On a Nonlinear Helmholtz System

(Über ein nichtlineares Helmholtz-System)

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Dominic Scheider

Referent:	Dr. Rainer Mandel
1. Korreferent:	Prof. Dr. Wolfgang Reichel
2. Korreferent:	Prof. Dr. Tobias Weth

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Contents

Chapter 1.

Introduction

Abstract

This thesis aims at developing and applying methods to solve coupled systems of nonlinear Helmholtz equations. A prototypical example of such a system might be

$$\begin{cases} -\Delta u - \mu u = u(u^2 + b\, v^2) & \text{on } \mathbb{R}^3, \\ -\Delta v - \nu v = v(v^2 + b\, u^2) & \text{on } \mathbb{R}^3 \end{cases} \tag{H}$$

where $\mu, \nu > 0$ are positive constants and $b \in \mathbb{R}$ denotes the coupling. Under suitable additional assumptions, the results presented in the following chapters are concerned with

(I) the existence of real-valued solutions $u, v : \mathbb{R}^3 \to \mathbb{R}$ of systems as (H) which are localized in the sense that $u(x), v(x) \to 0$ as $|x| \to \infty$, and

(II) the formulation of criteria whether or not these solutions are fully nontrivial, that is, whether or not they satisfy $u \neq 0$ and $v \neq 0$.

To the best of the author's knowledge, the results in this thesis are the first ones to address the aforementioned questions for nonlinear Helmholtz systems, which is why, apart from the final part of this thesis, the focus lies on two-component systems as the one above. In Chapter 2, a variational ansatz will be introduced which, at least partially, answers the questions (I) and (II) for a nonlinear Helmholtz system similar to (H) but in space dimension $N \geq 2$, with a nonlinearity of the power $p \in \left(\frac{2(N+1)}{N-1}, \frac{2N}{N-2}\right)$ and under the restriction $0 \leq b \leq p-1$. Chapter 3 provides an existence result for fully nontrivial and radially symmetric solutions of the cubic system (H) based on bifurcation theory and a thorough analysis of the far field of the solutions, that is, of the form of $|x|u(x)$ resp. $|x|v(x)$ in the limit $|x| \to \infty$. As a further application of the techniques developed in Chapter 3, time-periodic solutions of the wave-type equations

$$\partial_t^2 U - \Delta U \mp U = U^3 \quad \text{on } \mathbb{R} \times \mathbb{R}^3 \tag{W}$$

will be constructed in the final Chapter 4. In particular, the analysis in this chapter involves the bifurcation methods developed in the previous one and thus provides an exemplary

demonstration of the transfer of results for the two-component Helmholtz system (H) to an infinite one, see equation (4.7).

On the notion of "Helmholtz" and "Schrödinger" case

In contrast to the Helmholtz case, much more knowledge exists about coupled systems of nonlinear Schrödinger equations, prototypically

$$\begin{cases} -\Delta u + \mu u = u(u^2 + b\, v^2) & \text{on } \mathbb{R}^3, \\ -\Delta v + \nu v = v(v^2 + b\, u^2) & \text{on } \mathbb{R}^3 \end{cases} \tag{S}$$

where again $\mu, \nu > 0$. It should be pointed out that the main difference between the systems (H) and (S) are the signs appearing in the linear part. As a consequence, in the Schrödinger case, 0 belongs to the resolvent set of suitable self-adjoint realizations of the linear differential operators $-\Delta + \mu, -\Delta + \nu$ on $L^2(\mathbb{R}^3)$ whereas 0 is an element of the essential spectrum of $-\Delta - \mu, -\Delta - \nu$ in the Helmholtz case.
The Schrödinger system (S) can be analyzed using a functional analytic setting in the Sobolev space $H^1(\mathbb{R}^3)$. A direct variational approach in $H^1(\mathbb{R}^3)$ is possible via the functional

$$(u,v) \mapsto \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \mu u^2 + |\nabla v|^2 + \nu v^2 \, \mathrm{d}x - \frac{1}{4} \int_{\mathbb{R}^3} u^4 + 2b\, u^2 v^2 + v^4 \, \mathrm{d}x$$

(where $u, v \in H^1(\mathbb{R}^3)$) the quadratic part of which is positive definite since $\mu, \nu > 0$. A suitable reformulation of (S) for bifurcation theory can be established when applying the resolvent operators $(-\Delta + \mu)^{-1}, (-\Delta + \nu)^{-1} : H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$. This provides powerful tools for the study of the system (S) and numerous variants. When working with the Helmholtz system (H), on the other hand, the corresponding functional is strongly indefinite and resolvent operators as above do not exist due to the negative signs preceding μ and ν. Thus the introduction of the technical framework tends to be more complex and restrictive, as will be explained in detail in the following chapters. In contrast to the Schrödinger case, where solutions of (S) typically decay exponentially and have, in the radial case, profiles with a finite number of nodes, it will be shown that solutions of (H) are oscillating with power-type decay, $u(x), v(x) = O\left(|x|^{-1}\right)$ as $|x| \to \infty$; in particular, such solutions do not belong to $H^1(\mathbb{R}^3)$.
Among many others, Schrödinger systems such as (S) have been studied by Ambrosetti and Colorado [7] as well as Maia, Montefusco and Pellacci [47] using variational methods and by Bartsch, Wang and Wei [9] as well as Bartsch, Dancer and Wang [8] using bifurcation techniques. More detailed information about these results will be provided in the introductory sections of Chapters 2 and 3, respectively, establishing relations to the new findings on Helmholtz systems.

The examples above focus on systems with constant potentials, following the knowledge and methods available for Helmholtz equations. More generally, however, when considering an operator on a suitable subspace of $L^2(\mathbb{R}^N)$, $N \in \mathbb{N}$, given by $-\Delta + V(x)$ for some appropriate potential $V : \mathbb{R}^N \to \mathbb{R}$, we will speak of the Schrödinger case if $0 \notin \sigma(-\Delta + V(x))$ and of the Helmholtz case else. In the special case of constant potentials (and $N = 3$), this matches the above nomenclature.

Some remarks...

- ▷ ... concerning bounded domains:
  In the case of bounded domains $\Omega \subseteq \mathbb{R}^3$, the contrast between the Schrödinger and the Helmholtz case is not as sharp as in an unbounded setting and the mathematical challenges are of a different nature, which is why it will not be discussed in this

thesis. For instance, there are no obvious analogues of far field patterns and slow decay rates, which are characteristic features of (full-space) solutions of (H). On a formal level, a major difference is that the spectrum of $-\Delta - \lambda$ on a bounded domain is discrete whereas, on $\mathbb{R}^3$, 0 belongs to the essential spectrum. If the underlying domain is bounded, solutions can still be expected to belong to the Hilbert space $H^1(\Omega)$ (possibly involving suitable boundary conditions), and a direct variational approach using linking techniques might be possible. Indeed, in [27], Theorem 1.1, Evéquoz and Weth apply the classical Linking Theorem in order to solve a nonlinear Helmholtz equation on a ball as a part of a full-space problem with a compactly supported nonlinearity, see also the final paragraph of Chapter 1.4.1.

▷ ... concerning exterior and unbounded domains:
All results of this thesis will be obtained in the full-space case. The aim is not to conceal how the above-mentioned challenges such as slow decay rates and the non-existence of Hilbert space resolvents are dealt with by the discussion of boundary conditions on unbounded or exterior domains. Still, it might be an interesting aim for future studies to generalize the results of the following chapters to such settings.

▷ ... concerning the space dimension:
Even though generalizations of the system (H) to other space dimensions will be discussed, the case $N = 1$ will not appear. The reason is that, in one-dimensional settings, the behavior of solutions as $|x| \to \infty$ is expected to be qualitatively different; more precisely, solutions are in general not localized. This is motivated in the decoupled case, $b = 0$, by Theorem 1.2 of [54], quoted as Theorem 1.13 below. It shows that (radially symmetric) solutions of a more general class of nonlinear Helmholtz equations decay as $|x| \to \infty$ if and only if $N \geq 2$.

Outline of the introductory chapter

In this introductory chapter, we review results for (scalar) nonlinear Helmholtz resp. Schrödinger equations of the prototypical form

$$-\Delta u \pm \lambda u = Q(x)\,|u|^{p-2}u \quad \text{on } \mathbb{R}^N.$$

First, a short overview of classical results in the Schrödinger case is presented in Section 1.1. Corresponding results in the Helmholtz case will be given in Section 1.4. The intermediate chapters all concern the Helmholtz case. A short motivation in Part 1.2 indicates that Helmholtz equations naturally arise in various fields of physics mostly concerned with wave phenomena. After that, Part 1.3 introduces the technical methods from the theory of linear Helmholtz equations required in Part 1.4 when discussing nonlinear Helmholtz equations on the full space $\mathbb{R}^N$. Roughly speaking, solutions of the homogeneous Helmholtz equation $-\Delta u - \lambda u = 0$ on $\mathbb{R}^N$ will be characterized, and the construction of resolvent-type operators $(-\Delta - \lambda)^{-1}$ in suitable topologies will be presented. These results will frequently be referred to throughout this thesis, which is why they are given at the very beginning and in some detail. Finally, in Part 1.5 the general organization of the following chapters is presented.

For the notations and conventions that will be introduced and used throughout the introduction but also the subsequent chapters, we refer to the short overview at the end of this thesis.

1.1 A Short Review on the Schrödinger Case

When compared with the literature on nonlinear Helmholtz equations, classical results on nonlinear Schrödinger equations have been available much longer and in much more general form. As a particularly well-known example, we quote Theorem 1 in [14] and Theorem 6 in [15] by Berestycki and Lions.

Theorem 1.1 (Berestycki, Lions 1983).

Let $N \geq 3$, $\lambda > 0$ and $h : \mathbb{R} \to \mathbb{R}$ be continuous and odd with $\int_0^s h(\sigma)\,\mathrm{d}\sigma > \frac{\lambda}{2}s^2$ for some $s > 0$. Additionally, assume that

$$-\infty < \liminf_{s\to 0+} \frac{h(s)}{s} \leq \limsup_{s\to 0+} \frac{h(s)}{s} \leq 0 \quad \text{and} \quad -\infty \leq \limsup_{s\to\infty} \frac{h(s)}{s^{\frac{2N}{N-2}-1}} \leq 0.$$

Then there exists a sequence of distinct radially symmetric functions $u_k \in H^1(\mathbb{R}^N)$, $k \in \mathbb{N}_0$, of class C^2 which satisfy the Schrödinger equation

$$-\Delta u_k + \lambda u_k = h(u_k) \qquad \text{on } \mathbb{R}^N$$

and with the property that the radial profiles $u_k(x), u_k'(x), u_k''(x)$ decay exponentially as $|x| \to \infty$. Moreover, u_0 is positive and radially nonincreasing.

The positive solution u_0 is referred to as a ground state solution, whereas the solutions u_k, $k \in \mathbb{N}$, are said to be bound states. In particular, Theorem 1.1 covers the special cases

$$\begin{aligned} h(u) &= |u|^{p-2}u && \text{for } 2 < p < \frac{2N}{N-2}, \\ h(u) &= |u|^{p-2}u - \gamma|u|^{q-2}u && \text{for } \gamma > 0, \quad 2 < q < p < \frac{2N}{N-2}. \end{aligned}$$

The above result has been slightly generalized by Struwe to include the zero-mass case $\lambda = 0$, see [70], Theorem 3.1. For $\lambda > 0$ and $N = 2$, there is an existence result for ground and (infinitely many) bound states by Berestycki, Gallouët and Kavian in [13] under slightly different assumptions.

Berestycki and Lions prove the above result by means of variational methods in the space $H^1(\mathbb{R}^N)$, more precisely constraint minimization techniques. Working on $\mathbb{R}^N$, the characteristic loss of compactness is overcome using uniform decay properties of radially symmetric functions essentially due to Strauss, see the Radial Lemma 1 in [68]. Berestycki and Lions also point to earlier existence results by Strauss, see [68], Theorem 2 concerning ground states and [68], Theorem 5 for a multiplicity result under stronger assumptions on the nonlinearity, respectively.
The non-autonomous case, especially when assuming a non-radial dependence of the coefficients on the space variable x, therefore requires different techniques. We focus on the Pohozaev problem

$$-\Delta u + u = Q(x)\,|u|^{p-2}u \qquad \text{on } \mathbb{R}^N \tag{1.1}$$

and give a short and, inevitably, incomplete overview of available results. Early results in the case of possibly non-radial Q were achieved by Ding and Ni in [22]. They proved in [22], Corollary 3.19 the existence of a positive ground state of (1.1) under the assumption

that $\lim_{|x|\to\infty} Q(x) = \inf_{\mathbb{R}^N} Q \geq 0$ and $2 < p < \frac{2N}{N-2}, N \geq 3$. Their method consists of solving a similar problem on a ball and then controlling the limit as the radius increases. Working originally in a more general framework, Lions obtained similar results for (1.1) formulated as a minimization problem, see [46], Theorem I.2 and Remarks I.5, I.6. The central tool is Lions' principle of concentration-compactness, Lemma I.1 of [45], which allows to deal with a possible loss of compactness on $\mathbb{R}^N$ also in non-radial settings. In situations with $Q(x) \to \bar{Q}$ as $|x| \to \infty$, Lions verified the existence of ground state solutions of (1.1) and related problems by proving relative compactness of minimizing sequences up to translations. A slightly adapted version of Lions' Lemma will also be applied in the Helmholtz case, as will be explained in Section 1.4.1.
So far, the discussion of the non-autonomous problem (1.1) has focused on the existence of positive ground state solutions. Under additional symmetry assumptions, the existence of infinitely many bound states was proved e.g. by Bartsch and Willem, see Theorems 2.1 of [11], [10]. Working with less restrictive conditions, Clapp and Weth showed the existence of finitely many bound states in [18], Theorem 1.
Let us remark only briefly that some of the above-mentioned results also apply to a non-constant potential $V(x)$, that is, to equations of the form

$$-\Delta u + V(x)u = Q(x)\,|u|^{p-2}u \qquad \text{on } \mathbb{R}^N. \tag{1.2}$$

The analysis is mostly done under the assumption that $V(x) \geq V_0 > 0$. This causes the spectrum of $-\Delta + V(x)$ as an operator on $H^1(\mathbb{R}^N)$ to be a subset of the interval $[V_0, \infty) \subseteq (0, \infty)$, which in turn provides definiteness properties of the associated functional. In the case of periodic coefficients $V(x)$ and $Q(x)$, where the spectrum $\sigma(-\Delta+V(x))$ typically has a band-gap structure, the Schrödinger case is also realized if $0 \notin \sigma(-\Delta+V(x))$ but belongs to a band gap, leading to an indefinite functional. The existence of ground state solutions (and their exponential decay) was established e.g. by Pankov [61], Theorem 1.1 as well as by Szulkin and Weth [71], Theorem 1.1 using constrained minimization on a generalized Nehari set and suitable concentration-compactness techniques. Their techniques even admit a more general right-hand side satisfying certain growth assumptions.

These results confirm the announced characteristic features of Schrödinger-type equations, i.e. the occurrence of ground and bound state solutions in $H^1(\mathbb{R}^N)$ which decay exponentially and do not oscillate. In the following subchapters, we will return to the Helmholtz case. In particular, we explain the additional technical challenges and present methods to overcome these, most of which have been found only recently. Finally, in Part 1.4, we provide an overview of available results corresponding to those for the Schrödinger case mentioned above. We will see in particular that solutions of Helmholtz-type problems typically oscillate, have slow, power-type decay and do not belong to $H^1(\mathbb{R}^N)$. However, it will turn out that a multitude of questions which has been answered in the Schrödinger case is still open in the Helmholtz case; from the viewpoint of this thesis, this includes in particular results about the nonlinear Helmholtz system (H).

1.2 The Helmholtz Equation in Physics

Helmholtz equations appear in various fields of physics. Typically, they arise from more complex, time-dependent equations when inserting special ansatz functions. What follows is a brief and exemplary overview of this variety, indicating in what way the Helmholtz equation can be obtained in systems governed by (classical) wave equations and by quantum mechanical Schrödinger equations. As will be mentioned, more involved applications are provided at other parts of the thesis. In order to emphasize the fundamental role of the Helmholtz equation in physics, the guiding reference for this section are the books of

Nolting's *Basic Course: Theoretical Physics.*

The wave equation can be derived in the context of mechanics where it models e.g. the propagation of sound as well as in the framework of classical electrodynamics, which is governed by Maxwell's equations for the electric and magnetic fields. These fields can be expressed in terms of a "scalar" and a "vector" potential, and assuming the so-called Lorentz gauge condition, one finally finds a system of four equations of the form

$$\partial_t^2 a(t,x) - \Delta a(t,x) = j(t,x) \qquad (t \in \mathbb{R},\ x \in \mathbb{R}^3)$$

where $j(t,x)$ is a source term containing current resp. charge densities and $a(t,x)$ is a component of the vector resp. the scalar potential, cf. [58], equations (4.38), (4.39) with all constants set to 1. One standard problem in electrodynamics is the analysis of the fields generated by temporally oscillating sources, i.e. $j(t,x) = j_0 \mathrm{e}^{\mathrm{i}\omega t}$ for some $j_0, \omega \in \mathbb{R} \setminus \{0\}$. Letting $a(t,x) = a_0(x)\mathrm{e}^{\mathrm{i}\omega t}$, see [58], equations (4.444) and (4.446), one finds the Helmholtz equation as a reduced wave equation

$$-\Delta a_0(x) - \omega^2 a_0(x) = j_0(x) \qquad (x \in \mathbb{R}^3).$$

In [58], equation (4.447), this is solved by a convolution formula

$$a_0(x) = \int_{\mathbb{R}^3} \frac{\mathrm{e}^{\mathrm{i}\omega|x-y|}}{4\pi|x-y|} j_0(y)\,\mathrm{d}y \qquad (x \in \mathbb{R}^3),$$

which has been derived in [58], Chapter 4.5.1 using Fourier techniques and noting that solutions are not unique. We will recover similar strategies and formulas later in a rigorous mathematical setting, e.g. in Theorem 1.9 (ii). The next step for physicists is then to analyze the field a_0 radiated by the source j_0; the leading-order term as $|x| \to \infty$ is given by

$$a_0(x) \approx p_0 \cdot \frac{\mathrm{e}^{\mathrm{i}\omega|x|}}{4\pi|x|} \qquad (|x| \to \infty)$$

where p_0 denotes the so-called dipole moment of the charge distribution j_0, c.f. [58], equation (4.456). These leading-order terms of the far field (or: radiation zone) will be of high importance also in our analysis e.g. in Chapter 3, see the central auxiliary results in Propositions 3.13 and 3.18. It should be mentioned that, since the problem is linear and the coefficients of the wave equation are real, one can easily pass to real-valued solutions by simply taking the real part in all above identities; this is different in the presence of a nonlinearity. Nonlinear wave equations arise in models where the source j_0 resp. j depends (in a nonlinear way) on the field a_0 resp. a. In Chapter 4 we will study a cubic wave-type equation using a polychromatic ansatz

$$a(t,x) = \sum_{k \in \mathbb{Z}} a_k(x)\,\mathrm{e}^{\mathrm{i}k\omega t} \qquad (t \in \mathbb{R},\ x \in \mathbb{R}^3),$$

which is much more general than the monochromatic version $a(t,x) = a_0(x)\,\mathrm{e}^{\mathrm{i}\omega t}$ above. The nonlinearity causes a mixing of the modes a_k, which will lead to a coupled system of an infinite number of Helmholtz equations with a variety of solutions. We will make sure to obtain real-valued solutions by demanding $a_{-k} = a_k$.

Helmholtz equations also arise in quantum mechanics. In a well-known interpretation of this theory, the state of a physical system is described by its wave function $\psi(t,x)$, which is not a quantity accessible by experiment itself; its square $|\psi(t,x)|^2$, however, is usually interpreted as a probability density and can thus be related to measurements. The wave function in turn satisfies the time-dependent Schrödinger equation

$$\mathrm{i}\partial_t\psi(t,x) = H[\psi(t,x)],$$

see [59], equation (2.18). Here ψ is an element of a suitable Hilbert space of complex-valued functions, and H is the Hamilton operator; frequently, $H[\psi] = -\Delta\psi + V(x)\psi$ with some potential $V(x)$ and, again, all constants set to 1. The Schrödinger equation is not a consequence but rather an axiom in quantum theory. Unlike in all other chapters, the term "Schrödinger" equation refers here to this fundamental equation from quantum mechanics and not to the "Schrödinger case" which we set in contrast to the "Helmholtz case" in the previous chapter; we will establish a connection below.

If the Hamilton operator does not contain time-dependent coefficients or derivatives with respect to time, then a separation ansatz with $\psi(t,x) = \mathrm{e}^{-\mathrm{i}Et}\,\varphi(x)$ formally (not going into details about a suitable domain of H) yields the stationary Schrödinger equation

$$E\,\varphi(x) = H[\varphi(x)].$$

It is an eigenvalue equation for the Hamilton operator, and the eigenvalue parameter E is interpreted as the energy of the corresponding eigenstate of the system, cf. equations (2.15), (2.17) in [59]. In particular, for $H[\varphi] = -\Delta\varphi + V(x)\varphi$ and as a full-space problem, the equation becomes

$$-\Delta\varphi + (V(x) - E)\varphi = 0 \qquad \text{on } \mathbb{R}^3. \tag{1.3}$$

Here, in principle, φ might be a real-valued function again, which is the case we will study. In quantum mechanical literature, one now distinguishes regions of space with $V(x) - E > 0$, which are said to be "classically forbidden" (since the total energy E is below the potential) and where, typically, the density $|\varphi|^2$ decays exponentially. Regions with $V(x) - E < 0$ are "classically allowed", and the density $|\varphi|^2$ is expected to be oscillatory in nature. For details in a one-dimensional setting, cf. [59], pp. 238–240, and for a three-dimensional radially symmetric setting, cf. [60], pp.92–98. In case of a constant potential $V(x) \equiv V_0$, the author observes that there are only discrete values E with $V_0 - E > 0$ such that the equation (1.3) has a bounded solution; these discrete ground and bound states have strong decay. On the contrary, for any energy E with $V_0 - E < 0$, equation (1.3) possesses an (oscillating) solution, see e.g. the explanation after [60], equation (6.24).
Thus for a constant potential, the "classically forbidden" case for the stationary Schrödinger equation (1.3) matches the "Schrödinger case" introduced for nonlinear equations as (1.1), both being characterized by the occurrence of exponentially decaying discrete ground and bound states. The "classically allowed" case, on the other hand, corresponds to the "Helmholtz case" with its continuum of solutions lacking strong localization.
Nonlinear problems such as (1.1) are obtained e.g. when describing self-interactions using nonlinear Hamiltonians. In the following subchapter, the quantum mechanical description of a two-component system will be sketched which, under some idealized assumptions, leads to the cubic systems (S) or (H).

1.2.1 The Nonlinear Helmholtz System (H)

At the very beginning of this introductory chapter, the study of the nonlinear Helmholtz system (H) was motivated from a mathematical point of view by the fact that, to the author's best knowledge and in contrast to the corresponding Schrödinger system (S), it has not been investigated so far. Regarding applications, it would of course be possible to establish toy models described by (H) based on the derivation of the linear Helmholtz equation in the preceding section. However, this subchapter will provide a sketch of a more involved physical problem leading to the nonlinear Helmholtz system (H). For more details, we refer to the review [19].

The authors of [19] consider Bose-Einstein condensation of atomic gases in a trapping potential. Roughly speaking, Bose-Einstein condensation describes a quantum mechanical state where a large number (meaning about 10^{23}) of identical particles occupies the ground state of a system at temperatures differing significantly from zero. Experimental evidence is presented in Chapter 14 of [19].

The theoretical description of trapped Bose-Einstein condensates is outlined in Chapters 2.6, 9.1, 9.2 of [19]. The most important aspect is that, even though a Bose-Einstein condensate consists of a large number of particles, it can be described by one single quantum mechanical quantity φ. The authors outline two quantum statistical models based on different averaging and approximation procedures. In one case, φ denotes the so-called order parameter of the condensate; in the other case, φ is a quantity called the coherent field, see Chapters 9.2 and 9.1 of [19], respectively. In both models, at low temperatures, φ is governed by the same differential equation, see e.g. (2.45), (9.5), (9.10) in [19]. We adopt the term the "Gross-Pitaevskii equation" as in Chapter 9.2; it reads

$$\mathrm{i}\partial_t\varphi = \left(-\frac{1}{2m}\,\Delta + U(t,x) + N\int_{\mathbb{R}^3}\Phi(t,x-y)|\varphi(t,y)|^2\,\mathrm{d}y\right)\varphi.$$

Here $N \in \mathbb{N}$ is the number of particles, and $m > 0$ denotes their mass. U describes the trapping potential, which can prototypically be chosen as a harmonic oscillator $U(x) = V_0 + V_1 \cdot |x|^2$. Finally, Φ models the strength of the atomic interactions which are repulsive if $\Phi > 0$ and attractive if $\Phi < 0$. A common approximation are Fermi point interactions, Φ acting as a Dirac delta distribution $\Phi = g \cdot \delta_0$ with interaction strength $g \in \mathbb{R}$.

For a system of various trapped Bose-Einstein condensates, the authors derive a system of coupled Gross-Pitaevskii equations, see [19], equations (14.8), (14.9). Labeling the (finite number of) components by some index j, they obtain the coupled system

$$\mathrm{i}\partial_t\varphi_j = \left(-\frac{1}{2m_j}\,\Delta + U_j(t,x) + \sum_i N_i\int_{\mathbb{R}^3}\Phi_{ij}(t,x-y)|\varphi_i(t,y)|^2\,\mathrm{d}y\right)\varphi_j.$$

We consider a system of two trapped Bose-Einstein condensates with, for simplicity, $N_1 = N_2 = 1$ and $2m_1 = 2m_2 = 1$ and make the following approximations:

- ▷ point interactions, that is, $\Phi_{11} = \Phi_{22} = -a\cdot\delta_0$ and $\Phi_{12} = \Phi_{21} = -b\cdot\delta_0$ with strength of interaction modeled by $a, b \in \mathbb{R}$ (cf. [19], (14.11)),
- ▷ a stationary trapping potential varying only at large scales, i.e. we assume $U_j(t,x) \equiv V_0$ for some $V_0 \in \mathbb{R}$ (cf. [19], explanation between (14.17), (14.18)),
- ▷ time-periodic solutions of the form $\varphi_1(t,x) = \mathrm{e}^{-\mathrm{i}E_1 t}u(x)$ and $\varphi_2(t,x) = \mathrm{e}^{-\mathrm{i}E_2 t}v(x)$ (cf. [19], (14.14) for stationary $U(x)$) where $E_1, E_2 > V_0$ and $u, v : \mathbb{R}^3 \to \mathbb{R}$.

Then one line of calculation yields the stationary nonlinear Helmholtz system

$$\begin{cases} -\Delta u - (E_1 - V_0)u = (a\,u^2 + b\,v^2)\,u & \text{on } \mathbb{R}^3, \\ -\Delta v - (E_2 - V_0)v = (b\,u^2 + a\,v^2)\,v & \text{on } \mathbb{R}^3. \end{cases}$$

This is in fact the problem which will be discussed in most parts of this thesis the focus of which, however, will from now on be mathematical in nature, asking for existence results and suitable methods and frameworks to establish them. Throughout, the results also strive to answer the question whether the solutions (u, v) thus obtained are fully nontrivial, i.e.

whether both $u \neq 0$ and $v \neq 0$ can be guaranteed, which is clearly physically relevant in the sense that a true mixture of two components is described. It might be an interesting topic of further research whether these solutions have a certain physical interpretation, and whether the mathematical tools provided here also work under less restrictive assumptions in the physical model.

1.3 The Linear Helmholtz Equation on $\mathbb{R}^N$

In this chapter, some classical results concerning the linear Helmholtz equation

$$-\Delta w - \lambda w = f \quad \text{on } \mathbb{R}^N, \qquad w(x) \to 0 \text{ as } |x| \to \infty \tag{1.4}$$

where $N \geq 2$, $\lambda > 0$ and for suitable $f : \mathbb{R}^N \to \mathbb{C}$ are reviewed. In order to keep the presentation short, the discussion focuses on problems on the whole space $\mathbb{R}^N$ and on properties which will be of importance in the following chapters. For instance, it will be shown that the asymptotic behavior of solutions $w(x)$ as $|x| \to \infty$ is crucial when assessing existence and uniqueness questions. Indeed, in comparison to the Schrödinger equation $-\Delta w + \lambda w = f$ where, in most applications, exponentially decaying solutions are considered, solutions of (1.4) typically have power decay and exhibit a characteristic far field behavior, which will be derived and explained in the following subchapters. Moreover, also in contrast to the Schrödinger case, a particular challenge even of the linear Helmholtz equation (1.4) is that the notion of a resolvent $(-\Delta - \lambda)^{-1}$ is not well-defined on the Hilbert space $H^1(\mathbb{R}^N)$. The second subchapter will give an overview of methods to overcome this problem, which are commonly named Limiting Absorption Principles and provide solutions of (1.4) on appropriate Banach spaces.

Although we finally aim to find real-valued solutions of the Helmholtz system (H), the general theory reviewed next will mostly involve complex-valued functions. In order to emphasize the difference, we shall throughout this thesis explicitly label spaces of complex-valued functions in the form $L(\mathbb{R}^N, \mathbb{C})$ whereas $L(\mathbb{R}^N)$ will denote the real-valued case.

1.3.1 The Helmholtz Kernel, Fundamental Solutions

As mentioned before, in contrast to the Schrödinger case, the Helmholtz operator $-\Delta - \lambda$ does not possess a resolvent in the L^2 sense. Moreover, there exist nontrivial localized solutions of the homogeneous Helmholtz equation $-\Delta u - \lambda u = 0$ on $\mathbb{R}^N$. It will be shown that even smooth radial solutions exist but that they cannot be elements of $H^1(\mathbb{R}^N)$ due to their slow decay. This subchapter aims at general properties and characterizations of nontrivial solutions of $-\Delta u - \lambda u = 0$ on $\mathbb{R}^N$ in a suitable subspace of $L^2_{\text{loc}}(\mathbb{R}^N)$.

Remark 1.2.

Since the following results are based on a range of different concepts of solutions, it is worth mentioning some aspects concerning regularity. Indeed, given some distributional solution $u \in L^2_{\text{loc}}(\mathbb{R}^N)$ of $-\Delta u = \lambda u$ on $\mathbb{R}^N$, one can deduce that u is in fact a smooth classical solution as follows:

A regularity result by Zhang and Bao, Proposition 1.1 in [76], implies $u \in W^{2,2}_{\text{loc}}(\mathbb{R}^N)$ and that, in particular, u is a weak solution of $-\Delta u = \lambda u$. Then (higher-order) elliptic regularity theory yields iteratively $u \in W^{k,2}_{\text{loc}}(\mathbb{R}^N)$ for all $k \in \mathbb{N}$, and smoothness of u is a consequence of suitable Sobolev embeddings.

We start with a classical result due to Rellich, Satz 1 in [64], concerning the decay of solutions of the homogeneous linear Helmholtz equation. He assumes classical solutions of class C^2, which according to the previous Remark 1.2 is not a severe restriction. The following version of Rellich's theorem is slightly adapted to the viewpoint and notation of this thesis. In particular, Rellich's results include the case of boundary value problems for exterior domains, which is a usual setting in scattering theory but not presented here since it will not be touched in the subsequent chapters.

Theorem 1.3 (Rellich, 1942).

Let $N \geq 2$ and $u : \mathbb{R}^N \to \mathbb{C}$, $u \not\equiv 0$ be a twice continuously differentiable solution of $-\Delta u - \lambda u = 0$ on $\mathbb{R}^N$. Then there exists $\delta > 0$ such that, for all $r > 1$,

$$\int_{B_r(0)\setminus B_1(0)} |u(x)|^2 \, \mathrm{d}x \geq \delta \cdot r.$$

Rellich infers from this Theorem that $|x|^{\frac{N-1}{2}} u(x) \not\to 0$ uniformly with respect to $|x|$ as $|x| \to \infty$. He concludes that nontrivial solutions of $-\Delta u - \lambda u = 0$ on $\mathbb{R}^N$ cannot belong to $L^2(\mathbb{R}^N)$; in particular, exponentially decaying solutions do not occur. Moreover, he deduces on p. 58 of [64] a uniqueness property which is presented next in a slightly rephrased form.

Corollary 1.4 (Rellich, 1942).

Let $N \geq 2$. Up to multiplication with a constant, there is at most one twice continuously differentiable solution $u : \mathbb{R}^N \to \mathbb{C}$, $u \not\equiv 0$ of the homogeneous Helmholtz equation $-\Delta u - \lambda u = 0$ which satisfies Sommerfeld's outgoing radiation condition,

$$\lim_{|x|\to\infty} |x|^{\frac{N-1}{2}} \left(\frac{x}{|x|} \cdot \nabla u(x) - \mathrm{i}\sqrt{\lambda} u(x) \right) = 0.$$

A similar statement holds when imposing $\lim_{|x|\to\infty} |x|^{\frac{N-1}{2}} \left(\frac{x}{|x|} \cdot \nabla u(x) + \mathrm{i}\sqrt{\lambda} u(x) \right) = 0$, known as Sommerfeld's ingoing radiation condition.

More detailed characterizations of the full-space problem have later been provided by Agmon, e.g. in [5]. As motivated by Rellich's estimate in the previous Theorem, Agmon considers solutions in the space

$$B^*(\mathbb{R}^N, \mathbb{C}) := \left\{ u \in L^2_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{C}) \mid \|u\|_{B^*} < \infty \right\}$$
$$\text{where} \qquad \|u\|_{B^*} := \sup_{r>1} \left(\frac{1}{r} \int_{B_r(0)} |u(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}.$$

Agmon's key to a deeper understanding of the asymptotic behavior of solutions is a suitable characterization using an integral representation formula. We combine statements from Theorems 4.3 and 4.5 in [5].

Theorem 1.5 (Agmon, 1990).

Let $N \geq 2$. For $\lambda > 0$, the following statements are equivalent.

(i) *$u \in B^*(\mathbb{R}^N, \mathbb{C})$ solves the homogeneous Helmholtz equation $-\Delta u - \lambda u = 0$ on $\mathbb{R}^N$.*

(ii) *There exists $\phi \in L^2(\mathbb{S}^{N-1}, \mathbb{C})$ with*

$$u(x) = \int_{\mathbb{S}^{N-1}} \phi(\theta) \mathrm{e}^{-\mathrm{i} x \cdot \theta \sqrt{\lambda}} \, \mathrm{d}\sigma(\theta).$$

In this case, the asymptotic behavior of u is given by the formula

$$u(x) = \left(\frac{2\pi}{|x|\sqrt{\lambda}} \right)^{\frac{N-1}{2}} \left[\mathrm{e}^{\mathrm{i}(N-1)\frac{\pi}{4}} \, \mathrm{e}^{-\mathrm{i}|x|\sqrt{\lambda}} \, \phi\left(\frac{x}{|x|} \right) + \mathrm{e}^{-\mathrm{i}(N-1)\frac{\pi}{4}} \, \mathrm{e}^{\mathrm{i}|x|\sqrt{\lambda}} \, \phi\left(-\frac{x}{|x|} \right) \right] + \delta(x)$$

where $\frac{1}{r} \int_{B_r(0)} |\delta(x)|^2 \, \mathrm{d}x \to 0$ as $r \to \infty$.

In fact, this statement extends to $\lambda = k^2$ where $k \in \mathbb{C} \setminus \{0\}$ with $\operatorname{Im} k \geq 0$; then $\mathrm{e}^{\mathrm{i}k|\cdot|} u \in B^*(\mathbb{R}^N)$ is assumed in (i) and only the first term in the asymptotic expansion appears. This generalization is an important motivation for the following subchapter where in fact $\lambda > 0$ will be replaced by some complex $\lambda + \mathrm{i}\varepsilon$, and solutions of the linear inhomogeneous Helmholtz equation (1.4) will be recovered taking the limit $\varepsilon \searrow 0$.

Finally, the case of radially symmetric solutions of the homogeneous Helmholtz equation on $\mathbb{R}^N \setminus \{0\}$ shall be addressed. These will frequently appear in the kernels of convolution operators arising from the limiting process just mentioned, and they are given in terms of a class of special functions. Indeed, looking for radially symmetric solutions $u : \mathbb{R}^N \setminus \{0\} \to \mathbb{C}$ of the homogeneous Helmholtz equation which are smooth outside the origin, a short calculation shows that this is equivalent to $u(x) = |x|^{-\frac{N}{2}+1} \, w(|x|\sqrt{\lambda})$, $x \neq 0$ where $w \in C^2((0, \infty))$ solves Bessel's differential equation

$$r^2 \, w''(r) + r \, w'(r) + \left(r^2 - \left(\frac{N}{2} - 1 \right)^2 \right) w(r) = 0, \qquad r > 0.$$

Based on this observation, the following statements hold.

Remark 1.6 (The radial case. Fundamental solutions).

Assume that $u : \mathbb{R}^N \setminus \{0\} \to \mathbb{C}$ is a twice differentiable, radially symmetric solution of $-\Delta u - \lambda u = 0$ on $\mathbb{R}^N \setminus \{0\}$.

(a) *For some $\alpha, \beta \in \mathbb{C}$, u satisfies $u(x) = \alpha u_1(x) + \beta u_2(x)$ for $x \neq 0$. Here u_1, u_2 denote the fundamental solutions of the (homogeneous) Helmholtz equation,*

$$u_1(x) = \frac{1}{|x|^{\frac{N}{2}-1}} \cdot J_{\frac{N}{2}-1}\left(|x|\sqrt{\lambda} \right), \quad u_2(x) = \frac{1}{|x|^{\frac{N}{2}-1}} \cdot Y_{\frac{N}{2}-1}\left(|x|\sqrt{\lambda} \right) \tag{1.5}$$

with the Bessel functions $J_{\frac{N}{2}-1}, Y_{\frac{N}{2}-1}$ of order $\frac{N}{2} - 1$ of first resp. second kind.

(b) *As $|x| \to \infty$, the asymptotic expansions of the Bessel functions imply*

$$u_1(x) = \sqrt{\frac{2}{\pi\sqrt{\lambda}}}\,\frac{\cos\left(|x|\sqrt{\lambda} - \frac{(N-1)\pi}{4}\right)}{|x|^{\frac{N-1}{2}}}\left(1 + O\left(\frac{1}{|x|}\right)\right),$$
$$u_2(x) = \sqrt{\frac{2}{\pi\sqrt{\lambda}}}\,\frac{\sin\left(|x|\sqrt{\lambda} - \frac{(N-1)\pi}{4}\right)}{|x|^{\frac{N-1}{2}}}\left(1 + O\left(\frac{1}{|x|}\right)\right).$$

This is similar to the expansion in Theorem 1.5.

(c) *Asymptotic expansions of the Bessel functions for small arguments further show that u_1 can be smoothly extended to $x = 0$ but u_2 cannot. Indeed, approximation as $|x| \to 0$ yields in leading order*

$$u_2(x) \sim \begin{cases} \frac{\Gamma(\frac{N}{2}-1)}{\pi}\left(\frac{2}{\sqrt{\lambda}}\right)^{\frac{N}{2}-1} \cdot \frac{1}{|x|^{N-2}} & N \geq 3, \\ \frac{2}{\pi}\log(|x|\sqrt{\lambda}) & N = 2. \end{cases}$$

A straightforward calculation shows that u_1 but not u_2 provides a distributional solution of $-\Delta u - \lambda u = 0$ on all of $\mathbb{R}^N$. Moreover, the asymptotic expansions show $u_1 \in B^*(\mathbb{R}^N, \mathbb{C})$, and u_1 can be obtained from Theorem 1.5 by considering a constant density $\varphi \in L^2(\mathbb{S}^{N-1}, \mathbb{C})$ in (ii).

In fact, the singularity of u_2 at $x = 0$ leads to

$$(-\Delta u_2 - \lambda u_2)[\varphi] = \int_{\mathbb{R}^N} u_2 \cdot (-\Delta\varphi - \lambda\varphi)\,\mathrm{d}x = c(N, \lambda) \cdot \varphi(0) \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$$

with some constant $c(N, \lambda)$, as can be proved via Green's formula on $\mathbb{R}^N \setminus B_\varepsilon(0)$ with $\varepsilon \searrow 0$. This will be important to make the resolvent-type operators work which we present in the following chapter. These are convolution operators the kernels of which are linear combinations of u_1 and u_2.

The asymptotic expansions of the Bessel functions quoted above can be found in [2], (9.1.7) to (9.1.13) for small resp. (9.2.1) to (9.2.2) for large positive arguments. For the reader's convenience, they are listed in the appendix on Conventions and Abbreviations at the end of this thesis.

1.3.2 Limiting Absorption Principles

Having characterized all solutions $u \in B^*(\mathbb{R}^N, \mathbb{C})$ of the homogeneous Helmholtz equation $-\Delta u - \lambda u = 0$ on $\mathbb{R}^N$, it is next natural to aim at a way of constructing a solution of the inhomogeneous problem (1.4) for a suitable right-hand side $f : \mathbb{R}^N \to \mathbb{C}$. Generalizing (1.4), we consider the differential equation

$$-\Delta w - k^2 w = f \quad \text{on } \mathbb{R}^N \tag{1.6}$$

for $k \in \mathbb{C} \setminus \{0\}$ with $\operatorname{Im} k \geq 0$. Roughly, the idea is as follows. Assuming $f \in \mathcal{S}(\mathbb{R}^N, \mathbb{C})$ to be a Schwartz function, an application of the Fourier transformation to the differential equation (1.4) yields the algebraic equation

$$|\xi|^2\,\hat{w}(\xi) - k^2\,\hat{w}(\xi) = \hat{f}(\xi), \qquad \xi \in \mathbb{R}^N.$$

If even $\operatorname{Im} k > 0$, then the term $||\xi|^2 - k^2|$ has a positive lower bound, which allows to conclude that (1.6) has a unique solution $w \in \mathcal{S}(\mathbb{R}^N, \mathbb{C})$ given by

$$\begin{aligned} w(x) &= \mathcal{F}^{-1}\left(\frac{\hat{f}}{|\cdot|^2 - k^2}\right)(x) \\ &= \int_{\mathbb{R}^N} \frac{\mathrm{i}}{4}\left(\frac{k}{2\pi|x-y|}\right)^{\frac{N-2}{2}} H^{(1)}_{\frac{N}{2}-1}(k|x-y|) \cdot f(y)\,\mathrm{d}y, \end{aligned} \tag{1.7}$$

where $H^{(1)}_{\frac{N}{2}-1} = J_{\frac{N}{2}-1} + \mathrm{i}Y_{\frac{N}{2}-1}$ denotes the Hankel function of the first kind, referring to equation (2.7) in [5] for the convolution kernel. In the following, in the case $\operatorname{Im} k > 0$, the notation $\mathfrak{R}(k) := (-\Delta - k^2)^{-1}$ will be used for the convolution operator above. With a view to the asymptotic behavior of the Hankel functions given in the appendix on Conventions and Abbreviations, the condition $\operatorname{Im} k > 0$ ensures that the Hankel function $H^{(1)}_{\frac{N}{2}-1}(k|x-y|)$ decays exponentially as $|x-y| \to \infty$, hence the convolution integral above is well-defined for a large class of functions f, e.g. $f \in L^2(\mathbb{R}^N, \mathbb{C})$. When considering $\mathfrak{R}(k)$ as a conventional L^2 resolvent, Plancherel's identity applied to (1.7) implies

$$\|\mathfrak{R}(k)\|_{\mathcal{L}(L^2(\mathbb{R}^N,\mathbb{C}),L^2(\mathbb{R}^N,\mathbb{C}))} = \left\| \frac{1}{|\cdot|^2 - k^2} \right\|_{L^\infty(\mathbb{R}^N,\mathbb{C})} = \frac{1}{|\operatorname{Im} k^2|},$$

which diverges as $\operatorname{Im} k \to 0$. Limiting Absorption Principles now aim at evaluating the limit case $\operatorname{Im} k \searrow 0$, i.e. $k^2 \to \lambda$ for some $\lambda > 0$ in suitable topologies necessarily different from $(L^2(\mathbb{R}^N, \mathbb{C}), L^2(\mathbb{R}^N, \mathbb{C}))$. They are therefore based on uniform estimates of the form

$$\|\mathfrak{R}(k)f\|_Y \leq C \cdot \|f\|_X \qquad \text{for all } f \in \mathcal{S}(\mathbb{R}^N, \mathbb{C}) \text{ and } k \in \mathbb{C} \text{ with } \operatorname{Im} k > 0 \tag{1.8}$$

with appropriately chosen Banach spaces X, Y, some of which will be presented here. Usually, as $k^2 \to \lambda \in (0, \infty)$ with $\operatorname{Im} k > 0$, Limiting Absorption Principles provide different limit operators for $\operatorname{Im} k^2 \searrow 0$ from above (in other words, $\operatorname{Re} k \to \sqrt{\lambda}$) and $\operatorname{Im} k^2 \nearrow 0$ from below (i.e., $\operatorname{Re} k \to -\sqrt{\lambda}$), respectively; we will later see that these correspond to solutions of the Helmholtz equation satisfying outgoing resp. incoming radiation conditions or combinations of these.

An early result by Agmon using a range of weighted L^2 spaces can be found in [4], Theorem 4.1. It is based on uniform estimates of the type (1.8) in the spaces

$$\begin{aligned} X &= L^{2,s}(\mathbb{R}^N, \mathbb{C}) := \left\{ f \in L^2(\mathbb{R}^N, \mathbb{C}) \,\middle|\, (1 + |\cdot|^2)^{\frac{s}{2}} f \in L^2(\mathbb{R}^N, \mathbb{C}) \right\}, \\ Y &= H^{2,-s}(\mathbb{R}^N, \mathbb{C}) := \left\{ w \in L^{2,-s}(\mathbb{R}^N, \mathbb{C}) \,\middle|\, \nabla w, D^2 w \in L^{2,-s}(\mathbb{R}^N, \mathbb{C}) \right\} \end{aligned}$$

provided $s > \frac{1}{2}$. Still following pioneering work by Agmon and Hörmander in [5], [6], the first Limiting Absorption Principle presented here in detail is a refinement of the previously mentioned one with the additional advantage that $Y = B^*(\mathbb{R}^N, \mathbb{C})$, which matches Theorem 1.5 in the previous section characterizing the Helmholtz kernel. What follows is a version due to Agmon and Hörmander, see Theorem 3.1 in [5] and the references given there. It will be assumed that $f \in B(\mathbb{R}^N, \mathbb{C})$ with

$$\begin{aligned} &B(\mathbb{R}^N, \mathbb{C}) := \left\{ f \in L^2(\mathbb{R}^N, \mathbb{C}) \mid \|f\|_B < \infty \right\} \\ &\text{where} \qquad \|f\|_B := \sum_{j=0}^{\infty} \left(2^j \int_{B_{2^{j+1}}(0) \setminus B_{2^j}(0)} |f(x)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}. \end{aligned}$$

Then, as suggested by the notation, $B^*(\mathbb{R}^N, \mathbb{C})$ can be shown to be the dual of $B(\mathbb{R}^N, \mathbb{C})$, and the following estimate of type (1.8) with $X = B(\mathbb{R}^N,\mathbb{C}), Y = B^*(\mathbb{R}^N,\mathbb{C})$ holds: For any bounded subset $K \subseteq \{z \in \mathbb{C} \setminus \{0\} \mid \operatorname{Im} z > 0\}$ there exists a constant $C(K) > 0$ with

$$\|\mathfrak{R}(z)f\|_{B^*(\mathbb{R}^N,\mathbb{C})} \leq C(K) \cdot \|f\|_{B(\mathbb{R}^N,\mathbb{C})} \qquad \text{for all } z \in K,\ f \in B(\mathbb{R}^N,\mathbb{C}),$$

see [5], equation (3.3) and [6], Theorem 6.3. This estimate extends to $z \in \overline{K}$ with $\operatorname{Im} z = 0, z \neq 0$ and implies the following statement:

Theorem 1.7 (Agmon, Hörmander, 1976).

Let $N \geq 2$. The operator-valued mapping $\mathfrak{R}$ admits a weak∗-continuous extension to $k \in \mathbb{R} \setminus \{0\}$, that is, there exists a linear operator $\mathfrak{R}(k) \in \mathcal{L}(B(\mathbb{R}^N,\mathbb{C}), B^*(\mathbb{R}^N,\mathbb{C}))$ with

$$\langle \mathfrak{R}(k)f, g\rangle = \lim_{\substack{z\to k\\ \operatorname{Im} z>0}} \langle \mathfrak{R}(z)f, g\rangle \qquad \textit{for all } f,g \in B(\mathbb{R}^N,\mathbb{C}).$$

Remark 1.8. *(a) Using the same technique as in the proof of [4], Theorem 4.1 (ii), one can see that for all $k \in \mathbb{R} \setminus \{0\}$ the function $u := \mathfrak{R}(k)f$ from Theorem 1.7 is a distributional solution of $-\Delta u - k^2 u = f$ in the sense that*

$$\big\langle u, (-\Delta - k^2)\varphi\big\rangle = \langle f, \varphi\rangle \qquad \textit{for all } \varphi \in \mathcal{S}(\mathbb{R}^N,\mathbb{C}).$$

As in Remark 1.2, given $f \in L^2_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{C})$, one can conclude that u is a strong solution with $u \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{C})$.

(b) Observe that, given $\lambda = k^2 > 0$, we obtain two solutions

$$u_\pm := \lim_{\varepsilon\searrow 0} \mathfrak{R}(\pm\sqrt{\lambda} + \mathrm{i}\varepsilon)f \quad \textit{with} \quad -\Delta u_\pm - \lambda u_\pm = f \quad \textit{on } \mathbb{R}^N.$$

Since $u_{\varepsilon,\pm} := \mathfrak{R}(\pm\sqrt{\lambda} + \mathrm{i}\varepsilon)f$ solves $-\Delta u_{\varepsilon,\pm} - (\lambda \pm \mathrm{i}\,2\sqrt{\lambda}\varepsilon - \varepsilon^2)u_{\varepsilon,\pm} = f$, the notation $u_\pm = (-\Delta - (\lambda \pm \mathrm{i}0))^{-1}f$ is used frequently to indicate the respective limits as $\varepsilon \searrow 0$. In general, u_+ and u_- do not agree. The difference $u_+ - u_-$ solves the homogeneous Helmholtz equation and can hence be written as in Theorem 1.5 (ii), see also the equation on top of p. 25, [6]. Following Definition 6.5 in [6], u_+ is said to be a λ-outgoing solution, and u_- is called λ-incoming. These notions go back to Sommerfeld's outgoing resp. incoming radiation condition an averaged version of which is satisfied by u_+ resp. u_-. For details, see Theorem 7.8 in [6].

(c) Combining Theorems 1.5 and 1.7, it is evident that for $\lambda > 0$ and given $f \in B(\mathbb{R}^N,\mathbb{C})$, the following are equivalent:

(i) $u \in B^*(\mathbb{R}^N,\mathbb{C})$ solves $-\Delta u - \lambda u = f$ on $\mathbb{R}^N$.

(ii) There exists $\phi \in L^2(\mathbb{S}^{N-1},\mathbb{C})$ with $u(x) = (\mathfrak{R}(k)f)(x) + \int_{\mathbb{S}^{N-1}} \phi(\theta)\mathrm{e}^{\mathrm{i}x\cdot\theta\sqrt{\lambda}}\,\mathrm{d}\sigma(\theta)$.

The surface integral in (ii) is said to be the Herglotz wave associated to u.

Being interested in solutions of the system (H) with $u, v \in L^p(\mathbb{R}^N)$, suitable versions of the previous results are required. In [33] Gutiérrez proved a Limiting Absorption Principle which is based on the following estimate of the form (1.8): Let $N \geq 3$ and $p, q \in (1,\infty)$ with $q < \frac{2N}{N+1}$, $p > \frac{2N}{N-1}$, $\frac{2}{N+1} < \frac{1}{q} - \frac{1}{p} < \frac{2}{N}$. Then there exists a constant $c_{p,q} > 0$ such that, for every $\lambda > 0$ and $\varepsilon > 0$,

$$\left\|\mathfrak{R}(\sqrt{\lambda} + \mathrm{i}\varepsilon)f\right\|_{L^p(\mathbb{R}^N,\mathbb{C})} \leq c_{p,q} \cdot \|f\|_{L^q(\mathbb{R}^N,\mathbb{C})} \qquad \text{for all } f \in \mathcal{S}(\mathbb{R}^N,\mathbb{C}), \tag{1.9}$$

see Theorem 6 in [33]. This allows to pass to the limit $\varepsilon \searrow 0$. The following Theorem investigates the limit operator $\mathfrak{R}_\lambda := \lim_{\varepsilon\searrow 0} \mathfrak{R}(\sqrt{\lambda}+\mathrm{i}\varepsilon)$, collecting results from [33], Chapter 3.

Theorem 1.9 (Gutiérrez, 2004).

Let $N \geq 3$ and $p, q \in (1,\infty)$ with $q < \frac{2N}{N+1}$, $p > \frac{2N}{N-1}$, $\frac{2}{N+1} \leq \frac{1}{q} - \frac{1}{p} \leq \frac{2}{N}$. Then for $\lambda > 0$, the operator $\mathfrak{R}_\lambda$ is a continuous operator mapping $L^q(\mathbb{R}^N,\mathbb{C})$ into $L^p(\mathbb{R}^N,\mathbb{C})$. It has the following properties:

(i) For $f \in L^q(\mathbb{R}^N,\mathbb{C})$, the function $u := \mathfrak{R}_\lambda f \in L^p(\mathbb{R}^N,\mathbb{C})$ is a distributional solution of the Helmholtz equation $-\Delta u - \lambda u = f$ on $\mathbb{R}^N$. It satisfies a weaker version of Sommerfeld's outgoing radiation condition,

$$\lim_{R\to\infty} \frac{1}{R} \int_{B_R(0)} \left| \frac{x}{|x|} \cdot \nabla u(x) - \mathrm{i}\lambda u(x) \right|^2 \mathrm{d}x = 0.$$

Moreover, u is the only element of $L^p(\mathbb{R}^N,\mathbb{C})$ with these properties.

(ii) If $f \in \mathcal{S}(\mathbb{R}^N,\mathbb{C})$, the operator $\mathfrak{R}_\lambda$ acts as a convolution $\mathfrak{R}_\lambda f = \Phi_\lambda * f$ with kernel

$$\Phi_\lambda(x) := \frac{\mathrm{i}}{4} \left(\frac{\sqrt{\lambda}}{2\pi|x|} \right)^{\frac{N-2}{2}} H^{(1)}_{\frac{N}{2}-1}(\sqrt{\lambda}|x|) \qquad (x \in \mathbb{R}^N, x \neq 0).$$

For the applications to follow, the case of dual exponents $q = p'$ where $\frac{1}{p} + \frac{1}{p'} = 1$ will be crucial, which has been discussed earlier by Kenig, Ruiz and Sogge in [39], Theorem 2.3. In this case, Theorem 1.9 states that

$$\mathfrak{R}_\lambda \in \mathcal{L}(L^{p'}(\mathbb{R}^N,\mathbb{C}), L^p(\mathbb{R}^N,\mathbb{C})) \quad \text{if} \quad \frac{2(N+1)}{N-1} \leq p \leq \frac{2N}{N-2}. \tag{1.10}$$

In fact, without further symmetry assumptions, these bounds on p are optimal in the sense that there is no hope to extend (1.10) to cover the full subcritical and superlinear range $2 < p < \frac{2N}{N-2}$. This is due to an underlying optimal result in Fourier Restriction Theory, which will be commented on after the following Remark.

Remark 1.10. *(a) The statements of Theorem 1.9 extend to the case $N = 2$, as has been shown by Evéquoz around Theorem 2.1 in [26]; one then has to use strict upper estimates $\frac{2}{3} \leq \frac{1}{q} - \frac{1}{p} < 1$ in Theorem 1.9 resp. $6 \leq p < \infty$ in the dual case.*

(b) The bounds on p, q can be improved when restricting to spaces of radially symmetric functions. Indeed, one finds

$$\mathfrak{R}_\lambda \in \mathcal{L}(L^q_{\mathrm{rad}}(\mathbb{R}^N,\mathbb{C}), L^p_{\mathrm{rad}}(\mathbb{R}^N,\mathbb{C})) \qquad \text{if } q < \frac{2N}{N+1},\ p > \frac{2N}{N-1},\ \frac{3N-1}{2N^2} \leq \frac{1}{q} - \frac{1}{p} \leq \frac{2}{N},$$

$$\mathfrak{R}_\lambda \in \mathcal{L}(L^{p'}_{\mathrm{rad}}(\mathbb{R}^N,\mathbb{C}), L^p_{\mathrm{rad}}(\mathbb{R}^N,\mathbb{C})) \qquad \text{if } \frac{4N^2}{(N-1)(2N-1)} \leq p \leq \frac{2N}{N-2}$$

again with strict upper estimates for $N = 2$.

These bounds can be obtained, for $N \geq 3$, by a mere modification of Gutierrez' proof as presented in some detail in Remark 3.1 of [17]. Still, the main idea behind the optimality of (1.10) and its improvement in radially symmetric settings will be presented. Specifically, on p. 19 of [33] the Stein-Tomas Theorem is applied, see Theorem 1 in [73] and the following explanations. It states that, for any Schwartz function $f \in \mathcal{S}(\mathbb{R}^N, \mathbb{C})$,

$$\int_{\mathbb{S}^{N-1}} |\hat{f}(\vartheta)|^2 \,\mathrm{d}\sigma(\vartheta) \leq c_p \, \|f\|^2_{L^p(\mathbb{R}^N)} \qquad \text{for } 1 \leq p \leq \frac{2(N+1)}{N+3} \tag{1.11}$$

holds with some constants c_p independent of the function f. An example by Knapp shows that such an estimate fails for $p > \frac{2(N+1)}{N+3}$, see also [73] and [72], p. 5. The Stein-Tomas Theorem thus belongs to a field of Harmonic Analysis concerned with restriction problems. These ask, roughly speaking, for conditions on the exponent p which allow to restrict the Fourier transform $\hat{f}$ of $f \in L^p(\mathbb{R}^N, \mathbb{C})$ to certain submanifolds of $\mathbb{R}^N$ in a meaningful way; the Stein-Tomas Theorem provides an optimal result for restrictions to the sphere $\mathbb{S}^{N-1}$. For a detailed and more general introduction and overview, see e.g. Tao's article [72]. In the case of radial symmetry, the bounds in (1.11) can be improved. This can be proved explicitly by means of the following formula for the Fourier transform of radially symmetric functions $f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^N, \mathbb{C})$ (the profile of which will also be denoted by f), to be found e.g. in [32], Appendix B.5:

$$\hat{f}(\xi) = \frac{1}{|\xi|^{\frac{N}{2}-1}} \int_0^\infty f(r) J_{\frac{N}{2}-1}(r|\xi|)\, r^{\frac{N}{2}} \,\mathrm{d}r \tag{1.12}$$

where $\xi \in \mathbb{R}^N \setminus \{0\}$ and $J_{\frac{N}{2}-1}$ denotes as before the Bessel function of the first kind. Thus for $\vartheta \in \mathbb{S}^{N-1}$,

$$\hat{f}(\vartheta) = \int_0^\infty f(r) J_{\frac{N}{2}-1}(r)\, r^{\frac{N}{2}} \,\mathrm{d}r = \int_{\mathbb{R}^N} f(x) \frac{J_{\frac{N}{2}-1}(|x|)}{|x|^{\frac{N}{2}-1}} \,\mathrm{d}x,$$

and an estimate as in (1.11) holds if $|\cdot|^{1-\frac{N}{2}} J_{\frac{N}{2}-1}(|\cdot|) \in L^{p'}(\mathbb{R}^N)$. With a view to the asymptotics of Bessel functions for $r \searrow 0$ resp. $r \to \infty$,

$$\frac{J_{\frac{N}{2}-1}(|x|)}{|x|^{\frac{N}{2}-1}} \leq C_N \cdot \min\left\{1, \frac{1}{|x|^{\frac{N-1}{2}}}\right\} \qquad \text{for } x \neq 0,$$

this implies for radial Schwarz functions $f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^N, \mathbb{C})$ the improved Fourier restriction estimate

$$\int_{\mathbb{S}^{N-1}} |\hat{f}(\vartheta)|^2 \,\mathrm{d}\sigma(\vartheta) \leq \tilde{c}_p \, \|f\|^2_{L^p(\mathbb{R}^N)} \qquad \text{for } 1 \leq p < \frac{2N}{N+1}. \tag{1.13}$$

Using the bounds for p given in (1.13) instead of (1.11) in Gutiérrez' proof of Lemma 1, p. 19 of [33], and following the estimates and interpolation arguments carefully, one finally arrives at the statement of the above Remark 1.10 (b) in case $N \geq 3$. For $N = 2$, Evéquoz' proof in [26], Theorem 2.1 follows the ideas of Gutiérrez; hence, assuming radial symmetry, the admissible range of exponents can be extended in a similar way, starting from [26], equation (15).

1.4 The Nonlinear Helmholtz Equation on $\mathbb{R}^N$

1.4.1 Dual Variational Techniques

In the initial chapter, a short review of the literature on nonlinear Schrödinger equations has been provided, which includes (but often goes beyond) equations of the form

$$-\Delta w + \lambda w = Q(x)|w|^{p-2}w \qquad \text{on } \mathbb{R}^N$$

with some $\lambda > 0$; in particular, more general nonlinearities and non-constant potentials have been investigated. In contrast, the corresponding Helmholtz problem

$$-\Delta w - \lambda w = Q(x)|w|^{p-2}w \qquad \text{on } \mathbb{R}^N \tag{1.14}$$

has only been discussed during the past five years and in much less generality. Since $\lambda > 0$ belongs to the essential spectrum of $-\Delta$, the Helmholtz case requires different concepts in order to handle oscillating solutions with slow decay which, in general, are not elements of $H^1(\mathbb{R}^N)$. These concepts are built on the representation results and Limiting Absorption Principles presented in the previous subchapter. The announced oscillation and decay properties have been analyzed in detail in [54] under the additional assumption of radial symmetry, which will be presented in the next subchapter. In particular, that discussion will include the case $N = 1$.

In this section, the focus lies on a dual variational approach introduced by Evéquoz and Weth in [26, 28] for $N \geq 3$ and $N = 2$, respectively, which provides real-valued strong solutions of the equation (1.14). The following result appears in [28], Theorems 1.1 and 1.2 and in [26], Theorem 1.3. Here and in the following, we understand $\frac{2N}{N-2} = +\infty$ in the case $N = 2$.

Theorem 1.11 (Evéquoz, Weth 2014 and Evéquoz 2016).

Let $N \geq 2$ and $Q \in L^\infty(\mathbb{R}^N)$ be nonnegative and not identically zero.

(i) If Q is $\mathbb{Z}^N$-periodic and $\frac{2(N+1)}{N-1} < p < \frac{2N}{N-2}$, equation (1.14) admits a real-valued, nontrivial, strong solution $w \in W^{2,q}(\mathbb{R}^N) \cap C^{1,\alpha}(\mathbb{R}^N)$ for all $q \in [p, \infty)$ and $\alpha \in (0,1)$.

(ii) If $Q(x) \to 0$ as $|x| \to \infty$ and $\frac{2(N+1)}{N-1} \leq p < \frac{2N}{N-2}$, equation (1.14) admits a sequence of real-valued, nontrivial, strong solutions $w_n \in W^{2,q}(\mathbb{R}^N) \cap C^{1,\alpha}(\mathbb{R}^N)$ for all $q \in [p, \infty)$ and $\alpha \in (0,1)$ with $\|w_n\|_{L^p(\mathbb{R}^N)} \to 0$ as $n \to \infty$.

Moreover, Evéquoz and Weth provide asymptotic properties of the solutions thus obtained. These expansions take the form of radiation conditions as the ones already encountered, albeit in a complex-valued version, in Theorem 1.5 by Agmon. By Lemma 4.3 in [28], the solutions in Theorem 1.11 satisfy

$$\lim_{r\to\infty} \frac{1}{r} \int_{B_r(0)} \left| w(x) + 2\left(\frac{2\pi}{|x|\sqrt{\lambda}}\right)^{\frac{N-1}{2}} \operatorname{Re}\left(\mathrm{e}^{-\mathrm{i}(N-1)\frac{\pi}{4}} \mathrm{e}^{\mathrm{i}|x|\sqrt{\lambda}} g_w(x/|x|)\right)\right|^2 \mathrm{d}x = 0 \tag{1.15}$$

$$\text{where} \quad g_w(\xi) = -\frac{\mathrm{i}}{4}\left(\frac{\lambda}{2\pi}\right)^{\frac{N}{2}-1} \mathcal{F}(Q|w|^{p-2}w)(\xi\sqrt{\lambda}), \qquad \xi \in \mathbb{S}^{N-1},$$

and w is shown to be the real part of a function satisfying Sommerfeld's outgoing radiation condition in the form given in Theorem 1.9 (i). Aiming finally at an adaptation of these methods for Helmholtz systems as (H), the central ideas of Evéquoz and Weth will be outlined next. Afterwards, some extensions of Theorem 1.11 mainly due to Evéquoz deserve to be mentioned.

First, using the resolvent-type operator $\mathfrak{R}_\lambda \in \mathcal{L}(L^{p'}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ constructed in (1.10) by means of Gutiérrez' Limiting Absorption Principle, the Helmholtz equation (1.14) is transformed into $w = \mathfrak{R}_\lambda \left[Q(x)\,|w|^{p-2}w\right]$ on $\mathbb{R}^N$. As in Theorem 1.9 (ii), $\mathfrak{R}_\lambda$ is a convolution operator with kernel Φ_λ. It is worth noticing that the transformed equation is not an equivalent problem; its solutions satisfy not only the Helmholtz equation (1.14) but also (a complex version of) certain asymptotic conditions as described above in (1.15). Being interested in real-valued solutions, Evéquoz and Weth pass to $\mathcal{R}_\lambda \in \mathcal{L}(L^{p'}(\mathbb{R}^N), L^p(\mathbb{R}^N))$ defined via

$$\mathcal{R}_\lambda f := \Psi_\lambda * f \quad \text{with } \Psi_\lambda := \operatorname{Re}\Phi_\lambda \quad \text{for } f \in \mathcal{S}(\mathbb{R}^N), \tag{1.16}$$

and look for solutions of the problem

$$w = \mathcal{R}_\lambda \left[Q(x)\,|w|^{p-2}w\right], \qquad w \in L^p(\mathbb{R}^N) \tag{1.17}$$

where the range of p is restricted according to the requirements of Gutiérrez' Limiting Absorption Principle excluding the endpoints - this is necessary for the proof of the compactness result presented below as Theorem 1.12. The construction ensures that solutions of (1.17) provide real-valued solutions of the original equation (1.14) additionally satisfying (1.15). Solutions of (1.17) are now obtained using variational methods: Setting

$$\bar{w}(x) = Q(x)^{\frac{1}{p'}}|w(x)|^{p-2}w(x),$$

one obtains a dual equation

$$|\bar{w}|^{p'-2}\bar{w} = Q(x)^{\frac{1}{p}}\,\mathcal{R}_\lambda\left[Q(x)^{\frac{1}{p}}\bar{w}\right] \quad \text{on } \mathbb{R}^N,$$

which is variational and gives rise to the functional

$$\begin{aligned} I_\lambda :\quad & L^{p'}(\mathbb{R}^N) \to \mathbb{R}, \\ I_\lambda(\bar{w}) :=\ & \frac{1}{p'}\int_{\mathbb{R}^N} |\bar{w}|^{p'}\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} Q(x)^{\frac{1}{p}}\bar{w}\cdot\mathcal{R}_\lambda\left[Q(x)^{\frac{1}{p}}\bar{w}\right]\,\mathrm{d}x. \end{aligned} \tag{1.18}$$

For symmetry of the convolution operator $\mathcal{R}_\lambda$, see Lemma 4.1 of [28] (for $N \geq 3$, but the proof can be repeated verbatim for $N = 2$). I_λ can be shown to have Mountain Pass geometry. Finally, applying the Mountain Pass Theorem, the authors prove the existence of a ground state $\bar{w} \in L^{p'}(\mathbb{R}^N)$ of the dual problem (and the existence of bound states, via the Symmetric Mountain Pass Theorem), which yields the solutions mentioned in Theorem 1.11. In the case $Q(x) \to 0$ as $|x| \to \infty$, this is achieved by verifying the Palais-Smale condition. For periodic Q, the characteristic loss of compactness has to be dealt with; replacing the Palais-Smale condition, Evéquoz and Weth establish a concentration compactness argument. It is based on the following nonvanishing property placed central as Theorem 3.1 in [26, 28], which will be of similar importance for the application to corresponding Helmholtz systems, e.g. (H) with periodic coupling $b(x)$.

Theorem 1.12 (Evéquoz, Weth 2014; Evéquoz 2016. Nonvanishing Property).

Let $N \geq 2$ and $\frac{2(N+1)}{N-1} < p < \frac{2N}{N-2}$. Consider a bounded sequence $(w_n)_n$ in $L^{p'}(\mathbb{R}^N)$ with

$$\limsup_{n\to\infty} \left| \int_{\mathbb{R}^N} w_n \, \mathfrak{R}_\lambda w_n \, \mathrm{d}x \right| > 0.$$

Then there exist a subsequence $(w_{n_k})_k$ and $R > 0$, $\zeta > 0$, $x_k \in \mathbb{R}^N$ with

$$\int_{B_R(x_k)} |w_{n_k}|^{p'} \, \mathrm{d}x \geq \zeta \quad \textit{for all } k \in \mathbb{N}.$$

As already announced earlier, there are various generalizations and extensions of Theorem 1.11 mainly due to Evéquoz. He proves the existence of infinitely many solutions also for periodic Q in [25], Theorem 1.1. Moreover, it is shown that the dual problem possesses a gound state if Q is assumed to be the sum of a periodic and a decaying term. For the same form of Q, Evéquoz and Yeşil discuss the critical case $p = \frac{2N}{N-2}$ in [30]; they show the existence of a dual ground state for $N \geq 4$ and prove nonexistence for $N = 3$ in [30], Theorem 3.7 and Proposition 3.8, respectively. In [23], Evéquoz demonstrates the applicability of dual variational techniques for any $p \in \left(2, \frac{2N}{N-2}\right)$ provided Q satisfies suitable integrability conditions, see in particular [23], Corollary 1.2 (i). Finally, assuming Q to be continuous and nonnegative, Evéquoz proves in [24], Theorems 1.1 and 1.2, existence, concentration and multiplicity of ground states of the dual problem in the limit of high frequencies $\lambda \nearrow \infty$ for $\frac{2(N+1)}{N-1} < p < \frac{2N}{N-2}$ based on a comparison of energies with a suitable limit problem.

There are some non-dual methods due to Evéquoz and Weth which also deserve to be mentioned here concerning the case of compactly supported Q but allowing the full subcritical range $2 < p < \frac{2N}{N-2}$ and extending to more general forms of the nonlinearity (maintaining compact support in the variable x). In [27], the problem is split into a nonlinear equation on a bounded domain and a linear exterior problem. For the latter, classical results about existence and far-field expansion of solutions are available. Variational methods, specifically linking techniques, are then used to solve the remaining problem on a bounded domain, and both solutions are coupled by means of a Dirichlet-to-Neumann operator. The approach in [29] is based on Leray-Schauder continuation, which allows to extend certain branches of solutions (w, λ) of (1.14) in the Schrödinger case (here $\lambda < 0$) to parameters $\lambda > 0$, which produces solutions in the Helmholtz case belonging to suitable spaces $L^p(\mathbb{R}^N)$.

1.4.2 The Radially Symmetric Case

Under the assumption of radial symmetry, Mandel, Montefusco and Pellacci provide a detailed account [54] on solutions of the autonomous nonlinear Helmholtz problem

$$-\Delta w - \lambda w = h(w) \qquad \text{on } \mathbb{R}^N \tag{1.19}$$

where $\lambda > 0$ as before, and the notation has been slightly adapted. These results illustrate the contrast between the Schrödinger and the Helmholtz case when comparing the following result to the classical findings of Berestycki-Lions and Strauss, cf. Theorem 1.1.

Theorem 1.13 (Mandel, Montefusco, Pellacci 2017).

Let $N \geq 1$, $\lambda > 0$ and assume for some $\sigma \in (0,1)$ and $\alpha_0 \in (0, \infty]$

$$\begin{aligned} &h \in C^{1,\sigma}_{\text{loc}}(\mathbb{R}), \quad h \text{ is odd}, \quad h'(0) = 0, \\ &h(w) + \lambda w > 0 \ (0 < w < \alpha_0) \quad \text{and} \quad h(w) + \lambda w < 0 \ (w > \alpha_0). \end{aligned}$$

Then there is a maximal continuum $\{w_\alpha | -\alpha_0 < \alpha < \alpha_0\} \subseteq C^2_{\text{loc}}(\mathbb{R}^N)$ of radially symmetric and oscillating solutions of the autonomous nonlinear Helmholtz equation (1.19). Moreover, for $\alpha \in (-\alpha_0, \alpha_0)$,

(i) $w_\alpha(0) = \alpha$ *and* $\|w_\alpha\|_{L^\infty(\mathbb{R}^N)} = |\alpha|$.

(ii) For $N = 1$, all profiles w_α are periodic; for $N \geq 2$, they are localized and satisfy

$$\frac{c_\alpha}{|x|^{\frac{N-1}{2}}} \leq |w_\alpha(x)| + |\partial_r w_\alpha(x)| + |\partial_r^2 w_\alpha(x)| \leq \frac{C_\alpha}{|x|^{\frac{N-1}{2}}}$$

for some $c_\alpha, C_\alpha > 0$ and all $x \in \mathbb{R}^N$ with $|x| \geq 1$.

These results, stated in Theorem 1.2 of [54], are proved using ODE methods available thanks to the assumption of radial symmetry. In comparison with the Schrödinger case resp. Theorem 1.1, it is worth mentioning that the Helmholtz case admits a continuum of radially symmetric solutions whereas the result by Berestycki and Lions identifies a sequence of distinct solutions. The latter are shown to decay exponentially and belong to $H^1(\mathbb{R}^N)$; in contrast, Theorem 1.13 provides an explicit polynomial bound from below and above for solutions of the Helmholtz problem. Finally, in the Helmholtz case, radially symmetric solutions oscillate, and hence positive solutions cannot be expected. There is also a result concerning a non-autonomous version of equation (1.19), Theorem 2.10 of [54], which ensures the existence of a solution with properties similar to (i), (ii) of Theorem 1.13.

These features will reappear when discussing radially symmetric solutions of the Helmholtz system (H) and comparing them to the case of coupled Schrödinger equations.

1.4.3 Further Results

So far, the methods and results presented are concerned with Helmholtz equations with a constant potential. In the more general case

$$-\Delta u + V(x)u - \lambda u = Q(x)|u|^{p-1}u \quad \text{on } \mathbb{R}^N \tag{1.20}$$

with a periodic, non-constant potential $V(x)$ and $0 \in \sigma(-\Delta + V(x) - \lambda)$, the central challenge is the introduction of a suitable Limiting Absorption Principle. This has been successfully done by Mandel in [52]. Starting from results by Radosz [63] on Limiting Absorption Principles for periodic Schrödinger operators and combining these with ideas of Gutiérrez [33], he proved the existence of a strong solution of (1.20) under certain assumptions. Apart from the conditions $N \geq 2$, $\frac{2(N+1)}{N-1} < p < \frac{2N}{N-2}$ and from the assumption of periodicity, boundedness and nonnegativity of Q, the existence result in [52], Corollary 1 imposes geometric conditions on the Fermi surfaces induced by the periodic differential operator. These conditions were shown to be satisfied in the special case of low frequencies

$\lambda \in \sigma(-\Delta + V(x))$ and of a separated potential, i.e. $V(x) = \sum_{k=1}^{N} V_k(x_k)$, which is close to a constant one, see e.g. Lemma 1 in [52].

Another generalization by Mandel [53] goes beyond the power-type nonlinearity in (1.14) and considers

$$-\Delta u - \lambda u = f(x,u) \quad \text{on } \mathbb{R}^N \tag{1.21}$$

with $\lambda > 0$ and growth assumptions of the form

$$|f(x,z)| \le Q(x)|z|^{p-1}, \quad |f(x,z) - f(x,w)| \le Q(x)(|z|+|w|)^{p-2}\,|z-w|$$

for $x \in \mathbb{R}^N$ and $|z|,|w| \le 1$ and with $Q \in L^\infty(\mathbb{R}^N) \cap L^s(\mathbb{R}^N)$ for some $s \in [1,\infty]$ and $p > \max\left\{2, \frac{2s(n^2+2n-1)-2n(n+1)}{(n^2-1)s}\right\}$. According to Theorem 1 in [53], these assumptions already guarantee the existence of uncountably many small, nontrivial, strong solutions of (1.21). The proof is based on the contraction mapping principle; the multitude of solutions is parametrized by a subset of suitable differentiable densities $h \in C^m(\mathbb{S}^{N-1},\mathbb{C})$ for $m = \lfloor \frac{N-1}{2} \rfloor + 1$ appearing in Herglotz waves of the form $x \mapsto \int_{\mathbb{S}^{N-1}} h(\vartheta\sqrt{\lambda})\mathrm{e}^{\mathrm{i}x\cdot\vartheta\sqrt{\lambda}}\mathrm{d}\sigma(\vartheta)$. The general idea of finding small (in that case complex-valued) solutions of the nonlinear Helmholtz equation using a contraction mapping approach has already been introduced in a comparatively special situation by Gutiérrez, see Theorem 1 of [33].

