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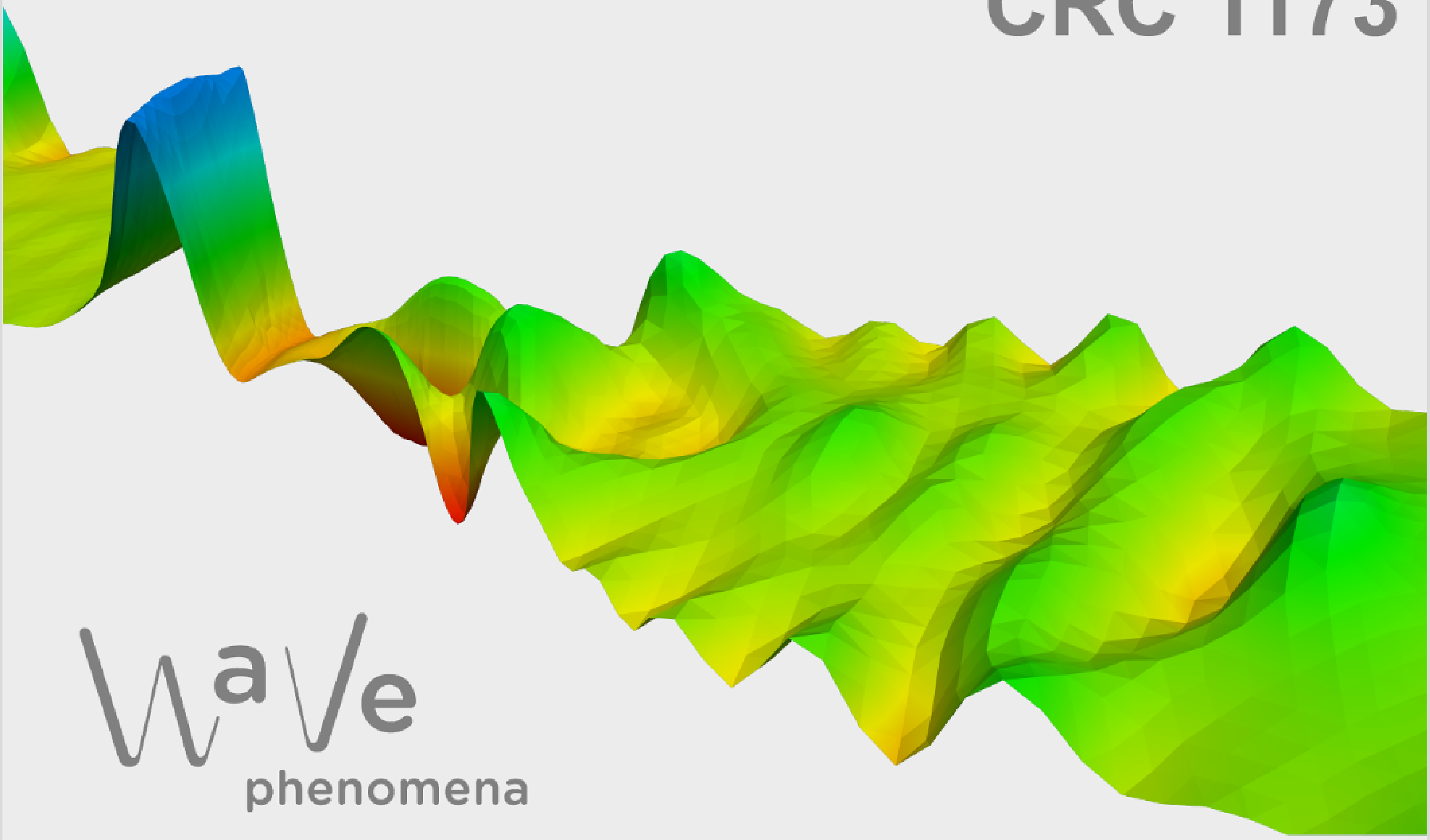

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# A LIMITING ABSORPTION PRINCIPLE FOR LINEAR AND NONLINEAR HELMHOLTZ EQUATIONS WITH A STEP POTENTIAL

RAINER MANDEL AND DOMINIC SCHEIDER

ABSTRACT. We consider the Helmholtz equation $-\Delta u + V\,u - \lambda\,u = f$ on $\mathbb{R}^n$ where the potential $V : \mathbb{R}^n \to \mathbb{R}$ is constant on each of the half-spaces $\mathbb{R}^{n-1} \times (-\infty, 0)$ and $\mathbb{R}^{n-1} \times (0, \infty)$. We prove an $L^p - L^q$-Limiting Absorption Principle for frequencies $\lambda > \max V$ and derive the existence of nontrivial solutions of linear and nonlinear Helmholtz equations.

In this paper we are interested in the Limiting Absorption Principle (LAP) for the Helmholtz equation on $\mathbb{R}^n$ involving a step potential of the form

$$V(x,y) = \begin{cases} V_1 & \text{if } x \in \mathbb{R}^{n-1}, y > 0, \\ V_2 & \text{if } x \in \mathbb{R}^{n-1}, y < 0 \end{cases} \tag{1}$$

where $V_1 \neq V_2$ are two fixed real numbers. We will without loss of generality assume $V_1 > V_2$ in the following. To explain the motivation behind our study, we recall the interesting phenomenon called "double scattering". In the context of the Schrödinger equation it means that for sufficiently regular and fast decaying right hand sides $f$ the unique solution of the initial value problem

$$\mathrm{i}\partial_t \psi - \Delta\psi + V\psi = f \quad \text{in } \mathbb{R}^n, \quad \psi(0) = \psi_0,$$

with $V$ as in (1) splits up into two pieces as $t \to \pm\infty$ that correspond to the two different values of $V$ at infinity. This phenomenon is mathematically understood in the one-dimensional case $n = 1$ [16, Theorem 1.2], see also [6, 7]. One byproduct of our results is that such a splitting into two pieces may as well be observed for the solutions of the corresponding Helmholtz equations in $\mathbb{R}^n$ which are obtained through the Limiting Absorption Principle, see for instance the formula (19) where the two parts $f(x,y)1_{(0,\infty)}(\pm y)$ of the right hand side contribute differently to the LAP-solution of the Helmholtz equation. Notice that solutions $u$ of such Helmholtz equations provide monochromatic solutions $\psi(x,t) = \mathrm{e}^{\mathrm{i}\lambda t}u(x)$ of the Schrödinger equation where $\lambda$ belongs to the $L^2$-spectrum of the selfadjoint operator $-\Delta + V$ with domain $H^2(\mathbb{R}^n)$. We prove our LAP in the topology of Lebesgue spaces in order to treat both linear and nonlinear Helmholtz equations. As far as we can see, the more classical results in weighted $L^2$ spaces resp. $B(\mathbb{R}^n), B^*(\mathbb{R}^n)$ (for the definition, cf. [4, page 4]) by Agmon [1–3] and Agmon-Hörmander [4] do not apply in the nonlinear setting. We refer to [8, Theorem 1], [9, Theorem 2] and [19] for recent contributions about linear and

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nonlinear elliptic problems on $\mathbb{R}^n$ involving interfaces modelled by potentials with different asymptotics at infinity.

Being interested in the LAP for the Helmholtz operator $-\Delta+V$ we fix the notation

$$\mathcal{R}(\mu) := (-\Delta+V-\mu)^{-1} \qquad \text{for } \mu\in\mathbb{C}\setminus\sigma(-\Delta+V)$$

A computation reveals $\sigma(-\Delta+V)=[\min\{V_1,V_2\},\infty)=[V_2,\infty)$. Our LAP reads as follows.

**Theorem 1.** *Let $V$ be given by* (1) *and assume $\lambda>V_1>V_2$. Then the resolvent estimate*

$$\sup_{0<|\varepsilon|\le 1}\|\mathcal{R}(\lambda+\mathrm{i}\varepsilon)\|_{L^p(\mathbb{R}^n;\mathbb{C})\to L^q(\mathbb{R}^n;\mathbb{C})}<\infty \tag{2}$$

*holds provided that $n\in\{2,3,4\}$ and $p,q\in[1,\infty]$ satisfy one of the following conditions:*

*(i) $n=2$ and $4<p',q\le\infty$, $\frac{2}{3}\le\frac{1}{p}-\frac{1}{q}<1$ and $(p,q)\neq(1,3)$,*

*(ii) $n=3$ and $\frac{10}{3}<p',q<30$, $\frac{1}{p}-\frac{1}{q}\le\frac{2}{3}$, $\frac{1}{p}\ge\frac{3}{q},\frac{1}{q'}\ge\frac{3}{p'}$,*

*(iii) $n=4$ and $\frac{10}{3}<p',q<5$, $\frac{7}{15}<\frac{1}{p}-\frac{1}{q}\le\frac{1}{2}$.*

*Moreover, the resolvent operators $\mathcal{R}(\lambda+\mathrm{i}\varepsilon)$ converge to nontrivial operators $\mathcal{R}(\lambda\pm\mathrm{i}0)$ as $\pm\varepsilon\searrow 0$ in the weak topology of bounded linear operators from $L^p(\mathbb{R}^n;\mathbb{C})$ to $L^q(\mathbb{R}^n;\mathbb{C})$.*

For an illustration of our conditions in (i),(ii),(iii) see the three respective panels in Figure 1 in Section 3. Our resolvent estimates for $n=2$ coincide with the corresponding estimates for the constant potential whereas the ones for $n\in\{3,4\}$ cover a smaller range of parameters, cf. Theorem 2. It is unclear to the authors whether there is a fundamental reason behind this or not. If the Restriction Conjecture was true, then our method would allow for larger ranges of $p,q$ than the present ones (but possibly still non-optimal). Let us mention that our result only covers frequencies in the range $\lambda>V_1>V_2$ and thus not all frequencies in the (essential) spectrum. We believe that in the larger range $\lambda>V_2$ we actually get the same estimates with some technical work. Especially regarding the treatment of Schrödinger or wave equations, uniform estimates with respect to all $\lambda\in\mathbb{C}$ would be very helpful and remain a challenging task for the future.

