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Approximate impedance for time-harmonic Maxwell's equations in a non planar domain with contrasted multi-thin layers

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Abstract

The aim of this paper is to give asymptotic models for the impedance of contrasted multi-thin layers for the harmonic Maxwell's equations. We start from a transmission problem which describes the scattering of electromagnetic waves by an obstacle covered with a thin coating (superposition of different thin layers of dielectric materials). We show how to model the effect of the thin coating by an impedance boundary condition on the boundary of the propagation domain. To this end, we use a technique of abstract differential equations.

Keyword: time-harmonic Maxwell's equations, scattering, thin dielectric layers, impedance operator, abstract differential equations.

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1 Introduction

The concept of impedance boundary condition is widely used in the study of scattering of electromagnetic waves by obstacles covered with a thin coating. In the literature, one can find many applications in different fields: biology, elasticity systems, inverse problems, telecommunication, industry and others... (see, e.g., [1, 2, 7, 8, 9, 10, 12, 13, 15, 16, 18, 19, 20, 25, 26, 27]). This boundary condition was introduced first for reasons linked to electrical engineering in the turn of the 18th and 19th century. Later, it was shown that this condition models other situations, such as a perfect conductor coated by a thin layer of dielectric material, which we are interested in in this research.

The numerical resolution of problems defined on domains containing thin layers reveals instabilities related to the parameter δ, which represents the thickness of the thin layer (see, e.g., [3, 4]). Hence the power of the impedance condition: it serves to replace the initial problem defined on the thin part of the domain by a boundary condition defined just on the boundary of the propagation domain. Thus, the impedance boundary condition (IBC) depends strongly on the parameter δ; and it models the effect of the thin layer on the exterior domain of propagation of the electromagnetic wave. This boundary condition is given through an operator defined on the exterior surface of the thin coating, called the impedance operator. In electromagnetism, the impedance operator couples tangential components of the electric and the magnetic fields (see, e.g., [14]).

The coating of the perfect conductor is made for two main purposes. The first one is to protect the obstacle from any degradation caused by nature and the second one is to reduce its radiation. In some cases, we need several contrasting thin layers for a better functionality of this coating. To our knowledge, in studies related to this subject, authors only considered perfect conductors

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covered by a single thin layer of dielectric substrate. So far, our investigation is to consider a coating consisting of several thin layers with different physical and geometrical characteristics. In [15] we studied the planar case using Fourier analysis, while in this paper we are interested in a smooth non planar geometry. Generally, we do not know exactly the impedance operator. However, we settle to find an approximate impedance condition by using techniques of abstract differential equations and asymptotic analysis.

In this work, we are studying the scattering of electromagnetic waves, with harmonic time dependency, by an obstacle coated with contrasted multi-thin layers of dielectric materials. Generally, a suitable expression of the exact impedance operator is not reachable by calculus. Taking advantages of tools from intrinsic differential geometry of thin layers (see, e.g., [22, 24]), we can write Maxwell's equations inside the thin coating in the form of a first-order (in the normal direction) abstract differential system, the coefficients of which are differential operators with respect to the tangential component (see, e.g., [14]). A Taylor expansion of the boundary condition of perfect conductor type is also used to get Padé-like approximations.

We highlight that the present study is based on an investigation made in [6]. The same problem was considered for exactly one thin dielectric layer. The authors checked simultaneously the validity of the derived efficient Padé-like IBCs and determined the domain of validity in terms of the thickness δ of the thin shell for a spherical geometry. In this paper, we also assess the performance of the approximate multi-layer IBC, in the case of a spherical obstacle. Our numerical examples indicate that the third order IBC consistently gives the most accurate results up to a value of $\delta = 0.09$. Important numerical evaluations concerning high order impedance boundary conditions are performed also in many references, we cite, e.g., [17, 28, 29].

The paper is structured as follows: in Sec. 2, we describe the geometrical and the physical characteristics of the propagation domain, then we recall the harmonic Maxwell's equations and the suitable boundary conditions. In Sec. 3, we first present a parametrization of one thin layer which allows us to describe the parametrization of multi-thin layers. After that, we introduce Maxwell's equations in the form of abstract differential equations. In Sec. 4, we recall a tensor notation that we used in [15] for the planar case; this notation allows us to give compact formulae of the approximate impedance conditions. The main contribution in this research is presented in Sec. 5, by Thm. 5.4, in which we give several approximations of the impedance operator of increasing order in the thickness of the layers, up to order three. These conditions explicitly express the effect of the surface curvature. In the last section, we present some numerical experiments to illustrate the performance of the derived IBCs. For a spherical obstacle, the scattered fields obtained with these conditions are compared with those obtained by solving full transmission problem for the Maxwell system with different material constants in each layer

2 General setting

2.1 Physical and geometrical hypothesis

We denote by D an obstacle of non planar, but smooth geometry which is assumed to be a perfect conductor. We assume that electromagnetic waves propagate in an exterior domain $\Omega^{ext} \subseteq \mathbb{R}^3$. We also suppose that the obstacle is covered by a thin coating constituting of N thin layers of disparate dielectric materials denoted by Ω_j^δ, $j = 1, ..., N$, so that $\Omega^{ext} = \mathbb{R}^3 \setminus (\overline{D} \cup \bigcup_{j=1}^N \overline{\Omega_j^\delta})$. The interface between the exterior domain and the thin coating is a smooth surface denoted by $\Gamma = \Gamma_0^\delta$, the interface between the thin coating and the metal obstacle is denoted by $\Gamma^\delta = \Gamma_N^\delta$. The unit normal vector $\mathbf{n}$ is directed outward from the obstacle (see Figure 1).

The dielectric materials covering the obstacle are assumed to be homogeneous and isotropic media. They are stratified and separated by the interfaces Γ_j^δ, $j = 1, ..., N-1$. We will assume that the boundary Γ is as smooth as we require, but let us remark that we assume at least C^2 smoothness of Γ, so that the interfaces Γ_j^δ, $j = 1, ..., N$, are well-defined as parallel surfaces. This will be made explicit in Sec. 3.

Each thin layer is characterized by :
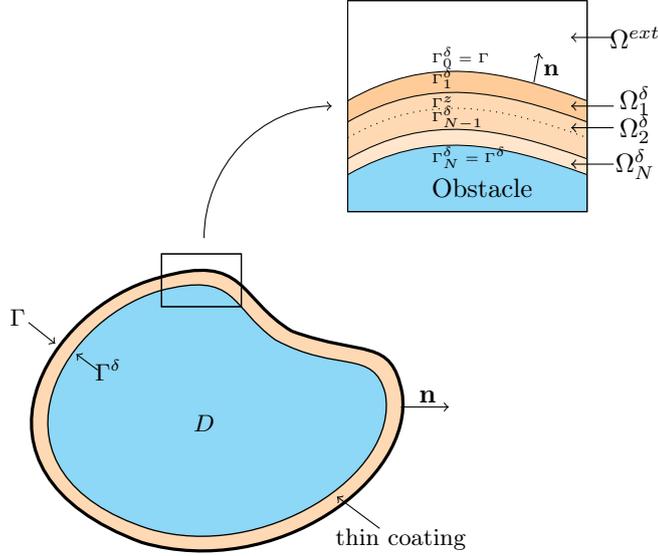
Figure 1: Illustration of an obstacle perfect conductor D of smooth geometry. The thin coating of D is constituted from N thin layers of different dielectric materials, denoted Ω_j^δ, $j = 1, ..., N$. The surfaces separating the thin layers are parallel surfaces denoted Γ_j^δ, $j = 1, ..., N-1$. The exterior surface is denoted by $\Gamma_0^\delta = \Gamma$ and the interface between the thin coating and the metal obstacle is denoted by $\Gamma_N^\delta = \Gamma^\delta$. The unit normal vector $\mathbf{n}$ is directed outward from the obstacle. The exterior domain is denoted Ω^{ext}.

- Geometric characteristic : the thickness $\delta_j = \alpha_j \delta$, $j = 1, ..., N$, where $\sum_{j=1}^N \delta_j = \delta > 0$ is very small compared to the other dimensions of the obstacle.

- Physical characteristics : each dielectric material is a perfect medium which is characterized by the relative electric permittivity ε_j and the relative magnetic permeability μ_j, for $j = 1, ..., N$, which are complex scalar parameters.

2.2 Maxwell's equations

In the absence of external charges and currents, the harmonic Maxwell's equations describing the propagation of electromagnetic waves in the domain described above are

$$
\begin{cases}
\nabla \times \mathbf{E} - i\kappa \mathbf{H} = 0, & \text{in } \Omega^{ext}, & \text{(1a)} \\
\nabla \times \mathbf{H} + i\kappa \mathbf{E} = 0, & \text{in } \Omega^{ext}, & \text{(1b)} \\
\nabla \times \mathbf{E}_j - i\kappa \mu_j \mathbf{H}_j = 0, & \text{in } \Omega_j^\delta, \ j = 1, \cdots, N, & \text{(1c)} \\
\nabla \times \mathbf{H}_j + i\kappa \varepsilon_j \mathbf{E}_j = 0, & \text{in } \Omega_j^\delta, \ j = 1, \cdots, N, & \text{(1d)}
\end{cases}
$$

where $\mathbf{E}$ is the electric field, $\mathbf{H}$ is the magnetic field and κ represents the wave number. The index j refers to each thin layer Ω_j^δ.

We complete the system of equations by the boundary conditions, which are categorized in three types.

Transmission conditions: Crossing the interfaces Γ_j^δ, $j = 0, ..., N-1$, the tangential components of the electromagnetic field are continuous

$$
\begin{cases}
[\mathbf{E} \times \mathbf{n}]\,|_{\Gamma_j^\delta} = 0, & j = 0, \cdots, N-1, & \text{(2a)} \\
[\mathbf{n} \times \mathbf{H}]\,|_{\Gamma_j^\delta} = 0, & j = 0, \cdots, N-1, & \text{(2b)}
\end{cases}
$$

where $[\cdot]$ stands for the jump across a surface.

Dirichlet boundary condition: On the boundary of the perfectly conducting obstacle, the tangential components of the electric field vanish,

$$\mathbf{E} \times \mathbf{n} = 0 \quad \text{on} \quad \Gamma^\delta. \tag{3}$$

Radiation condition: At infinity, we consider the Silver-Müller radiation condition

$$\lim_{|\mathbf{x}| \to \infty} |\mathbf{x}| \left(\nabla \times (\mathbf{E} - \mathbf{E}^{inc}) \times \frac{\mathbf{x}}{|\mathbf{x}|} - i\kappa(\mathbf{E} - \mathbf{E}^{inc}) \right) = 0, \tag{4}$$

uniformly in all directions, where $\mathbf{E}^{inc}$ and $\mathbf{H}^{inc}$ are incident waves satisfying Maxwell's system.

A formal proof of existence and uniqueness of solution for the problem (1)–(4) was obtained in the recent thesis [11].

A variational formulation of the problem (1)–(4) requires the introduction of Sobolev spaces and corresponding trace spaces. However, under the regularity assumptions we require for the derivation of approximate IBC, the solutions of the Maxwell system will be smooth functions as well. Therefore, we will refrain from formally introducing Sobolev spaces which allow the definition of tangential traces of the electric and the magnetic fields. For ample details on this subject, we refer the reader to the monographs [21], [23] or [24], for instance.

2.3 Formal definition of the impedance.

For a regular field φ defined on Γ_0^δ, the impedance operator Z_δ^{eff} is defined by

$$Z_\delta^{\text{eff}} \varphi = \mathbf{E}_{T|\Gamma_0^\delta},$$

where $\mathbf{E}_{T|\Gamma_0^\delta}$ is the tangential component of the radiating electric field $\mathbf{E}$ that satisfies Maxwell's equations (1), the transmission conditions (2), the Dirichlet boundary condition (3) and the boundary condition

$$\mathbf{n} \times \mathbf{H} = \varphi \text{ on } \Gamma_0^\delta.$$

2.4 Impedance problem

Making use of the impedance operator we can see that the transmission problem on Ω^{ext} and Ω_j^δ, $j = 1, ..., N$, is equivalent to the following impedance problem set on Ω^{ext}:

$$\begin{cases} \text{Maxwell's equations on } \Omega^{ext} \\ \text{Radiation condition at infinity} \\ \text{The exact impedance condition on } \Gamma_0^\delta : \ \mathbf{E}_{T|\Gamma_0^\delta} = Z^{\text{eff}} \left(\mathbf{n} \times \mathbf{H} \right). \end{cases}$$

The proof of existence and uniqueness for Maxwell's system with the IBC will be not addressed in this paper. We recall that our aim is to model the boundary condition of type impedance using approximation techniques. One can find such proof for Helmholtz equation with such boundary conditions in [27].

