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Abstract

The focus of the research described herein is the scattering of time-harmonic electromagnetic waves when encountering with impenetrable and penetrable obstacles. We study both the direct and inverse problems. In the case of an impenetrable obstacle, we assume perfectly conducting boundary condition and apply the integral equation method to show well-posedness of the direct problem. In the case of a penetrable obstacle, we assume conducting transmission conditions and apply both the integral equation and variational method to show well-posedness. The inverse problem we consider is determining the shape of an obstacle from the knowledge of the far field pattern. Specifically, we concentrated on uniqueness issues, that is, we examined under what conditions an obstacle can be identified from a knowledge of its far far field patterns for incident plane waves. We conclude this thesis with a discussion of an interior eigenvalue problem motivated by the penetrable case with conducting boundary conditions and show that the set of transmission eigenvalues form at most a discrete set.

Contents

1 Introduction

In mathematics and physics, scattering theory provides a framework for studying and understanding the scattering of waves and particles. Wave scattering occurs when a wave collides with some material object and scatters, and scattering theory provides a setting for studying and understanding the interaction or scattering of solutions to partial differential equations. In particular, scattering theory deals with two basic problems: the scattering of time-harmonic acoustic or electromagnetic waves by a bounded impenetrable obstacle and by a penetrable inhomogeneous medium of compact support. The focus of this research was the scattering of time-harmonic electromagnetic waves. We considered the case of an impenetrable obstacle with perfectly conducting boundary condition and the case of a penetrable obstacle with conductive transmission conditions. In the case of an impenetrable obstacle, we assumed the obstacle to be embedded in an inhomogeneous and bounded medium, whereas in the case of a penetrable obstacle, we assumed the scatterer to be an inhomogeneous medium situated in a homogeneous background. Moreover, we considered a transmission eigenvalue problem for the case of conductive boundary conditions.

In the discussion below we will consider both the *direct* (also called *forward*) and the *inverse* problems for both the impenetrable and penetrable cases. The *direct* scattering problem is to find the scattered wave given information on the boundary of the scatterer and the nature of the boundary condition. We will address the questions of uniqueness, existence and stability, i.e. the continuous dependence of the scattered fields E^s, H^s with respect to the corresponding incident fields E^i, H^i. In other words, we address the question of the well-posedness of the direct problem, as postulated by Hadamard in mathematical physics, which, generally speaking, can be studied either by applying the integral equation or the variational method. In Chapter Two, we apply the former in the case of an impenetrable obstacle, whereas, in Chapter Three, we apply both methods in the case of a penetrable obstacle.

The field of inverse scattering theory has been a particularly active area within applied mathematics for the past thirty years. The two basic aims of research in this field has been to detect and to identify unknown objects through the use of acoustic, electromagnetic or elastic waves. In many practical applications, the detection and identification aims are connected in a complicated way. For instance, a problem in medical imaging could involve the presence of a hard object inside a damaged human body. Here, we are interested both in locating the object and recovering the damage by exterior measurements and one possible model would be an object inside an inhomogeneous structure. The goal we would have in mind in this situation is to determine the shapes of the scatterers from the knowledge of the far field patterns E_∞, H_∞. In contrast to the *direct* problem, the *inverse* problem is improperly posed. As mentioned above, a problem is well-posed in the sense of Hadamard if the solution exists, the solution is unique and the solution depends continuously on the data. A problem is ill-posed if at least one of these three statements does not hold, as is the case for the inverse problems we will be considering. In particular, small perturbations of the far field pattern in any reasonable norm lead to a function that lies outside the class of far field patterns. Nevertheless, the *inverse* scattering problem is important in areas such as radar, sonar, geophysical exploration, medical imaging and nondestructive testing. With the knowledge of the *direct* scattering problem, the *inverse* problem is currently in the foreground of mathematical research in scattering theory.

The *transmission eigenvalue problem* refers to a family of spectral problems and is a class of non-selfadjoint eigenvalue problem. It is a boundary value problem for a set of equations defined in a bounded domain coinciding with the support of the scattering object. It is related to the scattering problem for inhomogeneous medium that has become an important area of research in inverse scattering theory. The solution of the *transmission eigenvalue problem* can be viewed as determining an incident field such that for a given inhomogeneous medium, the scattered field is zero. In recent years, particular attention has been given to the study of the frequencies for which this problem has non-unique solutions, i.e. the so-called transmission eigenvalues, whose values can, for instance, be used to determine the values of the medium physical parameters. This research involved the study of the *transmission eigenvalue problem* for Maxwell's equations where both physical parameters ε and μ differed from the ε_0 and μ_0 modelling the background medium, with conductive transmission conditions. As we will prove the problem is of Fredholm type, and the transmission eigenvalues form at most a discrete set.

1.1 Maxwell's equations

Let $\mathcal{E}$ and $\mathcal{H}$ denote the electric and magnetic fields, respectively. The electromagnetic wave satisfies Maxwell's equations

$$\operatorname{curl}\mathcal{E} + \mu\frac{\partial\mathcal{H}}{\partial t} = 0, \qquad \operatorname{curl}\mathcal{H} - \varepsilon\frac{\partial\mathcal{E}}{\partial t} = \sigma\mathcal{E},$$

where ε, μ and σ are the electric permittivity, magnetic permeability and conductivity, respectively. In the time-harmonic case, we assume that the electric and magnetic field can be decomposed into space-dependent and time-dependent parts as follows:

$$\mathcal{E}(x,t) = E(x)e^{-i\omega t}, \qquad \mathcal{H}(x,t) = H(x)e^{-i\omega t},$$

where $\omega > 0$ is the frequency. Then, the (complex-valued) fields E and H satisfy

$$\operatorname{curl} E - i\omega\mu H = 0, \qquad \operatorname{curl} H + i\omega\varepsilon E = \sigma E.$$

Both in the case of an impenetrable and a penetrable obstacle, we consider the scattering of an incident time-harmonic electromagnetic wave $\mathcal{E}^i(x,t) = E^i(x)e^{-i\omega t}$, $\mathcal{H}^i(x,t) = H^i(x)e^{-i\omega t}$, $x \in \mathbb{R}^3$, $t \in \mathbb{R}$, $t > 0$ with E^i and H^i satisfying

$$\operatorname{curl} E^i - i\omega\mu_0 H^i = 0, \qquad \operatorname{curl} H^i + i\omega\varepsilon_0 E^i = 0 \qquad \text{in } \mathbb{R}^3,$$

where ε_0 and μ_0 are real positive numbers, denoting the electric permittivity and magnetic permeability in a vacuum. This incident wave is assumed to be scattered by a medium, resulting in the total fields E and H, given as the sum $E = E^i + E^s$ and $H = H^i + H^s$, respectively, where E^s and H^s denote the scattered fields. For E^s and H^s to be outgoing, we require them to satisfy the Silver-Müller radiation condition

$$\sqrt{\frac{\mu_0}{\varepsilon_0}} H^s(x) \times x - |x| E^s(x) = \mathcal{O}\left(\frac{1}{|x|}\right) \qquad \text{as } |x| \to \infty$$

uniformly with respect to all directions $x/|x|$. The above radiation condition leads to an asymptotic behavior of the form

$$E^s(x) = \frac{e^{ik|x|}}{|x|}\left\{E_\infty\left(\frac{x}{|x|}\right) + \mathcal{O}\left(\frac{1}{|x|}\right)\right\}, \tag{1.1}$$

$$H^s(x) = \frac{e^{ik|x|}}{|x|}\left\{H_\infty\left(\frac{x}{|x|}\right) + \mathcal{O}\left(\frac{1}{|x|}\right)\right\} \tag{1.2}$$

as $|x| \to \infty$, where the vector fields E_∞ and H_∞, defined on the unit sphere $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$, are the electric far field pattern and magnetic far field pattern, respectively, of the scattered waves. The constant $k := \omega\sqrt{\varepsilon_0\mu_0} > 0$ is called the wave number, because

$k/2\pi$ tells us the number of wavelengths per unit length. The far field pattern satisfy

$$H_\infty = \frac{x}{|x|} \times E_\infty \quad \text{and} \quad \frac{x}{|x|} \cdot E_\infty = \frac{x}{|x|} \cdot H_\infty = 0,$$

with the unit outward normal $\hat{x} = x/|x|$ on S^2. A vanishing magnetic far field pattern on the unit sphere implies $H^s = E^s = 0$ in $\mathbb{R}^3 \setminus \overline{D}$ (see [6]), that is, the far field pattern uniquely determines the scattered wave and consequently the total wave in the exterior of the scatterer.

1.2 Spaces and traces

In order to formulate and study the scattering problems, we require appropriate function spaces that are based or built on the Sobolev spaces for scalar- and vector-valued functions. The theory of Sobolev spaces was originated by Russian mathematician S.L. Sobolev around 1938. These spaces were not introduced for purely theoretical reasons, but for the purposes of the theory of partial differential equations, see e.g. [18], [2].

A large part of our work concerns Maxwell's equations on bounded domains in $\mathbb{R}^3$ (unbounded domains will be reduced to bounded domains by a truncation problem). By a domain in $\mathbb{R}^3$ we mean an open connected set in 3-dimensional real Euclidean space $\mathbb{R}^3$. In this section, we let $G \subset \mathbb{R}^3$, be a bounded domain with unit outward normal ν and boundary $\partial G = \Sigma$. By a unit outward normal, we mean a normal vector, such that $|\nu| = 1$, pointing towards the exterior of G. Vector-valued functions are indicated by the superscript '3' and, unless otherwise stated, all functions are complex-valued. The dot '$\cdot$' denotes a complex scalar product. For a general Hilbert space X, we denote the dual space by X'. The support of a function is defined by

$$\text{supp } u = \overline{\{x \in G : \; u(x) \neq 0\}}.$$

We start by defining some standard spaces of continuous functions (see e.g. [18]) and classes of domains.

Continuous functions. For $0 \leq k < \infty$ we define:

$C^k(G)$: the set of k-times continuously differentiable functions on G;

$C_0^k(G)$: the set of functions $u \in C^k(G)$ having compact support in G;

$C^{k,\alpha}(G)$: the set of k times continuously differentiable functions on G such that the k'th partial derivatives are Hölder continuous with exponent α, where $0 < \alpha \leq 1$;

$C^k(\overline{G})$: the set of functions in $C^k(G)$ which have bounded and uniformly continuous derivatives up to order k on $\overline{G}$ (i.e. the restrictions of functions in $C_0^k(\mathbb{R}^3)$ to G);

$C^\infty(G) = \bigcap_{k=1}^\infty C^k(G)$: the set of smooth functions;

$C_0^\infty(G) = C^\infty(G) \cap C_0(G)$: the set of smooth functions having a compact support in G.

As mentioned above, in the case of vector fields, we will write '3' as a superscript to the space. So, for example $C^k(G)^3 = C^k(G, \mathbb{C}^3)$ denotes all k-times continuously differentiable vector fields $G \to \mathbb{C}^3$.

Definition 1.1. *The boundary $\Sigma = \partial G$ of a bounded domain G in $\mathbb{R}^3$ is C^k-smooth, $0 \leq k \leq \infty$, if for every $x \in \Sigma$ there is an open set $\mathcal{U} \subset \mathbb{R}^3$ with $x \in \mathcal{U}$ and an orthogonal coordinate system with coordinates $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ having the following properties. There is a vector $a \in \mathbb{R}^3$ with*

$$\mathcal{U} = \{\zeta : -a_j < \zeta_j < a_j,\ j = 1, 2, 3\}$$

and a C^k-continuous function g defined on

$$\mathcal{U}' = \{\zeta' \in \mathbb{R}^2 : -a_j < \zeta_j < a_j,\ j = 1, 2\}$$

with $|g(\zeta')| \leq a_3/2$ for all $\zeta' \in \mathcal{U}'$ such that

$$\begin{aligned} G \cap \mathcal{U} &= \{\zeta : \zeta_3 < g(\zeta'),\ \zeta' \in \mathcal{U}'\} \quad \textit{and} \\ \Sigma \cap \mathcal{U} &= \{\zeta : \zeta_3 = g(\zeta'),\ \zeta' \in \mathcal{U}'\}. \end{aligned}$$

The boundary Σ is said to be Lipschitz continuous if the function g which describes the boundary locally is Lipschitz continuous, that is, there exists a constant $L > 0$ such that $|g(\zeta') - g(\eta')| \leq L|\zeta' - \eta'|$ for all $\zeta', \eta' \in \mathcal{U}'$.

We will simply say that the domain is Lipschitz or C^k-smooth when we mean that it has a Lipschitz continuous or C^k-smooth boundary. We note that a C^m-smooth domain ($m \geq 1$) is also a Lipschitz domain. One key property of a Lipschitz domain is that it has a well-defined unit outward ν at almost every point on Σ.

L^p and standard Sobolev spaces. For $1 \leq p < \infty$, $L^p(G)$ denotes the set of functions ϕ on G for which $|\phi|^p$ is integrable, or, stated more exactly, functions ϕ such that

$$\int_G |\phi|^p dx < \infty.$$

The norm on $L^p(G)$ will be deonted by $\|\cdot\|_{L^p(G)}$. In the case of vector fields, we write $L^p(G)^3$. The most important case here is $p = 2$, which is the set of all square-integrable functions on G. In the case of $L^2(G)^3$, an inner product exists and will be denoted by $(\cdot,\cdot)_G$, that is

$$(U,V)_G = \int_G U \cdot \overline{V}\, dx = \int_G \sum_j^3 U_j \overline{V}_j\, dx$$

for $U, V \in L^2(G)^3$, and the norm induced by the inner product will be denoted by $\|\cdot\|_G$. For a bounded Lipschitz domain G, $C_0^\infty(G)$ is dense in $L^2(G)$ (see e.g., lemma 3.4 in [2]).

We use the standard multi-index notation for derivatives. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T \in \mathbb{Z}_+^3$ and $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$, where $\mathbb{Z}_+$ is the set of non-negative integers. We then set

$$|\alpha| = \sum_{j=1}^3 |\alpha_j|, \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}, \quad \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}}.$$

For $u \in L^1_{\text{loc}}(G)$, we call $v \in L^1_{\text{loc}}(G)$ the α-th weak derivative of u, written $\partial^\alpha u = v$, if

$$\int_G u \partial^\alpha \phi\, dx = (-1)^{|\alpha|} \int_G v\phi\, dx \quad \text{for all } \phi \in C_0^\infty(G).$$

If $|\alpha| = 1$, then $\partial u = \nabla u$ is the weak gradient of u. The weak derivative, if it exists, is uniquely defined up to a set of measure zero. For functions $u \in C^k(G)$, the weak and classical (or strong) derivatives of u agree provided $|\alpha| \leq k$.

For $s \in \mathbb{Z}_+$, the standard Sobolev spaces are denoted by $W^{s,p}(G)$. These spaces are defined by

$$W^{s,p}(G) = \{u \in L^p(G) : \partial^\alpha u \in L^p(G) \text{ for all } |\alpha| \leq s\}.$$

We equip $W^{s,p}(G)$ with the norm

$$\|u\|_{W^{s,p}(G)} = \left(\sum_{|\alpha| \leq s} \int_G |\partial^\alpha \phi(x)|^p\, dx \right)^{\frac{1}{p}}$$

and use the convention $W^{0,p}(G) = L^p(G)$. The most important case to us is when $p = 2$. Then $H^s(G) = W^{s,2}(G)$, even for Lipschitz domains, see [3]. The Sobolev spaces can also be defined for non-integers $s \in \mathbb{R}_{\geq 0}$. Write $s = m + r$ with $m \in \mathbb{N}_0$ and $0 < r < 1$. Then $W^{s,p}(G)$ is the space of functions $u \in W^{m,p}(G)$ with

$$\iint_G \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x-y|^{3+rp}}\, dxdy < \infty \quad \text{for all multi-indices } \alpha \text{ with } |\alpha| = m, \tag{1.3}$$

equipped with the norm

$$\|u\|_{W^{s,p}(G)} = \|u\|_{W^{m,p}(G)} + \sum_{|\alpha=m} \iint_G \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x-y|^{3+rp}}\, dxdy. \tag{1.4}$$

We still have $H^s(G) = W^{s,2}(G)$ for $p = 2$ and again, use the abbreviation $H^s(G)^3$ for the vector-valued case and $H^{-s}(G)$ for $s \geq 0$ denotes the dual space of $H^s(G)$ (with respect to L^2), i.e. $H^{-s}(G) := (H^s(G))'$. Moreover, we denote by $W^{s,p}_{\text{loc}}(G)$ the space of functions whose restrictions to any bounded subdomain B of G belonging to $W^{s,p}(B)$.

For functions in the Sobolev spaces to satisfy a Dirichlet boundary condition (i.e. vanishing on the boundary), the closure of $C_0^\infty(G)$ in the appropriate norm is used to define

$$W_0^{s,p}(G) = \text{ closure of } C_0^\infty(G) \text{ in the } W^{s,p}(G) \text{ norm.}$$

Again, the special case $p = 2$ has its own notation

$$H_0^s(G) = W_0^{s,2}(G).$$

As we will shortly see, functions u in $H_0^1(G)$ satisfy the boundary condition $u = 0$ on the boundary Σ in an appropriate sense.

We say that $W^{s,p}(G)$ is imbedded in a space X and write $W^{s,p}(G) \hookrightarrow X$ if $W^{s,p}(G)$ is a subset of X and if the identity map I from $W^{s,p}(G)$ to X is continuous. This is equivalent to saying that there exists a constant C independent of u such that $\|Iu\|_X \leq C\|u\|_{W^{s,p}(G)}$ for all $u \in W^{s,p}(G)$. The embedding is said to be compact if the embedding operator I is compact. The following theorem from [34], see also [3], summarizes some results on compact embeddings that will be useful for our purposes.

Theorem 1.2. *Let $G \subset \mathbb{R}^3$ be a bounded Lipschitz domain. The following embeddings are* *compact:*

$$W^{j+m,p}(G) \hookrightarrow W^{j,q}(G) \quad \text{if } 0 < n - mp \text{ and } j + m - \frac{n}{p} \geq j - \frac{n}{q}.$$

In particular, for the special case $p = q = 2$ we obtain compactness if $0 \leq m < \frac{3}{2}$.

Remark 1.3. *For C^1-smooth domains G, the embeddings $W^{j+m,p}(G) \hookrightarrow W^{j,q}(G)$ are compact if $m \geq 0$ (Kondrachov embedding theorem).*

Traces and trace spaces for standard Sobolev spaces. We start by defining Sobolev spaces on the boundary Σ of G. We follow [2], section 3.2.1, and recall from definition 1.1 that the boundary Σ is such that, for every $x \in \Sigma$, there is a Lipschitz continuous map $g : \mathcal{U}' \subset \mathbb{R}^2 \to \mathbb{R}$ with

$$\Sigma \cup \mathcal{U} = \{\zeta = (\zeta', g(\zeta') : \zeta' \in \mathcal{U}')\}$$

and thus locally Σ is a two-dimensional surface in $\mathbb{R}^3$. We define $\mathbf{g}$ via $\mathbf{g}(\zeta') = (\zeta', g(\zeta'))$. Then $\mathbf{g}^{-1}$ exists and is Lipschitz continuous on $\mathbf{g}(\mathcal{U}')$. This motivates the following definition.

Definition 1.4. *Let $|s| \leq 1$. u belongs to $W^{s,p}(\Sigma)$ if the composition $u \circ \boldsymbol{g}$ belongs to $W^{s,p}(\mathcal{U}' \cap \boldsymbol{g}^{-1}(\Sigma \cap \mathcal{U}))$ for all possible $\mathcal{U}$ and $\boldsymbol{g}$ fulfilling the criteria of definition 1.1.*

To define a norm on $W^{s,p}(\Sigma)$, we let $\left(\mathcal{U}_j, \mathbf{g}_j\right)_{j=1}^N$ be any local coordinate system of Σ such that the pairs $\left(\mathcal{U}_j, \mathbf{g}_j\right)$ satisfy the conditions of definition 1.1. Then

$$\|u\|_{W^{s,p}(\Sigma)} = \left(\sum_{j=1}^{N} \|u \circ \mathbf{g}_j\|^p_{W^{s,p}(\mathcal{U}_j' \cap \mathbf{g}_j^{-1}(\Sigma \cap \mathcal{U}_j'))} \right)^{\frac{1}{2}}.$$

In the particular case $s \in [0, 1)$, this definition is equivalent to

$$\|u\|_{W^{s,p}(\Sigma)} = \left(\int_\Sigma |u|^p \, ds + \int_\Sigma \int_\Sigma \frac{|u(x) - u(y)|^p}{|x-y|^{2+sp}} \, ds(x) \, ds(y) \right)^{\frac{1}{p}},$$

where ds is the surface measure on Σ. As usual, $H^s(\Sigma) = W^{s,2}(\Sigma)$.

It is well known (e.g. [35]) that there exists a linear, continuous trace operator $\gamma_0 : W^{s,p}(G) \to W^{s-\frac{1}{p},p}(\Sigma)$ such that

$$\gamma_0(\phi) = \phi|_\Sigma \tag{1.5}$$

provided $1/p \leq s \leq 1$. Using the trace operator, the space $W_0^{1,p}(G)$ consists of all functions in $W^{s,p}(G)$ with vanishing traces on the boundary. An alternative definition for $W_0^{1,p}(G)$, $p > 1$ is

$$W_0^{1,p}(G) = \left\{u \in L^p(G) : \nabla u \in L^p(G)^3 \text{ and } \gamma_0(u) = 0\right\},$$

where ∇ denotes the gradient defined by

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}\right)^T.$$

The most important trace space for us will be $H^{\frac{1}{2}}(\Sigma)$ and its dual space $H^{-\frac{1}{2}}(\Sigma)$. The norm on this space is the usual dual norm. In particular, for any Lipschitz surface S we define

$$< \varphi, \psi >_S := \int_S \varphi\overline{\psi}\, ds.$$

The norm on $H^{-\frac{1}{2}}(\Sigma)$ can be written as

$$\|\phi\|_{H^{-\frac{1}{2}}(\Sigma)} = \sup_{\psi \in H^{\frac{1}{2}}(\Sigma)} \frac{|\langle \phi, \psi \rangle_\Sigma|}{\|\psi\|_{H^{\frac{1}{2}}(\Sigma)}}$$

where we used the fact that $H^{-\frac{1}{2}}(\Sigma)$ can also be characterized as the completion of $L^2(\Sigma)$ in a suitable norm to show that we may identify the duality pairing with the $L^2(\Sigma)$ inner product (see [18] for more details). We also require the use of trace spaces for $s > 1$. The following definition is from [36] and agrees with the previous one for $0 \leq s \leq 1$. For $s > 1$, we define the normed space

$$H^s(\Sigma) = \left\{u \in L^2(\Sigma) : u = U|_\Sigma \text{ for some } U \in H^{s+\frac{1}{2}}(G)\right\},$$

and norm given by

$$\|u\|_{H^s(\Sigma)} = \inf_{U \in H^{s+\frac{1}{2}}(G), u = U|_\Sigma} \|U\|_{H^{s+\frac{1}{2}}(G)}.$$

In particular, $\|u\|_{H^s(\Sigma)} = \|U\|_{H^{s+\frac{1}{2}}(G)}$, where $U \in H^{s+\frac{1}{2}}(G)$ satisfies $U|_\Sigma = u$ and

$$(U, \phi)_{H^{s+\frac{1}{2}}(G)} = 0 \quad \text{for all } \phi \in H^{s+\frac{1}{2}}(G) \cap H_0^1(G).$$

This function exists by the Lax-Milgram theorem 1.15. Thus, we can see that $H^s(\Sigma)$ is complete since $H^{s+\frac{1}{2}}(G)$ is complete, and is, in fact, a Hilbert space.

Differential operators on a surface. Next, we define some differential operators related to tangential vector fields on Σ. We introduce the space of surface tangential vector fields in $L^2(\Sigma)$ by

$$L_t^2(\Sigma) = \{\varphi \in L^2(\Sigma)^3 : \varphi \cdot \nu = 0 \text{ on } \Sigma\}$$

where again ν is the unit outward normal to G. The norm on this space is the standard $L^2(\Sigma)^3$ norm. We start by defining two fundamental differential operators, the surface gradient and surface divergence. There exist several different, but equivalent, approaches for doing so. We follow the approach from [2]. For a function $p \in H^1(\Sigma)$, we define the surface gradient $\nabla_\Sigma p$ via a parametric representation of Σ. Suppose $\boldsymbol{x} \in \Sigma$ can be written as

$$\boldsymbol{x} = (x_1(u_1, u_2), x_2(u_1, u_2), x_3(u_1, u_2))^T$$

for some surface patch of Σ. Then, on this patch, $\nabla_\Sigma p \in L_t^2(\Sigma)$ is defined by

$$\nabla_\Sigma p = \sum_{i,j=1}^{2} f^{i,j} \frac{\partial p}{\partial u_i} \frac{\partial \boldsymbol{x}}{\partial u_j}$$

where $f^{i,j}$ is the (i,j)-th entry of the inverse of the matric F given by

$$F_{i,j} = \frac{\partial \boldsymbol{x}}{\partial u_i} \cdot \frac{\partial \boldsymbol{x}}{\partial u_j}, \quad i, j = 1, 2.$$

We note that the surface gradient and volume gradient are related for functions p that are differentiable in the neighborhood of Σ by

$$(\nabla p)\,|_\Sigma = \nabla_\Sigma p + \frac{\partial p}{\partial \nu} \nu.$$

With this observation, we see that $(\nu \times \nabla p) \times \nu = \nabla_\Sigma p$. Having defined the surface gradient, we can define the surface divergence $\mathrm{Div} : L_t^2(\Sigma) \to H^{-1}(\Sigma)$ by duality so that if $V \in L_t^2(\Sigma)$, then $\mathrm{Div}\, V \in H^{-1}(\Sigma)$ satisfies

$$\int_\Sigma \mathrm{Div}\, V p \, ds = -\int_\Sigma V \cdot \nabla_\Sigma p \, ds \quad \text{for all } p \in H^1(\Sigma).$$

The third fundamental operator we consider here is the surface vector curl denoted by

Curl : $H^1(\Sigma) \to L_t^2(\Sigma)$ and defined by

$$\mathbf{Curl}\, p = -\nu \times \nabla_\Sigma p.$$

Lastly, we define the scalar curl denoted by Curl : $L_t^2(\Sigma) \to H^{-1}(\Sigma)$ and defined via duality using Stokes theorem, so that if $V \in L_t^2(\Sigma)$ then

$$\int_\Sigma \mathrm{Curl}\, V p\, ds = \int_\Sigma V \cdot \mathbf{Curl}\, p\, ds \quad \text{for all } p \in H^(\Sigma).$$

By using the duality definitions, we see that for $V \in L_t^2(\Sigma)$ we have

$$\mathrm{Curl}\, V = -\mathrm{Div}\,(\nu \times V) \quad \text{and} \quad \mathrm{Div}\, V = \mathrm{Curl}\,(\nu \times V)\,.$$

Vector fields with weak curl and divergence. We recall that for a smooth vector field $U = (U_1, U_2, U_3)^T$, the divergence and the curl are defined as follows:

$$\mathrm{div}\, U := \sum_{j=1}^{3} \frac{\partial U_j}{\partial x_j} \quad \text{and} \quad \mathrm{curl}\, U := \left(\frac{\partial U_3}{\partial x_2} - \frac{\partial U_2}{\partial x_3}, \frac{\partial U_1}{\partial x_3} - \frac{\partial U_3}{\partial x_1}, \frac{\partial U_2}{\partial x_1} - \frac{\partial U_1}{\partial x_2} \right)^T.$$

Using partial integration, we can introduce a weak notation of these differential operators in the following way; see e.g. [2], section 3.5.

- For $U \in L^2(G)^3$, we call $\mathrm{div}\, U \in L^2(G)$ the weak divergence of U if it satisfies

$$\int_G \mathrm{div}\, U \phi\, dx = -\int_G U \cdot \nabla \phi\, dx \quad \text{for all } \phi \in C_0^\infty(G).$$

- For $U \in L^2(G)^3$, we call $\mathrm{curl}\, U \in L^2(G)^3$ the weak curl of U if it satisfies

$$\int_G \mathrm{curl}\, U \cdot \psi\, dx = \int_G U \cdot \mathrm{curl}\, \psi\, dx \quad \text{for all } \psi \in C_0^\infty(G)^3.$$

Using the weak divergence and curl, we define the following spaces:

$$H(\mathrm{div}, G) = \{U \in L^2(G)^3 : \mathrm{div}\ U \in L^2(G)\},$$
$$H(\mathrm{curl}\,, G) = \{U \in L^2(G)^3 : \mathrm{curl}\, U \in L^2(G)^3\}.$$

These spaces are endowed with the natural graph norms

$$\|U\|_{H(\mathrm{div}\,,G)} = \left(\|U\|_G^2 + \|\mathrm{div}\, U\|_G^2\right)^{\frac{1}{2}}$$

and

$$\|U\|_{H(\mathrm{curl}\,,G)} = \left(\|U\|_G^2 + \|\mathrm{curl}\, U\|_G^2\right)^{\frac{1}{2}}.$$

$H(\mathrm{curl}\,, G)$ and $H(\mathrm{div}\,, G)$ are Hilbert spaces and furthermore, the space of smooth vector fields $C_0^\infty(\overline{G})^3$ is a dense subspace of the previously mentioned spaces. This motivates the following definitions:

$H_0(\mathrm{curl}\,, G) =$ closure of $C_0^\infty(\overline{G})^3$ in the $H(\mathrm{curl}\,, G)$ norm,

$H_0(\mathrm{div}, G) =$ closure of $C_0^\infty(\overline{G})^3$ in the $H(\mathrm{div}, G)$ norm.

Normal trace. For a function $V \in C^\infty(\overline{G})^3$, the normal trace operator γ_n is defined by

$$\gamma_n(V) = V|_\Sigma \cdot \nu. \tag{1.6}$$

We have the following trace and Green's theorem (corresponding to theorem 3.24 in [2]).

Theorem 1.5. *(a) The trace operator γ_n, defined by (1.6) on $C^\infty(\overline{G})^3$, can be continuously extended to a continuous linear operator γ_n from $H(\mathrm{div}\,, G)$ onto $H^{-\frac{1}{2}}(\Sigma)$.*

(b) The following form of Green's theorem holds for functions $V \in H(\mathrm{div}\,, G)$ and $\phi \in H^1(G)$:

$$(\mathrm{div}\, V, \phi)_G + (V, \nabla\phi)_G =< \phi, \gamma_n V >_\Sigma. \tag{1.7}$$

Using the normal trace operator, we can give the following explicit characterization of the space $H_0(\mathrm{Div}\,, G)$:

$$H_0(\mathrm{div}\,, G) = \{V \in H(\mathrm{div}\,, G) : \gamma_n V = 0 \text{ on } \Sigma\}.$$

Tangential trace. Finally, we discuss the trace properties of functions in $H(\mathrm{curl}\,, G)$. For a smooth vector function $U \in C^\infty(\overline{G})^3$, we define the traces

$$\gamma_t U = \nu \times U|_\Sigma, \tag{1.8}$$

$$\gamma_T U = (\nu \times U|_\Sigma) \times \nu. \tag{1.9}$$

We have the following trace and Green's theorem (corresponding to theorem 3.29 in [2]).

Theorem 1.6. *(a) The trace operator γ_t defined by (1.8) on $C^\infty(\overline{G})^3$ can be continuously extended to a continuous linear operator γ_t from $H(\operatorname{curl}, G)$ into $H^{-\frac{1}{2}}(\Sigma)^3$.*

(b) The following form of Green's theorem holds for functions $U \in H(\operatorname{curl}, G)$ and $\psi \in H^1(G)^3$:

$$(\operatorname{curl} U, \psi)_G - (U, \operatorname{curl} \psi)_G = <\gamma_t U, \psi>_\Sigma . \tag{1.10}$$

Using the tangential trace operator, we can give the following explicit characterization of the space $H_0(\operatorname{curl}, G)$:

$$H_0(\operatorname{curl}, G) = \{U \in H(\operatorname{curl}, G) : \gamma_t U = 0 \text{ on } \Sigma\}.$$

We note that the map $\gamma_t : H(\operatorname{curl}, G) \to H^{-\frac{1}{2}}(\Sigma)^3$ is not surjective since for any U, the trace $\gamma_t U$ is tangential to Σ, whereas $H^{-\frac{1}{2}}(\Sigma)^3$ contains vectors that are not tangential to Σ. Furthermore, we would like to prove a similar result about γ_T, but this is not valid for Lipschitz domains because, even if $V \in H^1(G)^3$, γ_T is not necessarily an element of $H^{\frac{1}{2}}(\Sigma)^3$ (this only holds for smooth domains). To do so, and to obtain surjectivity of the trace operator γ_t, we define the trace space of $H(\operatorname{curl}, G)$ as follows:

$$H^{-\frac{1}{2}}(\mathrm{Div}, \Sigma) = \{U \in H^{-\frac{1}{2}}(\Sigma)^3 : \nu \cdot U|_\Sigma = 0, \mathrm{Div}\, U \in H^{-\frac{1}{2}}(\Sigma)\}$$

and its dual space

$$H^{-\frac{1}{2}}(\mathrm{Curl}, \Sigma) = \{U \in H^{-\frac{1}{2}}(\Sigma)^3 : \nu \cdot U|_\Sigma = 0, \mathrm{Curl}\, U \in H^{-\frac{1}{2}}(\Sigma)\}.$$

With the above trace spaces, we have the following theorem (corresponding to theorem 3.24 in [1]).

Theorem 1.7. *The trace operators $\gamma_t : H(\operatorname{curl}, G) \to H^{-1/2}(\mathrm{Div}, \Sigma)$ and $\gamma_T : H(\operatorname{curl}, G) \to H^{-1/2}(\mathrm{Curl}, \Sigma)$, given by (1.8) and (1.9), respectively, are well defined, linear and continuous. Moreover, they are surjective and, for any $U \in H(\operatorname{curl}, G)$ and $\psi \in H(\operatorname{curl}, G)$, the following form of Green's theorem holds:*

$$(U, \operatorname{curl} \psi)_G - (\operatorname{curl} U, \psi)_G = <\gamma_t U, \gamma_T \psi>_\Sigma . \tag{1.11}$$

Remark 1.8. *We can equivalently define the traces $\gamma_0, \gamma_n, \gamma_t$ and γ_T for functions on $\mathbb{R}^3 \setminus G$. That is,*

$$\begin{aligned}
&\gamma_0 : H^s_{\rm loc}(\mathbb{R}^3 \setminus G) \to H^{s-1/2}(\Sigma), \quad 1/2 < s \leq 1,\\
&\gamma_n : H_{\rm loc}(\mathrm{div}, \mathbb{R}^3 \setminus G) \to H^{-1/2}(\Sigma),\\
&\gamma_t : H_{\rm loc}(\mathrm{curl}\,, \mathbb{R}^3 \setminus G) \to H^{-1/2}(\mathrm{Div}, \Sigma),\\
&\gamma_T : H_{\rm loc}(\mathrm{curl}\,, \mathbb{R}^3 \setminus G) \to H^{-1/2}(\mathrm{Curl}\,, \Sigma)
\end{aligned}$$

are all bounded. When necessary, we will use the subscripts '+' and '-' to distinguish between the trace from the exterior domain $\mathbb{R}^3 \setminus G$ and interior of G, respectively.

1.3 Important theorems

All the theorems we recall here form the theoretical basis of the study of scattering problems for almost every type of obstacle. We start with three results that are essential to prove the uniqueness of the direct scattering problems.

Theorem 1.9. *(Rellich's lemma) Let $B_R = \{x \in \mathbb{R}^3 : |x| < R\}$ be a ball of radius $R > 0$ centered at the origin with unit outward normal $\hat{x}$, and suppose E^s,H^s are radiating solutions of Maxwell's equations in the exterior of B_R, that is,*

$$\begin{aligned}
&\mathrm{curl}\, E^s - i\omega\mu_0 H = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B}_R,\\
&\mathrm{curl}\, H^s + i\omega\varepsilon_0 E^s = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B}_R,\\
&\sqrt{\frac{\mu_0}{\varepsilon_0}} H^s(x) \times x - |x| E^s(x) = \mathcal{O}\left(\frac{1}{|x|}\right), \quad |x| \to \infty.
\end{aligned}$$

If either

$$Re\left(\int_{\partial B'_R} (\hat{x} \times E^s) \cdot \overline{H^s}\, ds\right) \leq 0 \quad \text{for all } R' > R$$

or

$$\int_{\partial B_R} |H^s|\, ds \to 0 \text{ as } R \to \infty$$

then $E^s = H^s = 0$ in $\mathbb{R}^3 \setminus \overline{B}_R$.

For a proof we refer to [2], [6]. The following two theorems correspond to theorem 4.38 and 4.39 in [1], respectively.

Theorem 1.10. *(Interior regularity principle) Let $G \subset \mathbb{R}^3$ be a bounded domain, $f \in L^2(G)$ and U be an open set with $\overline{U} \subseteq G$.*

(a) Let $u \in H^1(D)$ be a solution of the variational equation

$$\int_G \nabla u \cdot \nabla \psi \, dx = \int_G f \psi \, dx \quad \text{for all } \psi \in C_0^\infty(G).$$

Then $u|_U \in H^2(U)$ and $\Delta u = -f$ in U.

(b) Let $u \in L^2(G)$ be a solution of the variational equation

$$\int_G u \Delta \psi \, dx = -\int_G f \psi \, dx \quad \text{for all } \psi \in C_0^\infty(G).$$

Then $u|_U \in H^2(U)$ and $\Delta u = -f$ in U.

Theorem 1.11. *(Unique continuation property) Let $G \subset \mathbb{R}^3$ be a domain; that is, a nonempty, open and connected set, and $u_1, ..., u_m \in H^2(G)$ be real valued such that*

$$|\Delta u_j| \leq c \sum_{l=1}^{m} (|u_l| + |\nabla u_l|) \quad \text{in } G \text{ for } j = 1, ..., m.$$

If u_j vanish in some open set $U \subseteq G$ for all $j = 1, ..., m$, then u_j vanish identically in G for all $j = 1, ..., m$.

Next, we continue with results from functional analysis and start with two important theorems in Fredholm theory: the Fredholm and the analytic Fredholm theorem. These are well-known theorems in functional analysis for compact operators. The first theorem will be useful in the following chapters for demonstrating the existence of unique solutions to the scattering problems under consideration, whilst the second will be useful for showing the discretness of the set of transmission eigenvalues. The following results are from [1].

Theorem 1.12. *(Fredholm) Let $T : X \to Y$ be a linear and bounded operator between the normed spaces X and Y. Let T be of the form $A + K$ such that A is an isomorphism from X onto Y and $K : X \to Y$ is compact. If T is one-to-one, then T is also onto. Moreover, T^{-1} is bounded from Y onto X. In other words, if the homogeneuous equation*

$$Tx = 0$$

admits only the trivial solution $x = 0$, then the inhomogeneous equation

$$Tx = y$$

is uniquely solvable for all $y \in Y$ and the solution x depends continuously on y.

By writing $T = A\,(I + A^{-1}K)$ it is obvious that it is suffucient to consider the case $Y = X$ and $A = I$. We will demonstrate the well-posedness of the direct problems by showing that these problems, and in particular their variational formulations, are of Fredholm type. Hence, the existence of a solution follows if uniqueness holds. For the next theorem, we denote by $\mathcal{L}(X)$ the Banach space of bounded linear operators mapping the Banach space X into itself.

Theorem 1.13. *Let V be a domain in $\mathbb{C}$ and let $K : V \to \mathcal{L}(X)$ be an operator-valued analytic function such that $K(z)$ is compact for each $z \in V$. Then, one of the following holds:*

(i) $(I - K(z))^{-1}$ *does not exist for any $z \in V$;*

(ii) $(I - K(z))^{-1}$ *exists for all $z \in V \setminus S$ where S is a discrete subset of V.*

The above theorem states that, if we can find at least one z for which the analytic Fredholm operator is injective, then it is always injective except for a discrete set of values of z.

The next two results are used for proving the existence of solutions of boundary value problems, in particular those formulated as equivalent variational formulations.

Theorem 1.14. *(Representation theorem of Riesz) Let H be a Hilbert space with inner product $(\cdot,\cdot)_H$ and let $l : H \to \mathbb{C}$ be a linear and bounded functional. Then there exists a unique $z \in H$ with*

$$l(x) = (x, z)_H \quad \textit{for all } x \in H.$$

Furthermore, $\|l\| = \|x\|_H$.

An extension is given by the following theorem.

Theorem 1.15. *(Lax Milgram theorem) Let H be a Hilbert space over $\mathbb{C}$ with inner product $(\cdot,\cdot)_H$, let $l : X \to \mathbb{C}$ be linear and bounded, and let $a : H \times H :\to \mathbb{C}$ be a bounded and* *coercive sesquilinear form, that is, there exists* $c_1, c_2 > 0$ *with*

$$|a(u,v)| \leq c_1 \|u\|_H \|v\|_H \quad \textit{for all } u, v \in H$$
$$\text{Re } a(u,v) \geq c_2 \|u\|_H^2 \quad \textit{for all } u \in H.$$

Then there exists a unique $u \in H$ *with*

$$a(\psi, u) = l(\psi) \quad \textit{for all } \psi \in H.$$

Furthermore, there exists $c > 0$*, independent of* u*, such that* $\|u\|_H > c\|l\|_{H'}$*, where* H' *denotes the dual space of* H*.*

2 Scattering of electromagnetic waves from a perfect conductor in an inhomogeneous medium

In this section, we consider the scattering of electromagnetic time-harmonic fields from a perfect conductor embedded in an inhomogeneous medium. A perfect conductor is an idealized material having zero electrical resistance, thus allowing a steady current to flow within it without losing energy to resistance. While perfect conductors do not exist in nature, the concept provides a useful model for situatuions in which electrical resistance is negligible compared to other effects. One example is electrical circuit diagrams, in which the wires connecting components are implicitly assumed to have no resistance.