As an application of the above Limiting Absorption Principle we consider Helmholtz Equations on $\mathbb{R}^n$ involving the potential $V$ given by (1). We start with linear problems of the form

$$-\Delta u+Vu-\lambda u=f \quad \text{in } \mathbb{R}^n \tag{3}$$

where $f$ is supposed to belong to $L^p(\mathbb{R}^n)$. Theorem 1 allows, for $p,q$ as described there, to define the outgoing solution $u_+:=\mathcal{R}(\lambda+\mathrm{i}0)f\in L^q(\mathbb{R}^n;\mathbb{C})$ of this equation. Notice that in the context of Helmholtz equations the word "outgoing" is used to distinguish $u_+=\mathcal{R}(\lambda+\mathrm{i}0)f$ from the corresponding "incoming" solution $u_-:=\mathcal{R}(\lambda-\mathrm{i}0)f=\overline{u_+}$, see [4, Definition 6.5]. Combining this with local elliptic regularity theory we obtain the following result.

**Corollary 1.** *Let $V$ be given by* (1) *and assume $\lambda>V_1>V_2$. Moreover assume that $n,p,q$ satisfy one of the conditions (i),(ii),(iii) in Theorem 1. Then for any $f\in L^p(\mathbb{R}^n;\mathbb{C})$ the Helmholtz equation* (3) *has a nontrivial "outgoing" resp. "incoming" strong solution*

$u_+$ *(resp.* $u_-$*)* $\in L^q(\mathbb{R}^n;\mathbb{C})\cap W^{2,p}_{loc}(\mathbb{R}^n;\mathbb{C})$ *obtained by the Limiting Absorption Principle, and there holds an estimate of the form*

$$\|u_-\|_{L^q(\mathbb{R}^n)} + \|u_+\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}. \tag{4}$$

Here the symbol $\lesssim$ is used in the sense that there exists some constant $C > 0$ depending only on the parameters $V_1, V_2, n, p, q, \lambda$ such that

$$\|u_-\|_{L^q(\mathbb{R}^n)} + \|u_+\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}.$$

Several questions remain open. As in the context of Theorem 1 it is unclear whether the ranges for $p, q$ are optimal and whether the corresponding theory for frequencies $\lambda \in (V_2, V_1]$ is similar. Next, an analysis of the appropriate radiation conditions for "outgoing" resp. "incoming" solutions remains to be done. Here we believe that the results for the ranges $\lambda \in (V_2, V_1), \lambda = V_1$ and $\lambda \in (V_1, \infty)$ will be different. Moreover, it would be nice to provide a reasonable notion of Herglotz-type waves, i.e. localized solutions of the homogeneous Helmholtz equation (3) (i.e. $f = 0$) with the property that for each sufficiently regular right hand side $f$ the imaginary part of $\mathcal{R}(\lambda + \mathrm{i}0)f$ belongs to this class of functions.

In our final result we use the Limiting Absorption Principle from Theorem 1 to prove the existence of solutions to nonlinear Helmholtz equations following the dual variational approach developed by Evéquoz and Weth [14, Theorem 1.2]. We refer to the articles [13, 15, 20, 21, 23] for related results and other approaches to such equations.

**Corollary 2.** *Let* $V$ *be given by* (1) *and assume* $\lambda > V_1 > V_2$. *Let* $\Gamma \in L^\infty(\mathbb{R}^n)$ *satisfy* $\Gamma > 0$ *on* $\mathbb{R}^n$ *and* $\Gamma(x, y) \searrow 0$ *as* $|(x, y)| \to \infty$. *Then the nonlinear Helmholtz equation*

$$-\Delta u + Vu - \lambda u = \Gamma|u|^{q-2}u \quad \text{in } \mathbb{R}^n \tag{5}$$

*has a nontrivial solution in* $L^q(\mathbb{R}^n) \cap W^{2,r}_{loc}(\mathbb{R}^n)$ *for all* $r < \infty$ *provided that*

$$\text{(i) } n = 2, 6 \leq q < \infty, \qquad \text{(ii) } n = 3, 4 \leq q \leq 6 \quad \text{or} \quad \text{(iii) } n = 4, \frac{15}{4} < q \leq 4.$$

We stress that despite the limited range of exponents this result covers the physically relevant special cases of the cubic and quintic nonlinearities for $n = 3$. More refined dual variational techniques as in [10, 12, 14] might be applicable as well to get one or even infinitely many solutions for larger classes of nonlinearities. For the proof of Corollary 2 we concentrate on an adaptation of [14, Theorem 1.2] in order to keep the technicalities at a moderate level. Let us mention that the integrability properties of the solution at infinity are actually slightly better, which can be proved along the lines of [14, Theorem 4.4] with the aid of a nonlinear bootstrap procedure based on Theorem 1.

The outline of this paper is the following. In Section 1 we first present our approach in the technically easier one-dimensional setting. Here, the one-sided Fourier transforms are introduced and their use in the Limiting Absorption Principle is demonstrated. In Section 2 we generalize these ideas to the $n$-dimensional setting and derive the formula for $u_+ = \mathcal{R}(\lambda + i0)f$. In Section 3 we state all the essential estimates (Propositions 4, 5, 6) and combine them in order to prove our main results. The Propositions are proved in the following three sections; some technical parts of the required estimates are moved to the Appendix.

Before starting our analysis let us fix some notation and conventions. For $p \in [1,\infty]$, we write $L^p(\mathbb{R}^d;\mathbb{C})$ resp. $L^p(\mathbb{R}^d)$ for the (classical) Lebesgue space of $p$-integrable functions which are complex-valued resp. real-valued. The corresponding standard norms are in both cases denoted by $\|\cdot\|_{L^p(\mathbb{R}^d)}$. Moreover, we let $p' = \frac{p}{p-1} \in [1,\infty]$ be the conjugate exponent. The inner product in $L^2(\mathbb{R}^d;\mathbb{C})$ is given by $\langle f,g\rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} f(x)\overline{g(x)}\,\mathrm{d}x$. The $d$-dimensional Fourier transform is given by $\mathcal{F}_d g(\xi) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} g(x)\mathrm{e}^{-\mathrm{i}x\cdot\xi}\,\mathrm{d}x$ with inverse $\mathcal{F}_d^{-1}h(\xi) = (\mathcal{F}_d h)(-\xi)$ where $g,h:\mathbb{R}^d\to\mathbb{R}$ are sufficiently regular. At some points it will be necessary to slightly abuse the notation by writing $\mathcal{F}_d^{-1}(g(\xi))(x)$ in place of $\mathcal{F}_d^{-1}(g)(x)$. The sphere of radius $\mu$ in $\mathbb{R}^d$ is given by $\mathbb{S}^{d-1}_\mu = \{\xi\in\mathbb{R}^d : |\xi| = \mu\}$ along with its canonical surface measure $\sigma_\mu$. The corresponding Lebesgue spaces will be denoted by $L^s(\mathbb{S}^{d-1}_\mu;\mathbb{C}), L^s(\mathbb{S}^{d-1}_\mu), s\in[1,\infty]$.

## 1. The one-dimensional interface problem

In this section we discuss the Limiting Absorption Principle for the one-dimensional Helmholtz equation with a step potential $V$ given by

$$V(y) = \begin{cases} V_1 & \text{if } y>0,\\ V_2 & \text{if } y<0\end{cases} \qquad \text{where } V_1 > V_2.$$

Our motivation is to present the method in the technically simpler one-dimensional setting in order to treat the higher-dimensional case later in an efficient and reasonably quick manner. The differential operator $\psi\mapsto -\psi'' + V(y)\psi,\ \psi\in C_c^\infty(\mathbb{R};\mathbb{C})$ is symmetric, bounded from below and densely defined in $L^2(\mathbb{R};\mathbb{C})$ and thus possesses a unique selfadjoint extension with domain $H^2(\mathbb{R};\mathbb{C})$ and spectrum $[V_2,\infty)$. The Limiting Absorption Principle aims at constructing nontrivial solutions of the associated Helmholtz equation

$$-u'' + V(y)u - \lambda u = f \quad \text{in } \mathbb{R} \tag{6}$$

for frequencies $\lambda$ inside the spectrum. To this end one analyzes the uniquely determined functions $u_\varepsilon\in H^2(\mathbb{R};\mathbb{C})$ satisfying

$$-u_\varepsilon'' + V(y)u_\varepsilon - (\lambda+\mathrm{i}\varepsilon)u_\varepsilon = f \quad \text{in } \mathbb{R} \tag{7}$$

as $\pm\varepsilon\searrow 0$ for a given right hand side $f\in L^2(\mathbb{R};\mathbb{C})$. To keep the presentation simple, we concentrate on the case $\lambda > V_1$ and $\varepsilon\searrow 0$. The three remaining cases $\lambda = V_1, \lambda\in(V_2,V_1)$ and $\lambda = V_2$ will be commented on later, see Remark 1. A solution formula for $u_\varepsilon$ can be found with the aid of the Laplace-Fourier transform

$$\mathcal{F}_1^+ f(\eta) := \frac{1}{\sqrt{2\pi}}\int_0^\infty f(y)\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}y, \qquad \mathcal{F}_1^- f(\eta) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 f(y)\mathrm{e}^{-\mathrm{i}y\eta}\,\mathrm{d}y.$$

We collect a few elementary facts.