3 System of abstract equations

This work is part of applications of asymptotic expansion techniques, which are used to give models of the effect of thin layers by boundary conditions of impedance type. We consider a transmission problem modeling the propagation of electromagnetic waves in a domain containing thin coating. Before writing Maxwell's equations in the form of abstract equations, we need to parametrize the thin part of the domain. We show first how to parametrize one thin layer and then we generalize the approach for multi-thin layers.

Let Ω^δ denote a thin layer with an exterior boundary Γ of regularity C^2, defined by

$$\Omega^\delta = \left\{ \mathbf{x} \in \mathbb{R}^3 \setminus \overline{\Omega^{ext}} \ ; \ \text{dist} \left(\mathbf{x}, \Gamma \right) < \delta \right\}.$$

For sufficiently small δ, we define the isomorphism

$$\begin{array}{rcl} \Gamma \times (-\delta, 0) & \to & \Omega^\delta \\ (\mathbf{m}, z) & \to & \mathbf{x} = \mathbf{m} + z\mathbf{n}, \end{array} \qquad (6)$$

where $\mathbf{n}$ is the unit normal outwardly directed from Γ at the point $\mathbf{m}$. The fact that this map is indeed an isomorphism relies on the assumed smoothness of Γ. In this way we can write Ω^δ as the union of all the smooth surfaces Γ^z, also callel parallel surfaces, such that for all $z \in (-\delta, 0)$ we have

$$\Gamma^z = \{\mathbf{x} \in \Omega^\delta, \ \mathbf{x} = \mathbf{m} + z\mathbf{n}, \ \mathbf{m} \in \Gamma \ \}.$$

For the parametrization of the multi-thin layers, we adopt the notation

$$\tilde{\delta}_j = \sum_{l=1}^{j} \delta_l = \sum_{l=1}^{j} \alpha_l \delta = \tilde{\alpha}_j \delta, \qquad \text{with} \qquad \sum_{l=1}^{N} \alpha_l = 1.$$

For $j = N$, we have $\tilde{\delta}_N = \delta$ and for $j = 1$ we have $\sum_{l=1}^{j-1} \delta_j = \tilde{\delta}_0 = 0$. The parametrization of the multi-thin layers Ω_j^δ, $j = 1, ..., N$, separated by the interfaces Γ_j^δ, $j = 1, ..., N-1$, is then given by

$$\Omega_j^\delta = \bigcup_{z \in (-\tilde{\delta}_j, -\tilde{\delta}_{j-1})} \Gamma^z, \qquad j = 1, ..., N \,.$$

This parametrization allows us to write Maxwell's equations inside the thin layer in the form of an abstract differential equations of the first order in the variable z, where the coefficients are differential operators with respect to $\mathbf{m}$.

On each surface Γ^z, we first of all use the standard surface gradient ∇_{Γ^z} and surface divergence div_{Γ^z}. We furthermore use the scalar surface curl operator curl_{Γ^z} and the vectorial surface curl operator $\mathbf{curl}_{\Gamma^z}$, defined as

$$\mathrm{curl}_{\Gamma^z} \mathbf{v} = \mathrm{div}_{\Gamma^z}(\mathbf{v} \times \mathbf{n}), \qquad \mathbf{curl}_{\Gamma^z} v = -\mathbf{n} \times \nabla_{\Gamma^z} v \,.$$

Here, $\mathbf{v}$ and v refer to a vector- and a scalar field, respectively, defined in Ω^δ. All these four operators can finally be combined to define the Laplace-Beltrami operator

$$\Delta_{\Gamma^z} = \mathrm{div}_{\Gamma^z} \nabla_{\Gamma^z}, \qquad (7)$$

and vectorial Laplacian or Hodge operator

$$\overrightarrow{\Delta_{\Gamma^z}} = \nabla_{\Gamma^z} \mathrm{div}_{\Gamma^z} - \mathbf{curl}_{\Gamma^z} \mathrm{curl}_{\Gamma^z} \,. \qquad (8)$$

The curvature of the surface $\Gamma = \Gamma_0^\delta$ is described using the operators $\mathcal{C}$, $\mathcal{H}$ and $\mathcal{G}$, called: the curvature operator, the mean curvature operator and Gauss curvature, respectively (see, e.g., [24]). The operators on the surfaces Γ^z can be expressed as operators on the surface Γ by means of the following formulae

$$\nabla_{\Gamma^z} = (1 + z\mathcal{C})^{-1} \nabla_\Gamma, \qquad (9a)$$

$$\mathrm{div}_{\Gamma^z} = \frac{1}{1 + 2z\mathcal{H} + z^2\mathcal{G}} \, \mathrm{div}_\Gamma(1 - z(\mathcal{C} - 2\mathcal{H})), \qquad (9b)$$

$$\mathrm{curl}_{\Gamma^z} = \frac{1}{1 + 2z\mathcal{H} + z^2\mathcal{G}} \, \mathrm{curl}_\Gamma(1 + z\mathcal{C}), \qquad (9c)$$

$$\mathbf{curl}_{\Gamma^z} = (1 - z(\mathcal{C} - 2\mathcal{H}))^{-1} \mathbf{curl}_\Gamma. \qquad (9d)$$

The proof can be found in [14].

We show next how to write the Maxwell's system as an abstract first order equations. We decompose a vector field $\mathbf{v}$ on its tangential and normal components as follows

$$\mathbf{v} = \mathbf{v}_T + \mathbf{v_n}\mathbf{n}. \qquad (10)$$

The curl of $\mathbf{v}$ inside Ω^δ is given by (see, e.g., [24, Sec. 2.5])

$$\nabla \times \mathbf{v} = \mathbf{n}\ \mathrm{curl}_{\Gamma^z}\,\mathbf{v}_T + (\mathcal{C}_z - 2\mathcal{H}_z)\,(\mathbf{v} \times \mathbf{n}) + \mathbf{curl}_{\Gamma^z}(\mathbf{v} \cdot \mathbf{n}) - \partial_z \mathbf{v} \times \mathbf{n}. \tag{11}$$

On the surface Γ_0^δ, i.e., when $z = 0$, we can omit the index z from the curvature operators. Hence, Maxwell's equations inside a thin coating are written

$$\begin{cases} \mathbf{n}\ \mathrm{curl}_{\Gamma^z}\,\mathbf{E}_T + (\mathcal{C}_z - 2\mathcal{H}_z)\,(\mathbf{E} \times \mathbf{n}) + \mathbf{curl}_{\Gamma^z}(\mathbf{E} \cdot \mathbf{n}) - \partial_z \mathbf{E} \times \mathbf{n} - \mathrm{i}\kappa\mu\mathbf{H} = 0, & \text{(12a)} \\ \mathbf{n}\ \mathrm{curl}_{\Gamma^z}\,\mathbf{H}_T + (\mathcal{C}_z - 2\mathcal{H}_z)\,(\mathbf{H} \times \mathbf{n}) + \mathbf{curl}_{\Gamma^z}(\mathbf{H} \cdot \mathbf{n}) - \partial_z \mathbf{H} \times \mathbf{n} + \mathrm{i}\kappa\varepsilon\mathbf{E} = 0. & \text{(12b)} \end{cases}$$

We make use of the identity [24]

$$\mathbf{curl}_{\Gamma^z} v := \nabla_{\Gamma^z} v \times \mathbf{n}.$$

Then, by taking the projections of Eqs. (12a) and (12b) on the normal, we obtain the laws of the conservation of charges for the equivalent currents on Γ^z

$$\begin{cases} \mathrm{curl}_{\Gamma^z}\,\mathbf{E}_T - \mathrm{i}\kappa\mu\mathbf{H} \cdot \mathbf{n} = 0, & \text{(13a)} \\ \mathrm{curl}_{\Gamma^z}\,\mathbf{H}_T + \mathrm{i}\kappa\varepsilon\mathbf{E} \cdot \mathbf{n} = 0. & \text{(13b)} \end{cases}$$

From (12a) and (12b), the tangential components of Maxwell's equations inside the thin coating read

$$\begin{cases} (\mathcal{C}_z - 2\mathcal{H}_z)\,(\mathbf{E} \times \mathbf{n}) + \mathbf{curl}_{\Gamma^z}(\mathbf{E} \cdot \mathbf{n}) - \partial_z \mathbf{E} \times \mathbf{n} - \mathrm{i}\kappa\mu\mathbf{H}_T = 0, & \text{(14a)} \\ (\mathcal{C}_z - 2\mathcal{H}_z)\,(\mathbf{H} \times \mathbf{n}) + \mathbf{curl}_{\Gamma^z}(\mathbf{H} \cdot \mathbf{n}) - \partial_z \mathbf{H} \times \mathbf{n} + \mathrm{i}\kappa\varepsilon\mathbf{E}_T = 0. & \text{(14b)} \end{cases}$$

Therefore, we have

$$\begin{cases} (\mathcal{C}_z - 2\mathcal{H}_z)\,\mathbf{E} \times \mathbf{n} + \mathbf{curl}_{\Gamma^z}\left(-\dfrac{1}{\mathrm{i}\kappa\varepsilon}\,\mathrm{curl}_{\Gamma^z}\,\mathbf{H}_T\right) - \partial_z \mathbf{E} \times \mathbf{n} - \mathrm{i}\kappa\mu\mathbf{H}_T = 0, & \text{(15a)} \\[2ex] (\mathcal{C}_z - 2\mathcal{H}_z)\,\mathbf{H} \times \mathbf{n} + \mathbf{curl}_{\Gamma^z}\left(\dfrac{1}{\mathrm{i}\kappa\mu}\,\mathrm{curl}_{\Gamma^z}\,\mathbf{E}_T\right) - \partial_z \mathbf{H} \times \mathbf{n} + \mathrm{i}\kappa\varepsilon\mathbf{E}_T = 0. & \text{(15b)} \end{cases}$$

Now, we use the identities [24]

$$\mathrm{curl}_{\Gamma^z}\,\mathbf{v} := \mathrm{div}_{\Gamma^z}(\mathbf{v} \times \mathbf{n}), \qquad \mathbf{v}_T = (\mathbf{n} \times \mathbf{v}) \times \mathbf{n},$$

and

$$[(\mathcal{C}_z - 2\mathcal{H}_z)\,(\mathbf{v} \times \mathbf{n})] \times \mathbf{n} = \mathcal{C}_z \mathbf{v}.$$

Here, the last identity is obtained by a straightforward computation by means of the eigenexpansion of the operator $\mathcal{C}_z$, see, e.g., [14, 24]. We get the following abstract equations

$$\frac{\partial}{\partial z}\mathbf{X}_j = \mathbb{M}_j \mathbf{X}_j, \qquad j = 1, ..., N, \tag{16}$$

with

$$\mathbf{X}_j = \begin{bmatrix} \mathbf{n} \times (\mathbf{E} \times \mathbf{n}) \\ \mathbf{n} \times \mathbf{H} \end{bmatrix}_j = \begin{bmatrix} \mathbf{E}_T \\ \mathbf{n} \times \mathbf{H} \end{bmatrix}_j, \tag{17}$$

where $\mathbf{E}_T$ is the tangential component of the electric field. The coefficient-matrices $\mathbb{M}_j$ are defined by

$$\mathbb{M}_j = \mathbb{M}(z, \varepsilon_j, \mu_j) = \begin{bmatrix} -\mathcal{C}_z & -\mathrm{i}\kappa\mu_j A_z \\ \frac{1}{\mathrm{i}\kappa\mu_j} B_z & (\mathcal{C}_z - 2\mathcal{H}_z) \end{bmatrix}. \tag{18}$$

They are surface differential operators in the variable $\mathbf{m}$. The coefficients A_z and B_z are given by

$$A_z = A_z(\varepsilon_j, \mu_j) = 1 + \frac{1}{\kappa^2 \varepsilon_j \mu_j}\nabla_{\Gamma^z}\,\mathrm{div}_{\Gamma^z}, \tag{19}$$

$$B_z = B_z(\varepsilon_j, \mu_j) = \kappa^2 \varepsilon_j \mu_j - \mathbf{curl}_{\Gamma^z}\,\mathrm{curl}_{\Gamma^z}. \tag{20}$$

The index j, for $j = 1, ..., N$, refers to the thin layer on which the problem is defined and the index z refers to the surface Γ^z.