Initially considering the direct problem we show that it is well-posed, that is, that it has a unique radiating solution. Studying the inverse problem, we then show that, if the outside inhomogeneity is known, the scatterer is uniquely determined by the fixed energy far field data.

2.1 Direct problem

To demonstrate the existence of a unique solution to the direct problem, we apply the integral equation method. Generally speaking, the idea behind this method is to obtain an operator equation that is equivalent to the scattering problem. To establish the direct problem's well-posedness, we show that the operator equation is of Fredholm type. In the case of a homogeneous medium, the operator equation is a boundary integral equation, also referred to as an ansatz of a solution. In the case of a two-dimensional configuration, Green's represention theorem typically motivates this ansatz, whereas the Stratton-Chu formula typically provides this motivation in the three-dimensional configuration, [6], [1]. Since we will work with an inhomogeneous medium, we will derive an operator equation, known as a Lippmann-Schwinger equation in cases involving an inhomogeneous medium, by using a general representation formula.

For the analysis of the operator equation, coercivity and compactness are two important properties. For equations of the second kind (i.e., a inhomogeneous Fredholm equation of the seond kind), such as the Lippmann-Schwinger equation, exploiting the compactness of the integral operators by using Riesz theory is of course the more desirable strategy whenever it is possible. Unfortunately, in electromagnetic medium scattering, where both the electric permittivity ε and the magnetic permeability μ are space dependent, the operator in the corresponding Lippmann-Schwinger equation fails to be compact. To overcome this problem, we first consider the operator, which lacks compactness, in the case of a complex wavenumber. Specifically, we show that the sum of the identity operator and the operator appearing in the Lippmann-Schwinger equation is coercive for complex wavenumber. Afterwards, we use a compactness argument to show that the operator equation is of Fredholm type for all wavenumbers k as long as k^2 is not a so-called Dirichlet eigenvalue. This approach has been applied in cases involving a penetrable obstacle without any boundary condition in [4] and also in [5], where the lack of compactness was replaced by positivity, which is implicitly introduced via positivity constraints on the coefficients of the partial differential equations.

For a two-dimensional configuration, the corresponding electromagnetic model reduces to a scalar problem involving the operators $-\mathrm{div}\,(\tau\nabla\cdot)$ with a Dirichlet or Neumann boundary condition, and τ being equal to ε^{-1} or μ^{-1}; see [8]. Although the integral equation method and the scattering of Maxwell's equations from a perfect conductor are certainly known to the experts, we were not able to find an application of the above-named approach to the scattering problem in the (mathematical) literature. However, [1] employed the integral equation method to show the existence of a unique weak solution in the case of a homogeneous medium, and [2] applied the variational mehtod in case of an inhomogeneous medium and continuously differentiable electric permittivity and constant magnetic permeability. Thus, we were able to weaken the assumptions on the data.

2.1.1 Problem statement and uniqueness

An incident electromagnetic field E^i, H^i, which satisfies the reduced Maxwell system

$$\mathrm{curl}\, E^i - i\omega\mu_0 H^i = 0, \quad \mathrm{curl}\, H^i + i\omega\varepsilon_0 E^i = 0 \quad \text{in } \mathbb{R}^3 \tag{2.1}$$

is scattered by a perfect conductor occupying a bounded domain $D \subset \mathbb{R}^3$ with boundary ∂D and connected exterior $\mathbb{R}^3 \setminus \overline{D}$. We assume that D is surrounded by a medium with space-dependent electric permittivity $\varepsilon(x)$, magnetic permeability $\mu(x)$ and conductivity $\sigma(x)$. We assume the scatterer to be bounded, that is, we assume that there exists a ball $B_{R_0} = \{x \in \mathbb{R}^3 : |x| \leq R_0\}$ of radius $R_0 > 0$ centered at the origin, such that $\overline{D} \subset B_{R_0}$, $\varepsilon(x) \equiv \varepsilon_0$, $\mu(x) \equiv \mu_0$ and $\sigma(x) \equiv 0$ for $|x| > R_0$. The total fields are superpositions of the incident and scattered fields, i.e. $E = E^i + E^s$ and $H = H^i + H^s$ and satisfy Maxwell system

$$\operatorname{curl} E - i\omega\mu H = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \tag{2.2}$$

$$\operatorname{curl} H + i\omega\varepsilon E = \sigma E \quad \text{in } \mathbb{R}^3 \setminus \overline{D}. \tag{2.3}$$

On the boundary ∂D of D, we impose the perfect conducting boundary condition,

$$E \times \nu = 0 \quad \text{on } \partial D, \tag{2.4}$$

where ν is the unit outward normal to ∂D. To ensure that the scattered field is outgoing, it has to satisfy the Silver-Müller radiation condition

$$\sqrt{\frac{\mu_0}{\varepsilon_0}} H^s(x) \times x - |x| E^s(x) = \mathcal{O}\left(\frac{1}{|x|}\right) \quad \text{as } |x| \to \infty \tag{2.5}$$

uniformly with respect to all directions $x/|x|$.

We will work with magnetic field H only. This is motivated by the fact that for the important case of non-magnetic media (i.e. $\mu = \mu_0$), the magnetic field is divergence free as seen from (2.2) and the fact that $\operatorname{div}\operatorname{curl} = 0$. Therefore, eliminating the electric field E from (2.3) and substituting into (2.2) leads to

$$\operatorname{curl}\left(\frac{1}{\sigma - i\omega\varepsilon}\operatorname{curl} H\right) - i\omega\mu H = 0,$$

that is,

$$\operatorname{curl}\left(\frac{1}{\varepsilon_r}\operatorname{curl} H\right) - k^2\mu_r H = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \tag{2.6}$$

where $k = \omega\sqrt{\varepsilon_0\mu_0} > 0$ is the wave number, ε_r denotes the (complex valued) relative permittivity and μ_r is the relative permeability given by

$$\varepsilon_r(x) = \frac{\varepsilon(x)}{\varepsilon_0} + i\frac{\sigma(x)}{\omega\varepsilon_0}, \quad \mu_r(x) = \frac{\mu(x)}{\mu_0}$$

respectively. We note that $\varepsilon_r \equiv 1$ and $\mu_r \equiv 1$ outside the ball B_{R_0}. On ∂D, we have

$$\frac{1}{\varepsilon_r}\operatorname{curl} H \times \nu = 0 \tag{2.7}$$

and the Silver-Müller radiation condition reads

$$\operatorname{curl} H^s(x) \times \frac{x}{|x|} - ikH^s(x) = \mathcal{O}\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty. \tag{2.8}$$

For $R > 0$ we set

$$\Omega_R := \{x \in \mathbb{R}^3 \setminus \overline{D} : |x| \leq R\}.$$

To simplify notation, we omit the subscript R for the special case $R = R_0$ and write Ω instead of Ω_{R_0}. To avoid making smooth assumptions on the data ε_r, μ_r, we need to derive a variational formulation of problem (2.6)-(2.8). To do so, we multiply equation (2.6) by a sufficiently smooth test function ψ, integrate over Ω_R, $R \geq R_0$, and formally use integration by parts:

$$\begin{aligned} 0 &= \iint_{\Omega_R} \left(\operatorname{curl}\left[\frac{1}{\varepsilon_r}\operatorname{curl} H\right] \cdot \psi - k^2\mu_r H \cdot \psi\right) dx \\ &= \iint_{\Omega_R} \left(\frac{1}{\varepsilon_r}\operatorname{curl} H \cdot \operatorname{curl}\psi - k^2\mu_r H \cdot \psi\right) dx + \int_{\partial\Omega_R} \left(\nu \times \frac{1}{\varepsilon_r}\operatorname{curl} H\right) \cdot \psi \, ds \\ &= \iint_{\Omega_R} \left(\frac{1}{\varepsilon_r}\operatorname{curl} H \cdot \operatorname{curl}\psi - k^2\mu_r H \cdot \psi\right) dx - \int_{\partial B_R} (\nu \times \operatorname{curl} H) \cdot \psi \, ds. \end{aligned}$$

where we used the fact that $\partial\Omega_R = \partial D \cup \partial B_R$, boundary condition (2.7) and the fact that $\varepsilon_r = 1$ on ∂B_{R_0}. Assuming $\psi(x) = 0$ for $|x| \geq R_0$ and letting $R \to \infty$ yields

$$\iint_{\mathbb{R}^3 \setminus \overline{D}} \left(\frac{1}{\varepsilon_r}\operatorname{curl} H \cdot \operatorname{curl}\psi - k^2\mu_r H \cdot \psi\right) dx = 0. \tag{2.9}$$

Introducing

$$p = \mu_r - 1 \quad \text{and} \quad q = 1 - \frac{1}{\varepsilon_r}$$

and noting that $p \equiv q \equiv 0$ outside B_{R_0}, we can write equation (2.9) as

$$\iint_{\mathbb{R}^3 \setminus \overline{D}} \left(\operatorname{curl} H \cdot \operatorname{curl}\psi - k^2 H \cdot \psi\right) dx = k^2 \iint_{\Omega} pH \cdot \psi \, dx + \iint_{\Omega} q \operatorname{curl} H \cdot \operatorname{curl}\psi \, dx. \tag{2.10}$$

The above variational formulation (2.9) holds for $\mu_r \in L^\infty(\mathbb{R}^3 \setminus \overline{D})$ and $\varepsilon_r \in L^\infty(\mathbb{R}^3 \setminus \overline{D})$ such that $\frac{1}{\varepsilon_r} \in L^\infty(\mathbb{R}^3 \setminus \overline{D})$. Thus we can state our problem.

Problem statement (P1):

Determine $H \in H_{\rm loc}(\mathrm{curl}\,, \mathbb{R}^3 \setminus \overline{D})$ such that the variational equation (2.9) holds for all $\psi \in H_{\rm loc}(\mathrm{curl}\,, \mathbb{R}^3 \setminus \overline{D})$ with $\psi(x) = 0$ for $|x| \geq R_0$, and $H^s = H - H^i$ satisfies the Silver-Müller radiation condition (2.8).

We note the following regularity result for the solution H outside B_{R_0}. For a proof, we refer the reader to e.g., [26].

Remark 2.1. *Outside B_{R_0}, the solution H is smooth and satisfies*

$$\mathrm{curl}\,\mathrm{curl}\, H - k^2 H = 0.$$

Taking the divergence of the above equation and using the identities $\mathrm{div}\,\mathrm{curl}\; = 0$ *and* $\mathrm{curl}^{\,2} = -\Delta + \nabla\mathrm{div}$*, the above system is equivalent to the pair of equations*

$$\Delta H + k^2 H = 0 \quad \textit{and} \quad \mathrm{div} H = 0.$$

Using interior regularity results, we can show H to be analytic outside B_{R_0}.

The next lemma shows equivalence between the scattering problem (2.2)-(2.4) and the variational problem $(P1)$ for the magnetic field H.

Lemma 2.2. *Let $H \in H_{loc}(\mathrm{curl}\,, \mathbb{R}^3 \setminus \overline{D})$ satisfy the variational problem $(P1)$. Set*

$$E = \frac{1}{\sigma - iw\varepsilon} \mathrm{curl}\, H \quad \textit{in } \mathbb{R}^3 \setminus \overline{D}.$$

Then, $H \in H_{loc}(\mathrm{curl}\,, \mathbb{R}^3 \setminus \overline{D})$ and $E \in H_{loc}(\mathrm{curl}\,, \mathbb{R}^3 \setminus \overline{D})$ satisfy Maxwell's system (2.2)-(2.2), E satisfies the perfectly conducting boundary condition (2.4) on ∂D and $E^s = E - E^i$, $H^s = H - H^i$ satisfy radiation condition (2.5). Furthermore, $E \in H_{loc}(\mathrm{curl}\,, \mathbb{R}^3 \setminus \overline{D})$ satisfies

$$\iint_{\mathbb{R}^3 \setminus \overline{D}} \left(\frac{1}{\mu_r} \mathrm{curl}\, E \cdot \mathrm{curl}\, \psi - k^2 \varepsilon_r E \cdot \psi \right) dx = 0$$

for all $\psi \in H(\mathrm{curl}\,, \mathbb{R}^3 \setminus \overline{D})$ with compact support, $E \times \nu = 0$ on ∂D and $E^s = E - E^i$ satisfies the Silver-Müller radiation condition (2.8).

The same statement holds for E and H interchanged.

Proof. Let $H \in H_{\rm loc}(\mathrm{curl}\,, \mathbb{R}^3 \setminus \overline{D})$ satisfy the variational problem $(P1)$. We note that $E = \frac{1}{\sigma - iw\varepsilon} \mathrm{curl}\, H \in L^2_{\rm loc}(\mathbb{R}^3 \setminus \overline{D})^3$. Substituting the definition of E into the variational equation (2.9) yields

$$\iint_{\mathbb{R}^3\setminus\overline{D}} \left(\underbrace{\frac{\sigma - i\omega\varepsilon}{\varepsilon_r}}_{=-i\omega\varepsilon_0} E \cdot \operatorname{curl} \psi - k^2 \mu_r H \cdot \psi \right) dx = 0 \tag{2.11}$$

which is the variational form of

$$-i\omega\varepsilon_0 \operatorname{curl} E - k^2 \mu_r H = 0.$$

Division by $-i\omega\varepsilon_0$ and substituting for $k^2 = \omega^2\varepsilon_0\mu_0$ and $\mu_r = \frac{\mu}{\mu_0}$ yields (2.2). Eliminating H from (2.2) and substituting into variational equation (2.11) yields

$$\begin{aligned}
0 &= \iint_{\mathbb{R}^3\setminus\overline{D}} \left(-i\omega\varepsilon_0 E \cdot \operatorname{curl} \psi - k^2 \mu_r \frac{1}{i\omega\mu} \operatorname{curl} E \cdot \psi \right) dx \\
&= \iint_{\mathbb{R}^3\setminus\overline{D}} \left(-i\omega\varepsilon_0 E \cdot \operatorname{curl} \psi + i\omega\varepsilon_0 \operatorname{curl} E \cdot \psi \right) dx \\
&= -(i\omega\varepsilon_0) \iint_{\mathbb{R}^3\setminus\overline{D}} \left(E \cdot \operatorname{curl} \psi - \operatorname{curl} E \cdot \psi \right) dx.
\end{aligned}$$

Dividing by $-(i\omega\varepsilon_0)$ and using Green's theorem (1.11) gives

$$0 = \langle \gamma_t E, \gamma_T \psi \rangle_{\partial D}$$

for all $\psi \in H(\operatorname{curl}, \mathbb{R}^3 \setminus \overline{D})$ with $\psi(x) = 0$ for $|x| \geq R_0$, where γ_t and γ_T are the traces given by (1.8) and (1.9), respectively. Thus we conclude $\gamma_t E = E \times \nu = 0$ on ∂D. We can derive the variational formulation for the electric field E by taking the dot product of (2.3) with a test function $\psi \in H(\operatorname{curl}, \mathbb{R}^3 \setminus \overline{D})$ of compact support, integrating over $\mathbb{R}^3 \setminus \overline{D}$, inserting for H from (2.2) and using Green's theorem (1.11). □

With the above lemma, we note that if H is a solution to $(P1)$, then $\operatorname{curl} H \in H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3 \setminus \overline{D})$ and choosing sufficiently smooth test functions allows it to be easily verified that H solves the differential equation (2.6) outside D and the boundary condition (2.7) on ∂D.

Next we state the assumptions on the data and prove the uniqueness of problem $(P1)$.

Assumption 2.3. *We assume that:*

- $D \subset \mathbb{R}^3$ *is a bounded Lipschitz domain with connected exterior* $\mathbb{R}^3 \setminus \overline{D}$.

- $\mu_r \in W^{2,\infty}(\mathbb{R}^3 \setminus \overline{D})$ *real-valued and* $\mu_r \equiv 1$ *outside* B_{R_0}.

- $\varepsilon_r \in W^{1,\infty}(\mathbb{R}^3 \setminus \overline{D})$, $\operatorname{Im} \varepsilon_r \geq 0$ *and* $\varepsilon_r \equiv 1$ *outside* B_{R_0}. *Consequently* $\operatorname{Im} \frac{1}{\varepsilon_r} = \operatorname{Im} \frac{\overline{\varepsilon}_r}{|\varepsilon_r|^2} \leq 0$.

- *There exists a constant* $c_0 > 0$ *with* $\operatorname{Re} \varepsilon_r \geq c_0$ *and* $\mu_r \geq c_0$ *on* Ω. *Then, in particular* $\frac{1}{\varepsilon_r} \in L^\infty(\mathbb{R}^3 \setminus \overline{D})$ *and* $\frac{1}{\mu_r} \in L^\infty(\mathbb{R}^3 \setminus \overline{D})$.

Theorem 2.4. *Under assumption 2.3, there exists at most one solution to problem (P1).*

Proof. Let H be a solution corresponding to $H^i = 0$, i.e. H itself satisfies radiation condition (2.8). Further, let $\phi \in C_0^\infty(\mathbb{R}^3)$ be real-valued with $\phi(x) = 1$ for $x \in \Omega_R$ and $\phi(x) = 0$ for $|x| > R+1$ where $R \geq R_0$. Substituting $\psi = \phi \overline{H}$ into variational equation (2.9) yields

$$\iint_{\Omega_R} \left(\frac{1}{\varepsilon_r} |\operatorname{curl} H|^2 - k^2 \mu_r |H|^2 \right) dx + \iint_{R<|x|<R+1} \left(\operatorname{curl} H \cdot \operatorname{curl} (\phi \overline{H}) - k^2 \phi |H|^2 \right) dx = 0. \tag{2.12}$$

By Remark (2.1), H is a smooth solution to $\operatorname{curl}^2 H - k^2 H = 0$ in the region $\{x \in \mathbb{R}^3 : R < |x| < R+1\}$, and thus we compute for the second term on left hand side of (2.12):

$$\begin{aligned} &\iint_{R<|x|<R+1} \left(\operatorname{curl} H \cdot \operatorname{curl} (\phi \overline{H}) - k^2 \phi |H|^2 \right) dx \\ &= \iint_{R<|x|<R+1} \phi \overline{H} \cdot \underbrace{\left(\operatorname{curl}^2 H - k^2 H \right)}_{=0} dx + \int_{|x|=R} (\nu \times \operatorname{curl} H) \cdot \overline{H} \, ds \\ &= - \int_{|x|=R} (\operatorname{curl} H \times \nu) \cdot \overline{H} \, ds, \end{aligned} \tag{2.13}$$

where we used that $\phi(x) = 0$ for $|x| = R+1$ and $\phi(x) = 1$ for $|x| = R$. Substituting (2.13) into (2.12) and taking the imaginary part yields

$$\operatorname{Im} \left(\int_{|x|=R} (\operatorname{curl} H \times \nu) \cdot \overline{H} \, ds \right) = \iint_{\Omega_R} \left[\operatorname{Im} \left(\frac{1}{\varepsilon_r} \right) |\operatorname{curl} H|^2 - k^2 \mu_r |H|^2 \right] dx \leq 0. \tag{2.14}$$

From the radiation condition (2.8) we obtain

$$
\begin{aligned}
0 \overset{R\to\infty}{\longleftarrow} & \int_{|x|=R} |\mathrm{curl}\, H \times \nu - ikH|^2 \; ds \\
&= \int_{|x|=R} \left(|\mathrm{curl}\, H \times \nu|^2 + k^2 |H|^2\right) \; ds - 2k\mathrm{Im} \int_{|x|=R} (\mathrm{curl}\, H \times \nu) \cdot \overline{H} \, ds \\
&\overset{(2.14)}{\geq} \int_{|x|=R} \left(|\mathrm{curl}\, H \times \nu|^2 + k^2 |H|^2\right) \; ds
\end{aligned}
$$

and thus conclude $\lim_{R\to\infty} \int_{|x|=R} |H|^2 \, ds = 0$. Rellich's lemma 1.9 now implies that $H = 0$ in $\mathbb{R}^3 \setminus B_R$.

Next we want to apply the unique continuation principle 1.11 to conclude that H vanishes in the exterior of D. We start by substituting $\psi = \nabla \xi$, for some $\xi \in H_0^1(\mathbb{R}^3 \setminus \overline{D})$, into the variational equation (2.9). Using Green's theorem (1.7), and the fact that $\xi = 0$ on ∂D (in the trace sense) and $\xi(x) = 0$ for some $R > 0$ with $|x| > R$, yields

$$
0 = -k^2 \iint_{\mathbb{R}^3 \setminus \overline{D}} \mu_r H \cdot \nabla \xi \, dx = k^2 \iint_{\mathbb{R}^3 \setminus \overline{D}} \mathrm{div}(\mu_r H) \xi \, dx,
$$

that is

$$
\iint_{\mathbb{R}^3 \setminus \overline{D}} \mathrm{div}(\mu_r H) \xi \, dx = 0. \tag{2.15}
$$

The above result holds for all $\xi \in H_0^1(\mathbb{R}^3 \setminus \overline{D})$, and thus (2.15) is the variational form of $\mathrm{div}(\mu_r H) = 0$ in $\mathbb{R}^3 \setminus \overline{D}$. (We note that taking the divergence of (2.6) yields $\mathrm{div}(\mu_r H) = 0$ and thus $\mu_r H \in H_{\mathrm{loc}}(\mathrm{div}, \mathbb{R}^3 \setminus \overline{D})$, thereby justifying the use of Green's theorem (1.7)). If ε_r, μ_r and H were sufficiently smooth, we could rewrite equation (2.6) as

$$
0 = \varepsilon_r \mathrm{curl} \left(\frac{1}{\varepsilon_r} \mathrm{curl}\, H \right) - k^2 \mu_r \varepsilon_r H = \mathrm{curl}^{\,2} H + \varepsilon_r \nabla \left(\frac{1}{\varepsilon_r} \right) \times \mathrm{curl}\, H - k^2 \mu_r \varepsilon_r H.
$$

From $\mathrm{div}(\mu_r H) = 0$, we obtain $\mathrm{div} H = -\frac{\nabla \mu_r}{\mu_r} \cdot H$ and thus

$$
\Delta H = -\nabla \left(\frac{\nabla \mu_r}{\mu_r} \cdot H \right) + \varepsilon_r \nabla \left(\frac{1}{\varepsilon_r} \right) \times \mathrm{curl}\, H - k^2 \mu_r \varepsilon_r H,
$$

where we have used the identity $\mathrm{curl}^{\,2} = -\Delta + \nabla \mathrm{div}$. Next we will derive this formula by the variational equation. We set $\psi = \varepsilon_r \overline{\varphi}$ for some $\varphi \in C_0^\infty(\Omega_R)^3$, $R \geq R_0$, and extend ψ

by zero into $\mathbb{R}^3$. Then, $\psi \in H_0(\mathrm{curl}\,,\mathbb{R}^3)$ with compact support and, because

$$\mathrm{curl}\,\psi = \varepsilon_r \mathrm{curl}\,\overline{\varphi} + \nabla\varepsilon_r \times \overline{\varphi}$$

we obtain, by substituting for ψ into (2.9),

$$\iint_{\Omega_R} \left(\mathrm{curl}\, H \cdot \mathrm{curl}\,\overline{\varphi} + \frac{1}{\varepsilon_r} \mathrm{curl}\, H \cdot (\nabla\varepsilon_r \times \overline{\varphi}) - k^2 \mu_r \varepsilon_r H \cdot \overline{\varphi} \right) dx = 0$$

for all $\varphi \in C_0^\infty(\Omega_R)^3$. The last equation we write as

$$\iint_{\Omega_R} \mathrm{curl}\, H \cdot \mathrm{curl}\,\overline{\varphi}\, dx = \iint_{\Omega_R} G \cdot \overline{\varphi}\, dx$$

where $G = -\frac{1}{\varepsilon_r}\mathrm{curl}\, H \times \nabla\varepsilon_r + k^2\mu_r\varepsilon_r H \in L^2(\Omega_R, \mathbb{C}^3)$. Further, we compute

$$\begin{aligned}
\iint_{\Omega_R} G \cdot \overline{\varphi}\, dx &= \iint_{\Omega_R} \mathrm{curl}\, H \cdot \mathrm{curl}\,\overline{\varphi}\, dx \\
&\overset{(1.11)}{=} \iint_{\Omega_R} H \cdot \mathrm{curl}\,^2 \overline{\varphi}\, dx \\
&= -\iint_{\Omega_R} H \cdot \Delta\overline{\varphi}\, dx + \iint_{\Omega_R} H \cdot \nabla \mathrm{div}\overline{\varphi}\, dx \\
&\overset{(1.7)}{=} -\iint_{\Omega_R} H \cdot \Delta\overline{\varphi}\, dx - \iint_{\Omega_R} \mathrm{div} H \ \mathrm{div}\overline{\varphi}\, dx \\
&\overset{(1.7)}{=} -\iint_{\Omega_R} H \cdot \Delta\overline{\varphi}\, dx + \iint_{\Omega_R} \nabla \mathrm{div} H \cdot \overline{\varphi}\, dx \\
&= -\iint_{\Omega_R} H \cdot \Delta\overline{\varphi}\, dx - \iint_{\Omega_R} \nabla \left(\frac{\nabla\mu_r}{\mu_r} \cdot H \right) \cdot \overline{\varphi}\, dx,
\end{aligned}$$

that is

$$\iint_{\Omega_R} H \cdot \Delta\overline{\varphi}\, dx = -\iint_{\Omega_R} \underbrace{\left(G + \nabla\left(\frac{\nabla\mu_r}{\mu_r} \cdot H \right) \right)}_{\in L^2(\Omega_R)^3} \cdot \overline{\varphi}\, dx$$

where we used that $\varphi = 0$ on $\partial\Omega_R$, $\mathrm{div}\varphi \in C_0^\infty(\Omega_R)$ and $\mathrm{curl}\,\varphi \in C_0^\infty(\Omega_R)^3$. The above equation holds for all $\varphi \in C_0^\infty(\Omega_R)^3$. By the interior regularity property 1.10, we conclude that $H \in H^2(U)^3$ where U is an open set with $\overline{U} \subseteq \Omega_R$ and

$$\Delta H = -G - \nabla\left(\frac{\nabla\mu_r}{\mu_r} \cdot H \right) = -\frac{1}{\varepsilon_r}\mathrm{curl}\, H \times \nabla\varepsilon_r + k^2\mu_r\varepsilon_r H - \nabla\left(\frac{\nabla\mu_r}{\mu_r} \cdot H \right)$$

in U. Since every component of curl$H$ is a combination of partial derivatives of H_j, $j = 1, 2, 3$, we conclude that there exists a constant $c > 0$ such that

$$|\Delta H_j| \leq c \sum_{l=1}^{3} (|\nabla H_l| + |H_l|) \quad \text{in } U$$

for $j = 1, 2, 3$. Therefore $H = 0$ in all of U by the unique continuation principle 1.11. Because U is an arbitrary domain with $\overline{U} \subseteq \Omega_R$, we conclude that $H = 0$ in Ω_R and thus $H = 0$ outside D. □

2.1.2 Potentials

The key ingredient needed to apply the integral equation method are potentials. In the case of the scalar Helmholtz equation in a homogeneous medium, we can present the solution via a combination of single- and double-layer potentials, following from Green's representation theorem; see [6], [1]. For Maxwell's system, we obtain with the help of the Stratton-Chu formular a representation for a solution via the curl and curl ^{2} of the vector-valued single-layer potential; see [1] in case of Lipschitz domains working in the Sobolev space $H(\text{curl}\,,\cdot)$. In case of an inhomogeneous medium, the volume potential will be needed. Both the single-layer and volume potentials are integral operators with a weakly singular kernel.

The derivation of a Lippmann-Schwinger equation is based on a representation formula for the magnetic field inside and outside D. The equation will not be an integral equation, since it will contain the derivatives of both the volume and single-layer potential. In [4], the term *integro-differential equation* was used, which we find suitable.

In this section we define the necessary potentials and state some of their properties in case of a Lipschitz domain.

Lemma 2.5. *Let Φ_k be the fundamental solution of the scalar Helmholtz equation $\Delta\Phi_k + k^2\Phi_k = 0$ in $\mathbb{R}^3$, defined by*

$$\Phi_k(x, y) = \frac{e^{ik|x-y|)}}{4\pi|x - y|}, \quad x \neq y.$$

(a) *For $g \in L^2(\Omega)^3$ the volume potential*

$$v(x) = \iint_\Omega g(y)\Phi_k(x,y)\,dy, \quad x \in \mathbb{R}^3 \tag{2.16}$$

defines a function in $H^2_{\mathrm{loc}}(\mathbb{R}^3)^3$ which satisfies $\Delta v + k^2 v = -\widetilde{g}$ in $\mathbb{R}^3$, where $\widetilde{g}$ is the extension of g by zero into $\mathbb{R}^3$. Furthermore, v satisfies the Silver-Müller radiation condition (2.8) and the restriction $v|_\Omega$ of v to Ω defines a bounded operator from $L^2(\Omega)^3$ into $H^2(\Omega)^3$.

(b) *For $g \in L^2(\Omega)^3$ the vector fields*

$$u(x) = \operatorname{curl} \iint_\Omega g(y)\Phi_k(x,y)\,dy, \quad x \in \mathbb{R}^3, \tag{2.17}$$

$$w(x) = \left(k^2 + \nabla \operatorname{div}\right) \iint_\Omega g(y)\Phi_k(x,y)\,dy, \quad x \in \mathbb{R}^3 \tag{2.18}$$

define functions in $H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3)$ which satisfy $\operatorname{curl}^2 u - k^2 u = \operatorname{curl} g$ and $\operatorname{curl}^2 w - k^2 w = k^2 g$ in the variational sense, respectively, i.e.

$$\iint_{\mathbb{R}^3} \left(\operatorname{curl} u \cdot \operatorname{curl} \psi - k^2 u \cdot \psi\right)\,dx = \iint_\Omega g \cdot \operatorname{curl} \psi\,dx, \tag{2.19}$$

$$\iint_{\mathbb{R}^3} \left(\operatorname{curl} w \cdot \operatorname{curl} \psi - k^2 w \cdot \psi\right)\,dx = k^2 \iint_\Omega g \cdot \psi\,dx \tag{2.20}$$

for all $\psi \in H(\operatorname{curl}, \mathbb{R}^3)$ with compact support. Furthermore, u and w satisfy the Silver-Müller radiation condition (2.8) and the restriction $u|_\Omega$ of u to Ω and the restriction $w|_\Omega$ of w to Ω define bounded operators from $L^2(\Omega)^3$ into $H(\operatorname{curl}, \Omega)$.

Proof. This is lemma 2.2 from [4] in case of C^2-smooth domain Ω. Since Green's theorem is also valid for Lipschitz domains, see [1], the above statements remains true for Lipschitz domains. □

The single layer potential $\widetilde{\mathcal{S}}$ is defined by

$$\left(\widetilde{\mathcal{S}}a\right)(x) := \int_{\partial D} a(y)\Phi_k(x,y)\,ds(y), \quad x \in D \cup \left(\mathbb{R}^3 \setminus \overline{D}\right). \tag{2.21}$$

For $s \in \left[-\frac{1}{2}, \frac{1}{2}\right]$,

$$\widetilde{\mathcal{S}} : H^{-\frac{1}{2}+s}(\partial D) \to H^{1+s}(D) \cup H^{1+s}_{\text{loc}}(\mathbb{R}^3 \setminus \overline{D})$$

is linear and bounded, see [9], [10].

We define a potential $\widetilde{\mathcal{N}}$ generated by an electric current $a \in H^{-\frac{1}{2}}(\text{Div}\,, \partial D)$ by

$$\widetilde{\mathcal{N}} a = \text{curl}\,^2 \int_{\partial D} a(y) \Phi_k(\cdot, y)\, ds(y) \quad x \in D \cup \left(\mathbb{R}^3 \setminus \overline{D}\right). \tag{2.22}$$

This can also be written as $\widetilde{\mathcal{N}} a = \nabla \widetilde{\mathcal{S}} \text{Div}\, a + k^2 \widetilde{\mathcal{S}} a$ because of the Helmholtz equation and the identity $-\Delta = \text{curl}\,\text{curl}\, - \nabla \text{div}\,$. We define a "magnetic analogue" generated by $a \in H^{-\frac{1}{2}}(\text{Div}\,, \partial D)$ as

$$\widetilde{\mathcal{M}} a = \text{curl}\, \int_{\partial D} a(y) \Phi_k(\cdot, y)\, ds(y) \quad x \in D \cup \left(\mathbb{R}^3 \setminus \overline{D}\right). \tag{2.23}$$

The following lemma, stating properties of the operators $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{M}}$, corresponds to lemma 5.52 in [1].

Lemma 2.6. *Let Q be a bounded domain with $\partial D \subseteq Q$.*

(a) The operators $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{M}}$, defined by (2.22) and (2.23), respectively, are well-defined and bounded from $H^{-\frac{1}{2}}(\text{Div}, \partial D)$ into $H(\text{curl}\,, D)$ and into $H(\text{curl}\,, Q \setminus \overline{D})$.

(b) For $a \in H^{-\frac{1}{2}}(\text{Div}, \partial D)$, the fields $u = \widetilde{\mathcal{M}} a$ and $\text{curl}\; u = \widetilde{\mathcal{N}} a$ satisfy $u|_D, \text{curl}\, u|_D \in H(\text{curl}\,, D)$ and $u|_{Q \setminus \overline{D}}, \text{curl}\, u|_{Q \setminus \overline{D}} \in H(\text{curl}\,, Q \setminus \overline{D})$ and $\gamma_t u_- - \gamma_t u_+ = a$ and $\gamma_t \text{curl}\, u_- - \gamma_t \text{curl}\, u_+ = 0$. In particular, $u \in C^\infty(\mathbb{R}^3 \setminus \partial D)^3$ and u satisfies the equation $\text{curl}\,^2 u - k^2 u = 0$ in $\mathbb{R}^3 \setminus \partial D$. Furthermore, u and $\text{curl}\, u$ satisfy the Silver-Müller radiation condition (2.8); that is

$$\text{curl}\, u \times \hat{x} - iku = \mathcal{O}\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty,$$

uniformly with respect to $\hat{x}$.

(c) The traces

$$\mathcal{N} = \gamma_t \widetilde{\mathcal{N}} \quad \text{on } \partial D,$$
$$\mathcal{M} = \frac{1}{2}\left(\gamma_t \widetilde{\mathcal{M}}_- + \gamma_t \widetilde{\mathcal{M}}_+\right) \quad \text{on } \partial D,$$

are bounded from $H^{-\frac{1}{2}}(\mathrm{Div}, \partial D)$ into itself. With these notations the following jump condition hold for $u = \widetilde{\mathcal{M}}a$ and $a \in H^{-\frac{1}{2}}(\mathrm{Div}, \partial D)$

$$\gamma_t u_\pm = \mp \frac{1}{2} a + \mathcal{M}a, \quad \gamma_t \mathrm{curl}\, u_\pm = \mathcal{N}a. \tag{2.24}$$

(d) $\mathcal{N}$ is the sum $\mathcal{N} = \hat{\mathcal{N}} + \hat{\mathcal{K}}$ of an isomorphism $\hat{\mathcal{N}}$ from $H^{-\frac{1}{2}}(\mathrm{Div}, \partial D)$ onto itself and a compact operator $\hat{\mathcal{K}}$.

2.1.3 Existence

To apply the integral equation method, we need to derive a Lippmann-Schwinger equation, that is an operator equation that is equivalent to the scattering problem $(P1)$. The idea is first to argue classically, that is, to use a representation formula that holds for general vector fields $H, E \in C^1(\Omega)^3 \cap C(\overline{\Omega})^3$ such that $\mathrm{curl}\, H \in C^1(\overline{\Omega})^3$ and $\mathrm{div}\, E \in C^1(\overline{\Omega})$. Using Maxwell's system, we will be able to derive a Lippmann-Schwinger operator equation. We start by stating the representation formula (the result for the electric field can be find in [1], theorem 3.25; see also [6]).

Lemma 2.7. *Let $H, E \in C^1(\Omega)^3 \cap C(\overline{\Omega})^3$ such that $\mathrm{curl}\, H \in C^1(\overline{\Omega})^3$ and $\mathrm{div}\, E \in C^1(\overline{\Omega})$. Then we have for $x \in \Omega$:*

$$\begin{aligned} H(x) =& \mathrm{curl} \iint_\Omega (\mathrm{curl}\, H(y) + i\omega\varepsilon_0 E(y))\, \Phi_k(x,y)\, dy - \nabla \iint_\Omega \mathrm{div}\, H(y) \Phi_k(x,y)\, dy \\ &- i\omega\varepsilon_0 \iint_\Omega (\mathrm{curl}\, E(y) - i\omega\mu_0 H(y))\, \Phi_k(x,y)\, dy \\ &- \mathrm{curl} \int_{\partial\Omega} (\nu(y) \times H(y))\, \Phi_k(x,y)\, ds(y) + \nabla \int_{\partial\Omega} (\nu(y) \cdot H(y))\, \Phi_k(x,y)\, ds(y) \\ &+ i\omega\varepsilon_0 \int_{\partial\Omega} (\nu(y) \times E(y))\, \Phi_k(x,y)\; ds(y). \end{aligned} \tag{2.25}$$

Furthermore, the right-hand side of this equation vanishes for $x \notin \overline{\Omega}$.