**Proposition 1.** *For all $v\in H^2(\mathbb{R};\mathbb{C})$ and $\eta\in\mathbb{R}$ we have $\mathcal{F}_1 = \mathcal{F}_1^+ + \mathcal{F}_1^-$ and*

$$\mathcal{F}_1^+(v'')(\eta) = -\eta^2\mathcal{F}_1^+ v(\eta) - (2\pi)^{-\frac12}(v'(0) + \mathrm{i}\eta v(0)),$$
$$\mathcal{F}_1^-(v'')(\eta) = -\eta^2\mathcal{F}_1^- v(\eta) + (2\pi)^{-\frac12}(v'(0) + \mathrm{i}\eta v(0)).$$

*Moreover, the spaces $\mathrm{ran}(\mathcal{F}_1^+), \mathrm{ran}(\mathcal{F}_1^-)$ are $L^2(\mathbb{R};\mathbb{C})$-orthogonal to each other.*

We only comment on the orthogonality property. For $f, g \in L^2(\mathbb{R};\mathbb{C})$ set $F := f \cdot 1_{(0,\infty)}$ and $G := g \cdot 1_{(-\infty,0)}$. Then Plancherel's identity implies

$$\langle \mathcal{F}_1^+ f, \mathcal{F}_1^- g\rangle_{L^2(\mathbb{R})} = \langle \mathcal{F}_1 F, \mathcal{F}_1 G\rangle_{L^2(\mathbb{R})} = \langle F, G\rangle_{L^2(\mathbb{R})} = 0$$

since the supports of $F, G$ intersect only in a null set. We now use Proposition 1 in order to find a solution formula for the solutions $u_\varepsilon$ in (7). To this end we introduce the complex numbers $\mu_{1,\varepsilon}, \mu_{2,\varepsilon}$ by requiring

$$\mu_{1,\varepsilon}^2 := \lambda - V_1 + \mathrm{i}\varepsilon, \qquad \mu_{2,\varepsilon}^2 := \lambda - V_2 + \mathrm{i}\varepsilon, \qquad \mathrm{Im}(\mu_{1,\varepsilon}), \mathrm{Im}(\mu_{2,\varepsilon}) > 0. \tag{8}$$

Notice that $\mu_{j,\varepsilon} \to \mu_j := \sqrt{\lambda - V_j}$ as $\varepsilon \searrow 0$ because $\lambda > V_1 > V_2$.

**Proposition 2.** *Let $\lambda > V_1 > V_2$ and $f \in L^2(\mathbb{R};\mathbb{C})$. Then, for any given $\varepsilon > 0$, the unique solution $u_\varepsilon \in H^2(\mathbb{R};\mathbb{C})$ of (7) is given by*

$$\begin{aligned} u_\varepsilon(y) &= \frac{\mathrm{i}}{2\mu_{1,\varepsilon}} \int_0^\infty \mathrm{e}^{\mathrm{i}|y-z|\mu_{1,\varepsilon}} f(z)\,\mathrm{d}z + \frac{\mathrm{i}}{2\mu_{2,\varepsilon}} \int_{-\infty}^0 \mathrm{e}^{\mathrm{i}|y-z|\mu_{2,\varepsilon}} f(z)\,\mathrm{d}z \\ &\quad + \alpha_{f,\varepsilon} \cdot \frac{\mathrm{sign}(y)}{2} \left(\mathrm{e}^{\mathrm{i}|y|\mu_{1,\varepsilon}} - \mathrm{e}^{\mathrm{i}|y|\mu_{2,\varepsilon}}\right) - \beta_{f,\varepsilon} \cdot \frac{\mathrm{i}}{2} \left( \frac{\mathrm{e}^{\mathrm{i}|y|\mu_{1,\varepsilon}}}{\mu_{1,\varepsilon}} - \frac{\mathrm{e}^{\mathrm{i}|y|\mu_{2,\varepsilon}}}{\mu_{2,\varepsilon}} \right) \end{aligned}$$

*where the complex numbers $\alpha_{f,\varepsilon}, \beta_{f,\varepsilon}$ satisfy*

$$\begin{pmatrix} \alpha_{f,\varepsilon} \\ \beta_{f,\varepsilon} \end{pmatrix} = \frac{\sqrt{2\pi}}{\mu_{1,\varepsilon} + \mu_{2,\varepsilon}} \begin{pmatrix} \mathrm{i} & \mathrm{i} \\ \mu_{2,\varepsilon} & -\mu_{1,\varepsilon} \end{pmatrix} \begin{pmatrix} \mathcal{F}_1^+ f(-\mu_{1,\varepsilon}) \\ \mathcal{F}_1^- f(\mu_{2,\varepsilon}) \end{pmatrix}. \tag{9}$$

*Proof.* Without loss of generality we prove the statement only for $f \in C_c^\infty(\mathbb{R}^n)$. We set $\alpha_{f,\varepsilon} := u_\varepsilon(0), \beta_{f,\varepsilon} := u_\varepsilon'(0)$. Applying $\mathcal{F}_1^+, \mathcal{F}_1^-$ to equation (7) we obtain from Proposition 1 and (8)

$$\begin{aligned} \mathcal{F}_1^+ u_\varepsilon(\eta) &= \frac{\mathcal{F}_1^+ f(\eta) - (2\pi)^{-\frac12}(u_\varepsilon'(0) + \mathrm{i}\eta u_\varepsilon(0))}{\eta^2 + V_1 - \lambda - \mathrm{i}\varepsilon} = \frac{\mathcal{F}_1^+ f(\eta) - (2\pi)^{-\frac12}(\beta_{f,\varepsilon} + \mathrm{i}\eta\alpha_{f,\varepsilon})}{\eta^2 - \mu_{1,\varepsilon}^2}, \\ \mathcal{F}_1^- u_\varepsilon(\eta) &= \frac{\mathcal{F}_1^- f(\eta) + (2\pi)^{-\frac12}(u_\varepsilon'(0) + \mathrm{i}\eta u_\varepsilon(0))}{\eta^2 + V_2 - \lambda - \mathrm{i}\varepsilon} = \frac{\mathcal{F}_1^- f(\eta) + (2\pi)^{-\frac12}(\beta_{f,\varepsilon} + \mathrm{i}\eta\alpha_{f,\varepsilon})}{\eta^2 - \mu_{2,\varepsilon}^2}. \end{aligned} \tag{10}$$

The function $g_p(y) := \sqrt{2\pi}\mathrm{e}^{-p|y|}, p > 0$ satisfies $\mathcal{F}_1^\pm g_p(\eta) = \frac{1}{\pm\mathrm{i}\eta + p}$. So the orthogonality relation from Proposition 1 yields

$$\begin{aligned} 0 &= \langle \mathcal{F}_1^- u_\varepsilon, \mathcal{F}_1^+ g_p\rangle_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}} \left( \frac{\mathcal{F}_1^- f(\eta) + (2\pi)^{-\frac12}(\beta_{f,\varepsilon} + \mathrm{i}\eta\alpha_{f,\varepsilon})}{\eta^2 - \mu_{2,\varepsilon}^2} \cdot \frac{1}{-\mathrm{i}\eta + p} \right) \mathrm{d}\eta \\ &= \int_{\mathbb{R}} \frac{\mathcal{F}_1^- f(\eta)}{(\eta^2 - \mu_{2,\varepsilon}^2)(-\mathrm{i}\eta + p)}\,\mathrm{d}\eta + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\beta_{f,\varepsilon} + \mathrm{i}\eta\alpha_{f,\varepsilon}}{(\eta^2 - \mu_{2,\varepsilon}^2)(-\mathrm{i}\eta + p)}\,\mathrm{d}\eta \\ &= \mathcal{F}_1^- f(\mu_{2,\varepsilon}) \cdot \frac{\mathrm{i}\pi}{\mu_{2,\varepsilon}(p - \mathrm{i}\mu_{2,\varepsilon})} + \frac{\beta_{f,\varepsilon}}{\sqrt{2\pi}} \cdot \frac{\mathrm{i}\pi}{\mu_{2,\varepsilon}(p - \mathrm{i}\mu_{2,\varepsilon})} - \frac{\alpha_{f,\varepsilon}}{\sqrt{2\pi}} \cdot \frac{\pi}{p - \mathrm{i}\mu_{2,\varepsilon}}. \end{aligned}$$

The last equality holds because of the Residue Theorem. Similarly we get

$$\begin{aligned}
0 &= \langle \mathcal{F}_1^+ u_\varepsilon, \mathcal{F}_1^- g_p\rangle_{L^2(\mathbb{R})} \\
&= \int_{\mathbb{R}} \left( \frac{\mathcal{F}_1^+ f(\eta) - (2\pi)^{-\frac12}(\beta_{f,\varepsilon} + \mathrm{i}\eta\alpha_{f,\varepsilon})}{\eta^2 - \mu_{1,\varepsilon}^2} \cdot \frac{1}{\mathrm{i}\eta + p} \right) \mathrm{d}\eta \\
&= \int_{\mathbb{R}} \frac{\mathcal{F}_1^+ f(\eta)}{(\eta^2 - \mu_{1,\varepsilon}^2)(\mathrm{i}\eta + p)} \,\mathrm{d}\eta - \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\beta_{f,\varepsilon} + \mathrm{i}\eta\alpha_{f,\varepsilon}}{(\eta^2 - \mu_{1,\varepsilon}^2)(\mathrm{i}\eta + p)} \,\mathrm{d}\eta \\
&= \mathcal{F}_1^+ f(-\mu_{1,\varepsilon}) \cdot \frac{\mathrm{i}\pi}{\mu_{1,\varepsilon}(p - \mathrm{i}\mu_{1,\varepsilon})} - \frac{\beta_{f,\varepsilon}}{\sqrt{2\pi}} \cdot \frac{\mathrm{i}\pi}{\mu_{1,\varepsilon}(p - \mathrm{i}\mu_{1,\varepsilon})} - \frac{\alpha_{f,\varepsilon}}{\sqrt{2\pi}} \cdot \frac{\pi}{p - \mathrm{i}\mu_{1,\varepsilon}}.
\end{aligned}$$