4 Tensorial notation

We recall that the impedance operator gives information about the propagation of electromagnetic waves inside the thin coating, thereafter many parameters are present: the physical characteristics $\varepsilon = (\varepsilon_j)_{1 \leq j \leq N}$, $\mu = (\mu_j)_{1 \leq j \leq N}$ and the geometric characteristics given by the thicknesses $(\delta_j)_{1 \leq j \leq N}$ and the curvature operators. For the sake of simplicity and in order to have a compact notation, we introduce the following forms that we used when studying the planar case (see [15]).

Let A be a ring, and let $X, Y, Z \in A^n$. Then, we define:

- The linear form $T_1(X)$ by

$$
\begin{aligned}
T_1(X): \quad \mathbb{R}^N \quad &\to A \\
\delta \quad &\to T_1(X)(\delta) = \sum_{i=1}^{N} \delta_i X_i.
\end{aligned}
\tag{21}
$$

- The bilinear form $T_2(X, Y)$ by

$$
\begin{aligned}
T_2(X, Y): \left(\mathbb{R}^N\right)^2 &\to A \\
(\delta, \eta) \to T_2(X, Y)(\delta, \eta) &= \sum_{i=1}^{N} \delta_i \eta_i X_i Y_i + \sum_{1 \leq j < i \leq N} 2 \delta_i \eta_j X_i Y_j.
\end{aligned}
\tag{22}
$$

- The trilinear form $T_3(X, Y, Z)$ by

$$
\begin{aligned}
T_3(X, Y, Z): \left(\mathbb{R}^N\right)^3 &\to A \\
(\delta, \eta, \zeta) \to T_3(X, Y, Z)(\delta, \eta, \zeta) &= \sum_{i,j,k} (T_3(X, Y, Z))_{i,j,k} \, \delta_i \eta_j \zeta_k,
\end{aligned}
\tag{23}
$$

where

$$
(T_3(X, Y, Z))_{i,j,k} = \begin{cases}
X_i Y_i Z_i, & \text{if} \quad i = j = k, \\
3 X_i Y_i Z_k, & \text{if} \quad i = j > k, \\
3 X_i Y_j Z_j, & \text{if} \quad i > j = k, \\
6 X_i Y_j Z_k, & \text{if} \quad i > j > k, \\
0, & \text{else.}
\end{cases}
$$

5 Construction of the approximate impedance operator

We recall that the impedance operator links the tangential components of the electromagnetic field, which gives the IBC defined on the boundary of the exterior domain. In this investigation, we give an approximate IBC by following the approach of Bendali *et al.* [6], in which the authors gave an approximate impedance operator at the third order of Padé type for a coating one thin layer.

A Padé-like approximation is a rational approximation that gives an impedance operator of the form

$$
Z_\delta^{\text{eff}} = \sum_{l=1}^{N} R_{\delta l} Q_{\delta l}^{-1} P_{\delta l},
\tag{24}
$$

where $R_{\delta l}$, $Q_{\delta l}$ and $P_{\delta l}$ are differential operators defined on the exterior surface of the thin coating and $Q_{\delta l}$ is an invertible elliptic operator.

This approach is developed to overcome difficulties related to numerical instabilities in the context of a fourth order operator appearing in the third order approximation of the impedance (see Bendali *et al.* [5]). The procedure consists in writing Maxwell's equation inside the thin coating in the form of abstract differential equations (see Sec. 3) in the variable z, the coefficients of which are surface differential operators. Then, by writing a Taylor expansion of the boundary condition of perfect conductor type, we can derive the approximate impedance boundary condition.

In the sequel, we will have an impedance condition of the form

$$
\mathcal{P} \mathbf{E}_{T|\Gamma_0^\delta} = \mathcal{Q}(\mathbf{n} \times \mathbf{H})_{|\Gamma_0^\delta},
\tag{25}
$$

where $\mathcal{P}$ and $\mathcal{Q}$ are differential operators defined on the surface Γ_0^δ. Then we invert the operator $\mathcal{P}$ in order to have an impedance condition of the form

$$
\mathbf{E}_{T|\Gamma_0^\delta} = Z_\delta^{\text{eff}} (\mathbf{n} \times \mathbf{H})_{|\Gamma_0^\delta}, \quad \text{where} \quad Z_\delta^{\text{eff}} = \mathcal{P}^{-1} \mathcal{Q}.
\tag{26}
$$

Remark 5.1. *For multi-thin layers, the analysis starts by applying Taylor expansion on the boundary condition of perfect conductor type (3) over δ_N, i.e., the thickness of the last thin layer Ω_N^δ. In order to be able to pass from one thin layer to the next one, we use interface conditions (2a) and (2b). Then, we do the same process for the thicknesses δ_j, $j = N-1, ..., 1$, until arriving to the exterior surface Γ_0^δ. We are interested in deriving a third order approximate impedance boundary condition, so we eliminate terms of order greater than three.*

The next result represents a technical tool, that we use to compute the different orders of the approximate impedance. In this lemma, we use a non-standard notation for products,

$$\prod_{j=N}^{1} a_j = a_N \, a_{N-1} \, \cdots \, a_1 \,.$$

Lemma 5.2. *The restriction of the vector field* $\mathbf{X}_N = \begin{bmatrix} \mathbf{E}_T \\ \mathbf{n} \times \mathbf{H} \end{bmatrix}_N$ *to the surface Γ^δ (i.e. $z = -\delta$) is related to the restriction of $\mathbf{X}_1$ to the surface Γ_0^δ (i.e. $z = 0$) by the operator $\mathcal{N}_n$ as follows*

$$\mathbf{X}_N(-\delta) = \mathcal{N}_n \mathbf{X}_1(0), \tag{27}$$

where the expansion of the operator $\mathcal{N}_n$ is defined by

$$\mathcal{N}_n = \prod_{j=N}^{1} \left(\begin{array}{c} I - \delta_j \mathbb{M} + \frac{1}{2}\delta_j^2(\frac{\partial}{\partial z}\mathbb{M} + \mathbb{M}^2) + \delta_j \left(\tilde{\delta}_{j-1}\right)\frac{\partial}{\partial z}\mathbb{M} \\ -\frac{1}{6}\delta_j^3 \left(\frac{\partial^2}{\partial z^2}\mathbb{M} + \mathbb{M}\frac{\partial}{\partial z}\mathbb{M} + 2(\frac{\partial}{\partial z}\mathbb{M})\mathbb{M} + \mathbb{M}^3 \right) \\ -\frac{1}{2}\delta_j \left(\tilde{\delta}_{j-1}\right)^2 \frac{\partial^2}{\partial z^2}\mathbb{M} \\ -\frac{1}{2}\delta_j^2 \left(\tilde{\delta}_{j-1}\right)\left(\frac{\partial^2}{\partial z^2}\mathbb{M} + \mathbb{M}\frac{\partial}{\partial z}\mathbb{M} + (\frac{\partial}{\partial z}\mathbb{M})\mathbb{M} \right) \end{array} \right)(0, \varepsilon_j, \mu_j). \tag{28}$$

The index n refers to the different approximation orders.

Proof. For $z \in \left(-\tilde{\delta}_j, -\tilde{\delta}_{j-1}\right)$, we have a medium corresponding to the coefficients ε_j and μ_j, $j = 1, 2, ..., N$.
On each thin layer of dielectric substrate, we apply a Taylor approximation over the corresponding thickness. In the jth thin layer, for example, we are located in the interval $\left(-\tilde{\delta}_j, -\tilde{\delta}_{j-1}\right)$, then we have

$$\mathbf{X}_j(-\tilde{\delta}_j) = \mathbf{X}_{j-1}(-\tilde{\delta}_{j-1}) - \delta_j\frac{\partial}{\partial z}\mathbf{X}_{j-1}(-\tilde{\delta}_{j-1}) + \frac{\delta_j^2}{2}\frac{\partial^2}{\partial z^2}\mathbf{X}_{j-1}(-\tilde{\delta}_{j-1}) - \frac{\delta_j^3}{6}\frac{\partial^3}{\partial z^3}\mathbf{X}_{j-1}(-\tilde{\delta}_{j-1}) + \text{h.o.t.} \tag{29}$$

By means of the first-order abstract differential system (16), we can replace the derivatives of $\mathbf{X}_j$ by the matrices $\mathbb{M}_j$ and their derivatives as follows

$$\mathbf{X}_j(-\tilde{\delta}_j) = \mathbf{X}_{j-1}(-\tilde{\delta}_{j-1}) - \delta_j\mathbb{M}\left(z, \varepsilon_j, \mu_j\right)\mathbf{X}_{j-1}(-\tilde{\delta}_{j-1}) + \frac{\delta_j^2}{2}(\frac{\partial}{\partial z}\mathbb{M} + \mathbb{M}^2)\left(z, \varepsilon_j, \mu_j\right)\mathbf{X}_{j-1}(-\tilde{\delta}_{j-1})$$

$$- \frac{\delta_j^3}{6}\left(\frac{\partial^2}{\partial z^2}\mathbb{M} + \mathbb{M}\frac{\partial}{\partial z}\mathbb{M} + 2(\frac{\partial}{\partial z}\mathbb{M})\mathbb{M} + \mathbb{M}^3 \right)\left(z, \varepsilon_j, \mu_j\right)\mathbf{X}_{j-1}(-\tilde{\delta}_{j-1}) + \text{h.o.t.} \tag{30}$$

In order to pass to the next dielectric substrate, we use the transmission conditions (2a) and (2b) defined on the surface Γ_{j-1}^δ. Then, we apply a Taylor expansion over the thickness δ_{j-2} defined on the interval $\left(-\tilde{\delta}_{j-1}, -\tilde{\delta}_{j-2}\right)$ and we replace the derivatives of the vector $\mathbf{X}_{j-1}$ by the matrices $\mathbb{M}_{j-1}$ and their derivatives. We use again the transmission condition to pass to the next thin layer and so on until arriving to the exterior surface Γ_0^δ on which we have $z = 0$.
We emphasize that each time we apply the Taylor expansion, we keep just terms of order less than or equal to three, by this way we have the following formula

$$\mathbf{X}_N(-\tilde{\delta}_N) = \prod_{j=N}^{1} \left(\begin{array}{c} I - \delta_j\mathbb{M} + \frac{\delta_j^2}{2}(\frac{\partial}{\partial z}\mathbb{M} + \mathbb{M}^2) - \\ \frac{\delta_j^3}{6}\left(\frac{\partial^2}{\partial z^2}\mathbb{M} + \mathbb{M}\frac{\partial}{\partial z}\mathbb{M} + 2(\frac{\partial}{\partial z}\mathbb{M})\mathbb{M} + \mathbb{M}^3 \right) \end{array} \right)(\tilde{\delta}_j, \varepsilon_j, \mu_j)\mathbf{X}_1(0, \varepsilon_1, \mu_1). \tag{31}$$