1.5 On the Structure of this Thesis

In the following Chapters, new results concerning nonlinear Helmholtz systems will be presented. As already announced, in Chapter 2 it will be demonstrated in what way the dual variational methods introduced by Evéquoz and Weth can also be applied to systems of equations, and under which additional assumptions they provide fully nontrivial dual ground state solutions. Chapter 3 provides an ansatz based on bifurcation theory which works in the practically important case of $N = 3$ space dimensions but requires a radially symmetric setup, and Chapter 4 contains an application of these bifurcation methods to construct time-periodic solutions of wave and Klein-Gordon equations.

All these chapters are organized similarly: The very first section is dedicated to the presentation of the main results along with an overview of literature which focuses on closely related results. It thus goes beyond the more general survey in this introductory part and aims at comparing the new results for Helmholtz systems e.g. with corresponding Schrödinger-type problems. Subsequently, the main technical tools are defined, explained and motivated but not proved yet in order not to obscure the main lines of thought. They will then be applied directly in the proof of the main results of the respective chapter. The proofs of the technical results are given after that. Each chapter closes with a short summary and a collection of interesting aspects for future research.

As already mentioned, one finds a collection of the notational conventions and of the abbreviations which are used throughout at the very end of the thesis.

CHAPTER 2.

Dual Ground States of a Nonlinear Helmholtz System

2.1 Introduction and Main Results

This chapter aims at the existence and characterization of (dual) ground state solutions of a more general version of the nonlinear Helmholtz system (H) on $\mathbb{R}^N$, $N \geq 2$. All major results of this chapter have been published in [55], and we will, without further mentioning, present most of the statements and proofs verbatim but add more explanations and details. In particular, we will be more careful in distinguishing the terms of ground state solutions (of a dual problem) resp. dual ground state solutions (of the original problem), as will be explained in this introduction. Some notation will have to be adjusted, too. Concerning the mathematical content, the most important difference to the published version is that we also discuss the occurrence of diagonal dual ground states (u, u) in the special case $\mu = \nu$, see the extended statement in Corollary 2.6 (i) below.

Throughout, 2^* denotes the critical Sobolev exponent, $2^* = \frac{2N}{N-2}$ for $N \geq 3$, $2^* = \infty$ for $N = 2$. Given $\mu, \nu > 0$, $\frac{2(N+1)}{N-1} < p < 2^*$ and nonnegative, $\mathbb{Z}^N$-periodic coefficients $a, b \in L^\infty(\mathbb{R}^N)$, we intend to find a pair of real-valued and strong solutions $(u, v) \in W^{2,p}(\mathbb{R}^N) \setminus \{0\} \times W^{2,p}(\mathbb{R}^N) \setminus \{0\}$ of

$$\begin{cases} -\Delta u - \mu u = a(x) \left(|u|^{\frac{p}{2}} + b(x)|v|^{\frac{p}{2}} \right) |u|^{\frac{p}{2}-2} u & \text{on } \mathbb{R}^N, \\ -\Delta v - \nu v \, = a(x) \left(|v|^{\frac{p}{2}} + b(x)|u|^{\frac{p}{2}} \right) |v|^{\frac{p}{2}-2} v & \text{on } \mathbb{R}^N, \\ u, v \in L^p(\mathbb{R}^N). \end{cases} \tag{2.1}$$

At first glance, this excludes the special case (H); however, assuming in addition radial symmetry and hence constant coefficients a, b, the case $N = 3$ and $p = 4$ can also be covered, see Remark 2.3 below.

We will show that, under suitable assumptions on the coefficients a and b, the dual variational approach and the existence results by Evéquoz and Weth in [26,28], see Chapter 1.4.1, extend to the case of the system (2.1) and provide the existence of dual ground state solutions of (2.1) provided $0 \le b \le p-1$. Such a solution (u,v) of (2.1) is said to be semitrivial if either $u \equiv 0$ or $v \equiv 0$ and fully nontrivial if both $u \not\equiv 0$ and $v \not\equiv 0$. Our aim is further to find conditions ensuring that the dual ground states are fully nontrivial. In brief, under suitable additional assumptions on b and p, we will demonstrate that the dual functional of the system (2.1) attains a ground state at some level $c_{\mu\nu}$ which is strictly below those ones obtained in the same way for dual ground states of the single Helmholtz equations $\Delta u - \mu u = a(x)|u|^{p-2}u$ resp. $-\Delta v - \nu v = a(x)|v|^{p-2}v$; from this fact we will infer that, under the aforementioned additional conditions, dual ground states cannot be semitrivial.

In order to prove the existence of dual ground state solutions, we will introduce a dual formulation for the system (2.1) of the form

$$\begin{cases} \partial_{\bar{s}} h(x,\bar{u},\bar{v}) = \mathcal{R}_\mu[\bar{u}] & \text{on } \mathbb{R}^N, \\ \partial_{\bar{t}} h(x,\bar{u},\bar{v}) = \mathcal{R}_\nu[\bar{v}] & \text{on } \mathbb{R}^N, \\ \bar{u},\bar{v} \in L^{p'}(\mathbb{R}^N). \end{cases} \tag{2.2}$$

Here $\mathcal{R}_\mu, \mathcal{R}_\nu$ denote the convolution operators with real-valued kernels Ψ_μ, Ψ_ν arising in the dual variational method by Evéquoz and Weth from Gutierrez' Limiting Absorption Principle as explained in Chapter 1.4.1, see equation (1.16). In the following chapter, we will explain and justify in detail the introduction of the dual variables $\bar{u}, \bar{v} \in L^{p'}(\mathbb{R}^N)$ via

$$\begin{aligned} \bar{u}(x) &:= a(x)\left(|u(x)|^{\frac{p}{2}} + b(x)|v(x)|^{\frac{p}{2}}\right)|u(x)|^{\frac{p}{2}-2}u(x), \\ \bar{v}(x) &:= a(x)\left(|v(x)|^{\frac{p}{2}} + b(x)|u(x)|^{\frac{p}{2}}\right)|v(x)|^{\frac{p}{2}-2}v(x) \end{aligned} \tag{2.3}$$

as well as the definition of the function $h : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, cf. Proposition 2.9 below. Notice that we use the notation $\bar{u}, \bar{v} \in L^{p'}(\mathbb{R}^N)$ in place of $u, v \in L^p(\mathbb{R}^N)$ whenever we are working in the dual setting; it does not denote complex conjugation, which does not occur in this chapter dealing with spaces of real-valued functions only.

As for the scalar case presented in Chapter 1.4.1, in view of the symmetry properties of the convolution operators $\mathcal{R}_\mu, \mathcal{R}_\nu$, the dual system (2.2) is variational. We introduce the corresponding energy functional

$$\begin{aligned} J_{\mu\nu} :& \quad L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N) \to \mathbb{R}, \\ J_{\mu\nu}(\bar{u},\bar{v}) :=& \int_{\mathbb{R}^N} h(x,\bar{u},\bar{v})\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} \bar{u}\,\mathcal{R}_\mu[\bar{u}] + \bar{v}\,\mathcal{R}_\nu[\bar{v}]\,\mathrm{d}x \end{aligned} \tag{2.4}$$

with mountain pass level

$$\begin{aligned} & c_{\mu\nu} := \inf_{\gamma\in\Gamma_{\mu\nu}} \sup_{0\le t\le 1} J_{\mu\nu}(\gamma(t)) \\ & \text{where } \Gamma_{\mu\nu} := \left\{\gamma \in C([0,1], L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)) :\ \gamma(0) = 0, J_{\mu\nu}(\gamma(1)) < 0\right\}. \end{aligned} \tag{2.5}$$

The main results will be proved under the following assumptions:

$$\begin{gathered} N \ge 2, \quad \mu,\nu > 0, \quad \frac{2(N+1)}{N-1} < p < 2^*, \\ a, b \in L^\infty(\mathbb{R}^N) \text{ are } [0,1]^N\text{-periodic with } 0 \le b(x) \le p-1,\ a(x) \ge a_0 > 0. \end{gathered} \tag{2.6}$$

We denote by a_-, b_- the (essential) infimum and by a_+, b_+ the (essential) supremum of the functions a and b, respectively.

Theorem 2.1 (Existence).

Assuming (2.6), there exists a nontrivial critical point $(\bar{u}, \bar{v}) \in L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)$ of the functional $J_{\mu\nu}$ on the mountain pass level $c_{\mu\nu} > 0$ and $(u, v) := \nabla_{\bar{s},\bar{t}} h(\,\cdot\,, \bar{u}, \bar{v})$ is a strong solution of the nonlinear Helmholtz system (2.1) with $u, v \in W^{2,q}(\mathbb{R}^N) \cap C^{1,\alpha}(\mathbb{R}^N)$ for all $q \in [p, \infty)$ and $\alpha \in (0, 1)$.

Remark 2.2 (The scalar case).

The scalar functional $I_\mu : L^{p'}(\mathbb{R}^N) \to \mathbb{R}$ from equation (1.18),

$$I_\mu(\bar{u}) = \frac{1}{p'} \int_{\mathbb{R}^N} a(x)^{1-p'} |\bar{u}|^{p'} \,\mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^N} \bar{u}\, \mathcal{R}_\mu[\bar{u}] \,\mathrm{d}x \tag{2.7}$$

satisfies $I_\mu(\bar{u}) := J_{\mu\nu}(\bar{u}, 0)$ due to Lemma 2.10 (iv) below. The results by Evéquoz and Weth in [26, 28] yield a critical point $\bar{u} \in L^{p'}(\mathbb{R}^N)$ of I_μ at the scalar mountain pass level c_μ and a corresponding solution $u = a^{1-p'} |\bar{u}|^{p'-2} \bar{u} \in W^{2,q}(\mathbb{R}^N) \cap C^{1,\alpha}(\mathbb{R}^N)$, $q \in [p, \infty)$ and $\alpha \in (0, 1)$, of the scalar Helmholtz equation

$$-\Delta u - \mu u = a(x) |u|^{p-2} u \quad \text{on } \mathbb{R}^N. \tag{2.8}$$

In view of Remark 1.10 (b), one can weaken assumption (2.6) imposing radial symmetry.

Remark 2.3 (Radial symmetry).

If we consider spaces of radially symmetric functions and constant coefficients a, b, all statements of this chapter requiring the assumptions (2.6) hold under the weaker condition $\frac{4N^2}{(N-1)(2N-1)} < p < 2^*$ instead of $\frac{2(N+1)}{N-1} < p < 2^*$.

Any critical point $(\bar{u}, \bar{v}) \in L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)$ of the functional $J_{\mu\nu}$ on the level $c_{\mu\nu}$ will henceforth be referred to as a ground state of $J_{\mu\nu}$ resp. of the dual system (2.2). Proposition 2.9 below ensures that there is a unique corresponding pair $(u, v) \in L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ according to the transformation (2.3), which we name a dual ground state of the system (2.1). In view of the definition of the mountain pass level in (2.5), ground states $(\bar{u}, \bar{v})$ are nontrivial critical points of $J_{\mu\nu}$ of minimal energy. They provide solutions (u, v) of the system (2.1); however, as in the scalar case in Chapter 1.4.1, the method resp. the definition of the functional only takes into account those solutions of (2.1) with asymptotic behavior as in equation (1.15), namely

$$\lim_{r\to\infty} \frac{1}{r} \int_{B_r(0)} \left| u(x) + 2 \left(\frac{2\pi}{|x|\sqrt{\mu}} \right)^{\frac{N-1}{2}} \operatorname{Re}\left(\mathrm{e}^{-\mathrm{i}(N-1)\frac{\pi}{4}} \mathrm{e}^{\mathrm{i}|x|\sqrt{\mu}} g_u(x/|x|) \right) \right|^2 \mathrm{d}x = 0,$$

$$\lim_{r\to\infty} \frac{1}{r} \int_{B_r(0)} \left| v(x) + 2 \left(\frac{2\pi}{|x|\sqrt{\nu}} \right)^{\frac{N-1}{2}} \operatorname{Re}\left(\mathrm{e}^{-\mathrm{i}(N-1)\frac{\pi}{4}} \mathrm{e}^{\mathrm{i}|x|\sqrt{\nu}} g_v(x/|x|) \right) \right|^2 \mathrm{d}x = 0$$

$$\text{where} \quad g_u(\xi) = -\frac{\mathrm{i}}{4} \left(\frac{\mu}{2\pi} \right)^{\frac{N}{2}-1} \mathcal{F}\left[a \left(|u|^{\frac{p}{2}} + b|v|^{\frac{p}{2}} \right) |u|^{\frac{p}{2}-2} u \right] (\xi\sqrt{\mu}), \qquad \xi \in \mathbb{S}^{N-1},$$

$$g_v(\xi) = -\frac{\mathrm{i}}{4} \left(\frac{\nu}{2\pi} \right)^{\frac{N}{2}-1} \mathcal{F}\left[a \left(|v|^{\frac{p}{2}} + b|u|^{\frac{p}{2}} \right) |v|^{\frac{p}{2}-2} v \right] (\xi\sqrt{\nu}), \qquad \xi \in \mathbb{S}^{N-1}.$$

More solutions with asymptotic behavior differing from the one just mentioned can be obtained by using modified versions of the convolution operators $\mathcal{R}_\mu$ resp. $\mathcal{R}_\nu$. We will not follow this idea at this point of the thesis in order to keep the presentation of the dual variational method clear. However, in Chapter 3, the variation of the convolution operator (see equation (3.12)) will be a central technique in demonstrating that, roughly speaking, Helmholtz equations admit various solutions due to this freedom of choice, which is in contrast to Schrödinger equations. This will, then, be done in the case $N = 3$ since it is especially illustrative and relevant in many applications.

A short calculation using (2.3) and the assumptions $a(x) > 0$ and $b(x) \geq 0$ in (2.6) shows that (u, v) is semitrivial (resp. fully nontrivial) if and only if $(\bar{u}, \bar{v})$ is semitrivial (resp. fully nontrivial). Theorem 2.1 yields the existence of a nontrivial dual ground state (u, v) of (2.1); it does not, however, exclude the semitrivial case where either $u \equiv 0$ or $v \equiv 0$. As we will prove in Lemma 2.15 (ii), a semitrivial (dual) ground state corresponds to a (dual) ground state solution of the scalar problem (see Remark 2.2); thus we now discuss under which conditions ground states of $J_{\mu\nu}$ resp. dual ground state solutions of (2.1) are fully nontrivial.

Theorem 2.4 (Fully nontrivial ground states).

Assume conditions (2.6) *to hold.*

(i) If $2 < p < 4$ *and* $b_- > 0$, *then every ground state of the functional* $J_{\mu\nu}$ *is fully nontrivial.*

(ii) If $p \geq 4$ *and* $b_- > \frac{a_+}{a_-} 2^{\frac{p-2}{2}} - 1$, *then there exists* $\delta > 0$ *with the property that, for* $\mu, \nu > 0$ *with* $\left|\sqrt{\frac{\mu}{\nu}} - 1\right| < \delta$, *every ground state of* $J_{\mu\nu}$ *is fully nontrivial.*

Theorem 2.5 (Semitrivial ground states).

Assume (2.6) *as well as*

$$p \geq 4 \quad \text{and} \quad 0 \leq b_+ < 2^{\frac{p-2}{2}} - 1.$$

Then every ground state of the functional $J_{\mu\nu}$ *is semitrivial.*

In the special case of constant coefficients a, b and $\mu = \nu$ we provide a full characterization of the parameter ranges where semitrivial and fully nontrivial dual ground state solutions occur. For $\mu = \nu$, it is also interesting to investigate the occurrence of fully nontrivial (dual) ground states which are diagonal, that is, they satisfy $|u| = |v| \not\equiv 0$ or, equivalently, $|\bar{u}| = |\bar{v}| \not\equiv 0$. (Indeed, the equivalence follows from equation (2.3) and from the fact that it defines a one-to-one correspondence, which we will explain in detail in the following section.) As for semitrivial solutions, diagonal solutions essentially solve the scalar problem

$$-\Delta u - \mu u = a(1 + b)\,|u|^{p-2}u \quad \text{on } \mathbb{R}^N,$$

which is why, at least in the case of constant coefficients, we provide criteria whether or not such diagonal ground states occur among the fully nontrivial ones.

Corollary 2.6.

Assume that conditions (2.6) hold with constant coefficients $a(x) \equiv a > 0$ and $b(x) \equiv b \in [0, p-1]$. Then we have the following:

(i) $J_{\mu\mu}$ attains the level $c_{\mu\mu}$ in a fully nontrivial ground state if and only if

$$2 < p < 4 \text{ and } b > 0 \qquad \text{or} \qquad p \geq 4 \text{ and } b \geq 2^{\frac{p-2}{2}} - 1.$$

In addition, these ground state solutions are non-diagonal if $b < \frac{p}{2} - 1$, and diagonal if $b \geq \frac{p}{2} - 1$ with $(p, b) \neq (4, 1)$. For $(p, b) = (4, 1)$, there exist both fully nontrivial non-diagonal and diagonal (as well as semitrivial) ground state solutions, and for every such ground state $(\bar{u}, \bar{v}) \in L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)$, the functions $|\bar{u}|, |\bar{v}|$ are linearly dependent.

(ii) $J_{\mu\mu}$ attains the level $c_{\mu\mu}$ in a semitrivial ground state if and only if

$$2 < p < 4 \text{ and } b = 0 \qquad \text{or} \qquad p \geq 4 \text{ and } 0 \leq b \leq 2^{\frac{p-2}{2}} - 1.$$

The proofs of these results will be given in Chapter 2.5. They essentially consist of a comparison of the energy levels $c_{\mu\nu}$ and $\min\{c_\mu, c_\nu\}$, cf. Lemma 2.15 in Chapter 2.4. Indeed, we will derive the conclusion in Theorem 2.4 from the strict inequality $c_{\mu\nu} < \min\{c_\mu, c_\nu\}$; in the situation of Theorem 2.5, however, we show that $c_{\mu\nu} = \min\{c_\mu, c_\nu\}$.

Remark 2.7.

If (2.6) holds and $p > 8$, we have $2^{\frac{p-2}{2}} - 1 > p - 1\ (\geq b_+)$ and thus by Theorem 2.5, only semitrivial (dual) ground states occur.

In the situation of Corollary 2.6 with $\mu = \nu$ and constant coefficients, fully nontrivial ground states occur only if $p \leq 8$, and fully nontrivial non-diagonal ground states occur only if $p < 4$ (if $b \neq 1$) resp. $p \leq 4$ (if $b = 1$) since, for $p \geq 4$, $2^{\frac{p-2}{2}} - 1 \geq \frac{p}{2} - 1$.

Finally, let us briefly compare our results concerning the occurrence of fully nontrivial dual ground state solutions of (2.1) to those available in the case of Schrödinger systems, i.e. $\mu, \nu < 0$ in (2.1). We assume $2 < p < 2^*$ and constant coupling $b(x) \equiv \beta \neq 0$. With a new parameter $\omega := \sqrt{\frac{\nu}{\mu}}$ obtained by rescaling, we discuss

$$\begin{cases} -\Delta u + \quad u = \left(|u|^{\frac{p}{2}} + \beta\, |v|^{\frac{p}{2}}\right) |u|^{\frac{p}{2}-2} u & \text{on } \mathbb{R}^N, \\ -\Delta v + \omega^2 v = \left(|v|^{\frac{p}{2}} + \beta\, |u|^{\frac{p}{2}}\right) |v|^{\frac{p}{2}-2} v & \text{on } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N). \end{cases}$$

Sharp characterizations of the occurrence of fully nontrivial ground state solutions have been provided by Mandel in [51] for the cooperative case $\beta > 0$, following pioneering work by Ambrosetti and Colorado [7], Maia, Montefusco and Pellacci [47] and others. In contrast to the Helmholtz case, the parameter p can be chosen from the full superlinear and subcritical range $2 < p < 2^*$ whereas in the Helmholtz case, we use mapping properties of the resolvent available only for $\frac{2(N+1)}{N-1} < p < 2^*$ with slight improvements in the case of constant coefficients and radially symmetric solutions. Moreover, in order to obtain a suitable dual formulation, our discussion for the Helmholtz system only covers the range $0 \leq \beta \leq p-1$; in particular, we only study cooperative systems. In the Schrödinger case, results for the repulsive case $\beta < 0$ are available as well, see for instance [49] and the references therein. Notice that in the special case $\omega = 1$ and $\mu = \nu$ in (2.2), the ranges for p and β in Theorem 1 and Remark 1(a) of [51] for the Schrödinger case agree with those from Corollary 2.6 above for the Helmholtz case. Finally, in the situation of Theorem 2.4 (ii) of the Helmholtz case, the question remains open whether there are threshold values for the existence and non-existence of fully nontrivial ground state solutions such as in Theorem 1 in [51].

2.2 The Dual Formulation. Convexity and the Legendre Transform

In this chapter we intend to explain and justify the transition from the nonlinear Helmholtz system (2.1) to its dual form (2.2). Let us first note that the system (2.1) can be written in the form

$$\begin{cases} -\Delta u - \mu u = \partial_s f(x,u,v) & \text{on } \mathbb{R}^N, \\ -\Delta v - \nu v = \partial_t f(x,u,v) & \text{on } \mathbb{R}^N, \\ u, v \in L^p(\mathbb{R}^N) \end{cases}$$

where

$$f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad f(x,s,t) = \frac{a(x)}{p}\left(|s|^p + 2b(x)|s|^{\frac{p}{2}}|t|^{\frac{p}{2}} + |t|^p\right). \tag{2.9}$$

Solutions of this system are provided by solving

$$\begin{cases} u = \mathcal{R}_\mu[\partial_s f(x,u,v)] & \text{on } \mathbb{R}^N, \\ v = \mathcal{R}_\nu[\partial_t f(x,u,v)] & \text{on } \mathbb{R}^N, \\ u, v \in L^p(\mathbb{R}^N) \end{cases} \tag{2.10}$$

where the convolution operators $\mathcal{R}_\mu, \mathcal{R}_\nu$ are constructed using Gutiérrez' Limiting Absorption Principle, see the explanations in Chapter 1.4.1. In this part, we focus on the transformation (2.3), which contains difficulties that do not appear in the case of a single equation as in [26, 28] but seem to be inevitable for coupled systems. As mentioned earlier, we aim to reformulate the system (2.10) by, roughly speaking, replacing the functions $u, v \in L^p(\mathbb{R}^N)$ by a corresponding pair

$$\bar{u}, \bar{v} \in L^{p'}(\mathbb{R}^N) \quad \text{via} \quad \bar{u} := \partial_s f(\,\cdot\,, u, v), \;\; \bar{v} := \partial_t f(\,\cdot\,, u, v),$$

see also (2.3), such that the convolutions occur in the linear part of the transformed equations. We will see in Proposition 2.9 that, under suitable assumptions on the coefficients a and b, this transformation is invertible and preserves the variational structure in the sense that

$$u = \partial_{\bar{s}} h(\,\cdot\,, \bar{u}, \bar{v}), \quad v = \partial_{\bar{t}} h(\,\cdot\,, \bar{u}, \bar{v})$$

with a suitable function $h : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, which then finally provides a one-to-one correspondence between solutions of the systems (2.10) and (2.2). It turns out that we have to choose $h(x, \,\cdot\,, \,\cdot\,)$ to be the Legendre transform of $f(x, \,\cdot\,, \,\cdot\,)$ for every fixed $x \in \mathbb{R}^N$. We remark that, in the case of a single nonlinear Helmholtz equation (2.8),

$-\Delta u - \mu u = a(x)|u|^{p-2}u$ on $\mathbb{R}^N$, the associated change of variables can be done explicitly,

$$\bar{u}(x) := a(x)|u(x)|^{p-2}u(x) \quad \text{and hence} \quad u(x) = a(x)^{1-p'}|\bar{u}(x)|^{p'-2}\bar{u}(x).$$

Notice that we have chosen to include the whole term $a(x)$ into the substitution, which is slightly different from the definition of $\bar{u}$ in [26, 28] and due to the fact that we aim for the scalar analogue of the transformation (2.3) where such a choice seems natural. In the case of a system of coupled equations, special situations admit to calculate the Legendre transform h explicitly. For instance, in the case $b(x) \equiv 1$, we have for all $x \in \mathbb{R}^N$ and $s, t \in \mathbb{R}$

$$\begin{aligned}
f(x,s,t) &= \frac{a(x)}{p}\left(|s|^{\frac{p}{2}} + |t|^{\frac{p}{2}}\right)^2, \\
\nabla_{s,t} f(x,s,t) &= a(x)\left(|s|^{\frac{p}{2}} + |t|^{\frac{p}{2}}\right)\begin{pmatrix} |s|^{\frac{p}{2}-2}s \\ |t|^{\frac{p}{2}-2}t \end{pmatrix}, \\
h(x,\bar{s},\bar{t}) &= \frac{a(x)^{1-p'}}{p'}\left(|\bar{s}|^{\frac{p}{p-2}} + |\bar{t}|^{\frac{p}{p-2}}\right)^{1-\frac{1}{p-1}}, \\
\nabla_{\bar{s},\bar{t}} h(x,\bar{s},\bar{t}) &= \left(a(x)\left(|\bar{s}|^{\frac{p}{p-2}} + |\bar{t}|^{\frac{p}{p-2}}\right)\right)^{1-p'}\begin{pmatrix} |\bar{s}|^{\frac{2}{p-2}-1}\bar{s} \\ |\bar{t}|^{\frac{2}{p-2}-1}\bar{t} \end{pmatrix}.
\end{aligned} \tag{2.11}$$

In particular, by the boundedness assumptions on $a(x)$ in (2.6), this calculation shows that $\left(\int_{\mathbb{R}^N} h(x,\bar{u},\bar{v})\,\mathrm{d}x\right)^{\frac{1}{p'}}$ defines an equivalent norm in the space $L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)$, which will also be an important consequence of the general discussion following next. In the general case, to the author's knowledge, no such explicit form is available. The transformation can still be done using the following classical result of convex analysis, see Theorems 26.5 and 26.6 in [65]:

Theorem 2.8 (The Legendre transform).

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be differentiable, strictly convex and co-finite. Then $\nabla F : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism, and the Legendre transform of F,

$$H : \mathbb{R}^2 \to \mathbb{R}, \quad H(\bar{s},\bar{t}) := \sup_{(s,t)\in\mathbb{R}^2} (s\bar{s} + t\bar{t} - F(s,t))$$

is well-defined, differentiable, strictly convex, co-finite and satisfies $\nabla H = (\nabla F)^{-1}$.

Let us remark that, for a convex function $F : \mathbb{R}^2 \to \mathbb{R}$, co-finiteness is characterized by

$$\lim_{\lambda\to\infty} \frac{F(\lambda s, \lambda t)}{\lambda} = \infty \quad \text{for all } (s,t) \neq (0,0),$$

cf. the equation before Theorem 26.6 in [65]. We check that, under the assumptions (2.6) and for fixed $x \in \mathbb{R}^N$, Theorem 2.8 applies to the function

$$f(x,\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,s,t) = \frac{a(x)}{p}\left(|s|^p + 2b(x)|s|^{\frac{p}{2}}|t|^{\frac{p}{2}} + |t|^p\right)$$

so that a dual variational formulation for (2.1) is available and given by (2.2).

Proposition 2.9 (Existence of the Legendre transform).

Let $p > 2$ and $x \in \mathbb{R}^N$ with $a(x) > 0$ and $0 \le b(x) \le p-1$. Then the function $f(x,\,\cdot\,,\,\cdot\,)$ is differentiable, strictly convex and co-finite. Hence, its Legendre transform $h(x,\,\cdot\,,\,\cdot\,)$ is well-defined, differentiable, strictly convex, co-finite and satisfies $\nabla_{\bar{s},\bar{t}} h(x,\,\cdot\,,\,\cdot\,) = (\nabla_{s,t} f(x,\,\cdot\,,\,\cdot\,))^{-1}$.

As for the following auxiliary results, the proof will be given at the very end of this chapter in Section 2.6.1. Let us emphasize that Proposition 2.9 is the only auxiliary result which requires the assumption $0 \le b(x) \le p-1$ in (2.6). We next provide some properties of the abstract transform $h(x,\,\cdot\,,\,\cdot\,)$.

Lemma 2.10 (Properties of the Legendre transform).

Let $p > 2$ and $x \in \mathbb{R}^N$ with $a(x) > 0$, $0 \le b(x) \le p-1$. Then, for $\bar{s}, \bar{t} \in \mathbb{R}$,

(i) $h(x,\bar{s},\bar{t}) = \dfrac{a(x)^{1-p'}}{p'} \left[\sup_{\sigma > 0} \dfrac{|\bar{s}| + \sigma|\bar{t}|}{(1 + 2b(x)\,\sigma^{\frac{p}{2}} + \sigma^p)^{\frac{1}{p}}} \right]^{p'}$,

(ii) $h(x,\bar{s},\bar{t}) = h(x,\bar{t},\bar{s}) = h(x,-\bar{s},\bar{t})$,

(iii) $h(x,\bar{s},\bar{t}) = \dfrac{1}{p'} \nabla_{\bar{s},\bar{t}} h(x,\bar{s},\bar{t}) \cdot \begin{pmatrix} \bar{s} \\ \bar{t} \end{pmatrix}$,

(iv) $h(x,\bar{s},0) = \dfrac{a(x)^{1-p'}}{p'} |\bar{s}|^{p'}$ *as well as* $h(x,\bar{s},\bar{s}) = \dfrac{2a(x)^{1-p'}}{p'} (1+b(x))^{1-p'} |\bar{s}|^{p'}$,

(v) $\dfrac{1}{p'} (a(x)(1+b(x)))^{1-p'} \big(|\bar{s}|^{p'} + |\bar{t}|^{p'} \big) \le h(x,\bar{s},\bar{t}) \le \dfrac{1}{p'} a(x)^{1-p'} \big(|\bar{s}|^{p'} + |\bar{t}|^{p'} \big)$.

If we additionally impose that the coefficients $a, b : \mathbb{R}^N \to \mathbb{R}$ are measurable, we conclude that for $\bar{u}, \bar{v} \in L^{p'}(\mathbb{R}^N)$ the mapping

$$\begin{aligned} x \mapsto h(x, \bar{u}(x), \bar{v}(x)) &= \sup_{s,t \in \mathbb{R}} \big(s\bar{u}(x) + t\bar{v}(x) - f(x,s,t) \big) \\ &= \sup_{s,t \in \mathbb{Q}} \big(s\bar{u}(x) + t\bar{v}(x) - f(x,s,t) \big) \end{aligned}$$

is measurable since it is a pointwise supremum of countably many measurable functions. Moreover, when combined with (v) of the previous Lemma, we have $h(\,\cdot\,,\bar{u},\bar{v}) \in L^1(\mathbb{R}^N)$ and the functional $J_{\mu\nu}$ as introduced in equation (2.4) is well-defined. Even more, we have the following auxiliary result:

Lemma 2.11.

Let $\mu, \nu > 0$, $p > 2$ and $a, b \in L^\infty(\mathbb{R}^N)$ with $a(x) \geq a_0 > 0$, $0 \leq b(x) \leq p-1$ almost everywhere. Then, the functional $J_{\mu\nu}$ in equation (2.4) is continuously Fréchet differentiable; in particular, for $\bar{u}, \bar{v} \in L^{p'}(\mathbb{R}^N)$,

$$J'_{\mu\nu}(\bar{u}, \bar{v})[\bar{u}, \bar{v}] = \int_{\mathbb{R}^N} p'\, h(x, \bar{u}, \bar{v})\, \mathrm{d}x - \int_{\mathbb{R}^N} \bar{u}\, \mathcal{R}_\mu[\bar{u}] + \bar{v}\, \mathcal{R}_\nu[\bar{v}]\, \mathrm{d}x.$$

2.3 Existence of Dual Ground States. Proof of Theorem 2.1

In this chapter, we give the proof of Theorem 2.1. This will be achieved using the Mountain Pass Theorem and following the ideas in [28], in particular involving the Nonvanishing Theorem 1.12 to account for the loss of compactness in the case of periodic coefficients at hand. We endow the product space $L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)$ with the norm denoted by $\|(\bar{u}, \bar{v})\|_{L^{p'}(\mathbb{R}^N)} := \|\bar{u}\|_{L^{p'}(\mathbb{R}^N)} + \|\bar{v}\|_{L^{p'}(\mathbb{R}^N)}$ for $\bar{u}, \bar{v} \in L^{p'}(\mathbb{R}^N)$ and, as a first step, collect the following two major auxiliary results:

Lemma 2.12 (Mountain Pass Geometry, see Lemma 4.2 in [28]).

Assuming (2.6), the functional $J_{\mu\nu}$ has the following properties:

(i) There exist $\delta > 0$ and $\rho \in (0, 1)$ with the property that, for $(\bar{u}, \bar{v}) \in L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)$, $J_{\mu\nu}(\bar{u}, \bar{v}) > 0$ if $0 < \|(\bar{u}, \bar{v})\|_{L^{p'}(\mathbb{R}^N)} \leq \rho$ and $J_{\mu\nu}(\bar{u}, \bar{v}) \geq \delta$ if $\|(\bar{u}, \bar{v})\|_{L^{p'}(\mathbb{R}^N)} = \rho$.

(ii) There exists $(\bar{u}_1, \bar{v}_1) \in L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)$ with $\|(\bar{u}_1, \bar{v}_1)\|_{L^{p'}(\mathbb{R}^N)} > 1$ and with $J_{\mu\nu}(\bar{u}_1, \bar{v}_1) < 0$.

(iii) Every Palais-Smale sequence for $J_{\mu\nu}$ is bounded in $L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)$.

Lemma 2.13 (Existence of Palais-Smale sequence, see Lemma 6.1 in [28]).

Assuming (2.6), there exists a bounded Palais-Smale sequence $(\bar{u}_n, \bar{v}_n)_{n\in\mathbb{N}}$ in $L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)$ for $J_{\mu\nu}$ at the level $c_{\mu\nu}$ as given in equation (2.5).

Proof of Lemmas 2.12, 2.13

Up to minor modifications, both results can be proved in the same way as the corresponding scalar results, Lemma 4.2 and Lemma 6.1 in [28] for $N \geq 3$. The same argumentation can be applied for $N = 2$, see the explanation at the beginning of the proof of Theorem 1.3 (b) of [26]. For the convenience of the reader, we present the central ideas with a focus on the required modifications.

We start with Lemma 2.12 on the Mountain Pass Geometry.

(i) Let $\rho \in (0,1)$ and consider $(\bar{u},\bar{v}) \in L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)$ with $\|(\bar{u},\bar{v})\|_{L^{p'}(\mathbb{R}^N)} = \rho$. Choosing $\kappa(\lambda) := \|\mathcal{R}_\lambda\|_{\mathcal{L}(L^{p'}(\mathbb{R}^N),L^p(\mathbb{R}^N))}$ for $\lambda \in \{\mu,\nu\}$ and using (v) of Lemma 2.10 with $\gamma_p := (a_+(1+b_+))^{1-p'}$, we estimate

$$\begin{aligned} J_{\mu\nu}(\bar{u},\bar{v}) &\geq \frac{\gamma_p}{p'} \int_{\mathbb{R}^N} |\bar{u}|^{p'} + |\bar{v}|^{p'} \,\mathrm{d}x \\ &\quad - \frac{1}{2}\Big(\|\bar{u}\|_{L^{p'}(\mathbb{R}^N)} \|\mathcal{R}_\mu[\bar{u}]\|_{L^p(\mathbb{R}^N)} + \|\bar{v}\|_{L^{p'}(\mathbb{R}^N)} \|\mathcal{R}_\nu[\bar{v}]\|_{L^p(\mathbb{R}^N)}\Big) \\ &\geq \frac{\gamma_p}{p'}\Big(\|\bar{u}\|^{p'}_{L^{p'}(\mathbb{R}^N)} + \|\bar{v}\|^{p'}_{L^{p'}(\mathbb{R}^N)}\Big) - \frac{1}{2}\Big(\kappa(\mu)\,\|\bar{u}\|^2_{L^{p'}(\mathbb{R}^N)} + \kappa(\nu)\,\|\bar{v}\|^2_{L^{p'}(\mathbb{R}^N)}\Big) \\ &\geq \frac{\gamma_p}{p'} \cdot \frac{1}{2^{p'}}\,\|(\bar{u},\bar{v})\|^{p'}_{L^{p'}(\mathbb{R}^N)} - \frac{\kappa(\mu)+\kappa(\nu)}{2}\,\|(\bar{u},\bar{v})\|^2_{L^{p'}(\mathbb{R}^N)} \\ &= \frac{\gamma_p}{2^{p'}p'}\rho^{p'} - \frac{\kappa(\mu)+\kappa(\nu)}{2}\rho^2 \end{aligned}$$

and since $p' < 2$, we find $\rho \in (0,1)$ sufficiently small with

$$\delta := \frac{\gamma_p}{2^{p'}p'}\rho^{p'} - \frac{\kappa(\mu)+\kappa(\nu)}{2}\rho^2 > 0.$$

(ii) Here we immediately use the result from the scalar case. For $N \geq 3$, by Lemma 4.2 (ii) in [28], there exists $\bar{u}_0 \in L^{p'}(\mathbb{R}^N)$ with $\|\bar{u}_0\|_{L^{p'}(\mathbb{R}^N)} = \|(\bar{u}_0,0)\|_{L^{p'}(\mathbb{R}^N)} > 1$ and $I_\mu(\bar{u}_0) = J_{\mu\nu}(\bar{u}_0,0) < 0$. We then let $(\bar{u}_1,\bar{v}_1) := (\bar{u}_0,0)$. In the case $N = 2$ the corresponding scalar result is mentioned at the beginning of the proof of Theorem 1.3 (b) of [26].

(iii) We consider a Palais-Smale sequence $(\bar{u}_n,\bar{v}_n)_{n\in\mathbb{N}} \subseteq L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)$ for the functional $J_{\mu\nu}$, i.e.

$$J'_{\mu\nu}(\bar{u}_n,\bar{v}_n) \to 0 \quad \text{and} \quad \sup_{n\in\mathbb{N}} J_{\mu\nu}(\bar{u}_n,\bar{v}_n) < \infty.$$

Using the identity in Lemma 2.11, we find for $\bar{u},\bar{v} \in L^{p'}(\mathbb{R}^N)$

$$\begin{aligned} J'_{\mu\nu}(\bar{u},\bar{v})[\bar{u},\bar{v}] &= \int_{\mathbb{R}^N} p' \cdot h(x,\bar{u},\bar{v})\,\mathrm{d}x - \int_{\mathbb{R}^N} \bar{u}\,\mathcal{R}_\mu[\bar{u}] + \bar{v}\,\mathcal{R}_\nu[\bar{v}]\,\mathrm{d}x \\ &= \begin{cases} p' \cdot J_{\mu\nu}(\bar{u},\bar{v}) - \left(1 - \frac{p'}{2}\right) \int_{\mathbb{R}^N} \bar{u}\,\mathcal{R}_\mu[\bar{u}] + \bar{v}\,\mathcal{R}_\nu[\bar{v}]\,\mathrm{d}x \\ 2 \cdot J_{\mu\nu}(\bar{u},\bar{v}) - (2-p') \int_{\mathbb{R}^N} h(x,\bar{u}_n,\bar{v}_n)\,\mathrm{d}x \end{cases} \end{aligned} \tag{2.12}$$

and estimate with $\alpha_n := \left\|J'_{\mu\nu}(\bar{u}_n,\bar{v}_n)\right\|_{\mathcal{L}(L^{p'}(\mathbb{R}^N)\times L^{p'}(\mathbb{R}^N),\mathbb{R})}$, (v) of Lemma 2.10 and $\gamma_p > 0$ as above:

$$\begin{aligned} J_{\mu\nu}(\bar{u}_n,\bar{v}_n) &\overset{(2.12)}{=} \frac{1}{2} J'_{\mu\nu}(\bar{u}_n,\bar{v}_n)[\bar{u}_n,\bar{v}_n] + \left(1 - \frac{p'}{2}\right) \int_{\mathbb{R}^N} h(x,\bar{u}_n,\bar{v}_n)\,\mathrm{d}x \\ &\geq -\frac{1}{2}\alpha_n \,\|(\bar{u}_n,\bar{v}_n)\|_{L^{p'}(\mathbb{R}^N)} + \left(1 - \frac{p'}{2}\right) \cdot \frac{\gamma_p}{p'}\big(\,\|\bar{u}\|^{p'}_{L^{p'}(\mathbb{R}^N)} + \|\bar{v}\|^{p'}_{L^{p'}(\mathbb{R}^N)}\big) \\ &\geq -\frac{1}{2}\alpha_n \,\|(\bar{u}_n,\bar{v}_n)\|_{L^{p'}(\mathbb{R}^N)} + \left(\frac{1}{p'} - \frac{1}{2}\right) \cdot \frac{\gamma_p}{2^{p'}}\,\|(\bar{u}_n,\bar{v}_n)\|^{p'}_{L^{p'}(\mathbb{R}^N)}. \end{aligned}$$

Then as in the proof of Lemma 4.2 (iii) for the scalar case (which is given for $N \geq 3$, but in fact the argument does not depend of the space dimension), the fact that $\alpha_n \to 0$ as $n \to \infty$ and the boundedness of $(J_{\mu\nu}(\bar{u}_n,\bar{v}_n))_{n\in\mathbb{N}}$, together with $1 < p' < 2$, imply that the sequence $(\bar{u}_n,\bar{v}_n)_{n\in\mathbb{N}}$ is bounded in $L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)$.

With that, Lemma 2.12 is proved. The proof of Lemma 2.13 is abstract; in the scalar case, it relies on the Deformation Lemma applied in the Banach space $L^{p'}(\mathbb{R}^N)$, cf. Lemma 6.1 in [28] for $N \geq 3$ and (the proof of) Theorem 1.3 (b) in [26] for $N = 2$, respectively. Taking instead the Banach space $L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)$, we can follow the proof in the scalar case line by line. □

Starting from these results, we now complete the proof of the Existence Theorem 2.1.

Proof of THEOREM 2.1

This proof mainly follows the lines of the proof of Theorem 1.3(b) in [26] for $N = 2$ and Theorem 6.2 in [28] for $N \geq 3$, respectively, which we will again refer to as the scalar case. We will therefore focus on those parts which differ due to the fact that we discuss a system of equations. Let $(\bar{u}_n, \bar{v}_n)_{n\in\mathbb{N}}$ denote a bounded Palais-Smale sequence at the level $c_{\mu\nu}$ which exists by Lemma 2.13; then w.l.o.g. $\bar{u}_n \rightharpoonup \bar{u}$ and $\bar{v}_n \rightharpoonup \bar{v}$ as $n \to \infty$ weakly in $L^{p'}(\mathbb{R}^N)$. We perform a concentration compactness argument which relies on the periodicity of the coefficients a, b.

▷ <u>STEP 1</u>: *(Nonvanishing.) There exists a ball $B \subseteq \mathbb{R}^N$ such that, up to a subsequence and up to translations,*

$$\inf_{n\in\mathbb{N}} \int_B h(x, \bar{u}_n, \bar{v}_n) \,\mathrm{d}x > 0.$$

As in the scalar case, definition (2.4) and identity (2.12) imply, as $n \to \infty$,

$$\begin{aligned} \int_{\mathbb{R}^N} \bar{u}_n \,\mathcal{R}_\mu[\bar{u}_n] + \bar{v}_n \,\mathcal{R}_\nu[\bar{v}_n] \,\mathrm{d}x &= \frac{2p}{p-2}\left[J_{\mu\nu}(\bar{u}_n, \bar{v}_n) - \frac{1}{p'} J'_{\mu\nu}(\bar{u}_n, \bar{v}_n)[\bar{u}_n, \bar{v}_n]\right] \\ &\to \frac{2p}{p-2} \cdot c_{\mu\nu}. \end{aligned}$$

As $c_{\mu\nu} > 0$ due to (2.5) and Lemma 2.12 (i), we conclude

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \bar{u}_n \,\mathcal{R}_\mu[\bar{u}_n] \,\mathrm{d}x > 0 \quad \text{or} \quad \limsup_{n\to\infty} \int_{\mathbb{R}^N} \bar{v}_n \,\mathcal{R}_\nu[\bar{v}_n] \,\mathrm{d}x > 0.$$

We apply the (scalar) Nonvanishing Theorem 1.12 by Evéquoz and Weth either to $\bar{u}_n$ or to $\bar{v}_n$, respectively. We remark that this is possible since $\bar{u}_n$, $\bar{v}_n$ are real-valued functions and hence $\bar{u}_n \,\mathcal{R}_\mu[\bar{u}_n] = \mathrm{Re}\,(\bar{u}_n \mathfrak{R}_\mu[\bar{u}_n])$, $\bar{v}_n \,\mathcal{R}_\nu[\bar{v}_n] = \mathrm{Re}\,(\bar{v}_n \mathfrak{R}_\nu[\bar{v}_n])$ due to definition (1.16). In any case, we thus find $R, \zeta > 0$ and $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^N$ such that, up to a subsequence,

$$\int_{B_R(x_n)} |\bar{u}_n|^{p'} + |\bar{v}_n|^{p'} \,\mathrm{d}x \geq \zeta \quad \text{for all } n \in \mathbb{N}.$$

Possibly enlarging the radius R, we may w.l.o.g. assume $x_n \in \mathbb{Z}^N$ for all $n \in \mathbb{N}$. By Lemma 2.10 (v), and with $\gamma_p := \frac{1}{p'}(a_+(1 + b_+))^{1-p'}$, we have

$$\int_{B_R(x_n)} h(x, \bar{u}_n, \bar{v}_n) \,\mathrm{d}x \geq \gamma_p \,\zeta \quad \text{for all } n \in \mathbb{N}. \tag{2.13}$$

Next, we introduce the shifted functions $\bar{U}_n(x) := \bar{u}(x_n + x)$ and $\bar{V}_n(x) := \bar{v}(x_n + x)$ for $n \in \mathbb{N}, x \in \mathbb{R}^N$. We note that, due to the periodicity of the coefficients a, b and since

$x_n \in \mathbb{Z}^N$, the Legendre transform is invariant under such translations in the sense that

$$\begin{aligned} h(x, \bar{U}_n(x), \bar{V}_n(x)) &= \sup_{s,t\in\mathbb{R}} \big(s\bar{U}_n(x) + t\bar{V}_n(x) - f(x,s,t)\big) \\ &= \sup_{s,t\in\mathbb{R}} \big(s\bar{U}_n(x) + t\bar{V}_n(x) - f(x_n + x,s,t)\big) \\ &= h(x_n + x, \bar{U}_n(x), \bar{V}_n(x)) \\ &= h(x_n + x, \bar{u}_n(x_n + x), \bar{v}_n(x_n + x)) \end{aligned}$$

for almost all $x \in \mathbb{R}^N$ and every $n \in \mathbb{N}$. Thus, and due to (2.13),

$$\int_{\mathbb{R}^N} h(x, \bar{U}_n, \bar{V}_n)\,\mathrm{d}x = \int_{\mathbb{R}^N} h(x, \bar{u}_n, \bar{v}_n)\,\mathrm{d}x \quad \text{and} \quad \int_{B_R(0)} h(x, \bar{U}_n, \bar{V}_n)\,\mathrm{d}x \geq \gamma_p\,\zeta. \tag{2.14}$$

With that, arguing as in the scalar case, we obtain that $(\bar{U}_n, \bar{V}_n)_{n\in\mathbb{N}}$ is a bounded Palais-Smale sequence for $J_{\mu\nu}$ to the level $c_{\mu\nu}$. Hence, w.l.o.g., there exist $\bar{U}, \bar{V} \in L^{p'}(\mathbb{R}^N)$ with $\bar{U}_n \rightharpoonup \bar{U}$ and $\bar{V}_n \rightharpoonup \bar{V}$ as $n \to \infty$ weakly in $L^{p'}(\mathbb{R}^N)$.

We intend to prove that

$$\int_{B_R(0)} h(x, \bar{U}_n, \bar{V}_n)\,\mathrm{d}x \to \int_{B_R(0)} h(x, \bar{U}, \bar{V})\,\mathrm{d}x \qquad \text{as } n \to \infty$$

and that hence, due to the inequality in (2.14), $(\bar{U}, \bar{V}) \neq (0,0)$. To this end, we need the following auxiliary result:

$\triangleright$ <u>Step 2</u>: *We have, as $n \to \infty$,*

$$\mathbb{1}_{B_R(0)} \cdot \nabla_{\bar{s},\bar{t}} h(\,\cdot\,, \bar{U}_n, \bar{V}_n) \to \mathbb{1}_{B_R(0)} \cdot \nabla_{\bar{s},\bar{t}} h(\,\cdot\,, \bar{U}, \bar{V}) \quad \textit{strongly in} \quad L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N).$$

Let $\varphi, \psi \in L^{p'}(\mathbb{R}^N)$ and $\tilde{\varphi} := \varphi \cdot \mathbb{1}_{B_R(0)}, \tilde{\psi} := \psi \cdot \mathbb{1}_{B_R(0)}$. We estimate for $m, n \in \mathbb{N}$

$$\begin{aligned} &\left|\int_{\mathbb{R}^N} \big(\mathbb{1}_{B_R(0)} \cdot \nabla_{\bar{s},\bar{t}} h(\,\cdot\,, \bar{U}_n, \bar{V}_n) - \mathbb{1}_{B_R(0)} \cdot \nabla_{\bar{s},\bar{t}} h(\,\cdot\,, \bar{U}_m, \bar{V}_m)\big) \cdot \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \mathrm{d}x\right| \\ &= \Bigg| J'_{\mu\nu}(\bar{U}_n, \bar{V}_n)[\tilde{\varphi}, \tilde{\psi}] - J'_{\mu\nu}(\bar{U}_m, \bar{V}_m)[\tilde{\varphi}, \tilde{\psi}] \\ &\qquad + \int_{\mathbb{R}^N} \tilde{\varphi} \cdot \mathcal{R}_\mu\left[\bar{U}_n - \bar{U}_m\right] \mathrm{d}x + \int_{\mathbb{R}^N} \tilde{\psi} \cdot \mathcal{R}_\nu\left[\bar{V}_n - \bar{V}_m\right] \mathrm{d}x \Bigg| \\ &\leq C_{nm} \left\|(\varphi, \psi)\right\|_{L^{p'}(\mathbb{R}^N)}, \end{aligned}$$

where

$$\begin{aligned} C_{nm} = &\left\|J'_{\mu\nu}(\bar{U}_n, \bar{V}_n)\right\|_{\mathcal{L}(L^{p'}(\mathbb{R}^N)\times L^{p'}(\mathbb{R}^N),\mathbb{R})} + \left\|J'_{\mu\nu}(\bar{U}_m, \bar{V}_m)\right\|_{\mathcal{L}(L^{p'}(\mathbb{R}^N)\times L^{p'}(\mathbb{R}^N),\mathbb{R})} \\ &+ \left\|\mathbb{1}_{B_R(0)} \cdot \mathcal{R}_\mu\left[\bar{U}_n - \bar{U}_m\right]\right\|_{L^p(\mathbb{R}^N)} + \left\|\mathbb{1}_{B_R(0)} \cdot \mathcal{R}_\nu\left[\bar{V}_n - \bar{V}_m\right]\right\|_{L^p(\mathbb{R}^N)}. \end{aligned}$$

Then, we have $C_{nm} \to 0$ as $m, n \to \infty$ since $\left\|J'_{\mu\nu}(\bar{U}_n, \bar{V}_n)\right\|_{\mathcal{L}(L^{p'}(\mathbb{R}^N)\times L^{p'}(\mathbb{R}^N),\mathbb{R})} \to 0$ by the Palais-Smale property and since the operators

$$L^{p'}(\mathbb{R}^N) \to L^p(\mathbb{R}^N), \quad g \mapsto \mathbb{1}_{B_R(0)} \cdot \mathcal{R}_\mu[g] \quad \text{and} \quad g \mapsto \mathbb{1}_{B_R(0)} \cdot \mathcal{R}_\nu[g]$$

are compact, cf. Lemma 4.1 in [28] for $N \geq 3$ and the corresponding result at the beginning of Section 3 of [26] for $N = 2$.

By duality, $(\mathbb{1}_{B_R(0)} \cdot \nabla_{\bar{s},\bar{t}} h(\,\cdot\,, \bar{U}_n, \bar{V}_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$. We thus find $U, V \in L^p(B_R(0))$ with

$$\nabla_{\bar{s},\bar{t}} h(\,\cdot\,, \bar{U}_n, \bar{V}_n) \to (U, V) \quad \text{as } n \to \infty \quad \text{in } L^p(B_R(0)) \times L^p(B_R(0)) \tag{2.15}$$

and, up to a subsequence, pointwise almost everywhere on $B_R(0)$.

As $\nabla_{s,t} f(x, \,\cdot\,, \,\cdot\,)$ is a homeomorphism on $\mathbb{R}^2$ for almost all $x \in \mathbb{R}^N$, we have

$$(\bar{U}_n, \bar{V}_n) \to \nabla_{s,t} f(\,\cdot\,, U, V) \quad \text{almost everywhere on } B_R(0)$$

as $n \to \infty$. Since the sequences $(\bar{U}_n)_{n\in\mathbb{N}}, (\bar{V}_n)_{n\in\mathbb{N}}$ are bounded in $L^{p'}(B_R(0))$, Theorem 1 in [37] implies that

$$(\bar{U}_n, \bar{V}_n) \rightharpoonup \nabla_{s,t} f(\,\cdot\,, U, V) \quad \text{weakly in } L^{p'}(B_R(0)) \times L^{p'}(B_R(0))$$

as $n \to \infty$. However, from the end of Step 1, we know that $(\bar{U}_n, \bar{V}_n) \rightharpoonup (\bar{U}, \bar{V})$ weakly in $L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)$ as $n \to \infty$. Uniqueness of the weak limit now implies $(\bar{U}, \bar{V}) = \nabla_{s,t} f(\,\cdot\,, U, V)$ in $L^{p'}(B_R(0)) \times L^{p'}(B_R(0))$, hence $(U, V) = \nabla_{\bar{s},\bar{t}} h(\,\cdot\,, \bar{U}, \bar{V})\big|_{B_R(0)}$ and due to (2.15)

$$\mathbb{1}_{B_R(0)} \cdot \nabla_{\bar{s},\bar{t}} h(\,\cdot\,, \bar{U}_n, \bar{V}_n) \to \mathbb{1}_{B_R(0)} \cdot \nabla_{\bar{s},\bar{t}} h(\,\cdot\,, \bar{U}, \bar{V}) \quad \text{as } n \to \infty \text{ in } L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N).$$

▷ <u>Step 3</u>: *Conclusion.*

We find with Lemma 2.10 (iii)

$$\begin{aligned}
&\left| \int_{B_R(0)} h(x, \bar{U}_n, \bar{V}_n) - h(x, \bar{U}, \bar{V}) \, \mathrm{d}x \right| \\
&\quad = \frac{1}{p'} \left| \int_{B_R(0)} \nabla_{\bar{s},\bar{t}} h(x, \bar{U}_n, \bar{V}_n) \cdot \begin{pmatrix} \bar{U}_n \\ \bar{V}_n \end{pmatrix} - \nabla_{\bar{s},\bar{t}} h(x, \bar{U}, \bar{V}) \cdot \begin{pmatrix} \bar{U} \\ \bar{V} \end{pmatrix} \, \mathrm{d}x \right| \\
&\quad \le \frac{1}{p'} \left\| \mathbb{1}_{B_R(0)} \left(\nabla_{\bar{s},\bar{t}} h(\,\cdot\,, \bar{U}_n, \bar{V}_n) - \nabla_{\bar{s},\bar{t}} h(\,\cdot\,, \bar{U}, \bar{V}) \right) \right\|_{L^p(\mathbb{R}^N)} \left\| (\bar{U}_n, \bar{V}_n) \right\|_{L^{p'}(\mathbb{R}^N)} \\
&\qquad + \frac{1}{p'} \left| \int_{B_R(0)} \nabla_{\bar{s},\bar{t}} h(x, \bar{U}, \bar{V}) \cdot \left[\begin{pmatrix} \bar{U}_n \\ \bar{V}_n \end{pmatrix} - \begin{pmatrix} \bar{U} \\ \bar{V} \end{pmatrix} \right] \, \mathrm{d}x \right|
\end{aligned}$$

and both terms tend to zero by Step 2 and by the weak convergence $\bar{U}_n \rightharpoonup \bar{U}, \bar{V}_n \rightharpoonup \bar{V}$ in $L^{p'}(\mathbb{R}^N)$, respectively. Hence, in view of the inequality in (2.14), we have

$$\int_{B_R(0)} h(x, \bar{U}, \bar{V}) \, \mathrm{d}x = \lim_{n\to\infty} \int_{B_R(0)} h(x, \bar{U}_n, \bar{V}_n) \, \mathrm{d}x \ge \gamma_p \, \zeta > 0,$$

which shows (via (v) of Lemma 2.10) that the weak limit satisfies $(\bar{U}, \bar{V}) \neq (0, 0)$.

What remains to prove is that indeed $J_{\mu\nu}(\bar{U}, \bar{V}) = c_{\mu\nu}$ and $J'_{\mu\nu}(\bar{U}, \bar{V}) = 0$. As in the scalar case, this is a consequence of the fact that $(\bar{U}_n, \bar{V}_n)_{n\in\mathbb{N}}$ is a Palais-Smale sequence which converges weakly to $(\bar{U}, \bar{V})$; for details cf. the last lines of the proof of Theorem 6.2 in [28] and of Theorem 1.3 (b) in [26], respectively.

Finally, letting $u := \partial_{\bar{s}} h(\,\cdot\,, \bar{U}, \bar{V})$ and $v := \partial_{\bar{t}} h(\,\cdot\,, \bar{U}, \bar{V})$, it can be shown as in Lemma 4.3 in [28] that this provides a strong solution of (2.1) and that $u, v \in W^{2,q}(\mathbb{R}^N) \cap C^{1,\alpha}(\mathbb{R}^N)$ for all $p \le q < \infty, 0 < \alpha < 1$. □

2.4 Energy Levels. An inf-sup Characterization of a Dual Ground State

As announced earlier, the proofs of Theorems 2.4 and 2.5 essentially consist of a comparison of energy levels. In brief, we will demonstrate in Lemma 2.15 that ground states of $J_{\mu\nu}$ which are semitrivial correspond to ground states of one of the scalar functionals I_μ resp. I_ν, and in particular that the associated mountain pass levels are the same. Hence once we prove that the mountain pass levels satisfy $c_{\mu\nu} \neq c_\mu$ and $c_{\mu\nu} \neq c_\nu$ (in fact, this always implies $c_{\mu\nu} < \min\{c_\mu, c_\nu\}$), we infer that the ground state of $J_{\mu\nu}$ cannot be semitrivial.

First of all, the following alternative characterization of the mountain pass level is a crucial ingredient. In preparation, we define $F_{\mu\nu} : L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N) \to (0, \infty]$ by

$$F_{\mu\nu}(\bar{u}, \bar{v}) := \frac{p-2}{2p} \left(\frac{\left[\int_{\mathbb{R}^N} p'\, h(x, \bar{u}, \bar{v})\, \mathrm{d}x\right]^{\frac{1}{p'}}}{\left[\int_{\mathbb{R}^N} \bar{u}\, \mathcal{R}_\mu[\bar{u}] + \bar{v}\, \mathcal{R}_\nu[\bar{v}]\, \mathrm{d}x\right]_+^{\frac{1}{2}}} \right)^{\frac{2p}{p-2}} \tag{2.16}$$

for $(\bar{u}, \bar{v}) \in L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)$, where we understand $F_{\mu\nu}(\bar{u}, \bar{v}) = +\infty$ in case the denominator vanishes, in particular $F_{\mu\nu}(0, 0) = +\infty$. With definition (2.4) and Lemma 2.10 (i), we have

$$J_{\mu\nu}(\tau\bar{u}, \tau\bar{v}) = \frac{\tau^{p'}}{p'} \int_{\mathbb{R}^N} p'\, h(x, \bar{u}, \bar{v})\, \mathrm{d}x - \frac{\tau^2}{2} \int_{\mathbb{R}^N} \bar{u}\, \mathcal{R}_\mu[\bar{u}] + \bar{v}\, \mathcal{R}_\nu[\bar{v}]\, \mathrm{d}x \tag{2.17}$$

for $\tau > 0$. For $(\bar{u}, \bar{v}) \neq (0, 0)$, the mapping $\tau \mapsto J_{\mu\nu}(\tau\bar{u}, \tau\bar{v})$ possesses a critical point on $(0, \infty)$ if and only if $\int_{\mathbb{R}^N} \bar{u}\, \mathcal{R}_\mu[\bar{u}] + \bar{v}\, \mathcal{R}_\nu[\bar{v}]\, \mathrm{d}x > 0$; in this case, the critical point is unique and a global maximum. A straightforward calculation shows

$$\sup_{\tau>0} J_{\mu\nu}(\tau\bar{u}, \tau\bar{v}) = F_{\mu\nu}(\bar{u}, \bar{v}), \tag{2.18}$$

which continues to hold if the maximum is not attained in the sense that the supremum equals $+\infty$. As a result of one-dimensional calculus, we obtain the following Lemma which provides an inf-sup characterization of the mountain pass level $c_{\mu\nu}$ defined in equation (2.5). Variants of it can be found in the literature, e.g. the first lines of the proof of Proposition 2.1 in [50] and the text before Lemma 2.1 in [24]. Again, we postpone the proofs to the end of the chapter, Section 2.6.2.

Lemma 2.14 (inf-sup characterization).

Under the assumptions given in (2.6), the mountain pass level $c_{\mu\nu}$ as defined in equation (2.5) can be characterized as follows:

$$\begin{aligned} c_{\mu\nu} &= \inf \left\{ F_{\mu\nu}(\bar{u}, \bar{v}) : \; (\bar{u}, \bar{v}) \in L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N) \right\} \\ &= \inf_{\substack{\bar{u}, \bar{v} \in L^{p'}(\mathbb{R}^N) \\ (\bar{u}, \bar{v}) \neq (0,0)}} \sup_{\tau>0} J_{\mu\nu}(\tau\bar{u}, \tau\bar{v}). \end{aligned}$$

Moreover, $(\bar{u}_0, \bar{v}_0) \in L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)$ is a minimizer of the functional $F_{\mu\nu}$ if and only if it is a nonzero multiple of a critical point of $J_{\mu\nu}$ on the mountain pass level $c_{\mu\nu}$.

These results also apply in the scalar case discussed in Remark 2.2; we define the functional

$$E_\mu : L^{p'}(\mathbb{R}^N) \to (0,\infty], \quad E_\mu(\bar{u}) := \frac{p-2}{2p} \left(\frac{\left[\int_{\mathbb{R}^N} a(x)^{1-p'} |\bar{u}|^{p'} \,\mathrm{d}x\right]^{\frac{1}{p'}}}{\left[\int_{\mathbb{R}^N} \bar{u}\, \mathcal{R}_\mu[\bar{u}] \,\mathrm{d}x\right]_+^{\frac{1}{2}}} \right)^{\frac{2p}{p-2}} \tag{2.19}$$

again with $E_\mu(\bar{u}) := +\infty$ if the denominator is zero. Then $E_\mu(\bar{u}) = F_{\mu\nu}(\bar{u}, 0)$ for $\bar{u} \in L^{p'}(\mathbb{R}^N)$ by Lemma 2.10 (iv) and the scalar mountain pass level c_μ is the infimum of E_μ, attained in particular at critical points of I_μ on the level c_μ.

We derive some direct consequences describing the relation between the mountain pass level associated with the system (2.2) and the scalar mountain pass level. Recall that a critical point of $J_{\mu\nu}$ on the mountain pass level $c_{\mu\nu}$ is said to be a ground state, and we adopt the same nomenclature in the scalar case.

Lemma 2.15.

We assume that conditions (2.6) *hold. Then we have the following:*

(i) The inequality $c_{\mu\nu} \leq \min\{c_\mu, c_\nu\}$ holds.

(ii) If $(\bar{u}_0, 0)$ is a semitrivial ground state of $J_{\mu\nu}$, then $c_{\mu\nu} = c_\mu$ and $\bar{u}_0$ is a ground state of the scalar functional I_μ.

2.5 Structure of Dual Ground States. Proof of Theorems 2.4, 2.5 and Corollary 2.6

We recall that $a_- \leq a(x) \leq a_+$ and $b_- \leq b(x) \leq b_+$ hold for almost all $x \in \mathbb{R}^N$. By $h_\pm : \mathbb{R}^2 \to \mathbb{R}$ we denote the Legendre transforms of the functions

$$f_\pm : \mathbb{R}^2 \to \mathbb{R}, \quad f_\pm(s,t) := \frac{1}{p}\left(|s|^p + 2b_\pm\, |s|^{\frac{p}{2}} |t|^{\frac{p}{2}} + |t|^p \right). \tag{2.20}$$

As a direct consequence of Lemma 2.10 (i), we have for $\bar{s}, \bar{t} \in \mathbb{R}$

$$h_\pm(\bar{s}, \bar{t}) = \frac{1}{p'} \left[\sup_{\sigma > 0} \frac{|\bar{s}| + \sigma |\bar{t}|}{(1 + 2b_\pm\, \sigma^{\frac{p}{2}} + \sigma^p)^{\frac{1}{p}}} \right]^{p'} \tag{2.21}$$

and obtain the following chain of inequalities for all $\bar{s}, \bar{t} \in \mathbb{R}$ and $x \in \mathbb{R}^N$:

$$a_+^{1-p'} h_+(\bar{s}, \bar{t}) \leq a(x)^{1-p'} h_+(\bar{s}, \bar{t}) \leq h(x, \bar{s}, \bar{t}) \leq a(x)^{1-p'} h_-(\bar{s}, \bar{t}) \leq a_-^{1-p'} h_-(\bar{s}, \bar{t}). \tag{2.22}$$

Moreover, we will need the following auxiliary result, the proof of which will be given in Section 2.6.3:

Lemma 2.16.

Assuming (2.6), *we have for* $\bar{u}, \bar{v} \in L^{p'}(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} h(x, \bar{u}, \bar{v})\, \mathrm{d}x \geq h_+\left(\left\| a^{-1/p}\, \bar{u} \right\|_{L^{p'}(\mathbb{R}^N)}, \left\| a^{-1/p}\, \bar{v} \right\|_{L^{p'}(\mathbb{R}^N)} \right).$$

For $b_+ > 0$, *equality can hold only if* $|\bar{u}|, |\bar{v}|$ *are linearly dependent.*

Proof of Theorem 2.4 (i)

We consider a minimizer $\bar{w} \in L^{p'}(\mathbb{R}^N)$, $\bar{w} \neq 0$ of the scalar functional E_μ, that is, by definition (2.19)

$$c_\mu = E_\mu(\bar{w}) = \frac{p-2}{2p} \left(\frac{\left[\int_{\mathbb{R}^N} a(x)^{1-p'} |\bar{w}|^{p'}\, \mathrm{d}x \right]^{\frac{1}{p'}}}{\left[\int_{\mathbb{R}^N} \bar{w}\, \mathcal{R}_\mu[\bar{w}]\, \mathrm{d}x \right]^{\frac{1}{2}}} \right)^{\frac{2p}{p-2}}. \tag{2.23}$$

(Notice that, for a minimizer, the denominator is strictly positive.) The idea is now to prove $c_{\mu\nu} < c_\mu$ by showing that $(c_{\mu\nu} \leq)\ F_{\mu\nu}(\bar{w}, \bar{w}_\eta) < E_\mu(\bar{w})$ for some suitable "small" $\bar{w}_\eta \in L^{p'}(\mathbb{R}^N)$. More precisely, we will choose $\bar{w}_\eta = \eta \cdot \bar{w} \mathbb{1}_B$ with some $B \subseteq \mathbb{R}^N$ and some small $\eta > 0$.

Theorem 1.1 in [26] for $N = 2$ and Lemma 4.3 in [28] for $N \geq 3$, respectively, ensure that $\bar{w}$ is continuous. Since $\bar{w} \not\equiv 0$, we can choose $x_0 \in \mathbb{R}^N$ with $\bar{w}(x_0) \neq 0$, and due to continuity there exists $r_0 > 0$ such that either $\bar{w} > 0$ on $B_{r_0}(x_0)$ or $\bar{w} < 0$ on $B_{r_0}(x_0)$. Moreover, equations (6) in [26] and (11), (12) in [28] imply that $\Psi_\nu > 0$ near zero. Hence we can choose $r_1 > 0$ with $\Psi_\nu > 0$ on $B_{2r_1}(0)$. Then, with $r := \min\{r_0, r_1\}$,

$$q := \frac{\int_{\mathbb{R}^N} \bar{w} \mathbb{1}_{B_r(x_0)}\, \mathcal{R}_\nu[\bar{w} \mathbb{1}_{B_r(x_0)}]\, \mathrm{d}x}{\int_{\mathbb{R}^N} \bar{w}\, \mathcal{R}_\mu[\bar{w}]\, \mathrm{d}x} = \frac{\int_{B_r(x_0)} \int_{B_r(x_0)} \bar{w}(y) \bar{w}(z) \Psi_\nu(y-z)\, \mathrm{d}y \mathrm{d}z}{\int_{\mathbb{R}^N} \bar{w}\, \mathcal{R}_\mu[\bar{w}]\, \mathrm{d}x} > 0$$

and we estimate for sufficiently small $\eta > 0$:

$$\begin{aligned}
c_{\mu\nu} &\overset{\text{Lem. 2 14}}{=} \inf_{\bar{u}, \bar{v} \in L^{p'}(\mathbb{R}^N)} F_{\mu\nu}(\bar{u}, \bar{v}) \\
&\leq F_{\mu\nu}(\bar{w}, \eta\, \bar{w} \mathbb{1}_{B_r(x_0)}) \\
&= \frac{p-2}{2p} \left(\frac{\left(\int_{\mathbb{R}^N} p'\, h(x, \bar{w}, \eta\, \bar{w} \mathbb{1}_{B_r(x_0)})\, \mathrm{d}x \right)^{\frac{1}{p'}}}{\left(\int_{\mathbb{R}^N} \bar{w}\, \mathcal{R}_\mu[\bar{w}] + \eta^2\, \bar{w} \mathbb{1}_{B_r(x_0)}\, \mathcal{R}_\nu[\bar{w} \mathbb{1}_{B_r(x_0)}]\, \mathrm{d}x \right)_+^{\frac{1}{2}}} \right)^{\frac{2p}{p-2}} \\
&= \frac{p-2}{2p} \left(\frac{\left(\int_{\mathbb{R}^N} p'\, h(x, \bar{w}, \eta\, \bar{w} \mathbb{1}_{B_r(x_0)})\, \mathrm{d}x \right)^{\frac{1}{p'}}}{\left(\int_{\mathbb{R}^N} \bar{w}\, \mathcal{R}_\mu[\bar{w}]\, \mathrm{d}x \right)^{\frac{1}{2}} \cdot (1 + \eta^2 q)^{\frac{1}{2}}} \right)^{\frac{2p}{p-2}} \\
&\overset{(2.22)}{\leq} \frac{p-2}{2p} \left(\frac{\left(\int_{\mathbb{R}^N} p'\, a(x)^{1-p'} h_-(\bar{w}, \eta\, \bar{w} \mathbb{1}_{B_r(x_0)})\, \mathrm{d}x \right)^{\frac{1}{p'}}}{\left(\int_{\mathbb{R}^N} \bar{w}\, \mathcal{R}_\mu[\bar{w}]\, \mathrm{d}x \right)^{\frac{1}{2}} \cdot (1 + \eta^2 q)^{\frac{1}{2}}} \right)^{\frac{2p}{p-2}}
\end{aligned}$$

$$\overset{(2.21)}{=} \frac{p-2}{2p}\left(\frac{\left(\int_{\mathbb{R}^N} a(x)^{1-p'}\left(\sup_{\sigma>0}\frac{|\bar w(x)|+\sigma\eta|\bar w(x)|\mathbb{1}_{B_r(x_0)}(x)}{(1+2b_-\sigma^{\frac p2}+\sigma^p)^{\frac1p}}\right)^{p'}\mathrm{d}x\right)^{\frac{1}{p'}}}{\left(\int_{\mathbb{R}^N}\bar w\,\mathcal{R}_\mu[\bar w]\,\mathrm{d}x\right)^{\frac12}\cdot(1+\eta^2 q)^{\frac12}}\right)^{\frac{2p}{p-2}}$$

$$\le \frac{p-2}{2p}\left(\frac{\left(\int_{\mathbb{R}^N} a(x)^{1-p'}|\bar w(x)|^{p'}\,\mathrm{d}x\right)^{\frac{1}{p'}}\cdot\sup_{\sigma>0}\frac{1+\sigma\eta}{(1+2b_-\sigma^{\frac p2}+\sigma^p)^{\frac1p}}}{\left(\int_{\mathbb{R}^N}\bar w\,\mathcal{R}_\mu[\bar w]\,\mathrm{d}x\right)^{\frac12}\cdot(1+\eta^2 q)^{\frac12}}\right)^{\frac{2p}{p-2}}$$

$$\overset{(2.23)}{=} c_\mu\cdot\left(\sup_{\sigma>0}\frac{1+\sigma\eta}{(1+2b_-\sigma^{\frac p2}+\sigma^p)^{\frac1p}(1+\eta^2q)^{\frac12}}\right)^{\frac{2p}{p-2}}$$

$$< c_\mu.$$

The latter estimate holds for sufficiently small positive η because we have, with $\tilde b_- := \min\{1,b_-\}>0$ and Taylor's Theorem,

$$\begin{aligned}
\sup_{\sigma>0}\frac{1+\sigma\eta}{(1+2b_-\sigma^{\frac p2}+\sigma^p)^{\frac1p}(1+\eta^2q)^{\frac12}} &\le \sup_{\sigma>0}\frac{1+\sigma\eta}{(1+\tilde b_-\sigma^{\frac p2})^{\frac2p}(1+\eta^2q)^{\frac12}}\\
&= \frac{\left(1+\eta^{\frac{p}{p-2}}\tilde b_-^{-\frac{2}{p-2}}\right)^{\frac{p-2}{p}}}{(1+\eta^2q)^{\frac12}}\\
&= \frac{1+\frac{p-2}{p}\eta^{\frac{p}{p-2}}\tilde b_-^{-\frac{2}{p-2}}+o\left(\eta^{\frac{p}{p-2}}\right)}{1+\frac12\eta^2q+o(\eta^2)}\\
&= 1-\frac12\eta^2q+o(\eta^2) \qquad\qquad \text{as } \eta\searrow 0
\end{aligned}$$

where we used that $\frac{p}{p-2}>2$ since $2<p<4$. We have shown that $c_{\mu\nu}<c_\mu$. Similarly, one proves that $c_{\mu\nu}<c_\nu$. Lemma 2.15 (ii) implies that $J_{\mu\nu}$ cannot have a semitrivial dual ground state. □

The motivation for the proof of Theorem 2.4 (ii) is the following observation: Given $\mu=\nu$, we have for $\bar w\in L^{p'}(\mathbb{R}^N)$

$$F_{\mu\mu}(\bar w,\bar w) = \frac{p-2}{2p}\left(\frac{\left[\int_{\mathbb{R}^N} p'\,h\,(x,\bar w,\bar w)\,\mathrm{d}x\right]^{\frac{1}{p'}}}{\left[\int_{\mathbb{R}^N} 2\,\bar w\,\mathcal{R}_\mu[\bar w]\,\mathrm{d}x\right]_+^{\frac12}}\right)^{\frac{2p}{p-2}} \overset{\text{Lem. 2 10(iv)}}{\le} \frac{2}{(1+b_-)^{\frac{2}{p-2}}}\cdot E_\mu(\bar w);$$

hence $c_{\mu\mu}\le \frac{2}{(1+b_-)^{\frac{2}{p-2}}}\cdot c_\mu < c_\mu$ by assumption on b and p. This will be extended to the case $\mu\ne\nu$, $\mu\approx\nu$ by a continuity argument which requires additional knowledge of the scalar case for $a\equiv 1$. Here we let

$$D_\lambda(\bar w) := \frac{p-2}{2p}\left(\frac{\left(\int_{\mathbb{R}^N}|\bar w|^{p'}\,\mathrm{d}x\right)^{\frac{1}{p'}}}{\left(\int_{\mathbb{R}^N}\bar w\,\mathcal{R}_\lambda[\bar w]\,\mathrm{d}x\right)_+^{\frac12}}\right)^{\frac{2p}{p-2}}, \qquad d_\lambda := \inf_{\bar w\in L^{p'}(\mathbb{R}^N)} D_\lambda(\bar w)$$

and, in view of definition (2.19), immediately note that

$$a_+^{-\frac{2}{p-2}}d_\lambda \le c_\lambda \le a_-^{-\frac{2}{p-2}}d_\lambda. \tag{2.24}$$

Remark 2.2 guarantees that the functional I_1 with $a(x) \equiv 1$ admits a ground state $\bar{z} \in L^{p'}(\mathbb{R}^N)$ which, by the remarks following Lemma 2.14, is a minimizer of the functional D_1. We fix such a minimizer $\bar{z}$ and introduce for $\lambda > 0$ the rescaled functions

$$\bar{z}_\lambda \in L^{p'}(\mathbb{R}^N), \qquad \bar{z}_\lambda(x) := \lambda^{\frac{N+2}{4}} \bar{z}(\sqrt{\lambda}\, x), \qquad x \in \mathbb{R}^N. \tag{2.25}$$

Then $\bar{z}_\lambda$ is a minimizer of the functional D_λ, and we have

$$d_\lambda = \lambda^{\frac{p}{p-2}-\frac{N}{2}} \cdot d_1 = \lambda^{\frac{p}{p-2}-\frac{N}{2}} \cdot \frac{p-2}{2p} \left(\frac{\left[\int_{\mathbb{R}^N} |\bar{z}|^{p'}\,\mathrm{d}x\right]^{\frac{1}{p'}}}{\left[\int_{\mathbb{R}^N} \bar{z}\,\mathcal{R}_1[\bar{z}]\,\mathrm{d}x\right]^{\frac{1}{2}}} \right)^{\frac{2p}{p-2}}. \tag{2.26}$$

The proof of (2.26) is based on the observations

$$\Psi_\lambda(x) = \lambda^{\frac{N-2}{2}} \Psi_1\left(\sqrt{\lambda} x\right) \quad \text{and hence} \quad \int_{\mathbb{R}^N} \bar{z}_\lambda\, \mathcal{R}_\lambda[\bar{z}_\lambda]\,\mathrm{d}x = \int_{\mathbb{R}^N} \bar{z}\,\mathcal{R}_1[\bar{z}]\,\mathrm{d}x \tag{2.27}$$

for $x \in \mathbb{R}^N$ and $\lambda > 0$, which is a direct consequence of the form of the kernel $\Psi_\lambda = \operatorname{Re}\Phi_\lambda$, see Theorem 1.9 (ii).