Rearranging the previous equations we find

$$\begin{pmatrix} -\mathrm{i}\mu_{1,\varepsilon} & 1 \\ -\mathrm{i}\mu_{2,\varepsilon} & -1 \end{pmatrix} \begin{pmatrix} \alpha_{f,\varepsilon} \\ \beta_{f,\varepsilon} \end{pmatrix} = \sqrt{2\pi} \begin{pmatrix} \mathcal{F}_1^+ f(-\mu_{1,\varepsilon}) \\ \mathcal{F}_1^- f(\mu_{2,\varepsilon}) \end{pmatrix},$$

which proves (9). From this and (10) we get

$$\begin{aligned}
u_\varepsilon(y) &= (\mathcal{F}_1^+ + \mathcal{F}_1^-)^{-1}(\mathcal{F}_1^+ u_\varepsilon + \mathcal{F}_1^- u_\varepsilon)(y) \\
&= \mathcal{F}_1^{-1} \left( \frac{\mathcal{F}_1^+ f(\eta) - (2\pi)^{-\frac12}(\beta_{f,\varepsilon} + \mathrm{i}\eta\alpha_{f,\varepsilon})}{\eta^2 - \mu_{1,\varepsilon}^2} + \frac{\mathcal{F}_1^- f(\eta) + (2\pi)^{-\frac12}(\beta_{f,\varepsilon} + \mathrm{i}\eta\alpha_{f,\varepsilon})}{\eta^2 - \mu_{2,\varepsilon}^2} \right)(y) \\
&= \mathcal{F}_1^{-1} \left( \frac{\mathcal{F}_1^+ f(\eta)}{\eta^2 - \mu_{1,\varepsilon}^2} + \frac{\mathcal{F}_1^- f(\eta)}{\eta^2 - \mu_{2,\varepsilon}^2} \right)(y) \\
&\quad - \frac{\alpha_{f,\varepsilon}}{\sqrt{2\pi}} \cdot \mathcal{F}_1^{-1} \left( \frac{\mathrm{i}\eta}{\eta^2 - \mu_{1,\varepsilon}^2} - \frac{\mathrm{i}\eta}{\eta^2 - \mu_{2,\varepsilon}^2} \right)(y) - \frac{\beta_{f,\varepsilon}}{\sqrt{2\pi}} \cdot \mathcal{F}_1^{-1} \left( \frac{1}{\eta^2 - \mu_{1,\varepsilon}^2} - \frac{1}{\eta^2 - \mu_{2,\varepsilon}^2} \right)(y) \\
&= \frac{1}{2\pi} \int_0^\infty f(z) \left( \int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}(y-z)\eta}}{\eta^2 - \mu_{1,\varepsilon}^2} \,\mathrm{d}\eta \right) \mathrm{d}z + \frac{1}{2\pi} \int_{-\infty}^0 f(z) \left( \int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}(y-z)\eta}}{\eta^2 - \mu_{2,\varepsilon}^2} \,\mathrm{d}\eta \right) \mathrm{d}z \\
&\quad - \frac{\alpha_{f,\varepsilon}}{2\pi} \int_{\mathbb{R}} \left( \frac{\mathrm{i}\eta}{\eta^2 - \mu_{1,\varepsilon}^2} - \frac{\mathrm{i}\eta}{\eta^2 - \mu_{2,\varepsilon}^2} \right) \mathrm{e}^{\mathrm{i}y\eta} \,\mathrm{d}\eta - \frac{\beta_{f,\varepsilon}}{2\pi} \int_{\mathbb{R}} \left( \frac{1}{\eta^2 - \mu_{1,\varepsilon}^2} - \frac{1}{\eta^2 - \mu_{2,\varepsilon}^2} \right) \mathrm{e}^{\mathrm{i}y\eta} \,\mathrm{d}\eta \\
&= \frac{\mathrm{i}}{2\mu_{1,\varepsilon}} \int_0^\infty \mathrm{e}^{\mathrm{i}|y-z|\mu_{1,\varepsilon}} f(z) \,\mathrm{d}z + \frac{\mathrm{i}}{2\mu_{2,\varepsilon}} \int_{-\infty}^0 \mathrm{e}^{\mathrm{i}|y-z|\mu_{2,\varepsilon}} f(z) \,\mathrm{d}z \\
&\quad + \alpha_{f,\varepsilon} \cdot \frac{\operatorname{sign}(y)}{2} \left( \mathrm{e}^{\mathrm{i}|y|\mu_{1,\varepsilon}} - \mathrm{e}^{\mathrm{i}|y|\mu_{2,\varepsilon}} \right) - \beta_{f,\varepsilon} \cdot \frac{\mathrm{i}}{2} \left( \frac{\mathrm{e}^{\mathrm{i}|y|\mu_{1,\varepsilon}}}{\mu_{1,\varepsilon}} - \frac{\mathrm{e}^{\mathrm{i}|y|\mu_{2,\varepsilon}}}{\mu_{2,\varepsilon}} \right).
\end{aligned}$$

Again, the Residue Theorem was used. This proves the claim. □

Given the formula from Proposition 2 we may pass to the limit $\varepsilon \searrow 0$. The only reasonable limit function is given by

$$\begin{aligned}
u_+(y) &:= \frac{\mathrm{i}}{2\mu_1} \int_0^\infty \mathrm{e}^{\mathrm{i}|y-z|\mu_1} f(z) \,\mathrm{d}z + \frac{\mathrm{i}}{2\mu_2} \int_{-\infty}^0 \mathrm{e}^{\mathrm{i}|y-z|\mu_2} f(z) \,\mathrm{d}z \\
&\quad + \alpha_f \cdot \frac{\operatorname{sign}(y)}{2} \left( \mathrm{e}^{\mathrm{i}|y|\mu_1} - \mathrm{e}^{\mathrm{i}|y|\mu_2} \right) - \beta_f \cdot \frac{\mathrm{i}}{2} \left( \frac{\mathrm{e}^{\mathrm{i}|y|\mu_1}}{\mu_1} - \frac{\mathrm{e}^{\mathrm{i}|y|\mu_2}}{\mu_2} \right)
\end{aligned} \tag{11}$$

where $\alpha_f := \alpha_{f,0} = u_+(0), \beta_f := \beta_{f,0} = u_+'(0)$. One can check that $u_+$ is a strong solution of the Helmholtz equation (6). In the following section we will generalize the above approach to define the corresponding solution in the higher-dimensional case, see (19). Notice that lengthy computations show that $u_+$ can be rewritten as

$$u_+(y) = \begin{cases} \alpha_f \cos(\mu_1 y) + \frac{\beta_f}{\mu_1}\sin(\mu_1 y) + \frac{1}{\mu_1}\int\limits_0^y \sin(\mu_1(z-y))f(z)\mathrm{d}z & \text{if } y > 0, \\ \alpha_f \cos(\mu_2 y) + \frac{\beta_f}{\mu_2}\sin(\mu_2 y) + \frac{1}{\mu_2}\int\limits_0^y \sin(\mu_2(z-y))f(z)\mathrm{d}z & \text{if } y < 0. \end{cases}$$

**Remark 1.** *In the case $\lambda = V_1$ we have $\mu_1 = 0$ so that the formula* (11) *does not make sense a priori; nevertheless, a lengthy computation allows to evaluate the limit $\varepsilon \searrow 0$ in the formula for $u_\varepsilon$ given in Proposition 2. In the case $\lambda \in (V_2, V_1)$ it suffices to replace $\mu_1 = \sqrt{\lambda - V_1}$ by $\tilde{\mu}_1 := \mathrm{i}\sqrt{V_1 - \lambda}$ in* (11) *so that exponentially decaying terms show up. The case $\lambda = V_2$ may then be treated like the case $\lambda = V_1$.*

## 2. The higher-dimensional interface problem

In this section we generalize the one-dimensional approach to the higher-dimensional case. Once again we consider the simplest possible step potential $V$ given by

$$V(x,y) = \begin{cases} V_1 & \text{if } x \in \mathbb{R}^{n-1}, y > 0, \\ V_2 & \text{if } x \in \mathbb{R}^{n-1}, y < 0 \end{cases} \qquad \text{where } V_1 > V_2.$$

In the following we will always write $x, \xi \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}, \eta \in \mathbb{C}$. The first task is to find a solution formula for $u_\varepsilon \in H^2(\mathbb{R}^n; \mathbb{C})$ solving the perturbed Helmholtz equation

$$-\Delta u_\varepsilon + V(x,y)u_\varepsilon - (\lambda + \mathrm{i}\varepsilon)u_\varepsilon = f \quad \text{in } \mathbb{R}^n \tag{12}$$

where $\lambda > V_1 > V_2$. Again we use the one-sided (one-dimensional) Fourier transforms $\mathcal{F}_1^\pm$ and define the corresponding full one-sided Fourier transforms by $\mathcal{F}_n^\pm := \mathcal{F}_{n-1} \otimes \mathcal{F}_1^\pm$. Using the shorthand notations $f_\pm(x,y) := f(x,y) \cdot 1_{(0,\infty)}(\pm y)$ we may rewrite this definition as $(\mathcal{F}_n^\pm f)(\xi,\eta) = \mathcal{F}_n(f_\pm)(\xi,\eta)$ for $\xi \in \mathbb{R}^{n-1}, \eta \in \mathbb{C}$. Furthermore, we need the complex-valued functions $\nu_{j,\varepsilon} : \mathbb{R}^{n-1} \to \mathbb{C}$ defined via

$$\nu_{j,\varepsilon}(\xi)^2 = \mu_{j,\varepsilon}^2 - |\xi|^2 = \lambda - V_j - |\xi|^2 + \mathrm{i}\varepsilon \quad \text{and} \quad \mathrm{Im}(\nu_{j,\varepsilon}(\xi)) > 0.$$

Notice that $\nu_{j,\varepsilon}(\xi) \to \nu_j(\xi)$ as $\varepsilon \searrow 0$ where

$$\nu_j(\xi) = \begin{cases} (\mu_j^2 - |\xi|^2)^{\frac{1}{2}} & \text{if } |\xi| \leq \mu_j, \\ \mathrm{i}(|\xi|^2 - \mu_j^2)^{\frac{1}{2}} & \text{if } |\xi| \geq \mu_j. \end{cases} \tag{13}$$