The matrices $\mathbb{M}_j$ are characterized by the physical characteristics of each thin layer, so they are dependent on the domains Ω_j^δ as well. Then, we have to apply a Taylor expansion for $\mathbb{M}_j$ and their derivatives

$$\mathbb{M}\left(\tilde{\delta}_j, \varepsilon_j, \mu_j\right) = \mathbb{M}(0, \varepsilon_j, \mu_j) - \tilde{\delta}_j \frac{\partial}{\partial z}\mathbb{M}(0, \varepsilon_j, \mu_j) + \frac{\tilde{\delta}_j^2}{2}\frac{\partial^2}{\partial z^2}\mathbb{M}(0, \varepsilon_j, \mu_j) + \text{h.o.t}, \tag{32}$$

$$(\frac{\partial}{\partial z}\mathbb{M}+\mathbb{M}^2)\left(\tilde{\delta}_j, \varepsilon_j, \mu_j\right) = (\frac{\partial}{\partial z}\mathbb{M}+\mathbb{M}^2)(0, \varepsilon_j, \mu_j) - \tilde{\delta}_j\left(\frac{\partial^2}{\partial z^2}\mathbb{M} + \mathbb{M}\frac{\partial}{\partial z}\mathbb{M} + (\frac{\partial}{\partial z}\mathbb{M})\mathbb{M}\right)(0, \varepsilon_j, \mu_j)+\text{h.o.t}, \tag{33}$$

and

$$\left(\frac{\partial^2}{\partial z^2}\mathbb{M} + \mathbb{M}\frac{\partial}{\partial z}\mathbb{M} + 2(\frac{\partial}{\partial z}\mathbb{M})\mathbb{M} + \mathbb{M}^3\right)\left(\tilde{\delta}_j, \varepsilon_j, \mu_j\right) = \left(\begin{array}{c}\frac{\partial^2}{\partial z^2}\mathbb{M} + \mathbb{M}\frac{\partial}{\partial z}\mathbb{M}+\\ 2(\frac{\partial}{\partial z}\mathbb{M})\mathbb{M} + \mathbb{M}^3\end{array}\right)(0, \varepsilon_j, \mu_j) + \text{h.o.t.} \tag{34}$$

Finally, we compute the product of all the obtained terms by keeping always terms of order less than or equal to three and we get the final formula

$$\mathbf{X}_N(-\tilde{\delta}_N) = \prod_{j=N}^{1}\left(\begin{array}{c} I - \delta_j\mathbb{M} + \frac{1}{2}\delta_j^2(\frac{\partial}{\partial z}\mathbb{M} + \mathbb{M}^2) + \delta_j\left(\tilde{\delta}_{j-1}\right)\frac{\partial}{\partial z}\mathbb{M} \\ -\frac{1}{6}\delta_j^3\left(\frac{\partial^2}{\partial z^2}\mathbb{M} + \mathbb{M}\frac{\partial}{\partial z}\mathbb{M} + 2(\frac{\partial}{\partial z}\mathbb{M})\mathbb{M} + \mathbb{M}^3\right) \\ -\frac{1}{2}\delta_j\left(\tilde{\delta}_{j-1}\right)^2\frac{\partial^2}{\partial z^2}\mathbb{M} \\ -\frac{1}{2}\delta_j^2\left(\tilde{\delta}_{j-1}\right)\left(\frac{\partial^2}{\partial z^2}\mathbb{M} + \mathbb{M}\frac{\partial}{\partial z}\mathbb{M} + (\frac{\partial}{\partial z}\mathbb{M})\mathbb{M}\right) \end{array}\right)(0, \varepsilon_j, \mu_j)\mathbf{X}_1(0, \varepsilon_1, \mu_1). \tag{35}$$

$\square$

Remark 5.3. *The operator $\mathcal{N}_n$ is a square matrix of dimension two whose coefficients are surface differential operators. We write the coefficients in rows j and columns k as $[\mathcal{N}_n]_{jk}$.*

Different approximations are given by using Eq. (27) in the lemma 5.2. By means of boundary condition (3), we get

$$0 = [\mathcal{N}_n]_{11}\,\mathbf{E}_{T|\Gamma_0^\delta} + [\mathcal{N}_n]_{12}\,(\mathbf{n} \times \mathbf{H})_{|\Gamma_0^\delta}. \tag{36}$$

We write the different approximations as follows

$$\mathbf{X}_{N|\Gamma^\delta} = \mathcal{N}_0(\varepsilon, \mu)\mathbf{X}_{1|\Gamma_0^\delta}, \tag{37}$$

$$\mathbf{X}_{N|\Gamma^\delta} = \mathcal{N}_1(\varepsilon, \mu)\mathbf{X}_{1|\Gamma_0^\delta}, \tag{38}$$

$$\mathbf{X}_{N|\Gamma^\delta} = \mathcal{N}_2(\varepsilon, \mu)\mathbf{X}_{1|\Gamma_0^\delta}, \tag{39}$$

$$\mathbf{X}_{N|\Gamma^\delta} = \mathcal{N}_3(\varepsilon, \mu)\mathbf{X}_{1|\Gamma_0^\delta}. \tag{40}$$

Then, for each approximation order, we need to compute $\mathcal{N}_n$ for $n = 0, 1, 2, 3$, and we have also to invert $[\mathcal{N}_n]_{11}$.

The next theorem represents the main contribution of this research, in which we give all the different approximations written in compact forms using the tensorial notation introduced in Sec. 4, which combines the physical and geometric characteristics of the contrasted thin layers.

In what follows, for the sake of brevity, we omit the subscript 0 and the superscript z in the surface differential operators defined on the surface Γ_0^z.

Theorem 5.4. *For a propagation domain containing contrasted thin layers, we have the following approximate impedance boundary conditions:*

- *At order zero, we have*

$$E_{T|\Gamma_0^\delta} = 0. \tag{41}$$

- *At order one, we have*

$$0 = E_{T|\Gamma_0^\delta} + ik\delta \left(T_1(\mu)(\alpha) + \frac{1}{k^2} T_1(\frac{1}{\varepsilon})(\alpha)\nabla_\Gamma \operatorname{div}_\Gamma \right)(\mathbf{n} \times H)_{|\Gamma_0^\delta}. \tag{42}$$

- *At order two, we have*

$$0 = E_{T|\Gamma_0^\delta} + i\kappa\delta \left(\begin{array}{c} T_1(\mu)(\alpha) + \frac{1}{\kappa^2} T_1\left(\frac{1}{\varepsilon}\right)(\alpha)\nabla_\Gamma \operatorname{div}_\Gamma \\ +\delta\frac{1}{\kappa^2}\nabla_\Gamma(T_2(\frac{1}{\varepsilon},\mathbf{1})(\alpha,\alpha)\mathcal{H}\operatorname{div}_\Gamma) \\ -\delta T_2(\mu,\mathbf{1})(\alpha,\alpha)(\mathcal{C}-\mathcal{H}) \end{array} \right)(\mathbf{n} \times H)_{|\Gamma_0^\delta}. \tag{43}$$

*By **1**, we refer to the identity vector $(1_j)_{j=1}^N$.*

- *At order three, we have*

$$0 = \mathbf{E}_{T|\Gamma_0^\delta} + i\kappa\delta \left(\begin{array}{c} T_1(\mu)(\alpha)\left(1 + \delta Q_\delta - \frac{1}{3}\delta^2 \frac{T_3(\mu,\mu,\frac{1}{\mu})(\alpha,\alpha,\alpha)}{T_1(\mu)(\alpha)}\overrightarrow{\Delta_\Gamma}\right)^{-1} \\ +\frac{1}{\kappa^2}\nabla_\Gamma T_1\left(\frac{1}{\varepsilon}\right)(\alpha)\left(1 + \delta b_\delta - \frac{1}{3}\delta^2 \frac{T_3(\frac{1}{\varepsilon},\frac{1}{\varepsilon},\varepsilon)(\alpha,\alpha,\alpha)}{T_1(\frac{1}{\varepsilon})(\alpha)}\Delta_\Gamma\right)^{-1}\operatorname{div}_\Gamma \end{array} \right)(\mathbf{n} \times \mathbf{H})_{|\Gamma_0^\delta}. \tag{44}$$

The coefficients Q_δ and b_δ are given by

$$Q_\delta = \frac{T_2(\mu,\mathbf{1})(\alpha,\alpha)}{T_1(\mu)(\alpha)}(\mathcal{C}-\mathcal{H}) + \delta\left(\frac{T_2(\mu,\mathbf{1})(\alpha,\alpha)}{T_1(\mu)(\alpha)}(\mathcal{C}-\mathcal{H})\right)^2$$
$$- \delta\left(\frac{2}{3}\frac{T_3(\mu,\mathbf{1},\mathbf{1})(\alpha,\alpha,\alpha)}{T_1(\mu)(\alpha)}(\mathcal{C}-\mathcal{H})(\mathcal{C}-2\mathcal{H}) + \frac{1}{3}\kappa^2\frac{T_3(\mu,\mu,\varepsilon)(\alpha,\alpha,\alpha)}{T_1(\mu)(\alpha)}\right), \tag{45}$$

and

$$b_\delta = -\frac{T_2(\frac{1}{\varepsilon},\mathbf{1})(\alpha,\alpha)}{T_1(\frac{1}{\varepsilon})(\alpha)}\mathcal{H} + \delta\left(\frac{T_2(\frac{1}{\varepsilon},\mathbf{1})(\alpha,\alpha)}{T_1(\frac{1}{\varepsilon})(\alpha)}\mathcal{H}\right)^2$$
$$- \frac{\delta}{3}\frac{T_3(\frac{1}{\varepsilon},\mathbf{1},\mathbf{1})(\alpha,\alpha,\alpha)}{T_1\left(\frac{1}{\varepsilon}\right)(\alpha)}(4\mathcal{H}^2-\mathcal{G}) - \frac{1}{3}\delta\kappa^2\frac{T_3(\frac{1}{\varepsilon},\mu,\varepsilon)(\alpha,\alpha,\alpha)}{T_1\left(\frac{1}{\varepsilon}\right)(\alpha)}$$
$$- \frac{1}{3}\delta\kappa^2\frac{T_3(\mu,\frac{1}{\varepsilon},\varepsilon)(\alpha,\alpha,\alpha)}{T_1\left(\frac{1}{\varepsilon}\right)(\alpha)} + \frac{1}{3}\delta\kappa^2\frac{T_3(\mu,\mu,\frac{1}{\mu})(\alpha,\alpha,\alpha)}{T_1\left(\frac{1}{\varepsilon}\right)(\alpha)}. \tag{46}$$

The proof of this theorem is given in the following paragraphs with discussions of the obtained results.

5.1 Impedance condition of order zero

For $n = 0$, from the formula (28) it is clear that $\mathcal{N}_0$ is the identity matrix. By using the boundary condition (3), we get the approximate impedance condition of order 0 defined on Γ_0^δ

$$\mathbf{E}_{T|\Gamma_0^\delta} = 0. \tag{47}$$

This impedance boundary condition means that the effect of the thin coating on the exterior domain Ω^{ext} is neglected.

5.2 Impedance condition of order one

First form: For $n = 1$, from the formula (28) we obtain that

$$\mathcal{N}_1 = \mathcal{N}_0 + \text{``term of order 1''}.$$

Developing the product in (28), we get

$$\mathcal{N}_1 = I - \delta \sum_{j=1}^{N} \alpha_j \mathbb{M}_j. \tag{48}$$

Using the tensorial notation defined in Sec. 4, we can write

$$\mathcal{N}_1 = I - \delta T_1(\mathbb{M})(\alpha). \tag{49}$$

Using the boundary condition (3), we get

$$0 = [\mathcal{N}_1]_{11} \, \mathbf{E}_{T|\Gamma_0^\delta} + [\mathcal{N}_1]_{12} \, (\mathbf{n} \times \mathbf{H})_{|\Gamma_0^\delta} \tag{50}$$

$$= [I - \delta T_1(\mathbb{M})(\alpha)]_{11} \, \mathbf{E}_{T|\Gamma_0^\delta} - [\delta T_1(\mathbb{M})(\alpha)]_{12} \, (\mathbf{n} \times \mathbf{H})_{|\Gamma_0^\delta}. \tag{51}$$

Second form: We remark that the term $[\mathcal{N}_1]_{12}$ depends on δ, so we have no need to develop $[\delta T_1(\mathbb{M})(\alpha)]_{11}$ because at the end we will have a second order term.
The second form of the impedance condition at the first order is given by

$$0 = \mathbf{E}_{T|\Gamma_0^\delta} - \delta \left[T_1(-i\kappa\mu A)(\alpha) \right] (\mathbf{n} \times \mathbf{H})_{|\Gamma_0^\delta}, \tag{52}$$

where

$$A = A(\varepsilon, \mu) = 1 + \frac{1}{\kappa^2 \mu} \nabla_\Gamma \left(\frac{1}{\varepsilon} \operatorname{div}_\Gamma \right), \quad \varepsilon = (\varepsilon_i)_{1 \le i \le N} \, , \, \mu = (\mu_i)_{1 \le i \le N} \, . \tag{53}$$

For the sake of brevity, we omitted the index j from the surface vector-operators A and B.