Proof of Theorem 2.4 (ii)

We aim to prove $c_{\mu\nu} < \min\{c_\mu, c_\nu\}$ for sufficiently small values of $\left|\sqrt{\frac{\mu}{\nu}} - 1\right|$, which again yields the assertion when applying Lemma 2.15 (ii).

With scalar minimizers $\bar{z}, \bar{z}_\mu, \bar{z}_\nu \in L^{p'}(\mathbb{R}^N)$ as above, we estimate as follows:

$$\begin{aligned}
c_{\mu\nu} &\le F_{\mu\nu}(\bar{z}_\mu, \bar{z}_\nu) \\
&= \frac{p-2}{2p} \left(\frac{\left[\int_{\mathbb{R}^N} p'\, h(x, \bar{z}_\mu, \bar{z}_\nu)\,\mathrm{d}x\right]^{\frac{1}{p'}}}{\left[\int_{\mathbb{R}^N} \bar{z}_\mu\,\mathcal{R}_\mu[\bar{z}_\mu]\,\mathrm{d}x + \int_{\mathbb{R}^N} \bar{z}_\nu\,\mathcal{R}_\nu[\bar{z}_\nu]\,\mathrm{d}x\right]_+^{\frac{1}{2}}} \right)^{\frac{2p}{p-2}} \\
&\overset{(2.27)}{=} \frac{p-2}{2p} \left(\frac{\left[\int_{\mathbb{R}^N} p'\, h(x, \bar{z}_\mu, \bar{z}_\nu)\,\mathrm{d}x\right]^{\frac{1}{p'}}}{\left[2\int_{\mathbb{R}^N} \bar{z}\,\mathcal{R}_1[\bar{z}]\,\mathrm{d}x\right]^{\frac{1}{2}}} \right)^{\frac{2p}{p-2}} \\
&\overset{(2.26)}{=} \frac{d_1}{2^{\frac{p}{p-2}}} \left(\frac{\int_{\mathbb{R}^N} p'\, h(x, \bar{z}_\mu, \bar{z}_\nu)\,\mathrm{d}x}{\int_{\mathbb{R}^N} |\bar{z}|^{p'}\,\mathrm{d}x} \right)^{\frac{2(p-1)}{p-2}} \\
&\overset{(2.22)}{\le} \frac{d_1}{2^{\frac{p}{p-2}}} \left(\frac{a_-^{1-p'} \int_{\mathbb{R}^N} p'\, h_-(\bar{z}_\mu, \bar{z}_\nu)\,\mathrm{d}x}{\int_{\mathbb{R}^N} |\bar{z}|^{p'}\,\mathrm{d}x} \right)^{\frac{2(p-1)}{p-2}} \\
&\overset{\text{Lem.2 10}(iv)}{\le} \frac{d_1}{2^{\frac{p}{p-2}} a_-^{\frac{2}{p-2}}} \left(\frac{\int_{\mathbb{R}^N} h_-(\bar{z}_\mu, \bar{z}_\nu)\,\mathrm{d}x}{\frac{1}{2}(1+b_-)^{\frac{1}{p-1}} \int_{\mathbb{R}^N} h_-(\bar{z}, \bar{z})\,\mathrm{d}x} \right)^{\frac{2(p-1)}{p-2}} \\
&= d_1 \cdot \left(\frac{2^{\frac{p-2}{2}}}{1+b_-} \frac{1}{a_-} \right)^{\frac{2}{p-2}} \left(\frac{\int_{\mathbb{R}^N} h_-(\bar{z}_\mu, \bar{z}_\nu)\,\mathrm{d}x}{\int_{\mathbb{R}^N} h_-(\bar{z}, \bar{z})\,\mathrm{d}x} \right)^{\frac{2(p-1)}{p-2}}.
\end{aligned}$$

We now introduce $\lambda := \sqrt{\frac{\mu}{\nu}}$. Then, with $h_-(\alpha\bar{s}, \alpha\bar{t}) = |\alpha|^{p'} h_-(\bar{s}, \bar{t})$ (see equation (2.21)) and substitution:

$$\left(\int_{\mathbb{R}^N} h_-(\bar{z}_\mu, \bar{z}_\nu)\,\mathrm{d}x \right)^{\frac{2(p-1)}{p-2}} = \left(\int_{\mathbb{R}^N} h_-\left(\mu^{\frac{N+2}{4}} \bar{z}(\sqrt{\mu}x), \nu^{\frac{N+2}{4}} \bar{z}(\sqrt{\nu}x)\right)\,\mathrm{d}x \right)^{\frac{2(p-1)}{p-2}}$$

$$
\begin{aligned}
&= \left(\int_{\mathbb{R}^N} h_- \left(\lambda^{\frac{N+2}{2}} \nu^{\frac{N+2}{4}} \bar{z}(\lambda\sqrt{\nu}x), \nu^{\frac{N+2}{4}} \bar{z}(\sqrt{\nu}x) \right) \mathrm{d}x \right)^{\frac{2(p-1)}{p-2}} \\
&= \nu^{\frac{p}{p-2}-\frac{N}{2}} \left(\int_{\mathbb{R}^N} h_- \left(\lambda^{\frac{N+2}{2}} \bar{z}(\lambda y), \bar{z}(y) \right) \mathrm{d}y \right)^{\frac{2(p-1)}{p-2}} \\
&= \nu^{\frac{p}{p-2}-\frac{N}{2}} \left(\int_{\mathbb{R}^N} h_- \left(\bar{z}_{\lambda^2}, \bar{z} \right) \mathrm{d}y \right)^{\frac{2(p-1)}{p-2}} .
\end{aligned}
$$

We insert this into the previous estimate and find

$$
\begin{aligned}
c_{\mu\nu} &\leq d_1 \, \nu^{\frac{p}{p-2}-\frac{N}{2}} \cdot \left(\frac{2^{\frac{p-2}{2}}}{1+b_-} \frac{1}{a_-} \right)^{\frac{2}{p-2}} \left(\frac{\int_{\mathbb{R}^N} h_- \left(\bar{z}_{\lambda^2}, \bar{z} \right) \mathrm{d}x}{\int_{\mathbb{R}^N} h_-(\bar{z}, \bar{z}) \, \mathrm{d}x} \right)^{\frac{2(p-1)}{p-2}} \\
&\overset{(2.26)}{=} d_\nu \cdot \left(\frac{2^{\frac{p-2}{2}}}{1+b_-} \frac{1}{a_-} \right)^{\frac{2}{p-2}} \left(\frac{\int_{\mathbb{R}^N} h_- \left(\bar{z}_{\lambda^2}, \bar{z} \right) \mathrm{d}x}{\int_{\mathbb{R}^N} h_-(\bar{z}, \bar{z}) \, \mathrm{d}x} \right)^{\frac{2(p-1)}{p-2}} \\
&\overset{(2.24)}{\leq} c_\nu \cdot \left(\frac{2^{\frac{p-2}{2}}}{1+b_-} \frac{a_+}{a_-} \right)^{\frac{2}{p-2}} \left(\frac{\int_{\mathbb{R}^N} h_- \left(\bar{z}_{\lambda^2}, \bar{z} \right) \mathrm{d}x}{\int_{\mathbb{R}^N} h_-(\bar{z}, \bar{z}) \, \mathrm{d}x} \right)^{\frac{2(p-1)}{p-2}} .
\end{aligned}
$$

Similarly,

$$
c_{\mu\nu} \leq c_\mu \cdot \left(\frac{2^{\frac{p-2}{2}}}{1+b_-} \frac{a_+}{a_-} \right)^{\frac{2}{p-2}} \left(\frac{\int_{\mathbb{R}^N} h_- \left(\bar{z}_{\lambda^{-2}}, \bar{z} \right) \mathrm{d}x}{\int_{\mathbb{R}^N} h_-(\bar{z}, \bar{z}) \, \mathrm{d}x} \right)^{\frac{2(p-1)}{p-2}} .
$$

Notice that the terms on the right depend continuously on the parameter λ since $\lambda \mapsto \lambda^{\frac{N+2}{2}} \bar{z}(\lambda \,\cdot\,)$ is continuous in $L^{p'}(\mathbb{R}^N)$. Hence,

$$
\frac{\int_{\mathbb{R}^N} h_- \left(\bar{z}_{\lambda^{\pm 2}}, \bar{z} \right) \mathrm{d}x}{\int_{\mathbb{R}^N} h_-(\bar{z}, \bar{z}) \, \mathrm{d}x} \to 1 \qquad \text{as } \lambda \to 1.
$$

As we have assumed $\frac{2^{\frac{p-2}{2}}}{1+b_-} \cdot \frac{a_+}{a_-} < 1$, we find $\delta > 0$ such that $|\lambda - 1| < \delta$ implies

$$
\left(\frac{2^{\frac{p-2}{2}}}{1+b_-} \frac{a_+}{a_-} \right)^{\frac{2}{p-2}} \left(\frac{\int_{\mathbb{R}^N} h_- \left(\bar{z}_{\lambda^{\pm 2}}, \bar{z} \right) \mathrm{d}x}{\int_{\mathbb{R}^N} h_-(\bar{z}, \bar{z}) \, \mathrm{d}x} \right)^{\frac{2(p-1)}{p-2}} < 1 \quad \text{and hence} \quad c_{\mu\nu} < \min\{c_\mu, c_\nu\}.
$$

Lemma 2.15 (ii) ensures that, for such μ and ν, every ground state is fully nontrivial. □

Proof of Theorem 2.5

We consider a ground state $(\bar{u}, \bar{v}) \in L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)$ of the functional $J_{\mu\nu}$, hence a minimizer of $F_{\mu\nu}$. Knowing that $(\bar{u}, \bar{v}) \neq (0, 0)$, we consider the case $\bar{u} \neq 0$. We write $\left\| a^{-1/p} \bar{v} \right\|_{L^{p'}(\mathbb{R}^N)} = \eta_0 \left\| a^{-1/p} \bar{u} \right\|_{L^{p'}(\mathbb{R}^N)}$ for some $\eta_0 \geq 0$ and aim to show that necessarily $\eta_0 = 0$, hence $\bar{v} = 0$.
Recalling that c_μ, c_ν are the minima of E_μ, E_ν, respectively, we estimate with (2.19)

$$
\begin{aligned}
\left(\int_{\mathbb{R}^N} \bar{u} \, \mathcal{R}_\mu[\bar{u}] \, \mathrm{d}x \right)_+ &\leq \left(\frac{2p}{p-2} c_\mu \right)^{-\frac{p-2}{p}} \left\| a^{-1/p} \bar{u} \right\|^2_{L^{p'}(\mathbb{R}^N)} , \\
\left(\int_{\mathbb{R}^N} \bar{v} \, \mathcal{R}_\nu[\bar{v}] \, \mathrm{d}x \right)_+ &\leq \left(\frac{2p}{p-2} c_\nu \right)^{-\frac{p-2}{p}} \left\| a^{-1/p} \bar{v} \right\|^2_{L^{p'}(\mathbb{R}^N)} .
\end{aligned}
$$

Combining this with Lemma 2.16, we have

$$
\begin{aligned}
c_{\mu\nu} &= F_{\mu\nu}(\bar{u},\bar{v}) \\
&\overset{\text{Lem.2 16}}{\geq} \frac{p-2}{2p} \frac{\left[p'\, h_+\left(\left\|a^{-1/p}\,\bar{u}\right\|_{L^{p'}(\mathbb{R}^N)}, \eta_0 \left\|a^{-1/p}\,\bar{u}\right\|_{L^{p'}(\mathbb{R}^N)}\right)\right]^{\frac{2(p-1)}{p-2}}}{\left[\left(\frac{2p}{p-2}c_\mu\right)^{-\frac{p-2}{p}} \left\|a^{-1/p}\bar{u}\right\|^2_{L^{p'}(\mathbb{R}^N)} + \left(\frac{2p}{p-2}c_\nu\right)^{-\frac{p-2}{p}} \eta_0^2 \left\|a^{-1/p}\bar{u}\right\|^2_{L^{p'}(\mathbb{R}^N)}\right]^{\frac{p}{p-2}}} \\
&\overset{\text{Lem.2 10}(i)}{=} \frac{\left[p' \left\|a^{-1/p}\,\bar{u}\right\|^{p'}_{L^{p'}(\mathbb{R}^N)} h_+(1,\eta_0)\right]^{\frac{2(p-1)}{p-2}}}{\left[\left((c_\mu)^{-\frac{p-2}{p}} + (c_\nu)^{-\frac{p-2}{p}}\eta_0^2\right) \left\|a^{-1/p}\bar{u}\right\|^2_{L^{p'}(\mathbb{R}^N)}\right]^{\frac{p}{p-2}}} \\
&\geq \frac{\left[p' \left\|a^{-1/p}\,\bar{u}\right\|^{p'}_{L^{p'}(\mathbb{R}^N)} h_+(1,\eta_0)\right]^{\frac{2(p-1)}{p-2}}}{\left[\min\{c_\mu,c_\nu\}^{-\frac{p-2}{p}} \cdot \left(1+\eta_0^2\right) \left\|a^{-1/p}\bar{u}\right\|^2_{L^{p'}(\mathbb{R}^N)}\right]^{\frac{p}{p-2}}} \\
&= \min\{c_\mu,c_\nu\} \left(\frac{\left[p'\,h_+(1,\eta_0)\,\right]^{\frac{1}{p'}}}{\left[1+\eta_0^2\right]^{\frac{1}{2}}}\right)^{\frac{2p}{p-2}} \\
&\overset{(2.21)}{=} \min\{c_\mu,c_\nu\} \left(\sup_{\sigma>0} \frac{(1+\sigma\eta_0)}{(1+2b_+\sigma^{\frac{p}{2}}+\sigma^p)^{\frac{1}{p}}(1+\eta_0^2)^{\frac{1}{2}}}\right)^{\frac{2p}{p-2}} \\
&\geq \min\{c_\mu,c_\nu\} \left(\frac{(1+\eta_0^2)^{\frac{1}{2}}}{(1+2b_+\eta_0^{\frac{p}{2}}+\eta_0^p)^{\frac{1}{p}}}\right)^{\frac{2p}{p-2}} \\
&\geq \min\{c_\mu,c_\nu\} \left(\frac{(1+\eta_0^2)^{\frac{1}{2}}}{(1+(2^{\frac{p}{2}}-2)\eta_0^{\frac{p}{2}}+\eta_0^p)^{\frac{1}{p}}}\right)^{\frac{2p}{p-2}} \\
&\geq \min\{c_\mu,c_\nu\}.
\end{aligned}
$$

The latter estimate holds since

$$
\forall\,\eta \geq 0 \qquad \frac{(1+\eta^2)^{\frac{1}{2}}}{(1+(2^{\frac{p}{2}}-2)\eta^{\frac{p}{2}}+\eta^p)^{\frac{1}{p}}} \geq 1, \tag{2.28}
$$

which we will prove in Section 2.6.3. With that, the assertion can be concluded as follows: Lemma 2.15 (ii) yields $c_{\mu\nu} \leq \min\{c_\mu,c_\nu\}$, and thus we have $c_{\mu\nu} = \min\{c_\mu,c_\nu\}$ and equality must hold in all above estimates. But then, since we assume $b_+ < 2^{\frac{p-2}{2}} - 1$, we infer $\eta_0 = 0$. Hence, the ground state is semitrivial with $\bar{v} = 0$. □

Proof of Corollary 2.6

Concerning semitrivial vs. fully nontrivial solutions, the previously proved Theorems cover most cases: If $2 < p < 4$ and $b > 0$, Theorem 2.4 (i) states that every ground state of $J_{\mu\mu}$ is fully nontrivial; so does Theorem 2.4 (ii) in case $p \geq 4$ and $b > 2^{\frac{p-2}{2}} - 1$. (Notice that we assume $\nu = \mu$.) If, however, $p \geq 4$ and $0 \leq b < 2^{\frac{p-2}{2}} - 1$, Theorem 2.5 ensures that ground states of $J_{\mu\mu}$ are semitrivial. So only two cases remain open. After that, we discuss the question of diagonal ground states, which has not been addressed before but can be answered using similar techniques.

▷ Step 1: *Assume* $2 < p < 4$ *and* $b = 0$.

The proof then follows the lines of that of Theorem 2.5. Considering a ground state $(\bar{u}, \bar{v}) \in L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)$ of $J_{\mu\nu}$ with $\bar{u} \neq 0$ and assuming $\|a^{-1/p}\bar{v}\|_{L^{p'}(\mathbb{R}^N)} = \eta_0 \|a^{-1/p}\bar{u}\|_{L^{p'}(\mathbb{R}^N)}$, we again aim to prove that $\eta_0 = 0$. The same estimate as in the previous proof preceding (2.28) yields

$$\begin{aligned}
c_{\mu\nu} &\geq \min\{c_\mu, c_\nu\} \left(\sup_{\sigma>0} \frac{(1+\sigma\eta_0)}{(1+\sigma^p)^{\frac{1}{p}}(1+\eta_0^2)^{\frac{1}{2}}} \right)^{\frac{2p}{p-2}} \\
&\geq \min\{c_\mu, c_\nu\} \left(\frac{(1+\eta_0^{p'})^{\frac{1}{p'}}}{(1+\eta_0^2)^{\frac{1}{2}}} \right)^{\frac{2p}{p-2}} \\
&\overset{(p'<2)}{\geq} \min\{c_\mu, c_\nu\}
\end{aligned}$$

with equality if and only if $\eta_0 = 0$. Since $c_{\mu\nu} \leq \min\{c_\mu, c_\nu\}$ by Lemma 2.15 (i), this implies $\eta_0 = 0$.

▷ Step 2: *Assume* $p \geq 4$ *and* $b = 2^{\frac{p-2}{2}} - 1$.

In this case, one can show as in the proof of Theorem 2.5 that we have $c_{\mu\mu} = c_\mu$. For any scalar ground state $\bar{w} \in L^{p'}(\mathbb{R}^N)$ of I_μ, we calculate

$$\begin{aligned}
F_{\mu\mu}(\bar{w}, \bar{w}) &\overset{(2.16)}{=} \frac{p-2}{2p} \left(\frac{\left[\int_{\mathbb{R}^N} p'\, h(x, \bar{w}, \bar{w})\, \mathrm{d}x\right]^{\frac{1}{p'}}}{\left[\int_{\mathbb{R}^N} \bar{w}\, \mathcal{R}_\mu[\bar{w}] + \bar{w}\, \mathcal{R}_\mu[\bar{w}]\, \mathrm{d}x\right]_+^{\frac{1}{2}}} \right)^{\frac{2p}{p-2}} \\
&\overset{\text{Lem. } 2.10(iv)}{=} \frac{p-2}{2p} \left(\frac{\left[\int_{\mathbb{R}^N} 2(1+b)^{1-p'} a^{1-p'} |\bar{w}(x)|^{p'}\, \mathrm{d}x\right]^{\frac{1}{p'}}}{\left[2 \int_{\mathbb{R}^N} \bar{w}\, \mathcal{R}_\mu[\bar{w}]\, \mathrm{d}x\right]_+^{\frac{1}{2}}} \right)^{\frac{2p}{p-2}} \\
&= \frac{2}{(1+b)^{\frac{2}{p-2}}} \frac{p-2}{2p} \left(\frac{\left[\int_{\mathbb{R}^N} a^{1-p'} |\bar{w}(x)|^{p'}\, \mathrm{d}x\right]^{\frac{1}{p'}}}{\left[\int_{\mathbb{R}^N} \bar{w}\, \mathcal{R}_\mu[\bar{w}]\, \mathrm{d}x\right]_+^{\frac{1}{2}}} \right)^{\frac{2p}{p-2}} \\
&\overset{(2.19)}{=} \frac{2}{(1+b)^{\frac{2}{p-2}}} E_\mu(\bar{w}) \\
&= E_\mu(\bar{w}) \\
&= c_\mu
\end{aligned}$$

and $F_{\mu\mu}(\bar{w}, 0) = E_\mu(\bar{w}) = c_\mu$. Hence, $F_{\mu\mu}(\bar{w}, \bar{w}) = F_{\mu\mu}(\bar{w}, 0) = c_{\mu\mu}$ and by Lemma 2.14, this provides (up to multiplication with suitable constants) both a semitrivial and a fully nontrivial ground state of $J_{\mu\mu}$.

▷ Step 3: *Diagonal ground states:* $b \geq \frac{p}{2} - 1$ *with* $(p, b) \neq (4,1)$.

For any $\bar{w} \in L^{p'}(\mathbb{R}^N)$, we have just calculated

$$F_{\mu\mu}(\bar{w}, \bar{w}) = \frac{2}{(1+b)^{\frac{2}{p-2}}} E_\mu(\bar{w}). \tag{2.29}$$

Passing to the infimum with respect to $\bar{w} \in L^{p'}(\mathbb{R}^N)$, we deduce

$$c_{\mu\mu} \leq \inf_{\bar{w} \in L^{p'}(\mathbb{R}^N)} F_{\mu\mu}(\bar{w}, \bar{w}) = \frac{2}{(1+b)^{\frac{2}{p-2}}} c_\mu =: c_\mu^{\mathrm{diag}}. \tag{2.30}$$

From now on, we additionally assume $b \geq \frac{p}{2} - 1\,(> 0)$. We show that equality holds in the above estimate. The first part of (i) guarantees that the level $c_{\mu\mu}$ is attained by a fully nontrivial ground state $(\bar{u}, \bar{v}) \in L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)$. Defining $\eta_0 > 0$ via

$$\left\| a^{-\frac{1}{p}} \bar{v} \right\|_{L^{p'}(\mathbb{R}^N)} = \eta_0 \left\| a^{-\frac{1}{p}} \bar{u} \right\|_{L^{p'}(\mathbb{R}^N)},$$

we estimate as in the proof of Theorem 2.5 using Lemma 2.16

$$\begin{aligned}
c_{\mu\mu} &= F_{\mu\mu}(\bar{u}, \bar{v}) \\
&\overset{\text{Lem.2 16}}{\geq} \frac{p-2}{2p} \frac{\left[p'\, h_+ \left(\left\| a^{-1/p}\, \bar{u} \right\|_{L^{p'}(\mathbb{R}^N)}, \eta_0 \left\| a^{-1/p}\, \bar{u} \right\|_{L^{p'}(\mathbb{R}^N)} \right) \right]^{\frac{2(p-1)}{p-2}}}{\left[\left(\frac{2p}{p-2} c_\mu \right)^{-\frac{p-2}{p}} \left\| a^{-1/p} \bar{u} \right\|^2_{L^{p'}(\mathbb{R}^N)} + \left(\frac{2p}{p-2} c_\mu \right)^{-\frac{p-2}{p}} \eta_0^2 \left\| a^{-1/p} \bar{u} \right\|^2_{L^{p'}(\mathbb{R}^N)} \right]^{\frac{p}{p-2}}} \\
&\geq c_\mu \left(\frac{\left[p'\, h_+ (1, \eta_0) \right]^{\frac{1}{p'}}}{\left[1 + \eta_0^2 \right]^{\frac{1}{2}}} \right)^{\frac{2p}{p-2}} \\
&\overset{(2.21)}{=} c_\mu \left(\sup_{\sigma > 0} \frac{(1 + \sigma \eta_0)}{(1 + 2b\sigma^{\frac{p}{2}} + \sigma^p)^{\frac{1}{p}} (1 + \eta_0^2)^{\frac{1}{2}}} \right)^{\frac{2p}{p-2}} \\
&\geq c_\mu \left(\inf_{\eta > 0} \sup_{\sigma > 0} \frac{(1 + \sigma \eta)}{(1 + 2b\sigma^{\frac{p}{2}} + \sigma^p)^{\frac{1}{p}} (1 + \eta^2)^{\frac{1}{2}}} \right)^{\frac{2p}{p-2}}.
\end{aligned}$$

We will show in Section 2.6.3 that the assumption $b \geq \frac{p}{2} - 1$ implies

$$\inf_{\eta > 0} \sup_{\sigma > 0} \frac{(1 + \sigma\eta)^p}{(1 + 2b\sigma^{\frac{p}{2}} + \sigma^p)(1 + \eta^2)^{\frac{p}{2}}} = \frac{2^{\frac{p-2}{2}}}{1+b} \quad \text{attained only at } \eta = 1. \tag{2.31}$$

Then the above estimate yields

$$c_{\mu\mu} \geq c_\mu \frac{2}{(1+b)^{\frac{2}{p-2}}} = c_\mu^{\mathrm{diag}},$$

and we infer from equation (2.30) that, in fact, $c_{\mu\mu} = c_\mu^{\mathrm{diag}}$. Thus equality holds in all estimates above, in particular when applying Lemma 2.16. Knowing that the ground state $(\bar{u}, \bar{v})$ is fully nontrivial and that $b > 0$, the lemma states that $|\bar{v}| = \eta_0 |\bar{u}|$ holds almost everywhere on $\mathbb{R}^N$. The uniqueness statement of equation (2.31) implies $\eta_0 = 1$, i.e. the (arbitrarily chosen) ground state $(\bar{u}, \bar{v})$ is diagonal.

▷ <u>Step 4</u>: *Non-diagonal ground states:* $b < \frac{p}{2} - 1$.

We follow a procedure similar to that in the proof of Theorem 2.4 (i), showing by Taylor expansion and elementary estimates that for any $\bar{w} \in L^{p'}(\mathbb{R}^N), \bar{w} \neq 0$ with $F_{\mu\mu}(\bar{w}, \bar{w}) < \infty$

$$c_{\mu\mu} \leq F_{\mu\mu}(\bar{w}, (1+\kappa)\bar{w}) < F_{\mu\mu}(\bar{w}, \bar{w})$$

for sufficiently small $|\kappa|$, whence we conclude that minimizers of $F_{\mu\mu}$ and hence ground states of $J_{\mu\mu}$ cannot be diagonal. Again, we estimate for $\bar{w} \in L^{p'}(\mathbb{R}^N), \bar{w} \neq 0$ with $F_{\mu\mu}(\bar{w},\bar{w}) < \infty$ and for small $|\kappa|$

$$\begin{aligned}
&F_{\mu\mu}(\bar{w},(1+\kappa)\bar{w}) \\
&\overset{(2.16)}{=} \frac{p-2}{2p}\left(\frac{\left[\int_{\mathbb{R}^N} p'\, h(x,\bar{w},(1+\kappa)\bar{w})\,\mathrm{d}x\right]^{\frac{1}{p'}}}{\left[\int_{\mathbb{R}^N} \bar{w}\,\mathcal{R}_\mu[\bar{w}] + (1+\kappa)^2\,\bar{w}\,\mathcal{R}_\mu[\bar{w}]\,\mathrm{d}x\right]_+^{\frac12}}\right)^{\frac{2p}{p-2}} \\
&\overset{\text{Lem. 2 10}(i)}{=} \frac{p-2}{2p}\left(\frac{\left[\int_{\mathbb{R}^N} a^{1-p'}|\bar{w}(x)|^{p'}\,\mathrm{d}x\right]^{\frac{1}{p'}}}{\left[\int_{\mathbb{R}^N} \bar{w}\,\mathcal{R}_\mu[\bar{w}]\,\mathrm{d}x\right]_+^{\frac12}} \cdot \sup_{\sigma>0} \frac{1+\sigma(1+\kappa)}{(1+2b\sigma^{\frac p2}+\sigma^p)^{\frac1p}(1+(1+\kappa)^2)^{\frac12}}\right)^{\frac{2p}{p-2}} \\
&\overset{(2.19)}{=} E_\mu(\bar{w})\cdot\left(\sup_{\sigma>0}\frac{(1+\sigma(1+\kappa))^p}{(1+2b\sigma^{\frac p2}+\sigma^p)(1+(1+\kappa)^2)^{\frac p2}}\right)^{\frac{2}{p-2}} \\
&\overset{(2.29)}{=} F_{\mu\mu}(\bar{w},\bar{w})\cdot\left(\frac{1+b}{2^{\frac{p-2}{2}}}\cdot\sup_{\sigma>0}\frac{(1+\sigma(1+\kappa))^p}{(1+2b\sigma^{\frac p2}+\sigma^p)(1+(1+\kappa)^2)^{\frac p2}}\right)^{\frac{2}{p-2}}.
\end{aligned}$$

As announced, we prove in Section 2.6.3 by Taylor expansion for $\kappa \to 0$

$$\sup_{\sigma>0}\frac{(1+\sigma(1+\kappa))^p}{(1+2b\sigma^{\frac p2}+\sigma^p)(1+(1+\kappa)^2)^{\frac p2}} = \frac{2^{\frac{p-2}{2}}}{1+b}\left[1-\kappa^2\,\frac{p\left(\frac p2-1-b\right)}{4(p-1-b)}+O(\kappa^3)\right]. \qquad (2.32)$$

Recalling that we assume $b < \frac p2 - 1$, this reveals that minimizers of $F_{\mu\mu}$ are non-diagonal.

$\triangleright$ <u>Step 5</u>: *All sorts of ground states: The case $p=4, b=1$.*

Step 2 ensures the existence of both a semitrivial and a fully nontrivial (diagonal) minimizer of the functional $F_{\mu\mu}$, $(\bar{w},0)$ resp. $(\bar{w},\bar{w}) \in L^{p'}(\mathbb{R}^N)\times L^{p'}(\mathbb{R}^N) = L^{\frac43}(\mathbb{R}^N)\times L^{\frac43}(\mathbb{R}^N)$. Using the explicit formula for the Legendre transform in (2.11), one can easily see that

$$F_{\mu\mu}(\bar{u},\bar{v}) \overset{(2.16),(2.11)}{=} \frac14\left(\frac{\left[\int_{\mathbb{R}^N} a^{-1/3}(|\bar{u}|^2+|\bar{v}|^2)^{\frac23}\,\mathrm{d}x\right]^{\frac34}}{\left[\int_{\mathbb{R}^N}\bar{u}\,\mathcal{R}_\mu[\bar{u}]+\bar{v}\,\mathcal{R}_\mu[\bar{v}]\,\mathrm{d}x\right]_+^{\frac12}}\right)^4 \qquad (\bar{u},\bar{v}\in L^{\frac43}(\mathbb{R}^N))$$

and hence $F_{\mu\mu}(\bar{w}\cos\vartheta,\bar{w}\sin\vartheta) = F_{\mu\mu}(\bar{w},\bar{w}) = c_{\mu\mu}$, $\vartheta \notin \frac{\pi}{4}\mathbb{Z}$, provides non-diagonal minimizers and hence non-diagonal ground states of $J_{\mu\mu}$.

Given any fully nontrivial minimizer $(\bar{u},\bar{v}) \in L^{\frac43}(\mathbb{R}^N)\times L^{\frac43}(\mathbb{R}^N)$ of $F_{\mu\mu}$, we can proceed as in Step 3 and find $|\bar{v}| = \eta_0|\bar{u}|$ for some $\eta_0 > 0$. The only point which requires a small change is the adaptation of (2.31), which can now be done explicitly:

$$\inf_{\eta>0}\sup_{\sigma>0}\frac{(1+\sigma\eta)^p}{(1+2b\sigma^{\frac p2}+\sigma^p)(1+\eta^2)^{\frac p2}} = \inf_{\eta>0}\underbrace{\sup_{\sigma>0}\frac{(1+\sigma\eta)^4}{(1+\sigma^2)^2(1+\eta^2)^2}}_{=1} = 1 = \frac{2^{\frac{p-2}{2}}}{1+b},$$

which is, now, attained at $\sigma = \eta$ and arbitrary $\eta > 0$. This is why we cannot conclude that $\eta_0 = 1$ and thus the ground state need not be diagonal. $\square$

2.6 Proofs of the Auxiliary Results

Finally, we give the proofs of various auxiliary statements which were stated and applied throughout the previous sections.

2.6.1 Results about the Legendre Transform

Proof of Proposition 2.9

Fix $x \in \mathbb{R}^N$ and recall for $s, t \in \mathbb{R}$

$$f(x,s,t) = \frac{a(x)}{p}\left(|s|^p + 2b(x)\,|s|^{\frac{p}{2}}|t|^{\frac{p}{2}} + |t|^p\right).$$

Differentiability and co-finiteness of $f(x,\,\cdot\,,\,\cdot\,)$ are a straightforward consequence of the assumption $p > 2$. We will show below that $f(x,\,\cdot\,,\,\cdot\,)$ is strictly convex; with that, the existence and the asserted properties of the Legendre transform $h(x,\cdot,\cdot)$ of $f(x,\cdot,\cdot)$ are guaranteed by Theorem 2.8. To verify strict convexity, we show that, for all $s_1, s_2, t_1, t_2 \in \mathbb{R}$ with $s_2 \neq 0$ or $t_2 \neq 0$,

$$f(x, s_1+s_2, t_1+t_2) > f(x,s_1,t_1) + s_2\,\partial_s f(x,s_1,t_1) + t_2\,\partial_t f(x,s_1,t_1). \tag{2.33}$$

We denote the difference by

$$\mathcal{I} := f(x,s_1+s_2,t_1+t_2) - \Big[f(x,s_1,t_1) + s_2\,\partial_s f(x,s_1,t_1) + t_2\,\partial_t f(x,s_1,t_1)\Big].$$

So if we prove $\mathcal{I} > 0$, we conclude (2.33). We introduce the line segment

$$\ell := \left\{(s_1,t_1) + \theta(s_2,t_2) \in \mathbb{R}^2 :\ 0 \le \theta \le 1\right\}.$$

▷ <u>Step 1</u>: *Let us assume that ℓ is a subset of either of the sets*

$$\begin{aligned} &\{(s,0) \in \mathbb{R}^2 :\ s \in \mathbb{R}\}, \quad \{(0,t) \in \mathbb{R}^2 :\ t \in \mathbb{R}\},\\ &\{(s,s) \in \mathbb{R}^2 :\ s \in \mathbb{R}\}, \quad \{(s,-s) \in \mathbb{R}^2 :\ s \in \mathbb{R}\}. \end{aligned} \tag{2.34}$$

We then conclude $\mathcal{I} > 0$ since the functions

$$\begin{aligned} &s \mapsto f(x,s,0) = \frac{a(x)}{p}|s|^p, && t \mapsto f(x,0,t) = \frac{a(x)}{p}|t|^p,\\ &s \mapsto f(x,s,s) = \frac{2a(x)(1+b(x))}{p}|s|^p, && s \mapsto f(x,s,-s) = \frac{2a(x)(1+b(x))}{p}|s|^p, \end{aligned}$$

respectively, are strictly convex.

▷ <u>Step 2</u>: *We now assume that ℓ intersects none of the sets in* (2.34).

Then $f(x,\cdot,\cdot)$ is twice continuously differentiable with respect to (s,t) in a neighborhood of ℓ, and the Fundamental Theorem of Calculus yields the integral representation

$$\mathcal{I} = \int_0^1\int_0^1 \tau\,(s_2,t_2) D^2_{s,t} f(x, s_1+\tau\sigma s_2, t_1+\tau\sigma t_2)\begin{pmatrix} s_2\\ t_2\end{pmatrix}\,\mathrm{d}\sigma\,\mathrm{d}\tau. \tag{2.35}$$

We show that the Hessian $D^2_{s,t} f(x,s,t)$ is strictly positive definite for all $(s,t) \in \ell$.

Let $(s,t) \in \ell$, i.e. in particular $s \neq 0$, $t \neq 0$ and $|s| \neq |t|$, see (2.34). Recall that we assume $a(x) > 0$ and $0 \le b(x) \le p-1$. We calculate the trace and the determinant of the Hessian:

$$\begin{aligned}
&\operatorname{tr} D^2_{s,t} f(x,s,t) \\
&\quad = a(x)(p-1)\left(|s|^{p-2} + |t|^{p-2}\right) + a(x)\frac{b(x)}{2}(p-2)\left(|s|^{\frac{p}{2}-2}|t|^{\frac{p}{2}} + |t|^{\frac{p}{2}-2}|s|^{\frac{p}{2}}\right), \\
&\det D^2_{s,t} f(x,s,t) \\
&\quad = a(x)^2(p-1)\left[(p-1-b(x)^2)|s|^{\frac{p}{2}}|t|^{\frac{p}{2}} + \frac{b(x)}{2}(p-2)\left(|s|^p + |t|^p\right)\right]|s|^{\frac{p}{2}-2}|t|^{\frac{p}{2}-2}.
\end{aligned}$$

Since $a(x) > 0$ $b(x) \ge 0$, $s \neq 0$ and $t \neq 0$, we always have $\operatorname{tr} D^2_{s,t} f(x,s,t) > 0$. If $0 \le b(x) \le \sqrt{p-1}$, we infer $\det D^2_{s,t} f(x,s,t) > 0$ and hence $D^2_{s,t} f(s,t)$ is strictly positive definite. Else if $\sqrt{p-1} < b(x) \le p-1$, we recall that $|s| \neq |t|$ by assumption on ℓ, which gives the strict estimate $|s|^{\frac{p}{2}}|t|^{\frac{p}{2}} < \frac{1}{2}\left(|s|^p + |t|^p\right)$. Thus,

$$\begin{aligned}
\det D^2_{s,t} f(x,s,t) &> a(x)^2\frac{p-1}{2}\left((p-1-b(x)^2) + b(x)(p-2)\right)\left(|s|^p + |t|^p\right)|s|^{\frac{p}{2}-2}|t|^{\frac{p}{2}-2} \\
&= a(x)^2\frac{(p-1)(b(x)+1)}{2}(p-1-b(x))\left(|s|^p + |t|^p\right)|s|^{\frac{p}{2}-2}|t|^{\frac{p}{2}-2} \\
&\ge 0,
\end{aligned}$$

which proves strict positive definiteness of $D^2_{s,t} f(x,s,t)$.

$\triangleright$ <u>Step 3</u>: *Finally, in all remaining cases, ℓ intersects the sets of* (2.34) *in at most finitely many points.*

The integrand in (2.35) now contains weakly singular terms with powers $\frac{p}{2} - 2 > -1$ (which is due to $p > 2$). Thus the expression in (2.35) converges in the sense of an indefinite integral. This justifies that the integral representation from (2.35) continues to hold and the previous step gives $\mathcal{I} > 0$.

Hence, $f(x,\,\cdot\,,\,\cdot\,)$ is strictly convex, which concludes the proof. $\square$

Proof of Lemma 2.10

For $\bar{s}, \bar{t} \in \mathbb{R}$, we recall the definition of the Legendre transform:

$$h(x,\bar{s},\bar{t}) = \sup_{s,t\in\mathbb{R}}\left(s\bar{s} + t\bar{t} - f(x,s,t)\right)$$

where $f(x,s,t) = \frac{a(x)}{p}\left(|s|^p + 2b(x)\,|s|^{\frac{p}{2}}|t|^{\frac{p}{2}} + |t|^p\right)$. We note that, since $f(x,s,t) \ge 0$, this immediately yields $h(x,0,0) = 0$.

(i) We assume w.l.o.g. that $\bar{s} \neq 0$. With that, using $f(x,s,t) \ge 0$ as well as the symmetry relations $f(x,-s,t) = f(x,s,t) = f(x,s,-t)$, we calculate

$$\begin{aligned}
h(x,\bar{s},\bar{t}) &= \sup_{s,t\in\mathbb{R}}\left[s\bar{s} + t\bar{t} - f(x,s,t)\right] \\
&= \sup_{s,t>0}\left[s|\bar{s}| + t|\bar{t}| - f(x,s,t)\right] \\
&= \sup_{s,\sigma>0}\left[s(|\bar{s}| + \sigma|\bar{t}|) - s^p f(x,1,\sigma)\right] \\
&= \sup_{\sigma>0}\frac{1}{p'}\left(\frac{(|\bar{s}| + \sigma|\bar{t}|)^p}{pf(x,1,\sigma)}\right)^{\frac{1}{p-1}}
\end{aligned}$$

$$= \frac{a(x)^{1-p'}}{p'} \left[\sup_{\sigma>0} \frac{|\bar{s}| + \sigma|\bar{t}|}{(1 + 2b(x)\,\sigma^{\frac{p}{2}} + \sigma^p)^{\frac{1}{p}}} \right]^{p'}$$

where the supremum with respect to $s > 0$ has been evaluated explicitly.

(ii) This is a direct consequence of the symmetry of $f(x, \cdot\,, \cdot\,)$, i.e. $f(x, s, t) = f(x, t, s)$ and of the fact that $f(x, -s, t) = f(x, s, t)$, respectively, for all $s, t \in \mathbb{R}$.

(iii) As a consequence of part (i), we have $h(x, \alpha\bar{s}, \alpha\bar{t}) = \alpha^{p'} h(x, \bar{s}, \bar{t})$ for $\alpha > 0$. We differentiate with respect to α and find

$$\nabla_{\bar{s},\bar{t}} h(x, \alpha\bar{s}, \alpha\bar{t}) \cdot \begin{pmatrix} \bar{s} \\ \bar{t} \end{pmatrix} = p'\alpha^{p'-1} h(x, \bar{s}, \bar{t}).$$

Evaluating the latter identity at $\alpha = 1$, the assertion of (iii) is proved.

(iv) We only prove the second identity, the first one can be shown in the same way. By direct computation we find $\nabla_{s,t} f(x, s, s) = a(x)(1 + b(x))|s|^{p-2}s\,(1, 1)$ for $s \in \mathbb{R}$. Recalling that $\nabla_{\bar{s},\bar{t}} h(x, \cdot\,, \cdot\,)$ is a diffeomorphism on $\mathbb{R}^2$ with inverse $\nabla_{s,t} f(x, \cdot\,, \cdot\,)$, this implies $\nabla_{\bar{s},\bar{t}} h(x, \bar{s}, \bar{s}) = \big(a(x)(1+b(x))\big)^{-(p'-1)} |\bar{s}|^{p'-2}\bar{s}\,(1, 1)$, and hence using (iii)

$$h(x, \bar{s}, \bar{s}) = \frac{1}{p'} \nabla_{\bar{s},\bar{t}} h(x, \bar{s}, \bar{s}) \cdot \begin{pmatrix} \bar{s} \\ \bar{s} \end{pmatrix} = \frac{2}{p'} \big(a(x)(1 + b(x))\big)^{1-p'} |\bar{s}|^{p'}.$$

(v) We have by definition of the Legendre transform and due to $a(x), b(x) \geq 0$

$$\begin{aligned} h(x, \bar{s}, \bar{t}) &= \sup_{s,t\in\mathbb{R}} \left(s\bar{s} + t\bar{t} - \frac{a(x)}{p} \left(|s|^p + 2b(x)\,|s|^{\frac{p}{2}}|t|^{\frac{p}{2}} + |t|^p \right) \right) \\ &\leq \sup_{s,t\in\mathbb{R}} \left(s\bar{s} + t\bar{t} - \frac{a(x)}{p} \left(|s|^p + |t|^p \right) \right) \\ &= \frac{1}{p'} a(x)^{1-p'} \left(|\bar{s}|^{p'} + |\bar{t}|^{p'} \right) \end{aligned}$$

where we calculated the latter supremum explicitly. On the other hand, defining $s_x \in \mathbb{R}$ via

$$s_x := (a(x)(1 + b(x)))^{1-p'} \cdot |\bar{s}|^{p'-2}\bar{s} \in \mathbb{R},$$

we notice that s_x maximizes the map $\mathbb{R} \to \mathbb{R}$, $s \mapsto s\bar{s} - \frac{1}{p} a(x)(1 + b(x))|s|^p$ and that $\bar{s} = a(x)(1 + b(x))|s_x|^{p-2}s_x$. Defining similarly

$$t_x := (a(x)(1 + b(x)))^{1-p'} \cdot |\bar{t}|^{p'-2}\bar{t} \in \mathbb{R},$$

we estimate

$$\begin{aligned} &\frac{1}{p'} (a(x)(1 + b(x)))^{1-p'} \left(|\bar{s}|^{p'} + |\bar{t}|^{p'} \right) \\ &= \left(1 - \frac{1}{p}\right) (s_x\bar{s} + t_x\bar{t}) \\ &\leq s_x\bar{s} + t_x\bar{t} - \frac{1}{p} \left(s_x \cdot a(x)(1 + b(x))|s_x|^{p-2}s_x + t_x \cdot a(x)(1 + b(x))|t_x|^{p-2}t_x \right) \\ &= s_x\bar{s} + t_x\bar{t} - \frac{a(x)}{p} \left(|s_x|^p + 2b(x)|s_x|^{\frac{p}{2}}|t_x|^{\frac{p}{2}} + |t_x|^p \right) \end{aligned}$$

$$\begin{aligned}
&\le \sup_{s,t\in\mathbb{R}} \left(s\bar{s} + t\bar{t} - \frac{a(x)}{p}\left(|s|^p + 2b(x)|s|^{\frac{p}{2}}|t|^{\frac{p}{2}} + |t|^p\right)\right) \\
&= h(x,\bar{s},\bar{t}).
\end{aligned}$$

□

Proof of Lemma 2.11

We focus on proving continuous differentiability of the non-quadratic part of the functional, which is the new ingredient when compared with the case of a single Helmholtz equation. We let

$$H : L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N) \to \mathbb{R}, \quad H(\bar{u},\bar{v}) := \int_{\mathbb{R}^N} h(x,\bar{u},\bar{v})\,\mathrm{d}x$$

and show that H is of class C^1 with derivative

$$H'(\bar{u},\bar{v})[(\bar{k},\bar{l})] = \int_{\mathbb{R}^N} \nabla h_{\bar{s},\bar{t}}(x,\bar{u},\bar{v}) \cdot \begin{pmatrix}\bar{k}\\ \bar{l}\end{pmatrix}\,\mathrm{d}x.$$

Then in particular, making use of Lemma 2.10 (iii),

$$H'(\bar{u},\bar{v})[(\bar{u},\bar{v})] = \int_{\mathbb{R}^N} p'\,h(x,\bar{u},\bar{v})\,\mathrm{d}x,$$

which verifies the corresponding part of the asserted formula. The proof uses standard arguments based on pointwisely convergent, uniformly majorized subsequences which arise from the Riesz-Fischer Theorem. First, for almost all $x \in \mathbb{R}^N$ and all $\bar{s},\bar{t} \in \mathbb{R}$, we prove the estimates

$$|\partial_{\bar{s}} h(x,\bar{s},\bar{t})| \le \left(\frac{1}{a_0}\right)^{p'-1} |\bar{s}|^{p'-1}, \quad |\partial_{\bar{t}} h(x,\bar{s},\bar{t})| \le \left(\frac{1}{a_0}\right)^{p'-1} |\bar{t}|^{p'-1}. \tag{2.36}$$

Let $\bar{s},\bar{t} \in \mathbb{R}$ and define $(s,t) := \nabla_{\bar{s},\bar{t}} h(x,\bar{s},\bar{t})$. Then, we have for almost every $x \in \mathbb{R}^N$ since $a(x) \ge a_0 > 0$ and $b(x) \ge 0$

$$\begin{aligned}
|\bar{s}| = |\partial_s f(x,s,t)| = a(x)\left(|s|^{\frac{p}{2}} + b(x)|t|^{\frac{p}{2}}\right)|s|^{\frac{p}{2}-1} \ge a_0|s|^{p-1}, \\
|\bar{t}| = |\partial_t f(x,s,t)| = a(x)\left(|t|^{\frac{p}{2}} + b(x)|s|^{\frac{p}{2}}\right)|t|^{\frac{p}{2}-1} \ge a_0|t|^{p-1},
\end{aligned}$$

which proves (2.36) since $s = \partial_{\bar{s}} h(x,\bar{s},\bar{t})$ and $t = \partial_{\bar{t}} h(x,\bar{s},\bar{t})$.

Now, we prove differentiability. For $\bar{u},\bar{v},\bar{k}_n,\bar{l}_n \in L^{p'}(\mathbb{R}^N)$ with $\bar{k}_n,\bar{l}_n \to 0$ as $n \to \infty$, we find using the Fundamental Theorem, Fubini's Theorem and Hölder's inequality

$$\begin{aligned}
&\left| H(\bar{u}+\bar{k}_n, \bar{v}+\bar{l}_n) - H(\bar{u},\bar{v}) - \int_{\mathbb{R}^N} \nabla_{\bar{s},\bar{t}} h(x,\bar{u},\bar{v}) \cdot \begin{pmatrix}\bar{k}_n\\ \bar{l}_n\end{pmatrix}\,\mathrm{d}x \right| \\
&= \left| \int_{\mathbb{R}^N}\int_0^1 \left[\nabla_{\bar{s},\bar{t}} h(x,\bar{u}+\tau\bar{k}_n,\bar{v}+\tau\bar{l}_n) - \nabla_{\bar{s},\bar{t}} h(x,\bar{u},\bar{v})\right] \cdot \begin{pmatrix}\bar{k}_n\\ \bar{l}_n\end{pmatrix}\,\mathrm{d}\tau\,\mathrm{d}x \right| \\
&\le \|\bar{k}_n\|_{L^{p'}(\mathbb{R}^N)} \cdot \int_0^1 \left(\int_{\mathbb{R}^N} \left|\partial_{\bar{s}} h(x,\bar{u}+\tau\bar{k}_n,\bar{v}+\tau\bar{l}_n) - \partial_{\bar{s}} h(x,\bar{u},\bar{v})\right|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\,\mathrm{d}\tau \\
&\quad + \|\bar{l}_n\|_{L^{p'}(\mathbb{R}^N)} \cdot \int_0^1 \left(\int_{\mathbb{R}^N} \left|\partial_{\bar{t}} h(x,\bar{u}+\tau\bar{k}_n,\bar{v}+\tau\bar{l}_n) - \partial_{\bar{t}} h(x,\bar{u},\bar{v})\right|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\,\mathrm{d}\tau.
\end{aligned}$$

The Riesz-Fischer Theorem guarantees that every subsequence of $(\bar{k}_n,\bar{l}_n)_{n\in\mathbb{N}}$ possesses a sub-subsequence $(\bar{k}_{n_j}, \bar{l}_{n_j})_{j\in\mathbb{N}}$ which converges to zero pointwise almost everywhere and satisfies $|\bar{k}_{n_j}| \le \bar{k}^*, |\bar{l}_{n_j}| \le \bar{l}^*$ for some nonnegative $\bar{k}^*, \bar{l}^* \in L^{p'}(\mathbb{R}^N)$. Then, however, we have due to continuity of $h(x, \cdot, \cdot)$

$$\nabla_{\bar{s},\bar{t}} h(x, \bar{u} + \tau\bar{k}_{n_j}, \bar{v} + \tau\bar{l}_{n_j}) - \nabla_{\bar{s},\bar{t}} h(x, \bar{u}, \bar{v}) \to 0 \quad \text{as } j \to \infty \text{ pointwise a. e. on } \mathbb{R}^N,$$

and moreover due to (2.36)

$$\begin{aligned}
&\left|\partial_{\bar{s}} h(x, \bar{u} + \tau\bar{k}_{n_j}, \bar{v} + \tau\bar{l}_{n_j}) - \partial_{\bar{s}} h(x, \bar{u}, \bar{v})\right|^p \\
&\qquad \le 2^p \left(\left|\partial_{\bar{s}} h(x, \bar{u} + \tau\bar{k}_{n_j}, \bar{v} + \tau\bar{l}_{n_j})\right|^p + |\partial_{\bar{s}} h(x, \bar{u}, \bar{v})|^p\right) \\
&\qquad \le 2^p \left(\frac{1}{a_0}\right)^{p'} \left(|\bar{u} + \tau\bar{k}_{n_j}|^{(p'-1)p} + |\bar{u}|^{(p'-1)p}\right) \\
&\qquad \le 2^p \left(\frac{1}{a_0}\right)^{p'} \left((|\bar{u}| + |\bar{k}^*|)^{p'} + |\bar{u}|^{p'}\right), \\
&\left|\partial_{\bar{t}} h(x, \bar{u} + \tau\bar{k}_{n_j}, \bar{v} + \tau\bar{l}_{n_j}) - \partial_{\bar{t}} h(x, \bar{u}, \bar{v})\right|^p \\
&\qquad \le 2^p \left(\frac{1}{a_0}\right)^{p'} \left((|\bar{v}| + |\bar{l}^*|)^{p'} + |\bar{v}|^{p'}\right).
\end{aligned}$$

The latter terms are integrable, and dominated convergence gives

$$\int_0^1 \int_{\mathbb{R}^N} \left|\nabla_{\bar{s},\bar{t}} h(x, \bar{u} + \tau\bar{k}_{n_j}, \bar{v} + \tau\bar{l}_{n_j}) - \nabla_{\bar{s},\bar{t}} h(x, \bar{u}, \bar{v})\right|^p \, \mathrm{d}x \, \mathrm{d}\tau \to 0 \quad \text{as } j \to \infty.$$

This proves

$$H(\bar{u} + \bar{k}, \bar{v} + \bar{l}) - H(\bar{u}, \bar{v}) - \int_{\mathbb{R}^N} \nabla_{\bar{s},\bar{t}} h(x, \bar{u}, \bar{v}) \cdot \begin{pmatrix} \bar{k} \\ \bar{l} \end{pmatrix} \mathrm{d}x = o\left(\|\bar{k}\|_{L^{p'}(\mathbb{R}^N)} + \|\bar{l}\|_{L^{p'}(\mathbb{R}^N)}\right)$$

and hence differentiability with the asserted derivative. We yet have to prove continuity of the derivative. Consider $\bar{u}_n, \bar{v}_n, \bar{u}, \bar{v} \in L^{p'}(\mathbb{R}^N)$ with $\bar{u}_n \to \bar{u}$, $\bar{v}_n \to \bar{v}$ in $L^{p'}(\mathbb{R}^N)$ as $n \to \infty$. We have

$$\begin{aligned}
&\left\|H'(\bar{u}_n, \bar{v}_n) - H'(\bar{u}, \bar{v})\right\|_{\mathcal{L}(L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N), L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N))} \\
&\quad = \left\|\nabla_{\bar{s},\bar{t}} h(\,\cdot\,, \bar{u}_n, \bar{v}_n) - \nabla h_{\bar{s},\bar{t}}(\,\cdot\,, \bar{u}, \bar{v})\right\|_{L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)}
\end{aligned}$$

and aim to prove that this norm vanishes as $n \to \infty$. This can be done as in the previous step by the Riesz-Fischer Theorem and dominated convergence. □

2.6.2 Results about the inf-sup Characterization of Dual Ground States

Proof of Lemma 2.14

Let us observe that

$$\inf_{(\bar{u},\bar{v})\neq(0,0)} \sup_{\tau>0} J_{\mu\nu}(\tau\bar{u}, \tau\bar{v}) \overset{(2.18)}{=} \inf_{(\bar{u},\bar{v})\neq(0,0)} F_{\mu\nu}(\bar{u}, \bar{v}) \overset{F_{\mu\nu}(0,0)=\infty}{=} \inf_{\bar{u},\bar{v}} F_{\mu\nu}(\bar{u}, \bar{v}).$$

First, we prove that this quantity equals the mountain pass level $c_{\mu\nu}$. For short, we let for $\bar{u}, \bar{v} \in L^{p'}(\mathbb{R}^N)$

$$B(\bar{u}, \bar{v}) := \int_{\mathbb{R}^N} \bar{u}\, \mathcal{R}_\mu[\bar{u}] + \bar{v}\, \mathcal{R}_\nu[\bar{v}] \, \mathrm{d}x$$

$$= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \bar{u}(x)\Psi_\mu(x-y)\bar{u}(y) + \bar{v}(x)\Psi_\nu(x-y)\bar{v}(y) \, \mathrm{d}y \, \mathrm{d}x.$$

▷ STEP 1: *We show that* $c_{\mu\nu} \leq \inf_{(\bar{u},\bar{v})\neq(0,0)} \sup_{\tau>0} J_{\mu\nu}(\tau\bar{u},\tau\bar{v})$.

Let $(\bar{u},\bar{v}) \in L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N) \setminus \{(0,0)\}$. In case that $B(\bar{u},\bar{v}) \leq 0$, equation (2.17) shows

$$\sup_{\tau>0} J_{\mu\nu}(\tau\bar{u},\tau\bar{v}) = \infty;$$

we thus only discuss $B(\bar{u},\bar{v}) > 0$. Again due to equation (2.17), we find $0 < t_0 < t_1$ with

$$J_{\mu\nu}(t_0\bar{u},t_0\bar{v}) = \max_{0<t<\infty} J_{\mu\nu}(t\bar{u},t\bar{v}) \quad \text{and} \quad J_{\mu\nu}(t_1\bar{u},t_1\bar{v}) < 0.$$

We then let, for $0 \leq \tau \leq 1$, $\gamma(\tau) := \tau t_1 \cdot (\bar{u},\bar{v})$ and, by definition of the mountain pass level in equation (2.5), conclude that $\gamma \in \Gamma_{\mu\nu}$ as well as

$$c_{\mu\nu} \leq \max_{0\leq\tau\leq 1} J(\gamma(\tau)) = J_{\mu\nu}(t_0\bar{u},t_0\bar{v}) = \sup_{\tau>0} J_{\mu\nu}(\tau\bar{u},\tau\bar{v}).$$

Passing to the infimum with respect to $(\bar{u},\bar{v}) \in L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)$, the asserted inequality is proved.

▷ STEP 2: *We show that* $c_{\mu\nu} \geq \inf_{(\bar{u},\bar{v})\neq(0,0)} \sup_{\tau>0} J_{\mu\nu}(\tau\bar{u},\tau\bar{v})$, *and that critical points of* $J_{\mu\nu}$ *on the mountain pass level minimize* $F_{\mu\nu}$.

We denote by $(\bar{u}_0,\bar{v}_0) \in L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)$ any critical point of $J_{\mu\nu}$ on the level $c_{\mu\nu}$, e.g. the one given by Theorem 2.1. Then the Chain Rule implies

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} J_{\mu\nu}(t\bar{u}_0,t\bar{v}_0) = 0,$$

hence, as an application of equation (2.17) as in the first step, $t_0 = 1$ is the only stationary point and thus the unique global maximum of the map $(0,\infty) \to \mathbb{R}$, $\tau \mapsto J_{\mu\nu}(\tau\bar{u}_0,\tau\bar{v}_0)$. We conclude $c_{\mu\nu} = J_{\mu\nu}(t_0\bar{u}_0,t_0\bar{v}_0) = \sup_{\tau>0} J_{\mu\nu}(\tau\bar{u}_0,\tau\bar{v}_0)$ and therefore

$$c_{\mu\nu} \geq \inf_{(\bar{u},\bar{v})\neq(0,0)} \sup_{\tau>0} J_{\mu\nu}(\tau\bar{u},\tau\bar{v}).$$

which yields the asserted estimate. Combining with the first step, we conclude equality as stated in the Lemma.

In particular, we have just shown

$$F_{\mu\nu}(\bar{u}_0,\bar{v}_0) \overset{(2.18)}{=} \sup_{\tau>0} J_{\mu\nu}(\tau\bar{u}_0,\tau\bar{v}_0) = c_{\mu\nu} = \inf_{(\bar{u},\bar{v})\neq(0,0)} \sup_{\tau>0} J_{\mu\nu}(\tau\bar{u},\tau\bar{v}) \overset{(2.18)}{=} \inf_{(\bar{u},\bar{v})\neq(0,0)} F_{\mu\nu}(\bar{u},\bar{v})$$

and thus critical points of $J_{\mu\nu}$ on the mountain pass level minimize $F_{\mu\nu}$.

▷ STEP 3: *We show that minimizers of* $F_{\mu\nu}$ *provide critical points of* $J_{\mu\nu}$ *on the mountain pass level when multiplied with a suitable positive constant.*

Conversely, we let $(\bar{u}_0,\bar{v}_0) \in L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N) \setminus \{(0,0)\}$ be a minimizer of $F_{\mu\nu}$. Since $F_{\mu\nu}(t\bar{u}_0,t\bar{v}_0) = F_{\mu\nu}(\bar{u}_0,\bar{v}_0)$ for all $t > 0$, we may assume without loss of generality that

$$p' \int_{\mathbb{R}^N} h(x,\bar{u}_0,\bar{v}_0) \, \mathrm{d}x = B(\bar{u}_0,\bar{v}_0) \tag{2.37}$$

and note that this is a positive quantity. Due to the formula in Lemma 2.11, this immediately implies $J'_{\mu\nu}(\bar u_0, \bar v_0)[\bar u_0, \bar v_0] = 0$ and, moreover,

$$c_{\mu\nu} = F_{\mu\nu}(\bar u_0, \bar v_0) \overset{(2.16),(2.37)}{=} \left[\frac{1}{p'} - \frac{1}{2}\right] B(\bar u_0, \bar v_0) = J_{\mu\nu}(\bar u_0, \bar v_0).$$

In order to see that $J'_{\mu\nu}(\bar u_0, \bar v_0) = 0$, we exploit firstly that $F_{\mu\nu}$ is differentiable near $(\bar u_0, \bar v_0)$ due to $B(\bar u_0, \bar v_0) > 0$, and secondly that $(\bar u_0, \bar v_0)$ is assumed to be a minimizer. Thus for $\phi, \chi \in L^{p'}(\mathbb{R}^N)$, a straightforward calculation repeatedly using (2.37) shows

$$\begin{aligned} 0 &= F'_{\mu\nu}(\bar u_0, \bar v_0)[\phi, \chi] \\ &= \int_{\mathbb{R}^N} \nabla_{\bar s, \bar t} h(x, \bar u_0, \bar v_0) \cdot \begin{pmatrix} \phi \\ \chi \end{pmatrix} \, \mathrm{d}x - \int_{\mathbb{R}^N} \phi \, \mathcal{R}_\mu[\bar u_0] + \chi \, \mathcal{R}_\nu[\bar v_0] \, \mathrm{d}x \\ &= J'_{\mu\nu}(\bar u_0, \bar v_0)[\phi, \chi]. \end{aligned}$$

Thus $(\bar u_0, \bar v_0)$ is a critical point of $J_{\mu\nu}$ at the mountain pass level $c_{\mu\nu}$. □

Proof of Lemma 2.15

(i) This is a consequence of the fact that $I_\mu(\bar u) = J_{\mu\nu}(\bar u, 0)$ for $\bar u \in L^{p'}(\mathbb{R}^N)$ resp. $I_\nu(\bar v) = J_{\mu\nu}(0, \bar v)$ for $\bar v \in L^{p'}(\mathbb{R}^N)$.

(ii) Assume that $(\bar u_0, 0) \in L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N)$ is a ground state of $J_{\mu\nu}$, i.e.

$$J'_{\mu\nu}(\bar u_0, 0) = 0 \quad \text{and} \quad J_{\mu\nu}(\bar u_0, 0) = c_{\mu\nu}.$$

As we have $I_\mu(\bar w) = J_{\mu\nu}(\bar w, 0)$ for all $\bar w \in L^{p'}(\mathbb{R}^N)$, this implies $I'_\mu(\bar u_0) = 0$ and $I_\mu(\bar u_0) = c_{\mu\nu}$. Then, with (2.19) and Lemma 2.14,

$$c_\mu = \inf_{\bar u \in L^{p'}(\mathbb{R}^N)} E_\mu(\bar u) \le E_\mu(\bar u_0) = F_{\mu\nu}(\bar u_0, 0) = c_{\mu\nu} \overset{(i)}{\le} c_\mu,$$

we conclude $c_{\mu\nu} = c_\mu$ and therefore $\bar u_0$ is a ground state of I_μ.

□

2.6.3 Further Results

Proof of Lemma 2.16

First of all, if $\bar u \equiv 0$ or $\bar v \equiv 0$, the asserted estimate follows directly from the explicit formula in Lemma 2.10 (iv), and linear dependence is trivially satisfied. We thus focus on the case $\bar u \not\equiv 0$ and $\bar v \not\equiv 0$. By definition of the Legendre transform, we have for $x \in \mathbb{R}^N$

$$h(x, \bar u(x), \bar v(x)) = \sup_{s,t \in \mathbb{R}} \left[s\bar u(x) + t\bar v(x) - f(x, s, t) \right].$$

In order to estimate the supremum, we insert explicitly

$$\begin{aligned} s_x &:= \frac{\sigma}{\left\| a^{-1/p}\, \bar u \right\|^{p'-1}_{L^{p'}(\mathbb{R}^N)}} \cdot a(x)^{1-p'} |\bar u(x)|^{p'-2} \bar u(x), \\ t_x &:= \frac{\tau}{\left\| a^{-1/p}\, \bar v \right\|^{p'-1}_{L^{p'}(\mathbb{R}^N)}} \cdot a(x)^{1-p'} |\bar v(x)|^{p'-2} \bar v(x) \end{aligned}$$

where $\sigma, \tau \in \mathbb{R}$ are arbitrary. With that, we integrate, estimate $b(x) \le b_+$ and apply Hölder's inequality:

$$\begin{aligned}
&\int_{\mathbb{R}^N} h(x, \bar{u}(x), \bar{v}(x))\,\mathrm{d}x \\
&\ge \int_{\mathbb{R}^N} s_x \bar{u}(x) + t_x \bar{v}(x) - \frac{a(x)}{p}\left(|s_x|^p + 2b_+|s_x|^{\frac{p}{2}}|t_x|^{\frac{p}{2}} + |t_x|^p\right)\mathrm{d}x \\
&= \sigma \left\|a^{-1/p}\,\bar{u}\right\|_{L^{p'}(\mathbb{R}^N)} + \tau \left\|a^{-1/p}\,\bar{v}\right\|_{L^{p'}(\mathbb{R}^N)} \\
&\quad - \frac{1}{p}\left(|\sigma|^p + 2b_+|\sigma\tau|^{\frac{p}{2}} \cdot \int_{\mathbb{R}^N} \frac{\left(a(x)^{-\frac{1}{p}}|\bar{u}|\right)^{\frac{p'}{2}}}{\left\|a^{-\frac{1}{p}}\,\bar{u}\right\|^{\frac{p'}{2}}_{L^{p'}(\mathbb{R}^N)}} \frac{\left(a(x)^{-\frac{1}{p}}|\bar{v}|\right)^{\frac{p'}{2}}}{\left\|a^{-\frac{1}{p}}\,\bar{v}\right\|^{\frac{p'}{2}}_{L^{p'}(\mathbb{R}^N)}}\,\mathrm{d}x + |\tau|^p\right) \\
&\ge \sigma \left\|a^{-1/p}\,\bar{u}\right\|_{L^{p'}(\mathbb{R}^N)} + \tau \left\|a^{-1/p}\,\bar{v}\right\|_{L^{p'}(\mathbb{R}^N)} - \frac{1}{p}\left(|\sigma|^p + 2b_+|\sigma\tau|^{\frac{p}{2}} + |\tau|^p\right) \\
&\overset{(2.20)}{=} \sigma \left\|a^{-1/p}\,\bar{u}\right\|_{L^{p'}(\mathbb{R}^N)} + \tau \left\|a^{-1/p}\,\bar{v}\right\|_{L^{p'}(\mathbb{R}^N)} - f_+(\sigma, \tau).
\end{aligned}$$

Passing to the supremum with respect to $\sigma, \tau \in \mathbb{R}$, we find the asserted inequality. To discuss the necessary condition for the case of equality, let us observe first that, due to strict convexity of f_+, the above-mentioned supremum is attained for the unique choice $(\sigma, \tau) = (\sigma_0, \tau_0)$ where

$$\begin{aligned}
\left\|a^{-1/p}\,\bar{u}\right\|_{L^{p'}(\mathbb{R}^N)} &= \partial_s f_+(\sigma_0, \tau_0) = |\sigma_0|^{\frac{p}{2}-2}\sigma_0\left(|\sigma_0|^{\frac{p}{2}} + b_+|\tau_0|^{\frac{p}{2}}\right), \\
\left\|a^{-1/p}\,\bar{v}\right\|_{L^{p'}(\mathbb{R}^N)} &= \partial_t f_+(\sigma_0, \tau_0) = |\tau_0|^{\frac{p}{2}-2}\tau_0\left(|\tau_0|^{\frac{p}{2}} + b_+|\sigma_0|^{\frac{p}{2}}\right).
\end{aligned}$$

So let us now assume $b_+ > 0$ and that equality holds in the above estimate; we aim to prove that $|\bar{u}|$ and $|\bar{v}|$ are linearly dependent. Since $\bar{u} \not\equiv 0$ and $\bar{v} \not\equiv 0$, we infer $\sigma_0, \tau_0 > 0$ thanks to the previous considerations; hence equality in the above estimate implies in particular that equality must hold in the Cauchy-Schwarz type estimate

$$\int_{\mathbb{R}^N} \left(a(x)^{-\frac{1}{p}}|\bar{u}|\right)^{\frac{p'}{2}}\left(a(x)^{-\frac{1}{p}}|\bar{v}|\right)^{\frac{p'}{2}}\mathrm{d}x = \left\|a^{-\frac{1}{p}}\,\bar{u}\right\|^{\frac{p'}{2}}_{L^{p'}(\mathbb{R}^N)}\left\|a^{-\frac{1}{p}}\,\bar{v}\right\|^{\frac{p'}{2}}_{L^{p'}(\mathbb{R}^N)}.$$

Due to positivity of a, this proves the asserted linear dependence of $|\bar{u}|$ and $|\bar{v}|$. □

Proof of (2.28)

We have to show that, for $p > 4$,

$$\forall\,\eta \ge 0 \qquad \frac{(1+\eta^2)^{\frac{1}{2}}}{(1+(2^{\frac{p}{2}}-2)\eta^{\frac{p}{2}}+\eta^p)^{\frac{1}{p}}} \ge 1.$$

We let $p > 4$ and consider, for $\eta \ge 0$,

$$\psi(\eta) := \frac{(1+\eta^2)^{\frac{1}{2}}}{(1+(2^{\frac{p}{2}}-2)\eta^{\frac{p}{2}}+\eta^p)^{\frac{1}{p}}}.$$

We assert that ψ has exactly three critical points on $(0,\infty)$ which are given by $\{\eta_1, 1, \eta_1^{-1}\}$ for some $\eta_1 \in (0,1)$, and that ψ attains its minimum on $(0,\infty)$ uniquely at $\eta = 1 =: \eta_0$.

We note first that ψ is smooth on $(0,\infty)$, and that $\psi(\eta) \to 1$ as $\eta \searrow 0$ or $\eta \nearrow \infty$. Moreover,

$\psi(\eta^{-1}) = \psi(\eta)$ holds for all $\eta > 0$. Critical points of ψ satisfy

$$0 = \psi'(\eta), \text{ equivalently } 1 + (2^{\frac{p-2}{2}} - 1)\eta^{\frac{p}{2}} = \eta^{p-2} + (2^{\frac{p-2}{2}} - 1)\eta^{\frac{p-4}{2}}. \tag{2.38a}$$

Obviously, this is satisfied for $\eta = \eta_0 = 1$. Moreover, $p > 4$ implies that $\psi''(1) = \frac{2^{\frac{p}{2}} - p}{2 \cdot 2^{\frac{p}{2}}} > 0$, which proves that $\psi(1) = 1$ is a strict local minimum. Once we have established that ψ has a unique critical point η_1 in the interval $(0,1)$, we conclude that ψ attains local maxima at η_1 and at η_1^{-1} and hence that the local minimum in $\eta_0 = 1$ is in fact global.

We substitute $\kappa := 2^{\frac{p-2}{2}} - 1 (> 1)$, $\sigma := \frac{p-4}{p} \in (0,1)$, $y := \eta^{\frac{p}{2}}$ and (2.38a) gives

$$0 = \psi'\left(y^{\frac{2}{p}}\right) \quad \Leftrightarrow \quad \frac{1 + \kappa y}{\kappa + y} = y^{\sigma}. \tag{2.38b}$$

Existence of η_1: This is guaranteed by the Mean Value Theorem since $\psi(0) = \psi(1) = 1$.

Uniqueness of η_1: Now assume that ψ possesses (at least) two critical points η_1, η_2 in $(0,1)$ with $0 < \eta_1 < \eta_2 < 1$; then $\frac{1}{\eta_2}, \frac{1}{\eta_1} \in (1, \infty)$ are two more critical points. We denote $y_j := \eta_j^{\frac{p}{2}}$ for $j = 0,1,2$. Notice that, by (2.38b), we have

$$\frac{1 + \kappa y}{\kappa + y} - y^{\sigma} = 0 \quad \text{for } y \in \left\{ y_1, y_2, 1, \frac{1}{y_1}, \frac{1}{y_2} \right\}.$$

The Mean Value Theorem yields $z_1 \in (y_1, y_2), z_2 \in (y_2, 1), z_3 \in \left(1, \frac{1}{y_2}\right), z_4 \in \left(\frac{1}{y_2}, \frac{1}{y_1}\right)$ with

$$\left.\frac{\mathrm{d}}{\mathrm{d}y}\right|_{y=z_j} \left(\frac{1 + \kappa y}{\kappa + y} - y^{\sigma} \right) = 0, \quad \text{equivalently} \quad \sqrt{\frac{\sigma}{\kappa^2 - 1}}(\kappa + z_j) = z_j^{\frac{1-\sigma}{2}}.$$

Then again, we find $z_1^* \in (z_1, z_2)$ and $z_2^* \in (z_3, z_4)$ satisfying

$$\left.\frac{\mathrm{d}}{\mathrm{d}y}\right|_{y=z_j^*} \left(\sqrt{\frac{\sigma}{\kappa^2 - 1}}(\kappa + y) - y^{\frac{1-\sigma}{2}} \right) = 0, \quad \text{equivalently} \quad \frac{(1-\sigma)^2(\kappa^2 - 1)}{4\sigma} = (z_j^*)^{\sigma+1}.$$

The latter equation, however, possesses a unique positive solution; since we have found two distinct ones $z_1^* \in (0,1), z_2^* \in (1, \infty)$, we have a contradiction. This verifies (2.28). □

Proof of (2.31)

For b, p as in (2.6), we aim to prove that

$$\inf_{\eta>0} \sup_{\sigma>0} \frac{(1 + \sigma\eta)^p}{(1 + 2b\sigma^{\frac{p}{2}} + \sigma^p)(1 + \eta^2)^{\frac{p}{2}}} = \frac{2^{\frac{p-2}{2}}}{1 + b} \quad \text{attained only at } \eta = 1$$

under the additional conditions

$$2 < p < 4 \text{ and } b \geq \frac{p}{2} - 1 \qquad \text{or} \qquad p \geq 4 \text{ and } b \geq \max\left\{ 2^{\frac{p-2}{2}} - 1, \frac{p}{2} - 1 \right\} = 2^{\frac{p-2}{2}} - 1$$

with $(p, b) \neq (4, 1)$. Throughout, we denote

$$g : (0, \infty)^2 \to \mathbb{R}, \quad g(\eta, \sigma) := \frac{(1 + \sigma\eta)^p}{(1 + 2b\sigma^{\frac{p}{2}} + \sigma^p)(1 + \eta^2)^{\frac{p}{2}}}$$

and aim to show that the infimum $\inf_{\eta>0} \sup_{\sigma>0} g(\eta, \sigma)$ is attained uniquely at $\eta = 1$.