We will need the following elementary estimate:

$$1 + |\xi| \lesssim |\nu_j(\xi)|\sqrt{1 + |\nabla \nu_j(\xi)|^2} \lesssim 1 + |\xi|. \tag{14}$$

**Proposition 3.** *Let $\lambda > V_1 > V_2$ and $f \in L^2(\mathbb{R}^n;\mathbb{C})$. Then, for any given $\varepsilon > 0$, the unique solution $u_\varepsilon \in H^2(\mathbb{R}^n;\mathbb{C})$ of (12) is given by*

$$\begin{aligned} u_\varepsilon(x,y) &= \mathcal{F}_n^{-1}\left(\frac{\mathcal{F}_n^+ f}{|\cdot|^2-\mu_{1,\varepsilon}^2} + \frac{\mathcal{F}_n^- f}{|\cdot|^2-\mu_{2,\varepsilon}^2}\right)(x,y) \\ &\quad + \frac{\operatorname{sign}(y)}{2}\mathcal{F}_{n-1}^{-1}\left(\left(\mathrm{e}^{\mathrm{i}|y|\nu_{1,\varepsilon}} - \mathrm{e}^{\mathrm{i}|y|\nu_{2,\varepsilon}}\right)\alpha_{f,\varepsilon}\right)(x) \\ &\quad - \frac{\mathrm{i}}{2}\mathcal{F}_{n-1}^{-1}\left(\left(\frac{\mathrm{e}^{\mathrm{i}|y|\nu_{1,\varepsilon}}}{\nu_{1,\varepsilon}} - \frac{\mathrm{e}^{\mathrm{i}|y|\nu_{2,\varepsilon}}}{\nu_{2,\varepsilon}}\right)\beta_{f,\varepsilon}\right)(x), \end{aligned} \tag{15}$$

*where, for all $\xi \in \mathbb{R}^{n-1}$,*

$$\begin{pmatrix}\alpha_{f,\varepsilon}(\xi)\\ \beta_{f,\varepsilon}(\xi)\end{pmatrix} = \frac{\sqrt{2\pi}}{\nu_{1,\varepsilon}(\xi)+\nu_{2,\varepsilon}(\xi)}\begin{pmatrix}\mathrm{i} & \mathrm{i}\\ \nu_{2,\varepsilon}(\xi) & -\nu_{1,\varepsilon}(\xi)\end{pmatrix}\begin{pmatrix}\mathcal{F}_n^+ f(\xi,-\nu_{1,\varepsilon}(\xi))\\ \mathcal{F}_n^- f(\xi,\nu_{2,\varepsilon}(\xi))\end{pmatrix}. \tag{16}$$

*Proof.* Without loss of generality we prove this only for $f \in C_c^\infty(\mathbb{R}^n)$. We define

$$\alpha_{f,\varepsilon}(\xi) := \mathcal{F}_{n-1}(u_\varepsilon(\cdot,0))(\xi), \qquad \beta_{f,\varepsilon}(\xi) := \frac{\mathrm{d}}{\mathrm{d}y}\Big|_{y=0}\mathcal{F}_{n-1}(u_\varepsilon(\cdot,y))(\xi).$$

Notice that these point evaluations are possible since $u_\varepsilon$ is continuously differentiable by local elliptic regularity theory. Applying first $\mathcal{F}_{n-1}$ to (12) with respect to $x$ we get for all $\xi \in \mathbb{R}^{n-1}$

$$\begin{aligned} (|\xi|^2 - \partial_{yy} + V_1 - \lambda - \mathrm{i}\varepsilon)(\mathcal{F}_{n-1}u_\varepsilon(\cdot,y))(\xi) &= (\mathcal{F}_{n-1}f(\cdot,y))(\xi) \qquad (y>0),\\ (|\xi|^2 - \partial_{yy} + V_2 - \lambda - \mathrm{i}\varepsilon)(\mathcal{F}_{n-1}u_\varepsilon(\cdot,y))(\xi) &= (\mathcal{F}_{n-1}f(\cdot,y))(\xi) \qquad (y<0). \end{aligned}$$

Applying now the one-sided Fourier transforms $\mathcal{F}_1^\pm$ with respect to the $y$-variable and using $\mu_{j,\varepsilon}^2 = \lambda - V_j + \mathrm{i}\varepsilon$, we get

$$\begin{aligned} (|\xi|^2 + \eta^2 - \mu_{1,\varepsilon}^2)\mathcal{F}_n^+ u_\varepsilon(\xi,\eta) + (2\pi)^{-\frac12}(\mathrm{i}\eta\alpha_{f,\varepsilon}(\xi) + \beta_{f,\varepsilon}(\xi)) &= \mathcal{F}_n^+ f(\xi,\eta),\\ (|\xi|^2 + \eta^2 - \mu_{2,\varepsilon}^2)\mathcal{F}_n^- u_\varepsilon(\xi,\eta) - (2\pi)^{-\frac12}(\mathrm{i}\eta\alpha_{f,\varepsilon}(\xi) + \beta_{f,\varepsilon}(\xi)) &= \mathcal{F}_n^- f(\xi,\eta). \end{aligned}$$

This and $\nu_{j,\varepsilon}(\xi)^2 = \mu_{j,\varepsilon}^2 - |\xi|^2$ imply the analogue of (10) in the one-dimensional case,

$$\begin{aligned} \mathcal{F}_n^+ u_\varepsilon(\xi,\eta) &= \frac{\mathcal{F}_n^+ f(\xi,\eta) - (2\pi)^{-\frac12}\left(\beta_{f,\varepsilon}(\xi) + \mathrm{i}\eta\alpha_{f,\varepsilon}(\xi)\right)}{\eta^2 - \nu_{1,\varepsilon}(\xi)^2},\\ \mathcal{F}_n^- u_\varepsilon(\xi,\eta) &= \frac{\mathcal{F}_n^- f(\xi,\eta) + (2\pi)^{-\frac12}\left(\beta_{f,\varepsilon}(\xi) + \mathrm{i}\eta\alpha_{f,\varepsilon}(\xi)\right)}{\eta^2 - \nu_{2,\varepsilon}(\xi)^2}. \end{aligned} \tag{17}$$

As in the one-dimensional case we may exploit $\operatorname{ran}(\mathcal{F}_1^+) \perp \operatorname{ran}(\mathcal{F}_1^-)$ in order to compute $\alpha_{f,\varepsilon}, \beta_{f,\varepsilon}$. The Residue Theorem gives for all $\phi \in \mathcal{S}(\mathbb{R}^{n-1};\mathbb{C})$ and $g_p(z) := \sqrt{2\pi}\mathrm{e}^{-p|z|}$ for $z \in \mathbb{R}$ and some $p > 0$

$$\begin{aligned} 0 &= \langle \mathcal{F}_n^- u_\varepsilon, \mathcal{F}_n^+(\phi \otimes g_p)\rangle_{L^2(\mathbb{R}^n;\mathbb{C})}\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\left(\frac{\mathcal{F}_n^- f(\xi,\eta) + (2\pi)^{-\frac12}(\beta_{f,\varepsilon}(\xi) + \mathrm{i}\eta\alpha_{f,\varepsilon}(\xi))}{\eta^2 - \nu_{2,\varepsilon}(\xi)^2}\cdot\frac{\overline{\hat{\phi}(\xi)}}{-\mathrm{i}\eta + p}\right)\mathrm{d}\xi\,\mathrm{d}\eta \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^{n-1}} \overline{\hat{\phi}(\xi)} \Bigg[ \left( \int_{\mathbb{R}} \frac{\mathcal{F}_n^- f(\xi,\eta)}{(\eta^2 - \nu_{2,\varepsilon}(\xi)^2)(-\mathrm{i}\eta + p)} \, \mathrm{d}\eta \right) \\
&\qquad + \frac{\beta_{f,\varepsilon}(\xi)}{\sqrt{2\pi}} \left( \int_{\mathbb{R}} \frac{1}{(\eta^2 - \nu_{2,\varepsilon}(\xi)^2)(-\mathrm{i}\eta + p)} \, \mathrm{d}\eta \right) \\
&\qquad + \frac{\alpha_{f,\varepsilon}(\xi)}{\sqrt{2\pi}} \left( \int_{\mathbb{R}} \frac{\mathrm{i}\eta}{(\eta^2 - \nu_{2,\varepsilon}(\xi)^2)(-\mathrm{i}\eta + p)} \, \mathrm{d}\eta \right) \Bigg] \, \mathrm{d}\xi \\
&= \int_{\mathbb{R}^{n-1}} \frac{\mathrm{i}\pi \overline{\hat{\phi}(\xi)}}{\nu_{2,\varepsilon}(\xi)(p - \mathrm{i}\nu_{2,\varepsilon}(\xi))} \cdot \left( \mathcal{F}_n^- f(\xi, \nu_{2,\varepsilon}(\xi)) + (2\pi)^{-\frac{1}{2}} (\beta_{f,\varepsilon}(\xi) + \mathrm{i}\nu_{2,\varepsilon}(\xi)\alpha_{f,\varepsilon}(\xi)) \right) \mathrm{d}\xi
\end{aligned}
$$

and similarly

$$
\begin{aligned}
0 &= \langle \mathcal{F}_n^+ u_\varepsilon, \mathcal{F}_n^-(\phi \otimes g_p) \rangle_{L^2(\mathbb{R}^n)} \\
&= \int_{\mathbb{R}^{n-1}} \frac{\mathrm{i}\pi \overline{\hat{\phi}(\xi)}}{\nu_{1,\varepsilon}(\xi)(p - \mathrm{i}\nu_{1,\varepsilon}(\xi))} \cdot \left( \mathcal{F}_n^+ f(\xi, -\nu_{1,\varepsilon}(\xi)) - (2\pi)^{-\frac{1}{2}} (\beta_{f,\varepsilon}(\xi) - \mathrm{i}\nu_{1,\varepsilon}(\xi)\alpha_{f,\varepsilon}(\xi)) \right) \mathrm{d}\xi.
\end{aligned}
$$

