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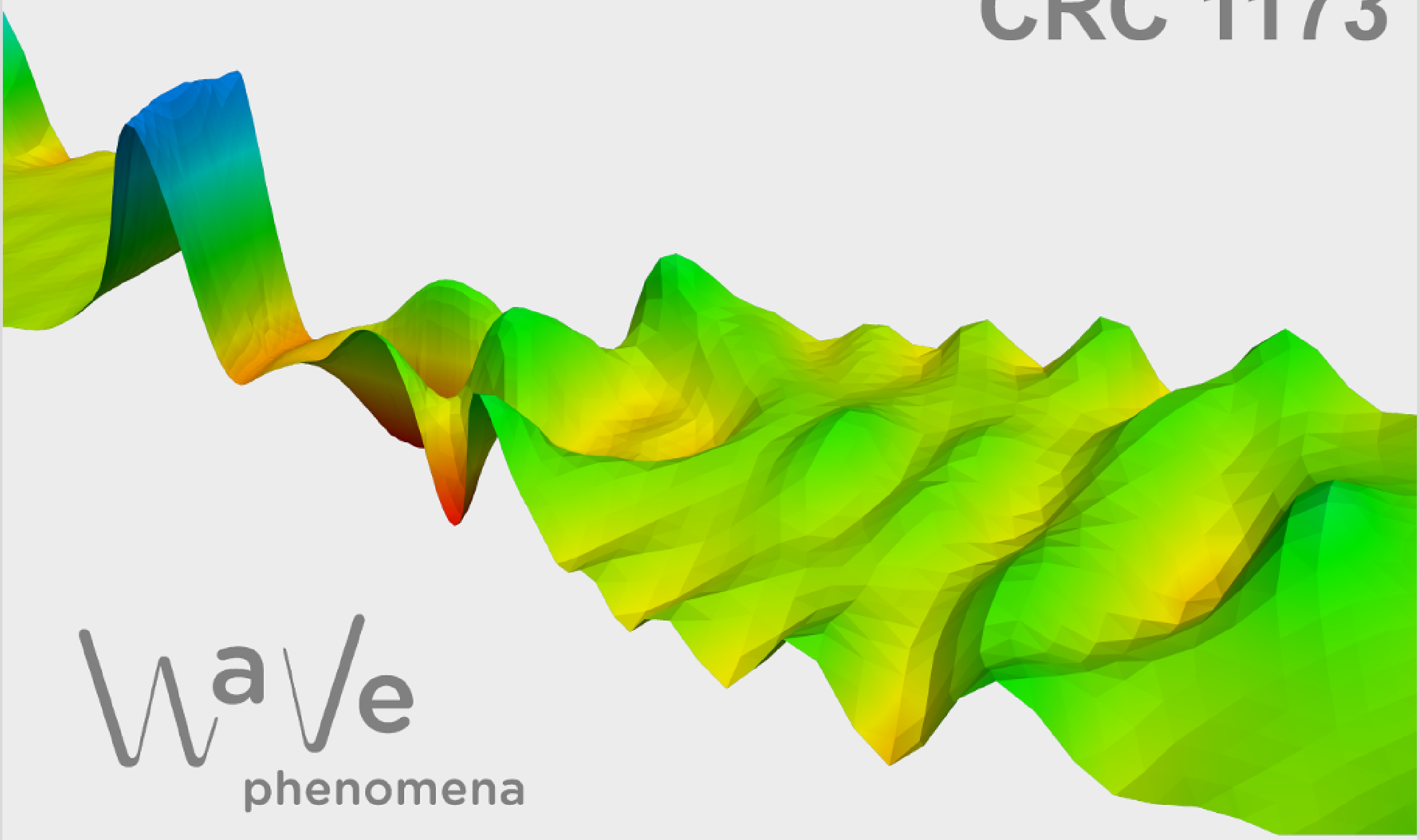
Irfan Glogić, Birgit Schörkhuber

CRC Preprint 2019/23, November 2019

KARLSRUHE INSTITUTE OF TECHNOLOGY

CRC 1173



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Funded by

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ISSN 2365-662X

NONLINEAR ASYMPTOTIC STABILITY OF HOMOTHETICALLY SHRINKING YANG-MILLS SOLITONS IN THE EQUIVARIANT CASE

IRFAN GLOGIĆ AND BIRGIT SCHÖRKHUBER

ABSTRACT. We study the heat flow for Yang-Mills connections on $\mathbb{R}^d \times SO(d)$ for $5 \leq d \leq 9$. It is well-known that for this model homothetically shrinking solitons exist and an explicit example was found by Weinkove [21]. In this paper, we prove the nonlinear asymptotic stability of this solution under small $SO(d)-$equivariant perturbations and extend the results of [8] for $d = 5$ to higher space dimensions. Also, we substantially simplify proof and provide new techniques to rigorously solve the spectral problem for the linearization, which turns out to be more involved in higher space dimensions.

1. INTRODUCTION

In this paper, we study connection 1-forms $A_j : \mathbb{R}^d \to \mathfrak{so}(d)$, $j = 1, \dots, d$, where $\mathfrak{so}(d)$ denotes the Lie algebra of the Lie group $SO(d)$, i.e., $\mathfrak{so}(d)$ can be considered as the set of skew-symmetric $(d \times d)$-matrices endowed with the commutator bracket. In the following, Einstein's summation convention is in force. The associated covariant derivative acting on $\mathfrak{so}(d)$-valued functions is defined by $\mathbf{D}_j := \partial_j + [A_j, \cdot]$ and the curvature tensor amount to

$$F_{jk} := \partial_j A_k - \partial_j A_k + [A_j, A_k].$$

The Yang-Mills functional is then defined as

$$\mathcal{F}[A] = \frac{1}{2} \int_{\mathbb{R}^d} \mathrm{tr} \langle F_{jk}, F^{jk} \rangle dx. \tag{1.1}$$

The associated Euler-Lagrange equations read

$$\mathbf{D}^j F_{jk}(x) = 0 \tag{1.2}$$

and solutions are referred to as Yang-Mills connections. By introducing an artificial time-dependence, the gradient flow associated to Eq. (1.2) yields

$$\partial_t A_j(x,t) = \mathbf{D}^j F_{jk}(x,t), \quad t > 0, \tag{1.3}$$

for some initial condition $A_j(0, x) = A_j(x)$. This model is referred to as the *Yang-Mills heat flow* for connections on the trivial bundle $\mathbb{R}^d \times SO(d)$. The natural question concerns the existence of solutions to this initial value problem and the possibility of the formation of singularities in finite time.

Eq. (1.3) enjoys scale invariance, $A_j \mapsto A_j^\lambda$,

$$A_j^\lambda(x,t) := \lambda A_j(\lambda x, \lambda^2 t), \quad \lambda > 0$$

and the model is *critical* for $d = 4$. In this case, global existence of solutions in the equivariant setting was shown by Schlatter, Struwe and Tahvildar-Zadeh [17]. For more general geometric situations, this was a long-standing problem, which has been resolved only very recently by Waldron [19]. In the supercritical case $d \geq 5$ the picture is more clear and it is well-known that the Yang-Mills heat flow can develop singularities in finite time, see the works of Naito [15], Grotowski [12] and Gastel [11]. Weinkove [21] investigated the nature of singularities for the Yang-Mills heat flow over a compact d-dimensional manifold and proved that under some assumption on the blowup rate, solutions converge in a suitable sense to *homothetically shrinking solitons*, locally around the blowup point. These objects correspond to self-similar solutions of the Yang-Mills heat flow on the trivial bundle over $\mathbb{R}^n$. An explicit example was provided for Eq. (1.3) and is given by

$$A_j^T(x,t) = u_T(|x|,t)\sigma_j(x), \tag{1.4}$$

where $\sigma_j^{ik}(x) = \delta_j^i x^k - \delta_j^k x^i$,

$$u_T(r,t) = \tfrac{1}{T-t} W\left(\tfrac{r}{\sqrt{T-t}}\right), \quad W(\rho) = -(a\rho^2 + b)^{-1}, \tag{1.5}$$

for some $T > 0$, with constants

$$a = \tfrac{\sqrt{d-2}}{2\sqrt{2}}, \quad b = \tfrac{1}{2}(6d - 12 - (d+2)\sqrt{2d-4}), \tag{1.6}$$

for $5 \leq d \leq 9$. In this paper, we investigate the stabilty of the Weinkove soliton in the $SO(d)-$equivariant setting, i.e., we only consider connections of the form

$$A_j(x,t) = u(|x|,t)\sigma_j(x). \tag{1.7}$$

It is well-known that this symmetry is preserved by the flow, see for example [12], [11]. Furthermore, Eq. (1.3) reduces to a single equation for the function $u : [0,\infty) \times [0,\infty) \to \mathbb{R}$, given by

$$\begin{aligned} \partial_t u(r,t) - \partial_r^2 u(r,t) - \frac{d+1}{r}\partial_r u(r,t) \\ + 3(d-2)u^2(r,t) + (d-2)r^2 u^3(r,t) = 0, \end{aligned} \tag{1.8}$$

with initial condition $u(\cdot,0) = u_0$. The scale invariance of Eq. (1.3) implies that Eq. (1.8) is invariant under $u \mapsto u_\lambda$,

$$u_\lambda(r,t) = \lambda^2 u(\lambda r, \lambda^2 t), \quad \lambda > 0.$$

In analogy to the related heat flow of harmonic maps, infinitely many self-similar profiles are expected to exist in dimension $5 \leq d \leq 9$ with W as a ground state, see Biernat and Bizoń [1]. In higher space dimensions $d \geq 10$, the existence of self-similar solutions to Eq. (1.8) was excluded by Bizoń and Wasserman [3].

1.1. **The main result.** In view of (1.8) it makes sense to consider u as a radial function on $\mathbb{R}^{d+2}$. Since (1.8) is basically a quadratic heat equation, scaling implies that the $\dot{H}^{s_c}(\mathbb{R}^{d+2})$-norm for $s_c = \frac{d}{2} - 1$ is scale invariant. Now, for the blowup profile, we have $W \in \dot{H}^s(\mathbb{R}^{d+2})$ for every $s > s_c$ but it fails to be in the critical space. Furthermore, the corresponding blowup solution behaves like

$$\|u_T(|\cdot|, t)\|_{\dot{H}^s(\mathbb{R}^{d+2})} \simeq (T-t)^{\frac{1}{2}(\frac{d}{2}-1-s)}, \tag{1.9}$$

for $s > s_c$, i.e., u_T blows up in $\dot{H}^s(\mathbb{R}^{d+2})$. In the following we set $n := d+2$ and define

$$\mathcal{S}_{\text{rad}}(\mathbb{R}^n) := \{f \in \mathcal{S}(\mathbb{R}^n) : f \text{ is radial}\}$$

and set

$$\|f\|_X := \|f\|_{\dot{H}^{\kappa_0}(\mathbb{R}^n)} + \|f\|_{\dot{H}^{\kappa_1}(\mathbb{R}^n)}, \tag{1.10}$$

where

$$\kappa_0 = \begin{cases} \frac{n-3}{2}, & \text{for } n \text{ odd} \\ \frac{n-2}{2}, & \text{for } n \text{ even} \end{cases}, \quad \kappa_1 = \kappa_0 + 2. \tag{1.11}$$

We denote by $(X, \|\cdot\|_X)$ the completion of $\mathcal{S}_{rad}(\mathbb{R}^n)$ with respect to $\|\cdot\|_X$. Note that in the $X-$norm, the blow rate of u_T is given by

$$\|u_T(|\cdot|, t)\|_X \simeq (T-t)^{-\frac{1}{2}(\kappa_1+2-\frac{n}{2})}. \tag{1.12}$$

In the following, we fix $T = 1$ and consider the time evolution governed by (1.8) for perturbations of the blowup initial data

$$u(\cdot, 0) = u_1(\cdot, 0) + v \tag{1.13}$$

where v denotes a free radial function.