Third form: The final formula of the impedance condition at the first order for multi-thin layers is given by

$$0 = \mathbf{E}_{T|\Gamma_0^\delta} + \delta i\kappa \left[T_1(\mu)(\alpha) + \frac{1}{\kappa^2} T_1 \left(\frac{1}{\varepsilon} \right) (\alpha) \nabla_\Gamma \operatorname{div}_\Gamma \right] (\mathbf{n} \times \mathbf{H})_{|\Gamma_0^\delta}. \tag{54}$$

Remark 5.5. *We observe that we got the same result as in the planar case that we treated by Fourier analysis (see [15]). The effect of the curvature does not appear, therefore we need to add the second order terms to include geometric characteristics of the thin coating.*

5.3 Impedance condition of order two

First form: For $n = 2$, from the formula (28) we obtain

$$\mathcal{N}_2 = \mathcal{N}_1 + \text{``term of order 2''}.$$

Then, by developing the product in (28) we get

$$\mathcal{N}_2 = \mathcal{N}_1 + \delta^2 \sum_{N \ge i > j \ge 1} \alpha_i \alpha_j \mathbb{M}_i \mathbb{M}_j + \delta^2 \sum_{i=1}^{N} \frac{1}{2} \alpha_i^2 \left(\mathbb{M}_i^{(1)} + \mathbb{M}_i^2 \right) + \delta^2 \sum_{i=1}^{N} \alpha_i \tilde{\alpha}_{i-1} \mathbb{M}_i^{(1)}. \tag{55}$$

Here and below, we use the notation $\mathbb{M}^{(j)} = \frac{\partial^j}{\partial z^j} \mathbb{M}$. Using the tensorial notation defined in Sec. 4, we obtain

$$\mathcal{N}_2 = \mathcal{N}_1 + \delta^2 \frac{1}{2} T_2(\mathbb{M}, \mathbb{M})(\alpha, \alpha) + \delta^2 \frac{1}{2} T_2(\mathbb{M}^{(1)}, \mathbf{1})(\alpha, \alpha). \tag{56}$$

Now, using the boundary condition (3) we get

$$0 = [\mathcal{N}_2]_{11} \, \mathbf{E}_{T|\Gamma_0^\delta} + [\mathcal{N}_2]_{12} \, (\mathbf{n} \times \mathbf{H})_{|\Gamma_0^\delta} \tag{57}$$

$$= \left[I - \delta T_1(\mathbb{M})(\alpha) + \delta^2 \frac{1}{2} T_2(\mathbb{M}, \mathbb{M})(\alpha, \alpha) + \delta^2 \frac{1}{2} T_2(\mathbb{M}^{(1)}, \mathbf{1})(\alpha, \alpha) \right]_{11} \mathbf{E}_{T|\Gamma_0^\delta}$$

$$+ \left[-\delta T_1(\mathbb{M})(\alpha) + \delta^2 \frac{1}{2} T_2(\mathbb{M}, \mathbb{M})(\alpha, \alpha) + \delta^2 \frac{1}{2} T_2(\mathbb{M}^{(1)}, \mathbf{1})(\alpha, \alpha) \right]_{12} (\mathbf{n} \times \mathbf{H})_{|\Gamma_0^\delta}. \tag{58}$$

Second form: The term $[\mathcal{N}_2]_{12}$ is of order δ^2, therefor left multiplication by $[I + \delta T_1(\mathbb{M})(\alpha)]_{11}$ suffices to obtain an approximation of order 2 with respect to δ of the impedance condition

$$0 = \mathbf{E}_{T|\Gamma_0^\delta} + \left\{ \begin{bmatrix} -\delta T_1(\mathbb{M})(\alpha) + \delta^2 \frac{1}{2} T_2(\mathbb{M}, \mathbb{M})(\alpha, \alpha) \\ +\delta^2 \frac{1}{2} T_2(\mathbb{M}^{(1)}, \mathbf{1})(\alpha, \alpha) \\ -\delta^2 [T_1(\mathbb{M})(\alpha)]_{11} [T_1(\mathbb{M})(\alpha)]_{12} \end{bmatrix}_{12} \right\} (\mathbf{n} \times \mathbf{H})_{|\Gamma_0^\delta}. \tag{59}$$

The coefficient of order one with respect to δ is the same one as in the first-order case, by developing the coefficients in dependency on δ^2 we get the second form

$$0 = \mathbf{E}_{T|\Gamma_0^\delta} + \begin{pmatrix} \mathrm{i}\kappa\delta T_1(\mu A)(\alpha) \\ +\mathrm{i}\kappa\delta^2 \frac{1}{\kappa^2} \nabla_\Gamma \left(\frac{1}{2} T_2(\frac{1}{\varepsilon}, \mathbf{1})(\alpha, \alpha) \, 2\mathcal{H} \operatorname{div}_\Gamma \right) \\ -\mathrm{i}\kappa\delta^2 T_2(\mu, \mathbf{1})(\alpha, \alpha)(\mathcal{C} - \mathcal{H}) \end{pmatrix} (\mathbf{n} \times \mathbf{H})_{|\Gamma_0^\delta}, \tag{60}$$

where the operator $A = A(\varepsilon, \mu)$ is given by (53).

Third form : The final formula of the impedance condition at second-order for multi-thin layers is obtained by inserting the expression for A and is given by

$$0 = \mathbf{E}_{T|\Gamma_0^\delta} + \mathrm{i}\kappa\delta \begin{pmatrix} T_1(\mu)(\alpha) + \frac{1}{\kappa^2} T_1\left(\frac{1}{\varepsilon}\right)(\alpha) \nabla_\Gamma \operatorname{div}_\Gamma \\ +\delta \frac{1}{2} \frac{1}{\kappa^2} \nabla_\Gamma (T_2(\frac{1}{\varepsilon}, \mathbf{1})(\alpha, \alpha) \, 2\mathcal{H} \operatorname{div}_\Gamma) \\ -\delta T_2(\mu, \mathbf{1})(\alpha, \alpha)(\mathcal{C} - \mathcal{H}) \end{pmatrix} (\mathbf{n} \times \mathbf{H})_{|\Gamma_0^\delta}. \tag{61}$$

Remark 5.6. *The effect of the curvature operators appears in this approximation, which is of order two.*

5.4 Impedance condition of order three

First form : For $n = 3$, from the formula (28) we obtain

$$\mathcal{N}_3 = \mathcal{N}_2 + \text{"term of order 3"}.$$

Then, by developing the product in (28), we get

$$\mathcal{N}_3 = \mathcal{N}_2 - \delta^3 \sum_{N \geq i > j > k \geq 1} \alpha_i \alpha_j \alpha_k \mathbb{M}_i \mathbb{M}_j \mathbb{M}_k - \delta^3 \sum_{N \geq i > j \geq 1} \alpha_i \alpha_j \left(\sum_{l=1}^{i-1} \alpha_l \right) \mathbb{M}_i^{(1)} \mathbb{M}_j$$

$$- \delta^3 \sum_{N \geq i > j \geq 1} \alpha_i \alpha_j \left(\sum_{l=1}^{j-1} \alpha_l \right) \mathbb{M}_i \mathbb{M}_j^{(1)} - \delta^3 \sum_{N \geq i > j \geq 1} \frac{1}{2} \alpha_i^2 \alpha_j \left(\mathbb{M}_i^{(1)} + \mathbb{M}_i^2 \right) \mathbb{M}_j$$

$$- \delta^3 \sum_{N \geq i > j \geq 1} \frac{1}{2} \alpha_i \alpha_j^2 \mathbb{M}_i \left(\mathbb{M}_j^{(1)} + \mathbb{M}_j^2 \right) - \delta^3 \sum_{i=N}^{1} \alpha_i \frac{1}{2} \left(\sum_{j=1}^{i-1} \alpha_j \right)^2 M_i^{(2)}$$

$$- \delta^3 \sum_{i=1}^{N} \frac{1}{2} \alpha_i^2 \left(\sum_{j=1}^{i-1} \alpha_j \right) \left(\mathbb{M}_i^{(2)} + \mathbb{M}_i^{(1)} \mathbb{M}_i + \mathbb{M}_i \mathbb{M}_i^{(1)} \right)$$

$$- \delta^3 \sum_{i=1}^{N} \frac{1}{6} \alpha_i^3 \left(\mathbb{M}_i^{(2)} + 2\mathbb{M}_i^{(1)} \mathbb{M}_i + \mathbb{M}_i \mathbb{M}_i^{(1)} + \mathbb{M}_i^3 \right). \tag{62}$$

Using the tensorial notation defined in Sec. 4, we obtain

$$\mathcal{N}_3 = \mathcal{N}_2 - \delta^3 \frac{1}{6} T_3(\mathbb{M}, \mathbb{M}, \mathbb{M}) (\alpha, \alpha, \alpha) \ - \delta^3 \frac{1}{6} T_3(\mathbb{M}^{(2)}, \mathbf{1}, \mathbf{1}) (\alpha, \alpha, \alpha)$$

$$- \delta^3 \frac{1}{6} T_3(\mathbb{M}^{(1)}, \mathbb{M}, \mathbf{1}) (\alpha, \alpha, \alpha) - \delta^3 \frac{1}{6} T_3(\mathbb{M}, \mathbb{M}^{(1)}, \mathbf{1}) (\alpha, \alpha, \alpha) - \delta^3 \frac{1}{6} T_3(\mathbb{M}^{(1)}, \mathbf{1}, \mathbb{M}) (\alpha, \alpha, \alpha) . \quad (63)$$

Now, using the boundary condition (3) we get

$$0 = [\mathcal{N}_3]_{11} \mathbf{E}_{T|\Gamma_0^\delta} + [\mathcal{N}_3]_{12} (\mathbf{n} \times \mathbf{H})_{|\Gamma_0^\delta}. \quad (64)$$

Remark 5.7. *To invert the operator* $[\mathcal{N}_3]_{11}$, *we use the same argument as for the approximate impedance of* 1st *and* 2nd *orders. That means we don't take into consideration the coefficients of order* δ^3.

