)] \, \mathrm{d}\xi. \end{aligned}$$

Inserting the definition of $u_{nl}(t)$ into this and pulling out the integrals yields

$$\begin{aligned} g(u(t)) = &\sum_{k=0}^{n-1} \frac{1}{k!} \int_0^t \cdots \int_0^t g^{(k)}(u_l(t))[\big(T(t-s_r)g(u(s_r))\big)_{r=1}^{k}] \, \mathrm{d}s_k \cdots \mathrm{d}s_1 \\ &+ \frac{1}{n!} \int_0^t \cdots \int_0^t \int_0^1 g^{(n)}(u_l(t) + \xi u_{nl}(t))[\big(T(t-s_r)g(u(s_r))\big)_{r=1}^{n}] \, \mathrm{d}\xi \, \mathrm{d}s_n \cdots \mathrm{d}s_1. \end{aligned} \tag{3.13}$$

We plug this into (3.8) while renaming (k, t, s_j) by $(k_{1,1}, t_{1,1}, t_{2,j})$ to arrive at

$$\begin{aligned} u(h) = u_l(h) + &\sum_{k_{1,1}=0}^{n-1} \frac{1}{k_{1,1}!} \int_0^h \int_0^{t_{1,1}} \cdots \int_0^{t_{1,1}} \frac{g^{(k)}(u_l(t))[\big(T(t_{1,1}-t_{2,r})g(u(t_{2,r}))\big)_{r=1}^{k_{1,1}}]}{k_{1,1}!} \\ &\mathrm{d}t_{2,k_{1,1}} \cdots t_{2,1} + \frac{1}{n!} \int_0^h \int_0^{t_{1,1}} \cdots \int_0^{t_{1,1}} \int_0^1 g^{(k_{1,1})}(u_l(t_{1,1}) + \xi u_{nl}(t_{1,1}))) \\ &\Big[\big(T(t_{1,1}-t_{2,r})g(u(t_{2,r}))\big)_{r=1}^{n} \, \mathrm{d}\xi \, \mathrm{d}t_{2,k_{1,1}} \cdots t_{2,1}, \end{aligned}$$

which is (3.10) for $p = 1$. Now we assume that (3.10) is true for some fixed $p \in \{1, \dots, n-1\}$ and show it for $p+1$. We leave the first and the third term ($u_l(h)$ and the sum over m) be, since they are the same for $p+1$. For the second term, we first notice that

$$\begin{aligned} &\widetilde{F}_{(k_{j,r}),p}\Big[\Big(g^{(k_{p,r})}(u_l(t_{p,r}))\Big[\big(T(t_{p,r}-t_{p+1,s})g(u(t_{p+1,s}))\big)_{s=K_p^{(r-1)}+1}^{K_p^{(r)}}\Big]\Big)_{r=1}^{K_{p-1}}\Big]((t_{j,r})) \\ &\qquad = \widetilde{F}_{(k_{j,r}),p+1}\Big[\big(g(u(t_{p+1,s}))\big)_{s=1}^{K_p}\Big]((t_{j,r})). \end{aligned}$$

Now, we replace the terms $g(u(t_{p+1,s}))$ by (3.13), this time with $n+1-S_p$ instead of n, while splitting the result according to whether $S_{p+1} \leqslant n$ (which means there are no remainder terms involved) or $S_{p+1} > n$ (which means there might be remainder terms involved, see (3.9)). This yields

$$\sum_{\substack{k_{j,r}=0\\ j=1,\dots,p;r=1,\dots,K_{j-1}\\ S_p\leqslant n}}^{n-S_{j-1}} \frac{1}{\prod_{j=1}^{p}\prod_{r=1}^{K_{j-1}} k_{j,r}!} \int_{N_{p+1,h}^{(k_{j,r})}} \widetilde{F}_{(k_{j,r}),p+1}\Big[g(u(t_{p+1,s}))\Big]_{s=1}^{K_p}((t_{j,r}))\,\mathrm{d}t_{p+1,K_p}\cdots \mathrm{d}t_{1,1}$$

$$= \sum_{\substack{k_{j,r}=0\\ j=1,\dots,p;r=1,\dots,K_{j-1}\\ S_p\leqslant n}}^{n-S_{j-1}} \sum_{\substack{k_{p+1,1},\dots,k_{p+1,K_p}=0\\ S_{p+1}\leqslant n}}^{n-S_p} \frac{1}{\prod_{j=1}^{p}\prod_{r=1}^{K_{j-1}} k_{j,r}!}\frac{1}{\prod_{j=1}^{K_p} k_{p+1,j}!}$$

$$\int_{N_{p+1,h}^{(k_{j,r})}} \underbrace{\int_0^{t_{p,1}}\cdots\int_0^{t_{p,1}}}_{k_{p,1}\text{ times}}\cdots\cdots \underbrace{\int_0^{t_{p,K_{p-1}}}\cdots\int_0^{t_{p,K_{p-1}}}}_{k_{p,K_{p-1}}\text{ times}} \widetilde{F}_{(k_{j,r}),p+1}\Big[\Big(g^{(k_{p+1,r})}(u_l(t_{p+1,r}))$$

$$\Big[\Big(T(t_{p+1,r}-t_{p+2,s})g(u(t_{p+2,s}))\Big)_{s=K_{p+1}^{(r-1)}+1}^{K_{p+1}^{(r)}}\Big]\Big)_{r=1}^{K_p}\Big]((t_{j,r}))\,\mathrm{d}t_{p+2,K_{p+1}}\cdots \mathrm{d}t_{1,1}$$

$$+ \sum_{\substack{k_{j,r}=0\\ j=1,\dots,p;r=1,\dots,K_{j-1}\\ S_p\leqslant n}}^{n-S_{j-1}} \sum_{\substack{k_{p+1,1},\dots,k_{p+1,K_p}=0\\ S_{p+1}> n}}^{n-S_p} \frac{1}{\prod_{j=1}^{p}\prod_{r=1}^{K_{j-1}} k_{j,r}!}\frac{1}{\prod_{j=1}^{K_p} k_{p+1,j}!}\int_{N_{p+1,h}^{(k_{j,r})}}$$

$$\underbrace{\int_0^{t_{p,1}}\cdots\int_0^{t_{p,1}}}_{k_{p,1}\text{ times}}\cdots\cdots \underbrace{\int_0^{t_{p,K_{p-1}}}\cdots\int_0^{t_{p,K_{p-1}}}}_{k_{p,K_{p-1}}\text{ times}} \widetilde{F}_{(k_{j,r}),p+1}\Big[\Big(\int_0^1 g^{(k_{p+1,r})}(A_{k_{p+1,s}}^{n-S_p}(t_{p+1,s}))$$

$$\Big[\Big(T(t_{p+1,r}-t_{p+2,s})g(u(t_{p+2,s}))\Big)_{s=K_{p+1}^{(r-1)}+1}^{K_{p+1}^{(r)}}\Big]\,\mathrm{d}\xi\Big)_{r=1}^{K_p}\Big]((t_{j,r}))\,\mathrm{d}t_{p+2,K_{p+1}}\cdots \mathrm{d}t_{1,1}$$

$$= \sum_{\substack{k_{j,r}=0\\ j=1,\dots,p+1;r=1,\dots,K_{j-1}\\ S_{p+1}\leqslant n}}^{n-S_{j-1}} \frac{1}{\prod_{j=1}^{p+1}\prod_{r=1}^{K_{j-1}} k_{j,r}!} \int_{N_{p+2,h}^{(k_{j,r})}} \widetilde{F}_{(k_{j,r}),p+1}\Big[\Big(g^{(k_{p+1,r})}(u_l(t_{p+1,r}))$$

$$\Big[\Big(T(t_{p+1,r}-t_{p+2,s})g(u(t_{p+2,s}))\Big)_{s=K_{p+1}^{(r-1)}+1}^{K_{p+1}^{(r)}}\Big]\Big)_{r=1}^{K_p}\Big]((t_{j,r}))\,\mathrm{d}t_{p+2,K_{p+1}}\cdots \mathrm{d}t_{1,1}$$

$$+\sum_{\substack{k_{j,r}=0\\ j=1,\dots,p+1;r=1,\dots,K_{j-1}\\ S_p\leqslant n;S_{p+1}>n+1}}^{n-S_{j-1}} \frac{1}{\prod_{j=1}^{p+1}\prod_{r=1}^{K_{j-1}} k_{j,r}!}\int_{N_{p+2,h}^{(k_{j,r})}} \widetilde{F}_{(k_{j,r}),p+1}\Big[\Big(\int_0^1 g^{(k_{p+1,r})}(A_{k_{p+1,s}}^{n-S_p}(t_{p+1,s}))$$

$$\Big[\Big(T(t_{p+1,r}-t_{p+2,s})g(u(t_{p+2,s}))\Big)_{s=K_{p+1}^{(r-1)}+1}^{K_{p+1}^{(r)}}\Big]\,\mathrm{d}\xi\Big)_{r=1}^{K_p}\Big]((t_{j,r}))\,\mathrm{d}t_{p+2,K_{p+1}}\cdots\,\mathrm{d}t_{1,1}$$

Plugging this into (3.10), the second part becomes the new summand ($m=p+1$) for the third term in and we end up with

$$u(h)=u_l(h)+\sum_{\substack{k_{j,r}=0\\ j=1,\dots,p+1;r=1,\dots,K_{j-1}\\ S_{p+1}\leqslant n}}^{n-S_{j-1}} \frac{1}{\prod_{j=1}^{p+1}\prod_{r=1}^{K_{j-1}} k_{j,r}!}\int_{N_{p+2,h}^{(k_{j,r})}} \widetilde{F}_{(k_{j,r}),p}\Big[\Big(g^{(k_{p+1,r})}(u_l(t_{p+1,r}))$$

$$\Big[\Big(T(t_{p+1,r}-t_{p+2,s})g(u(t_{p+2,s}))\Big)_{s=K_{p+1}^{(r-1)}+1}^{K_{p+1}^{(r)}}\Big]\Big)_{r=1}^{K_p}\Big]((t_{j,r}))\,\mathrm{d}t_{p+2,K_{p+1}}\cdots\,\mathrm{d}t_{1,1}$$

$$+\sum_{m=1}^{p+1}\sum_{\substack{k_{j,r}=0\\ j=1,\dots,m;r=1,\dots,K_{j-1}\\ S_{m-1}\leqslant n;S_m>n}}^{n+1-S_{j-1}} \frac{1}{\prod_{j=1}^{m}\prod_{r=1}^{K_{j-1}} k_{j,r}!}\int_{N_{m+1,h}^{(k_{j,r})}} \widetilde{F}_{(k_{j,r}),m}\Big[\Big(\int_0^1 g^{(k_{m,r})}(A_{k_{m,s}}^{n-S_{m-1}}(t_{m,s}))$$

$$\Big[\Big(T(t_{m,r}-t_{m+1,s})g(u(t_{m+1,s}))\Big)_{s=K_m^{(r-1)}+1}^{K_m^{(r)}}\Big]\,\mathrm{d}\xi\Big)_{r=1}^{K_{m-1}}\Big]((t_{j,r}))\,\mathrm{d}t_{m+1,K_m}\cdots\,\mathrm{d}t_{1,1},$$

which is (3.10) for $p+1$ instead of p. This concludes the induction and therefore the proof. ■

Remark 3.8

Although the system behind the inductive definition of the integrand is quite straightforward, the structure of the nesting might not be clear at first glance. For this reason, we want to explicitly write down the terms for n up to four. For $n=1$*, we obtain*

$$u(h)=u_l(h)+\int_0^h T(h-t_{1,1})g(u_l(t_{1,1}))\,\mathrm{d}t_{1,1}+\int_0^h\int_0^{t_{1,1}} T(h-t_{1,1})g'(A_1^1(t_{1,1}))$$
$$[T(t_{2,1}-t_{1,1})g(u(t_{2,1}))]\,\mathrm{d}t_{2,1}\,\mathrm{d}t_{1,1}, \tag{3.14}$$

the last term being the remainder.

For $n = 2$, we get

$$
\begin{aligned}
u(h) = u_l(h) &+ \int_0^h T(h-t_{1,1}) g(u_l(t_{1,1}))\,\mathrm{d}t_{1,1} \\
&+ \int_0^h \int_0^{t_{1,1}} T(h-t_{1,1}) g'(u_l(t_{1,1}))\Big[T(t_{1,1}-t_{2,1}) g(u_l(t_{2,1}))\Big]\,\mathrm{d}t_{2,1}\,\mathrm{d}t_{1,1} \\
&+ \int_0^h \int_0^{t_{1,1}} \int_0^{t_{2,1}} T(h-t_{1,1}) g'(u_l(t_{1,1}))\Big[T(t_{1,1}-t_{2,1}) g'(A_1^1(t_{2,1})) \\
&\quad [T(t_{3,1}-t_{2,1}) g(u(t_{3,1}))]\Big]\,\mathrm{d}t_{3,1}\,\mathrm{d}t_{2,1}\,\mathrm{d}t_{1,1} \\
&+ \frac{1}{2}\int_0^h T(h-t_{1,1}) \int_0^1 g''(A_2^2(t_{1,1})) \\
&\quad [T(t_{2,1}-t_{1,1}) g(u(t_{2,1})), T(t_{2,2}-t_{1,1}) g(u(t_{2,2}))]\,\mathrm{d}\xi\,\mathrm{d}t_{2,2}\,\mathrm{d}t_{2,1}\,\mathrm{d}t_{1,1},
\end{aligned}
\tag{3.15}
$$

the last two being remainder terms. For $n = 3$, we end up with

$$
\begin{aligned}
u(h) = u_l(h) &+ \int_0^h T(h-t_{1,1}) g(u_l(t_{1,1}))\,\mathrm{d}t_{1,1} \\
&+ \int_0^h \int_0^{t_{1,1}} T(h-t_{1,1}) g'(u_l(t_{1,1}))\Big[T(t_{1,1}-t_{2,1}) g(u_l(t_{2,1}))\Big]\,\mathrm{d}t_{2,1}\,\mathrm{d}t_{1,1} \\
&+ \int_0^h \int_0^{t_{1,1}} \int_0^{t_{2,1}} T(h-t_{1,1}) g'(u_l(t_{1,1})) \\
&\quad \Big[T(t_{1,1}-t_{2,1}) g'(u_l(t_{2,1}))\Big[T(t_{2,1}-t_{3,1}) g(u_l(t_{3,1}))\Big]\Big]\,\mathrm{d}t_{3,1}\,\mathrm{d}t_{2,1}\,\mathrm{d}t_{1,1} \\
&+ \frac{1}{2}\int_0^h \int_0^{t_{1,1}} \int_0^{t_{1,1}} T(h-t_{1,1}) g''(u_l(t_{1,1})) \\
&\quad \Big[T(t_{1,1}-t_{2,1}) g(u_l(t_{2,1})), T(t_{1,1}-t_{2,2}) g(u_l(t_{2,2}))\Big]\,\mathrm{d}t_{2,2}\,\mathrm{d}t_{2,1}\,\mathrm{d}t_{1,1} \\
&+ \int_0^h \int_0^{t_{1,1}} \int_0^{t_{2,1}} T(h-t_{1,1}) g'(u_l(t_{1,1}))\Big[T(t_{1,1}-t_{2,1}) g'(u_l(t_{2,1})) \\
&\quad \Big[\int_0^1 g'(A_1^1(t_{3,1}))[T(t_{4,1}-t_{3,1}) g(u(t_{4,1}))]\Big]\,\mathrm{d}\xi\Big]\,\mathrm{d}t_{4,1}\,\mathrm{d}t_{3,1}\,\mathrm{d}t_{2,1}\,\mathrm{d}t_{1,1} \\
&+ \sum_{k_{1,1}=1}^{2} \sum_{\substack{k_{2,1},\dots,k_{2,k_{1,1}}=0 \\ k_{1,1}+k_{2,1}+\cdots+k_{2,k_{1,1}} \geqslant 3}}^{3-k_{1,1}} \frac{1}{k_{1,1}!k_{2,1}!\cdots k_{2,k_{1.1}}!} \int_0^h \int_0^{t_{1,1}} \cdots \int_0^{t_{1,1}} T(h-t_{1,1}) \\
&\quad g^{(k_{1,1})}(u_l(t_{1,1}))\Big[\Big(T(t_{2,r}-t_{1,1}) \int_0^1 g^{(k_{2,r})}(A_{k_{2,r}}^{3-k_{1,1}})\Big[\Big(T(t_{3,s}-t_{2,r})
\end{aligned}
$$

$$g(u(t_{3,s}))\Big)_{s=K_2^{(r-1)}+1}^{K_2^{(r)}}\Big]\,\mathrm{d}\xi\Big)_{r=1}^{k_{1,1}}\Big]\,\mathrm{d}t_{3,K_2}\cdots\,\mathrm{d}t_{1,1}$$

$$+\frac{1}{n!}\int_0^h T(h-t_{1,1})\int_0^1 g'''(A_3^3(t_{1,1}))[T(t_{2,1}-t_{1,1})g(u(t_{2,1})),$$

$$T(t_{2,2}-t_{1,1})g(u(t_{2,2})),T(t_{2,3}-t_{1,1})g(u(t_{2,3}))]\,\mathrm{d}\xi\,\mathrm{d}t_{1,1}, \tag{3.16}$$

where the last three summands (including the double sum which gives four terms) are remainder terms. For $n=4$, we have

$$u(h)=u_l(h)+\int_0^h T(h-t_{1,1})g(u_l(t_{1,1}))\,\mathrm{d}t_{1,1}$$

$$+\int_0^h\int_0^{t_{1,1}} T(h-t_{1,1})g'(u_l(t_{1,1}))\Big[T(t_{1,1}-t_{2,1})g(u_l(t_{2,1}))\Big]\,\mathrm{d}t_{2,1}\,\mathrm{d}t_{1,1}$$

$$+\int_0^h\int_0^{t_{1,1}}\int_0^{t_{2,1}} T(h-t_{1,1})g'(u_l(t_{1,1}))\Big[T(t_{1,1}-t_{2,1})g'(u_l(t_{2,1}))$$

$$\Big[T(t_{2,1}-t_{3,1})g(u_l(t_{3,1}))\Big]\Big]\,\mathrm{d}t_{3,1}\,\mathrm{d}t_{2,1}\,\mathrm{d}t_{1,1}$$

$$+\frac{1}{2}\int_0^h\int_0^{t_{1,1}}\int_0^{t_{1,1}} T(h-t_{1,1})g''(u_l(t_{1,1}))\Big[T(t_{1,1}-t_{2,1})g(u_l(t_{2,1})),$$

$$T(t_{1,1}-t_{2,2})g(u_l(t_{2,2}))\Big]\,\mathrm{d}t_{2,2}\,\mathrm{d}t_{2,1}\,\mathrm{d}t_{1,1}$$

$$+\int_0^h\int_0^{t_{1,1}}\int_0^{t_{2,1}}\int_0^{t_{3,1}} T(h-t_{1,1})g'(u_l(t_{1,1}))\Big[T(t_{1,1}-t_{2,1})g'(u_l(t_{2,1}))$$

$$\Big[T(t_{2,1}-t_{3,1})g'(u_l(t_{3,1}))[T(t_{3,1}-t_{4,1})g(u_l(t_{4,1}))]\Big]\Big]\,\mathrm{d}t_{4,1}\,\mathrm{d}t_{3,1}\,\mathrm{d}t_{2,1}\,\mathrm{d}t_{1,1}$$

$$+\frac{1}{2}\int_0^h\int_0^{t_{1,1}}\int_0^{t_{2,1}}\int_0^{t_{2,1}} T(h-t_{1,1})g'(u_l(t_{1,1}))\Big[T(t_{1,1}-t_{2,1})g''(u_l(t_{2,1}))$$

$$\Big[T(t_{2,1}-t_{3,1})g(u_l(t_{3,1})),T(t_{2,1}-t_{3,2})g(u_l(t_{3,2}))\Big]\Big]\,\mathrm{d}t_{3,2}\,\mathrm{d}t_{3,1}\,\mathrm{d}t_{2,1}\,\mathrm{d}t_{1,1}$$

$$+\frac{1}{2}\int_0^h\int_0^{t_{1,1}}\int_0^{t_{1,1}}\int_0^{t_{2,1}} T(h-t_{1,1})g''(u_l(t_{1,1}))\Big[T(t_{1,1}-t_{2,1})g'(u_l(t_{2,1}))$$

$$[T(t_{2,1}-t_{3,1})g(u_l(t_{3,1}))],T(t_{1,1}-t_{2,2})g(u_l(t_{2,2}))\Big]\,\mathrm{d}t_{3,1}\,\mathrm{d}t_{2,2}\,\mathrm{d}t_{2,1}\,\mathrm{d}t_{1,1}$$

$$+\frac{1}{2}\int_0^h\int_0^{t_{1,1}}\int_0^{t_{1,1}}\int_0^{t_{2,2}} T(h-t_{1,1})g''(u_l(t_{1,1}))\Big[T(t_{1,1}-t_{2,1})g(u_l(t_{2,1})),$$

$$T(t_{1,1}-t_{2,2})g'(u_l(t_{2,2}))[T(t_{2,2}-t_{3,1})g(u_l(t_{3,1}))]\Big]\,\mathrm{d}t_{3,1}\,\mathrm{d}t_{2,2}\,\mathrm{d}t_{2,1}\,\mathrm{d}t_{1,1}$$

$$+\frac{1}{6}\int_0^h\int_0^{t_{1,1}}\int_0^{t_{1,1}}\int_0^{t_{1,1}} T(h-t_{1,1})g'''(u_l(t_{1,1}))\Big[T(t_{1,1}-t_{2,1})g(u_l(t_{2,1})),$$

$$T(t_{1,1}-t_{2,2})g(u_l(t_{2,2})),T(t_{1,1}-t_{2,3})g(u_l(t_{2,3}))\Big]\,\mathrm{d}t_{2,3}\,\mathrm{d}t_{2,2}\,\mathrm{d}t_{2,1}\,\mathrm{d}t_{1,1}$$

$$+\sum_{m=1}^{4} \sum_{\substack{k_{j,r}=0 \\ j=1,\dots,m;r=1,\dots,K_{j-1} \\ S_{m-1}\leqslant n;S_m>n}}^{n-S_{j-1}} \frac{h^{S_{m-1}}}{\prod_{j=1}^{m}\prod_{r=1}^{K_{j-1}} k_{j,r}!} \int_{N_{m+1}^{(k_{j,r})}} \widetilde{F}_{(k_{j,r}),m}$$

$$\Big[\Big(\int_0^1 g^{(k_{m,r})}(A_{k_{m,s}}^{n-S_{m-1}}(t_{m,s}))\Big[\Big(T((t_{m,r}-t_{m+1,s})h)g(u(t_{m+1,s}h))\Big)_{s=K_m^{(r-1)}+1}^{K_m^{(r)}}\Big]$$

$$\mathrm{d}\xi\Big)_{r=1}^{K_{m-1}}\Big]((t_{j,r}h))\,\mathrm{d}t_{m+1,K_m}\cdots\,\mathrm{d}t_{1,1}, \tag{3.17}$$

where we didn't elaborate on the remainder term, since its details bear too little importance for the space they would use.

We now move on to developing the numerical approximation of the exact solution at time h in terms of orders of h. For $q \geqslant 2$, our splitting scheme is given by

$$S_q^h(u_0) = T(\alpha_q h)\psi_{S_{q-1}^h(u_0)}(\beta_{q-1}h), \tag{3.18}$$

see (3.3). We begin by noticing that by (3.2b), the nonlinear solution from above fulfils the equation

$$\psi_{S_{q-1}^h(u_0)}(\beta_{q-1}h) = S_{q-1}^h(u_0) + \underbrace{\int_0^{\beta_{q-1}h} g(\psi_{S_{q-1}^h(u_0)}(t))\,\mathrm{d}t}_{=:u_a^{(q-1)}(\beta_{q-1}h)} \tag{3.19}$$

by the variation of constants formula. Replacing $u(t)$ in the integral by the same formula, using Taylor's Theorem and iterating this process, we get the following representation.

Proposition 3.9

If g is n-times Fréchet differentiable on some X_s, then

$$S_q^h(u_0) = u_l(h) + \sum_{\substack{k_{j,r}=0 \\ j=1,\ldots,n-1;r=1,\ldots,K_{j-1} \\ S_{n-1}\leqslant n}}^{n-S_{j-1}} \frac{h^{S_{n-1}}}{\prod_{j=1}^{n-1}\prod_{r=1}^{K_{j-1}} k_{j,r}!} \sum_{i_{1,1}=1}^{q-1} \sum_{\substack{i_{j,s}=1;\,j=2,\ldots,n \\ r=1\ldots,K_{j-2};s=K_{j-1}^{(r-1)}+1,\ldots,K_{j-1}^{(r)}}}^{i_{j-1,r}}$$
$$\Big(\prod_{j=1}^{n}\prod_{s=1}^{K_{j-1}} \beta_{i_{j,s}}\Big)|M_n^{(k_{j,r}),(i_{j,s})}|\ F_{(k_{j,r},n)}((c_{i_{j,s}}h))$$
$$+\sum_{m=1}^{n} \sum_{\substack{k_{j,r}=0 \\ j=1,\ldots,m;r=1,\ldots,K_{j-1} \\ S_{m-1}\leqslant n;S_m>n}}^{n-S_{j-1}} \frac{1}{\prod_{j=1}^{m}\prod_{r=1}^{K_{j-1}} k_{j,r}!} \sum_{i_{1,1}=1}^{q-1} \sum_{\substack{i_{j,s}=1;\,j=2,\ldots,m+1 \\ r=1\ldots,K_{j-2};s=K_{j-1}^{(r-1)}+1,\ldots,K_{j-1}^{(r)}}}^{i_{j-1,r}}$$
$$\Big(\prod_{j=1}^{m+1}\prod_{s=1}^{K_{j-1}} \beta_{i_{j,s}}\Big)\int_{M_{m+1}^{(k_{j,r}),(i_{j,s})}} \widetilde{F}_{(k_{j,r}),m}\Big[\Big(\int_0^1 g^{(k_{m,r})}(B_{m+1}^{(i_{j,s})}(t_{m,r}h))$$
$$\Big[\Big(T(c_{i_{m+1,s}+1,i_{m,r}}h)g(\psi_{S^h_{i_{m+1,s}}(u_0)}(t_{m+1,s}))\Big)_{s=K_m^{(r-1)}+1}^{K_m^{(r)}}\Big]\,\mathrm{d}\xi\Big)_{r=1}^{K_{m-1}}\Big]$$
$$((c_{i_{j,s}}h))\,\mathrm{d}t_{m+1,K_m}\cdots\,\mathrm{d}t_{1,1},$$

where we recall the definitions made in subSection 3.3 and Proposition 3.7 and define

$$B_{m+1}^{(k_{j,r}),(i_{j,s})}(t_{m,r}) := \begin{cases} u_l(c_{i_{m,r}}h) & ,k_{m,r} < n-S_{m-1} \\ T(c_{i_0+1,i_{m,r}}h)(S_{i_0}^h(u_0)+\xi u_a^{(i_0)}(\beta_{i_0}h)) & ,k_{m,r} = n-S_{m-1},\ i_0 < i_{m,r}, \\ S_{i_{m,r}}^h(u_0)+\xi u_a^{(i_{m,r})}(t_{m,r}) & ,k_{m,r} = n-S_{m-1},\ i_0 = i_{m,r} \end{cases} \tag{3.20}$$

for $m \in \{1,\ldots,n\}$, where $i_0 := \min\{i_{m+1,s} \mid s \in \{K_m^{(r-1)}+1,\ldots,K_m^{(r)}\}\}$.

Proof. Again, we show the result inductively, meaning we want to prove

$$S_q^h(u_0) = u_l(h) + \sum_{\substack{k_{j,r}=0 \\ j=1,\ldots,p;r=1,\ldots,K_{j-1} \\ S_p\leqslant n}}^{n-S_{j-1}} \frac{1}{\prod_{j=1}^{p}\prod_{r=1}^{K_{j-1}} k_{j,r}!} \sum_{i_{1,1}=1}^{q-1} \sum_{\substack{i_{j,s}=1;\,j=2,\ldots,p+1 \\ r=1\ldots,K_{j-2};s=K_{j-1}^{(r-1)}+1,\ldots,K_{j-1}^{(r)}}}^{i_{j-1,r}}$$

$$\int_{M^{(i_{j,s})}_{p+1,(\beta_{i_{j,s}}h)}} \widetilde{F}_{(k_{j,r}),p}\Big[\Big(g^{(k_{p,r})}(u_l(c_{i_{p,r}}h))\Big[\Big(T(c_{i_{p+1,s}+1,i_{p,r}}h)$$

$$g(\psi_{S^h_{i_{p+1,s}}(u_0)}(t_{p+1,s}))\Big)_{s=K_p^{(r-1)}+1}^{K_p^{(r)}}\Big]\Big)_{r=1}^{K_{p-1}}\Big]((c_{i_{j,s}}h))\,\mathrm{d}t_{p+1,K_p}\cdots\,\mathrm{d}t_{1,1}$$

$$+\sum_{m=1}^{p}\sum_{\substack{k_{j,r}=0\\ j=1,\dots,m;r=1,\dots,K_{j-1}\\ S_{m-1}\leqslant n;S_m>n}}^{n+1-S_{j-1}} \frac{1}{\prod_{j=1}^m\prod_{r=1}^{K_{j-1}}k_{j,r}!}\sum_{i_{1,1}=1}^{q-1}\sum_{\substack{i_{j,s}=1;\,j=2,\dots,m+1\\ r=1\dots,K_{j-2};s=K_{j-1}^{(r-1)}+1,\dots,K_{j-1}^{(r)}}}^{i_{j-1,r}}$$

$$\int_{M^{(i_{j,s})}_{m+1,(\beta_{i_{j,s}}h)}} \widetilde{F}_{(k_{j,r}),m}\Big[\Big(\int_0^1 g^{(k_{m,r})}(B^{(k_{j,r}),(i_{j,s})}_{m+1}(t_{m,r}))\Big[\Big(T(c_{i_{m+1,s}+1,i_{m,r}}h)$$

$$g(\psi_{S^h_{i_{m+1,s}}(u_0)}(t_{m+1,s}))\Big)_{s=K_m^{(r-1)}+1}^{K_m^{(r)}}\Big]\,\mathrm{d}\xi\Big)_{r=1}^{K_{m-1}}\Big]((c_{i_{j,s}}h))\,\mathrm{d}t_{m+1,K_m}\cdots\,\mathrm{d}t_{1,1},\tag{3.21}$$

for $p \in \{1,\dots,n\}$, where

$$M^{(k_{j,r}),(i_{j,s})}_{m,(\beta_{i_{j,s}}h)} := \{0 \leqslant t_{1,1} \leqslant \beta_{i_{1,1}}h,\;\; 0 \leqslant t_{j,s} \leqslant t_{j-1,r}\;(i_{j,s} = i_{j-1,r}),\;\; 0 \leqslant t_{j,s} \leqslant \beta_{i_{j,s}}h$$
$$(i_{j,s} < i_{j-1,r}), j = 1,\dots,m,\;\; r = 1,\dots,K_{j-2},\;\; s = K_{j-1}^{(r-1)} + 1,\dots,K_{j-1}^{(r)}\}.$$

For $p = n$, this gives the desired formula. Comparing the two, we see that this is the case since on the one hand, we obtain the integration set $M_m^{(k_{j,r}),(i_{j,s})}$ from $M^{(i_{j,s})}_{m,(\beta_{i_{j,s}}h)}$ through the simple substitution $(t_{j,s}) \mapsto \Big(\frac{t_{j,s}}{\beta_{i_{j,s}}h}\Big)$, giving us again a power of h for every t as well as the corresponding β and a factor βh in front of every t. On the other hand, we have $k_{n,1} = \cdots = k_{n,K_{n-1}} = 0$, which means we do not need to sum over those indices, $S_n = S_{n-1}$ and the $t_{n+1,r}$ as well as the $i_{n+1,s}$ do not occur. Hence, the integrands in the second term are constant, so that the integral is actually just the volume of the integral set $M_n^{(i_{j,s})}$. The fact that this is true follows with the exact same argument as in the beginning of the proof of Proposition 3.7.

Starting off with $p = 1$, we actually need a second induction in that we show

$$S^h_q(u_0) = T(c_{\tilde{p}+1,q}h)S^h_{\tilde{p}}(u_0) + \sum_{i_{1,1}=\tilde{p}}^{q-1}\sum_{\substack{k^{(\tilde{p})}_{1,1},\dots,k^{(i_{1,1})}_{1,1}\geqslant 0\\ \sum_{j=\tilde{p}}^{i_{1,1}}k^{(j)}_{1,1}<n}} \frac{1}{\prod_{j=\tilde{p}}^{i_{1,1}}k^{(j)}_{1,1}!}\int_0^{\beta_{i_{1,1}}h} T(c_{i_{1,1}+1,q}h)$$

$$g^{(\sum_{j=\tilde p}^{i_{1,1}} k_{1,1}^{(j)})}(T(c_{\tilde p+1,i_{1,1}}h)S_{\tilde p}^h(u_0))[\underbrace{u_a^{(i_{1,1})}(t_{1,1})}_{k_{1,1}^{(i_{1,1})}\text{ times}},\underbrace{T(\alpha_{i_{1,1}}h)u_a^{(i_{1,1}-1)}(\beta_{i_{1,1}-1}h)}_{k_{1,1}^{(i_{1,1}-1)}\text{ times}},\dots,$$

$$\underbrace{T(c_{\tilde p+1,i_{1,1}}h)u_a^{(\tilde p)}(\beta_{\tilde p}h)}_{k_{1,1}^{(\tilde p)}\text{ times}}]\,\mathrm{d}t_{1,1}+\sum_{i_{1,1}=\tilde p}^{q-1}\sum_{\substack{k_{1,1}^{(\tilde p)},\dots,k_{1,1}^{(i_{1,1})}\geqslant 0\\ \sum_{j=\tilde p}^{i_{1,1}} k_{1,1}^{(j)}=n}}\frac{1}{\prod_{j=\tilde p}^{i_{1,1}} k_{1,1}^{(j)}!}\int_0^{\beta_{i_{1,1}}h}$$

$$T(c_{i_{1,1}+1,q}h)\int_0^1 g^{(\sum_{j=\tilde p}^{i_{1,1}} k_{1,1}^{(j)})}(\widetilde{B}_{1,\tilde p}^{i_{1,1}})[\underbrace{u_a^{(i_{1,1})}(t_{1,1})}_{k_{1,1}^{(i_{1,1})}\text{ times}},\underbrace{T(\alpha_{i_{1,1}}h)u_a^{(i_{1,1}-1)}(\beta_{i_{1,1}-1}h)}_{k_{1,1}^{(i_{1,1}-1)}\text{ times}},\dots,$$

$$\underbrace{T(c_{\tilde p+1,i_{1,1}}h)u_a^{(\tilde p)}(\beta_{\tilde p}h)}_{k_{1,1}^{(\tilde p)}\text{ times}}]\,\mathrm{d}\xi\,\mathrm{d}t_{1,1} \tag{3.22}$$

for $\tilde p \in \{1,\dots,q-1\}$, where

$$\widetilde{B}_{1,\tilde p}^{i_{1,1}} := \begin{cases} T(c_{j_0+1,i_{1,1}}h)(S_{j_0}^h(u_0)+\xi u_a^{(j_0)}(\beta_{j_0}h)) & ,j_0 < i_{1,1}, \\ S_{i_{1,1}}^h(u_0)+\xi u_a^{(i_{1,1})}(t_{1,1}) & ,j_0 = i_{1,1}, \end{cases}$$

with $j_0 := \min\{j \in \{\tilde p,\dots,i_{1,1}\} \mid k_{1,1}^{(j)} > 0\}$. This we show with an inversed induction, so starting with $\tilde p = q-1$. We use Taylor's Theorem (see (3.12), this time with $f(s) = g(S_{q-1}^h(u_0)+su_a^{(q-1)}(t))$, to see that

$$f^{(k)}(s) = g^{(k)}(S_{q-1}^h(u_0)+su_a^{(q-1)}(t))[\underbrace{u_a^{(q-1)}(t)}_{k\text{ times}}],$$

and therefore

$$\begin{aligned} g(\psi_{S_{q-1}^h(u_0)}(t)) = f(1) &= \sum_{k=0}^{n-1}\frac{1}{k!}g^{(k)}(S_{q-1}^h(u_0))[\underbrace{u_a^{(q-1)}(t)}_{k\text{ times}}] \\ &+\frac{1}{n!}\int_0^1 g^{(n)}(S_{q-1}^h(u_0)+\xi u_a^{(q-1)}(t))[\underbrace{u_a^{(q-1)}(t)}_{n\text{ times}}]\,\mathrm{d}\xi. \end{aligned} \tag{3.23}$$

Plugging in (3.23) into (3.19) and this into (3.18), we arrive at

$$S_q^h(u_0) = T(\alpha_q h)S_{q-1}^h(u_0) + \sum_{0\leqslant k_{1,1}^{(q-1)}<n} \frac{1}{k_{1,1}^{(q-1)}!}\int_0^{\beta_{q-1}h} T(\alpha_q h)g^{(k_{1,1}^{(q-1)})}(S_{q-1}^h(u_0))$$
$$\underbrace{[u_a^{(q-1)}(t_{1,1})]}_{k_{1,1}^{(q-1)}\text{ times}}\,\mathrm{d}t_{1,1} + \frac{1}{n!}\int_0^{\beta_{q-1}h} T(\alpha_q h)\int_0^1 g^{(n)}(S_{q-1}^h(u_0)+\xi u_a^{(q-1)}(t_{1,1}))$$
$$\underbrace{[u_a^{(q-1)}(t_{1,1})]}_{n\text{ times}}\,\mathrm{d}\xi\,\mathrm{d}t_{1,1},$$

which is exactly (3.22) for $\tilde{p} = q-1$. Now we assume that (3.22) holds for some $\tilde{p} \in \{2,\dots,q-1\}$ and show the same for $\tilde{p}-1$. This is done by replacing $T(c_{\tilde{p}+1,q}h)S_{\tilde{p}}^h(u_0)$ and $g^{(\sum_{j=\tilde{p}}^{i_{1,1}} k_{1,1}^{(j)})}(T(c_{\tilde{p}+1,i_{1,1}}h)S_{\tilde{p}}^h(u_0))$, respectively. To this end, we see that

$$T(c_{\tilde{p}+1,i_{1,1}}h)S_{\tilde{p}}^h(u_0) = T(c_{\tilde{p},i_{1,1}}h)S_{\tilde{p}-1}^h(u_0) + T(c_{\tilde{p},i_{1,1}}h)\underbrace{\int_0^{\beta_{\tilde{p}-1}h} g(\psi_{S_{\tilde{p}-1}^h(u_0)}(t_{1,1}))\,\mathrm{d}t_{1,1}}_{=u_a^{(\tilde{p}-1)}(\beta_{\tilde{p}-1}h)}$$

by definition in (3.3) and therefore, using (3.23) with $\tilde{p}-1$ instead of $q-1$ and $i_{1,1} = q$ above,

$$T(c_{\tilde{p}+1,q}h)S_{\tilde{p}}^h(u_0) = T(c_{\tilde{p},q}h)S_{\tilde{p}-1}^h(u_0) + \sum_{k_{1,1}^{(\tilde{p}-1)}=0}^{n-1} \frac{1}{k_{1,1}^{(\tilde{p}-1)}!}\int_0^{\beta_{\tilde{p}-1}h} T(c_{\tilde{p},q}h)$$
$$g^{(k_{1,1}^{(\tilde{p}-1)})}(S_{\tilde{p}-1}^h(u_0))\underbrace{[u_a^{(\tilde{p}-1)}(t_{1,1})]}_{k_{1,1}^{(\tilde{p}-1)}\text{ times}}\,\mathrm{d}t_{1,1}$$
$$+\frac{1}{n!}\int_0^{\beta_{\tilde{p}-1}h} T(c_{\tilde{p},q}h)\int_0^1 g^{(n)}(S_{\tilde{p}-1}^h(u_0)+\xi u_a^{(\tilde{p}-1)}(t_{1,1}))$$
$$\underbrace{[u_a^{(\tilde{p}-1)}(t_{1,1})]}_{n\text{ times}}\,\mathrm{d}\xi\,\mathrm{d}t_{1,1},$$

as well as

$$g^{(\sum_{j=\tilde{p}}^{i_{1,1}} k_{1,1}^{(j)})}(T(c_{\tilde{p}+1,i_{1,1}}h)S_{\tilde{p}}^h(u_0))[\underbrace{u_a^{(i_{1,1})}(t_{1,1})}_{k_{1,1}^{(i_{1,1})}\text{ times}}, \underbrace{T(\alpha_{i_{1,1}}h)u_a^{(i_{1,1}-1)}(\beta_{i_{1,1}-1}h)}_{k_{1,1}^{(i_{1,1}-1)}\text{ times}}, \dots,$$

$$\underbrace{T(c_{\tilde{p}+1,i_{1,1}}h)u_a^{(\tilde{p})}(\beta_{\tilde{p}}h)}_{k_{1,1}^{(\tilde{p})}\text{ times}}]$$

$$= \sum_{k_{1,1}^{(\tilde{p}-1)}=0}^{n-1-\sum_{j=\tilde{p}}^{i_{1,1}} k_{1,1}^{(j)}} \frac{1}{k_{1,1}^{(\tilde{p}-1)}!} g^{(\sum_{j=\tilde{p}-1}^{i_{1,1}} k_{1,1}^{(j)})}(T(c_{\tilde{p},i_{1,1}}h)S_{\tilde{p}-1}^h(u_0))[\underbrace{u_a^{(i_{1,1})}(t_{1,1})}_{k_{1,1}^{(i_{1,1})}\text{ times}},$$

$$\underbrace{T(\alpha_{i_{1,1}}h)u_a^{(i_{1,1}-1)}(\beta_{i_{1,1}-1}h)}_{k_{1,1}^{(i_{1,1}-1)}\text{ times}}, \dots, \underbrace{T(c_{\tilde{p},i_{1,1}}h)u_a^{(\tilde{p}-1)}(\beta_{\tilde{p}-1}h)}_{k_{1,1}^{(\tilde{p}-1)}\text{ times}}]$$

$$+ \frac{1}{(n-\sum_{j=\tilde{p}-1}^{i_{1,1}} k_{1,1}^{(j)})!} \int_0^1 g^{(n)}(T(c_{\tilde{p},i_{1,1}}h)(S_{\tilde{p}-1}^h(u_0) + \xi u_a^{(\tilde{p}-1)}(\beta_{\tilde{p}-1})))$$

$$[\underbrace{u_a^{(i_{1,1})}(t_{1,1})}_{k_{1,1}^{(i_{1,1})}\text{ times}}, \underbrace{T(\alpha_{i_{1,1}}h)u_a^{(i_{1,1}-1)}(\beta_{i_{1,1}-1}h)}_{k_{1,1}^{(i_{1,1}-1)}\text{ times}}, \dots, \underbrace{T(c_{\tilde{p},i_{1,1}}h)u_a^{(\tilde{p}-1)}(\beta_{\tilde{p}-1}h)}_{k_{1,1}^{(\tilde{p}-1)}\text{ times}}]\,\mathrm{d}\xi,$$

by Taylor's Theorem. Inserting these observations into (3.22), we obtain

$$S_q^h(u_0) = T(c_{\tilde{p},q}h)S_{\tilde{p}-1}^h(u_0) + \sum_{i_{1,1}=\tilde{p}-1}^{q-1} \sum_{\substack{k_{1,1}^{(\tilde{p}-1)},\dots,k_{1,1}^{(i_{1,1})}\geqslant 0 \\ \sum_{j=\tilde{p}-1}^{i_{1,1}} k_{1,1}^{(j)}<n}} \frac{1}{\prod_{j=\tilde{p}-1}^{i_{1,1}} k_{1,1}^{(j)}!} \int_0^{\beta_{i_{1,1}}h} T(c_{i_{1,1}+1,q}h)$$

$$g^{(\sum_{j=\tilde{p}-1}^{i_{1,1}} k_{1,1}^{(j)})}(T(c_{\tilde{p},i_{1,1}}h)S_{\tilde{p}-1}^h(u_0))[\underbrace{u_a^{(i_{1,1})}(t_{1,1})}_{k_{1,1}^{(i_{1,1})}\text{ times}}, \underbrace{T(\alpha_{i_{1,1}}h)u_a^{(i_{1,1}-1)}(\beta_{i_{1,1}-1}h)}_{k_{1,1}^{(i_{1,1}-1)}\text{ times}}, \dots,$$

$$\underbrace{T(c_{\tilde{p},i_{1,1}}h)u_a^{(\tilde{p}-1)}(\beta_{\tilde{p}-1}h)}_{k_{1,1}^{(\tilde{p}-1)}\text{ times}}]\,\mathrm{d}t_{1,1} + \sum_{i_{1,1}=\tilde{p}-1}^{q-1} \sum_{\substack{k_{1,1}^{(\tilde{p}-1)},\dots,k_{1,1}^{(i_{1,1})}\geqslant 0 \\ \sum_{j=\tilde{p}-1}^{i_{1,1}} k_{1,1}^{(j)}=n}} \frac{1}{\prod_{j=\tilde{p}-1}^{i_{1,1}} k_{1,1}^{(j)}!}$$

$$\int_0^{\beta_{i_{1,1}}h} T(c_{i_{1,1}+1,q}h) \int_0^1 g^{(\sum_{j=\tilde{p}-1}^{i_{1,1}} k_{1,1}^{(j)})}(\widetilde{B}_{1,\tilde{p}-1}^{(i_{1,1})})[\underbrace{u_a^{(i_{1,1})}(t_{1,1}),}_{k_{1,1}^{(i_{1,1})}\text{ times}}$$

$$\underbrace{T(\alpha_{i_{1,1}}h)u_a^{(i_{1,1}-1)}(\beta_{i_{1,1}-1}h),}_{k_{1,1}^{(i_{1,1}-1)}\text{ times}}\dots,\underbrace{T(c_{\tilde{p},i_{1,1}}h)u_a^{(\tilde{p}-1)}(\beta_{\tilde{p}-1}h)}_{k_{1,1}^{(\tilde{p}-1)}\text{ times}}]\,\mathrm{d}\xi\,\mathrm{d}t_{1,1},$$

where the summands for $i_{1,1} = \tilde{p}-1$ are the two last summands from $T(c_{\tilde{p}+1,q}h)$ $S_{\tilde{p}}^h(u_0)$ above. This is (3.22) for $\tilde{p}$ replaced by $\tilde{p}-1$. For $\tilde{p} = 1$, we get the representation

$$\begin{aligned} S_q^h(u_0) = u_l(c_qh) + \sum_{i_{1,1}=1}^{q-1} \sum_{\substack{k_{1,1}^{(1)},\dots,k_{1,1}^{(i_{1,1})}\geqslant 0 \\ \sum_{j=1}^{i_{1,1}} k_{1,1}^{(j)}<n}} \frac{1}{\prod_{j=1}^{i_{1,1}} k_{1,1}^{(j)}!} \int_0^{\beta_{i_{1,1}}h} T(c_{i_{1,1}+1,q}h) \\ g^{(\sum_{j=1}^{i_{1,1}} k_{1,1}^{(j)})}(u_l(c_{i_{1,1}}h))[\underbrace{u_a^{(i_{1,1})}(t_{1,1}),}_{k_{1,1}^{(i_{1,1})}\text{ times}}\underbrace{T(\alpha_{i_{1,1}}h)u_a^{(i_{1,1}-1)}(\beta_{i_{1,1}-1}h),}_{k_{1,1}^{(i_{1,1}-1)}\text{ times}}\dots, \\ \underbrace{T(c_{2,i_{1,1}}h)u_a^{(1)}(\beta_1h)}_{k_{1,1}^{(1)}\text{ times}}]\,\mathrm{d}t_{1,1} \\ + \sum_{i_{1,1}=1}^{q-1} \sum_{\substack{k_{1,1}^{(1)},\dots,k_{1,1}^{(i_{1,1})}\geqslant 0 \\ \sum_{j=1}^{i_{1,1}} k_{1,1}^{(j)}=n}} \frac{1}{\prod_{j=1}^{i_{1,1}} k_{1,1}^{(j)}!} \int_0^{\beta_{i_{1,1}}h} T(c_{i_{1,1}+1,q}h) \int_0^1 g^{(n)}(\widetilde{B}_{1,1}^{(i_{1,1})}) \\ [\underbrace{u_a^{(i_{1,1})}(t_{1,1}),}_{k_{1,1}^{(i_{1,1})}\text{ times}}\underbrace{T(\alpha_{i_{1,1}}h)u_a^{(i_{1,1}-1)}(\beta_{i_{1,1}-1}h),}_{k_{1,1}^{(i_{1,1}-1)}\text{ times}}\dots,\underbrace{T(c_{2,i_{1,1}}h)u_a^{(1)}(\beta_1h)}_{k_{1,1}^{(1)}\text{ times}}]\,\mathrm{d}\xi\,\mathrm{d}t_{1,1}. \end{aligned} \tag{3.24}$$