First, we estimate the infimum by $\eta = 1$, i.e. $\inf_{\eta>0} \sup_{\sigma>0} g(\eta,\sigma) \le \sup_{\sigma>0} g(1,\sigma)$. We now demonstrate that this supremum is a strict global maximum, attained only at $\sigma = 1$. To this end, we let

$$q : (0,\infty) \to \mathbb{R}, \quad q(\sigma) := g(1,\sigma) = 2^{-\frac{p}{2}} \frac{(1+\sigma)^p}{1+2b\sigma^{\frac{p}{2}}+\sigma^p},$$

and make the following observations concerning local and global extrema of q:

(i) q is a smooth function with

$$q'(\sigma) = \frac{2^{-\frac{p}{2}}\, p(1+\sigma)^{p-1}}{(1+2b\sigma^{\frac{p}{2}}+\sigma^p)^2} \cdot \left(1 + b\sigma^{\frac{p}{2}} - b\sigma^{\frac{p}{2}-1} - \sigma^{p-1}\right). \tag{2.39}$$

In particular, $q'(\sigma) = 0$ if and only if $q'(\sigma^{-1}) = 0$.

(ii) We have $q(\sigma) \to 2^{-\frac{p}{2}}$ as $\sigma \to 0$ and as $\sigma \to \infty$. Moreover, in view of (2.39) and of $p > 2$, hence $p - 1 > \frac{p}{2}$, q is strictly increasing at small arguments and strictly decreasing at large arguments.

(iii) We have $q'(1) = 0$. In view of equation (2.39) and the overall condition $b \le p - 1$, see (2.6), the Taylor expansion

$$\begin{aligned}&\left(1 + b(1+h)^{\frac{p}{2}} - b(1+h)^{\frac{p}{2}-1} - (1+h)^{p-1}\right)\\ &= (b-p+1)\,h + \frac{p-2}{2}(b-p+1)h^2 + \frac{p-2}{24}(3b(p-4) - 4(p-1)(p-3))h^3 + O(h^4)\end{aligned}$$

shows that $q'(1+h)$ is positive for $h < 0, h \approx 0$ and negative for $h > 0, h \approx 0$. Thus $q(1) = \frac{2^{\frac{p}{2}-1}}{1+b}$ is a strict local maximum of q.

(iv) We show that there are at most two other critical points of q.
To this end, we assume that there are four critical points $\sigma_4 > \sigma_3 > \sigma_2 > \sigma_1 > 0$ and find a contradiction by twice applying the Mean Value Theorem as in the proof of equation (2.28). Indeed, equation (2.39) implies

$$1 + b\sigma_j^{\frac{p}{2}} - b\sigma_j^{\frac{p}{2}-1} - \sigma_j^{p-1} = 0, \qquad j \in \{1,2,3,4\}.$$

We obtain $\tau_j \in (\sigma_j, \sigma_{j+1})$ and $\zeta_j \in (\tau_j, \tau_{j+1})$ with

$$\begin{aligned}&b\frac{p}{2}\tau_j^{\frac{p}{2}-1} - b\left(\frac{p}{2}-1\right)\tau_j^{\frac{p}{2}-2} - (p-1)\tau_j^{p-2} = 0, \qquad j \in \{1,2,3\}\\ &\text{or, equivalently,} \quad b\frac{p}{2}\tau_j - b\left(\frac{p}{2}-1\right) - (p-1)\tau_j^{\frac{p}{2}} = 0, \qquad j \in \{1,2,3\};\\ &b\frac{p}{2} - (p-1)\frac{p}{2}\zeta_j^{\frac{p}{2}-1} = 0, \qquad j \in \{1,2\}.\end{aligned}$$

Since the latter equation has a unique positive solution, this is contradictory.

(v) We show that q attains a unique global maximum in $\sigma = 1$.
If 1 is the only critical point of q, this follows from (iii) and

$$q(1) = \frac{2^{\frac{p}{2}-1}}{1+b} > 2^{-\frac{p}{2}} = \lim_{\sigma\to 0} q(\sigma^{\pm 1}).$$

(Indeed, we have $1+b \le p < 2^{p-1}$ for $p > 2$.) Else, (i) and (iv) imply that q possesses exactly three critical points $\sigma_0 < 1 < \sigma_0^{-1}$. But then, $q(\sigma_0^{\pm 1})$ cannot be local maxima - in view of (ii), they are indeed saddle points. Hence, we conclude that q is strictly increasing on $(0,1)$ and strictly decreasing on $(1,\infty)$. We summarize

$$\max_{\sigma>0} q(\sigma) = q(1) = \frac{2^{\frac{p}{2}-1}}{1+b} \quad \text{and hence } \inf_{\eta>0}\sup_{\sigma>0} g(\eta,\sigma) \leq \frac{2^{\frac{p}{2}-1}}{1+b}.$$

Second, we estimate the supremum by $\sigma = \eta$, i.e. $\inf_{\eta>0}\sup_{\sigma>0} g(\eta,\sigma) \geq \inf_{\eta>0} g(\eta,\eta)$. This time, we show that the infimum is a strict global minimum, attained only at $\eta = 1$, and thus introduce

$$r : (0,\infty) \to \mathbb{R}, \quad r(\eta) := g(\eta,\eta) = \frac{(1+\eta^2)^{\frac{p}{2}}}{1+2b\eta^{\frac{p}{2}}+\eta^p}.$$

Again, we analyze the extrema of r in steps analogous to those above. We recall $b \geq \frac{p}{2}-1$.

(i) r is a smooth function with

$$r'(\eta) = \frac{p\eta(1+\eta^2)^{\frac{p}{2}-1}}{(1+2b\eta^{\frac{p}{2}}+\eta^p)^2} \cdot \left(1 + b\eta^{\frac{p}{2}} - b\eta^{\frac{p}{2}-2} - \eta^{p-2}\right). \tag{2.40}$$

In particular, $r'(\eta) = 0$ if and only if $r'(\eta^{-1}) = 0$.

(ii) We have $r(\eta) \to 1$ as $\eta \to 0$ and as $\eta \to \infty$. Moreover, in view of (2.40), we have the following cases:

(a) If $p = 4$, we have by assumption $b \geq 2^{\frac{p-2}{2}} - 1 = 1, b \neq 1$ and immediately conclude that r is strictly decreasing for $\eta < 1$ and strictly increasing for $\eta > 1$, hence as asserted $r(1)$ is the unique global minimum.

(b) If $p > 4$ and hence $p-2 > \frac{p}{2}$, $\frac{p}{2}-2 > 0$, we infer that r is strictly increasing at small arguments and strictly decreasing at large arguments.

(c) If $p < 4$, we find similarly that r is strictly decreasing at small arguments and strictly increasing at large arguments.

(iii) We have $r'(1) = 0$ and, whenever the strict inequality $b > \frac{p}{2}-1$ holds,

$$r''(1) = \frac{p2^{\frac{p}{2}}}{8(1+b)^2} \cdot (2(1+b)-p) > 0.$$

Thus in these cases $r(1)$ is a strict local minimum. We demonstrate that the same holds in all relevant situations where $b = \frac{p}{2}-1$. Indeed, for $p > 4$, the assumption $b \geq 2^{\frac{p-2}{2}} - 1$ shows that the case of equality does not have to be considered since $2^{\frac{p-2}{2}} > \frac{p}{2}$ for $p > 4$; and the occurrence of a strict minimum $r(1)$ for $p = 4$ has already been derived in (ii) (a). For $p \in (2,4)$ and $b = \frac{p}{2}-1 \in (0,1)$, Taylor expansion shows as $\eta \to 1$

$$\begin{aligned} r'(\eta) &= \frac{p\eta(1+\eta^2)^{\frac{p}{2}-1}}{(1+2b\eta^{\frac{p}{2}}+\eta^p)^2} \cdot \left(1 + b\eta^{b+1} - b\eta^{b-1} - \eta^{2b}\right) \\ &= \frac{p\eta(1+\eta^2)^{\frac{p}{2}-1}}{(1+2b\eta^{\frac{p}{2}}+\eta^p)^2} \cdot \left(\frac{b(1-b^2)}{3}(\eta-1)^3 + O((\eta-1)^4)\right), \end{aligned}$$

and hence the changing sign of the derivative implies that $r(1)$ is a strict local minimum.

(iv) For $p \neq 4$, we show that there are at most two other critical points of r. (Note that, for $p = 4$, this is covered by (ii) (a).)

To this end, we assume that there are four critical points $\eta_4 > \eta_3 > \eta_2 > \eta_1 > 0$ and find a contradiction as before. Indeed, equation (2.40) implies

$$1 + b\eta_j^{\frac{p}{2}} - b\eta_j^{\frac{p}{2}-2} - \eta_j^{p-2} = 0, \qquad j \in \{1,2,3,4\}.$$

We obtain $\tau_j \in (\eta_j, \eta_{j+1})$ and $\zeta_j \in (\tau_j, \tau_j + 1)$ with

$$b\frac{p}{2}\tau_j^{\frac{p}{2}-1} - b\left(\frac{p}{2}-2\right)\tau_j^{\frac{p}{2}-3} - (p-2)\tau_j^{p-3} = 0, \qquad j \in \{1,2,3\}$$

or, equivalently, $$b\frac{p}{2}\tau_j^2 - b\left(\frac{p}{2}-2\right) - (p-2)\tau_j^{\frac{p}{2}} = 0, \qquad j \in \{1,2,3\};$$

$$bp\zeta_j - (p-2)\frac{p}{2}\zeta_j^{\frac{p}{2}-1} = 0, \qquad j \in \{1,2\}.$$

Given $p \neq 4$, the latter equation has a unique positive solution, which is contradictory.

(v) We show that r attains a unique global minimum at $\eta = 1$.
For $p = 4$, this has been settled in (ii)(a). Let us consider $p \neq 4$. Then by (iii) $r(1)$ is a strict local minimum, and (iv) and (i) guarantee that there are at most two other critical points $\eta_0^{\pm 1}$ which, then, cannot be local minima. We thus compare $r(1)$ with the boundary values, recalling the general assumptions $2^{\frac{p-2}{2}}, \frac{p}{2} \leq 1 + b$:

$$r(1) = \frac{2^{\frac{p-2}{2}}}{1+b} \leq 1 = \lim_{\eta \to 0} r(\eta^{\pm 1}).$$

If $p > 4$, the monotonicity statement in (ii)(b) now shows that $r(1)$ is a global minimum even if equality holds in the estimate above. If $p < 4$, we have $\frac{p}{2} > 2^{\frac{p-2}{2}}$ and hence even

$$r(1) = \frac{2^{\frac{p-2}{2}}}{1+b} \leq \frac{2^{\frac{p-2}{2}}}{\frac{p}{2}} < 1 = \lim_{\eta \to 0} r(\eta^{\pm 1}),$$

which also ensures a global minimum. We summarize

$$\min_{\eta>0} r(\eta) = r(1) = \frac{2^{\frac{p}{2}-1}}{1+b} \quad \text{and hence} \quad \inf_{\eta>0} \sup_{\sigma>0} g(\eta,\sigma) \geq \frac{2^{\frac{p}{2}-1}}{1+b}.$$

It remains to justify the uniqueness statement in equation (2.31). For any fixed $\eta > 0$, the uniqueness statement in the second discussion reveals that

$$\sup_{\sigma>0} g(\eta,\sigma) \geq g(\eta,\eta) = r(\eta) > \frac{2^{\frac{p}{2}-1}}{1+b} \quad \text{for all } \eta \neq 1.$$

Thus the infimum can be (and is) realized only at $\eta = 1$. □

Proof of (2.32)

We assume $b < \frac{p}{2} - 1$ and aim to prove that, as $|\kappa| \searrow 0$,

$$\sup_{\sigma>0} \frac{(1+\sigma(1+\kappa))^p}{(1+2b\sigma^{\frac{p}{2}} + \sigma^p)(1+(1+\kappa)^2)^{\frac{p}{2}}} = \frac{2^{\frac{p-2}{2}}}{1+b}\left[1 - \kappa^2\,\frac{p\left(\frac{p}{2}-1-b\right)}{4(p-1-b)} + O(\kappa^3)\right].$$

First, we verify that the supremum is attained at some unique $\sigma = \sigma(\kappa)$ provided $|\kappa|$ is sufficiently small. We introduce

$$g_\kappa : [0,\infty) \to \mathbb{R}, \quad g_\kappa(\sigma) := \frac{(1+\sigma(1+\kappa))^p}{1+2b\sigma^{\frac{p}{2}}+\sigma^p}.$$

From (v) in the first part of the proof of (2.31), we know that g_0 attains a unique global maximum at $\sigma(0)=1$ with $g_0(\sigma(0)) = \frac{2^{p-1}}{1+b}$, and that g_0 is strictly increasing on $(0,1)$ and decreasing on $(1,\infty)$. For sufficiently small $|\kappa|$, we still have (due to $2^p > p$)

$$g_\kappa(1) = \frac{(2+\kappa)^p}{2(1+b)} > \frac{(2+\kappa)^p}{p} > \max\{1,(1+\kappa)^p\} = \max\left\{g_\kappa(0), \lim_{\sigma\to\infty} g_\kappa(\sigma)\right\},$$

and hence g_κ attains its maximum at some point(s) with $g_\kappa'(\sigma)=0$, i.e.

$$(1+\kappa)\cdot\left(1+b\sigma^{\frac{p}{2}}\right) = b\sigma^{\frac{p}{2}-1}+\sigma^{p-1}. \tag{2.41}$$

The Implicit Function Theorem yields $\delta>0$ and a (smooth) curve of solutions $(\kappa,\sigma(\kappa))$, $-\delta<\kappa<\delta$, of equation (2.41) with $(0,\sigma(0))=(0,1)$; in particular, the derivative of $\sigma(\kappa)$ with respect to the parameter is given by

$$\sigma'(0) = \frac{1+b}{p-1-b}.$$

(We recall that $p-1-b > \frac{p}{2}-1-b > 0$.) It is worth noticing that we can find such $\delta^* \in (0,\delta]$ that g_κ attains its global maximum at $\sigma(\kappa)$ provided $|\kappa|\le\delta^*$. This can be seen as follows: The Implicit Function Theorem asserts that, for some $\varepsilon>0$, the points $(\kappa,\sigma(\kappa))$ are the only solutions (κ,σ) of (2.41) in $(-\delta,\delta)\times(1-\varepsilon,1+\varepsilon)$. For $\sigma\in(0,\infty)\setminus(1-\varepsilon,1+\varepsilon)$ and any $|\kappa|\le\delta^*<\delta$ (where δ^* can still be chosen), we estimate using the monotonicity properties of g_0 recalled above

$$\begin{aligned}
g_\kappa(\sigma) &= \frac{(1+\sigma(1+\kappa))^p}{1+2b\sigma^{\frac{p}{2}}+\sigma^p} \\
&\le \frac{((1+|\kappa|)+\sigma(1+|\kappa|))^p}{1+2b\sigma^{\frac{p}{2}}+\sigma^p} \\
&\le (1+|\kappa|)^p\cdot g_0(\sigma) \\
&\le (1+|\kappa|)^p\cdot \max\{g_0(1\pm\varepsilon)\} \\
&= (1+|\kappa|)^p\cdot\frac{\max\{g_0(1\pm\varepsilon)\}}{g_0(1)}\cdot\frac{g_0(1)}{g_\kappa(1)}\cdot g_\kappa(1) \\
&= (1+|\kappa|)^p\cdot\frac{\max\{g_0(1\pm\varepsilon)\}}{g_0(1)}\cdot\frac{2^p}{(2+\kappa)^p}\cdot g_\kappa(1) \\
&\le \frac{(1+|\kappa|)^p}{(1-|\kappa|)^p}\cdot\frac{\max\{g_0(1\pm\varepsilon)\}}{g_0(1)}\cdot g_\kappa(1) \\
&\le \frac{(1+\delta^*)^p}{(1-\delta^*)^p}\cdot\frac{\max\{g_0(1\pm\varepsilon)\}}{g_0(1)}\cdot g_\kappa(1).
\end{aligned}$$

Since $\max\{g_0(1\pm\varepsilon)\}<g_0(1)$, we can choose $\delta^*=\delta^*(\varepsilon)\in(0,\delta]$ in such way that $g_\kappa(\sigma)<g_\kappa(1)$ for all $|\kappa|\le\delta^*$ and all $\sigma\in(0,\infty)\setminus(1-\varepsilon,1+\varepsilon)$. Hence, for fixed $|\kappa|\le\delta^*$, g_κ attains its global maximum on the interval $(1-\varepsilon,1+\varepsilon)$, and the uniqueness assertion in the Implicit Function Theorem guarantees that

$$\max_{\sigma>0} g_\kappa(\sigma) = g_\kappa(\sigma(\kappa)) \qquad \text{for all } |\kappa|\le\delta^*.$$

Moreover, since $\kappa\mapsto\sigma(\kappa)$ is smooth, a lengthy but straightforward calculation based on

Taylor's theorem allows to approximate the maximum of g_κ as $\kappa \to 0$ via

$$\begin{aligned}
\max_{\sigma>0} g_\kappa(\sigma) &= g_\kappa(\sigma(\kappa)) \\
&= g_0(\sigma(0)) + \kappa \left.\frac{\mathrm{d}}{\mathrm{d}\kappa}\right|_{\kappa=0} g_\kappa(\sigma(\kappa)) + \frac{\kappa^2}{2} \left.\frac{\mathrm{d}^2}{\mathrm{d}\kappa^2}\right|_{\kappa=0} g_\kappa(\sigma(\kappa)) + O(\kappa^3) \\
&= \frac{2^{p-1}}{1+b} \left[1 + \kappa \frac{p}{2} + \kappa^2 \frac{p}{8} \left(\frac{b-p+1}{1+b} \sigma'(0)^2 + 2\sigma'(0) + (p-1) \right) + O(\kappa^3) \right] \\
&= \frac{2^{p-1}}{1+b} \left[1 + \kappa \frac{p}{2} + \kappa^2 \frac{p\,(-p^2 + (b+2)p - 2(1+b))}{8(b-p+1)} + O(\kappa^3) \right].
\end{aligned}$$

(We only remark here that all terms containing $\sigma''(0)$ drop out, which is why we did not provide its value above.) When combined with $(1+(1+\kappa)^2)^{-\frac{p}{2}} = 2^{-\frac{p}{2}} \left[1 - \kappa \frac{p}{2} + \kappa^2 \frac{p^2}{8} + O(\kappa^3)\right]$, we finally conclude

$$\begin{aligned}
&\sup_{\sigma>0} \frac{(1+\sigma(1+\kappa))^p}{(1+2b\sigma^{\frac{p}{2}} + \sigma^p)(1+(1+\kappa)^2)^{\frac{p}{2}}} \\
&= g_\kappa(\sigma(\kappa)) \cdot (1+(1+\kappa)^2)^{-\frac{p}{2}} \\
&= \frac{2^{\frac{p-2}{2}}}{1+b} \left[1 + \kappa \frac{p}{2} + \kappa^2 \frac{p\,(-p^2+(b+2)p-2(1+b))}{8(b-p+1)} \right] \left[1 - \kappa \frac{p}{2} + \kappa^2 \frac{p^2}{8} \right] + O(\kappa^3) \\
&= \frac{2^{\frac{p-2}{2}}}{1+b} \left[1 - \kappa^2 \frac{p}{4(p-1-b)} \left(\frac{p}{2} - 1 - b \right) \right] + O(\kappa^3).
\end{aligned}$$

□

2.7 Summary

In this chapter we investigated dual ground state solutions of the nonlinear Helmholtz system (2.1),

$$\begin{cases} -\Delta u - \mu u = a(x) \left(|u|^{\frac{p}{2}} + b(x)|v|^{\frac{p}{2}} \right) |u|^{\frac{p}{2}-2} u & \text{on } \mathbb{R}^N, \\ -\Delta v - \nu v = a(x) \left(|v|^{\frac{p}{2}} + b(x)|u|^{\frac{p}{2}} \right) |v|^{\frac{p}{2}-2} v & \text{on } \mathbb{R}^N, \\ u, v \in L^p(\mathbb{R}^N) \end{cases}$$

where $N \geq 2$ and the coefficient functions $a(x), b(x)$ were assumed periodic and bounded with $a(x) \geq a_0 > 0$. A dual variational setting could be established under the additional assumptions $\frac{2(N+1)}{N-1} < p < 2^*$ and $0 \leq b(x) \leq p-1$. The former condition is related to the theory of linear Helmholtz equations, see (1.10), and also appears in the study of corresponding single Helmholtz equations by Evéquoz and Weth [28], [26]; the latter condition only arises when considering systems of equations. It is a convexity condition which ensures a one-to-one correspondence between $u, v \in L^p(\mathbb{R}^N)$ and the dual variables $\bar{u}, \bar{v} \in L^{p'}(\mathbb{R}^N)$. Under the aforementioned assumptions, the existence of a dual ground state of (2.1) was proved in Theorem 2.1. Moreover, we provided criteria whether or not such dual ground states are fully nontrivial in Theorems 2.4 and 2.5.

Naturally, there are open questions some of which might be worth future research. First of all, it seems likely that similar existence results can be obtained for more general nonlinearities, replacing the explicit power-type expression $f(x,s,t)$ in (2.9) by a strictly convex, x-periodic function of class C^1 which is homogeneous of degree p in the variables s, t. Alternatively, one might strive for multiplicity results as obtained by Evéquoz and Weth for the single equation, cf. Theorem 1.11 and the remarks following Theorem 1.12. In both cases, however, the question whether fully nontrivial ground resp. bound state solutions occur will require novel ideas; indeed, the proofs of Theorems 2.4 and 2.5 heavily rely on the explicit form of $f(x, s, t)$ and on the mountain-pass type saddle point geometry of a ground state in the dual setting.

Moreover, generalizations to systems with a larger number of components might be interesting. Another extension of our results could be achieved by allowing periodic potentials in place of μ, ν. An appropriate Limiting Absorption Principles and suitable resolvent-type operators are constructed in [52] under certain assumptions on the potential terms.

The most urgent question, however, concerns the dual variational ansatz itself. It would be most interesting to find a "physical" interpretation of the notion of dual ground states, and to assess whether (possibly different) variational techniques can be applied in parameter ranges beyond the convexity condition $0 \leq b(x) \leq p - 1$. At least, we will demonstrate in the following chapter, in a radially symmetric setting and for the practically relevant choice $N = 3, p = 4$, that it is possible to construct fully nontrivial solutions of nonlinear Helmholtz systems with arbitrarily large positive and negative couplings.

Chapter 3.

Bifurcations of a Cubic Helmholtz System

3.1 Introduction and Main Results

In this chapter, we analyze in detail the physically most relevant case of the nonlinear Helmholtz system (H) in $N = 3$ space dimensions with a Kerr nonlinearity, $p = 4$. Aiming for radially symmetric solutions, we further assume constant coefficients $a \equiv 1, b \in \mathbb{R}$. That is, we discuss the system (H),

$$\begin{cases} -\Delta u - \mu u = \left(u^2 + b\, v^2\right) u & \text{on } \mathbb{R}^3, \\ -\Delta v - \nu v = \left(v^2 + b\, u^2\right) v & \text{on } \mathbb{R}^3, \\ u(x), v(x) \to 0 \quad \text{as } |x| \to \infty \end{cases} \tag{3.1}$$

for given $\mu, \nu > 0$. We are mostly interested in existence results for fully nontrivial radially symmetric solutions $u, v \in L^4_{\text{rad}}(\mathbb{R}^3)$ of this system. These will be obtained using bifurcation theory where the coupling b acts as a bifurcation parameter. Such an approach is new in the context of nonlinear Helmholtz equations or systems. In contrast to the previous chapter, we construct fully nontrivial, radially symmetric solutions also for negative coupling $b < 0$ and for arbitrarily large values of $|b|$. The motivation for assuming radial symmetry is twofold: Firstly, it ensures the validity of Limiting Absorption Principles for the cubic nonlinearity on $\mathbb{R}^3$ as explained in Remark 1.10 (b). Secondly, the symmetry assumption reduces the number of solutions of the linearized equations and hence will be one of the essential ingredients in the verification of bifurcation from simple eigenvalues.

Let us give an outline of this chapter. We first present bifurcation results for Schrödinger systems corresponding to (3.1) and proceed with our new counterparts in the Helmholtz case. Afterwards, we state the Crandall-Rabinowitz Bifurcation Theorem and Rabinowitz' Global Bifurcation Theorem, which are the essential ingredients of our proofs; we then close the introductory section with some auxiliary results of technical character concerning fundamental properties of the underlying vector spaces, the fundamental solutions in the special case $N = 3$ and the regularity of solutions of (3.1). In Section 3.2, we introduce the concepts and technical results we apply in order to verify the assumptions of the bifurcation theorems in the proofs of Theorems 3.2 and 3.4. Parts 3.3 and 3.4 are then devoted to the proofs of our main theorems. In an abridged version, our findings on bifurcations of cubic Helmholtz systems have recently been accepted for publication in ANONA. For a preliminary version, cf. [56].

3.1.1 Bifurcation in the Schrödinger Case

Our results are inspired by known bifurcation results for the nonlinear Schrödinger system

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + b\, uv^2 & \text{on } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + b\, vu^2 & \text{on } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \quad u > 0,\ v > 0 \end{cases} \tag{3.2}$$

where one assumes $\lambda_1, \lambda_2 > 0$ in contrast to (3.1). We focus on bifurcation results by Bartsch, Wang and Wei in [9] and Bartsch, Dancer and Wang in [8] and refer to the respective introductory sections for a general overview of methods and results for (3.2) available in the years 2007 resp. 2010. In Theorem 1.1 of the first-mentioned paper the authors show that a continuum consisting of positive radially symmetric solutions $(u, v, \lambda_1, \lambda_2, \mu_1, \mu_2, b)$ of (3.2) of topological dimension at least 5 bifurcates from a two-dimensional set of semipositive solutions $(u, v) = (u_{\lambda_1,\mu_1}, 0)$ parametrized by $\lambda_1, \mu_1 > 0$. For a similar system with $\mu_1 = \mu_2 = 1$ and more general exponents, the existence of countably many bifurcation points giving rise to sign-changing radially symmetric solutions was proved by Mandel in his dissertation thesis (Satz 2.1.6 of [48]).

In the special case $N = 2{,}3$ and $\lambda_2 = \lambda_1 > 0$, $\mu_2, \mu_1 > 0$ with (w.l.o.g.) $\mu_2 \geq \mu_1$, $\lambda_2 = \lambda_1 = 1$, Bartsch, Dancer and Wang proved in [8] the existence of countably many mutually disjoint global continua of solutions bifurcating from some diagonal solution family

$$\{(u_b, v_b, b) : b \in (-\sqrt{\mu_1\mu_2}, \mu_1) \cup (\mu_2, \infty)\} \subset H^1_{\text{rad}}(\mathbb{R}^N) \times H^1_{\text{rad}}(\mathbb{R}^N) \times \mathbb{R}$$

with a concentration of bifurcation points as $b \searrow -\sqrt{\mu_1\mu_2}$. Here

$$u_b = \left(\frac{\mu_2 - b}{\mu_1\mu_2 - b^2}\right)^{\frac{1}{2}} u_0, \qquad v_b = \left(\frac{\mu_1 - b}{\mu_1\mu_2 - b^2}\right)^{\frac{1}{2}} u_0$$

where $u_0 \in H^1_{\text{rad}}(\mathbb{R}^N)$ is a nondegenerate positive solution of the nonlinear Schrödinger equation $-\Delta u + u = u^3$. (Strictly speaking, the term "diagonal family" is only justified in the special case $\mu_1 = \mu_2$, as we assume in the Helmholtz system in (3.1).) Moreover, having introduced a suitable labeling of the continua, the authors showed that the k-th bifurcating continuum consists of solutions where the radial profile of $u - v$ has exactly $k - 1$ nodes, cf. Theorem 2.3 in [8].

In the Helmholtz case, we will analyze the corresponding cases of bifurcations from semitrivial and diagonal solutions in Theorems 3.2 and 3.4, respectively. In contrast to the Schrödinger case, we will show (in a suitable topology) that bifurcation occurs at every point. Looking more closely, we find the same structure of discrete bifurcation points as in the Schrödinger case when fixing a set of asymptotic parameters τ_1, ω prescribing the oscillatory behavior of solutions as $|x| \to \infty$. This is what will be done in Theorems 3.2 and 3.4 using the asymptotic conditions (3.6) resp. (3.7). Moreover, in the Schrödinger case, the bifurcating solutions are characterized by their nodal structure; we will see that in the Helmholtz case this characterization is replaced by a condition on the "asymptotic phase" of the solution (an integral quantity), which at least close to the k-th bifurcation point takes the value $\omega + k\pi$, see Proposition 3.18 and the explanations following it.

3.1.2 Bifurcation in the Helmholtz Case. Main Results

Motivated by the decay properties of radial solutions of nonlinear Helmholtz equations in [54], e.g. Theorem 1.2 (iii), we look for solutions in the space $X_1 \times X_1$ where, for $q \geq 1$,

$$X_q := \left\{ w \in C_{\text{rad}}(\mathbb{R}^3, \mathbb{R}) \;\middle|\; \sup_{x \in \mathbb{R}^3} (1 + |x|^2)^{\frac{q}{2}} |w(x)| < \infty \right\}. \tag{3.3}$$

For completeness and embedding properties of these spaces, we refer to Lemma 3.8 at the end of the introductory section. Working on X_1, we will be able to derive compactness properties which are crucial when proving our bifurcation results. Throughout, we discuss classical, smooth radially symmetric solutions (u, v) with $u, v \in X_1 \cap C^2_{\text{loc}}(\mathbb{R}^3)$ of the system (3.1) and related equations. Let us remark here only briefly that, in fact, all distributional solutions with $u, v \in L^4_{\text{rad}}(\mathbb{R}^3)$ actually belong to $X_1 \cap C^2_{\text{loc}}(\mathbb{R}^3)$; so the regularity and decay assumption is not too restrictive and in particular allows to relate the bifurcation results to those on the existence of dual ground states obtained in the previous chapter (where we have not investigated higher regularity). For more details on regularity, we refer to Lemma 3.11, also to be found at the end of the introductory section. A comparison of our main bifurcation results with the variational results follows Theorem 3.2 and Remark 3.3 below.

In our first result, we study bifurcation of solutions (u, v, b) of the nonlinear Helmholtz system (3.1) from a branch of semitrivial solutions of the form

$$\mathcal{T}_{u_0} := \{(u_0, 0, b) \mid b \in \mathbb{R}\} \subseteq X_1 \times X_1 \times \mathbb{R}$$

in the Banach space $X_1 \times X_1 \times \mathbb{R}$. In contrast to the Schrödinger case, we will see that for each of the uncountably many radial solutions $u_0 \in X_1 \cap C^2_{\text{loc}}(\mathbb{R}^3)$ of the scalar problem

$$-\Delta u_0 - \mu u_0 = u_0^3 \qquad \text{on } \mathbb{R}^3 \tag{3.4}$$

(see Theorem 1.13 in the Introduction) we have that every point in $\mathcal{T}_{u_0}$ is a bifurcation point for fully nontrivial solutions of (3.1). In order to formulate the precise statement, we need the following result on the scalar Helmholtz equation, which will be proved at the end of that chapter using the methods developed for the system in Part 3.2.

Proposition 3.1.

Let $\mu > 0$ and let $u_0 \in X_1 \cap C^2_{\text{loc}}(\mathbb{R}^3), u_0 \not\equiv 0$ be any radially symmetric solution of the nonlinear Helmholtz equation (3.4). Then u_0 satisfies

$$u_0(x) = c_0 \frac{\sin(|x|\sqrt{\mu} + \sigma_0)}{|x|} + O\left(\frac{1}{|x|^2}\right) \qquad \text{as } |x| \to \infty$$

for some constants $c_0 \neq 0$ and $\sigma_0 \in [0, \pi)$, and there exists a unique $\tau_0 \in [0, \pi)$ such that

$$\begin{cases} -\Delta w - \mu w = 3u_0^2(x)\, w & \text{on } \mathbb{R}^3, \\ w(x) = \frac{\sin(|x|\sqrt{\mu} + \tau_0)}{|x|} + O\left(\frac{1}{|x|^2}\right) & \text{as } |x| \to \infty \end{cases}$$

admits a nontrivial solution $w_0 \in X_1 \cap C^2_{\text{loc}}(\mathbb{R}^3)$. Moreover, this solution w_0 is unique.

Here and in the following we fix $\mu, \nu > 0$ and $u_0 \in X_1 \cap C^2_{\text{loc}}(\mathbb{R}^3), u_0 \not\equiv 0$ with associated constants $\sigma_0, \tau_0 \in [0,\pi)$ as in Proposition 3.1. For $\tau_1 \in [0,\pi)\setminus\{\tau_0\}$, we observe in particular that

$$\begin{cases} -\Delta w - \mu w = 3u_0^2(x)\, w & \text{on } \mathbb{R}^3, \\ w(x) = \frac{\sin(|x|\sqrt{\mu}+\tau_1)}{|x|} + O\left(\frac{1}{|x|^2}\right) & \text{as } |x| \to \infty \end{cases} \quad \text{has no solution.} \tag{3.5}$$

This nondegeneracy property will be used later to prove that the linearization of the system (3.1) at points $(u_0, 0, b) \in \mathcal{T}_{u_0}$ admits at most one-dimensional kernels. Our strategy will be to use bifurcation from simple eigenvalues with b acting as a bifurcation parameter. The existence of isolated and algebraically simple eigenvalues will be ensured by assuming radial symmetry and by imposing suitable conditions on the asymptotic behavior of the solutions u, v. For $\tau_1, \omega \in [0,\pi)$ with $\tau_1 \neq \tau_0$, we define $\mathcal{S}(\omega) \subseteq X_1 \times X_1 \times \mathbb{R} \setminus \mathcal{T}_{u_0}$ as the set of all solutions $(u,v,b) \in X_1 \times X_1 \times \mathbb{R} \setminus \mathcal{T}_{u_0}$ of (3.1) satisfying the asymptotic conditions

$$\begin{aligned} u(x) - u_0(x) &= c_u \, \frac{\sin(|x|\sqrt{\mu}+\tau_1)}{|x|} + O\left(\frac{1}{|x|^2}\right) \\ v(x) &= c_v \, \frac{\sin(|x|\sqrt{\nu}+\omega)}{|x|} + O\left(\frac{1}{|x|^2}\right) \end{aligned} \qquad \text{as } |x| \to \infty \tag{3.6}$$

for some $c_u, c_v \in \mathbb{R}$. Proposition 3.1 above as well as Propositions 3.13 and 3.18 in the following section show that an asymptotic behavior of such form is natural to assume for solutions of the system (3.1). We emphasize that we do not explicitly indicate the dependence of the set $\mathcal{S}(\omega)$ and of the asymptotic conditions (3.6) on the choice $\tau_1 \in [0,\pi) \setminus \{\tau_0\}$. With these preparations, we formulate the following

Theorem 3.2 (Bifurcation from a semitrivial family).

Let $\mu, \nu > 0$, fix any $u_0 \in X_1 \setminus \{0\}$ solving the nonlinear Helmholtz equation (3.4) and choose $\tau_1 \in [0,\pi) \setminus \{\tau_0\}$ with τ_0 as in Proposition 3.1. Then, for every $\omega \in [0,\pi)$, there exists a strictly increasing sequence $(b_k(\omega))_{k\in\mathbb{Z}}$ such that $(u_0, 0, b_k(\omega)) \in \overline{\mathcal{S}(\omega)}$ where $\mathcal{S}(\omega)$ denotes the set of all solutions $(u,v,b) \in X_1 \times X_1 \times \mathbb{R} \setminus \mathcal{T}_{u_0}$ of (3.1) satisfying (3.6). Moreover,

(i) the respective connected components $\mathcal{C}_k(\omega)$ of $(u_0, 0, b_k(\omega))$ in $\overline{\mathcal{S}(\omega)}$ are unbounded in $X_1 \times X_1 \times \mathbb{R}$; and

(ii) each bifurcation point $(u_0, 0, b_k(\omega))$ has a neighborhood where the set $\mathcal{C}_k(\omega)$ is a smooth curve in $X_1 \times X_1 \times \mathbb{R}$ which, except for the bifurcation point, consists of fully nontrivial solutions.

We add some remarks the proof of which will be given after having proved Theorem 3.2 in Part 3.3.

Remark 3.3. *(a) We will also see that fully nontrivial solutions of (3.1) satisfying the asymptotic condition (3.6) bifurcate from some point $(u_0, 0, b) \in \mathcal{T}_{u_0}$ if and only if $b = b_k(\omega)$ for some $k \in \mathbb{Z}$.*

(b) Furthermore, we will prove that the map $\mathbb{R} \to \mathbb{R}, k\pi + \omega \mapsto b_k(\omega)$ where $0 \leq \omega < \pi, k \in \mathbb{Z}$ is strictly increasing and onto with $b_k(\omega) \to \pm\infty$ as $k \to \pm\infty$. Thus, in particular, every point $(u_0, 0, b) \in \mathcal{T}_{u_0}$, $b \in \mathbb{R}$, is a bifurcation point for fully nontrivial radial solutions of (3.1), which is in contrast to the case of Schrödinger systems where bifurcation points are isolated, cf. [48], Satz 2.1.6.

(c) *Close to the respective bifurcation point* $(u_0, 0, b_k(\omega)) \in \mathcal{T}_{u_0}$*, each continuum* $\mathcal{C}_k(\omega)$ *is characterized by a phase parameter* $\omega + k\pi$ *derived from the asymptotic behavior of* v*. It seems that, in the Helmholtz case of oscillating solutions, the integer* k *takes the role of the nodal characterizations in the Schrödinger case, cf. Satz 2.1.6 in [48]. That phase parameter is constant on connected subsets of the continuum until it possibly runs into another family of semitrivial solutions* $\mathcal{T}_{u_1}$ *with* $u_1 \neq u_0$*; unfortunately we cannot provide criteria deciding whether or not this happens. For this reason we cannot claim that* $\mathcal{C}_k(\omega)$ *contains an unbounded sequence of fully nontrivial solutions.*

(d) *The condition* $\tau_1 \neq \tau_0$ *is a nondegeneracy condition which ensures the existence of simple kernels as required in the above-mentioned bifurcation theorems. If we additionally impose* $\tau_1 \neq \sigma_0$*, we infer* $u \neq 0$ *for any solution* $(u, v, b) \in \mathcal{C}_k(\omega)$*. Moreover, the proof will show that the values* $b_k(\omega)$ *do not depend on the choice of* τ_1*.*

(e) *Fully nontrivial solutions in* $\mathcal{C}_k(\omega)$ *satisfy the asymptotic condition* (3.6) *with* $c_v \neq 0$ *but possibly* $c_u = 0$*. Thus the theorem provides no information whether a different choice of* $\tau_1 \in [0, \pi) \setminus \{\tau_0\}$ *leads to different bifurcating continua or not.*

Following Remark 2.3, the additional assumptions of radial symmetry and constant coefficients allow to compare these results with those on the existence of dual ground states in Chapter 2. It is then natural to ask whether the (fully nontrivial) solutions we obtain here can be dual ground states, which can be answered at least partly. First and foremost, we are going to construct solutions for arbitrarily large positive and negative values of the coupling b; however, unless $0 \leq b \leq 3$, a dual variational formulation of (3.1) as constructed in the previous chapter is not available and hence there is no meaningful concept of a dual ground state (yet). Second, according to Theorem 2.5, fully nontrivial solutions with $0 \leq b < 1$ cannot be dual ground states since the latter are then semitrivial. Only in the range $1 < b \leq 3$ both the solutions we obtain here and dual ground states are fully nontrivial (at least for $\mu \approx \nu$, see Theorem 2.4 (ii)) and might agree; however, since the methods are strikingly different, we point out that there is no evidence that this might actually happen. On the contrary, consulting the proof of Theorem 2.4 (ii) again, we see that the estimate of the mountain pass levels

$$c_{\mu\nu} \leq F_{\mu\nu}\,(\bar{z}_\mu, \bar{z}_\nu) \leq (\ \dots\ ) < \min\{c_\mu, c_\nu\},$$

which ensures that dual ground states are fully nontrivial, has been realized using "almost diagonal" pairs $(\bar{z}_\mu, \bar{z}_\nu) \in L^{\frac{4}{3}}(\mathbb{R}^3) \times L^{\frac{4}{3}}(\mathbb{R}^3)$. Here the components are dual ground states of the scalar problems and satisfy $\bar{z}_\mu \to \bar{z}_\nu$ as $\mu \to \nu$, see (2.25). This at least suggests that the dual ground states in the situation of Theorem 2.4 (ii) with $\mu \approx \nu$ have two components of comparable size. The local bifurcation result above, however, guarantees fully nontrivial solutions where the v component is typically small, $(u, v) \approx (u_0, 0)$, which may be interpreted as a hint that at least close to the bifurcation points, the continua of Theorem 3.2 are not likely to contain dual ground states. The interpretation is reversed when considering bifurcations from families of diagonal solutions, which is what we do next.

In our second result we provide a counterpart of the global bifurcation result by Bartsch, Dancer and Wang [8] described earlier. Using the same functional analytical setup as in Theorem 3.2, we find an analogue of their results for the nonlinear Helmholtz system (3.1). For u_0 as in Proposition 3.1 and $\tau_1, \omega \in [0, \pi)$, $\tau_1 \neq \tau_0$, we introduce the diagonal solution family

$$\mathfrak{T}_{u_0} := \left\{(u_b, u_b, b) \,\middle|\, b > -1\right\} \subseteq X_1 \times X_1 \times \mathbb{R} \qquad \text{with } u_b := (1+b)^{-1/2}\, u_0$$

and denote by $\mathfrak{S}(\omega)$ the set of all solutions $(u, v, b) \in X_1 \times X_1 \times \mathbb{R} \setminus \mathfrak{T}_{u_0}$ of the nonlinear Helmholtz system (3.1) with

$$\begin{aligned} u(x) + v(x) &= 2u_b(x) + \tilde{c}\,\frac{\sin(|x|\sqrt{\mu} + \tau_1)}{|x|} + O\left(\frac{1}{|x|^2}\right) \\ u(x) - v(x) &= c\,\frac{\sin(|x|\sqrt{\mu} + \omega)}{|x|} + O\left(\frac{1}{|x|^2}\right) \end{aligned} \qquad \text{as } |x| \to \infty \tag{3.7}$$

for some $\tilde{c}, c \in \mathbb{R}$. Our existence result for fully nontrivial solutions of (3.1) bifurcating from $\mathfrak{T}_{u_0}$ with asymptotics (3.7) reads as follows.

Theorem 3.4 (Bifurcation from a diagonal family).

Let $\mu > 0$, fix any $u_0 \in X_1 \setminus \{0\}$ solving the nonlinear Helmholtz equation (3.4) and choose $\tau_1 \in [0, \pi) \setminus \{\tau_0\}$ with τ_0 as in Proposition 3.1. Then, for every $\omega \in [0, \pi)$, there exist $k_\omega \in \mathbb{Z}$ and a sequence $(\mathfrak{b}_k(\omega))_{k \geq k_\omega}$ such that $(u_{\mathfrak{b}_k(\omega)}, u_{\mathfrak{b}_k(\omega)}, \mathfrak{b}_k(\omega)) \in \overline{\mathfrak{S}(\omega)}$ where $\mathfrak{S}(\omega)$ denotes the set of all solutions $(u, v, b) \in X_1 \times X_1 \times \mathbb{R} \setminus \mathfrak{T}_{u_0}$ of (3.1) satisfying (3.7). Moreover,

(i) the respective connected components $\mathfrak{C}_k(\omega)$ of $(u_{\mathfrak{b}_k(\omega)}, u_{\mathfrak{b}_k(\omega)}, \mathfrak{b}_k(\omega))$ in $\overline{\mathfrak{S}(\omega)}$ are unbounded in $X_1 \times X_1 \times \mathbb{R}$; and

(ii) each bifurcation point $(u_{\mathfrak{b}_k(\omega)}, u_{\mathfrak{b}_k(\omega)}, \mathfrak{b}_k(\omega))$ has a neighborhood where the set $\mathfrak{C}_k(\omega)$ is a smooth curve in $X_1 \times X_1 \times \mathbb{R}$ which, except for the bifurcation point, consists of fully nontrivial, non-diagonal solutions.

We will see in the proof that $\mathfrak{b}_k(\omega) = \frac{3 - b_k(\omega)}{1 + b_k(\omega)} = \frac{4}{1 + b_k(\omega)} - 1$ with $b_k(\omega)$ as in Theorem 3.2. The index k_ω is defined to be the smallest integer satisfying $b_k(\omega) > -1$. (Here we exploit that the sequence $(b_k(\omega))_{k \in \mathbb{Z}}$ is strictly increasing.)

Again, similar statements as in Remark 3.3 can be proved. For instance, one can check that every point on $\mathfrak{T}_{u_0}$ is a bifurcation point by a suitable choice of ω, in particular $\{\mathfrak{b}_k(\omega) \mid \omega \in [0, \pi), k \geq k_\omega\} = (-1, \infty)$.

3.1.3 Some Classical Bifurcation Theorems

The main tools in proving Theorem 3.2 are the two Bifurcation Theorems cited next, to be found in a slightly more general form in [20], Theorem 1.7 and in [62], Theorem 1.3, respectively. We first present the Crandall-Rabinowitz Bifurcation Theorem, which will be used to show the local statement (ii) of Theorem 3.2.

Theorem 3.5 (Crandall-Rabinowitz, 1971).

Let X be a real Banach space, $b_0 \in \mathbb{R}$ and $F : X \times \mathbb{R} \to X$ such that

(a) $F(0, b) = 0$ *for all* $b \in \mathbb{R}$, $F \in C^1(X \times \mathbb{R})$ *and* $\partial_b D_x F \in C(X \times \mathbb{R})$*;*

(b) $D_x F(0, b_0)$ *is a Fredholm operator of index zero with* $\ker D_x F(0, b_0) = \operatorname{span}\{v_0\}$ *for some* $v_0 \in X$, $v_0 \neq 0$ *(simplicity);*

(c) $\partial_b D_x F(0, b_0)[v_0] \notin \operatorname{ran} D_x F(0, b_0)$ *(transversality).*

Then $(0, b_0)$ *is a bifurcation point for* $F(x, b) = 0$*, and for some neighborhood* $U \subseteq X \times \mathbb{R}$ *of* $(0, b_0)$ *and some* $\delta > 0$*,*

$$F^{-1}(\{0\}) \cap U = \{(0, b) \mid (0, b) \in U\} \cup \{(tv_0 + tz(t), b_0 + b(t)) \mid |t| < \delta\}$$

where $b : (-\delta, \delta) \to \mathbb{R}$, $z : (-\delta, \delta) \to X$ *are continuous with* $b(0) = 0$ *and* $z(0) = 0$*. If additionally* $D_x^2 F \in C(X \times \mathbb{R})$*, they are continuously differentiable.*

This bifurcation result is local in nature, which is why we will study linearized versions of the equations in (3.1) in some detail in the following Part 3.2. What follows is a result on the global structure of bifurcating solutions due to Rabinowitz, the proof of which involves topological concepts from degree theory.

Theorem 3.6 (Rabinowitz, 1971).

Let X be a real Banach space and $G : X \times \mathbb{R} \to X$ *continuous and compact with* $G(x, b) = o(\|x\|)$ *as* $x \to 0$*, locally uniformly w.r.t* $b \in \mathbb{R}$*. Let* $K : X \to X$ *be linear and compact and*

$$F : X \times \mathbb{R} \to X, \quad F(x, b) := x - b\,Kx + G(x, b)$$

as well as $S := \{(x, b) \in X \times \mathbb{R} \mid x \neq 0,\ F(x, b) = 0\}$*. Then, if* $\frac{1}{b_0}$ *is an eigenvalue of* K *with odd algebraic multiplicity, one can conclude that* $(0, b_0) \in \overline{S}$ *and that the connected component* C *of* $\overline{S}$ *containing* $(0, b_0)$ *has one of the following properties:*

(a) C *is unbounded, or*

(b) the set $\{b \in \mathbb{R} \setminus \{b_0\} \mid (0, b) \in C\}$ *has an odd number of elements.*

As Kielhöfer shows in Theorem II.3.3 in [40], using essentially the same methods, one can prove the following version we will exploit later to prove the global statement (i) of Theorem 3.3. We remark that, in [40], the concept of odd crossing numbers appears in place of the index condition we give below; the relation between both is given in [40], equation (II.3.4).

Theorem 3.7.

Let X be a real Banach space, $b_0 \in \mathbb{R}$ and $f : X \times \mathbb{R} \to X$ compact with continuous derivative $D_x f(0, \cdot\,) \in C(\mathbb{R}, \mathcal{L}(X))$ and with $f(0,b) = 0$ for all $b \in \mathbb{R}$. Define

$$F : X \times \mathbb{R} \to X, \quad F(x,b) := x - f(x,b)$$

as well as $S := \{(x,b) \in X \times \mathbb{R} \mid x \neq 0,\ F(x,b) = 0\}$. Then, if 0 is an isolated eigenvalue of $D_x f(0,b_0)$ of finite algebraic multiplicity and if the index $\operatorname{ind} D_x F(0,b)$ changes sign at $b = b_0$, $(0,b_0) \in \overline{S}$ and the connected component C of $\overline{S}$ containing $(0,b_0)$ has one of the following properties:

(a) C is unbounded, or

(b) there exists $b_1 \neq b_0$ with $(0,b_1) \in C$.

3.1.4 Some Technical Aspects

The following technical results will be proved in Section 3.5.1. We start with elementary properties of the spaces X_q defined in equation (3.3).

Lemma 3.8.

For any $q \geq 1$, the space X_q endowed with the weighted maximum norm

$$\|w\|_{X_q} := \sup_{x \in \mathbb{R}^3} (1 + |x|^2)^{\frac{q}{2}} |w(x)| \qquad (w \in X_q)$$

is a Banach space. Given $1 \leq p \leq \infty$, the space X_q embeds continuously into $L^p_{\mathrm{rad}}(\mathbb{R}^3)$ if and only if $pq > 3$.

These embeddings into L^p spaces are important at various places since the functional analytic framework will be set up using the resolvent-type operators $\mathcal{R}_\lambda$ from the L^p-version of the Limiting Absorption Principle of Gutiérrez, see Theorem 1.9. We will demonstrate that they can be redefined in the X_q spaces, and prove that they enjoy stronger compactness properties in these topologies, see Proposition 3.13 (i).

As initially announced, the fact that we restrict our study to the case of three space dimensions leads to a number of explicit formulae, some of which will be derived next. We first note that there is an explicit expression for the convolution kernel Φ_λ of $\mathfrak{R}_\lambda$, which is in turn due to explicit formulae for the Bessel and Hankel functions of order $\frac{N}{2} - 1 = \frac{1}{2}$. These are summarized next; we also consider the real-valued kernels $\Psi_\lambda := \operatorname{Re} \Phi_\lambda$ and $\tilde{\Psi}_\lambda := \operatorname{Im} \Phi_\lambda$, which are the ones we will use throughout the chapter.

Lemma 3.9.

For $r > 0$, the Bessel (resp. Hankel) functions of order $\frac{1}{2}$ are given by

$$J_{\frac{1}{2}}(r) = \sqrt{\frac{2}{\pi r}}\sin(r), \quad Y_{\frac{1}{2}}(r) = -\sqrt{\frac{2}{\pi r}}\cos(r), \quad H^{(1)}_{\frac{1}{2}}(r) = -\mathrm{i}\sqrt{\frac{2}{\pi r}}\mathrm{e}^{\mathrm{i}r}.$$

Therefore, given $\lambda > 0$, the fundamental solutions $\Phi_\lambda, \Psi_\lambda = \operatorname{Re}\Phi_\lambda, \tilde{\Psi}_\lambda = \operatorname{Im}\Phi_\lambda$ of the Helmholtz equation $-\Delta w - \lambda w = 0$ satisfy for every $x \in \mathbb{R}^3 \setminus \{0\}$

$$\Phi_\lambda(x) = \frac{\mathrm{e}^{\mathrm{i}|x|\sqrt{\lambda}}}{4\pi|x|}, \quad \Psi_\lambda(x) = \frac{\cos(|x|\sqrt{\lambda})}{4\pi|x|}, \quad \tilde{\Psi}_\lambda(x) = \frac{\sin(|x|\sqrt{\lambda})}{4\pi|x|}.$$

Although this is far from being a new result, we sketch an elementary proof. The Bessel functions of order $\frac{1}{2}$ are solutions of the ODE

$$r^2\phi''(r) + r\phi'(r) + \left(r^2 - \left(\frac{1}{2}\right)^2\right)\phi(r) = 0 \qquad \text{for } r \in (0,\infty).$$

The general solution can be calculated explicitly by substituting $\psi(r) := \sqrt{r}\cdot\phi(r)$; then due to the appearance of the term $\left(\frac{1}{2}\right)^2$, the equation simplifies to $\psi''(r) + \psi(r) = 0$ on $(0,\infty)$. This implies for some $a, b \in \mathbb{C}$ and all $r > 0$

$$\psi(r) = a\cos(r) + b\sin(r) \quad \text{and} \quad \phi(r) = a\,\frac{\cos(r)}{\sqrt{r}} + b\,\frac{\sin(r)}{\sqrt{r}}.$$

Comparing with the asymptotic behavior of $J_{\frac{1}{2}}(r)$, $Y_{\frac{1}{2}}(r)$, $H^{(1)}_{\frac{1}{2}}(r)$ as given in Chapter 9 of [2] or in the appendix, the first assertion is proved. The fundamental solutions are then obtained from the formula for Φ_λ in Theorem 1.9 (ii) and taking the real resp. imaginary part.

For the convolution operator $\mathfrak{R}_\lambda$, see Theorem 1.9, we prove:

Lemma 3.10.

For $f \in L^{\frac{4}{3}}_{\mathrm{rad}}(\mathbb{R}^3,\mathbb{C})$ and almost all $x \in \mathbb{R}^3 \setminus \{0\}$, we have

$$\begin{aligned}\mathfrak{R}_\lambda f(x) &= \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}|x|}}{|x|}\cdot\int_0^{|x|}\frac{\sin(\sqrt{\lambda}r)}{\sqrt{\lambda}r}\cdot f(r)\,r^2\,\mathrm{d}r + \frac{\sin(\sqrt{\lambda}|x|)}{|x|}\cdot\int_{|x|}^\infty\frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}r}}{\sqrt{\lambda}r}\cdot f(r)\,r^2\,\mathrm{d}r \\ &= \sqrt{\frac{\pi}{2}}\,\hat{f}(\sqrt{\lambda})\cdot\frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}|x|}}{|x|} + \int_{|x|}^\infty f(r)\cdot\frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}r}\sin(\sqrt{\lambda}|x|) - \mathrm{e}^{\mathrm{i}\sqrt{\lambda}|x|}\sin(\sqrt{\lambda}r)}{\sqrt{\lambda}r|x|}\,r^2\mathrm{d}r.\end{aligned}$$

For simplicity, we have adopted the shorthand notation $\hat{f}(\sqrt{\lambda}) := \hat{f}(\sqrt{\lambda}\,\xi)$ for some $\xi \in \mathbb{S}^2$, which denotes the (profile of the) Fourier transform on $\mathbb{R}^3$:

$$\hat{f}(\sqrt{\lambda}) \overset{(1.12)}{=} \frac{1}{\sqrt[4]{\lambda}}\int_0^\infty f(r)J_{\frac{1}{2}}(r\sqrt{\lambda})\,r^{\frac{3}{2}}\,\mathrm{d}r \overset{\text{Lem. 3.9}}{=} \sqrt{\frac{2}{\pi\lambda}}\int_0^\infty f(r)\sin(r\sqrt{\lambda})\,r\,\mathrm{d}r. \tag{3.8}$$

Throughout this chapter, we will be working mostly on the level of classical solutions in the space X_1, which will be justified in the remaining part of this introduction.

Lemma 3.11 (Regularity and decay I).

Assume $(u,v) \in L^4_{\mathrm{rad}}(\mathbb{R}^3) \times L^4_{\mathrm{rad}}(\mathbb{R}^3)$ is a distributional solution of

$$-\Delta u - \mu u = \big(u^2 + b\, v^2\big)\, u \text{ on } \mathbb{R}^3, \qquad -\Delta v - \nu v = \big(v^2 + b\, u^2\big)\, v \text{ on } \mathbb{R}^3.$$

Then $u,v \in X_1 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$, and the pair (u,v) solves the differential equations in the classical sense; moreover, the profiles satisfy the ODE system

$$-u'' - \frac{2}{r}u' - \mu u = \big(u^2 + b\, v^2\big)\, u \text{ on } [0,\infty), \qquad -v'' - \frac{2}{r}v' - \nu v = \big(v^2 + b\, u^2\big)\, v \text{ on } [0,\infty).$$

The proof will show that, in fact, $u, v \in C^\infty(\mathbb{R}^3)$. A corresponding result holds for the linearized equations. Here we consider some fixed $g \in X_2 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$; typically $g(x) = bu_0^2(x)$ for $b \in \mathbb{R}$ and a fixed distributional solution $u_0 \in L^4_{\mathrm{rad}}(\mathbb{R}^3)$ of the nonlinear Helmholtz equation $-\Delta u_0 - \mu u_0 = u_0^3$ on $\mathbb{R}^3$, which by the previous result is smooth, belongs to X_1 and satisfies the differential equation in a classical sense.

Lemma 3.12 (Regularity and decay II).

Assume $w \in L^4_{\mathrm{rad}}(\mathbb{R}^3)$ is a distributional solution of the linearized problem

$$-\Delta w - \lambda w = g(x)\, w \qquad \text{on } \mathbb{R}^3$$

for some $g \in X_2 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$. Then $w \in X_1 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$ is a classical solution of the above equation and its profile satisfies

$$-w'' - \frac{2}{r}w' - \lambda w = g(r)\, w.$$

3.2 On the Scalar Problem. Spectral Properties

The main challenge in proving the bifurcation results in Theorems 3.2 and 3.4 is a thorough analysis of the linearized problem which we provide in this chapter. Throughout, we fix $\lambda > 0$ and discuss the linear Helmholtz equation

$$-\Delta w - \lambda w = f \qquad \text{on } \mathbb{R}^3 \tag{3.9}$$

for some $f \in X_3$, where X_3 is defined in (3.3). We will frequently identify radially symmetric functions $x \mapsto w(x)$ with their profiles; in particular, we denote by $w' := \partial_r w, w'' = \partial_r^2 w$ the radial derivatives. The results we establish in this section will demonstrate how to rewrite the system (3.1) in a way suitable for Bifurcation Theory. In order not to interrupt the presentation, we postpone the proofs to the end of this chapter into Section 3.5.2.

3.2.1 Representation Formulas

First, we discuss a representation formula for solutions of the linear and inhomogeneous Helmholtz equation (3.9). The statements resemble Agmon's representation results as summarized in Remark 1.8 (c); in the radial setting on $\mathbb{R}^3$, however, the proof is much easier thanks to the explicit formulas in Lemmas 3.9 and 3.10. Aside from continuity, we are especially interested in compactness and pointwise asymptotic expansions of resolvent-type operators $\mathfrak{R}_\lambda, \mathcal{R}_\lambda$ (see e.g. Remark 1.10 (b) and equation (1.16)) for the Helmholtz equation, which we are able to establish working in the spaces X_3 resp. X_1.

In a slight abuse of notation, we now (re-)define $\mathfrak{R}_\lambda$ and $\mathcal{R}_\lambda, \tilde{\mathcal{R}}_\lambda$ as convolution operators with kernels Φ_λ resp. Ψ_λ resp. $\tilde{\Psi}_\lambda$ (see Lemma 3.9) on the domain X_3. We recall that, due to Lemma 3.8, X_3 embeds into $L^{\frac{4}{3}}_{\text{rad}}(\mathbb{R}^3)$ where the operators are known to be well-defined and to satisfy the formula given in Lemma 3.10.

Proposition 3.13 ("Resolvent" operators).

The linear convolution operators $\mathcal{R}_\lambda : X_3 \to X_1, f \mapsto \Psi_\lambda * f$ and $\tilde{\mathcal{R}}_\lambda : X_3 \to X_1, f \mapsto \tilde{\Psi}_\lambda * f$ have the following properties:

(i) $\mathcal{R}_\lambda$ and $\tilde{\mathcal{R}}_\lambda$ are well-defined, continuous and compact.

(ii) For $f \in X_3$, we have $w := \mathcal{R}_\lambda[f], \tilde{w} := \tilde{\mathcal{R}}_\lambda[f] \in X_1 \cap C^2_{\text{loc}}(\mathbb{R}^3)$ with

$$-\Delta w - \lambda w = f, \quad -\Delta \tilde{w} - \lambda \tilde{w} = 0 \qquad \text{on } \mathbb{R}^3.$$

(iii) For $f \in X_3$ and $w, \tilde{w} \in X_1$ as in (ii), the profiles satisfy the asymptotic identities

$$\left.\begin{aligned} w(r) &= \sqrt{\frac{\pi}{2}}\, \hat{f}(\sqrt{\lambda}) \cdot \frac{\cos(r\sqrt{\lambda})}{r} + \frac{\delta_f(r)}{r^2} \\ \tilde{w}(r) &= \sqrt{\frac{\pi}{2}}\, \hat{f}(\sqrt{\lambda}) \cdot \frac{\sin(r\sqrt{\lambda})}{r} + \frac{\tilde{\delta}_f(r)}{r^2}, \\ w'(r) &= -\sqrt{\frac{\pi}{2}}\, \hat{f}(\sqrt{\lambda}) \cdot \frac{\sqrt{\lambda}\sin(r\sqrt{\lambda})}{r} + \frac{\delta^*_f(r)}{r^2} \\ \tilde{w}'(r) &= \sqrt{\frac{\pi}{2}}\, \hat{f}(\sqrt{\lambda}) \cdot \frac{\sqrt{\lambda}\cos(r\sqrt{\lambda})}{r} + \frac{\tilde{\delta}^*_f(r)}{r^2} \end{aligned}\right\} \quad \text{as } r \to \infty$$

where $|\delta_f(r)|, |\tilde{\delta}_f(r)| \leq \frac{2}{\sqrt{\lambda}} \cdot \|f\|_{X_3}$ and $|\delta^*_f(r)|, |\tilde{\delta}^*_f(r)| \leq \frac{\pi}{2\sqrt{\lambda}} + 2\left(1 + \frac{1}{\sqrt{\lambda}r}\right) \cdot \|f\|_{X_3}$. In particular, $\tilde{w} = \tilde{\mathcal{R}}_\lambda[f] = 4\pi\sqrt{\frac{\pi}{2}}\, \hat{f}(\sqrt{\lambda}) \cdot \tilde{\Psi}_\lambda$.

The occurrence of improved properties of the convolution operators $f \mapsto \Phi_\lambda * f$ when imposing radial symmetry has already been studied by Evéquoz. Applied to the case of $N = 3$ space dimensions, his yet unpublished results yield a constant $C(\lambda) > 0$ with

$$\left\| \min\{|\cdot|, |\cdot|^{\frac{3}{2}}\} \cdot |\Phi_\lambda * f| \right\|_{L^\infty(\mathbb{R}^3)} \leq C(\lambda) \cdot \|f\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \quad \text{for all } f \in L^{\frac{4}{3}}_{\text{rad}}(\mathbb{R}^3)$$

and guarantee the pointwise asymptotic expansion

$$(\Phi_\lambda * f)(x) = \sqrt{\frac{\pi}{2}} \hat{f}(\sqrt{\lambda}) \cdot \frac{e^{i|x|\sqrt{\lambda}}}{|x|} + o(|x|^{-1}) \quad \text{as } |x| \to \infty \quad \text{for all } f \in L^{\frac{4}{3}}_{\text{rad}}(\mathbb{R}^3).$$

The fact that we choose the stronger topology of X_3 instead of $L^{\frac{4}{3}}_{\mathrm{rad}}(\mathbb{R}^3)$ ensures that the convolution even maps to $L^\infty_{\mathrm{rad}}(\mathbb{R}^3)$ without additional weight at the origin and leads to asymptotic expansions with an explicit error estimate of the form $\leq C|x|^{-2} \cdot \|f\|_{X_3}$. This will be used to prove compactness of $f \mapsto \Phi_\lambda * f$ as a map from X_3 to X_1. In the L^p topologies, Evéquoz and Weth have shown compactness only in a local sense, i.e. for $f \mapsto \mathbb{1}_B \cdot \Phi_\lambda * f$ with bounded measurable B, see Lemma 4.1 (i) in [28]. We will derive a local compactness result which is slightly stronger and at the same time more elementary, applying the Arzelà-Ascoli Theorem and explicitly using the assumption of radial symmetry. The advantage is that it can be generalized in order to prove the following Remark 3.14 and a compactness statement for the Schrödinger case in the next chapter. Together with the explicit asymptotic estimates in (iii), such a local compactness result can be used to verify the (global) compactness property in (i) above. This property will be essential later on when applying the Bifurcation Theorems 3.5 and 3.7.

In view of the fact that we aim to discuss cubic nonlinearities in place of the right-hand side f, the choice of the space X_3 is the natural one in the proposition above. However, the decay rate prescribed by the X_3 space is not the optimal one yielding continuity and compactness as in (i). We generalize:

Remark 3.14 (Optimal decay rates).

Let $\varepsilon > 0$. The convolution operators $\mathcal{R}_\lambda, \tilde{\mathcal{R}}_\lambda$ are well-defined, continuous and compact as operators from $X_{2+\varepsilon}$ to X_1. This is optimal in the sense that there is no continuous linear operator $X_2 \to X_1$ which extends $\mathcal{R}_\lambda$.

We can now study the set of all radially symmetric, twice differentiable solutions of the inhomogeneous linear Helmholtz equation (3.9) with some fixed right-hand side $f \in X_3$. The existence of such solutions is guaranteed by Proposition 3.13 (ii); we intend to introduce asymptotic conditions which ensure uniqueness.

Remark 3.15 (The linear inhomogeneous Helmholtz equation).

For $f \in X_3$, the following holds:

(a) Every radial, twice differentiable solution of $-\Delta w - \lambda w = f$ on $\mathbb{R}^3$ satisfies

$$w(x) = \mathcal{R}_\lambda[f](x) + C \cdot \frac{\sin(|x|\sqrt{\lambda})}{|x|} \qquad \text{for all } x \in \mathbb{R}^3 \setminus \{0\} \tag{3.10}$$

for some $C \in \mathbb{R}$, and vice versa. In particular $w \in X_1$.

(b) If $\hat{f}(\sqrt{\lambda}) = 0$, then every solution w as in (a) satisfies

$$w(x) = C \cdot \frac{\sin(|x|\sqrt{\lambda})}{|x|} + O\left(\frac{1}{|x|^2}\right) \qquad \text{as } |x| \to \infty.$$

In particular, there is exactly one radial C^2 solution of $-\Delta w - \lambda w = f$ with $w(x) = O\left(\frac{1}{|x|^2}\right)$ as $|x| \to \infty$, and it is given by $w = \mathcal{R}_\lambda[f]$. Moreover, $\tilde{\mathcal{R}}_\lambda[f] \equiv 0$.

(c) If $\hat{f}(\sqrt{\lambda}) \neq 0$, then every solution w as in (a) satisfies

$$w(x) = \sqrt{\frac{\pi}{2}} \hat{f}(\sqrt{\lambda}) \cdot \frac{\sin(|x|\sqrt{\lambda} + \omega)}{\sin(\omega)|x|} + O\left(\frac{1}{|x|^2}\right) \qquad \text{as } |x| \to \infty$$

for some unique asymptotic phase parameter $\omega \in (0, \pi)$ *given by*

$$C = \sqrt{\frac{\pi}{2}} \hat{f}(\sqrt{\lambda}) \cdot \cot(\omega).$$

In this case, $w = \mathcal{R}_\lambda[f] + \cot(\omega)\,\tilde{\mathcal{R}}_\lambda[f]$. *In particular, there is no radial* C^2 *solution of* $-\Delta w - \lambda w = f$ *with* $w(x) = O\left(\frac{1}{|x|^2}\right)$ *and none with* $w(x) = C \cdot \frac{\sin(|x|\sqrt{\lambda})}{|x|} + O\left(\frac{1}{|x|^2}\right)$ *as* $|x| \to \infty$.

Aiming for uniqueness, equation (3.10) shows that there is essentially one free parameter which has to be fixed. (Here again, the assumption of radial symmetry simplifies the analysis since all twice differentiable radial solutions of $-\Delta w - \lambda w = 0$ on $\mathbb{R}^3$ are multiples of $|\cdot|^{-1} \sin(|\cdot|\sqrt{\lambda})$; otherwise, a much larger number of Herglotz waves would have to be considered.) Motivated by (b) and (c) of the above remark, we will consider solutions of (3.9) which additionally satisfy an asymptotic conditions of the following form:

$$\text{There is } c \in \mathbb{R} \text{ with} \quad w(x) = c \cdot \frac{\sin(|x|\sqrt{\lambda} + \omega)}{|x|} + O\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty. \tag{3.11}$$

For $\omega \in (0, \pi)$, we have just seen that the combined problem (3.9), (3.11) admits a unique solution. Below, we construct a suitable operator $\mathcal{R}_\lambda^\omega : X_3 \to X_1$ mapping $f \in X_3$ to the corresponding solution $w \in X_1$ of (3.9), (3.11). A clear advantage of choosing the phase parameter ω (and not, for instance, C in equation (3.10)) is that it actually describes a global property of the respective solution w. At first glance, one drawback of this choice is that the role of ω as a phase parameter becomes meaningless in cases where $c = 0$ in (3.11) or, equivalently, whenever $\hat{f}(\sqrt{\lambda}) = 0$. However, in Proposition 3.18 below we will see that this cannot occur whenever we study equations of the form $-\Delta w - \lambda w = gw$ with some $g \in X_2$, which is the case in the nonlinear Helmholtz system (3.1).

We will also see that we can extend the analysis to include the case of asymptotic conditions (3.11) with $\omega = 0$ (resp. $\omega = \pi$). However, we cannot hope to find a "solution operator" $\mathcal{R}_\lambda^0 : X_3 \to X_1$ in the above sense since, firstly, solutions with this asymptotic condition only exist if $\hat{f}(\sqrt{\lambda}) = 0$ and, secondly, in this case they are not unique. Even if the analysis in this case is not as elegant, it is worth the effort in order to obtain the complete picture of bifurcations of the nonlinear Helmholtz system as presented in Theorems 3.2 and 3.4.

As announced, we introduce the technical framework in the case $\omega \in (0, \pi)$. Inspired by Remark 3.15 (c), we define the compact linear convolution operators

$$\mathcal{R}_\lambda^\omega : X_3 \to X_1, \quad f \mapsto \mathcal{R}_\lambda[f] + \cot(\omega)\,\tilde{\mathcal{R}}_\lambda[f] = \Psi_\lambda * f + \cot(\omega)\,\tilde{\Psi}_\lambda * f. \tag{3.12}$$

We observe that the operator $\mathcal{R}_\lambda^\omega$ is not well-defined for $\omega = 0$ due to the pole of the cotangent (and, similarly, for $\omega = \pi$, which we do not consider due to periodicity), which is in accordance with the previous considerations concerning this case. For $\omega \in (0, \pi)$, the convolution operators $\mathcal{R}_\lambda^\omega$ provide solutions of the Helmholtz equation (3.9) the asymptotic behavior of which is described by the phase parameter ω as in (3.11) and summarized in the following precise statement.

Corollary 3.16 (Representation formulas I).

Let $\omega \in (0,\pi)$ and $f \in X_3$. Then there is a unique solution w of

$$\begin{cases} -\Delta w - \lambda w = f & \text{on } \mathbb{R}^3, \\ w(x) = c \cdot \frac{\sin(|x|\sqrt{\lambda}+\omega)}{|x|} + O\left(\frac{1}{|x|^2}\right) & \text{as } |x| \to \infty \text{ for some } c \in \mathbb{R}. \end{cases}$$

This solution is given by $w = \mathcal{R}_\lambda^\omega[f]$.

Next, we introduce a technical framework which allows to include the case of solutions satisfying the asymptotic condition (3.11) with $\omega = 0$. First, by the Hahn-Banach Theorem, we construct continuous linear functionals $\alpha^{(\lambda)}, \beta^{(\lambda)} \in X_1'$ as follows. On the linear subspace

$$U_1(\lambda) := \left\{ w \in X_1 \,\middle|\, w(x) = \alpha_w \frac{\sin(|x|\sqrt{\lambda})}{4\pi|x|} + \beta_w \frac{\cos(|x|\sqrt{\lambda})}{4\pi|x|} + O\left(\frac{1}{|x|^2}\right) \right. \\ \left. \text{as } |x| \to \infty \text{ for some } \alpha_w, \beta_w \in \mathbb{R} \right\}$$

we let, for $w \in U_1(\lambda)$ with $w(r) = \alpha_w \frac{\sin(r\sqrt{\lambda})}{4\pi r} + \beta_w \frac{\cos(r\sqrt{\lambda})}{4\pi r} + O\left(\frac{1}{r^2}\right)$ as $r = |x| \to \infty$,

$$\begin{aligned} \alpha^{(\lambda)}(w) &:= \alpha_w = \lim_{n\to\infty} \left[4\pi \cdot \frac{2\pi n + \frac{\pi}{2}}{\sqrt{\lambda}} \cdot w\left(\frac{2\pi n + \frac{\pi}{2}}{\sqrt{\lambda}} \right) \right], \\ \beta^{(\lambda)}(w) &:= \beta_w = \lim_{n\to\infty} \left[4\pi \cdot \frac{2\pi n}{\sqrt{\lambda}} \cdot w\left(\frac{2\pi n}{\sqrt{\lambda}} \right) \right]. \end{aligned} \tag{3.13}$$

But then $|\alpha^{(\lambda)}(w)|, |\beta^{(\lambda)}(w)| \leq \limsup_{r\to\infty} |4\pi\sqrt{1+r^2} \cdot w(r)| \leq 4\pi \|w\|_{X_1}$ for $w \in U_1(\sqrt{\lambda})$; hence, after continuous extension, $\alpha^{(\lambda)}, \beta^{(\lambda)} \in X_1'$. In particular, for any $f \in X_3$, $\lambda > 0$ and $\omega \in (0,\pi)$, Proposition 3.13 (iii) implies $\mathcal{R}_\lambda[f], \tilde{\mathcal{R}}_\lambda[f], \mathcal{R}_\lambda^\omega[f] \in U_1(\lambda)$ with

$$\begin{aligned} &\alpha^{(\lambda)}(\mathcal{R}_\lambda[f]) = \beta^{(\lambda)}(\tilde{\mathcal{R}}_\lambda[f]) = 0, \\ &\alpha^{(\lambda)}(\tilde{\mathcal{R}}_\lambda[f]) = \beta^{(\lambda)}(\mathcal{R}_\lambda[f]) = 4\pi\sqrt{\frac{\pi}{2}} \cdot \hat{f}(\sqrt{\lambda}), \\ &\alpha^{(\lambda)}(\mathcal{R}_\lambda^\omega[f]) = \cot(\omega) \cdot \beta^{(\lambda)}(\mathcal{R}_\lambda^\omega[f]) = \cot(\omega) \cdot 4\pi\sqrt{\frac{\pi}{2}} \cdot \hat{f}(\sqrt{\lambda}). \end{aligned} \tag{3.14}$$

We find characterizations of solutions of the inhomogeneous Helmholtz equation (3.9) both without any asymptotic condition and in all cases $\omega \in [0,\pi)$:

Corollary 3.17 (Representation formulas II).

Let $f \in X_3$ and $w \in X_1$, and consider continuous linear functionals $\alpha^{(\lambda)}, \beta^{(\lambda)} \in X_1'$ satisfying (3.13). Then the following characterizations hold:

(i) w is twice continuously differentiable and solves $-\Delta w - \lambda w = f$ on $\mathbb{R}^3$ if and only if $w = \mathcal{R}_\lambda[f] + \alpha^{(\lambda)}(w) \cdot \tilde{\Psi}_\lambda$.

(ii) *Let $\sigma \neq 0$ and $\omega \in [0, \pi)$. w is twice continuously differentiable, solves $-\Delta w - \lambda w = f$ on $\mathbb{R}^3$ and satisfies*

$$w(x) = c\,\frac{\sin(|x|\sqrt{\lambda} + \omega)}{|x|} + O\left(\frac{1}{|x|^2}\right) \qquad \text{as } |x| \to \infty$$

for some $c \in \mathbb{R}$ if and only if

$$w = \mathcal{R}_\lambda[f] + \left[(1 - \sigma \sin(\omega))\alpha^{(\lambda)}(w) + \sigma \cos(\omega)\beta^{(\lambda)}(w)\right] \cdot \tilde{\Psi}_\lambda$$

In this case, $\sin(\omega)\alpha^{(\lambda)}(w) = \cos(\omega)\beta^{(\lambda)}(w)$.