Theorem 1.1. *Fix $5 \leq d \leq 9$. There exist $\delta > 0$ amd $M > 0$ such that the following holds: For every $v \in \mathcal{S}_{\text{rad}}(\mathbb{R}^{d+2})$ satisfying $\|v\|_X \leq \frac{\delta}{M}$, there is a $T = T(v) \in [1-\delta, 1+\delta]$ such that the Cauchy problem given by* (1.8) *and* (1.13) *has a unique solution $u : [0, \infty) \times [0, T) \to \mathbb{R}$. Furthermore, the solution blows up at $t = T$ and converges to u_T in the sense that*

$$\frac{\|u(|\cdot|, t) - u_T(|\cdot|, t)\|_X}{\|u_T(|\cdot|, t)\|_X} \lesssim \delta(T-t)^\omega, \tag{1.14}$$

and

$$(T-t)\|u(\cdot, t) - u_T(\cdot, t)\|_{L^\infty(\mathbb{R}^+)} \lesssim \delta(T-t)^\omega, \tag{1.15}$$

for all $t \in [0, T)$ and some $\omega > 0$.

The case $d = 5$ has been addressed by Donninger and the second author in [8]. Theorem 1.1 extends this result and shows that the blowup described the Weinkove soliton is nonlinear asymptotically stable under small equivariant perturbations in all space dimensions where the solution is defined.

We note that notions of variational stability of homothetically shrinking solitons have been investigated by Kelleher and Streets [13] as well as by

Chen and Zhang [4]. However, to the best of our knowledge, there is no result other than Theorem 1.1 that would imply the stability of the Weinkove solution (in any sense).

1.2. **Some comments on the method of proof.** For the proof of Theorem 1.1 we generalize techniques developed in previous works by the second author with Donninger [8] addressing the case $d = 5$, respectively, with Biernat and Donninger [2] concerning the related heat flow of harmonic maps in $d = 3$.

The general idea is to consider the problem in adapted coordinates and to study small perturbations of self-similar blowup solutions, which correspond to static solutions in the new coordinates. The aim is to investigate the linearized problem by means of semigroup theory and to treat the nonlinearity as a perturbation by using fixed point arguments.

For the implementation of this strategy a suitable functional analytic setup has to be found. For the linearized problem, there is a canonical choice provided by a weighted L^2-space, which we denote by $\mathcal{H}$, where the problem is self-adjoint. However, the weight function decays at infinity which renders this setting useless for the nonlinear problem. Instead, we work in the intersection Sobolev space $X = \dot{H}^{\kappa_0} \cap \dot{H}^{\kappa_1}(\mathbb{R}^n)$, $s_c < \kappa_0 < \frac{n}{2} < \kappa_1$.

In [8], the problem was considered in a non-selfadjoint formulation on X. This approach uses very little structure but comes at the price of some technical difficulties that have to be overcome in order to prove a spectral mapping result. In [2], a different point of view was taken; by exploiting the continuous embedding of X into $\mathcal{H}$, it was shown that the semgroup on X can be defined by restriction (for this the explicit form of the free semigroup was used). The bounds for the linearized time-evolution on X are then obtained by exploiting the decay of the potential at infinity in order split the problem into a problem on a bound domain, where the self-adjoint growth bounds can be utilizied, and a remainder that can be made small in a suitable sense.

Both methods crucially rely on spectral theory for a self-adjoint Schrödinger operator $A : \mathcal{D}(A) \subset L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$, which corresponds to the linearization around the blowup profile. More precisely, one has to show that $\sigma(A) \subset \{-1\} \cup (0, \infty)$, where the unstable eigenvalue arises as a result of time-translation symmetry. For the Yang-Mills heat flow in $d = 5$, this was established in [8] by using ideas from supersymmetric quantum mechanics in order to remove the unstable eigenvalue and to show that $(A_S f|f) > 0$ for the corresponding supersymmetric operator $A_S : \mathcal{D}(A_S) \subset L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$.

In this paper, we implement the strategy of [2] for the Yang-Mills heat flow and generalize the approach to arbitrary space dimensions (although it is applied only to $5 \leq d \leq 9$ where the blowup solution exists). This yields a simple and general framework to investigate the stability of self-similar solutions in semilinear heat equations. The spectral analysis for the problem at hand turns out to be more involved in higher space dimensions and cannot be solved by the techniques used [8]. Instead, we resort to more advanced tools from the theory of Schrödinger operators.

1.3. **Notation and Conventions.** We write $\mathbb{N}$ for the natural numbers $\{1, 2, 3, \dots\}$, $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Furthermore, $\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$. The notation $a \lesssim b$ means $a \leq Cb$ for an absolute constant $C > 0$ and we write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$. If $a \leq C_\varepsilon b$ for a constant $C_\varepsilon > 0$ depending on some parameter ε, we write $a \lesssim_\varepsilon b$. We use the common notation $\langle x \rangle := \sqrt{1 + |x|^2}$ also known as the *Japanese bracket*. For a function $x \mapsto g(x)$, we denote by $g^{(n)}(x) = \frac{d^n g(x)}{dx^n}$ the derivatives of order $n \in \mathbb{N}$. For $n = 1, 2$, we also write $g'(x)$ and $g''(x)$, respectively. The spaces $L^2(\Omega)$ and $H^k(\Omega)$ for $k \in \mathbb{N}_0$ and $\Omega \subseteq \mathbb{R}^n$ some domain, denote the standard Lebesgue and Sobolev spaces with the usual norm

$$\|u\|^2_{H^k(\Omega)} := \sum_{\alpha : |\alpha| \leq k} \|\partial^\alpha u\|^2_{L^2(\Omega)}.$$

On $\mathbb{R}^n$, inhomogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^n)$ for $s > 0$ can be defined via Fourier transform, which we denote by $f \mapsto \mathcal{F}f$. For a closed linear operator $(L, \mathcal{D}(L))$, we write $\sigma(L)$ for the spectrum. The resolvent set is defined as $\rho(L) := \mathbb{C} \setminus \sigma(L)$ and we write $R_L(\lambda) := (\lambda - L)^{-1}$ for $\lambda \in \rho(L)$.

2. Formulation of the problem in similarity coordinates

Fix $5 \leq d \leq 9$, $d \in \mathbb{N}$. We rewrite the initial value problem given by (1.8) and (1.13) in similarity coordinates $(\rho, \tau) \in [0, \infty) \times [0, \infty)$ defined by

$$\tau = -\log(T - t) + \log T, \quad \rho = \frac{r}{\sqrt{T - t}}.$$

The blowup time $T > 0$ enters the analysis as a free parameter that will be fixed only at the very end of the argument. By setting

$$\psi(\tfrac{r}{\sqrt{T-t}}, -\log(T - t) + \log T) := (T - t)u(r, t),$$

we obtain

$$\begin{aligned} \partial_\tau \psi(\rho, \tau) = &\big(\partial_\rho^2 + \tfrac{d+1}{\rho}\partial_\rho - \tfrac{1}{2}\rho\partial_\rho - 1\big)\psi(\rho, \tau) \\ &- 3(d - 2)\psi(\rho, \tau)^2 - (d - 2)\rho^2\psi(\rho, \tau)^3 \end{aligned} \tag{2.1}$$

with initial condition

$$\psi(\rho, 0) = TW\left(\sqrt{T}\rho\right) + Tv(\sqrt{T}\rho) \tag{2.2}$$

The differential operator on the right hand side of Eq. (2.1) has a natural extension to $\mathbb{R}^n$, $n = d+2$. In fact, the evolution equation can be formulated as

$$\frac{d}{d\tau}\Psi(\tau) = L_0\Psi(\tau) + F(\Psi(\tau)) \tag{2.3}$$

with the formal differential operator

$$L_0 f(x) := \Delta f(x) - \frac{1}{2} x \cdot \nabla f(x) - f(x) \tag{2.4}$$

acting on radial functions. Note that here Δ denotes the Laplace operator of $\mathbb{R}^n$. Inserting the ansatz $\Psi(\tau) = W + \Phi(\tau)$, yields

$$\begin{aligned} \frac{d}{d\tau}\Phi(\tau) &= (L_0 + L')\Phi(\tau) + N(\Phi(\tau)), \quad \tau > 0 \\ \Phi(0) &= \mathcal{U}(v,T). \end{aligned} \tag{2.5}$$

with

$$L := L_0 + L', \quad L' f(x) := V(|x|) f(x) \tag{2.6}$$

where the potential is given by

$$V(\rho) = \frac{3(n-4)[2b + (2a-1)\rho^2]}{(b + a\rho^2)^2} \tag{2.7}$$

for $a = \frac{\sqrt{n-4}}{2\sqrt{2}}$, $b = n(3 - \frac{1}{2}\sqrt{2n-8}) - 12$. The remaining nonlinearity is given by

$$N(\Phi(\tau)) = -3(d-2)\big[1 + |\cdot|^2 W(|\cdot|)\big]\Phi(\tau)^2 - (d-2)|\cdot|^2\Phi(\tau)^3$$

and the initial data transforms to

$$\mathcal{U}(v,T) := Tv(\sqrt{T}|\cdot|) + TW\left(\sqrt{T}|\cdot|\right) - W(|\cdot|).$$

2.1. **Preliminaries.** In the following, we restrict ourselves to radial functions and use the same symbol for the function an its radial representative, i.e. $f(x) = f(|x|)$. On $\mathcal{S}_{rad}(\mathbb{R}^n)$ we define the scaling operator

$$\Lambda f(x) := \frac{1}{2} x \cdot \nabla f(x) + f(x)$$

and the (radial) derivative operators

$$D^k := \begin{cases} \Delta^{k/2}, & \text{for } k \in \mathbb{N}_0 \text{ even} \\ \nabla \Delta^{(k-1)/2}, & \text{for } k \in \mathbb{N}_0 \text{ odd.} \end{cases} \tag{2.8}$$

Then we have the commutator relation

$$D^k \Lambda = \Lambda D^k + \frac{k}{2} D^k, \tag{2.9}$$

which will be crucial later on. We define the Hilbert space $\mathcal{H}$ as a weighted L^2-space of radial functions

$$\mathcal{H} := \{f \in L^2_\sigma(\mathbb{R}^n) : f \text{ is radial}\},$$

with $\sigma(x) = e^{-|x|^2/4}$ and note that $\mathcal{S}_{rad}(\mathbb{R}^n) \subset \mathcal{H}$ is a dense. This space is the natural environment for the linear operator L_0 as it can be defined in a self-adjoint manner. However, this setting is not suited to study the nonlinear time evolution because the weight function decays at infinity. Hence, the goal of the next section is to utilize the self-adjoint framework to derive suitable bounds for the linearized time evolution in X.

3. Operator analysis and semigroup theory

3.1. **Self-adjoint theory.** It is easy to see that L_0 is symmetric on $\mathcal{S}_{rad}(\mathbb{R}^n) \subset \mathcal{H}$. The following Lemma summarizes some well-known results.