Now, let $E, H \in C^1(\mathbb{R}^3 \setminus \overline{D})^3 \cap C(\mathbb{R}^3 \setminus D)^3$ satisfy Maxwell's system (2.2)-(2.3), $E \times \nu = 0$ on ∂D and the scattered field E^s, H^s satisfies the Silver-Müller radiation condition (2.5). We rewrite (2.2)-(2.3) as

$$\operatorname{curl} E - i\omega\mu_0 H = i\omega\mu_0 pH, \qquad \operatorname{curl} H + i\omega\varepsilon_0 E = q\operatorname{curl} H \qquad \text{in } \mathbb{R}^3 \setminus \overline{D} \tag{2.26}$$

where we recall $q = 1 - \frac{1}{\varepsilon_r}$, $p = \mu_r - 1$ and $q \equiv p \equiv 0$ outside B_{R_0}. Substituting (2.26) into the representation formula (2.25) yields for all $x \in \Omega$:

$$\begin{aligned} H(x) =& \operatorname{curl} \iint_\Omega q(y)\operatorname{curl} H(y)\Phi_k(x,y)\,dy - \nabla \iint_\Omega \operatorname{div} H(y)\Phi_k(x,y)\,dy \\ &+ k^2 \iint_\Omega p(y)H(y)\Phi_k(x,y)\,dy \\ &- \operatorname{curl} \int_{\partial\Omega} (\nu(y) \times H(y))\,\Phi_k(x,y)\,ds(y) + \nabla \int_{\partial\Omega} (\nu(y) \cdot H(y))\,\Phi_k(x,y)\,ds(y) \\ &+ i\omega\varepsilon_0 \int_{\partial\Omega} (\nu(y) \times E(y))\,\Phi_k(x,y)\;ds(y). \end{aligned} \tag{2.27}$$

Dividing $\operatorname{curl} E - i\omega\mu H = 0$ by μ_0 and taking the divergence yields

$$\operatorname{div}(\mu_r H) = 0 \qquad \text{or equivalently} \qquad \operatorname{div}(pH) = -\operatorname{div} H.$$

Substituting the above relation into the second term of the right-hand side of (2.27) yields

$$\begin{aligned} &-\nabla \iint_\Omega \operatorname{div} H(y)\Phi_k(x,y)\,dy \\ =&\nabla \iint_\Omega \operatorname{div}\Big(p(y)H(y)\Big)\Phi_k(x,y)\,dy \\ =&\nabla \iint_\Omega \Big[\operatorname{div}\big(p(y)H(y)\Phi_k(x,y)\big) - \nabla_y\Phi_k(x,y)\cdot p(y)H(y)\Big]\,dy \\ =&\nabla \left[\int_{\partial\Omega} p(y)\big(\nu(y)\cdot H(y)\big)\Phi_k(x,y)\,ds(y) + \iint_\Omega \nabla_x\Phi_k(x,y)\cdot p(y)H(y)\,dy\right] \\ =&\nabla \left[\int_{\partial\Omega} (\mu_r(y)-1)\big(\nu(y)\cdot H(y)\big)\Phi_k(x,y)\,ds(y) + \iint_\Omega \operatorname{div}_x\Big(p(y)H(y)\Phi_k(x,y)\Big)\,dy\right]. \end{aligned} \tag{2.28}$$

By substituting (2.28) back into equation (2.27), we obtain

$$\begin{aligned}
H(x) =&\mathrm{curl} \iint_\Omega q(y)\mathrm{curl} H(y)\Phi_k(x,y)\,dy + (k^2+\nabla\mathrm{div})\iint_\Omega p(y)H(y)\phi_k(x,y)\,dy \\
&\underbrace{-\mathrm{curl}\int_{\partial\Omega}(\nu(y)\times H(y))\,\Phi_k(x,y)\,ds(y)}_{J_1}\underbrace{+\nabla\int_{\partial\Omega}\mu_r\,(\nu(y)\cdot H(y))\,\Phi_k(x,y)\,ds(y)}_{J_2} \\
&\underbrace{+i\omega\varepsilon_0\int_{\partial\Omega}(\nu(y)\times E(y))\,\Phi_k(x,y)\;ds(y)}_{J_3}. \qquad (2.29)
\end{aligned}$$

Next, we consider the terms J_1, J_2 and J_3. We have

$$\int_{\partial\Omega}\nu(y)\cdot\mathrm{curl}(E(y)\phi_k(x,y))\,ds(y) = 0 \qquad (2.30)$$

due to the divergence theorem and the fact that the divergence of the curl of any vector is zero. Then using

$$\begin{aligned}
\mathrm{div}_x\Big(\nu(y)\times E(y)\Phi_k(x,y)\Big) &= \nu(y)\cdot\Big(\mathrm{curl}_y\big[E(y)\Phi_k(x,y)\big]\Big) - \Phi_k(x,y)\nu(y)\cdot\mathrm{curl}_y E(y) \\
&= \nu(y)\cdot\Big(\mathrm{curl}_y\big[E(y)\Phi_k(x,y)\big]\Big) - \Phi_k(x,y)\nu(y)\cdot i\omega\mu(y)H(y)
\end{aligned}$$

we arrive at

$$\mathrm{div}\int_{\partial\Omega}\nu(y)\times E(y)\Phi_k(x,y)\,ds(y) \overset{(2.30)}{=} -i\omega\mu_0\int_{\partial\Omega}\nu(y)\cdot\mu_r(y)H(y)\Phi_k(x,y)\,ds(y)$$

that is

$$\int_{\partial\Omega}\nu(y)\cdot\mu_r(y)H(y)\Phi_k(x,y)\,ds(y) = -\frac{1}{i\omega\mu_0}\mathrm{div}\int_{\partial\Omega}\nu(y)\times E(y)\Phi_k(x,y)\,ds(y). \qquad (2.31)$$

Furthermore,

$$
\begin{aligned}
&J_2 + J_3 \\
&= \nabla \int_{\partial\Omega} \mu_r \left(\nu(y) \cdot H(y)\right) \Phi_k(x,y)\, ds(y) + i\omega\varepsilon_0 \int_{\partial\Omega} \left(\nu(y) \times E(y)\right) \Phi_k(x,y)\ ds(y) \\
&\overset{(2.31)}{=} -\frac{1}{i\omega\mu_0} \nabla \mathrm{div} \int_{\partial\Omega} \nu(y) \times E(y) \Phi_k(x,y)\, ds(y) + i\omega\varepsilon_0 \int_{\partial\Omega} \left(\nu(y) \times E(y)\right) \Phi_k(x,y)\ ds(y) \\
&= -\frac{1}{i\omega\mu_0} \left(\nabla \mathrm{div} \int_{\partial\Omega} \nu(y) \times E(y) \Phi_k(x,y)\, ds(y) + k^2 \int_{\partial\Omega} \left(\nu(y) \times E(y)\right) \Phi_k(x,y)\ ds(y) \right) \\
&= -\frac{1}{i\omega\mu_0} \left(\nabla \mathrm{div} - \Delta\right) \int_{\partial\Omega} \big(\nu(y) \times E(y)\big) \Phi_k(x,y)\, ds(y) \\
&= -\frac{1}{i\omega\mu_0} \mathrm{curl}^2 \int_{\partial\Omega} \big(\nu(y) \times E(y)\big) \Phi_k(x,y)\, ds(y)
\end{aligned}
$$

where we used the fact that $k^2 \Phi_k(x,y) = -\Delta \Phi_k(x,y)$, $x \neq y$. Our considerations so far imply that

$$
\begin{aligned}
&J_1 + J_2 + J_3 \\
&= -\mathrm{curl} \int_{\partial\Omega} \left(\nu(y) \times H(y)\right) \Phi_k(x,y)\, ds(y) - \frac{1}{i\omega\mu_0} \mathrm{curl}^2 \int_{\partial\Omega} \big(\nu(y) \times E(y)\big) \Phi_k(x,y)\, ds(y).
\end{aligned}
$$

We note that $\partial\Omega = \partial D \cup \partial B_{R_0}$. Inserting $E = E^i + E^s$, $H = H^i + H^s$ into the right-hand side of the above equation for the boundary part ∂B_{R_0} and using the Stratton-Chu formula for $x \in B_{R_0}$ yields

$$
\begin{aligned}
-\mathrm{curl} \int_{|x|=R_0} \big(\nu(y) \times H^i(y)\big) \Phi_k(x,y)\, ds(y) - \frac{1}{i\omega\mu_0} \mathrm{curl}^2 \int_{|x|=R_0} \big(\nu(y) \times E^i(y)\big) \Phi_k(x,y)\, ds(y) \\
= H^i(x)
\end{aligned}
$$

and

$$
-\mathrm{curl} \int_{|x|=R_0} \left(\nu(y) \times H^s(y)\right) \Phi_k(x,y)\, ds(y) - \frac{1}{i\omega\mu_0} \mathrm{curl}^2 \int_{|x|=R_0} \big(\nu(y) \times E^s(y)\big) \Phi_k(x,y)\, ds(y) = 0.
$$

Here, we used the fact that the scattered field E^s, H^s satisfies the Maxwell system

$$
\mathrm{curl} E - i\omega\mu H = 0, \quad \mathrm{curl} H + i\omega\varepsilon E = 0
$$

outside B_{R_0} and the Silver-Müller radiation condition at infinity. Summing, we obtain

$$\begin{aligned}
&J_1 + J_2 + J_3 \\
&= -\mathrm{curl} \int_{\partial D} (\nu(y) \times H(y))\, \Phi_k(x,y)\, ds(y) - \frac{1}{i\omega\mu_0} \mathrm{curl}^2 \int_{\partial D} \big(\underbrace{\nu(y) \times E(y)}_{=0}\big) \Phi_k(x,y)\, ds(y) \\
&= -\mathrm{curl} \int_{\partial D} (\nu(y) \times H(y))\, \Phi_k(x,y)\, ds(y).
\end{aligned}$$

Finally, substituting for $J_1 + J_2 + J_3$ into (2.29) yields the following operator equation:

$$\begin{aligned}
H(x) = H^i(x) &+ \mathrm{curl} \iint_\Omega q(y) \mathrm{curl} H(y) \Phi_k(x,y)\, dy \\
&+ (k^2 + \nabla \mathrm{div}) \iint_\Omega p(y) H(y) \Phi_k(x,y)\, dy \\
&- \mathrm{curl} \int_{\partial D} (\nu \times H(y))\, \Phi_k(x,y)\, ds(y), \quad x \in \Omega. \qquad (2.32)
\end{aligned}$$

To simplify notation, we define the operators $L_k, T_k : H(\mathrm{curl}, \Omega) \to H(\mathrm{curl}, \Omega)$ by

$$(L_k g)(x) := \mathrm{curl} \iint_\Omega g(y) \Phi_k(x,y)\, dy, \quad x \in \Omega, \qquad (2.33)$$

$$(T_k g)(x) := (k^2 + \nabla \mathrm{div}) \iint_\Omega g(y) \Phi_k(x,y)\, dy, \quad x \in \Omega \qquad (2.34)$$

for $g \in L^2(\Omega)^3$. By lemmas 2.5 and 2.6, the right hand side of (2.32) belongs to $H(\mathrm{curl}\,, \Omega)$. Moreover, using operator notation, (2.32) can be written as

$$H = H^i + L_k(q \mathrm{curl} H) + T_k(pH) - \widetilde{\mathcal{M}}(\gamma_t H) \quad \in H(\mathrm{curl}, \Omega)$$

where $\widetilde{\mathcal{M}}$ is given by (2.23) and where γ_t is the trace operator $H(\mathrm{curl}\,, \Omega) \to H^{-\frac{1}{2}}(\mathrm{Div}\,, \partial D)$, $H \to \nu \times H$.

Now that we have derived the Lippmann-Schwinger operator equation, we will prove the equivalence between (2.32) and problem $(P1)$. To do so, we need to assume that k^2 is not a Dirichlet eigenvalue to the Maxwell's problem inside D, that is, that the interior Maxwell problem with Dirichlet boundary value admits at most one solution.

Theorem 2.8. *We assume that k^2 is not an eigenvalue of*

$$\begin{aligned} \text{curl}^2 w - k^2 w &= 0 \quad \text{in } D, \\ \nu \times w &= 0 \quad \text{on } \partial D. \end{aligned}$$

(a) Let $H \in H(\text{curl}, \Omega)$ solve the integro-differential equation (2.32). Then H can be extended by the right hand side of (2.32) to a solution to problem $(P1)$.

(b) Let $H \in H_{\text{loc}}(\text{curl}, \mathbb{R}^3 \setminus \overline{D})$ be a solution to problem $(P1)$. Then the restriction $H|_\Omega \in H(\text{curl}, \Omega)$ of H to Ω solves the integro-differential equation (2.32).

Proof. (a) Let A be the right-hand side of (2.32) in all of $\mathbb{R}^3$. Then $A = H$ in $\mathbb{R}^3 \setminus \overline{D}$ and thus $\gamma_t A|_+ = \gamma_t H|_+$ on ∂D. Furthermore, using the jump condition (2.24) yields

$$\gamma_t A|_\pm = H^i + L_k(q\text{curl}H) + T_k(pH) \pm \frac{1}{2}\gamma_t H - \mathcal{M}(\gamma_t H) \quad \text{on } \partial D.$$

From

$$\gamma_t H|_+ = H^i + L_k(q\text{curl}H) + T_k(pH) + \frac{1}{2}\gamma_t H - \mathcal{M}(\gamma_t H)$$

that is

$$\frac{1}{2}\gamma_t H|_+ = H^i + L_k(q\text{curl}H) + T_k(pH) - \mathcal{M}(\gamma_t H) \quad \text{on } \partial D$$

we conclude $\gamma_t A|_- = 0$ on ∂D. The assumption on k^2 now implies $A = 0$ in D. Let $\psi \in H(\text{curl}, \mathbb{R}^3)$ with $\psi(x) = 0$ for $|x| \geq R_0$ and let

$$u = L_k(q\text{curl}H), \quad v = T_k(pH) \quad \text{in } \mathbb{R}^3, \quad w = \widetilde{\mathcal{M}}(\gamma_t H) \quad \text{in } \mathbb{R}^3 \setminus \partial D.$$

Then

$$\begin{aligned} &\iint_{\mathbb{R}^3 \setminus \overline{D}} \left(\text{curl}A \cdot \text{curl}\psi - k^2 A \cdot \psi\right) dx \\ &= \iint_{\mathbb{R}^3} \left(\text{curl}A \cdot \text{curl}\psi - k^2 A \cdot \psi\right) dx \\ &= \iint_{\mathbb{R}^3} \left(\text{curl}H^i \cdot \text{curl}\psi - k^2 H^i \cdot \psi\right) dx + \iint_\Omega q\text{curl}H \cdot \text{curl}\psi \, dx + k^2 \iint_\Omega pH \cdot \psi \, dx \\ &\quad - \iint_{\mathbb{R}^3} \left(\text{curl}w \cdot \text{curl}\psi - k^2 w \cdot \psi\right) dx \end{aligned} \tag{2.35}$$

where we used lemma 2.5. Let B_R be a ball of radius $R \geq R_0$ that contains the support of ψ. With Green's theorem (1.11) we compute

$$\iint_{\mathbb{R}^3} \left(\mathrm{curl} H^i \cdot \mathrm{curl}\psi - k^2 H^i \cdot \psi\right) \, dx = \iint_{B_R} \left(\mathrm{curl}^2 H^i - k^2 H^i\right) \cdot \psi \, dx = 0 \qquad (2.36)$$

and

$$\begin{aligned}
&\iint_{\mathbb{R}^3} \left(\mathrm{curl} w \cdot \mathrm{curl}\psi - k^2 w \cdot \psi\right) \, dx \\
=& \iint_{D} \left(\mathrm{curl} w \cdot \mathrm{curl}\psi - k^2 w \cdot \psi\right) \, dx + \iint_{B_R \setminus \overline{D}} \left(\mathrm{curl} w \cdot \mathrm{curl}\psi - k^2 w \cdot \psi\right) \, dx \\
=& \langle \gamma_t \mathrm{curl}\, w|_- - \gamma_t \mathrm{curl}\, w|_+, \gamma_T \psi \rangle_{\partial D} \\
=& 0
\end{aligned}$$

where we used that $w \in C^\infty(\mathbb{R}^3 \setminus \partial D)^3$, $\mathrm{curl}^2 w - k^2 w = 0$ in $\mathbb{R}^3 \setminus \partial D$, and the jump condition (2.24). Substituting the above identities into (2.35) yields

$$\iint_{\mathbb{R}^3 \setminus \overline{D}} \left(\mathrm{curl} A \cdot \mathrm{curl}\psi - k^2 A \cdot \psi\right) \, dx = \iint_{\Omega} q \mathrm{curl} H \cdot \mathrm{curl}\psi \, dx + k^2 \iint_{\Omega} p H \cdot \psi \, dx$$

for all $\psi \in H(\mathrm{curl}, \mathbb{R}^3)$ with $\psi(x) = 0$ for $|x| \geq R_0$. The above variational formulation is equivalent to (2.10). By lemmas 2.5 and 2.6, $H - H^i$ satisfies the Silver-Müller radiation condition (2.8).

(b) Let $H \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \overline{D})$ be a solution to (2.9), in particular, to (2.10) such that $H - H^i$ satisfies the Silver-Müller radiation condition (2.8). Further, let A be the right-hand side of (2.32) in $\mathbb{R}^3$. By part (a), it holds that

$$\gamma_t A|_+ - \gamma_t A|_- = \gamma_t H|_+, \quad \text{that is,} \quad \gamma_t (A - H)|_+ = \gamma_t A|_-.$$

Set

$$\hat{A} := \begin{cases} A - H, & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ A, & \text{in } D. \end{cases}$$

Then, $\gamma_t \hat{A}|_- = \gamma_t \hat{A}|_+$ on ∂D, and thus $\hat{A} \in H(\mathrm{curl}, \mathbb{R}^3)$, and, again by part (a), we obtain

$$\iint_{\mathbb{R}^3} \left(\text{curl}A \cdot \text{curl}\psi - k^2 A \cdot \psi\right)\, dx = \iint_{\Omega} q\text{curl}H \cdot \text{curl}\psi\, dx + k^2 \iint_{\Omega} pH \cdot \psi\, dx$$
$$\stackrel{(2.10)}{=} \iint_{\mathbb{R}^3\setminus\overline{D}} \left(\text{curl}H \cdot \text{curl}\psi - k^2 H \cdot \psi\right)\, dx$$

for all $\psi \in H(\text{curl}, \mathbb{R}^3)$ with $\psi(x) = 0$ for $|x| \geq R_0$, and hence

$$\iint_{\mathbb{R}^3} \left(\text{curl}\hat{A} \cdot \text{curl}\psi - k^2 \hat{A} \cdot \psi\right)\, dx = 0$$

for all $\psi \in H(\text{curl}, \mathbb{R}^3)$ with $\psi(x) = 0$ for $|x| \geq R_0$, which is the variational formulation of Maxwell's equation

$$\text{curl}^{\,2}\hat{A} - k^2\hat{A} = 0 \quad \text{in } \mathbb{R}^3.$$

Moreover, $\hat{A}$ satisfies radiation condition (2.8), and thus we conclude that $\hat{A} = 0$ in $\mathbb{R}^3$ (see [6], page 156).

□

Now we are in a position to prove the existince of a solution to problem $(P1)$. By the above theorem, we consider the operator equation (2.32) and show that it is of Fredholm type.

Theorem 2.9. *Let assumption (2.3) hold, and we make the same assumption on k^2 as in theorem 2.8. Then there exists a unique solution $H \in H_{\text{loc}}(\text{curl}, \mathbb{R}^3 \setminus \overline{D})$ to problem $(P1)$.*

Proof. By theorem (2.8) we can consider equation (2.32), which we rewrite as

$$\underbrace{H - L_i(q\text{curl}H) - T_i(pH) + \widetilde{\mathcal{M}}_i(\gamma_t H)}_{I_1}$$
$$- \underbrace{(L_k - L_i)\,(q\text{curl}H)}_{I_2} - \underbrace{(T_k - T_i)\,(pH)}_{I_3} + \underbrace{\left(\widetilde{\mathcal{M}} - \widetilde{\mathcal{M}}_i\right)(\gamma_t H)}_{I_4} = H^i \quad \text{in } \Omega, \tag{2.37}$$

where L_i, T_i and $\widetilde{\mathcal{M}}_i$ are the operators L_k, T_k and $\widetilde{\mathcal{M}}$ respectively, for the special case $k = i$. We want to apply the Fredholm theory to (2.37) to conclude the existence of a solution. To do so, we need to show that I_1 is an isomorphism and that I_2, I_3 and I_4 are compact.

(I_1): The operator $H \to H - L_i(q\text{curl}H) - T_i(pH) + \widetilde{\mathcal{M}}_i(\gamma_t H)$ is an isomorphism from $H(\text{curl}, \Omega)$ into itself:
Let $F \in H(\text{curl}, \Omega)$ be given and consider

$$H - L_i(q\text{curl}H) - T_i(pH) + \widetilde{\mathcal{M}}_i(\gamma_t H) = F \quad \text{in } \Omega,$$

which is equivalent to $H = F + U$ and

$$\begin{aligned} U &= L_i(q\text{curl}H) + T_i(pH) - \widetilde{\mathcal{M}}_i(\gamma_t H) \\ &= L_i(q\text{curl}U) + T_i(pU) - \widetilde{\mathcal{M}}_i(\gamma_t U) + L_i(q\text{curl}F) + T_i(pF) - \widetilde{\mathcal{M}}(\gamma_t F) \quad \text{in } \Omega \end{aligned}$$

that is,

$$U = L_i(q\text{curl}\, U) + T_i(pU) - \widetilde{\mathcal{M}}_i(\gamma_t U) + G \quad \text{in } \Omega \tag{2.38}$$

with $G = L_i(q\text{curl}\, F) + T_i(pF) - \widetilde{\mathcal{M}}(\gamma_t F) \in H(\text{curl}, \Omega)$. We note that the above equation is of the form (2.32) for $k = i$ and G instead of H^i. Extending U by the right hand side of (2.38) to all $\mathbb{R}^3$ and using similar arguments as in the proof of theorem (2.8) for $k = i$, we conclude that $U = 0$ in D, and U satisfies the variational equation (replacing ψ with its complex conjugate $\overline{\psi}$),

$$\begin{aligned} \iint_{\mathbb{R}^3} \left(\text{curl}U \cdot \text{curl}\overline{\psi} + U \cdot \overline{\psi}\right) dx &= \iint_{\Omega} q\text{curl}U \cdot \text{curl}\overline{\psi}\, dx - \iint_{\Omega} pU \cdot \overline{\psi}\, dx \\ &\quad + \iint_{\mathbb{R}^3} \left(\text{curl}G \cdot \text{curl}\overline{\psi} + G \cdot \overline{\psi}\right) \end{aligned}$$

for all $\psi \in H(\text{curl}\,, \mathbb{R}^3)$ with $\psi(x) = 0$ for $|x| \geq R_0$, where we have extended G by zero into $\mathbb{R}^3$. That is, U solves

$$\begin{aligned} \iint_D \left(\text{curl}\, U \cdot \text{curl}\, \overline{\psi} + U \cdot \overline{\psi}\right) dx + \iint_{\mathbb{R}^3 \setminus \overline{D}} \left(\frac{1}{\varepsilon_r} \text{curl}\, U \cdot \text{curl}\, \overline{\psi} + \mu_r U \cdot \overline{\psi}\right) dx \\ = \iint_{\mathbb{R}^3} \left(\text{curl}\, G \cdot \text{curl}\, \overline{\psi} + G \cdot \overline{\psi}\right) dx \end{aligned} \tag{2.39}$$

for all $\psi \in H(\text{curl}\,, \mathbb{R}^3)$ with $\psi(x) = 0$ for $|x| \geq R_0$. By the form $U = L_i(q\text{curl}\, H) + T_i(pH) - \widetilde{\mathcal{M}}_i(\gamma_t H)$ and the definition of Φ_i we observe that U decays exponentially as $|x|$ tends to infinity. Therefore, $U \in H(\text{curl}, \mathbb{R}^3)$ and the variational equation (2.39) holds for all $\psi \in H(\text{curl}, \mathbb{R}^3)$. The left-hand side of (2.39) defines a coercive sesquilinear form on $H(\text{curl}, \mathbb{R}^3)$. Indeed, by the assumptions made on μ_r and ε_r, there exist constants $c_0 > 0$ and $c_1 > 0$ such that $\mu_r \geq c_0$ and $\frac{1}{\varepsilon_r} \geq c_1$ on Ω. Thus

$$\begin{aligned}
&\iint_{\mathbb{R}^3} \left(\frac{1}{\varepsilon_r} \text{curl}\, U \cdot \text{curl}\, \overline{U} + \mu_r U \cdot \overline{U} \right) dx \\
&= \iint_{\Omega} \left(\frac{1}{\varepsilon_r} \text{curl}\, U \cdot \text{curl}\, \overline{U} + \mu_r U \cdot \overline{U} \right) dx + \iint_{\mathbb{R}^3 \setminus \overline{B}_{R_0}} \left(\text{curl}\, U \cdot \text{curl}\, \overline{U} + U \cdot \overline{U} \right) dx \\
&\geq \min\{c_0, c_1, 1\} \iint_{\mathbb{R}^3 \setminus \overline{D}} \left(|\text{curl}\, U|^2 + |U|^2 \right) dx \\
&= c \|U\|^2_{H(\text{curl}, \mathbb{R}^3)}
\end{aligned}$$

where $c := \min\{c_0, c_1, 1\}$ and where we assumed that $U = 0$ in D. The right-hand side of (2.39) defines a bounded conjugate-linear functional on $H(\text{curl}, \mathbb{R}^3)$. Consequently the theorem of Lax-Milgram implies the existence of a unique solution U of the variational equation (2.39).

(I_2),(I_3): The operators $H \to (L_k - L_i)(q\text{curl}\, H)$ and $H \to (T_k - T_i)(pH)$ are compact in $H(\text{curl}\,, \Omega)$:
Using the power series expansion for the exponential function, one obtains that the difference $\Phi_k - \Phi_i$ has the form

$$\Phi_k(x, y) - \Phi_i(x, y) = F_1(|x - y|^2) + |x - y| F_2(|x - y|^2)$$

with some analytic functions $F_1, F_2 : \mathbb{R}^3 \to \mathbb{C}$. With some ornate but elementary calculations, it can be verified that

$$\Big| \nabla_x \Big(\Phi_k(x, y) - \Phi_i(x, y) \Big) \Big| \leq k_1 \quad \text{and} \quad \Big| \nabla_{xx} \Big(\Phi_k(x, y) - \Phi_i(x, y) \Big) \Big| \leq \frac{k_2}{|x - y|}$$

for some constants $k_1 > 0$ and $k_2 > 0$, where $\nabla_{xx} := \left(\frac{\partial^2}{\partial x_1 \partial x_1}, \frac{\partial^2}{\partial x_2 \partial x_2}, \frac{\partial^2}{\partial x_3 \partial x_3} \right)^\top$. Thus, the kernels $\nabla_x\big(\Phi_k - \Phi_i\big)$ and $\nabla_{xx}\big(\Phi_k - \Phi_i\big)$ are weakly singular and hence

$$\begin{aligned}\big(L_k - L_i\big)g(x) &= \operatorname{curl} \iint_\Omega g(y)\big(\Phi_k(x,y) - \Phi_i(x,y)\big)\,dy \\ &= \iint_\Omega g(y) \times \nabla_x\big(\Phi_k(x,y) - \Phi_i(x,y)\big)\,dy\end{aligned}$$

and

$$\begin{aligned}\operatorname{curl}\big(L_k - L_i\big)g(x) &= \operatorname{curl}^2 \iint_\Omega g(y)\big(\Phi_k(x,y) - \Phi_i(x,y)\big)\,dy \\ &= (\nabla\operatorname{div} - \Delta) \iint_\Omega g(y)\big(\Phi_k(x,y) - \Phi_i(x,y)\big)\,dy \\ &= \big(\nabla\operatorname{div} + k^2\big) \iint_\Omega g(y)\big(\Phi_k(x,y) - \Phi_i(x,y)\big)\,dy \\ &= \iint_\Omega g(y) \cdot \nabla_{xx}\big(\Phi_k(x,y) - \Phi_i(x,y)\big)\,dy \\ &\quad + k^2 \iint_\Omega g(y)\big(\Phi_k(x,y) - \Phi_i(x,y)\big)\,dy\end{aligned}$$

are compact on $L^2(\Omega)^3$. From this and the boundedness of the mapping $H(\operatorname{curl},\Omega) \to L^2(\Omega)^3$, $H \to q\operatorname{curl} H$ we conclude the compactness of $H \to (L_k - L_i)\,(q\operatorname{curl} H)$ on $H(\operatorname{curl},\Omega)$. We note that $(T_k - T_i)\,g = \operatorname{curl}\big(\,(L_k - L_i)\,g\big)$ and $\operatorname{curl}\big(\,(T_k - T_i)\,g\big) = k^2\,(L_k - L_i)\,g$. From this and the boundedness of $H(\operatorname{curl},\Omega) \to L^2(\Omega)^3$, $H \to pH$ we also conclude the compactness of $H \to (T_k - T_i)\,(pH)$ in $H(\operatorname{curl},\Omega)$.

(I_4): The operator $H \to \Big(\widetilde{\mathcal{M}} - \widetilde{\mathcal{M}}_i\Big)\,(\gamma_t H)$ is compact on $H(\operatorname{curl},\Omega)$:
From the boundedness of the trace operator $\gamma_t : H(\operatorname{curl},\Omega) \to H^{-\frac{1}{2}}(\operatorname{Div},\partial D)$ and the fact that

$$\begin{aligned}\Big(\widetilde{M} - \widetilde{M}_i\Big)\,a(x) &= \operatorname{curl} \int_{\partial D} a(y)\,(\Phi_k(x,y) - \Phi_i(x,y))\,ds \\ &= \int_{\partial D} a(y)\,(\nabla_x\Phi_k(x,y) - \nabla_x\Phi_i(x,y))\,ds, \quad a \in H^{-\frac{1}{2}}(\operatorname{Div},\partial D)\end{aligned}$$

has a weakly singular kernel, we conclude compactness of $H \to \Big(\widetilde{\mathcal{M}} - \widetilde{\mathcal{M}}_i\Big)\,(\gamma_t H)$ on $H(\operatorname{curl},\Omega)$.

By Fredholm theorem 1.12 and the uniqueness result of theorem 2.4, we conclude that there exists a unique solution $H \in H_{\mathrm{loc}}(\operatorname{curl},\mathbb{R}^3 \setminus \overline{D})$ to problem $(P1)$. $\square$

2.2 Inverse problem

An important example of incident fields are plane waves

$$E^i(x,d,p) = ik(d \times p) \times de^{ikx\cdot d}, \quad H^i(x,d,p) = d \times pe^{ikx\cdot d}, \quad x \in \mathbb{R}^3$$

with propagation direction $d \in S^2$ and polarization $p \perp d$. In the following discussion, we denote the scattered wave and far field pattern corresponding to the incident plane wave by $E^s(\cdot,d,p)$, $H^s(\cdot,d,p)$, and $H_\infty(\cdot,d,p)$, $E_\infty(\cdot,d,p)$, respectively, indicating the dependence on the direction d and polarization p of the incident field. The inverse problem we will investigate here is, under what conditions an obstacle can be uniquely identified from a knowledge of the far field pattern $H_\infty(\cdot,d,p)$ (or the electric far field pattern $E_\infty(\cdot,d,p)$) for one or several incident plane waves with incident direction d and polarization p.

By Rellich's lemma 1.9, the scattered wave, and thus the total field in the exterior of the scatterer, is uniquely determined by the far field pattern. Consequently, showing the uniqueness of the inverse problem is equivalent to showing that the total field can not satisfy the perfectly conducting boundary condition (2.4) for two different domains D_1 and D_2. Assuming that the two scatterers are disjoint, that it, $\overline{D}_1 \cap \overline{D}_2 = \emptyset$, the scattered wave is defined in all of $\mathbb{R}^3$ because it is defined in the exterior of both D_1 and D_2. This implies that the scattered field E^s, H^s constitutes an entire solution to the Maxwell's equation satisfying the radiation condition and therefore must vanish. In particular, the scattered field vanishes outside of the inhomogeneity in which the domains D_1 and D_2 are imbedded. Using unique continuation, it is easily verified that the scattered field vanishes everywhere. However, then the total field coincides with the incident field, and therefore the incident field itself satisfies the perfectly conducting boundary condition $\nu \times E^i = 0$ on ∂D, an impossibility for a closed surface. As a consequence, nonuniqueness can occur only when $\overline{D}_1 \cap \overline{D}_2 \neq \emptyset$.

The idea behind unique determination of an obstacle is from Kirsch and Kress [19] (see theorem 7.1 in [2]) for the acoustic case. In [6], the idea was generalized for the electromagnetic case in a homogeneous medium. The idea is to assume that we have overdetermined data in the sense that the far field pattern is known for all incident directions and polarizations. We follow Potthast [11] and simplify the approach of Kirsch and Kress through the use of a mixed reciprocity relation. Determining the scatterer from the knowledge of the far field pattern for one incident plane wave is still an open problem. Partial progress has been made in inverse acoustic obstacle scattering; see [12] and [13]. In [14], the result was extended to the electromagentic case for sound-hard balls.

Let D_1 and D_2 be two obstacles imbedded in an inhomogeneous and bounded medium described by ε_r and μ_r. We assume hereinafter that μ_r satisfies assumption (2.3) and $\varepsilon_r \equiv 1$, in particular, $\varepsilon \equiv \varepsilon_0$ and $\sigma \equiv 0$. Furthermore, we assume that the corresponding far field patterns $H_{1,\infty}(\hat{x}, d, p)$ and $H_{2,\infty}(\hat{x}, d, p)$ with respect to D_1 and D_2 respectively, coincide for all $\hat{x}, d \in S^2$ and all $p \perp d$. We begin by showing that, under these assumptions, the scattered fields coincide also for those fields that are responses of electric dipoles

$$H_e^i(x; z) = \operatorname{curl}_x p\Phi_k(x, z), \qquad E_e^i(x; z) = -\frac{1}{ik}\operatorname{curl}_x \operatorname{curl}_x p\Phi_k(x, z)$$

with source point z in the unbounded component G of $\mathbb{R}^3 \setminus (\overline{D_1} \cup \overline{D_3})$. We will then use this result, together with the mixed electromagntic reciprocity relation, to show uniqueness of the inverse problem. We note the symmetric and asymmetric properties of the electric and magnetic dipoles, respectively, that is,

$$E_e^i(x; z) = E_e^i(z; x) \quad \text{and} \quad H_e^i(x; z) = -H_e^i(z; x), \quad x \neq z. \tag{2.40}$$

2.2.1 Scattering of electric dipoles

In this section, we will discuss the well-posedness of the scattering problem in case of incident electric dipole with source point $z \in \mathbb{R}^3 \setminus \overline{D}$ and polarization $a \in \mathbb{R}^3$:

$$H_e = H_e(\cdot; z) = \operatorname{curl}\,(a\Phi_k(\cdot, z)) + H_e^s \qquad \text{in } \mathbb{R}^3 \setminus \left(\overline{D} \cup \{z\}\right), \tag{2.41}$$

$$\operatorname{curl}\operatorname{curl} H_e - k^2\mu_r H_e = 0 \qquad \text{in } \mathbb{R}^3 \setminus \left(\overline{D} \cup \{z\}\right), \tag{2.42}$$

$$\operatorname{curl} H_e \times \nu = 0 \qquad \text{on } \partial D, \tag{2.43}$$

$$H_e^s \times \frac{x}{|x|} - ikH_e^s = \mathcal{O}\left(\frac{1}{|x|^2}\right) \qquad \text{as } |x| \to \infty. \tag{2.44}$$

We note that $H_e^i(\cdot; z) = \operatorname{curl} a\Phi_k(\cdot; z)$ satisfies the Silver-Müller radiation condition (2.44); see lemma 3.29 (a) in [1]. Consequently H_e satisfies the Silver-Müller radiation condition (2.44). The variational formulation of the above problem is given by

$$\iint_{\mathbb{R}^3 \setminus \overline{D}} \left(\operatorname{curl} H_e \cdot \operatorname{curl} \psi - k^2\mu_r H_e \cdot \psi\right)\, dx = 0 \tag{2.45}$$

for all $\psi \in H(\text{curl}\,, \mathbb{R}^3 \setminus \overline{D})$ with $\psi(x) = 0$ for $|x| \geq R_0$.

To show the existence of a unique solution, we would like to argue as we did previously, and thus we consider the operator equation (2.32) in case of incident electric dipole $H_e^i(\cdot; z)$:

$$H_e = H_e^i(\cdot; z) + T(pH_e) - \widetilde{\mathcal{M}}(\gamma_t H_e) \quad \text{in } \Omega. \tag{2.46}$$

Here T and $\widetilde{\mathcal{M}}$ are given by (2.34) and (2.23), respectively. We note that $q = 0$ since $\varepsilon_r = 1$, and thus the operator L given by (2.33) in (2.32) disappears. Next we need to establish equivalence between the scattering problem (2.46)-(2.44) and the operator (2.46) equation. Unfortunately, we cannot argue as we did in theorem 2.8 to show equivalence. This is due to computation (2.36) in the proof of theorem 2.8, that is, we need

$$\iint_{\mathbb{R}^3} \left(\text{curl} H_e^i(\cdot; z) \cdot \text{curl}\psi - k^2 H_e^i(\cdot; z) \cdot \psi\right)\, dx = \iint_{B_R} \left(\text{curl}^2 H_e^i(\cdot; z) - k^2 H_e^i(\cdot; z)\right) \cdot \psi \, dx \stackrel{!}{=} 0$$

for all $\psi \in H(\text{curl}\,, \mathbb{R}^3)$ with $\psi(x) = 0$ for $|x| \geq R_0$. The above equation only holds when the source point z lies outside B_R, where B_R is a ball of radius $R \geq R_0$ that contains the support of ψ, because then the dipole $H_e^i(\cdot; z)$ has no singularity. If $z \in \Omega$, then $H_e^i(\cdot; z)$ has a singularity at z and $\text{curl}\, H_e^i(\cdot; z)$ a singularity of order two and thus is not even a L^2-function. Therefore, we need to investigate the case when $z \in \Omega$ in more detail.

We isolate $z \in \Omega$ by taking a small ball $B_\epsilon(z)$ of radius $\epsilon > 0$ centered at z such that $B_\epsilon(z) \subset \Omega$. Further, let $\varphi \in C^\infty(\mathbb{R}^3)$ with

$$\varphi(x) = \begin{cases} 1, & \text{for } |x| > \frac{2\epsilon}{3}, \\ 0, & \text{for } |x| < \frac{\epsilon}{3} \end{cases}$$

and define $h : \mathbb{R}^3 \to \mathbb{C}$ by

$$h(x) = \Phi_k(x, z)\varphi(|x - z|^2).$$

Then, $h \in C^\infty(\mathbb{R}^3)$ and, for $x \notin B_\epsilon(z)$, we have that $h(x) = \Phi_k(x, z)$. Moreover, the trace of h on ∂D equals $\Phi_k(\cdot, z)|_{\partial D}$. Next we set

$$\hat{H} = H_e^s + \text{curl}\ (ah) = H_e + \text{curl}\ \left(a[\varphi(|\cdot - z|^2) - 1]\Phi_k(\cdot, z)\right) \quad \text{in } \mathbb{R}^3 \setminus \overline{D}. \tag{2.47}$$

To simplify notation we define

$$\hat{\varphi}_z(x) := \varphi(|x-z|^2) - 1.$$

If H_e solves problem (2.41)-(2.44), then

$$\operatorname{curl}\operatorname{curl}\hat{H} - k^2\mu_r\hat{H} = \operatorname{curl}^3(a\hat{\varphi}_z\phi_k(\cdot,z)) - k^2\mu_r\operatorname{curl}(a\hat{\varphi}_z\phi_k(\cdot,z)) \qquad \text{in } \mathbb{R}^3\setminus\overline{D}, \quad (2.48)$$
$$\operatorname{curl}\hat{H}\times\nu = \operatorname{curl}^2(a\hat{\varphi}_z\phi_k(\cdot,z))\times\nu \qquad \text{on } \partial D. \quad (2.49)$$

Remark 2.10. *(a) We note that $\hat{\varphi}_z(x) = \varphi(|x-z|^2) - 1 = 0$ for $|x-z|^2 > \frac{2\epsilon}{3}$ so* curl $(a\hat{\varphi}_z\Phi_k(\cdot,z)) = 0$ *away from z.*

(b) Since $h(x) = \Phi_k(x,z)$ for x away from z, curl (ah) *satisfies the Silver-Müller radiation condition (2.44). Thus, $\hat{H}$ also satisfies radiation condition (2.44).*

By the above remark, in case of $z\in\Omega$, in particular $z\notin\partial D$, the right-hand side of (2.49) equals zero. Next we need to examine the right-hand side of (2.48):

$$\begin{aligned}
&\operatorname{curl}^3(a\hat{\varphi}_z\phi_k(\cdot,z)) - k^2\mu_r\operatorname{curl}(a\hat{\varphi}_z\phi_k(\cdot,z))\\
=&-\operatorname{curl}\Delta(a\hat{\varphi}_z\phi_k(\cdot,z)) - k^2\mu_r\operatorname{curl}(a\hat{\varphi}_z\phi_k(\cdot,z))\\
=&-\operatorname{curl}\left(a\{\Phi_k(\cdot,z)\Delta\hat{\varphi}_z + 2\nabla\Phi_k(\cdot,z)\cdot\nabla\hat{\varphi}_z + \hat{\varphi}_z\underbrace{\Delta\Phi_k(\cdot,z)}_{=-k^2\Phi_k(\cdot,z)}\}\right) - k^2\mu_r\operatorname{curl}(a\hat{\varphi}_z\phi_k(\cdot,z))\\
=&k^2\underbrace{(1-\mu_r)}_{=-p}\operatorname{curl}(a\hat{\varphi}_z\Phi_k(\cdot,z)) - \operatorname{curl}(a\{\Phi_k(\cdot,z)\Delta\hat{\varphi}_z + 2\nabla\Phi_k(\cdot,z)\cdot\nabla\hat{\varphi}_z\})\\
=&-k^2\operatorname{curl}(pa\hat{\varphi}_z\Phi_k(\cdot,z)) + k^2\nabla p\times\{a\hat{\varphi}_z\Phi_k(\cdot,z)\} - \operatorname{curl}(a\{\Phi_k(\cdot,z)\Delta\hat{\varphi}_z + 2\nabla\Phi_k(\cdot,z)\cdot\nabla\hat{\varphi}_z\}).
\end{aligned}$$

For $x\in\Omega$ set

$$f(x) = -k^2ap(x)\hat{\varphi}_z(x)\Phi_k(x,z), \quad (2.50)$$
$$g(x) = k^2\nabla p(x)\times\{a\hat{\varphi}_z(x)\Phi_k(x,z)\} - \operatorname{curl}(a\{\Phi_k(x,z)\Delta\hat{\varphi}_z(x) + 2\nabla\Phi_k(x,z)\cdot\nabla\hat{\varphi}_z(x)\}). \quad (2.51)$$

Then $f,g\in L^2(\Omega)^3$ (we note that $\Delta\hat{\phi}_z(x) = \Delta\phi(|x-z|^2)$, $\nabla\hat{\phi}_z(x) = \nabla\phi(|x-z|^2)$ which vanish near the singularity of $\Phi_k(x,z)$) and due to remark 2.10 (a), both f and g vanish on ∂D. Summing, we obtain the following problem for $\hat{H}$:

$$\operatorname{curl}\operatorname{curl}\hat{H} - k^2\mu_r\hat{H} = \operatorname{curl} f + g \qquad \text{in } \mathbb{R}^3 \setminus \overline{D}, \tag{2.52}$$

$$\operatorname{curl}\hat{H} \times \nu = 0 \qquad \text{on } \partial D, \tag{2.53}$$

$$\hat{H} \times \frac{x}{|x|} - ik\hat{H} = \mathcal{O}\left(\frac{1}{|x|^2}\right) \qquad \text{as } |x| \to \infty \tag{2.54}$$

where we have extended f and g by zero outside into $\mathbb{R}^3 \setminus \overline{D}$.

We note that, if $\hat{H}$ solves problem (2.52)-(2.54), then by substituting $H_e = \hat{H} - \operatorname{curl}\,(a\hat{\varphi}_z\Phi_k(\cdot,z))$ into (2.52)-(2.54) and letting f and g be given by (2.50) and (2.51), respectively, solution H_e of problem (2.42)-(2.44) is obtained.

The corresponding variational formulation of problem (2.52)-(2.54) is given by

$$\iint_{\mathbb{R}^3\setminus\overline{D}} \left(\operatorname{curl}\hat{H}\cdot\operatorname{curl}\psi - k^2\mu_r\hat{H}\cdot\psi\right)\,dx = \iint_\Omega f\cdot\operatorname{curl}\psi\,dx + \iint_\Omega g\cdot\psi\,dx \tag{2.55}$$

for all $\psi \in H(\operatorname{curl},\mathbb{R}^3 \setminus \overline{D})$ with compact support. Further, set

$$\hat{H} = L_k(f) + \frac{1}{k^2}T_k(g) + T_k(p\hat{H}) - \widetilde{\mathcal{M}}(\gamma_t\hat{H}) \quad \text{in } \Omega \tag{2.56}$$

where L_k, T_k and $\widetilde{\mathcal{M}}$ are the opertors given by (2.33), (2.34) and (2.23), respectively. We can argue exactly as in theorem 2.8 to obtain the equivalence of the scattering problem (2.52)-(2.54) and the Lippmann-Schwinger operator equation (2.56). Moreover, writing

$$H - T_i(p\hat{H}) + \widetilde{\mathcal{M}}(\gamma_t\hat{H}) - (T_k - T_i)\,(p\hat{H}) + \left(\widetilde{\mathcal{M}} - \widetilde{\mathcal{M}}_i\right)(\gamma_t\hat{H}) = L(f) + \frac{1}{k^2}T(g) \quad \text{in } \Omega$$

and assuming that k^2 is not a Dirichlet eigenvalue in D, we obtain by theorem 2.9 and the uniqueness result of theorem 2.4 the existence of a unique solution $\hat{H} \in H_{\text{loc}}(\operatorname{curl},\mathbb{R}^3 \setminus \overline{D})$ to problem (2.55). Thus, there exists a unique solution $H_e \in H_{\text{loc}}\left(\operatorname{curl},\mathbb{R}^3 \setminus \left(\overline{D} \cup \{z\}\right)\right)$ to the variational problem (2.45).