3.2.2 The Asymptotic Phase

Frequently, equations of interest will take the form (3.9) with $f = g \cdot w$ for some $g \in X_2 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$, see (3.3). Lemma 3.12 then allows to apply ODE methods, more specifically the Prüfer transformation, to discuss the corresponding initial value problem for the profiles,

$$-w'' - \frac{2}{r}w' - \lambda w = g(r)\, w \qquad \text{on } (0, \infty) \qquad \text{with } w(0) = 1,\ w'(0) = 0. \tag{3.15}$$

Proposition 3.18 (The asymptotic phase).

Assume $g \in X_2$. Then the ODE initial value problem (3.15) *has a unique (global) solution $w : [0, \infty) \to \mathbb{R}$ which asymptotically satisfies*

$$w(r) = \varrho_\lambda(g)\,\frac{\sin(r\sqrt{\lambda} + \omega_\lambda(g))}{r} + O\left(\frac{1}{r^2}\right),$$
$$w'(r) = \varrho_\lambda(g)\sqrt{\lambda}\,\frac{\cos(r\sqrt{\lambda} + \omega_\lambda(g))}{r} + O\left(\frac{1}{r^2}\right)$$

as $r \to \infty$ for some $\varrho_\lambda(g) > 0$ and $\omega_\lambda(g) \in \mathbb{R}$. Here, the value of the asymptotic phase *$\omega_\lambda(g)$ is given by*

$$\omega_\lambda(g) = \frac{1}{\sqrt{\lambda}} \int_0^\infty g(r) \sin^2(\phi(r)\sqrt{\lambda})\,\mathrm{d}r$$
$$\text{where } \phi : [0, \infty) \to \mathbb{R} \text{ solves } \begin{cases} \phi' = 1 + \frac{1}{\lambda} g(r) \sin^2(\phi\sqrt{\lambda}), \\ \phi(0) = 0. \end{cases} \tag{3.16}$$

We will refer to the term $\omega_\lambda(g)$ as the asymptotic phase of the solution w of (3.15); we suggest to think of it as a way of quantifying the effect of the right-hand side of equation (3.15) (that is, of g) on the solution w in a situation where solutions typically oscillate. In equation (3.51) in the proof of the proposition, we will see that w can be written in the form

$$r \cdot w(r) = \varrho(r) \cdot \sin(\phi(r)\sqrt{\lambda}) = \varrho(r) \cdot \sin(r\sqrt{\lambda} + (\phi(r) - r)\sqrt{\lambda})$$

with $\varrho > 0$ and ϕ as in (3.16). In particular, $(\phi(r) - r)\sqrt{\lambda} \to \omega_\lambda(g)$ as $r \to \infty$. In the special case of vanishing right-hand side, $g \equiv 0$, we have $(\phi(r)-r)\sqrt{\lambda} \equiv 0$, which is why we say that $\omega_\lambda(g)$ describes the accumulated phase difference of the solution w which arises in the presence of g.
More precisely, writing $\omega_\lambda(g) = \omega + k\pi$ for some $k \in \mathbb{Z}$ and $\omega \in [0, \pi)$, the parameter ω describes the shift of phase between the profile $r \cdot w(r)$ and $\sin(r\sqrt{\lambda})$ at large radii; and the profile $r \cdot w(r)$ attains k additional nodes when compared with $\sin(r\sqrt{\lambda})$ in sufficiently large intervals of the form $[0, R]$, $R > 0$. In the special case $g = bu_0^2$ with $b \in \mathbb{R}$ and $u_0 \in X_1$ studied in Proposition 3.21 below, we will see that the asymptotic phase $\omega_\lambda(bu_0^2)$ depends on b in a monotone way; loosely speaking, "large" g cause "large" phase shifts.

Looking back to the asymptotic conditions imposed in Corollaries 3.16 and 3.17, we see that they are of the form

$$-\Delta w - \lambda w = g \cdot w \quad \text{on } \mathbb{R}^3, \qquad \omega_\lambda(g) \in \omega + \pi\mathbb{Z}.$$

Such boundary conditions at infinity will provide operators with spectral properties suitable for building the functional analytic framework in which to prove Theorem 3.2.

Remark 3.19.

The previous results are closely related to those in Corollary 3.16 in the special case of equation (3.9) *with right-hand side $f = g \cdot w$. In fact, comparing the asymptotic expansions in Corollary 3.16 and Proposition 3.18, we identify $\omega_\lambda(g) \in \omega + \pi\mathbb{Z}$ and $\varrho_\lambda(g) = |c|$.*

We point out two aspects in which Proposition 3.18 provides stronger statements: First, there is no singularity in case $\omega = 0$ as it appears in the definition (3.12) *of the convolution operators $\mathcal{R}_\lambda^\omega$. Second, we explicitly have $\varrho_\lambda(g) > 0$. However, in order to construct the functional analytic setting when proving Theorem 3.2, we will use the convolution operators $\mathcal{R}_\lambda^\omega$ due to their differentiability and compactness properties, see Proposition 3.13. The ODE results will then be helpful to extract spectral properties.*

As a first auxiliary result, we prove the following continuity property.

Proposition 3.20.

The asymptotic phase is continuous as a map $\omega_\lambda : X_2 \to \mathbb{R}, g \mapsto \omega_\lambda(g)$.

When studying eigenvalue problems of a linearization of (3.1) as often required in Bifurcation Theory, it will be helpful to know the dependence of the asymptotic phase $\omega_\lambda(b\,u_0^2)$ on the (eigenvalue) parameter $b \in \mathbb{R}$. Here we denote by $u_0 \in X_1 \cap C^2_{\text{loc}}(\mathbb{R}^3)$ some nonzero solution of $-\Delta u_0 - \mu u_0 = u_0^3$ on $\mathbb{R}^3$.

Proposition 3.21.

Let $u_0 \in X_1 \cap C^2_{\text{loc}}(\mathbb{R}^3)$ be a nontrivial solution of (3.4)*. Then the map $\mathbb{R} \to \mathbb{R}, b \mapsto \omega_\lambda(bu_0^2)$ is continuous, strictly increasing and onto with $\omega_\lambda(0) = 0$.*

3.2.3 The Spectrum of the Linearization

We provide a short glimpse on the central idea in proving Theorem 3.2 in order to explain the larger role of the spectral result established next. We will, slightly simplifying at this point, rewrite the nonlinear Helmholtz system (3.1) in the form

$$u = \mathcal{R}_\mu^\tau[u(u^2 + b\,v^2)], \qquad v = \mathcal{R}_\nu^\omega[v(v^2 + b\,u^2)], \qquad u, v \in X_1$$

for some $\tau, \omega \in (0, \pi)$, which additionally imposes a certain asymptotic behavior on the solutions, see Corollary 3.16. In order to analyze the linearized problem, we fix some nontrivial $u_0 \in X_1 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$ with $-\Delta u_0 - \mu u_0 = u_0^3$ on $\mathbb{R}^3$ and study the spectra of the linear operators

$$\mathbf{R}_\lambda^\omega : X_1 \to X_1, \qquad w \mapsto \mathcal{R}_\lambda^\omega[u_0^2\, w] = \Big(\Psi_\lambda + \cot(\omega)\,\tilde{\Psi}_\lambda\Big) * [u_0^2\, w], \tag{3.17}$$

which are compact thanks to Proposition 3.13 (i). Adjusting the parameters suitably, we will then find bifurcation from simple eigenvalues. Their existence and characterization is contained in the final result of this part:

Proposition 3.22 (The spectrum of $\mathbf{R}_\lambda^\omega$).

Let $\omega \in (0, \pi)$ and u_0 as before. For each $k \in \mathbb{Z}$, there is a unique $b_k(\omega, \lambda, u_0^2) \in \mathbb{R}$ with $\omega_\lambda(b_k(\omega, \lambda, u_0^2)\, u_0^2) = \omega + k\pi$. Then the spectrum of $\mathbf{R}_\lambda^\omega$ is

$$\sigma(\mathbf{R}_\lambda^\omega) = \{0\} \cup \sigma_{\mathrm{p}}(\mathbf{R}_\lambda^\omega), \quad \sigma_{\mathrm{p}}(\mathbf{R}_\lambda^\omega) = \left\{ \frac{1}{b_k(\omega, \lambda, u_0^2)} \;\middle|\; k \in \mathbb{Z} \right\}.$$

Moreover, all eigenvalues are algebraically simple, and the sequence $(b_k(\omega, \lambda, u_0^2))_{k \in \mathbb{Z}}$ is strictly increasing and unbounded below and above.

This excludes the case $\omega = 0$, even though the values $b_k(0, \lambda, u_0^2) \in \mathbb{R}$, $k \in \mathbb{Z}$, can be defined accordingly. Indeed, the first step of the proof of Proposition 3.22 above provides the following statement for all $\omega \in [0, \pi)$:

Remark 3.23.

Fix $\omega \in [0, \pi)$. Then the problem

$$-\Delta w - \lambda w = b u_0^2\, w \quad \text{on } \mathbb{R}^3, \qquad w(x) = c\,\frac{\sin(|x|\sqrt{\lambda} + \omega)}{|x|} + O\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty$$

for some $c \in \mathbb{R}$ has a nontrivial radial solution $w \in X_1 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$ if and only if $b = b_k(\omega, \lambda, u_0^2)$ for some $k \in \mathbb{Z}$, and this solution is unique up to a multiplicative constant.

3.3 Bifurcation from a Semitrivial Family. Proof of Theorem 3.2

We will first present the proof in case of asymptotic parameters $0 < \omega, \tau_1 < \pi$, $\tau_1 \neq \tau_0$, which more clearly exhibits the main ideas of exploiting suitable asymptotic properties of solutions. Afterwards, we demonstrate the modifications required in order to cover the cases $\omega = 0$ resp. $\tau_1 = 0$. Essentially, these changes are caused by the nonexistence of resolvent-type operators $\mathcal{R}^0_\mu$ resp. $\mathcal{R}^0_\nu$. We refer to the conclusions following Remark 3.15, where we have seen that asymptotic conditions of the form (3.6) lead to unique solvability of linear Helmholtz equations if and only if $\omega, \tau_1 \in (0, \pi)$. In order to overcome these difficulties for $\omega = 0$ resp. $\tau_1 = 0$, we will use the characterization of solutions given in Corollary 3.17; in that framework, the verification of the index condition in Theorem 3.7 will be more involved since it cannot be reduced to show algebraic simplicity of the eigenvalues of some operator. We comment on an alternative approach based on a lemma by Whyburn, which would serve to show (i) but not (ii) of Theorem 3.2, in Remark 3.24 (a).

The case $\omega \in (0, \pi)$ and $\tau_1 \in (0, \pi) \setminus \{\tau_0\}$.

▷ STEP 1: *The Setting.*

Let $\omega \in (0, \pi)$. We define the map

$$F: \quad X_1 \times X_1 \times \mathbb{R} \to X_1 \times X_1,$$
$$F(w, v, b) := \begin{pmatrix} w - \mathcal{R}^{\tau_1}_\mu[w^3 + 3u_0 w^2 + 3u_0^2 w + b\,(u_0 + w)v^2] \\ v - \mathcal{R}^\omega_\nu[v^3 + bv(u_0 + w)^2] \end{pmatrix}$$

with the convolution operators $\mathcal{R}^{\tau_1}_\mu, \mathcal{R}^\omega_\nu : X_3 \to X_1$ from Definition (3.12). Observe that F is well-defined since $u, v, w \in X_1$ implies $uvw \in X_3$. First, recalling Corollary 3.16 and (3.4), we have

$$F(w, v, b) = 0 \quad \Leftrightarrow \quad (u, v, b) := (u_0 + w, v, b) \text{ satisfies (3.1) with asymptotics (3.6).}$$

So we aim to find nontrivial zeros of F. Second, we observe that F has a trivial solution family, that is $F(0, 0, b) = 0$ holds for every $b \in \mathbb{R}$. Third, $F(\,\cdot\,, b)$ is a compact perturbation of the identity on $X_1 \times X_1$ since the operators $\mathcal{R}^{\tau_1}_\mu, \mathcal{R}^\omega_\nu : X_3 \to X_1$ are compact thanks to Proposition 3.13 (i). Moreover, F is twice continuously Fréchet differentiable; we have for $\varphi, \psi \in X_1$ and $b \in \mathbb{R}$, denoting by D the Fréchet derivative w.r.t. the w and v components,

$$DF(0, 0, b)[(\varphi, \psi)] = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} 3\,\mathcal{R}^{\tau_1}_\mu[u_0^2\,\varphi] \\ b\,\mathcal{R}^\omega_\nu[u_0^2\,\psi] \end{pmatrix} = \begin{pmatrix} \varphi - 3\,\mathbf{R}^{\tau_1}_\mu \varphi \\ \psi - b\,\mathbf{R}^\omega_\nu \psi \end{pmatrix} \tag{3.18}$$

with compact linear operators $\mathbf{R}^{\tau_1}_\mu, \mathbf{R}^\omega_\nu : X_1 \to X_1$ as in equation (3.17). We deduce that, due to (3.5) and $\tau_1 \neq \tau_0$, $DF(0, 0, b)[(\varphi, \psi)] = 0$ implies $\varphi = 0$. So nontrivial elements of $\ker DF(0, 0, b)$ are of the form $(0, \psi)$ where ψ satisfies $\psi = b\,\mathbf{R}^\omega_\nu \psi$. Proposition 3.22 reveals that such nontrivial ψ exists if and only if $b = b_k(\omega, \nu, u_0^2)$, i.e. $\omega_\nu(b\,u_0^2) = k\pi + \omega$ for some $k \in \mathbb{Z}$, and that the associated eigenspaces are one-dimensional. We write $b_k(\omega)$ instead of $b_k(\omega, \nu, u_0^2)$. Thus $b \in \{b_k(\omega) \mid k \in \mathbb{Z}\}$ is a necessary condition for bifurcation of solutions of $F(w, v, b) = 0$ from $(0, 0, b)$. We show in the following that it is also sufficient.

▷ STEP 2: *Local Bifurcation.*

We apply the Crandall-Rabinowitz Bifurcation Theorem and, to this end, verify its simplicity and transversality assumptions at the point $(0, 0, b_k(\omega))$. As $F(\,\cdot\,, b)$ is a compact perturbation of the identity on $X_1 \times X_1$, the Riesz-Schauder Theorem implies that

$DF(0,0,b_k(\omega))$ is a Fredholm operator of index zero. By the previous step,

$$\ker DF(0,0,b_k(\omega)) = \operatorname{span}\left\{\begin{pmatrix} 0 \\ \psi_k \end{pmatrix}\right\}$$

for some $\psi_k \in X_1 \setminus \{0\}$. To see that the transversality condition holds, we first compute

$$\partial_b DF(0,0,b_k(\omega))[(0,\psi_k)] \overset{(3.18)}{=} -\begin{pmatrix} 0 \\ \mathbf{R}_\nu^\omega \psi_k \end{pmatrix} = -\frac{1}{b_k(\omega)}\begin{pmatrix} 0 \\ \psi_k \end{pmatrix}.$$

Then, assuming there was $v \in X_1$ with $D_v F(0,0,b_k(\omega))[v] = v - b_k(\omega)\,\mathbf{R}_\nu^\omega v = \psi_k$, we conclude

$$v \in \ker(I - b_k(\omega)\,\mathbf{R}_\nu^\omega)^2 \setminus \ker(I - b_k(\omega)\,\mathbf{R}_\nu^\omega),$$

which contradicts the algebraic simplicity of the eigenvalue $b_k(\omega)^{-1}$ of $\mathbf{R}_\nu^\omega$ proved in Proposition 3.22. Thus $\partial_b DF(0,0,b_k(\omega))[(0,\psi_k)] \notin \operatorname{ran} DF(0,0,b_k(\omega))$, and the Crandall-Rabinowitz Bifurcation Theorem provides a curve of solutions of $F(w,v,b)=0$ as described in (ii). We remark that it is smooth since F is of class C^∞. Further, possibly shrinking the neighborhood where the local result holds, we may w.l.o.g. assume fully nontrivial solutions $(u,v) = (u_0 + w, v)$ of (3.1) since the direction of bifurcation with respect to $X_1 \times X_1$ is given by $(0,\psi_k)$ and thus solutions along the given curve are of the form

$$\begin{pmatrix} u(s) \\ v(s) \end{pmatrix} = \begin{pmatrix} u_0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ \psi_k \end{pmatrix} + o(s) \qquad \text{as } s \to 0.$$

$\triangleright$ <u>Step 3</u>: *Global Bifurcation.*

We have already seen that $F(\,\cdot\,,b)$, $b \in \mathbb{R}$, is a compact perturbation of the identity on $X_1 \times X_1$. Thus the application of Rabinowitz' Global Bifurcation Theorem only requires to verify that the index of $F(\,\cdot\,,b)$ in $(0,0)$ changes sign at each value $b = b_k(\omega)$, $k \in \mathbb{Z}$. By the identity (3.18), for $b \notin \{b_k(\omega) \mid k \in \mathbb{Z}\}$,

$$\begin{aligned}\operatorname{ind}_{X_1\times X_1}\big(F(\,\cdot\,,b),(0,0)\big) &= \operatorname{ind}_{X_1\times X_1}\big(DF(0,0,b),(0,0)\big) \\ &\overset{(3.18)}{=} \operatorname{ind}_{X_1}\big(I - 3\,\mathbf{R}_\mu^{\tau_1},0\big)\cdot \operatorname{ind}_{X_1}\big(I - b\,\mathbf{R}_\nu^\omega,0\big),\end{aligned}$$

and hence $\operatorname{ind}_{X_1\times X_1}\big(F(\,\cdot\,,b),(0,0)\big)$ changes sign at $b = b_k(\omega)$ if and only if so does $\operatorname{ind}_{X_1}\big(I - b\mathbf{R}_\nu^\omega,0\big)$. The latter change of index occurs since $b_k(\omega)^{-1}$ is an isolated eigenvalue of algebraic multiplicity 1 of $\mathbf{R}_\nu^\omega$, see Proposition 3.22.

Let us recall that $\mathcal{S}(\omega)$ has been introduced as the set of all solutions $(u,v,b) \in X_1 \times X_1 \times \mathbb{R}$ of (3.1), (3.6) which do not belong to the semitrivial family $\mathcal{T}_{u_0}$. By Step 2, $(u_0,0,b_k(\omega)) \in \overline{\mathcal{S}(\omega)}$. The Global Bifurcation Theorem by Rabinowitz asserts that the associated connected component $\mathcal{C}_k(\omega)$ of $(u_0,0,b_k(\omega))$ in $\overline{\mathcal{S}(\omega)}$ is unbounded or returns to the trivial branch at some point $(u_0,0,b^*) \in \mathcal{T}_{u_0}$. We prove that, in any case, the component is unbounded.

To see this, we recall the asymptotic phase ω_ν as introduced in Proposition 3.18. It satisfies $\omega_\nu(b_k(\omega)u_0^2) = \omega + k\pi$ by definition of $b_k(\omega)$, see Step 1. Moreover, we can conclude $\omega_\nu(v^2 + bu^2) \in \omega + \pi\mathbb{Z}$ for all $(u,v,b) \in \mathcal{C}_k(\omega)$ with $v \neq 0$ as follows: Since v solves the differential equation $-\Delta v - \nu v = (u^2 + b\,v^2)v$, Proposition 3.18 applies (with $g = u^2 + b\,v^2$) and yields $c \neq 0$ with

$$v(x) = c\,\frac{\sin(|x|\sqrt{\nu} + \omega_\nu(v^2 + bu^2))}{|x|} + O\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty.$$

Since the leading-order term does not vanish, the asymptotic condition (3.6) implies $\omega_\nu(v^2+bu^2) \in \omega + \pi\mathbb{Z}$, as asserted.

So if all elements $(u,v,b) \in \mathcal{C}_k(\omega)\backslash\mathcal{T}_{u_0}$ satisfy $v \neq 0$, then as a consequence of the continuity of ω_ν as stated in Proposition 3.20 and of the fact that $\mathcal{C}_k(\omega)$ is connected by definition, we infer that $\omega_\nu(v^2+bu^2) = \omega + k\pi$ for all $(u,v,b) \in \mathcal{C}_k(\omega)$. Let us now assume that $\mathcal{C}_k(\omega)$ returns to the trivial family in some point $(u_0, 0, b^*) \in \mathcal{T}_{u_0}$, $b^* \neq b_k(\omega)$. Then $\omega_\nu(b^* u_0^2) \neq \omega + k\pi$, hence $(u,v,b) \mapsto \omega_\nu(v^2+bu^2)$ is not constant on $\mathcal{C}_k(\omega)$. Thus, there exists a semitrivial element $(u_1, 0, b_1) \in \mathcal{C}_k(\omega) \setminus \mathcal{T}_{u_0}$, $u_1 \neq u_0$. Since $\mathcal{C}_k(\omega)$ is maximal connected, it contains the unbounded semitrivial family $\mathcal{T}_{u_1} = \{(u_1, 0, b) \mid b \in \mathbb{R}\}$. In any case, $\mathcal{C}_k(\omega)$ is unbounded.

The case $\omega = 0$ and $\tau_1 \in (0,\pi) \setminus \{\tau_0\}$.

$\triangleright$ Step 1: *The Setting.*

For $\omega = 0$, we consider instead of F the maps

$$
\begin{aligned}
G_\sigma : \quad & X_1 \times X_1 \times \mathbb{R} \to X_1 \times X_1, \\
G_\sigma(w,v,b) := & \begin{pmatrix} w - \mathcal{R}_\mu^{\tau_1}[w^3 + 3u_0 w^2 + 3u_0^2 w + b\,(u_0+w)v^2] \\ v - \mathcal{R}_\nu[v\,(v^2 + b\,(w+u_0)^2)] - (\alpha^{(\nu)}(v) + \sigma\beta^{(\nu)}(v)) \cdot \tilde{\Psi}_\nu \end{pmatrix}
\end{aligned}
$$

with the functionals $\alpha^{(\nu)}, \beta^{(\nu)}$ as in Corollary 3.17 and for $\sigma = \pm 1$. We will prove the local bifurcation result for each map G_σ but, in order to find global bifurcation, we require G_+ resp. G_- in order to verify the change of the index at $(0,0,b)$ with $b \geq 0$ resp. $b \leq 0$. Part (ii) of that Corollary states that $G_\sigma(w,v,b) = 0$ if and only if the point $(u_0 + w, v, b)$ solves the nonlinear Helmholtz system (3.1) with asymptotics (3.6), $\omega = 0$. In particular, $G_+(w,v,b) = 0$ if and only if $G_-(w,v,b) = 0$. Due to (3.5) and Corollaries 3.16, 3.17 (ii), $(\varphi,\psi) \in \ker DG_\sigma(0,0,b)$ if and only if

$$
\varphi \equiv 0, \quad -\Delta\psi - \nu\psi = b\,u_0^2\,\psi,\ \ \psi(x) = c_\psi \frac{\sin(|x|\sqrt{\nu})}{|x|} + O\left(\frac{1}{|x|^2}\right) \tag{3.19}
$$

for some $c_\psi \in \mathbb{R} \setminus \{0\}$. Proposition 3.18 and Remark 3.23 tell us that a nontrivial solution $\psi = \psi_k \in X_1$ exists if and only if the asymptotic phase satisfies $\omega_\nu(b\,u_0^2) \in \pi\mathbb{Z}$, equivalently $b = b_k(0,\nu,u_0^2) =: b_k(0)$ for some $k \in \mathbb{Z}$, and that the eigenspace is one-dimensional. Thus solutions of (3.1), (3.6) for $\omega = 0$ bifurcate from a point $(u_0, 0, b) \in \mathcal{T}_{u_0}$ only if $b = b_k(0)$ for some $k \in \mathbb{Z}$. We show that it happens indeed by checking the assumptions of the Crandall-Rabinowitz Theorem.

$\triangleright$ Step 2: *Local Bifurcation.*

First, we infer that, $G_\sigma(\,\cdot\,, b)$ being a compact perturbation of the identity, $DG_\sigma(0,0,b_k(0))$ is a 1-1-Fredholm operator. It remains to check transversality. We compute

$$
\partial_b DG_\sigma(0,0,b_k(0))[(0,\psi_k)] = -\begin{pmatrix} 0 \\ \mathcal{R}_\nu[u_0^2\,\psi_k] \end{pmatrix}
$$

and assume by contradiction that there exist $\varphi, \psi \in X_1$ with $DG_\sigma(0,0,b_k(0))[(\varphi,\psi)] = \partial_b DG_\sigma(0,0,b_k(0))[(0,\psi_k)]$. Then $\varphi \equiv 0$ due to (3.5), and

$$
\psi = b_k(0)\,\mathcal{R}_\nu[u_0^2\,\psi] + (\alpha^{(\nu)}(\psi) + \sigma\beta^{(\nu)}(\psi)) \cdot \tilde{\Psi}_\nu - \mathcal{R}_\nu[u_0^2\,\psi_k]. \tag{3.20}
$$

Thus, applying the functional $\alpha^{(\nu)}$ to (3.20), we find

$$
\alpha^{(\nu)}(\psi) = b_k(0)\alpha^{(\nu)}(\mathcal{R}_\nu[u_0^2\,\psi]) + (\alpha^{(\nu)}(\psi) + \sigma\beta^{(\nu)}(\psi)) \cdot \alpha^{(\nu)}(\tilde{\Psi}_\nu) - \alpha^{(\nu)}(\mathcal{R}_\nu[u_0^2\,\psi_k])
$$

$$\overset{(3.14)}{=} \alpha^{(\nu)}(\psi) + \sigma\beta^{(\nu)}(\psi);$$

hence, since $\sigma \neq 0$, we conclude $\beta^{(\nu)}(\psi) = 0$. Equation (3.20) and $DG_\sigma(0,0,b_k(0))[(0,\psi_k)] = (0,0)$ further provide, due to Proposition 3.13 (ii), the differential equations

$$-\psi_k'' - \frac{2}{r}\psi_k' - \nu\psi_k = b_k(0)\, u_0^2\, \psi_k, \quad -\psi'' - \frac{2}{r}\psi' - \nu\psi = b_k(0)\, u_0^2\, \psi - u_0^2\, \psi_k \tag{3.21}$$

for $r > 0$. Moreover, we have $\beta^{(\nu)}(\psi) = 0$ as shown above and $\beta^{(\nu)}(\psi_k) = 0$, which holds since $(0, \psi_k) \in \ker DG_\sigma(0,0,b_k(0))$ and hence satisfies (3.19). Thus by Proposition 3.13 (iii), the profiles satisfy

$$\begin{aligned} \psi_k(r) &= c_k \cdot \frac{\sin(r\sqrt{\nu})}{r} + O\left(\frac{1}{r^2}\right), \quad \psi_k'(r) = c_k\sqrt{\nu} \cdot \frac{\cos(r\sqrt{\nu})}{r} + O\left(\frac{1}{r^2}\right), \\ \psi(r) &= c \cdot \frac{\sin(r\sqrt{\nu})}{r} + O\left(\frac{1}{r^2}\right), \quad \psi'(r) = c\sqrt{\nu} \cdot \frac{\cos(r\sqrt{\nu})}{r} + O\left(\frac{1}{r^2}\right) \end{aligned} \tag{3.22}$$

as $r \to \infty$ for some $c, c_k \in \mathbb{R}$, where Proposition 3.18 guarantees $c_k \neq 0$. Multiplying the differential equations (3.21) by ψ resp. ψ_k and taking the difference yields

$$(r^2(\psi_k\psi' - \psi\psi_k'))' = r^2 u_0^2 \psi_k^2,$$

hence for $R > 0$

$$\int_0^R r^2 u_0^2(r)\psi_k^2(r)\,\mathrm{d}r = R^2\left(\psi_k(R)\psi'(R) - \psi(R)\psi_k'(R)\right) \overset{(3.22)}{=} O\left(\frac{1}{R}\right).$$

Thus letting $R \nearrow \infty$, we infer $u_0\psi_k \equiv 0$, a contradiction. Hence

$$\partial_b DG_\sigma(0,0,b_k(0))[(0,\psi_k)] \notin \operatorname{ran} DG_\sigma(0,0,b_k(0)),$$

as asserted, proving transversality and thus bifurcation from a simple eigenvalue.

▷ Step 3: *Global Bifurcation.*

Having already mentioned that $G_\sigma(\,\cdot\,, b)$ is a compact perturbation of the identity on $X_1 \times X_1$, Rabinowitz' Global Bifurcation Theorem applies and yields unbounded connected components $\mathcal{C}_k(0) \subseteq \overline{\mathcal{S}(0)}$ bifurcating from $(0,0,b_k(0))$ once we show that the index

$$\begin{aligned} \operatorname{ind}_{X_1\times X_1}\big(G_\sigma(\,\cdot\,,b),(0,0)\big) &= \operatorname{ind}_{X_1\times X_1}\big(DG_\sigma(0,0,b),(0,0)\big) \\ &= \operatorname{ind}_{X_1}\big(I - 3\,\mathbf{R}_\mu^{\tau_1}, 0\big)\cdot \operatorname{ind}_{X_1}\big(I - K_b, 0\big) \\ \text{where } K_b &:= b\,\mathcal{R}_\nu[u_0^2\,\cdot\,] + (\alpha^{(\nu)} + \sigma\beta^{(\nu)})\cdot\tilde{\Psi}_\nu \end{aligned} \tag{3.23}$$

changes sign at $b = b_k(0)$, $k \in \mathbb{Z}$ for a suitable choice of $\sigma \in \{-1, +1\}$. As initially announced, we analyze bifurcation at $b_k(0) \geq 0$ using the map G_+ and at $b_k(0) < 0$ using G_-.

In the following, we verify that 1 is an algebraically simple eigenvalue of $K_{b_k(0)}$ and, moreover, the corresponding perturbed eigenvalue $\lambda_b \approx 1$ of K_b for $b \approx b_k(0)$ has the property that $\lambda_b - 1$ changes sign as b crosses $b_k(0)$. For the existence, algebraic simplicity and continuous dependence of the perturbed eigenvalue λ_b on b we refer to Kielhöfer's book [40], p. 203. Rabinowitz' Global Bifurcation Theorem in the version of [40], Theorem II.3.3 then applies, and unboundedness of the component can then be proved as in Step 3 above.

▷▷ STEP 3 (a): *Algebraic Simplicity*

Here we adapt the proof of algebraic simplicity in Proposition 3.22 to the case $\omega = 0$ resp. to the map G_σ. Let us assume that $\ker(I - K_{b_k(0)}) = \operatorname{span}\{w\}$ and $v \in \ker(I - K_{b_k(0)})^2 \setminus \ker(I - K_{b_k(0)})$. Then $v - K_{b_k(0)}v \in \ker(I - K_{b_k(0)})$, and without loss of generality, we have $v - K_{b_k(0)}v = w = K_{b_k(0)}w$, hence

$$\begin{aligned} w &= b_k(0)\, \mathcal{R}_\nu[u_0^2\, w] + (\alpha^{(\nu)}(w) + \sigma\beta^{(\nu)}(w)) \cdot \tilde{\Psi}_\nu, \\ v &= b_k(0)\, \mathcal{R}_\nu[u_0^2\,(v + w)] + (\alpha^{(\nu)}(v + w) + \sigma\beta^{(\nu)}(v + w)) \cdot \tilde{\Psi}_\nu. \end{aligned} \tag{3.24}$$

Corollary 3.17 implies that the profiles satisfy

$$-w'' - \frac{2}{r}w' - \nu w = b_k(0)u_0^2\, w, \quad -v'' - \frac{2}{r}v' - \nu v = b_k(0)u_0^2\,(v + w) \tag{3.25}$$

on $(0, \infty)$ as well as $\beta^{(\nu)}(w) = 0$. Applying $\alpha^{(\nu)}$ to the second identity in (3.24) and recalling the identities (3.14), we further have $\beta^{(\nu)}(v) = -\sigma\alpha^{(\nu)}(w)$. By definition of $\alpha^{(\nu)}, \beta^{(\nu)}$ in (3.13), and in view of Proposition 3.13 (iii), we find the asymptotic expansions

$$\begin{aligned} w(r) &= \alpha^{(\nu)}(w)\, \frac{\sin(r\sqrt{\nu})}{4\pi r} + O\left(\frac{1}{r^2}\right), \\ w'(r) &= \alpha^{(\nu)}(w)\sqrt{\nu}\, \frac{\cos(r\sqrt{\nu})}{4\pi r} + O\left(\frac{1}{r^2}\right), \\ v(r) &= \alpha^{(\nu)}(v)\, \frac{\sin(r\sqrt{\nu})}{4\pi r} - \sigma\alpha^{(\nu)}(w)\, \frac{\cos(r\sqrt{\nu})}{4\pi r} + O\left(\frac{1}{r^2}\right), \\ v'(r) &= \alpha^{(\nu)}(v)\sqrt{\nu}\, \frac{\cos(r\sqrt{\nu})}{4\pi r} + \sigma\alpha^{(\nu)}(w)\sqrt{\nu}\, \frac{\sin(r\sqrt{\nu})}{4\pi r} + O\left(\frac{1}{r^2}\right). \end{aligned} \tag{3.26}$$

For $r \geq 0$, we introduce $q(r) := r^2\,(w(r)v'(r) - v(r)w'(r))$. Using the differential equations in (3.25), we find after a short calculation

$$q'(r) = -r^2 b_k(0)u_0^2(r)w^2(r) \qquad (r > 0),$$

hence q is nondecreasing if $b_k(0) \leq 0$ and nonincreasing if $b_k(0) \geq 0$. On the other hand, $q(0) = 0$, and the asymptotic expansions (3.26) imply as $r \to \infty$

$$q(r) = \sigma \cdot \frac{\alpha^{(\nu)}(w)^2}{(4\pi)^2}\sqrt{\nu} + O\left(\frac{1}{r}\right).$$

Since $\alpha^{(\nu)}(w) \neq 0$ according to Proposition 3.18, and since we choose $\sigma = +1$ to discuss $b_k(0) \geq 0$ and $\sigma = -1$ for $b_k(0) < 0$, this contradicts the monotonicity derived before. Hence $\ker(I - K_{b_k(0)}) = \ker(I - K_{b_k(0)})^2$, as claimed.

▷▷ STEP 3 (b): *Perturbation of the Eigenvalue*

We now discuss the perturbation of the simple eigenvalue $\lambda_{b_k(0)} = 1$ of $K_{b_k(0)}$. Throughout this step, we consider a perturbed value $b \approx b_k(0), b \neq b_k(0)$ and the corresponding eigenpair $\lambda_b \approx 1$ and $w_b \in X_1 \setminus \{0\}$ with $K_b w_b = \lambda_b w_b$. It satisfies

$$-\Delta w_b - \nu w_b = \frac{b}{\lambda_b}u_0^2(x)\, w_b \quad \text{on } \mathbb{R}^3, \qquad (\lambda_b - 1)\alpha^{(\nu)}(w_b) = \sigma\beta^{(\nu)}(w_b). \tag{3.27}$$

Since $b \approx b_k(0), b \neq b_k(0)$ and hence $\beta^{(\nu)}(w_b) \neq 0$ due to the strict monotonicity of the asymptotic phase, see Proposition 3.21, this immediately implies that $\lambda_b \neq 1$. By Corollary 3.16,

$$w_b = \mathcal{R}_\nu^{\omega_b}\left[\frac{b}{\lambda_b}u_0^2 w_b\right] \quad \text{where } \omega_b \in (0,\pi) \text{ with } \omega_\nu(b\lambda_b^{-1}u_0^2) \in \omega_b + \pi\mathbb{Z}.$$

Further, by the identities in (3.14),

$$\frac{\alpha^{(\nu)}(w_b)}{\beta^{(\nu)}(w_b)} = \cot(\omega_\nu(b\lambda_b^{-1}u_0^2)) \quad (b \neq b_k(0), b \approx b_k(0)), \tag{3.28}$$

and we have $\omega_\nu(b_k(0)u_0^2) \in \pi\mathbb{Z}$ by definition of $b_k(0)$, see Proposition 3.22 and the following Remark 3.23.

We now discuss the values $b_k(0) \geq 0$, where we chose $\sigma = +1$. In case $b > b_k(0)$ we show that $\lambda_b > 1$. Assuming $\lambda_b < 1$, we infer from the second identity in (3.27) that $\operatorname{sgn} \alpha^{(\nu)}(w_b) \neq \operatorname{sgn} \beta^{(\nu)}(w_b)$ and thus $\omega_\nu(b\lambda_b^{-1}u_0^2) \in \left(-\frac{\pi}{2}, 0\right) + \pi\mathbb{Z}$ due to (3.28). But since $b\lambda_b^{-1} > b_k(0)\lambda_b^{-1} \geq b_k(0)$, the monotonicity stated in Proposition 3.21 implies $\omega_\nu(b\lambda_b^{-1}u_0^2) \in \omega_\nu(b_k(0)u_0^2) + \left(0, \frac{\pi}{2}\right) \subseteq \left(0, \frac{\pi}{2}\right) + \pi\mathbb{Z}$, a contradiction. Hence (recalling $\lambda_b \neq 1$), we infer $\lambda_b > 1$ as claimed. In the same way, for $b < b_k(0)$, we can show that $\lambda_b < 1$.

We still have to consider $b_k(0) < 0$; here we took $\sigma = -1$. Then, for $b > b_k(0)$, we show that $\lambda_b < 1$. We assume for contradiction that $\lambda_b > 1$, which implies $\operatorname{sgn} \alpha^{(\nu)}(w_b) \neq \operatorname{sgn} \beta^{(\nu)}(w_b)$ and $\omega_\nu(b\lambda_b^{-1}u_0^2) \in \left(-\frac{\pi}{2}, 0\right) + \pi\mathbb{Z}$. On the other hand, $b\lambda_b^{-1} > b_k(0)$, which leads to $\omega_\nu(b\lambda_b^{-1}u_0^2) > \omega_\nu(b_k(0)u_0^2) \in \pi\mathbb{Z}$, a contradiction. This proves $\lambda_b < 1$. Similarly, for $b < b_k(0)$, we find $\lambda_b > 1$.

We have thus proved that, as b crosses $b_k(0)$, the perturbed eigenvalue λ_b crosses $\lambda_{b_k(0)} = 1$. Thus, by equation (3.23), the sign of the Leray-Schauder index $\operatorname{ind}_{X_1\times X_1}\left(G_\sigma(\,\cdot\,, b), (0,0)\right)$ changes at $b = b_k(0)$ for all $k \in \mathbb{Z}$ and for $\sigma \in \{\pm 1\}$ chosen as above.

The case $\tau_1 = 0$.

This is covered by redefining the first components of F resp. G_σ,

$$\begin{aligned}(w,v,b) \mapsto \quad & w - \mathcal{R}_\mu[w^3 + 3u_0w^2 + 3u_0^2w + b\,(u_0+w)v^2] - \left[\alpha^{(\mu)}(w) + \beta^{(\mu)}(w)\right]\cdot\tilde{\Psi}_\mu \\ & =: h(w,v,b)\end{aligned}$$

instead of

$$(w,v,b) \mapsto \quad w - \mathcal{R}_\mu^{\tau_1}[w^3 + 3u_0w^2 + 3u_0^2w + b\,(u_0+w)v^2].$$

This redefinition is similar to the changes in the second component when passing from F resp. parameters $\omega \in (0,\pi)$ to G_σ suitable for $\omega = 0$. Then still, F resp. G_σ is a compact perturbation of the identity. The redefinition ensures that, due to part (ii) of Corollary 3.17, $h(w,v,b) = 0$ implies that

$$\begin{aligned} -\Delta w - \mu w &= (u_0+w)^3 - u_0^3 + b\,(u_0+w)v^2 \quad \text{on } \mathbb{R}^3, \\ w(x) &= c_w\frac{\sin(|x|\sqrt{\mu})}{|x|} + O\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty \end{aligned}$$

for some $c_w \in \mathbb{R}$, i.e. that the w component of zeros of F resp. G_σ satisfies (3.1), (3.6). Similarly, for $\varphi, \psi \in X_1$ with $Dh(0,0,b)[(\varphi,\psi)] = (0,0)$, we obtain

$$\begin{aligned} -\Delta\varphi - \mu\varphi &= 3u_0^2(x)\varphi \quad \text{on } \mathbb{R}^3, \\ \varphi(x) &= c_\varphi\frac{\sin(|x|\sqrt{\mu})}{|x|} + O\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty \end{aligned}$$

for some $c_\varphi \in \mathbb{R}$, which implies $\varphi = 0$ thanks to the nondegeneracy condition (3.5) (with $\tau_1 = 0$). These are the only properties of the first component of F resp. G_σ required in the proof for $\tau_1 \neq 0$, which we can now again follow line by line, closing the proof of Theorem 3.2. □

Proof of Remark 3.3

(a) Step 1 of the proof above in fact shows that solutions of (3.1), (3.6) bifurcate from $(u_0, 0, b) \in \mathcal{T}_{u_0}$ only if $b = b_k(\omega)$ for $k \in \mathbb{Z}$; Step 2 shows that this condition is also sufficient.

(b) By Proposition 3.21, the map $q : \mathbb{R} \to \mathbb{R}$, $q(b) := \omega_\nu(b\,u_0^2)$ is strictly increasing and onto. Having chosen $b_k(\omega) = q^{-1}(\omega + k\pi)$ for $\omega \in [0,\pi), k \in \mathbb{Z}$, see Proposition 3.22 and its application following equation (3.18), we infer strict monotonicity and surjectivity of the map $\mathbb{R} \to \mathbb{R}$, $\omega + k\pi \mapsto b_k(\omega)$.

(c) In Step 2 we have seen that in a neighborhood of the bifurcation point $(u_0, 0, b_k(\omega))$, the continuum $\mathcal{C}_k(\omega)$ contains only fully nontrivial solutions apart from $(u_0, 0, b_k(\omega))$ itself. Following the argumentation which was given in detail for the case $\omega \in (0,\pi)$ at the end of Step 3 (and also holds for $\omega = 0$), we infer for all $(u, v, b) \in \mathcal{C}_k(\omega)$ from this neighborhood that the asymptotic phase of v satisfies $\omega_\nu(v^2 + bu^2) = \omega + k\pi$. More generally, $\omega_\nu(v^2 + bu^2) = \omega + k\pi$ holds on every connected subset of $\mathcal{C}_k(\omega) \setminus \{(u, 0, b) \,|\, u \in X_1, u \neq u_0, b \in \mathbb{R}\}$ containing $(u_0, 0, b_k(\omega))$.

(d) Assuming $\tau_1 \neq \sigma_0$, any solution (u, v, b) of (3.1), (3.6) satisfies

$$u(x) = u_0(x) + w(x) = c_0\,\frac{\sin(|x|\sqrt{\mu} + \sigma_0)}{|x|} + c_w\,\frac{\sin(|x|\sqrt{\mu} + \tau_1)}{|x|} + O\left(\frac{1}{|x|^2}\right)$$

as $|x| \to \infty$ for some $c_w \in \mathbb{R}, c_0 \in \mathbb{R} \setminus \{0\}$ by Proposition 3.1 and by the asymptotic condition (3.6). Hence, comparing the leading-order terms, we see that $u \not\equiv 0$. Moreover, as recalled in (b), the values $b_k(\omega) = q^{-1}(\omega + k\pi)$ do not change when choosing another asymptotic parameter τ_1 in (3.6).

(e) For every $\omega \in (0,\pi)$ and $\tau_1 \in [0,\pi) \setminus \{\tau_0\}$, Theorem 3.2 provides continua of solutions $(u_0 + w, v, b)$ of (3.1) with asymptotics

$$\begin{aligned} w(x) &= c_u\,\frac{\sin(|x|\sqrt{\mu} + \tau_1)}{|x|} + O\left(\frac{1}{|x|^2}\right) \\ v(x) &= c_v\,\frac{\sin(|x|\sqrt{\nu} + \omega)}{|x|} + O\left(\frac{1}{|x|^2}\right) \end{aligned} \qquad \text{as } |x| \to \infty.$$

Since v satisfies $-\Delta v - \nu v = g_v v$ with $g_v = (u_0 + w)^2 + b\,v^2 \in X_2$, Proposition 3.18 guarantees $c_v \neq 0$ if $v \not\equiv 0$. For w, this line of argumentation does not apply since $-\Delta w - \mu w = (w^2 + 3u_0 w + 3u_0^2 + bv^2)\,w + b\,u_0 v^2$, and thus possibly $c_u = 0$. In this case, the asymptotic condition for u resp. w in (3.6) is satisfied for every $\tau_1 \in [0,\pi) \setminus \{\tau_0\}$. This is why we cannot ensure that we obtain different continua of solutions when changing the value of τ_1 (but not of ω).

□

Remark 3.24 (On the proof of Theorem 3.2).

Finally, we wish to comment on some extensions and variants of the proof, explaining why we have chosen to present it in the way above.

(a) *A close look on the argumentation above shows that the distinction of the cases $\omega \neq 0$ and $\omega = 0$ could be avoided by treating the former also in the framework of Corollary 3.17 (ii). We have, however, decided to present the more elegant approach based on the spectral analysis of the operators $\mathbf{R}_\lambda^\omega$ for $\omega \in (0, \pi)$.*

Alternatively, the existence of bifurcating continua in case $\omega = 0$ could have been shown using an abstract result by Li and Sun [44], Lemma 1.4, which is based on Whyburn's Lemma. It yields a connected component of solutions of (3.1), (3.6) *with $\omega = 0$ provided the existence of unbounded continua $\mathcal{C}_k(\omega)$ for $\omega \in (0, \pi)$ has been proved. However, the local statement (ii) of Theorem 3.2 could not have been recovered for $\omega = 0$ when choosing this method, which is why we opted against it.*

(b) *At first glance, it would be much more natural not to reparametrize $w := u - u_0$ but instead consider a map of the form*

$$F:\ X_1 \times X_1 \times \mathbb{R} \to X_1 \times X_1, \quad F(u, v, b) := \begin{pmatrix} u - \mathcal{R}_\mu^\tau[u^3 + b\,uv^2] \\ v - \mathcal{R}_\nu^\omega[v^3 + bvu^2] \end{pmatrix}$$

for suitable $\omega, \tau \in (0, \pi)$ (and suitable modifications in case $\tau = 0$ resp. $\omega = 0$), which would correspond to asymptotic conditions

$$u(x) = c_u\,\frac{\sin(|x|\sqrt{\mu} + \tau)}{|x|} + O\left(\frac{1}{|x|^2}\right), \quad v(x) = c_v\,\frac{\sin(|x|\sqrt{\nu} + \omega)}{|x|} + O\left(\frac{1}{|x|^2}\right)$$

as $|x| \to \infty$. In particular, both u and v would then satisfy equations of the form discussed in Proposition 3.18, and in view of Remark 3.3 (e) it would guarantee nonvanishing leading-order terms in the asymptotic expansion, i.e. $c_u \neq 0$ resp. $c_v \neq 0$, whenever $u \not\equiv 0$ resp. $v \not\equiv 0$.

As we require F to vanish on the semitrivial family $\mathcal{T}_{u_0}$, we would be obliged to choose $\tau = \sigma_0$ in view of Proposition 3.1 if $\sigma_0 \in (0, \pi)$ (and once again consider the case $\sigma_0 = 0$ separately). The lack of freedom in choosing τ, however, has consequences regarding the nondegeneracy property (3.5). *Either we would have to show $\tau_0 \neq \sigma_0$, then* (3.5) *holds and bifurcation from simple eigenvalues can be verified as before. Or, in case $\tau_0 = \sigma_0$, we would have to discuss bifurcation from double eigenvalues. (Indeed, aiming for fully nontrivial solutions, we would not consider one-dimensional kernels of the form span $\{(\varphi_0, 0)\}$.) It is, however, not clear whether any of these alternatives is realistic.*

3.4 Bifurcation from a Diagonal Family. Proof of Theorem 3.4

We now prove the occurence of bifurcations from the diagonal solution family

$$\mathfrak{T}_{u_0} := \left\{ (u_b, u_b, b) \;\middle|\; u_b = \frac{1}{\sqrt{1+b}}\,u_0,\ b > -1 \right\}$$

as stated in Theorem 3.4. To this end we first rewrite the system (3.1) in an equivalent but more convenient way. Looking for solutions $(u, v, b) \in X_1 \times X_1 \times \mathbb{R} \setminus \mathfrak{T}_{u_0}$, we introduce the functions $w_1, w_2 \in X_1$ via

$$u =: u_b + w_1 - w_2, \qquad v =: u_b + w_1 + w_2.$$

A few computations then yield that bifurcation at the point (u_b, u_b, b) occurs if and only if we have bifurcation from the trivial solution of the nonlinear Helmholtz system

$$\begin{cases} -\Delta w_1 - \mu w_1 = (1+b)\big((w_1+u_b)^3 - u_b^3\big) + (3-b)(w_1+u_b)w_2^2 & \text{on } \mathbb{R}^3, \\ -\Delta w_2 - \mu w_2 = (1+b)w_2^3 + (3-b)(w_1+u_b)^2 w_2 & \text{on } \mathbb{R}^3, \end{cases} \tag{3.29}$$

and the asymptotic conditions (3.7) are equivalent to

$$\begin{aligned} w_1(x) &= c_1 \, \frac{\sin(|x|\sqrt{\mu} + \tau_1)}{|x|} + O\left(\frac{1}{|x|^2}\right), \\ w_2(x) &= c_2 \, \frac{\sin(|x|\sqrt{\mu} + \omega)}{|x|} + O\left(\frac{1}{|x|^2}\right) \end{aligned} \tag{3.30}$$

as $|x| \to \infty$ for some $c_1, c_2 \in \mathbb{R}$. As in the proof of Theorem 3.2, the functional analytical setting in the special cases $\omega = 0$ or $\tau_1 = 0$ is different from the general one since a substitute for the operators $\mathcal{R}_\mu^{\tau_1}, \mathcal{R}_\mu^{\omega}$ has to be found, see the definition of G_σ in the proof of Theorem 3.2. In order to keep the presentation short we only discuss the case $\tau_1, \omega \in (0, \pi)$ and refer to the proof of Theorem 3.2 for the modifications in the remaining cases. So we introduce the map $F : X_1 \times X_1 \times (-1, \infty) \to X_1 \times X_1$ via

$$F(w_1, w_2, b) := \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} - \begin{pmatrix} \mathcal{R}_\mu^{\tau_1}\left[(1+b)\big((w_1+u_b)^3 - u_b^3\big) + (3-b)(w_1+u_b)w_2^2\right] \\ \mathcal{R}_\mu^{\omega}\left[(1+b)w_2^3 + (3-b)(w_1+u_b)^2 w_2\right] \end{pmatrix}.$$

Then $F(0,0,b) = 0$ for all $b > -1$, $F(\,\cdot\,, b)$ is a compact perturbation of the identity on $X_1 \times X_1$ and it remains to find bifurcation points for this equation. First we identify candidates for bifurcation points by computing those $b \in (-1, \infty)$ where $\ker DF(0,0,b)$ is nontrivial. Using

$$DF(0,0,b)[(\phi_1, \phi_2)] = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} - \begin{pmatrix} \mathcal{R}_\mu^{\tau_1}\left[3(1+b)u_b^2\phi_1\right] \\ \mathcal{R}_\mu^{\omega}\left[(3-b)u_b^2\phi_2\right] \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} - \begin{pmatrix} 3\mathbf{R}_\mu^{\tau_1}\phi_1 \\ \frac{3-b}{1+b} \cdot \mathbf{R}_\mu^{\omega}\phi_2 \end{pmatrix},$$

we get that nontrivial kernels occur exactly if $\frac{3-b}{1+b} = b_k(\omega)$ for some $k \in \mathbb{Z}$, cf. Step 1 in the previous proof. For the analogous result in the Schrödinger case we refer to Lemma 3.1 in [8]. So we find

$$\ker DF(0,0,b) = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ \psi_k \end{pmatrix} \right\} \quad \text{provided } b = \frac{3 - b_k(\omega)}{b_k(\omega) + 1} > -1$$

for some $\psi_k \in X_1 \setminus \{0\}$. Notice that the first component of the kernel element is zero by choice of τ_1, see Proposition 3.1. Using the algebraic simplicity of ψ_k proved in Proposition 3.22 we infer exactly as in the proof of Theorem 3.2 that the transversality condition holds and that the Leray-Schauder index changes at the bifurcation point.

So, choosing $\mathfrak{b}_k(\omega) := \frac{3-b_k(\omega)}{1+b_k(\omega)}$ for all $k \in \mathbb{Z}$ with $b_k(\omega) > -1$, the Crandall-Rabinowitz Theorem and Rabinowitz' Global Bifurcation Theorem yield statements (ii) and (i) of the Theorem, respectively, where $k_\omega \in \mathbb{Z}$ is the unique integer satisfying

$$b_{k_\omega}(\omega) > -1 \geq b_{k_\omega - 1}(\omega), \tag{3.31}$$

referring to the fact that the sequence $(b_k(\omega))_{k\in\mathbb{Z}}$ is strictly increasing, cf. Remark 3.3 (b).

Unboundedness of the components can also be deduced as before. Indeed, assuming that $\mathfrak{C}_k(\omega)$ is bounded, it returns to $\mathfrak{T}_{u_0}$ at some point $(u_{b^*}, u_{b^*}, b^*) \neq (u_{\mathfrak{b}_k(\omega)}, u_{\mathfrak{b}_k(\omega)}, \mathfrak{b}_k(\omega))$ by Rabinowitz' Theorem. We then infer that the phase $\omega_\nu((1+b)w_2^2 + (3-b)(w_1+u_b)^2)$ cannot be constant along $\mathfrak{C}_k(\omega)$. Due to Proposition 3.18 applied to w_2 in (3.29), this requires the existence of some element $(u,v,b) \in \mathfrak{C}_k(\omega) \setminus \mathfrak{T}_{u_0}$ with $w_2 = \frac{1}{2}(v-u) = 0$, and hence the associated unbounded diagonal family belongs to $\mathfrak{C}_k(\omega)$, contradicting the assumption of boundedness. $\square$

3.5 Proofs of the Auxiliary Results

3.5.1 Technical Results

Proof of Lemma 3.8

$\triangleright$ Step 1: *Completeness.*

Consider a Cauchy sequence $(w_n)_{n\in\mathbb{N}}$ in X_q. Then in particular, it is a Cauchy sequence in the complete space $\big(C_0(\mathbb{R}^3), \|\cdot\|_\infty\big)$ of continuous functions vanishing at infinity, and hence there exists $w \in C_0(\mathbb{R}^3)$ with $w_n \to w$ uniformly as $n \to \infty$. Then $w \in X_q$ since, $(w_n)_{n\in\mathbb{N}}$ being a Cauchy sequence and hence bounded, there exists $C > 0$ with

$$(1+|x|^2)^{\frac{q}{2}}|w_n(x)| \leq C \quad \text{for all } n \in \mathbb{N} \text{ and } x \in \mathbb{R}^3$$

and therefore $\|w\|_{X_q} \leq C$. Similarly, we even have $w_n \to w$ in X_q. Indeed, taking $\varepsilon > 0$, we can fix $n_0 \in \mathbb{N}$ with

$$(1+|x|^2)^{\frac{q}{2}}|w_n(x) - w_m(x)| < \varepsilon \quad \text{for all } n,m \geq n_0 \text{ and } x \in \mathbb{R}^3.$$

Letting $m \to \infty$ in this estimate, we deduce $\|w_n - w\|_{X_q} \leq \varepsilon$ for all $n \geq n_0$.

$\triangleright$ Step 2: *Continuous embeddings.*

We assume $f \in X_q$ and $p \in [1,\infty]$ with $pq > 3$. The case $p = \infty$ is obvious since $\|f\|_{L^\infty(\mathbb{R}^3)} \leq \|f\|_{X_q}$ holds for any $q \geq 1$; otherwise we estimate

$$\|f\|^p_{L^p(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |f(x)|^p \,\mathrm{d}x \leq \int_{\mathbb{R}^3} \frac{\|f\|^p_{X_q}}{(1+|x|^2)^{\frac{pq}{2}}} \,\mathrm{d}x = \|f\|^p_{X_q} \cdot \int_0^\infty \frac{4\pi\, r^2}{(1+r^2)^{\frac{pq}{2}}} \,\mathrm{d}r < \infty.$$

If, on the other hand, $pq \leq 3$, the function $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x) := (1+|x|^2)^{-\frac{q}{2}}$ belongs to X_q but not to $L^p_{\mathrm{rad}}(\mathbb{R}^3)$. $\square$

Proof of Lemma 3.10

First, we let $f \in \mathcal{S}(\mathbb{R}^3,\mathbb{C})$ and $x \neq 0$; then $\mathfrak{R}_\lambda f = \Phi_\lambda * f$. Observing that the singular function Φ_λ is locally integrable in $\mathbb{R}^3$, the convolution is well-defined and we compute using spherical coordinates with respect to the x direction

$$\begin{aligned}
\mathfrak{R}_\lambda[f](x) &= (\Phi_\lambda * f)(x) \\
&= \int_{\mathbb{R}^3} \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}|x-y|}}{4\pi|x-y|} f(y) \,\mathrm{d}y \\
&= \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}\sqrt{|x|^2+r^2-2|x|r\cos(\vartheta)}}}{4\pi\sqrt{|x|^2+r^2-2|x|r\cos(\vartheta)}} f(r)\, r^2 \sin(\vartheta) \,\mathrm{d}\varphi \mathrm{d}\vartheta \mathrm{d}r \\
&= \int_0^\infty \left[\frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}\sqrt{|x|^2+r^2-2|x|r\cos(\vartheta)}}}{2\mathrm{i}\sqrt{\lambda}|x|r} \right]_{\vartheta=0}^{\vartheta=\pi} f(r)\, r^2 \,\mathrm{d}r
\end{aligned}$$

$$= \int_0^\infty \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}||x|+r|} - \mathrm{e}^{\mathrm{i}\sqrt{\lambda}||x|-r|}}{2\mathrm{i}\sqrt{\lambda}|x|r} \cdot f(r)\, r^2 \,\mathrm{d}r$$
$$= \int_0^{|x|} \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}|x|}\sin(\sqrt{\lambda}r)}{\sqrt{\lambda}|x|r} \cdot f(r)\, r^2 \,\mathrm{d}r + \int_{|x|}^\infty \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}r}\sin(\sqrt{\lambda}|x|)}{\sqrt{\lambda}|x|r} \cdot f(r)\, r^2 \,\mathrm{d}r. \quad (3.32)$$

When combined with formula (3.8), we can conclude as claimed

$$\mathfrak{R}_\lambda[f](x) = \sqrt{\frac{\pi}{2}}\, \hat{f}(\sqrt{\lambda})\frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}|x|}}{|x|} + \int_{|x|}^\infty f(r)\frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}r}\sin(\sqrt{\lambda}|x|) - \mathrm{e}^{\mathrm{i}\sqrt{\lambda}|x|}\sin(\sqrt{\lambda}r)}{\sqrt{\lambda}r|x|}\, r^2 \mathrm{d}r \quad (3.33)$$

for a Schwartz function f. More generally, given $f \in L^{\frac{4}{3}}_{\mathrm{rad}}(\mathbb{R}^3, \mathbb{C})$, the formulas extend for almost all $x \in \mathbb{R}^3$ by continuous extension. Indeed, we use here continuity of $\mathfrak{R}_\lambda : L^{\frac{4}{3}}_{\mathrm{rad}}(\mathbb{R}^3, \mathbb{C}) \to L^4_{\mathrm{rad}}(\mathbb{R}^3, \mathbb{C})$ guaranteed by the Limiting Absorption Principle in Remark 1.10 (b), and the fact that, for every fixed $x \neq 0$, the right-hand sides of equations (3.32), (3.8) resp. (3.33) define continuous linear functionals on $L^{\frac{4}{3}}_{\mathrm{rad}}(\mathbb{R}^3, \mathbb{C})$. □

Proof of Lemmas 3.11, 3.12

We only prove the former, the latter can be shown in exactly the same way.

▷ Step 1: *Regularity.*

We iteratively apply a regularity result of Zhang and Bao [76]. By assumption, $u, v \in L^4(\mathbb{R}^3) \subseteq L^1_{\mathrm{loc}}(\mathbb{R}^3)$ are distributional solutions of

$$-\Delta u = f,\ -\Delta v = g \quad \text{on } \mathbb{R}^3 \quad (3.34)$$
$$\text{where} \quad f = \mu u + u^3 + b\, uv^2,\ g = \nu v + v^3 + b\, vu^2 \in L^{\frac{4}{3}}_{\mathrm{loc}}(\mathbb{R}^3).$$

Thus Proposition 1.1 in [76] implies that $u, v \in W^{2,\frac{4}{3}}_{\mathrm{loc}}(\mathbb{R}^3)$. Sobolev embedding, cf. Theorem 4.12 in [3], now gives $u, v \in L^{12}_{\mathrm{loc}}(\mathbb{R}^3)$ and hence $f, g \in L^4_{\mathrm{loc}}(\mathbb{R}^3)$. As above, this implies $u, v \in W^{2,4}_{\mathrm{loc}}(\mathbb{R}^3)$ and in particular, due to Sobolev embedding, $u, v \in L^\infty_{\mathrm{loc}}(\mathbb{R}^3)$. Thus $f, g \in L^\infty_{\mathrm{loc}}(\mathbb{R}^3)$, and a final application of the regularity result by Zhang and Bao ensures $u, v \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^3)$ for all $q \in [1, \infty)$; thus by the product rule $f, g \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^3)$ for all $q \in [1, \infty)$.

We pass to higher-order differentiability by means of elliptic regularity, see e.g. Theorem 9.19 in [31], which guarantees $u, v \in W^{4,q}_{\mathrm{loc}}(\mathbb{R}^3)$ for all $q \in [1, \infty)$. Then again, since we have integrability of arbitrary order $q < \infty$, the product rule gives $f, g \in W^{4,q}_{\mathrm{loc}}(\mathbb{R}^3)$ for all $q \in [1, \infty)$. This procedure can be iterated; finally $u, v \in W^{k,q}_{\mathrm{loc}}(\mathbb{R}^3)$ for all $k \in \mathbb{N}_0$ and $q \in [1, \infty)$. Sobolev embedding then guarantees that u, v are, in particular, classical solutions of class C^∞.

The ODE satisfied by the profiles can thus be obtained by a straightforward calculation.

▷ Step 2: *Pointwise decay.*

To analyze the decay of u, we intend to make use of the asymptotic expansion in Lemma 3.10. To this end, we observe that by assumption $\tilde{f} := (u^2 + b\, v^2)u \in L^{\frac{4}{3}}_{\mathrm{rad}}(\mathbb{R}^3)$. Hence, with Gutiérrez' Limiting Absorption Principle in Theorem 1.9 (i) resp. Remark 1.10 (b), the function

$$\tilde{u} := \mathcal{R}_\mu[\tilde{f}] = \operatorname{Re}\Big(\mathfrak{R}_\mu[\tilde{f}]\Big) \in L^4_{\mathrm{rad}}(\mathbb{R}^3) \quad \Big(\text{with} \quad \tilde{f} = (u^2 + b\,v^2)u\Big)$$

is a distributional solution of $-\Delta\tilde{u} - \mu\tilde{u} = (u^2 + b\,v^2)u$ on $\mathbb{R}^3$. Exploiting the smoothness and integrability properties of the right-hand side, and arguing as above, we infer that $\tilde{u}$ is in fact smooth and a classical solution. Thus

$$(-\Delta - \mu)(u - \tilde{u}) = 0 \quad \text{on } \mathbb{R}^3 \quad \text{with } u - \tilde{u} \in L^4_{\mathrm{rad}}(\mathbb{R}^3) \cap C^2_{\mathrm{loc}}(\mathbb{R}^3).$$

A direct calculation shows that $u(x) - \tilde{u}(x) = c \cdot \frac{\sin(|x|\sqrt{\mu})}{|x|}$ for some $c \in \mathbb{R}$, hence $u - \tilde{u} \in X_1$. Moreover $\tilde{u} \in X_1$: Having already seen that $\tilde{u}$ is continuous and hence in particular locally bounded, it is sufficient to show that

$$\sup_{|x|\geq 1} \left|(1 + |x|^2)^{\frac{1}{2}}\tilde{u}(x)\right| < \infty.$$

This can be derived from the expansion in Lemma 3.10. We have for $|x| \geq 1$, using Hölder's inequality,

$$\begin{aligned}
&\left|(1 + |x|^2)^{\frac{1}{2}}\tilde{u}(x)\right| \\
&= \left|(1 + |x|^2)^{\frac{1}{2}} \operatorname{Re}\mathfrak{R}_\mu[\tilde{f}](x)\right| \\
&\leq \left|(1 + |x|^2)^{\frac{1}{2}} \mathfrak{R}_\mu[\tilde{f}](x)\right| \\
&\leq \frac{(1 + |x|^2)^{\frac{1}{2}}}{|x|} \cdot \int_0^{|x|} \left|\frac{\sin(\sqrt{\lambda}r)}{\sqrt{\lambda}r} \cdot \tilde{f}(r)\right| r^2 \,\mathrm{d}r + \frac{(1 + |x|^2)^{\frac{1}{2}}}{|x|} \cdot \int_{|x|}^\infty \left|\frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}r}}{\sqrt{\lambda}r} \cdot \tilde{f}(r)\right| r^2 \,\mathrm{d}r \\
&\leq \sqrt{2} \int_0^{|x|} \left|\frac{\sin(\sqrt{\lambda}r)}{\sqrt{\lambda}r} \cdot \tilde{f}(r)\right| r^2 \,\mathrm{d}r + \sqrt{2} \int_{|x|}^\infty \frac{|\tilde{f}(r)|}{\sqrt{\lambda}r} r^2 \,\mathrm{d}r \\
&\leq \sqrt{2} \left[\left(\int_0^\infty \left|\frac{\sin(\sqrt{\lambda}r)}{\sqrt{\lambda}r}\right|^4 r^2 \,\mathrm{d}r\right)^{\frac{1}{4}} + \left(\int_1^\infty \frac{\mathrm{d}r}{\lambda^2 r^2}\right)^{\frac{1}{4}}\right] \cdot \left(\int_0^\infty |\tilde{f}(r)|^{\frac{4}{3}} r^2 \,\mathrm{d}r\right)^{\frac{3}{4}},
\end{aligned}$$

and this expression is bounded since $\tilde{f} \in L^{\frac{4}{3}}_{\mathrm{rad}}(\mathbb{R}^3)$. Hence $\tilde{u} \in X_1$; we conclude $u = \tilde{u} + (u - \tilde{u}) \in X_1$. Likewise, $v \in X_1$. □

3.5.2 Results on the Scalar Problem

Proof of Proposition 3.13

We now prove the assertions of Proposition 3.13 for convolutions with $\Phi_\lambda = \Psi_\lambda + \mathrm{i}\,\tilde{\Psi}_\lambda$ in place of Ψ_λ resp. $\tilde{\Psi}_\lambda$. The latter (real-valued) case can be deduced from the former by taking the real resp. imaginary part of

$$\mathfrak{R}_\lambda[f] = \mathcal{R}_\lambda[f] + \mathrm{i}\,\tilde{\mathcal{R}}_\lambda[f]$$

since we throughout assume $f \in X_3$ to be real-valued. In general, $w = \mathfrak{R}_\lambda[f]$ will take complex values; we thus tacitly assume in this proof that X_1 is extended to complex-valued functions. We consider $f \in X_3$ and introduce

$$\begin{aligned} &u := \mathfrak{R}_\lambda[f] = \Phi_\lambda * f, \\ &w := \mathcal{R}_\lambda[f] = \Psi_\lambda * f = \operatorname{Re} u \quad \text{and} \quad \tilde{w} := \tilde{\mathcal{R}}_\lambda[f] = \tilde{\Psi}_\lambda * f = \operatorname{Im} u. \end{aligned} \tag{3.35}$$

▷ STEP 1: *Proof of (ii), solution properties.*

We show that $u \in C^2_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C})$ with $-\Delta u - \lambda u = f$ in $\mathbb{R}^3$; then, since f takes values in $\mathbb{R}$, we infer as asserted from (3.35)

$$-\Delta w - \lambda w = f \quad \text{and} \quad -\Delta \tilde{w} - \lambda \tilde{w} = 0.$$

By Lemma 3.10, we have for $x \in \mathbb{R}^3 \setminus \{0\}$

$$u(x) = \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}|x|}}{|x|} \cdot \int_0^{|x|} \frac{\sin(\sqrt{\lambda} r)}{\sqrt{\lambda} r} \cdot f(r)\, r^2 \,\mathrm{d}r + \frac{\sin(\sqrt{\lambda}|x|)}{\sqrt{\lambda}|x|} \cdot \int_{|x|}^\infty \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda} r}}{r} \cdot f(r)\, r^2 \,\mathrm{d}r. \tag{3.36}$$

On $\mathbb{R}^3 \setminus \{0\}$, twice continuous differentiability is a consequence of the Fundamental Theorem of Calculus and of the Chain Rule, and a direct (but lengthy) calculation confirms $-\Delta u(x) - \lambda u(x) = f(x)$ for $x \in \mathbb{R}^3 \setminus \{0\}$. In particular, we will frequently need the explicit formula for the radial derivative $u' := \partial_r u$ at some $x \neq 0$:

$$\begin{aligned} u'(x) = &\left(\mathrm{i} \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}|x|}}{|x|} - \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}|x|}}{\sqrt{\lambda}|x|^2} \right) \cdot \int_0^{|x|} \frac{\sin(\sqrt{\lambda} r)}{r} \cdot f(r)\, r^2 \,\mathrm{d}r \\ &+ \left(\frac{\cos(\sqrt{\lambda}|x|)}{|x|} - \frac{\sin(\sqrt{\lambda}|x|)}{\sqrt{\lambda}|x|^2} \right) \cdot \int_{|x|}^\infty \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda} r}}{r} \cdot f(r)\, r^2 \,\mathrm{d}r. \end{aligned} \tag{3.37}$$

At the point $x = 0$, continuity of f and the mean value theorem for definite integrals provide the expansions

$$\begin{aligned} &\int_0^{|h|} \frac{\sin(\sqrt{\lambda} r)}{\sqrt{\lambda} r} \cdot f(r)\, r^2 \,\mathrm{d}r = \frac{1}{3}|h|^3 \cdot f(0) + o(|h|^3), \\ &\int_{|h|}^\infty \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda} r}}{r} \cdot f(r)\, r^2 \,\mathrm{d}r = u(0) - \int_0^{|h|} \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda} r}}{r} \cdot f(r)\, r^2 \,\mathrm{d}r = u(0) - \frac{1}{2}|h|^2 \cdot f(0) + o(|h|^2) \end{aligned}$$

as $|h| \searrow 0$. Then, inserting these into the expressions for $u(h), \partial_r u(h)$ obtained from equations (3.36), (3.37) for $h \neq 0$, a short calculation yields as $|h| \searrow 0$

$$\begin{aligned} u(h) &= u(0) + \frac{1}{2}|h|^2 \cdot \frac{1}{3}\left(-\lambda u(0) - f(0)\right) + o(|h|^2), \\ \nabla u(h) &= \frac{h}{|h|}\, \partial_r u(h) = h \cdot \frac{1}{3}(-\lambda u(0) - f(0)) + o(|h|). \end{aligned}$$

We conclude that u is twice differentiable at the origin with

$$\nabla u(0) = 0, \qquad D^2 u(0) = \frac{1}{3}\left(-\lambda u(0) - f(0)\right) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and hence $-\Delta u(0) = -\operatorname{tr} D^2 u(0) = \lambda u(0) + f(0)$. We omit the proof of continuity of the second derivative at 0, which can be immediately verified using the same expansions.

▷ Step 2: *Proof of (i), first part. Continuity.*

Using Young's convolution inequality, we first prove boundedness:

$$\begin{aligned}
|u(x)| &\le \left\|\left(\mathbb{1}_{B_1(0)}\Phi_\lambda\right) * f\right\|_{L^\infty(\mathbb{R}^3)} + \left\|\left(\mathbb{1}_{\mathbb{R}^3\setminus B_1(0)}\Phi_\lambda\right) * f\right\|_{L^\infty(\mathbb{R}^3)} \\
&\le \left\|\mathbb{1}_{B_1(0)}\Phi_\lambda\right\|_{L^1(\mathbb{R}^3)} \|f\|_{L^\infty(\mathbb{R}^3)} + \left\|\mathbb{1}_{\mathbb{R}^3\setminus B_1(0)}\Phi_\lambda\right\|_{L^4(\mathbb{R}^3)} \|f\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \\
&\le \|f\|_{L^\infty(\mathbb{R}^3)} \int_{B_1(0)} \frac{\mathrm{d}y}{4\pi|y|} + \|f\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \left(\int_{\mathbb{R}^3\setminus B_1(0)} \frac{\mathrm{d}y}{(4\pi|y|)^4}\right)^{\frac{1}{4}} \\
&= \frac{1}{2}\|f\|_{L^\infty(\mathbb{R}^3)} + (4\pi)^{-\frac{3}{4}} \|f\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \\
&\le \frac{1}{2}\|f\|_{X_3} + (4\pi)^{-\frac{3}{4}} \left(\int_{\mathbb{R}^3} \frac{\|f\|_{X_3}^{\frac{4}{3}}}{(1+|y|^2)^2}\,\mathrm{d}y\right)^{\frac{3}{4}} \\
&\le \left(\frac{1}{2} + \left(\int_0^\infty \frac{1}{1+r^2}\,\mathrm{d}r\right)^{\frac{3}{4}}\right)\|f\|_{X_3} \\
&\le \frac{1+\pi}{2}\cdot\|f\|_{X_3}.
\end{aligned}$$

Next, by means of Lemma 3.10, we estimate for $x \in \mathbb{R}^3\setminus\{0\}$ in the weighted norm

$$\begin{aligned}
||x|\cdot u(x)| &= \left|\mathrm{e}^{\mathrm{i}\sqrt{\lambda}|x|}\cdot\int_0^{|x|}\frac{\sin(\sqrt{\lambda}r)}{\sqrt{\lambda}r}\,f(r)\,r^2\,\mathrm{d}r + \sin(\sqrt{\lambda}|x|)\cdot\int_{|x|}^\infty\frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}r}}{\sqrt{\lambda}r}\,f(r)\,r^2\,\mathrm{d}r\right| \\
&\le \int_0^{|x|}\frac{1}{\sqrt{\lambda}r}\cdot\frac{\|f\|_{X_3}}{(1+r^2)^{\frac{3}{2}}}\,r^2\,\mathrm{d}r + \int_{|x|}^\infty\frac{1}{\sqrt{\lambda}r}\cdot\frac{\|f\|_{X_3}}{(1+r^2)^{\frac{3}{2}}}\,r^2\,\mathrm{d}r \\
&\le \frac{1}{\sqrt{\lambda}}\cdot\|f\|_{X_3}\cdot\int_0^\infty\frac{\mathrm{d}r}{1+r^2} \\
&= \frac{\pi}{2\sqrt{\lambda}}\cdot\|f\|_{X_3}.
\end{aligned}$$

Combining both estimates, we have shown that

$$\|u\|_{X_1} = \sup_{x\in\mathbb{R}^3}\sqrt{1+|x|^2}\,|u(x)| \le \left[\frac{1+\pi}{2}+\frac{\pi}{2\sqrt{\lambda}}\right]\cdot\|f\|_{X_3}. \tag{3.38}$$

▷ Step 3: *Proof of (iii). Asymptotics of u and u'.*

Here again, we frequently identify (radially symmetric) functions with their profiles. For $f \in X_3$ and $r = |x| > 0$, Lemma 3.10 implies

$$\begin{aligned}
\left|u(r) - \sqrt{\frac{\pi}{2}}\,\hat{f}(\sqrt{\lambda})\frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}r}}{r}\right| &= \left|\int_r^\infty f(s)\frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}s}\sin(\sqrt{\lambda}r) - \mathrm{e}^{\mathrm{i}\sqrt{\lambda}r}\sin(\sqrt{\lambda}s)}{\sqrt{\lambda}sr}\,s^2\mathrm{d}s\right| \\
&\le \int_r^\infty\frac{\|f\|_{X_3}}{(1+s^2)^{\frac{3}{2}}}\cdot\frac{2}{\sqrt{\lambda}r}\,s\mathrm{d}s \\
&\le \|f\|_{X_3}\cdot\frac{2}{\sqrt{\lambda}r}\int_r^\infty\frac{1}{s^2}\,\mathrm{d}s \\
&= \|f\|_{X_3}\cdot\frac{2}{\sqrt{\lambda}r^2}.
\end{aligned} \tag{3.39}$$

Thus, using (3.35) and $\hat{f}(\sqrt{\lambda}) \in \mathbb{R}$ (see (3.8)),

$$\left| w(r) - \sqrt{\frac{\pi}{2}}\,\hat{f}(\sqrt{\lambda})\frac{\cos(\sqrt{\lambda}r)}{r} \right| \leq \|f\|_{X_3} \cdot \frac{2}{\sqrt{\lambda}r^2},$$
$$\left| \tilde{w}(r) - \sqrt{\frac{\pi}{2}}\,\hat{f}(\sqrt{\lambda})\frac{\sin(\sqrt{\lambda}r)}{r} \right| \leq \|f\|_{X_3} \cdot \frac{2}{\sqrt{\lambda}r^2}.$$

We deduce the formula stated for $\tilde{w} = \tilde{\Psi}_\lambda * f$. Due to (ii), $\tilde{w}$ is a radial solution of the homogeneous Helmholtz equation $-\Delta\tilde{w} - \lambda\tilde{w} = 0$ on $\mathbb{R}^3$ of class C^2 and therefore a scalar multiple of $\tilde{\Psi}_\lambda$ itself. Hence in the asymptotic expansion just proved only the leading-order term occurs, i.e. $\tilde{w}(r) = \sqrt{\frac{\pi}{2}}\,\hat{f}(\sqrt{\lambda})\frac{\sin(\sqrt{\lambda}r)}{r}$.