Next, we want to get rid of the asymmetry in the arguments of the g derivatives. For fixed $i_{1,1}$ and $k_{1,1}^{(j)}$, if instead of arranging the arguments as above, we let the r-th argument be

$$T(c_{i_{2,r}+1,i_{1,1}}h)u_a^{(i_{2,r})}(\beta_{i_{2,r}}h)$$

for some $i_{2,r} < i_{1,1}$, and $u_a^{(i_{1,1})}(t_{1,1})$ if $i_{2,r} = i_{1,1}$, we just need to assume that $k_{1,1}^{(j)}$ of the $i_{2,r}$ take the value j to obtain the same value from the g derivative, since it is actually symmetric in its arguments. Combinatorially, there are

$$\frac{\left(\sum_{j=1}^{i} k_{1,1}^{(j)}\right)!}{\prod_{j=1}^{i} k_{1,1}^{(j)}!}$$

ways to achieve this, which is why we must divide by this number in order to obtain the equivalent of one term of the g derivative. Naming the sum $\sum_{j=1}^{i_{1,1}} k_{1,1}^{(j)}$ as $k_{1,1}$ and summing over all possibilities for the $k_{1,1}^{(j)}$, which gives us all possibilities for the $i_{2,r}$, changes (3.24) into

$$\begin{aligned} S_q^h(u_0) = u_l(c_q h) &+ \sum_{k_{1,1}=0}^{n-1} \frac{1}{k_{1,1}!} \sum_{i_{1,1}=1}^{q-1} \sum_{i_{2,1},\ldots,i_{2,k_{1,1}}=1}^{i_{1,1}} \int_{M_{2,(\beta_{i_{j,s}} h)}^{(i_{j,s})}} T(c_{i_{1,1}+1,q} h) g^{(k_{1,1})}(u_l(c_{i_{1,1}} h)) \\ &[(T(c_{i_{2,r}+1,i_{1,1}} h) g(\psi_{S_{i_{2,r}}^h(u_0)}(t_{2,r})))_{r=1}^{k_{1,1}}] \, \mathrm{d}t_{2,k_{1,1}} \cdots \mathrm{d}t_{2,1} \, \mathrm{d}t_{1,1} \\ &+ \frac{1}{n!} \sum_{i_{1,1}=1}^{q-1} \sum_{i_{2,1},\ldots,i_{2,n}=1}^{i_{1,1}} \int_{M_{2,(\beta_{i_{j,s}} h)}^{(i_{j,s})}} T(c_{i_{1,1}+1,q} h) \int_0^1 g^{(n)}(B_2^{(k_{j,r}),(i_{j,s})}(t_{1,1})) \\ &[(T(c_{i_{2,r}+1,i_{1,1}} h) g(\psi_{S_{i_{2,r}}^h(u_0)}(t_{2,r})))_{r=1}^{n}] \, \mathrm{d}t_{2,n} \cdots \mathrm{d}t_{2,1} \, \mathrm{d}\xi \, \mathrm{d}t_{1,1}, \end{aligned} \tag{3.25}$$

where we also inserted the definition of the $u_a^{(j)}$. This is (3.21) for $p = 1$, using that for all $n \in \mathbb{N}$, the first condition coming from Assumption 3.5 is $c_q = 1$ (see Remark (3.12)).

We now assume that (3.21) is true for some $p \in \{1, \ldots, n-1\}$ and show the same for p replaced by $p + 1$. We do that by replacing the innermost appearances of g in the first big sum, the rest will remain untouched. Again, we need a second induction showing that for fixed $k_{j,r}$ ($j = 1, \ldots, p$, $r = 1, \ldots, K_{j-1}$) and $i_{j,r}$ ($j = 1, \ldots, p+1$, $r = 1, \ldots, K_{j-1}$) as well as for fixed $s \in \{1, \ldots, K_p\}$, we have

$$g(\psi_{S_{i_{p+1,s}}^h(u_0)}(t_{p+1,s})) = \sum_{\substack{k_{p+1,s}^{(\tilde{p})},\ldots,k_{p+1,s}^{(i_{p+1,s})} \geq 0 \\ \sum_{j=\tilde{p}}^{i_{p+1,s}} k_{p+1,s}^{(j)} < n - S_p}} \frac{1}{\prod_{j=\tilde{p}}^{i_{p+1,s}} k_{p+1,s}^{(j)}!} g^{(\sum_{j=\tilde{p}}^{i_{p+1,s}} k_{p+1,s}^{(j)})}(T(c_{\tilde{p}+1,i_{p+1,s}} h) S_{\tilde{p}}^h(u_0))$$

$$
\underbrace{[u_a^{(i_{p+1,s})}(t_{p+1,s}),}_{k_{p+1,s}^{(i_{p+1,s})}\text{ times}}\underbrace{T(\alpha_{i_{p+1,s}}h)u_a^{(i_{p+1,s}-1)}(\beta_{i_{p+1,s}-1}h)}_{k_{p+1,s}^{(i_{p+1,s}-1)}\text{ times}}\ldots,\underbrace{T(c_{\tilde{p}+1,i_{p+1,s}}h)u_a^{(\tilde{p})}(\beta_{\tilde{p}}h)]}_{k_{p+1,s}^{(\tilde{p})}\text{ times}}
$$

$$
+\sum_{\substack{k_{p+1,s}^{(\tilde{p})},\ldots,k_{p+1,s}^{(i_{p+1,s})}\geqslant 0\\ \sum_{j=\tilde{p}}^{i_{p+1,s}}k_{p+1,s}^{(j)}=n-S_p}}\frac{1}{\prod_{j=\tilde{p}}^{i_{p+1,s}}k_{p+1,s}^{(j)}!}\int_0^1 g^{(\sum_{j=\tilde{p}}^{i_{p+1,s}}k_{p+1,s}^{(j)})}(\widetilde{B}_{p+1,\tilde{p}}^{(k_{j,r}),(i_{j,s})})\underbrace{[u_a^{(i_{p+1,s})}(t_{p+1,s}),}_{k_{p+1,s}^{(i_{p+1,s})}\text{ times}}
$$

$$
\underbrace{T(\alpha_{i_{p+1,s}}h)u_a^{(i_{p+1,s}-1)}(\beta_{i_{p+1,s}-1}h)}_{k_{p+1,s}^{(i_{p+1,s}-1)}\text{ times}}\ldots,\underbrace{T(c_{\tilde{p}+1,i_{p+1,s}}h)u_a^{(\tilde{p})}(\beta_{\tilde{p}}h)]}_{k_{p+1,s}^{(\tilde{p})}\text{ times}}\,\mathrm{d}\xi \tag{3.26}
$$

for $\tilde{p}\in\{1,\ldots,i_{p+1,s}\}$, where

$$
\widetilde{B}_{p+1,\tilde{p}}^{(k_{j,r}),(i_{j,s})}:=\begin{cases} T(c_{j_0+1,i_{p+1,s}}h)(S_{j_0}^h(u_0)+\xi u_a^{(j_0)}(\beta_{j_0}h)) & ,j_0<i_{p+1,s},\\ S_{i_{p+1,s}}^h(u_0)+\xi u_a^{(i_{p+1,s})}(t_{p+1,s}) & ,j_0=i_{p+1,s}.\end{cases}
$$

where $j_0:=\min\{j\in\{\tilde{p},\ldots,i_{p+1,s}\}\mid k_{p+1,s}^{(j)}>0\}$. We show this via an inversed induction, so starting with $\tilde{p}=i_{p+1,s}$. We have

$$
\begin{aligned}
g(\psi_{S_{i_{p+1,s}}^h(u_0)}(t_{p+1,s})) &= g(S_{i_{p+1,s}}^h(u_0)+u_a^{(i_{p+1,s})}(t_{p+1,s}))\\
&=\sum_{k_{p+1,s}^{(i_{p+1,s})}=0}^{n-S_p-1}\frac{1}{k_{p+1,s}^{(i_{p+1,s})}!}g^{(k_{p+1,s}^{(i_{p+1,s})})}(S_{i_{p+1,s}}^h(u_0))\underbrace{[u_a^{(i_{p+1,s})}(t_{p+1,s})]}_{k_{p+1,s}^{(i_{p+1,s})}\text{ times}}\\
&\quad+\frac{1}{(n-S_p)!}\int_0^1 g^{(n-S_p)}(\widetilde{B}_{p+1,i_{p+1,s}}^{(k_{j,r}),(i_{j,s})})\underbrace{[u_a^{(i_{p+1,s})}(t_{p+1,s})]}_{k_{p+1,s}^{(i_{p+1,s})}\text{ times}}\,\mathrm{d}\xi,
\end{aligned}
$$

by Taylor's Theorem, which is (3.26) for $\tilde{p}=i_{p+1,s}$. Now we assume it holds for $\tilde{p}\in\{2,\ldots,i_{p+1,s}\}$ and also show it for $\tilde{p}-1$. This is done by replacing

$$
g^{(\sum_{j=\tilde{p}}^{i_{p+1,s}}k_{p+1,s}^{(j)})}(T(c_{\tilde{p}+1,i_{p+1,s}}h)S_{\tilde{p}}^h(u_0)).
$$

To this end, we see that

$$T(c_{\tilde p+1,i_{p+1,s}}h)S^h_{\tilde p}(u_0) = T(c_{\tilde p,i_{p+1,s}}h)S^h_{\tilde p-1}(u_0) + T(c_{\tilde p,i_{p+1,s}}h)\underbrace{\int_0^{\beta_{\tilde p-1}h} g(\psi_{S^h_{\tilde p-1}(u_0)}(t_{1,1}))\,\mathrm{d}t_{1,1}}_{=u_a^{(\tilde p-1)}(\beta_{\tilde p-1}h)}$$

by definition in (3.3) and therefore

$$\begin{aligned}
&g^{(\sum_{j=\tilde p}^{i_{p+1,s}} k^{(j)}_{p+1,s})}(T(c_{\tilde p+1,i_{p+1,s}}h)S^h_{\tilde p}(u_0))[\underbrace{u_a^{(i_{p+1,s})}(t_{p+1,s})}_{k^{(i_{p+1,s})}_{p+1,s}\text{ times}},\underbrace{T(\alpha_{i_{p+1,s}}h)u_a^{(i_{p+1,s}-1)}(\beta_{i_{p+1,s}-1}h)}_{k^{(i_{p+1,s}-1)}_{p+1,s}\text{ times}},\dots,\\
&\qquad\underbrace{T(c_{\tilde p+1,i_{p+1,s}}h)u_a^{(\tilde p)}(\beta_{\tilde p}h)}_{k^{(\tilde p)}_{p+1,s}\text{ times}}]\\
&= \sum_{k^{(\tilde p-1)}_{p+1,s}=0}^{n-1-\sum_{j=\tilde p}^{i_{p+1,s}} k^{(j)}_{p+1,s}} \frac{1}{k^{(\tilde p-1)}_{p+1,s}!} g^{(\sum_{j=\tilde p-1}^{i_{p+1,s}} k^{(j)}_{p+1,s})}(T(c_{\tilde p,i_{p+1,s}}h)S^h_{\tilde p-1}(u_0))\\
&\quad[\underbrace{u_a^{(i_{p+1,s})}(t_{p+1,s})}_{k^{(i_{p+1,s})}_{p+1,s}\text{ times}},\underbrace{T(\alpha_{i_{p+1,s}}h)u_a^{(i_{p+1,s}-1)}(\beta_{i_{p+1,s}-1}h)}_{k^{(i_{p+1,s}-1)}_{p+1,s}\text{ times}},\dots,\underbrace{T(c_{\tilde p,i_{p+1,s}}h)u_a^{(\tilde p-1)}(\beta_{\tilde p-1}h)}_{k^{(\tilde p-1)}_{p+1,s}\text{ times}}]\\
&\quad+\frac{1}{(n-\sum_{j=\tilde p-1}^{i_{p+1,s}} k^{(j)}_{p+1,s})!}\int_0^1 g^{(n)}(T(c_{\tilde p,i_{p+1,s}}h)(S^h_{\tilde p-1}(u_0)+\xi u_a^{(\tilde p-1)}(\beta_{\tilde p-1})))\\
&\quad[\underbrace{u_a^{(i_{p+1,s})}(t_{p+1,s})}_{k^{(i_{p+1,s})}_{p+1,s}\text{ times}},\underbrace{T(\alpha_{i_{p+1,s}}h)u_a^{(i_{p+1,s}-1)}(\beta_{i_{p+1,s}-1}h)}_{k^{(i_{p+1,s}-1)}_{p+1,s}\text{ times}},\dots,\underbrace{T(c_{\tilde p,i_{p+1,s}}h)u_a^{(\tilde p-1)}(\beta_{\tilde p-1}h)}_{k^{(\tilde p-1)}_{p+1,s}\text{ times}}]\,\mathrm{d}\xi.
\end{aligned}$$

Plugging this into (3.26) gives us

$$\begin{aligned}
g(\psi_{S^h_{i_{p+1,s}}(u_0)}(t_{p+1,s})) &= \sum_{\substack{k^{(\tilde p-1)}_{p+1,s},\dots,k^{(i_{p+1,s})}_{p+1,s}\geqslant 0\\ \sum_{j=\tilde p-1}^{i_{p+1,s}} k^{(j)}_{p+1,s}<n+1-S_p}} \frac{1}{\prod_{j=\tilde p-1}^{i_{p+1,s}} k^{(j)}_{p+1,s}!} g^{(\sum_{j=\tilde p-1}^{i_{p+1,s}} k^{(j)}_{p+1,s})}(T(c_{\tilde p,i_{p+1,s}}h)S^h_{\tilde p-1}(u_0))\\
&\quad[\underbrace{u_a^{(i_{p+1,s})}(t_{p+1,s})}_{k^{(i_{p+1,s})}_{p+1,s}\text{ times}},\underbrace{T(\alpha_{i_{p+1,s}}h)u_a^{(i_{p+1,s}-1)}(\beta_{i_{p+1,s}-1}h)}_{k^{(i_{p+1,s}-1)}_{p+1,s}\text{ times}}\dots,\underbrace{T(c_{\tilde p,i_{p+1,s}}h)u_a^{(\tilde p-1)}(\beta_{\tilde p-1}h)}_{k^{(\tilde p-1)}_{p+1,s}\text{ times}}]
\end{aligned}$$

$$+ \sum_{\substack{k_{p+1,s}^{(\tilde{p}-1)},\dots,k_{p+1,s}^{(i_{p+1,s})}\geqslant 0 \\ \sum_{j=\tilde{p}-1}^{i_{p+1,s}} k_{p+1,s}^{(j)}=n+1-S_p}} \frac{1}{\prod_{j=\tilde{p}-1}^{i_{p+1,s}} k_{p+1,s}^{(j)}!} \int_0^1 g^{(\sum_{j=\tilde{p}-1}^{i_{p+1,s}} k_{p+1,s}^{(j)})}(\widetilde{B}_{p+1,\tilde{p}-1}^{(k_{j,r}),(i_{j,s})})[\underbrace{u_a^{(i_{p+1,s})}(t_{p+1,s}),}_{k_{p+1,s}^{(i_{p+1,s})} \text{ times}}$$

$$\underbrace{T(\alpha_{i_{p+1,s}}h)u_a^{(i_{p+1,s}-1)}(\beta_{i_{p+1,s}-1}h)}_{k_{p+1,s}^{(i_{p+1,s}-1)} \text{ times}}\dots,\underbrace{T(c_{\tilde{p},i_{p+1,s}}h)u_a^{(\tilde{p}-1)}(\beta_{\tilde{p}-1}h)}_{k_{p+1,s}^{(\tilde{p}-1)} \text{ times}}]\,\mathrm{d}\xi,$$

where the first sum just changed into the sum over the additional variable and the old second sum is still part of the second sum for $k_{p+1,s}^{(\tilde{p}-1)} = 0$. This is (3.26) for $\tilde{p}$ replaced by $\tilde{p}-1$, which ends this induction. Setting $\tilde{p} = 1$ and using the symmetrization argument that gave us (3.25), replacing $k_{1,1}^{(j)}$ by $k_{p+1,s}^{(j)}$, transforms this into

$$g(\psi_{S^h_{i_{p+1,s}}(u_0)}(t_{p+1,s})) = \sum_{\substack{k_{p+1,s}=0 \\ k_{p+1,s}<n+1-S_p}}^{n-S_p} \frac{1}{k_{p+1,s}!} \sum_{\substack{i_{p+2,\tilde{s}}=1 \\ \tilde{s}=K_{p+1}^{(s-1)}+1,\dots,K_{p+1}^{(s)}}}^{i_{p+1,s}} \int_{M^{(i_{p+2,\tilde{s}})}} g^{(k_{p+1,s})}(u_l(c_{i_{p+1,s}}h))$$

$$[(T(c_{i_{p+2,\tilde{s}}+1,i_{p+1,s}}h)g(\psi_{S^h_{i_{p+2,\tilde{s}}}(u_0)}(t_{p+2,\tilde{s}}))_{\tilde{s}=K_{p+1}^{(s-1)}+1}^{K_{p+1}^{(s)}}]\,\mathrm{d}t_{p+2,K_{p+1}^{(s)}}\cdots t_{p+2,K_{p+1}^{(s-1)}+1}$$

$$+\frac{1}{(n+1-S_p)!} \sum_{\substack{i_{p+2,\tilde{s}}=1 \\ \tilde{s}=K_{p+1}^{(s-1)}+1,\dots,K_{p+1}^{(s)}}}^{i_{p+1,s}} \int_{\widetilde{M}^{(i_{p+2,\tilde{s}})}} \int_0^1 g^{(n-S_p)}(B_{p+2}^{(k_{j,r}),(i_{j,s})}(t_{p+1,\tilde{s}}))$$

$$[(T(c_{i_{p+2,\tilde{s}}+1,i_{p+1,s}}h)g(\psi_{S^h_{i_{p+2,\tilde{s}}}(u_0)}(t_{p+2,\tilde{s}}))_{\tilde{s}=K_{p+1}^{(s-1)}+1}^{K_{p+1}^{(s)}}]\,\mathrm{d}\xi\,\mathrm{d}t_{p+2,K_{p+1}^{(s)}}\cdots t_{p+2,K_{p+1}^{(s-1)}+1},$$

where $M^{(i_{p+2,s})}$ is the canonic subset of $M_{p+2,h}^{(k_{j,r}),(i_{j,s})}$. We plug this into (3.21) for every $s \in \{1,\dots,K_p\}$ where we notice that

$$\widetilde{F}_{(k_{j,r}),p}\Big[\Big(g^{(k_{p,r})}(u_l(c_{i_{p,r}}h))\Big[\Big(T(c_{i_{p+1,s}+1,i_{p,r}}h)g(\psi_{S^h_{i_{p+1,s}}(u_0)}(t_{p+1,s}))\Big)_{s=K_p^{(r-1)}+1}^{K_p^{(r)}}\Big]\Big)_{r=1}^{K_{p-1}}\Big]((c_{i_{j,s}}h))$$
$$= \widetilde{F}_{(k_{j,r}),p+1}\Big[\Big(g(\psi_{S^h_{i_{p+1,s}}(u_0)}(t_{p+1,s}))\Big)_{s=1}^{K_p}\Big]((c_{i_{j,s}}h))$$

to obtain

$$
\begin{aligned}
S^h_q(u_0) = u_l(h) + &\sum_{\substack{k_{j,r}=0\\ j=1,\dots,p+1;r=1,\dots,K_{j-1}\\ S_{n-1}\leqslant n}}^{n-S_{j-1}} \frac{1}{\prod_{j=1}^{p+1}\prod_{r=1}^{K_{j-1}} k_{j,r}!} \sum_{i_{1,1}=1}^{q-1} \sum_{\substack{i_{j,s}=1;\, j=2,\dots,p+2\\ r=1\dots,K_{j-2};s=K^{(r-1)}_{j-1}+1,\dots,K^{(r)}_{j-1}}}^{i_{j-1,r}} \\
&\int_{M^{(i_{j,s})}_{p+2,(\beta_{i_{j,s}}h)}} \widetilde{F}_{(k_{j,r}),p+1}\Big[\Big(g^{(k_{p+1,s})}(u_l(c_{i_{p+1,s}}h))[(T(c_{i_{p+2,\tilde{s}}+1,i_{p+1,s}}h) \\
&g(\psi_{S^h_{i_{p+2,\tilde{s}}}(u_0)}(t_{p+2,\tilde{s}}))^{K^{(s)}_{p+1}}_{\tilde{s}=K^{(s-1)}_{p+1}+1}]\Big)^{K_p}_{s=1}\Big]((c_{i_{j,s}}h))\,\mathrm{d}t_{p+2,K_{p+1}}\cdots \mathrm{d}t_{1,1} \\
+ &\sum_{\substack{k_{j,r}=0\\ j=1,\dots,p+1;r=1,\dots,K_{j-1}\\ S_p\leqslant n;S_{p+1}>n}}^{n+1-S_{j-1}} \frac{1}{\prod_{j=1}^{p+1}\prod_{r=1}^{K_{j-1}} k_{j,r}!} \sum_{i_{1,1}=1}^{q-1} \sum_{\substack{i_{j,s}=1;\, j=2,\dots,p+2\\ r=1\dots,K_{j-2};s=K^{(r-1)}_{j-1}+1,\dots,K^{(r)}_{j-1}}}^{i_{j-1,r}} \\
&\int_{M^{(k_{j,r}),(i_{j,s})}_{p+2,(\beta_{i_{j,s}}h)}} \widetilde{F}_{(k_{j,r}),p+1}\Big[\Big(\int_0^1 g^{(k_{p+1,s})}(B^{(k_{j,r}),(i_{j,s})}_{p+2}(t_{p+1,r}))[(T(c_{i_{p+2,\tilde{s}}+1,i_{p+1,s}}h) \\
&g(\psi_{S^h_{i_{p+2,\tilde{s}}}(u_0)}(t_{p+2,\tilde{s}}))^{K^{(s)}_{p+1}}_{\tilde{s}=K^{(s-1)}_{p+1}+1}]\,\mathrm{d}\xi\Big)^{K_p}_{s=1}\Big]((c_{i_{j,s}}h))\,\mathrm{d}t_{p+2,K_{p+1}}\cdots \mathrm{d}t_{1,1} \\
+ &\sum_{m=1}^{p}\sum_{\substack{k_{j,r}=0\\ j=1,\dots,m;r=1,\dots,K_{j-1}\\ S_{m-1}\leqslant n;S_m>n}}^{n+1-S_{j-1}} \frac{1}{\prod_{j=1}^{m}\prod_{r=1}^{K_{j-1}} k_{j,r}!} \sum_{i_{1,1}=1}^{q-1} \sum_{\substack{i_{j,s}=1;\, j=2,\dots,m+1\\ r=1\dots,K_{j-2};s=K^{(r-1)}_{j-1}+1,\dots,K^{(r)}_{j-1}}}^{i_{j-1,r}} \\
&\int_{M^{(k_{j,r}),(i_{j,s})}_{m+1,(\beta_{i_{j,s}}h)}} \widetilde{F}_{(k_{j,r}),m}\Big[\Big(\int_0^1 g^{(k_{m,r})}(B^{(k_{j,r}),(i_{j,s})}_{m+1}(t_{m,r}))\Big[\Big(T(c_{i_{m+1,s}+1,i_{m,r}}h) \\
&g(\psi_{S^h_{i_{m+1,s}}(u_0)}(t_{m+1,s}))\Big)^{K^{(r)}_m}_{s=K^{(r-1)}_m+1}\Big]\,\mathrm{d}\xi\Big)^{K_m}_{r=1}\Big]((c_{i_{j,s}}h))\,\mathrm{d}t_{m+1,K_m}\cdots \mathrm{d}t_{1,1},
\end{aligned}
\tag{3.27}
$$

where we split up the sum according to whether $S_{p+1} \leqslant n$ or $S_{p+1} > n$ and the definition of $B^{(k_{j,r}),(i_{j,s})}_{m+1}(t_{m,r})$ has exactly the right cases to cover the resulting terms. Moreover, we put together all the $M^{(i_{p+2,s})}$ with $M^{(k_{j,r}),(i_{j,s})}_{p+1,(\beta_{i_{j,s}}h)}$ to obtain $M^{(k_{j,r}),(i_{j,s})}_{p+2,(\beta_{i_{j,s}}h)}$ and collected all the sums over $i_{p+2,r}$ as well as the factorials. This way the second sum becomes the last summand ($m = p + 1$) in the third sum and since. Therefore, (3.27) is (3.21) for p replaced by $p+1$. This concludes the induction and therefore this proof. ■

Remark 3.10

Again, we want to write down the explicit formulas for n up to four. Let $k_{j,r} \in \mathbb{N}_0$ for $j \in \{1,\ldots,n\}$, $r \in \{1,\ldots,K_{j-1}\}$ with $S_{n-1} < n$. We define the function $\widetilde{F}_{(k_{j,r})}$ by

$$\widetilde{F}_{(k_{j,r})_{j=1,\ldots,n;r=1,\ldots,K_{j-1}}}((t_{j,r})_{j=1,\ldots,n;r=1,\ldots,K_{j-1}}) := u_n\Big[\Big(g(u_l(t_{n,s})\Big)_{s=1}^{K_{n-1}}\Big] \quad \forall (t_{j,r}) \in N_n^{(k_{j,r})}$$

For $n=1$, we obtain

$$\begin{aligned} S_q^h(u_0) = u_l(h) + h\sum_{i_{1,1}=1}^{q-1} \beta_{i_{1,1}} F_{(0)}(c_{i_{1,1}}h) + \sum_{i_{2,1}\leqslant i_{1,1}\leqslant q-1} \int_{M_{2,(\beta_{i_{j,s}}h)}^{(k_{j,r}),(i_{j,s})}} T(c_{i_{1,1}+1,q}h) \\ \int_0^1 g'(B_2^{(k_{j,r}),(i_{j,s})}(t_{1,1}))\Big[T(c_{i_{2,1}+1,i_{1,1}}h)g(\psi_{S_{i_{2,1}}^h(u_0)}(t_{2,1}))\Big]\,\mathrm{d}\xi\,\mathrm{d}t_{2,1}\,\mathrm{d}t_{1,1}, \end{aligned} \tag{3.28}$$

the last term being the remainder. For $n=2$, we get

$$\begin{aligned} S_q^h(u_0) = u_l(h) + h\sum_{i_{1,1}=1}^{q-1} \beta_{i_{1,1}} F_{(0)}(c_{i_{1,1}}h) + \frac{h^2}{2}\Big(\sum_{i_{1,1}\leqslant q-1} \beta_{i_{1,1}}^2 F_{(1,0)}(c_{i_{1,1}}h, c_{i_{1,1}}h) \\ + \sum_{i_{2,1}<i_{1,1}\leqslant q-1} 2\beta_{i_{1,1}}\beta_{i_{2,1}} F_{(1,0)}(c_{i_{1,1}}h, c_{i_{2,1}}h)\Big) \\ + \frac{1}{2}\sum_{i_{2,1},i_{2,2}\leqslant i_{1,1}\leqslant q-1} \int_{M_{2,(\beta_{i_{j,s}}h)}^{(k_{j,r}),(i_{j,s})}} T(c_{i_{1,1}+1,q}h)g''(B_2^{(k_{j,r}),(i_{j,s})}(t_{1,1}))\Big[T(c_{i_{2,1}+1,i_{1,1}}h) \\ g(\psi_{S_{i_{2,1}}^h(u_0)}(t_{2,1})), T(c_{i_{2,2}+1,i_{1,1}}h)g(\psi_{S_{i_{2,2}}^h(u_0)}(t_{2,2}))\Big]\,\mathrm{d}t_{2,2}\,\mathrm{d}t_{2,1}\,\mathrm{d}t_{1,1} \\ + \sum_{i_{3,1}\leqslant i_{2,1}\leqslant i_{1,1}\leqslant q-1} \int_{M_{3,(\beta_{i_{j,s}}h)}^{(k_{j,r}),(i_{j,s})}} T(c_{i_{1,1}+1,q}h)g'(u_l(c_{i_{1,1}}h))\Big[T(c_{i_{2,1}+1,i_{1,1}}h) \\ \int_0^1 g'(B_3^{(k_{j,r}),(i_{j,s})}(t_{2,1}))[T(c_{i_{3,1}+1,i_{2,1}}h)g(\psi_{S_{i_{3,1}}^h(u_0)}(t_{3,1}))]\Big]\,\mathrm{d}\xi\,\mathrm{d}t_{3,1}\,\mathrm{d}t_{2,1}\,\mathrm{d}t_{1,1}, \end{aligned} \tag{3.29}$$

the last two being remainder terms. For $n = 3$, we end up with

$$
\begin{aligned}
S_q^h(u_0) &= u_l(h) + h \sum_{i_{1,1}=1}^{q-1} \beta_{i_{1,1}} F_{(0)}(c_{i_{1,1}} h) \\
&\quad + \frac{h^2}{2}\Big(\sum_{i_{1,1}\leqslant q-1} \beta_{i_{1,1}}^2 F_{(1,0)}(c_{i_{1,1}} h, c_{i_{1,1}} h) + \sum_{i_{2,1}<i_{1,1}\leqslant q-1} 2\beta_{i_{1,1}}\beta_{i_{2,1}} F_{(1,0)}(c_{i_{1,1}} h, c_{i_{2,1}} h)\Big) \\
&\quad + \frac{h^3}{6}\Big(\sum_{i_{1,1}\leqslant q-1} \beta_{i_{1,1}}^3 F_{(1,1,0)}(c_{i_{1,1}} h, c_{i_{1,1}} h, c_{i_{1,1}} h) \\
&\qquad + \sum_{i_{3,1}<i_{1,1}\leqslant q-1} 3\beta_{i_{1,1}}^2 \beta_{i_{3,1}} F_{(1,1,0)}(c_{i_{1,1}} h, c_{i_{1,1}} h, c_{i_{3,1}} h) \\
&\qquad + \sum_{i_{2,1}<i_{1,1}\leqslant q-1} 3\beta_{i_{1,1}} \beta_{i_{2,1}}^2 F_{(1,1,0)}(c_{i_{1,1}} h, c_{i_{2,1}} h, c_{i_{2,1}} h) \\
&\qquad + \sum_{i_{3,1}<i_{2,1}<i_{1,1}\leqslant q-1} 6\beta_{i_{1,1}}\beta_{i_{2,1}}\beta_{i_{3,1}} F_{(1,1,0)}(c_{i_{1,1}} h, c_{i_{2,1}} h, c_{i_{3,1}} h)\Big) \\
&\quad + \frac{h^3}{3}\cdot\frac{1}{2}\Big(\sum_{i_{1,1}\leqslant q-1} \beta_{i_{1,1}}^3 F_{(2,0,0)}(c_{i_{1,1}} h, c_{i_{1,1}} h, c_{i_{1,1}} h) \\
&\qquad + \sum_{i_{2,1}<i_{1,1}\leqslant q-1} \frac{3}{2}\beta_{i_{1,1}}^2 \beta_{i_{2,1}} F_{(2,0,0)}(c_{i_{1,1}} h, c_{i_{2,1}} h, c_{i_{1,1}} h) \\
&\qquad + \sum_{i_{2,2}<i_{1,1}\leqslant q-1} \frac{3}{2}\beta_{i_{1,1}}^2 \beta_{i_{2,2}} F_{(2,0,0)}(c_{i_{1,1}} h, c_{i_{1,1}} h, c_{i_{2,2}} h) \\
&\qquad + \sum_{i_{2,1},i_{2,2}<i_{1,1}\leqslant q-1} 3\beta_{i_{1,1}}\beta_{i_{2,1}}\beta_{i_{2,2}} F_{(2,0,0)}(c_{i_{1,1}} h, c_{i_{2,1}} h, c_{i_{2,2}} h)\Big) \\
&\quad + \frac{1}{6} \sum_{i_{2,1},i_{2,2},i_{2,3}\leqslant i_{1,1}\leqslant q-1} \int_{M_{2,(\beta_{i_{j,s}} h)}^{(k_{j,r}),(i_{j,s})}} T(c_{i_{1,1}+1,q} h) g'''(B_2^{(k_{j,r}),(i_{j,s})}(t_{1,1})) \\
&\qquad \Big[T(c_{i_{2,1}+1,i_{1,1}} h) g(\psi_{S_{i_{2,1}}^h(u_0)}(t_{2,1})), T(c_{i_{2,2}+1,i_{1,1}} h) g(\psi_{S_{i_{2,2}}^h(u_0)}(t_{2,2})), \\
&\qquad T(c_{i_{2,3}+1,i_{1,1}} h) g(\psi_{S_{i_{2,3}}^h(u_0)}(t_{2,3}))\Big] \,\mathrm{d}t_{2,3}\,\mathrm{d}t_{2,2}\,\mathrm{d}t_{2,1}\,\mathrm{d}t_{1,1} \\
&\quad + \sum_{k_{1,1}=1}^{2} \sum_{\substack{k_{2,1},\dots,k_{2,k_{1,1}}=0 \\ k_{1,1}+k_{2,1}+\dots+k_{2,k_{1,1}}\geqslant 3}}^{3-k_{1,1}} \frac{1}{k_{1,1}!k_{2,1}!\cdots k_{2,k_{1.1}}!} \sum_{i_{1,1}=1}^{q-1} \sum_{i_{2,1},\dots,i_{2,k_{1,1}}=1}^{i_{1,1}}
\end{aligned}
$$

$$\sum_{\substack{i_{3,s}=1 \\ r=1\dots,k_{1,1};s=K_2^{(r-1)}+1,\dots,K_2^{(r)}}}^{i_{2,r}} \int_{M_{3,(\beta_{i_{j,s}}h)}^{(k_{j,r}),(i_{j,s})}} T(c_{i_{1,1}+1,q}h) g^{(k_{1,1})}(u_l(c_{i_{1,1}}h))$$

$$\Big[\Big(T(c_{i_{2,r}+1,i_{1,1}}h)\int_0^1 g^{(k_{2,r})}(B_3^{(k_{j,r}),(i_{j,s})}(t_{2,r}))\Big[\Big(T(c_{i_{3,s}+1,i_{2,r}}h)$$

$$g(\psi_{S^h_{i_{3,s}}(u_0)}(t_{3,s}))\Big)_{s=K_2^{(r-1)}+1}^{K_2^{(r)}}\Big]\,\mathrm{d}\xi\Big)_{r=1}^{k_{1,1}}\Big]\,\mathrm{d}t_{3,K_2}\cdots\mathrm{d}t_{2,1}\,\mathrm{d}t_{1,1}$$

$$+\sum_{i_{4,1}\leqslant i_{3,1}\leqslant i_{2,1}\leqslant i_{1,1}\leqslant q-1}\int_{M_{4,(\beta_{i_{j,s}}h)}^{(k_{j,r}),(i_{j,s})}} T(c_{i_{1,1}+1,q}h) g'(u_l(c_{i_{1,1}}h))\Big[T(c_{i_{2,1}+1,i_{1,1}}h)$$

$$g'(u_l(c_{i_{2,1}}h))\Big[T(c_{i_{3,1}+1,i_{2,1}}h)\int_0^1 g'(B_4^{(k_{j,r}),(i_{j,s})}(t_{3,1}))[g(\psi_{S^h_{i_{4,1}}(u_0)}(t_{4,1}))]\Big]\Big]$$

$$\mathrm{d}\xi\,\mathrm{d}t_{4,1}\,\mathrm{d}t_{3,1}\,\mathrm{d}t_{2,1}\,\mathrm{d}t_{1,1} \tag{3.30}$$

where the last three summands (including the double sum which gives four terms) are remainder terms. For $n = 4$, we get the terms from $n = 3$ minus the remainder terms plus the terms

$$+\frac{h^4}{24}\Big(\sum_{i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^4 F_{(1,1,1,0)}(c_{i_{1,1}}h, c_{i_{1,1}}h, c_{i_{1,1}}h, c_{i_{1,1}}h)$$

$$+4\sum_{i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}^3 F_{(1,1,1,0)}(c_{i_{1,1}}h, c_{i_{2,1}}h, c_{i_{2,1}}h, c_{i_{2,1}}h)$$

$$+4\sum_{i_{4,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^3\beta_{i_{4,1}} F_{(1,1,1,0)}(c_{i_{1,1}}h, c_{i_{1,1}}h, c_{i_{1,1}}h, c_{i_{4,1}}h)$$

$$+6\sum_{i_{3,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^2\beta_{i_{3,1}}^2 F_{(1,1,1,0)}(c_{i_{1,1}}h, c_{i_{1,1}}h, c_{i_{3,1}}h, c_{i_{3,1}}h)$$

$$+12\sum_{i_{4,1}<i_{3,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^2\beta_{i_{3,1}}\beta_{i_{4,1}} F_{(1,1,1,0)}(c_{i_{1,1}}h, c_{i_{1,1}}h, c_{i_{3,1}}h, c_{i_{4,1}}h)$$

$$+12\sum_{i_{4,1}<i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}^2\beta_{i_{4,1}} F_{(1,1,1,0)}(c_{i_{1,1}}h, c_{i_{2,1}}h, c_{i_{2,1}}h, c_{i_{4,1}}h)$$

$$+12\sum_{i_{3,1}<i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}\beta_{i_{3,1}}^2 F_{(1,1,1,0)}(c_{i_{1,1}}h, c_{i_{2,1}}h, c_{i_{3,1}}h, c_{i_{3,1}}h)$$

$$+24\sum_{i_{4,1}<i_{3,1}<i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}\beta_{i_{3,1}}\beta_{i_{4,1}} F_{(1,1,1,0)}(c_{i_{1,1}}h, c_{i_{2,1}}h, c_{i_{3,1}}h, c_{i_{4,1}}h)$$

$$+\frac{1}{2}\cdot\frac{h^4}{12}\Big(\sum_{i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^4 F_{(1,2,0,0)}(c_{i_{1,1}}h, c_{i_{1,1}}h, c_{i_{1,1}}h, c_{i_{1,1}}h)$$

$$+2 \sum_{i_{3,1}<i_{1,1}\leqslant q-1} \beta_{i_{1,1}}^3 \beta_{i_{3,1}} F_{(1,2,0,0)}(c_{i_{1,1}}h, c_{i_{1,1}}h, c_{i_{3,1}}h, c_{i_{1,1}}h)$$

$$+2 \sum_{i_{3,2}<i_{1,1}\leqslant q-1} \beta_{i_{1,1}}^3 \beta_{i_{3,2}} F_{(1,2,0,0)}(c_{i_{1,1}}h, c_{i_{1,1}}h, c_{i_{1,1}}h, c_{i_{3,2}}h)$$