Lemma 3.1. *The free operator has a self-adjoint realization $\mathcal{L}_0 : \mathcal{D}(\mathcal{L}_0) \subset \mathcal{H} \to \mathcal{H}$, $\mathcal{L}_0 f = L_0 f$ with compact resolvent and $\mathcal{S}_{\text{rad}}(\mathbb{R}^n) \subset \mathcal{D}(\mathcal{L}_0)$ a core. Furthermore, $\mathcal{L}_0$ generates a strongly continuous one-parameter semigroup $\{S_0(\tau) : \tau \geq\}$ which is explicitly given by*

$$[S_0(\tau)f](x) = e^{-\tau}(G_{\alpha(\tau)} * f)(e^{-\tau/2}x) \tag{3.1}$$

for all $\tau \geq 0$, where $G_{\alpha(\tau)}(x) = [4\pi\alpha(\tau)]^{\frac{n}{2}} e^{-|x|^2/4\alpha(\tau)}$ and $\alpha(\tau) = (1 - e^{-\tau})$.

Proof. The first part of the result is immediate by noting that L_0 is unitarily equivalent to the Schrdinger operator

$$A_0 u(\rho) = -u''(\rho) + q(\rho)u(\rho) \tag{3.2}$$

with

$$q(\rho) := \frac{\rho^2}{16} + \frac{(n-3)(n-1)}{4\rho^2} - \frac{n-4}{4}$$

via the map $U : L(\mathbb{R}^+) \to \mathcal{H}$, $u \mapsto Uu = |\mathbb{S}^{n-1}|^{-\frac{1}{2}} \rho^{-\frac{n-1}{2}} e^{\rho^2/8} u(|\cdot|)$, i.e., $-L_0 = U A_0 U^{-1}$. Obviously, A_0 corresponds to the quantum harmonic oscillator. First, we define A_0 on $C_c^\infty(0,1)$ and apply standard criteria to see that A_0 is limit-point at both endpoints of the interval $(0,\infty)$, see [20], Th. 6.6, p. 96 and Th. 6.4, p. 91 (we have $\lim_{\rho\to\infty} q(\rho) = \infty$ and $q(\rho) \geq \frac{3}{4\rho^2}$ for ρ close to zero). As a consequence, the maximal operator

$$\mathcal{D}(\mathcal{A}_0) = \{u \in L^2(\mathbb{R}^+) : u, u' \in AC_{\text{loc}}(\mathbb{R}^+), A_0 u \in L^2(\mathbb{R}^+)\}, \tag{3.3}$$

$\mathcal{A}_0 u = A_0 u$ for $u \in \mathcal{D}(\mathcal{A}_0) \subset L^2(\mathbb{R}^+)$ is self-adjoint and $C_c^\infty(0,1)$ is a core. Furthermore, the growth of q at infinity implies that $\mathcal{A}_0$ has compact resolvent. Hence, by defining $\mathcal{D}(\mathcal{L}_0) := U\mathcal{D}(\mathcal{A}_0) \subset \mathcal{H}$, $\mathcal{L}_0 f = L_0 f$, the same holds for the self-adjoint operator $(\mathcal{L}_0, \mathcal{D}(\mathcal{L}_0))$. It is easy to see that $(\mathcal{L}_0 f|f)_{L^2_\sigma(\mathbb{R}^n)} \leq -1$; hence $\mathcal{L}_0$ generates a strongly continuous one-parameter semigroup $\{S_0(\tau) : \tau \geq 0\}$. In fact, since we are simply dealing the the heat equation in rescaled variables, the semigroup can be given explicitly by just transforming the usual heat semigroup, which gives (3.1). It is easy to see that $\mathcal{S}_{\text{rad}}(\mathbb{R}^n)$ is invariant under the semigroup and by the density of $\mathcal{S}_{\text{rad}}(\mathbb{R}^n) \subset \mathcal{H}$ we conclude that the radial Schwarz functions are a core for $\mathcal{L}_0$. □

We note that the above formula for the semigroup is also referred to as the Ornstein-Uhlenbeck semigroup.

Proposition 3.2. *The operator $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset \mathcal{H} \to \mathcal{H}$, $\mathcal{L} = L_0 + L'$, $\mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{L}_0)$ is self-adjoint, has compact resolvent and generates a strongly* *continuous semigroup* $\{S(\tau) : \tau \geq 0\}$. *For the spectrum, which consists only of eigenvalues, we have*

$$\sigma(\mathcal{L}) \subset (-\infty, 0) \cup \{1\}.$$

The spectral point $\lambda = 1$ *is a simple eigenvalue with eigenfunction* $\mathbf{g} = g/\|g\|_{L^2_\sigma(\mathbb{R}^n)}$, *where*

$$g(\rho) = (a\rho^2 + b)^{-2}.$$

Proof. The decay of the potential implies that it gives rise to a bounded operator on $\mathcal{H}$ which implies that $\mathcal{L}$ as defined as above is self-adjoint Kato-Rellich theorem and has compact resolvent. By the bounded perturbation theorem $\mathcal{L}$ generates a strongly continuous semigroup $\{S(\tau) : \tau \geq 0\}$.
For the structure of the spectrum, it suffices to investigate the Schrödinger operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$ defined by $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}_0)$, $\mathcal{A}u = A_0 u + Vu$ where $\mathcal{A}_0$ is given by (3.2) and (3.3). First, one can easily check that $\lambda = -1$ is an eigenvalue of $\mathcal{A}$ with eigenfunction

$$\tilde{g}(\rho) = \rho^{\frac{n-1}{2}} e^{-\frac{\rho^2}{8}} (a\rho^2 + b)^{-2}.$$

The aim is to show that this is the only non-positive spectral point. For this, we exploit the fact that $\tilde{g}$ is strictly positive which leads to the factorization $\mathcal{A} = A^+A^- - 1$ such that the kernel of A^- is spanned by $\tilde{g}$. The corresponding supersymmetric expression $A^-A^+ - 1$ gives rise to a (maximally defined) self-adjoint operator $\mathcal{A}_S : \mathcal{D}(\mathcal{A}_S) \subset L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$. On the $C_c^\infty(0, \infty)$, which is a core for $\mathcal{A}_S$, it is given by

$$\mathcal{A}_{\mathcal{S}} u(\rho) = -u''(\rho) + \frac{n^2 - 1}{4\rho^2} u(\rho) + Q(\rho)u(\rho), \tag{3.4}$$

with

$$Q(r) = \frac{\rho^2}{16} - \frac{n}{4} + \frac{3}{2} - 2\frac{a(2a(n-4) + b)r^2 + b(2a(n-2) + b)}{(a\rho^2 + b)^2}.$$

We show that $\mathcal{A}_S$ has no postive eigenvalues, which implies the result. For $n = 7$ the claim is proved in [8]. However, the method used there does not carry over to higher dimensions. We therefore take a different approach. Namely we resort to integral bounds for the number of negative eigenvalues for Schrödinger operators. We first define

$$B(n,p) := \frac{(p-1)^{p-1}\Gamma(2p)}{n^{2p-1}p^p\Gamma(p)} \int_0^{+\infty} r^{2p-1} Q_-(\rho)^p d\rho \tag{3.5}$$

where

$$Q_-(\rho) := \begin{cases} -Q(\rho), & Q(\rho) < 0, \\ 0, & \text{otherwise.} \end{cases}$$

For $\frac{3}{2} \leq p < \infty$, the number of negative eigenvalues of $\mathcal{A}_S$ is bounded by $B(n,p)$, see e.g. [16], Theorem XIII.9. It is therefore enough to prove that

$$B(n,p) < 1 \tag{3.6}$$

to rule out negative eigenvalues. However, we need to show that $\lambda = 0$ is not an eigenvalue as well. In fact, for operators of the form (3.4), in [6], Sec. A, authors use a perturbative argument to show that having (3.6) is already enough.
The proof therefore reduces to showing that given $n \in \{8, 9, 10, 11\}$ there is a choice of p, such that (3.6) holds. We now fix $n = 8$, and show that $B(8, 4) < 1$. Since $Q(\rho) > 0$ for $\rho \geq \frac{47}{10}$, we have

$$B(8,4) < \frac{945}{8^9} \int_0^{\frac{47}{10}} \rho^7 W(\rho)^4 d\rho. \tag{3.7}$$

Now we observe that the integrand has a unique partial fraction decomposition of the following form

$$\rho^7 W(\rho)^4 = \sum_{i=0}^{6} a_i \rho^{2i+1} + \sum_{i=1}^{8} \frac{b_i \rho}{(a\rho^2 + b)^i},$$

for some real a_i, b_i. Hence, the integral in (3.7) can be explicitly computed, and this yields $B(8, 4) < 1$. In the same way we prove that the same holds for $B(9, 4), B(10, 6)$ and $B(11, 6)$. $\square$

In the following, we use Proposition 3.2 to derive growth estimates for the semigroup on the graphs of fractional powers of the generator. For this, we summarize some important properties.

Lemma 3.3. *There is a unique self-adjoint, positive operator* $(1-\mathcal{L})^{\frac{1}{2}}$ *such* $\mathcal{S}_{\text{rad}}(\mathbb{R}^n) \subset \mathcal{D}((1-\mathcal{L})^{\frac{1}{2}})$ *is a core and*

$$\|(1-\mathcal{L})^{\frac{1}{2}}\|_{L^2_\sigma(\mathbb{R}^n)} = \|Bf\|_{L^2_\sigma(\mathbb{R}^n)}$$

for all $f \in \mathcal{S}_{\text{rad}}(\mathbb{R}^n)$, *where*

$$Bf(\rho) = \mathbf{g}(\rho) \frac{d}{d\rho}\left(\mathbf{g}(\rho)^{-1} f(\rho)\right).$$

Furthermore, $[(1-\mathcal{L})^{\frac{1}{2}}]^2 = 1 - \mathcal{L}$ *and the square root commutes with any bounded operator that commutes with* $\mathcal{L}$.