2.2.2 Uniqueness

To show the uniqueness of the inverse problem, we are going to work with the electric field only instead of the magnetic field. We note that, by lemma 2.2, the E field can be computed from the H field (and vice versa). The reason why we choose to work with the electric field is because of the symmetry property of the electric dipole (2.40), that is,

$$E_e^i(x; z, p) = E_e^i(z; x, p) \quad \text{for all } x \neq z$$

which is necessary to prove a mixed electromagnetic reciprocity relation that we state in the following lemma. The problem for the electric field states: determine $E : \mathbb{R}^3 \setminus D \to \mathbb{C}^3$, $E = E^s + E^i$ such that

$$\begin{aligned}
&\operatorname{curl}\left(\frac{1}{\mu_r}\operatorname{curl} E\right) - k^2 E = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \\
&E \times \nu = 0 \quad \text{on } \partial D, \\
&E^s \text{ satisfies the Silver-Müller radiation condition.}
\end{aligned}$$

Lemma 2.11. *Let $E_{e,\infty}$ denote the electric far field pattern in case of an incident electric dipole with source point $z \in \mathbb{R}^3 \setminus \overline{D}$ and E^s the scattered wave corresponding to an incident plane wave. Then following relations holds*

$$p \cdot E_{e,\infty}(\hat{x}, z, q) = \frac{1}{4\pi} q \cdot E^s(z, -\hat{x}, p) \tag{2.57}$$

for all $z \in \mathbb{R}^3 \setminus \overline{D}$, for all incident directions $\hat{x} \in S^2$ and all polarizations $p \perp \hat{x}$ and $q \perp \hat{x}$.

Proof. We remove the electric dipole $E_e^i(\cdot; x, q)$ at $x = \hat{x}|x|$, $\hat{x} \in S^2$, to infinity in the direction of $\hat{x}$. Then for $z \in \mathbb{R}^3 \setminus \overline{D}$ fixed we obtain:

$$
\begin{aligned}
p \cdot E_e^i(z; x, q) &= p \cdot \left(-\frac{1}{ik} \text{curl}_z \text{curl}_z \left[q \Phi_k(z, x) \right] \right) \\
&= -\frac{1}{ik} p \cdot \text{curl}_z \text{curl}_z \left[q \Phi_k(x, z) \right] \\
&= -\frac{1}{ik} p \cdot \text{curl}_z \text{curl}_z q \left(\frac{e^{ik|x|}}{4\pi|x|} e^{-ik\hat{x}\cdot z} + \mathcal{O}\left(\frac{1}{|x|} \right) \right) \\
&= -\frac{1}{ik} \frac{e^{ik|x|}}{4\pi|x|} p \cdot \text{curl}_z \left(-ik[\hat{x} \times q] e^{-ik\hat{x}\cdot z} \right) + \mathcal{O}\left(\frac{1}{|x|} \right) \\
&= \frac{e^{ik|x|}}{4\pi|x|} p \cdot \left(\nabla_z e^{-ik\hat{x}\cdot z} \times [\hat{x} \times q] \right) + \mathcal{O}\left(\frac{1}{|x|} \right) \\
&= -ik \frac{e^{ik|x|}}{4\pi|x|} p \cdot (\hat{x} \times [\hat{x} \times q]) \, e^{-ik\hat{x}\cdot z} + \mathcal{O}\left(\frac{1}{|x|} \right) \\
&= -ik \frac{e^{ik|x|}}{4\pi|x|} q \cdot (-[\hat{x} \times p] \times \hat{x}) \, e^{-ik\hat{x}\cdot z} + \mathcal{O}\left(\frac{1}{|x|} \right) \\
&= \frac{e^{ik|x|}}{4\pi|x|} q \cdot (ik[(-\hat{x}) \times p] \times (-\hat{x})) \, e^{-ik\hat{x}\cdot z} + \mathcal{O}\left(\frac{1}{|x|} \right) \\
&= \frac{e^{ik|x|}}{4\pi|x|} q \cdot E^i(z, -\hat{x}, p) + \mathcal{O}\left(\frac{1}{|x|} \right), \quad |x| \to \infty,
\end{aligned}
$$

that is,

$$
p \cdot E_e^i(x; z, q) = \frac{e^{ik|x|}}{4\pi|x|} q \cdot E^i(z, -\hat{x}, p) + \mathcal{O}\left(\frac{1}{|x|} \right), \quad |x| \to \infty
$$

where we used the symmetry property (2.40) of the electric dipole. In other words, when removing the electric dipole to infinity, it acts like a plane wave, this implies

$$
p \cdot E_e^s(x; z, q) = \frac{e^{ik|x|}}{4\pi|x|} q \cdot E^s(z, -\hat{x}, p) + \mathcal{O}\left(\frac{1}{|x|} \right), \quad |x| \to \infty. \tag{2.58}
$$

Substituting

$$
p \cdot E_e^s(x; z, q) = \frac{e^{ik|x|}}{4\pi|x|} \left\{ p \cdot E_{e,\infty}(\hat{x}; z, q) + \mathcal{O}\left(\frac{1}{|x|} \right) \right\}
$$

into (2.58) yields the mixed reciprocity relation (2.57). □

Now we are in a position to prove our first result.

Theorem 2.12. *Let G be the unboudned component of $\mathbb{R}^3 \setminus (\overline{D_1} \cup \overline{D_3})$ and $E_{1,\infty}(\hat{x}, d, p) = E_{2,\infty}(\hat{x}, d, p)$ for all $\hat{x}, d \in S^2$ and all $p \perp d$. Let $z \in G$ and $E_{e,j}(\cdot; z) = E_{e,j}$, $j = 1, 2$ be the* *unique solution of*

$$E_{e,j} = -\frac{1}{ik}\operatorname{curl}\operatorname{curl} p\Phi_k(x,z) + E^s_{e,j} \quad \textit{in } \mathbb{R}^3 \setminus \overline{D}_j, \tag{2.59}$$

$$\operatorname{curl}\left(\frac{1}{\mu_r}\operatorname{curl} E_{e,j}\right) - k^2 E_{e,j} = 0 \quad \textit{in } \mathbb{R}^3 \setminus \overline{D}_j, \tag{2.60}$$

$$\frac{1}{\mu_r}\operatorname{curl} E_{e,j} \times \nu = 0 \quad \textit{on } \partial D_j, \tag{2.61}$$

$$E^s_{e,j} \textit{ satisfies the Silver-Müller radiation condition} \tag{2.62}$$

in the variational sense. Then $E^s_{e,1}(x;z) = E^s_{e,2}(x;z)$ *for all* $x \in \overline{G}$, $x \neq z$.

Proof. By Rellich's lemma, from the coincidence of the far field patterns for plane wave incidence, it follows that the corresponding scattered waves satisfy $E^s_1(\cdot,d,p) = E^s_2(\cdot,d,p)$ for all $d \in S^2$ and all $p \perp d$ in the exterior of the ball B_R of radius $R > 0$, where R is choosen large enough such that $\overline{D}_1 \cup \overline{D}_2 \subset B_R$ and $\mu_r \equiv 1$ outside B_R. By lemma 2.2, it also holds that $H^s_1(\cdot,d,p) = H^s_2(\cdot,d,p)$ for all $d \in S^2$ and all $p \perp d$ in $\mathbb{R}^3 \setminus \overline{B}_R$. Applying the unique continuation principle to $H^s = H^s_1 - H^s_2$ in a similar way as was done in the proof of theorem 2.4, we conclude that $H^s_1(\cdot,d,p) = H^s_2(\cdot,d,p)$ in G for all $d \in S^2$ and all $p \perp d$, which by lemma 2.2 implies

$$E^s_1(\cdot,d,p) = E^s_2(\cdot,d,p) \quad \text{in } G$$

for all $d \in S^2$ and all $p \perp d$. Now, from the mixed reciprocity relation (2.57) for scattering of electric dipole fields, we conclude that

$$E_{e,1,\infty}(\cdot;z,q) = E_{e,2,\infty}(\cdot;z,q) \quad \text{on } S^2,$$

for $z \in G$ and all polarizations q. Again by Rellich's lemma, this implies that the corresponding scattered waves coincide $E^s_{e,1}(x;z,q) = E^s_{e,2}(x;z,q)$ for all $x \in \mathbb{R}^3 \setminus \overline{B}_R$, $z \in G$ and all polarizations q. With the unique continuation principle applied as in the proof of theorem 2.4, we conclude that

$$E^s_{e,1}(x;z,q) = E^s_{e,2}(x;z,q) \quad \text{in } G$$

for all $z \in G$ and all polarizations q. □

Finally we prove uniqueness of the inverse problem.

Theorem 2.13. *Let $E^s(\cdot, d, p)$ and $E_\infty(\cdot, d, p)$ be the scattered wave and far field pattern, respectively, corresponding to the plane wave $E^i(x, d, p) = ik(d \times p) \times de^{ikx\cdot d}$, $x \in \mathbb{R}^3$ with propagation direction $d \in S^2$ and polarization $p \perp d$. If the far field patterns $E_{\infty,1}(\hat{x}, d, p)$ and $E_{\infty,2}(\hat{x}, d, p)$ for the obstacles D_1 and D_2 coincide for all incident directions d, all polarizations $p \perp d$ and all observations $\hat{x}$, then $D_1 = D_2$.*

Proof. We prove the claim by contradiction and assume that $D_1 \neq D_2$. Then without loss of generality, there exists $z^* \in \partial G$ such that $z^* \in \partial D_1$ and $z^* \notin \overline{D}_2$. We can choose $h > 0$ small enough such that the sequence

$$z_n := z^* + \frac{h}{n}\nu(z^*), \quad n = 1, 2, 3, ...$$

is contained in G, where $\nu(z^*)$ is the outward normal vector to ∂D_1 at z^*. Consider the (variational) solution $E^s_{e,n,j}$, $j = 1, 2$ to the boundary value problem (2.59)-(2.62) with z replaced by z_n. By theorem 2.12, it holds that $E^s_{e,n,1} = E^s_{e,n,2}$ in $\overline{G}$. We denote by $E^i_n(\cdot; z_n) = E^i_n(\cdot; z_n, p) = -\frac{1}{ik}\operatorname{curl}\operatorname{curl} p\Phi_k(\cdot, z_n)$ the incident electric dipole with source point z_n and polarization p.

Consider $E^s_{e,n} = E^s_{e,n,2}$ as the scattered field corresponding to D_2. In view of the well-posedness of the direct scattering problem for the scatterer D_2, since z^* has positive distance from $\overline{D}_2$, we obtain on one hand that

$$\|E^s_{e,n}(\cdot; z_n)\|_{H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3 \setminus \overline{D}_2)} < \infty \quad \text{for sufficiently large } n \in \mathbb{N}.$$

On the other hand, considering $E^s_{e,n} = E^s_{e,n,1}$ as the scattered field corresponding to D_1 we obtain, due to the singular behavior of $\Phi_k(\cdot, z^*)$, that

$$\|\nu \times E^s_{e,n}(\cdot; z_n)\|_{\partial D_1} = \|\nu \times E^i_n(\cdot; z_n)\|_{\partial D_1} \to \infty \quad \text{as } n \to \infty.$$

Hence

$$\|E^s(\cdot; z_n)\|_{H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3 \setminus \overline{D}_1)} \geq \widetilde{c}\|\nu \times E^s_{e,n}(\cdot; z_n)\|_{\partial D_1} \to \infty \quad \text{as } n \to \infty.$$

This is a contradiction, and thus we conclude $D_1 = D_2$. □

We will end this section by showing in the case of a homogeneous medium, that is, $\mu_r = 1$, where the scatterer is a ball, the uniqueness of the inverse problem can be proven from the knowledge of the far field pattern for the scattering of one incident plane wave only. The following result is from [14].

The solution of the scattering problem for a ball in a homegenous medium, has an explicit solution in terms of radiating spherical vector wave functions; see [1], [2]. This implies that the scattered field can be extended across the boundary of the ball into the interior, with the exception of the center of the ball. Consequently, if two different scattering balls have far field patterns that coincide for one incident plane wave, they must have the same center, since otherwise the scattered wave would be an entire solution to the Maxwell's equation. By symmetry, the electric far field pattern for the scattering of plane waves at a ball centered at the origin satisfies $E_\infty(R\hat{x}, Rd, Rp) = RE_\infty(\hat{x}, d, p)$ for all $\hat{x}, d \in S^2$, all $p \perp d$, and all rotations R (that is, for all orthogonal transformations with $\det R = 1$). Hence, knowledge of the far field pattern for one incident direction and polarization implies knowledge of the far field pattern for all incident directions and polarizations. Now the uniqueness of the inverse problem follows from theorem 2.12.

3 Scattering of electromagnetic waves with conductive transmission condition

In this section, we consider the scattering of time-harmonic electromagnetic waves in case of a penetrable obstacle. We assume the scatterer to be an inhomogeneous medium surrounded by a homogeneous setting and on the boundary of the scatterer we assume conductive transmission conditions. These transmission conditions model the occurrence of a thin layer of very high conductivity for, while the electric field does not penetrate into an ideal conductor of positive thickness, such a field certainly will penetrate into the medium beyond that conductor if the latter is infinitely thin. We will briefly mention the motivation of considering obstacles surrounded by a thin layer of very high conductivity and the advantage of approximating this kind of model via conductive transmission conditions.

In many practical applications, electronic devices are surrounded by casings or other layers of a highly conductive material to protect them from external electromagnetic fields (e.g., data cables) or to protect the environment from the electromagnetic fields generated by devices. To minimize the cost, size and weight, these layers have to be thin. This leads to a non-perfect shielding where the electromagnetic fields partly penetrate the shields and, e.g., external fields have a small but significant effect on the encased electronic devices. The large ratio of characteristic lengths (width of the device against thickness of the layer) leads to serious numerical problems. Even thought we are not going to consider any numerical results in this work, we would briefly like to point out where some of the main problems occur. The classical numerical methods such as finite differences or finite elements require a small mesh size. As the layers have to be solved by the mesh in thickness direction, the number of cells in the mesh increase with the decreasing layer thickness. The numerical modelling is much simplified if the thin conducting layers are replaced by transmission conditions on an interface, which is usually its mid-surface. Using conductive transmission conditions, which relate the electric and magnetic fields on both sides of the interface, meshes with much larger cells can be used. Providing an accurate prediction of the electromagnetic fields, those transmission conditions are called equivalent. Moreover, there are several different ways of deriving equivalent transmission conditions, e.g., with the scaled asymptotic expansions technique it can be shown that the model with the conductive transmission conditions can be used as a first order approximation for the full model; see [23] for the two-dimensional configuration and [21], [22] for the three-dimensional.

3.1 Problem statement

Let $D \subset \mathbb{R}^3$ be a bounded domain. We assume that D is surrounded by a homogeneous medium with constant electric permittivity ε_0 and constant magnetic permeability μ_0. Inside D, the electric permittivity ε, magnetic permeability μ and conductivity σ are assumed to be scalar functions. By ν we denote the unit outward normal to ∂D.

We consider the scattering of an incident time-harmonic electromagnetic wave E^i, H^i satisfying $\operatorname{curl} E^i - i\omega\mu_0 H^i = 0$ and $\operatorname{curl} H^i + i\omega\varepsilon_0 E^i = 0$ in all of $\mathbb{R}^3$. Inside D, the field E, H satisfies

$$\operatorname{curl} E - i\omega\mu H = 0, \qquad \text{in } D, \tag{3.1}$$

$$\operatorname{curl} H + (i\omega\varepsilon - \sigma)E = 0 \qquad \text{in } D \tag{3.2}$$

while outside D it satsfies

$$\operatorname{curl} E - i\omega\mu_0 H = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{D}, \tag{3.3}$$

$$\operatorname{curl} H + i\omega\varepsilon_0 E = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{D}. \tag{3.4}$$

Moreover, the total (exterior) field consists of the sum of incident and scattered fields

$$E = E^s + E^i, \quad H = H^s + H^i,$$

where E^s, H^s is an outgoing wave satisfying the Silver-Müller radiation condition

$$\lim_{|x|\to\infty} (H^s \times x - |x| E^s) = 0, \tag{3.5}$$

uniformly with respect to all directions $\hat{x} = x/|x|$. On the boundary ∂D, we have the conductive transmission conditions given by

$$\nu \times E|_+ - \nu \times E|_- = 0 \qquad \text{on } \partial D, \tag{3.6}$$

$$\nu \times H|_+ - \nu \times H|_- - \beta\nu \times (E \times \nu) = 0 \qquad \text{on } \partial D \tag{3.7}$$

where the subscripts "+" and "−" denote the traces from the outside and inside of D, respectively, and β is a strictly positive, real-valued function of position on ∂D.

We will present two different ways to show well-posedness of the direct problem. The first one is the integral equation method that we introduced in the previous section. Here we have to be careful with the spaces due to the transmission condition (3.7). As a first choice we seek a solution E, H in the space $H(\mathrm{curl}\,, D) \cup H_{\mathrm{loc}}(\mathrm{curl}\,, \mathbb{R}^3 \setminus \overline{D})$. But for a Lipschitz domain D, the traces $\nu \times H|_{\pm}$ belong to $H^{-\frac{1}{2}}(\mathrm{Div}\,, \partial D)$ while the trace $E_T := \nu \times (E \times \nu)$ belongs to $H^{-\frac{1}{2}}(\mathrm{Curl}\,, \partial D)$, thus we encounter trouble with the transmission condition (3.7). To overcome this problem we will need to assume more smoothness on the domain and data. Moreover, we will seek a solution E, H in the space $H^1(D)^3 \cup H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{D})^3$ that will allow us to consider the transmission condition (3.7) in $H^{\frac{1}{2}}(\partial D)^3$.

Here we will argue differently than for the previous scattering problem. We start by considering the case when the electric permitivity is constant and make an ansatz for E, H that solves Maxwell's system (3.1)-(3.4) via integral equations, in particular, boundary integrals, with unknown densities. Requiring the traces of the ansatz to satisfy the transmission conditions (3.6)-(3.7) will lead us to a system. To obtain a compactness result, we will consider the boundary intgeral operators appearing in the system, in particular, their traces, for different wavenumbers.

The second method we will apply is the variational approach. The idea is to derive an equivalent variational formulation of the full scattering problem on a bounded subdomain B_R. The formulation uses the electromagnetic analogue of the Dirichlet-to-Neumann map called the electric-to-magnetic Calderon operator described in section 3.3.2. Generally speaking, the electric-to-magnetic Calderon operator is used to replace the radiating conditions with transmission conditions on the artificial boundary ∂B_R, thereby enabling us to truncate the problem to the subdomain B_R. The advantage of using the variational method is, that we will be able to show well-posedness of the direct problem for Lipschitz domains and require less regularity on the data compared to the integral equation method. The general approach of using a Calderon operator in a non-local boundary condition for a weak formulation of unbounded electromagnetic scattering problems for bounded obstacles was introduced by A. Kirsch and P. Monk [15] in 1995.

Coercivity and compactness are not only important properties for the integral equation method, but also for the analysis of variational formulations. Applying the Lax Milgram theorem to obtain the existence of a unique solution is of course favorable whenever possible. Unfortunately, in electromagnetic medium scattering where the medium is inhomogeneous, the variational formulation fails to be coercive. Moreover, the solution space fails to be compactly imbedded in the space of all square integrable vector functions, thus making it difficult to write the variational formulation as the sum 'coercive + compact'. To overcome this problem, we will use decomposition of vector fields, that is, a Helmholtz decomposition [1].

In the following, we will only work with the electric field. We recall the relative permittivity and relative permeability given by

$$\varepsilon_r(x) = \frac{\varepsilon(x)}{\varepsilon_0} + i\frac{\sigma(x)}{\omega\varepsilon_0}, \quad \mu_r(x) = \frac{\mu(x)}{\mu_0}, \quad x \in D$$

respectively. Eliminating H from equation (3.1) and (3.3), and substituting into (3.2) and (3.4) yields the problem of determining $E : \mathbb{R}^3 \to \mathbb{C}^3$ such that

$$\operatorname{curl}\left(\frac{1}{\mu_r}\operatorname{curl} E\right) - k^2\varepsilon_r E = 0 \qquad \text{in } D, \tag{3.8}$$

$$\operatorname{curl}\operatorname{curl} E - k^2 E = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{D}, \tag{3.9}$$

$$E = E^i + E^s \qquad \text{in } \mathbb{R}^3 \setminus \overline{D}, \tag{3.10}$$

$$\nu \times E|_+ - \nu \times E|_- = 0 \qquad \text{on } \partial D, \tag{3.11}$$

$$\nu \times \operatorname{curl} E|_+ - \nu\frac{1}{\mu_r} \times \operatorname{curl} E|_- - \lambda E_T = 0 \qquad \text{on } \partial D, \tag{3.12}$$

$$\lim_{|x|\to\infty} |x| \left(\operatorname{curl} E^s \times \nu - ikE^s\right) = 0, \tag{3.13}$$

where $E_T := \nu \times (E \times \nu)$ and $\lambda := i\omega\mu_0\beta$.

3.2 Well-posedness of the direct problem via the integral equation method

Let $D \subset \mathbb{R}^3$ be a bounded $C^{2,1}$-smooth domain with connected boundary ∂D such that the complement $\mathbb{R}^3 \setminus \overline{D}$ is connected. By a $C^{2,1}$-smooth domain, we mean a domain that satisfies definition 1.1 where the function g, which describs the boundary locally, is two-times continuously differentiable such that each derivative is Hölder continuous with exponent 1. We will consider the case when $\mu \equiv \mu_0$ and $\beta \in \mathbb{R}$, $\beta > 0$ are constants, in particular, $\lambda = i\omega\mu_0\beta$ is constant with $\text{Im}(\lambda) > 0$. Define

$$X(D) = H^1(D)^3 \cap H(\text{curl}^2, D),$$
$$X(\mathbb{R}^3 \setminus \overline{D}) = H^1_{\text{loc}}(\mathbb{R}^3 \setminus \overline{D})^3 \cap H_{\text{loc}}(\text{curl}^2, \mathbb{R}^3 \setminus \overline{D})$$

where

$$H(\text{curl}^2, D) = \{U \in L^2(D)^3 : \text{curl}^2 U \in L^2(D)^3\},$$
$$H_{\text{loc}}(\text{curl}^2, \mathbb{R}^3 \setminus \overline{D}) = \{U \in L^2_{\text{loc}}(\mathbb{R}^3 \setminus \overline{D})^3 : \text{curl}^2 U \in L^2_{\text{loc}}(\mathbb{R}^3 \setminus \overline{D})^3\}.$$

We seek a solution E in the space $X(D) \cup X(\mathbb{R}^3 \setminus \overline{D})$ and consider transmission conditions (3.11)-(3.12) in $H^{\frac{1}{2}}(\partial D)^3$.

We note that if $E|_D \in X(D)$, $E|_{\mathbb{R}^3 \setminus \overline{D}} \in X(\mathbb{R}^3 \setminus \overline{D})$ solve (3.8)-(3.13), then it is easily verified that E and $H = \frac{1}{i\omega\mu_0} \text{curl}\, E$ solve Maxwell's system (3.1)-(3.7). Moreover, since $\text{div}\, H = 0$ and $\nu \times H|_+ - \nu \times H|_- \in H^{\frac{1}{2}}(\partial D)^3$, we conclude that $H \in H^1(D)^3 \cup H^1_{\text{loc}}(\mathbb{R}^3 \setminus \overline{D})$. Thus problem (3.1)-(3.7) is equivalent to (3.8)-(3.13), and the problem we will study reads:

Problem statement (P2):

Determine $E \in X(D) \cup X(\mathbb{R}^3 \setminus \overline{D})$, which satisfies (3.8)-(3.13).

Next we summerize the assumptions on the data and prove uniqueness.

Assumption 3.1. *We assume that:*

- $D \subset \mathbb{R}^3$ *is a bounded* $C^{2,1}$*-smooth domain with connected exterior* $\mathbb{R}^3 \setminus \overline{D}$.
- $\lambda = i\omega\mu_0\beta \in \mathbb{C}$ *with* $\beta > 0$ *and* $\mu = \mu_0 \in \mathbb{R}$ *constants.*
- $k = \omega\sqrt{\varepsilon_0\mu_0} > 0$.
- $\varepsilon_r \in W^{2,\infty}(D)$ *with* $\text{Im}(\varepsilon_r) \geq 0$ *and such that there exists a* $c_0 >$ *with* $\text{Re}(\varepsilon_r) > c_0$. *Then in particular* $\frac{1}{\varepsilon_r} \in L^\infty(D)$.

We note that by definition $\varepsilon_r = 1$ outside D.

Theorem 3.2. *There exists at most one solution to problem (P2).*

Proof. Let $E^i = 0$, i.e. E satisfies radiation condition (3.13). Let B_R be a ball of radius $R > 0$ such that $\overline{D} \subset B_R$ and set $\Omega_R = B_R \setminus \overline{D}$. As noted in remark 2.1, the solution E

is smooth outside D, and thus $E_T = \nu \times (E \times \nu) = E$ on $\partial B_R = \{x \in \mathbb{R}^3 : |x| = R\}$. Multiplying (3.8) by $\overline{E}$ and using Green's theorem (1.11) in D yields

$$\iint_D \left(|\operatorname{curl} E|^2 - k^2 \varepsilon_r |E|^2\right) dx + \langle \nu \times \operatorname{curl} E, E_T \rangle_{\partial D} = 0. \tag{3.14}$$

Similarly, multiplying (3.9) by $\overline{E}$ and using Green's theorem (1.11) in $\Omega_R = B_R \setminus \overline{D}$ yields

$$\iint_{\Omega_R} \left(|\operatorname{curl} E|^2 - k^2 |E|^2\right) dx - \langle \nu \times \operatorname{curl} E, E_T \rangle_{\partial D} + \underbrace{\langle \nu \times \operatorname{curl} E, E \rangle_{|x|=R}}_{= -\langle \operatorname{curl} E \times \nu, E \rangle_{|x|=R}} = 0. \tag{3.15}$$

By adding (3.14) to (3.15) and using transmission condition (3.7), we obtain

$$\begin{aligned} \langle \operatorname{curl} E \times \nu, E \rangle_{|x|=R} = &- i\omega\mu_0 \int_{\partial D} \beta |E_T|^2 \, ds + \iint_{B_R} |\operatorname{curl} E|^2 \, dx \\ &- k^2 \iint_D \varepsilon_r |E|^2 \, dx - k^2 \iint_{\Omega_R} |E|^2 \, dx. \end{aligned}$$

Moreover,

$$\operatorname{Im} \langle \operatorname{curl} E \times \nu, E \rangle_{|x|=R} \leq 0. \tag{3.16}$$

From the radiation condition (3.13), we obtain

$$\begin{aligned} 0 &\overset{R\to\infty}{\longleftarrow} \int_{|x|=R} |\operatorname{curl} E \times \nu - ikE|^2 \, ds \\ &= \int_{|x|=R} \left(|\operatorname{curl} E \times \nu|^2 + |E|^2\right) ds - 2\operatorname{Im} \int_{|x|=R} (\operatorname{curl} E \times \nu) \cdot \overline{E} \, ds \\ &\overset{(3.16)}{\geq} \int_{|x|=R} \left(|\operatorname{curl} E \times \nu|^2 + |E|^2\right) ds. \end{aligned}$$

Rellich's lemma 1.9 now implies that $E = 0$ outside B_R. Outside D the solution is smooth and analytic, hence $E = 0$ outside D. Now, taking the divergence of (3.8) yields $\operatorname{div}(\varepsilon_r E) = 0$ in D, that is, $\operatorname{div} E = -\frac{1}{\varepsilon_r} E \cdot \nabla \varepsilon_r = -E \cdot \nabla \ln(\varepsilon_r)$. Using the vector identity $\operatorname{curl}^2 =$

$\Delta + \nabla\text{div}$ and substituting for $\operatorname{div} E$, we obtain

$$\begin{aligned}\Delta E &= \nabla\text{div}\, E - k^2\varepsilon_r E \\ &= -\nabla\left(E\cdot\nabla\ln(\varepsilon_r)\right) - k^2\varepsilon_r E \\ &= -\sum_{l=1}^{3}\left(E_l\nabla\frac{\partial\ln(\varepsilon_r)}{\partial x_l} + \nabla E_l\frac{\partial\ln(\varepsilon_r)}{\partial x_l}\right) - k^2\varepsilon_r E.\end{aligned}$$

Here we argued classically and next we derive this formula in the variational sense. We multiply equation (3.8) by a test function $\phi \in C_0^\infty(D)^3$ and integrate over D, this yields:

$$\begin{aligned}0 &= \iiint_D \left(\text{curl}^2 E - k^2\varepsilon_r E\right)\cdot\phi\, dx \\ &\overset{(1.11)}{=} \iiint_D \left(E\cdot\text{curl}^2\phi - k^2\varepsilon_r E\cdot\phi\right)\, dx \\ &= \iiint_D \left(E\Delta\phi + E\cdot\nabla\text{div}\,\phi - k^2\varepsilon_r E\cdot\phi\right)\, dx \\ &\overset{(1.7)}{=} \iiint_D \left(E\Delta\phi + \text{div}\, E\text{div}\,\phi - k^2\varepsilon_r E\cdot\phi\right)\, dx \\ &= \iiint_D \left(E\Delta\phi - \left(E\cdot\nabla\ln(\varepsilon_r)\right)\text{div}\,\phi - k^2\varepsilon_r E\cdot\phi\right)\, dx \\ &\overset{(1.7)}{=} \iiint_D \left(E\Delta\phi - \nabla\left(E\cdot\nabla\ln(\varepsilon_r)\right)\cdot\phi - k^2\varepsilon_r E\cdot\phi\right)\, dx\end{aligned}$$

that is,

$$\iiint_D E\Delta\phi\, dx = \iiint_D \left(\nabla\left(E\cdot\nabla\ln(\varepsilon_r)\right) - k^2\varepsilon_r E\right)\cdot\phi\, dx$$

where we used that $\phi = 0$ on ∂D, $\operatorname{div}\phi \in C_0^\infty(D)$ and $\operatorname{curl}\phi \in C_0^\infty(D)^3$. The above identity holds for all $\phi \in C_0^\infty(D)^3$. By the interior regularity property 1.10, we conclude that $E \in H^2(U)^3$, where U is an open set with $\widetilde{U} \subseteq D$. Now we can argue as we did in the uniqueness result of theorem 2.4 and apply the unique continuation principle 1.11 to conclude that $E = 0$ in $\mathbb{R}^3$. □

Before we make an ansatz for a solution, we recall several integral operators and their mapping properties for smooth domains.

Surface potentials. We re-introduce the single layer potential from (2.21):

$$\left(\widetilde{S}_k a\right)(x) = \int_{\partial D} a(y)\Phi_k(x,y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D$$

and denote its restriction to the boundary ∂D by S_k, that is

$$\big(S_k a\big)(x) = \int_{\partial D} a(y)\Phi_k(x,y)\,ds(y), \quad x \in \partial D.$$

It is known that (see e.g., [18], [25])

$$\widetilde{S}_k : H^{s-\frac{1}{2}}(\partial D)^3 \to H^{s+1}(D)^3 \cup H^{s+1}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{D})^3$$
$$S_k : H^{s-\frac{1}{2}}(\partial D)^3 \to H^{s+\frac{1}{2}}(\partial D)^3$$

are bounded for $-1 \leq s \leq 2$. Furthermore, $U = \widetilde{S}_k a$ satisfies the vector Helmholtz equation $\Delta U + k^2 U = 0$ in $\mathbb{R}^3 \setminus \partial D$ and since each component $U_j,\ j = 1,2,3$ satisfies the Sommerfeld radiation condition, U satisfies the Silver-Müller radiation condition (this follows from theorem 6.7 in [6]). On ∂D, we have

$$\nu \times \widetilde{S}_k|_{\partial D} = \nu \times S_k. \tag{3.17}$$

Next we define the space

$$H^s_t(\partial D) := \{a \in H^s(\partial D) : \nu \cdot a = 0\}$$

of tangential fields and re-introduce the bounded operators from (2.22), (2.23):

$$\widetilde{M}_k = \operatorname{curl} \widetilde{S}_k : H_t^{s-\frac{1}{2}}(\partial D)^3 \to H^s(D)^3 \cap H^s_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{D})^3, \quad 0 \leq s \leq 3,$$
$$\widetilde{N}_k = \operatorname{curl}^2 \widetilde{S}_k : H_t^{s-\frac{1}{2}}(\partial D)^3 \to H^{s-1}(D)^3 \cap H^{s-1}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{D})^3, \quad 1 \leq s \leq 2$$

where the mapping properties follow from the mapping property of the single layer potential $\widetilde{S}_k$. We note that

$$\operatorname{curl} \widetilde{M}_k = \widetilde{N}_k, \quad \operatorname{curl} \widetilde{N}_k = k^2 \widetilde{M}_k.$$

The restrictions of $\widetilde{M}_k$ and $\widetilde{N}_k$ to the boundary ∂D will be denoted by M_k and N_k, respectively. Then, for a $C^{2,1}$-smooth boundary ∂D,

$$M_k : H_t^{s-\frac{1}{2}}(\partial D)^3 \to H_t^{s-\frac{1}{2}}(\partial D)^3$$

is compact and

$$N_k : H_t^{s-\frac{1}{2}}(\partial D)^3 \to H_t^{s-\frac{3}{2}}(\partial D)^3$$

is bounded, for $0 \le s \le 2$ (see [6], [25]). Moreover, the following jump condition hold for $a \in H_t^{s-\frac{1}{2}}(\partial D)$, $0 \le s \le 2$ (corresponding to lemma 2.6):

$$\nu \times \widetilde{M}_k a|_{|\pm} = M_k a \mp \frac{1}{2} a \qquad \in H_t^{s-\frac{1}{2}}(\partial D)^3, \tag{3.18}$$

$$\nu \times \widetilde{N}_k a|_{|\pm} = N_k a \qquad \in H_t^{s-\frac{3}{2}}(\partial D)^3. \tag{3.19}$$

Lastly, we define the double layer potential with density $\phi \in H^{\frac{1}{2}}(\partial D)$ by

$$\left(\overline{D}_k \phi\right)(x) = \int_{\partial D} \phi(y) \frac{\partial \Phi_k}{\partial \nu(y)}(x, y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D$$

and denote its restriction to the boundary ∂D by D_k. By theorem 5.46 in [1], $\overline{D}_k$ is a bounded map into $H^1(D)$ and into $H^1_{\text{loc}}(\mathbb{R}^3 \setminus \overline{D})$. Furthermore, $u = \overline{D}_k \phi$ satisfies the Helmholtz equation $\Delta u + k^2 u = 0$ in $\mathbb{R}^3 \setminus \partial D$ and the Sommerfeld radiation condition. Moreover, $D : H^{\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is bounded and the following jump condition holds

$$\nu \times \overline{D}_k \phi|_{\pm} = \pm \frac{1}{2} \phi + D\phi. \tag{3.20}$$

We end this section by stating one very important property that will be useful in the next section.

Lemma 3.3. *Let Φ_1 and Φ_2 denote the fundamental solution for the wave numbers k_1 and k_2, respectively. The operators*

$$S_1 - S_2 : H^{s-\frac{1}{2}}(\partial D)^3 \to H^{s+\frac{1}{2}}(\partial D)^3, \quad -1 \le s \le 2,$$
$$M_1 - M_2 : H_t^{s-\frac{1}{2}}(\partial D)^3 \to H_t^{s-\frac{1}{2}}(\partial D)^3, \quad 0 \le s \le 2,$$
$$N_1 - N_2 : H_t^{s-\frac{1}{2}}(\partial D)^3 \to H_t^{s-\frac{3}{2}}(\partial D)^3, \quad 0 \le s \le 2$$

are compact.

Proof. Let Φ_i be the fundamental solution for the special case $k = i$. By theorem 2.9, the kernels $\Phi_l - \Phi_i$, $\nabla_x(\Phi_l - \Phi_i)$ and $\nabla_{xx}(\Phi_l - \Phi_i)$, $l = 1, 2$ are weakly singular, where $\nabla_{xx} = \left(\frac{\partial^2}{\partial x_1 \partial x_1}, \frac{\partial^2}{\partial x_2 \partial x_2}, \frac{\partial^2}{\partial x_3 \partial x_3}\right)^T$. Thus writing $\Phi_1 - \Phi_2 = (\Phi_1 - \Phi_i) - (\Phi_2 - \Phi_i)$ proves the claim. ☐

3.2.1 Integral equation method for constant electric permitivity

Let $\varepsilon_r > 0$ be a constant such that $\varepsilon_r \neq 1$. In the following, we set

$$k_1 := k\sqrt{\varepsilon_r}.$$

Ansatz for a solution. We make an ansatz motivated from the exterior impedance boundary value problem considered in [6] for Hölder continuous densities. Generally speaking, the impedance problem can be considered as the "exterior conductive transmission condition problem". We set

$$E = \widetilde{S}_{k_1}(a) + \widetilde{M}_{k_1}(b) + \nabla\widetilde{S}_{k_1}(\phi - \phi_1) + \lambda\widetilde{S}_{k_1}(\phi_1\nu) \quad \text{in } D, \tag{3.21}$$

$$E = E^i + \widetilde{S}_k(a) + \widetilde{M}_k(b) + \nabla\widetilde{S}_k(\phi - \phi_1) + \lambda\widetilde{S}_k(\phi\nu) \quad \text{in } \mathbb{R}^3 \setminus \overline{D} \tag{3.22}$$

for unknown densities $a \in L^2(\partial D)^3$, $b \in H_t^{\frac{1}{2}}(\partial D)^3$ and $\phi, \phi_1 \in L^2(\partial D)$. W.l.o.g. we write $E = E|_D$ and $E^s = E|_{\mathbb{R}^3\setminus\overline{D}} = E - E^i$. We note that $\widetilde{M}_{k_1}$ and $\widetilde{M}_k$ satisfiy (3.8) and (3.9), respectively. Moreover, the single layer potential and the gradient of the single layer potential satisfy the vector Helmholtz equation in D and in $\mathbb{R}^3\setminus\overline{D}$ and the cartesian components of E^s satisfy the Sommerfeld radiation condition. Using the identity $\operatorname{curl}^2 = -\Delta + \nabla\operatorname{div}$ and insisting that $\operatorname{div} E = 0$ in D and $\operatorname{div} E^s = 0$ in $\mathbb{R}^3\setminus\overline{D}$, the ansatz (3.21)-(3.22) satisfies Maxwell's system (3.8)-(3.9) and E^s satisfies the Silver-Müller radiation condition. Since $\operatorname{div} E^s$ satisfies the scalar Helmholtz equation and the Sommerfeld radiation condition, by the uniqueness for the exterior Dirichlet problem it suffices to impose $\operatorname{div} E^s = 0$ only on ∂D. Under the assumption that k_1^2 is not a Dirichlet eigenvalue we similarly can deduce, by the uniqueness of the interior Dirichlet problem, that it suffices to impose $\operatorname{div} E^s = 0$ only on ∂D. Thus, under the assumption on k_1^2, ansatz (3.21)-(3.22) solves problem $(P2)$ if we can determine a, b, ϕ_1 and ϕ such that transmission conditions (3.11)-(3.12) are satisfied and $\operatorname{div} E = \operatorname{div} E^s = 0$ on ∂D.