$$+4 \sum_{i_{2,1}<i_{1,1}\leqslant q-1} \beta_{i_{1,1}} \beta_{i_{2,1}}^3 F_{(1,2,0,0)}(c_{i_{1,1}}h, c_{i_{2,1}}h, c_{i_{2,1}}h, c_{i_{2,1}}h)$$

$$+6 \sum_{i_{3,1},i_{3,2}<i_{1,1}\leqslant q-1} \beta_{i_{1,1}}^2 \beta_{i_{3,1}} \beta_{i_{3,2}} F_{(1,2,0,0)}(c_{i_{1,1}}h, c_{i_{1,1}}h, c_{i_{3,1}}h, c_{i_{3,2}}h)$$

$$+6 \sum_{i_{3,1}<i_{2,1}<i_{1,1}\leqslant q-1} \beta_{i_{1,1}} \beta_{i_{2,1}}^2 \beta_{i_{3,1}} F_{(1,2,0,0)}(c_{i_{1,1}}h, c_{i_{2,1}}h, c_{i_{3,1}}h, c_{i_{2,1}}h)$$

$$+6 \sum_{i_{3,2}<i_{2,1}<i_{1,1}\leqslant q-1} \beta_{i_{1,1}} \beta_{i_{2,1}}^2 \beta_{i_{3,2}} F_{(1,2,0,0)}(c_{i_{1,1}}h, c_{i_{2,1}}h, c_{i_{2,1}}h, c_{i_{3,2}}h)$$

$$+12 \sum_{i_{3,1},i_{3,2}<i_{2,1}<i_{1,1}\leqslant q-1} \beta_{i_{1,1}} \beta_{i_{2,1}} \beta_{i_{3,1}} \beta_{i_{3,2}} F_{(1,2,0,0)}(c_{i_{1,1}}h, c_{i_{2,1}}h, c_{i_{3,1}}h, c_{i_{3,2}}h)\Big)$$

$$+\frac{1}{2}\cdot\frac{h^4}{8}\Big(\sum_{i_{1,1}\leqslant q-1} \beta_{i_{1,1}}^4 F_{(2,1,0,0)}(c_{i_{1,1}}h, c_{i_{1,1}}h, c_{i_{1,1}}h, c_{i_{1,1}}h)$$

$$+\frac{8}{3} \sum_{i_{3,1}<i_{1,1}\leqslant q-1} \beta_{i_{1,1}}^3 \beta_{i_{3,1}} F_{(2,1,0,0)}(c_{i_{1,1}}h, c_{i_{1,1}}h, c_{i_{1,1}}h, c_{i_{3,1}}h)$$

$$+\frac{4}{3} \sum_{i_{2,2}<i_{1,1}\leqslant q-1} \beta_{i_{1,1}}^3 \beta_{i_{2,2}} F_{(2,1,0,0)}(c_{i_{1,1}}h, c_{i_{1,1}}h, c_{i_{2,2}}h, c_{i_{1,1}}h)$$

$$+2 \sum_{i_{2,1}<i_{1,1}\leqslant q-1} \beta_{i_{1,1}}^2 \beta_{i_{2,1}}^2 F_{(2,1,0,0)}(c_{i_{1,1}}h, c_{i_{2,1}}h, c_{i_{1,1}}h, c_{i_{2,1}}h)$$

$$+4 \sum_{i_{2,2},i_{3,1}<i_{1,1}\leqslant q-1} \beta_{i_{1,1}}^2 \beta_{i_{2,2}} \beta_{i_{3,1}} F_{(2,1,0,0)}(c_{i_{1,1}}h, c_{i_{1,1}}h, c_{i_{2,2}}h, c_{i_{3,1}}h)$$

$$+4 \sum_{i_{3,1}<i_{2,1}<i_{1,1}\leqslant q-1} \beta_{i_{1,1}}^2 \beta_{i_{2,1}} \beta_{i_{3,1}} F_{(2,1,0,0)}(c_{i_{1,1}}h, c_{i_{2,1}}h, c_{i_{1,1}}h, c_{i_{3,1}}h)$$

$$+4 \sum_{i_{2,1},i_{2,2}<i_{1,1}\leqslant q-1} \beta_{i_{1,1}} \beta_{i_{2,1}}^2 \beta_{i_{2,2}} F_{(2,1,0,0)}(c_{i_{1,1}}h, c_{i_{2,1}}h, c_{i_{2,2}}h, c_{i_{2,1}}h)$$

$$+8 \sum_{i_{3,1}<i_{2,1}<i_{1,1}\leqslant q-1;\, i_{2,2}<i_{1,1}} \beta_{i_{1,1}} \beta_{i_{2,1}} \beta_{i_{2,2}} \beta_{i_{3,1}} F_{(2,1,0,0)}(c_{i_{1,1}}h, c_{i_{2,1}}h, c_{i_{2,2}}h, c_{i_{3,1}}h)\Big)$$

$$+\frac{1}{2}\cdot\frac{h^4}{8}\Big(\sum_{i_{1,1}\leqslant q-1} \beta_{i_{1,1}}^4 F_{(2,0,1,0)}(c_{i_{1,1}}h, c_{i_{1,1}}h, c_{i_{1,1}}h, c_{i_{1,1}}h)$$

$$+\frac{8}{3} \sum_{i_{3,1}<i_{1,1}\leqslant q-1} \beta_{i_{1,1}}^3 \beta_{i_{3,1}} F_{(2,0,1,0)}(c_{i_{1,1}}h, c_{i_{1,1}}h, c_{i_{1,1}}h, c_{i_{3,1}}h)$$

$$
\begin{aligned}
&+\frac{4}{3}\sum_{i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^{3}\beta_{i_{2,1}}F_{(2,0,1,0)}(c_{i_{1,1}}h,c_{i_{2,1}}h,c_{i_{1,1}}h,c_{i_{1,1}}h)\\
&+2\sum_{i_{2,2}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^{2}\beta_{i_{2,2}}^{2}F_{(2,0,1,0)}(c_{i_{1,1}}h,c_{i_{1,1}}h,c_{i_{2,2}}h,c_{i_{2,2}}h)\\
&+4\sum_{i_{2,2},i_{3,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^{2}\beta_{i_{2,2}}\beta_{i_{3,1}}F_{(2,0,1,0)}(c_{i_{1,1}}h,c_{i_{1,1}}h,c_{i_{2,2}}h,c_{i_{3,1}}h)\\
&+4\sum_{i_{3,1}<i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^{2}\beta_{i_{2,1}}\beta_{i_{3,1}}F_{(2,0,1,0)}(c_{i_{1,1}}h,c_{i_{2,1}}h,c_{i_{1,1}}h,c_{i_{3,1}}h)\\
&+4\sum_{i_{2,1},i_{2,2}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}\beta_{i_{2,2}}^{2}F_{(2,0,1,0)}(c_{i_{1,1}}h,c_{i_{2,1}}h,c_{i_{2,2}}h,c_{i_{2,2}}h)\\
&+8\sum_{i_{3,1}<i_{2,2}<i_{1,1}\leqslant q-1;\,i_{2,1}<i_{1,1}}\beta_{i_{1,1}}\beta_{i_{2,1}}\beta_{i_{2,2}}\beta_{i_{3,1}}F_{(2,0,1,0)}(c_{i_{1,1}}h,c_{i_{2,1}}h,c_{i_{2,2}}h,c_{i_{3,1}}h)\Big)\\
+&\frac{1}{6}\cdot\frac{h^{4}}{4}\Big(\sum_{i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^{4}F_{(3,0,0,0)}(c_{i_{1,1}}h,c_{i_{1,1}}h,c_{i_{1,1}}h,c_{i_{1,1}}h)\\
&+\frac{4}{3}\sum_{i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^{3}\beta_{i_{2,1}}F_{(3,0,0,0)}(c_{i_{1,1}}h,c_{i_{2,1}}h,c_{i_{1,1}}h,c_{i_{1,1}}h)\\
&+\frac{4}{3}\sum_{i_{2,2}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^{3}\beta_{i_{2,2}}F_{(3,0,0,0)}(c_{i_{1,1}}h,c_{i_{1,1}}h,c_{i_{2,2}}h,c_{i_{1,1}}h)\\
&+\frac{4}{3}\sum_{i_{2,3}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^{3}\beta_{i_{2,3}}F_{(3,0,0,0)}(c_{i_{1,1}}h,c_{i_{1,1}}h,c_{i_{1,1}}h,c_{i_{2,3}}h)\\
&+2\sum_{i_{2,1},i_{2,2}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^{2}\beta_{i_{2,1}}\beta_{i_{2,2}}F_{(3,0,0,0)}(c_{i_{1,1}}h,c_{i_{2,1}}h,c_{i_{2,2}}h,c_{i_{1,1}}h)\\
&+2\sum_{i_{2,1},i_{2,3}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^{2}\beta_{i_{2,1}}\beta_{i_{2,3}}F_{(3,0,0,0)}(c_{i_{1,1}}h,c_{i_{2,1}}h,c_{i_{1,1}}h,c_{i_{2,3}}h)\\
&+2\sum_{i_{2,2},i_{2,3}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^{2}\beta_{i_{2,2}}\beta_{i_{2,3}}F_{(3,0,0,0)}(c_{i_{1,1}}h,c_{i_{1,1}}h,c_{i_{2,2}}h,c_{i_{2,3}}h)\\
&+4\sum_{i_{2,1},i_{2,2},i_{2,3}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}\beta_{i_{2,2}}\beta_{i_{2,3}}F_{(3,0,0,0)}(c_{i_{1,1}}h,c_{i_{2,1}}h,c_{i_{3,1}}h,c_{i_{3,2}}h)\Big)\\
+&\sum_{m=1}^{4}\sum_{k_{1,1}=0}^{n}\cdots\sum_{\substack{k_{m,1},\ldots,k_{m,K_{m-1}}=0\\ S_{m-1}<n,\quad S_m\geqslant n}}^{n-S_{m-1}}\frac{1}{\prod_{j=1}^{m}\prod_{r=1}^{K_{j-1}}k_{j,r}!}\sum_{i_{1,1}=1}^{q-1}\sum_{\substack{i_{j,s}=1;\,j=2,\ldots,m+1\\ r=1\ldots,K_{j-2};\,s=K_{j-1}^{(r-1)}+1,\ldots,K_{j-1}^{(r)}}}^{i_{j-1,r}}\\
&\int_{M_{m+1,(\beta_{i_{j,s}}h)}^{(k_{j,r}),(i_{j,s})}}u_{(k_{j,r})_{j=1,\ldots,m-1},(c_{i_{j,s}}h)_{j=1,\ldots,m}}\Big[\Big(\int_{0}^{1}g^{(k_{m,r})}(B_{m+1}^{(k_{j,r}),(i_{j,s})}(t_{m,r}))
\end{aligned}
$$

$$\Big[\Big(T(c_{i_{m+1,s}+1,i_{m,r}}h)g(\psi_{S^h_{i_{m+1,s}}(u_0)}(\beta_{i_{m+1,s}}t_{m+1,s}))\Big)_{s=K_m^{(r-1)}+1}^{K_m^{(r)}}\Big]\,\mathrm{d}\xi\Big)_{r=1}^{K_{m-1}}\Big]$$
$$\mathrm{d}t_{m+1,K_m}\cdots\,\mathrm{d}t_{1,1}, \tag{3.31}$$

where we didn't elaborate on the remainder term, since its details bear too little importance for the space they would use. We see that the number of terms grows rapidly in n. In fact, the number of terms newly added for n is the $(n-1)$th Catalan number C_{n-1}, where

$$C_n := \frac{(2n)!}{(n+1)!n!}.$$

In each of these terms, we obtain 2^{n-1} sums with possibly different coefficients, since we have $n-1$ indices that may or may not be equal to their respective predecessor in the hierarchy.

We continue by considering the difference of the two representations we just obtained. Using Propositions 3.7 and 3.9, we arrive at

$$\begin{aligned}
u(h)-S_q^h(u_0) = &\sum_{\substack{k_{j,r}=0\\ j=1,\dots,n-1;r=1,\dots,K_{j-1}\\ S_{n-1}\leqslant n}}^{n-S_{j-1}} \frac{h^{S_{n-1}}}{\prod_{j=1}^{n-1}\prod_{r=1}^{K_{j-1}}k_{j,r}!}\Big(\int_{N_n^{(k_{j,r})}} F_{(k_{j,r})}((t_{j,r}h))\,\mathrm{d}t_{n,K_{n-1}}\cdots\,\mathrm{d}t_{1,1}\\
&-\sum_{i_{1,1}=1}^{q-1}\sum_{\substack{i_{j,s}=1;j=2,\dots,n\\ r=1\dots,K_{j-2};s=K_{j-1}^{(r-1)}+1,\dots,K_{j-1}^{(r)}}}^{i_{j-1,r}} |M_n^{(k_{j,r}),(i_{j,s})}|\Big(\prod_{j=1}^{n}\prod_{s=1}^{K_{j-1}}\beta_{i_{j,s}}\Big)F_{(k_{j,r})}((c_{i_{j,s}}h))\Big) \qquad (3.32)\\
&+\sum_{m=1}^{n}\sum_{\substack{k_{j,r}=0\\ j=1,\dots,m;r=1,\dots,K_{j-1}\\ S_{m-1}\leqslant n;S_m>n}}^{n+1-S_{j-1}} \frac{h^{S_m}}{\prod_{j=1}^{m}\prod_{r=1}^{K_{j-1}}k_{j,r}!}\Big(\int_{N_{m+1}^{(k_{j,r})}}\widetilde{F}_{(k_{j,r}),m}\Big[\Big(g^{(k_{m,r})}(A_{k_{m,r}}^{n-S_{m-1}})\\
&\Big[\Big(T((t_{m,r}-t_{m+1,s})h)g(u(t_{m+1,s}h))\Big)_{s=K_m^{(r-1)}+1}^{K_m^{(r)}}\Big]\Big)_{r=1}^{K_{m-1}}\Big]((t_{j,r}h))\,\mathrm{d}t_{m+1,K_m}\cdots\,\mathrm{d}t_{1,1}\\
&-\sum_{i_{1,1}=1}^{q-1}\sum_{\substack{i_{j,s}=1;j=2,\dots,m+1\\ r=1\dots,K_{j-2};s=K_{j-1}^{(r-1)}+1,\dots,K_{j-1}^{(r)}}}^{i_{j-1,r}} \int_{M_{m+1}^{(k_{j,r}),(i_{j,s})}}\Big(\prod_{j=1}^{n}\prod_{s=1}^{K_{j-1}}\beta_{i_{j,s}}\Big)\widetilde{F}_{(k_{j,r}),m}\\
&\Big[\Big(g^{(k_{m,r})}(B_{m+1}^{(k_{j,r}),(i_{j,s})}(\beta_{i_{m,r}}t_{m,r}h))\Big[\Big(T(c_{i_{m+1,s}+1,i_{m,r}}h)
\end{aligned}$$

$$g(\psi_{S^h_{i_{m+1,s}}(u_0)}(\beta_{i_{m,r}} t_{m+1,s} h))\Big)_{s=K_m^{(r-1)}+1}^{K_m^{(r)}}\Big]\Big)_{r=1}^{K_{m-1}}\Big]((c_{i_{j,s}} h))\, \mathrm{d}t_{m+1,K_m} \cdots \mathrm{d}t_{1,1}\Big),$$

From the remainder terms, we obtain at least $n+1$ orders of h. We want to achieve orders in $(n, n+1]$ for the terms in the first sum, which gives us conditions on α_i and β_i. The result looks as follows.

Lemma 3.11

Let $n \in \mathbb{N}$ and $k_{j,r} \in \mathbb{N}_0$ for $j \in \{2, \dots, n\}$, $r \in \{1, \dots, K_{j-1}\}$ with $S_{n-1} \leqslant n$. If $F_{(k_{j,r})}$ is $n - S_{n-1}$ times Fréchet differentiable and Assumption 3.5 holds, then

$$\begin{aligned}
&\Big\| \int_{N_n^{(k_{j,r})}} F_{(k_{j,r})}((t_{j,r} h))\, \mathrm{d}t_{n,K_{n-1}} \cdots \mathrm{d}t_{1,1} \\
&\quad - \sum_{i_{1,1}=1}^{q-1} \sum_{\substack{i_{j,s}=1;\, j=2,\dots,n \\ r=1\dots,K_{j-2};\, s=K_{j-1}^{(r-1)}+1,\dots,K_{j-1}^{(r)}}}^{i_{j-1,r}} |M_n^{(k_{j,r}),(i_{j,s})}| \Big(\prod_{j=1}^{n} \prod_{s=1}^{K_{j-1}} \beta_{i_{j,s}} \Big) F_{(k_{j,r})}((c_{i_{j,s}} h)) \Big\|_Y \\
&\leqslant h^{n-S_{n-1}} \sum_{|\gamma|=n-S_{n-1}} C_{\gamma,q}^{(k_{j,r}),(\alpha_j),(\beta_j)} \sup_{t_{j,r}, s_{j,r} \in N_n^{(k_{j,r})}} \| D^\gamma F_{(k_{j,r})}((t_{j,r} h)) - D^\gamma F_{(k_{j,r})}((s_{j,r} h)) \|_Y
\end{aligned}$$

with

$$C_{\gamma,q}^{(k_{j,r}),(\alpha_j),(\beta_j)} = \frac{(q-1)^{S_{n-1}} (\max_{j \in \{1,\dots,q-1\}} |\beta_j|)^{S_{n-1}} (\max_{j \in \{1,\dots,q\}} |c_j|)^{n-S_{n-1}}}{\gamma! \int_{N_n^{(k_{j,r})}} (t_{j,r})^\gamma\, \mathrm{d}t_{n,K_{n-1}} \cdots \mathrm{d}t_{1,1}}$$

Proof. We use Taylor's Theorem in S_{n-1}-dimensions around the origin up to order $n - S_{n-1}$, to see

$$\begin{aligned}
&\int_{N_n^{(k_{j,r})}} F_{(k_{j,r})}((t_{j,r} h))\, \mathrm{d}t_{n,K_{n-1}} \cdots \mathrm{d}t_{1,1} \\
&\quad - \sum_{i_{1,1}=1}^{q-1} \sum_{\substack{i_{j,s}=1;\, j=2,\dots,n \\ r=1\dots,K_{j-2};\, s=K_{j-1}^{(r-1)}+1,\dots,K_{j-1}^{(r)}}}^{i_{j-1,r}} |M_n^{(k_{j,r}),(i_{j,s})}| \Big(\prod_{j=1}^{n} \prod_{s=1}^{K_{j-1}} \beta_{i_{j,s}} \Big) F_{(k_{j,r})}((c_{i_{j,s}} h)) \\
&= \sum_{|\gamma|<n-S_{n-1}} \frac{h^{|\gamma|} D^\gamma F_{(k_{j,r})}(0)}{\gamma!} \Big(\int_{N_n^{(k_{j,r})}} (t_{j,r})^\gamma\, \mathrm{d}t_{n,K_{n-1}} \cdots \mathrm{d}t_{1,1}
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i_{1,1}=1}^{q-1} \sum_{\substack{i_{j,s}=1;\, j=2,\dots,n \\ r=1\dots,K_{j-2};\, s=K_{j-1}^{(r-1)}+1,\dots,K_{j-1}^{(r)}}}^{i_{j-1,r}} |M_n^{(k_{j,r}),(i_{j,s})}| \Big(\prod_{j=1}^{n} \prod_{s=1}^{K_{j-1}} \beta_{i_{j,s}} \Big) (c_{i_{j,s}})^\gamma \Big) \\
&+ \sum_{|\gamma|=n-S_{n-1}} \frac{h^{n-S_{n-1}}}{\gamma!} \Big(\int_{N_n^{(k_{j,r})}} (t_{j,r})^\gamma \int_0^1 D^\gamma F_{(k_{j,r})}((\xi t_{j,r} h))\, \mathrm{d}\xi\, \mathrm{d}t_{n,K_{n-1}} \cdots \mathrm{d}t_{1,1} \\
&- \sum_{i_{1,1}=1}^{q-1} \sum_{\substack{i_{j,s}=1;\, j=2,\dots,n \\ r=1\dots,K_{j-2};\, s=K_{j-1}^{(r-1)}+1,\dots,K_{j-1}^{(r)}}}^{i_{j-1,r}} |M_n^{(k_{j,r}),(i_{j,s})}| \Big(\prod_{j=1}^{n} \prod_{s=1}^{K_{j-1}} \beta_{i_{j,s}} \Big) (c_{i_{j,s}})^\gamma D^\gamma F_{(k_{j,r})}((\xi c_{i_{j,s}} h))\, \mathrm{d}\xi \Big) \\
&= h^{n-S_{n-1}} \sum_{|\gamma|=n-S_{n-1}} \frac{1}{\gamma! \int_{N_n^{(k_{j,r})}} (t_{j,r})^\gamma\, \mathrm{d}t_{n,K_{n-1}} \cdots \mathrm{d}t_{1,1}} \sum_{i_{1,1}=1}^{q-1} \sum_{\substack{i_{j,s}=1;\, j=2,\dots,n \\ r=1\dots,K_{j-2};\, s=K_{j-1}^{(r-1)}+1,\dots,K_{j-1}^{(r)}}}^{i_{j-1,r}} \\
&\qquad |M_n^{(k_{j,r}),(i_{j,s})}| (c_{i_{j,s}})^\gamma \Big(\prod_{j=1}^{n} \prod_{s=1}^{K_{j-1}} \beta_{i_{j,s}} \Big) \int_{N_n^{(k_{j,r})}} (t_{j,r})^\gamma \Big(\int_0^1 D^\gamma F_{(k_{j,r})}((\xi t_{j,r} h)) \\
&\qquad\quad - D^\gamma F_{(k_{j,r})}((\xi c_{i_{j,s}} h))\, \mathrm{d}\xi \Big)\, \mathrm{d}t_{n,K_{n-1}} \cdots \mathrm{d}t_{1,1},
\end{aligned}
$$

where in the last step, the terms of lower order vanished by the assumption, which we also inserted in the highest order terms to match the coefficients and receive the difference between the derivatives of F in the integral. Applying $\|\cdot\|_Y$ and using very rough estimates gives the desired result: We pull out said difference with the supremum, estimate the integral and the volume by one, the β and c by their maximum and lastly the sum over i by its number of summands, which is bounded by $(q-1)^{S_{n-1}}$.

If we wanted, we would be able to obtain better constants here, but for the sake of simplifying the formulas, we stick with the estimates we mentioned. ■

Remark 3.12

Neither are we able to prove the existence of a q and (α_j), (β_j) which fulfil the equations from Proposition 3.11, nor can we reduce the number of equations for general $n \in \mathbb{N}$. But we strongly suspect that the set of equations can be drastically reduced down to

$$(k_{j,r}) = (k_{1,1}, \dots, k_{N,1}) = (1, \dots, 1) \qquad \forall N \in \{1, \dots, n\}$$

and the pure monomials in $t_{1,1}$, namely $P((t_{j,s})) = t_{1,1}^M$ for all $M \in \{0,\dots,n-1\}$. To justify this conjecture, lets look again at small n up to four. For $n = 1$, we obtain

$$c_q = 1, \tag{3.33a}$$
$$\sum_{i_{1,1}\leqslant q-1} \beta_{i_{1,1}} = 1. \tag{3.33b}$$

The most popular example for this is the Lie splitting ($q = 2$, $\alpha_1 = 0$, $\alpha_2 = 1$, $\beta_1 = 1$). For $n = 2$, we obtain the additional equation

$$\sum_{i_{1,1}\leqslant q-1} \beta_{i_{1,1}} c_{i_{1,1}} = \frac{1}{2} \tag{3.34}$$

as well as

$$2\sum_{i_{2,1}<i_{1,1}\leqslant q-1} \beta_{i_{1,1}}\beta_{i_{2,1}} + \sum_{i_{1,1}\leqslant q-1} \beta_{i_{1,1}}^2 = 1,$$

which is just (3.33b)^2. The most famous example for this would be the Strang splitting ($q = 2$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{2}$, $\beta_1 = 1$). For $n = 3$, we obtain

$$\sum_{i_{1,1}\leqslant q-1} \beta_{i_{1,1}} c_{i_{1,1}}^2 = \frac{1}{3}, \tag{3.35a}$$
$$2\sum_{i_{2,1}<i_{1,1}\leqslant q-1} \beta_{i_{1,1}}\beta_{i_{2,1}} c_{i_{1,1}} + \sum_{i_{1,1}\leqslant q-1} \beta_{i_{1,1}}^2 c_{i_{1,1}} = \frac{2}{3}. \tag{3.35b}$$

as well as

$$6\sum_{i_{3,1}<i_{2,1}<i_{1,1}\leqslant q-1} \beta_{i_{1,1}}\beta_{i_{2,1}}\beta_{i_{3,1}} + 3\sum_{i_{2,1}<i_{1,1}\leqslant q-1} (\beta_{i_{1,1}}^2\beta_{i_{2,1}} + \beta_{i_{1,1}}\beta_{i_{2,1}}^2) + \sum_{i_{1,1}\leqslant q-1} \beta_{i_{1,1}}^3 = 1,$$
$$2\sum_{i_{2,1}<i_{1,1}\leqslant q-1} \beta_{i_{1,1}}\beta_{i_{2,1}} c_{i_{2,1}} + \sum_{i_{1,1}\leqslant q-1} \beta_{i_{1,1}}^2 c_{i_{1,1}} = \frac{1}{3}.$$

Here, the first equation is just (3.33b)^3, while for the second one, we rearrange (3.33b) into

$$\sum_{i_{2,1}<i_{1,1}} \beta_{i_{2,1}} = 1 - \sum_{i_{1,1}\leqslant i_{2,1}\leqslant q-1} \beta_{i_{2,1}},$$

plug this into (3.35b) using (3.34) on the term that comes from the 1 *above to get*

$$2\sum_{i_{1,1}\leqslant i_{2,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}c_{i_{1,1}}-\sum_{i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^2c_{i_{1,1}}=\frac{1}{3},$$

and finally combine the case $i_{1,1}=i_{2,1}$ *with the second sum and swap the names of the indeces in the remaining sum. Those four equations are just coming from the case* $(k_{1,1},k_{2,1})=(1,1)$*. We get another equation from* $k_{1,1}=2$ *(see (3.30)), namely*

$$3\sum_{i_{2,1},i_{2,2}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}\beta_{i_{2,2}}+\frac{3}{2}\sum_{i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^2\beta_{i_{2,1}}$$
$$+\frac{3}{2}\sum_{i_{2,2}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^2\beta_{i_{2,2}}+\sum_{i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^3=1.$$

To show that this equation is also superfluous, we split up the first sum by $i_{2,1}<i_{2,2}$, $i_{2,2}<i_{2,1}$ *and* $i_{2,1}=i_{2,2}$*. The latter is artificially split in half and combined with the second and third sum, respectively. The fourth sum is also split in half, so that after rearranging, we receive*

$$3\sum_{i_{2,1}<i_{2,2}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}\beta_{i_{2,2}}+\frac{3}{2}\sum_{i_{2,2}<i_{1,1}\leqslant q-1}(\beta_{i_{1,1}}^2\beta_{i_{2,2}}+\beta_{i_{1,1}}\beta_{i_{2,2}}^2)+\frac{1}{2}\sum_{i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^3$$
$$+3\sum_{i_{2,2}<i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}\beta_{i_{2,2}}+\frac{3}{2}\sum_{i_{2,1}<i_{1,1}\leqslant q-1}(\beta_{i_{1,1}}^2\beta_{i_{2,1}}+\beta_{i_{1,1}}\beta_{i_{2,1}}^2)+\frac{1}{2}\sum_{i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^3$$
$$=1,$$

which we know is true by the first already superfluous equation from above, which times one half is added twice here, $(i_{3,1},i_{2,1},i_{1,1})$ *replaced by* $(i_{2,1},i_{2,2},i_{1,1})$ *and* $(i_{2,2},i_{2,1},i_{1,1})$*, respectively. One example for coefficients fulfilling these equations would be* $q=3$, $\alpha_1=\frac{1}{4}+\mathrm{i}\frac{\sqrt{3}}{12}$, $\alpha_2=\frac{1}{2}$, $\alpha_3=\frac{1}{4}-\mathrm{i}\frac{\sqrt{3}}{12}$, $\beta_1=\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{6}$, $\beta_2=\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{6}$*, if A generates a fitting analytic semigroup and the nonlinear flow is defined for complex times as well.*

Otherwise, we can take an option which also fits for $n=4$*. In this case, we have the equations*

$$\sum_{i_{1,1}\leqslant q-1}\beta_{i_{1,1}}c_{i_{1,1}}^3=\frac{1}{4},\qquad(3.36a)$$

$$2\sum_{i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}c_{i_{1,1}}^2+\sum_{i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^2c_{i_{1,1}}^2=\frac{1}{2}, \tag{3.36b}$$

$$6\sum_{i_{3,1}<i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}\beta_{i_{3,1}}c_{i_{1,1}}+3\sum_{i_{2,1}<i_{1,1}\leqslant q-1}(\beta_{i_{1,1}}^2\beta_{i_{2,1}}+\beta_{i_{1,1}}\beta_{i_{2,1}}^2)c_{i_{1,1}}$$
$$+\sum_{i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^3c_{i_{1,1}}=\frac{3}{4}, \tag{3.36c}$$

as well as

$$24\sum_{i_{4,1}<i_{3,1}<i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}\beta_{i_{3,1}}\beta_{i_{4,1}}+12\sum_{i_{3,1}<i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}\beta_{i_{3,1}}(\beta_{i_{1,1}}+\beta_{i_{2,1}}+\beta_{i_{3,1}})$$
$$+\sum_{i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}(4\beta_{i_{1,1}}^2+6\beta_{i_{1,1}}\beta_{i_{2,1}}+4\beta_{i_{2,1}}^2)+\sum_{i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^4=1,$$

$$2\sum_{i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}c_{i_{1,1}}c_{i_{2,1}}+\sum_{i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^2c_{i_{1,1}}c_{i_{2,1}}=\frac{1}{4},$$

$$2\sum_{i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}c_{i_{2,1}}^2+\sum_{i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^2c_{i_{2,1}}^2=\frac{1}{6},$$

$$6\sum_{i_{3,1}<i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}\beta_{i_{3,1}}c_{i_{2,1}}+3\sum_{i_{2,1}<i_{1,1}\leqslant q-1}(\beta_{i_{1,1}}^2\beta_{i_{2,1}}c_{i_{1,1}}+\beta_{i_{1,1}}\beta_{i_{2,1}}^2c_{i_{2,1}})$$
$$+\sum_{i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^3c_{i_{1,1}}=\frac{1}{2}$$

$$6\sum_{i_{3,1}<i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}\beta_{i_{3,1}}c_{i_{3,1}}+3\sum_{i_{2,1}<i_{1,1}\leqslant q-1}(\beta_{i_{1,1}}^2\beta_{i_{2,1}}+\beta_{i_{1,1}}\beta_{i_{2,1}}^2)c_{i_{2,1}}$$
$$+\sum_{i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^3c_{i_{1,1}}=\frac{1}{4}.$$

The first one is just (3.33b)^4, while the second one is (3.34)^2. The third one follows similarly as before by using $\sum_{i_{2,1}<i_{1,1}}\beta_{i_{2,1}}=1-\sum_{i_{1,1}\leqslant i_{2,1}\leqslant q-1}\beta_{i_{2,1}}$ in (3.36b), swapping the indeces and using (3.35a). For the fourth one, we use the same fact in the the term $3\sum_{i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^2\beta_{i_{2,1}}c_{i_{1,1}}$ of (3.36c), giving us amongst other terms $3\sum_{i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^2c_{i_{1,1}}$, which we replace using (3.35b). We combine the term that arises from that with the first sum in (3.36c), allowing us to use (3.33b) again to simplify. In the end, this gives us the fact that $\frac{5}{4}$ equals the left side of the forth equation plus the left side of (3.36c), giving us the desired result. For the fifth one, we use (3.33b)^2 on the term $3\sum_{i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}^2c_{i_{1,1}}$ of (3.36c).

Using (3.33b), this gives us the fact that $\frac{3}{4}$ equals the sum of the fifth and the forth left side, which yields the fifth equation because of the forth one.
Again, we also obtain more equations from the other cases. Firstly, there are

$$3\sum_{i_{2,1},i_{2,2}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}\beta_{i_{2,2}}c_{i_{1,1}}+3\sum_{i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^2\beta_{i_{2,1}}c_{i_{1,1}}+\sum_{i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^3c_{i_{1,1}}=\frac{1}{4},$$

$$3\sum_{i_{2,1},i_{2,2}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}\beta_{i_{2,2}}c_{i_{2,1}}+\frac{3}{2}\sum_{i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^2\beta_{i_{2,1}}(c_{i_{2,1}}+c_{i_{1,1}})+\sum_{i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^3c_{i_{1,1}}=\frac{1}{8},$$

from the case $k_{1,1}=2$ and polynomials of degree one, both of which trace back to (3.36b) and the forth and fifth equation from above by breaking up first sums into the cases $i_{2,1}<i_{2,2}$, $i_{2,1}=i_{2,2}$, $i_{2,1}>i_{2,2}$ and rearranging cleverly. From the cases $(k_{j,r})\in\{(1,1,1),(1,2),(2,1,0),(2,0,1),(3)\}$ for constant polynomials (see (3.31), we obtain the equations

$$12\sum_{i_{3,1},i_{3,2}<i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}\beta_{i_{3,1}}\beta_{i_{3,2}}+12\sum_{i_{3,1}<i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}^2\beta_{i_{3,1}}$$
$$+6\sum_{i_{3,1},i_{3,2}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^2\beta_{i_{3,1}}\beta_{i_{3,2}}+4\sum_{i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}(\beta_{i_{2,1}}^2+\beta_{i_{1,1}}^2)+\sum_{i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^4=1,$$

$$8\sum_{i_{3,1}<i_{2,1},i_{2,2}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}\beta_{i_{3,1}}\beta_{i_{3,2}}+4\sum_{i_{3,1}<i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^2\beta_{i_{2,1}}\beta_{i_{3,1}}$$
$$+4\sum_{i_{2,1},i_{2,2}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}\beta_{i_{2,2}}(\beta_{i_{1,1}}+\beta_{i_{2,1}})$$
$$+2\sum_{i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^2\beta_{i_{2,1}}(2\beta_{i_{1,1}}+1\beta_{i_{2,1}})+\sum_{i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^4=1,$$

$$4\sum_{i_{2,1},i_{2,2},i_{2,3}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}\beta_{i_{2,1}}\beta_{i_{3,1}}\beta_{i_{3,2}}+6\sum_{i_{2,1},i_{2,2}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^2\beta_{i_{2,1}}\beta_{i_{2,2}}$$
$$+4\sum_{i_{2,1}<i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^3\beta_{i_{2,1}}+\sum_{i_{1,1}\leqslant q-1}\beta_{i_{1,1}}^4=1,$$

which all trace back to $(3.33b)^4$ in the same way. Popular solutions to these equations are either $q=6$, $\alpha_1=\alpha_6=\frac{1}{8-2\cdot 4^{\frac{1}{3}}}$, $\alpha_2=\alpha_5=\frac{1}{4-4^{\frac{1}{3}}}$, $\alpha_3=\alpha_4=\frac{1-4^{\frac{1}{3}}}{8-2\cdot 4^{\frac{1}{3}}}$,

$\beta_1 = \beta_2 = \beta_4 = \beta_5 = \frac{1}{4-4^{\frac{1}{3}}}, \beta_3 = -\frac{4^{\frac{1}{3}}}{4-4^{\frac{1}{3}}}$ *or* $q = 5$, $\alpha_1 = \alpha_5 = \frac{1}{10} - \mathrm{i}\frac{1}{30}$, $\alpha_2 = \alpha_4 = \frac{4}{15} + \mathrm{i}\frac{2}{15}$, $\alpha_3 = \frac{4}{15} - \mathrm{i}\frac{1}{5}$, $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \frac{1}{4}$, *depending on whether* A *generates a* C_0 *group or an analytic semigroup.*

We can now finally put everything together to get to a local error estimate in Y. To this end, we combine Propositions 3.7 and 3.9 to obtain (3.32) and use Lemma 3.11 to start estimating it. Taking a close look at the appearing terms gives us sufficient conditions on the derivatives of g and the regularity of X_r compared to Y in order to end up with the desired order of convergence.

Proposition 3.13

Let $r = n - 1 + \theta > 0$ *and* $\|u(t)\|_{X_r} \leqslant R$ *for all* $t \in [0, T]$. *Moreover, assume that Assumption 3.5 and Assumption 3.2 hold for* n. *Then, the one-step error between the exact solution of (3.1) and its numerical approximation fulfils the estimate*

$$\|u(h) - S_q^h(u_0)\|_Y \leqslant \widetilde{C} h^{r+1},$$

where $\widetilde{C}$ *only depends on* R, T, n *and the maximum of the* β_i *and* c_i.

of Proposition 3.13. We start by restating (3.32), which followed from Propositions 3.7 and 3.9.

$$\begin{aligned} u(h) - S_q^h(u_0) = & \sum_{\substack{k_{j,r}=0 \\ j=1,\dots,n-1; r=1,\dots,K_{j-1} \\ S_{n-1}\leqslant n}}^{n-S_{j-1}} \frac{h^{S_{n-1}}}{\prod_{j=1}^{n-1}\prod_{r=1}^{K_{j-1}} k_{j,r}!} \Big(\int_{N_n^{(k_{j,r})}} F_{(k_{j,r})}((t_{j,r}h)) \,\mathrm{d}t_{n,K_{n-1}} \cdots \mathrm{d}t_{1,1} \\ & - \sum_{i_{1,1}=1}^{q-1} \sum_{\substack{i_{j,s}=1; j=2,\dots,n \\ r=1\dots,K_{j-2}; s=K_{j-1}^{(r-1)}+1,\dots,K_{j-1}^{(r)}}}^{i_{j-1,r}} |M_n^{(k_{j,r}),(i_{j,s})}| \Big(\prod_{j=1}^{n}\prod_{s=1}^{K_{j-1}} \beta_{i_{j,s}} \Big) F_{(k_{j,r})}((c_{i_{j,s}}h)) \Big) \\ & + \sum_{m=1}^{n} \sum_{\substack{k_{j,r}=0 \\ j=1,\dots,m; r=1,\dots,K_{j-1} \\ S_{m-1}\leqslant n; S_m>n}}^{n-S_{j-1}} \frac{h^{S_m}}{\prod_{j=1}^{m}\prod_{r=1}^{K_{j-1}} k_{j,r}!} \Big(\int_{N_{m+1}^{(k_{j,r})}} u_{(k_{j,r})_{j=1,\dots,m-1},(t_{j,r}h)_{j=1,\dots,m}} \\ & \Big[\Big(\int_0^1 g^{(k_{m,r})}(A_{k_{m,r}}^{n-S_{m-1}}) \Big[\Big(T((t_{m,r} - t_{m+1,s})h) g(u(t_{m+1,s}h)) \Big)_{s=K_m^{(r-1)}+1}^{K_m^{(r)}} \Big] \,\mathrm{d}\xi \Big)_{r=1}^{K_{m-1}} \Big] \end{aligned} \tag{3.37}$$

$$
\mathrm{d}t_{m+1,K_m}\cdots\,\mathrm{d}t_{1,1} - \sum_{i_{1,1}=1}^{q-1} \sum_{\substack{i_{j,s}=1;\,j=2,\dots,m+1 \\ r=1\dots,K_{j-2};\,s=K_{j-1}^{(r-1)}+1,\dots,K_{j-1}^{(r)}}}^{i_{j-1,r}} \int_{M_{m+1}^{(k_{j,r}),(i_{j,s})}} \Big(\prod_{j=1}^{n}\prod_{s=1}^{K_{j-1}} \beta_{i_{j,s}}\Big)
$$

$$
u_{(k_{j,r})_{j=1,\dots,m-1},(c_{i_{j,s}}h)_{j=1,\dots,m}}\Big[\Big(\int_0^1 g^{(k_{m,r})}(B_{m+1}^{(k_{j,r}),(i_{j,s})}(\beta_{i_{m,r}}t_{m,r}h))\Big[\Big(T(c_{i_{m+1,s}+1,i_{m,r}}h)
$$

$$
g(\psi_{S^h_{i_{m+1,s}}(u_0)}(\beta_{i_{m,r}}t_{m+1,s}h))\Big)_{s=K_m^{(r-1)}+1}^{K_m^{(r)}}\Big]\,\mathrm{d}\xi\Big)_{r=1}^{K_{m-1}}\Big]\,\mathrm{d}t_{m+1,K_m}\cdots\,\mathrm{d}t_{1,1}\Big)
$$

$$
=: D + R
$$

R is relatively easy to treat, since it doesn't force any loss of regularity. By Assumption 3.2, for some $\tilde{r} \in [0, r]$, all derivatives of g up to $g^{(n)}$ exist on $X_{\tilde{r}}$ and we start off with

$$
\|R\|_Y \leqslant \sum_{m=1}^{n} \sum_{\substack{k_{j,r}=0 \\ j=1,\dots,m;r=1,\dots,K_{j-1} \\ S_{m-1}\leqslant n;S_m>n}}^{n-S_{j-1}} \frac{h^{S_m}}{\prod_{j=1}^m \prod_{r=1}^{K_{j-1}} k_{j,r}!}\Big(|N_{m+1}^{(k_{j,r})}| \sup_{(t_{j,r}h)_{j=1,\dots,m}\in N_{m+1}^{(k_{j,r})}}
$$

$$
\Big\|u_{(k_{j,r})_{j=1,\dots,m-1},(t_{j,r}h)_{j=1,\dots,m}}\Big[\Big(\int_0^1 g^{(k_{m,r})}(A_{k_{m,r}}^{n-S_{m-1}}(t))\Big[\Big(T((t_{m,r}-t_{m+1,s})h)
$$

$$
g(u(t_{m+1,s}h))\Big)_{s=K_m^{(r-1)}+1}^{K_m^{(r)}}\Big]\,\mathrm{d}\xi\Big)_{r=1}^{K_{m-1}}\Big]\Big\|_{\tilde{r}}
$$

$$
+\sum_{i_{1,1}=1}^{q-1} \sum_{\substack{i_{j,s}=1;\,j=2,\dots,m+1 \\ r=1\dots,K_{j-2};\,s=K_{j-1}^{(r-1)}+1,\dots,K_{j-1}^{(r)}}}^{i_{j-1,r}} |M_{m+1}^{(k_{j,r}),(i_{j,s})}|\Big(\prod_{j=1}^{n}\prod_{s=1}^{K_{j-1}}|\beta_{i_{j,s}}|\Big) \sup_{(t_{j,r}h)_{j=1,\dots,m}\in M_{m+1}^{(k_{j,r}),(i_{j,s})}}
$$

$$
\Big\|u_{(k_{j,r})_{j=1,\dots,m-1},(c_{i_{j,s}}h)_{j=1,\dots,m}}\Big[\Big(\int_0^1 g^{(k_{m,r})}(B_{m+1}^{(k_{j,r}),(i_{j,s})}(\beta_{i_{m,r}}t_{m,r}h))\Big[\Big(T(c_{i_{m+1,s}+1,i_{m,r}}h)
$$

$$
g(\psi_{S^h_{i_{m+1,s}}(u_0)}(\beta_{i_{m,r}}t_{m+1,s}h))\Big)_{s=K_m^{(r-1)}+1}^{K_m^{(r)}}\Big]\,\mathrm{d}\xi\Big)_{r=1}^{K_{m-1}}\Big]\Big\|_{\tilde{r}}\Big)
$$

$$
\leqslant h^{n+\theta}\sum_{m=1}^{n} \sum_{\substack{k_{j,r}=0 \\ j=1,\dots,m;r=1,\dots,K_{j-1} \\ S_{m-1}\leqslant n;S_m>n}}^{n-S_{j-1}} \frac{T^{S_m-n-\theta}}{\prod_{j=1}^m \prod_{r=1}^{K_{j-1}} k_{j,r}!}\Big(\sup_{(t_{j,r}h)_{j=1,\dots,m}\in N_{m+1}^{(k_{j,r})}} \Big\|u_{(k_{j,r})_{j=1,\dots,m-1},(t_{j,r}h)_{j=1,\dots,m}}
$$

$$
\Big[\Big(\int_0^1 g^{(k_{m,r})}(A_{k_{m,r}}^{n-S_{m-1}})\Big[\Big(T((t_{m,r}-t_{m+1,s})h)g(u(t_{m+1,s}h))\Big)_{s=K_m^{(r-1)}+1}^{K_m^{(r)}}\Big]\,\mathrm{d}\xi\Big)_{r=1}^{K_{m-1}}\Big]\Big\|_{\tilde{r}}
$$

$$+\sum_{i_{1,1}=1}^{q-1} \sum_{\substack{i_{j,s}=1;\, j=2,\dots,m+1 \\ r=1\dots,K_{j-2};\, s=K_{j-1}^{(r-1)}+1,\dots,K_{j-1}^{(r)}}}^{i_{j-1,r}} (\max_{j\in\{1,\dots,q-1\}} |\beta_j|)^{S_m} \sup_{(t_{j,r}h)_{j=1,\dots,m}\in M_{m+1}^{(k_{j,r}),(i_{j,s})}}$$

$$\Big\| u_{(k_{j,r})_{j=1,\dots,m-1},(c_{i_{j,s}}h)_{j=1,\dots,m}} \Big[\Big(\int_0^1 g^{(k_{m,r})}(B_{m+1}^{(k_{j,r}),(i_{j,s})}(\beta_{i_{m,r}} t_{m,r}h))\Big[\Big(T(c_{i_{m+1,s}+1,i_{m,r}}h)$$

$$g(\psi_{S^h_{i_{m+1,s}}(u_0)}(\beta_{i_{m,r}} t_{m+1,s}h))\Big)_{s=K_m^{(r-1)}+1}^{K_m^{(r)}}\Big] \,\mathrm{d}\xi\Big)_{r=1}^{K_{m-1}}\Big]\Big\|_{\tilde r}\Big)$$