Proof. The existence and the basic properties of the square root are standard results. Let $f \in \mathcal{D}(((1-\mathcal{L})^{\frac{1}{2}}$ and $\varepsilon > 0$ be arbitrary. The fact that $\mathcal{D}(\mathcal{L})$ is core for $(1-\mathcal{L})^{\frac{1}{2}}$ and $\mathcal{S}_{\text{rad}}(\mathbb{R}^n)$ is a core for $\mathcal{D}(\mathcal{L})$ implies that there is a $\tilde{f} \in \mathcal{S}_{\text{rad}}(\mathbb{R}^n)$ such that $\|f - \tilde{f}\|_{L^2_\sigma(\mathbb{R}^n)} + \|(1-\mathcal{L})^{\frac{1}{2}}(f - \tilde{f})\|_{L^2_\sigma(\mathbb{R}^n)} < \varepsilon$ by using that $\|(1-\mathcal{L})^{\frac{1}{2}} f\|_{L^2_\sigma(\mathbb{R}^n)} \lesssim \|(1-\mathcal{L})f\|_{L^2_\sigma(\mathbb{R}^n)} + \|f\|_{L^2_\sigma(\mathbb{R}^n)}$. On $\mathcal{S}_{\text{rad}}(\mathbb{R}^n)$, we can write $(1-\mathcal{L}) = B^*B$, where $\mathcal{B}$ is defined above and

$$B^* f(\rho) = -\frac{1}{\mathbf{g}(\rho)\mu(\rho)} \frac{d}{d\rho}\left(\mu(\rho)\mathbf{g}(\rho)f(\rho)\right),$$

where $\mu(\rho) := \rho^{n-1}e^{-\rho^2/4}$. It is easy to check that B and B^* are formally adjoint to each other. This implies that for all $f \in \mathcal{S}_{\text{rad}}(\mathbb{R}^n)$,

$$\|(1-\mathcal{L})^{\frac{1}{2}}f\|^2_{L^2_\sigma(\mathbb{R}^n)} = ((1-\mathcal{L})f|f)_{L^2_\sigma(\mathbb{R}^n)} = (B^*Bf|f)_{L^2_\sigma(\mathbb{R}^n)} = \|Bf\|^2_{L^2_\sigma(\mathbb{R}^n)}.$$

□

In the following, work with graph norms associated to powers of $(1-\mathcal{L})^{\frac{1}{2}}$, i.e.,

$$\|f\|_{\mathcal{G}((1-\mathcal{L})^{k/2})} = \|f\|_{L^2_\sigma(\mathbb{R}^n)} + \|(1-\mathcal{L})^{k/2}f\|_{L^2_\sigma(\mathbb{R}^n)} \tag{3.8}$$

for $k \in \mathbb{N}_0$.
Furthermore, we denote by $\mathcal{P}$ the orthogonal projection onto $\mathbf{g}$,

$$\mathcal{P}f := (f|\mathbf{g})_{L^2_\sigma(\mathbb{R}^n)}\mathbf{g}, \tag{3.9}$$

which commutes with $S(\tau)$ for all $\tau \geq 0$.

Proposition 3.4. *There is an $\omega_0 > 0$ such that*

$$\|S(\tau)(1-\mathcal{P})f\|_{\mathcal{G}((1-\mathcal{L})^{k/2})} \leq e^{-\omega_0\tau}\|(1-\mathcal{P})f\|_{\mathcal{G}((1-\mathcal{L})^{k/2})}$$

for all $k \in \mathbb{N}_0$, all $f \in \mathcal{D}((1-\mathcal{L})^{k/2})$ and all $\tau \geq 0$.

Proof. The operators $\mathcal{P}$, $\mathcal{L}$ and $S(\tau)$ mutually commute. This implies that $(1-\mathcal{L})^{\frac{1}{2}}$ commutes with the projection and the semigroup and thus, the same holds for $(1-\mathcal{L})^{\frac{k}{2}}$, $k \in \mathbb{N}$. By Proposition 3.2, there is an $\omega_0 > 0$, such that for all $f \in \mathcal{H}$,

$$\|S(\tau)(1-\mathcal{P})f\|_{L^2_\sigma(\mathbb{R}^n)} \leq e^{-\omega_0\tau}\|(1-\mathcal{P})f\|_{L^2_\sigma(\mathbb{R}^n)},$$

for all $\tau \geq 0$. Hence, for all $f \in \mathcal{D}((1-\mathcal{L})^{k/2})$,

$$\begin{aligned}\|(1-\mathcal{L})^{k/2}S(\tau)(1-\mathcal{P})f\|_{L^2_\sigma(\mathbb{R}^n)} &= \|S(\tau)(1-\mathcal{P})(1-\mathcal{L})^{k/2}f\|_{L^2_\sigma(\mathbb{R}^n)} \\ &\leq e^{-\omega_0\tau}\|(1-\mathcal{P})(1-\mathcal{L})^{k/2}f\|_{L^2_\sigma(\mathbb{R}^n)} \\ &= e^{-\omega_0\tau}\|(1-\mathcal{L})^{k/2}(1-\mathcal{P})f\|_{L^2_\sigma(\mathbb{R}^n)},\end{aligned}$$

which implies the claim. □

The graph norms turn out to be extremely useful in order to control local Sobolev norms.

Lemma 3.5. *Fix $k \in \mathbb{N}_0$. Then for all $R > 0$,*

$$\|D^k f\|_{L^2(\mathbb{B}^n_R)} \leq C_{R,k}\sum_{j=0}^{k}\|f\|_{\mathcal{G}((1-\mathcal{L})^{j/2})}$$

for all $f \in \mathcal{S}_{\text{rad}}(\mathbb{R}^n)$ and some constant $C_{R,k} > 0$.

Proof. First, we note that $\mathbf{g}^{-1}(\rho)\mathbf{g}'(\rho) = \mathcal{O}(\langle\rho\rangle^{-2})$. We proceed by induction. For $k = 1$, we estimate

$$\begin{aligned}\|\nabla f\|_{L^2(\mathbb{B}^n_R)} &\simeq \||\cdot|^{\frac{n-1}{2}} f'(|\cdot|)\|_{L^2(0,R)} \lesssim_R \||\cdot|^{\frac{n-1}{2}} e^{-|\cdot|^2/8} f'(|\cdot|)\|_{L^2(0,\infty)} \\ &\lesssim_R \|Bf\|_{L^2_\sigma(\mathbb{R}^n)} + \|f\|_{L^2_\sigma(\mathbb{R}^n)} = \|(1-\mathcal{L})^{1/2} f\|_{L^2_\sigma(\mathbb{R}^n)} + \|f\|_{L^2_\sigma(\mathbb{R}^n)}.\end{aligned}$$

We assume that the statement holds up to some $k \in \mathbb{N}$. Since,

$$\begin{aligned}\|D^{k-1}(1-\mathcal{L})f\|^2_{L^2(\mathbb{B}^n_R)} &= \|D^{k+1}f - D^{k-1}(\Lambda - V + 1)f\|^2_{L^2(\mathbb{B}^n_R)} \\ &= \|D^{k+1}f\|^2_{L^2(\mathbb{B}^n_R)} - 2(D^{k+1}f|D^{k-1}(\Lambda - V + 1)f)_{L^2(\mathbb{B}^n_R)} \\ &+ \|D^{k-1}(\Lambda - V + 1)f\|^2_{L^2(\mathbb{B}^n_R)}\end{aligned}$$

we obtain

$$\begin{aligned}\|D^{k+1}f\|^2_{L^2(\mathbb{B}^n_R)} &\lesssim \|D^{k-1}(1-\mathcal{L})f\|^2_{L^2(\mathbb{B}^n_R)} + \|D^{k-1}(\Lambda - V + 1)f\|^2_{L^2(\mathbb{B}^n_R)} \\ &\lesssim \|D^{k-1}(1-\mathcal{L})f\|^2_{L^2(\mathbb{B}^n_R)} + \|(\Lambda + \tfrac{k+1}{2})D^{k-1}f\|^2_{L^2(\mathbb{B}^n_R)} \\ &+ \|D^{k-1}(Vf)\|^2_{L^2(\mathbb{B}^n_R)} \lesssim_{R,k} \|D^{k-1}(1-\mathcal{L})f\|^2_{L^2(\mathbb{B}^n_R)} + \sum_{j=0}^{k} \|D^j f\|^2_{L^2(\mathbb{B}^n_R)}.\end{aligned}$$

By assumption,

$$\|D^{k-1}(1-\mathcal{L})f\|_{L^2(\mathbb{B}^n_R)} \lesssim_{R,k} \sum_{j=0}^{k-1} \|(1-\mathcal{L})f\|_{\mathcal{G}((1-\mathcal{L})^{j/2})} \lesssim_{R,k} \sum_{j=0}^{k+1} \|f\|_{\mathcal{G}((1-\mathcal{L})^{j/2})}$$

and $\sum_{j=0}^{k} \|D^j f\|^2_{L^2(\mathbb{B}^n_R)} \lesssim \sum_{j=0}^{k} \|f\|_{\mathcal{G}((1-\mathcal{L})^{j/2})}$ which implies the claim. □

With these observations, we can now turn to the investigation of the problem on X.

3.2. **Some properties of X.** We start with some basic observations.

Lemma 3.6. *For all $f \in \mathcal{S}_{rad}(\mathbb{R}^n)$, we have*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_X \tag{3.10}$$

Consequently, X is a Banach algebra and

$$\|fg\|_X \lesssim \|f\|_X \|g\|_X.$$

for all $f, g \in X$.

Proof. Eq. (3.10) follows from Fourier transform by noting that

$$\begin{aligned}\|f\|_{L^\infty(\mathbb{R}^n)} &\lesssim \|\mathcal{F}f\|_{L^1(\mathbb{B}^n)} + \|\mathcal{F}f\|_{L^1(\mathbb{R}^n\setminus\mathbb{B}^n)} \\ &\lesssim \||\cdot|^{-\kappa_0}\mathcal{F}f\|_{L^2(\mathbb{B}^n)} \||\cdot|^{\kappa_0}\mathcal{F}f\|_{L^2(\mathbb{B}^n)} \\ &+ \||\cdot|^{-\kappa_1}\|_{L^2(\mathbb{R}^n\setminus\mathbb{B}^n)} \||\cdot|^{\kappa_1}\mathcal{F}f\|_{L^2(\mathbb{R}^n\setminus\mathbb{B}^n)} \lesssim \|f\|_X.\end{aligned}$$

To show the Banach Algebra property, we proceed as usual and use that

$$D^k(fg) = fD^k g + gD^k f + \sum_{j=1}^{k-1} c_j D^j f D^{k-j} g \tag{3.11}$$

for some constants $c_j > 0$ and all $f, g \in \mathcal{S}_{rad}(\mathbb{R}^n)$. For $k = \kappa_0$, the control of the first to terms is immediate. Hölder's inequality and Proposition 3.5 in [18] imply that

$$\begin{aligned}\|D^j f D^{k-j} g\|_{L^2(\mathbb{R}^n)} &\leq \|D^j f\|_{L^{\frac{2k}{j}}(\mathbb{R}^n)} \|D^{k-j} g\|_{L^{\frac{2k}{k-j}}(\mathbb{R}^n)} \\ &\lesssim \left(\|g\|_{L^\infty(\mathbb{R}^n)} \|D^k f\|_{L^2(\mathbb{R}^n)}\right)^{j/k} \left(\|f\|_{L^\infty(\mathbb{R}^n)} \|D^k g\|_{L^2(\mathbb{R}^n)}\right)^{1-j/k} \\ &\lesssim \|f\|_X \|g\|_X,\end{aligned}$$

and for $k = \kappa_1$ the argument is the same, which implies the claim. □

Corollary 3.7. *The function space X is continuously embedded into $\mathcal{H}$, i.e., $X \hookrightarrow \mathcal{H}$.*

Proof. In view of Lemma 3.6 and the exponential decay of the weight function, we immediately obtain that

$$\|f\|_{L^2_\sigma(\mathbb{R}^n)} \lesssim \|f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_X$$

for all $f \in \mathcal{S}_{rad}(\mathbb{R}^n)$. Now, let $f \in X$. Then there is a sequence $(f_j)_{j\in\mathbb{N}} \subset \mathcal{S}_{rad}(\mathbb{R}^n)$ such that $f_j \to f$ in X. By the above inequality $(f_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}$ and we denote its limit by g. We define $\iota : X \to \mathcal{H}$ by $\iota(f) := g$ and show that it is injective. In fact, if $\iota(f) = 0$, there is a sequence $(f_j)_{j\in\mathbb{N}} \subset \mathcal{S}_{rad}(\mathbb{R}^n)$ such that $f_j \to f$ in X and $f_j \to 0$ in $\mathcal{H}$. Assume that $f \neq 0$. Then $(D^{\kappa_0} f_j)_{j\in\mathbb{N}}$ and $(D^{\kappa_1} f_j)_{j\in\mathbb{N}}$ are Cauchy sequences in $L^2(\mathbb{R}^d)$ converging to some $\tilde{f}_0$, $\tilde{f}_1$ strongly and thus also in the sense of distributions. The assumption on f implies that at least one of two limit functions has to be nonzero. However, for every test function φ and every $k \in \mathbb{N}$ we obtain that

$$|(D^k f_j|\varphi)_{L^2(\mathbb{R}^n)}| = |(f_j|D^k\varphi)_{L^2(\mathbb{R}^n)}| \lesssim \|f_j\|_{L^2_\sigma(\mathbb{R}^n)} \to 0.$$

By uniqueness of distributional limits we have a contradiction, which shows that $f = 0$. The continuity of the embedding now follows from the above inequality. □

By a straightforward approximation argument, see e.g. the proof of Lemma 4.7 in [2], yields the next useful result.