Differentiating the formula in Lemma 3.10 (as in (3.37)), the radial derivative $u' := \partial_r u$ is for $|x| = r > 0$ given by

$$u'(r) = \sqrt{\frac{\pi}{2}}\,\hat{f}(\sqrt{\lambda}) \cdot \left(\mathrm{i}\sqrt{\lambda}\frac{e^{\mathrm{i}\sqrt{\lambda}r}}{r} - \frac{e^{\mathrm{i}\sqrt{\lambda}r}}{r^2} \right)$$
$$+ \int_r^\infty f(s) \cdot \left[e^{\mathrm{i}\sqrt{\lambda}s} \left(\frac{\cos(\sqrt{\lambda}r)}{r} - \frac{\sin(\sqrt{\lambda}r)}{\sqrt{\lambda}r^2} \right) - \sin(\sqrt{\lambda}s) \left(\frac{\mathrm{i}e^{\mathrm{i}\sqrt{\lambda}r}}{r} - \frac{e^{\mathrm{i}\sqrt{\lambda}r}}{\sqrt{\lambda}r^2} \right) \right] s\mathrm{d}s.$$

Further, we notice that the identity (3.8) yields

$$|\hat{f}(\sqrt{\lambda})| \leq \sqrt{\frac{2}{\pi\lambda}} \int_0^\infty |f(r)\sin(r\sqrt{\lambda})|\, r\,\mathrm{d}r \leq \sqrt{\frac{2}{\pi\lambda}} \int_0^\infty \frac{\|f\|_{X_3}}{1+r^2}\,\mathrm{d}r \leq \sqrt{\frac{\pi}{2\lambda}}\,\|f\|_{X_3} \qquad (3.40)$$

and conclude

$$\left| u'(r) - \mathrm{i}\sqrt{\lambda} \cdot \sqrt{\frac{\pi}{2}}\,\hat{f}(\sqrt{\lambda})\,\frac{e^{\mathrm{i}\sqrt{\lambda}r}}{r} \right|$$
$$\leq \sqrt{\frac{\pi}{2}}\,|\hat{f}(\sqrt{\lambda})| \cdot \frac{1}{r^2} + \int_r^\infty \frac{\|f\|_{X_3}}{s^2}\,\mathrm{d}s \cdot \left[\left(\frac{1}{r} + \frac{1}{\sqrt{\lambda}r^2} \right) + \left(\frac{1}{r} + \frac{1}{\sqrt{\lambda}r^2} \right) \right]$$
$$\leq \|f\|_{X_3} \cdot \frac{1}{r^2} \cdot \left[\frac{\pi}{2}\frac{1}{\sqrt{\lambda}} + 2\left(1 + \frac{1}{\sqrt{\lambda}r}\right) \right].$$

As in the previous step, we infer by passing to the real resp. to the imaginary part

$$\left| w'(r) + \sqrt{\lambda} \cdot \sqrt{\frac{\pi}{2}}\,\hat{f}(\sqrt{\lambda})\frac{\sin(\sqrt{\lambda}r)}{r} \right| \leq \|f\|_{X_3} \cdot \frac{1}{r^2} \cdot \left[\frac{\pi}{2\sqrt{\lambda}} + 2\left(1 + \frac{1}{\sqrt{\lambda}r}\right) \right],$$
$$\left| \tilde{w}'(r) - \sqrt{\lambda} \cdot \sqrt{\frac{\pi}{2}}\,\hat{f}(\sqrt{\lambda})\frac{\cos(\sqrt{\lambda}r)}{r} \right| \leq \|f\|_{X_3} \cdot \frac{1}{r^2} \cdot \left[\frac{\pi}{2\sqrt{\lambda}} + 2\left(1 + \frac{1}{\sqrt{\lambda}r}\right) \right].$$

▷ <u>Step 4</u>: *Proof of (i), second part. Compactness.*

We consider a bounded sequence $(f_n)_n$ in the space X_3 and aim to prove convergence of a subsequence of $(u_n)_n$ where $u_n := \Phi_\lambda * f_n$ in the space X_1.

We denote $C_* := \sup_{n\in\mathbb{N}} \|f_n\|_{X_3}$. Thanks to the estimate (3.38), we conclude

$$\sup_{n\in\mathbb{N}} \|u_n\|_{X_1} \leq \|\mathfrak{R}_\lambda\|_{\mathcal{L}(X_3,X_1)} \sup_{n\in\mathbb{N}} \|f_n\|_{X_3} \leq \left[\frac{1+\pi}{2} + \frac{\pi}{2\sqrt{\lambda}} \right] C_*$$

and thus, in particular, the sequence $(u_n)_n$ is pointwise bounded. As we will verify below using (3.37), it is also equicontinuous, and thus the Theorem of Arzelà-Ascoli can be applied. In combination with a suitable diagonalization technique this provides a subsequence $(u_{n_k})_k$ which converges locally uniformly to some radially symmetric, continuous function $u : \mathbb{R}^3 \to \mathbb{R}$. In order to improve this to a global convergence result in X_1, we combine the local compactness statement with the uniform asymptotic estimate (3.39). We observe that the estimate (3.40) implies that $(\hat{f}_{n_k}(\sqrt{\lambda}))_k$ is a bounded sequence in $\mathbb{C}$ since $(f_n)_n$ is bounded in X_3; hence without loss of generality we may assume that the subsequence is chosen in such way that $(\hat{f}_{n_k}(\sqrt{\lambda}))_k$ converges as $k \to \infty$.

With that, we will prove that $(u_{n_k})_k$ is a Cauchy sequence in X_1. We let $\varepsilon > 0$ and choose

$$R := \max\left\{ \frac{8\sqrt{2}}{\varepsilon\sqrt{\lambda}} \cdot C_*, 1 \right\}. \tag{3.41}$$

Then due to the local compactness result, we can choose $k_1(\varepsilon) \in \mathbb{N}$ with

$$\sup_{|x|\leq R} (1+|x|^2)^{\frac{1}{2}} |u_{n_k}(x) - u_{n_l}(x)| < \varepsilon \qquad \text{for all } k, l \geq k_1(\varepsilon). \tag{3.42}$$

Convergence of $(\hat{f}_{n_k}(\sqrt{\lambda}))_k$ in $\mathbb{C}$ provides $k_2(\varepsilon) \in \mathbb{N}$ with the property that

$$\sqrt{\pi}|\hat{f}_{n_k}(\sqrt{\lambda}) - \hat{f}_{n_l}(\sqrt{\lambda})| < \frac{\varepsilon}{2} \qquad \text{for all } k, l \geq k_2(\varepsilon). \tag{3.43}$$

Since $f_{n_k} \in X_3$, we estimate for $|x| > R \geq 1$ and $k, l \geq k_2(\varepsilon)$ using the asymptotic estimate (3.39) for u_{n_k} in Step 3

$$\begin{aligned}
&(1+|x|^2)^{\frac{1}{2}} |u_{n_k}(x) - u_{n_l}(x)| \\
&\overset{(3.39)}{\leq} \sqrt{\frac{\pi}{2}} |\hat{f}_{n_k}(\sqrt{\lambda}) - \hat{f}_{n_l}(\sqrt{\lambda})| \frac{(1+|x|^2)^{\frac{1}{2}}}{|x|} + \frac{2(1+|x|^2)^{\frac{1}{2}}}{\sqrt{\lambda}|x|^2} \cdot \|f_{n_k} - f_{n_l}\|_{X_3} \\
&\leq \sqrt{\pi}|\hat{f}_{n_k}(\sqrt{\lambda}) - \hat{f}_{n_l}(\sqrt{\lambda})| + \frac{2\sqrt{2}}{\sqrt{\lambda}R} \cdot 2C_* \\
&\overset{(3.41),(3.43)}{<} \varepsilon.
\end{aligned}$$

Combining this with (3.42), we have

$$\|u_{n_k} - u_{n_l}\|_{X_1} = \sup_{x\in\mathbb{R}^3} (1+|x|^2)^{\frac{1}{2}} |u_{n_k}(x) - u_{n_l}(x)| < \varepsilon \qquad \text{for all } k, l \geq \max\{k_1(\varepsilon), k_2(\varepsilon)\}.$$

Hence $(u_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence in X_1, which implies $u_{n_k} \to u$ strongly in X_1.

It remains to verify the equicontinuity of the sequence $(u_n)_n$. This is a consequence of a uniform bound on the radial derivatives u_n'; indeed, starting from equation (3.37), we estimate for $n \in \mathbb{N}$ and $r > 0$

$$\begin{aligned}
&|u_n'(r)| \\
&\leq \left| \mathrm{i}\frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}r}}{r} - \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}r}}{\sqrt{\lambda}r^2} \right| \int_0^r t|\sin(\sqrt{\lambda}t) f_n(t)| \,\mathrm{d}t + \left| \frac{\cos(\sqrt{\lambda}r)}{r} - \frac{\sin(\sqrt{\lambda}r)}{\sqrt{\lambda}r^2} \right| \int_r^\infty t|\mathrm{e}^{\mathrm{i}\sqrt{\lambda}t} f_n(t)| \,\mathrm{d}t \\
&\leq \frac{\sqrt{\lambda}r + 1}{\sqrt{\lambda}r^2} \int_0^r \frac{C_* \min\{\sqrt{\lambda}t^2, t\}}{(1+t^2)^{\frac{3}{2}}} \,\mathrm{d}t + \frac{\left|\sqrt{\lambda}r\cos(\sqrt{\lambda}r) - \sin(\sqrt{\lambda}r)\right|}{\sqrt{\lambda}r^2} \int_r^\infty \frac{C_* t}{(1+t^2)^{\frac{3}{2}}} \,\mathrm{d}t
\end{aligned}$$

$$\leq \left[\frac{\sqrt{\lambda}r+1}{\sqrt{\lambda}r^2}\min\{\sqrt{\lambda}r^2, r\} + \frac{\left|\sqrt{\lambda}r\cos(\sqrt{\lambda}r) - \sin(\sqrt{\lambda}r)\right|}{\sqrt{\lambda}r^2}\right]\int_0^\infty \frac{C_*}{1+t^2}\,\mathrm{d}t$$
$$\leq \left[\min\left\{\sqrt{\lambda}r+1, \frac{1}{\sqrt{\lambda}r}+1\right\} + \frac{\left|\sqrt{\lambda}r\cos(\sqrt{\lambda}r) - \sin(\sqrt{\lambda}r)\right|}{\sqrt{\lambda}r^2}\right]\frac{\pi}{2}\,C_*.$$

The terms in brackets are continuous on $(0,\infty)$ with

$$\left.\begin{aligned} &\min\left\{\sqrt{\lambda}r+1, \frac{1}{\sqrt{\lambda}r}+1\right\} \to 1 \\ &\frac{\sqrt{\lambda}r\cos(\sqrt{\lambda}r) - \sin(\sqrt{\lambda}r)}{\sqrt{\lambda}r^2} \to 0 \end{aligned}\right. \qquad \text{as } r\to 0 \text{ or as } r\to\infty.$$

This shows $\sup_{n\in\mathbb{N}}\sup_{r>0}|u_n'(r)| = \sup_{n\in\mathbb{N}}\sup_{x\in\mathbb{R}^3}|\nabla u_n(x)| < \infty$, hence $(u_n)_n$ is equicontinuous and the proof is complete. □

Proof of Remark 3.14

It is sufficient to study some $\varepsilon\in(0,1]$. We show that $\mathfrak{R}_\lambda : X_{2+\varepsilon}\to X_1$ is well-defined, continuous and compact by modifying the respective steps of the proof of Proposition 3.13, which discusses the case $\varepsilon=1$. We introduce the conjugate exponents

$$p_\varepsilon := \frac{3}{1-\varepsilon/4}\in(3,4] \quad\text{and}\quad p_\varepsilon' = \frac{3}{2+\varepsilon/4}\in\left[\frac{4}{3},\frac{3}{2}\right).$$

(Taking $\varepsilon/4$ in the denominator is not the only option here but yields $p_1 = 4$ as in Proposition 3.13.) By Lemma 3.8, the embedding $X_{2+\varepsilon}\hookrightarrow L^{p_\varepsilon'}_{\mathrm{rad}}(\mathbb{R}^3)$ is continuous since $(2+\varepsilon)\cdot p_\varepsilon' > 3$; we denote the associated embedding constant by D_ε. As for $\varepsilon=1$, the formulas in Lemma 3.10 extend to $f\in L^{p_\varepsilon'}_{\mathrm{rad}}(\mathbb{R}^3)$.

▷ Step 1: *Continuity.*

Proceeding exactly as in Step 2, we then find for $x\in\mathbb{R}^3$

$$\begin{aligned}
|u(x)| &\leq \left\|\mathbb{1}_{B_1(0)}\Phi_\lambda\right\|_{L^1(\mathbb{R}^3)}\|f\|_{L^\infty(\mathbb{R}^3)} + \left\|\mathbb{1}_{\mathbb{R}^3\setminus B_1(0)}\Phi_\lambda\right\|_{L^{p_\varepsilon}(\mathbb{R}^3)}\|f\|_{L^{p_\varepsilon'}(\mathbb{R}^3)} \\
&\leq \frac{1}{2}\|f\|_{L^\infty(\mathbb{R}^3)} + (p_\varepsilon-3)^{-\frac{1}{p_\varepsilon}}(4\pi)^{-\frac{1}{p_\varepsilon'}}\|f\|_{L^{p_\varepsilon'}(\mathbb{R}^3)} \\
&\leq \left(\frac{1}{2} + D_\varepsilon(p_\varepsilon-3)^{-\frac{1}{p_\varepsilon}}(4\pi)^{-\frac{1}{p_\varepsilon'}}\right)\|f\|_{X_{2+\varepsilon}}\,; \\
||x|\cdot u(x)| &= \left|\mathrm{e}^{\mathrm{i}\sqrt{\lambda}|x|}\cdot\int_0^{|x|}\frac{\sin(\sqrt{\lambda}r)}{\sqrt{\lambda}r}\,f(r)\,r^2\,\mathrm{d}r + \sin(\sqrt{\lambda}|x|)\cdot\int_{|x|}^\infty\frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}r}}{\sqrt{\lambda}r}\,f(r)\,r^2\,\mathrm{d}r\right| \\
&\leq \int_0^{|x|}\frac{1}{\sqrt{\lambda}r}\cdot\frac{\|f\|_{X_{2+\varepsilon}}}{(1+r^2)^{1+\varepsilon/2}}\,r^2\,\mathrm{d}r + \int_{|x|}^\infty\frac{1}{\sqrt{\lambda}r}\cdot\frac{\|f\|_{X_{2+\varepsilon}}}{(1+r^2)^{1+\varepsilon/2}}\,r^2\,\mathrm{d}r \\
&\leq \frac{1}{\sqrt{\lambda}}\cdot\|f\|_{X_{2+\varepsilon}}\cdot\int_0^\infty\frac{\mathrm{d}r}{(1+r^2)^{1/2+\varepsilon/2}}
\end{aligned}$$

and hence there exists a constant $C_\varepsilon>0$ independent of f,u with

$$\|u\|_{X_1} = \sup_{x\in\mathbb{R}^3}\sqrt{1+|x|^2}\,|u(x)| \leq C_\varepsilon\,\|f\|_{X_{2+\varepsilon}}. \tag{3.44}$$

▷ <u>Step 2</u>: *Compactness.*

Next, we show how to generalize the proof of compactness. We consider a sequence $(f_n)_n$ of continuous radial functions with

$$C_*(\varepsilon) := \sup_{n\in\mathbb{N}} \|f_n\|_{X_{2+\varepsilon}} < \infty$$

and let $u_n := \mathfrak{R}_\lambda[f_n] \in X_1$. In order to prove the local compactness result derived from the Arzelà-Ascoli Theorem, we have to ensure that we still obtain a uniform estimate of the radial derivatives. This can be done verbatim as in Step 4 by using the estimate $|f_n(r)| \leq C_*(\varepsilon)(1+r^2)^{-(1+\varepsilon/2)}$ instead of $|f_n(r)| \leq C_*(1+r^2)^{-3/2}$ when evaluating the integrands; we have for $r > 0$

$$\begin{aligned}
&|u_n'(r)| \\
&\leq \frac{\sqrt{\lambda}r+1}{\sqrt{\lambda}r^2}\int_0^r \frac{C_*(\varepsilon)\min\{\sqrt{\lambda}t^2, t\}}{(1+t^2)^{1+\varepsilon/2}}\,\mathrm{d}t + \frac{\left|\sqrt{\lambda}r\cos(\sqrt{\lambda}r)-\sin(\sqrt{\lambda}r)\right|}{\sqrt{\lambda}r^2}\int_r^\infty \frac{C_*(\varepsilon)t}{(1+t^2)^{1+\varepsilon/2}}\,\mathrm{d}t \\
&\leq \left[\min\left\{\sqrt{\lambda}r+1, \frac{1}{\sqrt{\lambda}r}+1\right\} + \frac{\left|\sqrt{\lambda}r\cos(\sqrt{\lambda}r)-\sin(\sqrt{\lambda}r)\right|}{\sqrt{\lambda}r^2}\right]\int_0^\infty \frac{C_*(\varepsilon)}{(1+t^2)^{1/2+\varepsilon/2}}\,\mathrm{d}t.
\end{aligned}$$

Then again, the Arzelà-Ascoli Theorem and a diagonalization technique provide a locally uniformly convergent subsequence $(u_{n_k})_k$. In order to pass to a global compactness result, we again assess the asymptotic behavior based on the expressions in Lemma 3.10. In view of identity (3.8), we may still assume that $(\hat{f}_{n_k}(\sqrt{\lambda}))_k$, which is still bounded due to the uniform bound on $(f_{n_k})_k$ in $X_{2+\varepsilon}$, converges in $\mathbb{C}$. Finally, we need to adapt the uniform asymptotic estimate (3.39); Lemma 3.10 allows to estimate

$$\begin{aligned}
\left|u_n(r) - \sqrt{\frac{\pi}{2}}\,\hat{f}_n(\sqrt{\lambda})\frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}r}}{r}\right| &= \left|\int_r^\infty f_n(s)\frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda}s}\sin(\sqrt{\lambda}r)-\mathrm{e}^{\mathrm{i}\sqrt{\lambda}r}\sin(\sqrt{\lambda}s)}{\sqrt{\lambda}sr}\,s^2\mathrm{d}s\right| \\
&\leq \int_r^\infty \frac{\|f_n\|_{X_{2+\varepsilon}}}{(1+s^2)^{1+\varepsilon/2}}\cdot\frac{2}{\sqrt{\lambda}r}\,s\mathrm{d}s \\
&\leq \|f_n\|_{X_{2+\varepsilon}}\cdot\frac{2}{\sqrt{\lambda}r}\int_r^\infty \frac{1}{s^{1+\varepsilon}}\,\mathrm{d}s \\
&= \|f_n\|_{X_{2+\varepsilon}}\cdot\frac{2}{\sqrt{\lambda}\varepsilon r^{1+\varepsilon}}. \qquad (3.45)
\end{aligned}$$

Given $\delta > 0$, we let

$$R := \max\left\{\left(\frac{8\sqrt{2}}{\delta\varepsilon\sqrt{\lambda}}\cdot C_*(\varepsilon)\right)^{\frac{1}{\varepsilon}}, 1\right\} \qquad (3.46)$$

and find such $k(\delta) \in \mathbb{N}$ that, for all $k \geq k(\delta)$,

$$\sup_{|x|\leq R}(1+|x|^2)^{\frac{1}{2}}|u_{n_k}(x)-u_{n_l}(x)| < \delta \quad \text{and} \quad \sqrt{\pi}|\hat{f}_{n_k}(\sqrt{\lambda})-\hat{f}_{n_l}(\sqrt{\lambda})| < \frac{\delta}{2} \qquad (3.47)$$

and hence for $|x| \geq R$

$$(1+|x|^2)^{\frac{1}{2}}|u_{n_k}(x)-u_{n_l}(x)|$$

$$\begin{aligned}
&\overset{(3.45)}{\leq} \sqrt{\frac{\pi}{2}}|\hat{f}_{n_k}(\sqrt{\lambda}) - \hat{f}_{n_l}(\sqrt{\lambda})|\frac{(1+|x|^2)^{\frac{1}{2}}}{|x|} + \frac{2(1+|x|^2)^{\frac{1}{2}}}{\sqrt{\lambda}\varepsilon|x|^{1+\varepsilon}} \cdot \|f_{n_k} - f_{n_l}\|_{X_{2+\varepsilon}} \\
&\leq \sqrt{\pi}|\hat{f}_{n_k}(\sqrt{\lambda}) - \hat{f}_{n_l}(\sqrt{\lambda})| + \frac{2\sqrt{2}}{\sqrt{\lambda}\varepsilon R^{\varepsilon}} \cdot 2C_*(\varepsilon) \\
&\overset{(3.46),(3.47)}{<} \delta.
\end{aligned}$$

This shows $\|f_{n_k} - f_{n_l}\|_{X_1} < \delta$ for $k, l \geq k(\delta)$.

▷ Step 3: *Optimality.*

For $m \in \mathbb{N}$, we consider the continuous radial functions $f_m \in C_c(\mathbb{R}^3)$ given by

$$f_m : \mathbb{R}^3 \to \mathbb{R}, \qquad f_m(t) := \frac{\sin(\sqrt{\lambda}t)}{t^2} \cdot \mathbb{1}_{[\pi/\sqrt{\lambda}, 2\pi m/\sqrt{\lambda}]}(t).$$

One can easily convince oneself that the sequence $(f_m)_m$ is bounded in the norm of X_2. We show that $(\Psi_\lambda * f_m)_m$ is unbounded in X_1; more precisely, we prove

$$\|\Psi_\lambda * f_m\|_{X_1} \geq \left(1 + \left(\frac{2\pi m}{\sqrt{\lambda}}\right)^2\right)^{\frac{1}{2}} \left|(\Psi_\lambda * f_m)\left(\frac{2\pi m}{\sqrt{\lambda}}\,\xi\right)\right| \to \infty \qquad \text{as } m \to \infty \tag{3.48}$$

for some unit vector $\xi \in \mathbb{S}^2$. Indeed, for $m \in \mathbb{N}$, the formula in Lemma 3.10 yields

$$\begin{aligned}
&(\Psi_\lambda * f_m)\left(\frac{2\pi m}{\sqrt{\lambda}}\,\xi\right) \\
&= \frac{\cos(2\pi m)}{2\pi m} \cdot \int_0^{\frac{2\pi m}{\sqrt{\lambda}}} \frac{\sin(\sqrt{\lambda}t)}{t} \cdot f_m(t)\, t^2\, \mathrm{d}t + \frac{\sin(2\pi m)}{2\pi m} \cdot \int_{\frac{2\pi m}{\sqrt{\lambda}}}^{\infty} \frac{\cos(\sqrt{\lambda}t)}{t} \cdot f_m(t)\, t^2\, \mathrm{d}t \\
&= \frac{1}{2\pi m} \cdot \sum_{k=1}^{2m-1} \int_{k\pi}^{(k+1)\pi} \frac{\sin^2(\tau)}{\tau}\, \mathrm{d}\tau \\
&\geq \frac{1}{2\pi m} \cdot \sum_{k=1}^{2m-1} \int_{k\pi}^{(k+1)\pi} \frac{\sin^2(\tau)}{(k+1)\pi}\, \mathrm{d}\tau \\
&= \frac{1}{4\pi m} \cdot \sum_{k=1}^{2m-1} \frac{1}{k+1},
\end{aligned}$$

and we conclude the asserted divergence property (3.48) from the lower estimate

$$\left(1 + \left(\frac{2\pi m}{\sqrt{\lambda}}\right)^2\right)^{\frac{1}{2}} \left|(\Psi_\lambda * f_m)\left(\frac{2\pi m}{\sqrt{\lambda}}\,\xi\right)\right| \geq \frac{2\pi m}{\sqrt{\lambda}} \cdot \frac{1}{4\pi m} \sum_{k=1}^{2m-1} \frac{1}{k+1} = \frac{1}{2\sqrt{\lambda}} \sum_{k=1}^{2m-1} \frac{1}{k+1}.$$

□

Proof of Remark 3.15

(a) Let $f \in X_3$ and w be a radial, twice differentiable solution of $-\Delta w - \lambda w = f$ on $\mathbb{R}^3$. By Proposition 3.13 (ii), the function $v := w - \mathcal{R}_\lambda[f]$ is radial, twice differentiable and solves the homogeneous Helmholtz equation $-\Delta v - \lambda v = 0$ on $\mathbb{R}^3$. Thus $v(x) =$

$C \cdot \frac{\sin(|x|\sqrt{\lambda})}{|x|}$ $(x \neq 0)$ for some $C \in \mathbb{R}$, see Remark 1.6 and Lemma 3.9. This shows

$$w(x) = \mathcal{R}_\lambda[f](x) + C \cdot \frac{\sin(|x|\sqrt{\lambda})}{|x|} \qquad \text{for all } x \in \mathbb{R}^3 \setminus \{0\},$$

and in particular $w \in X_1$. The converse is an immediate consequence of Proposition 3.13 (ii).

(b) Let $f \in X_3$ with $\hat{f}(\sqrt{\lambda}) = 0$. Then $\mathcal{R}_\lambda[f](x) = O(|x|^{-2})$ as $|x| \to \infty$ due to the asymptotic expansion derived in Proposition 3.13 (iii); the assertion thus is a consequence of (a). Moreover, Proposition 3.13 also yields $\tilde{\mathcal{R}}_\lambda[f] \equiv 0$. We observe in particular that $w(x) = O(|x|^{-2})$ if and only if $C = 0$ in (3.10).

(c) Let $f \in X_3$ with $\hat{f}(\sqrt{\lambda}) \neq 0$. We consider a solution w of (3.9) with associated constant C as in equation (3.10) of (a) and choose (the unique) $\omega \in (0, \pi)$ with

$$C = \sqrt{\frac{\pi}{2}} \hat{f}(\sqrt{\lambda}) \cdot \cot(\omega).$$

Then, by means of Proposition 3.13 (iii), we find for $x \in \mathbb{R}^3 \setminus \{0\}$

$$\begin{aligned}
w(x) &= \mathcal{R}_\lambda[f](x) + C \cdot \frac{\sin(|x|\sqrt{\lambda})}{|x|} \\
&= \sqrt{\frac{\pi}{2}} \hat{f}(\sqrt{\lambda}) \cdot \frac{\cos(|x|\sqrt{\lambda})}{|x|} + C \cdot \frac{\sin(|x|\sqrt{\lambda})}{|x|} + O\left(\frac{1}{|x|^2}\right) \\
&= \sqrt{\frac{\pi}{2}} \hat{f}(\sqrt{\lambda}) \cdot \frac{\cos(|x|\sqrt{\lambda})}{|x|} + \sqrt{\frac{\pi}{2}} \hat{f}(\sqrt{\lambda}) \cot(\omega) \cdot \frac{\sin(|x|\sqrt{\lambda})}{|x|} + O\left(\frac{1}{|x|^2}\right) \\
&= \sqrt{\frac{\pi}{2}} \hat{f}(\sqrt{\lambda}) \cdot \frac{\cos(|x|\sqrt{\lambda}) \sin(\omega) + \sin(|x|\sqrt{\lambda}) \cos(\omega)}{\sin(\omega)\, |x|} + O\left(\frac{1}{|x|^2}\right) \\
&= \sqrt{\frac{\pi}{2}} \hat{f}(\sqrt{\lambda}) \cdot \frac{\sin(|x|\sqrt{\lambda} + \omega)}{\sin(\omega)\, |x|} + O\left(\frac{1}{|x|^2}\right).
\end{aligned}$$

Moreover, the very last identity in Proposition 3.13 yields

$$\begin{aligned}
w(x) &= \mathcal{R}_\lambda[f](x) + C \cdot \frac{\sin(|x|\sqrt{\lambda})}{|x|} \\
&= \mathcal{R}_\lambda[f](x) + 4\pi \sqrt{\frac{\pi}{2}} \hat{f}(\sqrt{\lambda}) \cot(\omega) \cdot \frac{\sin(|x|\sqrt{\lambda})}{4\pi|x|} \\
&= \mathcal{R}_\lambda[f](x) + \cot(\omega) \cdot \tilde{\mathcal{R}}_\lambda[f](x).
\end{aligned}$$

□

Proof of Corollary 3.16

Let $f \in X_3$ and $\omega \in (0, \pi)$. In order to prove existence, we let $w := \mathcal{R}_\lambda^\omega[f] \in X_1$. Proposition 3.13 (ii) implies that w is twice differentiable and $(-\Delta - \lambda)\, w = f$ on $\mathbb{R}^3$. Proposition 3.13 (iii) further states

$$\begin{aligned}
w(x) &= \mathcal{R}_\lambda[f](x) + \cot(\omega)\, \tilde{\mathcal{R}}_\lambda[f](x) \\
&= \sqrt{\frac{\pi}{2}} \hat{f}(\sqrt{\lambda}) \cdot \frac{\sin(\omega) \cos(|x|\sqrt{\lambda}) + \cos(\omega) \sin(|x|\sqrt{\lambda})}{\sin(\omega)|x|} + O\left(\frac{1}{|x|^2}\right)
\end{aligned}$$

$$= \sqrt{\frac{\pi}{2}} \frac{\hat{f}(\sqrt{\lambda})}{\sin(\omega)} \cdot \frac{\sin(|x|\sqrt{\lambda} + \omega)}{|x|} + O\left(\frac{1}{|x|^2}\right)$$

as $|x| \to \infty$, as asserted. Concerning uniqueness, we distinguish two cases. If $\hat{f}(\sqrt{\lambda}) = 0$, then Remark 3.15 (b) implies that there is exactly one solution of $(-\Delta - \lambda)\, w = f$ on $\mathbb{R}^3$ with $w(x) = O(|x|^{-2})$ as $|x| \to \infty$, and this is the only one with the asserted asymptotic behavior (here $c = 0$). If $\hat{f}(\sqrt{\lambda}) \neq 0$, then Remark 3.15 (c) guarantees uniqueness. □

Proof of Corollary 3.17

(i) By Remark 3.15 (a), w is a radial C^2 solution of $-\Delta w - \lambda w = f$ on $\mathbb{R}^3$ if and only if $w = \mathcal{R}_\lambda[f] + 4\pi C \cdot \tilde{\Psi}_\lambda$ for some $C \in \mathbb{R}$. This proves the "if" part, and the "only if" statement can be seen by applying $\alpha^{(\lambda)}$ to this identity; in view of (3.13) and (3.14), this yields $\alpha^{(\lambda)}(w) = 0 + 4\pi C \cdot 1$.

(ii) If we assume that $w \in X_1$ is a C^2 solution of

$$-\Delta w - \lambda w = f \quad \text{on } \mathbb{R}^3, \qquad w(x) = c\,\frac{\sin(|x|\sqrt{\lambda} + \omega)}{|x|} + O\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty,$$

we infer $w = \mathcal{R}_\lambda[f] + \alpha^{(\lambda)}(w) \cdot \tilde{\Psi}_\lambda$ from (i). Moreover, this implies

$$w(x) = 4\pi c\, \cos(\omega) \frac{\sin(|x|\sqrt{\lambda})}{4\pi|x|} + 4\pi c\, \sin(\omega) \frac{\cos(|x|\sqrt{\lambda})}{4\pi|x|} + O\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty,$$

hence the defining equations (3.13) imply $\alpha^{(\lambda)}(w) = 4\pi c\, \cos(\omega)$ and $\beta^{(\lambda)}(w) = 4\pi c\, \sin(\omega)$. Hence $\sin(\omega)\alpha^{(\lambda)}(w) = \cos(\omega)\beta^{(\lambda)}(w)$ and therefore

$$w = \mathcal{R}_\lambda[f] + \Big[(1 - \sigma \sin(\omega))\alpha^{(\lambda)}(w) + \sigma \cos(\omega)\beta^{(\lambda)}(w)\Big] \cdot \tilde{\Psi}_\lambda. \tag{3.49}$$

Conversely, assuming (3.49), Remark 3.15 (a) shows that w is a C^2 solution of $-\Delta w - \lambda w = f$ on $\mathbb{R}^3$. Applying the functional $\alpha^{(\lambda)}$ to the identity (3.49), we find with equations (3.13), (3.14)

$$\alpha^{(\lambda)}(w) = 0 + \Big[(1 - \sigma \sin(\omega))\alpha^{(\lambda)}(w) + \sigma \cos(\omega)\beta^{(\lambda)}(w)\Big] \cdot 1.$$

Due to $\sigma \neq 0$, we infer $\sin(\omega)\alpha^{(\lambda)}(w) = \cos(\omega)\beta^{(\lambda)}(w)$, and in view of

$$w(x) = \alpha^{(\lambda)}(w)\, \frac{\sin(|x|\sqrt{\lambda})}{4\pi|x|} + \beta^{(\lambda)}(w)\, \frac{\cos(|x|\sqrt{\lambda})}{4\pi|x|} + O\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty,$$

the asserted asymptotic behavior holds: Indeed, given $\omega = 0$, we infer $\beta^{(\lambda)}(w) = 0$ and hence the second term disappears; otherwise we insert $\alpha^{(\lambda)}(w) = \cot(\omega)\beta^{(\lambda)}(w)$ and use the trigonometric angle sum identities as in the previous proofs.

□

Proof of Proposition 3.18

Let $g \in X_2$. Then the profile $w : [0, \infty) \to \mathbb{R}$ is a (global) solution of the initial value problem (3.15) if and only if $y : [0,\infty) \to \mathbb{R}, y(r) = r \cdot w(r)$ solves

$$\begin{cases} -y'' - \lambda y = g(r) \cdot y & \text{on } (0,\infty), \\ y(0) = 0, y'(0) = 1. \end{cases} \tag{3.50}$$

Moreover, $w \in X_1$ if y is a bounded solution of this initial value problem. Global existence and uniqueness of such $y \in C^2_{\mathrm{loc}}([0,\infty))$ are consequences of the Picard-Lindelöf Theorem and of Gronwall's Lemma since $g \in L^1([0,\infty))$. Our proof of boundedness of y and of the asserted asymptotic expansions is inspired by perturbation results of Hartman in [34]. It is an application of the Prüfer transformation, see equation (2.1) in [34]. Since $y \not\equiv 0$, uniqueness implies that $y(r)^2 + y'(r)^2 > 0$ for all $r \geq 0$. We thus parametrize using polar coordinates in the phase space

$$y(r) = \varrho(r) \cdot \sin(\phi(r)\sqrt{\lambda}), \qquad y'(r) = \varrho(r) \cdot \sqrt{\lambda}\cos(\phi(r)\sqrt{\lambda}) \qquad (r \geq 0) \tag{3.51}$$

with functions $\varrho : [0,\infty) \to (0,\infty)$ and $\phi : [0,\infty) \to \mathbb{R}$. A short calculation shows that we thus obtain a solution of (3.50) if and only if ϱ and ϕ satisfy the first-order system

$$\begin{cases} (\log \varrho)' = -\frac{g(r)}{2\sqrt{\lambda}} \sin(2\phi\sqrt{\lambda}) & \text{on } (0,\infty), \\ \phi' = 1 + \frac{g(r)}{\lambda} \sin^2(\phi\sqrt{\lambda}) & \text{on } (0,\infty), \\ \varrho(0) = \frac{1}{\sqrt{\lambda}}, \quad \phi(0) = 0. \end{cases} \tag{3.52}$$

Equivalently, for $r \geq 0$,

$$\begin{aligned} \varrho(r) &= \frac{1}{\sqrt{\lambda}} \cdot \exp\left(- \int_0^r \frac{g(t)}{2\sqrt{\lambda}} \sin(2\phi(t)\sqrt{\lambda})\,\mathrm{d}t \right), \\ \phi(r) &= r + \int_0^r \frac{g(t)}{\lambda} \sin^2(\phi(t)\sqrt{\lambda})\,\mathrm{d}t. \end{aligned} \tag{3.53}$$

We will frequently refer to the following estimate, which holds for all $r \geq 0$:

$$\begin{aligned} &\frac{1}{\sqrt{\lambda}} \exp\left(\left| \int_0^r \frac{g(t)}{2\sqrt{\lambda}} \sin(2\phi(t)\sqrt{\lambda})\,\mathrm{d}t \right| \right) \\ &\quad \leq \frac{1}{\sqrt{\lambda}} \exp\left(\int_0^r \frac{\|g\|_{X_2}}{2\sqrt{\lambda}(1+t^2)}\,\mathrm{d}t \right) \leq \frac{1}{\sqrt{\lambda}} \exp\left(\frac{\pi}{4\sqrt{\lambda}}\, \|g\|_{X_2} \right) =: C_g. \end{aligned} \tag{3.54}$$

Indeed, it immediately yields boundedness of the solution since, due to $g \in X_2$,

$$|y(r)| \overset{(3.51)}{\leq} |\varrho(r)| \overset{(3.53)}{\leq} \frac{1}{\sqrt{\lambda}} \cdot \exp\left(\left| \int_0^r \frac{g(t)}{2\sqrt{\lambda}} \sin(2\phi(t)\sqrt{\lambda})\,\mathrm{d}t \right| \right) \overset{(3.54)}{\leq} C_g.$$

Analogously, we see that the improper integrals in

$$\begin{aligned} \omega_\lambda(g) &:= \int_0^\infty \frac{g(t)}{\sqrt{\lambda}} \sin^2(\phi(t)\sqrt{\lambda})\,\mathrm{d}t = \sqrt{\lambda} \cdot \lim_{r\to\infty} (\phi(r) - r), \\ \varrho_\lambda(g) &:= \frac{1}{\sqrt{\lambda}} \cdot \exp\left(- \int_0^\infty \frac{g(t)}{2\sqrt{\lambda}} \sin(2\phi(t)\sqrt{\lambda})\,\mathrm{d}t \right) = \lim_{r\to\infty} \varrho(r) \end{aligned} \tag{3.55}$$

converge, observe $C_g \geq \varrho_\lambda(g) > 0$ due to (3.54), and verify the asserted asymptotic behavior of y as $r \to \infty$:

$$\left| y(r) - \varrho_\lambda(g) \sin(r\sqrt{\lambda} + \omega_\lambda(g)) \right|$$

$$
\begin{aligned}
&\overset{(3.51)}{=} \left|\varrho(r)\sin(\phi(r)\sqrt{\lambda}) - \varrho_\lambda(g)\sin(r\sqrt{\lambda}+\omega_\lambda(g))\right| \\
&\leq \left|[\varrho(r)-\varrho_\lambda(g)]\sin(\phi(r)\sqrt{\lambda})\right| + \left|\varrho_\lambda(g)\left[\sin(\phi(r)\sqrt{\lambda}) - \sin(r\sqrt{\lambda}+\omega_\lambda(g))\right]\right| \\
&\leq |\varrho(r)-\varrho_\lambda(g)| + C_g\cdot\left|\phi(r)\sqrt{\lambda} - r\sqrt{\lambda} - \omega_\lambda(g)\right| \\
&\overset{(3.53)}{=} \frac{1}{\sqrt{\lambda}}\cdot\left|\exp\left(-\int_0^r \frac{g(t)}{2\sqrt{\lambda}}\sin(2\phi(t)\sqrt{\lambda})\,\mathrm{d}t\right) - \exp\left(-\int_0^\infty \frac{g(t)}{2\sqrt{\lambda}}\sin(2\phi(t)\sqrt{\lambda})\,\mathrm{d}t\right)\right| \\
&\qquad + C_g\cdot\left|\int_r^\infty \frac{g(t)}{\sqrt{\lambda}}\sin^2(\phi(t)\sqrt{\lambda})\,\mathrm{d}t\right| \\
&\leq \frac{1}{\sqrt{\lambda}}\cdot\sup_{\xi>0}\left[\exp\left(-\int_0^\xi \frac{g(t)}{2\sqrt{\lambda}}\sin(2\phi(t)\sqrt{\lambda})\,\mathrm{d}t\right)\right]\cdot\left|\int_r^\infty \frac{g(t)}{2\sqrt{\lambda}}\sin(2\phi(t)\sqrt{\lambda})\,\mathrm{d}t\right| \\
&\qquad + C_g\cdot\left|\int_r^\infty \frac{g(t)}{\sqrt{\lambda}}\sin^2(\phi(t)\sqrt{\lambda})\,\mathrm{d}t\right| \\
&\overset{(3.54)}{\leq} C_g\cdot\left|\int_r^\infty \frac{g(t)}{2\sqrt{\lambda}}\sin(2\phi(t)\sqrt{\lambda})\,\mathrm{d}t\right| + C_g\cdot\left|\int_r^\infty \frac{g(t)}{\sqrt{\lambda}}\sin^2(\phi(t)\sqrt{\lambda})\,\mathrm{d}t\right| \\
&\leq C_g\cdot\frac{\|g\|_{X_2}}{2\sqrt{\lambda}}\cdot\int_r^\infty \frac{\mathrm{d}t}{1+t^2} + C_g\cdot\frac{\|g\|_{X_2}}{\sqrt{\lambda}}\cdot\int_r^\infty\frac{\mathrm{d}t}{1+t^2} \\
&\leq C_g\cdot\frac{3\,\|g\|_{X_2}}{2\sqrt{\lambda}}\cdot\frac{1}{r}.
\end{aligned}
$$

Thus $y(r) - \varrho_\lambda(g)\sin(r\sqrt{\lambda}+\omega_\lambda(g)) = O\left(\frac{1}{r}\right)$ as $r\to\infty$. Similarly one can show that $y'(r) - \varrho_\lambda(g)\sqrt{\lambda}\cos(r\sqrt{\lambda}+\omega_\lambda(g)) = O\left(\frac{1}{r}\right)$ as $r\to\infty$. Since $y(r) = r\cdot w(r)$, the proposition is proved. ☐

Proof of Proposition 3.20

We consider $g_n, g_0 \in X_2$ with $g_n \to g_0 \in X_2$ and aim to show that $\omega_\lambda(g_n)\to\omega_\lambda(g_0)$. By ϕ_n we denote the unique solution of

$$\phi_n' = 1 + \frac{g_n(r)}{\lambda}\sin^2(\phi_n\sqrt{\lambda}), \qquad \phi_n(0) = 0.$$

Then we have pointwise convergence, $\phi_n(r)\to\phi_0(r)$ for all $r\geq 0$. Indeed, let us fix any $R>0$ and estimate for $0\leq r\leq R$ and $n\in\mathbb{N}$

$$
\begin{aligned}
&|\phi_n(r)-\phi_0(r)| \\
&\quad = \left|\int_0^r \frac{g_n(t)}{\lambda}\sin^2(\phi_n(t)\sqrt{\lambda}) - \frac{g_0(t)}{\lambda}\sin^2(\phi_0(t)\sqrt{\lambda})\,\mathrm{d}t\right| \\
&\quad \leq \frac{1}{\lambda}\int_0^r |g_n(t)-g_0(t)|\,\mathrm{d}t + \frac{1}{\lambda}\int_0^r |g_0(t)|\left|\sin^2(\phi_n(t)\sqrt{\lambda}) - \sin^2(\phi_0(t)\sqrt{\lambda})\right|\,\mathrm{d}t \\
&\quad \leq \frac{1}{\lambda}\int_0^\infty \|g_n-g_0\|_{X_2}\,\frac{\mathrm{d}t}{1+t^2} + \frac{2\,\|g_0\|_\infty}{\sqrt{\lambda}}\int_0^r |\phi_n(t)-\phi_0(t)|\,\mathrm{d}t \\
&\quad \leq \frac{\pi}{2\lambda}\|g_n-g_0\|_{X_2} + \frac{2\,\|g_0\|_\infty}{\sqrt{\lambda}}\int_0^r |\phi_n(t)-\phi_0(t)|\,\mathrm{d}t.
\end{aligned}
$$

Thus Gronwall's Lemma yields $\phi_n\to\phi_0$ uniformly on $[0,R]$. Now we can deduce the convergence of the asymptotic phase,

$$\omega_\lambda(g_n) = \frac{1}{\sqrt{\lambda}}\int_0^\infty g_n(r)\sin^2(\phi_n(r)\sqrt{\lambda})\,\mathrm{d}r \to \frac{1}{\sqrt{\lambda}}\int_0^\infty g_0(r)\sin^2(\phi_0(r)\sqrt{\lambda})\,\mathrm{d}r = \omega_\lambda(g_0),$$

which is a consequence of the Dominated Convergence Theorem. Indeed, the integrands converge pointwise and are integrably majorized by

$$\left| g_n(r) \sin^2(\phi_n(r)\sqrt{\lambda}) \right| \le \frac{\sup_{n\in\mathbb{N}} \|g_n\|_{X_2}}{1+r^2} \qquad (r \ge 0).$$

□

Proof of Proposition 3.21

Given the assumptions of Proposition 3.21 and $b \in \mathbb{R}$, we recall the definition of the asymptotic phase in equation (3.55),

$$\omega_\lambda(b\,u_0^2) = \frac{b}{\sqrt{\lambda}} \int_0^\infty u_0^2(r) \sin^2(\phi_b(r)\sqrt{\lambda})\,\mathrm{d}r = \lim_{r\to\infty} (\phi_b(r) - r) \cdot \sqrt{\lambda}. \tag{3.56}$$

Here ϕ_b satisfies $\phi_b' = 1 + \frac{b}{\lambda} u_0^2(r) \sin^2(\phi_b\sqrt{\lambda})$ on $(0,\infty)$, $\phi_b(0) = 0$. We immediately see that $\omega_\lambda(0) = 0$ and $\operatorname{sgn} \omega_\lambda(b\,u_0^2) = \operatorname{sgn}(b)$ for all $b \in \mathbb{R} \setminus \{0\}$. Further, continuity of $b \mapsto \omega_\lambda(b\,u_0^2)$ is a consequence of Proposition 3.20. The assertions are proved once we show that $b \mapsto \omega_\lambda(b\,u_0^2)$ is strictly increasing with

$$\omega_\lambda(b\,u_0^2) \to \pm\infty \qquad \text{as } b \to \pm\infty.$$

▷ Step 1: *Strict monotonicity.*

We let $b_1 < b_2$, define

$$\chi(r) := \begin{cases} \frac{\sin^2(\phi_{b_2}(r)\sqrt{\lambda}) - \sin^2(\phi_{b_1}(r)\sqrt{\lambda})}{\phi_{b_2}(r)\sqrt{\lambda} - \phi_{b_1}(r)\sqrt{\lambda}} & \text{if } \phi_{b_2}(r) \neq \phi_{b_1}(r), \\ 2\sin(\phi_{b_1}(r)\sqrt{\lambda})\cos(\phi_{b_1}(r)\sqrt{\lambda}) & \text{else} \end{cases}$$

and observe that, due to continuous differentiability of $\xi \mapsto \sin^2(\xi)$, χ is bounded with $|\chi(r)| \le 2$ and continuous. The difference $\psi := \phi_{b_2} - \phi_{b_1}$ satisfies

$$\psi' = \frac{b_2 - b_1}{\lambda} u_0^2(r) \sin^2(\phi_{b_2}(r)\sqrt{\lambda}) + \frac{b_1}{\sqrt{\lambda}} u_0^2(r)\chi(r)\psi, \qquad \psi(0) = 0.$$

The unique solution is given by the Variation of Constants formula. In view of (3.56), we have

$$\begin{aligned} \omega_\lambda(b_2\,u_0^2) - \omega_\lambda(b_1\,u_0^2) &= \lim_{r\to\infty} \psi(r) \cdot \sqrt{\lambda} \\ &= \int_0^\infty \frac{b_2 - b_1}{\sqrt{\lambda}} u_0^2(\varrho) \sin^2(\phi_{b_2}(\varrho)\sqrt{\lambda}) \mathrm{e}^{\int_\varrho^\infty \frac{b_1}{\sqrt{\lambda}} u_0^2(\tau)\chi(\tau)\,\mathrm{d}\tau}\,\mathrm{d}\varrho \\ &> 0 \end{aligned}$$

since the integrand is nonnegative and not identically zero. Indeed, this is a consequence of the following three observations. Firstly, we assume $b_2 > b_1$. Secondly, since u_0 is a nontrivial solution of $-\Delta u_0 - \mu u_0 = u_0^3$ on $\mathbb{R}^3$, Lemma 2.5 (iii) in [54] implies that $u_0^2 > 0$ almost everywhere. Thirdly, (3.52) implies $\phi_{b_2}'(r) \to 1$ as $r \to \infty$ and hence $\sin^2(\phi_{b_2}(\varrho)\sqrt{\lambda}) > 0$ on a set of positive measure.

▷ Step 2: *Asymptotic behavior as $b \to \infty$.*

By the uniqueness statement of the Picard-Lindelöf Theorem, $u_0 \not\equiv 0$ requires $u_0(0) \neq 0$.

We can thus choose $r_0 > 0$ with

$$u_0^2(r) > \frac{1}{2} u_0^2(0) \qquad \text{for all } r \in [0, r_0]. \tag{3.57}$$

Roughly speaking, we are going to prove that the phase difference $\phi_b(r_0) - r_0$ becomes arbitrarily large as $b \to \infty$; due to $\phi_b' \geq 1$, this implies for the asymptotic phase

$$\omega_\lambda(b\, u_0^2) \overset{(3.56)}{=} \lim_{r\to\infty} (\phi_b(r) - r) \cdot \sqrt{\lambda} \geq (\phi_b(r_0) - r_0) \cdot \sqrt{\lambda} \to \infty,$$

as asserted. It remains to verify that $\phi_b(r_0) \to \infty$ as $b \to \infty$, which we intend to achieve via a comparison technique. To keep notation short, we let here $\xi := \frac{1}{2\lambda} u_0^2(0)$. We have for $b > 0$

$$\begin{aligned} \phi_b' &= 1 + \frac{b}{\lambda} u_0^2(r) \sin^2(\phi_b \sqrt{\lambda}) \quad \text{ on } [0, r_0], \qquad \phi_b(0) = 0, \\ &\overset{(3.57)}{\geq} 1 + b \cdot \xi \sin^2(\phi_b \sqrt{\lambda}). \end{aligned} \tag{3.58}$$

We now study the modified initial value problem

$$\psi_b' = 1 + b \cdot \xi \sin^2(\psi_b \sqrt{\lambda}) \quad \text{ on } [0, r_0], \qquad \psi_b(0) = 0. \tag{3.59}$$

Then $\phi_b \geq \psi_b$ on $[0, r_0]$ (see e.g. §9, Corollary to Satz VIII in [74]), in particular $\phi_b(r_0) \geq \psi_b(r_0)$. For $0 \leq r \leq r_0$ with $r \notin \frac{\pi}{2} + \pi\mathbb{Z}$, the unique solution of (3.59) is given by the expression

$$\psi_b(r) = \frac{1}{\sqrt{\lambda}} \left[n\pi + \arctan\left(\frac{\tan\left(r\sqrt{\lambda}\, \sqrt{1 + b \cdot \xi} \right)}{\sqrt{1 + b \cdot \xi}} \right) \right] \quad \text{for } \left| \sqrt{1 + b \cdot \xi} \sqrt{\lambda}\, r - n\pi \right| < \frac{\pi}{2}$$

where $n \in \mathbb{N}_0$. We deduce immediately $\psi_b(r_0) \to \infty$ as $b \to \infty$. Since by construction $\phi_b \geq \psi_b$ on $[0, r_0]$, this implies $\phi_b(r_0) \to \infty$ as $b \to \infty$, which is all we had to show.

$\triangleright$ <u>Step 3</u>: *Asymptotic behavior as $b \to -\infty$.*

Roughly speaking, we will prove that for large negative values of b, there is a large interval $[0, r_b)$ where ϕ_b cannot stay permanently above a certain small value $\ell_b := \frac{1}{\sqrt{\lambda}} \arcsin(|b|^{-\frac{1}{4}})$. Thus a large negative phase difference $\phi_b(r) - r$ occurs on $[0, r_b)$, which due to $\phi_b' \leq 1$ finally leads to large negative values of $\omega_\lambda(b\, u_0^2) = \sqrt{\lambda} \cdot \lim_{r\to\infty} (\phi_b(r) - r)$, cf. (3.56).

To be precise, for $b < -1$, we introduce the radius where ϕ_b finally leaves the level ℓ_b behind,

$$r_b := \max \left\{ r > 0 \;\middle|\; \phi_b(r) = \ell_b = \frac{1}{\sqrt{\lambda}} \arcsin(|b|^{-\frac{1}{4}}) \right\}.$$

Then $r_b \in (0, \infty)$ is well-defined since, due to $1 - \frac{|b|}{\lambda} \frac{\|u_0\|_{X_1}^2}{1+r^2} \leq \phi_b' \leq 1$ and $\phi_b(0) = 0$, we have $r - \frac{|b|}{\lambda} \|u_0\|_{X_1}^2 \arctan(r) \leq \phi_b(r) \leq r$ for $r \geq 0$. In particular, we have

$$\phi_b(r_b) = \ell_b \qquad \text{and} \qquad \phi_b(r) > \ell_b \text{ for all } r > r_b. \tag{3.60}$$

We observe that $\ell_b \to 0$ as $b \to -\infty$ and prove below $r_b \to \infty$ as $b \to -\infty$. Then for $r \geq r_b$, (3.60) and $\phi_b' \leq 1$ imply

$$\phi_b(r) \leq \phi_b(r_b) + (r - r_b) = r + (\ell_b - r_b).$$

Then the asymptotic phase satisfies

$$\omega_\lambda(b\,u_0^2) \overset{(3.56)}{=} \sqrt{\lambda} \cdot \lim_{r\to\infty}(\phi_b(r) - r) \le \sqrt{\lambda}\cdot(\ell_b - r_b) \to -\infty \quad \text{as } b \to -\infty.$$

This is the asserted asymptotic property. It remains to prove that $r_b \to \infty$ as $b \to -\infty$. We assume by contradiction that we find a subsequence $(b_k)_{k\in\mathbb{N}}$ and $\tilde{r} > 0$ with $b_k \searrow -\infty$, $r_{b_k} \to \tilde{r}$ as $k \to \infty$. Then, since $\phi'_{b_k} \le 1$ and due to (3.60), we have

$$\ell_{b_k} = \frac{1}{\sqrt{\lambda}}\arcsin(|b_k|^{-\frac{1}{4}}) \le \phi_{b_k}(r) \le \ell_{b_k} + \frac{1}{\sqrt{\lambda}} \quad \text{for } r_{b_k} \le r \le r_{b_k} + \frac{1}{\sqrt{\lambda}},\ k \in \mathbb{N}.$$

Since $\ell_{b_k} \to 0$ as $k \to \infty$, the upper estimate guarantees $\sqrt{\lambda}\phi_{b_k}(r) \le \frac{\pi}{2}$ for sufficiently large $k \in \mathbb{N}$. Hence, for these k, the estimate from below is equivalent to

$$\sin(\phi_{b_k}(r)\sqrt{\lambda}) \ge |b_k|^{-\frac{1}{4}} \qquad \text{for } r_{b_k} \le r \le r_{b_k} + \frac{1}{\sqrt{\lambda}}. \tag{3.61}$$

We conclude, as $k \to \infty$,

$$\begin{aligned}
&\phi_{b_k}\left(r_{b_k} + \frac{1}{\sqrt{\lambda}}\right) \\
&= \phi_{b_k}(r_{b_k}) + \int_0^{\frac{1}{\sqrt{\lambda}}} \phi'_{b_k}(r_{b_k} + \tau)\,\mathrm{d}\tau \\
&\overset{(3.60)}{=} \ell_{b_k} + \int_0^{\frac{1}{\sqrt{\lambda}}} \left[1 - \frac{|b_k|}{\lambda}u_0^2(r_{b_k} + \tau)\sin^2(\phi_{b_k}(r_{b_k}+\tau)\sqrt{\lambda})\right]\mathrm{d}\tau \\
&\overset{(3.61)}{\le} \ell_{b_k} + \frac{1}{\sqrt{\lambda}} - \frac{\sqrt{|b_k|}}{\lambda}\cdot\int_0^{\frac{1}{\sqrt{\lambda}}} u_0^2(r_{b_k}+\tau)\,\mathrm{d}\tau \\
&= o(1) + \frac{1}{\sqrt{\lambda}} - \frac{\sqrt{|b_k|}}{\lambda}\cdot\left(\int_0^{\frac{1}{\sqrt{\lambda}}} u_0^2(\tilde{r}+\tau)\,\mathrm{d}\tau + o(1)\right) \\
&\to -\infty
\end{aligned}$$

since $u_0^2 > 0$ almost everywhere. On the other hand that, for every $k \in \mathbb{N}$, the differential equation $\phi' = 1 + \frac{b_k}{\lambda}u_0^2(r)\sin^2(\phi\sqrt{\lambda})$ states that $\phi_{b_k}(r) = 0$ implies $\phi'_{b_k}(r) = 1$ and thus ϕ_{b_k} cannot attain negative values, which contradicts the limit calculated before. □

Proof of Proposition 3.22 (and Remark 3.23)

For $\omega \in (0,\pi)$ and $\lambda > 0$, we compute the spectrum of the linear operator

$$\mathbf{R}_\lambda^\omega : X_1 \to X_1, \quad w \mapsto \mathcal{R}_\lambda^\omega[u_0^2\,w] = \left(\Psi_\lambda + \cot(\omega)\tilde{\Psi}_\lambda\right) * [u_0^2\,w].$$

Compactness of $\mathbf{R}_\lambda^\omega$ is a consequence of Proposition 3.13 (i). Then immediately $\sigma(\mathbf{R}_\lambda^\omega) = \{0\} \cup \sigma_p(\mathbf{R}_\lambda^\omega)$ with discrete eigenvalues of finite multiplicity. In fact, $0 \notin \sigma_p(\mathbf{R}_\lambda^\omega)$; the argumentation in Step 1 below shows in particular that $\ker \mathbf{R}_\lambda^\omega = \{0\}$.
Existence and uniqueness of the values $b_k(\omega,\lambda,u_0^2) \in \mathbb{R}$ defined via $\omega_\lambda(b_k(\omega,\lambda,u_0^2)\,u_0^2) = \omega + k\pi$ (where $k \in \mathbb{Z}$) is guaranteed by the bijectivity of $\mathbb{R} \to \mathbb{R}$, $b \mapsto \omega_\lambda(b\,u_0^2)$, see Proposition 3.21. (We observe in particular that $b_k(\omega,\lambda,u_0^2) \ne 0$ since $\omega_\lambda(b\,u_0^2) = 0$ if and only if $b = 0$.)

▷ Step 1: *Eigenvalues.*

We find the eigenfunctions of $\mathbf{R}_\lambda^\omega$, that is, we look for such $\eta \in \mathbb{R}, \eta \ne 0$ and $w \in X_1 \setminus \{0\}$

that

$$\mathbf{R}_\lambda^\omega w = \eta \cdot w.$$

Corollary 3.16 implies that this is equivalent to finding $\eta \in \mathbb{R}, \eta \neq 0$ and $w \in X_1 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3) \setminus \{0\}$ with

$$\begin{cases} -\Delta w - \lambda w = \frac{1}{\eta} \cdot u_0^2(x)\, w & \text{on } \mathbb{R}^3, \\ w(x) = c_w \frac{\sin(|x|\sqrt{\lambda}+\omega)}{|x|} + O\left(\frac{1}{|x|^2}\right) & \text{as } |x| \to \infty \end{cases} \tag{3.62}$$

for some $c_w \in \mathbb{R}$. By Proposition 3.18, such an eigenfunction exists if and only if

$$\omega_\lambda \left(\frac{1}{\eta}\, u_0^2 \right) = \omega + k\pi \quad \text{for some } k \in \mathbb{Z}.$$

In this case, $c_w \neq 0$ and every eigenspace is one-dimensional because the radially symmetric solution w is unique up to multiplication by a constant. Since we have seen in Proposition 3.21 that $\mathbb{R} \to \mathbb{R}$, $b \mapsto \omega_\lambda(b\, u_0^2)$ is strictly increasing and onto, we can define $b_k(\omega, \lambda, u_0^2)$ via $\omega_\lambda(b_k(\omega, \lambda, u_0^2)\, u_0^2) = \omega + k\pi$ for all $k \in \mathbb{Z}$, and conclude

$$\sigma_p(\mathbf{R}_\lambda^\omega) = \left\{ \frac{1}{b_k(\omega, \lambda, u_0^2)} \;\middle|\; k \in \mathbb{Z} \right\}.$$

Starting from (3.62) with $\omega = 0$, one finds nontrivial solutions if and only if $\omega_\lambda \left(\frac{1}{\eta}\, u_0^2 \right) = k\pi$ for some $k \in \mathbb{Z}$. As above, we obtain values $b_k(0, \lambda, u_0^2)$ as asserted in Remark 3.23.

▷ <u>Step 2</u>: *Simplicity.*

It remains to show that the eigenvalues are algebraically simple - geometric simplicity has been proved in Step 1. We consider an eigenvalue $\eta := \frac{1}{b_k(\omega,\lambda,u_0^2)}$ of $\mathbf{R}_\lambda^\omega$ with eigenspace $\ker(\mathbf{R}_\lambda^\omega - \eta I_{X_1}) = \mathrm{span}\, \{w\}$, $w \not\equiv 0$. We have to prove that

$$\ker(\mathbf{R}_\lambda^\omega - \eta I_{X_1})^2 = \ker(\mathbf{R}_\lambda^\omega - \eta I_{X_1}).$$

So let now $v \in \ker(\mathbf{R}_\lambda^\omega - \eta I_{X_1})^2$. We assume for contradiction that $v \notin \ker(\mathbf{R}_\lambda^\omega - \eta I_{X_1})$.

By assumption on v, we have $\mathbf{R}_\lambda^\omega v - \eta v \in \ker(\mathbf{R}_\lambda^\omega - \eta I_{X_1}) \setminus \{0\}$, and since $\eta \neq 0$ we may assume without loss of generality $\mathbf{R}_\lambda^\omega v - \eta v = -\eta w = -\mathbf{R}_\lambda^\omega w$, hence

$$w = \mathcal{R}_\lambda^\omega \left[\frac{1}{\eta} u_0^2 w \right] \quad \text{and} \quad v = \mathcal{R}_\lambda^\omega \left[\frac{1}{\eta} u_0^2 (v + w) \right].$$

We observe that $v, w \in C^2_{\mathrm{loc}}(\mathbb{R}^3)$ by Proposition 3.13 (ii) as well as

$$\begin{aligned} -w'' - \frac{2}{r} w' - \lambda w &= \frac{1}{\eta}\, u_0^2(r) \cdot w \\ -v'' - \frac{2}{r} v' - \lambda v &= \frac{1}{\eta}\, u_0^2(r) \cdot (v + w) \end{aligned} \qquad \text{on } (0, \infty). \tag{3.63}$$

Furthermore, Proposition 3.13 (iii) implies

$$\begin{aligned}
w(r) &= c_w \cdot \frac{\sin(r\sqrt{\lambda}+\omega)}{r} + O\left(\frac{1}{r^2}\right), \\
w'(r) &= c_w\sqrt{\lambda} \cdot \frac{\cos(r\sqrt{\lambda}+\omega)}{r} + O\left(\frac{1}{r^2}\right), \\
v(r) &= c_v \cdot \frac{\sin(r\sqrt{\lambda}+\omega)}{r} + O\left(\frac{1}{r^2}\right), \\
v'(r) &= c_v\sqrt{\lambda} \cdot \frac{\cos(r\sqrt{\lambda}+\omega)}{r} + O\left(\frac{1}{r^2}\right)
\end{aligned} \tag{3.64}$$

for some $c_w, c_v \in \mathbb{R}$. Let us define $q(r) = r^2(w'(r)v(r) - v'(r)w(r))$ for $r \geq 0$. Then, using the differential equations (3.63), we find $q'(r) = \frac{1}{\eta}\, r^2 u_0^2(r) \cdot w^2(r)$ for $r \geq 0$. Hence, depending on the sign of η, q is monotone on $[0,\infty)$ with $q(0) = 0$. On the other hand, the asymptotic expansions in (3.64) imply that $q(r) = O\left(\frac{1}{r}\right)$ as $r \to \infty$. We conclude $q(r) = 0$ for all $r \geq 0$, hence also $q'(r) = 0$ for all $r \geq 0$.
In view of $q'(r) = \frac{1}{\eta}\, r^2 u_0^2(r) \cdot w^2(r)$ $(r \geq 0)$, this shows that $u_0\, w \equiv 0$ and hence $w = \mathcal{R}_\lambda^\omega\left[\frac{1}{\eta} u_0^2 w\right] \equiv 0$, a contradiction. □

Proof of Proposition 3.1

The existence of a continuum of radially symmetric solutions $u_0 \in X_1 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$ of the nonlinear Helmholtz equation (3.4) has been shown in Theorem 1.2 of [54]. Given such a solution u_0, the asymptotic expansion is a consequence of Proposition 3.18 applied to equation (3.15) with $g := u_0^2 \in X_2$, $\lambda := \mu$ and with unique solution $w := \frac{u_0}{u_0(0)}$. Proposition 3.18 also provides a unique radially symmetric solution $w_1 \in X_1 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$ of $-\Delta w - \mu w = 3u_0^2(x)\, w$ on $\mathbb{R}^3$, $w(0) = 1$. As for u_0, the asymptotic behavior of w_1 is

$$w_1(x) = c \cdot \frac{\sin(|x|\sqrt{\mu}+\tau_0)}{|x|} + O\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty$$

for some $c \neq 0$ and $\tau_0 \in [0,\pi)$; then, $w_0 := \frac{w_1}{c}$ has the asserted properties. □

3.6 Summary

This chapter was devoted to the study of bifurcations of the cubic Helmholtz system (3.1)

$$\begin{cases}
-\Delta u - \mu u = \left(u^2 + b\, v^2\right) u & \text{on } \mathbb{R}^3, \\
-\Delta v - \nu v = \left(v^2 + b\, u^2\right) v & \text{on } \mathbb{R}^3, \\
u(x), v(x) \to 0 \quad \text{as } |x| \to \infty
\end{cases}$$

from given families of semitrivial resp. diagonal solutions. We saw in Theorems 3.2 and 3.4 that unbounded global continua of solutions bifurcate at every point of these families, and that these continua form smooth curves of fully nontrivial solutions in a neighborhood of the respective bifurcation point.

A first idea might be to extend these results to arbitrary space dimensions $N \geq 2$ and suitable powers p in the nonlinearity, which would require to replace the spaces X_1, X_3 by $X_{\frac{N-1}{2}}, X_{\frac{p(N-1)}{2}}$ in accordance with the decay rates in Theorem 1.13. We expect qualitatively similar results but a considerably enlarged technical effort involving general Bessel functions

$J_{\frac{N}{2}-1}, Y_{\frac{N}{2}-1}$ and thus less explicit formulas.

In order to verify bifurcation from simple eigenvalues, we imposed radial symmetry as well as asymptotic conditions fixing phase parameters of the solutions as $|x| \to \infty$. For future research projects, it would certainly be interesting (and most likely challenging) to drop the assumption of radial symmetry and replace it, for instance, by suitable periodic settings. Regarding the strategy of the proof, it would then either be necessary to find another way of reducing the eigenspace dimensions (if possible) or to use more complex tools from bifurcation theory (if available). Moreover, compactness questions (Proposition 3.13) would have to be reconsidered and a substitute for all the auxiliary results using ODE analysis (e.g. Proposition 3.18) would have to be found.

Interestingly, our results show that, for a fixed set of asymptotic phase parameters, the bifurcation picture is the same as in the Schrödinger case - a sequence of discrete bifurcation points. In Schrödinger systems, the bifurcating solutions are characterized by their nodal structure. We found an analogue in the Helmholtz case, which is the value of an integral quantity called the asymptotic phase, see Remark 3.3 (c). It is constant on bifurcating continua in a neighborhood of the bifurcation point; however, it is possible that this parameter is not globally constant. One might ask: Is it possible to find unbounded continua of solutions where the asymptotic phase (and not only the asymptotic parameter ω) is globally constant?

Finally, our results indicate that infinitely many branches of solutions bifurcate at every point of the semitrivial resp. diagonal families since we can vary the second asymptotic parameter τ_1, see Remark 3.3 (e). It would be interesting to analyze whether these branches are mutually different and, in this case, to investigate the local structure of the bifurcating continuum.

Chapter 4.

Time-periodic Solutions of a Cubic Wave Equation

4.1 Introduction and Main Results

We discuss real-valued solutions $U(t,x)$ of the equation

$$\partial_t^2 U - \Delta U - U = \Gamma(x)\, U^3 \qquad \text{on } \mathbb{R}\times\mathbb{R}^3 \tag{4.1}$$

where $\Gamma \in L^\infty_{\mathrm{rad}}(\mathbb{R}^3)\cap C^1_{\mathrm{loc}}(\mathbb{R}^3)$. Again we restrict ourselves to the case of three space dimensions which is the most relevant one for applications in physics and which allows to use the tools established in the previous chapter. Throughout, the notations $\partial_{1,2,3}, \nabla, \Delta, D^2$ refer to differential operators acting on the space variables. The solutions we aim to construct are polychromatic, that is, they take the form

$$U(t,x) = u_0(x) + \sum_{k=1}^{\infty} 2\,\cos(kt) u_k(x) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}kt} u_k(x) \tag{4.2}$$

$$\text{where } u_k \in X_1 = \left\{ u \in C_{\mathrm{rad}}(\mathbb{R}^3) \middle| \; \left\| (1+|\cdot|^2)^{\frac{1}{2}} u \right\|_\infty < \infty \right\} \subseteq L^4_{\mathrm{rad}}(\mathbb{R}^3), u_{-k} = u_k.$$

Such solutions are periodic in time and localized as well as radially symmetric in space. They are sometimes referred to as breather solutions, see e.g. a breather solution for the Sine-Gordon equation in [1], equation (28). We will construct polychromatic solutions of (4.1) with $u_k \not\equiv 0$ for at least two distinct integers $k \in \mathbb{N}_0$ by rewriting it into an infinite system of cubic Helmholtz equations for the functions u_k, see (4.7), similar to the problem studied in detail in Chapter 3. The solutions we construct bifurcate from a given stationary solution, that is, they are of the form

$$U(t,x) = w_0(x) + V(t,x) = w_0(x) + v_0(x) + \sum_{k=1}^{\infty} 2\,\cos(kt) v_k(x) \tag{4.3}$$

where $w_0 \in X_1 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$, $w_0 \not\equiv 0$ is some fixed radially symmetric solution of the stationary nonlinear Helmholtz equation

$$-\Delta w_0 - w_0 = \Gamma(x)\, w_0^3 \quad \text{on } \mathbb{R}^3. \tag{4.4}$$

The existence of such solutions can be guaranteed under certain additional assumptions on Γ, see e.g. [54], Theorem 2.10. In particular, our main result presented next holds for any constant Γ.

Theorem 4.1 (Polychromatic solutions I).

Let $\Gamma \in L^\infty_{\mathrm{rad}}(\mathbb{R}^3) \cap C^1_{\mathrm{loc}}(\mathbb{R}^3)$ and assume there is some stationary solution $U^0(t,x) = w_0(x)$, $w_0 \not\equiv 0$ of the cubic wave equation (4.1), i.e. $w_0 \in X_1 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$ solving (4.4). Then for every $s \in \mathbb{N}$ there exist an open interval $J_s \subseteq \mathbb{R}$ containing 0 and a family $(U^\alpha)_{\alpha \in J_s} \subseteq C^2(\mathbb{R}, X_1) \cap C^2_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^3)$ with the following properties:

(i) All U^α are classical solutions of (4.1) of the polychromatic form (4.2),

$$U^\alpha(t,x) = u_0^\alpha(x) + \sum_{k=1}^{\infty} 2\,\cos(kt) u_k^\alpha(x).$$

They are time-periodic with period 2π.

(ii) The map $\alpha \mapsto (u_k^\alpha)_{k \in \mathbb{N}_0}$ is smooth in the topology of $\ell^1(\mathbb{N}_0, X_1)$ with

$$\left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\right|_{\alpha=0} u_k^\alpha \not\equiv 0 \quad \text{if and only if} \quad k = s$$

("excitation of the s-th mode"). In particular, for different values of s, these families consist of non-stationary solutions and mutually differ close to U^0.

(iii) If we assume additionally that $\Gamma(x) \neq 0$ for almost all $x \in \mathbb{R}^3$, then for every $\alpha \in J_s \setminus \{0\}$ the polychromatic solution U^α possesses infinitely many nonvanishing modes u_k^α.

We require continuity of Γ since we use the functional analytic framework of Chapter 3 based on spaces of continuous functions. The existence and continuity of $\nabla\Gamma$ will be exploited in proving that U is twice differentiable. This assumption as well as $\Gamma \neq 0$ almost everywhere in (iii) might be relaxed; however, this study does not aim at the most general setting for the coefficients but rather focuses on the introduction of the setup for the existence result.

Remark 4.2.

There are some generalizations of the wave equation (4.1) which can also be treated in the framework we establish below.

(a) For any $\xi > 0$, it is possible to construct polychromatic solutions $U(t,x) = u_0(x) + \sum_{k=1}^\infty 2\,\cos(kt) u_k(x)$ of the equation

$$\partial_t^2 U - \Delta U - \xi\, U = \Gamma(x)\, U^3 \qquad \text{on} \quad \mathbb{R} \times \mathbb{R}^3.$$

(b) *For any $\omega > 0$, it is possible to construct polychromatic solutions with modified period*

$$U(t,x) = u_0(x) + \sum_{k=1}^{\infty} 2\cos(\omega k t) u_k(x)$$

of the equation $\partial_t^2 U - \Delta U - U = \Gamma(x)\, U^3$ on $\mathbb{R} \times \mathbb{R}^3$.

(c) *Similar results hold for the equation*

$$\partial_t^2 U - \Delta U + \xi U = \Gamma(x)\, U^3 \qquad \text{on} \quad \mathbb{R} \times \mathbb{R}^3$$

where $\xi > 0$, commonly called the Klein-Gordon equation *with mass $\xi^{\frac{1}{2}}$. This requires to extend some of the linear theory in Chapter 3 to stationary Schrödinger-type equations. We give details in Section 4.3.*

Moreover, there are the following extreme cases:

(d) *The proof of (i) and (ii) also works in the case $s = 0$, where the direction of bifurcation points along the stationary solutions. Indeed, according to [54], there exists a continuum of solutions of the stationary equation* (4.4)*, and hence we might find a family of the form $U^\alpha(t,x) = w_0(x) + v_0^\alpha(x)$.*

(e) *In the special case of a linear problem $\Gamma \equiv 0$, a verbatim transfer of the proof of Theorem 4.1 provides families of solutions of the form*

$$U^\alpha(t,x) = c \cdot \frac{\sin(|x|)}{|x|} + c_\alpha \cdot \cos(st) \cdot \frac{\sin(|x|\sqrt{s^2+1})}{|x|}$$

for $t \in \mathbb{R}, x \neq 0$ and $c, c_\alpha \in \mathbb{R}$.

Concerning (e), a more common and more direct way of finding solutions of the linear equation is Fourier expansion in the time variable; this yields solutions of the general form

$$U(t,x) = \sum_{k=0}^{\infty} \tilde{c}_k \cdot \cos(kt) \cdot \frac{\sin(|x|\sqrt{k^2+1})}{|x|}$$

where $(\tilde{c}_k)_{k\in\mathbb{N}_0}$ is a suitable sequence of coefficients (e.g. in $\ell^1(\mathbb{N}_0)$). This is to illustrate that our method provides families of solutions with a (large) constant contribution, followed by a dominant excitation of the s-th mode.

Observe that we do not discuss the classical wave equation $\partial_t^2 U - \Delta U = \Gamma(x)\, U^3$ and thus, in particular, avoid stationary equations of the form $-\Delta u_0 = f_0$.
Indeed, this would not fit into the framework of the spaces X_1 and X_3; for instance, given $f_0 \in X_3$ with $f_0(x) = (1+|x|^2)^{-\frac{3}{2}}$, integration provides a unique localized solution $u_0(x) = |x|^{-1} \cdot \operatorname{arsinh}|x|$. But then, $u_0 \notin X_1$. We will not strive to include this case at the cost of extending the technical framework to another pair of suitable Banach spaces with appropriate decay rates.