The terms in the norms are nested (semi)groups T followed by derivatives $g^{(k)}$. We lose the (semi)groups easily via their uniform boundedness in t and estimate the multilinear maps $g^{(k)}$ by the fact that they map $(X_{\tilde r})^k$ to $X_{\tilde r}$. The only thing left are the respective operator norms, which depend on the X_r-norm of their arguments. These arguments are mostly of the form $u_l(t) = T(t)u_0$, which makes them bounded by R in X_r since $u_0 = u(0)$.
If in the first norm, the argument has the form $u_l(t) + \xi u_{nl}(t)$ or $u(t)$ for some $\xi \in (0,1)$ and $t \in [0,h]$, hence $u_l(t)+\xi u_{nl}(t) = \xi u(t)+(1-\xi)u_l(t)$ for some $\xi \in (0,1]$ and and $t \in [0,h]$ (see (3.8)), it is in the same way bounded by R in X_r.
If in the second norm, the argument has the form $B_{m+1}^{(k_{j,r}),(i_{j,s})}(t)$, hence $T(t_1)[S_i^h(u_0)+\xi u_a^{(i_0)}(t_2)] = T(t_1)[\xi\psi_{S_i^h(u_0)}(t_2) + (1-\xi)S_i^h(u_0)]$ for some $t_1,t_2 \in [0,\max\{|\beta_i|\}h]$, $i \in \{1,\dots,q\}$ and $\xi \in [0,1]$ (see (3.20) and (3.19)). By Proposition 3.6, those terms are bounded by constants depending on R and T in X_r.
After this, we can estimate everything very roughly: The norms by the biggest value out of all the above estimates, the factorials by 1, the sum over i by $(q-1)^{S_{m-1}}$, the sum over k by the sum of the first $n-1$ Catalan numbers (see Remark 3.10), S_{m-1} by n and S_m by e.g. n^2. This shows that

$$\|R\|_Y \leqslant Ch^n,$$

where C depends on R, T, n, q and the maximum of the β_i and c_i. We move on to the term D. By Proposition 3.11, we have

$$\|D\|_Y \leqslant \sum_{\substack{k_{j,r}=0 \\ j=1,\dots,n-1;r=1,\dots,K_{j-1} \\ S_{n-1}\leqslant n}}^{n-S_{j-1}} \frac{h^{S_{n-1}}}{\prod_{j=1}^{n-1}\prod_{r=1}^{K_{j-1}} k_{j,r}!} \Big\| \int_{N_n^{(k_{j,r})}} F_{(k_{j,r})}((t_{j,r}h)) \,\mathrm{d}t_{n,K_{n-1}} \cdots \mathrm{d}t_{1,1}$$

$$
\begin{aligned}
&- \sum_{i_{1,1}=1}^{q-1} \sum_{\substack{i_{j,s}=1;\, j=2,\dots,n \\ r=1\dots,K_{j-2};\, s=K_{j-1}^{(r-1)}+1,\dots,K_{j-1}^{(r)}}}^{i_{j-1,r}} |M_n^{(k_{j,r}),(i_{j,s})}| \Big(\prod_{j=1}^{n} \prod_{s=1}^{K_{j-1}} \beta_{i_{j,s}} \Big) F_{(k_{j,r})}((c_{i_{j,s}} h)) \Big\|_Y \\
&\leqslant h^n \sum_{\substack{k_{j,r}=0 \\ j=1,\dots,n-1;\, r=1,\dots,K_{j-1} \\ S_{n-1}\leqslant n}}^{n-S_{j-1}} \frac{1}{\prod_{j=1}^{n-1} \prod_{r=1}^{K_{j-1}} k_{j,r}!} \sum_{|\gamma|=n-S_{n-1}} C_{\gamma,q}^{(k_{j,r}),(\alpha_j),(\beta_j)} \\
&\qquad \sup_{t_{j,r},s_{j,r}\in N_n^{(k_{j,r})}} \|D^\gamma F_{(k_{j,r})}((t_{j,r}h)) - D^\gamma F_{(k_{j,r})}((s_{j,r}h))\|_Y
\end{aligned}
$$

At this point, we arrive at the crucial part of the proof. The functions $F_{(k_{j,r})}$ consist of a nesting of (semi)groups T and derivatives g^k. Differentiating once with respect to an arbitrary variable gives a sum of terms, each of which differs from the the original function in that is has a $\pm AT$ instead of T, or a g^{k+1} with added argument Au_l instead of g^k. Repeating the process gives the additional possibility of changing such a Au_l into a A^2u_l and so on.
Therefore, an arbitrary derivative of order $n - S_{n-1}$ consists of the standard nesting with a summed total of $S_{n-1} - 1$ to $n - 1$ orders in the derivatives of g, as well as $n - S_{n-1}$ times A in the form of either $A^l u_l$ in an argument of $g^{(k)}(u_l)$ or as $\pm A^l$ in front of a (semi)group T.
In the next step, we have to build the difference of this derivative at two arbitrary points. The standard procedure of inserting a telescopic sum, where two consecutive summands only have different variables in one of the terms (that is, in one T or one u_l) gives basically the same expression as before, the only difference being the single occurrence of either

$$
[T((t_1-t_2)h)-T((s_1-s_2)h)]v, \quad u_l(th)-u_l(sh), \quad \text{or} \quad g^{(k)}(u_l(th))-g^{(k)}(u_l(sh))[v_1,\dots,v_k].
$$

By the mean value Theorem for Fréchet differentiable functions (see [AP95, Theorem 1.8]), we can write

$$
\begin{aligned}
&g^{(k)}(u_l(th)) - g^{(k)}(u_l(sh))[v_1,\dots,v_k] \\
&\qquad = \int_0^1 g^{(k+1)}(\xi u_l(th) + (1-\xi)u_l(sh))[v_1,\dots,v_k,u_l(th)-u_l(sh)]\,\mathrm{d}\xi,
\end{aligned}
$$

reducing the third difference to the second one while gaining another derivative order. Since $u_l(th) = T(th)u_0$, the second and first difference are of the same structure. Estimating this gives us θ orders of h while loosing the same order of regularity, that is

$$\|[T(th) - T(sh)]v\|_{\hat{p}} \leqslant h^{\theta} \|v\|_{\hat{p}+\theta}.$$

This result is justified by part **a)** of the Theorem on page 77 of [Tri95] ($p = \infty$, $m = 1$), which connects the above norm to a real interpolation space, combined with part **d)** of the Theorem on page 101 of [Tri95], where this interpolation space is compared to the fitting fractional domain. We also need to use an easy reiteration argument (see [Tri95, 1.15.4]).
Now, estimating $\|D^{\gamma} F_{(k_{j,r})}((t_{j,r}h)) - D^{\gamma} F_{(k_{j,r})}((s_{j,r}h))\|_Y$, we start off with a difference of $r = n - 1 + \theta$ in regularity orders compared to the space X_r and at the same time at most $n-1$ occurrences of A worth one order of regularity plus one occurrence of a difference (in our case) worth θ orders of regularity. Since by assumption, we can distribute the order difference to r amongst all arguments whenever estimating $g^{(k)}$, we do this exactly in the way that keeps the necessary distance to process the terms A and the difference while still staying under r orders of regularity. By this we mean that in every argument $v_1, \ldots, v_k$ of $g^{(k)}(u_l(th))[v_1, \ldots, v_k]$, we count the number of times an A occurs ($l \in \{0, \ldots, n-1\}$) and the number of differences ($\hat{l} \in \{0, 1\}$) so that we can stay in the space $X_{p-l-\hat{l}}$) with this argument.
Lastly, we justify that the higher derivatives only need to exist and allow the required estimates in bigger spaces. For $g^{(k)}$ to show up, S_{n-1} must at least be $k+1$ by definition. Hence, we take at most $n-k-1$ derivative, giving us at most $n-k-1$ occurrences of A. $m \in \{0, \ldots, n-k-1\}$ of those derivatives are taken with respect to the variable involved in the argument, increasing the order of derivatives to $g^{(k+m)}$. As we have seen above, the resulting m occurrences of A appear inside one of the arguments that $g^{(k+m)}$ linearly depends on, so at most $n-k-m-1$ of them can appear in front of said $g^{(k+m)}$. Including the difference, this makes for a loss of regularity of at most $n - 1 + \theta - (k+m)$ before reaching $g^{(k+m)}$, meaning we only need the estimates mentioned in Assumption 3.2 for g^l on spaces bigger than X_{r-l}). This finally yields

$$\|u(h) - S_q^h(u_0)\|_Y \leqslant \widetilde{C} h^{n+\theta},$$

where $\widetilde{C}$ only depends on R, T, n, q and the maximum of the β_i and c_i. ■

3.6 Uniform bounds

The last thing we need to ensure is the uniform boundedness in X_s of all the terms on which we want to use Proposition 3.6 **a)** in the proof of Theorem 3.4. This is done in the following Lemma.

Lemma 3.14

If $\|u(t)\|_{X_r} \leqslant R$ for all $t \in [0,T]$. Put $h_0 = \min\{\left(\frac{R}{T e^{C(4R)|\beta|T} C_{loc}}\right)^{\frac{1}{r-s}}, \frac{\log(2)}{C(4R)|\beta|}, T\}$, where $C(R)$ is the the constant from Proposition 3.6 and C_{loc} is the constant from Proposition 3.13 with Y replaced by X_s. Then, for $h \in (0,h_0]$, $Nh \leqslant T$, $k \in \{0,\dots,N\}$ and $j \in \{1,\dots,k\}$, we have

$$\|(S^h)^{N-j}(u(kh))\|_{X_s} \leqslant 2R.$$

Proof. We show a stronger result by induction over N, namely

$$\|(S^h)^{N-j}(u(kh)) - u((N-j+k)h)\|_{X_s} \leqslant T e^{C(4R)|\beta|T} \widetilde{C} h^{r-s}. \tag{3.38}$$

for $Nh \leqslant T$ and all $k \in \{0,\dots,N\}$, $j \in \{k,\dots,N\}$. Indeed, by the triangle inequality and $h \leqslant h_0$, our bound on u in X_r and (3.38), we get

$$\|(S^h)^{N-j}(u(kh))\|_{X_s} \leqslant 2R. \tag{3.39}$$

We start with $N = 0$, for which the difference in (3.38) is 0 and hence the estimate is trivial. Assume (3.38) for some $N \in \mathbb{N}_0$ with $(N+1)h \leqslant T$ and $k \in \{0,\dots,N\}$. For $k = N+1$, we get $j = N+1$ and the estimate is once again trivial. For $k \in \{0,\dots,N\}$ and $j \in \{k+1,\dots N+1\}$, the resulting term is already covered by the induction assumption. Let $k \in \{0,\dots,N\}$ and $j = k$. We compute that

$$\begin{aligned}\|(S^h)^{N+1-k}(u(kh)) - u((N+1)h)\|_{X_s} \\ \leqslant \sum_{l=0}^{N-k} \|(S^h)^{N+1-k-l}(u((k+l)h)) - (S^h)^{N-k-l}(u((k+l+1)h))\|_{X_s}\end{aligned}$$

Now, we can use the stability property from Proposition 3.6 **b)** $N-k-l$ times. R_s can be taken to be $2R$ by our induction assumption (see (3.39)). This yields

$$\begin{aligned}&\|(S^h)^{N+1-k}(u(kh)) - u((N+1)h)\|_{X_s}\\&\qquad\leqslant \sum_{l=0}^{N-k} \mathrm{e}^{C(4R)|\beta|h(N-k-l)} \|u((k+l)h))) - S^h(u((k+l+1)h))\|_{X_s}.\end{aligned}$$

Now, we can use Proposition 3.13 with Y replaced by X_s, $n+\theta$ replaced by $r-s$ and u_0 by $u(kh)$, that is

$$\|S^h_{\mathrm{Lie}}(u(kh)) - u((k+1)h)\|_{X_s} \leqslant C_{\mathrm{loc}} h^{1+r-s}.$$

All the necessary estimates are covered by Assumption 3.2 once again. Therefore, we finally obtain

$$\begin{aligned}\|(S^h)^{N+1-k}(u(kh)) - u((N+1)h)\|_{X_s} &\leqslant \sum_{l=0}^{N-k} \mathrm{e}^{C(4R)|\beta|h(N-k-l)} C_{\mathrm{loc}} h^{1+r-s}\\&\leqslant N\mathrm{e}^{C(4R)|\beta|h(N-k-l)} C_{\mathrm{loc}} h^{1+r-s} \leqslant T\mathrm{e}^{C(4R)|\beta|T} C_{\mathrm{loc}} h^{r-s},\end{aligned}$$

which is (3.38) for N replaced by $N+1$ and $j=k$. This concludes the induction as well as the proof by (3.39). ∎

4 Applications

After establishing the theoretical framework, we now turn to several examples. We recall the definition of the splitting scheme: For fixed $h > 0$ and $q \in \mathbb{N}$, let

$$\begin{aligned} S_1^h(u_0) &:= T(\alpha_1 h)u_0, \\ S_{i+1}^h(u_0) &:= T(\alpha_{i+1}h)\psi_{S_i^h(u_0)}(\beta_i h), \qquad i \in \{1,\dots,q-1\}, \\ S^h(u_0) &:= S_q^h(u_0), \end{aligned} \tag{4.1}$$

where T was the solution to the linear part and ψ the one to the purely nonlinear part. Possible schemes for orders up to four were given in Remark 3.3, namely

- $r \leqslant 1$: $q = 2$, $\alpha_1 = 0$, $\alpha_2 = 1$, $\beta_1 = 1$ (Lie Splitting),
- $r \leqslant 2$: $q = 2$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{2}$, $\beta_1 = 1$ (Strang Splitting),
- $r \leqslant 3$: $q = 3$, $\alpha_1 = \frac{1}{4} + \mathrm{i}\frac{\sqrt{3}}{12}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{1}{4} - \mathrm{i}\frac{\sqrt{3}}{12}$, $\beta_1 = \frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{6}$, $\beta_2 = \frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{6}$, if A generates a fitting analytic semigroup and ψ admits complex times (otherwise, see $r \leqslant 4$),
- $r \leqslant 4$: $q = 6$, $\alpha_1 = \alpha_6 = \frac{1}{8-2\cdot 4^{\frac{1}{3}}}$, $\alpha_2 = \alpha_5 = \frac{1}{4-4^{\frac{1}{3}}}$, $\alpha_3 = \alpha_4 = \frac{1-4^{\frac{1}{3}}}{8-2\cdot 4^{\frac{1}{3}}}$, $\beta_1 = \beta_2 = \beta_4 = \beta_5 = \frac{1}{4-4^{\frac{1}{3}}}$, $\beta_3 = -\frac{4^{\frac{1}{3}}}{4-4^{\frac{1}{3}}}$, if A generates a C_0 group, or $q = 5$, $\alpha_1 = \alpha_5 = \frac{1}{10} - \mathrm{i}\frac{1}{30}$, $\alpha_2 = \alpha_4 = \frac{4}{15} + \mathrm{i}\frac{2}{15}$, $\alpha_3 = \frac{4}{15} - \mathrm{i}\frac{1}{5}$, $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \frac{1}{4}$, if A generates a fitting analytic semigroup.

For the general Assumption 3.5 and more details, we refer to Section 3.2. A result on polynomial nonlinearities which will be used throughout this chapter concerns Assumption 3.2.

Lemma 4.1

Assume that Y admits the definition of pointwise multiplication and there exists an $s_\infty > 0$ such that for $s > s_\infty$, it holds that for $u, v \in D(A^s)$, we have $uv \in D(A^s)$ and for $u \in D(A^s)$, $v \in Y = D(A^0)$, we have $uv \in Y$ with

$$\|uv\|_s \lesssim \|u\|_s \|v\|_s, \qquad \|uv\|_0 \lesssim \|u\|_s \|v\|_0,$$

respectively. Further assume that the fractional domains of A form a complex interpolation scale and complex conjugation does not change the norms.

a) *Let $n \in \mathbb{N}$ as well as $a_i, b \in [0,s]$ with $\sum_{i=1}^n a_i = b$. If $u_i \in H^{s-a_i}$, then $\prod_{i=1}^k u_i \in H^{s-b}$ and*

$$\Big\|\prod_{i=1}^n u_i\Big\|_{s-b} \lesssim \prod_{i=1}^n \|u_i\|_{s-a_i}.$$

b) *Let $k \in \mathbb{N}$ be odd. Then $g : H^s \to H^s$, $g(u) = |u|^{k-1}u$ is infinitely often real Fréchet differentiable with*

$$g^{(n)}(u)[v_1,\dots,v_n] = \begin{cases} \displaystyle\sum_{\substack{w_1,\dots w_k \in \{u,v_1,\dots,v_n\} \\ \#\{i: w_i = v_j\} = 1 \forall j}} \prod_{j=1}^{\frac{k+1}{2}} w_j \prod_{j=\frac{k+3}{2}}^{k} \overline{w_j} & , n \leqslant k \\ 0 & , n > k. \end{cases}$$

for all $n \in \mathbb{N}_0$. For all $u \in H^s$ and $n \in \mathbb{N}$ with $a_i, b \in [0,s]$ and $\sum_{i=1}^n a_i = b$, the derivatives are extendable to a multilinear operator

$$g^{(n)}(u) : \bigotimes_{i=1}^n H^{s-a_i} \to H^{s-b}$$

with $\|g^{(n)}(u)\| \leqslant C(R)$ for $\|u\|_s \leqslant R$.

c) *For $s > \tilde{s} > s_\infty$ and $u, v \in H^s$, we obtain*

$$\|g(u) - g(v)\|_0 \leqslant C(\widetilde{R}) \|u - v\|_0$$

for $\|u\|_{\tilde{s}}, \|v\|_{\tilde{s}} \leqslant \widetilde{R}$ *and any* $\widetilde{R} > 0$ *as well as*

$$\|g(u) - g(v)\|_s \leqslant C(R)\|u - v\|_s \quad \textit{and} \quad \|g(u) - g(v)\|_{\tilde{s}} \leqslant C(R)\|u - v\|_{\tilde{s}}$$

for $\|u\|_s, \|v\|_s \leqslant R$ *and any* $R > 0$*.*

Remark

a) *Notice that we have shown all of Assumption 3.2 in this lemma. Also notice that if we choose a spacer* $\widetilde{Y}$ *bigger than* Y *on the same interpolation scale and have the basic estimate on* $\widetilde{Y}$ *instead of* Y*, the estimate on* Y *follows directly by interpolation. This is relevant in the subsequent examples if we take* Y *to be for example a Sobolev space* H^s *for* $s > 0$ *and only show the multiplication property on* L^2*.*

b) *We mention here that the nonlinearities can be generalized to so called algebraic nonlinearities (see [Cha18][Definition 4.4]) without complications. These are defined by*

$$g(u) = \sum_{k \in \mathbb{N}_0} c_k |u|^{2k} u$$

with $c_k \in \mathbb{C}$ *and* $\lim_{k\to\infty} \sqrt[k]{|c_k|} = 0$*, that is, an appropriate series with the above nonlinearities as summands. The decay of* (c_k) *allows for no problems concerning convergence.*

Proof of Lemma 4.1. **a)** We use induction over $n \in \mathbb{N}$. $n = 1$ is trivial and we also need $n = 2$ in the induction assumption, so that is where we start. Let $a_1, a_2, b \in [0, s]$ with $a_1 + a_2 = b$. The two required estimates mean that for fixed $u \in H^s$, the mapping $v \mapsto uv$ is linear and bounded on H^0 and H^s. Interpolating between the two (see e.g. [BL76, Theorem 4.1.2], $\theta = \frac{b}{s}$) gives

$$\|uv\|_{s-b} \lesssim \|u\|_s \|v\|_{s-b}$$

for all $u \in H^s$ and $v \in H^{s-b}$. Symmetry obviously also gives the mirrored version

$$\|uv\|_{s-b} \lesssim \|u\|_{s-b} \|v\|_s$$

for all $u \in H^{s-b}$ and $v \in H^s$. Interpreting $(u,v) \mapsto uv$ as a bilinear map, we can use multilinear interpolation (see [BL76, Theorem 4.4.1] for $\theta = \frac{a_1}{b}$) to obtain

$$\|uv\|_{s-b} \lesssim \|u\|_{s-a_1} \|v\|_{s-a_2}$$

for all $u \in H^{s-a_1}$ and $v \in H^{s-a_2}$ from those two estimates (using $a_1 + a_2 = b$), giving us the desired result for $n = 2$.

Now we suppose the result holds for some $n \in \mathbb{N}$. Let $a_i, b \in [0,s]$ with $\sum_{i=1}^{n+1} a_i = b$ and $u_i \in H^{s-a_i}$. By induction assumption, we have $\prod_{i=1}^{n} u_i \in H^{s-(b-a_{n+1})}$ and

$$\Big\|\prod_{i=1}^{n} u_i\Big\|_{s-(b-a_{n+1})} \lesssim \prod_{i=1}^{n} \|u_i\|_{s-a_i}.$$

Now we only need the bilinear result once more to see that $\prod_{i=1}^{n+1} u_i \in H^{s-b}$ with

$$\Big\|\prod_{i=1}^{n+1} u_i\Big\|_{s-b} \lesssim \Big\|\prod_{i=1}^{n} u_i\Big\|_{s-(b-a_{n+1})} \|u_{n+1}\|_{s-a_{n+1}} \lesssim \prod_{i=1}^{n+1} \|u_i\|_{s-a_i}.$$

b) We once again use induction over $n \in \mathbb{N}_0$. $n = 0$ is trivial, since the given formula just reproduces g itself. Now assume that g is n times differentiable and $g^{(n)}$ has the given form. If $n > k$, the result is trivial, since the derivative of the zero function is again zero. For $n = k$, $g^{(n)}(u)$ does not depend on u, so $g^{(n+1)}(u) = 0$ by definition. Hence, let now be $n < k$. For $u, v_1, \dots, v_{n+1} \in H^s$, we have by induction assumption

$$\begin{aligned} g^{(n)}(u + v_{n+1})[v_1, \dots, v_n] &= \sum_{\substack{w_1,\dots w_k \in \{u+v_{n+1}, v_1,\dots,v_n\} \\ \#\{i: w_i = v_j\} = 1 \forall j \leqslant n}} \prod_{j=1}^{\frac{k+1}{2}} w_j \prod_{j=\frac{k+3}{2}}^{k} \overline{w_j} \\ &= \sum_{\substack{w_1,\dots w_k \in \{u, v_{n+1}, v_1,\dots,v_n\} \\ \#\{i: w_i = v_j\} = 1 \forall j \leqslant n}} \prod_{j=1}^{\frac{k+1}{2}} w_j \prod_{j=\frac{k+3}{2}}^{k} \overline{w_j}, \end{aligned}$$

expanding the product regarding the terms $u + v_{n+1}$ in the second step. Comparing this to

$$g^{(n)}(u)[v_1, \dots, v_n] = \sum_{\substack{w_1, \dots w_k \in \{u, v_1, \dots, v_n\} \\ \#\{i : w_i = v_j\} = 1 \forall j \leqslant n}} \prod_{j=1}^{\frac{k+1}{2}} w_j \prod_{j=\frac{k+3}{2}}^{k} \overline{w_j},$$

again by induction assumption. We see that the terms are identical up to the missing choice of the term v_{n+1}, hence the terms missing v_{n+1} in the first sum cancel out with the second sum, that is

$$g^{(n)}(u+v_{n+1})[v_1, \dots, v_n] - g^{(n)}(u)[v_1, \dots, v_n] = \sum_{\substack{w_1, \dots w_k \in \{u, v_1, \dots, v_{n+1}\} \\ \#\{i : w_i = v_j\} = 1 \forall j \leqslant n \\ \#\{i : w_i = v_{n+1} \geqslant 1\}}} \prod_{j=1}^{\frac{k+1}{2}} w_j \prod_{j=\frac{k+3}{2}}^{k} \overline{w_j}.$$

Comparing this to the alleged formula for $g^{(n+1)}(u)[v_1, \dots, v_{n+1}]$, that is,

$$g^{(n+1)}(u)[v_1, \dots, v_{n+1}] = \sum_{\substack{w_1, \dots w_k \in \{u, v_1, \dots, v_{n+1}\} \\ \#\{i : w_i = v_j\} = 1 \forall j \leqslant n+1}} \prod_{j=1}^{\frac{k+1}{2}} w_j \prod_{j=\frac{k+3}{2}}^{k} \overline{w_j},$$

we see that it cancels out the part of the sum above where $\#\{i : w_i = v_{n+1} = 1\}$, leaving us with

$$\begin{aligned}
\|g^{(n)}(u + v_{n+1})[v_1, \dots, v_n] - g^{(n)}(u)[v_1, \dots, v_n] - g^{(n+1)}(u)[v_1, \dots, v_{n+1}]\|_s \\
&= \Big\| \sum_{\substack{w_1, \dots w_k \in \{u, v_1, \dots, v_{n+1}\} \\ \#\{i : w_i = v_j\} = 1 \forall j \leqslant n \\ \#\{i : w_i = v_{n+1} \geqslant 2\}}} \prod_{j=1}^{\frac{k+1}{2}} w_j \prod_{j=\frac{k+3}{2}}^{k} \overline{w_j} \Big\|_s \\
&\leqslant \sum_{\substack{w_1, \dots w_k \in \{u, v_1, \dots, v_{n+1}\} \\ \#\{i : w_i = v_j\} = 1 \forall j \leqslant n \\ \#\{i : w_i = v_{n+1} \geqslant 2\}}} \Big\| \prod_{j=1}^{\frac{k+1}{2}} w_j \prod_{j=\frac{k+3}{2}}^{k} \overline{w_j} \Big\|_s \\
&\lesssim \|v_{n+1}\|_s^2 \prod_{i=1}^{n} \|v_i\|_s \sum_{w_1, \dots w_{k-(n+2)} \in \{u, v_{n+1}\}} \prod_{j=1}^{k-(n+2)} \|w_j\|_s.
\end{aligned}$$

If $n = k-2$, the last sum is empty and set to be one. We used the algebra property of the multiplication in H^s as well as the fact that $\|\overline{w}\|_s = \|w\|_s$ for the norms at hand. Divided by $\|v_{n+1}\|_s$, the result converges to zero as z_{n+1} goes to zero in H^s, which shows the validity of the formula for $g^{(n+1)}(u)[v_1,\dots,v_{n+1}]$.
Regarding the extendability, we note that for $n \leqslant k$ ($n > k$ is again the trivial case), every summand in $g^{(n)}(u)[v_1,\dots,v_n]$ contains $v_1,\dots,v_n$ (or the complex conjugate) once and u (or its complex conjugate) $k-n \geqslant 0$ times. We split off all occurrences of u by the bilinear version of **a)** for $a_1 = 0$, $a_2 = b$ and then use the multilinear version plus the algebra property to obtain

$$\|g^{(n)}(u)[v_1,\dots,v_n]\|_{s-b} \lesssim \prod_{i=1}^{n} \|v_i\|_{s-a_i} \|u\|_s^{k-n} \leqslant R^{k-n} \prod_{i=1}^{n} \|v_i\|_{s-a_i}$$

c) The previous parts hold true for s replaced by $\tilde{s}$. Let $u, v \in H^s$ with either $\|u\|_{\tilde{s}}, \|v\|_{\tilde{s}} \leqslant \widetilde{R}$ or $\|u\|_s, \|v\|_s \leqslant R$ for some $R, \widetilde{R} > 0$. Using the representation

$$g(u) - g(v) = \int_0^1 g'(\xi u + (1-\xi)v)[u-v]\, \mathrm{d}\xi,$$

we first observe that the H^s or $H^{\tilde{s}}$ norms of $\xi u + (1-\xi)v$ are also bounded by R or $\widetilde{R}$ respectively. From the we obtain from **b)** that

$$\|g(u) - g(v)\|_0 \leqslant \sup_{\xi \in [0,1]} \|g'(\xi u + (1-\xi)v)[u-v]\|_0 \leqslant C(\widetilde{R}) \|u - v\|_0$$

as well as

$$\|g(u) - g(v)\|_s \leqslant \sup_{\xi \in [0,1]} \|g'(\xi u + (1-\xi)v)[u-v]\|_s \leqslant C(R) \|u - v\|_s$$

and

$$\|g(u) - g(v)\|_{\tilde{s}} \leqslant \sup_{\xi \in [0,1]} \|g'(\xi u + (1-\xi)v)[u-v]\|_{\tilde{s}} \leqslant C(R) \|u - v\|_{\tilde{s}}$$

■

4.1 Nonlinearities self-mapping on fractional domains

First we show we can recover a (generalized) result by Ostermann and Hansen from our general method. We still work with the equation

$$\left.\begin{aligned} u'(t) &= (-Au)(t) + g(u(t)), \\ u(0) &= u_0. \end{aligned}\right\} \tag{4.2}$$

and keep the assumption on the operator A in all its generality:

Assumption 3.1

Let $(Y, \|\cdot\|_Y)$ be a Banach space and $A : D(A) \subseteq Y \to Y$ a linear operator such that $-A$ generates by $T(t) := \mathrm{e}^{-tA} : Y \to Y$

- *$r \leqslant 2$: A C_0 semigroup for $t \geqslant 0$*
- *$r > 2$: A C_0 group for $t \in \mathbb{R}$ or an analytic semigroup for t in a sector $\Sigma_\varphi := \{z \in \mathbb{C} \mid |\arg(z)| \leqslant \varphi\}$ for some $\varphi \in [0, \frac{\pi}{2})$.*

As for the nonlinearity g, we assume

Assumption 3.2*

g is locally Lipschitz on $X_r = D(A^r)$ as well as $\lceil b \rceil$ times differentiable on X_{r-b} for $b \in \{k, k-1+\theta\}$ ($k = 1, \ldots, n-1$) and $b = r$ with

$$\|g^{(l)}(x)\|_{B(X_{r-b}\times\cdots\times X_{r-b}, X_{r-b}))} \leqslant C(R)$$

for $l = 0, \ldots, \lceil b \rceil$, as long as $\|x\|_{X_{r-b}} \leqslant R$ for all $R \geqslant 0$. For $l = 1$ and $b = r$, it should also hold as long as $\|x\|_{X_0} \leqslant R$.

This is obviously a stronger assumption than Assumption 3.2 (we have $s(b) = r - b$ and we can choose $s = 0$ here). Hence, we derive the following result directly from Theorem 3.4.

Corollary 4.2

Let $r > 0$ and $u_0 \in X_r$. Let assumptions 3.1, 3.2 and 3.5 (see Remark 3.3) hold and assume the solution $u \in C([0,T], X_r)$ of (4.2) fulfils $\|u(t)\|_{X_r} \leqslant R$ for all $t \in [0,T]$ and some $R \geqslant 0$. Then, we conclude that there exists an $h_0 \in (0,T]$ such that*

$$\|(S^h)^N(u_0) - u(Nh)\|_Y \leqslant Ch^r$$

for all $h \in (0, h_0]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T$, where C only depends on R and T.

For $r \in \mathbb{N}_{\leqslant 4}$, this result is contained in [HO16], which was also the inspiration for our generalization. The fact that Assumption 3.5 the exact same as in [HO16] is shown in Remark 3.12 (see also Remark 3.3). The boundedness on bounded sets is not mentioned in this paper, but we strongly suggest it (or a slightly different version as mentioned in the Remark after Assumption 3.2) is implicitly used and necessary.

4.2 Schrödinger equations

In the following examples, we work with operators related to the Laplacian as well as polynomial nonlinearities. We use H^s (with $\|\cdot\|_s := \|\cdot\|_{H^s}$) to denote the fractional Sobolev spaces (on $\mathbb{R}^d$, $\mathbb{T}^d$ and compact manifolds M), which will be defined in the respective examples. The following Lemma is important for all examples.

4.2.1 On $L^2(\mathbb{R}^d)$, $L^2(\mathbb{T}^d)$ and $L^2(M)$ for manifolds

a) We take a look at the equation

$$\left.\begin{aligned} \mathrm{i}u'(x,t) &= ((-\Delta)^\sigma u)(x,t) \pm |u(x,t)|^{k-1} u(x,t), \\ u(x,0) &= u^0(x), \end{aligned}\right\} \tag{4.3}$$

for $\sigma > 0$ and $k \in \mathbb{N}$ odd as well as $(x,t) \in \mathbb{R}^d \times \mathbb{R}$, where $u^0 \in L^2(\mathbb{R}^d)$. To adapt the main Theorem to this equation, we set $A_\sigma = -\mathrm{i}(-\Delta)^\sigma$ and define the fractional Sobolev spaces by the Bessel potential spaces

$$H^s := H^s(\mathbb{R}^d) := \{f \in L^2(\mathbb{R}^d) \mid (1+|\xi|^2)^{\frac{s}{2}} \mathcal{F}f \in L^2(\mathbb{R}^d)\}$$

for $s \geqslant 0$ with their natural norm. We see that for $r > 0$ and $f \in H^{2s}$,

$$(\|(-\Delta)^s f\|_{L^2}^2 + \|f\|_{L^2}^2)^{\frac{1}{2}} \sim_s \|(I-\Delta)^s f\|_{L^2} = \|(1+|\xi|^2)^s \mathcal{F}f\|_{L^2} = \|f\|_{2s},$$

where we used the main result in [DG08] for the equivalence. Therefore, if we take $Y = H^s$ as a our base space, we obtain $D(A^r) = H^{s+2\sigma r}$. Since $(-\Delta)^\sigma$ is positive and self-adjoint on $L^2(\mathbb{R}^d)$, $-A$ generates a C_0 group on all H^s, so Assumption 3.1 is fulfilled. We now check the assumptions in Lemma 4.1.

- For $s > \frac{d}{2}$, if $f \in H^s(\mathbb{R}^d)$, then $f \in L^\infty(\mathbb{R}^d)$ by the Sobolev embedding Theorem. Indeed, by the Cauchy-Schwarz inequality

  $$\|f\|_{L^\infty} \lesssim \|\mathcal{F}f\|_{L^1} \leqslant \|(1+|\cdot|^2)^{-\frac{s}{2}}\|_{L^2} \|f\|_s.$$

  Hence, a trivial estimate gives

  $$\|fg\|_{L^2} \leqslant \|f\|_{L^\infty} \|g\|_{L^2} \lesssim \|f\|_s \|g\|_{L^2}$$

  for $f \in H^s(\mathbb{R}^d)$ and $g \in L^2(\mathbb{R}^d)$.

- For $s > \frac{d}{2}$, if $f, g \in H^s(\mathbb{R}^d)$, then $fg \in H^s(\mathbb{R}^d)$ with $\|fg\|_s \lesssim \|f\|_s \|g\|_s$. To see this, we observe that

  $$(1+|\xi|^2)^{\frac{s}{2}} \leqslant (1+3|\xi-\eta|^2+3|\eta|^2)^{\frac{s}{2}} \lesssim_s (1+|\xi-\eta|^2)^{\frac{s}{2}} + (1+|\eta|^2)^{\frac{s}{2}},$$

  seen for natural numbers s through estimating the mixed terms by the pure terms with highest order when multiplying (similar to the trick in the first estimate). Together with $\mathcal{F}(fg) = (\mathcal{F}f) * (\mathcal{F}g)$, this yields

  $$(1+|\xi|^2)^{\frac{s}{2}} |\mathcal{F}(fg)(\xi)| \lesssim_s (|(1+|\cdot|^2)^{\frac{s}{2}}\mathcal{F}f| * |\mathcal{F}g|)(\xi) + (|\mathcal{F}f| * |(1+|\cdot|^2)^{\frac{s}{2}}\mathcal{F}g|)(\xi)$$

and therefore by Young's inequality and the same estimate as in the first estimate gives

$$\|fg\|_s \lesssim_s \|f\|_s \|\mathcal{F}g\|_{L^1} + \|\mathcal{F}f\|_{L^1} \|g\|_s \lesssim \|f\|_s \|g\|_s.$$

Therefore, we choose $Y = H^s(\mathbb{R}^d)$ for some $s \geqslant 0$ such that $s + 2\sigma r > \frac{d}{2}$. This lets us obtain the required estimates in Assumption 3.2 from Lemma 4.1, where s can be chosen freely in $(\frac{d}{2}, s + 2r)$. We now obtain the following result from Theorem 3.4.

Corollary 4.3

Let $\sigma > 0$, $r > 0$ and $s \geqslant 0$ such that $s + 2\sigma r > \frac{d}{2}$. Let Assumption 3.5 (see Remark 3.3) hold for the splitting scheme (4.1). If $u_0 \in H^{s+2\sigma r}$ and the solution $u \in C([0,T], H^{s+2\sigma r})$ of (4.3) fulfils $\|u(t)\|_{s+2\sigma r} \leqslant R$ for all $t \in [0,T]$ and some $R \geqslant 0$, there exists an $h_0 \in (0,T]$ such that

$$\|(S^h)^N(u_0) - u(Nh)\|_s \leqslant Ch^r$$

for all $h \in (0, h_0]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T$, where C only depends on R and T.

Cases of interest are for example the square root of the Laplacian ($\sigma = \frac{1}{2}$), the Laplacian itself ($\sigma = 1$) or the Bilaplacian ($\sigma = 2$). We rewrite the result for those examples and add some values for s and orders up to two. Starting with the Laplacian, we have the equation

$$\left.\begin{aligned} \mathrm{i}u'(x,t) &= (-\Delta u)(x,t) \pm |u(x,t)|^{k-1} u(x,t), \\ u(x,0) &= u^0(x), \end{aligned}\right\} \tag{4.4}$$

with the result

Corollary 4.4

Let $r > 0$ be arbitrary and $s \geqslant 0$ such that $s + 2r > \frac{d}{2}$. Let Assumption 3.5 (see Remark 3.3) hold for the splitting scheme (4.1). If $u_0 \in H^{s+2r}$ and the solution $u \in C([0,T], H^{s+2r})$ of (4.4) fulfils $\|u(t)\|_{s+2r} \leqslant R$ for all $t \in [0,T]$

and some $R \geqslant 0$, there exists an $h_0 \in (0, T]$ such that

$$\|(S^h)^N(u_0) - u(Nh)\|_s \leqslant Ch^r$$

for all $h \in (0, h_0]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T$, where C only depends on R and T.

This result for $d \leqslant 3$, $k = 3$, $s = 0$ and $r \leqslant 2$ can be found in [ESS16] and before for $r \in \{1, 2\}$ in [Lub08]. For the Strang splitting (or for $r \leqslant 1$ also the Lie splitting), we have for example

Dim.	Y	Conv. Order	Initial Val.	Remarks
1	L^2	$r \in (\frac{1}{4}, 2]$	H^{2r}	**b) c)**
1	H^2	$r \in (0, 2]$	$H^{2(1+r)}$	**a)**
2	L^2	$r \in (\frac{1}{2}, 2]$	H^{2r}	**b) c)**
2	H^2	$r \in (0, 2]$	$H^{2(1+r)}$	**a)**
3	L^2	$r \in (\frac{3}{4}, 2]$	H^{2r}	**b) c)**
3	H^2	$r \in (0, 2]$	$H^{2(1+r)}$	**a)**

Next, we take the square root of the Laplacian for the equation

$$\left.\begin{aligned} \mathrm{i}u'(x,t) &= (\sqrt{-\Delta}u)(x,t) \pm |u(x,t)|^{k-1}u(x,t), \\ u(x,0) &= u^0(x), \end{aligned}\right\} \tag{4.5}$$

and the result

Corollary 4.5

Let $r > 0$ be arbitrary and $s \geqslant 0$ such that $s + r > \frac{d}{2}$. Let Assumption 3.5 (see Remark 3.3) hold for the splitting scheme (4.1). If $u_0 \in H^{s+r}$ and the solution $u \in C([0, T], H^{s+r})$ of (4.5) fulfils $\|u(t)\|_{s+r} \leqslant R$ for all $t \in [0, T]$ and some $R \geqslant 0$, there exists an $h_0 \in (0, T]$ such that

$$\|(S^h)^N(u_0) - u(Nh)\|_s \leqslant Ch^r$$

for all $h \in (0, h_0]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T$, where C only depends on R and T.

For the Strang splitting (or for $r \leqslant 1$ also the Lie splitting), we have the possible values

Dim.	Y	Conv. Order	Initial Val.	Remark
1	L^2	$r \in (\frac{1}{2}, 2]$	H^r	**b)**
1	H^1	$r \in (0, 2]$	H^{1+r}	**a)**
2	L^2	$r \in (1, 2]$	H^r	**b)**
2	H^1	$r \in (0, 2]$	H^{1+r}	**a)**
3	L^2	$r \in (\frac{3}{2}, 2]$	H^r	**b)**
3	H^1	$r \in (\frac{1}{4}, 2]$	H^{1+r}	**b)**

Finally, we take the Bilaplacian with equation

$$\left.\begin{aligned} \mathrm{i}u'(x,t) &= ((-\Delta)^2 u)(x,t) \pm |u(x,t)|^{k-1} u(x,t), \\ u(x,0) &= u^0(x), \end{aligned}\right\} \tag{4.6}$$

and the result

Corollary 4.6

Let $r > 0$ be arbitrary and $s \geqslant 0$ such that $s + 4r > \frac{d}{2}$. Let Assumption 3.5 (see Remark 3.3) hold for the splitting scheme (4.1). If $u_0 \in H^{s+4r}$ and the solution $u \in C([0,T], H^{s+4r})$ of (4.6) fulfils $\|u(t)\|_{s+4r} \leqslant R$ for all $t \in [0,T]$ and some $R \geqslant 0$, there exists an $h_0 \in (0,T]$ such that

$$\|(S^h)^N(u_0) - u(Nh)\|_s \leqslant Ch^r$$

for all $h \in (0, h_0]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T$, where C only depends on R and T.