Lemma 3.8. *Let $f \in C^\infty_{\mathrm{rad}}(\mathbb{R}^n)$ and assume that*

$$|D^k f(x)| \lesssim \langle x\rangle^{-2-k}$$

for all $x \in \mathbb{R}^n$ and all $k \in \{0, \dots, \kappa_1\}$. Then $f \in X$.

The next statement is crucial and again relies on the strong decay of the exponential weight.

Lemma 3.9. *For* $k \in \{0, \dots, \kappa_1\}$*, we have*

$$\|(1-\mathcal{L})^{k/2} f\|_{L^2_\sigma(\mathbb{R}^n)} \lesssim \|f\|_X \tag{3.12}$$

for all $f \in \mathcal{S}_{\text{rad}}(\mathbb{R}^n)$*.*

Proof. First, we show that for polynomially bounded functions $w \in C^\infty(\mathbb{R}^n)$,

$$\|w D^k f\|_{L^2_\sigma(\mathbb{R}^n)} \lesssim \|f\|_X, \tag{3.13}$$

for $k \in \{0, \dots, \kappa_1\}$. In fact, by exploiting the decay of the exponential weight and Hardy's inequality, see e.g. [14], Theorem 9.5, we get for $k \in \{0, \dots, \kappa_0\}$,

$$\|w D^k f\|_{L^2_\sigma(\mathbb{R}^n)} \lesssim \||\cdot|^{-\kappa_0 + k} D^k f\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{\dot{H}^{\kappa_0}(\mathbb{R}^n)} \tag{3.14}$$

since $\kappa_0 < \frac{n}{2}$. Furthermore, for $k \in \{\kappa_1 - 1, \kappa_1\}$,

$$\|w D^k f\|_{L^2_\sigma(\mathbb{R}^n)} \lesssim \||\cdot|^{-\kappa_1 + k} D^k f\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{\dot{H}^{\kappa_1}(\mathbb{R}^n)}. \tag{3.15}$$

Now, it is easy to see that for $f \in \mathcal{S}_{\text{rad}}(\mathbb{R}^n)$;

$$\|(1-\mathcal{L})^{k/2} f\|_{L^2_\sigma(\mathbb{R}^n)} \lesssim \|D^k f\|_{L^2_\sigma(\mathbb{R}^n)} + \sum_{j=0}^{k-1} \|w_j D^j f\|_{L^2_\sigma(\mathbb{R}^n)}$$

for smooth, polynomially bounded functions w_j, which implies

$$\|(1-\mathcal{L})^{k/2} f\|_{L^2_\sigma(\mathbb{R}^n)} \lesssim \|f\|_X$$

for $k \in \{1, \dots, \kappa_1\}$ and all $f \in \mathcal{S}_{\text{rad}}(\mathbb{R}^n)$. □

Corollary 3.10. *We have*

$$\|f\|_{H^k(\mathbb{B}^n_R)} \leq C_{R,k} \|f\|_X \tag{3.16}$$

for all $f \in \mathcal{S}_{\text{rad}}(\mathbb{R}^n)$*, all* $R > 0$ *and* $k = \{0, \dots \kappa_1\}$*.*

Proof. Lemma 3.9 and Lemma 3.5 imply that

$$\|D^k f\|_{L^2(\mathbb{B}^n_R)} \leq C_{R,k} \|f\|_X$$

for all $f \in \mathcal{S}_{\text{rad}}(\mathbb{R}^n)$, all $R > 0$ and $k \in \{0, \dots \kappa_1\}$. In the radial context, one can easily show that the full Sobolev norm is controlled by radial derivatives, i.e., that

$$\|f\|_{H^k(\mathbb{B}^n_R)} \leq C_{R,k} \sum_{j=0}^{k} \|D^k f\|_{L^2(\mathbb{B}^n_R)},$$

see for example Appendix B in [9] for $R = 1$, or Lemma 3.4 in [2] for the case $k = 2$, $n = 5$. The main idea is to define a suitable, compactly supported extension from $\overline{\mathbb{B}^n_R}$ to $\mathbb{R}^n$ and to use that $\|f\|_{H^k(\mathbb{R}^n)} \lesssim \|\langle \cdot \rangle \mathcal{F} f\|_{L^2(\mathbb{R}^d)}$.
By density, Eq. (3.16) extends to all of X and Sobolev embedding implies the claim. □

3.3. **The time evolution on X.** By using Lemma 3.6 and the explict form of the semigroup, we obtain the following result.

Lemma 3.11. *The restriction of $\{S(\tau) : \tau \geq 0\}$ to X defines a strongly continuous one-parameter semigroup $\{S_X(\tau) : \tau \geq 0\}$ on X. Its generator is given by the part of $\mathcal{L}$ in X,*

$$\mathcal{L}_X f := \mathcal{L} f, \quad \mathcal{D}(\mathcal{L}_X) := \{f \in \mathcal{D}(\mathcal{L}) \cap X : \mathcal{L} f \in X\}.$$

Furthermore, $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^n)$ is a core of $\mathcal{L}_X$.

In fact, by the above results it is clear that $X \subset \mathcal{D}(\mathcal{L})$.

Proof. First, we show that the semigroup leaves X invariant and is strongly continuous with respect to the norm on X. We first establish these properties for the free semigroup $\{S_0(\tau) : \tau \geq 0\}$, which is known explicitly and given by Eq. (3.1). We proceed as in the proof of Lemma 3.8 in [2]. Since $G_{\alpha(\tau)} \in L^1(\mathbb{R}^n)$ for every $\tau > 0$, Young's inequality yields

$$\|D^k S_0(\tau) f\|^2_{L^2(\mathbb{R}^n)} \lesssim e^{\frac{1}{2}(\frac{n}{2}-2-k)\tau} \|D^k f\|_{L^2(\mathbb{R}^n)} \tag{3.17}$$

for all $k \in \mathbb{N}_0$. In particular, X is invariant under $S_0(\tau)$ for all $\tau \geq 0$. Also, this shows that the free time evolution is growing exponentially in homogeneous Sobolev norms below scaling. By using again the explicit form of the semigroup, rescaling and Minkowski's inequality yield

$$\begin{aligned}&\|D^k[S_0(\tau) f - f]\|_{L^2(\mathbb{R}^n)} \\ &\qquad \lesssim e^{-\frac{k}{2}\tau} \int_{\mathbb{R}^n} \|(D^k f)(e^{-\frac{\tau}{2}}(\cdot) - \alpha(\tau)^{\frac{1}{2}} y) - D^k f\|_{L^2(\mathbb{R}^n)} dy\end{aligned}$$

and by dominated convergence we infer that $\|D^k[S_0(\tau) f - f]\|_{L^2(\mathbb{R}^n)} \to 0$ as $\tau \to 0^+$. This shows that the free semigroup is strongly continuous on X. By Lemma 3.9 and a standard result from semigroup theoy, see p.60, II.2.3 in [10], we infer that the part of the operator $\mathcal{L}_0$ in X defined as $\mathcal{L}_0|_X f := \mathcal{L}_0 f$,

$$\mathcal{D}(\mathcal{L}_0|_X) = \{f \in \mathcal{D}(\mathcal{L}_0) \cap X : \mathcal{L}_0 f \in X\}$$

generates the restricted semigroup $\{S_0|_X(\tau) : \tau \geq 0\}$. An application of Corollary 3.10 shows that $V \in X$ and thus

$$\|V f\|_X \lesssim \|V\|_X \|f\|_X \tag{3.18}$$

by Lemma 3.6. By the bounded perturbation theorem, $\mathcal{L}_X$, $\mathcal{D}(\mathcal{L}_X) = \mathcal{D}(\mathcal{L}_0|_X)$ generates the strongly continuous semigroup $\{S_X(\tau) : \tau \geq 0\}$. The density of $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^n)$ in X and the fact that $S_0(\tau)$ leaves radial Schwartz functions invariant implies last statement. □

Lemma 3.12. *The projection operator $\mathcal{P}$ defined in Eq.* (3.9) *induces a (non-orthogonal) projection $\mathcal{P}_X$ on X,*

$$\mathcal{P}_X f = (f|\mathbf{g})_{L^2_\sigma(\mathbb{R}^n)} \mathbf{g} \quad \textit{for } f \in X,$$

which commutes with operator $\mathcal{L}_X$ and the semigroup $S_X(\tau)$ for all $\tau \geq 0$. Furthermore,

$$\ker \mathcal{P}_X = \{f \in X : (f|\mathbf{g})_{L^2_\sigma(\mathbb{R}^n)} = 0\}.$$

Proof. The decay of $\mathbf{g}$ and Corollary 3.10 imply that $\mathbf{g} \in X$. By Cauchy-Schwarz and Corollary 3.7

$$\|\mathcal{P}_X f\|_X = \left|(f|\mathbf{g})_{L^2_\sigma(\mathbb{R}^n)}\right| \|\mathbf{g}\|_X \lesssim \|f\|_{L^2_\sigma(\mathbb{R}^n)} \|\mathbf{g}\|_X \lesssim \|f\|_X.$$

The other properties follow from the properties of the semigroup on $\mathcal{H}$. $\square$

In the following, we drop the subscript for $\mathcal{L}_X$, $S_X(\tau)$ and $\mathcal{P}_X$ for the sake of readability. To derive a suitable growth bound for the semigroup on X, the following Lemma is crucial.

Lemma 3.13. *For all $f \in \mathcal{S}_{rad}(\mathbb{R}^n)$ and all $R \geq 1$ we have*

$$(\mathcal{L}f|f)_X \leq (-\tilde{\omega} + \tfrac{C}{R^2})\|f\|_X^2 + C_R \sum_{j=0}^{\kappa_1} \|f\|^2_{\mathcal{G}((1-\mathcal{L})^{j/2})}. \tag{3.19}$$

with $\tilde{\omega} := \frac{1}{2}(\kappa_0 - \frac{n}{2} + 2) > 0$.