Since $\phi \in \mathcal{S}(\mathbb{R}^{n-1})$ was arbitrary, we get for almost all $\xi \in \mathbb{R}^{n-1}$

$$
\begin{pmatrix} -\mathrm{i}\nu_{1,\varepsilon}(\xi) & 1 \\ -\mathrm{i}\nu_{2,\varepsilon}(\xi) & -1 \end{pmatrix} \begin{pmatrix} \alpha_{f,\varepsilon}(\xi) \\ \beta_{f,\varepsilon}(\xi) \end{pmatrix} = \sqrt{2\pi} \begin{pmatrix} \mathcal{F}_n^+ f(\xi, -\nu_{1,\varepsilon}(\xi)) \\ \mathcal{F}_n^- f(\xi, \nu_{2,\varepsilon}(\xi)) \end{pmatrix},
$$

which implies (16). With these formulas we may now solve (17) for $u_\varepsilon$. From $\mathcal{F}_n^+ + \mathcal{F}_n^- = \mathcal{F}_n$ we get for $x \in \mathbb{R}^{n-1}, y \in \mathbb{R}$

$$
\begin{aligned}
u_\varepsilon(x,y) &= \mathcal{F}_n^{-1}(\mathcal{F}_n^+ u_\varepsilon + \mathcal{F}_n^- u_\varepsilon)(x,y) \\
&\overset{(17)}{=} \mathcal{F}_n^{-1} \left( \frac{\mathcal{F}_n^+ f(\xi,\eta)}{\eta^2 - \nu_{1,\varepsilon}(\xi)^2} + \frac{\mathcal{F}_n^- f(\xi,\eta)}{\eta^2 - \nu_{2,\varepsilon}(\xi)^2} \right)(x,y) \\
&\quad - \frac{1}{\sqrt{2\pi}} \mathcal{F}_n^{-1} \left[ \left( \frac{\mathrm{i}\eta}{\eta^2 - \nu_{1,\varepsilon}(\xi)^2} - \frac{\mathrm{i}\eta}{\eta^2 - \nu_{2,\varepsilon}(\xi)^2} \right) \cdot \alpha_{f,\varepsilon}(\xi) \right](x,y) \\
&\quad - \frac{1}{\sqrt{2\pi}} \mathcal{F}_n^{-1} \left[ \left( \frac{1}{\eta^2 - \nu_{1,\varepsilon}(\xi)^2} - \frac{1}{\eta^2 - \nu_{2,\varepsilon}(\xi)^2} \right) \cdot \beta_{f,\varepsilon}(\xi) \right](x,y) \\
&= \mathcal{F}_n^{-1} \left( \frac{\mathcal{F}_n^+ f}{|\cdot|^2 - \mu_{1,\varepsilon}^2} + \frac{\mathcal{F}_n^- f}{|\cdot|^2 - \mu_{2,\varepsilon}^2} \right)(x,y) \\
&\quad - \frac{1}{2\pi} \mathcal{F}_{n-1}^{-1} \left( \int_{\mathbb{R}} \left( \frac{\mathrm{i}\eta}{\eta^2 - \nu_{1,\varepsilon}(\xi)^2} - \frac{\mathrm{i}\eta}{\eta^2 - \nu_{2,\varepsilon}(\xi)^2} \right) \mathrm{e}^{\mathrm{i}y\eta} \, \mathrm{d}\eta \cdot \alpha_{f,\varepsilon}(\xi) \right)(x) \\
&\quad - \frac{1}{2\pi} \mathcal{F}_{n-1}^{-1} \left( \int_{\mathbb{R}} \left( \frac{1}{\eta^2 - \nu_{1,\varepsilon}(\xi)^2} - \frac{1}{\eta^2 - \nu_{2,\varepsilon}(\xi)^2} \right) \mathrm{e}^{\mathrm{i}y\eta} \, \mathrm{d}\eta \cdot \beta_{f,\varepsilon}(\xi) \right)(x) \\
&= \mathcal{F}_n^{-1} \left( \frac{\mathcal{F}_n^+ f}{|\cdot|^2 - \mu_{1,\varepsilon}^2} + \frac{\mathcal{F}_n^- f}{|\cdot|^2 - \mu_{2,\varepsilon}^2} \right)(x,y) \\
&\quad + \frac{\operatorname{sign}(y)}{2} \mathcal{F}_{n-1}^{-1} \left( \left( \mathrm{e}^{\mathrm{i}|y|\nu_{1,\varepsilon}} - \mathrm{e}^{\mathrm{i}|y|\nu_{2,\varepsilon}} \right) \alpha_{f,\varepsilon} \right)(x)
\end{aligned}
$$

$$- \frac{\mathrm{i}}{2}\mathcal{F}_{n-1}^{-1}\left(\Big(\frac{\mathrm{e}^{\mathrm{i}|y|\nu_{1,\varepsilon}}}{\nu_{1,\varepsilon}} - \frac{\mathrm{e}^{\mathrm{i}|y|\nu_{2,\varepsilon}}}{\nu_{2,\varepsilon}}\Big)\beta_{f,\varepsilon}\right)(x),$$

which is all we had to show. □

Notice that the last two lines may be rewritten as

$$\frac{1}{2}\sum_{j=1,2}(-1)^j\mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_j}\left(i\beta_{f,\varepsilon}\nu_{j,\varepsilon}^{-1} - \mathrm{sign}(y)\alpha_{f,\varepsilon}\right)\right)(x) = \sum_{j=1,2}\big(w_{j,\varepsilon}(x,y) + W_{j,\varepsilon}(x,y)\big)$$

where the small frequencies are collected in $w_{j,\varepsilon}$ and the big ones in $W_{j,\varepsilon}$. More precisely,

$$\begin{aligned} w_{j,\varepsilon}(x,y) &:= \frac{(-1)^j}{2}\mathcal{F}_{n-1}^{-1}\left(1_{|\cdot|\le\mu_j}\mathrm{e}^{\mathrm{i}|y|\nu_{j,\varepsilon}(\cdot)}m_{j,\varepsilon,y}(\cdot)\right)(x), \\ W_{j,\varepsilon}(x,y) &:= \frac{(-1)^j}{2}\mathcal{F}_{n-1}^{-1}\left(1_{|\cdot|\ge\mu_j}\mathrm{e}^{\mathrm{i}|y|\nu_{j,\varepsilon}(\cdot)}m_{j,\varepsilon,y}(\cdot)\right)(x), \\ m_{j,\varepsilon,y}(\xi) &:= \mathrm{i}\beta_{f,\varepsilon}(\xi)\nu_{j,\varepsilon}(\xi)^{-1} - \mathrm{sign}(y)\alpha_{f,\varepsilon}(\xi). \end{aligned} \tag{18}$$

In the following section we will present estimates for $w_{j,\varepsilon}, W_{j,\varepsilon}$ that evetually lead to the proof of Theorem 1. Before going on with this we first identify the limit of $u_\varepsilon$ as $\varepsilon \searrow 0$. The above representation formula for $u_\varepsilon$ leads to the definition

$$\begin{aligned} \big(\mathcal{R}(\lambda+\mathrm{i}0)f\big)(x,y) := u_+(x,y) &:= \lim_{\varepsilon\searrow 0}\mathcal{F}_n^{-1}\left(\frac{\mathcal{F}_n^+ f}{|\cdot|^2-\mu_1^2-\mathrm{i}\varepsilon} + \frac{\mathcal{F}_n^- f}{|\cdot|^2-\mu_2^2-\mathrm{i}\varepsilon}\right)(x,y) \\ &+ \frac{\mathrm{sign}(y)}{2}\mathcal{F}_{n-1}^{-1}\left(\Big(\mathrm{e}^{\mathrm{i}|y|\nu_1} - \mathrm{e}^{\mathrm{i}|y|\nu_2}\Big)\alpha_f\right)(x) \\ &- \frac{\mathrm{i}}{2}\mathcal{F}_{n-1}^{-1}\left(\Big(\frac{\mathrm{e}^{\mathrm{i}|y|\nu_1}}{\nu_1} - \frac{\mathrm{e}^{\mathrm{i}|y|\nu_2}}{\nu_2}\Big)\beta_f\right)(x), \end{aligned} \tag{19}$$

where $\alpha_f := \alpha_{f,0}, \beta_f := \beta_{f,0}$ and the limit in the first line has to be understood in the distributional sense. Notice that

$$\begin{pmatrix}\alpha_f(\xi)\\ \beta_f(\xi)\end{pmatrix} = \frac{\sqrt{2\pi}}{\nu_1(\xi)+\nu_2(\xi)}\begin{pmatrix}\mathrm{i} & \mathrm{i}\\ \nu_2(\xi) & -\nu_1(\xi)\end{pmatrix}\begin{pmatrix}\mathcal{F}_n^+ f(\xi,-\nu_1(\xi))\\ \mathcal{F}_n^- f(\xi,\nu_2(\xi))\end{pmatrix}. \tag{20}$$

As above, the last two lines of (19) can be rewritten as

$$\frac{1}{2}\sum_{j=1,2}(-1)^j\mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_j}\left(i\beta_f\nu_j^{-1} - \mathrm{sign}(y)\alpha_f\right)\right)(x) = \sum_{j=1,2}\big(w_j(x,y) + W_j(x,y)\big)$$

where

$$\begin{aligned} w_j(x,y) &:= \frac{(-1)^j}{2}\mathcal{F}_{n-1}^{-1}\left(1_{|\cdot|\le\mu_j}\mathrm{e}^{\mathrm{i}|y|\nu_j(\cdot)}m_{j,y}(\cdot)\right)(x), \\ W_j(x,y) &:= \frac{(-1)^j}{2}\mathcal{F}_{n-1}^{-1}\left(1_{|\cdot|\ge\mu_j}\mathrm{e}^{\mathrm{i}|y|\nu_j(\cdot)}m_{j,y}(\cdot)\right)(x); \\ m_{j,y}(\xi) &:= \mathrm{i}\beta_f(\xi)\nu_j(\xi)^{-1} - \mathrm{sign}(y)\alpha_f(\xi). \end{aligned} \tag{21}$$

The essential new ingredients for the proofs of Theorem 1 and Corollary 1 are estimates for $w_1, w_2, W_1, W_2$ and $w_{1,\varepsilon}, w_{2,\varepsilon}, W_{1,\varepsilon}, W_{2,\varepsilon}$, see Propositions 4, 5, 6 in the following section.