For the Strang splitting (or for $r \leqslant 1$ also the Lie splitting), we obtain

Dim.	Y	Conv. Order	Initial Val.	Remark
1	L^2	$r \in (\frac{1}{8}, 2]$	H^{4r}	**b)**
1	H^4	$r \in (0, 2]$	$H^{4(1+r)}$	**a)**
2	L^2	$r \in (\frac{1}{4}, 2]$	H^{4r}	**b)**
2	H^4	$r \in (0, 2]$	$H^{4(1+r)}$	**a)**
3	L^2	$r \in (\frac{3}{8}, 2]$	H^{4r}	**b)**
3	H^4	$r \in (0, 2]$	$H^{4(1+r)}$	**a)**

Remark

a) *These results are optimal in the sense that the required smoothness of the initial value is apparently necessary to obtain the convergence order r in the norm Y and there are no restrictions on the order r (except for $r \leqslant 2$)*

b) *Here, the required smoothness of the initial value is still optimal for r starting from a lower threshold, but not for all $r > 0$.*

c) *In these cases, results by [ESS16] and [Liu13a] are contained.*

b) The Laplacian on the torus $\mathbb{T}^d$ can be treated very similarly to the one on the full space. The equation is

$$\left.\begin{aligned} \mathrm{i}u_t(x,t) &= ((-\Delta)^\sigma u)(x,t) \pm |u(x,t)|^{k-1}u(x,t), \\ u(x,0) &= u^0(x), \end{aligned}\right\} \tag{4.7}$$

for $\sigma > 0$, $k \in \mathbb{N}$ odd and $(x,t) \in \mathbb{T}^d \times \mathbb{R}$, where $u^0 \in L^2(\mathbb{T}^d)$. We set $A = -\mathrm{i}\Delta$, and define the fractional Sobolev spaces by the Bessel potential spaces

$$H^s := H^s(\mathbb{T}^d) := \{u \in L^2(\mathbb{T}^d) \mid ((1+n^2)^{\frac{s}{2}}\hat{u}(n))_{n\in\mathbb{Z}^d} \in \ell^2\}$$

for $s \geqslant 0$ with their natural norm. Compared to the full space, we therefore just replaced the Fourier transform $\mathcal{F}u$ by the sequence $(\hat{u}(n))_n$ of Fourier coefficients. For a thorough treatment of these spaces and more references, see [BO13]. We get $D(A^r) = H^{s+2\sigma r}$ for a base space $Y = H^s$ by [DG08] if we just extend the functions canonically from $\mathbb{T}^d$ to $\mathbb{R}^d$ by zero. Since $-\Delta$ is positive and self-adjoint on $L^2(\mathbb{T}^d)$, $-A$ generates a C_0 group on all H^s, so Assumption 3.1 is fulfilled.
The assumptions in Lemma 4.1 follow exactly as in the case of the full space, since we again have the inversion formula and Young's inequality for discrete convolutions. We just need to replace $\mathcal{F}u$ by $(\hat{u}(n))_n$, ξ by n as well as the L^1 and L^2 by the ℓ^1 and ℓ^2 norms, respectively.
We now obtain the following version of Theorem 3.4.

Corollary 4.7

Let $r > 0$ be arbitrary and $s \geqslant 0$ such that $s + 2\sigma r > \frac{d}{2}$. Let Assumption 3.5 (see Remark 3.3) hold for the splitting scheme (4.1). If $u_0 \in H^{s+2\sigma r}$ and the solution $u \in C([0,T], H^{s+2\sigma r})$ of (4.7) fulfils $\|u(t)\|_{s+2\sigma r} \leqslant R$ for all $t \in [0,T]$ and some $R \geqslant 0$, there exists an $h_0 \in (0,T]$ such that

$$\|(S^h)^N(u_0) - u(Nh)\|_s \leqslant Ch^r$$

for all $h \in (0, h_0]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T$, where C only depends on R and T.

This result for $d \leqslant 3$, $k = 3$, $s = 0$ and $1 < r \leqslant 2$ can be found in [ESS16]

c) Finally, we take the equation

$$\left.\begin{aligned} \mathrm{i}u_t(x,t) &= (-\Delta_M u)(x,t) \pm |u(x,t)|^{k-1} u(x,t), \\ u(x,0) &= u^0(x), \end{aligned}\right\} \tag{4.8}$$

for $k \in \mathbb{N}$ odd and $(x,t) \in M \times \mathbb{R}$ with $u^0 \in L^2(M)$. Here, (M,g) is a smooth, d-dimensional complete Riemannian manifold which is closed and $A = -\mathrm{i}\Delta_M$ is a multiple of the Laplace-Beltrami operator.
As fractional Sobolev spaces, we use the analogue of the Bessel potential spaces for manifolds, that is

$$H^s := \{u \in L^2(M) \mid (1 - \Delta_M)^{\frac{s}{2}} u \in L^2(M)\}$$

with their natural norm. In [Str83, Thm 4.4], the same norm equivalence as in the two previous examples is shown, therefore, using $Y = H^s$, we again obtain $D(A^r) = H^{s+2r}$. Since $-\Delta_M$ is positive and self-adjoint on $L^2(M)$, $-A$ generates a C_0 group on all H^s, so Assumption 3.1 is fulfilled. We now check the assumptions of Lemma 4.1.

- For $s > \frac{d}{2}$, if $f \in H^s$, then $f \in L^\infty(M)$ and we have

$$\|f\|_{L^\infty} \lesssim_{M,s} \|f\|_s.$$

This is shown in [Aub98, Lemma 2.22], mentioning that the result we need is not exactly stated, but proven in 2.23. Hence, a trivial estimate gives

$$\|fg\|_{L^2} \leqslant \|f\|_{L^\infty}\|g\|_{L^2} \lesssim \|f\|_s\|g\|_{L^2}$$

for $f \in H^s$ and $g \in L^2$.

- For $s > \frac{d}{2}$, if $f, g \in H^s$, then $fg \in H^s$ with $\|fg\|_s \lesssim_{M,s} \|f\|_s\|g\|_s$. To see this, we need to take a finite Atlas $\{(U_i, \varphi_i)\}_{i=1,\dots,n}$ and a corresponding partition of unity $\{\alpha_i\}_{i=1,\dots,n}$. We cite results from [vdB02] below, where charts are used to define the Sobolev spaces $H^s(M)$ via the Bessel potential spaces $H^s(\mathbb{R}^d)$. This definition coincides with the one we use, since the Laplace-Beltrami operator can be defined via charts and the Laplacian on $\mathbb{R}^d$, which gives exactly the same correspondence of $D((-\Delta_M)^{s/2})$ to the the Bessel potential spaces $H^s(\mathbb{R}^d)$. We now see that

$$\begin{aligned}
\|fg\|_s &= \left\|\left(\sum_{i=1}^n \alpha_i f\right)\left(\sum_{i=1}^n \alpha_i g\right)\right\|_s \leqslant \sum_{i,j=1}^n \|(\alpha_i f)(\alpha_j g)\|_{H^s(U_i)} \\
&\overset{10.3}{\lesssim_{M,s}} \sum_{i,j=1}^n \|(\alpha_i f \circ \varphi_i^{-1})(\alpha_j g \circ \varphi_i^{-1})\|_{H^s(\varphi_i(U_i))} \\
&= \sum_{i,j=1}^n \|(\alpha_i f \circ \varphi_i^{-1})(\alpha_j g \circ \varphi_i^{-1})\|_{H^s(\mathbb{R}^d)} \\
&\lesssim_{d,s} \sum_{i,j=1}^n \|\alpha_i f \circ \varphi_i^{-1}\|_{H^s(\mathbb{R}^d)}\|\alpha_j g \circ \varphi_i^{-1}\|_{H^s(\mathbb{R}^d)} \\
&= \sum_{i,j=1}^n \|\alpha_i f \circ \varphi_i^{-1}\|_{H^s(\varphi_i(U_i))}\|\alpha_j g \circ \varphi_i^{-1}\|_{H^s(\varphi_i(U_i))} \\
&\overset{10.4}{\lesssim_{M,s}} \sum_{i,j=1}^n \|\alpha_i f\|_{H^s(U_i)}\|\alpha_j g\|_{H^s(U_i)} \leqslant \sum_{i,j=1}^n \|\alpha_i f\|_s\|\alpha_j g\|_s \\
&\overset{10.5}{\lesssim_{M,s}} \|f\|_s\|g\|_s,
\end{aligned}$$

where we extended $\alpha_i f \circ \varphi_i^{-1}$ and $\alpha_j g \circ \varphi_i^{-1}$ by zero outside of $\varphi_i(U_i)$.

This gives the same conditions on smoothness as before. Theorem 3.4 translates here into the following corollary

Corollary 4.8

Let $\sigma > 0$, M be a d-dimensional, closed Riemannian manifold, $r > 0$ be arbitrary and $s \geqslant 0$ such that $s + 2\sigma r > \frac{d}{2}$. Let Assumption 3.5 (see Remark 3.3) hold for the splitting scheme (4.1). If $u_0 \in H^{s+2\sigma r}$ and the solution $u \in C([0,T], H^{s+2\sigma r})$ of (4.8) fulfils $\|u(t)\|_{s+2\sigma r} \leqslant R$ for all $t \in [0,T]$ and some $R \geqslant 0$, there exists an $h_0 \in (0,T]$ such that

$$\|(S^h)^N(u_0) - u(Nh)\|_s \leqslant Ch^r$$

for all $h \in (0, h_0]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T$, where C only depends on R and T.

4.2.2 Nonlinear harmonic oscillator on $L^2(\mathbb{R}^d)$

We set $A = \mathrm{i}(-\Delta + |x|^2)$ and get the equation

$$\left.\begin{aligned} \mathrm{i}u_t(x,t) &= (-\Delta u)(x,t) + |x|^2 u(x,t) \pm |u(x,t)|^{k-1} u(x,t), \\ u(x,0) &= u^0(x), \end{aligned}\right\} \tag{4.9}$$

for $k \in \mathbb{N}$ odd and $(x,t) \in \mathbb{R}^d \times \mathbb{R}$, where $u^0 \in L^2(\mathbb{R}^d)$. The fact that A is self-adjoint follows from [Mik01, Theorem 2.6.8] or [DL90, p. 38-39] since it is symmetric, linear, unbounded and has a compact resolvent, see [DL90, p. 64-67]. Therefore, we once again work with fractional orders. As fractional Sobolev spaces, we once again use the Bessel potential spaces, that is

$$H^s := \{u \in L^2(\mathbb{R}^d) \mid (1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F} u \in L^2(\mathbb{R}^d)\}$$

with their natural norm. Furthermore, we define

$$\mathcal{H}^s := \{u \in L^2(\mathbb{R}^d) \mid A^{\frac{s}{2}} u \in L^2(\mathbb{R}^d)\}$$

for $s \geqslant 0$. For $u \in \mathcal{H}^s$, we have

$$\begin{aligned}\|f\|_{\mathcal{H}^s} &= (\|(-\Delta+|x|^2)^{\frac{s}{2}}f\|_{L^2}^2+\|f\|_{L^2}^2)^{\frac{1}{2}} \sim_{d,s} \|(-\Delta)^{\frac{s}{2}}f\|_{L^2}+\||x|^s f\|_{L^2}+\|f\|_{L^2} \\ &\sim_{d,s} \|(I-\Delta)^{\frac{s}{2}}f\|_{L^2}+\||x|^s f\|_{L^2} = \|f\|_{H^s}+\||x|^s f\|_{L^2} \geqslant \|f\|_{H^s}\end{aligned} \tag{4.10}$$

where we used the main result in [DG08] for the equivalences. This renders the $\mathcal{H}^\int$ norms equivalent to something comparable to the Bessel potential space norms with the inclusion $\mathcal{H}^s \hookrightarrow H^s$ for all $s \geqslant 0$. We now check the assumptions from Lemma 4.1, which we then may use for the spaces $\mathcal{H}_s$ according to the Remark given after it. This works since A has spectral multipliers (see [DOS02][Theorem 7.10 and 7.11]) and therefore has bounded imaginary powers, which implies that the fractional domains of A define a complex interpolation scale (see [Tri95][1.15.3]).

- For $s > \frac{d}{2}$, if $f \in \mathcal{H}^s$, then $f \in L^\infty(\mathbb{R}^d)$ and by same computation as for the Laplacian on $\mathbb{R}^d$, using 4.10, we obtain

  $$\|f\|_{L^\infty} \lesssim \|\mathcal{F}f\|_{L^1} \leqslant \|(1+|\cdot|^2)^{-s}\|_{L^2}\|f\|_s \lesssim_{d,s} \|f\|_{\mathcal{H}^s}.$$

  Hence, a trivial estimate gives

  $$\|fg\|_{L^2} \leqslant \|f\|_{L^\infty}\|g\|_{L^2} \lesssim \|f\|_{\mathcal{H}^s}\|g\|_{L^2}$$

  for $f \in \mathcal{H}^s$ and $g \in L^2$.

- For $s > \frac{d}{2}$, if $f, g \in \mathcal{H}^s$, then $fg \in \mathcal{H}^s$ with $\|fg\|_{\mathcal{H}^s} \lesssim_{d,s} \|f\|_{\mathcal{H}^s}\|g\|_{\mathcal{H}^s}$. To see this, we once again use what we already know from the results for H^s and 4.10 to see that

  $$\begin{aligned}\|fg\|_{\mathcal{H}^s} &\lesssim_{d,s} \|fg\|_s + \||x|^s fg\|_{L^2} \lesssim_{d,s} \|f\|_s\|g\|_{L^\infty}+\|f\|_{L^\infty}\|g\|_s+\||x|^s f\|_{L^2}\|g\|_{L^\infty} \\ &\lesssim_{d,s} (\|f\|_s+\||x|^s f\|_{L^2})\|g\|_s+\|f\|_s\|g\|_s \\ &\lesssim (\|f\|_s+\||x|^s f\|_{L^2})(\|g\|_s+\||x|^s g\|_{L^2}) \overset{(4.10)}{\lesssim_{d,s}} \|f\|_{\mathcal{H}^s}\|g\|_{\mathcal{H}^s}.\end{aligned}$$

Therefore, we choose $Y = \mathcal{H}^s$ for some $s \geqslant 0$ such that for our desired order r, $D(A^r) = \mathcal{H}^{s+2r}$ is smaller than $\mathcal{H}^{\frac{d}{2}}$. Theorem 3.4 translates here into

Corollary 4.9

Let $r > 0$ be arbitrary and $s \geqslant 0$ such that $s + 2r > \frac{d}{2}$. Let Assumption 3.5 (see Remark 3.3) hold for the splitting scheme (4.1). If $u_0 \in H^{s+2r}$ and the solution $u \in C([0,T], \mathcal{H}^{s+2r})$ of (4.9) fulfils $\|u(t)\|_{\mathcal{H}^{s+2r}} \leqslant R$ for all $t \in [0,T]$ and some $R \geqslant 0$, there exists an $h_0 \in (0,T]$ such that

$$\|(S^h)^N(u_0) - u(Nh)\|_{\mathcal{H}^s} \leqslant Ch^r$$

for all $h \in (0,h_0]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T$, where C only depends on R and T.

A similar result, albeit with different methods and only for $k = 3$, $s > d$ even and $r = 1$ as well as $s = 0$, $r = 2$ and a loss of regularity of strictly more than $d + 2$, can be found in [Gau11, Theorem 3.1].

4.2.3 On modulation spaces

In this subsection, we once again work with the equation

$$\left.\begin{aligned} \mathrm{i}u'(x,t) &= ((-\Delta)^\sigma u)(x,t) \pm |u(x,t)|^{k-1} u(x,t), \\ u(x,0) &= u^0(x), \end{aligned}\right\} \tag{4.11}$$

for $\sigma > 0$ and $k \in \mathbb{N}$ odd as well as $(x,t) \in \mathbb{R}^d \times \mathbb{R}$. Instead of $L^2(\mathbb{R}^d)$, we are now looking at modulation spaces, which are a useful tool in time frequency analysis, theoretical harmonic analysis and the theory of evolution equations.

Definition 4.10

Let $\psi \in C^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\psi \subseteq \{|x| \leqslant \sqrt{d}\}$, $\psi(x) > c > 0$ for $x \in [-\frac{1}{2}, \frac{1}{2}]^d$ and $\sum_{n\in\mathbb{Z}^d} \psi(x - n) = 1$ for all $x \in \mathbb{R}^d$. A function $u : \mathbb{R}^d \to \mathbb{C}$ belongs to the modulation space $M^s_{p,q}(\mathbb{R}^d)$ for $s \in \mathbb{R}$ and $p \in [1,\infty]$, $q \in [1,\infty)$, if

$$\|u\|_{M^s_{p,q}} := \left(\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{sq} \|\mathcal{F}^{-1}(\psi_n \mathcal{F} u)\|^q_{L^p(\mathbb{R}^d)} \right)^{\frac{1}{q}} < \infty$$

with $\langle n\rangle := (1 + |n|^2)^{\frac{1}{2}}$ and $\psi_n = \psi(\cdot - n)$.

Here are some useful properties of the $M^s_{p,q}(\mathbb{R}^d)$ spaces.

a) $M^s_{p,q}(\mathbb{R}^d)$ are Banach spaces with respect to the norm $\|\cdot\|_{M^s_{p,q}}$ (see [Cha18] [Proposition 2.11]).

b) They form a complex interpolation scale (see [Cha18][Proposition 2.19]).

c) (i) $M^{s_1}_{p_1,q_1}(\mathbb{R}^d) \hookrightarrow M^{s_2}_{p_2,q_2}(\mathbb{R}^d)$ for $p_1 \leqslant p_2$, $q_1 \leqslant q_2$, $s_1 \geqslant s_2$ (see [Fei03] [Proposition 6.5],

(ii) $M^{s_1}_{p,q_1}(\mathbb{R}^d) \hookrightarrow M^{s_2}_{p,q_2}(\mathbb{R}^d)$ for $s_1 - s_2 > d(\frac{1}{q_2} - \frac{1}{q_1})$ (see [Cha18] [Proposition 2.31],

(iii) $M^0_{\infty,1}(\mathbb{R}^d) \hookrightarrow C_b(\mathbb{R}^d)$ (see [Cha18][Proposition 2.32],

(iv) $M^0_{p,p'}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ (see [Cha18][Proposition 2.34]).

d) $M^s_{2,2}(\mathbb{R}^d) \cong H^s(\mathbb{R}^d)$ (see [Cha18][Proposition 2.33].

e) $M^s_{p,p'}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ for $p > \frac{d}{s}$, $p \geqslant 2$ (follows from [Fei03][Proposition 6.7] and the Sobolev embedding Theorem).

f) The Bessel potential $(I - \Delta)^{-r}$ defines an isomorphism

$$J^r : M^{s+2r}_{p,q}(\mathbb{R}^d) \to M^s_{p,q}(\mathbb{R}^d)$$

(see [Cha18][Proposition 2.35]), which means that if we define the Laplacian on $M^s_{p,q}(\mathbb{R}^d)$, then $D((-\Delta)^r) = M^{s+2r}_{p,q}(\mathbb{R}^d)$ with the graph norm.

g) A big advantage of the modulation spaces is that the Schrödinger group is defined on them, namely

$$\|\mathrm{e}^{\mathrm{i}t\Delta}\|_{B(M^s_{p,q}(\mathbb{R}^d))} \leqslant C\langle t\rangle^{d|\frac{1}{2}-\frac{1}{p}|}$$

for all $t \in \mathbb{R}$, $s \in \mathbb{R}$ and $p, q \in [1, \infty]$ (see [Cha18][Theorem 3.4]). For $q < \infty$, this group is strongly continuous (see [Cha18][Proposition 3.5]). The proofs allow to get the same result for the group generated by $-\mathrm{i}(-\Delta)^\sigma$.

h) The modulation spaces also enjoy nice multiplication properties: The space $M^s_{p,q}(\mathbb{R}^d)$ is an algebra for $s > \frac{d}{q'}$ (see [Fei03][Proposition 6.9 and Remark 6.4]), while $M^s_{p,q}(\mathbb{R}^d) \cap M^0_{\infty,1}(\mathbb{R}^d)$ is an algebra for $s \geqslant 0$ and $p, q \in [1, \infty]$ (see [Cha18][Proposition 4.2]).

Finally, by **c)**(i) and (ii), we have $M^s_{p,q}(\mathbb{R}^d) \hookrightarrow M^0_{\infty,1}(\mathbb{R}^d)$ for $s > \frac{d}{q'}$, which together with the continuous multiplication

$$\cdot : M^0_{p,q}(\mathbb{R}^d) \times M^0_{\infty,1}(\mathbb{R}^d) \to M^0_{p,q}(\mathbb{R}^d)$$

(see [CN09][Proposition 3.5]) gives us the continuous multiplication

$$\cdot : M^0_{p,q}(\mathbb{R}^d) \times M^s_{p,q}(\mathbb{R}^d) \to M^0_{p,q}(\mathbb{R}^d)$$

for $s > \frac{d}{q'}$.

Now we are in a position to state our results. We fix $\sigma > 0$, $p \in [1,\infty]$ and $q \in [1,\infty)$. By **g)**, $A = -\mathrm{i}(-\Delta)^\sigma$ generates a C_0 group on $Y = M^s_{p,q}(\mathbb{R}^d)$ for any $s \geqslant 0$, hence Assumption 3.1 is fulfilled. By **f)**, the fractional domains are given by $D(A^r) = M^{s+2\sigma r}_{p,q}(\mathbb{R}^d)$.
As for Assumption 3.2, **h)** gives both estimates needed in Lemma 4.1 and by **b)**, they form a complex interpolation scale, so that said Lemma gives all needed results. Hence, Theorem 3.4 takes the form

Corollary 4.11

For fixed $\sigma > 0$, $p \in [1,\infty]$ and $q \in [1,\infty)$, let $r > 0$ be arbitrary and $s \geqslant 0$ such that $s + 2\sigma r > \frac{d}{q'}$. Let Assumption 3.5 (see Remark 3.3) hold for the splitting scheme (4.1). If $u_0 \in M^{s+2\sigma r}_{p,q}(\mathbb{R}^d)$ and the solution $u \in C([0,T], M^{s+2\sigma r}_{p,q}(\mathbb{R}^d))$ of (4.11) fulfils $\|u(t)\|_{M^{s+2\sigma r}_{p,q}} \leqslant R$ for all $t \in [0,T]$ and some $R \geqslant 0$, there exists an $h_0 \in (0,T]$ such that

$$\|(S^h)^N(u_0) - u(Nh)\|_{M^s_{p,q}} \leqslant Ch^r$$

for all $h \in (0,h_0]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T$, where C only depends on R and T.

For the second result, we notice that by **g)**, A also generates a C_0 group on $Y = M^s_{p,q}(\mathbb{R}^d) \cap M^0_{\infty,1}(\mathbb{R}^d)$ for any $s \geqslant 0$, hence Assumption 3.1 is fulfilled. By **f)**, the fractional domains are given by $D(A^r) = M^{s+2\sigma r}_{p,q}(\mathbb{R}^d) \cap M^{2\sigma r}_{\infty,1}(\mathbb{R}^d)$ (see Lemma 4.16).
By **h)**, all fractional domains are Banach algebras, hence Lemma 4.1 gives

the differentiability and the estimates in Assumption 3.2 follow by inclusion. Theorem 3.4 now states the following (note that $M^0_{\infty,1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ by **c)** (iii)).

Corollary 4.12

For fixed $\sigma > 0$, $p \in [1,\infty]$ and $q \in [1,\infty)$, let $s \geqslant 0$ and $r > 0$ be arbitrary. Let Assumption 3.5 (see Remark 3.3) hold for the splitting scheme (4.1). If $u_0 \in M^{s+2\sigma r}_{p,q}(\mathbb{R}^d) \cap M^{2\sigma r}_{\infty,1}(\mathbb{R}^d)$ and the solution $u \in C([0,T], M^{s+2\sigma r}_{p,q}(\mathbb{R}^d) \cap M^{2\sigma r}_{\infty,1}(\mathbb{R}^d))$ of (4.11) fulfils $\|u(t)\|_{M^{s+2\sigma r}_{p,q} \cap M^{2\sigma r}_{\infty,1}} \leqslant R$ for all $t \in [0,T]$ and some $R \geqslant 0$, there exists an $h_0 \in (0,T]$ such that

$$\|(S^h)^N(u_0) - u(Nh)\|_{M^s_{p,q} \cap L^\infty} \leqslant Ch^r$$

for all $h \in (0,h_0]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T$, where C only depends on R and T.

For $s = 0$ and $p = q = 2$, we obtain a error estimate of order $r > 0$ in $L^2(\mathbb{R}^d)$ by **d)** even if $2\sigma r$ is not larger than $\frac{d}{2}$, as long as the initial value is not just in in $H^{2\sigma r}$, but also in the modulation space $M^{2\sigma r}_{\infty,1}(\mathbb{R}^d) \hookrightarrow H^{2\sigma r}_\infty(\mathbb{R}^d)$.

4.2.4 Discrete Schrödinger equation on ℓ^p

We now mention discrete evolution equations

$$\left.\begin{aligned} \mathrm{i}(u_n'(t)) &= (A(u_n))(x,t) \pm |(u_n(t))|^{k-1}(u_n(t)), \\ (u_n(0)) &= (u_n^0), \end{aligned}\right\} \tag{4.12}$$

where A is a bounded operator on $Y = \ell^p$ for $1 \leqslant p \leqslant \infty$. An interesting case is the discrete Laplace operator defined by

$$A(u_n) = (u_{n+1} - 2u_n + u_{n-1})$$

for $(u_n) \in \ell^p$. All such operators generate C_0 groups on Y.

By the generalized Hölder inequality, the space Y and all spaces $X_s \cong Y$ are Banach algebras with respect to pointwise multiplication. Indeed, for $f,g \in \ell^p$, we get $fg \in \ell^{\frac{p}{2}} \subseteq \ell^p$. Hence, Assumption 3.2 is true with Lemma 4.1 (no interpolation is needed since we have the stronger algebra properties). Therefore, Theorem 3.4 amounts to

Corollary 4.13

Let $r > 0$ and $1 \leqslant p \leqslant \infty$ be arbitrary. Let Assumption 3.5 (see Remark 3.3) hold for the splitting scheme (4.1). If $u_n^0 \in \ell^p$ and the solution $(u_n) \in C([0,T],\ell^p)$ of (4.12) fulfils $\|(u_n(t))\|_{\ell^p} \leqslant R$ for all $t \in [0,T]$ and some $R \geqslant 0$, there exists an $h_0 \in (0,T]$ such that

$$\|(S^h)^N(u_n^0) - (u_n(Nh))\|_{\ell^p} \leqslant Ch^r$$

for all $h \in (0,h_0]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T$, where C only depends on R and T.

4.3 Parabolic equations

4.3.1 On $L^p(\mathbb{R}^d)$, $L^p(\mathbb{T}^d)$ and $L^p(M)$ for manifolds $(1 < p \leqslant \infty)$

We now take a look at parabolic equations on $\Omega \in \{\mathbb{R}^d, \mathbb{T}^d, M\}$ for a compact d-dimensional Riemannian manifold M and $d \in \mathbb{N}$. Note that $\mathbb{T}^d$ is covered by the those types of manifolds. To this end, let A be the negative Laplace or Laplace-Beltrami operator on Ω. For $\Omega = \mathbb{R}^d$, it is also possible to use a uniformly elliptic second order differential operator

$$A := -\sum_{i,j=1}^{d} \partial_i a_{ij}(x)\partial_j + \sum_{i=1}^{d} b_i(x)\partial_i + cu$$

with real, continuous coefficients a_{ij}, b_i and $c \in \mathbb{R}$, where

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geqslant c|\xi|^2$$

for some $c > 0$ and all $x, \xi \in \mathbb{R}^d$.

Remark

Our approach covers more general situations which we only quote here to avoid lengthy explanations.

- *Elliptic systems on $\mathbb{R}^d$ as in [KW04][Section 6]*

- *Complete Riemannian manifolds with finite geometry and positive injectivity radius and the Laplace Beltrami operator (see [Tri92][Chapter 7], [CRTN01])*

- *Connected unimodular Liegroups endowed with a family of left-invariant Hörmander vector fields and the associated sublaplacian (see [CRTN01][Theorem 1 and Theorem 2]*

- *Doubling metric measure spaces (M, d, μ) and the associated Markov generator $-A$ endowed with a "carré de champs" (see [BF18])*

Note that if A has a sectoriality angle $\omega(A) \neq 0$, then A^σ is only defined for $\sigma < \frac{\pi}{\omega(A)}$ and defines an analytic semigroup for $\sigma < \frac{\pi}{2\omega(A)}$ (see [KW04][Theorem 15.16]). The equations we treat have the form

$$\left.\begin{aligned} u'(x,t) &= (-A^\sigma u)(x,t) \pm |u(x,t)|^{k-1} u(x,t), \\ u(x,0) &= u^0(x), \end{aligned}\right\} \tag{4.13}$$

for $\sigma \in (0, \frac{\pi}{2\omega(A)}]$ on Ω. We will study the convergence of the splitting method for these equations in three different spaces: $L^p(\Omega)$ with $1 < p < \infty$, $L^\infty(\Omega)$ and $L^p(\Omega) \cap L^\infty(\Omega)$. The advantage of moving away from the L^2 scale is that the function algebra properties of these spaces allow in various ways to reduce the smoothness assumptions in our convergence estimates.

4.3.1.1 On L^p $(1 < p < \infty)$

On $L^p(\mathbb{R}^d)$ for $1 < p < \infty$, $-A$ is the generator of an analytic semigroup (see [KW04][8.1]). On $L^p(M, \mu)$, where μ is the Riemannian measure, the same holds for A being the Laplace-Beltrami operator (see [Dav89] or [Str83][Theorem 3.5 and Section 4]).
We again work with the Bessel potential spaces, defined by

$$H^s_p(\mathbb{R}^d) := \{f \in L^p(\mathbb{R}^d) \mid \mathcal{F}^{-1}(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F} f \in L^p(\mathbb{R}^d)\}$$

for $\Omega = \mathbb{R}^d$ and

$$H^s_p(M) := \{u \in L^p(M) \mid (1 - \Delta_M)^{\frac{s}{2}} u \in L^p(M)\}$$

for $\Omega = M$, both with their natural norm $\|\cdot\|_{s,p}$. It follows that for $Y = H_p^s(\Omega)$, $D(A^r) = H_p^{s+2r}(\Omega)$ if we use the graph norm. Now let $s > \frac{d}{p}$. We get the Sobolev embedding $H_p^s(\Omega) \subseteq L^\infty(\Omega)$ and hence

$$\|fg\|_{L^p} \leqslant \|f\|_{L^\infty}\|g\|_{L^p} \lesssim \|f\|_{s,p}\|g\|_{L^p}$$

for $f \in H_p^s(\Omega)$, $g \in L^p(\Omega)$. We also have that $H_p^s(\Omega)$ is a function algebra as well, that is

$$\|fg\|_{s,p} \lesssim \|f\|_{s,p}\|g\|_{s,p}$$

for $f, g \in H_p^s(\Omega)$. For $\Omega = \mathbb{R}^d$, this is shown in [RS96][2.4.4 and 4.6.4]. The same holds for $H_p^s(M)$ which can be deduced from the above via charts as in 4.2.1 **c)**. For direct quotes see [Tri92][Chapter 7] or [CRTN01][Section 4], since compact manifolds have a positive injectivity radius and bounded geometry. Since the spaces $H_p^s(\Omega)$ define a complex interpolation scale (see [Tri92][1.6.4 and 7.4.5], we can use Lemma 4.1 in order to get Assumption 3.2. Hence Theorem 3.4 amounts for (4.13), we obtain

Corollary 4.14

Let $\sigma \in (0, \frac{\pi}{2\omega(A)}]$, $r > 0$ be arbitrary and $s \geqslant 0$ such that $s + 2\sigma r > \frac{d}{p}$. Let Assumption 3.5 (see Remark 3.3) hold for the splitting scheme (4.1). If $u_0 \in H_p^{s+2\sigma r}(\Omega)$ and the solution $u \in C([0,T], H_p^{s+2\sigma r}(\Omega))$ of (4.13) fulfils $\|u(t)\|_{s+2\sigma r,p} \leqslant R$ for all $t \in [0,T]$ and some $R \geqslant 0$, there exists an $h_0 \in (0,T]$ such that

$$\|(S^h)^N(u_0) - u(Nh)\|_{s,p} \leqslant Ch^r$$

for all $h \in (0, h_0]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T$, where C only depends on R and T.

4.3.1.2 On L^∞

If we choose $Y = L^\infty(\mathbb{R}^d)$ as the norm for the error estimate, we encounter the problem that the domain of A in $L^\infty(\mathbb{R}^d)$ takes the form

$$D(A) = \{u \in \bigcap_{p \geqslant 1} W^{2,p}_{\text{loc}} \mid u, Au \in L^\infty(\mathbb{R}^d)\},$$

see [Lun95][Theorem 3.1.7, Theorem 3.1.9]. Then $\overline{D(A)} = UC(\mathbb{R}^d)$ and the part of $-A$ on $UC(\mathbb{R}^d)$ generates an analytic semigroup. However, the fractional domains of A are complicated and most certainly not function algebras.
As a way out of this impass we propose to choose the slightly smaller Besov space $Y = B^0_{\infty,1}(\Omega) \subseteq L^\infty(\Omega)$ (see [Tri78][Theorem 1, p.133] for $\Omega = \mathbb{R}^d$ and [Tri85][Theorem 5] for $\Omega = M$) whose elements have the same regularity (in terms of derivatives) as $L^\infty(\Omega)$. For $\Omega = M$, $B^s_{p,q}(M)$ is defined as the real interpolation space $(F^{s_0}_{p,p}(M), F^{s_1}_{p,p}(M))_{\theta,q}$ for $1 \leqslant p, q \leqslant \infty$ and $-\infty < s_0 < s < s_1 < \infty$ with $s = (1-\theta)s_0 + \theta s_1$ (see [Tri85][Definition 3]).
Then $-A^\sigma$ generates an analytic semigroup on $B^0_{\infty,1}(\Omega)$ in the sector $\Sigma_{\frac{\pi}{2}}$ and $X_\alpha = D(A^\alpha) = B^{2\alpha}_{\infty,1}(\Omega)$, since $(-\Delta)^\alpha$ and therefore A^α maps $B^s_{1,\infty}(\mathbb{R}^d)$ onto $B^{s-2\alpha}_{1,\infty}(\mathbb{R}^d)$ (see [RS96][2.1.4] for $\Omega = \mathbb{R}^d$ and [Tri85][Theorem 6] for $\Omega = M$).
Because for $\Omega = \mathbb{R}^d$, X_α is a function algebra for all $\alpha > 0$ (see [RS96][4.6.4]), our nonlinearity is infinitely often differentiable on those spaces (see Lemma 4.1 **b)**) and all estimates from Assumption 3.2 (see Lemma 4.1 **b)** and **c)**) follow by inclusion, except for the local Lipschitz continuity in $Y = X_0$ on bounded sets in some X_s and all estimates for $b = r$ on the derivatives of g. For those, we use the continuous multiplication

$$\cdot : B^0_{\infty,1}(\mathbb{R}^d) \times B^\varepsilon_{\infty,1}(\mathbb{R}^d) \to B^0_{\infty,1}(\mathbb{R}^d)$$

for $\varepsilon > 0$ (see [RS96][4.6.1, Theorem 2]) to obtain local Lipschitz continuity in X_0 on bounded sets in any X_ε. For $b = r$ and $a_i < r$ for all i, we can just use the inclusion of X_0 in $X_{r-\max\{a_i\}}$ followed by the algebra property of this space and further inclusions in the spaces in question. If without loss of generality, $a_1 = r$ and hence the rest of the a_i are zero, we use the above estimate for $\varepsilon = r$ to keep the first variable in X_0 and then the algebra property on X_r. We mention that the above multiplication property is necessary, since $B^0_{\infty,1}$ is in fact not an function algebra (see [ST95][Remark 4.3.5].
For $\Omega = \mathbb{T}^d$, both the algebra property and the multiplication estimate still hold. For $\Omega = M$, we restrict ourselves to compact manifolds for which the same is true, but we conjecture that this is the case for all compact Riemannian manifolds. Hence, we obtain from Theorem 3.4 that for (4.13)

Corollary 4.15

Let $\sigma \in (0, \frac{\pi}{2\omega(A)}]$, $r > 0$ be arbitrary and let Assumption 3.5 (see Remark 3.3) hold for the splitting scheme (4.1). If $u_0 \in B^{2\sigma r}_{\infty,1}(\Omega)$ and the solution $u \in C([0,T], B^{2\sigma r}_{\infty,1}(\Omega))$ of (4.13) fulfils $\|u(t)\|_{B^{2\sigma r}_{\infty,1}} \leqslant R$ for all $t \in [0,T]$ and some $R \geqslant 0$, there exists an $h_0 \in (0,T]$ such that

$$\|(S^h)^N(u_0) - u(Nh)\|_{L^\infty} \leqslant Ch^r$$

for all $h \in (0, h_0]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T$, where C only depends on R and T.

Note that $C^\alpha(\mathbb{R}^d) \subseteq B^\alpha_{\infty,1}(\mathbb{R}^d)$ for all $\alpha > 0$.

4.3.1.3 On $L^p \cap L^\infty$ $(1 < p < \infty)$

We are interested in $L^p(\Omega) \cap L^\infty(\Omega)$ estimates, because these spaces are function algebras for all $1 < p < \infty$ and $B^0_{\infty,1}(\Omega) \subseteq L^\infty(\Omega)$. This provides an opportunity to find a way around Sobolev embeddings. Given the difficulties surrounding the fractional domains of A in L^∞ mentioned in **b)** we again replace $L^\infty(\Omega)$ by $B^0_{\infty,1}(\Omega)$, that is, we fix $p \in (1,\infty)$ and put

$$X_s := H^{2s}_p(\Omega) \cap B^{2s}_{\infty,1}(\Omega)$$

for $s \geqslant 0$. Note that

(i) X_s is a function algebra for all $s > 0$ and the multiplication from $X_0 \times X_s \to X_0$ is continuous for all $s > 0$.

(ii) $-A$ with $D(A) = X_1$ generates an analytic semigroup on all X_s with $D(A^\alpha) = X_\alpha$ for $\alpha \geqslant 0$.

(iii) g is infinitely often Freéchet differentiable on X_s for all $s > 0$, where $g(u) = |u|^{k-1}u$ for odd $k \geqslant 3$. Additionally, all estimates from Assumption 3.2 are true as well.

Proof. (i) $X_s = (H^{2s}_p(\Omega) \cap L^\infty(\Omega)) \cap B^{2s}_{\infty,1}(\Omega)$ is a function algebra for $s > 0$ since both $H^{2s}_p(\Omega) \cap L^\infty(\Omega)$ and $B^{2s}_{\infty,1}(\Omega)$ are function algebras (see [RS96][4.6.4] for $\Omega = \mathbb{R}^d$ and [Tri92][Chapter 7] or [CRTN01] for $\Omega = M$). Since the additional multiplication estimate also holds true on both spaces of the intersection, it also holds for X_s.

(ii) Since we know from **a)** and **b)** that $-A$ generates analytic semigroups on $L^p(\Omega)$ and on $B^0_{\infty,1}(\Omega)$ with fractional domains $H^{2s}_p(\Omega)$ and $B^{2s}_{\infty,1}(\Omega)$, respectively, the claim follows from the next Lemma.

(iii) The differentiability follows from (i) with the same proof as in Lemma 4.1 **b)**. The estimates from Assumption 3.2 follow from (i) as in **b)**.

■

Lemma 4.16

Let X_1, X_2 be Banach spaces continuously embedded into $L^1(\Omega) + L^\infty(\Omega)$. Let $T_i(t)$ be a strongly continuous semigroup on X_i for $i = 1, 2$ so that on $X = X_1 \cap X_2$, $\|x\|_X = \|x\|_{X_1} + \|x\|_{X_2}$, $T_1(t) = T_2(t)$ for $t \geqslant 0$. Then

a) *$T(t) = T_1(t) = T_2(t)$ is a strongly continuous semigroup on X.*

b) *If $-A_i$ and $-A$ are the generators of $T_i(t)$ and $T(t)$ on X_i and X, respectively, then*

$$D(A) = D(A_1) \cap D(A_2) \qquad \text{and} \qquad A_1 x = A_2 x = Ax \quad \text{for } x \in D(A).$$

c) *$D(A^\alpha) = D(A_1^\alpha) \cap D(A_2^\alpha)$.*

Proof. **a)** This follows from the assumptions and the definition of $\|\cdot\|_X$.

b) $D(A) = D(A_1) \cap D(A_2)$ again follows directly from the definition of a generator and of $\|\cdot\|_X$.

c) By [KW04][Prop. 15.23], for $\mu > 0$ large enough, there are isomorphic maps

$$(\mu + A)^{-\alpha} : X \to D(A^\alpha), \qquad (\mu + A_i)^{-\alpha} : X_i \to D(A_i^\alpha) \quad (i = 1, 2)$$

with $(\mu{+}A)^{-\alpha} y = (\mu{+}A_i)^{-\alpha} y$ for $y \in X$ and $i = 1, 2$. by the functional calculus and furthermore $(\mu + A)^\alpha x = (\mu + A_i)^\alpha x$ for $x \in D(A^\alpha)$ and $i = 1, 2$. Hence, if $x \in D(A^\alpha)$, then there is a $y \in X$ with $x = (\mu + A)^{-\alpha} y$ and $x = (\mu + A_i)^{-\alpha} y_i$ for some $y_i \in X_i$, so that $x \in D(A_1^\alpha) \cap D(A_2^\alpha)$. The reverse argument also holds.

■

We are now in the position to deduce from Theorem 3.4 that for (4.13), we have

Corollary 4.17

Let $\sigma \in (0, \frac{\pi}{2\omega(A)}]$ and $s \geqslant 0$, $r > 0$ be arbitrary. Let Assumption 3.5 (see Remark 3.3) hold for the splitting scheme (4.1). If $u_0 \in H_p^{s+2\sigma r}(\Omega) \cap B_{\infty,1}^{s+2\sigma r}(\Omega)$ and the solution $u \in C([0,T], H_p^{s+2\sigma r} \cap B_{\infty,1}^{s+2\sigma r})$ of (4.13) fulfils $\|u(t)\|_{H_p^{s+2\sigma r}(\Omega)\cap B_{\infty,1}^{s+2\sigma r}(\Omega)} \leqslant R$ for all $t \in [0,T]$ and some $R \geqslant 0$, there exists an $h_0 \in (0,T]$ such that

$$\|(S^h)^N(u_0) - u(Nh)\|_{H_p^s \cap B_{\infty,1}^s} \leqslant Ch^r$$

for all $h \in (0,h_0]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T$, where C only depends on R and T.

In particular, for $s = 0$, we obtain error estimates in the norm of $L^p(\Omega) \cap L^\infty(\Omega)$ under assumptions on the regularity of u_0 which are optimal also for small r and therefore initial values with little regularity.