Next we consider the traces of (3.21)-(3.22). Using jump conditions (3.17) and (3.18)-(3.19) we obtain for $x \in \partial D$:

$$\begin{aligned}
\nu(x) \times E(x) &= \nu(x) \times (S_{k_1} a)(x) + (M_{k_1} b)(x) + \frac{1}{2} b(x) \\
&\quad + \nu(x) \times \int_{\partial D} [\phi(y) - \phi_1(y)] \nabla_x \Phi_{k_1}(x,y)\, ds + \lambda S_{k_1}(\phi_1 \nu), \\
\nu(x) \times E^s(x) &= \nu(x) \times (S_k a)(x) + (M_k b)(x) - \frac{1}{2} b(x) \\
&\quad + \nu(x) \times \int_{\partial D} [\phi(y) - \phi_1(y)] \nabla_x \Phi_k(x,y)\, ds + \lambda \nu(x) \times S_k(\phi \nu), \\
\nu \times \operatorname{curl} E(x) &= (M_{k_1} a)(x) + \frac{1}{2} a(x) + (N_{k_1} b)(x) \\
&\quad + \lambda\, \nu(x) \times \int_{\partial D} \phi_1(y) \nu(y) \times \nabla_x \Phi_{k_1}(x,y)\, ds, \\
\nu \times \operatorname{curl} E^s(x) &= (M_k a)(x) - \frac{1}{2} a(x) + (N_k b)(x) \\
&\quad + \lambda\, \nu(x) \times \int_{\partial D} \phi(y) \nu(y) \times \nabla_x \Phi_k(x,y)\, ds.
\end{aligned}$$

Transmission condition (3.12) is satisfied if there exists $a \in L^2_t(\partial D)$, $b \in H_t^{\frac{1}{2}}(\partial D)^3$ and $\phi, \phi_1 \in L^2(\partial D)$ such that:

$$\begin{aligned}
&- a(x) + (M_k - M_{k_1}) a(x) + \lambda \nu(x) \times (S_{k_1} a)(x) \\
&+ (N_k - N_{k_1}) b(x) + \lambda \nu(x) \times \left(M_{k_1} + \frac{1}{2} I \right) b(x) \\
&+ \lambda \nu(x) \times \int_{\partial D} \phi_1(y) [\nu(y) - \nu(x)] \times \nabla_x \Phi_{k_1}(x,y)\, ds(y) + \lambda \nu(x) \times (S_{k_1}(\phi_1 \nu))(x) \\
&+ \lambda \nu(x) \times \int_{\partial D} \phi(y) [\nu(x) \times \nabla_x \Phi_{k_1}(x,y) - \nu(y) \times \nabla_x \Phi_k(x,y)]\, ds(y) = -\nu(x) \times \operatorname{curl} E^i(x).
\end{aligned} \tag{3.23}$$

By lemma 3.3, the operators $M_k - M_{k_1} : L^2_t(\partial D) \to L^2_t(\partial D)$ and $N_k - N_{k_1} : H_t^{\frac{1}{2}}(\partial D)^3 \to L^2_t(\partial D)$ are compact. Due to the compact embeddings $H^1(\partial D)^3 \hookrightarrow L^2(\partial D)^3$ and $H^{\frac{1}{2}}(\partial D)^3 \hookrightarrow L^2(\partial D)^3$, we conclude that the operators S and $M_{k_1} + \frac{1}{2} I$ are compact from $L^2_t(\partial D)$ into itself. Moreover,

$$\hat{S}_{k_1}(\cdot) := S_{k_1}(\cdot \nu) : L^2(\partial D)^3 \to L^2_t(\partial D)$$

is compact. Set

$$\left(P_{k_1}\phi_1\right)(x) := \nu(x) \times \int_{\partial D} \phi_1(y)\left[\nu(y) - \nu(x)\right] \times \nabla_x \Phi_{k_1}(x,y)\, ds(y)$$

and

$$\left(Q\phi\right)(x) := \nu(x) \times \int_{\partial D} \phi(y)\left[\nu(x) \times \nabla_x \Phi_{k_1}(x,y) - \nu(y) \times \nabla_x \Phi_{k}(x,y)\right]\, ds(y)$$

for $\phi_1, \phi \in L^2(\partial D)$. We note that the factor $\nu(y) - \nu(x)$ makes the kernel of P_{k_1} weakly singular; see corollary 2.9 in [7], thus $P_{k_1} : L^2(\partial D) \to L_t^2(\partial D)$ is compact. The operator Q we can rewrite as

$$\begin{aligned}\left(Q\phi\right)(x) &= \nu(x) \times \int_{\partial D} \phi(y)\left[\nu(x) \times \nabla_x \Phi_{k_1}(x,y) - \nu(y) \times \nabla_x \Phi_{k}(x,y)\right]\, ds(y) \\ &= -\left(P_{k_1}\phi\right)(x) + \int_{\partial D} \phi(y)\nu(y) \times \left[\nabla_x \Phi_{k}(x,y) - \nabla_x \Phi_{k_1}(x,y)\right]\, ds(y).\end{aligned}$$

Since the kernel of the second boundary integral is weakly singular and the operator P_{k_1} compact, we obtain that $Q : L^2(\partial D) \to L_t^2(\partial D)$ is compact. So far, we have that (3.23) is of Fredholm type considered in $L_t^2(\partial D)$.

Now we turn to transmission condition (3.11). It is satisfied if there exists $a \in L_t^2(\partial D)$, $b \in H_t^{\frac{1}{2}}(\partial D)^3$ and $\phi_1, \phi \in L^2(\partial D)$ such that:

$$\begin{aligned}&\left(S_k - S_{k_1}\right) a(x) + \left(M_k - M_{k_1}\right) b(x) - b(x) \\ &- \lambda\left(S_{k_1}(\phi_1\nu)\right)(x) - \nu(x) \times \int_{\partial D} \phi_1(y)\left[\nabla_x \Phi_{k}(x,y) - \nabla_x \Phi_{k_1}(x,y)\right]\, ds(y) \\ &+ \lambda\left(S_{k}(\phi\nu)\right)(x) + \nu(x) \times \int_{\partial D} \phi(y)\left[\nabla_x \Phi_{k}(x,y) - \nabla_x \Phi_{k_1}(x,y)\right]\, ds(y) = -\nu(x) \times E^i(x).\end{aligned} \tag{3.24}$$

Due to the identity map $b \to b$ and since $b \in H_t^{\frac{1}{2}}(\partial D)^3$, we have to consider the above equation in $H_t^{\frac{1}{2}}(\partial D)^3$. By the mapping properties of S and M and lemma 3.3, the operators $S_k - S_{k_1} : L_t^2(\partial D)^3 \to H_t^{\frac{1}{2}}(\partial D)$ and $M_k - M_{k_1} : H_t^{\frac{1}{2}}(\partial D)^3 \to H_t^{\frac{1}{2}}(\partial D)^3$ are compact. Moreover,

$$\hat{S}_{k_1} = S_{k_1}(\cdot\nu) : L^2(\partial D)^3 \to H_t^1(\partial D)^3 \overset{\text{compact}}{\hookrightarrow} H_t^{\frac{1}{2}}(\partial D)^3$$

is compact. The operator

$$(W\phi)(x) := \nu(x) \times \int_{\partial D} \phi(y) \left[\nabla_x \Phi_k(x,y) - \nabla_x \Phi_{k_1}(x,y)\, ds\right] = \nu(x) \times \left(\nabla_x \left[S_k - S_{k_1}\right] \phi\right)(x)$$

has a weakly singular kernel and is thus compact from $L^2(\partial D)$ to $H_t^{\frac{1}{2}}(\partial D)^3$. We thus have that (3.24) is of Fredholm type considered in $H_t^{\frac{1}{2}}(\partial D)^3$.

Finally, we consider the divergence-free conditions $\operatorname{div} E = \operatorname{div} E^s = 0$ on ∂D. The first one is satisfied if there exists $a \in L_t^2(\partial D)$, $b \in H_t^{\frac{1}{2}}(\partial D)^3$ and $\phi_1, \phi \in L^2(\partial D)$ such that:

$$\int_{\partial D} a(y) \cdot \nabla_x \Phi_{k_1}(x,y)\, ds(y) - k_1^2 S_{k_1}(\phi_1 - \phi) - \lambda D_{k_1} \phi_1 + \frac{1}{2} \lambda \phi_1 = 0 \tag{3.25}$$

where we used that

$$\operatorname{div} \nabla S_{k_1} = \Delta S_{k_1} = -k_1^2 S_{k_1}, \quad \operatorname{div} \widetilde{S}_{k_1}(\phi_1 \nu) = \widetilde{D}_{k_1} \phi_1$$

and the jump condition for the double layer potential (3.20). The single layer and double layer potentials are, for smooth domains, compact on $L^2(\partial D)$. Analogously, the condition $\operatorname{div} E^s = 0$ on ∂D is satisfied if there exists $a \in L_t^2(\partial D)^3$, $b \in H_t^{\frac{1}{2}}(\partial D)^3$ and $\phi_1, \phi \in L^2(\partial D)$ such that:

$$\int_{\partial D} a(y) \cdot \nabla_x \Phi_k(x,y)\, ds(y) - k^2 S_k(\phi_1 - \phi) - \lambda D_k \phi_1 - \frac{1}{2} \lambda \phi = 0. \tag{3.26}$$

It remains to consider the operator

$$(La)(x) := \int_{\partial D} a(y) \cdot \nabla_x \Phi(x,y)\, ds(y)$$

where Φ is either Φ_k or Φ_{k_1}. In [6], section 9.5, it is stated that $L : C_t^{0,\alpha}(\partial D)^3 \to C^{0,\alpha}(\partial D)$ is bounded. It is easily verified that the adjoint $L^* : C^{0,\alpha}(\partial D) \to C_t^{0,\alpha}(\partial D)^3$ of L is given by

$$(L^*\psi)(x) = \nu(x) \times \left(\int_{\partial D} \psi(y) \nabla_x \Phi(x,y)\, ds(y) \times \nu(x) \right)$$

which is bounded. Thus, by Lax theorem $L : L_t^2(\partial D)^3 \to L^2(\partial D)$ is bounded.

Using matrix-vector notation, we obtain that ansatz (3.21)-(3.22) solves problem $(P2)$, if there exists $a \in L_t(\partial D)^3$, $b \in H_t^{\frac{1}{2}}(\partial D)^3$ and $\phi_1, \phi \in L^2(\partial D)$ such that

$$JU + KU = R \tag{3.27}$$

where

$$U = \begin{pmatrix} a \\ b \\ \phi_1 \\ \phi \end{pmatrix}, \quad R = \begin{pmatrix} -\nu \times \operatorname{curl} E^i \\ -\nu \times E^i \\ 0 \\ 0 \end{pmatrix}, \quad J = \begin{pmatrix} -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ L_{k_1} & 0 & \frac{1}{2}I & 0 \\ L_k & 0 & 0 & -\frac{1}{2}I \end{pmatrix},$$

$$K = \begin{pmatrix} M_k - M_{k_1} + \lambda\nu \times S_{k_1} & N_k - N_{k_1} + \lambda\nu \times \left(M_{k_1} + \frac{1}{2}I\right) & \lambda P_{k_1} + \lambda \hat{S}_{k_1} & \lambda Q \\ S_k - S_{k_1} & M_k - M_{k_1} & -\lambda \hat{S}_{k_1} - W & \lambda \hat{S}_k + W \\ 0 & 0 & -k_1^2 S_{k_1} - \lambda D_{k_1} & k_1^2 S_{k_1} - \lambda D_{k_1} \\ 0 & 0 & -k^2 S_k - \lambda D_k & k^2 S_k - \lambda D_k \end{pmatrix}.$$

We consider (3.27) in the product space

$$L_t^2(\partial D)^3 \times H_t^{\frac{1}{2}}(\partial D)^3 \times L^2(\partial D) \times L^2(\partial D).$$

All of the entries of the matrix K are compact and matrix J has a bounded inverse because of its triangular form. Hence, we can apply the Riesz-Fredholm theory to (3.27). To show uniqueness of solutions of (3.27), we assume that the scattering problem (3.8)-(3.13) itself has at most one solution, which holds under assumptions 3.1. Let a, b, ϕ_1 and ϕ be solutions to the homogeneous equation corresponding to (3.27), i.e. $R = 0$ and thus $E^i = 0$. For $j \in \{k, k_1\}$, we define

$$E_j = \widetilde{S}_j(a) + \widetilde{M}_j(b) + \nabla \widetilde{S}_j(\phi - \phi_1) + \lambda \widetilde{S}_j(\phi_j \nu) \quad \text{in } \mathbb{R}^3 \setminus \partial D$$

where $\phi_k = \phi$ and $\phi_{k_1} = \phi_1$. Use of standard potential-theoretic arguments and the jump conditions, see (37), then lead to the following main theorem.

Theorem 3.4. *Under the assumptions made in 3.8 and the fact that k_1 is not a Dirichlet eigenvalue in D, the boundary value problem $(P2)$ has exactly one solution.*

3.2.2 General inhomogeneous medium

In the more general case when $\varepsilon_r = \varepsilon_r(x)$ in D, we can prove the same result as in theorem 3.4 by replacing the fundamental solution $\Phi_{k_1}(\cdot, y)$ with the free space fundamental solution

$\mathcal{G}(\cdot, y)$ of

$$\Delta \mathcal{G}(\cdot, y) + k^2 \varepsilon_r(x) \mathcal{G}(\cdot, y) = -\delta_y \quad \text{in } \mathbb{R}^3$$

in the distributional sense together with the Sommerfeld radiation condition, where $\varepsilon_r(x)$ is extended by one to the whole space $\mathbb{R}^3$. This follows because the mapping properties of the boundary operators appearing in (3.21)-(3.22) with kernels $\mathcal{G}$ and Φ are the same. Unfortunately we were not able to find the proof of this statement. The following result is from Andreas Kirsch.

It is the aim to construct the Green's function $\mathcal{G}$ for the equation

$$\Delta u + k^2 n u = 0$$

where $n(x) = 1$ outside of D. We make the ansatz $\mathcal{G}(x, y) = \Phi_k(x, y) + \hat{\Phi}(x, y)$ where Φ_k is, as before, the fundamental solution of $\Delta + k^2$ in $\mathbb{R}^3$ and $\hat{\Phi}$ unkown. We fix $y \in \mathbb{R}^3$ and from

$$\Delta_x \hat{\Phi}(x, y) + k^2 n(x) \hat{\Phi}(x, y) = -\Delta_x \Phi_k(x, y) - k^2 n(x) \phi_k(x, y) = -k^2 \left(n(x) - 1\right) \Phi_k(x, y)$$

we consider the right-hand side as a source and define $f_y(x) := k^2 \left(n(x) - 1\right) \Phi_k(x, y)$.

Lemma 3.5. *$f_y \in L^2(D)$ and $y \to f_y$ is continuous from $\mathbb{R}^3$ into $L^2(D)$.*

Proof. $f_y \in L^2(D)$ is obvious because $\int_D \frac{dz}{|y-z|^2}$ exists and is finite. Furthermore, for any $\eta > 0$, we write ($c > 0$ generic constant)

$$\begin{aligned} \|f_{y_1} - f_{y_2}\|_D^2 &\leq c \int_D \left| \frac{1}{|y_1 - z|} - \frac{1}{|y_2 - z|} \right|^2 dz \\ &\leq c \int_{|z-y_1|<\eta} \left[\frac{1}{|y_1 - z|^2} - \frac{1}{|y_2 - z|^2} \right] dz + c \int_{|z-y_1|>\eta} \left| \frac{1}{|y_1 - z|} - \frac{1}{|y_2 - z|} \right|^2 dz. \end{aligned}$$

From $\int_{|z|<\eta} \frac{dz}{|z|^2} = 4\pi\eta$ and $|z - y_2| \leq |z - y_1| + |y_1 - y_2| < \eta + |y_1 - y_2|$ for $|z - y_1| < \eta$ we conclude that the first integral is estimated by $4\pi\eta + 4\pi(\eta + |y_1 - y_2|)$. For the second integral we write

$$\begin{aligned}\int_{|z-y_1|>\eta} \left| \frac{1}{|y_1 - z|} - \frac{1}{|y_2 - z|} \right|^2 dz &\leq \int_{|z-y_1|>\eta} \frac{|y_1 - y_2|^2}{|y_1 - z|^2 |y_2 - z|^2} \, dz \\ &\leq \frac{|y_1 - y_2|^2}{\eta^2} \int_D \frac{dz}{|y_2 - z|^2} \\ &\leq c \frac{|y_1 - y_2|^2}{\eta^2}.\end{aligned}$$

Altogether we have

$$\|f_{y_1} - f_{y_2}\|_D^2 \leq c \left[\eta + |y_1 - y_2| + \frac{|y_1 - y_2|^2}{\eta^2} \right].$$

Now we choose $\eta = |y_1 - y_2|^{\frac{2}{3}}$. Then $\|f_{y_1} - f_{y_2}\|_D^2 \leq |y_1 - y_2|^{\frac{2}{3}}$. □

The equation

$$\Delta_x \hat{\Phi}(x, y) + k^2 n(x) \hat{\Phi}(x, y) = -f_y(x)$$

can be rewritten as

$$\Delta_x \hat{\Phi}(x, y) + k^2 \hat{\Phi}(x, y) = -f_y(x) - k^2 \left(n(x) - 1 \right) \hat{\Phi}(x, y)$$

which is equivalent to the Lippmann-Schwinger equation

$$\hat{\Phi}(\cdot, y) - V\hat{\Phi}(\cdot, y) = \int_D f_y(z) \Phi_k(\cdot, z) \, dz \quad \text{in } D$$

where $V : L^2(D) \to H^2_{\text{loc}}(\mathbb{R}^3)$ is the volume potential defined

$$\big(Vf\big)(x) = k^2 \int_D \left(n(z) - 1 \right) f(z) \Phi_k(x, z) \, dz \quad x \in \mathbb{R}^3.$$

The right-hand side of the Lippmann-Schwinger equation is in $H^2(D)$ and depends continuously on y. There, the same holds for the solution, and we have the following result.

Theorem 3.6. $\hat{\Phi}(\cdot, y) \in H^2_{loc}(\mathbb{R}^3)$ *and* $y \to \hat{\Phi}(\cdot, y)$ *is continuous from* $\mathbb{R}^3$ *into* $H^2(K)$ *for any bounded* $K \subset \mathbb{R}^3$.

Therefore, the mapping properties of the boundary operators with kernels $\mathcal{G}$ and Φ are the same.

3.3 Well-posedness of the direct problem via the variational approach

We will apply the variational approach to show the existence of a unique solution. We will start by deriving a variational formulation on a reduced domain by introducing the Calderon operator on the auxiliary boundary. The solution space to the reduced problem is motivated from the variational formulation. Unfortunately, the space fails to be compact in L^2, hence we can not apply directly the Fredholm theory to the reduced problem. To overcome this difficulty, we use decompositions of vector fields to factor out the null space of the curl operator. Due to the Calderon operator and its properties, any solution to the reduced problem also solves the scattering problem.

In the following, we combine techniques from sections 4 and 10 in [2]. In section 4, the cavity problem with impedance boundary condition was considered and it will be helpful to us to study the conductive transmission conditions. In section 10, the exterior scattering problem with perfectly conducting boundary condition using Calderon maps was considered. This will assist us to derive a variational formulation to our scattering problem on a reduced domain. We will also adopt some of the notation used in sections 4 and 10 in [2].

3.3.1 Variational formulation and uniqueness

In the following, we no longer assume μ and λ to be constants. Moreover, we let $D \subset \mathbb{R}^3$ be a bounded Lipschitz domain with a connected boundary ∂D such that the complement $\mathbb{R}^3 \setminus \overline{D}$ is simply connected. To obtain a variational formulation, we introduce a ball $B_R = \{x \in \mathbb{R}^3; |x| < R\}$ of radius R, where $R > 0$ is choosen such that $\overline{D} \subset B$ and we set as before $\Omega = B_R \setminus \overline{D}$. This will be the computational domain. The auxiliary boundary is the boundary of B_R denoted by $\Sigma = \partial B_R$. We note that the incident field is assumed to satisfy

$$\operatorname{curl}\operatorname{curl} E^i - k^2 E^i = 0 \quad \text{in } \mathbb{R}^3.$$

In Ω we will solve for the total field E, while in the exterior of B_R we will solve for the scattered field E^s. Using Ω and matching the fields across Σ, we obtain the following problem:

$$\operatorname{curl}\left(\frac{1}{\mu_r}\operatorname{curl} E\right) - k^2\varepsilon_r E = 0 \qquad \text{in } D, \tag{3.28}$$

$$\operatorname{curl}\operatorname{curl} E - k^2 E = 0 \qquad \text{in } \Omega, \tag{3.29}$$

$$\operatorname{curl}\operatorname{curl} E^s - k^2 E^s = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{B}_R, \tag{3.30}$$

$$\nu \times E|_+ - \nu \times E|_- = 0 \qquad \text{on } \partial D, \tag{3.31}$$

$$\nu \times \operatorname{curl} E|_+ - \nu \times \frac{1}{\mu_r}\operatorname{curl} E|_- - \lambda E_T = 0 \qquad \text{on } \partial D, \tag{3.32}$$

$$\nu \times E = \nu \times E^i + \nu \times E^s \qquad \text{on } \Sigma, \tag{3.33}$$

$$\nu \times \operatorname{curl} E = \nu \times \operatorname{curl}(E^i + E^s) \qquad \text{on } \Sigma, \tag{3.34}$$

$$\lim_{|x|\to\infty} |x| \left(\operatorname{curl} E^s \times \nu - ikE^s\right) = 0. \tag{3.35}$$

We set

$$\widetilde{\mu}_r = \begin{cases} \mu_r, & \text{in } D, \\ 1, & \text{in } \mathbb{R}^3 \setminus D, \end{cases} \qquad \widetilde{\varepsilon}_r = \begin{cases} \varepsilon_r, & \text{in } D, \\ 1, & \text{in } \mathbb{R}^3 \setminus D \end{cases}. \tag{3.36}$$

We multiply (3.28) and (3.29) by a smooth test function ψ, integrate over D and Ω, respectively, and formally use integration by parts. This yields:

$$0 = \iint_{B_R} \left(\frac{1}{\widetilde{\mu}_r}\operatorname{curl} E, \operatorname{curl}\psi - k^2\widetilde{\varepsilon}_r E, \psi\right)\, dx$$
$$+ \int_{\partial D} \left(\nu \times \frac{1}{\mu_r}\operatorname{curl} E|_- - \nu \times \operatorname{curl} E|_+\right) \cdot \widetilde{(\nu \times \psi)} \times \nu \; ds + \int_{\Sigma} (\nu \times \operatorname{curl} E) \cdot \widetilde{(\nu \times \psi)} \times \nu \, ds.$$

Now using boundary condition (3.32) on ∂D and (3.34) on Σ, we may write

$$0 = \left(\frac{1}{\widetilde{\mu}_r}\operatorname{curl} E, \operatorname{curl}\psi\right)_{B_R} - k^2\left(\widetilde{\varepsilon}_r E, \psi\right)_{B_R} - \langle \lambda E_T, \psi_T\rangle_{\partial D} + \left\langle \nu \times \operatorname{curl}(E^s + E^i), \psi_T\right\rangle_{\Sigma}.$$

To complete the derivation of the variational formulation, we need to specify how $\nu \times \operatorname{curl}(E^s + E^i)$ depends on $\nu \times E$. We use the so-called electric-to-magnetic Calderon operator G_e which is, for a given tangential vector field α on Σ, defined by

$$G_e : H^{-\frac{1}{2}}(\mathrm{Div}\,,\Sigma) \to H^{-\frac{1}{2}}(\mathrm{Div}\,,\Sigma), \quad G_e\alpha = \nu \times \frac{1}{ik}\mathrm{curl}\,E^s$$

where E^s satisfies the homogeneous isotropic Maxwell equation (3.30) in $\mathbb{R}^3 \setminus \overline{B}_R$. A brief introduction on the electric-to-magnetic Calderon operator will be given in the next subsection. Choosing $\alpha = \nu \times E^s = \nu \times (E - E^i)$, we see that the definition of G_e implies that we have $\nu \times \frac{1}{ik}\mathrm{curl}\,E^s = G_e\,(\nu \times (E - E^i))$ on Σ. Our variational problem now becomes the problem of finding E such that $\nu \times E|_- = \nu \times E|_+$ on ∂D and

$$\begin{aligned}\left(\frac{1}{\widetilde{\mu}_r}\mathrm{curl}\,E, \mathrm{curl}\,\psi\right)_{B_R} &- k^2\,(\widetilde{\varepsilon}_r E, \psi)_{B_R} - \langle \lambda E_T, \psi_T\rangle_{\partial D} \\ &+ ik\,\langle G_e(\nu \times E), \psi_T\rangle_{\partial D} = \left\langle ikG_e(\nu \times E^i) - \nu \times \mathrm{curl}\,E^i, \psi_T\right\rangle_\Sigma \end{aligned} \tag{3.37}$$

for all smooth test functions ψ defined on $\mathbb{R}^3$.

Remark 3.7. *We note that ψ_T, E_T and $\nu \times E$ in the above variational equation are meant in the trace sense. That is, defining*

$$\begin{aligned}&\gamma_t : H(\mathrm{curl}\,, B) \to H^{-\frac{1}{2}}(\mathrm{Div}\,, \partial D), \quad \gamma_t u = \nu \times u|_{\partial D}, \\ &\gamma_T : H(\mathrm{curl}\,, B) \to H^{-\frac{1}{2}}(\mathrm{Curl}\,, \partial D), \quad \gamma_t u = \nu \times (\nu \times u|_{\partial D})\,, \\ &\pi_t : H(\mathrm{curl}\,, B) \to H^{-\frac{1}{2}}(\mathrm{Div}\,, \Sigma), \quad \pi_t u = \nu \times u|_\Sigma, \\ &\pi_T : H(\mathrm{curl}\,, B) \to H^{-\frac{1}{2}}(\mathrm{Curl}\,, \Sigma), \quad \pi_t u = \nu \times (\nu \times u|_\Sigma)\end{aligned}$$

the precise formulation of the variational equation (3.37) is

$$\begin{aligned}\left(\frac{1}{\widetilde{\mu}_r}\mathrm{curl}\,E, \mathrm{curl}\,\psi\right)_{B_R} &- k^2\,(\widetilde{\varepsilon}_r E, \psi)_{B_R} - \langle \gamma_T E, \gamma_T \psi\rangle_{\partial D} \\ &+ ik\,\langle G_e(\pi_t E), \pi_T\psi\rangle_\Sigma = \left\langle ikG_e(\pi_t E^i) - \pi_t \mathrm{curl}\,E^i, \pi_T\psi\right\rangle_\Sigma .\end{aligned}$$

To simplify notation, we omit writing the traces γ_t, γ_T, π_t and π_T, since it is clear from the context which trace is meant, i.e. $\nu \times u$ and $u_T = \nu \times (u \times \nu)$ on ∂D is meant in the trace sense $\gamma_t u$ and $\gamma_T u$ respectively, while $\nu \times u$ and $u_T = \nu \times (u \times \nu)$ on Σ is meant in the trace sense $\pi_t u$ and $\pi_T u$, respectively.

In order for all the integrals of the variational equation (3.37) to be well defined, we define the solution space X by

$$X := \left\{ u \in H(\operatorname{curl}, B_R) : u_T \in L^2_t(\partial D, \mathbb{C}^3) \text{ on } \partial D \right\}.$$

We note that $E \in H(\operatorname{curl}, D) \cup H(\operatorname{curl}, \Omega)$ with $\nu \times E|_- = \nu \times E|_+$ on ∂D implies $E \in H(\operatorname{curl}, B_R)$. Indeed, for $\psi \in C_0^\infty(B_R)$ using integration by party (1.11) yields

$$\begin{aligned}
\int_{B_R} E \cdot \operatorname{curl} \psi \, dx &= \int_D E \cdot \operatorname{curl} \psi \, dx + \int_\Omega E \cdot \operatorname{curl} \psi \, dx \\
&= \int_D \operatorname{curl} E \cdot \psi \, dx + \int_\Omega \operatorname{curl} E \cdot \psi \, dx + \langle \nu \times E|_- - \nu \times E|_+, \psi \rangle_{\partial D} \\
&= \int_{B_R} \operatorname{curl} E \cdot \psi \, dx
\end{aligned}$$

where we used that ψ vanishes on $\partial B_R = \Sigma$ and $\nu \times E|_- = \nu \times E|_+$ on ∂D. Thus E possesses a variational curl in B_R. We equip X with the following inner product, defined for each $u, v \in X$, by

$$(u, v)_X := (u, v)_{B_R} + (\operatorname{curl} u, \operatorname{curl} v)_{B_R} + \langle u_T, v_T \rangle_{\partial D}.$$

Then $(X, (\cdot, \cdot)_X)$ is a Hilbert space. The proof is similar to the proof of theorem 4.1 in [2]. Summing, we can now state our problem.

Problem statement (P3):

Determine $E \in X$ such that the variational formulation (3.37), that is,

$$\begin{aligned}
\left(\frac{1}{\widetilde{\mu}_r} \operatorname{curl} E, \operatorname{curl} \psi \right)_{B_R} &- k^2 \left(\widetilde{\varepsilon}_r E, \psi \right)_{B_R} - \langle \lambda E_T, \psi_T \rangle_{\partial D} \\
&+ ik \left\langle G_e(\nu \times E), \psi_T \right\rangle_\Sigma = \left\langle ikG_e(\nu \times E^i) - \nu \times \operatorname{curl} E^i, \psi_T \right\rangle_\Sigma
\end{aligned}$$

holds for all $\psi \in X$.

We note that, if E is a solution of $(P3)$, then it is easy to show by choosing sufficiently smooth test functions that E satisfies the differential equation (3.28) in D and (3.29) in Ω, the transmission conditions (3.31)-(3.32) on ∂D and $\nu \times (\operatorname{curl} E^s) = ikG_e \left(\nu \times (E - E^i) \right)$ on Σ.

Before we continue, we make the following assumptions regarding the data.

Assumption 3.8. • *$D \subset \mathbb{R}^3$ is a bounded simply connected Lipschitz domain with* *connected exterior* $\mathbb{R}^3 \setminus \overline{D}$.

- $k = \omega\sqrt{\varepsilon_0 \mu_0} > 0$.
- $\beta \in L^\infty(\partial D)$ *real-valued and strictly positive, implying that* $\lambda = i\omega\mu_o\beta \in L^\infty(\partial D)$ *is pure imaginary with* $\mathrm{Im}\,\lambda$ *stricly positive.*
- $\mu_r \in W^{1,\infty}(D)$ *real valued and* $\mu_r \geq c_0$ *on* D *for some constant* $c_0 > 0$. *We note that this implies* $\frac{1}{\mu_r} \in W^{1,\infty}(D)$.
- $\varepsilon_r \in W^{2,\infty}(D)$ *with* $\mathrm{Im}\,\varepsilon_r \geq 0$ *and* $\mathrm{Re}\,\varepsilon_r \geq c_1$ *on* D *for some constant* $c_1 > 0$.

Further we define

$$\varepsilon^+ := \sup_{x\in B_R}|\varepsilon_r(x)|, \quad \varepsilon^- := \inf_{x\in B_R}|\varepsilon_r(x)|, \quad \mu^+ := \sup_{x\in B_R}|\widetilde{\mu}_r(x)|, \quad \mu^- := \inf_{x\in B_R}|\mu_r(x)|$$

and

$$\lambda^+ := \sup_{x\in\partial D}|\lambda(x)|, \quad \lambda^- := \inf_{x\in\partial D}|\lambda(x)|.$$

Next we discuss uniqueness. We note that it is sufficient to show uniqueness for the problem (3.8)-(3.13).

Theorem 3.9. *Under assumption 3.8, problem (3.8)-(3.13) has at most one solution.*

Proof. By linearity, we need only to consider the case $E^i = 0$. Hence $E = E^s$ in $\mathbb{R}^3 \setminus \overline{D}$ and thus E is a radiationg solution of Maxwell's equations in $\mathbb{R}^3 \setminus \overline{D}$. By taking the dot product of (3.8) and (3.9) with $\overline{E}$, integrating over D and Ω, respectively, and using integration by parts (1.11) and boundary condition (3.12), we obtain

$$0 = \iint_{B_R} \left(\frac{1}{\widetilde{\mu}_r} |\mathrm{curl}\, E|^2 - k^2 \widetilde{\varepsilon}_r |E|^2 \right) dx - \int_{\partial D} \lambda |E_T|^2\, ds + \int_{|x|=R} (\nu \times \mathrm{curl}\, E) \cdot \overline{E}_T\, ds$$

that is

$$\int_{|x|=R} (\mathrm{curl}\, E \times \nu) \cdot \overline{E}_T\, ds = \iint_{B_R} \left(\frac{1}{\widetilde{\mu}_r} |\mathrm{curl}\, E|^2 - k^2 \widetilde{\varepsilon}_r |E|^2 \right) dx - \int_{\partial D} \lambda |E_T|^2\, ds.$$

Taking the imaginary part yields

$$\mathrm{Im} \left(\int_{|x|=R} (\mathrm{curl}\, E \times \nu) \cdot \overline{E}_T\ ds \right) = -k^2 \iint_{B_R} \mathrm{Im}(\widetilde{\varepsilon}_r) |E|^2\, dx - \int_{\partial D} \mathrm{Im}(\lambda) |E_T|^2\ ds < 0. \tag{3.38}$$

From the radiation condition (3.35) we obtain

$$\begin{aligned}
0 \overset{R\to\infty}{\longleftarrow} & \int_{|x|=R} |\operatorname{curl} E \times \nu - ikE|^2 \, ds \\
&= \int_{|x|=R} \left(|\operatorname{curl} E \times \nu|^2 + k^2|H|^2\right) ds - 2k\operatorname{Im} \int_{|x|=R} (\operatorname{curl} E \times \nu) \cdot \overline{E} \, ds \\
&\overset{(3.38)}{\geq} \int_{|x|=R} \left(|\operatorname{curl} E \times \nu|^2 + k^2|E|^2\right) ds
\end{aligned}$$

and thus conclude $\lim_{R\to\infty} \int_{|x|=R} |E|^2 \, ds = 0$. Rellich's lemma 1.9 now implies that $E = 0$ in $\mathbb{R}^3 \setminus B_R$. Outside D the solution E is smooth and satisfies $\operatorname{curl}\operatorname{curl} E - k^2 E = 0$. As noted in remark 2.1, the solution is analytic outside D, and thus $E = 0$ in $\mathbb{R}^3 \setminus \overline{D}$. We can now argue as in the proof of theorem 2.4, with the roles of μ_r and ε_r interchanged, to conclude that

$$\Delta E = -\nabla \left(\frac{\nabla \varepsilon_r}{\varepsilon_r} \cdot E \right) + \mu_r \nabla \left(\frac{1}{\mu_r} \right) \times \operatorname{curl} E - k^2 \mu_r \varepsilon_r E \quad \in L^2(D)^3$$

in the variational sense. Applying the interior regularity property 1.10 and the unique continuation principle 1.11 as in theorem 2.4 yields $E = 0$ in $\mathbb{R}^3$. □

Now, assuming that the reduced problem $(P3)$ has a solution, we can construct an extension of this solution from the bounded domain B_R to $\mathbb{R}^3 \setminus \overline{B}_R$ (for details we refer the reader to section 9.3.3 in [2]). Due to the use of the Calderon map G_e, this extended solution satisfies the Maxwell equation (3.9) in the weak sense in $H_{\text{loc}}(\operatorname{curl}, \mathbb{R}^3 \setminus \overline{B_R})$ together with the Silver-Müller radiation condition. The above uniqueness result then implies that this extension is the only solution of $(P3)$. Hence, once we have proven the existence of a solution of $(P3)$, we also have then verified that the transmission problem (3.8)-(3.13) has a unique solution $E \in H_{\text{loc}}(\operatorname{curl}, \mathbb{R}^3)$ with $E_T \in L^2_t(\partial D)^3$.

3.3.2 The electric-to-magnetic Calderon operator

Before introducing the electric-to-magnetic Calderon operator and listing some of the operators properties, we briefly need to discuss the series solution of the exterior Maxwell problem, that is, the problem of determining E^s and H^s such that

$$\operatorname{curl} E^s - ikH^s = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{B}_R, \tag{3.39}$$

$$\operatorname{curl} H^s + ikE^s = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{B}_R, \tag{3.40}$$

$$\nu \times E^s = \alpha \qquad \text{on } \partial B_R = \Sigma, \tag{3.41}$$

$$|x| \left(H^s \times \hat{x} - E^s\right) \to 0 \qquad \text{as } |x| \to \infty, \tag{3.42}$$

where $\hat{x} = x/|x|$. In particular, working only with the electric field, we obtain the following problem for E^s:

$$\operatorname{curl}\operatorname{curl} E^s - k^2 E^s = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{B}_R, \tag{3.43}$$

$$\nu \times E^s = \alpha \qquad \text{on } \Sigma, \tag{3.44}$$

$$|x| \left(\operatorname{curl} E^s \times \hat{x} - ikE^s\right) \to 0 \qquad \text{as } |x| \to \infty. \tag{3.45}$$

Here, α is suitable given tangential vector field on Σ. We adopt the notation from [2] and refer to this literature for more detailed reading.

Let Y_n^m for $n = 0, 1, \cdots$ and $m = -n, \cdots, n$, denote an orthonormal sequence of *spherical harmonics* on the unit sphere S^2. The basis functions for tangential fields on Σ are then the *vector spherical harmonics* of order n given by

$$U_n^m = \frac{1}{\sqrt{n(n+1)}} \nabla_{S^2} Y_n^m \quad \text{and} \quad V_n^m = \hat{x} \times U_n^m$$

for $n = 1, 2, \cdots$ and $m = -n, \cdots, n$, where ∇_{S^2} denotes the surface gradient on the surface of the unit sphere S^2. By lemma 9.15, in [2] the vector spherical harmonics U_n^m and V_n^m form a complete orthonormal basis for $L_t^2(S^2)$. Thus, we can expand any function $\alpha \in L_t^2(\Sigma)$ by

$$\alpha = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} c_{n,m} U_n^m + \widetilde{c}_{n,m} V_n^m. \tag{3.46}$$

Next we define the *vector wave functions*

$$M_n^m = \operatorname{curl}\left\{x h_n^{(1)}\left(k|x|\right) Y_n^m(\hat{x})\right\} \quad \text{and} \quad N_n^m = \frac{1}{ik} \operatorname{curl} M_n^m$$

for $n = 1, 2, \cdots$ and $m = -n, \cdots, n$, where $h_n^{(1)}$ is the spherical Hankel function of first kind and order n (presented in section 9.3.2 in [2]). Then, by theorem 9.16 in [2] the functions M_n^m and N_n^m are radiating solutions of Maxwell's equations in $\mathbb{R}^3 \setminus \{0\}$.

The following theorem corresponds to theorem 9.17 in [2].

Theorem 3.10. *Let E^s be a radiating solution of Maxwell's equations for $|x| > R > 0$. Then E^s has the representation*

$$E^s(x) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \{d_{n,m} M_n^m(x) + \overline{D}_{n,m} N_n^m(x)\}.$$

The series converges uniformly (together with its derivative) on compact subsets of $|x| > R$. The corresponding series for $H^s = \frac{1}{ik}\operatorname{curl} E^s$ is

$$H^s(x) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \{d_{n,m} N_n^m(x) - \overline{D}_{n,m} M_n^m(x)\}.$$

Now, let us suppose the boundary data $\alpha \in H^{-\frac{1}{2}}(\mathrm{Div}\,, \Sigma)$ has representation (3.46). Next, we want to compute the scattered field satisfying (3.43)-(3.45) in terms of the coefficients of this expansion. With the representation from theorem 3.10 for E^s we can express $\hat{x} \times E^s$ on $|x| = R$ in terms of the coefficients of the expansion,

$$\hat{x} \times E^s = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \{d_{n,m} \hat{x} \times M_n^m(x) + \overline{D}_{n,m} \hat{x} \times N_n^m(x)\}.$$

Using the definition of M_n^m, a suitable vector identity and the definition of U_n^m we obtain

$$\hat{x} \times M_n^m(x) = h^{(1)}(kR)\sqrt{n(n+1)}U_n^m(\hat{x}) \quad \text{on } |x| = R. \tag{3.47}$$

Further, using the definition of N_n^m, suitable vector identities, the fact that $\hat{x}$ and x are parall and the definition of V_n^m, we obtain

$$\hat{x} \times N_n^m(x) = \frac{1}{ikR}\left[h^{(1)}(k|x|) + |x|\frac{\partial}{\partial r}h^{(1)}(k|x|)\right]\sqrt{n(n+1)}V_n^m(\hat{x}).$$

Using this equality and (3.47), shows that, on $|x| = R$,

$$\begin{aligned}\hat{x} \times E^s =& \sum_{n=1}^{\infty} \sum_{m=-n}^{n} d_{n,m} h^{(1)}(kR) \sqrt{n(n+1)} U_n^m \\ &+ \frac{1}{ikR} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \overline{D}_{n,n} \{h^{(1)}(kR) + kR(h^{(1)})'(kR)\} \sqrt{n(n+1)} V_n^m. \end{aligned} \tag{3.48}$$

Analogously we obtain

$$\begin{aligned}\hat{x} \times H^s =& \frac{1}{ikR} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} d_{n,n} \{h^{(1)}(kR) + kR(h^{(1)})'(kR)\} \sqrt{n(n+1)} V_n^m \\ &- \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \overline{D}_{n,m} h^{(1)}(kR) \sqrt{n(n+1)} U_n^m. \end{aligned} \tag{3.49}$$

Now we can solve the boundary value problem (3.39)-(3.42) for arbitrary tangential boundary data α and obtain a series for each field E^s and H^s that converges in $H_{loc}(\text{curl}\,, \mathbb{R}^3 \backslash \overline{B}_R)$. The following lemma is lemma 9.19 in [2].

Lemma 3.11. *For* $\alpha \in H^{-\frac{1}{2}}(\text{Div}\,, \Sigma)$ *given by (3.46), the unique solution* E^s, $H^s \in H_{\text{loc}}(\text{curl}\,, \mathbb{R}^3 \setminus \overline{B}_R)$ *of (3.39)-(3.42) is given by*

$$E^s = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[\frac{c_{n,m} M_n^m}{h_n^{(1)}(kR) \sqrt{n(n+1)}} + \frac{ikR \widetilde{c}_{n,m} N_n^m}{[h_n^{(1)}(kR) + kR(h_n^{(1)})'(kR)] \sqrt{n(n+1)}} \right],$$

$$H^s = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[\frac{c_{n,m} N_n^m}{h_n^{(1)}(kR) \sqrt{n(n+1)}} - \frac{ikR \widetilde{c}_{n,m} M_n^m}{[h_n^{(1)}(kR) + kR(h_n^{(1)})'(kR)] \sqrt{n(n+1)}} \right].$$

The electric-to-magnetic Calderon operator G_e takes the electric field boundary data to magnetic field boundary data. In particular, for a given tangential vector field α on Σ we define

$$G_e \alpha = \hat{x} \times H^s$$

where E^s and H^s satisfy (3.39)-(3.42). For $\alpha \in H^{-\frac{1}{2}}(\text{Div}\,, \Sigma)$ given by (3.46), we can use (3.49) to obtain an explicit representation for the map G_e,

$$G_e \alpha = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left\{ -ikR \frac{\widetilde{c}_{n,m}}{\delta_n} U_n^m + \frac{c_{n,m} \delta_n}{ikR} V_n^m \right\} \tag{3.50}$$

where

$$\delta_n = kR\frac{\left(h_n^{(1)}\right)'(kR)}{h_n^{(1)}(kR)} + 1.$$

The Calderon operator $G_e : H^{-\frac{1}{2}}(\mathrm{Div}\,,\Sigma) \to H^{-\frac{1}{2}}(\mathrm{Div}\,,\Sigma)$ is continuous (theorem 9.21 in [2]). We denote by $\widetilde{G}_e$ the Calderon operator for pure imaginary wavenumbers, that is, $\widetilde{G}_e$ is defined by (3.50) with $k = i$. We have the following property of $\widetilde{G}_e$.