The first form of the impedance condition at order three is given by

$$0 = \left[\begin{array}{c} I - \delta T_1(\mathbb{M}) (\alpha) + \frac{1}{2}\delta^2 T_2(\mathbb{M}, \mathbb{M}) (\alpha, \alpha) \\ + \frac{1}{2}\delta^2 T_2(\mathbb{M}^{(1)}, \mathbf{1}) (\alpha, \alpha) \end{array} \right]_{11} \mathbf{E}_{T|\Gamma_0^\delta}$$

$$+ \left[\begin{array}{c} -\delta T_1(\mathbb{M}) (\alpha) + \frac{1}{2}\delta^2 T_2(\mathbb{M}, \mathbb{M}) (\alpha, \alpha) + \frac{1}{2}\delta^2 T_2(\mathbb{M}^{(1)}, \mathbf{1}) (\alpha, \alpha) \\ \\ -\frac{1}{6}\delta^3 \left(\begin{array}{c} T_3(\mathbb{M}, \mathbb{M}, \mathbb{M}) (\alpha, \alpha, \alpha) + T_3(\mathbb{M}^{(2)}, \mathbf{1}, \mathbf{1}) (\alpha, \alpha, \alpha) \\ + T_3(\mathbb{M}^{(1)}, \mathbb{M}, \mathbf{1}) (\alpha, \alpha, \alpha) + T_3(\mathbb{M}, \mathbb{M}^{(1)}, \mathbf{1}) (\alpha, \alpha, \alpha) \\ + T_3(\mathbb{M}^{(1)}, \mathbf{1}, \mathbb{M}) (\alpha, \alpha, \alpha) \end{array} \right) \end{array} \right]_{12} (\mathbf{n} \times \mathbf{H})_{|\Gamma_0^\delta}. \quad (65)$$

Second form: To get the second form of the impedance condition at order three, we follow three steps:

- <u>Step 1</u>: Left multiplying by $(I + \delta [T_1(\mathbb{M}) (\alpha)]_{11})$ gives

$$0 = \left(1 + \frac{1}{2}\delta^2 \left(\begin{array}{c} [T_2(\mathbb{M}, \mathbb{M}) (\alpha, \alpha)]_{11} + [T_2(\mathbb{M}^{(1)}, \mathbf{1}) (\alpha, \alpha)]_{11} \\ -2 ([T_1(\mathbb{M}) (\alpha)]_{11})^2 \end{array} \right) \right) \mathbf{E}_{T|\Gamma_0^\delta}$$
$$\qquad + \frac{1}{2}\delta^3 [T_1(\mathbb{M}) (\alpha)]_{11} \left([T_2(\mathbb{M}, \mathbb{M}) (\alpha, \alpha)]_{11} + [T_2(\mathbb{M}^{(1)}, \mathbf{1}) (\alpha, \alpha)]_{11} \right) \right)$$

$$+ \left(\begin{array}{c} -\delta [T_1(\mathbb{M}) (\alpha)]_{12} \\ +\delta^2 \frac{1}{2} \left(\begin{array}{c} [T_2(\mathbb{M}, \mathbb{M}) (\alpha, \alpha)]_{12} + [T_2(\mathbb{M}^{(1)}, \mathbf{1}) (\alpha, \alpha)]_{12} \\ -2 [T_1(\mathbb{M}) (\alpha)]_{11} [T_1(\mathbb{M}) (\alpha)]_{12} \end{array} \right) \\ \\ -\frac{\delta^3}{6} \left(\begin{array}{c} [T_3(\mathbb{M}, \mathbb{M}, \mathbb{M}) (\alpha, \alpha, \alpha)]_{12} + [T_3(\mathbb{M}^{(2)}, \mathbf{1}, \mathbf{1}) (\alpha, \alpha, \alpha)]_{12} \\ + [T_3(\mathbb{M}^{(1)}, \mathbb{M}, \mathbf{1}) (\alpha, \alpha, \alpha)]_{12} + [T_3(\mathbb{M}, \mathbb{M}^{(1)}, \mathbf{1}) (\alpha, \alpha, \alpha)]_{12} \\ + [T_3(\mathbb{M}^{(1)}, \mathbf{1}, \mathbb{M}) (\alpha, \alpha, \alpha)]_{12} \\ -3 [T_1(\mathbb{M}) (\alpha)]_{11} \left(\begin{array}{c} [T_2(\mathbb{M}, \mathbb{M}) (\alpha, \alpha)]_{12} \\ + [T_2(\mathbb{M}^{(1)}, \mathbf{1}) (\alpha, \alpha)]_{12} \end{array} \right) \end{array} \right) \end{array} \right) (\mathbf{n} \times \mathbf{H})_{|\Gamma_0^\delta}. \quad (66)$$

We emphasize that we are looking at the end to invert the operator applied on $\mathbf{E}_{T|\Gamma_0^\delta}$ by left multiplying with a suitable operator. Since the term of least order in the operator applied on $(\mathbf{n} \times \mathbf{H})_{|\Gamma_0^\delta}$ is of order δ, then the term of order δ^3 in the operator applied on $\mathbf{E}_{T|\Gamma_0^\delta}$, given by $\frac{1}{2}\delta^3 [T_1(\mathbb{M}) (\alpha)]_{11} \left([T_2(\mathbb{M}, \mathbb{M}) (\alpha, \alpha)]_{11} + [T_2(\mathbb{M}^{(1)}, \mathbf{1}) (\alpha, \alpha)]_{11} \right)$, will be omitted in the next step because terms of order greater than 3 will be not considered.

- <u>Step 2</u> : We apply a second left multiplication, on the last formula, by

$$1 - \frac{1}{2}\delta^2 \left(\left[T_2(\mathbb{M}, \mathbb{M}) (\alpha, \alpha) + T_2(\mathbb{M}^{(1)}, \mathbf{1}) (\alpha, \alpha) \right]_{11} - 2 ([T_1(\mathbb{M}) (\alpha)]_{11})^2 \right). \quad (67)$$

Then, we obtain

$$0 \;=\; \mathbf{E}_{T|\Gamma_0^\delta} + (\cdots)\,(\mathbf{n}\times\mathbf{H})_{|\Gamma_0^\delta}, \tag{68}$$

$$(\cdots) \;=\; \left\{\begin{array}{l} -\delta\,[T_1(\mathbb{M})\,(\alpha)]_{12} \\[4pt] +\tfrac{1}{2}\delta^2 \left(\begin{array}{l} [T_2(\mathbb{M},\mathbb{M})\,(\alpha,\alpha)]_{12} + [T_2(\mathbb{M}^{(1)},\mathbf{1})\,(\alpha,\alpha)]_{12} \\ -2\,[T_1(\mathbb{M})\,(\alpha)]_{11}\,[T_1(\mathbb{M})\,(\alpha)]_{12} \end{array}\right) \\[16pt] -\tfrac{1}{6}\delta^3 \left(\begin{array}{l} [T_3(\mathbb{M},\mathbb{M},\mathbb{M})\,(\alpha,\alpha,\alpha)]_{12} + [T_3(\mathbb{M}^{(2)},\mathbf{1},\mathbf{1})\,(\alpha,\alpha,\alpha)]_{12} \\ + [T_3(\mathbb{M}^{(1)},\mathbb{M},\mathbf{1})\,(\alpha,\alpha,\alpha)]_{12} + [T_3(\mathbb{M},\mathbb{M}^{(1)},\mathbf{1})\,(\alpha,\alpha,\alpha)]_{12} \\ + [T_3(\mathbb{M}^{(1)},\mathbf{1},\mathbb{M})\,(\alpha,\alpha,\alpha)]_{12} \\ -3\,[T_1(\mathbb{M})\,(\alpha)]_{11}\left(\begin{array}{l}[T_2(\mathbb{M},\mathbb{M})\,(\alpha,\alpha)]_{12}\\ + [T_2(\mathbb{M}^{(1)},\mathbf{1})\,(\alpha,\alpha)]_{12}\end{array}\right) \\ -3\left(\begin{array}{l}[T_2(\mathbb{M},\mathbb{M})\,(\alpha,\alpha)]_{11}\\ + [T_2(\mathbb{M}^{(1)},\mathbf{1})\,(\alpha,\alpha)]_{11}\\ -2\,([T_1(\mathbb{M})\,(\alpha)]_{11})^2\end{array}\right)[T_1(\mathbb{M})\,(\alpha)]_{12} \end{array}\right) \end{array}\right..$$

- <u>Step 3</u>: The coefficients of orders 1 and 2 with respect to δ are the same as in the approximate impedance condition of order two. We develop coefficients with dependency on δ^3 in the formula (68), we obtain

$$\mathrm{i}\kappa\frac{1}{3}\left(\begin{array}{c} T_3(\mu A,\mu A,\frac{1}{\mu}B)\,(\alpha,\alpha,\alpha) \\ +2T_3(\mu,\mathbf{1},\mathbf{1})\,(\alpha,\alpha,\alpha)\,(\mathcal{C}-\mathcal{H})(\mathcal{C}-2\mathcal{H}) \\ +\frac{1}{\kappa^2}\nabla_\Gamma\left(T_3(\frac{1}{\varepsilon},\mathbf{1},\mathbf{1})\,(\alpha,\alpha,\alpha)\,(4\mathcal{H}^2-\mathcal{G})\,\mathrm{div}_\Gamma\right) \end{array}\right). \tag{69}$$

The second form of the impedance condition of order three for multi-thin layers is given by

$$0 = \mathbf{E}_{T|\Gamma_0^\delta} + \mathrm{i}\kappa\delta T_1(\mu A)(\alpha)\,(\mathbf{n}\times\mathbf{H})_{|\Gamma_0^\delta}$$
$$+\tfrac{1}{2}\mathrm{i}\kappa\delta^2\left(\begin{array}{c}\frac{1}{\kappa^2}\nabla_\Gamma\left(T_2(\frac{1}{\varepsilon},\mathbf{1})\,(\alpha,\alpha)\,2\mathcal{H}\,\mathrm{div}_\Gamma\right)\\ -2T_2(\mu,\mathbf{1})\,(\alpha,\alpha)\,(\mathcal{C}-\mathcal{H})\end{array}\right)(\mathbf{n}\times\mathbf{H})_{|\Gamma_0^\delta}$$
$$+\tfrac{1}{3}\mathrm{i}\kappa\delta^3\left(\begin{array}{c} T_3(\mu A,\mu A,\frac{1}{\mu}B)\,(\alpha,\alpha,\alpha) \\ +2T_3(\mu,\mathbf{1},\mathbf{1})\,(\alpha,\alpha,\alpha)\,(\mathcal{C}-\mathcal{H})(\mathcal{C}-2\mathcal{H}) \\ +\frac{1}{\kappa^2}\nabla_\Gamma\left(T_3(\frac{1}{\varepsilon},\mathbf{1},\mathbf{1})\,(\alpha,\alpha,\alpha)\,(4\mathcal{H}^2-\mathcal{G})\,\mathrm{div}_\Gamma\right) \end{array}\right)(\mathbf{n}\times\mathbf{H})_{|\Gamma_0^\delta}. \tag{70}$$

Third form: By substituting the operators A and B by their formulas in (70), and using simplifications of the multi-linear forms for scalar arguments, we obtain

$$0 = \mathbf{E}_{T|\Gamma_0^\delta} + \mathrm{i}\kappa\delta\,(\cdots)\,(\mathbf{n}\times\mathbf{H})_{|\Gamma_0^\delta},$$

$$(\cdots) = \left\{\begin{array}{l} T_1(\mu)\,(\alpha) - \delta T_1(\mu)\,(\alpha)\,(\mathcal{C}-\mathcal{H}) \\ +\tfrac{2}{3}\delta^2 T_1(\mu)\,(\alpha)\,(\mathcal{C}-\mathcal{H})(\mathcal{C}-2\mathcal{H}) \\ +\tfrac{1}{3}\delta^2\kappa^2 T_3(\mu,\mu,\varepsilon)\,(\alpha,\alpha,\alpha) \\ +\tfrac{1}{3}\delta^2 T_3(\mu,\mu,\frac{1}{\mu})\,(\alpha,\alpha,\alpha)\,(\nabla_\Gamma\,\mathrm{div}_\Gamma - \mathbf{curl}_\Gamma\,\mathrm{curl}_\Gamma) \\ +\frac{1}{\kappa^2}\nabla_\Gamma\left(\begin{array}{l} T_1\left(\frac{1}{\varepsilon}\right)(\alpha) \\ +\tfrac{1}{2}\delta T_1(\frac{1}{\varepsilon})\,(\alpha)\,2\mathcal{H} \\ +\tfrac{1}{3}\delta^2 T_1(\frac{1}{\varepsilon})\,(\alpha)\,(4\mathcal{H}^2-\mathcal{G}) \\ +\tfrac{1}{3}\delta^2 T_3(\frac{1}{\varepsilon},\frac{1}{\varepsilon},\varepsilon)\,(\alpha,\alpha,\alpha)\,\mathrm{div}_\Gamma\,\nabla_\Gamma \\ +\tfrac{1}{3}\kappa^2\delta^2 T_3(\frac{1}{\varepsilon},\mu,\varepsilon)\,(\alpha,\alpha,\alpha) + \tfrac{1}{3}\kappa^2\delta^2 T_3(\mu,\frac{1}{\varepsilon},\varepsilon)\,(\alpha,\alpha,\alpha) \\ -\tfrac{1}{3}\kappa^2\delta^2 T_3(\mu,\mu,\frac{1}{\mu})\,(\alpha,\alpha,\alpha) \end{array}\right)\mathrm{div}_\Gamma \end{array}\right. \tag{71}$$