4.3.2 On uniform L^p spaces

We once again treat the equations

$$\left.\begin{aligned} u'(x,t) &= (-Au)(x,t) \pm |u(x,t)|^{k-1}u(x,t), \\ u(x,0) &= u^0(x), \end{aligned}\right\} \tag{4.14}$$

where A is the Laplacian or more generally speaking a second order differential operator

$$A = \sum_{i,j=1}^{d} \partial_i(a_{ij}\partial_j + a_i(x)) + \sum_{i=1}^{d} b_i(x)\partial_i + c(x),$$

with real coefficients in $L^\infty(\mathbb{R}^d)$ or

$$A = \sum_{i,j=1}^{d} a_{ij}\partial_i\partial_j + \sum_{i=1}^{d} b_i(x)\partial_i + c(x),$$

with real, Hölder continuous coefficients. In both cases, we assume the ellipticity condition

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geqslant c|\xi|^2$$

for some $c > 0$ and all $x, \xi \in \mathbb{R}^d$. As an alternative to the spaces $L^p(\mathbb{R}^d)$, we now use the uniform L^p spaces and their Bessel potential spaces $UH_p^k(\mathbb{R}^d)$ defined by

$$UH_p^k(\mathbb{R}^d) := \{u \in H_{p,\mathrm{loc}}^k(\mathbb{R}^d) \mid \sup_{x\in\mathbb{R}^d} \|u\|_{H_p^k(B(x,1))}\}$$

for $k \in \mathbb{N}_0$ and $1 \leqslant p \leqslant \infty$. For $s \geqslant 0$, that is $s = \theta k + (1-\theta)(k+1)$ for some $k \in \mathbb{N}_0$, we define $UH_p^s(\mathbb{R}^d)$ as the complex interpolation space

$$UH_p^s(\mathbb{R}^d) := (UH_p^k(\mathbb{R}^d), UH_p^{k+1}(\mathbb{R}^d))_\theta.$$

Notice that $UL^\infty(\mathbb{R}^d) = L^\infty(\mathbb{R}^d)$. The advantage of these spaces is that constant and periodic functions belong to them, but clearly not to $H_p^s(\mathbb{R}^d)$. We now collect the necessary properties of the uniform L^p scale.

- $-A^\sigma$ generates an analytic semigroup $UL^p(\mathbb{R}^d)$ for and for $Y = UH_p^s(\mathbb{R}^d)$, we have $D(A^r) = UH_p^{s+2r}(\mathbb{R}^d)$ (see [ACDRB04][Theorem 2.1, Theorem 2.2, Theorem 2.3])

- Since $H_p^s(B(x,1))$ is defined by means of an extension to $H_p^s(\mathbb{R}^d)$ (see [Tri92][5.1], all multiplication estimates from Subsection 4.3.1 can be transferred over to the $UH_p^s(\mathbb{R}^d)$ spaces.

Hence, we can copy the results from Subsection 4.3.1 to obtain the following results.

Corollary 4.18

Let $\sigma > 0$, $r > 0$ be arbitrary and $s \geqslant 0$ such that $s + 2\sigma r > \frac{d}{p}$. Let Assumption 3.5 (see Remark 3.3) hold for the splitting scheme (4.1). If $u_0 \in UH_p^{s+2\sigma r}(\mathbb{R}^d)$ and the solution $u \in C([0,T], UH_p^{s+2\sigma r}(\mathbb{R}^d))$ of (4.14) fulfils $\|u(t)\|_{UH_p^s} \leqslant R$ for all $t \in [0,T]$ and some $R \geqslant 0$, there exists an $h_0 \in (0,T]$ such that

$$\|(S^h)^N(u_0) - u(Nh)\|_{UH_p^s} \leqslant Ch^r$$

for all $h \in (0, h_0]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T$, where C only depends on R and T.

Corollary 4.19

Let $\sigma > 0$ and $s \geqslant 0$, $r > 0$ be arbitrary. Let Assumption 3.5 (see Remark 3.3) hold for the splitting scheme (4.1). If $u_0 \in UH_p^{s+2\sigma r}(\mathbb{R}^d) \cap B_{\infty,1}^{s+2\sigma r}(\mathbb{R}^d)$ and the solution $u \in C([0,T], UH_p^{s+2\sigma r}(\mathbb{R}^d) \cap B_{\infty,1}^{s+2\sigma r}(\mathbb{R}^d))$ of (4.14) fulfils $\|u(t)\|_{H_p^{s+2\sigma r} \cap B_{\infty,1}^{s+2\sigma r}} \leqslant R$ for all $t \in [0,T]$ and some $R \geqslant 0$, there exists an $h_0 \in (0,T]$ such that

$$\|(S^h)^N(u_0) - u(Nh)\|_{UH_p^s \cap B_{\infty,1}^s} \leqslant Ch^r$$

for all $h \in (0, h_0]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T$, where C only depends on R and T.

4.3.3 On modulation spaces

In analogy to Subsection 4.2.3, we take the equation

$$\left.\begin{aligned} u'(x,t) &= (-(-\Delta)^\sigma u)(x,t) \pm |u(x,t)|^{k-1} u(x,t), \\ u(x,0) &= u^0(x), \end{aligned}\right\} \qquad (4.15)$$

for $\sigma > 0$ and $k \in \mathbb{N}$ odd on the modulation spaces $M_{p,q}^s(\mathbb{R}^d)$. All needed properties have been mentionned before except for the fact that Δ (and similarly $-(-\Delta)^\sigma$) generates a (contractive) C_0 semigroup on all $M_{p,q}^s(\mathbb{R}^d)$. This is shown in [Iwa10][Proposition 2.10] and we get the similar (up to the inclusion of $q = \infty$) results on $M_{p,q}^s$ and $M_{p,q}^s \cap L^\infty$ by Theorem 3.4.

Corollary 4.20

For fixed $\sigma > 0$, $p, q \in [1, \infty]$, let $r > 0$ be arbitrary and $s \geqslant 0$ such that $s + 2r > \frac{d}{q'}$. Let Assumption 3.5 (see Remark 3.3) hold for the splitting scheme (4.1). If $u_0 \in M_{p,q}^{s+2\sigma r}(\mathbb{R}^d)$ and the solution $u \in C([0,T], M_{p,q}^{s+2\sigma r}(\mathbb{R}^d))$ of (4.15) fulfils $\|u(t)\|_{M_{p,q}^{s+2\sigma r}} \leqslant R$ for all $t \in [0,T]$ and some $R \geqslant 0$, there exists an $h_0 \in (0,T]$ such that

$$\|(S^h)^N(u_0) - u(Nh)\|_{M_{p,q}^s} \leqslant Ch^r$$

for all $h \in (0, h_0]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T$, where C only depends on R and T.

Corollary 4.21

For fixed $\sigma > 0$, $p, q \in [1, \infty]$, let $s \geqslant 0$ and $r > 0$ be arbitrary. Let Assumption 3.5 (see Remark 3.3) hold for the splitting scheme (4.1). If $u_0 \in M^{s+2\sigma r}_{p,q}(\mathbb{R}^d) \cap M^{2\sigma r}_{\infty,1}(\mathbb{R}^d)$ and the solution $u \in C([0,T], M^{s+2\sigma r}_{p,q}(\mathbb{R}^d) \cap M^{2\sigma r}_{\infty,1}(\mathbb{R}^d))$ of (4.15) fulfils $\|u(t)\|_{M^{s+2\sigma r}_{p,q} \cap M^{2\sigma r}_{\infty,1}} \leqslant R$ for all $t \in [0,T]$ and some $R \geqslant 0$, there exists an $h_0 \in (0,T]$ such that

$$\|(S^h)^N(u_0) - u(Nh)\|_{M^s_{p,q} \cap L^\infty} \leqslant Ch^r$$

for all $h \in (0, h_0]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T$, where C only depends on R and T.

4.3.4 Discrete Laplacian on ℓ^p

Complete analogously to Subsection 4.2.4, we have the equation

$$\left.\begin{aligned} (u_n'(t)) &= (A(u_n))(x,t) \pm |(u_n(t))|^{k-1}(u_n(t)), \\ (u_n(0)) &= (u_n^0), \end{aligned}\right\} \tag{4.16}$$

where A is a bounded operator on $Y = \ell^p$ for $1 \leqslant p \leqslant \infty$. An interesting case is the discrete Laplace operator defined by

$$A(u_n) = (u_{n+1} - 2u_n + u_{n-1})$$

for $(u_n) \in \ell^p$. All such operators generate C_0 groups on Y.
By the generalized Hölder inequality, the space Y and all spaces $X_s \cong Y$ are Banach algebras with respect to pointwise multiplication. Indeed, for $f, g \in \ell^p$, we get $fg \in \ell^{\frac{p}{2}} \subseteq \ell^p$. Hence, Assumption 3.2 is true with Lemma 4.1 (no interpolation is needed since we have the stronger algebra properties). Therefore, Theorem 3.4 amounts to

Corollary 4.22

Let $r > 0$ and $1 \leqslant p \leqslant \infty$ be arbitrary. Let Assumption 3.5 (see Remark 3.3) hold for the splitting scheme (4.1). If $u_n^0 \in \ell^p$ and the solution $(u_n) \in C([0,T], \ell^p)$ of (4.16) fulfils $\|(u_n(t))\|_{\ell^p} \leqslant R$ for all $t \in [0,T]$ and some $R \geqslant 0$, there exists an $h_0 \in (0,T]$ such that

$$\|(S^h)^N(u_n^0) - (u_n(Nh))\|_{\ell^p} \leqslant Ch^r$$

for all $h \in (0, h_0]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T$, where C only depends on R and T.

4.4 Equations with random initial values

In our analysis, Sobolev embeddings and the Banach algebra properties of a Sobolev space play a big role. The order of for example the Sobolev embeddings are optimal, but it is reasonable to expect that the counterexamples of functions that show that inequalities are sharp are 'rare' among a large class of 'generic' functions of the same smoothness that enjoy better Sobolev embeddings. One way to quantify these better behaved functions is through randomization. Roughly speaking, given a decomposition $u = \sum_n u_n$ of a function $u \in H^s$, for example with respect to a basis of H^s, we introduce an appropriate sequence of independent, identically distributed random variables g_n on a probability space $(\Omega, \mathbb{P})$ and show that

$$u^\omega = \sum_n g_n(\omega) u_n \qquad \forall \omega \in \Omega$$

belongs almost surely to H_p^s for all $p \geqslant 2$. In this way, we obtain functions much better behaved than an arbitrary element of H^s and we can obtain stronger results for such initial values. Also note that in numerical experiments, one often uses random initial data of some kind. We are working with the equation

$$\left.\begin{aligned} u'(x,t) &= (-\Delta u)(x,t) \pm |u(x,t)|^{k-1} u(x,t), \\ u(x,0) &= u^0(x), \end{aligned}\right\} \tag{4.17}$$

on either $\mathbb{T}^d$ or $\mathbb{R}^d$.

4.4.1 On $L^p(\mathbb{T}^d)$

Let $\{e_n\}$ be the trigonometric basis of $L^2(\mathbb{T}^d)$ and for $u \in L^2(\mathbb{T}^d)$, denote the Fourier coefficients of u by $\hat{u}(n)$. Hence,

$$u = \sum_{n \in \mathbb{Z}^d} \hat{u}(n) e_n$$

Let now (g_n) be a sequence of complex, independent random variables which satisfy

$$\exists \delta > 0: \quad \mathbb{E}(\mathrm{e}^{\alpha g_n}) \leqslant \mathrm{e}^{\delta \alpha^2} \;\; \forall \alpha \in \mathbb{R}, n \in \mathbb{Z}^d.$$

To avoid trivialities, we also require (g_n) to not accumulate in 0, that is

$$\exists c, \delta > 0 : \mathbb{P}(|g_n| > c) \geqslant \delta \quad \forall n \in \mathbb{Z}^d.$$

Notice that this holds for Bernoulli random variables or more precisely for all families of random variables with mean zero and a support uniformly bounded in n as well as for standard Gaussian random variables. Now define

$$u^\omega = \sum_{n \in \mathbb{Z}^d} g_n(\omega) \hat{u}(n) e_n \qquad \forall \omega \in \Omega.$$

We obtain the following result.

Lemma 4.23

Let $u \in H^s(\mathbb{T}^d)$ for some $s \geqslant 0$.

a) *For $1 \leqslant p < \infty$, we have $u^w \in H^s_p(\mathbb{T}^d)$ almost surely. Moreover, the following large deviation estimate holds:*

$$\exists \alpha, a > 0 : \quad \mathbb{P}(\|u^w\|_{H^s_p(\mathbb{T}^d)} > \Lambda) \leqslant \mathrm{e}^{-\alpha \Lambda^2 / \|u\|_{H^s(\mathbb{T}^d)}} \;\; \forall \Lambda \geqslant a.$$

b) *If $u \notin H^{\tilde{s}}(\mathbb{T}^d)$ for some $\tilde{s} > s$, then $u^w \notin H^{\tilde{s}}(\mathbb{T}^d)$ almost surely.*

The first part shows that randomization of the Fourier expansion of a $u \in H^s(\mathbb{T}^d)$ improves the integrability of almost all u^ω drastically, while the second part shows that it does not improve on the regularity of u^ω over u. It also shows that

$\{u^\omega \mid \omega \in \Omega\}$ is not a 'thin' subset of H^s as for example a dense subset of $C^\infty(\mathbb{T}^d)$, but rather a relatively large set of 'true' H^s functions.

of Lemma 4.23. **a)** For $u \in L^2(\mathbb{T}^d)$ this result is shown in [Bur11][Theorem 2.2] (for $d = 1$) or [BT08][Lemma 3.1] (for general manifolds). For $u \in H^s(\mathbb{T}^d)$, put $v := (I-\Delta)^{\frac{s}{2}} \in L^2(\mathbb{T}^d)$ and apply the result to v. Then $v^\omega \in L^p(\mathbb{T}^d)$ almost surely and hence $\tilde{u}^w := (I-\Delta)^{-\frac{s}{2}} v^\omega \in H^s_p(\mathbb{T}^d)$ almost surely. Finally we see that

$$\tilde{u}^\omega = (I-\Delta)^{-\frac{s}{2}} v^\omega = \sum_{n\in\mathbb{Z}} g_n(\omega)(1+|n|^2)^{-\frac{s}{2}}\hat{v}(n)e_n = \sum_{n\in\mathbb{Z}} g_n(\omega)\hat{u}(n)e_n = u^\omega.$$

b) This is shown in [Bur11][Theorem 2.5] (for $d = 1$) or [BT08][Lemma B.1] (for general manifolds).

∎

From Subsection 4.3.1, we now obtain the following result

Corollary 4.24

Let $s \geqslant 0$ and $u_0 \in H^s(\mathbb{T}^d)$ and $0 < r < \frac{s}{2}$. Let Assumption 3.5 (see Remark 3.3) hold for the splitting scheme (4.1). Then for almost $\omega \in \Omega$ and the initial data u_0^ω, there exist $T(\omega), h_0(\omega) > 0$ and a unique mild solution of 4.17 in $C([0,T(\omega)], H^{2r}_p(\mathbb{T}^d) \cap B^{2r}_{\infty,1}(\mathbb{T}^d))$ for $p > \frac{2d}{s-2r}$ and

$$\|(S^h)^N(u_0) - u(Nh)\|_{L^\infty} \leqslant C(\omega)h^r$$

for all $h \in (0, h_0(\omega)]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T(\omega)$.

Proof. Set $\varepsilon := \frac{s}{2} - r > 0$ and let $p > \frac{d}{\varepsilon}$. By Lemma 4.23, $u_0^\omega \in H^s_p(\mathbb{T}^d)$ almost surely. We have

$$H^s_p(\mathbb{T}^d) \hookrightarrow H^{2r+\varepsilon}_\infty(\mathbb{T}^d) \hookrightarrow B^{2r+\varepsilon}_{\infty,\infty}(\mathbb{T}^d) \hookrightarrow B^{2r}_{\infty,1}(\mathbb{T}^d).$$

The first inclusion is the usual Sobolev embedding, the second is proven in [BL76][Theorem 6.2.4] and third is shown in [RS96][2.2.1] for $\mathbb{R}^d$ which transfers naturally to $\mathbb{T}^d$. Hence,

$$u_0^\omega \in H^{2r}_p(\mathbb{T}^d) \cap B^{2r}_{\infty,1}(\mathbb{T}^d)$$

almost surely. Since this space is an algebra, the existence of a local solution follows as usual by [Paz92, Theorem 6.1.4]. The norm of u_0^ω and hence the upper bound on the norm of the solution depends on ω and we obtain from Corollary 4.17, that

$$\|(S^h)^N(u_0) - u(Nh)\|_{L^\infty} \lesssim \|(S^h)^N(u_0) - u(Nh)\|_{L^p \cap B^0_{\infty,1}} \leqslant C(\omega)h^r$$

for all $h \in (0, h_0(\omega)]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T(\omega)$. ■

Remark

Looking at Lemma 4.23, we see that $\|u^\omega\|_{H^s_p} \leqslant C\|u\|_{H^s}$ on a set of measure at least $1 - \mathrm{e}^{-\alpha C^2}$. This norm determines $T(\omega)$ and $h_0(\omega)$ as well as an upper bound on the norm of the solution in $H^{2r}_p(\mathbb{T}^d) \cap B^{2r}_{\infty,1}(\mathbb{T}^d)$. The size of ε determines p and hence also influences the constants.

4.4.2 On $L^p(\mathbb{R}^d)$

For $L^p(\mathbb{R}^d)$, a different randomization, the so called Wiener randomization, is needed. To this end, let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be a 'window function', that is $\operatorname{supp}\psi \subset [-1,1]^d$, $\sum_{n\in\mathbb{Z}^d} \psi(\xi - n) = 1$ for all $\xi \in \mathbb{R}^d$. Since $\mathcal{F}^{-1}\psi \in \mathcal{S}(\mathbb{R}^d)$ the operators $\psi(D-n)$ defined by

$$(\psi(D-n)u)(x) = \int_{\mathbb{R}^d} \mathrm{e}^{2\pi \mathrm{i} x\cdot\xi} \psi(\xi - n) \mathcal{F}u \, \mathrm{d}\xi$$

are bounded on $L^p(\mathbb{R}^d)$ and for $u \in L^p(\mathbb{R}^d)$,

$$\sum_{n\in\mathbb{Z}^d} \psi(D-n)u = u.$$

The random variables g_n are as in the preceding subsection and we define

$$u^\omega := \sum_{n\in\mathbb{Z}^d} g_n(\omega)\psi(D-n)u \qquad \forall \omega \in \Omega.$$

Analogously to before, we obtain the following result.

Lemma 4.25

Let $u \in H^s(\mathbb{R}^d)$ for some $s \geqslant 0$.

a) *For $1 \leqslant p < \infty$, we have $u^w \in H^s_p(\mathbb{R}^d)$ almost surely. Moreover, the following large deviation estimate holds:*

$$\exists \alpha, a > 0 : \quad \mathbb{P}(\|u^w\|_{H^s_p(\mathbb{T})} > \Lambda) \leqslant \mathrm{e}^{-\alpha\Lambda^2/\|u\|_{H^s(\mathbb{R}^d)}} \ \forall \Lambda \geqslant a.$$

b) *If $u \notin H^{\tilde{s}}(\mathbb{R}^d)$ for some $\tilde{s} > s$, then $u^w \notin H^{\tilde{s}}(\mathbb{R}^d)$ almost surely.*

Again, the randomization improves the integrability but not the smoothness of the given function u.

of Lemma 4.25. **a)** For $u \in L^2(\mathbb{R}^d)$ this result is shown in [BOP14][Lemma 2.3]. For $u \in H^s(\mathbb{R}^d)$, put $v := (I-\Delta)^{\frac{s}{2}} \in L^2(\mathbb{R}^d)$ and apply the result to v. Then $v^\omega \in L^p(\mathbb{R}^d)$ almost surely and hence $\tilde{u}^w := (I-\Delta)^{-\frac{s}{2}} v^\omega \in H^s_p(\mathbb{R}^d)$ almost surely. Finally we see that

$$\tilde{u}^\omega = (I-\Delta)^{-\frac{s}{2}} v^\omega = \sum_{n\in\mathbb{Z}} g_n(\omega)(1+|n|^2)^{-\frac{s}{2}}\hat{v}(n)e_n = \sum_{n\in\mathbb{Z}} g_n(\omega)\hat{u}(n)e_n = u^\omega.$$

b) This is proven in the same way as on $\mathbb{T}^d$, see [Bur11][Theorem 2.5] or [BT08][Lemma B.1].

■

From Subsection 4.3.1, we now obtain the following result

Corollary 4.26

Let $s \geqslant 0$ and $u_0 \in H^s(\mathbb{R}^d)$ and $0 < r < \frac{s}{2}$. Let Assumption 3.5 (see Remark 3.3) hold for the splitting scheme (4.1). Then for almost $\omega \in \Omega$ and the initial data u_0^ω, there exist $T(\omega), h_0(\omega) > 0$ and a unique mild solution of 4.17 in $C([0,T(\omega)], H^{2r}_p(\mathbb{R}^d) \cap B^{2r}_{\infty,1}(\mathbb{R}^d))$ for $p > \frac{2d}{s-2r}$ and

$$\|(S^h)^N(u_0) - u(Nh)\|_{L^\infty} \leqslant C(\omega)h^r$$

for all $h \in (0, h_0(\omega)]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T(\omega)$.

Proof. Set $\varepsilon := \frac{s}{2} - r > 0$ and let $p > \frac{d}{\varepsilon}$. By Lemma 4.25, $u_0^\omega \in H_p^s(\mathbb{R}^d)$ almost surely. We have

$$H_p^s(\mathbb{R}^d) \hookrightarrow H_\infty^{2r+\varepsilon}(\mathbb{R}^d) \hookrightarrow B_{\infty,\infty}^{2r+\varepsilon}(\mathbb{R}^d) \hookrightarrow B_{\infty,1}^{2r}(\mathbb{R}^d).$$

The first inclusion is the usual Sobolev embedding, the second is proven in [BL76][Theorem 6.2.4] and third is shown in [RS96][2.2.1]. Hence,

$$u_0^\omega \in H_p^{2r}(\mathbb{R}^d) \cap B_{\infty,1}^{2r}(\mathbb{R}^d)$$

almost surely. Since this space is an algebra, the existence of a local solution follows as usual by [Paz92, Theorem 6.1.4]. The norm of u_0^ω and hence the upper bound on the norm of the solution depends on ω and we obtain from Corollary 4.17, that

$$\|(S^h)^N(u_0) - u(Nh)\|_{L^\infty} \lesssim \|(S^h)^N(u_0) - u(Nh)\|_{L^p \cap B_{\infty,1}^0} \leqslant C(\omega) h^r$$

for all $h \in (0, h_0(\omega)]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T(\omega)$. ∎

4.4.3 On uniform L^p spaces

To introduce randomization on locally uniform L^p spaces, choose $\varphi \in C^\infty(\mathbb{R}^d)$ with $\operatorname{supp} \varphi \subset (-\pi, \pi)^d$, $\sum_{j \in \mathbb{Z}^d} \varphi(x - j) = 1$ for all $x \in \mathbb{R}^d$. For $UL^2(\mathbb{R}^d)$, defining $u_j(x) := \varphi(x - j) u(x)$ and using the trigonometric functions $e_n(x) = \mathrm{e}^{2\pi \mathrm{i} j \cdot x}$, we hence have

$$u(x) = \sum_{j \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} \hat{u}_j(n) e_n(x) \qquad \forall x \in \mathbb{R}^d.$$

Choosing a sequence of random variables $g_{j,n}$ as before, we define

$$u^\omega(x) = \sum_{j \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} \hat{u}_j(n) 2^{-|j|} g_{j,n}(w) e_n(x) \qquad \forall x \in \mathbb{R}^d, \omega \in \Omega,$$

where $|j| = \sum_{i=1}^d |j_i|$. By applying the reasoning for $L^p(\mathbb{T}^d)$ to each summand for a fixed $j \in \mathbb{Z}^d$, we obtain

Lemma 4.27

Let $u \in UH^s(\mathbb{R}^d)$ for some $s \geqslant 0$.

a) *For $1 \leqslant p < \infty$, we have $u^w \in UH_p^s(\mathbb{R}^d)$ almost surely.*

b) *If $u_j \notin H^{\tilde{s}}(\mathbb{R}^d)$ for some $\tilde{s} > s$, then $u^w \notin UH^{\tilde{s}}(\mathbb{R}^d)$ almost surely.*

Proof. **a)** Note that a ball $B(x,1)$ can be covered by finitely many intervals $j + (-\pi,\pi)^d$ for $j \in \mathbb{Z}^d$ and vice versa. Also, every ball only intersects with finitely many such intervals and vice versa. Hence, the norm on $UH_p^s(\mathbb{R}^d)$ is equivalent to the norm taking the supremum over all intervals $j + (-\pi,\pi)^d$. We obtain

$$\begin{aligned}
\mathbb{E}\|u^w\|_{UH_p^s(\mathbb{R}^d)} &\lesssim \mathbb{E}\sup_{j\in\mathbb{Z}^d} 2^{-|j|}\|u_j^w\|_{H_p^s(\mathbb{R}^d)} \leqslant \mathbb{E}\sum_{j\in\mathbb{Z}^d} 2^{-|j|}\|u_j^w\|_{H_p^s(\mathbb{R}^d)} \\
&\leqslant \sum_{j\in\mathbb{Z}^d} 2^{-|j|}\mathbb{E}\|u_j^w\|_{H_p^s(\mathbb{R}^d)} \lesssim \sum_{j\in\mathbb{Z}^d} 2^{-|j|}\|u_j\|_{H^s(R^d)} \\
&\lesssim \sup_{j\in\mathbb{Z}^d}\|u_j\|_{H^s(R^d)} \sum_{j\in\mathbb{Z}^d} 2^{-|j|} \lesssim \|u_j\|_{UH^s(R^d)},
\end{aligned}$$

where we used the Lemma 4.23 **a)** to obtain the estimate on $\mathbb{E}\|u_j^w\|_{H_p^s(\mathbb{R}^d)}$.

b) Let $u_j \notin H^{\tilde{s}}(\mathbb{R}^d)$ for some $j \in \mathbb{Z}^d$. Then, we can use the result on the torus to see that $u_j^\omega \notin H^{\tilde{s}}(\mathbb{R}^d)$ almost surely. Hence $u^w \notin UH^{\tilde{s}}(\mathbb{R}^d)$ almost surely. If it was, we could cover the support of u_j^ω by finitely many balls and obtain that $u_j^\omega \in H^{\tilde{s}}(\mathbb{R}^d)$.

■

From Subsection 4.3.2, we now obtain the following result

Corollary 4.28

Let $s \geqslant 0$ and $u_0 \in UH^s(\mathbb{R}^d)$ and $0 < r < \frac{s}{2}$. Let Assumption 3.5 (see Remark 3.3) hold for the splitting scheme (4.1). Then for almost $\omega \in \Omega$ and the initial data u_0^ω, there exist $T(\omega), h_0(\omega) > 0$ and a unique mild solution of 4.17 in $C([0,T(\omega)], UH_p^{2r}(\mathbb{R}^d) \cap B_{\infty,1}^{2r}(\mathbb{R}^d))$ for $p > \frac{2d}{s-2r}$ and

$$\|(S^h)^N(u_0) - u(Nh)\|_{L^\infty} \leqslant C(\omega)h^r$$

for all $h \in (0, h_0(\omega)]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T(\omega)$.

Proof. Set $\varepsilon := \frac{s}{2} - r > 0$ and let $p > \frac{d}{\varepsilon}$. By Lemma 4.27, $u_0^\omega \in UH_p^s(\mathbb{R}^d)$ almost surely. We have

$$UH_p^s(\mathbb{R}^d) \hookrightarrow UH_\infty^{2r+\varepsilon}(\mathbb{R}^d) = H_\infty^{2r+\varepsilon}(\mathbb{R}^d) \hookrightarrow B_{\infty,\infty}^{2r+\varepsilon}(\mathbb{R}^d) \hookrightarrow B_{\infty,1}^{2r}(\mathbb{R}^d).$$

The inclusions follow as in the case $H_p^s(\mathbb{T}^d)$. Hence,

$$u_0^\omega \in UH_p^{2r}(\mathbb{R}^d) \cap B_{\infty,1}^{2r}(\mathbb{R}^d)$$

almost surely. Since this space is an algebra, the existence of a local solution follows as usual by [Paz92, Theorem 6.1.4]. The norm of u_0^ω and hence the upper bound on the norm of the solution depends on ω and we obtain from Corollary 4.17, that

$$\|(S^h)^N(u_0) - u(Nh)\|_{L^\infty} \lesssim \|(S^h)^N(u_0) - u(Nh)\|_{L^p \cap B_{\infty,1}^0} \leqslant C(\omega) h^r$$

for all $h \in (0, h_0(\omega)]$ and all $N \in \mathbb{N}$ with $Nh \leqslant T(\omega)$. ■

5 Lie Splitting for the stochastic Schrödinger equation

5.1 The equation

In this part, we are working with the formal stochastic evolution equation

$$\left.\begin{aligned} \mathrm{i}\,\mathrm{d}u &= -Au\,\mathrm{d}t + g(u)\,\mathrm{d}t + B(u)\circ\,\mathrm{d}W, \\ u(0) &= u_0. \end{aligned}\right\} \tag{5.1}$$

on a Hilbert space $Y \hookrightarrow L^2(U,\mu)$ for some measure space (U,μ). We start off by stating the assumptions on the equation, followed by the Ito form of the above Stratonovic equation as defined in the literature.

Assumption 5.1

- *Let $A : D(A) \subseteq Y \to Y$ be linear operator such that $T(t) := \mathrm{e}^{\mathrm{i}tA} : Y \to Y$ defines a C_0 group for $t \in \mathbb{R}$.*

- *Let $g : Y \to Y$ be given by $g(u) = \pm|u|^{k-1}u$ for some odd $k \in \mathbb{N}$.*

- *Let (β_k) be an independent sequence of Brownian motions associated with the filtration $\{\mathcal{F}_t \mid t \geqslant 0\}$ of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\{e_k\}$ be an orthonormal basis of a separable Hilbert space $\widetilde{Y}$ and define the white noise W by*
$$W(t,\omega,x) = \sum_{k\in\mathbb{Z}} \beta_k(t,\omega)e_k(x)$$
for $t \geqslant 0$, $\omega \in \Omega$ and $x \in U$.

- *For the convergence order $\theta \in (0,1]$, let Y and $D(A^\theta)$ (with the graph norm) be an algebra with respect to the pointwise multiplication in*

$L^2(U,\mu)$. Also, let

$$\|uw\|_\theta \lesssim \|u\|_0 \|w\|_\theta + \|u\|_\theta \|w\|_0$$

for $u, w \in D(A^\theta)$.

- *Let $B : Y \to \mathcal{L}_2(\widetilde{Y}, Y)$ be defined by*

  $$B(u)v = u \cdot \Phi v$$

  where $\Phi \in \mathcal{L}_2(\widetilde{Y}, D(A^\sigma)) \subseteq \mathcal{L}_2(\widetilde{Y}, Y)$. By $\mathcal{L}_2(\widetilde{Y}, Y)$, we denote the space of Hilbert-Schmidt operators with their norm

  $$\|\Psi\|^2_{\mathcal{L}_2(\widetilde{Y},Y)} = \sum_{k\in\mathbb{Z}} \|\Psi e_k\|^2_Y.$$

Equation (5.1) is the Stratonovic version, which we will only use in its Ito form

$$\left.\begin{aligned} \mathrm{i}\,\mathrm{d}u &= Au\,\mathrm{d}t + (g(u) + \tfrac{\mathrm{i}}{2} F_\Phi u)\,\mathrm{d}t + B(u)\,\mathrm{d}W, \\ u(0) &= u_0, \end{aligned}\right\} \tag{5.2}$$

with $F_\Phi = \sum_{k\in\mathbb{Z}} (\Phi e_k)^2$. With our assumptions, this makes sense of the equation in the interpretation by Da Prato and Zabczyk (see [DPZ14][Chapter 6]).
Because of problems arising when trying to prove convergence results for equation (5.2), we need a version of it in which we cut off the nonlinearity according to the size of the norm in Y: Let $\Theta \in C^\infty(\mathbb{R})$ with $\Theta(x) = 1$ ($x \in [0,1]$) and $\Theta(x) = 0$ ($x \geqslant 4$) as well as $\Theta_R(w) := \Theta(\frac{\|w\|_0^2}{R^2})$ for $R > 0$ and $w \in Y$. With $g_R := \theta_R g$, we obtain the equation

$$\left.\begin{aligned} \mathrm{i}\,\mathrm{d}u_R &= Au_R\,\mathrm{d}t + (g_R(u_R) + \tfrac{\mathrm{i}}{2} F_\Phi u_R)\,\mathrm{d}t + B(u_R)\,\mathrm{d}W, \\ u_R(0) &= u_0. \end{aligned}\right\} \tag{5.3}$$

Since this makes the nonlinearity Lipschitz continuous and both g_R and B fulfil the linear growth condition, [DPZ14][Theorem 7.4] gives a unique mild solution of (5.3) among the processes which are almost surely in $L^2([0,T], Y)$ for arbitrary $T > 0$ as long as u_0 is an $\mathcal{F}_0$-measurable Y-valued random variable.

Remark

The operator A could be one of the following.

- *The Laplacian $-\Delta$ on $L^2(\mathbb{R}^d)$, $L^2(\mathbb{T}^d)$ or $L^2(M)$ for a d-dimensional Riemannian manifold M which is complete, smooth and closed*
- *The fractional Laplacian $(-\Delta)^\sigma$ on the above spaces for $\sigma > 0$*
- *The harmonic oscillator on $L^2(\mathbb{R}^d)$*

These operators all generate C_0 groups. Their fractional domains are the usual Bessel potential spaces H^s (or comparable spaces $\mathcal{H}^s$ in the case of the harmonic oscillator), see Section 4.2 for details. There, it was also shown that these spaces are algebras for $s > \frac{d}{2}$. Observing the estimates that gave us the algebra property more precisely even gives us the stronger estimate

$$\|fg\|_{s'} \lesssim \|f\|_{s'}\|g\|_s + \|f\|_s\|g\|_{s'}$$

holds for $s' \geqslant s > \frac{d}{2}$ and f, g in $H^{s'}$ or $\mathcal{H}^{s'}$. Hence, by choosing $Y = H^s$ or $Y = \mathcal{H}^s$ for $s > \frac{d}{2}$, we know the multiplication estimates from Assumption 5.1 to be true.

5.2 The splitting method

In order to define the Lie splitting, we first need to split off the operator A from the rest of equation (5.1).

$$\mathrm{i}\,\mathrm{d}v = \begin{cases} -Av\,\mathrm{d}t, & (5.4a)\\ (g(v) + \frac{\mathrm{i}}{2}F_\Phi v)\,\mathrm{d}t + B(v)\,\mathrm{d}W, & (5.4b)\end{cases}$$

both having initial value $v(t_0) = v_0$ for a $t_0 \in \mathbb{R}$. Equation (5.4a) has the solution $T(t)v_0$ for all $v_0 \in Y$ and $t \in \mathbb{R}$. Equation (5.4b) has the solution

$$v^{u_0,t_0} = v_0 \exp(-\mathrm{i}[(t-t_0)|v_0|^{k-1} + W(t) - W(t_0)]).$$

This follows by Ito's formula (see [DPZ14][Theorem 4.17]) as seen in [Liu13a][Theorem 2.1].
The idea behind the splitting is the same as for deterministic equations, namely

alternately following the linear solution of (5.4a) and then the solution of (5.4b). For fixed $h > 0$, we therefore define $v^0 = u_0$ and

$$v^n := T(h)\exp(-\mathrm{i}[h|v^{n-1}|^{k-1} + \Phi W(nh) - \Phi W((n-1)h)])v^{n-1} \qquad \forall n \in \mathbb{N}. \quad (5.5)$$

We also split up equation (5.3) to obtain

$$\left.\begin{aligned} \mathrm{i}\, \mathrm{d}v_R &= g_R(v_R)\,\mathrm{d}t + B(v_R) \circ \,\mathrm{d}W, \\ v_R(0) &= v_0. \end{aligned}\right\} \quad (5.6)$$

with solution

$$v_R^{u_0,t_0} = v_0 \exp(-\mathrm{i}[(t-t_0)\theta_R(v_0)|v_0|^{k-1} + \Phi W(t) - \Phi W(t_0)]).$$

The adapted Lie splitting now reads $v_R^0 = u_0$ and

$$v_R^n := T(h)\exp(-\mathrm{i}[h\theta_R(v_R^{n-1})|v_R^{n-1}|^{k-1} + \Phi W(nh) - \Phi W((n-1)h)])v_R^{n-1} \quad (5.7)$$

for all $n \in \mathbb{N}$.

5.3 The result

We now go on stating the result for the cut off equation (5.3).

Proposition 5.2

Let Assumption 5.1 hold and let $R, T > 0$. For $\mathcal{F}_0$-measurable initial values u_0 with $\mathbb{E}\|u_0\|_\theta^p \leqslant M_\theta < \infty$ for $p \in \{2,4\}$, there exists a constant C_R depending on R, T, M_θ and $\|\Phi\|_{\mathcal{L}_2(Y,D(A^\theta))}$ so that

$$\mathbb{E} \max_{0 \leqslant n \leqslant \frac{T}{h}} \|v_R(nh) - v_R^n\|_0 \leqslant C_R h^\theta.$$

From this, we will be able to derive the result for the original equation (5.1).

Theorem 5.3

Let Assumption 5.1 hold. For $\mathcal{F}_0$-measurable initial values u_0 with $\mathbb{E}\|u_0\|_\theta^4 \leqslant M_\theta < \infty$, $L, T > 0$ and the stopping time $\tau_L = \inf\{t \leqslant T, \|v(t)\|_0 \geqslant L\}$, we have

$$\lim_{K\to\infty} \mathbb{P}\big(\max_{0\leqslant n\leqslant \frac{\tau_L}{h}} \|v(nh) - v^n\|_0 \geqslant Kh^\theta \big) = 0.$$

5.4 Auxiliary results

Before we start with the proof, we need some auxiliary results. The first rather trivial one concerns the operator B.

Lemma 5.4

If Assumption 5.1 holds, we have that

$$\begin{aligned} \|T(t)B(u)\|_{\mathcal{L}_2(\widetilde{Y},Y)} &\lesssim \|\Phi\|_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))} \|u\|_0 \\ \|T(t)B(u)\|_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))} &\lesssim \|\Phi\|_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))} \|u\|_\theta \\ \|(T(t)-T(s))B(u)\|_{\mathcal{L}_2(\widetilde{Y},Y)} &\lesssim |t-s|^\theta \|\Phi\|_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))} \|u\|_\theta \end{aligned}$$

for $t, s \in \mathbb{R}$ and u in the respective space.

Proof. Using Assumption 5.1 and the fact that $T(t)$ operates on $D(A^s)$, we obtain

$$\begin{aligned} \|T(t)B(u)\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^s))} &= \sum_{j\in\mathbb{N}} \|T(t)B(u)e_j\|_s^2 = \sum_{j\in\mathbb{N}} \|T(t)(u\cdot\Phi e_j)\|_s^2 \\ &\lesssim \sum_{j\in\mathbb{N}} \|u\cdot\Phi e_j\|_s^2 \lesssim \sum_{j\in\mathbb{N}} \|u\|_s^2 \|\Phi e_j\|_\theta^2 \\ &= \|u\|_s^2 \sum_{j\in\mathbb{N}} \|\Phi e_j\|_\theta^2 = \|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))} \|u\|_s^2 \end{aligned}$$

for $s \in \{0, \theta\}$, which gives the first two inequalities. For the third one, we see that

$$\begin{aligned} \|(T(t)-T(s))B(u)\|^2_{\mathcal{L}_2(\widetilde{Y},Y)} &= \sum_{j\in\mathbb{N}} \|(T(t)-T(s))(u\cdot\Phi e_j)\|_0^2 \\ &\lesssim \sum_{j\in\mathbb{N}} |t-s|^{2\theta} \|u\cdot\Phi e_j\|_\theta^2 \\ &= |t-s|^{2\theta} \|B(u)\|_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))} \end{aligned}$$

$$\lesssim |t-s|^{2\theta} \|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))} \|u\|^2_\theta$$

■

Next, we check some estimates for g and its derivatives.

Lemma 5.5

Let $g(u) = |u|^{k-1}u$ for some odd $k \in \mathbb{N}$ and $\theta \in (0,1]$. Then g is infinitely often Fréchet differentiable on both Y and D^θ) and for all u, v and w in the respective spaces and $\|u\|_0 \leqslant M$ for $M \geqslant 0$, we have

$$\left.\begin{aligned} \|g(u)\|_s &\leqslant C(M)\|u\|_s \\ \|g'(u)[v]\|_0 &\leqslant C(M)\|v\|_0 \\ \|g''(u)[v,w]\|_0 &\leqslant C(M)\|v\|_0\|w\|_0 \end{aligned}\right\} \tag{5.8}$$

for $s \in \{0,\theta\}$. Additionally, if $u \in L^2(\Omega, D(A^\theta))$ is $\mathcal{F}_{t_0}$ measurable, $\|u(\omega)\|_0 \leqslant M$ for all $\omega \in \Omega$ and $0 \leqslant t_0 \leqslant t_1 \leqslant T$, we have

$$\mathbb{E}\left\|g'(u)\left[\int_{t_0}^{t_1} B(u)\,\mathrm{d}W(s)\right]\right\|^2_\theta \leqslant C(M)T\|\Phi\|_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\|u\|^2_\theta. \tag{5.9}$$

Proof. We recall from Lemma 4.1 that g is infinitely often Fréchet differentiable on any algebra with a norm that does not change upon conjugation, which is given here for Y. We have also already proven the estimates on g, g' and g'' in Y. For the other two estimates, we recall from Assumption 5.1 that

$$\|fg\|_\theta \lesssim \|f\|_\theta\|g\|_0 + \|f\|_0\|g\|_\theta$$

holds. An easy induction yields

$$\left\|\prod_{i=1}^k u_i\right\|_\theta \lesssim \sum_{j=1}^k \|u_j\|_\theta \prod_{\substack{i=1\\ i\neq j}}^k \|u_i\|_0 \tag{5.10}$$

for $u_i \in D(A^\theta)$ and $k \in \mathbb{N}$. Using this, we see that

$$\||u|^{k-1}u\|_\theta \lesssim k\|u\|_0^{k-1}\|u\|_\theta \leqslant kM^{k-1}\|u\|_\theta.$$

With the formula for g' from Lemma 4.1 as well as (5.10), we obtain, using the Ito isometry, that

$$
\begin{aligned}
&\mathbb{E}\left\| g'(u)\left[\int_{t_0}^{t_1} B(u)\,\mathrm{d}W(s)\right]\right\|_\theta^2 \\
&\quad\lesssim \mathbb{E}\left\| \frac{k-1}{2} u^{\frac{k-1}{2}} \overline{u}^{\frac{k-1}{2}} \overline{\int_{t_0}^{t_1} B(u)\,\mathrm{d}W(s)} + \frac{k+1}{2} u^{\frac{k-3}{2}} \overline{u}^{\frac{k+1}{2}} \int_{t_0}^{t_1} B(u)\,\mathrm{d}W(s)\right\|_\theta^2 \\
&\quad\lesssim_k \mathbb{E}\left\| \int_{t_0}^{t_1} B(u)\,\mathrm{d}W(s)\right\|_0^2 \|u\|_0^{2(k-1)} \|u\|_\theta^2 + \mathbb{E}\,\|u\|_0^{2(k-1)} \left\| \int_{t_0}^{t_1} B(u)\,\mathrm{d}W(s)\right\|_\theta^2 \\
&\quad= \mathbb{E}\left\| \int_{t_0}^{t_1} B(\|u\|_0^{k-1}\,\|u\|_\theta u)\,\mathrm{d}W(s)\right\|_0^2 + \mathbb{E}\left\| \int_{t_0}^{t_1} B(\|u\|_0^{k-1} u)\,\mathrm{d}W(s)\right\|_\theta^2 \\
&\quad\lesssim \int_{t_0}^{t_1} \mathbb{E}\,\|B(\|u\|_0^{k-1}\,\|u\|_\theta u)\|_{\mathcal{L}_2(\widetilde{Y},D(A^s))}^2\,\mathrm{d}W(s) \\
&\qquad + \int_{t_0}^{t_1} \mathbb{E}\,\|B(\|u\|_0^{k-1} u)\|_{\mathcal{L}_2(\widetilde{Y},D(A^s))}^2\,\mathrm{d}W(s) \\
&\quad\lesssim T\,\|\Phi\|_{\mathcal{L}_2(\widetilde{Y},H^\theta)} \mathbb{E}\,\|u\|_0^{k-1}\,\|u\|_\theta^2 \\
&\quad\leqslant C(M) T\,\|\Phi\|_{\mathcal{L}_2(\widetilde{Y},H^\theta)} \mathbb{E}\,\|u\|_\theta^2
\end{aligned}
$$

■

We go on by deducing the differentiability of and estimates for g_R from the respective properties for g.