Proof. Let $f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^n)$. The commutator relation (2.9) yields

$$D^k \mathcal{L} f = D^{k+2} f - D^k \Lambda f + D^k(Vf) = D^{k+2} f - \Lambda D^k f - \frac{k}{2} D^k f + D^k(Vf).$$

We use that $(\Lambda f|f)_{L^2(\mathbb{R}^n)} = (1 - \frac{n}{4})\|f\|^2_{L^2(\mathbb{R}^n)}$ to estimate

$$\begin{aligned}(D^k \mathcal{L} f|D^k f)_{L^2(\mathbb{R}^n)} &\leq -(\Lambda D^k f|D^k f)_{L^2(\mathbb{R}^n)} - \tfrac{k}{2}\|D^k f\|^2_{L^2(\mathbb{R}^n)} \\ &\quad + (D^k(Vf)|D^k f)_{L^2(\mathbb{R}^n)} \leq -\tfrac{1}{2}(k - \tfrac{n}{2} + 2)\|D^k f\|^2_{L^2(\mathbb{R}^n)} \\ &\quad + (D^k(Vf)|D^k f)_{L^2(\mathbb{R}^n)}.\end{aligned}$$

This yields the bound

$$(\mathcal{L}f|f)_X \leq -\tilde{\omega}\|f\|_X^2 + (Vf|f)_X.$$

To estimate the last term, we exploit the decay of the potential at infinity and Lemma 3.5. We use the Leibnitz formula (3.11) and estimate for $j = 0, \dots, k$,

$$\begin{aligned}|(D^j V D^{k-j} f|D^k f)_{L^2(\mathbb{R}^n)}| &\leq \||D^j V|^{\frac{j+1}{j+2}} D^{k-j} f\|^2_{L^2(\mathbb{R}^n)} \\ &\quad + \||D^j V|^{\frac{1}{j+2}} D^k f\|^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

For the last term, we get for every $R \geq 1$,

$$\begin{aligned}\||D^j V|^{\frac{1}{j+2}} D^k f\|_{L^2(\mathbb{R}^n)} &\leq C_R \|D^k f\|_{L^2(\mathbb{B}^n_R)} + \tfrac{C}{R}\|D^k f\|_{L^2(\mathbb{R}^n \setminus \mathbb{B}^n_R)} \\ &\leq C_R \sum_{i=0}^{k} \|f\|_{\mathcal{G}((1-\mathcal{L})^{i/2})} + \tfrac{C}{R}\|D^k f\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

For the first term we argue similarly and use Eq. (3.14) to obtain

$$\||D^jV|^{\frac{j+1}{j+2}}D^{k-j}f\|_{L^2(\mathbb{R}^n)} \leq C_R\sum_{i=0}^{k-j}\|f\|_{\mathcal{G}((1-\mathcal{L})^{i/2})} + C\||\cdot|^{-j-1}D^{k-j}f\|_{L^2(\mathbb{R}^n\setminus\mathbb{B}^n_R)}.$$

For $k=\kappa_0<\frac{n}{2}$, Hardy's inequality yields

$$|(D^{\kappa_0}(Vf)|D^{\kappa_0}f)_{L^2(\mathbb{R}^n)}| \leq C_R\sum_{j=0}^{\kappa_0}\|f\|^2_{\mathcal{G}((1-\mathcal{L})^{j/2})} + \tfrac{C}{R^2}\|D^{\kappa_0}f\|^2_{L^2(\mathbb{R}^n)}.$$

For $k=\kappa_1>\frac{n}{2}$, we estimate

$$\||\cdot|^{-j}D^{\kappa_1-j}f\|_{L^2(\mathbb{R}^n)} \lesssim \|D^{\kappa_1}f\|_{L^2(\mathbb{R}^n)}$$

for $j=0,\dots\kappa_0$ and treat separately the cases $j\in\{\kappa_1-1,\kappa_1\}$ for which we get $\||\cdot|^{-j}D^{\kappa_1-j}f\|_{L^2(\mathbb{R}^n\setminus\mathbb{B}^n_R)} \lesssim \|D^{\kappa_0}f\|_{L^2(\mathbb{R}^n)}$ This implies that

$$|(Vf|f)_X| \leq C_R\sum_{j=0}^{\kappa_1}\|f\|^2_{\mathcal{G}((1-\mathcal{L})^{j/2})} + \tfrac{C}{R^2}\|f\|^2_X$$

for some constants $C, C_R>0$ and Eq. (3.19) follows. ☐

Finally, we obtain the desired growth bounds for the linearized time evolution on X.

Proposition 3.14. *There is a $\omega>0$ such that*

$$\begin{aligned}\|S(\tau)(1-\mathcal{P})f\|_X &\lesssim e^{-\omega\tau}\|(1-\mathcal{P})f\|_X \\ \|S(\tau)\mathcal{P}f\|_X &= e^{\tau}\|\mathcal{P}f\|\end{aligned} \tag{3.20}$$

for all $f\in X$ and all $\tau\geq 0$.

Proof. Let $f\in\mathcal{S}_{rad}(\mathbb{R}^n)\cap\ker\mathcal{P}_X$. We use Lemma 3.13, Proposition 3.4 and Lemma 3.9 and chose $R>0$ sufficiently large to obtain

$$\begin{aligned}
\tfrac{1}{2}\partial_\tau\|S(\tau)f\|^2_X &= (\partial_\tau S(\tau)f|S(\tau)f)_X = (\mathcal{L}S(\tau)f|S(\tau)f)_X \\
&\leq (-\tilde{\omega}+\tfrac{C}{R^2})\|S(\tau)f\|^2_X + C_R\sum_{j=0}^{\kappa_1}\|S(\tau)f\|^2_{\mathcal{G}((1-\mathcal{L})^{j/2})} \\
&\leq -\tfrac{\tilde{\omega}}{2}\|S(\tau)f\|^2_X + Ce^{-2\omega_0\tau}\sum_{j=0}^{\kappa_1}\|f\|^2_{\mathcal{G}((1-\mathcal{L})^{j/2})} \\
&\leq -\tfrac{\tilde{\omega}}{2}\|S(\tau)f\|^2_X + Ce^{-2\omega_0\tau}\|f\|^2_X \leq -2c_0\|S(\tau)f\|^2_X + Ce^{-4c_0\tau}\|f\|^2_X
\end{aligned}$$

for $c_0=\frac{1}{2}\min\{\omega_0,\frac{\tilde{\omega}}{2}\}$. This inequality can be written as

$$\tfrac{1}{2}\partial_\tau\left[e^{4c_0\tau}\|S(\tau)f\|^2_X\right] \leq C\|f\|^2_X$$

and integration yields

$$\|S(\tau)f\|^2_X \leq (1+2C\tau)e^{-4c_0\tau}\|f\|^2_X \lesssim e^{-2\omega\tau}\|f\|^2_X,$$

for some suitably chosen $\omega > 0$. By a density argument, this can be extended to all $f \in \ker \mathcal{P}_X \subset X$. □

4. Nonlnear time evolution

4.1. **Estimates for the nonlinearity.** The bounds for the nonlinearity follow from the Banach algebra property of X and a generalized version of Strauss' inequality for higher homogeneous Soblev spaces that can be found for example in [5]. More precisely, we need the following estimates.

Lemma 4.1. *We have*

$$\||\cdot|^{\frac{3}{2}+k}D^k f\|_{L^\infty(\mathbb{R}^n\setminus\mathbb{B}^n)} \lesssim \|f\|_X, \quad \text{for } n \text{ odd} \tag{4.1}$$

$$\||\cdot|^{1+k}D^k f\|_{L^\infty(\mathbb{R}^n\setminus\mathbb{B}^n)} \lesssim \|f\|_X,, \quad \text{for } n \text{ even} \tag{4.2}$$

for all $f \in \mathcal{S}_{\text{rad}}(\mathbb{R}^n)$ and $k = 0, \dots, \kappa_0 - 1$. Furthermore,

$$\||\cdot|^{\frac{n}{2}-2}D^{\kappa_0} f\|_{L^\infty(\mathbb{R}^n\setminus\mathbb{B}^n)} \lesssim \|f\|_X. \tag{4.3}$$

Proof. Proposition 1 in [5] implies that for all $n \geq 2$ and $\frac{1}{2} < s < \frac{n}{2}$,

$$\||\cdot|^{\frac{n}{2}-s} f\|_{L^\infty(\mathbb{R}^n\setminus\mathbb{B}^n)} \lesssim \|f\|_{\dot{H}^s(\mathbb{R}^n)}.$$

The proof follows from the Fourier transform of radial functions and the Cauchy-Schwarz inequality. From this it is immediate that for $j = 1, \dots, \kappa_0$,

$$\||\cdot|^{\frac{n}{2}-j}D^{\kappa_0-j} f\|_{L^\infty(\mathbb{R}^n\setminus\mathbb{B}^n)} \lesssim \|D^{\kappa_0} f\|.$$

Setting $k = \kappa_0 - j$ and inserting the definition of κ_0 yields the first two estimates. The last one follows from the fact that

$$\||\cdot|^{\frac{n}{2}-2}D^{\kappa_0} f\|_{L^\infty(\mathbb{R}^n\setminus\mathbb{B}^n)} \lesssim \|D^{\kappa_0+2} f\|_{L^2(\mathbb{R}^n)}.$$

□

In the following, we denote by $\mathcal{B}_X$ the unit ball in X. We set

$$N(f)(x) = -3(d-2)\big[1 + |x|^2 W(|x|)\big] f(x)^2 - (d-2)|x|^2 f(x)^3 \tag{4.4}$$

for $f \in \mathcal{S}_{rad}(\mathbb{R}^n)$.

Lemma 4.2. *The nonlinearity defined in* (4.4) *extends to a map $\mathcal{N} : X \to X$ satisfying*

$$\|\mathcal{N}(f) - \mathcal{N}(g)\|_X \lesssim (\|f\|_X + \|g\|_X)\|f - g\|_X \tag{4.5}$$

for all $f, g \in \mathcal{B}_X$ and $\mathcal{N}(f) = N(f)$ for all $f \in \mathcal{S}_{\text{rad}}(\mathbb{R}^n)$. Furthermore, $\mathcal{N}$ is differentiable at every $f \in X$ with Fréchet-derivative $DN(f) : X \to X$ bounded and the mapping $f \mapsto DN(f)$ is continuous.