Lemma 3.12. *The operator $\widetilde{G}_e$ is negative definite in the sense that*

$$\left\langle \widetilde{G}_e\alpha, \alpha \times \hat{x} \right\rangle_\Sigma < 0$$

for any $\alpha \in H^{-\frac{1}{2}}(\mathrm{Div}\,,\Sigma)$ with $\alpha \neq 0$. Furthermore,

$$\left| \left\langle \widetilde{G}_e\alpha, \alpha \right\rangle_\Sigma \right| \geq x\|\alpha\|_\Sigma^2 \quad \textit{for all } \alpha \in H^{-\frac{1}{2}}(\mathrm{Div}\,,\Sigma).$$

Proof. This is lemma 9.23 in [2]. □

Our final result of this section shows that a suitable combination of G_e and $\widetilde{G}_e$ is compact on a suitable set of functions on Σ. Let

$$H_{\mathrm{Div}}^{-\frac{1}{2}}(\mathrm{Div}\,,\Sigma) = \left\{ \alpha = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \widetilde{c}_{n,m} V_n^m : \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \frac{1}{\sqrt{1+n(n+1)}} |\widetilde{c}_{n,m}|^2 < \infty \right\}.$$

Then the following operator is well defined and bounded:

$$G_e + ik\widetilde{G}_e|_{H_{\mathrm{Div}}^{-\frac{1}{2}}(\mathrm{Div}\,,\Sigma)} : H_{\mathrm{Div}}^{-\frac{1}{2}}(\mathrm{Div}\,,\Sigma) \to H^{-\frac{1}{2}}(\mathrm{Div}\,,\Sigma). \tag{3.51}$$

For a proof, we refer the reader to lemma 9.24 in [2].

3.3.3 Existence

In this section, we will discuss the reduced problem $(P3)$ with the aim of showing that the problem is well-posed. We start by defining $A : X \times X \to \mathbb{C}$ and $B : X \to \mathbb{C}$ by

$$A(E,\psi) := \left(\frac{1}{\widetilde{\mu}_r}\operatorname{curl} E, \operatorname{curl}\psi\right)_{B_R} - k^2\left(\widetilde{\varepsilon}_r E,\psi\right)_{B_R} - \langle \lambda E_T, \psi_T\rangle_{\partial D} + ik\left\langle G_e(\nu\times E),\psi_T\right\rangle_{\Sigma}, \tag{3.52}$$

$$B(\psi) := \left\langle ikG_e(\nu\times E^i) - \nu\times\operatorname{curl} E^i, \psi_T\right\rangle_{\Sigma}. \tag{3.53}$$

Then, we can write problem $(P3)$ as the problem of finding $E \in X$ such that

$$A(E,\psi) = B(\psi), \quad \text{for all } \psi \in X.$$

To prove existence, we will use decomposition of vector fields, that is, a Helmholtz decomposition that is closely related to the Maxwell system. There are several forms of a Helmholtz decomposition. In our case, we will need to decompose the solution space $X = \{u \in H(\operatorname{curl}, B_R) : u_T \in L^2_t(\partial D, \mathbb{C}^3) \text{ on } \partial D\}$ to factor out the null space of the curl operator.

3.3.3.1 The scalar problem

To take into account the functions in X that have vanishing curl, we suppose $u \in X$ is such that $\operatorname{curl} u = 0$ in B_R. Since D is simply connected Lipschitz domain, by theorem 3.37 in [2], we know that $u = \nabla\xi$ for some $\xi \in H^1(B_R)$. We note that $u_T = (\nabla\xi)_T = \nabla_{\partial D}\xi$, where $\nabla_{\partial D}\xi$ is the surface gradient on ∂D. By definition of the space X, we have $u_T \in L^2_t(\partial D)^3$, this motivates us to define the following space:

$$S = \left\{\xi \in H^1(B_R) : \nabla_{\partial D}\xi \in L^2_t(\partial D)^3 \text{ on } \partial D \text{ and } \int_{\partial D}\xi ds = 0\right\}.$$

The condition $\int_{\partial D}\xi ds = 0$ for all elements $\xi \in S$, as we will see, is necessary to prove uniqueness. We note that, we have shown that there exists a scalar potential $\xi \in S$ with $u = \nabla\xi$. We equipp S with the following scalar product, defined for each $\xi,\eta \in S$:

$$(\xi,\eta)_S := (\xi,\eta)_{H^1(B_R)} + \langle\nabla_{\partial D}\xi, \nabla_{\partial D}\eta\rangle_{\partial D}.$$

Since the trace operator $\nabla\xi \to \nu\times(\nabla\xi\times\nu)$ is continuous, it is easily verified that $(S,(\cdot,\cdot)_S)$ is a Hilbert space.

Now, to factor out the null space of the curl operator, which is motivated by substituting $\psi = \nabla\xi$, $\xi \in S$ as a test function in the variational formulation (3.37), we have to consider the scalar problem of finding $p \in S$ such that

$$A(\nabla p, \nabla\xi) = B(\nabla\xi), \quad \text{for all } \xi \in S. \tag{3.54}$$

In particular,

$$-k^2\left(\widetilde{\varepsilon}_r \nabla p, \nabla\xi\right)_{B_R} - \langle\lambda\nabla_{\partial D} p, \nabla_{\partial D}\xi\rangle_{\partial D} + ik\,\langle G_e(\nu\times\nabla p), \nabla_\Sigma\xi\rangle_\Sigma$$
$$= \left\langle ikG_e(\nu\times E^i) - \nu\times \operatorname{curl} E^i, \nabla_\Sigma\xi\right\rangle_\Sigma$$

for all $\xi \in S$. We note that A is not coercive on $S\times S$, but we can show that it satsifes the Fredholm property. With the help of the Riesz representation theorem, we define the operator $\mathcal{A}$ from S to S such that

$$(\mathcal{A}p, \xi)_{H^1(B_R)} = A(\nabla p, \nabla\xi), \quad \text{for all } p, \xi \in S.$$

Theorem 3.13. *Under the conditions on the data outlined in assumptions (3.8) and assuming that $\varepsilon_r = \mu_r = 1$ in a neighborhood of Σ, the following hold:*

(a) The operator $\mathcal{A}$ associated with the scalar problem (3.54) satisfies the equality $\mathcal{A} = \mathcal{J}_1 + \mathcal{K}_1$, where $\mathcal{J}_1$ is an isomorphism on S and $\mathcal{K}_1$ is a compact operator on S.

(b) The operator $\mathcal{J}_1+\mathcal{K}_1$ is an isomorphism from S into itself. The scalar problem (3.54) is uniquely solvable in S and the solution is given by $p = (\mathcal{J}+\mathcal{K}_1)^{-1} z$, where $z \in S$ satisfies $B(\nabla\xi) = (z,\xi)_{H^1(B_R)}$ for all $\xi \in S$.

Proof. (a) Using the Calderon operator $\widetilde{G}_e$ for pure imaginary wavenumbers and writing $-k^2 = (ik)^2$, we can write

$$\begin{aligned} A(\nabla p, \nabla\xi) &= -k^2\left(\widetilde{\varepsilon}_r \nabla p, \nabla\xi\right)_{B_R} - \langle\lambda\nabla_{\partial D} p, \nabla_{\partial D}\xi\rangle_{\partial D} + ik\,\langle G_e(\nu\times\nabla p), \nabla_\Sigma\xi\rangle_\Sigma \\ &= -k^2\left(\widetilde{\varepsilon}_r \nabla p, \nabla\xi\right)_{B_R} - \langle\lambda\nabla_{\partial D} p, \nabla_{\partial D}\xi\rangle_{\partial D} \\ &\quad + k^2\left\langle \widetilde{G}_e(\nu\times\nabla p), \nabla_\Sigma\xi\right\rangle_\Sigma + ik\left\langle\left(G_e + ik\widetilde{G}_e\right)(\nu\times\nabla p), \nabla_\Sigma\xi\right\rangle_\Sigma . \end{aligned}$$

We decompose A as

$$A(\nabla p, \nabla \xi) = a_1(p, \xi) + b_1(p, \xi)$$

where

$$\begin{aligned} a_1(p,\xi) &= -k^2 \left(\widetilde{\varepsilon}_r \nabla p, \nabla \xi\right)_{B_R} - k^2 \left(\widetilde{\varepsilon}_r p, \xi\right)_{B_R} - \langle \lambda \nabla_{\partial D} p, \nabla_{\partial D} \xi \rangle_{\partial D} \\ &\quad + k^2 \left\langle \widetilde{G}_e(\nu \times \nabla p), \nabla_\Sigma \xi \right\rangle_\Sigma, \\ b_1(p,\xi) &= k^2 \left(\widetilde{\varepsilon}_r p, \xi\right)_{B_R} + ik \left\langle \left(G_e + ik\widetilde{G}_e\right)(\nu \times \nabla p), \nabla_\Sigma \xi \right\rangle_\Sigma . \end{aligned}$$

We first consider the sesquilinear form a_1. Using the Cauchy-Schwartz inequality, boundedness of the Calderon operator $\widetilde{G}_e : H^{-\frac{1}{2}}(\mathrm{Div}\,, \Sigma) \to H^{-\frac{1}{2}}(\mathrm{Div}\,, \Sigma)$ as well as boundedness of the trace operators π_t and π_T defined in remark 3.7, we obtain that there exists a constant $\widetilde{c} > 0$ with

$$\left| \left\langle \widetilde{G}_e(\nu \nabla p), \nabla_\Sigma \xi \right\rangle_\Sigma \right| \leq \widetilde{c} \|\nabla p\|_{B_R} \|\nabla \xi\|_{B_R} \leq \widetilde{c} \|p\|_S \|\xi\|_S.$$

Consequently, using the boundedness of λ and $\varepsilon_r \in L^\infty(D)$, in particular, $|\widetilde{\varepsilon}_r| \leq \max\{\varepsilon^+, 1\}$, we estimate:

$$\begin{aligned} |a_1(p,\xi)| &\leq \left|-k^2\right| \left| \left(\widetilde{\varepsilon}_r \nabla p, \nabla \xi\right)_{B_R} + \left(\widetilde{\varepsilon}_r p, \xi\right)_{B_R} + \frac{1}{k^2} \langle \lambda \nabla_{\partial D} p, \nabla_{\partial D} \xi \rangle_{\partial D} \right| \\ &\quad + \left| k^2 \left\langle \widetilde{G}_e(\nu \times \nabla p), \nabla_\Sigma \xi \right\rangle_\Sigma \right| \\ &\leq k^2 \max \left\{ \varepsilon^+, 1, \frac{\lambda^+}{k^2}, \widetilde{c} \right\} \left(|(p,\xi)_S| + \|p\|_S \|\xi\|_S \right) \\ &\leq C \|p\|_S \|\xi\|_S \end{aligned}$$

for $p, \xi \in S$, with $C = k^2 \max\left\{\varepsilon^+, 1, \frac{\lambda^+}{k^2}, \widetilde{c}\right\} > 0$, which implies boundedness of a_1. To show coercivity, we estimate:

$$
\begin{aligned}
&2|a_1(p,p)| \\
&\geq |\mathrm{Re}\,(a_1(p,p))| + |\mathrm{Im}\,(a_1(p,p))| \\
&= k^2 \left|(\widetilde{\varepsilon}_r \nabla p, \nabla p)_{B_R} + (\widetilde{\varepsilon}_r p, \xi)_{B_R}\right| + k^2 \left| \frac{1}{k^2} \langle \lambda \nabla_{\partial D} p, \nabla_{\partial D} \xi \rangle_{\partial D} - \left\langle \widetilde{G}_e(\nu \times \nabla p), \nabla_\Sigma p \right\rangle_\Sigma \right| \\
&\geq k^2 (\widetilde{\varepsilon}_r \nabla p, \nabla p)_{B_R} + k^2 (\widetilde{\varepsilon}_r p, p)_{B_R} + \frac{1}{k^2} \langle \lambda \nabla_{\partial D} p, \nabla_{\partial D} \xi \rangle_{\partial D} \\
&\geq \underbrace{k^2 \min\left\{\varepsilon^-, 1, \frac{\lambda^-}{k^2}\right\}}_{:=c>0} \Big((\nabla p, \nabla p)_{B_R} + (p,p)_{B_R} + \langle \nabla_{\partial D} p, \nabla_{\partial D} p \rangle_{\partial D} \Big) \\
&= c\|p\|_S^2
\end{aligned}
$$

where we used lemma 3.12 to estimate $\widetilde{G}_e$. That is, a_1 is a bounded, coercive sesquilinear form on $S \times S$. Applying the Lax-Milgram theorem yields the existence of a unique bijective bounded linear operator $\mathcal{J}_1 : S \to S$ with bounded inverse, that is, an isomorphism, satisfying

$$
a_1(p,\xi) = (\mathcal{J}_1 p, \xi)_{H^1(B_R)}, \quad \text{for all } \xi \in S.
$$

By the Riesz representation theorem, there exists a bounded linear operator $\mathcal{K}_1 : S \to S$ defined by

$$
b_1(p,\xi) = (\mathcal{K}_1 p, \xi)_S, \quad \text{for all } \xi \in S.
$$

To prove compactness, let $(p_n)_n \subset S$ be a sequence converging to zero weakly in S, that is $(p_n, p)_S \to 0$ for all $p \in S$. By the compact embeddings $H^1(B_R) \hookrightarrow L^2(B_R)$ and $H^1(B_R) \supset S \hookrightarrow L^2(B_R)$, we conclude that $\|p_n\|_{B_R} \to 0$. Using the compactness of $G_e + ik\widetilde{G}_e$ on $H^{-\frac{1}{2}}(\mathrm{Div}\,, \Sigma)$ and the boundedness of the trace operator we estimate

$$
\begin{aligned}
|b_1(p_n,\xi)| &= \left| k^2 (\widetilde{\varepsilon}_r p_n, \xi)_{B_R} + ik \left\langle \left(G_e + ik\widetilde{G}_e\right)(\nu \times \nabla p_n), \nabla_\Sigma \xi \right\rangle_\Sigma \right| \\
&\leq d_1 \|p_n\|_{B_R} \|\xi\|_{B_R} + d_2 \|\nabla p_n\|_{B_R} \|\nabla \xi\|_{B_R} \\
&\leq d_n \|\xi\|_{H^1(B_R)} \\
&\leq d_n \|\xi\|_S,
\end{aligned}
$$

with $d_1 > 0$, $d_2 > 0$ and $d_n \to 0$ as $n \to \infty$. Then,

$$\|\mathcal{K}_1 p_n\|_S^2 = b_1(\mathcal{K}_1 p_n, \mathcal{K}_1 p_n) \le d_n \|\mathcal{K}_1 p_n\|_S$$

which shows that $\|\mathcal{K}_1 p_n\|_S \to 0$. This proves that $\mathcal{K}_1$ is compact.

(b) Let $z \in S$ be such that $B(\nabla\xi) = (z, \xi)_{H^1(B_R)}$, for all $\xi \in S$. Now we can write the scalar problem (3.54) as

$$(\mathcal{J} + \mathcal{K}_1)\, p = z \tag{3.55}$$

which is a Fredholm equation, hence existence follows from uniqueness. To show uniqueness, it is sufficient to consider the case when $z = 0$. Hence p satisfies

$$(\mathcal{J} + \mathcal{K}_1)\, p = 0$$

that is

$$-k^2(\widetilde{\varepsilon}_r \nabla p, \nabla\xi)_{B_R} - \langle \lambda \nabla_{\partial D} p, \nabla_{\partial D}\xi\rangle_{\partial D} + ik\, \langle G_e(\nu \times \nabla p), \nabla_\Sigma \xi\rangle_\Sigma = 0, \quad \text{for all } \xi \in S.$$

Choosing $\xi = p$, we obtain

$$ik\, \langle G_e(\nu \times \nabla p), \nabla_\Sigma p\rangle_\Sigma = k^2(\widetilde{\varepsilon}_r \nabla p, \nabla p)_{B_R} + \langle \lambda \nabla_{\partial D} p, \nabla_{\partial D} p\rangle_{\partial D}\,. \tag{3.56}$$

Now, if $u \in H_{\text{loc}}(\text{curl}\,, \mathbb{R}^3 \setminus \overline{B}_R)$ is the weak solution of

$$\text{curl}\,\text{curl}\, u - k^2 u = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{B}_R, \tag{3.57}$$

$$\nu \times u = \nu \times \nabla p \qquad \text{on } \Sigma, \tag{3.58}$$

that is,

$$\iint_{\mathbb{R}^3 \setminus \overline{B}_R} \left(\text{curl}\, u \cdot \text{curl}\, \varphi - k^2 u \cdot \varphi\right)\, dx + \int_\Sigma (\nu \times \text{curl}\, u) \cdot \varphi \; ds = 0$$

for all $\varphi \in H^1_{\text{loc}}(\text{curl}\,, \mathbb{R}^3 \setminus \overline{B}_R)$ of bounded support, together with the Silver-Müller radiation condition, then by the definition of G_e we have

$$G_e(\nu \times \nabla p) = \nu \times w \quad \text{on } \Sigma \tag{3.59}$$

where $w = \frac{1}{ik}\operatorname{curl} u$. Thus u solves

$$\iint_{\mathbb{R}^3\setminus\overline{B}_R} \left(\operatorname{curl} u \cdot \operatorname{curl}\varphi - k^2 u\cdot\varphi\right)\,dx - ik\int_\Sigma (\nu\times\varphi)\cdot w\,ds = 0$$

for all $\varphi \in H^1_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3 \setminus \overline{B}_R)$ of bounded support. Let $\phi \in C_0^\infty(\mathbb{R}^3)$ be real-valued with $\phi(x) = 1$ for $x \in B_{R'} \setminus \overline{B}_R$ and $\phi(x) = 0$ for $|x| > R' + 1$ where $R' > R$. We substitute $\varphi = \overline{w}\phi$ in the above variatonal equation and argue as in the uniqueness proof of theorem 2.4. The integral (2.12) appearing in the proof of theorem 2.4 is given by

$$\begin{aligned}
\int_\Sigma (\nu\times u)\cdot\overline{w}\,ds &\overset{(3.58)}{=} -\langle\nabla_\Sigma p, \nu\times w\rangle_\Sigma \\
&\overset{(3.59)}{=} -\langle\nabla_\Sigma p, G_e(\nu\times\nabla p)\rangle_\Sigma \\
&= -\overline{\langle G_e(\nu\times\nabla p), \nabla_\Sigma p, \rangle}_\Sigma \\
&\overset{(3.56)}{=} -\overline{\frac{k}{i}\left(\widetilde{\varepsilon}_r\nabla p, \nabla p\right)_{B_R}} - \overline{\frac{1}{ik}\langle\lambda\nabla_{\partial D} p, \nabla_{\partial D} p\rangle_{\partial D}} \\
&= -ik\overline{(\widetilde{\varepsilon}_r\nabla p, \nabla p)}_{B_R} - \frac{i}{k}\overline{\langle\lambda\nabla_{\partial D} p, \nabla_{\partial D} p\rangle_{\partial D}}.
\end{aligned}$$

Taking the complex conjugate of both sides, we obtain

$$\operatorname{Re}\left(\int_\Sigma (\nu\times u)\cdot\overline{w}\,ds\right) = -k\iint_{B_R}\underbrace{\operatorname{Im}(\widetilde{\varepsilon}_r)}_{\geq 0}|\nabla p|^2\,dx - \frac{1}{k}\int_{\partial D}\underbrace{\operatorname{Im}(\lambda)}_{>0}|\nabla_{\partial D} p|^2\,ds \leq 0.$$

The Silver-Müller radiation condition and Rellich lemma imply that $u = 0$ in $\mathbb{R}^3\setminus\overline{B}_R$. From (3.58), we conclude $\nabla_\Sigma p = 0$ on Σ, that it p is constant on Σ. By (3.56) this implies $(\widetilde{\varepsilon}_r\nabla p, \nabla p)_{B_R} + \langle\lambda\nabla_{\partial D} p, \nabla_{\partial D} p\rangle_{\partial D} = 0$. Since

$$\begin{aligned}
0 = (\widetilde{\varepsilon}_r\nabla p, \nabla p)_{B_R} + \langle\lambda\nabla_{\partial D} p, \nabla_{\partial D} p\rangle_{\partial D} &\geq \min\{\varepsilon^-, 1, \lambda^-\}\left(\|\nabla p\|^2_{B_R} + \|\nabla_{\partial D} p\|^2_{\partial D}\right) \\
&\geq \min\{\varepsilon^-, 1, \lambda^-\}(\|\nabla p\|^2_{B_R}
\end{aligned}$$

it follows that $\nabla p = 0$ in B_R, i.e. p constant on B_R. By the definition of the space S we have $\int_{\partial D} p\,ds = 0$, hence $p = 0$ in B_R.

□

3.3.3.2 Helmholtz decomposition

Now that we have characterized the null space of the curl operator, we can remove, or factor out, this component from X. By substituting $\psi = \nabla\xi$ for some $\xi \in S$ into the variational equation (3.37) motivates us to introduce the space

$$X_0 = \left\{ u \in X : -k^2 \left(\widetilde{\varepsilon}_r u, \nabla\xi\right)_{B_R} - \langle \lambda u, \nabla_{\partial D}\xi\rangle_{\partial D} + ik \left\langle G_e(\nu \times u), \nabla_\Sigma \xi\right\rangle_\Sigma = 0, \text{ for all } \xi \in S \right\}.$$

Let us state a lemma characterizing the elements of X_0.

Lemma 3.14. *Let u be an element of X. Then, u belongs to X_0 if and only if* $\operatorname{div}(\widetilde{\varepsilon}_r u) = 0$ *in $B_R \setminus \partial D$, $k^2(\nu \cdot u|_+ - \varepsilon_r \nu \cdot u|_-) = \nabla_{\partial D} \cdot (\lambda u_T)$ on ∂D and $\nu \cdot u = -\frac{i}{k}\nabla_\Sigma \cdot G_e(\nu \times u)$ on Σ.*

Proof. Consider $u \in X_0$. By definition, we have

$$-k^2 \left(\widetilde{\varepsilon}_r u, \nabla\xi\right)_{B_R} - \langle \lambda u, \nabla_{\partial D}\xi\rangle_{\partial D} + ik \left\langle G_e(\nu \times u), \nabla_\Sigma \xi\right\rangle_\Sigma = 0, \quad \text{for all } \xi \in S.$$

Taking $\xi \in C_0^\infty(B_R)$ with $\nabla\xi = 0$ on ∂D, and applying Green's theorem (1.7), we obtain

$$\begin{aligned} 0 &= \left(\widetilde{\varepsilon}_r u, \nabla\xi\right)_{B_R} = \left(\varepsilon_r u, \nabla\xi\right)_D \left(\varepsilon_r u, \nabla\xi\right)_\Omega \\ &= -\left(\operatorname{div}(\varepsilon_r u), \xi\right)_D - \left(\operatorname{div}(\varepsilon_r u), \xi\right)_\Omega \\ &= -\left(\operatorname{div}(\widetilde{\varepsilon}_r u), \xi\right)_{B_R} \end{aligned}$$

where we used that $B_R = D \cup \Omega$ and the fact that $\nabla\xi = 0$ on ∂D and on Σ. Consequently, $\operatorname{div}(\widetilde{\varepsilon}_r u) = 0$ in B_R. Taking now $\xi \in C_0^\infty(B_R)$ and applying again Green's theorem (1.7), we obtain

$$\begin{aligned} 0 &= -k^2 \left(\widetilde{\varepsilon}_r u, \nabla\xi\right)_{B_R} - \langle \lambda u, \nabla_{\partial D}\xi\rangle_{\partial D} \\ &= -k^2 \left(\varepsilon_r u, \nabla\xi\right)_D - k^2 \left(u, \nabla\xi\right)_\Omega - \langle \lambda u, \nabla_{\partial D}\xi\rangle_{\partial D} \\ &= k^2(\operatorname{div}(\widetilde{\varepsilon}_r u), \xi)_{B_R} + k^2 \langle \nu \cdot \varepsilon_r u|_- - \nu \cdot u|_+, \xi\rangle_{\partial D} + \langle \nabla_{\partial D} \cdot (\lambda u), \xi\rangle_{\partial D} \\ &= k^2 \langle \nu \cdot \varepsilon_r u|_- - \nu \cdot u|_+, \xi\rangle_{\partial D} + \langle \nabla_{\partial D} \cdot (\lambda u), \xi\rangle_{\partial D}. \end{aligned}$$

Consequently, $k^2(\nu\cdot u|_+ - \varepsilon_r\nu\cdot u|_-) = \nabla_{\partial D}\cdot(\lambda u_T)$ on ∂D. Now, if $\xi\in S$, we can write

$$\begin{aligned}
k^2\langle\nu\cdot u,\xi\rangle_\Sigma &= k^2(\operatorname{div}(u),\xi)_\Omega + k^2(u,\nabla\xi)_\Omega + k^2\langle\nu\cdot u|_+,\xi\rangle_{\partial D}\\
&= k^2(\operatorname{div}(u),\xi)_\Omega + k^2(u,\nabla\xi)_\Omega + k^2\langle\nu\cdot\varepsilon_r u|_-,\xi\rangle_{\partial D} - \langle\nabla_{\partial D}\cdot(\lambda u),\xi\rangle_{\partial D}\\
&= k^2(\operatorname{div}(\widetilde{\varepsilon}_r u),\xi)_\Omega + k^2(u,\nabla\xi)_\Omega + k^2(\operatorname{div}(\varepsilon_r u),\xi)_D + (\varepsilon_r u,\nabla\xi)_D + \langle\lambda u,\nabla_{\partial D}\xi\rangle_{\partial D}\\
&= k^2(\operatorname{div}(\widetilde{\varepsilon}_r u),\xi)_{B_R} + k^2(\widetilde{\varepsilon}_r u,\nabla\xi)_{B_R} + \langle\lambda u,\nabla_{\partial D}\xi\rangle_{\partial D}\\
&= ik\langle G_e(\nu\times u),\nabla_\Sigma\xi\rangle_\Sigma = -ik\langle\nabla_\Sigma\cdot G_e(\nu\times u),\xi\rangle_\Sigma\,.
\end{aligned}$$

Now, let $u\in X$ satisfy $\operatorname{div}(\widetilde{\varepsilon}_r u) = 0$ in B_R, $k^2(\nu\cdot u|_+ - \varepsilon_r\nu\cdot u|_-) = \nabla_{\partial D}\cdot(\lambda u_T)$ on ∂D and $\nu\cdot u = -\frac{i}{k}\nabla_\Sigma\cdot G_e(\nu\times u)$ on Σ. Then

$$\begin{aligned}
-k^2(\widetilde{\varepsilon}_r u,\nabla\xi)_{B_R} - \langle\lambda u,\nabla_{\partial D}\xi\rangle_{\partial D} &= -k^2(\varepsilon_r u,\nabla\xi)_D - k^2(u,\nabla\xi)_\Omega - \langle\lambda u,\nabla_{\partial D}\xi\rangle_{\partial D}\\
&= k^2\langle\nu\cdot\varepsilon_r u|_- - \nu\cdot u|_+,\xi\rangle_{\partial D} - \langle\lambda u,\nabla_{\partial D}\xi\rangle_{\partial D} + k^2\langle\nu\cdot u,\xi\rangle_\Sigma\\
&= k^2\langle\nu\cdot u,\xi\rangle_\Sigma\\
&= -ik\langle\nabla_\Sigma\cdot G_e(\nu\times u),\xi\rangle_\Sigma
\end{aligned}$$

for all $\xi\in S$. Consequently, $u\in X_0$. □

Next we can state the Helmholtz decomposition of the space X.

Theorem 3.15. *(a) The spaces X_0 and $\nabla S = \{\nabla\xi : \xi\in S\}$ are closed subspaces of X.*

(b) We may write

$$X = X_0\oplus\nabla S. \tag{3.60}$$

Proof. (a) We note that for $p\in S$ we have $\operatorname{curl}(\nabla p) = 0$ in B_R, $(\nabla p)_T\in L^2_t(\partial D)$ on ∂D. Hence $\nabla p\in X$ and closedness of ∇S in X follows from the closedness of S in $H^1(B_R)$. To show closedness of X_0 in X, we let $\xi\in S$ be fixed and consider the linear functionals

$$l_1(u) = (\widetilde{\varepsilon}_r u,\nabla\xi)_{B_R} + \langle\lambda u,\nabla_{\partial D}\xi\rangle_{\partial D}\,,\quad l_2(u) = \langle G_e(\nu\times u),\nabla_\Sigma\xi\rangle_\Sigma\quad u\in X.$$

Using the Cauchy–Schwarz inequality and the boundedness of $\widetilde{\varepsilon}_r$, λ and the trace operator, we conclude that $l_1 : X\to\mathbb{C}$ is bounded. The boundedness of l_2 on X follows by the boundedness of the trace operator $H(\operatorname{curl},B_R)\to H^{-\frac{1}{2}}(\operatorname{Div},\Sigma)$. This implies that X_0 is closed subspace of X.

(b) Let $u \in X$ be fixed and let $p \in S$ be the unique solution of

$$A(\nabla p, \nabla \xi) = A(u, \nabla \xi), \quad \text{for all } \xi \in S$$

which exists by theorem 3.13. Set $w = u - \nabla p$. Clearly $w \in X$. Moreover $w \in X_0$, because

$$\begin{aligned} & - k^2(\widetilde{\varepsilon}_r w, \nabla\xi)_{B_R} - \langle \lambda w, \nabla_{\partial D}\xi\rangle_{\partial D} + ik\,\langle G_e(\nu \times w), \nabla_\Sigma \xi\rangle_\Sigma \\ =& - k^2(\widetilde{\varepsilon}_r u, \nabla\xi)_{B_R} - \langle \lambda u, \nabla_{\partial D}\xi\rangle_{\partial D} + ik\,\langle G_e(\nu \times u), \nabla_\Sigma \xi\rangle_\Sigma \\ & \quad + k^2(\widetilde{\varepsilon}_r \nabla p, \nabla\xi)_{B_R} + \langle \lambda \nabla_{\partial D} p, \nabla_{\partial D}\xi\rangle_{\partial D} + ik\,\langle G_e(\nu \times \nabla p), \nabla_\Sigma \xi\rangle_\Sigma \\ =& A(u, \nabla\xi) - A(\nabla p, \nabla \xi) \\ =& 0. \end{aligned}$$

Thus, we conclude that $u = w + \nabla p$ for $w \in X_0$ and $p \in S$. It remains to show that $X_0 \cap \nabla S = \{0\}$. Suppose $u \in X_0 \cap \nabla S$. Then $u = \nabla p$ for some $p \in S$ and, since also $u \in X_0$, it holds that

$$0 = A(u, \nabla\xi) = A(\nabla p, \nabla \xi), \quad \text{for all } \xi \in S.$$

Theorem 3.13 now implies that $p = 0$.

□

Next, we prove a compactness property of the space X_0.

Theorem 3.16. *The space X_0 is compactly imbedded in $L^2(B_R)^3$.*

Proof. Let $\{u_j\}_j \subset X_0$ be a bounded sequence. By solving the exterior Maxwell problem

$$\begin{aligned} \operatorname{curl}\operatorname{curl} v_j - k^2 v_j = 0 \quad & \text{in } \mathbb{R}^3 \setminus \overline{B}_R, \\ \nu \times v_j = \nu \times u_j \quad & \text{on } \Sigma \end{aligned}$$

together with the Silver-Müller radiation condition at infinity, we can extend every $u_j \in X_0$ to all of $\mathbb{R}^3$. Then the function u_j^e defined by

$$u_j^e = \begin{cases} u_j & \text{in } B_R \\ v_j & \text{in } \mathbb{R}^3 \setminus \overline{B}_R \end{cases}$$

for all j, is in $H_{\mathrm{loc}}(\mathrm{curl}\,,\mathbb{R}^3)$ since the tangential components are continous across Σ. Let $R_1 > R$ and $\xi \in H_0^1(B_{R_1})$ with $\nabla_{\partial D}\xi \in L_t^2(\partial D)^3$. By lemma 3.14 and the definition of G_e, we obtain

$$\begin{aligned} k^2 \left\langle \nu \cdot u_j, \xi \right\rangle_\Sigma &= -ik \left\langle \nabla_\Sigma \cdot G_e(\nu \times u_j), \xi \right\rangle_\Sigma = ik \left\langle G_e(\nu \times u_j), \nabla_\Sigma \xi \right\rangle_\Sigma \\ &= \left\langle \nu \times \mathrm{curl}\, v_j, \nabla_\Sigma \xi \right\rangle_\Sigma = \left(\mathrm{curl}^2 v_j, \nabla \xi \right)_{B_{R_1} \setminus B_R} \\ &= -k^2 \left(v_j, \nabla \xi \right)_{B_{R_1} \setminus B_R} = k^2 \left(\mathrm{div}\, v_j, \xi \right)_{B_{R_1} \setminus B_R} + k^2 \left\langle \nu \cdot v_j, \xi \right\rangle_\Sigma \\ &= k^2 \left\langle \nu \cdot v_j, \xi \right\rangle_\Sigma \end{aligned}$$

that is, $\nu \cdot u_j = \nu \cdot v_j$ on Σ, where we used the Maxwell equation and the divergence-free condition of v_j outside B_R. Thus, the normal component of u_j^e is continous on Σ. Moreover, for ξ as here above, we have

$$k^2 \int_{B_{R_1}} \widetilde{\varepsilon}_r \; u_j^e \cdot \nabla \xi \, dx \; + \int_{\partial D} \lambda \, u_{jT} \cdot \nabla_{\partial D} \xi \, ds = 0.$$

Now, let $\phi \in C_0^\infty(\mathbb{R}^3)$ with support in B_{R_1} and $\phi = 1$ on B_R and consider the bounded sequence $\{\phi u_j^e\}_j$. Next, we extract a subsequence of $\{\phi u_j^e\}_j$ converging strongly in $L^2(B_R)^3$. Set

$$w_j = \phi u_j^e + \nabla p_j$$

for all j, where $p_j \in H_0^1(B_{R_1})$ solves

$$\int_{B_{R_1}} \widetilde{\varepsilon}_r \, \nabla p_j \cdot \nabla \xi \, dx \; = \int_{B_{R_1}} \widetilde{\varepsilon}_r \, (\phi - 1) u_j^e \cdot \nabla \xi \, dx \; = \; - \int_{B_{R_1}} \nabla(\widetilde{\varepsilon}_r \phi) \cdot v_j \, \xi \, dx \quad \text{for all } \xi \in H_0^1(B_{R_1})$$

because $\nabla \cdot v_j = 0$. For $\xi \in H_0^1(B_{R_1})$ with $\nabla_{\partial D}\xi \in L_t^2(\partial D)^3$ the right hand side is

$$\int_{B_{R_1}} \widetilde{\varepsilon}_r \, (\phi - 1) u_j^e \cdot \nabla \xi \, dx \; = \int_{B_{R_1}} \widetilde{\varepsilon}_r \, \phi \, u_j^e \cdot \nabla \xi \, dx \; + \; \frac{1}{k^2} \int_{\partial D} \lambda \, u_{jT} \cdot \nabla_{\partial D} \xi \, ds \, .$$

Therefore, $\nabla \cdot (\widetilde{\varepsilon}_r \nabla p_j) = -\nabla \cdot (\widetilde{\varepsilon}_r \phi u_j^e) = -\nabla(\widetilde{\varepsilon}_r \phi) \cdot v_j$ in B_{R_1}. Thus $\nabla \cdot w_j = 0$ in B_{R_1}, w_j bounded in $H(\mathrm{curl}\,, B_{R_1})$ and $\nu \times w_j = 0$ on ∂B_{R_1}. Therefore, there exists convergence subsequence in $L^2(B_{R_1})$. Also, from $\nabla \cdot (\widetilde{\varepsilon}_r \nabla p_j) = -\nabla(\widetilde{\varepsilon}_r \phi) \cdot v_j$ we conclude that p_j is bounded in $H^2(B_{R_1}$, thus contains a convergent subsequence in $H^1(B_{R_1)})$. Therefore, ϕu_j^e contains convergent subsequence in $L^2(B_{R_1})$. Since $\phi = 1$ on B_R we have a convergent subsequence of u_j in $L^2(B_R)$. □

Finally we return to problem $(P3)$. To discuss existence in the solution space X, we use the Helmholtz decomposition to decompose $E = E_0 + \nabla p$ for uniquely $E_0 \in X_0$ and $p \in S$. Substituting this decomposition into the variational equation (3.37) yields

$$A(E_0, \psi) + A(\nabla p, \psi) = B(\psi) \quad \text{for all } \psi \in X.$$

Taking $\psi \in S$ as a test function and recalling that

$$-k^2 \left(\widetilde{\varepsilon}_r E_0, \nabla \psi\right)_{B_R} - \langle \lambda E_0, \nabla_{\partial D} \psi \rangle_{\partial D} + ik \left\langle G_e(\nu \times E_0), \nabla_\Sigma \psi \right\rangle_\Sigma = 0 \quad \text{for all } \psi \in S$$

yields

$$A(\nabla p, \nabla \psi) = B(\nabla \psi) \quad \text{for all } \psi \in S.$$

By theorem 3.13, the above scalar problem has a unique solution $p \in S$. Defining

$$F(\psi) = B(\psi) - A(\nabla p, \psi) \quad \text{for all } \psi \in X_0$$

it thus remains to show that there exists a $E_0 \in X_0$ that satisfies the equation

$$A(E_0, \psi) = F(\psi) \quad \text{for all } \psi \in X_0 \tag{3.61}$$

and is continuously dependent on the data. To do so, we decompose the sesquilinear form into a coercive and compact part. To motivate a decomposition, we need to examine the Calderon operator in more detal. The following results are from [2], page 269. Suppose $\alpha \in H^{-\frac{1}{2}}(\mathrm{Div}\,, \Sigma)$ has the expansion

$$\alpha = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[c_{n,m} U_n^m + \widetilde{c}_{n,m} V_n^m \right].$$

and define $\widetilde{\delta} = iR \frac{\left(h_n^{(1)}\right)'(iR)}{h_n^{(1)}(iR)} + 1$ for $n \in \mathbb{N}$. The expansion for G_e in (3.50) we can write as

$$G_e \alpha = G_e^1 \alpha + G_e^2 \alpha$$

with

$$G_e^1 \alpha = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[-ikR \frac{\widetilde{c}_{n,m}}{\delta_n} U_n^m + \frac{c_{n,m}(\delta_n - \widetilde{\delta}_n)}{ikR} V_n^m \right], \qquad G_e^2 \alpha = \frac{1}{ikR} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} c_{n,m} \widetilde{\delta}_n V_n^m.$$

Part (a) of the next lemma corresponds to lemma 10.5 in [2].

Lemma 3.17. *(a) Let $\pi_t : H(\operatorname{curl}, B) \to H^{-\frac{1}{2}}(\operatorname{Div}, \Sigma)$ be the trace operator defined in remark 3.7. The operator $G_e^1 \circ \pi_t$, that is, the mapping $u \to G_e^1(\nu \times u)$, is compact from X_0 into $H^{-\frac{1}{2}}(\operatorname{Div}, \Sigma)$.*

(b) The operator G_e^2 satisfies

$$ik \left\langle G_e^2(\nu \times \alpha), \alpha_T \right\rangle_\Sigma \geq 0$$

for all $\alpha \in H^{-\frac{1}{2}}(\operatorname{Div}, \Sigma)$.

The above lemma suggests the decomposition $A = a_2 + b_2$ where

$$a_2(u, \psi) = \left(\frac{1}{\widetilde{\mu}_r} \operatorname{curl} u, \operatorname{curl} \psi \right)_{B_R} + k^2 \left(\widetilde{\varepsilon}_r u, \psi \right)_{B_R} - \langle \lambda u_T, \psi_T \rangle_{\partial D} + ik \left\langle G_e^2(\nu \times u), \psi_T \right\rangle_\Sigma ,$$
$$b_2(u, \psi) = -2k^2 \left(\widetilde{\varepsilon}_r u, \psi \right)_{B_R} + ik \left\langle G_e^1(\nu \times u), \psi_T \right\rangle_\Sigma .$$

With the above decomposition we prove in the next theorem that problem $(P3)$ satisfies the Fredholm property.

Theorem 3.18. *Under the same hypptheses as in theorem 3.13, the following hold:*

(a) There exist an isomorphism $\mathcal{J}_2 : X_0 \to X_0$ and a compact operator $\mathcal{K}_2 : X_0 \to X_0$ such that $A(u, \psi) = a_2(u, \psi) + b_2(u, \psi) = ((\mathcal{J}_2 + \mathcal{K}_2)u, \psi)_X$ for all $u, \psi \in X_0$.

(b) The operator $\mathcal{J}_2 + \mathcal{K}_2$ is an isomorphism from X_0 onto itself. The variational problem

$$A(E_0, \psi) = F(\psi), \quad \textit{for all } \psi \in X_0$$

is uniquely solvable in X_0 and the solution is given by $E_0 = (I + \mathcal{K}_2)^{-1} w$ where $w \in X_0$ satisfies $F(\psi) = (w, \psi)_X$ for all $\psi \in X_0$.

Proof. We note that the operator $A : X_0 \times X_0 \to \mathbb{C}$ is bounded. This follows from the boundedness of the Calderon and trace operator and the assumptions made for μ_r and ε_r. Thus $a_2, b_2 : X_0 \times X_0 \to \mathbb{C}$ are bounded.