## 3. Proof of Theorem 1 and Corollary 2

We collect a few results that we will need in our estimates. For $n \geq 3$, the first line in (15) and (19) may be analyzed with the aid of Gutiérrez' Limiting Absorption Principle [18, Theorem 6] for the Helmholtz equation with constant coefficients in $\mathbb{R}^n$. The corresponding result for the case $n = 2$ was provided by Evéquoz [11, Theorem 2.1].

**Theorem 2** (Gutiérrez, Evéquoz)**.** *Assume $n \in \mathbb{N}, n \geq 3$ and $p', q > \frac{2n}{n-1}, \frac{2}{n+1} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{n}$. If $V_1 = V_2$ then the solutions $u_\varepsilon$ of (12) and $u_+, u_-$ from (19) satisfy*

$$\|u_+\|_{L^q(\mathbb{R}^n)} + \|u_-\|_{L^q(\mathbb{R}^n)} + \sup_{0<|\varepsilon|\leq 1} \|u_\varepsilon\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

*The same is true in case $n = 2$ if additionally $\frac{1}{p} - \frac{1}{q} < \frac{2}{n}$ is assumed.*

We will also need the Fourier restriction theorems by Stein and Tomas as well as its generalization due to Tao, see [24, Figure 3] and [25, p.1382]. Since we will use it for the Fourier transforms $\mathcal{F}_n, \mathcal{F}_{n-1}$ restricted to spheres in $\mathbb{R}^{n-1}$ respectively $\mathbb{R}^n$, we use $d$ as the dimensional parameter.

**Theorem 3** (Stein-Tomas)**.** *Let $d \in \mathbb{N}, d \geq 2$ and $1 \leq p \leq \frac{2(d+1)}{d+3}$, $\mu > 0$. Then*

$$\|\mathcal{F}_n f\|_{L^2(\mathbb{S}^{d-1}_\mu)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}, \qquad \|\mathcal{F}_n(g\, d\sigma_\mu)\|_{L^{p'}(\mathbb{R}^d)} \lesssim \|g\|_{L^2(\mathbb{S}^{d-1}_\mu)}.$$

**Theorem 4** (Tao)**.** *Let $d \in \mathbb{N}, d \geq 2$ and assume $p' > \frac{2(d+2)}{d}, q \geq \left(\frac{d-1}{d+1}\, p'\right)', \mu > 0$. Then*

$$\|\mathcal{F}_d h\|_{L^{q'}(\mathbb{S}^{d-1}_\mu)} \lesssim \|h\|_{L^p(\mathbb{R}^d)}, \qquad \|\mathcal{F}_d(g\, d\sigma_\mu)\|_{L^{p'}(\mathbb{R}^d)} \lesssim \|g\|_{L^q(\mathbb{S}^{d-1}_\mu)}.$$

Loosely speaking, Tao's result will play the role of the "best known approximation" to the Restriction Conjecture. Recall that the latter says that the above estimates even hold under the weaker assumption $p' > \frac{2d}{d-1}, q \geq \left(\frac{d-1}{d+1}\, p'\right)'$, which would yield larger ranges of admissible exponents $p, q$ in Propositions 4, 5 below and hence in Theorem 1. The above tools are used in our derivation of the estimates for the small frequency parts $w_1, w_2, w_{1,\varepsilon}, w_{2,\varepsilon}$, see (21),(18).

**Proposition 4.** *Let $n \in \{2, 3, 4\}$. For $p', q > \frac{2(n+2)}{n}$ and $\frac{1}{p} - \frac{1}{q} \geq \frac{2}{n+1}$ we have*

$$\|w_1\|_{L^q(\mathbb{R}^n)} + \|w_2\|_{L^q(\mathbb{R}^n)} + \sup_{0<|\varepsilon|\leq 1} \left(\|w_{1,\varepsilon}\|_{L^q(\mathbb{R}^n)} + \|w_{2,\varepsilon}\|_{L^q(\mathbb{R}^n)}\right) \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

The proof will be given in Section 4. Next we estimate the large frequency parts $W_1, W_2$ and $W_{1,\varepsilon}, W_{2,\varepsilon}$. In our first estimate we use Theorem 4 for $d = n - 1$ and thus only deal with dimensions $n \geq 3$.

**Proposition 5.** *Let $n \in \{3, 4\}$. For $1 < p < \frac{2(n+1)}{n+3}, \frac{2}{q} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{n}$ we have*

$$\|W_1\|_{L^q(\mathbb{R}^n)} + \|W_2\|_{L^q(\mathbb{R}^n)} + \sup_{0<|\varepsilon|\leq 1} \Big(\|W_{1,\varepsilon}\|_{L^q(\mathbb{R}^n)} + \|W_{2,\varepsilon}\|_{L^q(\mathbb{R}^n)}\Big) \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

Notice that the conditions on $p, q$ implicitly imply $n \in \{3, 4\}$ because

$$\frac{n+3}{2(n+1)} < \frac{1}{p} \leq \frac{1}{q} + \frac{2}{n} \leq \frac{3}{n} \quad \Rightarrow \quad n^2 - 3n - 6 < 0 \quad \Rightarrow \quad n \leq 4.$$

In order to cover the case $n = 2$ we circumvent Fourier Restriction Theory with the aid of the Hausdorff-Young inequality. The same technique actually applies in all space dimensions but the ranges for the exponents turn out to be worse than those obtained in Proposition 5.

**Proposition 6.** *Let $n = 2$. For $p, q' > 1, \frac{2}{3} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{n}$ or $p = 1, 3 < q < \infty$ or $q = \infty, 1 < p < \frac{3}{2}$ we have*

$$\|W_1\|_{L^q(\mathbb{R}^n)} + \|W_2\|_{L^q(\mathbb{R}^n)} + \sup_{0<|\varepsilon|\leq 1} \Big( \|W_{1,\varepsilon}\|_{L^q(\mathbb{R}^n)} + \|W_{2,\varepsilon}\|_{L^q(\mathbb{R}^n)} \Big) \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

The proof of Theorem 1 is now a matter of combining the previous results and to improve the resulting estimates with the aid of the corresponding dual estimates where $(p, q)$ is formally replaced by $(q', p')$. This is justified because of

$$\int_{\mathbb{R}^n} \mathcal{R}(\lambda + \mathrm{i}\varepsilon) f \cdot g \,\mathrm{d}(x, y) = \int_{\mathbb{R}^n} f \cdot \mathcal{R}(\lambda + \mathrm{i}\varepsilon) g \,\mathrm{d}(x, y) \quad \text{for all } f, g \in C_c^\infty(\mathbb{R}^n). \tag{22}$$

Finally, interpolating our estimates with their dual counterparts will prove Theorem 1. Instead of going into the tedious details of the interpolation procedure, we prefer to illustrate the situation with the aid of the corresponding Riesz diagrams in Figure 1.

**Proof of Theorem 1.** From the Proposition 3 and the representation formula (19) we get

$$\begin{aligned} &\|u_+\|_{L^p(\mathbb{R}^n)} + \|u_-\|_{L^p(\mathbb{R}^n)} + \sup_{0<|\varepsilon|\leq 1} \|u_\varepsilon\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \sup_{0<|\varepsilon|\leq 1} \left\| \mathcal{F}_n^{-1} \left( \frac{\mathcal{F}_n^+ f}{|\cdot|^2 - \mu_1^2 - \mathrm{i}\varepsilon} + \frac{\mathcal{F}_n^- f}{|\cdot|^2 - \mu_2^2 - \mathrm{i}\varepsilon} \right) \right\|_{L^q(\mathbb{R}^n)} \\ &\quad + \sum_{j=1,2} \Big( \|w_j\|_{L^q(\mathbb{R}^n)} + \|W_j\|_{L^q(\mathbb{R}^n)} \Big) \\ &\quad + \sum_{j=1,2} \sup_{0<|\varepsilon|\leq 1} \Big( \|w_{j,\varepsilon}\|_{L^q(\mathbb{R}^n)} + \|W_{j,\varepsilon}\|_{L^q(\mathbb{R}^n)} \Big). \end{aligned} \tag{23}$$

In the case $n = 2$ we get from Theorem 2, Proposition 4 and Proposition 6 that these terms can be bounded by $\|f\|_{L^p(\mathbb{R}^n)}$ for exponents $p, q$ such that

$$n = 2, \quad 4 < p', q < \infty, \quad \frac{2}{3} \leq \frac{1}{p} - \frac{1}{q} < 1.$$

This already proves part (i) of the Theorem. In the case $n = 3$, Theorem 2, Proposition 4 and Proposition 5 allow to estimate all terms in (23) by $\|f\|_{L^p(\mathbb{R}^n)}$ under the assumptions

$$n = 3, \quad 4 < p' < \infty, \quad q > \frac{10}{3}, \quad \max\left\{\frac{2}{q}, \frac{1}{2}\right\} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{3}.$$