Proof. To see this, we first prove that

$$\||\cdot|^2 f_1 f_2 f_3\|_X \lesssim \prod_{j=1}^{3} \|f_j\|_X \tag{4.6}$$

for all $f_1, f_2, f_3 \in \mathcal{S}_{\text{rad}}(\mathbb{R}^n)$. First, for a function $w \in C^\infty(\mathbb{R}^n)$ we write

$$\|D^k(wf)\|_{L^2(\mathbb{R}^n)} = \|D^k(wf)\|_{L^2(\mathbb{B}^n)} + \|D^k(wf)\|_{L^2(\mathbb{R}^n\setminus\mathbb{B}^n)} \tag{4.7}$$

For $k = \kappa_0 < \frac{n}{2}$, the Leibnitz rule (3.11) and Hardy's inequality imply that

$$\begin{aligned}\|D^{\kappa_0}(wf)\|_{L^2(\mathbb{B}^n)} &\lesssim \sum_{j=0}^{\kappa_0} \|D^{\kappa_0-j}f\|_{L^2(\mathbb{B}^n)} \lesssim \sum_{j=0}^{\kappa_0} \||\cdot|^{-j}D^{\kappa_0-j}f\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \|D^{\kappa_0}f\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_X.\end{aligned}$$

Similarly,

$$\begin{aligned}\|D^{\kappa_1}(wf)\|_{L^2(\mathbb{B}^n)} &\lesssim \sum_{j=0}^{\kappa_1} \|D^{\kappa_1-j}f\|_{L^2(\mathbb{B}^n)} \lesssim \|D^{\kappa_1}f\|_{L^2(\mathbb{B}^n)} \\ &\quad + \||\cdot|^{-1}D^{\kappa_1-1}f\|_{L^2(\mathbb{B}^n)} + \sum_{j=0}^{\kappa_0} \|D^{\kappa_0-j}f\|_{L^2(\mathbb{B}^n)} \lesssim \|f\|_X.\end{aligned}$$

Using this and the Banach algebra property yields

$$\begin{aligned}\||\cdot|^2 f_1f_2f_3\|_X &\lesssim \prod_{j=1}^{3}\|f_j\|_X + \|D^{\kappa_0}[|\cdot|^2 f_1f_2f_3]\|_{L^2(\mathbb{R}^n\setminus\mathbb{B}^n)} \\ &\quad + \|D^{\kappa_1}[|\cdot|^2 f_1f_2f_3]\|_{L^2(\mathbb{R}^n\setminus\mathbb{B}^n)}.\end{aligned} \tag{4.8}$$

Now,

$$\begin{aligned}&\|D^k[|\cdot|^2 f_1f_2f_3]\|_{L^2(\mathbb{R}^n\setminus\mathbb{B}^n)} \lesssim \||\cdot|^2 D^k[f_1f_2f_3]\|_{L^2(\mathbb{R}^n\setminus\mathbb{B}^n)} \\ &\quad + \||\cdot|D^{k-1}[f_1f_2f_3]\|_{L^2(\mathbb{R}^n\setminus\mathbb{B}^n)} + \|D^{k-2}[f_1f_2f_3]\|_{L^2(\mathbb{R}^n\setminus\mathbb{B}^n)}.\end{aligned} \tag{4.9}$$

To control $D^k(f_1f_2f_3)$ we have to estimate terms of the form $D^{j_1}f_1D^{j_2}f_2D^{j_3}f_3$ for $j \in \mathbb{N}_0^3$, $|j| = k$. First, we discuss $k = \kappa_0$. Assume that $j_i = \kappa_0$ for some $i \in \{1, 2, 3\}$ and without loss of generality we set $i = 3$. Then

$$\begin{aligned}&\||\cdot|^2 f_1f_2D^{\kappa_0}f_3\|_{L^2(\mathbb{R}^n\setminus\mathbb{B}^n)} \\ &\quad\lesssim \||\cdot|f_1\|_{L^\infty(\mathbb{R}^n\setminus\mathbb{B}^n)}\||\cdot|f_2\|_{L^\infty(\mathbb{R}^n\setminus\mathbb{B}^n)}\|D^{\kappa_0}f_3\|_{L^2(\mathbb{R}^n\setminus\mathbb{B}^n)} \lesssim \prod_{j=1}^{3}\|f_j\|_X\end{aligned}$$

by Lemma 4.1. If all $j_i \neq \kappa_0$ then

$$\begin{aligned}&\||\cdot|^2 D^{j_1}f_1D^{j_2}f_2D^{j_3}f_3\|_{L^2(\mathbb{R}^n\setminus\mathbb{B}^n)} \\ &= \||\cdot|^{2+j_1+j_2}|\cdot|^{-(j_1+j_2)}D^{j_1}f_1D^{j_2}f_2D^{\kappa_0-(j_1+j_2)}f_3\|_{L^2(\mathbb{R}^n\setminus\mathbb{B}^n)} \\ &\lesssim \||\cdot|^{-(j_1+j_2)}D^{\kappa_0-(j_1+j_2)}f_3\|_{L^2(\mathbb{R}^n)}\sum_{i=1}^{2}\||\cdot|^{1+j_i}D^{j_i}f_i\|_{L^\infty(\mathbb{R}^n\setminus\mathbb{B}^n)} \lesssim \prod_{j=1}^{3}\|f_j\|_X\end{aligned}$$

by Lemma 4.1 and Hardy's inequality. For the last term in Eq. (4.9), we similarly estimate

$$\|D^{\kappa_0-2}[f_1f_2f_3]\|_{L^2(\mathbb{R}^n\setminus\mathbb{B}^n)}$$
$$\lesssim \||\cdot|^{2+j_1+j_2}D^{j_1}f_1D^{j_2}f_2|\cdot|^{-2-(j_1+j_2)}D^{\kappa_0-2-(j_1+j_2)}f_3\|_{L^2(\mathbb{R}^n\setminus\mathbb{B}^n)} \lesssim \prod_{j=1}^{3}\|f_j\|_X$$

and the corresponding bound for the second term in Eq. (4.9) follows analogously. For $k=\kappa_1$ the last term in Eq. (4.9) can be controlled just by using the Banach algebra property of X. For the first term we distinguish again several cases. Let $j_3 \in \{\kappa_1, \kappa_1-1\}$ the bounds follow immediately by using the same arguments as above and similar for $j_3=\kappa_0$, $j_1=j_2=1$. If $j_1=0$ and $j_2=2$, then (4.1) and (4.2) imply the required bounds for $n\geq 8$, where $\kappa_0\geq 3$. However, for $n=7$ one has to use (4.3) in addition. Now, if all $j_i\leq \kappa_0-1$ and $0\leq j_1+j_2\leq\kappa_0$, the we can apply again Hardy's inequality to obtain

$$\||\cdot|^2D^{j_1}f_1D^{j_2}f_2D^{j_3}f_3\|_{L^2(\mathbb{R}^n\setminus\mathbb{B}^n)}$$
$$\lesssim \||\cdot|^{2+j_1+j_2}D^{j_1}f_1D^{j_2}f_2|\cdot|^{-(j_1+j_2)}D^{\kappa_1-(j_1+j_2)}f_3\|_{L^2(\mathbb{R}^n\setminus\mathbb{B}^n)} \lesssim \prod_{j=1}^{3}\|f_j\|_X.$$

For $j_1+j_2=\kappa_0+1$, we have $j_3=1$ such that

$$\||\cdot|^2Df_3|\cdot|^{j_1-1}D^{j_1}f_1|\cdot|^{-(j_1-1)}D^{\kappa_0-(j_1-1)}f_2\|_{L^2(\mathbb{R}^n\setminus\mathbb{B}^n)} \lesssim \prod_{j=1}^{3}\|f_j\|_X.$$

and for $j_1+j_2=\kappa_0+2$,

$$\||\cdot|f_3|\cdot|^{1+j_1}D^{j_1}f_1|\cdot|^{-j_1}D^{\kappa_0+2-j_1}f_2\|_{L^2(\mathbb{R}^n\setminus\mathbb{B}^n)} \lesssim \prod_{j=1}^{3}\|f_j\|_X,$$

which yields the bound for the first term in Eq. (4.9) . The corresponding estimate for the remaining second term follows from similar reasoning and we thus obtain Eq. (4.6). Now, $W\in X$ by Lemma 3.8 and the fact that $W^{(k)}(\rho)=\mathcal{O}(\langle\rho\rangle^{-2-k})$. Hence, we can use Eq. (4.6) to obtain the bound

$$\|\||\cdot|^2W(|\cdot|)f_1f_2\|_X \lesssim \|f_1\|_X\|f_2\|_X. \tag{4.10}$$

Finally, the first term in $\mathcal{N}$ can be controlled by simply using the Banach algebra property of X. In summary, we infer that $\mathcal{N}:\mathcal{S}_{\text{rad}}(\mathbb{R}^n)\to X$ and by using $a^p-b^p=(a-b)\sum_{j=0}^{p-1}a^{p-1-j}b^j$ together with Eq. (4.6) and Eq. (4.10) we infer that for all $f,g\in\mathcal{S}_{\text{rad}}(\mathbb{R}^n)$,

$$\|N(f)-N(g)\|\leq\gamma(\|f\|,\|g\|)\|f-g\|$$

for a continuous function $\gamma:[0,\infty)\times[0,\infty)\to[0,\infty)$ satisfying $\gamma(\|f\|,\|g\|)\lesssim \|f\|+\|g\|$ for all $f,g\in\mathcal{S}_{\text{rad}}(\mathbb{R}^n)\cap\mathcal{B}_X$. Hence, by density $\mathcal{N}$ extends to a continuous map $\mathcal{N}:X\to X$ satisfying the same bounds, see e.g. Lemma

3.2 in [7]. For the differentiabilty, we refer the reader to the proof Lemma 6.2 in [8]. □

The strategy of proof is the same as in [8], [2] and thus some details will be omitted in the following.

4.2. **The initial data operator.** We set $\mathcal{R}(v,T) := Tv(\sqrt{T}\cdot)$ we describe the properties of the initial data operator

$$\mathcal{U}(v,T) := \mathcal{R}(v,T) + \mathcal{R}(W,T) - \mathcal{R}(W,1) \tag{4.11}$$

and show that it is well-defined on X.

Lemma 4.3. *The map $\mathcal{U}(v,T)$: $\mathcal{B}_X \times [\frac{1}{2}, \frac{3}{2}] \to X$ is continuous. Furthermore, if $\|v\| \leq \delta$ then*

$$\|\mathcal{U}(v,T)\| \lesssim \delta$$

for all $T \in [1-\delta, 1+\delta]$.

Proof. First, note that for all $v_1, v_2 \in X$ and all $T \in [\frac{1}{2}, \frac{3}{2}]$

$$\|\mathcal{R}(v_1,T) - \mathcal{R}(v_2,T)\|_X \lesssim \|v_1 - v_2\|_X$$

such that $\mathcal{U}(\cdot,T) : X \to X$ is Lipschitz continuous. Next, for $v \in C^\infty_{\text{rad}}(\mathbb{R}^n)$, and $T_1, T_2 \in [\frac{1}{2}, \frac{3}{2}]$, the fundamental theorem of calculus implies that

$$v(\sqrt{T_1}\rho) - v(\sqrt{T_2}\rho) = (\sqrt{T_1} - \sqrt{T_2}) \int_0^1 \rho v'(\rho(\lambda_1 - \lambda_2)s + r\lambda_2)ds.$$

Now, the integral term can be controlled in X provided that v has sufficient decay at infinity. This in particular shows that

$$\|\mathcal{R}(W,T_1) - \mathcal{R}(W,T_2)\|_X \lesssim |T_1 - T_2|,$$

i.e., $T \mapsto \mathcal{R}(W,T)$ is Lipschitz continuous. For general $v \in X$, this is not the case. However, for given $\tilde{\varepsilon} > 0$ we find a $\tilde{v} \in \mathcal{S}_{\text{rad}}(\mathbb{R}^n)$ with $\|v - \tilde{v}\|_X < \tilde{\varepsilon}$ such that

$$\begin{aligned}\|v(\sqrt{T_1}\cdot) - v(\sqrt{T_2}\cdot)\|_X \leq& \|v(\sqrt{T_1}\cdot) - \tilde{v}(\sqrt{T_1}\cdot)\|_X + \|\tilde{v}(\sqrt{T_1}\cdot) - \tilde{v}(\sqrt{T_2}\cdot)\|_X \\ &+ \|\tilde{v}(\sqrt{T_2}\cdot) - v(\sqrt{T_2}\cdot)\|_X \lesssim \tilde{\varepsilon} + |T_1 - T_2|.\end{aligned}$$

Hence, for given $(v_1, T_1) \in \mathcal{B}_X \times [\frac{1}{2}, \frac{3}{2}]$ and $\varepsilon > 0$ let (v_2, T_2) be such that $\|v_1 - v_2\|_X + |T_1 - T_2| < \delta$ for $\delta > 0$. Furthermore, chose $\tilde{v}_1 \in \mathcal{S}_{\text{rad}}(\mathbb{R}^n)$ such that $\|v_1 - \tilde{v}\|_X < \delta$, then by the above considerations

$$\begin{aligned}\|\mathcal{U}(v_1,T_1) - \mathcal{U}(v_2,T_2)\|_X &\leq \|\mathcal{R}(v_1,T_1) - \mathcal{R}(v_1,T_2)\|_X \\ &+ \|\mathcal{R}(v_1,T_2) - \mathcal{R}(v_2,T_2)\|_X + \|\mathcal{R}(W,T_1) - \mathcal{R}(W,T_2)\|_X \lesssim \delta.\end{aligned}$$

This implies the claim provided that δ is chosen sufficiently small. Finally, for $v \in X$, $\|v\|_X \leq \delta$ we get

$$\|\mathcal{U}(v,T)\|_X \lesssim \|v\|_X + |T - 1| \lesssim \delta$$

for all $T \in [1-\delta, 1+\delta]$. □

4.3. **The nonlinear time evolution.** We consider the integral version of Eq. (2.5) by using the Duhamel formula and the above defined operators.

$$\Phi(\tau) = S(\tau)\mathcal{U}(v,T) + \int_0^\tau S(\tau-\tau')\mathcal{N}(\Phi(\tau'))d\tau'. \tag{4.12}$$

The aim of this section is to prove the following result.