4.1.1 An Overview of Literature

Polychromatic Solutions

The results in Theorem 4.1 (and Theorem 4.8 below) can and should be compared with recent findings on breather (that is to say, time-periodic and spatially localized) solutions of the wave equation with periodic potentials $V(x), q(x) = c \cdot V(x) \geq 0$,

$$V(x)\partial_t^2 U - \partial_x^2 U + q(x)U = \Gamma(x)U^3 \qquad \text{on } \mathbb{R} \times \mathbb{R}. \tag{4.5}$$

Such breather solutions have been constructed by Schneider et al., see Theorem 1.1 in [16], and Hirsch and Reichel, see Theorem 1.3 in [36], respectively. In brief, the main difference to the results in this thesis is that the authors of [16], [36] consider a setting in one space dimension and obtain strongly spatially localized solutions, which requires a comparably huge technical effort. We give some details: Both existence results are established using a polychromatic ansatz, which reduces the time-dependent equation to an infinite set of stationary problems with periodic coefficients, see [16], p. 823, resp. [36], equation (1.2). The authors of [16] apply spatial dynamics and center manifold reduction; their ansatz is based on a very explicit choice of the coefficients q, V, Γ. The approach in [36] incorporates more general potentials and nonlinearities and is based on variational techniques. It provides ground state solutions, which are possibly "large" - in contrast to the local bifurcation methods, which only yield solutions close to a given stationary one as described in Theorem 4.1, i.e. with a typically "small" time-dependent contribution.
Periodicity of the potentials is explicitly required since it leads to the occurrence of spectral gaps when analyzing the associated differential operators of the stationary equations. In contrast to the Helmholtz methods introduced here, the authors both of [16] and of [36] strive to construct the potentials in such way that 0 lies in the aforementioned spectral gaps, and moreover that the distance between 0 and the spectra has a positive lower bound. This is realized by assuming a certain "roughness" of the potentials, referring to the step potential defined in Theorem 1.1 of [16] and to the assumptions (P1)-(P3) in [36] which allow potentials with periodic spikes modeled by Dirac delta distributions, periodic step potentials or some specific, non-explicit potentials in $H^r_{\text{rad}}(\mathbb{R})$ with $1 \leq r < \frac{3}{2}$ (see [36], Lemma 2.8).
In a periodic setting, the required spectral properties can be proved using Floquet-Bloch theory, which provides a suitable description of the band spectra characteristic for periodic differential operators. In our (special) case of constant potentials and vanishing band gaps, it is much more convenient to use techniques based on Fourier transformation for the spectral analysis instead, which is what we do implicitly when applying Limiting Absorption Principles derived with Fourier techniques, see e.g. equation (1.7).

For technical reasons, in both articles, the polychromatic ansatz is chosen in such way that only odd modes u_k, $k \in 2\mathbb{Z}+1$ appear. This is clearly not compatible with our approach of bifurcation from trivial (that is to say, constant in time) solutions which forces us to admit the 0-th mode. Even more, our method allows to consider excitations of arbitrary higher modes in the bifurcating branches, see Theorem 4.1 (ii).

Let us point out that, on the one hand, the methods for constructing breather solutions of (4.5) outlined above can handle periodic potentials but require irregularity and are very restrictive concerning the form of the potentials. The Helmholtz ansatz presented in this thesis, on the other hand, allows constant potentials but does not, so far, work in the non-constant, periodic case. This might be an interesting extension of our results; for further research perspectives, see the summary at the end of the chapter.

The Klein-Gordon Equation as a Cauchy Problem

Possibly due to its relevance in physics, there is a number of classical results in the literature concerning the nonlinear Klein-Gordon equation as mentioned in Remark 4.2 (c). The fundamental difference to the results in this thesis is that the vast majority of these concerns the Cauchy problem of the Klein-Gordon equation, i.e.

$$\begin{aligned} &\partial_t^2 U - \Delta U + U = \pm U^3 \quad \text{on } [0,\infty)\times\mathbb{R}^3 \\ &U(0,x) = f(x),\ \partial_t U(0,x) = g(x) \quad \text{on } \mathbb{R}^3 \end{aligned} \tag{4.6}$$

for suitable initial data $f, g : \mathbb{R}^3 \to \mathbb{R}$. Usually, the dependence of the nonlinearity on U is much more general (allowing also derivatives of U) and the space dimension is not restricted to $N = 3$. On the other hand, most results in the literature only concern the autonomous case, which is why we set in this discussion $\Gamma \equiv \pm 1$. As we will see, there are fundamental differences depending on the sign.

An overview of the state of knowledge towards the end of the 1970s can be found e.g. in [69] by Strauss, who discusses among other topics global existence (Theorem 1.1), regularity and uniqueness (Theorem 1.2), blow-up (Theorem 1.4) and convergence to solutions of the free Klein-Gordon equation as $t \to \infty$ (Theorem 4.1). During the following decade, Klainerman [41, 42] and Shatah [66, 67] independently developed new techniques leading to significant improvements in the study of uniqueness questions and of the asymptotic behavior of solutions as $t \to \infty$. In the case of a cubic nonlinearity, these results only apply if the space dimension is at least 2. This is why, more recently, the question of corresponding uniqueness and convergence properties for cubic nonlinearities in $N = 1$ space dimensions has attracted attention; we wish to mention at least some of the related papers. For explicit choices of the cubic nonlinearity, there are results by Moriyama and by Delort, see Theorem 1.1 of [57] resp. Théorèmes 1.2, 1.3 in [21]. Only the latter result allows a nonlinearity of the form $\pm U^3$ not containing derivatives (see [21], Remarque 1.4); however, the initial data are assumed to have compact support. Global existence, uniqueness, decay rates and convergence to solutions of the free equation as $t \to \infty$ exclusively for the nonlinearity $\pm U^3$ can be found in Corollary 1.2 of [35] by Hayashi and Naumkin.
We provide some more details on the classical global existence result for (4.6) as presented by Strauss. In the case $\Gamma \equiv -1$, Theorem 1.2 and Example 2 of [69] ensure the existence of a (distributional) real-valued solution $U(t, x)$ of (4.6) which exists globally in time and has locally finite energy, i.e.

$$E_B[U(t,\cdot)] = \frac{1}{2}\int_B |\partial_t U(t,x)|^2 + |\nabla U(t,x)|^2 + |U(t,x)|^2 \,\mathrm{d}x + \frac{1}{4}\int_B |U(t,x)|^4 \,\mathrm{d}x < \infty$$

for every ball $B \subseteq \mathbb{R}^3$ and all $t \geq 0$ provided the initial conditions have locally finite energy. Moreover, this solution is shown to be unique and as smooth as the initial data permit. This global existence result has been proved by Jörgens [38] following a characteristic strategy in the field of evolution equations. He first shows local existence and uniqueness using a suitable fixed-point iteration, and then extends the so obtained solution globally in time by iterating the local construction on time intervals of a fixed minimal length; the latter can be guaranteed by conservation of energy. For $\Gamma \equiv +1$, however, Theorem 1.4 (a) in [69] shows that finite-time blow-up can occur for real-valued solutions of (4.6) even for compactly supported and smooth initial data f, g.
The main Theorem in [42] by Klainerman yields global existence and uniqueness of smooth solutions of (4.6) as well as a spatially uniform decay rate $\sim t^{-5/4}$ as $t \to \infty$ provided the initial data are smooth, compactly supported and sufficiently small. Application II.1 in [66] due to Shatah also ensures global existence and uniqueness of solutions $U \in C([0,\infty), H^\alpha(\mathbb{R}^3))$ with $\partial_t U \in C([0,\infty), H^{\alpha-1}(\mathbb{R}^3))$ (where $\alpha \in \mathbb{N}$ is typically large) for sufficiently regular initial values; it is also shown that U is asymptotically close to a solution of the linear (free) Klein-Gordon equation at large times t. The case of a power-type cubic nonlinearity depending only on U is contained as a special case - both results admit more general nonlinearities the growth rate of which is only prescribed at small arguments;

in particular, the sign of the nonlinearity does not play a role.

The relation to our results is not straightforward since the bifurcation methods automatically provide solutions U^α which exist globally in time irrespective of the sign (or even of a possible x-dependence) of Γ; in particular, thus, we do not construct the blow-up solutions mentioned in case $\Gamma \equiv +1$. On the contrary, Theorem 4.8 below illustrates that global existence is possible even in this case, which is also mentioned (but not in the form of a theorem) at the beginning of Strauss' article. Moreover, our ansatz does not put special emphasis on the role of the initial values

$$U^\alpha(0,x) = u_0^\alpha(x) + \sum_{k=1}^{\infty} 2u_k^\alpha(x); \quad \nabla U^\alpha(0,x) = \nabla u_0^\alpha(x) + \sum_{k=1}^{\infty} 2\nabla u_k^\alpha(x).$$

Noticing that the initial values satisfy the local finiteness assumption for the energy, we infer in the case $\Gamma \equiv -1$ that the polychromatic solution $U^\alpha(t,x)$ we construct is the only solution of the Klein-Gordon equation with these initial data. However, the ansatz presented here does not provide any further knowledge of the initial data along the bifurcating continua of solutions; its strong point is the description of several global properties of the solutions $U^\alpha(t,x)$ such as periodicity in time and localization as well as decay rates in space.

4.2 The Proof of Theorem 4.1

4.2.1 The Functional-Analytic Setting

We look for polychromatic solutions as in (4.2) with coefficients

$$\begin{aligned}
&\mathbf{u} = (u_k)_{k\in\mathbb{Z}} \in \mathcal{X}_1 \\
&\text{where } \mathcal{X}_1 := \ell^1_{\text{sym}}(\mathbb{Z}, X_1) \\
&\qquad := \left\{ (u_k)_{k\in\mathbb{Z}} \,\middle|\, u_k = u_{-k} \in X_1, \|(u_k)_{k\in\mathbb{Z}}\|_{\ell^1(\mathbb{Z},X_1)} := \sum_{k\in\mathbb{Z}} \|u_k\|_{X_1} < \infty \right\}.
\end{aligned}$$

In particular, we denote by $\mathbf{w} = (\delta_{k,0}w_0)_{k\in\mathbb{Z}} = (\ldots,0,w_0,0,\ldots)$ the stationary solution with $w_0 \in X_1 \cap C^2_{\text{loc}}(\mathbb{R}^3)$ fixed according to equation (4.4). We will find polychromatic solutions of (4.1) by solving the countably infinite Helmholtz system

$$\begin{aligned}
-\Delta u_k - (k^2+1)u_k &= \Gamma(x) \cdot \sum_{\substack{l,m,n\in\mathbb{Z} \\ l+m+n=k}} u_l \cdot u_m \cdot u_n \\
&= \Gamma(x) \cdot (\mathbf{u} * \mathbf{u} * \mathbf{u})_k \qquad \text{on } \mathbb{R}^3,
\end{aligned} \tag{4.7}$$

which is equivalent to (4.1), (4.2) on a formal level; for details including convergence of the polychromatic sum in (4.2), see Proposition 4.5. Our strategy is then as follows: We intend to apply the bifurcation techniques developed in the previous chapter, which is possible since the linearized version of the infinite-dimensional system (4.7) resembles closely the one of the two-component system in Chapter 3. We therefore recall, for the reader's convenience, a collection of results of the previous chapter concerning the linearized setting in Proposition 4.4. After that, we present a suitable setup for bifurcation theory; in particular, we introduce a bifurcation parameter which will be hidden in the asymptotic conditions. The fact that solutions of (4.7) obtained in this setting provide polychromatic, classical solutions of the wave equation (4.1) will be proved as a part of Proposition 4.5 below. Indeed, regarding differentiability, we will see that the choice of suitable asymptotic conditions will ensure uniform convergence and hence smoothness properties of the infinite sums defining the polychromatic states. Finally, in Proposition 4.7, we essentially verify the assumptions of the Crandall-Rabinowitz Bifurcation Theorem, which will subsequently be used to give a very short proof of Theorem 4.1. As in the previous chapters, the auxiliary results will be proved in the final Section 4.4.

Throughout, we denote the convolution in $\mathbb{R}^3$ by the symbol $*$ and use $\star$ in the convolution algebra ℓ^1.

Proposition 4.3.

The convolution of sequences $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{u}^{(3)} \in \mathcal{X}_1$ is well-defined in a pointwise sense and satisfies $\mathbf{u}^{(1)} \star \mathbf{u}^{(2)} \star \mathbf{u}^{(3)} \in \mathcal{X}_3 := \ell^1_{\text{sym}}(\mathbb{Z}, X_3)$. Moreover, we have the estimate

$$\left\|\mathbf{u}^{(1)} \star \mathbf{u}^{(2)} \star \mathbf{u}^{(3)}\right\|_{\ell^1(\mathbb{Z},X_3)} \leq \left\|\mathbf{u}^{(1)}\right\|_{\ell^1(\mathbb{Z},X_1)} \left\|\mathbf{u}^{(2)}\right\|_{\ell^1(\mathbb{Z},X_1)} \left\|\mathbf{u}^{(3)}\right\|_{\ell^1(\mathbb{Z},X_1)}.$$

We rewrite the system (4.7) using $\mathbf{u} = \mathbf{w} + \mathbf{v}$ with $\mathbf{w} = (..., 0, w_0, 0, ...)$; then,

$$-\Delta v_k - (k^2+1)\, v_k = \Gamma(x) \cdot \left[((\mathbf{w}+\mathbf{v}) \star (\mathbf{w}+\mathbf{v}) \star (\mathbf{w}+\mathbf{v}))_k - \delta_{k,0} w_0^3\right] \quad \text{on } \mathbb{R}^3. \tag{4.8}$$

As in the previous chapter, we will find solutions of this system of differential equations by solving instead a system of coupled convolution equations which, for $k \notin \{0, \pm s\}$, have the form $v_k = \mathcal{R}^{\tau_k}_{k^2+1}[f_k]$. Here f_k represents the right-hand side of (4.8), the operators $\mathcal{R}^\tau_\mu$ have been defined in equation (3.12), and the coefficients $\tau_k \in (0,\pi)$ have to be chosen properly according to a nondegeneracy condition. We recall, in short, the relevant facts from the previous chapter in the following Proposition.

Proposition 4.4.

Let $w_0 \in X_1 \cap C^2_{\text{loc}}(\mathbb{R}^3)$ be a solution of equation (4.4) with $\Gamma \in L^\infty_{\text{rad}}(\mathbb{R}^3) \cap C_{\text{loc}}(\mathbb{R}^3)$. For every $k \in \mathbb{Z}$, there exists (up to a multiplicative constant) a unique nontrivial and radially symmetric solution $q_k \in X_1$ of the problem

$$-\Delta q_k - (k^2+1)\, q_k = 3\,\Gamma(x) w_0^2(x)\, q_k \qquad \text{on } \mathbb{R}^3. \tag{4.9a}$$

It is twice continuously differentiable and satisfies, for some $c_k \neq 0$ and $\sigma_k \in [0,\pi)$,

$$q_k(x) = c_k \cdot \frac{\sin(|x|\sqrt{k^2+1} + \sigma_k)}{|x|} + O\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty. \tag{4.9b}$$

The equations (4.9a), (4.9b) are equivalent to the convolution identities

$$\begin{cases} q_k = 3\,\mathcal{R}^{\sigma_k}_{k^2+1}[\Gamma w_0^2\, q_k] = 3\,\left(\mathcal{R}_{k^2+1}[\Gamma w_0^2\, q_k] + \cot(\sigma_k)\tilde{\mathcal{R}}_{k^2+1}[\Gamma w_0^2\, q_k]\right) & \text{if } \sigma_k \in (0,\pi), \\ q_k = 3\,\mathcal{R}_{k^2+1}[\Gamma w_0^2\, q_k] + \left(\alpha^{(k^2+1)}(q_k) + \beta^{(k^2+1)}(q_k)\right) \cdot \tilde{\Psi}_{k^2+1} & \text{if } \sigma_k = 0. \end{cases}$$

For all $k \in \mathbb{Z}$, $\cos(\sigma_k)\,\beta^{(k^2+1)}(q_k) = \sin(\sigma_k)\,\alpha^{(k^2+1)}(q_k)$.

This can be proved using Proposition 3.18 for the former statement as well as Corollaries 3.16 and 3.17 (ii) for the convolution identities. For these results to apply we have assumed initially that Γ is continuous and bounded, whence $3\,\Gamma w_0^2 \in X_2$.

We now present the general assumptions valid throughout the following construction and the proof of Theorem 4.1. We let σ_k for $k \in \mathbb{Z}$ as in Proposition 4.4 above and fix $s \in \mathbb{N}$, recalling that we aim to "excite the s-th mode" in the sense of Theorem 4.1 (ii). With this, let us introduce

$$\tau_{\pm s} := \sigma_{\pm s}, \qquad \tau_k := \begin{cases} \frac{\pi}{4} & \text{if } \sigma_k \neq \frac{\pi}{4}, \\ \frac{3\pi}{4} & \text{if } \sigma_k = \frac{\pi}{4} \end{cases} \qquad \text{for } k \in \mathbb{Z} \setminus \{\pm s\}. \tag{4.10}$$

Thus in particular $\tau_k \neq \sigma_k$ for $k \in \mathbb{Z}\setminus\{\pm s\}$, and we conclude from the uniqueness statement in Proposition 4.4 the nondegeneracy property

$$k \in \mathbb{Z} \setminus \{\pm s\}, \quad q \in X_1, \quad q = 3\,\mathcal{R}^{\tau_k}_{k^2+1}[\Gamma w_0^2\, q] \qquad \Rightarrow \qquad q \equiv 0. \tag{4.11}$$

In place of (4.10), one could in fact consider a much more general setting of parameters τ_k. At this point, let us just say that we prefer the explicit values above in order to avoid more abstract parameters in the proof; for a deeper reason, we refer to Remark 4.6 (b) below.

We now introduce a map with the property that its zeros provide solutions of the system (4.8). Again, as in the previous chapter, we have to distinguish the cases $\tau_s \in (0, \pi)$ and $\tau_s = 0$. (In the following, please recall that we consider some fixed $s \neq 0$.)

Case 1: $0 < \tau_{\pm s} < \pi$.

We introduce the map $F : \mathcal{X}_1 \times \mathbb{R} \to \mathcal{X}_1$ with

$$\begin{aligned} F(\mathbf{v}, \lambda)_0 &:= v_0 - \mathcal{R}^{\tau_0}_1 \left[\Gamma\ ((\mathbf{w}+\mathbf{v}) \star (\mathbf{w}+\mathbf{v}) \star (\mathbf{w}+\mathbf{v}))_0 - \Gamma\, w_0^3\right], \\ F(\mathbf{v}, \lambda)_{\pm s} &:= v_{\pm s} - \mathcal{R}_{s^2+1} \left[\Gamma\ ((\mathbf{w}+\mathbf{v}) \star (\mathbf{w}+\mathbf{v}) \star (\mathbf{w}+\mathbf{v}))_{\pm s}\right] \\ &\qquad - (\cot(\tau_{\pm s}) - \lambda)\tilde{\mathcal{R}}_{s^2+1} \left[\Gamma\ ((\mathbf{w}+\mathbf{v}) \star (\mathbf{w}+\mathbf{v}) \star (\mathbf{w}+\mathbf{v}))_{\pm s}\right], \\ F(\mathbf{v}, \lambda)_k &:= v_k - \mathcal{R}^{\tau_k}_{k^2+1} \left[\Gamma\ ((\mathbf{w}+\mathbf{v}) \star (\mathbf{w}+\mathbf{v}) \star (\mathbf{w}+\mathbf{v}))_k\right] \end{aligned} \tag{4.12}$$

where in the last line $k \in \mathbb{Z} \setminus \{0, \pm s\}$.

Case 2: $\tau_{\pm s} = 0$.

Here we define $G : \mathcal{X}_1 \times \mathbb{R} \to \mathcal{X}_1$ with

$$\begin{aligned} G(\mathbf{v}, \lambda)_0 &:= v_0 - \mathcal{R}^{\tau_0}_1 \left[\Gamma\ ((\mathbf{w}+\mathbf{v}) \star (\mathbf{w}+\mathbf{v}) \star (\mathbf{w}+\mathbf{v}))_0 - \Gamma\, w_0^3\right], \\ G(\mathbf{v}, \lambda)_{\pm s} &:= v_{\pm s} - \mathcal{R}_{s^2+1} \left[\Gamma\ ((\mathbf{w}+\mathbf{v}) \star (\mathbf{w}+\mathbf{v}) \star (\mathbf{w}+\mathbf{v}))_{\pm s}\right] \\ &\qquad - (1-\lambda) \left(\alpha^{(s^2+1)}(v_{\pm s}) + \beta^{(s^2+1)}(v_{\pm s})\right) \tilde{\Psi}_{s^2+1}, \\ G(\mathbf{v}, \lambda)_k &:= v_k - \mathcal{R}^{\tau_k}_{k^2+1} \left[\Gamma\ ((\mathbf{w}+\mathbf{v}) \star (\mathbf{w}+\mathbf{v}) \star (\mathbf{w}+\mathbf{v}))_k\right] \end{aligned} \tag{4.13}$$

where again in the last line $k \in \mathbb{Z} \setminus \{0, \pm s\}$.

The following result collects some basic properties of the maps F and G and the polychromatic states related to their zeros. Since the statements mainly concern the convergence properties of infinite sums and since F and G only differ in two components, both can be discussed at the same time.

Proposition 4.5.

Let $s \in \mathbb{N}$ and $(\tau_k)_{k\in\mathbb{Z}}$ be chosen as in (4.10). The maps $F, G : \mathcal{X}_1 \times \mathbb{R} \to \mathcal{X}_1$ are well-defined and smooth with $F(\mathbf{0}, \lambda) = G(\mathbf{0}, \lambda) = \mathbf{0}$ for all $\lambda \in \mathbb{R}$. Further, if $F(\mathbf{v}, \lambda) = \mathbf{0}$ resp. $G(\mathbf{v}, \lambda) = \mathbf{0}$ for some $\mathbf{v} \in \mathcal{X}_1, \lambda \in \mathbb{R}$, then $\mathbf{v}$ solves the Helmholtz system (4.8) and

$$U(t,x) := w_0(x) + v_0(x) + \sum_{k=1}^{\infty} 2\cos(kt) v_k(x) \qquad (t \in \mathbb{R}, x \in \mathbb{R}^3)$$

defines a classical solution $U \in C^2(\mathbb{R}, X_1) \cap C^2_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^3)$ of the cubic wave equation (4.1).

Again, the proof can be found in Section 4.4. We will even show that $U \in C^\infty(\mathbb{R}, X_1)$. For the derivatives of F resp. G with respect to the Banach space component $\mathbf{v} \in \mathcal{X}_1$, we will verify the following explicit formulas: Letting $\mathbf{q} \in \mathcal{X}_1$ and abbreviating $\mathbf{u} := \mathbf{v} + \mathbf{w}$,

$$(DF(\mathbf{v},\lambda)[\mathbf{q}])_k = q_k - \begin{cases} 3\,\mathcal{R}^{\tau_k}_{k^2+1}\left[\Gamma\left(\mathbf{q} \star \mathbf{u} \star \mathbf{u}\right)_k\right] & k \neq \pm s, \\ 3\,\mathcal{R}_{s^2+1}\left[\Gamma\left(\mathbf{q} \star \mathbf{u} \star \mathbf{u}\right)_{\pm s}\right] & \\ \quad +3\left(\cot(\tau_s) - \lambda\right)\tilde{\mathcal{R}}_{s^2+1}\left[\Gamma\left(\mathbf{q} \star \mathbf{u} \star \mathbf{u}\right)_{\pm s}\right] & k = \pm s; \end{cases} \tag{4.14}$$

$$(DG(\mathbf{v},\lambda)[\mathbf{q}])_k = q_k - \begin{cases} 3\,\mathcal{R}^{\tau_k}_{k^2+1}\left[\Gamma\left(\mathbf{q} \star \mathbf{u} \star \mathbf{u}\right)_k\right] & k \neq \pm s, \\ 3\,\mathcal{R}_{s^2+1}\left[\Gamma\left(\mathbf{q} \star \mathbf{u} \star \mathbf{u}\right)_{\pm s}\right] & \\ \quad +(1-\lambda)\Big(\alpha^{(s^2+1)}(q_{\pm s}) + \beta^{(s^2+1)}(q_{\pm s})\Big)\tilde{\Psi}_{s^2+1} & k = \pm s. \end{cases} \tag{4.15}$$

Remark 4.6. *(a) Let us, already at this point, emphasize that other than in the previous chapter, the bifurcation parameter λ appears only in the asymptotic expansions of the s-th components $v_{\pm s}$ of the solutions and not in the differential equation (4.1).*

(b) As mentioned above, the choice of the parameters τ_k in equation (4.10) is far from unique. Indeed, one could instead consider any configuration satisfying

$$\tau_k = \tau_{-k} \neq \sigma_k \text{ for all } k \in \mathbb{Z} \setminus \{\pm s\}, \qquad \{\tau_k \mid k \in \mathbb{Z} \setminus \{\pm s\}\} \subseteq (\delta, \pi - \delta)$$

for some $\delta \in \left(0, \frac{\pi}{2}\right)$. The former condition is required for the nondegeneracy statement (4.11), and the latter will be used to obtain uniform decay estimates in the proof of Proposition 4.5, see Lemma 4.13.

However, as in the previous chapter, the question whether another choice of τ_k leads to different bifurcating families is still open; cf. Remark 3.3 (e). Hence we discuss only the explicit choice in (4.10).

We intend to apply the Crandall-Rabinowitz Bifurcation Theorem. The next result shows that its assumptions are satisfied.

Proposition 4.7 (Simplicity and transversality).

Let $s \in \mathbb{N}$ and $(\tau_k)_{k\in\mathbb{Z}}$ be chosen as in (4.10). The linear operator $DF(\mathbf{0}, 0) : \mathcal{X}_1 \to \mathcal{X}_1$ is a 1-1-Fredholm operator the kernel of which has the form

$$\ker DF(\mathbf{0}, 0) = \operatorname{span}\{\mathbf{q}\} \qquad \text{with } q_k \neq 0 \text{ if and only if } k = \pm s.$$

Moreover, the transversality condition is satisfied, that is,

$$\partial_\lambda DF(\mathbf{0}, 0)[\mathbf{q}] \notin \operatorname{ran} DF(\mathbf{0}, 0).$$

A corresponding statement holds true for $DG(\mathbf{0}, 0) : \mathcal{X}_1 \to \mathcal{X}_1$.

4.2.2 The Proof of Theorem 4.1

Let us fix some $s \in \mathbb{N}$, and choose $(\tau_k)_{k\in\mathbb{Z}}$ as in (4.10). We introduce the trivial family $\mathcal{T} := \{(\mathbf{0}, \lambda) \in \mathcal{X}_1 \times \mathbb{R} \mid \lambda \in \mathbb{R}\}$.

$\triangleright$ Step 1: *Proof of (i).*

By Proposition 4.5, the maps F resp. G are smooth and vanish on the trivial family $\mathcal{T}$. In view of Proposition 4.7, the Crandall-Rabinowitz Theorem 3.5 shows that $(\mathbf{0}, 0) \in \mathcal{T}$ is a bifurcation point and provides an open interval $J_s \subseteq \mathbb{R}$ containing 0 and a smooth curve

$$J_s \to \mathcal{X}_1 \times \mathbb{R}, \qquad \alpha \mapsto (\mathbf{v}^\alpha, \lambda^\alpha) = ((v_k^\alpha)_{k\in\mathbb{Z}}, \lambda^\alpha)$$

of zeros of F resp. G (we do not denote its dependence on s) with $\mathbf{v}^0 = \mathbf{0}, \lambda^0 = 0$ as well as $\frac{\mathrm{d}}{\mathrm{d}\alpha}\big|_{\alpha=0}\mathbf{v}^\alpha = \mathbf{q}$ where $\mathbf{q}$ is a nontrivial element of the kernel of $DF(\mathbf{0}, 0)$ resp. $DG(\mathbf{0}, 0)$. We let $\mathbf{u}^\alpha := \mathbf{v}^\alpha + \mathbf{w}$ and define polychromatic states U^α as in (i). Then U^α is a classical solution of the cubic wave equation (4.1) due to Proposition 4.5 since $F(\mathbf{v}^\alpha, \lambda^\alpha) = 0$ resp. $G(\mathbf{v}^\alpha, \lambda^\alpha) = 0$. By their very definition, the solutions U^α are time-periodic with period 2π (maybe less). This proves (i).

$\triangleright$ Step 2: *Proof of (ii).*

Since F resp. G are smooth, so is the map $J_s \to \mathcal{X}_1 \times \mathbb{R}$, $\alpha \mapsto (\mathbf{v}^\alpha, \lambda^\alpha)$. By Proposition 4.7, $q_k \neq 0$ if and only if $k = \pm s$, which implies that only the $\pm s$-th components of

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\bigg|_{\alpha=0} \mathbf{u}^\alpha = \frac{\mathrm{d}}{\mathrm{d}\alpha}\bigg|_{\alpha=0} \mathbf{v}^\alpha = \mathbf{q}$$

do not vanish. For sufficiently small nonzero values of α, the solutions U^α are thus non-stationary. In particular, the direction of bifurcation changes when changing the value of s, and the associated bifurcating curves are, at least locally, mutually different.

$\triangleright$ Step 3: *Proof of (iii).*

We show finally that, under the additional assumption that $\Gamma(x) \neq 0$ for almost all $x \in \mathbb{R}^3$,

$$U^\alpha(t, x) = w_0(x) + v_0^\alpha(x) + \sum_{k=1}^{\infty} 2\cos(kt)\, v_k^\alpha(x)$$

in fact possesses infinitely many nontrivial coefficients v_k^α. Indeed, assuming the contrary, we can choose a maximal $r \geq s$ (hence in particular $r > 0$) with $v_r^\alpha \not\equiv 0$ or equivalently $u_r^\alpha = v_r^\alpha + w_r = v_r^\alpha \not\equiv 0$. But then,

$$v_{3r}^\alpha = \sum_{l+m+n=3r} \mathcal{R}_{(3r)^2+1}^{\tau_{3r}}[\Gamma\, u_l^\alpha\, u_m^\alpha\, u_n^\alpha] = \mathcal{R}_{(3r)^2+1}^{\tau_{3r}}[\Gamma\, (v_r^\alpha)^3] \not\equiv 0$$

since the convolution identity implies $-\Delta v_{3r}^\alpha - ((3r)^2+1)v_{3r}^\alpha = \Gamma\,(v_r^\alpha)^3$, and $\Gamma\,(v_r^\alpha)^3 \not\equiv 0$ since $\Gamma(x) \neq 0$ almost everywhere by assumption. This contradicts the maximality of r.

□

Proof of Remark 4.2

We demonstrate how (a) and (b) can be derived using scaling arguments.

In order to prove (a), one replaces (k^2+1) by $(k^2+\xi)$ in the system (4.7) and modifies all proofs accordingly, which is possible since $k^2+\xi \geq \xi > 0$ is bounded uniformly away from 0. Most of all, this concerns the estimates in Proposition 4.5 discussed below. (b) can be shown similarly, replacing (k^2+1) by $(\omega^2 k^2+1)$. Alternatively, (b) can be reduced to (a) by instead solving for $\tilde{U}(t,x) := \omega^{-1}\,U(\omega^{-1}t, \omega^{-1}x)$, which then has to satisfy $\partial_t^2\tilde{U} - \Delta\tilde{U} - \omega^{-2}\,\tilde{U} = \Gamma(\omega^{-1}x)\,\tilde{U}^3$ and which is polychromatic with $\tilde{U}(t,x) = \tilde{u}_0(x) + \sum_{k=1}^\infty 2\,\cos(kt)\tilde{u}_k(x)$.

The generalization in (c) is, as announced, more involved and will be presented in the final section.

(d) is self-explanatory. In order to verify (e), we recall that for $\Gamma \equiv 0$ we necessarily have $w_0(x) = c\,\frac{\sin(|x|)}{|x|}$ which is the only radial and smooth solution of the (linear and homogeneous) Helmholtz equation (4.4). For the same reason, we have $\sigma_s = \tau_s = 0$ in Proposition 4.4 and equation (4.10). Thus for $\alpha \in \mathbb{R}$ and $k \in \mathbb{Z}$, $(\mathbf{v}^\alpha, \lambda^\alpha)$ solves according to (4.13)

$$v_k^\alpha = 0 \text{ for } k \in \mathbb{Z} \setminus \{\pm s\}, \quad v_{\pm s}^\alpha = (1-\lambda^\alpha)\cdot\Big(\alpha^{(s^2+1)}(v_{\pm s}^\alpha) + \beta^{(s^2+1)}(v_{\pm s}^\alpha)\Big)\cdot\tilde{\Psi}_{s^2+1}.$$

This yields, as asserted, for $\alpha \in \mathbb{R}$ and $t \in \mathbb{R}, x \neq 0$

$$U^\alpha(t,x) = w_0(x) + 2u_s^\alpha(x)\,\cos(st) = c\cdot\frac{\sin(|x|)}{|x|} + c_\alpha\cdot\frac{\sin(|x|\sqrt{s^2+1})}{|x|}\cdot\cos(st)$$

with $c_\alpha = \frac{1}{2\pi}\,(1-\lambda^\alpha)\cdot\Big(\alpha^{(s^2+1)}(v_{\pm s}^\alpha) + \beta^{(s^2+1)}(v_{\pm s}^\alpha)\Big)$. (In fact, it even implies $\beta^{(s^2+1)}(v_{\pm s}^\alpha) = 0$ and therefore $\lambda^\alpha = 0$ and $c_\alpha = \frac{1}{2\pi}\cdot\alpha^{(s^2+1)}(v_{\pm s}^\alpha)$.) □

4.3 On the Klein-Gordon Equation

In this part, we discuss Remark 4.2 (c); that is, we demonstrate how to apply our techniques in order to find polychromatic solutions of the cubic Klein-Gordon equation

$$\partial_t^2 U - \Delta U + U = \Gamma(x)\,U^3 \qquad \text{on} \quad \mathbb{R}\times\mathbb{R}^3. \tag{4.16}$$

This will eventually lead to a mixed stationary system as (4.7) consisting of (finitely many) Schrödinger-type and infinitely many Helmholtz-type equations. For technical reasons to be explained below, we aim at keeping the number of Schrödinger equations minimal and thus choose the ansatz

$$U(t,x) = u_0(x) + \sum_{k=1}^{\infty} 2\,\cos(2kt) u_k(x) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{2\mathrm{i}kt} u_k(x) \tag{4.17}$$

where $u_k = u_{-k} \in X_1$ for $k \in \mathbb{Z}$. A short and formal calculation then leads to a system of one Schrödinger and infinitely many Helmholtz equations

$$\begin{aligned} &-\Delta u_0 + u_0 &&= \Gamma(x)\ (\mathbf{u} \star \mathbf{u} \star \mathbf{u})_0\,, && && (4.18a)\\ &-\Delta u_k - (4k^2-1)u_k &&= \Gamma(x)\ (\mathbf{u} \star \mathbf{u} \star \mathbf{u})_k && \text{for } k \in \mathbb{Z}\setminus\{0\}. && (4.18b)\end{aligned}$$

In fact, (4.18b) includes (4.18a), but we intend to write the Schrödinger equation separately.

Again, we study bifurcation from a stationary solution $U^0(t,x) = w_0(x)$ where $w_0 \in X_1 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$ solves the cubic Schrödinger equation

$$-\Delta w_0 + w_0 = \Gamma(x)\, w_0^3 \qquad \text{on } \mathbb{R}^3. \tag{4.19}$$

We refer to the explanations following problem (1.1) in the introduction for a small selection of related existence results, e.g. concerning positive ground state solutions. However, even if there is much more knowledge about the existence of stationary solutions than in the Helmholtz case, we lose the flexibility of adding elements of the (nontrivial) Helmholtz kernel to guarantee nondegeneracy properties as in (4.11), which have proved useful in view of the simplicity condition of the Crandall-Rabinowitz Theorem. Thus, nondegeneracy will be imposed as an additional assumption, that is, we assume that for any $q_0 \in X_1 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$

$$-\Delta q_0 + q_0 = 3\Gamma(x)\, w_0^2\, q_0 \text{ on } \mathbb{R}^3 \qquad \text{implies} \qquad q_0 \equiv 0. \tag{4.20}$$

We comment on this assumption in Remark 4.9 (a) below. Assuming that it is satisfied, we can prove the analogue of Theorem 4.1.

Theorem 4.8 (Polychromatic solutions II).

Let $\Gamma \in L^\infty_{\mathrm{rad}}(\mathbb{R}^3) \cap C^1_{\mathrm{loc}}(\mathbb{R}^3)$ and assume there is some stationary solution $U^0(t,x) = w_0(x)$, $w_0 \not\equiv 0$ of the cubic Klein-Gordon equation (4.16), i.e. $w_0 \in X_1 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$ solving (4.19). Assume further that w_0 is nondegenerate in the sense of (4.20). Then for every $s \in \mathbb{N}$ there exist an open interval $J_s \subseteq \mathbb{R}$ containing 0 and a family $(U^\alpha)_{\alpha\in J_s} \subseteq C^2(\mathbb{R}, X_1) \cap C^2_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^3)$ with the following properties:

(i) All U^α are classical solutions of (4.16) of the polychromatic form (4.17),

$$U^\alpha(t,x) = u_0^\alpha(x) + \sum_{k=1}^{\infty} 2\,\cos(2kt) u_k^\alpha(x).$$

They are time-periodic with period 2π.

(ii) The map $\alpha \mapsto (u_k^\alpha)_{k\in\mathbb{N}_0}$ is smooth in the topology of $\ell^1(\mathbb{N}_0, X_1)$ with

$$\left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\right|_{\alpha=0} u_k^\alpha \not\equiv 0 \quad \textit{if and only if} \quad k = s$$

("excitation of the 2s-th mode"). In particular, for different values of s, these families consist of non-stationary solutions and mutually differ close to U^0.

(iii) *If we assume additionally that $\Gamma(x) \neq 0$ for almost all $x \in \mathbb{R}^3$, then for every $\alpha \in J_s \setminus \{0\}$ the polychromatic solution U^α possesses infinitely many nonvanishing modes u_k^α.*

Remark 4.9. (a) *In some special cases, nondegeneracy properties like* (4.20) *have been verified, e.g. by Bates and Shi [12] in Theorem 5.4 (6), or by Wei [75] in Lemma 4.1, both assuming that w_0 is a ground state solution of* (4.19) *in the autonomous case with constant positive Γ. It should be pointed out that, although the quoted results discuss nondegeneracy in a setting on the Hilbert space $H^1(\mathbb{R}^3)$, the statements can be adapted to the topology of $X_1 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$, as we will demonstrate in Lemma 4.11.*

(b) *An immediate generalization yields, for any given $\xi > 0$, polychromatic solutions to the equation*

$$\partial_t^2 U - \Delta U + \xi U = \Gamma(x)\, U^3 \qquad \text{on} \quad \mathbb{R} \times \mathbb{R}^3$$

which are of the form

$$U(t,x) = u_0(x) + \sum_{k=1}^{\infty} 2\,\cos(m_\xi k t) u_k(x) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{m_\xi i k t} u_k(x).$$

Here m_ξ is the least positive integer with $m_\xi > \sqrt{\xi}$ ("above the mass").

Indeed, such choice of m_ξ again leads to a coupled system of one Schrödinger and infinitely many Helmholtz equations as (4.18). Again only one nondegeneracy condition of the form (4.20) is required.
If one prefers to take $m_\xi = 1$ also if $\xi > 1$, a larger number of Schrödinger equations appears in the infinite system and hence more nondegeneracy assumptions have to be imposed. A disadvantage is here, however, that the result mentioned in (a) only applies for the 0-th component and thus cannot be used to guarantee nondegeneracy for these additional Schrödinger equations even in the special case described above.

We only sketch the proof of Theorem 4.8, focusing on the parts that differ from the discussion of Theorem 4.1. Essentially, this consists of a redefinition of the 0-th component of the map F resp. G using the Schrödinger resolvent $\mathcal{P}_1 : X_3 \to X_1$, which will be introduced next. Similar to equation (1.7), for radial Schwartz functions $u, f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^3)$ with $-\Delta u + u = f$ on $\mathbb{R}^3$, one can prove e.g. via formula (3.8) for the Fourier transform and the residue theorem

$$u(x) = \mathcal{F}^{-1}\left(\frac{\hat{f}}{|\cdot|^2 + 1}\right)(x) = \int_{\mathbb{R}^3} f(y)\, \frac{\mathrm{e}^{-|x-y|}}{4\pi|x-y|}\, \mathrm{d}y.$$

We thus define, on a suitable Banach space of radially symmetric functions $f : \mathbb{R}^3 \to \mathbb{R}$, the linear operator

$$\mathcal{P}_1 : f \mapsto \mathcal{P}_1[f] := \Lambda_1 * f \quad \text{with} \quad \Lambda_1(x) = \frac{\mathrm{e}^{-|x|}}{4\pi|x|} \quad (x \neq 0). \tag{4.21}$$

Some of the properties we require are presented in the next lemma.

Lemma 4.10 (On the Schrödinger resolvent).

The operator $\mathcal{P}_1 : X_3 \to X_1$ is well-defined, continuous and compact. Moreover, for $f \in X_3$, we have $w := \mathcal{P}_1[f] \in X_3 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$ with

$$-\Delta w + w = f \qquad \text{on } \mathbb{R}^3.$$

In this lemma, we do not aim for optimal regularity and decay properties but rather for an analogue of its Helmholtz counterpart in Proposition 3.13; the techniques in the proof will indeed be similar.

Let us remark that, in the Schrödinger case, we do not obtain a family of possible resolvent-type operators $\mathcal{R}_1^\tau = \mathcal{R}_1 + \cot(\tau)\tilde{\mathcal{R}}_1$, $0 < \tau < \pi$, as in the Helmholtz case. This is due to the fact that the homogeneous Helmholtz equation $-\Delta\psi - \psi = 0$ has a smooth and localized nontrivial solution $\tilde{\Psi}_1 \in X_1 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$ such that $\operatorname{ran} \tilde{\mathcal{R}}_1 = \operatorname{span}\{\tilde{\Psi}_1\}$ whereas the homogeneous Schrödinger equation does not. As a consequence, we have to impose nondegeneracy of w_0 as an assumption rather than, as in the Helmholtz case, generate it by choosing an appropriate resolvent $\mathcal{R}_1^\tau$.

Finally, as announced in Remark 4.9 (a), we verify the nondegeneracy assumption (4.20) for constant positive Γ.

Lemma 4.11 (Nondegeneracy, à la Bates and Shi [12]).

Let $\Gamma \equiv \Gamma_0$ for some $\Gamma_0 > 0$, and assume that $w_0 \in C^2_{\mathrm{rad}}(\mathbb{R}^3)$ is a radially symmetric solution of (4.19) the profile of which satisfies $w_0(r) > 0$, $w_0'(r) < 0$ for all $r > 0$, and both $w_0(r)$ and $w_0'(r)$ decay exponentially as $r \to \infty$. Then the nondegeneracy property (4.20) holds, i.e. for any radially symmetric, twice differentiable $q_0 \in X_1$

$$-\Delta q_0 + q_0 = 3\Gamma_0\, w_0^2\, q_0 \ \text{on } \mathbb{R}^3 \qquad \text{implies} \qquad q_0 \equiv 0.$$

We present the proof of both lemmas at the very end of this chapter in Section 4.4 and now turn to the main existence result for polychromatic solutions of the Klein-Gordon equation.

Proof of Theorem 4.8

As announced above, we replace the definitions (4.12), (4.13) of $F, G : \mathcal{X}_1 \times \mathbb{R} \to \mathcal{X}_1$ as follows. For $\mathbf{v} \in \mathcal{X}_1$ and $\lambda \in \mathbb{R}$, we let as usual $\mathbf{u} := \mathbf{v} + \mathbf{w}$ with $\mathbf{w} = (..., 0, 0, w_0, 0, 0, ...)$ for w_0 as described in the theorem and introduce τ_k, $k \in \mathbb{Z} \setminus \{0, \pm s\}$ analogous to (4.10), that is, $\tau_k \in \{\frac{\pi}{4}, \frac{3\pi}{4}\}$ with the nondegeneracy property

$$k \in \mathbb{Z} \setminus \{0, \pm s\}, \quad q \in X_1, \quad q = 3\,\mathcal{R}^{\tau_k}_{4k^2-1}[\Gamma w_0^2\, q] \qquad \Rightarrow \qquad q \equiv 0.$$

If $\sigma_s \in (0,\pi)$, we let $\tau_{\pm s} := \sigma_s$ and consider the map

$$F(\mathbf{v},\lambda)_k := v_k - \begin{cases} \mathcal{P}_1\left[\Gamma\ (\mathbf{u}\star\mathbf{u}\star\mathbf{u})_0 - \Gamma\, w_0^3\right] & k=0,\\ \mathcal{R}_{4s^2-1}\left[\Gamma\ (\mathbf{u}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right] & \\ \qquad +(\cot(\tau_{\pm s})-\lambda)\tilde{\mathcal{R}}_{4s^2-1}\left[\Gamma\ (\mathbf{u}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right] & k=\pm s,\\ \mathcal{R}^{\tau_k}_{4k^2-1}\left[\Gamma\ (\mathbf{u}\star\mathbf{u}\star\mathbf{u})_k\right] & \text{else.}\end{cases} \tag{4.22}$$

Similarly, if $\sigma_s = 0$, we introduce

$$G(\mathbf{v},\lambda)_k := v_k - \begin{cases} \mathcal{P}_1\left[\Gamma\ (\mathbf{u}\star\mathbf{u}\star\mathbf{u})_0 - \Gamma\, w_0^3\right] & k=0,\\ \mathcal{R}_{4s^2-1}\left[\Gamma\ (\mathbf{u}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right] & \\ \qquad +(1-\lambda)\left(\alpha^{(4s^2-1)}(v_{\pm s})+\beta^{(4s^2-1)}(v_{\pm s})\right)\tilde{\Psi}_{4s^2-1} & k=\pm s,\\ \mathcal{R}^{\tau_k}_{4k^2-1}\left[\Gamma\ (\mathbf{u}\star\mathbf{u}\star\mathbf{u})_k\right] & \text{else.}\end{cases} \tag{4.23}$$

With these redefinitions, and using in the 0-th component the mapping properties in Lemma 4.10 resp. the nondegeneracy properties (4.20) as a replacement for Proposition 3.13 resp. condition (4.11), it is possible to prove Theorem 4.8 along the lines of the proof of Theorem 4.1. □

4.4 Proofs of the Auxiliary Results

4.4.1 Results concerning Theorem 4.1

Proof of Proposition 4.3

Let $\mathbf{u}^{(j)} = (u_k^{(j)})_{k\in\mathbb{Z}} \in \mathcal{X}_1$ for $j=1,2,3$. We find the following chain of inequalities

$$\begin{aligned}
&\left\|\mathbf{u}^{(1)}\star\mathbf{u}^{(2)}\star\mathbf{u}^{(3)}\right\|_{\ell^1(\mathbb{Z},X_3)} = \sum_{k\in\mathbb{Z}}\left\|(\mathbf{u}^{(1)}\star\mathbf{u}^{(2)}\star\mathbf{u}^{(3)})_k\right\|_{X_3}\\
&= \sum_{k\in\mathbb{Z}}\left\|\sum_{\substack{l,m,n\in\mathbb{Z}\\ l+m+n=k}} u_l^{(1)}\,u_m^{(2)}\,u_n^{(3)}\right\|_{X_3}\\
&\le \sum_{k\in\mathbb{Z}}\sum_{\substack{l,m,n\in\mathbb{Z}\\ l+m+n=k}}\left\|u_l^{(1)}\,u_m^{(2)}\,u_n^{(3)}\right\|_{X_3}\\
&\le \sum_{k\in\mathbb{Z}}\sum_{\substack{l,m,n\in\mathbb{Z}\\ l+m+n=k}}\left\|u_l^{(1)}\right\|_{X_1}\left\|u_m^{(2)}\right\|_{X_1}\left\|u_n^{(3)}\right\|_{X_1}\\
&= \sum_{k\in\mathbb{Z}}\left(\left(\left\|u_l^{(1)}\right\|_{X_1}\right)_{l\in\mathbb{Z}}\star\left(\left\|u_m^{(2)}\right\|_{X_1}\right)_{m\in\mathbb{Z}}\star\left(\left\|u_n^{(3)}\right\|_{X_1}\right)_{n\in\mathbb{Z}}\right)_k\\
&= \left\|\left(\left\|u_l^{(1)}\right\|_{X_1}\right)_{l\in\mathbb{Z}}\star\left(\left\|u_m^{(2)}\right\|_{X_1}\right)_{m\in\mathbb{Z}}\star\left(\left\|u_n^{(3)}\right\|_{X_1}\right)_{n\in\mathbb{Z}}\right\|_{\ell^1(\mathbb{Z})}\\
&\le \left\|\left(\left\|u_l^{(1)}\right\|_{X_1}\right)_{l\in\mathbb{Z}}\right\|_{\ell^1(\mathbb{Z})}\left\|\left(\left\|u_m^{(2)}\right\|_{X_1}\right)_{m\in\mathbb{Z}}\right\|_{\ell^1(\mathbb{Z})}\left\|\left(\left\|u_n^{(3)}\right\|_{X_1}\right)_{n\in\mathbb{Z}}\right\|_{\ell^1(\mathbb{Z})}\\
&= \left\|\mathbf{u}^{(1)}\right\|_{\ell^1(\mathbb{Z},X_1)}\left\|\mathbf{u}^{(2)}\right\|_{\ell^1(\mathbb{Z},X_1)}\left\|\mathbf{u}^{(3)}\right\|_{\ell^1(\mathbb{Z},X_1)}
\end{aligned}$$

$$< \infty,$$

where Young's inequality for convolutions in $\ell^1(\mathbb{Z})$ has been applied. Since absolutely convergent sums in Banach spaces always have a limit, we infer $\mathbf{u}^{(1)} \star \mathbf{u}^{(2)} \star \mathbf{u}^{(3)} \in \mathcal{X}_3$. □

We turn to the proof of Proposition 4.5. This requires convergence properties in order to handle the infinite series in the definition of $U(t,x)$, which we first provide in the following two lemmas.

Lemma 4.12.

The convolution operators $\mathcal{R}_\omega^\tau : X_3 \to X_1$ satisfy for $\tau \in (0,\pi)$ and $\omega > 0$

$$\forall\, f \in X_3 \qquad \begin{aligned} \|\mathcal{R}_\omega^\tau[f]\|_{X_1} &\le \frac{C}{\sin(\tau)} \left(1 + \frac{1}{\sqrt{\omega}}\right) \cdot \|f\|_{X_3}\,, \\ \|\mathcal{R}_\omega^\tau[f]\|_{L^4(\mathbb{R}^3)} &\le \frac{C}{\sqrt[4]{\omega}\,\sin(\tau)} \cdot \|f\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}\,. \end{aligned}$$

The fact that a power of ω appears in the denominator is crucial since it will finally provide the convergence and regularity of the polychromatic sums where $\omega = \omega_k = k^2+1$ for $k \in \mathbb{Z}$.

Proof of Lemma 4.12

To see the estimate on the spaces X_3, X_1, we recall the explicit norm estimate (3.38) from the previous chapter telling us that

$$\forall\, f \in X_3 \qquad \|\mathfrak{R}_\omega[f]\|_{X_1} \le C \left(1 + \frac{1}{\sqrt{\omega}}\right) \cdot \|f\|_{X_3}\,.$$

Thus, due to $\mathfrak{R}_\omega = \mathcal{R}_\omega + \mathrm{i}\tilde{\mathcal{R}}_\omega$ where $\mathcal{R}_\omega, \tilde{\mathcal{R}}_\omega$ have real-valued kernels, we obtain for (real-valued) functions $f \in X_3$ the identity

$$\mathcal{R}_\omega^\tau[f] \overset{(3.12)}{=} \mathcal{R}_\omega[f] + \cot(\tau)\,\tilde{\mathcal{R}}_\omega[f] = \operatorname{Im}\left[\frac{\mathrm{e}^{\mathrm{i}\tau}}{\sin(\tau)} \cdot \mathfrak{R}_\omega[f]\right] \tag{4.24}$$

and therefore the estimate

$$\|\mathcal{R}_\omega^\tau[f]\|_{X_1} \le \frac{1}{\sin(\tau)}\, \|\mathfrak{R}_\omega[f]\|_{X_1} \le \frac{C}{\sin(\tau)} \left(1 + \frac{1}{\sqrt{\omega}}\right) \|f\|_{X_3}\,.$$

The estimate on L^p spaces can be proved by a rescaling technique involving Theorem 2.1 in [28], which provides $C > 0$ with

$$\forall\, f \in L^{\frac{4}{3}}(\mathbb{R}^3) \qquad \|\mathfrak{R}_1[f]\|_{L^4(\mathbb{R}^3)} \le C \cdot \|f\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}\,.$$

Since the functions under study have values in $\mathbb{R}$, we obtain as above

$$\forall\, f \in L^{\frac{4}{3}}(\mathbb{R}^3) \qquad \|\mathcal{R}_1^\tau[f]\|_{L^4(\mathbb{R}^3)} \le \frac{1}{\sin(\tau)}\, \|\mathfrak{R}_1[f]\|_{L^4(\mathbb{R}^3,\mathbb{C})} \le \frac{C}{\sin(\tau)}\, \|f\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}\,. \tag{4.25}$$

Then, for $\omega \neq 1$ and $f \in L^{\frac{4}{3}}_{\mathrm{rad}}(\mathbb{R}^3)$, we estimate via scaling

$$\begin{aligned}
\|\mathcal{R}^\tau_\omega[f]\|_{L^4(\mathbb{R}^3)} &= \left(\int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} \frac{\sin(|y|\sqrt{\omega}+\tau)}{4\pi|y|\sin(\tau)} f(x-y)\,\mathrm{d}y \right|^4 \mathrm{d}x \right)^{\frac{1}{4}} \\
&= \left(\int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} \frac{\sin(|y'|+\tau)}{4\pi|y'|\sin(\tau)} f\left(\omega^{-1/2}(x'-y')\right) \frac{\mathrm{d}y'}{\omega} \right|^4 \frac{\mathrm{d}x'}{\omega^{3/2}} \right)^{\frac{1}{4}} \\
&= \omega^{-11/8} \cdot \left\| \mathcal{R}^\tau_1 \left[f\left(\omega^{-1/2}\,\cdot\right)\right]\right\|_{L^4(\mathbb{R}^3)} \\
&\overset{(4.25)}{\leq} \omega^{-11/8} \cdot \frac{C}{\sin(\tau)} \cdot \left\| f\left(\omega^{-1/2}\,\cdot\right)\right\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \\
&= \omega^{-11/8} \cdot \frac{C}{\sin(\tau)} \cdot \omega^{9/8} \cdot \|f\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \\
&= \frac{C}{\omega^{1/4}\sin(\tau)} \cdot \|f\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}.
\end{aligned}$$

This proves the assertion. □

Lemma 4.13 (Decay and regularity estimates).

Assume that $M \subseteq \mathbb{Z}$ is finite and symmetric, $\Gamma \in L^\infty_{\mathrm{rad}}(\mathbb{R}^3) \cap C^1_{\mathrm{loc}}(\mathbb{R}^3)$ and $\mathbf{u} = (u_k)_{k\in\mathbb{Z}} \in \mathcal{X}_1$ satisfies the infinite system of convolution equations

$$\begin{aligned}
u_k &= \mathcal{R}^{\tau_k}_{k^2+1}[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_k] && \text{if } k \in \mathbb{Z}\setminus M, \\
u_k &= \mathcal{R}^{\tau_k}_{k^2+1}[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_k] + g_k && \text{if } k \in M
\end{aligned}$$

where $g_k = g_{-k} \in X_1 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$ and $\tau_k \in (\delta, \pi-\delta)$ for some fixed $\delta \in \left(0, \frac{\pi}{2}\right)$. Then the following holds:

(i) For every $\alpha \geq 0$, there exists a constant $C_\alpha \geq 0$ with

$$\|u_k\|_{L^4(\mathbb{R}^3)} + \|\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_k\|_{L^4(\mathbb{R}^3)} \leq C_\alpha \cdot (k^2+1)^{-\frac{\alpha}{2}} \qquad (k\in\mathbb{Z}).$$

(ii) All u_k are twice continuously differentiable, and for every centered ball $B = B_R(0) \subseteq \mathbb{R}^3$ and every $\alpha \geq 0$ there exists a constant $D_\alpha(B)$ with

$$|u_k(x)| + |\nabla u_k(x)| + |D^2 u_k(x)| \leq D_\alpha(B) \cdot (k^2+1)^{-\frac{\alpha}{2}} \qquad (k\in\mathbb{Z}, x \in B).$$

(iii) For every $\alpha \geq 0$, there exists a constant $E_\alpha \geq 0$ with

$$\|u_k\|_{X_1} \leq E_\alpha \cdot (k^2+1)^{-\frac{\alpha}{2}} \qquad (k\in\mathbb{Z}).$$

Proof of Lemma 4.13

For notational convenience, we define for $k \in \mathbb{Z}$

$$\langle k \rangle := \left(1+k^2\right)^{\frac{1}{2}}.$$

Let us remark that, for $k, l \in \mathbb{Z}$ and $\alpha \geq 0$,

$$\begin{aligned} \langle k+l\rangle^{\alpha} = \big(1+(k+l)^2\big)^{\frac{\alpha}{2}} &\leq \big(1+2k^2+2l^2\big)^{\frac{\alpha}{2}} \\ &\leq 2^{\frac{\alpha}{2}} \cdot \big((1+k^2)(1+l^2)\big)^{\frac{\alpha}{2}} = 2^{\frac{\alpha}{2}} \cdot \langle k\rangle^{\alpha} \langle l\rangle^{\alpha}. \end{aligned} \tag{4.26}$$

Throughout, we abbreviate $f_k := \Gamma\,(\mathbf{u} \star \mathbf{u} \star \mathbf{u})_k = \Gamma \sum_{l+m+n=k} u_l\, u_m\, u_n$ and set $g_k := 0$ for $k \in \mathbb{Z} \setminus M$. Then,

$$u_k = \mathcal{R}^{\tau_k}_{\langle k\rangle^2}[f_k] + g_k \qquad \text{for all } k \in \mathbb{Z} \tag{4.27}$$

with $f_k \in X_3, g_k \in X_1 \cap C^2_{\text{loc}}(\mathbb{R}^3)$ and $g_k = 0$ up to finitely many exceptions. Without loss of generality, we let $M = [-s, s] \cap \mathbb{Z}$ for some $s \in \mathbb{N}$. The previous Lemma 4.12 yields, since we assume $\delta < \tau_k < \pi - \delta$,

$$\begin{aligned} \|u_k\|_{L^4(\mathbb{R}^3)} &\leq \frac{C}{\langle k\rangle^{\frac{1}{2}}\, \sin(\delta)} \cdot \|f_k\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} + \frac{\langle s\rangle^{\frac{1}{2}}}{\langle k\rangle^{\frac{1}{2}}} \|g_k\|_{L^4(\mathbb{R}^3)} \\ &\leq \frac{C_1}{\langle k\rangle^{\frac{1}{2}}} \cdot \Big(\|f_k\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} + \|g_k\|_{L^4(\mathbb{R}^3)}\Big) \end{aligned} \tag{4.28}$$

for all $k \in \mathbb{Z}$, where we can choose $C_1 := \frac{C}{\sin(\delta)} + \langle s\rangle^{\frac{1}{2}}$.

$\triangleright$ <u>Step 1</u>: *Proof of (i): Bounds on the L^4 norms.*

For $\alpha \geq 0$, we first prove the existence of constants $C'_\alpha \geq 0$ with

$$\|u_k\|_{L^4(\mathbb{R}^3)} \leq C'_\alpha \cdot \langle k\rangle^{-\alpha} \qquad \text{for all } k \in \mathbb{Z} \tag{4.29}$$

by an iterative procedure based on the scaling property of the convolutions stated in Lemma 4.12; more precisely, we prove that one can choose

$$C'_\alpha := \sum_{k\in\mathbb{Z}} \langle k\rangle^{\alpha}\, \|u_k\|_{L^4(\mathbb{R}^3)} < \infty.$$

To start with, the asserted estimate holds true for $\alpha = 0$ since, by assumption, $\mathbf{u} \in \ell^1(\mathbb{Z}, X_1)$ and $X_1 \hookrightarrow L^4(\mathbb{R}^3)$. Assuming it holds for some $\alpha \geq 0$, we estimate

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \langle k\rangle^{\alpha+\frac{1}{2}} \cdot \|u_k\|_{L^4(\mathbb{R}^3)} &\overset{(4.28)}{\leq} \sum_{k\in\mathbb{Z}} \langle k\rangle^{\alpha} \cdot C_1 \cdot \Big(\|f_k\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} + \|g_k\|_{L^4(\mathbb{R}^3)}\Big) \\ &\leq C_1 \cdot \Bigg(\sum_{k\in\mathbb{Z}} \|\Gamma\|_\infty \sum_{l+m+n=k} \Big[\langle k\rangle^{\alpha}\, \|u_l\|_{L^4(\mathbb{R}^3)}\, \|u_m\|_{L^4(\mathbb{R}^3)}\, \|u_n\|_{L^4(\mathbb{R}^3)}\Big] \\ &\qquad + \sum_{k\in\mathbb{Z}} \langle k\rangle^{\alpha}\, \|g_k\|_{L^4(\mathbb{R}^3)}\Bigg) \\ &= C_1 \cdot \Bigg(\sum_{k\in\mathbb{Z}} \|\Gamma\|_\infty \sum_{l+m+n=k} \Big[\langle l+m+n\rangle^{\alpha}\, \|u_l\|_{L^4(\mathbb{R}^3)}\, \|u_m\|_{L^4(\mathbb{R}^3)}\, \|u_n\|_{L^4(\mathbb{R}^3)}\Big] \\ &\qquad + \sum_{k\in\mathbb{Z}} \langle k\rangle^{\alpha}\, \|g_k\|_{L^4(\mathbb{R}^3)}\Bigg) \\ &\overset{(4.26)}{\leq} C_1 \cdot \Bigg(\sum_{k\in\mathbb{Z}} \|\Gamma\|_\infty \sum_{l+m+n=k} \Big[2^{\alpha}\, \langle l\rangle^{\alpha}\, \|u_l\|_{L^4(\mathbb{R}^3)}\, \langle m\rangle^{\alpha}\, \|u_m\|_{L^4(\mathbb{R}^3)}\, \langle n\rangle^{\alpha}\, \|u_n\|_{L^4(\mathbb{R}^3)}\Big] \end{aligned}$$

$$
\begin{aligned}
&\qquad + \sum_{k\in M} \langle s\rangle^\alpha \max_{n\in M} \|g_n\|_{L^4(\mathbb{R}^3)} \Bigg) \\
&\le C_1 \cdot \left(2^\alpha \|\Gamma\|_\infty \left(\sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \|u_n\|_{L^4(\mathbb{R}^3)} \right)^3 + (2s+1)\ \langle s\rangle^\alpha \max_{n\in M} \|g_n\|_{L^4(\mathbb{R}^3)} \right),
\end{aligned}
$$

and this is finite by assumption. Iterating this, we see that the claimed estimate holds for all $\alpha \ge 0$. We now derive the estimate $\|f_k\|_{L^4(\mathbb{R}^3)} \le C''_\alpha \cdot \langle k\rangle^{-\alpha}$. We find for $k \in \mathbb{Z}$

$$
\begin{aligned}
&\langle k\rangle^\alpha\ \|f_k\|_{L^4(\mathbb{R}^3)} \\
&\quad \le \sum_{l+m+n=k} \|\Gamma\|_\infty \langle l+m+n\rangle^\alpha \|u_l\|_{L^{12}(\mathbb{R}^3)}\ \|u_m\|_{L^{12}(\mathbb{R}^3)}\ \|u_n\|_{L^{12}(\mathbb{R}^3)} \\
&\quad \overset{(4.26)}{\le} 2^\alpha \cdot \sum_{l+m+n=k} \|\Gamma\|_\infty \langle l\rangle^\alpha \langle m\rangle^\alpha \langle n\rangle^\alpha \cdot \Big(\|u_l\|_{L^4(\mathbb{R}^3)}\ \|u_m\|_{L^4(\mathbb{R}^3)}\ \|u_n\|_{L^4(\mathbb{R}^3)} \Big)^{\frac13} \\
&\qquad\qquad \cdot \Big(\|u_l\|_{L^\infty(\mathbb{R}^3)}\ \|u_m\|_{L^\infty(\mathbb{R}^3)}\ \|u_n\|_{L^\infty(\mathbb{R}^3)} \Big)^{\frac23} \\
&\quad \le 2^\alpha \cdot \sum_{l+m+n=k} \|\Gamma\|_\infty \langle l\rangle^\alpha \langle m\rangle^\alpha \langle n\rangle^\alpha \cdot \Big(\|u_l\|_{L^4(\mathbb{R}^3)}\ \|u_m\|_{L^4(\mathbb{R}^3)}\ \|u_n\|_{L^4(\mathbb{R}^3)} \Big)^{\frac13} \\
&\qquad\qquad \cdot \big(\|u_l\|_{X_1}\ \|u_m\|_{X_1}\ \|u_n\|_{X_1} \big)^{\frac23} \\
&\quad \le 2^\alpha \|\mathbf{u}\|^2_{\ell^1(\mathbb{Z},X_1)} \|\Gamma\|_\infty \sum_{l+m+n=k} \langle l\rangle^\alpha \langle m\rangle^\alpha \langle n\rangle^\alpha \cdot \Big(\|u_l\|_{L^4(\mathbb{R}^3)}\ \|u_m\|_{L^4(\mathbb{R}^3)}\ \|u_n\|_{L^4(\mathbb{R}^3)} \Big)^{\frac13} \\
&\quad \le 2^\alpha \|\mathbf{u}\|^2_{\ell^1(\mathbb{Z},X_1)} \|\Gamma\|_\infty \left(\sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \cdot \|u_n\|^{\frac13}_{L^4(\mathbb{R}^3)} \right)^3 \\
&\quad \overset{(4.29)}{\le} 2^\alpha \|\mathbf{u}\|^2_{\ell^1(\mathbb{Z},X_1)} \|\Gamma\|_\infty \left(\sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \cdot \Big[C'_{3\alpha+6} \cdot \langle n\rangle^{-3\alpha-6} \Big]^{\frac13} \right)^3 \\
&\quad \le 2^\alpha C'_{3\alpha+6} \|\mathbf{u}\|^2_{\ell^1(\mathbb{Z},X_1)} \|\Gamma\|_\infty \left(\sum_{n\in\mathbb{Z}} \frac{1}{n^2+1} \right)^3,
\end{aligned}
$$

which is finite and hence the assertion of (i) is verified.

▷ <u>Step 2</u>: *Proof of (ii): Local C^2 bounds.*

Let us recall (4.27), $u_k = \mathcal{R}^{\tau_k}_{\langle k\rangle^2}[f_k] + g_k$ with $g_k \in X_1 \cap C^2_{\text{loc}}(\mathbb{R}^3)$ and $g_k = 0$ for $|k| > s$. By Propositions 4.3 and 3.13, $f_k = \Gamma\,(\mathbf{u} \star \mathbf{u} \star \mathbf{u})_k \in X_3$ and hence $u_k \in X_1 \cap C^2_{\text{loc}}(\mathbb{R}^3)$.

We fix open balls $B := B_R(0)$ and $B' := B_{2R}(0)$. It is sufficient to prove the local estimate for indices $k \in \mathbb{Z}$ with $|k| > s$ since we have just seen that $u_k \in C^2(\bar{B}')$ and hence, for any $\alpha \ge 0$ and all $k \in \mathbb{Z} \cap [-s, s]$,

$$
\sup_{x\in\bar{B}'} \big(|u_k(x)| + |\nabla u_k(x)| + |D^2 u_k(x)| \big) \le \underbrace{\left[\langle s\rangle^\alpha \cdot \sup_{|j|\le s} \|u_j\|_{C^2(\bar{B}')} \right]}_{=:D^{(0)}_\alpha(B')<\infty} \cdot \langle k\rangle^{-\alpha}. \tag{4.30}
$$

For $k \in \mathbb{Z}$ with $|k| > s$, we apply elliptic regularity with the intention to exploit the strong bounds proved in part (i). We notice first that (the restriction of) u_k is the unique solution of the Dirichlet problem

$$-\Delta\varphi = \langle k\rangle^2 u_k + f_k \quad \text{in } B', \qquad \varphi = u_k(2R) \quad \text{on } \partial B' \tag{4.31}$$

in the space $W^{1,2}(B')$, see Theorem 8.3 in [31]. Here we have rearranged the equation in such way that the constants in Chapters 6, 9 of the book [31] by Gilbarg and Trudinger do not depend on the coefficient $\langle k\rangle^2$.

▷▷ Step 2 (a): *L^4 estimates and a bound in $W^{2,4}$.*

We first use L^4 estimates in order to benefit from the bounds derived in (i). Since u_k is an element of $W^{2,4}_{\text{loc}}(\mathbb{R}^3)$ and satisfies (4.31), Theorem 9.11 in [31] provides a constant $D(B')$ depending only on the size of the ball B' such that, for every $|k| > s$ and $\alpha \geq 0$,

$$\begin{aligned}
\|u_k\|_{W^{2,4}(B')} &\leq D(B') \cdot \left(\left\| f_k + \langle k\rangle^2 u_k \right\|_{L^4(\mathbb{R}^3)} + \|u_k\|_{L^4(\mathbb{R}^3)} \right) \\
&\leq D(B') \cdot 2 \langle k\rangle^2 \left(\|f_k\|_{L^4(\mathbb{R}^3)} + \|u_k\|_{L^4(\mathbb{R}^3)} \right) \\
&\overset{(i)}{\leq} D(B') \cdot 2 \cdot C_{\alpha+2} \cdot \langle k\rangle^{-\alpha} .
\end{aligned}$$

The Sobolev embedding $W^{2,4}(B') \hookrightarrow C^1(\overline{B}')$ combined with the estimate (4.30) (for terms with $|k| \leq s$) then provides some constant $D^{(1)}_\alpha(B') > 0$ with

$$\forall\, k \in \mathbb{Z} \qquad \|u_k\|_{C^1(\overline{B}')} \leq D^{(1)}_\alpha(B') \cdot \langle k\rangle^{-\alpha} . \tag{4.32}$$

▷▷ Step 2 (b): *Hölder estimates and a bound in C^2.*

Since we assume $\Gamma \in C^1_{\text{loc}}(\mathbb{R}^3)$, we can use (4.32) to estimate as follows:

$$\begin{aligned}
&\left\| \langle k\rangle^2 u_k + f_k \right\|_{C^1(\overline{B'})} \\
&\quad\leq \langle k\rangle^2 \|u_k\|_{C^1(\overline{B'})} + \|\Gamma\|_{C^1(\overline{B'})} \sum_{l+m+n=k} \|u_l\|_{C^1(\overline{B'})} \|u_m\|_{C^1(\overline{B'})} \|u_n\|_{C^1(\overline{B'})} \\
&\quad\overset{(4.32)}{\leq} D^{(1)}_{\alpha+2}(B') \cdot \langle k\rangle^{-\alpha} + \|\Gamma\|_{C^1(\overline{B'})} D^{(1)}_{\alpha+2}(B')^3 \sum_{l+m+n=k} \langle l\rangle^{-\alpha-2} \langle m\rangle^{-\alpha-2} \langle n\rangle^{-\alpha-2} \\
&\quad\overset{(4.26)}{\leq} D^{(1)}_{\alpha+2}(B') \cdot \langle k\rangle^{-\alpha} \\
&\qquad\quad + \|\Gamma\|_{C^1(\overline{B'})} D^{(1)}_{\alpha+2}(B')^3 \sum_{l+m+n=k} \langle l\rangle^{-2} \langle m\rangle^{-2} \langle n\rangle^{-2} \cdot 2^\alpha \langle l+m+n\rangle^{-\alpha} \\
&\quad\leq \left[D^{(1)}_{\alpha+2}(B') + \|\Gamma\|_{C^1(\overline{B'})} D^{(1)}_{\alpha+2}(B')^3 \left(\sum_{n\in\mathbb{Z}} \frac{1}{n^2+1} \right)^3 \cdot 2^\alpha \right] \cdot \langle k\rangle^{-\alpha} .
\end{aligned}$$

Then $\langle k\rangle^2 u_k + f_k \in C^1(\overline{B}') \subseteq C^{0,\gamma}(\overline{B}')$ for some fixed $\gamma \in (0,1)$ and all $k \in \mathbb{Z}$. For all $\alpha \geq 0$, this provides constants $D^{(2)}_\alpha(B') > 0$ with

$$\forall\, k \in \mathbb{Z} \qquad \left\| \langle k\rangle^2 u_k + f_k \right\|_{C^{0,\gamma}(\overline{B'})} \leq D^{(2)}_\alpha(B') \cdot \langle k\rangle^{-\alpha} . \tag{4.33}$$

But then, Corollary 6.9 in [31] ensures that $u_k \in C^{2,\gamma}(\overline{B'})$ as a solution of the Dirichlet problem (4.31). The Schauder interior estimates on $B \subset\subset B'$, see e.g. Corollary 6.3 in [31], provide constants $D(B,B')>0$ such that, for $|k|>s$,

$$\begin{aligned}\|u_k\|_{C^{2,\gamma}(\overline{B})} &\le D(B,B')\cdot\left(\|u_k\|_{C^0(\overline{B'})}+\left\|\langle k\rangle^2 u_k+f_k\right\|_{C^{0,\gamma}(\overline{B'})}\right)\\ &\overset{(4.32),(4.33)}{\le} D(B,B')\cdot\left(D_\alpha^{(1)}(B')\ \langle k\rangle^{-\alpha}+D_\alpha^{(2)}(B')\ \langle k\rangle^{-\alpha}\right).\end{aligned}$$

Combining this with (4.30), we infer in particular for $\alpha\ge 0$

$$D_\alpha(B):=\sup_{k\in\mathbb{Z}}\ \sup_{x\in\overline{B}}\ \left(|u_k(x)|+|\nabla u_k(x)|+|D^2u_k(x)|\right)\langle k\rangle^\alpha\le\sup_{k\in\mathbb{Z}}\ \|u_k\|_{C^{2,\gamma}(\overline{B})}\langle k\rangle^\alpha<\infty$$

and the proof is complete.