Using the definitions of Δ_Γ in (7) and $\overrightarrow{\Delta_\Gamma}$ in (8), we can write the impedance boundary condition of order three as follows

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$$0 = \mathbf{E}_{T|\Gamma_0^\delta} + i\kappa\delta \left(\begin{array}{c} T_1(\mu)\,(\alpha)\left(1 + \delta P_\delta + \frac{1}{3}\delta^2 g\overrightarrow{\Delta_\Gamma}\right) \\ +\frac{1}{\kappa^2}\nabla_\Gamma T_1\left(\frac{1}{\varepsilon}\right)(\alpha)\left(1 + \delta a_\delta + \frac{1}{3}\delta^2 h\Delta_\Gamma\right)\mathrm{div}_\Gamma \end{array} \right)(\mathbf{n}\times\mathbf{H})_{|\Gamma_0^\delta}, \tag{72}$$

with

$$P_\delta = -\frac{T_2(\mu,\mathbf{1})\,(\alpha,\alpha)}{T_1\,(\mu)\,(\alpha)}(\mathcal{C} - \mathcal{H}) + \delta\frac{2}{3}\frac{T_3(\mu,\mathbf{1},\mathbf{1})\,(\alpha,\alpha,\alpha)}{T_1\,(\mu)\,(\alpha)}(\mathcal{C} - \mathcal{H})(\mathcal{C} - 2\mathcal{H})$$
$$+ \delta\frac{1}{3}\kappa^2\frac{T_3(\mu,\mu,\varepsilon)\,(\alpha,\alpha,\alpha)}{T_1\,(\mu)\,(\alpha)}, \tag{73}$$

$$g = \frac{T_3(\mu,\mu,\frac{1}{\mu})\,(\alpha,\alpha,\alpha)}{T_1\,(\mu)\,(\alpha)}, \tag{74}$$

$$a_\delta = \frac{T_2(\frac{1}{\varepsilon},\mathbf{1})\,(\alpha,\alpha)}{T_1\left(\frac{1}{\varepsilon}\right)(\alpha)}\mathcal{H} + \frac{\delta}{3}\frac{T_3(\frac{1}{\varepsilon},\mathbf{1},\mathbf{1})\,(\alpha,\alpha,\alpha)}{T_1\left(\frac{1}{\varepsilon}\right)(\alpha)}(4\mathcal{H}^2 - \mathcal{G}) + \frac{1}{3}\delta\kappa^2\frac{T_3(\frac{1}{\varepsilon},\mu,\varepsilon)\,(\alpha,\alpha,\alpha)}{T_1\left(\frac{1}{\varepsilon}\right)(\alpha)}$$
$$+ \frac{1}{3}\delta\kappa^2\frac{T_3(\mu,\frac{1}{\varepsilon},\varepsilon)\,(\alpha,\alpha,\alpha)}{T_1\left(\frac{1}{\varepsilon}\right)(\alpha)} - \frac{1}{3}\delta\kappa^2\frac{T_3(\mu,\mu,\frac{1}{\mu})\,(\alpha,\alpha,\alpha)}{T_1\left(\frac{1}{\varepsilon}\right)(\alpha)}, \tag{75}$$

and

$$h = \frac{T_3(\frac{1}{\varepsilon},\frac{1}{\varepsilon},\varepsilon)\,(\alpha,\alpha,\alpha)}{T_1\left(\frac{1}{\varepsilon}\right)(\alpha)}. \tag{76}$$

In the formula (72), we have a fourth order differential operator which is undesirable from the numerical point of view, because it causes unstable solutions. To overcome this issue, we need to use a technique from asymptotic analysis. We replace the terms

$$\left(1 + \delta P_\delta + \frac{1}{3}\delta^2 g\overrightarrow{\Delta_\Gamma}\right), \tag{77}$$

and

$$\left(1 + \delta a_\delta + \frac{1}{3}\delta^2 h\Delta_\Gamma\right), \tag{78}$$

by their inverses. At the second order we have

$$\left(1 + \delta P_\delta + \frac{1}{3}\delta^2 g\overrightarrow{\Delta_\Gamma}\right) = \left(1 + \delta Q_\delta - \frac{1}{3}\delta^2 g\overrightarrow{\Delta_\Gamma}\right)^{-1}, \tag{79}$$

with

$$Q_\delta = \frac{T_2(\mu,\mathbf{1})\,(\alpha,\alpha)}{T_1\,(\mu)\,(\alpha)}(\mathcal{C} - \mathcal{H}) + \delta\left(\frac{T_2(\mu,\mathbf{1})(\alpha,\alpha)}{T_1(\mu)(\alpha)}(\mathcal{C} - \mathcal{H})\right)^2$$
$$- \delta\left(\frac{2}{3}\frac{T_3(\mu,\mathbf{1},\mathbf{1})(\alpha,\alpha,\alpha)}{T_1(\mu)(\alpha)}(\mathcal{C} - \mathcal{H})(\mathcal{C} - 2\mathcal{H}) + \frac{1}{3}\kappa^2\frac{T_3(\mu,\mu,\varepsilon)(\alpha,\alpha,\alpha)}{T_1(\mu)(\alpha)}\right). \tag{80}$$

We have also

$$\left(1 + \delta a_\delta + \frac{1}{3}\delta^2 h\Delta_\Gamma\right) = \left(1 + \delta b_\delta - \frac{1}{3}\delta^2 h\Delta_\Gamma\right)^{-1}, \tag{81}$$

with

$$b_\delta = -\frac{T_2(\frac{1}{\varepsilon},\mathbf{1})(\alpha,\alpha)}{T_1(\frac{1}{\varepsilon})(\alpha)}\mathcal{H} + \delta\left(\frac{T_2(\frac{1}{\varepsilon},\mathbf{1})(\alpha,\alpha)}{T_1(\frac{1}{\varepsilon})(\alpha)}\mathcal{H}\right)^2$$
$$- \frac{\delta}{3}\frac{T_3(\frac{1}{\varepsilon},\mathbf{1},\mathbf{1})\,(\alpha,\alpha,\alpha)}{T_1\left(\frac{1}{\varepsilon}\right)(\alpha)}(4\mathcal{H}^2 - \mathcal{G}) - \frac{1}{3}\delta\kappa^2\frac{T_3(\frac{1}{\varepsilon},\mu,\varepsilon)\,(\alpha,\alpha,\alpha)}{T_1\left(\frac{1}{\varepsilon}\right)(\alpha)}$$
$$- \frac{1}{3}\delta\kappa^2\frac{T_3(\mu,\frac{1}{\varepsilon},\varepsilon)\,(\alpha,\alpha,\alpha)}{T_1\left(\frac{1}{\varepsilon}\right)(\alpha)} + \frac{1}{3}\delta\kappa^2\frac{T_3(\mu,\mu,\frac{1}{\mu})\,(\alpha,\alpha,\alpha)}{T_1\left(\frac{1}{\varepsilon}\right)(\alpha)}. \tag{82}$$

The final formula of the impedance boundary condition of order 3 for coating multi-thin layers is given by

$$0 = \mathbf{E}_{T|\Gamma_0^\delta} + i\kappa\delta \left(\begin{array}{c} T_1(\mu)(\alpha)\left(1 + \delta Q_\delta - \frac{1}{3}\delta^2 \frac{T_3(\mu,\mu,\frac{1}{\mu})(\alpha,\alpha,\alpha)}{T_1(\mu)(\alpha)}\overrightarrow{\Delta_\Gamma}\right)^{-1} \\ + \frac{1}{\kappa^2}\nabla_\Gamma T_1\left(\frac{1}{\varepsilon}\right)(\alpha)\left(1 + \delta b_\delta - \frac{1}{3}\delta^2 \frac{T_3(\frac{1}{\varepsilon},\frac{1}{\varepsilon},\varepsilon)(\alpha,\alpha,\alpha)}{T_1(\frac{1}{\varepsilon})(\alpha)}\Delta_\Gamma\right)^{-1}\mathrm{div}_\Gamma \end{array} \right) (\mathbf{n}\times\mathbf{H})_{|\Gamma_0^\delta}.$$
(83)

The coefficients Q_δ and b_δ are given in (45) and (46), respectively.

6   Numerical Validation

As pointed out in Sec. 4 of [6], for general smooth surfaces, the impedance boundary conditions with second order surface differential operators can be readily used in a boundary integral formulation of the boundary value and solved by standard boundary element techniques. The implementation of such an approach, however, is beyond the scope of this paper.

To illustrate the performance of these conditions in practical calculations, we have carried out some calculations for the case that the obstacle D is a sphere. In this case, solutions of the Maxwell system can be written explicitly as Mie series, i.e. expanded in products of (vector) spherical harmonics and spherical Bessel functions.

Let us fix the notation: by j_n we denote the spherical Bessel function of order n, by $h_n^{(1)}$ the spherical Hankel function of the first kind of order n. We use the spherical harmonics

$$Y_n^m(\widehat{\mathbf{x}}) = \sqrt{\frac{(2n+1)(n-|m|)!}{4\pi(n+|m|)!}} P_n^{|m|}(\cos\vartheta)\,\mathrm{e}^{im\varphi}, \qquad n\in\mathbb{N}_0, \quad m=-n,\ldots,n,$$

where $\widehat{\mathbf{x}} = (\cos\varphi\sin\vartheta, \sin\varphi\sin\vartheta, \cos\vartheta)^\top \in S^2$ denotes the unit vector in $\mathbb{R}^3$ with angular coordinates $\vartheta\in[0,\pi]$ and $\varphi\in(-\pi,\pi]$, and P_n^m the associated Legendre functions. The functions $\{Y_n^m\}$ form a complete orthonormal system in $L^2(S^2)$ and their derivatives

$$U_n^m(\widehat{\mathbf{x}}) = \frac{1}{\sqrt{n(n+1)}}\nabla_{S^2}Y_n^m(\widehat{\mathbf{x}}), \qquad V_n^m(\widehat{\mathbf{x}}) = \widehat{\mathbf{x}}\times U_n^m(\widehat{\mathbf{x}}), \qquad n\in\mathbb{N}, \quad n=-m,\ldots,m,$$

a complete orthonormal system in $\mathbf{L}_t^2(S^2)$, the space of square integrable tangential vector fields on the unit sphere. For the expansions of the solutions to the Maxwell system we will use the functions

$$M_n^m(\mathbf{x};\kappa) = -j_n(\kappa|\mathbf{x}|)\,V_n^m(\widehat{\mathbf{x}}), \qquad N_n^m(\mathbf{x};\kappa) = -h_n^{(1)}(\kappa|\mathbf{x}|)\,V_n^m(\widehat{\mathbf{x}}), \qquad \mathbf{x}=|\mathbf{x}|\widehat{\mathbf{x}}.$$

See for example [21] for a detailed discussion of these functions and their properties. We will make use of the fact that a solution to the Maxwell system in an open ball $B_R(0)$ can be expanded as

$$\left.\begin{array}{l} \mathbf{E}(\mathbf{x}) = \displaystyle\sum_{n=1}^\infty\sum_{m=-n}^n\left[a_n^m M_n^m(\mathbf{x}) + b_n^m\frac{1}{i\kappa}\nabla\times M_n^m(\mathbf{x})\right], \\[2.5ex] \mathbf{H}(\mathbf{x}) = \displaystyle\sum_{n=1}^\infty\sum_{m=-n}^n\left[\frac{a_n^m}{i\kappa}\nabla\times M_n^m(\mathbf{x}) - b_n^m M_n^m(\mathbf{x})\right], \end{array}\right\} \qquad |\mathbf{x}|<R, \qquad (84)$$

while a radiating solution outside of $\overline{B_R(0)}$ can be expanded as

$$\left.\begin{array}{l} \mathbf{E}_{\mathrm{rad}}(\mathbf{x}) = \displaystyle\sum_{n=1}^\infty\sum_{m=-n}^n\left[c_n^m N_n^m(\mathbf{x};k) + d_n^m\frac{1}{ik}\nabla\times N_n^m(\mathbf{x};\kappa)\right], \\[2.5ex] \mathbf{H}_{\mathrm{rad}}(\mathbf{x}) = \displaystyle\sum_{n=1}^\infty\sum_{m=-n}^n\left[\frac{c_n^m}{i\kappa}\nabla\times N_n^m(\mathbf{x};\kappa) - d_n^m N_n^m(\mathbf{x};\kappa)\right], \end{array}\right\} \qquad |\mathbf{x}|>R. \qquad (85)$$

Explicit expressions for the boundary values of such expansions on $|\mathbf{x}|=R$ can readily be computed and yield expansions of the tangential traces in terms of the functions U_n^m and V_n^m. Inserting into

the various transmission and boundary conditions and comparing coefficients, we obtain simple linear systems for the coefficients in the expansions.