Lemma 5.6

Let $\theta \in (0,1]$ and Assumption 5.1 hold. If $\Theta \in C_c^2(\mathbb{R})$, Then $g_R : D(A^\theta) \to D(A^\theta)$ defined by $g_R(u) := \Theta(\frac{\|u\|_0^2}{R^2}) g(u)$ is two times Fréchet differentiable with

$$
g_R'(u)[v] = \frac{2}{R^2}\Theta'\Big(\frac{\|u\|_0^2}{R^2}\Big)\operatorname{Re}\langle u,v\rangle_0\, g(u) + \Theta\Big(\frac{\|u\|_0^2}{R^2}\Big) g'(u)[v]
$$

and

$$
\begin{aligned}
g_R''(u)[v,w] &= \frac{4}{R^4}\Theta''\Big(\frac{\|u\|_0^2}{R^2}\Big)\operatorname{Re}\langle u,v\rangle_0 \operatorname{Re}\langle u,w\rangle_0\, g(u) + \Theta\Big(\frac{\|u\|_0^2}{R^2}\Big) g''(u)[v,w] \\
&\quad + \frac{2}{R^2}\Theta'\Big(\frac{\|u\|_0^2}{R^2}\Big)\left[\operatorname{Re}\langle v,w\rangle_0\, g(u) + \operatorname{Re}\langle u,v\rangle_0\, g'(u)[w] + \operatorname{Re}\langle u,w\rangle_0\, g'(u)[v]\right]
\end{aligned}
$$

Additionally, we obtain the estimates

$$\left.\begin{aligned}
\|g_R(u)\|_s &\leqslant C(R)\,\|u\|_s\\
\|g_R'(u)[v]\|_0 &\leqslant C(R)\,\|v\|_0\\
\|g_R''(u)[v,w]\|_0 &\leqslant C(R)\,\|v\|_0\,\|w\|_0\\
\|g_R(u_1)-g_R(u_2)\|_0 &\leqslant C(R)\,\|u_1-u_2\|_0\\
\|g_R'(u_1)[v]-g_R'(u_2)[v]\|_0 &\leqslant C(R)\,\|v\|_0\,\|u_1-u_2\|_0
\end{aligned}\right\} \tag{5.11}$$

for $s\in\{0,\theta\}$ and all u,u_1,u_2,v and w in the respective spaces. Additionally, if $u\in L^2(\Omega,D(A^\theta))$ is $\mathcal{F}_{t_0}$ measurable and $0\leqslant t_0\leqslant t_1\leqslant T$,

$$\mathbb{E}\left\|g_R'(u)\left[\int_{t_0}^{t_1} B(u)\,\mathrm{d}W(s)\right]\right\|_\theta^2 \lesssim_{R,T,\|\Phi\|_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}} \|u\|_\theta^2. \tag{5.12}$$

Proof. Let $u,h\in D(A^\theta)$ with $h\neq 0$. We see that

$$\begin{aligned}
\|h\|_\theta^{-1}\,\|g_R(u+h)-g_R(u)-g_R'(u)[h]\|_\theta &\leqslant \|h\|_\theta^{-1}\,|\Theta(\frac{\|u\|_0^2}{R^2})|\,\|g(u+h)-g(u)-g'(u)[h]\|_\theta\\
&+\underbrace{\|h\|_\theta^{-1}}_{\leqslant\|h\|_0^{-1}}|\Theta(\frac{\|u+h\|_0^2}{R^2})-\Theta(\frac{\|u\|_0^2}{R^2})-\frac{2}{R^2}\Theta'(\frac{\|u\|_0^2}{R^2})\operatorname{Re}\langle u,h\rangle_0|\underbrace{\|g(u+h)\|_\theta}_{\leqslant C(h\text{ small})}\\
&+\|h\|_\theta^{-1}\frac{2}{R^2}|\Theta'(\frac{\|u\|_0^2}{R^2})|\underbrace{|\operatorname{Re}\langle u,h\rangle_0|}_{\lesssim\|u\|_\theta\|h\|_\theta}\underbrace{\|g(u+h)-g(u)\|_\theta}_{\xrightarrow{h\to 0}0}\xrightarrow{h\to 0}0,
\end{aligned}$$

which proves the existence and formula of the first derivative. Next, let $u,v,h\in D(A^\theta)$ with $h\neq 0$. We compute

$$\begin{aligned}
&\|h\|_\theta^{-1}\,\|g_R'(u+h)[v]-g_R'(u)[v]-g_R''(u)[v,h]\|_\theta\\
&\quad\leqslant \|h\|_\theta^{-1}\frac{2}{R^2}|\Theta'(\frac{\|u\|_0^2}{R^2})|\,|\operatorname{Re}\langle u,v\rangle_0|\,\|g(u+h)-g(u)-g'(u)[h]\|_\theta\\
&\quad\quad+\|h\|_\theta^{-1}\frac{2}{R^2}|\Theta'(\frac{\|u+h\|_0^2}{R^2})|\underbrace{|\operatorname{Re}\langle h,v\rangle_0|}_{\lesssim\|h\|_\theta\|v\|_\theta}\underbrace{\|g(u+h)-g(u)\|_\theta}_{\xrightarrow{h\to 0}0}
\end{aligned}$$

$$
\begin{aligned}
&+ \|h\|_\theta^{-1} \frac{2}{R^2} \underbrace{|\Theta'(\frac{\|u+h\|_0^2}{R^2}) - \Theta'(\frac{\|u\|_0^2}{R^2})|}_{\xrightarrow{h\to 0} 0} \underbrace{|\mathrm{Re}\langle h, v\rangle_0|}_{\lesssim \|h\|_\theta \|v\|_\theta} \|g(u)\|_\theta \\
&+ \underbrace{\|h\|_\theta^{-1}}_{\leqslant \|h\|_0^{-1}} \frac{2}{R^2} |\Theta'(\frac{\|u+h\|_0^2}{R^2}) - \Theta'(\frac{\|u\|_0^2}{R^2}) - \frac{2}{R^2}\Theta''(\frac{\|u\|_0^2}{R^2}) \mathrm{Re}\langle u, h\rangle_0| \\
&\qquad |\mathrm{Re}\langle u, v\rangle_0| \underbrace{\|g(u+h)\|_\theta}_{\leqslant C(h \text{ small})} \\
&+ \|h\|_\theta^{-1} |\Theta(\frac{\|u\|_0^2}{R^2})| \|g'(u+h)[v] - g'(u)[v] - g''(u)[v,h]\|_\theta \\
&+ \underbrace{\|h\|_\theta^{-1}}_{\leqslant \|h\|_0^{-1}} |\Theta(\frac{\|u+h\|_0^2}{R^2}) - \Theta(\frac{\|u\|_0^2}{R^2}) - \frac{2}{R^2}\Theta'(\frac{\|u\|_0^2}{R^2}) \mathrm{Re}\langle u, h\rangle_0| \underbrace{\|g'(u+h)[v]\|_\theta}_{\leqslant C(h \text{ small})} \\
&+ \|h\|_\theta^{-1} \frac{2}{R^2} |\Theta'(\frac{\|u\|_0^2}{R^2})| \underbrace{|\mathrm{Re}\langle u, h\rangle_0|}_{\lesssim \|u\|_\theta \|h\|_\theta} \underbrace{\|g'(u+h)[v] - g(u)[v]\|_\theta}_{\xrightarrow{h\to 0} 0} \xrightarrow{h\to 0} 0,
\end{aligned}
$$

which gives us the existence and formula for the second derivative. All convergences which were not explicitly stated follow from the differentiability of g and Θ including the chain rule. Coming to the estimates on g, g' and g'', the first three follow more or less directly from their counterparts in (5.8), estimating every Y norm by $2R$ if in the same term includes a factor of $\Theta^{(k)}$ with the same norm as part of its variable. We start with g_R and

$$
\|g_R(u)\|_\theta \leqslant |\Theta(\frac{\|u\|_0^2}{R^2})| \|g(u)\|_\theta \leqslant |\Theta(\frac{\|u\|_0^2}{R^2})| C(2R) \|u\|_\theta \leqslant C(R) \|u\|_\theta,
$$

go on with g_R' with

$$
\begin{aligned}
\|g_R'(u)[v]\|_0 &\leqslant \frac{2}{R^2} |\Theta'(\frac{\|u\|_0^2}{R^2})| |\mathrm{Re}\langle u, v\rangle_0| \|g(u)\|_0 + |\Theta(\frac{\|u\|_0^2}{R^2})| \|g'(u)[v]\|_0 \\
&\leqslant \frac{2}{R^2} |\Theta'(\frac{\|u\|_0^2}{R^2})| \|u\|_0 \|v\|_0 C(2R) \|u\|_0 + |\Theta(\frac{\|u\|_0^2}{R^2})| C(2R) \|v\|_0 \\
&\leqslant C(R) \|v\|_0
\end{aligned}
$$

followed by g_R'' with

$$
\begin{aligned}
\|g_R''(u)[v,w]\|_0 &\leqslant \frac{4}{R^4}|\Theta''(\frac{\|u\|_0^2}{R^2})||\mathrm{Re}\langle u,v\rangle_0||\mathrm{Re}\langle u,w\rangle_0|\|g(u)\|_0 \\
&\quad + |\Theta(\frac{\|u\|_0^2}{R^2})|\|g''(u)[v,w]\|_0 + \frac{2}{R^2}|\Theta'(\frac{\|u\|_0^2}{R^2})|[|\mathrm{Re}\langle v,w\rangle_0|\|g(u)\|_0 \\
&\quad + |\mathrm{Re}\langle u,v\rangle_0|\|g'(u)[w]\|_0 + |\mathrm{Re}\langle u,w\rangle_0|\|g'(u)[v]\|_0] \\
&\leqslant \frac{4}{R^4}|\Theta''(\frac{\|u\|_0^2}{R^2})|\|u\|_0^2\|v\|_0\|w\|_0 C(2R)\|u\|_0 + |\Theta(\frac{\|u\|_0^2}{R^2})|\|g''(u)[v,w]\|_0 \\
&\quad + \frac{2}{R^2}|\Theta'(\frac{\|u\|_0^2}{R^2})|[\|v\|_0\|w\|_0 C(2R)\|u\|_0 + \|u\|_0\|v\|_0 C(2R)\|w\|_0 \\
&\quad + \|u\|_0\|w\|_0 C(2R)\|v\|_0] \\
&\leqslant C(R)\|v\|_0\|w\|_\theta.
\end{aligned}
$$

For the last two estimates in (5.11), we use Taylor's Theorem and the above estimates on g_R' and g_R'' in Y to see that

$$
\begin{aligned}
\|g_R(u_1) - g_R(u_2)\|_0 &= \left\| \int_0^1 (1-\xi) g_R'(\xi u_1 + (1-\xi)u_2)[u_1 - u_2]\, \mathrm{d}\xi \right\|_0 \\
&\leqslant \sup_{\xi\in[0,1]} \|g_R'(\xi u_1 + (1-\xi)u_2)[u_1-u_2]\|_0 \leqslant C(R)\|u_1 - u_2\|_0
\end{aligned}
$$

as well as

$$
\begin{aligned}
\|g_R'(u_1)[v] - g_R'(u_2)[v]\|_0 &= \left\| \int_0^1 (1-\xi) g_R''(\xi u_1 + (1-\xi)u_2)[v, u_1 - u_2]\, \mathrm{d}\xi \right\|_0 \\
&\leqslant \sup_{\xi\in[0,1]} \|g_R''(\xi u_1 + (1-\xi)u_2)[v, u_1-u_2]\|_0 \leqslant C(R)\|v\|_\theta\|u_1 - u_2\|_0.
\end{aligned}
$$

Moving on to (5.12), we first define

$$
\tilde{u}(\omega) = \begin{cases} u(\omega) & , \|u(\omega)\|_0 \leqslant 2R, \\ 0 & , \|u(\omega)\|_0 > 2R. \end{cases}
$$

to see that $\|\tilde{u}(\omega)\|_0 \leqslant 2R$ for all $\omega \in \Omega$ and then estimate

$$
\begin{aligned}
\mathbb{E}\left\|g_R'(u)\left[\int_{t_0}^{t_1} B(u)\,\mathrm{d}W(s)\right]\right\|_\theta^2 &\lesssim \mathbb{E}\left|\Theta'(\frac{\|u\|_0^2}{R^2})\right|^2 \left|\left\langle u, \int_{t_0}^{t_1} B(u)\,\mathrm{d}W(s)\right\rangle_0\right|^2 \|g(u)\|_\theta^2 \\
&\quad + \mathbb{E}\left|\Theta(\frac{\|u\|_0^2}{R^2})\right|^2 \left\|g'(u)\left[\int_{t_0}^{t_1} B(u)\,\mathrm{d}W(s)\right]\right\|_\theta^2 \\
&\lesssim \mathbb{E}\,\|\tilde{u}\|_0^2 \left\|\int_{t_0}^{t_1} B(\tilde{u})\,\mathrm{d}W(s)\right\|_0^2 \|g(\tilde{u})\|_\theta^2 \\
&\quad + \mathbb{E}\left\|g'(\tilde{u})\left[\int_{t_0}^{t_1} B(\tilde{u})\,\mathrm{d}W(s)\right]\right\|_\theta^2 \\
&\lesssim \mathbb{E}\left\|\int_{t_0}^{t_1} B(\|\tilde{u}\|_0\,\|\tilde{u}\|_\theta \tilde{u})\,\mathrm{d}W(s)\right\|_0^2 + \mathbb{E}\,\|\tilde{u}\|_\theta^2 \\
&\lesssim \mathbb{E}\left\|\int_{t_0}^{t_1} B(\|\tilde{u}\|_0\,\|\tilde{u}\|_\theta \tilde{u})\,\mathrm{d}W(s)\right\|_0^2 + \mathbb{E}\,\|\tilde{u}\|_\theta^2 \\
&= \int_{t_0}^{t_1} \mathbb{E}\,\|B(\|\tilde{u}\|_0\,\|\tilde{u}\|_\theta \tilde{u})\|_{\mathcal{L}_2(\widetilde{Y},Y)}^2\,\mathrm{d}s + \mathbb{E}\,\|\tilde{u}\|_\theta^2 \\
&\lesssim \mathbb{E}\,\|\tilde{u}\|_0^4\,\|\tilde{u}\|_\theta + \mathbb{E}\,\|\tilde{u}\|_\theta^2 \lesssim \mathbb{E}\,\|\tilde{u}\|_\theta^2 \\
&\leqslant \mathbb{E}\,\|u\|_\theta^2,
\end{aligned}
$$

where we used Ito's isometry and Lemmata 5.4 and 5.5. ■

Next, we take a look at one very special function which seems to make the biggest trouble in the proof.

Lemma 5.7

For $u \in L^2(\Omega, Y)$ and $j \in \mathbb{N}$ with $(j+1)h \leqslant T$ and $t \in [jh,(j+1)h)$ fixed, define

$$
(F(u))(\omega) := g_R'(u(\omega))\left[\int_{jh}^{t} B(u(\cdot))\,\mathrm{d}W(s)\right](\omega).
$$

If (5.8) holds, Then $F : L^2(\Omega, Y) \to L^2(\Omega, Y)$ and for almost all $\omega \in \Omega$, $F_\omega : L^2(\Omega, Y) \to Y$ defined by $F_\omega(u) = (F(u))(\omega)$ is Gâteaux differentiable, $F_\omega'(u)[v]$ is given by

$$
F_\omega'(u)[v] = g_R''(u(\omega))\left[\left(\int_{jh}^{t} B(u(\cdot))\,\mathrm{d}W(s)\right)(\omega), v(\omega)\right] + g_R'(u(\omega))\left[\left(\int_{jh}^{t} B(v(\cdot))\,\mathrm{d}W(s)\right)(\omega)\right]
$$

and, with the same dependencies of the arising constants as in Proposition 5.2,

$$\mathbb{E}\|F(u_1)-F(u_2)\|_0^2 \lesssim \mathbb{E}\|u_1-u_2\|_0^2$$

Proof. We start off by showing that F is well-defined. For $u \in L^2(\Omega, Y)$, it holds that

$$\begin{aligned}\mathbb{E}\|F(u)\|_0^2 &= \mathbb{E}\left\|g_R'(u)\left[\int_{jh}^t B(u)\,\mathrm{d}W(s)\right]\right\|_0^2 \lesssim C(R)\mathbb{E}\left\|\int_{jh}^t B(u)\,\mathrm{d}W(s)\right\|_0^2 \\ &= C(R)\mathbb{E}\int_{jh}^t \|B(u)\|_{\mathcal{L}_2(\widetilde{Y},Y)}^2\,\mathrm{d}s \lesssim C(R)\|\Phi\|_{\mathcal{L}_2(\widetilde{Y},Y)}^2 h\mathbb{E}\|u\|_0^2 < \infty,\end{aligned}$$

hence $F(u) \in L^2(\Omega, Y)$ and therefore $(F(u))(\omega) \in Y$ for almost all $\omega \in \Omega$. Next, we estimate that for $u, v \in L^2(\Omega, Y)$,

$$\begin{aligned}\mathbb{E}\|F_\omega'(u)[v]\|_0^2 &= \mathbb{E}\left\|g_R''(u)\left[\int_{jh}^t B(u)\,\mathrm{d}W(s), v\right]\right\|_0^2 + \mathbb{E}\left\|g_R'(u)\left[\left(\int_{jh}^t B(v)\,\mathrm{d}W(s)\right)\right]\right\|_0^2 \\ &\lesssim \mathbb{E}\left(\mathbb{1}_{\{\|\cdot\|_0\leqslant 2R\}}(u)\left\|\int_{jh}^t B(u)\,\mathrm{d}W(s)\right\|_0^2\|v\|_0^2\right) + \mathbb{E}\left\|\int_{jh}^t B(v)\,\mathrm{d}W(s)\right\|_0^2 \\ &= \mathbb{E}\left(\left\|\int_{jh}^t B(\|v\|_0\mathbb{1}_{\{\|\cdot\|_0\leqslant 2R\}}(u)u)\,\mathrm{d}W(s)\right\|_0^2\right) + \mathbb{E}\left\|\int_{jh}^t B(v)\,\mathrm{d}W(s)\right\|_0^2 \\ &= \mathbb{E}\int_{jh}^t \|B(\|v\|_0\mathbb{1}_{\{\|\cdot\|_0\leqslant 2R\}}(u)u)\|_{\mathcal{L}_2(\widetilde{Y},Y)}^2\,\mathrm{d}s \\ &\quad + \mathbb{E}\int_{jh}^t \|B(v)\|_{\mathcal{L}_2(\widetilde{Y},Y)}^2\,\mathrm{d}s \\ &\lesssim T\|\Phi\|_{\mathcal{L}_2(\widetilde{Y},Y)}^2\mathbb{E}(\|v\|_0\mathbb{1}_{\{\|\cdot\|_0\leqslant 2R\}}(u)\|u\|_0^2) + T\|\Phi\|_{\mathcal{L}_2(\widetilde{Y},Y)}^2\mathbb{E}\|v\|_0^2 \\ &\lesssim 4R^2T\|\Phi\|_{\mathcal{L}_2(\widetilde{Y},Y)}^2\mathbb{E}\|v\|_0 + T\|\Phi\|_{\mathcal{L}_2(\widetilde{Y},Y)}^2\mathbb{E}\|v\|_0^2 \lesssim \mathbb{E}\|v\|_0^2,\end{aligned}$$

hence especially $F_\omega'(u)[v] \in L^2(\Omega, Y)$ and therefore $(F(u))(\omega) \in Y$ for almost all $\omega \in \Omega$. If we can show that F_ω' is the derivative of $F\omega$, then by virtue of Taylor's Theorem and Hoelder's inequality

$$\begin{aligned}\mathbb{E}\|F(u_1)-F(u_2)\|_0^2 &= \mathbb{E}\left\|\int_0^1 (1-\xi)F_\omega'(\xi u_1 + (1-\xi)u_2)[u_1-u_2]\,\mathrm{d}\xi\right\|_0^2 \\ &\leqslant \mathbb{E}\left(\int_0^1 (1-\xi)\|F_\omega'(\xi u_1+(1-\xi)u_2)[u_1-u_2]\|_0\,\mathrm{d}\xi\right)^2\end{aligned}$$

$$
\begin{aligned}
&\lesssim \mathbb{E}\bigg(\int_0^1 (1-\xi)^2 \,\mathrm{d}\xi\bigg)\bigg(\int_0^1 \|F'_\omega(\xi u_1 + (1-\xi)u_2)[u_1-u_2]\|_0^2 \,\mathrm{d}\xi\bigg) \\
&\lesssim \mathbb{E}\int_0^1 \|F'_\omega(\xi u_1 + (1-\xi)u_2)[u_1-u_2]\|_0^2 \,\mathrm{d}\xi \\
&= \int_0^1 \mathbb{E}\|F'_\omega(\xi u_1 + (1-\xi)u_2)[u_1-u_2]\|_0^2 \,\mathrm{d}\xi \\
&\lesssim \int_0^1 \|u_1-u_2\|_0^2 \,\mathrm{d}\xi = \|u_1-u_2\|_0^2.
\end{aligned}
$$

To show the differentiability, we note that for $u, v \in L^2(\Omega, Y)$, for almost all $\omega \in \Omega$, $u(\omega), v(\omega)$ and hence $(\int_{jh}^t B(u)\,\mathrm{d}W(s))(\omega)$ and $(\int_{jh}^t B(v)\,\mathrm{d}W(s))(\omega)$ lie in Y. We suppress ω now and deduce from the differentiability of g_R' as well as Taylor's Theorem, that

$$
\begin{aligned}
&\Big\|\frac{F(u+\varepsilon v)-F(u)}{\varepsilon} - F'(u)[v]\Big\|_0 \\
&\quad\lesssim \Big\|\frac{g_R'(u+\varepsilon v)[\int_{jh}^t B(u)\,\mathrm{d}W(s)] - g_R'(u)[\int_{jh}^t B(u)\,\mathrm{d}W(s)]}{\varepsilon} - g_R''(u)\Big[\int_{jh}^t B(u)\,\mathrm{d}W(s)\Big]\Big\|_0 \\
&\quad + \Big\|g_R'(u+\varepsilon v)\Big[\int_{jh}^t B(v)\,\mathrm{d}W(s)\Big] - g_R'(u)\Big[\int_{jh}^t B(v)\,\mathrm{d}W(s)\Big]\Big\|_0 \\
&\quad\lesssim \Big\|\frac{g_R'(u+\varepsilon v)[\int_{jh}^t B(u)\,\mathrm{d}W(s)] - g_R'(u)[\int_{jh}^t B(u)\,\mathrm{d}W(s)]}{\varepsilon} - g_R''(u)\Big[\int_{jh}^t B(u)\,\mathrm{d}W(s)\Big]\Big\|_0 \\
&\quad + \varepsilon \sup_{\xi\in[0,1]} \Big\|g_R''(u+\xi\varepsilon v)\Big[\int_{jh}^t B(v)\,\mathrm{d}W(s), v\Big]\Big\|_0 \\
&\quad\lesssim \Big\|\frac{g_R'(u+\varepsilon v)[\int_{jh}^t B(u)\,\mathrm{d}W(s)] - g_R'(u)[\int_{jh}^t B(u)\,\mathrm{d}W(s)]}{\varepsilon} - g_R''(u)\Big[\int_{jh}^t B(u)\,\mathrm{d}W(s)\Big]\Big\|_0 \\
&\quad + \varepsilon\Big\|\int_{jh}^t B(v)\,\mathrm{d}W(s)\Big\|_0^2 \|v\|_0 \xrightarrow{\varepsilon\to 0} 0.
\end{aligned}
$$

■

5.5 Proof of propostion 5.2

We now prepare the proof of Proposition 5.2 by finding a suitable representation of the exact solution and its numerical approximation. For the numerical approximation, we first define the right-continuous function φ_R by $\varphi_R(0) = v_0$, and for $0 < n \leqslant \frac{T}{h}$

$$\varphi_R(t) = \begin{cases} \exp\Big(-\mathrm{i}[(t-(n-1)h)\theta_R(\varphi_R((n-1)h))\,|\varphi_R((n-1)h)|^{k-1} + W(nh) - W((n-1)h)]\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot\varphi_R((n-1)h), \\ T(h)\lim_{t\to nh}\varphi_R(t) \end{cases}$$

for $t \in ((n-1)h, nh)$ and $t = nh$, respectively. We notice that $\varphi(nh) = v^n$ for all $n \in \mathbb{N}$ and since in $((n-1)h, nh)$, it moves along the solution of (5.4b), we also have the integral equality

$$\varphi_R(t) = \varphi_R((n-1)h) - \int_{(n-1)h}^{t} \mathrm{i}g_R(\varphi_R(s)) + \frac{1}{2}F_\Phi\varphi_R(s)\,\mathrm{d}s - \mathrm{i}\int_{(n-1)h}^{t} B(\varphi_R(s))\,\mathrm{d}W(s) \tag{5.13}$$

for $t \in ((n-1)h, nh)$. Using the definition of φ_R, we arrive at

$$\begin{aligned} \varphi_R(nh) = T(h)\varphi_R((n-1)h) &- \int_{(n-1)h}^{t} T(h)(\mathrm{i}g_R(\varphi_R(s)) + \frac{1}{2}F_\Phi\varphi_R(s))\,\mathrm{d}s \\ &- \mathrm{i}\int_{(n-1)h}^{t} T(h)B(\varphi_R(s))\,\mathrm{d}W(s). \end{aligned}$$

Replacing the first term in (5.13) by the above equation while changing n to $n-1$, we arrive at

$$\begin{aligned} \varphi_R(t) = T(h)\varphi_R((n-2)h) &- \int_{(n-2)h}^{t} (\mathbb{1}_{[(n-2)h,(n-1)h]}T(h) + \mathbb{1}_{[(n-1)h,nh]})(\mathrm{i}g_R(\varphi_R(s)) \\ &+ \frac{1}{2}F_\Phi\varphi_R(s))\,\mathrm{d}s - \mathrm{i}\int_{(n-1)h}^{t} (\mathbb{1}_{[(n-2)h,(n-1)h]}T(h) + \mathbb{1}_{[(n-1)h,nh]})B(\varphi_R(s))\,\mathrm{d}W(s). \end{aligned}$$

Repeating this process, that is, replacing the first term in the equation above by the one before with $n-2$ instead of n, we inductively end up with

$$\begin{aligned} v^n = \varphi_R(nh) = T(nh)v_0 &- \mathrm{i}\int_0^{nh} T_n(s)g_R(\varphi_R(s))\,\mathrm{d}s \\ &- \frac{1}{2}\int_0^{nh} T_n(s)F_\Phi\varphi_R(s)\,\mathrm{d}s - \mathrm{i}\int_0^{nh} T_n(s)B(\varphi_R(s))\,\mathrm{d}W(s) \end{aligned} \tag{5.14}$$

with $T_n(s) = \sum_{j=0}^{n-1}\mathbb{1}_{[jh,(j+1)h]}(s)T((n-j)h)$ and $0 \leqslant n \leqslant \frac{T}{h}$. We are also going to need a version of (5.13) with $n-1$ replaced by j. For $0 \leqslant j \leqslant n-1$ and $t \in [jh,(j+1)h)$, we end up with

$$\varphi_R(t) = \varphi_R(jh) - \underbrace{\underbrace{\int_{jh}^{t} \mathrm{i} g_R(\varphi_R(s)) + \frac{1}{2} F_\Phi \varphi_R(s)\,\mathrm{d}s}_{=:\varphi_R^{j,1}(t)} \underbrace{- \mathrm{i} \int_{jh}^{t} B(\varphi_R(s))\,\mathrm{d}W(s)}_{=:\varphi_R^{j,2}(t)}}_{=:\varphi_R^{j}(t)} \tag{5.15}$$

For the exact solution, things are easier, as we obtain the two integral equations

$$\begin{aligned} v_R(nh) = T(nh)v_0 - \mathrm{i} \int_0^{nh} T(nh-s) g_R(v_R(s))\,\mathrm{d}s \\ - \frac{1}{2} \int_0^{nh} T(nh-s) F_\Phi v_R(s)\,\mathrm{d}s - \mathrm{i} \int_0^{nh} T(nh-s) B(v_R(s))\,\mathrm{d}W(s) \end{aligned} \tag{5.16}$$

for $0 \leqslant n \leqslant \frac{T}{h}$ as well as

$$v_R(t) = S(t-jh)v_R(jh) - \underbrace{\underbrace{\int_{jh}^{t} T(t-s)(\mathrm{i} g_R(v_R(s)) + \frac{1}{2} F_\Phi v_R(s))\,\mathrm{d}s}_{=:v_R^{j,1}(t)} \underbrace{- \mathrm{i} \int_{jh}^{t} T(t-s) B(v_R(s))\,\mathrm{d}W(s)}_{=:v_R^{j,2}(t)}}_{=:v_R^{j}(t)} \tag{5.17}$$

for $0 \leqslant j \leqslant n-1$ and $t \in [jh, (j+1)h)$.
From (5.15) and (5.17), Taylor's Theorem gives us the equations

$$\begin{aligned} g_R(\varphi_R(t)) = g_R(T(t-jh)\varphi_R(jh)) + g_R'(T(t-jh)\varphi_R(jh)) \left[\varphi_R^j(t)\right] \\ + \frac{1}{2} \int_0^1 (1-\xi) g_R'' \Big(T(t-jh)\varphi_R(jh) + \xi \varphi_R^j(t)\Big) \left[\varphi_R^j(t), \varphi_R^j(t)\right] \mathrm{d}\xi \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} g_R(v_R(t)) = g_R(T(t-jh)v_R(jh)) + g_R'(T(t-jh)v_R(jh)) \left[v_R^j(t)\right] \\ + \frac{1}{2} \int_0^1 (1-\xi) g_R'' \Big(T(t-jh)v_R(jh) + \xi v_R^j(t)\Big) \left[v_R^j(t), v_R^j(t)\right] \mathrm{d}\xi \end{aligned} \tag{5.19}$$

for some $\xi \in [0,1]$. Starting off with (5.14), we obtain

$$\begin{aligned}
v^n &= T(nh)v_0 + \sum_{j=0}^{n-1}\Bigg[-\int_{jh}^{(j+1)h} T(t^{n-j})\Big(\mathrm{i}g_R(\varphi_R(t)) + \frac{1}{2}F_\Phi\varphi_R(t)\Big)\,\mathrm{d}t \\
&\quad -\mathrm{i}\int_{jh}^{(j+1)h} T(t^{n-j})B(\varphi_R(t))\,\mathrm{d}W(t)\Bigg] \\
&\overset{(5.15),(5.18)}{=} T(nh)v_0 + \sum_{j=0}^{n-1}\Bigg[-\int_{jh}^{(j+1)h} T(t^{n-j})\Big[\mathrm{i}g_R(\varphi_R(jh)) + \frac{1}{2}F_\Phi\varphi_R(jh)\Big]\,\mathrm{d}t \\
&\quad -\mathrm{i}\int_{jh}^{(j+1)h} T(t^{n-j})B(\varphi_R(jh))\,\mathrm{d}W(t) - \mathrm{i}\int_{jh}^{(j+1)h} T(t^{n-j})B(\varphi_R^{j,1}(t)+\varphi_R^{j,2}(t))\,\mathrm{d}W(t) \\
&\quad -\int_{jh}^{(j+1)h} T(t^{n-j})\Big(\mathrm{i}g_R'(\varphi_R(jh)) + \frac{1}{2}F_\Phi\varphi_R(jh)\Big)\Big[\varphi_R^{j,1}(t)+\varphi_R^{j,2}(t)\Big]\,\mathrm{d}t \\
&\quad -\frac{\mathrm{i}}{2}\int_{jh}^{(j+1)h} T(t^{n-j})\int_0^1 (1-\xi)g_R''(\varphi_R(jh)+\xi\varphi_R^j(t))\Big[\varphi_R^j(t),\varphi_R^j(t)\Big]\,\mathrm{d}\xi\,\mathrm{d}t\Bigg] \\
&\overset{\text{Def. }\varphi_R^{j,1},(5.15)}{=} T(nh)v_0 + \sum_{j=0}^{n-1}\Bigg[-\int_{jh}^{(j+1)h} T(t^{n-j})\Big[\mathrm{i}g_R(\varphi_R(jh)) + \frac{1}{2}F_\Phi\varphi_R(jh)\Big]\,\mathrm{d}t
\end{aligned} \tag{5.20a}$$

$$+\mathrm{i}\int_{jh}^{(j+1)h} T(t^{n-j})\Big(\mathrm{i}g_R'(\varphi_R(jh)) + \frac{1}{2}F_\Phi\varphi_R(jh)\Big)\Big[\int_{t_j}^t B(\varphi_R(jh))\,\mathrm{d}W(s)\Big]\,\mathrm{d}t \tag{5.20b}$$

$$-\mathrm{i}\int_{jh}^{(j+1)h} T(t^{n-j})B(\varphi_R(jh))\,\mathrm{d}W(t) \tag{5.20c}$$

$$-\mathrm{i}\int_{jh}^{(j+1)h} T(t^{n-j})B\Big(\int_{t_j}^t B(\varphi_R(jh)\Big)\,\mathrm{d}W(s))\,\mathrm{d}W(t) \tag{5.20d}$$

$$-\frac{\mathrm{i}}{2}\int_{jh}^{(j+1)h} T(t^{n-j})\int_0^1 (1-\xi)g_R''(\varphi_R(jh)+\xi\varphi_R^j(t))\Big[\varphi_R^j(t),\varphi_R^j(t)\Big]\,\mathrm{d}\xi\,\mathrm{d}t \tag{5.20e}$$

$$-\int_{jh}^{(j+1)h} T((n-j)h)\Big(\mathrm{i}g_R'(\varphi_R(jh)) + \frac{1}{2}F_\Phi\Big)\Big[\varphi_R^{j,1}(t)\Big]\,\mathrm{d}t \tag{5.20f}$$

$$-\mathrm{i}\int_{jh}^{(j+1)h} T((n-j)h)B(\varphi_R^{j,1}(t))\,\mathrm{d}W(t)\Bigg], \tag{5.20g}$$

where we used (5.15) on $\varphi_R(s)$ in the definition of $\varphi_R^{j,2}$. Similarly, starting from (5.16) and artificially inserting the same sum that naturally occurred above, we arrive at

$$v^n = T(nh)v_0 + \sum_{j=0}^{n-1}\Bigg[-\int_{jh}^{(j+1)h} T(nh-t)\Big[\mathrm{i}g_R(T(t-jh)v_R(jh)) + \frac{1}{2}F_\Phi v_R(jh)\Big]\,\mathrm{d}t \tag{5.21a}$$

$$+\,\mathrm{i}\int_{jh}^{(j+1)h} T(nh-t)\Big(\mathrm{i}g_R'(T(t-jh)v_R(jh)) + \frac{1}{2}F_\Phi v_R(jh)\Big) \Big[\int_{t_j}^{t} T(t-s)B(T(s-jh)v_R(jh))\,\mathrm{d}W(s)\Big]\,\mathrm{d}t \tag{5.21b}$$

$$-\,\mathrm{i}\int_{jh}^{(j+1)h} T(nh-t)B(T(t-jh)v_R(jh))\,\mathrm{d}W(t) \tag{5.21c}$$

$$-\,\mathrm{i}\int_{jh}^{(j+1)h} T(nh-t)B\Big(\int_{t_j}^{t} T(t-s)B(T(s-jh)v_R(jh)\Big)\,\mathrm{d}W(s))\,\mathrm{d}W(t) \tag{5.21d}$$

$$-\,\frac{\mathrm{i}}{2}\int_{jh}^{(j+1)h} T(nh-t)\int_0^1 (1-\xi)g_R''(T(t-jh)v_R(jh)+\xi v_R^j(t))\Big[v_R^j(t), v_R^j(t)\Big]\,\mathrm{d}\xi\,\mathrm{d}t \tag{5.21e}$$

$$-\int_{jh}^{(j+1)h} T(nh-t)\Big(\mathrm{i}g_R'(T(t-jh)v_R(jh)) + \frac{1}{2}F_\Phi\Big)\Big[v_R^{j,1}(t)\Big]\,\mathrm{d}t \tag{5.21f}$$

$$-\mathrm{i}\int_{jh}^{(j+1)h} T(nh-t)B(v_R^{j,1}(t))\,\mathrm{d}W(t)\Bigg]. \tag{5.21g}$$

The last thing we need to do before we start estimating is giving estimates on the terms defined in (5.15) and (5.17).

Lemma 5.8

If (5.8) holds, then, with the same dependencies of the arising constants as in Proposition 5.2 (plus p), we have

$$\mathbb{E}\|\varphi_R(t)\|_\theta^p \lesssim 1, \qquad \mathbb{E}\|v_R(t)\|_\theta^p \lesssim 1$$

and therefore

$$\mathbb{E}\|\varphi_R^j(t)\|_\theta^p \lesssim h^{\frac{p}{2}}, \qquad \mathbb{E}\|v_R^j(t)\|_\theta^p \lesssim h^{\frac{p}{2}}$$

for $p \in \{2,4\}$ as well as

$$\mathbb{E}\|\varphi_R^{j,1}(t)\|_\theta^2 \lesssim h^2, \qquad \mathbb{E}\|v_R^{j,1}(t)\|_\theta^2 \lesssim h^2$$

Proof. For the estimates on v_R, we look at the integral representations

$$v_R(t) = T(t)v_0 - \mathrm{i}\int_0^t T(t-s)g_R(v_R(s))\,\mathrm{d}s - \frac{1}{2}\int_0^t F_\Phi v_R(s)\,\mathrm{d}s - \mathrm{i}\int_0^t T(t-s)B(v_R(s))\,\mathrm{d}W(s)$$

for $t \in [0,T]$. From this, we deduce

$$\begin{aligned}
\mathbb{E}\|v_R(t)\|_\theta^p &\lesssim \mathbb{E}\|v_0\|_\theta^p + \mathbb{E}\Big\|\int_0^t T(t-s)g_R(v_R(s))\,\mathrm{d}s\Big\|_\theta^p + \mathbb{E}\Big\|\int_0^t T(t-s)(F_\Phi v_R(s))\,\mathrm{d}s\Big\|_\theta^p \\
&\quad + \mathbb{E}\Big\|\int_0^t T(t-s)B(v_R(s))\,\mathrm{d}W(s)\Big\|_\theta^p \\
&\leqslant \mathbb{E}\|v_0\|_\theta^p + \mathbb{E}\Big(\int_0^t \|T(t-s)g_R(v_R(s))\|_\theta\,\mathrm{d}s\Big)^p + \mathbb{E}\Big(\int_0^t \|T(t-s)(F_\Phi v_R(s))\|_\theta\,\mathrm{d}s\Big)^p \\
&\quad + \mathbb{E}\sup_{0\leqslant t'\leqslant t}\Big\|\int_0^{t'} T(t-s)B(v_R(s))\,\mathrm{d}W(s)\Big\|_\theta^p \\
&\lesssim \mathbb{E}\|v_0\|_\theta^p + \mathbb{E}\Big(\int_0^t \|g_R(v_R(s))\|_\theta\,\mathrm{d}s\Big)^p + \mathbb{E}\Big(\int_0^t \|F_\Phi\|_\theta\|v_R(s)\|_\theta\,\mathrm{d}s\Big)^p \\
&\quad + \mathbb{E}\Big(\int_0^t \|T(t-s)B(v_R(s))\|_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}^2\,\mathrm{d}s\Big)^{p/2} \\
&\lesssim \mathbb{E}\|v_0\|_\theta^p + (T^{p-1}C(R)^p + T^{p-1}\|\Phi\|_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}^p \\
&\quad + T^{\frac{p}{2}-1}\|\Phi\|_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}^p)\mathbb{E}\int_0^t \|v_R(s)\|_\theta^p\,\mathrm{d}s \\
&\lesssim 1 + \int_0^t \mathbb{E}\|v_R(s)\|_\theta^p\,\mathrm{d}s,
\end{aligned}$$

where we used Burkholder's inequality (see [BP99][Theorem 7.3]) followed by Hölder's inequality plus all estimates from the Lemmata before. Gronwall gives the desired result. For v_R^j, we repeat its definition

$$v_R^j(t) = -\int_{jh}^t T(t-s)(\mathrm{i}g_R(v_R(s)) + \frac{1}{2}F_\Phi v_R(s))\,\mathrm{d}s - \mathrm{i}\int_{jh}^t T(t-s)B(v_R(s))\,\mathrm{d}W(s)$$

for $t \in [jh,(j+1)h)$. Compared to v_R, the first term is missing and the integrals have a range bounded by h instead of T. Therefore, the exact same estimates deliver, using the first result at the end

$$\begin{aligned}
\mathbb{E}\|v_R^j(t)\|_\theta^p &\lesssim (h^{p-1}C(R)^p + h^{p-1}\|\Phi\|^p_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))} + h^{\frac{p}{2}-1}\|\Phi\|^p_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))})\int_{jh}^t \mathbb{E}\|v_R(s)\|_\theta^p\,\mathrm{d}s \\
&\lesssim (h^pC(R)^p + h^p\|\Phi\|^p_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))} + h^{\frac{p}{2}}\|\Phi\|^p_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}) \\
&\lesssim h^{\frac{p}{2}}(T^{\frac{p}{2}}C(R)^p + T^{\frac{p}{2}}\|\Phi\|^p_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))} + \|\Phi\|^p_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}) \lesssim h^{\frac{p}{2}}.
\end{aligned}$$

Finally, $v_R^{j,1}$ is defined by

$$v_R^{j,1}(t) = -\int_{jh}^t T(t-s)(\mathrm{i}g_R(v_R(s)) + \frac{1}{2}F_\Phi v_R(s))\,\mathrm{d}s$$

for $t \in [jh,(j+1)h)$. Hence it consists of the first two terms of v_R^j, meaning that the crucial term preventing an estimate by h^p vanished. This gives

$$\begin{aligned}
\mathbb{E}\|v_R^{j,1}(t)\|_\theta^2 &\lesssim (hC(R)^2 + h\|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))})\int_{jh}^t \mathbb{E}\|v_R(s)\|_\theta^2\,\mathrm{d}s \\
&\lesssim h^2(C(R)^2 + \|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}) \lesssim h^2.
\end{aligned}$$

For φ_R, we start with (compare (5.14))

$$\varphi_R(t) = T(jh)v_0 - \mathrm{i}\int_0^t T_j(s)g_R(\varphi_R(s))\,\mathrm{d}s - \frac{1}{2}\int_0^t T_j(s)F_\Phi\varphi_R(s)\,\mathrm{d}s - \mathrm{i}\int_0^t T_j(s)B(\varphi_R(s))\,\mathrm{d}W(s)$$

for $T_j(s) = \sum_{l=0}^j \mathbb{1}_{[lh,(l+1)h]}(s)T((j-l)h)$, $0 \leqslant j < n \leqslant \frac{T}{h}$ and $t \in [jh,(j+1)h)$. Hence we can use the same estimates as for v_R. The only difference is $T_j(s)$, which pointwise is just some $T((j-l)h)$ and therefore handled like $T(t-s)$ before. Again, Gronwall gives the result. φ_R^j and $\varphi_R^{j,1}$ are defined analogously to their counterparts v_R^j and $v_R^{j,1}$ without $T(t-s)$, meaning that the same arguments as above yield the last results. ■

Proof of Proposition 5.2. We start estimating the term in question (with T replaced by $a \in [0,T]$, namely

$$\mathbb{E} \max_{0 \leqslant n \leqslant \frac{a}{h}} \|v_R(nh) - v_R^n\|_0^2.$$