(a) Using lemma 3.17 (b) and the arithmetic-geometric mean inequality, we obtain for any $\delta > 0$

$$
\begin{aligned}
|a_2(u,u)|^2 &\geq \left(\|\mu_r^{-\frac{1}{2}}\operatorname{curl} u\|_{B_R}^2 + k^2\|\operatorname{Re}(\varepsilon_r)^{\frac{1}{2}}u\|_{B_R}^2\right)^2 + \left(k^2\|\operatorname{Im}(\varepsilon_r)^{\frac{1}{2}}u\|_{B_R}^2 - \|\lambda^{\frac{1}{2}}u_t\|_{\partial D}^2\right)^2 \\
&\geq \|\mu_r^{-\frac{1}{2}}\operatorname{curl} u\|_{B_R}^4 + k^4\|\operatorname{Re}(\varepsilon_r)^{\frac{1}{2}}u\|_{B_R}^4 + k^4\|\operatorname{Im}(\varepsilon_r)^{\frac{1}{2}}u\|_{B_R}^4 + \|\lambda^{\frac{1}{2}}u_t\|_{\partial D}^4 \\
&\qquad - 2k^2\|\operatorname{Im}(\varepsilon_r)^{\frac{1}{2}}u\|_{B_R}^2\|\lambda^{\frac{1}{2}}u_t\|_{\partial D}^2 \\
&\geq \|\mu_r^{-\frac{1}{2}}\operatorname{curl} u\|_{B_R}^4 + k^4\|\operatorname{Re}(\varepsilon_r)^{\frac{1}{2}}u\|_{B_R}^4 + k^4\|\operatorname{Im}(\varepsilon_r)^{\frac{1}{2}}u\|_{B_R}^4 + \|\lambda^{\frac{1}{2}}u_t\|_{\partial D}^4 \\
&\qquad - k^4\frac{1}{\delta}\|\operatorname{Im}(\varepsilon_r)^{\frac{1}{2}}u\|_{B_R}^4 - \delta\|\lambda^{\frac{1}{2}}u_t\|_{\partial D}^4 \\
&= \|\mu_r^{-\frac{1}{2}}\operatorname{curl} u\|_{B_R}^4 + k^4\|\operatorname{Re}(\varepsilon_r)^{\frac{1}{2}}u\|_{B_R}^4 + k^4\left(1 - \frac{1}{\delta}\right)\|\operatorname{Im}(\varepsilon_r)^{\frac{1}{2}}u\|_{B_R}^4 \\
&\qquad + (1-\delta)\|\lambda^{\frac{1}{2}}u_t\|_{\partial D}^4.
\end{aligned}
$$

By the assumptions on ε_r, there is a constant $c_1 > 0$ such that Re $\varepsilon_r \geq c_1$ on D, and, by the boundedness of ε_r, there exists a constant $c_2 \geq 0$ such that Im $\varepsilon_r \leq c_2$ on D. Now, if we choose $\delta < 1$, we may estimate

$$
\|\operatorname{Re}(\varepsilon_r)^{\frac{1}{2}}u\|^4 + \left(1 - \frac{1}{\delta}\right)\|\operatorname{Im}(\varepsilon_r)^{\frac{1}{2}}u\|^4 \geq \left(c_1^2 + c_2^2 - \frac{1}{\delta}c_2^2\right)\|u\|_{B_R}^4.
$$

Thus, choosing δ such that $\frac{c_2^2}{c_1^2+c_2^2} < \delta < 1$ yields the coercivity of $a_2 : X_0 \times X_0 \to \mathbb{C}$. By the Lax-Milgram lemma, there exists an isomorphism $\mathcal{J}_2 : X_0 \to X_0$ satisfying

$$
a_2(u,\psi) = (\mathcal{J}_2 u, \psi)_X\,, \quad \text{for all } \psi \in X_0.
$$

Furthermore, by the Riesz representation theorem there exists a bounded linear operator $\mathcal{K}_2 : X_0 \to X_0$ defined by

$$
b_2(u,\psi) = (\mathcal{K}_2 u, \psi)_X\,, \quad \text{for all } \psi \in X_0.
$$

Using that the embedding $X_0 \to L^2(B_R)^3$ is compact and lemma 3.17 (a), we can argue as we did in the proof of theorem 3.13 (b) to prove that $\mathcal{K}_2$ is compact.

(b) Let $w \in X_0$ be such that $F(\psi) = (w,\psi)_X$ for all $\psi \in X_0$. Now we can write the variational problem $A(E_0,\psi) = F(\psi)$ as

$$
(\mathcal{J}_2 + \mathcal{K}_2)\,E_0 = w
$$

which is a Fredholm equation, hence it remains to prove that $(\mathcal{J}_2 + \mathcal{K}_2)\, E_0 = 0$ has only the trivial solution $E_0 = 0$. If $(\mathcal{J}_2 + \mathcal{K}_2)\, E_0 = 0$, then E_0 satisfies

$$A(E_0, \psi) = 0, \quad \text{for all } \psi \in X_0.$$

But, since $E_0 \in X_0$, we have for any $p \in S$,

$$A(E_0, \psi + \nabla p) = A(E_0, \psi) + A(E_0, \nabla p) = 0,$$

so that E_0, extended to $\mathbb{R}^3 \setminus \overline{B}_R$ as a solution of Maxwell's equations, is a weak solution of the scattering problem with vanishing incoming wave. Hence, by the uniqueness theorem 3.9, we conclude $E_0 = 0$.

The Fredholm alternative shows the existence of E_0 for general data and completes the proof.

□

We now combine these results in the following main theorem of this section.

Theorem 3.19. *Under the same hypotheses as Theorem 3.13 the variational problem* $(P3)$ *is uniquely solvable in* X *for every incident field* E^i *that satisfies* $\operatorname{curl}\operatorname{curl} E^i - k^2 E^i = 0$ *in* B_R.

It will be useful for the inverse problems to have a generalization of this theorem, that is, of the scattering problem (3.28)-(3.35). Let $f, g \in H^{-\frac{1}{2}}(\operatorname{Div}, \partial D)$ be given and suppose that we wish to find $E^{int} \in H(\operatorname{curl}, D)$ and $E^{ext} \in H_{\text{loc}}(\operatorname{curl}, \mathbb{R}^3 \setminus \overline{D})$ such that

$$\operatorname{curl}\left(\frac{1}{\mu_r}\operatorname{curl} E^{int}\right) - k^2\varepsilon_r E^{int} = 0, \qquad \text{in } D,$$
$$\operatorname{curl}\operatorname{curl} E^{ext} - k^2 E^{ext} = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{D},$$

$$\nu \times E^{ext} - \nu \times E^{int} = f \qquad \text{on } \partial D,$$
$$\nu \times \operatorname{curl} E^{ext} - \nu \times \frac{1}{\mu_r}\operatorname{curl} E^{int} - \lambda\nu \times (E^{int} \times \nu) = g \qquad \text{on } \partial D,$$

$$\lim_{|x|\to\infty} |x| \left(\operatorname{curl} E^{ext} \times \nu - ikE^{ext}\right) = 0.$$

Moreover, we want to show that

$$\|E^{int}\|_{H(\operatorname{curl},D)} + \|E^{ext}\|_{H_{\text{loc}}(\operatorname{curl},\mathbb{R}^3\setminus\overline{D})} \leq c\left(\|f\|_{\partial D} + \|g\|_{\partial D}\right)$$

for some positive constand c independent of f and g.

By the surjectivity of the trace operator, there exists a $F \in H(\operatorname{curl}, D)$ such that $\nu \times F = f$ on ∂D. Next we define

$$\hat{E} := \begin{cases} E^{int} + F, & \text{in } D, \\ E^{ext}, & \text{in } \mathbb{R}^3 \setminus \overline{D}. \end{cases}$$

Introducing the computational domain $\Omega = B_R \setminus \overline{D}$ as we did before, we obtain the following problem for $\hat{E}$:

$$\operatorname{curl}\left(\frac{1}{\mu_r}\operatorname{curl}\hat{E}\right) - k^2\varepsilon_r\hat{E} = -\operatorname{curl}\left(\frac{1}{\mu_r}\operatorname{curl}F\right) + k^2\varepsilon_r F, \qquad \text{in } D,$$
$$\operatorname{curl}\operatorname{curl}\hat{E} - k^2\hat{E} = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{D},$$

$$\nu \times \hat{E}|_+ - \nu \times \hat{E}|_- = 0 \qquad \text{on } \partial D,$$
$$\nu \times \operatorname{curl}\hat{E}|_+ - \nu \times \frac{1}{\mu_r}\operatorname{curl}\hat{E}|_- - \lambda\hat{E}_T|_- = g - \nu \times \frac{1}{\mu_r}\operatorname{curl}F - \lambda F_T \qquad \text{on } \partial D,$$

$$\nu \times \hat{E}|_+ - \nu \times \hat{E}|_- = 0 \qquad \text{on } \Sigma,$$
$$\frac{ik}{\nu} \times \operatorname{curl}\hat{E}|_+ - \frac{1}{ik}\nu \times \operatorname{curl}\hat{E}|_- = 0 \qquad \text{on } \Sigma,$$

$$\lim_{|x|\to\infty} |x|\left(\operatorname{curl}\hat{E} \times \nu - ik\hat{E}\right) = 0.$$

Multiplying the above differential equations with a test function $\psi \in X$ and using integration by parts (1.11) causes the above boundary conditions, and the introduction of the Calderon operator $G_e(\nu\times\hat{E}) = \nu\times\frac{1}{ik}\operatorname{curl}\hat{E}$, to yield the variational problem of determining $\hat{E} \in X$ such that

$$\left(\frac{1}{\widetilde{\mu}_r}\operatorname{curl}\hat{E}, \operatorname{curl}\psi\right)_{B_R} - k^2\left(\widetilde{\varepsilon}_r\hat{E}, \psi\right)_{B_R} - \left\langle\lambda\hat{E}_T, \psi_T\right\rangle_{\partial D} + ik\left\langle G_e(\nu\times\hat{E}), \psi_T\right\rangle_{\Sigma} =$$
$$\langle g, \psi_T\rangle_\Sigma - \left(\frac{1}{\mu_r}\operatorname{curl}F, \operatorname{curl}\psi\right)_\Omega + k^2\left(\varepsilon_r F, \psi\right)_\Omega + \langle\lambda F_T, \psi_T\rangle_{\partial D} \tag{3.62}$$

for all $\psi \in X$. Extending F by zero outside D and denoting the extension by $\widetilde{F}$, the above variational problem is equivalent to

$$A(\hat{E}, \psi) = -A(\widetilde{F}, \psi) + l_g(\psi) \quad \text{for all } \psi \in X \tag{3.63}$$

where $l_g(\psi) = \langle g, \psi_T\rangle_{\partial D}$. By the previous results, there exists a unique solution $\hat{E} \in X$ and

$$\|\hat{E}\|_X \le \|\widetilde{F}\|_X + \|g\|_{\partial D}.$$

Thus, $E^{int} \in H(\operatorname{curl}, D)$ and $E^{ext} \in H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3 \backslash \overline{D})$ exists. Now, a norm on $H^{-\frac{1}{2}}(\operatorname{Div}, \partial D)$ is given by

$$\|f\|_{H^{-\frac{1}{2}}(\operatorname{Div}, \partial D)} = \inf_{F \in H(\operatorname{curl}, D),\ \nu \times F = f} \|F\|_{H(\operatorname{curl}, D)},$$

see e.g. [2], page 58, thus taking the infimum over all $F \in H(\operatorname{curl}, D)$ with $\nu \times F = f$ on ∂D yields the desired inequality

$$\|E^{int}\|_{H(\operatorname{curl}, D)} + \|E^{ext}\|_{H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3 \backslash \overline{D})} \leq c\left(\|f\|_{\partial D} + \|g\|_{\partial D}\right). \tag{3.64}$$

3.4 Inverse problem

The inverse problem we will consider in this subsection is to determine the support of the medium D given the far field pattern of the scattered field for many incident plane waves. To be more precise, given the far field pattern E_∞, H_∞ corresponding to all incident plane waves

$$E^i(x, d, p) = ik(d \times p) \times d e^{ikx \cdot d}, \qquad H^i(x, d, p) = d \times p e^{ikx \cdot d}$$

for $x \in \mathbb{R}^3$, with polarization $p \in \mathbb{R}^3$ and incident direction d on the unit sphere S^2, find the support of D. We note that the H field can be computed from the E field (and vice versa). As we mentioned in the previous section, showing uniqueness of the inverse problem is equivalent to showing that the total field cannot satisfy the conductive transmission condition (3.11)-(3.12) for two different domains D_1 and D_2. We will use the same idea as in section 2.2.

Let D_1 and D_2 be two obstacles described by $\varepsilon_{r,1}$ and $\mu_{r,1}$ respectively, where $\varepsilon_{r,2}$ and $\mu_{r,2}$ satsfiy assumption 3.8. We assume that the obstacles are surrounded by vaccum and that the far field patterns $E_{1,\infty}(\hat{x}, d, p)$ and $E_{2,\infty}(\hat{x}, d, p)$ coincide for all $\hat{x}, d \in S^2$ and all $p \in \mathbb{R}^3$, $p \perp d$. We start by considering the scattering of electric dipole fields

$$E_e^i(x, z, p) = -\frac{1}{ik} \operatorname{curl}_x \operatorname{curl}_x p \Phi_k(x, z), \qquad H_e^i(x, z, p) = \operatorname{curl}_x p \Phi_k(x, z)$$

due to an electric dipole with polarization p with source point z in the unbounded component G of $\mathbb{R}^3 \setminus \left(\overline{D}_1 \cup \overline{D}_2\right)$.

Theorem 3.20. *Let G be the unboudned component of $\mathbb{R}^3 \setminus (\overline{D_1} \cup \overline{D_2})$ and $E_{1,\infty}(\hat{x}, d, p) = E_{2,\infty}(\hat{x}, d, p)$ for all $\hat{x}, d \in S^2$ and all $p \perp d$. Let $z \in G$ and $E_{e,j}(\cdot; z) = E_{e,j}$, $j = 1, 2$ be the* *unique solution of*

$$\operatorname{curl}\left(\frac{1}{\mu_r}\operatorname{curl} E_{e,j}\right) - k^2 E_{e,j} = 0 \quad \textit{in } D_j, \tag{3.65}$$

$$\operatorname{curl}\operatorname{curl} E^s_{e,j} - k^2 E^s_{e,j} = 0 \quad \textit{in } \mathbb{R}^3 \setminus \overline{D}_j, \tag{3.66}$$

$$\tag{3.67}$$

$$\nu \times E^s_{e,j} - \nu \times E_{e,j} = -\nu \times E^i_e \quad \textit{on } \partial D_j \tag{3.68}$$

$$\nu \times \operatorname{curl} E^s_{e,j} - \nu \times \frac{1}{\mu_r}\operatorname{curl} E_{e,j} - \lambda\nu \times (E_{e,j} \times \nu) = -\nu \times \operatorname{curl} E^i_e \quad \textit{on } \partial D_j, \tag{3.69}$$

$$\tag{3.70}$$

$$E^s_{e,j} \textit{ satisfies the Silver-Müller radiation condition} \tag{3.71}$$

in the variational sense. Then $E^s_{e,1}(x;z) = E^s_{e,2}(x;z)$ *for all* $x \in \overline{G}$, $x \neq z$.

Proof. We note that the existence of a unique solution follows from the previous existence result (we do not encounter the same difficulty as we did for the impenetrable case). By Rellich's lemma 1.9, from the coincidence of the far field patterns for plane wave incidence, it follows that the corresponding scattered waves satisfy $E^s_1(\cdot, d, p) = E^s_2(\cdot, d, p)$ for all $d \in S^2$ and all $p \perp d$ in the exterior of the ball B_R of radius $R > 0$, where R is choosen large enough such that $\overline{D}_1 \cup \overline{D} \subset B_R$. Since the solutions E^s_1, E^s_2 are analytic outisde D_1 and D_2, respectively, we obtain that the scattered fields coincide outside $\overline{D}_1 \cup \overline{D}_2$. Applying the unique continuation principle 1.11 to $E^s = E^s_1 - E^s_2$ in a similar way as was done in the proof of theorem 3.9, we conclude that $E^s_1(\cdot, d, p) = E^s_2(\cdot, d, p)$ in G for all $d \in S^2$ and all $p \perp d$.

Now, from the mixed reciprocity relation (2.11) for scattering of electric dipole fields we conclude

$$E_{e,1,\infty}(\cdot; z, q) = E_{e,2,\infty}(\cdot; z, q) \quad \text{on } S^2,$$

for all $z \in G$ and all polarizations q. Again by Rellich's lemma and the analycity of the solutions, this implies that the corresponding scattered waves coincide $E^s_{e,1}(x; z, q) = E^s_{e,2}(x; z, q)$ for all x outside B_R, $z \in G$ and all polarizations q. □

Next we turn to the main result.

Theorem 3.21. *Let* $E^s(\cdot, d, p)$ *and* $E_\infty(\cdot, d, p)$ *be the scattered wave and far field pattern, respectively, corresponding to the plane wave* $E^i(x, d, p) = ik(d \times p) \times de^{ikx\cdot d}$, $x \in \mathbb{R}^3$ *with* *propagation direction $d \in S^2$ and polarization $p \perp d$. If the far field patterns $E_{\infty,1}(\hat{x}, d, p)$ and $E_{\infty,2}(\hat{x}, d, p)$ for the obstacles D_1 and D_2 coincide for all incident directions d, all polarizations $p \perp d$ and all observations $\hat{x}$, then $D_1 = D_2$.*

Proof. We prove the claim by contradiction and assume that $D_1 \neq D_2$. Then, without loss of generality, there exists $z^* \in \partial G$ such that $z^* \in \partial D_1$ and $z^* \notin \overline{D}_2$. We can choose $h > 0$ such that the sequence

$$z_n := z^* + \frac{h}{n}\nu(z^*), \quad n = 1, 2, 3, ...$$

is contained in G. Consider the (variational) solution $E_{e,n,j}$, $j = 1, 2$ to the boundary value problem (3.65)-(3.71) with z replaced by z_n. By theorem 3.20, it holds that $E^s_{e,n,1} = E^s_{e,n,2}$ in $\overline{G}$. Consider $E_{e,n} = E_{e,n,2}$ as the scattered field corresponding to D_2. Since z^* has positive distance from $\overline{D}_2$, we conclude from the well-posedness of the transmission problem and (3.64) that there exists a $C > 0$ with

$$\begin{aligned} &\|E_{e,n}(\cdot; z_n)\|_{H(\mathrm{curl}, D_2)} + \|E^s_{e,n}(\cdot; z_n)\|_{H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \overline{D}_2)} \\ &\leq \|\nu \times E^i_e(\cdot, z_n)\|_{\partial D_2} + \|\nu \times \mathrm{curl}\, E^i_e(\cdot, z_n)\|_{\partial D_2} \\ &\leq C \quad \text{for sufficiently large } n. \end{aligned}$$

On the other hand, considering $E_{e,n} = E_{e,n,1}$ as the scattered field corresponding to D_1 we conclude, due to the singularity of $\Phi_k(\cdot, z_n)$,

$$\begin{aligned} \infty \leftarrow &\|\nu \times E^i_e(\cdot; z_n)\|_{\partial D_1} + \|\nu \times \mathrm{curl}\, E^i_e(\cdot; z_n)\|_{\partial D_1} \\ = &\|\nu \times E_e(\cdot; z_n) - \nu \times E^s_e(\cdot; z_n)\|_{\partial D_1} + \|\nu \times \frac{1}{\mu_r}\mathrm{curl}\, E_e(\cdot; z_n) - \nu \times \mathrm{curl}\, E^s_e(\cdot; z_n) \\ &\quad + \lambda\nu \times (E_e(\cdot; z_n) \times \nu)\|_{\partial D_1} \\ \leq &\|\nu \times E_e(\cdot; z_n)\|_{\partial D_1} + \|\nu \times E^s_e(\cdot; z_n)\|_{\partial D_1} + c_1\|\nu \times \mathrm{curl}\, E_e(\cdot; z_n)\|_{\partial D_1} \\ &\quad + \|\nu \times \mathrm{curl}\, E^s_e(\cdot; z_n)\|_{\partial D_1} + c_2\|\nu \times (E_e(\cdot; z_n) \times \nu)\|_{\partial D_1} \\ \leq &\hat{C}\left(\|E_e(\cdot; z_n)\|_{H(\mathrm{curl}, D_1)} + \|E^s_e(\cdot; z_n)\|_{H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \overline{D}_1)}\right) \end{aligned}$$

where c_1, c_2 and $\hat{C}$ are positive constants (c_1 and c_2 are due to estimating μ_r and λ). Here we used transmission conditions (3.68)-(3.69), the triangular inequality and the boundedness of the trace operators. This is a contradiction, and thus we conclude $D_1 = D_2$. □

4 Electromagnetic transmission eigenvalue problem

To motivate the topic of our next chapter, we recall that, in both sections Two and Three, in order to demonstrate the existence of the scattering problems $(P1)$ and $(P2)$, we had to assume that k^2 is not a transmission eigenvalue for D. That is, we had to exclude particular frequencies for which there exists an incident wave that does not scatter, called transmission eigenvalues. Generally speaking, for the transmission eigenvalue problems that have been studied, the following results have been shown to hold:

- Transmission eigenvalues exist.
- They form a discrete set.
- Estimates on the parameters of the scatterer ε_r and μ_r with respect to transmission eigenvalues were found.

In the study of the inverse problem, the transmission eigenvalue problem plays an important role in determining the shape of a penetrable medium from knowledge of the time-harmonic incident waves and the far field patterns of the scattered waves. Thus, for example, it provied a powerful tool for establishing uniqueness in case of anisotropic media and also plays a fundametal role in reconstruction the support of penetrable objects, [28].

One possible approach solving the transmission eigenvalue problem is the use of an integral-type method. For instance, if the index of refraction is assumed to be smooth inside the medium and to have no jump across the boundary, this type of method has been successfully applied to the case of an inhomogeneous medium; [29]. However, as presented in [30], it gives only partial answeres in the case of anisotropic media. Another approach for solving the transmission eigenvalue problem is using a variational framework where less regularity on the index of refraction is required; [28].

In this section, the underlying scattering problem is the scattering problem (3.1)-(3.4) of section Three. That is, the scattering of electromagnetic waves by an inhomogeneous medium of bounded support D situated in a homogeneous backround with conductive transmission conditions, which is given by, in terms of the electric field:

$$
\begin{aligned}
&\operatorname{curl}\left(\frac{1}{\mu_r}\operatorname{curl} E\right) - k^2\varepsilon_r E = 0 && \text{in } D,\\
&\operatorname{curl}\operatorname{curl} E - k^2 E = 0 && \text{in } \mathbb{R}^3 \setminus \overline{D},\\
&E = E^i + E^s && \text{in } \mathbb{R}^3 \setminus \overline{D},\\
&\nu \times E_+ - \nu \times E_- = 0 && \text{on}\\
&\nu \times \operatorname{curl} E_+ - \nu \times \frac{1}{\mu_r}\operatorname{curl} E_- - i\omega\mu_0\beta E_T = 0 && \text{on } \partial D,\\
&\lim_{|x|\to\infty} |x| \left(\operatorname{curl} E^s \times \nu - ikE^s\right) = 0,
\end{aligned}
$$

where $E_T := \nu \times (E \times \nu)$, as before, but now with $\beta(x) < 0$ for all $x \in \partial D$. To solve the transmission eigenvalue problem, we use a variational framework, in particular, the T-coercivity approach, to show that our transmission problem is of Fredholm type and that the transmission eigenvalues form at most a discrete set when $\varepsilon_r - 1$ and $\mu_r - 1$ are either positive or negative in a neighborhood of the boundary and can change sign inside the domain D. The T-coercivity aproach was used initially to study metamaterials in [17]. In the case of 2D configurations, that is, in Helmholtz-like problems, this approach is a reformulation of the Banach-Neas-Babuska theory. Whereas the so-called BNB approach relies on an abstract inf-sup condition, T-coercivity uses explicit inf-sup operators. In case of non-conductive transmission conditions, the T-coercivity approach was successfully applied in [16] to show discreteness of the transmission eigenvalues when the contrast was either positive or negative in a neighborhood of the boundary and could change sign inside the domain D. Closely following their study, we adjusted it to fit our problem. [32] employed the T-coercivity approach to investigate the time-harmonic Maxwell problem in a composite material surrounded by a perfect conductor with sign changing coefficients.

Another important question related to transmission eigenvalue problems is proving the existence of transmission eigenvalues, which can then be used in determining the values of the physical parameters of the inclusion. This question will not be dealt with here. Indeed, up to now, the T-coercivity approach appears inefficient to show these kinds of results because the formulation on which we work, although it presents the useful property of positivity, is not symmetric and thus prevents using the *min-max* arguments; see [33].

To cover all aspects, we would like to mention that the transmission eigenvalue problem for scattering problems in case of impenetrable obstacles, that is, the transmission eigenvalue problem for regions with cavities, was studied in [27]. There the problem was reformulated into a fourth order boundary value problem to establish the Fredholm property of the problem and to show that the transmission eigenvalues exist and form a discrete set. This kind of reformulation, is in fact not new and was used previously in [31] to study the acoustic case of a nonabsorbing inhomogeneous medium and in [28] to study the electromagnetic case for anisotropic media.

4.1 Problem statement

Let $D \subset \mathbb{R}^3$ be a bounded, simply connected Lipschitz domain with boundary ∂D described by the electric permittivity $\varepsilon \in L^\infty(D)$ and magnetic permeability $\mu \in L^\infty(D)$. Further, let ν denote the unit outward normal to ∂D and set $\lambda = -i\omega\mu_0\beta$. The transmission eigenvalue problem is related to non-scattering incident fields. In particular, if E^i is such that $E^s = 0$, then E and $E_0 = E^i|_D$ satisfy the following homogeneous problem:

$$\operatorname{curl}\left(\frac{1}{\mu_r}\operatorname{curl} E\right) - k^2\varepsilon_r E = 0 \qquad \text{in } D, \tag{4.1}$$

$$\operatorname{curl}\operatorname{curl} E_0 - k^2 E_0 = 0 \qquad \text{in } D, \tag{4.2}$$

$$\nu \times E = \nu \times E_0 \qquad \text{on } \partial D, \tag{4.3}$$

$$\nu \times \frac{1}{\mu_r}\operatorname{curl} E - \lambda E_T = \nu \times \operatorname{curl} E_0 \qquad \text{on } \partial D \tag{4.4}$$

which is referred to as the *transmission eigenvalue problem*. Conversely, if (4.1)-(4.4) has a nontrivial solution E and E_0 and if E_0 can be extended outside D as a solution to $\operatorname{curl}\operatorname{curl} E_0 - k^2 E_0$, then if this extended solution E_0 is considered as the incident field, the corresponding scattered field is $E^s = 0$.

Definition 4.1. *Values of $k \in \mathbb{C}$ such that there exists a pair $(E, E_0) \neq (0, 0)$ solving the* *transmission eigenvalue problem (4.1)-(4.4), that is*

$$\begin{aligned}
&\operatorname{curl}\left(\frac{1}{\mu_r}\operatorname{curl} E\right) - k^2\varepsilon_r E = 0 && \textit{in } D,\\
&\operatorname{curl}\operatorname{curl} E_0 - k^2 E_0 = 0 && \textit{in } D,\\
&\nu \times E = \nu \times E_0 && \textit{on } \partial D,\\
&\nu \times \frac{1}{\mu_r}\operatorname{curl} E - \lambda E_T = \nu \times \operatorname{curl} E_0 && \textit{on } \partial D
\end{aligned}$$

are called transmission eigenvalues.

Multiplying (4.1)-(4.2) with two test functions ϕ and ψ, respectively, such that $\nu\times\phi = \nu\times\psi$ on ∂D, using integration by parts (1.11) and boundary condition (4.4) yields the variational equation:

$$\begin{aligned}
0 = \int_D \left(\frac{1}{\mu_r}\operatorname{curl} E\cdot\operatorname{curl}\phi - k^2\varepsilon_r E\cdot\phi\right)\, dx - \int_D \left(\operatorname{curl} E_0\cdot\operatorname{curl}\psi - k^2 E_0\cdot\psi\right)\, dx\\
+ \int_{\partial D} \lambda\left((\nu\times E)\times\nu\right)\cdot\phi_T\, ds.
\end{aligned}$$

For all the terms in the above equation to be well defined, we set

$$H_{\mathrm{imp}}(\operatorname{curl} D) = \left\{U \in H(\operatorname{curl}, D) : U_T \in L^2_t(\partial D)^3\right\}$$

and define our solution space by the following:

$$X(D) = \left\{(U,V) \in H_{\mathrm{imp}}(\operatorname{curl}, D)\times H(\operatorname{curl}, D) : U - V \in H_0(\operatorname{curl}, D)\right\}.$$

We equipp $X(D)$ with the inner product

$$((U,V),(U',V'))_{X(D)} = (U,U')_D + (\operatorname{curl} U, \operatorname{curl} U')_D + (V,V')_D + (\operatorname{curl} V, \operatorname{curl} V')_D + \langle U_T, U'_T\rangle_{\partial D}.$$

We note that the above inner product gives rise to the norm $\|\cdot\|_{X(D)}$ defined as follows, for every $(U,V)\in X(D)$:

$$\begin{aligned}
\|(U,V)\|^2_{X(D)} &= \|U\|^2_D + \|\operatorname{curl} U\|^2_D + \|V\|^2_D + \|\operatorname{curl} V\|^2_D + \|U_T\|^2_{\partial D}\\
&= \|U\|^2_{H(\operatorname{curl}, D)} + \|V\|^2_{H(\operatorname{curl}, D)} + \|U_T\|^2_{\partial D}.
\end{aligned}$$

Moreover, for any $(U,V) \in X(D)$ it holds that $\langle U_T, V_T \rangle_{\partial D} = \langle \nu \times U, \nu \times V \rangle_{\partial D}$, so that we can use $\|\nu \times U\|_{\partial D}$ in place of the corresponding norm on U_T in the definition of $X(D)$. $X(D)$ together with $(\cdot,\cdot)_{X(D)}$ defines a Hilbert space, and the proof is simular as the proof from theorem 4.1 in [2]. We state the variational problem:

Problem statement (P4):

Determine $(E, E_0) \in X(D)$ such that

$$\left(\frac{1}{\mu_r}\operatorname{curl} E, \operatorname{curl} \phi\right)_D - k^2 (\varepsilon_r E, \phi)_D - (\operatorname{curl} E_0, \operatorname{curl} \psi)_D + k^2 (E_0, \psi)_D + \langle \lambda E_T, \phi_T \rangle_{\partial D} = 0 \tag{4.5}$$

for all $(\phi, \psi) \in X(D)$.

We note that, if (E, E_0) solves problem $(P4)$, then it is easily verified, by choosing sufficiently smooth test functions, that (E, E_0) satisfies the differential equations (4.1) and (4.2) in D and boundary conditions (4.3) and (4.4) on ∂D.

We introduce the sesquilinear form on $X(D) \times X(D)$ as follows:

$$\begin{aligned} a_k((U,V),(U',V')) = {} & \left(\frac{1}{\mu_r}\operatorname{curl} U, \operatorname{curl} U'\right)_D - (\operatorname{curl} V, \operatorname{curl} V')_D - k^2\big[\,(\varepsilon_r U, U')_D - (V, V')_D\,\big] \\ & + \langle \lambda U_T, U'_T \rangle_{\partial D} \,. \end{aligned}$$

and make the following assumptions regarding the data.

Assumption 4.2.
- *$D \subset \mathbb{R}^3$ is a bounded and simply connected Lipschitz domain with connected boundary ∂D.*
- *$\varepsilon_r \in L^\infty(D)$, $\mu_r \in L^\infty(D)$ rela-valued such that $\frac{1}{\varepsilon_r} \in L^\infty(D)$ and $\frac{1}{\mu_r} \in L^\infty(D)$.*
- *λ pure imaginary with Im $\lambda > 0$.*

We note that, in the previous section we assumed Im λ to be negative. Why we need positiveness will be made clear in the proof of lemma 4.10. We define

$$\varepsilon_- = \inf_{x \in D} |\varepsilon_r(x)| < \infty, \quad \varepsilon_+ = \sup_{x \in D} |\varepsilon_r(x)| < \infty,$$
$$\mu_- = \inf_{x \in D} |\mu_r(x)| < \infty, \quad \mu_+ = \sup_{x \in D} |\mu_r(x)| < \infty$$

and let

$$\varepsilon_* > 1, \quad \varepsilon^* < 1 \quad \mu_* > 1, \quad \mu^* < 1$$

be constants, allowing us to state some assumptions on the values of ε_r and μ_r in a neighborhood of the boundary ∂D. We note that, under the assumptions on ε_r and μ_r, the sesquilinear form a_k is bounded on $X(D) \times X(D)$.

Let $(U, V) \in X(D)$, then $\nu \times U = \nu \times V$ on ∂D. If we assume that $\operatorname{curl} U = \operatorname{curl} V = 0$ in D and $\nu \times U = \nu \times V = 0$ on ∂D, then there exist scalar potential $(\xi, \eta) \in H^1(D) \times H^1(D)$ such that $(U, V) = (\nabla\xi, \nabla\eta)$. This follows from theorem 3.37 in [2]. The perfect conducting boundary condition implies that $\nu \times \nabla\xi = \nu \times \nabla\eta = 0$. Thus, ξ and η are constant on ∂D and we can choose the same constant for ξ and η on ∂D. Then set

$$S(D) = \left\{(\xi, \eta) \in H^1(D) \times H^1(D) : \xi = \eta = \text{constant on } \partial D \text{ and } \langle \xi, 1\rangle_{\partial D} = \langle \eta, 1\rangle_{\partial D} = 0\right\}.$$

The condition $\langle \xi, 1\rangle_{\partial D} = \langle \eta, 1\rangle_{\partial D} = 0$ for elements (ξ, η) of S is only used to set the constants, that is, if $(\varphi_1, \varphi_2) \in S \cap \mathbb{C}^2$, then $\varphi_1 = \varphi_2 = 0$. It is easily verified that $S(D)$ together with

$$\big((u, v), (\xi, \eta)\big)_{S(D)} = (\nabla u, \nabla\xi)_D + (\nabla v, \nabla\eta)_D$$

defines an inner product space. We note that $(\xi, \eta) \in S(D)$ implies $(\nabla\xi, \nabla\eta) \in X(D)$ because $(\nabla\xi - \nabla\eta) \times \nu = 0$ and $(\nabla\xi)_T = \nabla_{\partial D}\xi = 0$ on ∂D, where $\nabla_{\partial D}$ denotes the surface gradient.

Now, if (E, E_0) solves $(P4)$, then substituting $\phi = \nabla\xi$ and $\psi = \nabla\eta$ into the variational equation (4.5) yields

$$(\varepsilon_r E, \nabla\xi)_D - (E_0, \nabla\eta)_D = 0 \tag{4.6}$$

for all $(\xi, \eta) \in S$, leading us to define the space

$$X_0(D) = \{(U, V) \in X(D) : (\varepsilon_r U, \nabla\xi)_D - (V, \nabla\eta)_D = 0 \text{ for all } (\xi, \eta) \in S(D)\}$$

and the scalar problem of determining $(u, v) \in S(D)$ such that

$$(\varepsilon_r \nabla u, \nabla\xi)_D - (\nabla v, \nabla\eta)_D = 0 \quad \text{for all } (\xi, \eta) \in S(D).$$

The above scalar problem is a general case of the following problem: determine $(u, v) \in$

$S(D)$ such that

$$(\varepsilon_r u, \nabla\xi)_D - (v, \nabla\eta)_D = f\left((\xi,\eta)\right) \quad \text{for all } (\xi,\eta) \in S(D) \tag{4.7}$$

where $f \in S(D)'$. Here $S(D)'$ denotes the dual space of $S(D)$.

The solution space $X(D)$ is not compactly imbedded in $L^2(D) \times L^2(D)$, and so we can not apply the analytic Fredholm theorem. As we saw in the previous chapter, the compactness will be obtained by taking into account the divergence-free condition working in the space $X_0(D)$. We end this section by characterizing the elements of $X_0(D)$ and consider a compactness property of $X_0(D)$. We then study the scalar problem, which will be useful since doing so will provide information about the space $X_0(D) \cap \nabla S(D)$ and we will be able to present the T-coercivity approach on a simpler example.

Lemma 4.3. *Let $(U,V) \in X(D)$. Then, (U,V) belongs to $X_0(D)$ if and only if* $\operatorname{div}(\varepsilon_r U) = \operatorname{div} V = 0$ *in D and $\nu \cdot (\varepsilon_r U - V) = 0$ on ∂D.*

Proof. Let $(U,V) \in X_0(D)$. Then, for all $(\xi,\eta) \in S(D)$, by definition of $X_0(D)$, we obtain

$$0 = (\varepsilon_r U, \nabla\xi)_D - (V, \nabla\eta)_D = -\left(\operatorname{div}(\varepsilon_r U), \xi\right)_D + \left(\operatorname{div} V, \eta\right)_D + \langle \nu \cdot \varepsilon_r U, \xi\rangle_{\partial D} + \langle \nu \cdot V, \eta\rangle_{\partial D}$$

where we used (1.7). Choosing $\xi = 0$ and $\eta \in C_0^\infty(D)$ yields $\operatorname{div} V = 0$ in D. Analogously $\operatorname{div}(\varepsilon_r U) = 0$ in D when $\eta = 0$ and $\xi \in C_0^\infty(D)$. Now for $\xi \in H^1(D)$, using (1.7) we can write

$$\langle \nu \cdot (\varepsilon_r U - V), \xi\rangle_{\partial D} = \left(\operatorname{div}(\varepsilon_r U - V), \xi\right)_D + (\varepsilon_r U - V, \nabla\xi)_D = (\varepsilon_r U, \nabla\xi)_D - (V, \nabla\xi)_D = 0$$

and hence $\nu \cdot (\varepsilon_r U - V) = 0$ on ∂D.

If $(U,V) \in X(D)$ satisfies $\operatorname{div}(\varepsilon_r U) = \operatorname{div} V = 0$ in D and $\nu \cdot (\varepsilon_r U - V) = 0$ on ∂D, then for all $(\xi,\eta) \in S(D)$ we obtain

$$(\varepsilon_r U, \nabla\xi)_D - (V, \nabla\xi)_D = \langle \nu \cdot \varepsilon_r U, \xi\rangle_{\partial D} - \langle \nu \cdot V, \eta\rangle_{\partial D} = \langle \nu \cdot (\varepsilon_r U - V), \xi\rangle_{\partial D} = 0.$$

□

Set $\hat{X}(D) := \{(U,V) \in H(\operatorname{curl}, D) \times H(\operatorname{curl}, D) : U - V \in H_0(\operatorname{curl}, D)\}$ and let $\hat{X}_0(D)$ be the space $X_0(D)$ with elements from $\hat{X}(D)$. By theorem 4.1 in [16], the space $\hat{X}_0(D)$ is compactly imbedded in $L^2(D)^3 \times L^2(D)^3$ when $\varepsilon_r \geq \varepsilon_* > 1$ or $\varepsilon_r \leq \varepsilon^* < 1$. Since for any $U \in X_0(D)$, we have $\|U\|_{\hat{X}(D)} \leq \|U\|_{X(D)}$, we conclude that $X_0(D)$ is compactly imbedded in $L^2(D)^3 \times L^2(D)^3$.

4.2 The scalar problem

Define the sesquilinear form $b\left((u,v)(\xi,\eta)\right) = (\varepsilon_r \nabla u, \nabla \xi)_D - (\nabla v, \nabla \eta)_D$ and, with the help of the Riesz representation theorem, the operator $\mathcal{B} : S(D) \to S(D)$, such that

$$\big(\mathcal{B}(u,v),(\xi,\eta)\big)_{S(D)\times S(D)} = b\big((u,v),(\xi,\eta)\big) \quad \text{for all } (u,v),(\xi,\eta) \in S(D).$$

Due to the negative sign in front of the term $(\nabla v, \nabla \eta)_D$, it is not possible to directly show that the variational formulation leads to a Fredholm equation. In particular, the sequilinear form b is not coercive, nor 'coercive + compact' as was the case for the scalar problem in subsection 3.3.3.1. To overcome this problem and restore some property of positivity for the principal part of b, we apply the T-coercivity approach. We recall the general concept before we continue with the scalar problem.

T-coercivity approach

Let $\mathcal{H}$ denote a Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{H}}$.

Definition 4.4. *Let T be an isomorphism on $\mathcal{H}$. A sesquilinear form $b(\cdot,\cdot)$ is T-coercive on $\mathcal{H} \times \mathcal{H}$ if there exists a constant $\gamma > 0$ such that for all $v \in \mathcal{H}$ it holds that*

$$|b(v, Tv)| \geq \gamma \|v\|_{\mathcal{H}}^2.$$

We note that, if the sesquilinear form $b : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is T-coercive, then using the Lax-Milgram theorem and since T is an isomorphism on $\mathcal{H}$, the operator $B : \mathcal{H} \to \mathcal{H}$ defined by $(B(v), w)_{\mathcal{H}} = b(v,w)$ for all $v, w \in \mathcal{H}$ is an isomorphism. We can generalize this statement and state the next theorem. The proof can be found in [17].

Theorem 4.5. *Let $l : \mathcal{H} \to \mathbb{C}$ be a linear and continuous functional on $\mathcal{H}$ and let $a(\cdot,\cdot)$ be a continuous sesquilinear form on $\mathcal{H} \times \mathcal{H}$. Assume that a can be split as $a(\cdot,\cdot) = b(\cdot,\cdot) + c(\cdot,\cdot)$ where the sesqulinear forms $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are both continuous and linear on $\mathcal{H} \times \mathcal{H}$ and the bounded linear operator C on $\mathcal{H}$ associated with $c(\cdot,\cdot)$ is compact. Assume, moreover,* *that there exists an isomorphism T on $\mathcal{H}$ such that $b(\cdot,\cdot)$ is T-coercive on $\mathcal{H}\times\mathcal{H}$. Then the variational problem of determining $u\in\mathcal{H}$ such that*

$$a(u,v)=l(v) \quad \textit{for all } v\in\mathcal{H} \tag{4.8}$$

has a solution if and only if uniqueness holds, that is, the only solution of (4.8) with $l=0$ is $u=0$.