For our numerical experiments, we assume throughout that $\kappa = 2\pi$, $R = 1$ and that $\mu = 1$ in all layers. The incident field is a plane wave

$$\mathbf{E}^i(\mathbf{x}) = e^{(1)}\,\mathrm{e}^{-\mathrm{i}\kappa e^{(3)}\cdot\mathbf{x}}\,, \qquad \mathbf{H}^i(\mathbf{x}) = -e^{(2)}\,\mathrm{e}^{-\mathrm{i}\kappa e^{(3)}\cdot\mathbf{x}}\,.$$

Here $e^{(j)}$ denotes the j-th unit coordinate vector. The expansion (84) of a plane wave is easily obtained (see [23, Chapter 9] for an explicit calculation). For the first series of experiments we use the values $\varepsilon_1 = 4$, $\varepsilon_2 = 2$, $\varepsilon_3 = 7$ as well as $\alpha_1 = 1/3$, $\alpha_2 = 1/2$ and $\alpha_3 = 1/6$. All expansions are cut off at $N = 15$.

To assess the performance of the approximate impedance boundary condition, we solve the original transmission problem (1a)-(1d) and compute the far field pattern of the radiating field outside the ball. The far field pattern $\mathbf{E}^\infty$ of the radiating field is obtained from the asymptotic expansion of $\mathbf{E}_{\mathrm{rad}}$ for large $\mathbf{x}$,

$$\mathbf{E}_{\mathrm{rad}}(\mathbf{x}) = \frac{\mathrm{e}^{\mathrm{i}\kappa\,|\mathbf{x}|}}{4\pi\,|\mathbf{x}|}\left[\mathbf{E}^\infty(\widehat{\mathbf{x}}) + \mathrm{O}\left(\frac{1}{|\mathbf{x}|}\right)\right]\,, \qquad |\mathbf{x}| \to \infty\,,$$

and for the expansion in (85), it is given explicitly by

$$\mathbf{E}^\infty(\widehat{\mathbf{x}}) = \frac{4\pi}{\kappa} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{1}{\mathrm{i}^{n+1}}\left[d_n^m\, U_n^m(\widehat{\mathbf{x}}) - c_n^m\, V_n^m(\widehat{\mathbf{x}})\right]\,.$$

In a first experiment, we compare $|\mathbf{E}^\infty(\widehat{\mathbf{x}})|$ for the conditions of order $1-3$ for fixed $\widehat{\mathbf{x}}$ and varying δ. The results are shown in Figure 2. Indeed, the third order IBC consistently gives the most accurate results up to a value of $\delta = 0.09$.

(a) $\vartheta = 0$ (b) $\vartheta = \pi/4$

(c) $\vartheta = \pi/6$ (d) $\vartheta = \pi/2$
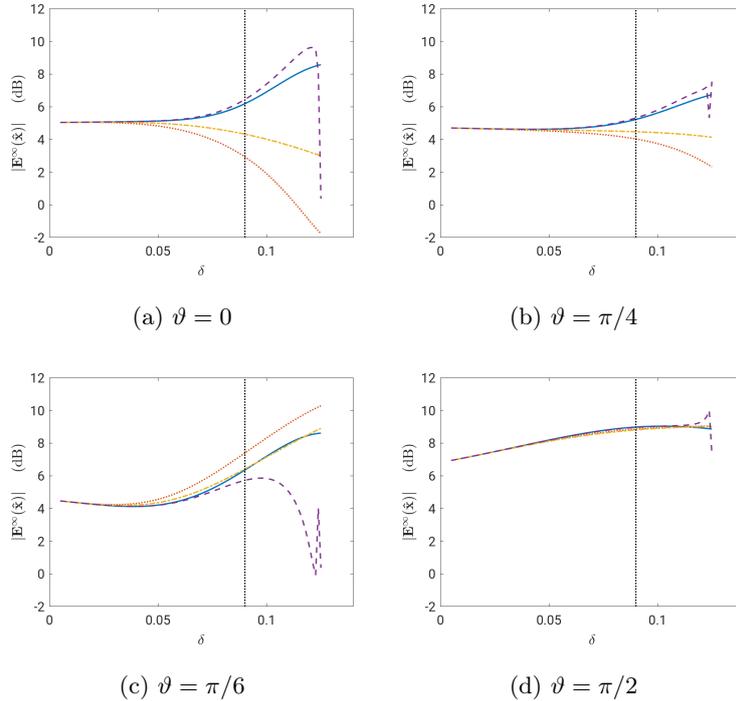
Figure 2: Plot of $|\mathbf{E}^\infty(\widehat{\mathbf{x}})|$ against δ for different values of ϑ. The plots are the exact scattered field (blue continuous line), the scattered field obtained by the first order IBC (red dotted), second order IBC (orange dash-dotted), third order IBC (purple dashed), respectively. The vertical dotted line indicates the value to which the third order IBC appears to be accurate.

The quality of the results obtained for the boundary conditions of different order are quite different for different values of ϑ. In Figure 3 we show plots of $|\mathbf{E}^\infty(\widehat{\mathbf{x}})|$ for fixed $\delta = 0.09$ for the
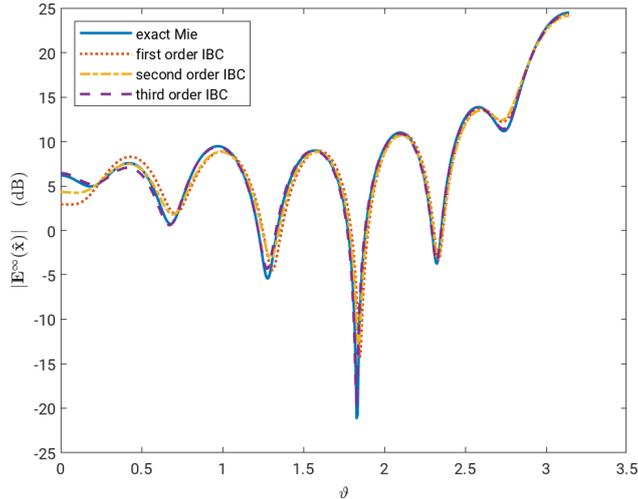
Figure 3: Plot of $|\mathbf{E}^{\infty}(\widehat{\mathbf{x}})|$ against ϑ for $\delta = 0.09$.

entire range of ϑ values. All these results are quite comparable to the results presented in [6] for a single layer coating of similar thickness with $\varepsilon = 4$.

Further numerical experiments indicate that the quality of the approximation particularly depends on the value of ε in the outer most layer. We present here a second series of experiments where we repeat all calculations with the value of $\varepsilon_1 = 8$. The plots of $|\mathbf{E}^{\infty}(\widehat{\mathbf{x}})|$ against δ for various values of ϑ are presented in Figure 4. Clearly, the range of validity of the impedance boundary conditions is reduced.

In Figure 5 we show $|\mathbf{E}^{\infty}(\widehat{\mathbf{x}})|$ plotted against $\vartheta \in (-\pi, \pi)$, in this case for $\delta = 0.05$. Again one observes that the third-order approximation is best suited to approximate the true field, in particular in directions where the far field pattern has local minima.

7 Conclusion

The aim of this paper is the derivation of a third order impedance boundary condition for a thin multilayer with a non planar geometry, i.e., enlightening the importance of the curvatures and the different thicknesses of the different layers; by means of parameterizing the thin coating and writing Maxwell's equations in the form of abstract differential equations. A tensor notation was used to give compact formulae of the approximate impedance conditions. We checked the validity of the derived IBCs for a sphere obstacle. Effectively, the third order approximate IBC gave the most accurate results for very small thicknesses of the thin shells as assumed.

The proof of existence and uniqueness for Maxwell's system with such an impedance boundary condition requires to write the problem as a system of integral equations on the exterior boundary of the obstacle. This issue is not considered in the present paper and it will be addressed in a forthcoming investigation.

Acknowledgements

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(a) $\vartheta = 0$ (b) $\vartheta = \pi/4$

(c) $\vartheta = \pi/6$ (d) $\vartheta = \pi/2$
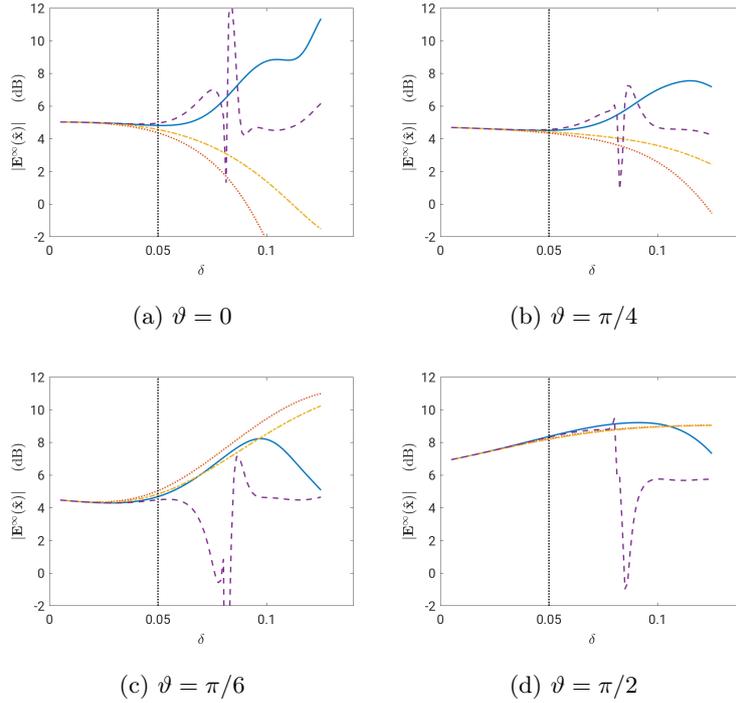
Figure 4: Plot of $|\mathbf{E}^\infty(\widehat{\mathbf{x}})|$ against δ for different values of ϑ for the three-layer sphere with $\varepsilon_1 = 8$. The plots are the exact problem (blue continuous line), first order IBC (red dotted), second order IBC (orange dash-dotted), third order IBC (purple dashed). The vertical dotted line indicates the value to which the third order IBC appears to be accurate.

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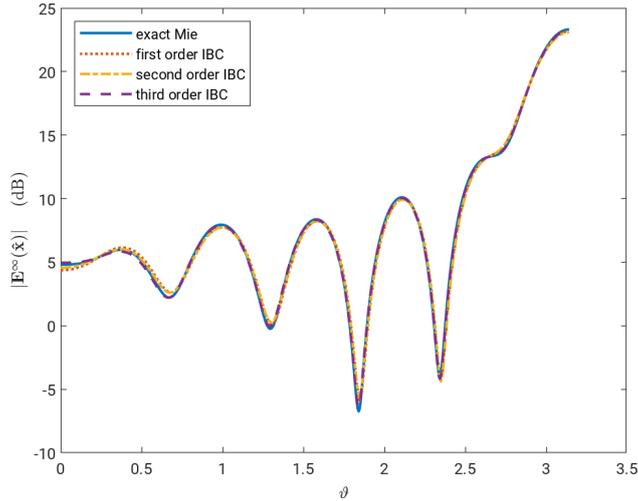
Figure 5: Plot of $|\mathbf{E}^{\infty}(\hat{\mathbf{x}})|$ against ϑ for $\varepsilon_1 = 8$ and $\delta = 0.05$.

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