First of all, we make some general observations. Building the difference $v_R(nh) - v_R^n$, the first term in (5.20 and (5.21) cancel each other out. Secondly, taking the norm $\|v_R(nh) - v_R^n\|_0^2$ lets us loose $T(nh)$, since it operates on Y. After that, we group together the terms (5.20a) to (5.20d) with their counterparts (5.21a) to (5.21d). The rest of the terms are left by themselves (all terms containing the sum over j), leaving us with ten terms. We then use the triangle inequality to get the norm inside those terms and then also pull the square inside, giving us a fixed constant for the estimate (since we are dealing with ten terms independently of any variable). Since the expectation and maximum are monotone and (sub-)linear, we also spread those two on all terms.
This procedure leaves us with two kinds of terms, depending on if the outermost integral is deterministic or stochastic. In the first case, we estimate, using Hölder's inequality,

$$\begin{aligned}
\mathbb{E} \max_{0 \leqslant n \leqslant \frac{a}{h}} \left\| \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} F_j(t)\,\mathrm{d}t \right\|_0^2 &= \mathbb{E} \max_{0 \leqslant n \leqslant \frac{a}{h}} \left\| \int_0^{nh} \sum_{j=0}^{n-1} \mathbb{1}_{[jh,(j+1)h]}(t) F_j(t)\,\mathrm{d}t \right\|_0^2 \\
&\leqslant \mathbb{E} \max_{0 \leqslant n \leqslant \frac{a}{h}} \left(\int_0^{nh} \left\| \sum_{j=0}^{n-1} \mathbb{1}_{[jh,(j+1)h]}(t) F_j(t) \right\|_0 \mathrm{d}t \right)^2 \\
&\leqslant \mathbb{E} \max_{0 \leqslant n \leqslant \frac{a}{h}} nh \int_0^{nh} \left\| \sum_{j=0}^{n-1} \mathbb{1}_{[jh,(j+1)h]}(t) F_j(t) \right\|_0^2 \mathrm{d}t \\
&\leqslant \mathbb{E} T \int_0^a \left\| \sum_{j=0}^{\lfloor \frac{a}{h} \rfloor - 1} \mathbb{1}_{[jh,(j+1)h]}(t) F_j(t) \right\|_0^2 \mathrm{d}t \\
&= T \int_0^a \sum_{j=0}^{\lfloor \frac{a}{h} \rfloor - 1} \mathbb{1}_{[jh,(j+1)h]}(t) \mathbb{E} \|F_j(t)\|_0^2 \,\mathrm{d}t. \qquad (5.22)
\end{aligned}$$

In the second case, we obtain, mainly using Burkholder's inequality (see [BP99] [Theorem 7.3])

$$
\begin{aligned}
\mathbb{E}\max_{0\leqslant n\leqslant \frac{a}{h}}\Big\|\sum_{j=0}^{n-1}\int_{jh}^{(j+1)h} G_j(t)\,\mathrm{d}W(t)\Big\|_0^2 &= \mathbb{E}\max_{0\leqslant n\leqslant \frac{a}{h}}\Big\|\int_0^{nh}\sum_{j=0}^{\lfloor \frac{a}{h}\rfloor-1}\mathbb{1}_{[jh,(j+1)h]}(t)G_j(t)\,\mathrm{d}W(t)\Big\|_0^2 \\
&\leqslant \mathbb{E}\sup_{0\leqslant s\leqslant a}\Big\|\int_0^{s}\sum_{j=0}^{\lfloor \frac{a}{h}\rfloor-1}\mathbb{1}_{[jh,(j+1)h]}(t)G_j(t)\,\mathrm{d}W(t)\Big\|_0^2 \\
&\lesssim \mathbb{E}\int_0^{a}\Big\|\sum_{j=0}^{\lfloor \frac{a}{h}\rfloor-1}\mathbb{1}_{[jh,(j+1)h]}(t)G_j(t)\Big\|_{\mathcal{L}_2(\widetilde{Y},Y)}^2\,\mathrm{d}t \\
&= \int_0^{a}\sum_{j=0}^{\lfloor \frac{a}{h}\rfloor-1}\mathbb{1}_{[jh,(j+1)h]}(t)\mathbb{E}\|G_j(t)\|_{\mathcal{L}_2(\widetilde{Y},Y)}^2\,\mathrm{d}t.
\end{aligned}
\tag{5.23}
$$

In both cases, the last equality follows pointwise and by Fubini's Theorem. F_j and G_j are appropriate functions, respectively.
Next, we take a look at the expressions $\mathbb{E}\|F_j(t)\|_0^2$ and $\mathbb{E}\|G_j(t)\|_{\mathcal{L}_2(\widetilde{Y},Y)}^2$ for the ten terms from (5.20) and (5.21) mentioned above. We regularly use (5.8) as well as Lemmata 5.4, 5.6 and 5.8. The constants in the estimates implicitly depend on $T, R, \|\Phi\|_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}^2, \mathbb{E}\|v_0\|_\theta^2, \mathbb{E}\|v_0\|_\theta^4$. t lies in $(jh,(j+1)+1]$.
For the difference of (5.20a) and (5.21a), we estimate

$$
\begin{aligned}
&\mathbb{E}\Big\|T(-jh)\Big(\mathrm{i}g_R(\varphi_R(jh))+\frac{1}{2}F_\Phi\varphi_R(jh)\Big)-T(-t)\Big(\mathrm{i}g_R(v_R(jh))+\frac{1}{2}F_\Phi v_R(jh)\Big)\Big\|_0^2 \\
&\quad\lesssim \mathbb{E}\Big\|(T(-jh)-T(-t))\Big(\mathrm{i}g_R(\varphi_R(jh))+\frac{1}{2}F_\Phi\varphi_R(jh)\Big)\Big\|_0^2 \\
&\qquad+\mathbb{E}\|T(-t)\,(g_R(\varphi_R(jh))-g_R(v_R(jh)))\|_0^2 \\
&\qquad+\mathbb{E}\|T(-t)\,(g_R(v_R(jh))-g_R(S(t-jh)v_R(jh)))\|_0^2 \\
&\qquad+\mathbb{E}\|T(-t)F_\Phi\,(\varphi_R(jh)-v_R(jh))\|_0^2 \\
&\quad\lesssim h^{2\theta}\mathbb{E}\Big\|\mathrm{i}g_R(\varphi_R(jh))+\frac{1}{2}F_\Phi\varphi_R(jh)\Big\|_\theta^2+\mathbb{E}\|g_R(\varphi_R(jh))-g_R(v_R(jh))\|_0^2 \\
&\qquad+\mathbb{E}\|g_R(v_R(jh))-g_R(S(t-jh)v_R(jh))\|_0^2+\mathbb{E}\|F_\Phi\,(\varphi_R(jh)-v_R(jh))\|_0^2 \\
&\quad\lesssim h^{2\theta}\mathbb{E}\|g_R(\varphi_R(jh))\|_\theta^2+h^{2\theta}\mathbb{E}\|F_\Phi\varphi_R(jh)\|_\theta^2+\mathbb{E}\|\varphi_R(jh))-v_R(jh))\|_0^2 \\
&\qquad+\mathbb{E}\|(I-S(t-jh))v_R(jh))\|_0^2+\mathbb{E}\|F_\Phi\,(\varphi_R(jh)-v_R(jh))\|_0^2
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim h^{2\theta}\mathbb{E}\|\varphi_R(jh)\|_\theta^2 + h^{2\theta}\|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\|\varphi_R(jh)\|_\theta^2 + \mathbb{E}\|\varphi_R(jh)) - v_R(jh))\|_0^2 \\
&\quad + \mathbb{E}\|v_R(jh))\|_\theta^2 + \|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\|\varphi_R(jh) - v_R(jh)\|_0^2 \\
&\lesssim h^{2\theta} + \mathbb{E}\|\varphi_R(jh)) - v_R(jh))\|_0^2.
\end{aligned}
$$

For the difference of (5.20b) and (5.21b), we have

$$
\begin{aligned}
&\mathbb{E}\Big\| T(-jh)\Big(\mathrm{i}g_R'(\varphi_R(jh)) + \frac{1}{2}F_\Phi\Big)\Big[\int_{jh}^t B(\varphi_R(jh))\,\mathrm{d}W(s)\Big] \\
&\qquad - T(-t)\Big(\mathrm{i}g_R'(S(t-jh)v_R(jh)) + \frac{1}{2}F_\Phi\Big)\Big[\int_{jh}^t S(t-s)B(S(s-jh)v_R(jh))\,\mathrm{d}W(s)\Big]\Big\|_0^2 \\
&\lesssim \mathbb{E}\Big\|(T(-jh) - T(-t))\Big(\mathrm{i}g_R'(\varphi_R(jh)) + \frac{1}{2}F_\Phi\Big)\Big[\int_{jh}^t B(\varphi_R(jh))\,\mathrm{d}W(s)\Big]\Big\|_0^2 \\
&\quad + \mathbb{E}\Big\|T(-t)\Big(\mathrm{i}g_R'(\varphi_R(jh))\Big[\int_{jh}^t B(\varphi_R(jh))\,\mathrm{d}W(s)\Big] - \mathrm{i}g_R'(v_R(jh))\Big[\int_{jh}^t B(v_R(jh))\,\mathrm{d}W(s)\Big]\Big)\Big\|_0^2 \\
&\quad + \mathbb{E}\Big\|T(-t)\,(\mathrm{i}g_R'(v_R(jh)) - \mathrm{i}g_R'(S(t-jh)v_R(jh)))\Big[\int_{jh}^t B(v_R(jh))\,\mathrm{d}W(s)\Big]\Big\|_0^2 \\
&\quad + \mathbb{E}\Big\|T(-t)F_\Phi\int_{jh}^t B(\varphi_R(jh) - v_R(jh))\,\mathrm{d}W(s)\Big\|_0^2 \\
&\quad + \mathbb{E}\Big\|T(-t)\Big(\mathrm{i}g_R'(v_R(jh)) + \frac{1}{2}F_\Phi\Big)\Big[\int_{jh}^t (I - T(t-s))B(v_R(jh))\,\mathrm{d}W(s)\Big]\Big\|_0^2 \\
&\quad + \mathbb{E}\Big\|T(-t)\Big(\mathrm{i}g_R'(v_R(jh)) + \frac{1}{2}F_\Phi\Big)\Big[\int_{jh}^t T(t-s)B((I - T(s-jh))v_R(jh))\,\mathrm{d}W(s)\Big]\Big\|_0^2 \\
&=: I + II + III + IV + V + VI.
\end{aligned}
$$

We follow this up by estimating the six terms, using $F(u) = \mathrm{i}g_R'(u)\Big[\int_{jh}^t B(u)\,\mathrm{d}W(s)\Big]$ and Lemma 5.7 for II, to estimate

$$
\begin{aligned}
I &\lesssim h^{2\theta}\mathbb{E}\Big\|\Big(\mathrm{i}g_R'(\varphi_R(jh)) + \frac{1}{2}F_\Phi\Big)\Big[\int_{jh}^t B(\varphi_R(jh))\,\mathrm{d}W(s)\Big]\Big\|_\theta^2 \\
&\lesssim h^{2\theta}\mathbb{E}(\|\varphi_R(jh)\|_\theta^2 + 1 + \frac{1}{2}\|F_\Phi\|_\theta^2)\Big\|\int_{jh}^t B(\varphi_R(jh))\,\mathrm{d}W(s)\Big\|_\theta^2 \\
&\lesssim h^{2\theta}\mathbb{E}\Big\|\int_{jh}^t B((\|\varphi_R(jh)\|_\theta^2 + 1)\varphi_R(jh))\,\mathrm{d}W(s)\Big\|_\theta^2 \\
&\lesssim h^{2\theta}\|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\int_{jh}^t \|\varphi_R(jh)\|_\theta^4 + \|\varphi_R(jh)\|_\theta^2\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim h^{2\theta}\|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}T(\mathbb{E}\|\varphi_R(jh)\|^4_\theta+\mathbb{E}\|\varphi_R(jh)\|^2_\theta)\lesssim h^{2\theta},\\
II &\lesssim \mathbb{E}\left\|\mathrm{i}g_R'(\varphi_R(jh))\left[\int_{jh}^t B(\varphi_R(jh))\,\mathrm{d}W(s)\right]-\mathrm{i}g_R'(v_R(jh))\left[\int_{jh}^t B(v_R(jh))\,\mathrm{d}W(s)\right]\right\|_0^2\\
&=\mathbb{E}\|F(\varphi_R(jh))-F(v_R(jh))\|_0^2\lesssim \mathbb{E}\|\varphi_R(jh)-v_R(jh)\|_0^2,\\
III &\lesssim \mathbb{E}\left\|(\mathrm{i}g_R'(v_R(jh))-\mathrm{i}g_R'(S(t-jh)v_R(jh)))\left[\int_{jh}^t B(v_R(jh))\,\mathrm{d}W(s)\right]\right\|_0^2\\
&\lesssim \mathbb{E}\left\|g_R''((1-\xi(\omega))v_R(jh)+\xi(\omega)T(t-jh)v_R(jh))\left[\int_{jh}^t B(v_R(jh))\,\mathrm{d}W(s),(I-T(t-jh)v_R(jh)\right]\right\|_0^2\\
&\lesssim \mathbb{E}\left\|\int_{jh}^t B(v_R(jh))\,\mathrm{d}W(s)\right\|_\theta^2\|(I-T(t-jh)v_R(jh)\|_0^2\\
&\lesssim h^{2\theta}\mathbb{E}\left\|\int_{jh}^t B(v_R(jh))\,\mathrm{d}W(s)\right\|_\theta^2\|v_R(jh)\|_\theta^2\lesssim h^{2\theta}\mathbb{E}\left\|\int_{jh}^t B(\|v_R(jh)\|_\theta^2 v_R(jh))\,\mathrm{d}W(s)\right\|_\theta^2\\
&\lesssim h^{2\theta}\mathbb{E}\int_{jh}^t\|B(\|v_R(jh)\|_\theta^2 v_R(jh))\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\,\mathrm{d}s\\
&\lesssim h^{2\theta}\|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\int_{jh}^t\|v_R(jh)\|_\theta^4\,\mathrm{d}s\lesssim h^{2\theta}\|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}T\mathbb{E}\|v_R(jh)\|_\theta^4\lesssim h^{2\theta},\\
IV &\lesssim \mathbb{E}\left\|\left(\mathrm{i}g_R'(v_R(jh))+\frac{1}{2}F_\Phi\right)\left[\int_{jh}^t (I-T(t-s))B(v_R(jh))\,\mathrm{d}W(s)\right]\right\|_0^2\\
&\qquad\lesssim \mathbb{E}\left\|\int_{jh}^t (I-T(t-s))B(v_R(jh))\,\mathrm{d}W(s)\right\|_0^2\\
&\lesssim \mathbb{E}\int_{jh}^t\|(I-T(t-s))B(v_R(jh))\|^2_{\mathcal{L}_2(\widetilde{Y},Y)}\,\mathrm{d}s\\
&\lesssim h^{2\theta}\mathbb{E}\int_{jh}^t\|B(v_R(jh))\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\,\mathrm{d}s\\
&\lesssim h^{2\theta}\|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}T\mathbb{E}\|v_R(jh)\|_\theta^2\lesssim h^{2\theta},\\
V &\lesssim \mathbb{E}\left\|F_\Phi\int_{jh}^t B(\varphi_R(jh)-v_R(jh))\,\mathrm{d}W(s)\right\|_0^2\\
&\lesssim \|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\left\|\int_{jh}^t B(\varphi_R(jh)-v_R(jh))\,\mathrm{d}W(s)\right\|_0^2\\
&\lesssim \|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\int_{jh}^t\|B(\varphi_R(jh)-v_R(jh))\|^2_{\mathcal{L}_2(\widetilde{Y},Y)}\,\mathrm{d}s\\
&\lesssim \|\Phi\|^4_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}T\mathbb{E}\|\varphi_R(jh)-v_R(jh))\|_0^2\lesssim \mathbb{E}\|\varphi_R(jh)-v_R(jh))\|_0^2,\\
VI &\lesssim \mathbb{E}\left\|\left(\mathrm{i}g_R'(v_R(jh))+\frac{1}{2}F_\Phi\right)\left[\int_{jh}^t T(t-s)B((I-T(s-jh))v_R(jh))\,\mathrm{d}W(s)\right]\right\|_0^2
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \mathbb{E}\left\| \int_{jh}^{t} T(t-s)B((I-T(s-jh))v_R(jh))\,\mathrm{d}W(s)\right\|_0^2 \\
&\qquad \lesssim \mathbb{E}\int_{jh}^{t} \|T(t-s)B((I-T(s-jh))v_R(jh))\|^2_{\mathcal{L}_2(\widetilde{Y},Y}\,\mathrm{d}s \\
&\lesssim \mathbb{E}\int_{jh}^{t} \|B((I-T(s-jh))v_R(jh))\|^2_{\mathcal{L}_2(\widetilde{Y},Y}\,\mathrm{d}s \\
&\lesssim \|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\int_{jh}^{t} \|(I-T(s-jh))v_R(jh)\|_0^2\,\mathrm{d}s \\
&\lesssim h^{2\theta}\|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))} T\,\mathbb{E}\|v_R(jh)\|_\theta^2 \lesssim h^{2\theta}.
\end{aligned}
$$

We then return to the original term to obtain

$$
\begin{aligned}
&\mathbb{E}\left\| T(-jh)\left(\mathrm{i}g_R'(\varphi_R(jh)) + \frac{1}{2}F_\Phi\right)\left[\int_{jh}^{t} B(\varphi_R(jh))\,\mathrm{d}W(s)\right]\right. \\
&\qquad \left. - T(-t)\left(\mathrm{i}g_R'(S(t-jh)v_R(jh)) + \frac{1}{2}F_\Phi\right)\left[\int_{jh}^{t} S(t-s)B(S(s-jh)v_R(jh))\,\mathrm{d}W(s)\right]\right\|_0^2 \\
&\quad \lesssim h^{2\theta} + \mathbb{E}\|\varphi_R(jh) - v_R(jh)\|_0^2.
\end{aligned}
$$

For the difference of (5.20c) and (5.21c), we obtain

$$
\begin{aligned}
&\mathbb{E}\|T(-jh)B(\varphi_R(jh)) - T(-t)B(T(t-jh)v_R(jh)\|^2_{\mathcal{L}_2(\widetilde{Y},Y)} \\
&\quad \lesssim \mathbb{E}\|(T(-jh)-T(-t))B(\varphi_R(jh))\|^2_{\mathcal{L}_2(\widetilde{Y},Y)} \\
&\qquad\quad + \mathbb{E}\|T(-t)B(\varphi_R(jh) - T(t-jh)v_R(jh))\|^2_{\mathcal{L}_2(\widetilde{Y},Y)} \\
&\quad \lesssim h^{2\theta}\mathbb{E}\|B(\varphi_R(jh))\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))} \\
&\qquad\quad + \mathbb{E}\|B(\varphi_R(jh) - T(t-jh)v_R(jh))\|^2_{\mathcal{L}_2(\widetilde{Y},Y)} \\
&\quad \lesssim h^{2\theta}\|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\|\varphi_R(jh)\|_\theta^2 \\
&\qquad\quad + \|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\|\varphi_R(jh) - T(t-jh)v_R(jh)\|_0^2 \\
&\quad \lesssim h^{2\theta} + \mathbb{E}\|(I-T(t-jh))\varphi_R(jh)\|_0^2 + \mathbb{E}\|T(t-jh)(v_R(jh) - \varphi_R(jh))\|_0^2 \\
&\quad \lesssim h^{2\theta} + h^{2\theta}\mathbb{E}\|\varphi_R(jh)\|_\theta^2 + \mathbb{E}\|v_R(jh) - \varphi_R(jh)\|_0^2 \\
&\quad \lesssim h^{2\theta} + \mathbb{E}\|v_R(jh) - \varphi_R(jh)\|_0^2.
\end{aligned}
$$

For the difference of (5.20d) and (5.21d), we compute

$$
\begin{aligned}
&\mathbb{E}\bigg\| T(-jh)B\bigg(\int_{jh}^{t} B(\varphi_R(jh))\,\mathrm{d}W(s)\bigg) \\
&\qquad\qquad - T(-t)B\bigg(\int_{jh}^{t} T(t-s)B(T(s-jh)v_R(jh))\,\mathrm{d}W(s)\bigg)\bigg\|^2_{\mathcal{L}_2(\widetilde{Y},Y)} \\
&\quad\lesssim \mathbb{E}\bigg\|(T(-jh)-T(-t))B\bigg(\int_{jh}^{t} B(\varphi_R(jh))\,\mathrm{d}W(s)\bigg)\bigg\|^2_{\mathcal{L}_2(\widetilde{Y},Y)} \\
&\qquad + \mathbb{E}\bigg\|T(-t)B\bigg(\int_{jh}^{t}(I-T(t-s))B(\varphi_R(jh))\,\mathrm{d}W(s)\bigg)\bigg\|^2_{\mathcal{L}_2(\widetilde{Y},Y)} \\
&\qquad + \mathbb{E}\bigg\|T(-t)B\bigg(\int_{jh}^{t} T(t-s)B((I-T(s-jh)\varphi_R(jh))\,\mathrm{d}W(s)\bigg)\bigg\|^2_{\mathcal{L}_2(\widetilde{Y},Y)} \\
&\qquad + \mathbb{E}\bigg\|T(-t)B\bigg(\int_{jh}^{t} T(t-s)B(T(s-jh)(\varphi_R(jh)-v_R(jh)))\,\mathrm{d}W(s)\bigg)\bigg\|^2_{\mathcal{L}_2(\widetilde{Y},Y)} \\
&\quad =: I + II + III + IV.
\end{aligned}
$$

We now turn to the four terms separately to obtain

$$
\begin{aligned}
I &\lesssim h^{2\theta}\mathbb{E}\bigg\|B\bigg(\int_{jh}^{t} B(\varphi_R(jh))\,\mathrm{d}W(s)\bigg)\bigg\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))} \\
&\lesssim h^{2\theta}\|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\bigg\|\int_{jh}^{t} B(\varphi_R(jh))\,\mathrm{d}W(s)\bigg\|^2_\theta \\
&= h^{2\theta}\|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\int_{jh}^{t}\|B(\varphi_R(jh))\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\,\mathrm{d}s \\
&\lesssim h^{2\theta}\|\Phi\|^4_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\int_{jh}^{t}\|\varphi_R(jh)\|^2_\theta\,\mathrm{d}s \\
&\lesssim h^{2\theta}\|\Phi\|^4_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}T\,\mathbb{E}\|\varphi_R(jh)\|^2_\theta \lesssim h^{2\theta}, \\
II &\lesssim \mathbb{E}\bigg\|B\bigg(\int_{jh}^{t}(I-T(t-s))B(\varphi_R(jh))\,\mathrm{d}W(s)\bigg)\bigg\|^2_{\mathcal{L}_2(\widetilde{Y},Y)} \\
&\lesssim \|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\bigg\|\int_{jh}^{t}(I-T(t-s))B(\varphi_R(jh))\,\mathrm{d}W(s)\bigg\|^2_0 \\
&= \|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\int_{jh}^{t}\|(I-T(t-s))B(\varphi_R(jh))\|^2_{\mathcal{L}_2(\widetilde{Y},Y)}\,\mathrm{d}s \\
&\lesssim h^{2\theta}\|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\int_{jh}^{t}\|B(\varphi_R(jh))\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim h^{2\theta}\|\Phi\|^4_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}T\mathbb{E}\|\varphi_R(jh)\|^2_\theta \lesssim h^{2\theta},\\
III &\lesssim \mathbb{E}\left\|B\left(\int_{jh}^t T(t-s)B((I-T(s-jh)\varphi_R(jh))\,\mathrm{d}W(s)\right)\right\|^2_{\mathcal{L}_2(\widetilde{Y},Y)}\\
&\qquad\lesssim \|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\left\|\int_{jh}^t T(t-s)B((I-T(s-jh)\varphi_R(jh))\,\mathrm{d}W(s)\right\|^2_0\\
&\lesssim \|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\int_{jh}^t \|T(t-s)B((I-T(s-jh)\varphi_R(jh))\|^2_{\mathcal{L}_2(\widetilde{Y},Y)}\,\mathrm{d}s\\
&\lesssim \|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\int_{jh}^t \|B((I-T(s-jh)\varphi_R(jh))\|^2_{\mathcal{L}_2(\widetilde{Y},Y)}\,\mathrm{d}s\\
&\lesssim \|\Phi\|^4_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\int_{jh}^t \|(I-T(s-jh)\varphi_R(jh)\|^2_0\,\mathrm{d}s\\
&\lesssim h^{2\theta}\|\Phi\|^4_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}T\mathbb{E}\|\varphi_R(jh)\|^2_0\,\mathrm{d}s \lesssim h^{2\theta},\\
IV &\lesssim \mathbb{E}\left\|B\left(\int_{jh}^t T(t-s)B(T(s-jh)(\varphi_R(jh)-v_R(jh)))\,\mathrm{d}W(s)\right)\right\|^2_{\mathcal{L}_2(\widetilde{Y},Y)}\\
&\lesssim \|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\left\|\int_{jh}^t T(t-s)B(T(s-jh)(\varphi_R(jh)-v_R(jh)))\,\mathrm{d}W(s)\right\|^2_0\\
&\lesssim \|\Phi\|^4_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\int_{jh}^t \|T(t-s)B(T(s-jh)(\varphi_R(jh)-v_R(jh)))\|^2_{\mathcal{L}_2(\widetilde{Y},Y)}\,\mathrm{d}s\\
&\lesssim \|\Phi\|^4_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\int_{jh}^t \|B(T(s-jh)(\varphi_R(jh)-v_R(jh)))\|^2_{\mathcal{L}_2(\widetilde{Y},Y)}\,\mathrm{d}s\\
&\lesssim \|\Phi\|^4_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\int_{jh}^t \|T(s-jh)(\varphi_R(jh)-v_R(jh))\|^2_0\,\mathrm{d}s\\
&\lesssim \|\Phi\|^4_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\int_{jh}^t \|\varphi_R(jh)-v_R(jh)\|^2_0\,\mathrm{d}s\\
&\lesssim \|\Phi\|^4_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}T\mathbb{E}\|\varphi_R(jh)-v_R(jh)\|^2_0 \lesssim \mathbb{E}\|\varphi_R(jh)-v_R(jh)\|^2_0.
\end{aligned}
$$

Returning to the original term, this means that

$$
\begin{aligned}
&\mathbb{E}\Bigg\|T(-jh)B\left(\int_{jh}^t B(\varphi_R(jh))\,\mathrm{d}W(s)\right)\\
&\qquad\qquad -T(-t)B\left(\int_{jh}^t T(t-s)B(T(s-jh)v_R(jh))\,\mathrm{d}W(s)\right)\Bigg\|^2_{\mathcal{L}_2(\widetilde{Y},Y)}\\
&\qquad\lesssim h^{2\theta}+\mathbb{E}\|\varphi_R(jh)-v_R(jh)\|^2_0.
\end{aligned}
$$

For (5.20e), we estimate

$$\begin{aligned}
&\mathbb{E}\left\|-\frac{\mathrm{i}}{2}T(-jh)\int_0^1(1-\xi)g_R''\Big(\varphi_R(jh)+\xi\varphi_R^j(t)\Big)\Big[\varphi_R^j(t),\varphi_R^j(t)\Big]\,\mathrm{d}\xi\right\|_0^2\\
&\quad\lesssim \mathbb{E}\sup_{\xi\in[0,1]}\left\|g_R''\Big(\varphi_R(jh)+\xi\varphi_R^j(t)\Big)\Big[\varphi_R^j(t),\varphi_R^j(t)\Big]\right\|_0^2\lesssim \mathbb{E}\|\varphi_R^j(t)\|_\theta^4\lesssim h^2.
\end{aligned}$$

For (5.20f), we obtain

$$\begin{aligned}
&\mathbb{E}\left\|-T(-jh)\Big(\mathrm{i}g_R'(\varphi_R(jh))+\frac{1}{2}F_\Phi\Big)\Big[\varphi_R^{j,1}(t)\Big]\right\|_0^2\\
&\qquad\lesssim \mathbb{E}\left\|g_R'(\varphi_R(jh))\Big[\varphi_R^{j,1}(t)\Big]\right\|_0^2+\mathbb{E}\left\|F_\Phi\varphi_R^{j,1}(t)\right\|_0^2\\
&\qquad\lesssim \mathbb{E}\|\varphi_R^{j,1}(t)\|_0^2+\mathbb{E}\|F_\Phi\|_\theta^2\|\varphi_R^{j,1}(t)\|_0^2\\
&\qquad\lesssim \mathbb{E}\|\varphi_R^{j,1}(t)\|_\theta^2+\|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\|\varphi_R^{j,1}(t)\|_\theta^2\\
&\qquad\lesssim h^2.
\end{aligned}$$

For (5.20g), we have

$$\begin{aligned}
\mathbb{E}\left\|-\mathrm{i}T(-jh)B(\varphi_R^{j,1}(t))\right\|_{\mathcal{L}_2(\widetilde{Y},Y)}&\lesssim \mathbb{E}\|B(\varphi_R^{j,1}(t))\|_{\mathcal{L}_2(\widetilde{Y},Y)}\\
&\lesssim \mathbb{E}\|B(\varphi_R^{j,1}(t))\|_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\\
&\lesssim \|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\|\varphi_R^{j,1}(t)\|_\theta^2\\
&\lesssim h^2.
\end{aligned}$$

For (5.21e), we compute

$$\begin{aligned}
&\mathbb{E}\left\|-\frac{\mathrm{i}}{2}T(-t)\int_0^1(1-\xi)g_R''\Big(T(t-jh)v_R(jh)+\xi v_R^j(t)\Big)\Big[v_R^j(t),v_R^j(t)\Big]\,\mathrm{d}\xi\right\|_0^2\\
&\qquad\lesssim \mathbb{E}\sup_{\xi\in[0,1]}\left\|g_R''\Big(T(t-jh)v_R(jh)+\xi v_R^j(t)\Big)\Big[v_R^j(t),v_R^j(t)\Big]\right\|_0^2\\
&\qquad\lesssim \mathbb{E}\|v_R^j(t)\|_\theta^4\lesssim h^2.
\end{aligned}$$

For (5.21f), we obtain

$$\mathbb{E}\left\|-T(-t)\Big(\mathrm{i}g_R'(T(t-jh)v_R(jh))+\frac{1}{2}F_\Phi\Big)\Big[v_R^{j,1}(t)\Big]\right\|_0^2$$

$$
\begin{aligned}
&\lesssim \mathbb{E}\left\| g_R'(T(t-jh)v_R(jh))\left[v_R^{j,1}(t)\right]\right\|_0^2 + \mathbb{E}\left\| F_\Phi v_R^{j,1}(t)\right\|_0^2 \\
&\lesssim \mathbb{E}\|v_R^{j,1}(t)\|_0^2 + \mathbb{E}\|F_\Phi\|_\theta^2 \|v_R^{j,1}(t)\|_0^2 \\
&\lesssim \mathbb{E}\|v_R^{j,1}(t)\|_\theta^2 + \|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\|v_R^{j,1}(t)\|_\theta^2 \\
&\lesssim h^2.
\end{aligned}
$$

For (5.21g), we estimate

$$
\begin{aligned}
\mathbb{E}\|-\mathrm{i}T(-t)B(v_R^{j,1}(t))\|^2_{\mathcal{L}_2(\widetilde{Y},Y)} \lesssim \mathbb{E}\|B(v_R^{j,1}(t))\|^2_{\mathcal{L}_2(\widetilde{Y},Y)} &\lesssim \mathbb{E}\|B(v_R^{j,1}(t))\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))} \\
&\lesssim \|\Phi\|^2_{\mathcal{L}_2(\widetilde{Y},D(A^\theta))}\mathbb{E}\|v_R^{j,1}(t)\|_\theta^2 \lesssim h^2.
\end{aligned}
$$

Plugging all of these results into (5.22) and (5.23), taking into account the explanation before, we obtain

$$
\begin{aligned}
\mathbb{E}\max_{0\leqslant n\leqslant \frac{a}{h}}\|v_R(nh)-v_R^n\|_0^2 &\lesssim \int_0^a \sum_{j=0}^{\lfloor\frac{a}{h}\rfloor-1} \mathbb{1}_{[jh,(j+1)h]}(t)\left(h^{2\theta} + \mathbb{E}\|\varphi_R(jh)-v_R(jh)\|_0^2\right)\,\mathrm{d}t \\
&\lesssim h^{2\theta} + \int_0^a \mathbb{E}\max_{0\leqslant n\leqslant \frac{t}{h}}\|v_R(nh)-v_R^n\|_0^2\,\mathrm{d}t.
\end{aligned}
$$

By Gronwall's inequality, we finally obtain

$$
\mathbb{E}\max_{0\leqslant n\leqslant \frac{a}{h}}\|v_R(nh)-v_R^n\|_0^2 \lesssim h^{2\theta},
$$

before Jensen's inequality yields the final result. ■

5.6 Proof of Theorem 5.3

We denote the errors of the original and cut off equation by $e^n := v^n - v(nh)$ and $e_R^n := v_R^n - v_R(nh)$. We start off with the set whose probability we want to investigate. We observe that for $L, K > 0$

$$
\begin{aligned}
\{\max_{0\leqslant n\leqslant \frac{\tau_L}{h}}\|e^n\|_0 \geqslant Kh^\theta\} \subseteq &\left[\{\max_{0\leqslant n\leqslant \frac{\tau_L}{h}}\|e^n\|_0 \geqslant Kh^\theta\} \cap \{\max_{0\leqslant n\leqslant \frac{\tau_L}{h}}\|v^n\|_0 \leqslant R\} \cap \{\sup_{0\leqslant t\leqslant \tau_L}\|v(t)\|_0 \leqslant R\}\right] \\
&\cup \{\sup_{0\leqslant t\leqslant \tau_L}\|v(t)\|_0 > R\} \cup \{\max_{0\leqslant n\leqslant \frac{\tau_L}{h}}\|v^n\|_0 > R\}
\end{aligned}
$$

$$\subseteq \{ \max_{0\leqslant n\leqslant \frac{\tau_L}{h}} \|e_R^n\|_0 \geqslant Kh^\theta\} \cup \{ \sup_{0\leqslant t\leqslant \tau_L} \|v(t)\|_0 > R\} \cup \{ \max_{0\leqslant n\leqslant \frac{\tau_L}{h}} \|v^n\|_0 > R\}. \tag{5.24}$$

While the first inclusion is obvious, the second one follows from the fact that for $\max_{0\leqslant n\leqslant \frac{\tau_L}{h}} \|v^n\|_0 \leqslant R$ and $\sup_{0\leqslant n\leqslant \tau_L} \|v(t)\|_0 \leqslant R$, respectively, we have $v^n = v_R^n$ for $0 \leqslant n \leqslant \frac{\tau_L}{h}$ as well as $v_R(t) = v(t)$ for $0 \leqslant n \leqslant \tau_L$: If $\max_{0\leqslant n\leqslant \frac{\tau_L}{h}} \|v^n\|_0 \leqslant R$, we obtain

$$\begin{aligned} v^n &= T(h)\exp(-\mathrm{i}[hV(|v^{n-1}|) + \Phi W(nh) - \Phi W((n-1)h)])v^{n-1} \\ &= T(h)\exp(-\mathrm{i}[h\underbrace{\theta_R(v^{n-1})}_{=1}V(|v^{n-1}|) + \Phi W(nh) - \Phi W((n-1)h)])v^{n-1}, \end{aligned}$$

which means that v_n obeys the same equation as v_R^n. Since both algorithms start with v_0, we have $v^n = v_R^n$ for $0 \leqslant n \leqslant \frac{\tau_L}{h}$. If $\sup_{0\leqslant n\leqslant \tau_L} \|v(t)\|_0 \leqslant R$, we see that

$$\begin{aligned} v(t) &= T(t)v_0 - \mathrm{i}\int_0^t T(t-s)g(v(s))\,\mathrm{d}s \\ &\quad - \frac{1}{2}\int_0^t T(t-s)F_\Phi v(s)\,\mathrm{d}s - \mathrm{i}\int_0^t T(t-s)B(v(s))\,\mathrm{d}W(s) \\ &= T(t)v_0 - \mathrm{i}\int_0^t T(t-s)\underbrace{\theta_R(v(s))}_{=1}g(v(s))\,\mathrm{d}s \\ &\quad - \frac{1}{2}\int_0^t T(t-s)F_\Phi v(s)\,\mathrm{d}s - \mathrm{i}\int_0^t T(t-s)B(v(s))\,\mathrm{d}W(s), \end{aligned}$$

which means that $v(t)$ obeys the same equation as $v_R(t)$ ans therefore $v_R(t) = v(t)$ for $0 \leqslant n \leqslant \tau_L$.

Next, we take a look at the probability of the three terms in (5.24). For the first term, by Chebychev's inequality and proposition 5.2, using $\tau_L \leqslant T$, we obtain

$$\mathbb{P}(\max_{0\leqslant n\leqslant \frac{\tau_L}{h}} \|e_R^n\|_0 \geqslant Kh^\theta) \leqslant \frac{1}{Kh^\theta}\mathbb{E}\max_{0\leqslant n\leqslant \frac{\tau_L}{h}} \|e_R^n\|_0 \leqslant \frac{1}{Kh^\theta}\mathbb{E}\max_{0\leqslant n\leqslant \frac{T}{h}} \|e_R^n\|_0 \leqslant \frac{C_R}{K}. \tag{5.25}$$

For the second term, taking $R > L$, by the definition of the stopping time, we have

$$\mathbb{P}(\sup_{0\leqslant t\leqslant \tau_L} \|v(t)\|_0 > R) = \mathbb{P}(\tau_R < \tau_L) = 0. \tag{5.26}$$

Concerning the last term, we want to show that

$$\mathbb{P}(\max_{0\leqslant n\leqslant \frac{\tau_L}{h}} \|v^n\|_0 > R) \xrightarrow{R\to\infty} 0, \tag{5.27}$$

uniformly in $h \in (0,T]$. To this end, we define

$$f_R(h) := \mathbb{P}(\max_{0\leqslant n\leqslant \frac{\tau_L}{h}} \|v^n\|_0 > R) \qquad \forall h \in (0,T].$$

This makes f_R a continuous function (amongst others things because of the continuity of the Brownian motion) which is monotonously falling. Moreover, we obtain the pointwise convergence

$$f_R(h) \xrightarrow{R\to\infty} \mathbb{P}(\max_{0\leqslant n\leqslant \frac{\tau_L}{h}} \|v^n\|_0 = \infty) = 0$$

by the σ-continuity, since for fixed h, we take the maximum over finitely many finite values (note that $\tau_L \leqslant T$ fixed). We now assume that $\sup_{h\in(0,T]} f_R(h)$ does not converge to zero for $R \to \infty$. Since we are interested in the limit, we impose the restriction $R > L+1$. We infer that

$$\exists \varepsilon > 0 \ \forall R > L+1 \ \exists h_R \in (0,T] : f_R(h_R) > \varepsilon.$$

This yields

$$\mathbb{P}(\max_{0\leqslant n\leqslant \frac{\tau_L}{h_R}} \|e^n\|_0 \geqslant 1) = \mathbb{P}(\max_{0\leqslant n\leqslant \frac{\tau_L}{h_R}} \|e^n\|_0 \geqslant 1) + \mathbb{P}(\sup_{0\leqslant t\leqslant \tau_L} \|v(t)\|_0 > R-1) \geqslant f_R(h_R) > \varepsilon \tag{5.28}$$

for all $R > L+1$. On the other hand, for $h \in (0,T]$, we have

$$\{\max_{0\leqslant n\leqslant \frac{\tau_L}{h}} \|e^n\|_0 \geqslant 1\} \subseteq \{\max_{0\leqslant n\leqslant \frac{\tau_L}{h}} \|e^n_{L+1}\|_0 \geqslant 1\},$$

since if $\max_{0\leqslant n\leqslant \frac{\tau_L}{h}} \|e^n\|_0 \geqslant 1$, there exists $n_1 := \min\{n \leqslant \frac{\tau_L}{h}, \|e^n\|_0 \geqslant 1\}$. Moreover, by the definition of τ_L, we have $\|v(t)\|_0 \leqslant L$ for $t \leqslant n_1 h \leqslant \tau_L$ and therefore $\|v^n\|_0 \leqslant \|v(nh)\|_0 + \|e^n\|_0 \leqslant L+1$ for $n < n_1$. As in the proof for the inclusion in (5.24), we conclude $e^n = e^n_{L+1}$ for $n \leqslant n_1$. Setting $n = n_1$, we obtain $\max_{0\leqslant n\leqslant \frac{\tau_L}{h}} \|e^n_{L+1}\|_0 \geqslant 1$.

Therefore, we arrive at

$$\mathbb{P}(\max_{0\leqslant n\leqslant \frac{\tau_L}{h_R}} \|e^n\|_0 \geqslant 1) \leqslant \mathbb{P}(\max_{0\leqslant n\leqslant \frac{\tau_L}{h_R}} \|e^n_{L+1}\|_0 \geqslant 1) \leqslant \mathbb{E} \max_{0\leqslant n\leqslant \frac{\tau_L}{h_R}} \|e^n_{L+1}\|_0 \leqslant C_{L+1} h_R^\theta, \quad (5.29)$$

where we use Proposition 5.2. Comparing (5.29) to (5.28), we obtain $h_R \in [(\frac{\varepsilon}{C_{L+1}})^{1/\theta}, T]$, which means that $\sup_{h\in[\frac{\varepsilon}{C_{L+1}},T]} f_R(h)$ does not converge to zero for $R \to \infty$ in contradiction to the uniform convergence of monotonously falling, pointwise converging function sequences. Hence, the assumption is wrong and (5.27) holds.

To end the proof, we return to (5.24). Let $\varepsilon > 0$ and take $R > L$ to be large enough for $\mathbb{P}(\max_{0\leqslant n\leqslant \frac{\tau_L}{h}} \|v^n\|_0 > R)$ to be smaller than $\frac{\varepsilon}{2}$ (possible due to (5.27)). Afterwards, take K large enough such that $\mathbb{P}(\max_{0\leqslant n\leqslant \frac{\tau_L}{h}} \|e^n_R\|_0 \geqslant Kh^\theta)$ is smaller than $\frac{\varepsilon}{2}$ (possible due to (5.25)). Finally, since $R > L$, using (5.26), (5.24) yields

$$\mathbb{P}(\max_{0\leqslant n\leqslant \frac{\tau_L}{h}} \|e^n\|_0 \geqslant Kh^\theta) \leqslant \mathbb{P}(\max_{0\leqslant n\leqslant \frac{\tau_L}{h}} \|v^n\|_0 > R) + \mathbb{P}(\max_{0\leqslant n\leqslant \frac{\tau_L}{h}} \|e^n_R\|_0 \geqslant Kh^\theta) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for K large enough.

Remark 5.9

As mentionned in the introduction, a similar result has been stated in [Liu13b]. In his much shorter proof of Proposition 5.2, Liu does use neither derivatives of the nonlinearity nor reiteration of the variation of constants formula, but at one point, he estimates terms of the form

$$\mathbb{E} \sup_{0\leqslant s\leqslant t} \left\| \int_0^s (S(s-\tau) - I) B(u(\tau)) \, \mathrm{d}W(\tau) \right\|_Y^2$$

by the Burkholder inequality (see [BP99][Theorem 7.3]), which is not possible due to the integrand depending on s. To get around this difficulty, we chose a significantly different approach in our proof. Moreover, the proof of Theorem 5.3 includes some technical mistakes which can be corrected. For one of them, the stopping time has to be bounded by an arbitrary T (which Liu does not demand), since otherwise, the use of the Proposition 5.2 is incorrect.

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