Case when (ε_r-1) is positive

As mentioned in the introduction, we closely follow chapter 2.2 in [16] where the case $(\varepsilon_r-1)<0$ was fully considered. For the sake of completion, we thus consider the case $(\varepsilon_r-1)>0$, in particular, $1<\varepsilon_-\leq\varepsilon_r$. Our goal is to apply theorem 4.5 to the scalar problem (4.7), and the key is to be able to construct an appropiate isomorphism T on $S(D)$. An obvious first idea would be to consider $T(\xi,\eta)=(\xi,-\eta)$ in order to change the sign of the term $(\nabla v,\nabla\eta)_D$ in the variational formulation (4.7). But, unfortunately, $(\xi,-\eta)$ is not in $S(D)$ because $\xi\neq-\eta$ on ∂D. Thus, we need to modify T so that it satisfies all the properties of $S(D)$. We define $T:S(D)\to S(D)$ by

$$T(\xi,\eta)=(\xi,-\eta+2\xi).$$

Then, $T(\xi,\eta)\in S(D)$ because $\xi+\eta-2\xi=-\xi+\eta=0$ on ∂D and $\langle\xi,1\rangle_{\partial D}=\langle\eta-2\xi,1\rangle_{\partial D}=0$. Moreover, since $T^2(\xi,\eta)=T(\xi,-\eta+2\xi)=(\xi,\eta-2\xi+2\xi)=(\xi,\eta)$, that is, $T^2=I$ where I denotes the identity operator, T is isomorphism of $S(D)$. With the help of T, we define a new sequilinear form

$$b^T\big((u,v),(\xi,\eta)\big)=b\big((u,v),T(\xi,\eta)\big).$$

We note that b^T is coercive if and only if b is T-coercive. In particular, (u,v) is a solution to the scalar problem $b\big((u,v),(\xi,\eta)\big)=f\big((\xi,\eta)\big)$ if and only if (u,v) is a solution to $b^T\big((u,v),(\xi,\eta)\big)=f\big(T(\xi,\eta)\big)$. In the next lemma, we examine the coercivity of the sesquilinear form b^T.

Lemma 4.6. *Assume that $\varepsilon_->1$. Then the operator $\mathcal{B}$ associated with the scalar problem (4.7) is an isomorphism of $S(D)$.*

Proof. Let $(\xi,\eta)\in S(D)$ and $\tau>0$. Then, using Young's inequality, we obtain the following estimate:

$$\begin{aligned}\left|b^T\big((\xi,\eta),(\xi,\eta)\big)\right| &= \left|b\big((\xi,\eta),(\xi,-\eta+2\xi)\big)\right| \\ &= |(\varepsilon_r\nabla\xi,\nabla\xi)_D+(\nabla\eta,\nabla\eta)_D-2(\nabla\eta,\nabla\xi)_D| \\ &\geq (\varepsilon_r\nabla\xi,\nabla\xi)_D+(\nabla\eta,\nabla\eta)_D-2|(\nabla\eta,\nabla\xi)_D| \\ &\geq (\varepsilon_r\nabla\xi,\nabla\xi)_D+(\nabla\eta,\nabla\eta)_D-\tau(\nabla\xi,\nabla\xi)_D-\frac{1}{\tau}(\nabla\eta,\nabla\eta)_D \\ &\geq [\varepsilon_- -\tau]\,(\nabla\xi,\nabla\xi)_D+\left[1-\frac{1}{\tau}\right](\nabla\eta,\nabla\eta)_D.\end{aligned}$$

Choosing $\varepsilon_- > \tau > 1$ yields $\varepsilon_- - \tau > 0$ and $1-\frac{1}{\tau} > 0$. Thus, we have proven that the sesquilinear form b is T-coercive on $S(D)\times S(D)$, hence, the operator $\mathcal{B}$ is an isomorphism of $S(D)$. □

Next we wish to weaken the assumption on ε_r. To this end we let $\mathcal{U}$ be a neighborhood of ∂D, that is an open set of $\mathbb{R}^3$ such that $\partial D \subset (\mathcal{U}\cap\overline{D})$. We furhter introduce a cut-off function$\chi \in C_0^\infty(\overline{D},[0;1])$ with support in $\mathcal{U}\cap\overline{D}$ and equal to 1 in the neighbourhood of ∂D and redefine the operator T as follows:

$$T(\xi,\eta) := (\xi,\eta-2\chi\eta), \quad \text{for all } (\xi,\eta)\in S(D).$$

Theorem 4.7. *Assume that ε_r satisfies $\varepsilon_r \geq \varepsilon_- > 1$ almost everywhere on $D\cap\mathcal{U}$. Then the sesquilinear form $b^T(\cdot,\cdot)$ satisfies the Fredholm property. In particular, the operator $\mathcal{B}$ associated with the scalar problem (4.7) is the sum $\mathcal{B}=\mathcal{I}+\mathcal{C}$, where $\mathcal{I}$ is an is an isomorphism of $S(D)$ and $\mathcal{K}$ is a compact operator of $S(D)$.*

Proof. For $(\xi,\eta)\in S(D)$ we have, by definition,

$$\begin{aligned}b^T\big((\xi,\eta),(\xi',\eta')\big) &= (\varepsilon_r\nabla\xi,\nabla\xi')_D+(\nabla\eta,\nabla\eta')_D-2(\nabla\eta,\nabla(\chi\xi'))_D \\ &= (\varepsilon_r\nabla\xi,\nabla\xi')_D+(\nabla\eta,\nabla\eta')_D-2(\chi\nabla\eta,\nabla\xi')_D-2(\nabla\eta,\xi'\nabla\chi)_D.\end{aligned}$$

With the help of the Riesz representation theorem, we define the continuous operator $\mathcal{I}: S(D)\to S(D)$ by

$$(\mathcal{I}(\xi,\eta),T(\xi',\eta'))_{S(D)} = (\varepsilon_r\nabla\xi,\nabla\xi')_D+(\nabla\eta,\nabla\eta')_D-2(\chi\nabla\eta,\nabla\xi')_D$$

for all $(\xi,\eta),(\xi',\eta')\in S(D)\times S(D)$. We claim that $\mathcal{I}$ is an isomorphism of $S(D)$. Indeed, using Young's inequality, we estiamte

$$2(\chi\nabla\eta,\nabla\xi)_D = 2(\chi\nabla\eta,\nabla\xi)_{\mathcal{U}} \leq \frac{1}{\tau}(\nabla\eta,\nabla\eta)_{\mathcal{U}} + \tau(\nabla\xi,\nabla\xi)_{\mathcal{U}}$$

with $\tau > 0$ and thus

$$\begin{aligned}|(\mathcal{I}(\xi,\eta),T(\xi,\eta))| &\geq (\varepsilon_r\nabla\xi,\nabla\xi)_D + (\nabla\eta,\nabla\eta)_D - 2\,|(\chi\nabla\eta,\nabla\xi)_D| \\ &\geq \varepsilon_-(\nabla\xi,\nabla\xi)_{D\setminus\overline{\mathcal{U}}} + (\nabla\eta,\nabla\eta)_{D\setminus\overline{\mathcal{U}}} + (\nabla\xi,\nabla\xi)_{\mathcal{U}} + (1-\tau)(\nabla\eta,\nabla\eta)_{\mathcal{U}}.\end{aligned}$$

Taking $\tau < 1$ yields that $((\xi,\eta),(\xi',\eta')) \to (\mathcal{I}(\xi,\eta),T(\xi',\eta'))$ is coercive, in particular, $\big((\xi,\eta),(\xi',\eta')\big) \to (\mathcal{I}(\xi,\eta),(\xi',\eta'))$ is T-coercive. From this, we conclude that $\mathcal{I} : S(D) \to S(D)$ is an isomorphism.

Next, we show that $\mathcal{K} := \mathcal{B}-\mathcal{I}$ defines a compact operator on $S(D)$. For all $((\xi,\eta),(\xi',\eta')) \in S(D)\times S(D)$, we have

$$(\mathcal{K}(\xi,\eta),T(\xi',\eta'))_{S(D)} = -2(\nabla\eta,\xi'\nabla\chi)_D.$$

Let $(\xi_n,\eta_n)_n$ and $(\xi'_n,\eta'_n)_n$ be two bounded sequences of elements of $S(D)$. Since every bounded sequence in a Hilbert space contains a weakly convergent subsequence, we can extract subsequences, still denoted by $(\xi_n,\eta_n)_n$ and$(\xi'_n,\eta'_n)_n$, which converge weakly to (ξ,η) and (ξ',η'), respectively. Moreover, since the imbedding of S in $L^2(D)\times L^2(D)$ is compact, there again exist subsequences, still denoted by $(\xi_n,\eta_n)_n$ and $(\xi'_n,\eta'_n)_n$, converging strongly to (ξ,η) and (ξ',η') in $L^2(D)\times L^2(D)$, respectively, i.e. $\|(\xi_n,\eta_n)\|_D \to \|(\xi,\eta)\|_D$ and $\|(\xi'_n,\eta'_n)\|_D \to \|(\xi',\eta')\|_D$. Hence, from the definition of $\mathcal{K}$, $\mathcal{K}(\xi_n,\eta_n)$ is weakly convergent in $S(D)$ and $(\mathcal{K}(\xi_n,\eta_n),T(\xi'_n,\eta'_n))_{S(D)} \to (\mathcal{K}(\xi,\eta),T(\xi',\eta'))_{S(D)}$. Consequently, setting $(\xi'_n,\eta'_n)_n = T^{-1}\mathcal{K}(\xi_n,\eta_n)_n$ and noting that $(\xi'_n,\eta'_n)_n$ is bounded in $S(D)$, because T^{-1} and $\mathcal{K}$ are continuous operators, we obtain

$$\big(\mathcal{K}(\xi_m,\eta_m),\mathcal{K}(\xi_m,\eta_m)\big)_{S(D)} = -2\left(\nabla\eta_n,\xi'_n\nabla\chi\right)_D \to -2\left(\nabla\eta,\xi'\nabla\chi\right)_D = \big(\mathcal{K}(\xi,\eta),\mathcal{K}(\xi,\eta)\big)_{S(D)}$$

that is,

$$\|\mathcal{K}(\xi_m,\eta_m)\|_{S(D)} \to \|\mathcal{K}(\xi,\eta)\|_{S(D)}.$$

Hence we have shown that $\mathcal{K}$ is compact. □

We end this section by showing that the space $X_0(D)\cap\nabla S(D)$ equals the gradients of the elements of the kernel of the scalar problem (4.7). We denote by ker$A$ the kernel of the operator A.

Theorem 4.8. *$X_0(D) \cap \nabla S(D) = \nabla ker\mathcal{B}$ where $\mathcal{B}$ is the operator associated with the scalar problem (4.7).*

Proof. Let $(U, V) \in X_0(D) \cap \nabla S(D)$. Then $(U, V) = (\nabla\varphi, \nabla\psi)$ for some $(\varphi, \psi) \in S(D)$ and $(\mathcal{B}(\varphi, \psi), (\xi, \eta)) = 0$ since $(U, V) \in X_0(D)$. Thus $(\varphi, \psi) \in \ker\mathcal{B}$, in particular $(U, V) \in \nabla\ker\mathcal{B}$.
Now, let $(U, V) \in \nabla\ker\mathcal{B}$, then $(U, V) = (\nabla\varphi, \nabla\psi)$ for $(\varphi, \psi) \in \ker\mathcal{B} \subset S(D)$ and $(\mathcal{B}(\varphi, \psi), (\xi, \eta)) = 0$, hence $(U, V) \in X_0(D)$. □

4.3 A sufficient condition for the discreteness of transmission eigenvalues

We return to problem $(P4)$. As for the scalar problem, if T is an isomorphism of $X(D)$, then (E, E_0) is a solution to problem $(P4)$, that is $a_k\big((E, E_0), (\phi, \psi)\big) = 0$ for all $(\phi, \psi) \in X(D)$, if and only if (E, E_0) satisfies

$$a_k^T\big((E, E_0), (\phi, \psi)\big) = a_k\big((E, E_0), T(\phi, \psi)\big) = 0 \quad \text{for all } (\phi, \psi) \in X(D). \tag{4.9}$$

Now, suppose k is a non-trivial transmission eigenvalue, that is, there exists $(E, E_0) \neq (0, 0)$ solving (4.1)-(4.4). Then according to (4.6), the associated pair of eigenvectors belong to $X_0(D)$. This leads us to introduce the problem of determining $(U, V) \in X_0(D)$ such that

$$a_k^T\big((U, V), (\phi, \psi)\big) = l\left((\phi, \psi)\right) \quad \text{for all } (\phi, \psi) \in X_0(D) \tag{4.10}$$

where $l \in X_0'(D)$ with $X_0'(D)$, the dual space of $X_0(D)$. With the help of Riesz representation theorem, we define the operator $\mathcal{A}_k^T : X_0(D) \to X_0(D)$ by

$$\big(\mathcal{A}_k^T(U, V), (\phi, \psi)\big)_{X(D)\times X(D)} = a_k^T\big((U, V), (\phi, \psi)\big).$$

If (U, V) is a pair of eigenvector, associated with the transmission eigenvalue $k \neq 0$, then $a_k^T\big((U, V), (\phi, \psi)\big) = 0$ for all $(\phi, \psi) \in X_0(D)$ and thus $\mathcal{A}_k^T(U, V) = 0$. Consequently, we can prove that the transmission eigenvalues form at most a discrete set by showing that the operator $\mathcal{A}_k^T$ is injective for all $k \in \mathbb{C} \setminus \mathcal{S}$, where $\mathcal{S}$ is a discrete set of the complex plane.

Remark 4.9. *Let us assume that the operator $\mathcal{B}$ associated with the scalar problem (4.7) is an isomorphism. By lemma 4.6 a sufficient condition for that is $\varepsilon_- > 1$. Using theorem 4.8, this implies $\nabla \ker \mathcal{B} = X_0(D) \cap \nabla S(D) = \{0\}$. Since the scalar problem is well-defined, we can argue similarly as in the proof of theorem 3.15 to conclude that $X(D) = X_0(D) \oplus \nabla S$. Thus, if (E, E_0) is a solution to (P4), that is*

$$a_k\big((E, E_0), (\phi, \psi)\big) = 0 \quad \textit{for all } (\phi, \psi) \in X(D),$$

we can substitute $E = E' + \nabla u$, $E_0 = E_0' + \nabla v$ in the above equation, for $E', E_0' \in X_0(D)$ and $u, v \in S(D)$, and choosing $\phi = \nabla \xi$ and $\psi = \nabla \eta$ with $\xi, \eta \in S(D)$ yields

$$(\varepsilon_r \nabla u, \nabla \xi)_D - (\nabla v, \nabla \eta)_D = 0 \quad \textit{for all } (\xi, \eta) \in S(D)$$

where we used the definition of $X_0(D)$. This is equivalent to $\mathcal{B}\big((u, v)\big) = 0$. Since the operator $\mathcal{B}$ is an isomorphism, this implies $(u, v) = (0, 0)$. Now choosing $\phi = \phi'$ and $\phi = \psi'$ with $\phi', \psi' \in X_0(D)$, we obtain

$$a_k\big((E', E_0'), (\phi', \psi')\big) = 0 \quad \textit{for all } (\phi', \psi') \in X_0(D).$$

Since it is equivalent to consider a_k^T instead of a_k, and thus $\mathcal{A}_k$ instead of $\mathcal{A}_k^T$, we conclude from the above that

$$\mathcal{A}_k^T\big((E', E_0')\big) = 0.$$

If $\mathcal{A}_k^T$ is not injective, then k is a transmission eigenvalue.

Consequently, under the assumption that the operator $\mathcal{B}$ is an isomorphism, if $\mathcal{A}_k^T$ is not injective, then k is a transmission eigenvalue. In this case, it even holds that $k \neq 0$ is a transmission eigenvalue if and only if $\mathcal{A}_k^T$ is not injective.

4.4 Case $\frac{1}{\mu_r} \geq \mu^* > 1$ in a neighbourhood of the boundary ∂D

We study the case $\frac{1}{\mu_r} \geq \mu^* > 1$, in particular, $\mu_r - 1 < 0$, in a neighbourhood of the boundary ∂D. In the following, we show that the operator $\mathcal{A}_k^T$ is of Fredholm type and discuss discreteness of the transmission eigenvalues.

4.4.1 Fredholm property of the operator $\mathcal{A}_k^T$

In the following, we define $T : X(D) \to X(D)$ by

$$T(U,V) = (U, -V + 2\chi U)$$

where, as before, $\chi \in C_0^\infty(\overline{D}, [0;1])$ is a cut-off function with support in $\mathcal{U} \cap D$ and equal to 1 in the neighbourhood of ∂D. As for the scalar case, we can easily verify that $T^2 = I$ where I is the identity operator, and hence T is an isomorphism of $X(D)$. The next lemma states that there exists wavenumbers k such that the operator $\mathcal{A}_k^T$ is an isomorphism of $X_0(D)$.

Lemma 4.10. *Assume that $\frac{1}{\mu_r} \geq \mu_* > 1$ and $\varepsilon_r \geq \varepsilon_* > 1$ almost everywhere on $D \cap \mathcal{U}$. Then there exists $k = i\kappa$ with $\kappa \in \mathbb{R}$, such that the operator $\mathcal{A}_k^T$ is an isomorphism of $X_0(D)$.*

Proof. The goal is to show that the sesquilinear form $a_{i\kappa}^T$ is coercive for some $\kappa \in \mathbb{R}$. Let $(U,V) \in X_0(D)$, then we have

$$\begin{aligned}
&\left|a_{i\kappa}^T\big((U,V),(U,V)\big)\right| = \left|a_{i\kappa}\big((U,V),(U,-V+2\chi U)\big)\right| \\
&= \Big| \left(\frac{1}{\mu_r}\operatorname{curl} U, \operatorname{curl} U\right)_D + (\operatorname{curl} V, \operatorname{curl} V)_D - 2\left(\operatorname{curl} V, \operatorname{curl}(\chi U)\right)_D + \kappa^2(\varepsilon_r U, U)_D \\
&\qquad + \kappa^2 (V,V)_D - 2\kappa^2(V,\chi U)_D + \langle \lambda U_T, U_T\rangle_{\partial D} \Big| \\
&\geq \left(\frac{1}{\mu_r}\operatorname{curl} U, \operatorname{curl} U\right)_D + (\operatorname{curl} V, \operatorname{curl} V)_D + \kappa^2\left[(\varepsilon_r U, U)_D + (V,V)_D\right] + \langle \lambda U_T, U_T\rangle_{\partial D} \\
&\qquad - 2\underbrace{\left|\left(\operatorname{curl} V, \operatorname{curl}(\chi U)\right)_D\right|}_{(I)} - \underbrace{2\kappa^2\left|(V,\chi U)_D\right|}_{(II)}.
\end{aligned} \tag{4.11}$$

We note that, here we used that λ is positive, which is necessary to later on obtain an estimate w.r.t to the $X(D)$-norm. We are going to estimate the terms (I) and (II) with the help of Young's inequality. We let τ, ρ and ς be strictly positive constants. Then

$$\begin{aligned}(I) &= 2\left|(\operatorname{curl} V, \operatorname{curl}(\chi U))_D\right| \\ &\leq 2\left|(\chi \operatorname{curl} V, \operatorname{curl} U)_D\right| + 2\left|(\operatorname{curl} V, \nabla\chi \times U)_D\right| \\ &\leq \tau\,(\operatorname{curl} V, \operatorname{curl} V)_{\mathcal{U}} + \frac{1}{\tau}\,(\operatorname{curl} U, \operatorname{curl} U)_{\mathcal{U}} + \rho\,(\operatorname{curl} V, \operatorname{curl} V)_{\mathcal{U}} \\ &\qquad + \frac{1}{\rho}\,((\nabla\chi \times U), \nabla\chi \times U)_{\mathcal{U}} \\ &\leq \tau\,(\operatorname{curl} V, \operatorname{curl} V)_{\mathcal{U}} + \frac{1}{\tau}\,(\operatorname{curl} U, \operatorname{curl} U)_{\mathcal{U}} + \rho\,(\operatorname{curl} V, \operatorname{curl} V)_{\mathcal{U}} \\ &\qquad + C\frac{1}{\rho}\,(U, U)_{\mathcal{U}} \end{aligned} \tag{4.12}$$

with $C = C(\chi) > 0$ depending only on χ. For the term (II), we estimate

$$(II) = 2\left|(V, \chi U)_D\right| \leq \varsigma (V, V)_{\mathcal{U}} + \frac{1}{\varsigma}(U, U)_{\mathcal{U}}. \tag{4.13}$$

By the assumptions on $\frac{1}{\mu_r}$ and ε_r we also obtain the estimate

$$\left(\frac{1}{\mu_r}\operatorname{curl} U, \operatorname{curl} U\right)_D + k^2(\varepsilon_r U, U)_D \geq \mu_*(\operatorname{curl} U, \operatorname{curl} U)_D + k^2\varepsilon_*(U, U). \tag{4.14}$$

Now, we write $D = \left(D \setminus \overline{\mathcal{U}}\right) \cap \mathcal{U}$ and substitute (4.12), (4.13) and (4.14) into (4.11). This yields

$$\begin{aligned}&\left|a^T_{i\kappa}\big((U,V),(U,V)\big)\right| \geq \\ &\mu_*\,(\operatorname{curl} U, \operatorname{curl} U)_{D\setminus\overline{\mathcal{U}}} + (\operatorname{curl} V, \operatorname{curl} V)_{D\setminus\overline{\mathcal{U}}} + \kappa^2\left[\varepsilon_*(U,U)_{D\setminus\overline{\mathcal{U}}} + (V,V)_{D\setminus\overline{\mathcal{U}}}\right] + \langle \lambda U_T, U_T\rangle_{\partial D} \\ &\quad + \left(\mu_* - \frac{1}{\tau}\right)(\operatorname{curl} U, \operatorname{curl} U)_{\mathcal{U}} + (1-\tau-\rho)\,(\operatorname{curl} V, \operatorname{curl} V)_{\mathcal{U}} + \left[\kappa^2\left(\varepsilon_* - \frac{1}{\varsigma}\right) - c\frac{1}{\rho}\right](U,U)_{\mathcal{U}} \\ &\quad + \kappa^2(1-\varsigma)(V,V)_{\mathcal{U}} + \langle \lambda U_T, U_T\rangle_{\partial D}\,. \end{aligned} \tag{4.15}$$

We recall that $\mu_* > 1$ and $\varepsilon_* > 1$. We choose $\tau \in \left(\frac{1}{\mu_*}, 1\right)$ to obtain $1-\tau > 0$ and $1-\tau > 0$. Next, we choose $\rho \in (0, 1-\tau)$, and hence $1-\tau-\rho > 0$. Finally we choose $\varsigma \in \left(\frac{1}{\varepsilon_*}, 1\right)$ so that both $1-\varsigma > 0$ and $\varepsilon_* - \frac{1}{\varsigma} > 0$. Now, choosing κ sufficiently large in absolute value, we obtain

$$\left|a^T_{i\kappa}\big((U,V),(U,V)\big)\right| \geq c\big(\|U\|^2_{H(\operatorname{curl},U)} + \|V\|^2_{H(\operatorname{curl},U)} + \|U_T\|^2_{\partial D}\big) = c\|(U,V)\|^2_{X(D)}$$

where $c > 0$ is a constant independent of $(U, V) \in X_0(D)$.
We have proven that, for sufficiently large values of κ, the sequilinear form $a^T_{i\kappa}$ is coercive. For those values of κ, with the help of the Lax-Milgram theorem, we conclude that the operator $\mathcal{A}^T_{i\kappa}$ is an isomorphism of $X_0(D)$. $\square$

With the above lemma, we can prove that the sesquilinear form a^T_k satisfies the Fredholm property, in particular, the operator $\mathcal{A}^T_k$ is of Fredholm type.

Theorem 4.11. *Assume that $\mu_r \geq \frac{1}{\mu^*} > 1$ and $\varepsilon_r \geq \varepsilon^* > 1$ almost everywhere on $D \cap \mathcal{U}$. Then, the operator $\mathcal{A}^T_k$ satsfies the equality $\mathcal{A}^T_k = \mathcal{I} + \mathcal{K}_k$ for all $k \in \mathbb{C}$, where $\mathcal{I}$ is an isomorphism of $X_0(D)$ that is independent of k, and $\mathcal{K}_k$ is a compact operator of $X_0(D)$.*

Proof. We note that, under the assumption $\varepsilon_r \geq \varepsilon^* > 1$, the space $X_0(D)$ is compactly embedded in $L^2(D)^3 \times L^2(D)^3$. Let $a^T_{i\kappa,1/2}$ with $\kappa \in \mathbb{R}$, be the sesquilinear form $a^T_{i\kappa}$ for the special case when $\varepsilon_r = \frac{1}{2}$, that is

$$\begin{aligned} a^T_{i\kappa,1/2}\big((U,V),(U',V')\big) &= \left(\frac{1}{\mu_r}\operatorname{curl} U, \operatorname{curl} U'\right)_D + (\operatorname{curl} V, \operatorname{curl} V')_D - 2\,(\operatorname{curl} V, \operatorname{curl} \chi U')_D \\ &\quad + \kappa^2 \left[\left(\frac{1}{2}U, U'\right)_D + (V, V')_D - 2\,(V, \chi U')_D\right] \quad + \langle \lambda U_T, U'_T\rangle_{\partial D} \end{aligned}$$

for $(U,V), (U',V') \in X_0(D)$. Next we define the sesquilinear form $c^T_k : X_0(D) \times X_0(D) \to \mathbb{C}$ by

$$\begin{aligned} c_k\big((U,V),(U',V')\big) &= a^T_k\big((U,V),(U',V')\big) - a^T_{i\kappa,1/2}\big((U,V),(U',V')\big) \\ &= a_k\big((U,V),T(U',V')\big) - a_{i\kappa,1/2}\big((U,V),T(U',V')\big) \\ &= -k^2\left[(\varepsilon_r U, U')_D + (V,V')_D - 2(V,\chi U')_D\right] \\ &\quad - \kappa^2\left[\left(\frac{1}{2}U, U'\right)_D + (V,V')_D - 2(V,\chi U')_D\right]. \end{aligned}$$

From Riesz's representation theorem, we define the bounded linear operators $\mathcal{I}$ and $\mathcal{K}_k$ from $X_0(D)$ into itself by

$$(\mathcal{I}(U,V),(U',V'))_{X(D)} = a_{i\kappa,1/2}\big((U,V),(U',V')\big), \tag{4.16}$$

$$(\mathcal{K}_k(U,V),(U',V'))_{X(D)} = c_k\big((U,V),(U',V')\big). \tag{4.17}$$

By lemma 4.10, we can choose $\kappa \in \mathbb{R}$ so that $\mathcal{I}$ is an isomorphism of $X_0(D)$. Moreover, since the space $X_0(D)$ is compactly embedded in $L^2(D)^3 \times L^2(D)^3$, the operator $\mathcal{K}_k$ is compact on $X_0(D)$. □

4.4.2 Discreteness of the transmission eigenvalues

Let us assume the assumptions of theorem 4.11 to be true and let us recall the operators $\mathcal{I}$ and $\mathcal{K}_k$ with $\mathcal{A}_k^T = \mathcal{I} + \mathcal{K}_k$. Since $\mathcal{K}_k$ depends analytically on $k \in \mathbb{C}$ and $\mathcal{J}$ does not, the eigenvalue problem becomes $(I + \mathcal{K}_k \mathcal{I}^{-1})(U, V) = 0$ where $\mathcal{K}_k \mathcal{I}^{-1} : X_0(D) \to X_0(D)$ is compact and I denotes the identity operator of $X_0(D)$. To see that the mapping $k \to \mathcal{K}_k \mathcal{I}^{-1}$ is analytic in $\mathbb{C}$, we define the operators $\mathcal{F}$ and $\mathcal{G}$ from $X_0(D)$ to $X_0(D)$ by

$$(\mathcal{F}(U, V), (U', V'))_{X(D)} = (\varepsilon_r U, U')_D + (V, V')_D - 2(V, \chi U')_D$$
$$(\mathcal{G}(U, V), (U', V'))_{X(D)} = -\kappa^2 \left[\left(\frac{1}{2} U, U' \right)_D + (V, V')_D - 2(V, \chi U')_D \right].$$

Then, $\mathcal{A}_k^T = \mathcal{I} + k^2 \mathcal{F} + \mathcal{G}$, that is $\mathcal{A}_k^T \mathcal{I}^{-1} = I + k^2 \mathcal{F} \mathcal{I}^{-1} + \mathcal{G} \mathcal{I}^{-1}$. Due to the compact embedding of $X_0(D)$ into $L^2(D)^3 \times L^2(D)^3$, it follows that $\mathcal{F}$,$\mathcal{G}$ and thus $\mathcal{F}\mathcal{I}^{-1}$, $\mathcal{K}_k \mathcal{I}^{-1}$ are compact operators from $X_0(D)$ to $X_0(D)$. Furthermore, the map $k \to k^2 \mathcal{F} \mathcal{I}^{-1} + \mathcal{G} \mathcal{I}^{-1}$ from $\mathbb{C}$ to the Banach space of bounded operators from $X_0(D)$ to $X_0(D)$ is polynomial and so analytic.

To apply the analytic Fredholm theory, it remains to show that there exists a $k \in \mathbb{C}$ for which $\mathcal{I} + \mathcal{K}_k$ is injective. By lemma 4.10, we know that there exists $\kappa \in \mathbb{R}$ such that $\mathcal{A}_{i\kappa}^T$ is an isomorphism of $X_0(D)$. Thus $\mathcal{A}_k^T$ is injective for all $k \in \mathbb{C} \setminus S$, where S is a discrete set of the complex plane. For $k \in \mathbb{C} \setminus S$, this implies that the only solution of problem (4.9), and consequently of problem $(P4)$, is the zero solution. So far, we have proven the following theorem.

Theorem 4.12. *Assume that $\mu_r \geq \frac{1}{\mu_*} > 1$ and $\varepsilon_r \geq \varepsilon_* > 1$ almost everywhere on $D \cap \mathcal{U}$. Then the set of transmission eigenvalues is at most a discrete set in $\mathbb{C}$.*

Theorem 4.13. *Assume that $\mu_+ < 1$, so that $\frac{1}{\mu_r} \geq \frac{1}{\mu_+} > 1$. Assume further that the operator $\mathcal{B}$ associated with the scalar problem (4.7) is injective. Then the set of transmission eigenvalues is at most a discrete set in $\mathbb{C}$.*

Proof. We write $a_k^T = a_{i\kappa}^T - (a_{i\kappa}^T - a_k^T)$ for $\kappa \in \mathbb{R}$. Then

$$|a_k^T| \geq |a_{i\kappa}^T| - |(a_{i\kappa}^T - a_k^T)|. \tag{4.18}$$

By lemma 4.10 the sesquilinear form $a^t_{i\kappa}$ is coercive, thus with $\chi = 1$, there exists a constant $c > 0$ such that

$$\begin{aligned}\left|a^T_{i\kappa}\big((U,V),(U,V)\big)\right| \geq& c\|(U,V)\|^2_{X(D)}\\ =& c\left(\|\operatorname{curl} U\|^2_D + \|\operatorname{curl} V\|^2_D + \|U\|^2_D + \|V\|^2_D + \|U_T\|^2_{\partial D}\right) \qquad (4.19)\end{aligned}$$

for all $(U,V) \in X_0(D)$. We note that a sufficient condition for the operator $\mathcal{B}$ associated with the scalar problem to be an isomorphism is $\varepsilon_- > 1$ (or $\varepsilon_+ < 1$ as shown in [16]). Now for $(U,V) \in X_0(D)$, we estimate

$$\begin{aligned}\left|a^T_{i\kappa}\big((U,V),(U,V)\big) - a^T_k\big((U,V),(U,V)\big)\right| &= \left|(k^2+\kappa^2)\left[(\varepsilon_r U,U)_D + (V,V)_D - 2(V,U)_D\right]\right|\\ &\leq |k^2+\kappa^2|\Big(|(\varepsilon_r U,U)_D| + |(V,V)_D| + 2|(V,U)_D|\Big).\end{aligned}$$

Using Young's Inequality $2|(V,U)_D| \leq \|U\|^2_D + \|V\|^2_D$ and estimating ε_r yields that there exists a constant $\widetilde{c} > 0$ such that

$$|(a^T_{i\kappa} - a^T_k)| \leq \widetilde{c}|k|^2\big(\|U\|^2_D + \|V\|^2_D\big). \qquad (4.20)$$

Putting everything together, substituting (4.19) and (4.20) into (4.18) yields that there exists constants $c_1, c_2 > 0$, independent of k, such that

$$\left|a^T_k\big((U,V),(U,V)\big)\right| \geq c_1\left(\|\operatorname{curl} U\|^2_D + \|\operatorname{curl} V\|^2_D + \|U_T\|^2_{\partial D}\right) - c_2|k|^2\left(\|U\|^2_D + \|V\|^2_D\right). \qquad (4.21)$$

Next we show that

$$(U,V) \to \left(\|\operatorname{curl} U\|^2_D + \|\operatorname{curl} V\|^2_D + \|U_T\|^2_{\partial D} D\right)^{\frac{1}{2}}$$

defines a norm on $X_0(D)$ that is equivalent to the $X(D)$-norm. Since

$$\left(\|\operatorname{curl} U\|^2_D + \|\operatorname{curl} V\|^2_D + \|U_T\|^2_{\partial D} D\right)^{\frac{1}{2}} \leq \|(U,V)\|_{X(D)}$$

it is sufficient to prove that there exists a constant $C_p > 0$ such that

$$\|U\|^2_D + \|V\|^2_D \leq C_p\left(\|\operatorname{curl} U\|^2_D + \|\operatorname{curl} V\|^2_D + \|\nu \times U\|^2_{\partial D}\right) \qquad (4.22)$$

for all $(U,V) \in X_0(D)$. We prove by contradiction and let (U_n, V_n) be a sequence of elements of $X_0(D)$ such that

$$\|U_m\|_D^2 + \|V_m\|_D^2 = 1 \quad \text{for all } m \in \mathbb{N},$$

and

$$\lim_{m\to\infty} \left(\|\operatorname{curl} U\|_D^2 + \|\operatorname{curl} V\|_D^2 + \|\nu \times U\|_{\partial D}^2\right) = 0.$$

By the compactness property of $X_0(D)$, there exists a subsequence, still denoted by (U_m, V_m), that converge to $(U, V) \in X_0(D)$ in $L^2(D)^3 \times L^2(D)^3$. By the properties of the sequence, we have that $\|U\|_D^2 + \|V\|_D^2 = 1$ and $\operatorname{curl} U = \operatorname{curl} V = 0$ in D and $\nu \times U = 0$ on ∂D. Moreover, $\nu \times V = 0$ on ∂D. By theorem 3.37 in [2], there exist scalar potentials $(\xi, \eta) \in S(D)$ such that $(U, V) = (\nabla\xi, \nabla\eta)$. Now, $\mathcal{B}(\nabla\xi, \nabla\eta) = 0$, and since $\mathcal{B}$ is injective, we conclude that $(\nabla\xi, \nabla\eta) = (U, V) = 0$, which is a contradiction to $\|U\|_D^2 + \|V\|_D^2 = 1$.

Now we return to (4.21). Since the map $(U, V) \to (\|\operatorname{curl} U\|_D^2 + \|\operatorname{curl} V\|_D^2 + \|U_T\|_{\partial D}^2)^{\frac{1}{2}}$ defines a norm on $X_0(D)$ equivalent to the $X(D)$-norm, we conclude that a_k^T is coercive on $X_0(D) \times X_0(D)$ for $|k|^2 < \frac{c_1}{c_2 C_p}$, where C_p is defined in (4.22). Thus, for those values of k the operator $\mathcal{A}_k^T$ defines an isomorphism of $X_0(D)$. The analytic Fredholm theorem now implies that the set of transmission eigenvalues is at most a discrete set in $\mathbb{C}$. □

4.4.3 Case $\frac{1}{\mu_r} \leq \mu^* < 1$ in a neighbourhood of the boundary ∂D

For the case $\frac{1}{\mu_r} \leq \mu^* < 1$, in particular, $\mu_r - 1 > 0$ in a neighbourhood of the boundary ∂D, we can argue as in the previous section by taking $T : X_0(D) \to X_0(D)$, $T(U, V) = (U - 2\chi V, -V)$ with the cut-off function $\chi \in C_0^\infty(\overline{D}, [0; 1])$ as before. Then, two issues need to be remarked on.

The first is that, we need to verify that $T(U, V)$ is an element of $X_0(D)$. For $(U, V) \in X_0(D)$, in particular $(U, V) \in X(D)$, we have by definition that $U \in H_{\text{imp}}(\operatorname{curl}, D)$, $V \in H(\operatorname{curl}, D)$ and $\nu \times (U - V) = 0$ on ∂D. Since the trace operator is linear and the tangential trace of U belongs to $L_t^2(\partial D)$, we conclude that $V \in H_{\text{imp}}(\operatorname{curl}, D)$, and consequently $T(U, V) \in X_0(D)$. It is easily verfied that T defines an isomorphism on $X_0(D)$.

The second issue is that we need to verify that the estimates of $a_{i\kappa}$ in the proof of lemma 4.10 hold. We compute,

$$
\begin{aligned}
\left|a^T_{i\kappa}\big((U,V),(U,V)\big)\right| &= \left|a_{i\kappa}\big((U,V),(U-2\chi V,-V)\big)\right| \\
&\geq \left(\frac{1}{\mu_r}\operatorname{curl}U,\operatorname{curl}U\right)_D + (\operatorname{curl}V,\operatorname{curl}V)_D + \kappa^2(\varepsilon_r U,U)_D \\
&+ \kappa^2(V,V)_D + \langle \lambda U_T, U_T\rangle_{\partial D} \\
&- \left|2\left(\frac{1}{\mu_r}\operatorname{curl}U,\operatorname{curl}(\chi V)\right)_D\right| - 2\left|\kappa^2(\varepsilon_r U,\chi V)_D\right| - 2\left|\langle \lambda U_T, V_T\rangle_{\partial D}\right|.
\end{aligned}
$$

Compared to (4.11), we have obtained the additional term $-2\left|\langle \lambda U_T, V_T\rangle_{\partial D}\right|$. Now, estimating $\left|a^T_{i\kappa}\big((U,V),(U,V)\big)\right|$ for all $(U,V)\in X_0(D)$, as was done in lemma (4.10), using Young's inequality to estiamte $2|\,\langle \lambda U_T, \chi V\rangle_{\partial D}\,| \leq \sigma\,\langle \lambda U_T, U_T\rangle + \frac{1}{\sigma}\,\langle \lambda V_T, V_T\rangle_{\partial D}$, for some $\sigma > 0$, and choosing κ sufficiently large in absolute value, we obtain

$$
\begin{aligned}
\left|a^T_{i\kappa}\big((U,V),(U,V)\big)\right| &\geq c\big(\|U\|^2_{H(\operatorname{curl},U)} + \|V\|^2_{H(\operatorname{curl},U)} + \|U_T\|^2_{\partial D}\big) - \frac{1}{\sigma}\,\langle \lambda V_T, V_T\rangle_{\partial D} \\
&\ngeq c\|(U,V)\|^2_{X(D)}
\end{aligned}
$$

Thus, for the case $\mu_r - 1 > 0$ in a neighbourhood of the boundary ∂D, we unfortunately can not conclude the same results as in the previous subsections.

5 Conclusion

We considered scattering of time-harmonic electromagnetic waves. We were able to show the existence of a unique solution to direct problems and introduced both the integral equation and variational methods. Despite the differing approaches of the methods, the basic general idea is somewhat the same, that is, the derivation of an equivalent formulation of the scattering problem. Well-posedness was then obtained by showing that the equivalent formulation is of Fredholm type. In the case of the scattering of an impenetrable obstacle with perfectly conducting boundary conditions, we derived a Lippmann-Schwinger operator equation and considered that equation in the space $H_{\mathrm{loc}}(\mathrm{curl}\,,\cdot)$. There we assumed a general setting, that is, the obstacle occupied a Lipschitz domain, and the data were described by Sobolev functions. We were also able to obtain well-posedness of the direct problem applying the integral equation method to the scattering problem in the case of a penetrable obstacle with conductive transmission conditions. However, we had to assume more regularity on the data. In particular, we assumed the obstacle to occupy a smooth domain and sought a solution in the space $H^1(\cdot)^3$. Uniqueness of a unique solution for the general setting was obtained by applying the variational method.

For the scattering problems, we also studied the inverse problem of determining the shape of the obstacle from the knowledge of the far field pattern for the scattering of incident plane waves. In particular, we showed uniqueness when we had overdetermined data, in the sense that the far field pattern was known for all incident directions and polarizations. The question of uniqueness in case of the knowledge of the far field pattern for one incident plane wave still remains open. Moreover, as a further study, we might investigate the question of uniqueness w.r.t. to the conductive transmission function λ of problem $(P3)$.

In the case of the scattering from a penetrable obstacle with conductive transmission conditions, we also studied the case of non-scattering incident fields and obtained the interior transmission eigenvalue problem. Applying the variational method, in particular, the T-coercivity approach, we were able to prove discreteness of the transmission eigenvalues when $\mu_r - 1$ was negative and $\varepsilon_r - 1$ was either positive or negative in a neighborhood of the boundary and $\mu_r - 1$ and $\varepsilon_r - 1$ can change sign inside the domain. As a further study, we might investigate the following questions:

- $\mu_r - 1$ positive in a neighborhood of the boundary;

- Discreteness of transmission eigenvalues applying an integral equation type method;
- Existence of transmission eigenvalues.

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