$\triangleright$ <u>Step 3</u>: *Proof of (iii): Bounds on the X_1 norms.*

Let $\alpha\ge 0$. For $k\in\mathbb{Z}$ with $|k|\le s$, we have

$$\|u_k\|_{X_1}\le E'_\alpha\cdot\langle k\rangle^{-\alpha}\quad\text{where } E'_\alpha:=\langle s\rangle^\alpha\cdot\sup_{|k|\le s}\|u_k\|_{X_1}<\infty. \tag{4.34}$$

From now on, we consider $k\in\mathbb{Z}$ with $|k|>s$. We recall $f_k=\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_k$ as well as

$$\begin{aligned}u_k(x)&=\mathcal{R}^{\tau_k}_{k^2+1}[f_k]\,(x)\\ &\overset{(4.24)}{=}\operatorname{Im}\left[\frac{e^{i\tau_k}}{\sin(\tau_k)}\cdot\mathfrak{R}_{\langle k\rangle^2}[f_k]\,(x)\right]\\ &=\int_{\mathbb{R}^3}\frac{\sin(|x-y|\,\langle k\rangle+\tau_k)}{4\pi|x-y|\sin(\tau_k)}\cdot f_k(y)\,\mathrm{d}y &(4.35a)\\ &=\frac{\sin(\langle k\rangle\,|x|+\tau_k)}{|x|\sin(\tau_k)}\int_0^{|x|}\frac{\sin(\langle k\rangle\,r)}{\langle k\rangle\,r}f_k(r)\,r^2\,\mathrm{d}r\\ &\qquad+\frac{\sin(\langle k\rangle\,|x|)}{|x|\sin(\tau_k)}\int_{|x|}^\infty\frac{\sin(\langle k\rangle\,r+\tau_k)}{\langle k\rangle\,r}f_k(r)\,r^2\,\mathrm{d}r &(4.35b)\end{aligned}$$

for $x\in\mathbb{R}^3$, where the last line(s) can be obtained by applying Lemma 3.10 to the second line. Further, for $x\in\mathbb{R}^3$,

$$\begin{aligned}|u_k(x)|&\overset{(4.35a)}{\le}\int_{\mathbb{R}^3}\left|\frac{\sin(|x-y|\,\langle k\rangle+\tau_k)}{4\pi|x-y|\sin(\tau_k)}\cdot\Gamma(y)\sum_{l+m+n=k}u_l(y)u_m(y)u_n(y)\right|\,\mathrm{d}y\\ &\le\frac{\|\Gamma\|_\infty}{4\pi\sin(\delta)}\sum_{l+m+n=k}\int_{\mathbb{R}^3}\frac{|u_l(y)u_m(y)u_n(y)|}{|x-y|}\,\mathrm{d}y.\end{aligned} \tag{4.36}$$

We split the integral and estimate further on $\mathbb{R}^3\setminus B_1(x)$ via Hölder's inequality

$$\begin{aligned}&\int_{\mathbb{R}^3\setminus B_1(x)}\frac{|u_l(y)u_m(y)u_n(y)|}{|x-y|}\,\mathrm{d}y\\ &\le\|u_l\|_{L^4(\mathbb{R}^3)}\,\|u_m\|_{L^4(\mathbb{R}^3)}\,\|u_n\|_{L^4(\mathbb{R}^3)}\left(\int_{\mathbb{R}^3\setminus B_1(x)}\frac{\mathrm{d}y}{|x-y|^4}\right)^{\frac14}\\ &=(4\pi)^{\frac14}\ \|u_l\|_{L^4(\mathbb{R}^3)}\,\|u_m\|_{L^4(\mathbb{R}^3)}\,\|u_n\|_{L^4(\mathbb{R}^3)}\\ &\overset{(4.29)}{\le}(4\pi)^{\frac14}\,(C'_{\alpha+2})^3\,\frac{\langle l\rangle^{-\alpha}\,\langle m\rangle^{-\alpha}\,\langle n\rangle^{-\alpha}}{(1+l^2)(1+m^2)(1+n^2)}\end{aligned}$$

$$\overset{(4.26)}{\leq} (4\pi)^{\frac{1}{4}} \, (C'_{\alpha+2})^3 \, 2^\alpha \, \frac{\langle l+m+n\rangle^{-\alpha}}{(1+l^2)(1+m^2)(1+n^2)}.$$

For the integral on $B_1(x)$, Hölder's inequality leads to

$$\begin{aligned}
&\int_{B_1(x)} \frac{|u_l(y)u_m(y)u_n(y)|}{|x-y|}\,\mathrm{d}y \\
&\leq \|u_l u_m u_n\|_\infty^{\frac{2}{3}} \int_{B_1(x)} \frac{|u_l(y)u_m(y)u_n(y)|^{\frac{1}{3}}}{|x-y|}\,\mathrm{d}y \\
&\leq \|u_l u_m u_n\|_\infty^{\frac{2}{3}} \left(\|u_l\|_{L^4(\mathbb{R}^3)} \|u_m\|_{L^4(\mathbb{R}^3)} \|u_n\|_{L^4(\mathbb{R}^3)}\right)^{\frac{1}{3}} \left(\int_{B_1(x)} \frac{\mathrm{d}y}{|x-y|^{\frac{4}{3}}}\right)^{\frac{3}{4}} \\
&= \left(4\pi\cdot\frac{3}{5}\right)^{\frac{3}{4}} \|u_l u_m u_n\|_\infty^{\frac{2}{3}} \left(\|u_l\|_{L^4(\mathbb{R}^3)} \|u_m\|_{L^4(\mathbb{R}^3)} \|u_n\|_{L^4(\mathbb{R}^3)}\right)^{\frac{1}{3}} \\
&\leq (4\pi)^{\frac{3}{4}} \, \|\mathbf{u}\|^2_{\ell^1(\mathbb{Z},X_1)} \left(\|u_l\|_{L^4(\mathbb{R}^3)} \|u_m\|_{L^4(\mathbb{R}^3)} \|u_n\|_{L^4(\mathbb{R}^3)}\right)^{\frac{1}{3}} \\
&\overset{(4.29)}{\leq} (4\pi)^{\frac{3}{4}} \, \|\mathbf{u}\|^2_{\ell^1(\mathbb{Z},X_1)} \, C'_{3\alpha+6} \, \frac{\langle l\rangle^{-\alpha}\,\langle m\rangle^{-\alpha}\,\langle n\rangle^{-\alpha}}{(1+l^2)(1+m^2)(1+n^2)} \\
&\overset{(4.26)}{\leq} (4\pi)^{\frac{3}{4}} \, \|\mathbf{u}\|^2_{\ell^1(\mathbb{Z},X_1)} \, C'_{3\alpha+6} \, 2^\alpha \, \frac{\langle l+m+n\rangle^{-\alpha}}{(1+l^2)(1+m^2)(1+n^2)}.
\end{aligned}$$

Inserting both estimates into (4.36), we conclude

$$\begin{aligned}
|u_k(x)| &\leq \frac{\|\Gamma\|_\infty}{4\pi\sin(\delta)} \sum_{l+m+n=k} \int_{\mathbb{R}^3} \frac{|u_l(y)u_m(y)u_n(y)|}{|x-y|}\,\mathrm{d}y \\
&\leq \frac{2^\alpha \, \|\Gamma\|_\infty}{\sin(\delta)} \left((C'_{\alpha+2})^3 + \|\mathbf{u}\|^2_{\ell^1(\mathbb{Z},X_1)} \, C'_{3\alpha+6}\right) \sum_{l+m+n=k} \frac{\langle k\rangle^{-\alpha}}{(1+l^2)(1+m^2)(1+n^2)} \\
&\leq \frac{2^\alpha \, \|\Gamma\|_\infty}{\sin(\delta)} \left((C'_{\alpha+2})^3 + \|\mathbf{u}\|^2_{\ell^1(\mathbb{Z},X_1)} \, C'_{3\alpha+6}\right) \left(\sum_{n\in\mathbb{Z}} \frac{1}{1+n^2}\right)^3 \cdot \langle k\rangle^{-\alpha} \\
&=: E''_\alpha \cdot \langle k\rangle^{-\alpha}\,.
\end{aligned} \tag{4.37}$$

Next, we estimate $|x||u_k(x)|$ for $|x|\geq 1$ using the radial formula for the convolution

$$\begin{aligned}
&|x||u_k(x)| \\
&\overset{(4.35b)}{=} \left|\frac{\sin(\langle k\rangle\,|x|+\tau_k)}{\sin(\tau_k)} \int_0^{|x|} \frac{\sin(\langle k\rangle\, r)}{\langle k\rangle\, r} f_k(r)\, r^2\,\mathrm{d}r \right. \\
&\qquad\qquad \left. + \frac{\sin(\langle k\rangle\,|x|)}{\sin(\tau_k)} \int_{|x|}^\infty \frac{\sin(\langle k\rangle\, r+\tau_k)}{\langle k\rangle\, r} f_k(r)\, r^2\,\mathrm{d}r \right| \\
&\leq \frac{1}{\sin(\delta)} \int_0^{|x|} \left|\frac{\sin(\langle k\rangle\, r)}{\langle k\rangle\, r}\right| |f_k(r)|\, r^2\,\mathrm{d}r + \frac{1}{\sin(\delta)} \int_{|x|}^\infty \frac{1}{\langle k\rangle\, r} |f_k(r)|\, r^2\,\mathrm{d}r \\
&= \frac{1}{4\pi\sin(\delta)} \int_{B_{|x|}(0)} \left|\frac{\sin(\langle k\rangle\,|y|)}{\langle k\rangle\,|y|}\right| |f_k(y)|\,\mathrm{d}y + \frac{1}{4\pi\sin(\delta)} \int_{\mathbb{R}^3\setminus B_{|x|}(0)} \frac{1}{\langle k\rangle\,|y|} |f_k(y)|\,\mathrm{d}y \\
&\leq \frac{\|f_k\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}}{4\pi\sin(\delta)} \left(\int_{B_{|x|}(0)} \left|\frac{\sin(\langle k\rangle\,|y|)}{\langle k\rangle\,|y|}\right|^4 \mathrm{d}y\right)^{\frac{1}{4}} + \frac{\|f_k\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}}{4\pi\sin(\delta)} \left(\int_{\mathbb{R}^3\setminus B_{|x|}(0)} \frac{1}{\langle k\rangle^4\,|y|^4}\,\mathrm{d}y\right)^{\frac{1}{4}}
\end{aligned}$$

$$\leq \frac{\|f_k\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \langle k\rangle^{-\frac{3}{4}}}{4\pi\sin(\delta)} \left(\int_{\mathbb{R}^3} \frac{\sin^4(|z|)}{|z|^4}\,\mathrm{d}z\right)^{\frac{1}{4}} + \frac{\|f_k\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \langle k\rangle^{-1}}{4\pi\sin(\delta)} \left(\frac{4\pi}{|x|}\right)^{\frac{1}{4}}$$
$$\leq \frac{\|f_k\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}}{\sin(\delta)} \left(\int_{\mathbb{R}^3} \frac{\sin^4(|z|)}{|z|^4}\,\mathrm{d}z\right)^{\frac{1}{4}} + \frac{\|f_k\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}}{\sin(\delta)}$$

where we exploited $|x| \geq 1$ in the final step; further

$$\begin{aligned}
\|f_k\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} &= \|\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_k\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \\
&\leq \|\Gamma\|_\infty \cdot \sum_{l+m+n=k} \|u_l\|_{L^4(\mathbb{R}^3)}\,\|u_m\|_{L^4(\mathbb{R}^3)}\,\|u_n\|_{L^4(\mathbb{R}^3)} \\
&\overset{(4.29)}{\leq} \|\Gamma\|_\infty \cdot \sum_{l+m+n=k} (C'_{\alpha+2})^3\,\frac{\langle l\rangle^{-\alpha}\,\langle m\rangle^{-\alpha}\,\langle n\rangle^{-\alpha}}{(1+l^2)(1+m^2)(1+n^2)} \\
&\overset{(4.26)}{\leq} 2^\alpha\,\|\Gamma\|_\infty \cdot \sum_{l+m+n=k} (C'_{\alpha+2})^3\,\frac{\langle l+m+n\rangle^{-\alpha}}{(1+l^2)(1+m^2)(1+n^2)} \\
&\leq 2^\alpha\,\|\Gamma\|_\infty \cdot (C'_{\alpha+2})^3 \left(\sum_{n\in\mathbb{Z}} \frac{1}{1+n^2}\right)^3 \cdot \langle k\rangle^{-\alpha}.
\end{aligned}$$

Hence we conclude for $|x| \geq 1$ (the case $|x| < 1$ is covered by (4.37))

$$\begin{aligned}
|x||u_k(x)| &\leq \frac{\|f_k\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}}{\sin(\delta)} \left[\left(\int_{\mathbb{R}^3} \frac{\sin^4(|z|)}{|z|^4}\,\mathrm{d}z\right)^{\frac{1}{4}} + 1\right] \\
&\leq 2^\alpha\,\|\Gamma\|_\infty\,(C'_{\alpha+2})^3 \left(\sum_{n\in\mathbb{Z}} \frac{1}{1+n^2}\right)^3 \langle k\rangle^{-\alpha} \left[\left(\int_{\mathbb{R}^3} \frac{\sin^4(|z|)}{|z|^4}\,\mathrm{d}z\right)^{\frac{1}{4}} + 1\right] \\
&=: E'''_\alpha \cdot \langle k\rangle^{-\alpha}.
\end{aligned} \tag{4.38}$$

Combining the estimates (4.37), (4.38) for $|k| > s$ with (4.34) for $|k| \leq s$, we find

$$\|u_k\|_{X_1} = \sup_{x\in\mathbb{R}^3} (1+|x|^2)^{\frac{1}{2}}|u_k(x)| \leq \begin{cases} E'_\alpha \cdot \langle k\rangle^{-\alpha} & |k| \leq s, \\ (2E''_\alpha + E'''_\alpha)\cdot\langle k\rangle^{-\alpha} & |k| > s, \end{cases}$$

and the assertion is proved with $E_\alpha := \max\{E'_\alpha, 2E''_\alpha + E'''_\alpha\}$. □

Proof of Proposition 4.5

▷ Step 1: *Mapping properties of F resp. G.*

For $\lambda \in \mathbb{R}$ and $\mathbf{v} \in \mathcal{X}_1$, we set $\mathbf{u} := \mathbf{w}+\mathbf{v}$ and recall the defining equations (4.12) and (4.13):

$$F(\mathbf{v},\lambda)_k = v_k - \begin{cases} \mathcal{R}_1^{\tau_0}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_0 - \Gamma\,w_0^3\right] & k=0, \\ \mathcal{R}_{s^2+1}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right] & \\ \qquad +(\cot(\tau_s)-\lambda)\tilde{\mathcal{R}}_{s^2+1}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right] & k=\pm s, \\ \mathcal{R}_{k^2+1}^{\tau_k}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_k\right] & \text{else;} \end{cases}$$

$$G(\mathbf{v},\lambda)_k = v_k - \begin{cases} \mathcal{R}_1^{\tau_0}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_0 - \Gamma\,w_0^3\right] & k=0, \\ \mathcal{R}_{s^2+1}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right] & \\ \qquad +(1-\lambda)\left(\alpha^{(s^2+1)}(v_{\pm s}) + \beta^{(s^2+1)}(v_{\pm s})\right)\tilde{\Psi}_{s^2+1} & k=\pm s, \\ \mathcal{R}_{k^2+1}^{\tau_k}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_k\right] & \text{else.} \end{cases}$$

Our main concern will be convergence of the infinite sums related to the space $\mathcal{X}_1 = \ell^1_{\mathrm{sym}}(\mathbb{Z}, X_1)$. We notice that

- $\triangleright$ G differs from F only in the $\pm s$-th components, and
- $\triangleright$ the scalar parameter λ only appears as a multiplicative factor in F resp. G.

Thus, in order to keep the focus of the presentation on the central issues, we will fix $\lambda \in \mathbb{R}$ and prove well-definedness and differentiability for the map $F(\,\cdot\,, \lambda) : \mathcal{X}_1 \to \mathcal{X}_1$; the generalization to $F \in C^\infty(\mathcal{X}_1 \times \mathbb{R}, \mathcal{X}_1)$ and the proof of the corresponding properties of G are then straightforward.

The main tool in our estimates is the following uniform norm estimate for the linear operators $\mathcal{R}^{\tau_k}_{k^2+1}$ appearing in the components of F. Recalling that $\tau_k \in \{\frac{\pi}{4}, \frac{3\pi}{4}\}$ for $k \neq \pm s$ by (4.10), Lemma 4.12 above (for $k \neq \pm s$) and Proposition 3.13 (i) (for $k = \pm s$) provide a constant $C_1 = C_1(\lambda, \tau_s) > 0$ with

$$\begin{aligned} &\left\| \mathcal{R}^{\tau_k}_{k^2+1} \right\|_{\mathcal{L}(X_3, X_1)} \le C_1 \quad (k \in \mathbb{Z} \setminus \{\pm s\}), \\ &\left\| \mathcal{R}_{s^2+1} \right\|_{\mathcal{L}(X_3, X_1)} \le \frac{C_1}{2}, \quad \left\| (\cot(\tau_s) - \lambda)\, \tilde{\mathcal{R}}_{s^2+1} \right\|_{\mathcal{L}(X_3, X_1)} \le \frac{C_1}{2}. \end{aligned} \tag{4.39}$$

We now let $\mathbf{v} \in \mathcal{X}_1$ and define $\mathbf{u} = \mathbf{v} + \mathbf{w}$. Since Γ is assumed to be continuous and bounded, Proposition 4.3 implies that $\Gamma\,(\mathbf{u} \star \mathbf{u} \star \mathbf{u}) \in \mathcal{X}_3$. Thus every component $F(\mathbf{v}, \lambda)_k$ is a well-defined element of X_1, and we estimate

$$\begin{aligned} &\|F(\mathbf{v}, \lambda)\|_{\ell^1(\mathbb{Z}, X_1)} = \sum_{k \in \mathbb{Z}} \|F(\mathbf{v}, \lambda)_k\|_{X_1} \\ &= \left\| v_0 - \mathcal{R}^{\tau_0}_1 \left[\Gamma\,(\mathbf{u} \star \mathbf{u} \star \mathbf{u})_0 - \Gamma\, w_0^3\right] \right\|_{X_1} \\ &\quad + \left\| v_s - \mathcal{R}_{s^2+1}\left[\Gamma\,(\mathbf{u} \star \mathbf{u} \star \mathbf{u})_s\right] - (\cot(\tau_s) - \lambda)\tilde{\mathcal{R}}_{s^2+1}\left[\Gamma\,(\mathbf{u} \star \mathbf{u} \star \mathbf{u})_s\right] \right\|_{X_1} \\ &\quad + \left\| v_{-s} - \mathcal{R}_{s^2+1}\left[\Gamma\,(\mathbf{u} \star \mathbf{u} \star \mathbf{u})_{-s}\right] - (\cot(\tau_s) - \lambda)\tilde{\mathcal{R}}_{s^2+1}\left[\Gamma\,(\mathbf{u} \star \mathbf{u} \star \mathbf{u})_{-s}\right] \right\|_{X_1} \\ &\quad + \sum_{k \in \mathbb{Z} \setminus \{0, \pm s\}} \left\| v_k - \mathcal{R}^{\tau_k}_{k^2+1}\left[\Gamma\,(\mathbf{u} \star \mathbf{u} \star \mathbf{u})_k\right] \right\|_{X_1} \\ &\overset{(4.39)}{\le} \|\mathbf{v}\|_{\ell^1(\mathbb{Z}, X_1)} + C_1 \left\| \Gamma w_0^3 \right\|_{X_3} + C_1 \sum_{k \in \mathbb{Z}} \left\| \Gamma\,(\mathbf{u} \star \mathbf{u} \star \mathbf{u})_k \right\|_{X_3} \\ &\overset{\text{Prop. 4 3}}{\le} \|\mathbf{v}\|_{\ell^1(\mathbb{Z}, X_1)} + C_1 \|\Gamma\|_\infty \|w_0\|^3_{X_1} + C_1 \|\Gamma\|_\infty \|\mathbf{u}\|^3_{\ell^1(\mathbb{Z}, X_1)}. \end{aligned}$$

This is finite, hence $F(\mathbf{v}, \lambda) \in \mathcal{X}_1$ as asserted. We next prove differentiability of $F(\,\cdot\,, \lambda)$ with derivative as in (4.14),

$$(DF(\mathbf{v}, \lambda)[\mathbf{q}])_k = q_k - \begin{cases} 3\, \mathcal{R}^{\tau_k}_{k^2+1}\left[\Gamma\,(\mathbf{q} \star \mathbf{u} \star \mathbf{u})_k\right] & k \neq \pm s, \\ 3\, \mathcal{R}_{s^2+1}\left[\Gamma\,(\mathbf{q} \star \mathbf{u} \star \mathbf{u})_{\pm s}\right] & \\ \;+3\,(\cot(\tau_s) - \lambda)\tilde{\mathcal{R}}_{s^2+1}\left[\Gamma\,(\mathbf{q} \star \mathbf{u} \star \mathbf{u})_{\pm s}\right] & k = \pm s \end{cases}$$

for $\mathbf{v}, \mathbf{q} \in \mathcal{X}_1$ still with $\mathbf{u} = \mathbf{v} + \mathbf{w}$. Using the asserted expressions for $DF(\mathbf{v}, \lambda)[\mathbf{q}]$, we calculate

$$\begin{aligned} &(F(\mathbf{v} + \mathbf{q}, \lambda) - F(\mathbf{v}, \lambda) - DF(\mathbf{v}, \lambda)[\mathbf{q}])_k \\ &= \begin{cases} -\mathcal{R}_{s^2+1}\left[\Gamma\,(3\mathbf{q} \star \mathbf{q} \star \mathbf{u} + \mathbf{q} \star \mathbf{q} \star \mathbf{q})_{\pm s}\right] & \\ \qquad -(\cot(\tau_s) - \lambda)\tilde{\mathcal{R}}_{s^2+1}\left[\Gamma\,(3\mathbf{q} \star \mathbf{q} \star \mathbf{u} + \mathbf{q} \star \mathbf{q} \star \mathbf{q})_{\pm s}\right] & k = \pm s, \\ -\mathcal{R}^{\tau_k}_{k^2+1}\left[\Gamma\,(3\mathbf{q} \star \mathbf{q} \star \mathbf{u} + \mathbf{q} \star \mathbf{q} \star \mathbf{q})_k\right] & \text{else} \end{cases} \end{aligned}$$

and estimate

$$\begin{aligned}
&\|F(\mathbf{v}+\mathbf{q},\lambda) - F(\mathbf{v},\lambda) - DF(\mathbf{v},\lambda)[\mathbf{q}]\|_{\ell^1(\mathbb{Z},X_1)} \\
&\quad = \sum_{k\in\mathbb{Z}} \|(F(\mathbf{v}+\mathbf{q},\lambda) - F(\mathbf{v},\lambda) - DF(\mathbf{v},\lambda)[\mathbf{q}])_k\|_{X_1} \\
&\quad \overset{(4.39)}{\leq} C_1 \sum_{k\in\mathbb{Z}} \|\Gamma\,(3\mathbf{q}\star\mathbf{q}\star\mathbf{u} + \mathbf{q}\star\mathbf{q}\star\mathbf{q})_k\|_{X_3} \\
&\quad \leq 3C_1\,\|\Gamma\|_\infty \sum_{k\in\mathbb{Z}} \Big(\|(\mathbf{q}\star\mathbf{q}\star\mathbf{u})_k\|_{X_3} + \|(\mathbf{q}\star\mathbf{q}\star\mathbf{q})_k\|_{X_3} \Big) \\
&\quad \overset{\text{Prop. 4 3}}{\leq} 3C_1\,\|\Gamma\|_\infty \cdot \|\mathbf{q}\|^2_{\ell^1(\mathbb{Z},X_1)} \Big(\|\mathbf{u}\|_{\ell^1(\mathbb{Z},X_1)} + \|\mathbf{q}\|_{\ell^1(\mathbb{Z},X_1)} \Big) \\
&\quad = O(\|\mathbf{q}\|^2_{\ell^1(\mathbb{Z},X_1)})
\end{aligned}$$

and thus conclude differentiability of $F(\,\cdot\,,\lambda)$ with the derivative as in formula (4.14). Since $F(\,\cdot\,,\lambda)$ is a combination of continuous linear operators and polynomials in the convolution algebra, essentially the same estimates can be used to show higher-order differentiability. One finds for $\mathbf{v}=\mathbf{u}-\mathbf{w},\mathbf{p},\mathbf{q},\mathbf{r}\in\mathcal{X}_1$

$$(D^2F(\mathbf{v},\lambda)[\mathbf{p},\mathbf{q}])_k = -\begin{cases} 6\,\mathcal{R}^{\tau_k}_{k^2+1}\left[\Gamma\,(\mathbf{p}\star\mathbf{q}\star\mathbf{u})_k\right] & k\neq\pm s, \\ 6\,\mathcal{R}_{s^2+1}\left[\Gamma\,(\mathbf{p}\star\mathbf{q}\star\mathbf{u})_{\pm s}\right] & \\ \;+6\,(\cot(\tau_s)-\lambda)\tilde{\mathcal{R}}_{s^2+1}\left[\Gamma\,(\mathbf{p}\star\mathbf{q}\star\mathbf{u})_{\pm s}\right] & k=\pm s; \end{cases}$$

$$(D^3F(\mathbf{v},\lambda)[\mathbf{p},\mathbf{q},\mathbf{r}])_k = -\begin{cases} 6\,\mathcal{R}^{\tau_k}_{k^2+1}\left[\Gamma\,(\mathbf{p}\star\mathbf{q}\star\mathbf{r})_k\right] & k\neq\pm s, \\ 6\,\mathcal{R}_{s^2+1}\left[\Gamma\,(\mathbf{p}\star\mathbf{q}\star\mathbf{r})_{\pm s}\right] & \\ \;+6\,(\cot(\tau_s)-\lambda)\tilde{\mathcal{R}}_{s^2+1}\left[\Gamma\,(\mathbf{p}\star\mathbf{q}\star\mathbf{r})_{\pm s}\right] & k=\pm s \end{cases}$$

and all higher derivatives vanish identically.

$\triangleright$ <u>Step 2</u>: *Solution properties.*

First of all, recalling that $\mathbf{w}=(...,0,w_0,0,...)$ and hence $(\mathbf{w}\star\mathbf{w}\star\mathbf{w})_k=\delta_{k,0}\,w_0^3$ for $k\in\mathbb{Z}$, one can immediately see that $F(\mathbf{0},\lambda)=G(\mathbf{0},\lambda)=\mathbf{0}$ for all $\lambda\in\mathbb{R}$. Let us now assume that $F(\mathbf{v},\lambda)=0$ resp. $G(\mathbf{v},\lambda)=0$ for some $\mathbf{v}\in\mathcal{X}_1=\ell^1_{\text{sym}}(\mathbb{Z},X_1)$ and $\lambda\in\mathbb{R}$. Again, we define $\mathbf{u}:=\mathbf{v}+\mathbf{w}$, and summarize

$$\begin{aligned}
u_0 - w_0 = v_0 &= \mathcal{R}^{\tau_0}_1\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_0 - \Gamma\,w_0^3\right], \\
u_{\pm s} = v_{\pm s} &= \mathcal{R}_{s^2+1}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right] + \begin{cases} (\cot(\tau_s)-\lambda)\tilde{\mathcal{R}}_{s^2+1}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right], \\ (1-\lambda)\left(\alpha^{(s^2+1)}(v_{\pm s}) + \beta^{(s^2+1)}(v_{\pm s})\right)\tilde{\Psi}_{s^2+1}, \end{cases} \\
u_k = v_k &= \mathcal{R}^{\tau_k}_{k^2+1}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_k\right] \qquad (k\in\mathbb{Z}\setminus\{0,\pm s\}).
\end{aligned}$$

By choice of τ_k in equation (4.10) and due to $\mathcal{R}_{s^2+1}=\mathcal{R}^{\frac{\pi}{2}}_{s^2+1}$, we observe in particular that the requirements of Lemma 4.13 are satisfied with any $\delta<\frac{\pi}{4}$ and $M=\{0,\pm s\}$, which we will rely on throughout the subsequent steps. But first, according to Proposition 3.13 and Corollaries 3.16, 3.17 in Chapter 3, $v_k,u_k\in X_1\cap C^2_{\text{loc}}(\mathbb{R}^3)$ satisfy the differential equations

$$-\Delta v_k - (k^2+1)v_k = \Gamma(x)\left[(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_k - \delta_{k,0}w_0^3\right] \quad \text{on } \mathbb{R}^3$$

or equivalently, in view of $\mathbf{w}=(...,0,w_0,0,...)$ and of the differential equation (4.4),

$$-\Delta u_k - (k^2+1)u_k = \Gamma(x)\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_k \quad \text{on } \mathbb{R}^3. \tag{4.40}$$

We now define formally for $t \in \mathbb{R}, x \in \mathbb{R}^3$

$$U(t,x) := w_0(x) + v_0(x) + \sum_{k=1}^{\infty} 2\cos(kt)\, v_k(x) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}kt}\, u_k(x). \tag{4.41}$$

Since by assumption $\mathbf{u} = \mathbf{v} + \mathbf{w} \in \ell^1(\mathbb{Z}, X_1)$, the Weierstrass M-test asserts that the sum in (4.41) converges in X_1 uniformly with respect to $t \in \mathbb{R}$, and hence the map $t \mapsto U(t,\,\cdot\,)$ is continuous as a map from $\mathbb{R}$ to X_1. We show stronger regularity properties of $U(t,x)$ in the following two steps.

▷ <u>Step 3</u>: *Regularity in time.*

We prove that the map $t \mapsto U(t,\,\cdot\,)$, when interpreted as a map from $\mathbb{R}$ to X_1, possesses two continuous time derivatives given by

$$\partial_t U(t,\,\cdot\,) = \sum_{k\in\mathbb{Z}} \mathrm{i}k\, \mathrm{e}^{\mathrm{i}kt}\, u_k, \qquad \partial_t^2 U(t,\,\cdot\,) = \sum_{k\in\mathbb{Z}} -k^2\, \mathrm{e}^{\mathrm{i}kt}\, u_k. \tag{4.42}$$

Indeed, we have just shown uniform convergence of the sum in (4.41), and can thus conclude continuity of $t \mapsto U(t,\,\cdot\,)$ as a map to X_1. Term-by-term differentiation, which yields (4.42), is justified since the sums in (4.42) also converge in X_1 uniformly with respect to time. This is a consequence of the Weierstraß M-test and Lemma 4.13 (iii). Hence, as asserted, the map $t \mapsto U(t,\,\cdot\,)$ is twice continuously differentiable as a map from $\mathbb{R}$ to X_1 - our proof even shows C^∞ regularity in time.

▷ <u>Step 4</u>: *Regularity in space and time.*

Similar to the previous step, Lemma 4.13 (ii) implies convergence of the formal term-by-term derivatives with respect to time in (4.42) as well as in spatial directions ($1 \le j, l \le 3$)

$$\begin{aligned} \partial_j U(t,x) &= \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}kt}\, \partial_j u_k(x), \\ \partial_j\partial_l U(t,x) &= \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}kt}\, \partial_j\partial_l u_k(x), \\ \partial_j\partial_t U(t,x) &= \sum_{k\in\mathbb{Z}} \mathrm{i}k\, \mathrm{e}^{\mathrm{i}kt}\, \partial_j u_k(x), \end{aligned} \tag{4.43}$$

which is uniform with respect to $t \in \mathbb{R}$ and $x \in B$, $B = B_R(0) \subseteq \mathbb{R}^3$ denoting a ball centered at the origin of arbitrary radius R. Then, using the Weierstraß M-test as above, we conclude $U \in C^2(\mathbb{R} \times B)$, and term-by-term differentiation holds true. Since the radius of the ball B was arbitrary, we conclude for $t \in \mathbb{R}$ and all $x \in \mathbb{R}^3$

$$\begin{aligned} \left[\partial_t^2 - \Delta - 1\right] U(t,x) &= \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}kt}\, \left[-k^2 - \Delta - 1\right] u_k(x) \\ &\overset{(4.40)}{=} \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}kt}\, \Gamma(x) \sum_{l+m+n=k} u_l(x)\, u_m(x)\, u_n(x) \\ &= \Gamma(x) \left(\sum_{l\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}lt}\, u_l(x)\right) \left(\sum_{m\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}mt}\, u_m(x)\right) \left(\sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}nt}\, u_n(x)\right) \\ &= \Gamma(x)\, U(t,x)^3 \end{aligned}$$

where the re-ordering of the summation is justified by absolute convergence of the sums.

Thus U is shown to be a classical solution of the wave equation (4.1). $\square$

Proof of Proposition 4.7

We prove the statement for the map F and then comment on the aspects that differ in case of G. Using formula (4.14) following Proposition 4.5, we find for $k \in \mathbb{Z}$ and $\mathbf{q} \in \mathcal{X}_1$, recalling that $w_k = 0$ for $k \in \mathbb{Z} \setminus \{0\}$ and that $\mathcal{R}^{\tau_s}_{s^2+1} = \mathcal{R}_{s^2+1} + \cot(\tau_s)\, \tilde{\mathcal{R}}_{s^2+1}$,

$$\begin{aligned} DF(\mathbf{0},0)[\mathbf{q}]_k &= q_k - 3\,\mathcal{R}^{\tau_k}_{k^2+1}[\Gamma\,(\mathbf{q} \star \mathbf{w} \star \mathbf{w})_k] \\ &= q_k - 3 \sum_{l+m+n=k} \mathcal{R}^{\tau_k}_{k^2+1}\,[\Gamma\, w_m\, w_n \cdot q_l] \\ &= q_k - 3\,\mathcal{R}^{\tau_k}_{k^2+1}\left[\Gamma\, w_0^2 \cdot q_k\right]. \end{aligned}$$

Hence $\mathbf{q} \in \ker DF(\mathbf{0},0)$ if and only if $q_k = 3\,\mathcal{R}^{\tau_k}_{k^2+1}\left[\Gamma\, w_0^2 \cdot q_k\right]$ for all $k \in \mathbb{Z}$. Having chosen $\tau_k \neq \sigma_k$ for $k \in \mathbb{Z}, k \neq \pm s$ in equation (4.10), the nondegeneracy property (4.11) implies $q_k \equiv 0$ for $k \in \mathbb{Z}, k \neq \pm s$. Since $\tau_{\pm s} = \sigma_s$, Proposition 4.4 guarantees the existence of a nontrivial solution $q_s \in X_1$ of

$$q_s = 3\,\mathcal{R}^{\tau_s}_{s^2+1}\left[\Gamma\, w_0^2 \cdot q_s\right] \tag{4.44}$$

which is unique up to a multiplicative factor. Hence $\ker DF(\mathbf{0},0)$ has the asserted form. (We recall here that we consider the subspace of symmetric sequences, i.e. $q_k = q_{-k}$ for $k \in \mathbb{N}_0$ and in particular $q_{-s} = q_s$.) Further, for all $k \in \mathbb{Z}$, the linear operators

$$X_1 \to X_1, \qquad q_k \mapsto q_k - 3\,\mathcal{R}^{\tau_k}_{k^2+1}\left[\Gamma\, w_0^2 \cdot q_k\right]$$

are known to be compact perturbations of the identity, see Proposition 3.13 (i). Hence $\ker DF(\mathbf{0},0)$ is 1-1-Fredholm.

In order to verify transversality, we compute for $k \in \mathbb{Z}$ and $\mathbf{q} \in \ker DF(\mathbf{0},0) \setminus \{\mathbf{0}\}$

$$\partial_\lambda DF(\mathbf{0},0)[\mathbf{q}]_k = \begin{cases} 3\,\tilde{\mathcal{R}}_{s^2+1}[\Gamma\, w_0^2\, q_s], & k = \pm s, \\ 0, & \text{else.} \end{cases}$$

Assuming for contradiction that $\partial_\lambda DF(\mathbf{0},0)[\mathbf{q}] = DF(\mathbf{0},0)[\mathbf{p}]$ for some $\mathbf{p} \in \mathcal{X}_1$, we infer in particular that the component p_s satisfies the convolution identity

$$p_s - 3\,\mathcal{R}^{\tau_s}_{s^2+1}\left[\Gamma\, w_0^2 \cdot p_s\right] = 3\,\tilde{\mathcal{R}}_{s^2+1}[\Gamma\, w_0^2 \cdot q_s] \tag{4.45}$$

and hence, see Proposition 3.13 (ii),

$$-\Delta p_s - (s^2+1)p_s = 3\,\Gamma(x)\, w_0^2(x)\, p_s \quad \text{on } \mathbb{R}^3,$$

which is also nontrivially solved by q_s as a consequence of (4.44). Due to the uniqueness statement in Proposition 3.18, this implies that $p_s = c \cdot q_s$ for some $c \in \mathbb{R}$. But then, applying equation (4.44) to equation (4.45), we obtain $\tilde{\mathcal{R}}_{s^2+1}[\Gamma\, w_0^2 \cdot q_s] = 0$. Hence by Proposition 3.13 (iii)

$$\widehat{\Gamma w_0^2 q_s}(\sqrt{s^2+1}) = 0$$

and therefore, due to $q_s = 3\,\mathcal{R}^{\tau_s}_{s^2+1}[\Gamma\, w_0^2\, q_s]$ and again Proposition 3.13 (iii),

$$q_s(x) = O\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty.$$

This contradicts Proposition 3.18 stating that the leading-order term as $|x| \to \infty$ of a nontrivial solution q_s of $-\Delta q_s - (s^2+1)q_s = 3\,\Gamma(x)\,w_0^2(x)\,q_s$ cannot vanish.

In the case $\tau_s = 0$, we see as above that $\mathbf{q} \in \ker DG(\mathbf{0}, 0)$ if and only if $q_k = 0$ for $k \neq \pm s$, and that $q_s = q_{-s}$ can be chosen to be the (nontrivial) solution of

$$q_s = 3\,\mathcal{R}_{s^2+1}\left[\Gamma\, w_0^2 \cdot q_s\right] + \alpha^{(s^2+1)}(q_s)\,\tilde{\Psi}_{s^2+1} \quad \text{with} \quad \beta^{(s^2+1)}(q_s) = 0. \tag{4.46}$$

Similarly, $\ker DG(\mathbf{0}, 0)$ is 1-1-Fredholm. We again assume for contradiction that there is $\mathbf{p} \in \mathcal{X}_1$ with $\partial_\lambda DG(\mathbf{0}, 0)[\mathbf{q}] = DG(\mathbf{0}, 0)[\mathbf{p}]$, which implies in particular

$$p_s - 3\,\mathcal{R}_{s^2+1}\left[\Gamma\, w_0^2 \cdot p_s\right] - \left(\alpha^{(s^2+1)}(p_s) + \beta^{(s^2+1)}(p_s)\right)\tilde{\Psi}_{s^2+1} = \alpha^{(s^2+1)}(q_s)\tilde{\Psi}_{s^2+1} \tag{4.47}$$

with $\beta^{(s^2+1)}(q_s) = 0$. Thus, according to Proposition 3.13 (ii), p_s solves the differential equation

$$-\Delta p_s - (s^2+1)p_s = 3\,\Gamma(x)\,w_0^2(x)\,p_s \quad \text{on } \mathbb{R}^3,$$

which is also solved by q_s, see equation (4.46). As before, the uniqueness property in Proposition 3.18 implies $p_s = c \cdot q_s$ for some $c \in \mathbb{R}$, and inserting this into the identity (4.47), comparison with (4.46) yields $\alpha^{(s^2+1)}(q_s) = 0$. Since also $\beta^{(s^2+1)}(q_s) = 0$, see (4.46), we infer from the definition of the functionals $\alpha^{(s^2+1)}, \beta^{(s^2+1)}$ around (3.13) that, again, $q_s(x) = O\left(\frac{1}{|x|^2}\right)$, contradicting Proposition 3.18. □

Looking back to the proof of Theorem 3.2 in Chapter 3, one would expect that the proof of transversality is much more involved in the case $\tau_s = 0$ when compared to $\tau_s \neq 0$. However, we have now seen that the techniques in the proposition above are essentially the same in both cases, which is due to the choice of the bifurcation parameter λ and its position in the maps F resp. G as a multiplicative factor of an element of the Helmholtz kernel: Thus dropping out of the differential equations solved by p_s resp. q_s, it simplifies calculations, but at the price of abandoning control of the asymptotic behavior of the s-th components along branches of bifurcating solutions.

4.4.2 Results concerning Theorem 4.8

Proof of Lemma 4.10

We let $f \in X_3$ and $w := \mathcal{P}_1[f] = \Lambda_1 * f$.

▷ Step 1: *$w \in X_3$ and continuity of $\mathcal{P}_1$.*

Since f is bounded and continuous, and since the kernel Λ_1 is integrable, dominated convergence implies continuity of w. Further, for $x \in \mathbb{R}^3$, we estimate as follows:

$$\begin{aligned}
(1+|x|^2)^{\frac{3}{2}}|w(x)| &\le \int_{\mathbb{R}^3} (1+|x|^2)^{\frac{3}{2}}\,|f(x-y)|\,\frac{\mathrm{e}^{-|y|}}{4\pi|y|}\,\mathrm{d}y \\
&\overset{\text{as (4.26)}}{\le} \int_{\mathbb{R}^3} 2^{\frac{3}{2}}(1+|x-y|^2)^{\frac{3}{2}}(1+|y|^2)^{\frac{3}{2}}\,|f(x-y)|\,\frac{\mathrm{e}^{-|y|}}{4\pi|y|}\,\mathrm{d}y \\
&\le 2^{\frac{3}{2}}\,\|f\|_{X_3} \cdot \int_0^\infty (1+r^2)^{\frac{3}{2}}\,r\,\mathrm{e}^{-r}\,\mathrm{d}r
\end{aligned}$$

and the integral converges. Hence we even conclude that $\mathcal{P}_1$ is continuous as a map from

X_3 to X_3 and have some constant $C_0 > 0$ with

$$\forall\, f \in X_3 \qquad \|\mathcal{P}_1[f]\|_{X_1} \le \|\mathcal{P}_1[f]\|_{X_3} \le C_0\, \|f\|_{X_3}\,. \tag{4.48}$$

▷ <u>Step 2</u>: *w is twice continuously differentiable on $\mathbb{R}^3 \setminus \{0\}$.*

We first derive a pointwise formula for the convolution. For notational simplicity, we once again identify radial functions and with their profiles, and compute at $x \in \mathbb{R}^3 \setminus \{0\}$ with $r = |x|$ as in the proof of Lemma 3.10

$$\begin{aligned}
w(r) &= \int_{\mathbb{R}^3} f(|y|)\, \frac{\mathrm{e}^{-|x-y|}}{4\pi|x-y|}\, \mathrm{d}y \\
&= \int_0^\infty \int_0^\pi f(t) \frac{\mathrm{e}^{-\sqrt{r^2+t^2-2rt\cos\vartheta}}}{2\sqrt{r^2+t^2-2rt\cos\vartheta}}\, t^2 \sin(\vartheta)\, \mathrm{d}\vartheta \mathrm{d}t \\
&= \int_0^\infty f(t)\, \left(\mathrm{e}^{-|r-t|} - \mathrm{e}^{-(r+t)}\right)\, \frac{t}{2r}\, \mathrm{d}t \\
&= \frac{\mathrm{e}^{-r}}{r} \int_0^r f(t)\sinh(t) t\, \mathrm{d}t + \frac{\sinh(r)}{r} \int_r^\infty f(t) \mathrm{e}^{-t} t\, \mathrm{d}t.
\end{aligned}$$

We can now proceed as in Step 1 of the proof of Proposition 3.13; hence we are very brief here. The above formula shows that w is of class C^2 on $\mathbb{R}^3 \setminus \{0\}$, and a short calculation using the Fundamental Theorem of Calculus verifies, for $r = |x| > 0$,

$$-\Delta w(x) + w(x) = -w''(r) - \frac{2}{r} w'(r) + w(r) = f(x).$$

In particular, we find

$$w'(r) = -\frac{r+1}{r^2}\mathrm{e}^{-r} \int_0^r f(t)\sinh(t) t\, \mathrm{d}t + \frac{r\cosh(r) - \sinh(r)}{r^2} \int_r^\infty f(t)\mathrm{e}^{-t} t\, \mathrm{d}t. \tag{4.49}$$

Still as in the proof of Proposition 3.13, the mean value theorem for definite integrals and continuity of f allow to expand

$$\begin{aligned}
&\int_0^{|h|} f(r)\sinh(r) r\, \mathrm{d}r = \frac{1}{3}|h|^3 \cdot f(0) + o(|h|^3), \\
&\int_{|h|}^\infty f(r)\mathrm{e}^{-r} r\, \mathrm{d}r = w(0) - \int_0^{|h|} f(r)\mathrm{e}^{-r} r\, \mathrm{d}r = w(0) - \frac{1}{2}|h|^2 \cdot f(0) + o(|h|^2)
\end{aligned}$$

as $|h| \to 0$, and following the line of arguments in Proposition 3.13, we find for $h \in \mathbb{R}^3 \setminus \{0\}$

$$w(h) = w(0) + \frac{1}{6}|h|^2 (w(0) - f(0)) + o(|h|^2), \qquad \nabla w(h) = \frac{1}{3} h (w(0) - f(0)) + o(|h|),$$

which yields (twice) differentiability of w at $x = 0$ with

$$\nabla w(0) = 0, \qquad D^2 w(0) = \frac{1}{3}(w(0) - f(0)) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Again, we only remark that continuity of $D^2 w$ can be shown using the same expansions, and infer $-\Delta w(0) + w(0) = -\mathrm{tr}\, D^2 w(0) + w(0) = f(0)$ as claimed.

$\triangleright$ STEP 3: *Compactness of $\mathcal{P}_1$.*

It remains to prove that $\mathcal{P}_1 : X_3 \to X_1$ is compact. Let us consider a bounded sequence $(f_n)_{n\in\mathbb{N}}$ in X_3, and let $w_n := \mathcal{P}_1[f_n]$. We aim to show that a subsequence of $(w_n)_{n\in\mathbb{N}}$ converges in X_1. Throughout, we let $C_* := \sup_{n\in\mathbb{N}} \|f_n\|_{X_3}$.

As in the proof of Proposition 3.13 (i), we combine a uniform decay estimate and a local compactness result based on the Theorem of Arzelà-Ascoli. Once we show that $(w_n)_{n\in\mathbb{N}}$ is pointwise bounded and equicontinuous, we can (using a suitable diagonalization technique) extract a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ which converges to some continuous function $w : \mathbb{R}^3 \to \mathbb{R}$ uniformly on every compact subset of $\mathbb{R}^3$. Then, fixing some $\varepsilon > 0$, the estimate (4.48) provides such $R = R(\varepsilon) > 0$ that, for all $k \in \mathbb{N}$ and $x \in \mathbb{R}^3$ with $|x| \geq R$

$$(1+|x|^2)^{\frac{1}{2}} |w_{n_k}(x)| \leq \frac{\|w_{n_k}\|_{X_3}}{1+R^2} \overset{(4.48)}{\leq} \frac{C_0 \|f_{n_k}\|_{X_3}}{1+R^2} \leq \frac{C_0 C_*}{1+R^2} < \frac{\varepsilon}{3}. \tag{4.50}$$

Furthermore, we have $w_{n_k} \to w$ uniformly on the compact set $\overline{B}_R(0)$, and hence we find $k_0 = k_0(\varepsilon) \in \mathbb{N}$ with the property that, for $k, l \geq k_0$ and $x \in \overline{B}_R(0)$,

$$|w_{n_k}(x) - w_{n_l}(x)| < \frac{\varepsilon}{3(R^2+1)^{\frac{1}{2}}}$$

and thus

$$(1+|x|^2)^{\frac{1}{2}} |w_{n_k}(x) - w_{n_l}(x)| \leq (1+R^2)^{\frac{1}{2}} |w_{n_k}(x) - w_{n_l}(x)| < \frac{\varepsilon}{3}. \tag{4.51}$$

Combining the inequalities (4.50) and (4.51), we infer $\|w_{n_k} - w_{n_l}\|_{X_1} < \varepsilon$ for all $k, l \geq k_0$. So $(w_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence in X_1 and we have found, as asserted, a convergent subsequence.

We still have to verify the assumptions of the Arzelà-Ascoli Theorem. Pointwise boundedness of $(w_n)_{n\in\mathbb{N}}$ is an immediate consequence of the uniform bound in X_1 and X_3 due to the norm estimate (4.48), which even shows that $\|w_n\|_\infty \leq C_0 C_*$ for all $n \in \mathbb{N}$. We prove equicontinuity as in Proposition 3.13 (i) by verifying that there exists $C_1 > 0$ with $\|\nabla w_n\|_\infty \leq C_1$ for all $n \in \mathbb{N}$. Using the explicit formula from (ii), we have for all $n \in \mathbb{N}$ and $r > 0$

$$\begin{aligned}
|w_n'(r)| &\leq \frac{r+1}{r^2}\mathrm{e}^{-r} \int_0^r |f_n(t)| \sinh(t) t \,\mathrm{d}t + \frac{r\cosh(r) - \sinh(r)}{r^2} \int_r^\infty |f_n(t)| \mathrm{e}^{-t} t \,\mathrm{d}t \\
&\leq \|f_n\|_{X_3} \cdot \left[\frac{r+1}{r^2}\mathrm{e}^{-r} \int_0^r \frac{t\sinh(t)}{(t^2+1)^{\frac{3}{2}}} \,\mathrm{d}t + \frac{r\cosh(r) - \sinh(r)}{r^2} \int_r^\infty \frac{t\mathrm{e}^{-t}}{(t^2+1)^{\frac{3}{2}}} \,\mathrm{d}t \right] \\
&\leq C_* \cdot \left[\frac{r+1}{r}\mathrm{e}^{-r} \sinh(r) \int_0^r \frac{\mathrm{d}t}{(t^2+1)^{\frac{3}{2}}} + \frac{r\cosh(r) - \sinh(r)}{r^2}\mathrm{e}^{-r} \int_r^\infty \frac{t\,\mathrm{d}t}{(t^2+1)^{\frac{3}{2}}} \right] \\
&\leq C_* \cdot \left[\frac{r+1}{r}(1-\mathrm{e}^{-2r}) + \frac{r(1+\mathrm{e}^{-2r}) - (1-\mathrm{e}^{-2r})}{r^2} \right] \cdot \frac{1}{2} \int_0^\infty \frac{\mathrm{d}t}{1+t^2} \\
&= \frac{\pi}{4} C_* \cdot \frac{1}{r^2 \mathrm{e}^{2r}} \left[\mathrm{e}^{2r}\left(r^2 + 2r - 1\right) + 1 - r^2 \right].
\end{aligned}$$

One can easily convince oneself that this expression is bounded on $(0, \infty)$, and hence $\|w_n'\|_\infty \leq C_1$ for some $C_1 > 0$. The Arzelà-Ascoli Theorem is applicable and completes the proof of compactness. □

Proof of Lemma 4.11

We closely follow the line of argumentation by Bates and Shi [12], Theorem 5.4 (6). The main difference is that they state the nondegeneracy result as a spectral property of the operator $-\Delta + 1 + 3\Gamma_0 w_0^2 : H^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ whereas we cannot use the Hilbert space setting but discuss solutions in $X_1 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$. However, the technique of Bates and Shi (and also of Wei's proof in [75]) is based of an expansion of the eigenfunctions at a fixed radius $r > 0$ in terms of the eigenfunctions of the Laplace-Beltrami operator on $L^2(\mathbb{S}^2)$. This provides coefficients depending on r, and the conclusions are obtained from the analysis of these profiles. It turns out that this analysis is also helpful in our situation; even more, considering only radially symmetric solutions, the proof is actually quite short.

To be consistent with the notation of Bates and Shi, we let

$$g : \mathbb{R} \to \mathbb{R}, \quad g(u) := \Gamma_0 u^3 - u$$

and observe that g satisfies the assumptions of [12], Theorems 5.3 and 5.4, and is of class (A) as described in [12], p. 258. Moreover, w_0 is a ground state solution of (4.19) as described in [12], equation (5.85). Let us now consider $q_0 \in X_1 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$ with

$$-\Delta q_0 + q_0 = 3\Gamma_0\, w_0^2\, q_0 \text{ on } \mathbb{R}^3$$

as assumed in Lemma 4.11. Then the profile satisfies

$$q_0'' + \frac{2}{r} q_0' + g'(w_0(r))\, q_0 = 0 \quad \text{on } (0,\infty), \quad q_0'(0) = 0, \quad \lim_{r\to\infty} q_0(r) = 0. \tag{4.52}$$

This is equation (5.95) in [12]. Referring on their part to a result by Kwong and Zhang [43] on the ordinary differential equation, the authors infer that $q_0 \equiv 0$.

For the reader's convenience, we give a guiding reference to [43] which allows to follow the line of argumentation. For $\alpha > 0$, Kwong and Zhang discuss the initial value problems

$$p'' + \frac{2}{r} p' + g(p) = 0 \quad \text{on } (0,\infty), \quad p'(0) = 0, \quad p(0) = \alpha \tag{4.53}$$

with unique solution $p(\,\cdot\,,\alpha)$, and

$$q'' + \frac{2}{r} q' + g'(p) q = 0 \quad \text{on } (0,\infty), \quad q'(0) = 0, \quad q(0) = 1, \tag{4.54}$$

with unique solution $q(\,\cdot\,,\alpha) = \frac{\mathrm{d}}{\mathrm{d}\alpha} p(\,\cdot\,,\alpha)$, see equations (2.1), (2.2) of [43] with $m = N - 1 = 2$ and q, p, g instead of w, u, f. The assumptions on g from [12] are now called [F1] - [F3]. In [43], equation (2.6), the authors introduce a set

$$G := \{\alpha > 0 \mid p(\,\cdot\,,\alpha) \text{ has no zero in } (0,\infty),\ p(r,\alpha) \to 0 \text{ as } r \to \infty\}.$$

On p. 593, they summarize several earlier results and conclude that G contains exactly one point α^*. Knowing this, Lemma 6 in [43] states that $q(\,\cdot\,,\alpha^*)$ has exactly one zero in $(0,\infty)$. In this situation, [43], Lemma 9 applies and ensures that there exists $K \neq 0$ (possibly $K = \pm\infty$) with

$$q(r,\alpha^*) \to K \qquad \text{as } r \to \infty.$$

Since we consider a ground state w_0, the definition of G and the fact that $G = \{\alpha^*\}$ imply $w_0 = p(\,\cdot\,,\alpha^*)$ and $w_0(0) = \alpha^*$ as well as $q_0 = c \cdot q(\,\cdot\,,\alpha^*)$ for some $c \in \mathbb{R}$. This is why the vanishing limit of $q_0(r)$ as $r \to \infty$ in (4.52) implies $c = 0$ and hence $q_0 \equiv 0$. $\square$

4.5 Summary

In this chapter, we constructed time-periodic, spatially localized and radially symmetric polychromatic solutions

$$U(t,x) = u_0(x) + \sum_k 2\cos(kt)u_k(x) \qquad (t \in \mathbb{R}, x \in \mathbb{R}^3)$$

of the cubic wave-type equations

$$\partial_t^2 U - \Delta U \mp U = \Gamma(x)U^3 \qquad \text{on } \mathbb{R} \times \mathbb{R}^3.$$

This was achieved by analyzing an infinite Helmholtz system satisfied by the functions u_k with bifurcation tools introduced in Chapter 3. Naturally, the first question to ask is whether a similar analysis of an infinite Helmholtz system is possible via dual variational methods as in Chapter 2. This might in particular be a way to drop the assumption of radial symmetry and possibly provide a glimpse on a physical interpretation of dual ground states. However, a direct transfer of the dual method as in the previous chapter would require to verify convexity for the infinite-dimensional system, and even if that worked, one would have to assess whether the results from convex analysis can be transferred from a finite-dimensional into an infinite-dimensional version.

In view of the results for Schrödinger systems in a periodic setting by Schneider et al. [16] and Hirsch and Reichel [36], a generalization to non-constant periodic potentials would also be of high interest. As already outlined in the previous Chapter 3 , this would firstly require to construct the resolvent-type operators using suitable Limiting Absorption Principle for periodic potentials e.g. as in [52]. Secondly, it would be necessary to extend the analytic setup of Chapter 3 from radial symmetry to periodic problems; it is not obvious at first glance whether this is possible maintaining the simplicity and compactness properties we exploit to verify bifurcation from simple eigenvalues.

Another interesting (and probably less ambitious) extension of the results of this chapter would be the application to the classical cubic wave equation

$$\partial_t^2 U - \Delta U = \Gamma(x)U^3 \qquad \text{on } \mathbb{R} \times \mathbb{R}^3,$$

which would at least require an extension of the functional analytic framework as mentioned after Remark 4.2.

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Conventions and Abbreviations

General Notation and Conventions

We collect some notation which is used throughout the thesis without further explanation. In the following subchapters, the important quantities of the respective chapters are listed for convenience.

$B_R(x)$	$= \{y \in \mathbb{R}^N \mid \lvert x-y\rvert < R\} \subseteq \mathbb{R}^N$ the open ball with center $x \in \mathbb{R}^N$ and radius $R > 0$
$\mathbb{S}^{N-1}$	$= \{\xi \in \mathbb{R}^N \mid \lvert\xi\rvert = 1\} \subseteq \mathbb{R}^N$ the unit sphere in $\mathbb{R}^N$
$\mathrm{d}\sigma$	the surface measure on the sphere $\mathbb{S}^{N-1}$
p'	$= \frac{p}{p-1}$ for given $p \in [1, \infty]$, i.e. satisfies $\frac{1}{p} + \frac{1}{p'} = 1$ the conjugate exponent
$\mathcal{S}(\mathbb{R}^N, \mathbb{C})$	$= \{f \in C^\infty(\mathbb{R}^N, \mathbb{C}) \mid \text{for all } \alpha, \beta \in \mathbb{N}_0^N \quad \sup_{x\in\mathbb{R}^N} \lvert x^\alpha D^\beta f(x)\rvert < \infty\}$ the Schwartz space
$\mathcal{F}f(\xi), \hat{f}(\xi)$	$= (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x)\,\mathrm{d}x$ for $f \in \mathcal{S}(\mathbb{R}^N, \mathbb{C})$ and $\xi \in \mathbb{R}^N$ the Fourier transform
$\mathcal{L}(X, Y)$	$= \{A : X \to Y \mid A \text{ is linear and bounded}\}$ for given Banach spaces X, Y the space of bounded linear operators
J_ν	the Bessel function of the first kind of order ν as $r > 0, r \to \infty$, by (9.2.1) in [2] $J_\nu(r) = \sqrt{\frac{2}{\pi r}}\left[\cos\left(r - \frac{2\nu+1}{4}\pi\right) + O\left(\frac{1}{r}\right)\right]$ as $r > 0, r \to 0$, by (9.1.7) in [2] $J_\nu(r) = \frac{1}{\Gamma(\nu+1)}\left(\frac{r}{2}\right)^\nu \cdot (1 + O(r^2))$
Y_ν	the Bessel function of the second kind of order ν as $r > 0, r \to \infty$, by (9.2.2) in [2] $Y_\nu(r) = \sqrt{\frac{2}{\pi r}}\left[\sin\left(r - \frac{2\nu+1}{4}\pi\right) + O\left(\frac{1}{r}\right)\right]$ as $r > 0, r \to 0$, by (9.1.8), (9.1.9) in [2] $Y_0(r) \sim \frac{2}{\pi}\log(r)$ and $Y_\nu(r) \sim -\Gamma(\nu) \cdot \frac{2^\nu}{\pi z^\nu}$ for $\nu > 0$
$H_\nu^{(1)}$	the Hankel function of the first kind of order ν, $H_\nu^{(1)} = J_\nu + \mathrm{i}Y_\nu$ as $r > 0, r \to \infty$, by (9.2.2) in [2] $H_\nu^{(1)}(r) = \sqrt{\frac{2}{\pi r}}\left[\mathrm{e}^{\mathrm{i}r - \mathrm{i}\frac{2\nu+1}{4}\pi} + O\left(\frac{1}{r}\right)\right]$

Further, for $1 \leq p \leq \infty$, $k \in \mathbb{N}$ and a measurable set $U \subseteq \mathbb{R}^N$, we use the Lebesgue and

Sobolev spaces

$$L^p(U,\mathbb{C}), \quad W^{k,p}(U,\mathbb{C}), \quad H^k(U,\mathbb{C}) = W^{k,2}(U,\mathbb{C})$$

with the usual norms. For spaces of real-valued functions, we denote $L^p(U) := L^p(U,\mathbb{R})$ for short (similarly for all other function spaces).

For $j \in \mathbb{N}_0$, $\gamma \in (0,1)$ and a subset $B \subseteq \mathbb{R}^N$ which is open or closed, we denote by $C^j(B,\mathbb{C})$ the spaces of j times continuously differentiable functions (with $C(B,\mathbb{C}) = C^0(B,\mathbb{C})$), and by $C^{j,\gamma}(B,\mathbb{C})$ the space of j times continuously differentiable functions with γ-Hölder continuous derivatives of order j. We use the usual (supremum) norms and Hölder seminorms. Further, we denote

by $C_c^{\cdots}(B,\mathbb{C})$ the space of compactly supported functions,

by $C_{\mathrm{rad}}^{\cdots}(B,\mathbb{C})$ the space of radially symmetric functions,

by $C^{\cdots}(B,\mathbb{R}) = C^{\cdots}(B)$ the space of real-valued functions,

and use the index "loc" if the associated norm is finite only on compact subsets of B.

Important Quantities in Chapter 2

We recall the main assumptions (2.6):

$$N \geq 2, \quad \mu, \nu > 0, \quad \frac{2(N+1)}{N-1} < p < 2^*,$$
$$a, b \in L^\infty(\mathbb{R}^N) \text{ are } [0,1]^N\text{-periodic with } 0 \leq b(x) \leq p-1,\ a(x) \geq a_0 > 0.$$

Throughout, we write u, v, w for functions in $L^p(\mathbb{R}^N)$ and $\bar{u}, \bar{v}, \bar{w}$ for functions in $L^{p'}(\mathbb{R}^N)$. Further:

2^* — the critical Sobolev exponent
$2^* = \frac{2N}{N-2}$ for $N \geq 3$, $2^* = \infty$ for $N = 2$,

a_+, b_+ — the essential supremum of a resp. b,
a_-, b_- — the essential infimum of a resp. b,

$\mathcal{R}_\lambda$ — the real part of the Helmholtz resolvent in Gutierrez' LAP, Thm. 1.9
$\mathcal{R}_\lambda : L^{p'}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ with $\mathcal{R}_\lambda f = \mathrm{Re}\left[\lim_{\varepsilon \to 0+}(-\Delta - (\lambda + \mathrm{i}\varepsilon))^{-1} f\right]$

Ψ_λ — the convolution kernel of the Helmholtz resolvent, $\mathcal{R}_\lambda f = \Psi_\lambda * f$
$\Psi_\lambda(x) = \mathrm{Re}\left[\frac{\mathrm{i}}{4}\left(\frac{\sqrt{\lambda}}{2\pi|x|}\right)^{\frac{N-2}{2}} H^{(1)}_{\frac{N}{2}-1}(\sqrt{\lambda}|x|)\right]$ for $x \neq 0$

f — the primitive of the nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, see (2.9)
$f(x,s,t) = \frac{a(x)}{p}\left(|s|^p + 2b(x)|s|^{\frac{p}{2}}|t|^{\frac{p}{2}} + |t|^p\right)$

h — the Legendre transform $h : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, see Prop. 2.9 and Lem. 2.10
$h(x, \bar{s}, \bar{t}) = \sup_{s,t \in \mathbb{R}}\left(s\bar{s} + t\bar{t} - f(x,s,t)\right)$

I_μ — the scalar functional $I_\mu : L^{p'}(\mathbb{R}^N) \to \mathbb{R}$, see (2.7)
$I_\mu(\bar{u}) = \frac{1}{p'}\int_{\mathbb{R}^N} a(x)^{1-p'}|\bar{u}|^{p'}\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} \bar{u}\,\mathcal{R}_\mu[\bar{u}]\,\mathrm{d}x$

E_μ — the scalar functional $E_\mu : L^{p'}(\mathbb{R}^N) \to [0, \infty]$, see (2.19)
$E_\mu(\bar{u}) = \frac{p-2}{2p}\left(\frac{\left[\int_{\mathbb{R}^N} a(x)^{1-p'}|\bar{u}|^{p'}\,\mathrm{d}x\right]^{\frac{1}{p'}}}{\left[\int_{\mathbb{R}^N} \bar{u}\,\mathcal{R}_\mu[\bar{u}]\,\mathrm{d}x\right]_+^{\frac{1}{2}}}\right)^{\frac{2p}{p-2}}$

c_μ — the mountain pass level of I_μ and minimum of E_μ

$J_{\mu\nu}$ — the functional $J_{\mu\nu} : L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N) \to \mathbb{R}$, see (2.4)
$J_{\mu\nu}(\bar{u}, \bar{v}) = \int_{\mathbb{R}^N} h(x, \bar{u}, \bar{v})\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} \bar{u}\,\mathcal{R}_\mu[\bar{u}] + \bar{v}\,\mathcal{R}_\nu[\bar{v}]\,\mathrm{d}x$

$F_{\mu\nu}$ — the functional $F_{\mu\nu} : L^{p'}(\mathbb{R}^N) \times L^{p'}(\mathbb{R}^N) \to [0, \infty]$, see (2.16)
$F_{\mu\nu}(\bar{u}, \bar{v}) = \frac{p-2}{2p}\left(\frac{\left[\int_{\mathbb{R}^N} p'\,h(x,\bar{u},\bar{v})\,\mathrm{d}x\right]^{\frac{1}{p'}}}{\left[\int_{\mathbb{R}^N} \bar{u}\,\mathcal{R}_\mu[\bar{u}] + \bar{v}\,\mathcal{R}_\nu[\bar{v}]\,\mathrm{d}x\right]_+^{\frac{1}{2}}}\right)^{\frac{2p}{p-2}}$

$c_{\mu\nu}$ — the mountain pass level of $J_{\mu\nu}$ and minimum of $F_{\mu\nu}$, see (2.5) and Lem. 2.14

Important Quantities in Chapter 3

Throughout, we consider radially symmetric, real-valued functions and write e.g. $L^p_{\mathrm{rad}}(\mathbb{R}^3)$ to denote the associated subspace of $L^p(\mathbb{R}^3)$. We frequently use the same symbol both for a radially symmetric $w : \mathbb{R}^3 \to \mathbb{R}$ and for its profile $w : [0,\infty) \to \mathbb{R}$; in case of existence, $w' := \partial_r w$ is the derivative in radial direction.

Throughout, $\lambda, \mu, \nu > 0$ and $\sigma \in (0,\pi)$.

X_q	the Banach space $X_q = \left\{ w \in C_{\mathrm{rad}}(\mathbb{R}^3) \mid \|w\|_{X_q} < \infty \right\}$ with norm $\|w\|_{X_q} = \sup_{x\in\mathbb{R}^3} (1+\|x\|^2)^{\frac{q}{2}} \|w(x)\| < \infty$, see (3.3)
$U_1(\lambda)$	the linear subspace of X_1 with elements of the form $w(x) = \alpha_w \frac{\sin(\|x\|\sqrt{\lambda})}{4\pi\|x\|} + \beta_w \frac{\cos(\|x\|\sqrt{\lambda})}{4\pi\|x\|} + O\left(\frac{1}{\|x\|^2}\right)$, see before (3.13)
$\alpha^{(\lambda)}$	$\alpha^{(\lambda)} \in X_1'$ satisfies $\alpha^{(\lambda)}(w) = \alpha_w$ for $w \in U_1(\lambda)$ as above
$\beta^{(\lambda)}$	$\beta^{(\lambda)} \in X_1'$ satisfies $\beta^{(\lambda)}(w) = \beta_w$ for $w \in U_1(\lambda)$ as above
u_0	$u_0 \in X_1 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$, a (given) radial solution of $-\Delta u_0 - \mu u_0 = u_0^3$ on $\mathbb{R}^3$
u_b	$u_b = (1+b)^{-\frac{1}{2}}\, u_0 \in X_1 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$ for $b > -1$, a radial solution of $-\Delta u_b - \mu u_b = (1+b)\, u_b^3$ on $\mathbb{R}^3$
$\mathcal{T}_{u_0}$	family of semitrivial solutions $(u_0, 0, b)$ where $b \in \mathbb{R}$, see before Thm. 3.2
$\mathfrak{T}_{u_0}$	family of diagonal solutions (u_b, u_b, b) where $b > -1$, see before Thm. 3.4
σ_0	$u_0(x) = c_0 \frac{\sin(\|x\|\sqrt{\mu}+\sigma_0)}{\|x\|} + O\left(\frac{1}{\|x\|^2}\right)$ as $\|x\| \to \infty$ for $\sigma_0 \in [0,\pi)$, see Prop. 3.1
τ_0	$w_0(x) = c\, \frac{\sin(\|x\|\sqrt{\mu}+\tau_0)}{\|x\|} + O\left(\frac{1}{\|x\|^2}\right)$ as $\|x\| \to \infty$ for $\tau_0 \in [0,\pi)$, see Prop. 3.1, with a nonzero solution of $-\Delta w_0 - \mu w_0 = 3u_0^2\, w_0$ on $\mathbb{R}^3$
$\mathcal{S}(\omega)$	set of all solutions $(u,v,b) \in X_1 \times X_1 \times \mathbb{R} \setminus \mathcal{T}_{u_0}$ of (3.1) with asymptotics (3.6), see before Thm. 3.2
$\mathfrak{S}(\omega)$	set of all solutions $(u,v,b) \in X_1 \times X_1 \times \mathbb{R} \setminus \mathfrak{T}_{u_0}$ of (3.1) with asymptotics (3.7), see before Thm. 3.2
$b_k(\omega)$	$(u_0, 0, b_k(\omega))$, $k \in \mathbb{Z}$, are bifurcation points in Thm. 3.2, i.e. in $\mathcal{T}_{u_0} \cap \overline{\mathcal{S}(\omega)}$
$\mathfrak{b}_k(\omega)$	$(u_{\mathfrak{b}_k(\omega)}, u_{\mathfrak{b}_k(\omega)}, \mathfrak{b}_k(\omega))$ are bifurcation points in Thm. 3.4, i.e. in $\mathfrak{T}_{u_0} \cap \overline{\mathfrak{S}(\omega)}$, $\mathfrak{b}_k(\omega) := \frac{3-b_k(\omega)}{1+b_k(\omega)}$ for all $k \in \mathbb{Z}$ with $b_k(\omega) > -1$, i.e. $k \geq k_\omega$
k_ω	unique integer satisfying (3.31)
$\mathcal{C}_k(\omega)$	continuum bifurcating from $(u_0, 0, b_k(\omega)) \in \mathcal{T}_{u_0}$
$\mathfrak{C}_k(\omega)$	continuum bifurcating from $(u_{\mathfrak{b}_k(\omega)}, u_{\mathfrak{b}_k(\omega)}, \mathfrak{b}_k(\omega)) \in \mathfrak{T}_{u_0}$
Φ_λ	a complex fundamental solution of the Helmholtz equation, see Lem. 3.9, $\Phi_\lambda(x) = \frac{e^{i\|x\|\sqrt{\lambda}}}{4\pi\|x\|}$ for $x \in \mathbb{R}^3 \setminus \{0\}$
Ψ_λ	real part of Φ_λ, i.e. $\Psi_\lambda(x) = \frac{\cos(\|x\|\sqrt{\lambda})}{4\pi\|x\|}$ for $x \in \mathbb{R}^3 \setminus \{0\}$
$\tilde{\Psi}_\lambda$	imaginary part of Φ_λ, i.e. $\tilde{\Psi}_\lambda(x) = \frac{\sin(\|x\|\sqrt{\lambda})}{4\pi\|x\|}$ for $x \in \mathbb{R}^3 \setminus \{0\}$

$\mathcal{R}_\lambda$ the convolution operator $\mathcal{R}_\lambda : X_3 \to X_1,\ f \mapsto \Psi_\lambda * f$, see Prop. 3.13
$\tilde{\mathcal{R}}_\lambda$ the convolution operator $\tilde{\mathcal{R}}_\lambda : X_3 \to X_1,\ f \mapsto \tilde{\Psi}_\lambda * f$, see Prop. 3.13
$\mathcal{R}_\lambda^\sigma$ the convolution operator $\mathcal{R}_\lambda^\sigma = \mathcal{R}_\lambda + \cot(\sigma)\,\tilde{\mathcal{R}}_\lambda$, see (3.12)
$\mathbf{R}_\lambda^\sigma$ the linearized operator $\mathbf{R}_\lambda^\sigma : X_1 \to X_1,\ w \mapsto \mathcal{R}_\lambda^\sigma[u_0^2\, w]$, see (3.17)

F see proof of Thm. 3.2, case $\omega \neq 0$; for $v, w \in X_1$ and $b \in \mathbb{R}$:
$F(w,v,b) = \begin{pmatrix} w - \mathcal{R}_\mu^{\tau_1}[w^3 + 3u_0w^2 + 3u_0^2w + b\,(u_0+w)v^2] \\ v - \mathcal{R}_\nu^{\omega}[v^3 + bv(u_0+w)^2] \end{pmatrix}$;
$F(w,v,b) = 0$ if and only if $(u_0 + w, v, b)$ solves (3.1) with asymptotics (3.6)

$G_\pm$ see proof of Thm. 3.2, case $\omega = 0$; for $v, w \in X_1$ and $b \in \mathbb{R}$:
$G_\pm(w,v,b) = \begin{pmatrix} w - \mathcal{R}_\mu^{\tau_1}[w^3 + 3u_0w^2 + 3u_0^2w + b\,(u_0+w)v^2] \\ v - \mathcal{R}_\nu[v\,(v^2 + b\,(w+u_0)^2)] - (\alpha^{(\nu)}(v) \pm \beta^{(\nu)}(v)) \cdot \tilde{\Psi}_\nu \end{pmatrix}$;
$G_\pm(w,v,b) = 0$ if and only if $(u_0 + w, v, b)$ solves (3.1) with asymptotics (3.6)

Important Quantities in Chapter 4

Aside from the maps F, G and the function w_0, we still rely on the notations in Chapter 3 and only add additional ones. Throughout, we use...

... capital letters for functions $U : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$, $(t, x) \mapsto U(t, x)$,

... lower-case letters for (radially symmetric) functions $u : \mathbb{R}^3 \to \mathbb{R}$, $x \mapsto u(x)$, and

... bold letters for (symmetric) sequences of functions $\mathbf{u} = (u_k)_{k\in\mathbb{Z}}$.

Further:

$\mathcal{X}_q$ — the Banach space $\mathcal{X}_q = \ell^1_{\mathrm{sym}}(\mathbb{Z}, X_q)$
$= \{(u_k)_{k\in\mathbb{Z}} \mid u_k = u_{-k} \in X_q,\ \sum_{k\in\mathbb{Z}} \|u_k\|_{X_q} < \infty\}$

Γ — $\Gamma \in L^\infty_{\mathrm{rad}}(\mathbb{R}^3)$,
a (given) radial, continuous, bounded function such that w_0 (below) exists

w_0 — $w_0 \in X_1 \cap C^2_{\mathrm{loc}}(\mathbb{R}^3)$,
a (given) radial solution of $-\Delta w_0 \mp w_0 = \Gamma(x) w_0^3$ on $\mathbb{R}^3$, see (4.4) resp. (4.19)

$\mathbf{w}$ — the sequence $\mathbf{w} = (w_k)_{k\in\mathbb{Z}}$ with $w_k = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$

$\langle k\rangle$ — $:= \sqrt{k^2+1}$ for $k \in \mathbb{Z}$, see proof of Lemma 4.13

σ_k, τ_k — asymptotic parameters, see around equation (4.11)

F — see proof of Thm. 4.1 resp. Thm. 4.8, case $\tau_s \neq 0$; for $\mathbf{v} \in \mathcal{X}_1$, $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$:

$$F(\mathbf{v},\lambda)_k = v_k - \begin{cases} \mathcal{R}_1^{\tau_0}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_0 - \Gamma\, w_0^3\right] & k = 0,\\ \mathcal{R}_{s^2+1}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right] & \\ \qquad +(\cot(\tau_s)-\lambda)\tilde{\mathcal{R}}_{s^2+1}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right] & k = \pm s,\\ \mathcal{R}^{\tau_k}_{k^2+1}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_k\right] & \text{else} \end{cases}$$

resp.

$$F(\mathbf{v},\lambda)_k = v_k - \begin{cases} \mathcal{P}_1\left[\Gamma\ (\mathbf{u}\star\mathbf{u}\star\mathbf{u})_0 - \Gamma\, w_0^3\right] & k = 0,\\ \mathcal{R}_{4s^2-1}\left[\Gamma\ (\mathbf{u}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right] & \\ \qquad +(\cot(\tau_s)-\lambda)\tilde{\mathcal{R}}_{4s^2-1}\left[\Gamma\ (\mathbf{u}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right] & k = \pm s,\\ \mathcal{R}^{\tau_k}_{4k^2-1}\left[\Gamma\ (\mathbf{u}\star\mathbf{u}\star\mathbf{u})_k\right] & \text{else} \end{cases}$$

Λ_1 — the fundamental solution of the Schrödinger equation, see (4.21)
$\Lambda_1(x) = \Lambda_1(x) = \frac{\mathrm{e}^{-|x|}}{4\pi|x|}$ for $x \in \mathbb{R}^3 \setminus \{0\}$

$\mathcal{P}_1$ — the convolution operator $\mathcal{P}_1 : X_3 \to X_1,\ f \mapsto \Lambda_1 * f$, see Lem. 4.10

G — see proof of Thm. 4.1 resp. Thm. 4.8, case $\tau_s \neq 0$; for $\mathbf{v} \in \mathcal{X}_1$, $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$:

$$G(\mathbf{v},\lambda)_k = v_k - \begin{cases} \mathcal{R}_1^{\tau_0}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_0 - \Gamma\, w_0^3\right] & k = 0,\\ \mathcal{R}_{s^2+1}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right] & \\ \qquad +(1-\lambda)\left(\alpha^{(s^2+1)}(v_{\pm s}) + \beta^{(s^2+1)}(v_{\pm s})\right)\tilde{\Psi}_{s^2+1} & k = \pm s,\\ \mathcal{R}^{\tau_k}_{k^2+1}\left[\Gamma\,(\mathbf{u}\star\mathbf{u}\star\mathbf{u})_k\right] & \text{else} \end{cases}$$

resp.

$$G(\mathbf{v},\lambda)_k = v_k - \begin{cases} \mathcal{P}_1\left[\Gamma\ (\mathbf{u}\star\mathbf{u}\star\mathbf{u})_0 - \Gamma\, w_0^3\right] & k = 0,\\ \mathcal{R}_{4s^2-1}\left[\Gamma\ (\mathbf{u}\star\mathbf{u}\star\mathbf{u})_{\pm s}\right] & \\ \qquad +(1-\lambda)\left(\alpha^{(4s^2-1)}(v_{\pm s}) + \beta^{(4s^2-1)}(v_{\pm s})\right)\tilde{\Psi}_{4s^2-1} & k = \pm s,\\ \mathcal{R}^{\tau_k}_{4k^2-1}\left[\Gamma\ (\mathbf{u}\star\mathbf{u}\star\mathbf{u})_k\right] & \text{else} \end{cases}$$

Eidesstattliche Erklärung:

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Arbeit selbstständig und nur unter Zuhilfenahme der ausgewiesenen Hilfsmittel angefertigt habe. Sämtliche Stellen der Arbeit, die im Wortlaut oder dem Sinn nach anderen gedruckten oder im Internet veröffentlichten Werken entnommen sind, habe ich durch genaue Quellenangaben kenntlich gemacht.

Karlsruhe, den 6. September 2019