Theorem 4.4. *Let $M>0$ be sufficiently large and $\delta>0$ sufficiently small. For every $v\in X$ with $\|v\|_X \leq \frac{\delta}{M^2}$, there exists a $T=T_v\in[1-\frac{\delta}{M},1+\frac{\delta}{M}]$ and a unique function $\Phi\in C([0,\infty),X)$ that satisfies Eq. (4.12) for all $\tau\geq 0$. Furthermore,*

$$\|\Phi(\tau)\| \leq \delta e^{-\omega\tau}, \quad \forall \tau\geq 0.$$

First, we introduce the Banach space

$$\mathcal{X} := \{\Phi\in C([0,\infty),\mathcal{H}) : \|\Phi\|_{\mathcal{X}} := \sup_{\tau\geq 0} e^{\omega\tau}\|\Phi(\tau)\| < \infty\} \tag{4.13}$$

and set $\mathcal{X}_\delta := \{\Phi\in\mathcal{X} : \|\Phi\|_{\mathcal{X}}\leq\delta\}$. To control the behavior of the semigroup on the unstable subspace $\mathcal{P}X$, we define the correction term

$$\mathcal{C}(\Phi,u) := \mathcal{P}u + \int_0^\infty e^{-\tau'}\mathcal{P}\mathcal{N}(\Phi(\tau'))d\tau' \tag{4.14}$$

and set

$$K(\Phi,u)(\tau) := S(\tau)u + \int_0^\tau S(\tau-\tau')\mathcal{N}(\Phi(\tau'))d\tau' - e^\tau\mathcal{C}(\Phi,u). \tag{4.15}$$

Lemma 4.5. *There is a $c>0$ such that for all $u\in X$ with $\|u\|_X\leq\frac{\delta}{c}$, $K(\cdot,u):\mathcal{X}_\delta\to\mathcal{X}_\delta$ provided that $\delta>0$ is sufficiently small. Furthermore*

$$\|K(\Phi,u)-K(\Psi,u)\|_{\mathcal{X}} \leq \frac{1}{2}\|\Phi-\Psi\|_X$$

for all $\Phi,\Psi\in\mathcal{X}_\delta$ and all $u\in X$.

Proof. We have

$$\mathcal{P}K(\Phi,u)(\tau) = -\int_\tau^\infty e^{-(\tau'-\tau)}\mathcal{P}\mathcal{N}(\Phi(\tau'))d\tau'$$

and

$$(1-\mathcal{P})K(\Phi,u)(\tau) = e^{-\omega_0\tau}(1-\mathcal{P})u + \int_0^\tau e^{-\omega(\tau-\tau')}(1-\mathcal{P}_X)\mathcal{N}(\Phi(\tau'))d\tau'$$

From this it is straightforward to see that Lemma 4.2 implies

$$\|\mathcal{P}K(\Phi,u)(\tau)\| \lesssim e^{-2\omega\tau}\delta^2,$$

and

$$\|(1-\mathcal{P})K(\Phi,u)(\tau)\| \lesssim e^{-\omega\tau}(\tfrac{\delta}{c}+\delta^2)$$

for all $\Phi\in\mathcal{X}_\delta$ and $u\in X$ satisfying $\|u\|_X\leq\frac{\delta}{c}$. This implies the first claim. For the Lipschitz estimate we use that

$$\|\mathcal{N}(\Phi(\tau))-\mathcal{N}(\Psi(\tau))\|_X \lesssim \delta e^{-\omega\tau}\|\Phi-\Psi\|_{\mathcal{X}}$$

by Lemma 4.2 to obtain

$$\|K(\Phi,u)(\tau)-K(\Psi,u)(\tau)\|_X \lesssim \delta e^{-\omega\tau}\|\Phi-\Psi\|_{\mathcal{X}}$$

which yields the result provided that $\delta>0$ is chosen sufficiently small. □

4.4. **Proof of Theorem 4.4.** Let $v\in X$ with $\|v\|_X \leq \frac{\delta}{M^2}$. By Lemma 4.3 we can chose $M>0$ large enough to guarantee that

$$\|\mathcal{U}(v,T)\|_X \leq \tfrac{\delta}{c}$$

for all $T\in I_{\delta,M} := [1-\frac{\delta}{M}, 1+\frac{\delta}{M}]$, where $c>0$ is the constant from Lemma 4.5. An application of the Banach fixed point theorem implies that for every $T\in I_{\delta,M}$ there exists a unique solution $\Phi_T\in\mathcal{X}_\delta$ to the equation

$$\Phi(\tau) = K(\Phi,\mathcal{U}(v,T))(\tau), \quad \tau\geq 0. \tag{4.16}$$

Furthermore, by Lemma 4.3 and continuity of the solution map, the map $T\mapsto\Phi_T$ is continuous. To prove Theorem 4.4, we show that there exists a $T=T(v)$ such that $\mathcal{C}(\Phi_{T(v)},\mathcal{U}(v,T))=0$. In fact, we show that

$$\big(\mathcal{C}(\Phi_{T(v)},\mathcal{U}(v,T))|\mathbf{g}\big)=0. \tag{4.17}$$

For this, we use that $\partial_T\mathcal{R}(W,T)|_{T=1}=\alpha\mathbf{g}$ for some $\alpha\in\mathbb{R}$ to write

$$\mathcal{U}(v,T)=\mathcal{R}(v,T)+\alpha(T-1)\mathbf{g}+(T-1)^2R(T,\cdot)$$

by Taylor expansion, where the error term depends continuously on T and satisfies $\|R(T,\cdot)\|_X\lesssim 1$ for all $T\in I_{\delta,M}$. Thus,

$$(\mathcal{P}\mathcal{U}(v,T)|\mathbf{g}) = C(1-T)+f(T),$$

where $|f(T)|\lesssim\frac{\delta}{M^2}+\delta^2$. By using the bounds of Lemma 4.2, Eq. (4.17) can be written as $T=F(T)+1$ for a continuous function F that satisfies $|F(T)|\lesssim\frac{\delta}{M^2}+\delta^2$. Hence, by choosing $M>0$ sufficiently large and $\delta=\delta(M)>0$ sufficiently small we obtain $|F(T)|\leq\frac{\delta}{M}$, hence $T\mapsto F(T)+1 : I_{\delta,M}\to I_{\delta,M}$. An application of Brower's fixed point argument shows that there is a $T\in I_{\delta,M}$ such that Eq. (4.17) is satisfied such that the corresponding $\Phi_T\in\mathcal{X}_\delta$ solves Eq. (4.12).

4.5. **Theorem 4.4 implies 1.1.** Fix $5\leq d\leq 9$ and set $n=d+2$. Let $\delta>0$ and $M>0$ be such that Theorem 4.4 holds and set $\delta' := \frac{\delta}{M}$. Let $v\in\mathcal{S}_{\text{rad}}(\mathbb{R}^n)$ such that $\|v\|_X\leq\frac{\delta'}{M}$. Then there exists a function $\Phi\in\mathcal{X}_\delta$ and a $T\in[1-\delta,1+\delta]$ such that (4.12) is satisfied for all $\tau\geq 0$ and decays exponentially to zero. Our assumption on the data imply that $\mathcal{U}(v,T)\in\mathcal{D}(\mathcal{L}_X)$ and thus, in view of Lemma 4.2, $\Phi\in C^1([0,\infty),X)$ solves

$$\partial_\tau\Phi(\tau)=\mathcal{L}_X\Phi(\tau)+\mathcal{N}(\Phi(\tau))$$

with $\Phi(0)=\mathcal{U}(v,T)$ and $\Phi(\tau)\in\mathcal{D}(\mathcal{L}_X)$ for all $\tau\geq 0$. In particular $(1-\mathcal{L})\Phi(\tau)\in X$. By combining Lemma 3.9 and Lemma 3.5 in a similar manner as in Corollary 3.10, we infer that $\Phi(\tau)\in H^{\kappa_1+2}(\mathrm{B}^{\mathrm{n}}_{\mathrm{R}})$ for every $R>0$ which implies $\Phi(\tau)\in C^2_{\text{rad}}(\mathbb{R}^n)$ by Sobolev embedding. Thus, $\mathcal{L}|_X$ acts as a classical differential operator and by setting $\psi(\tau,\cdot) := W(\cdot) + \Phi(\tau)(|\cdot|)$ we obtain a classical solution to the initial value problem (2.1)-(2.2). Finally,

$$u(r,t) = \frac{1}{T-t}\psi(\tfrac{r}{\sqrt{T-t}}, -\log(T-t) + \log T)$$

solves Eq. (1.8) with initial data given by Eq. (1.13). In view of Eq. (1.9), we have

$$\|u_T(|\cdot|,t)\|_X \simeq (T-t)^{-\frac{1}{2}(\kappa_1+2-\frac{n}{2})}$$

such that

$$\begin{aligned}
&(T-t)^{\frac{1}{2}(\kappa_1+2-\frac{n}{2})}\|u(|\cdot|,t) - u_T(|\cdot|,t)\|_X \\
&= (T-t)^{\frac{\kappa_1}{2}-\frac{n}{4}}\|\Phi(-\log(T-t)+\log T)(\tfrac{|\cdot|}{\sqrt{T-t}})\|_X \\
&\lesssim \|\Phi(-\log(T-t)+\log T))\|_X \leq \delta(T-t)^{\omega}
\end{aligned}$$

which implies Eq. (1.14). Convergence in L^∞ follows from Eq. (3.10),

$$\begin{aligned}
(T-t)\|u(\cdot,t) - u_T(\cdot,t)\|_{L^\infty(\mathbb{R}^+)} &\simeq \|\Phi(-\log(T-t)+\log T))\|_{L^\infty(\mathbb{R}^n)} \\
&\lesssim \|\Phi(-\log(T-t)+\log T))\|_X \leq \delta(T-t)^{\omega}.
\end{aligned}$$

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Department of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria
E-mail address: irfan.glogic@univie.ac.at

Karlsruhe Institute of Technology, Institute for Analysis, Englerstrae 2, 76131 Karlsruhe, Germany
E-mail address: birgit.schoerkhuber@kit.edu