This corresponds to the black region in the second picture of Figure 1 below. The corresponding dual estimates enlarge the range of admissible exponents by the grey region in the same picture. A tedious computation reveals that the corresponding region of admissible exponents is given by

$$n = 3, \quad \frac{10}{3} < p', q < 30, \quad \frac{1}{p} - \frac{1}{q} \le \frac{2}{3}, \quad \frac{1}{p} \ge \frac{3}{q}, \quad \frac{1}{q'} \ge \frac{3}{p'},$$

which is convex and hence cannot be extended further by interpolation. This proves (ii). Similarly the aforementioned results guarantee a bound for (23) in terms of $\|f\|_{L^p(\mathbb{R}^n)}$ provided

$$n = 4, \quad \frac{10}{3} < p' < \infty, \quad q > 3, \quad \max\left\{\frac{2}{q}, \frac{2}{5}\right\} \le \frac{1}{p} - \frac{1}{q} \le \frac{1}{2}$$

hold. Again referring to the Riesz diagrams from Figure 1, duality and interpolation yield the estimates for $p, q$ satisfying

$$n = 4, \quad \frac{10}{3} < p', q < 5, \qquad \frac{7}{15} < \frac{1}{p} - \frac{1}{q} \le \frac{1}{2}$$

so that claim (iii) is proved as well. □

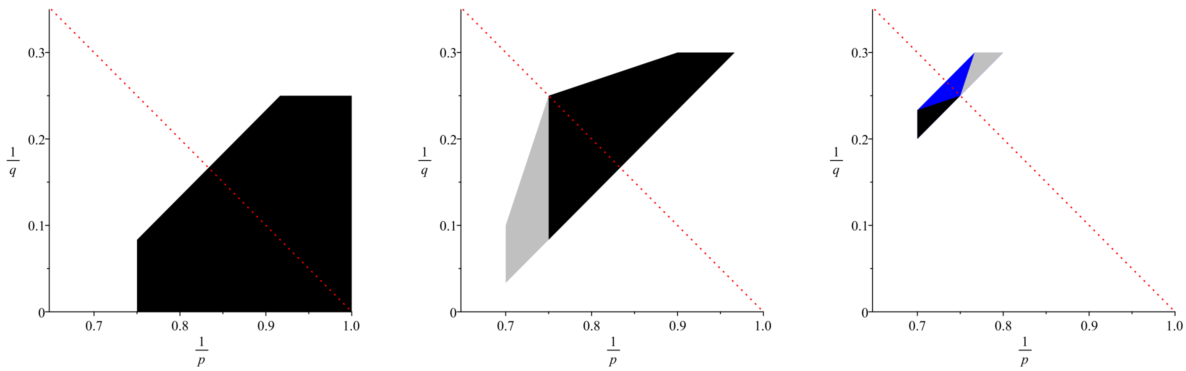


FIGURE 1. Riesz diagrams for the cases $n = 2$ (left), $n = 3$ (middle) as well as $n = 4$ (right) in Theorem 1. The $L^p - L^q$-boundedness for $(\frac{1}{p}, \frac{1}{q})$ in the black regions directly follows from the Propositions 4, 5, 6. The grey regions are obtained by duality. Then the combined region is symmetric to the red dotted line, which indicates the line of duality $\frac{1}{p} + \frac{1}{q} = 1$. The blue region in the diagram on the right is gained via interpolation.

**Proof of Corollary 2.** We briefly recall the dual variational technique for nonlinear Helmholtz equations from [14]. We aim at proving the existence of a real-valued function $u \in L^q(\mathbb{R}^n)$ satisfying

$$-\Delta u + Vu - \lambda u = \Gamma |u|^{q-2} u \qquad \text{in } \mathbb{R}^n \tag{24}$$

in the distributional sense. In view of elliptic regularity theory any distributional solution of such an equation will actually belong to $W^{2,r}_{loc}(\mathbb{R}^n)$ for all $r \in [1, \infty)$. Such solutions of the nonlinear PDE (24) will be obtained solving the integral equation $u = K(\Gamma |u|^{q-2} u)$ where

$K\phi := \mathrm{Re}(\mathcal{R}(\lambda+\mathrm{i}0)\phi)$ and $\mathcal{R}(\lambda+\mathrm{i}0)$ has the mapping properties stated in Theorem 1. We set $v := \Gamma^{\frac{1}{q'}}|u|^{q-2}u$ and thus look for $v \in L^{q'}(\mathbb{R}^n)$ satisfying

$$|v|^{q'-2}v = \Gamma^{\frac{1}{q}} K(\Gamma^{\frac{1}{q}} v).$$

Since $K$ is symmetric in the sense of (22), this equation has a variational structure. So we have to prove the existence of a nontrivial critical point of the functional

$$I(v) := \frac{1}{q'}\int_{\mathbb{R}^n} |v|^{q'} - \frac{1}{2}\int_{\mathbb{R}^n} \left(\Gamma^{\frac{1}{q}} v\right)\left[K\left(\Gamma^{\frac{1}{q}} v\right)\right].$$

This functional has the Mountain Pass geometry, as we will explain and verify below. Moreover, exploiting $\Gamma \to 0$ at infinity, it satisfies the Palais-Smale condition. This can be shown exactly as in [14, Lemma 5.2] where the corresponding statement is proved in the special case $V_1 = V_2$. With these two ingredients we may apply the Mountain Pass Theorem [5, Theorem 2.1] and obtain a nontrivial critical point $v$ of $I$. Transforming this function back according to $v = \Gamma^{\frac{1}{q'}}|u|^{q-2}u$, we get a nontrivial solution $u = \Gamma^{-\frac{1}{q}}|v|^{q'-2}v = K(\Gamma^{\frac{1}{q}}v) \in L^q(\mathbb{R}^n)$ of the nonlinear Helmholtz equation (5).

We now check that $I$ has the Mountain Pass geometry. First, by choice of $q$ in Corollary 2, the operator $\mathcal{R}(\lambda+\mathrm{i}0) : L^{q'}(\mathbb{R}^n) \to L^q(\mathbb{R}^n;\mathbb{C})$ is bounded and thus $K : L^{q'}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is bounded as well. Moreover,

$$I(v) \geq \frac{1}{q}\|v\|^{q'}_{L^{q'}(\mathbb{R}^n)} - \frac{1}{2}\|K\|_{L^{q'}(\mathbb{R}^n)\to L^q(\mathbb{R}^n)}\|\Gamma\|^{\frac{2}{q}}_{L^\infty(\mathbb{R}^n)}\|v\|^2_{L^{q'}(\mathbb{R}^n)}$$

and $q' < 2$ imply $I(0) = 0 < \inf_{S_\varrho} I$ for some sufficiently small $\varrho > 0$ where $S_\varrho$ denotes the sphere in $L^{q'}(\mathbb{R}^n)$ with radius $\varrho$. Finally, $I(tv) \to -\infty$ as $t \to \infty$ for some $v \in L^{q'}(\mathbb{R}^n)$, the proof of which will take the remainder of this section. We adapt an idea from [22, Section 3] and choose the ansatz $v = v_\delta$ where

$$v_\delta(x,y) := \Gamma(x,y)^{-\frac{1}{q}} w(x)\mathrm{e}^{-y} 1_{(\delta,\infty)}(\Gamma(x,y)) 1_{(0,\infty)}(y) \qquad (x \in \mathbb{R}^{n-1}, y \in \mathbb{R}, \delta > 0) \tag{25}$$

with sufficiently small $\delta > 0$ and with a nontrivial Schwartz function $w$ satisfying $\mathrm{supp}(\hat{w}) \subset B_{2\mu_2}(0) \setminus B_{\mu_2}(0) = \{\xi \in \mathbb{R}^{n-1} \,|\, \mu_2 \leq |\xi| < 2\mu_2\}$. Notice that $v_\delta \in L^{q'}(\mathbb{R}^n)$ because of $\delta > 0$ and

$$\Gamma^{\frac{1}{q}} v_\delta \to f \quad \text{in } L^{q'}(\mathbb{R}^n) \quad \text{as } \delta \searrow 0 \qquad \text{where } f(x,y) = w(x)\mathrm{e}^{-y}1_{(0,\infty)}(y).$$

Here we used $\Gamma > 0$ on $\mathbb{R}^n$. So we find with the aid of Plancherel's theorem

$$\begin{aligned}
&\lim_{\delta\searrow 0}\int_{\mathbb{R}^n} \left(\Gamma^{\frac{1}{q}} v_\delta\right)\left[K\left(\Gamma^{\frac{1}{q}} v_\delta\right)\right] \,\mathrm{d}(x,y) \\
&= \int_{\mathbb{R}^n} f(Kf)\,\mathrm{d}(x,y) \\
&= \mathrm{Re}\left(\int_{\mathbb{R}^n} (\mathcal{R}(\lambda+\mathrm{i}0)f)\cdot f\,\mathrm{d}(x,y)\right) \\
&\overset{(19)}{=} \mathrm{Re}\left(\int_{\mathbb{R}^n} \mathcal{F}_n^{-1}\left(\frac{\mathcal{F}_n^+ f}{|\cdot|^2-\mu_1^2-\mathrm{i}0} + \frac{\mathcal{F}_n^+ f}{|\cdot|^2-\mu_2^2-\mathrm{i}0}\right)(x,y)\cdot f(x,y)\,\mathrm{d}(x,y)\right)
\end{aligned}$$

$$
\begin{aligned}
&+\operatorname{Re}\left(\int_{\mathbb{R}^n} \frac{\operatorname{sign}(y)}{2}\mathcal{F}_{n-1}^{-1}\left(\left(\mathrm{e}^{\mathrm{i}|y|\nu_1}-\mathrm{e}^{\mathrm{i}|y|\nu_2}\right)\alpha_f\right)(x)\cdot f(x,y)\,\mathrm{d}(x,y)\right)\\
&-\operatorname{Re}\left(\int_{\mathbb{R}^n} \frac{\mathrm{i}}{2}\mathcal{F}_{n-1}^{-1}\left(\left(\frac{\mathrm{e}^{\mathrm{i}|y|\nu_1}}{\nu_1}-\frac{\mathrm{e}^{\mathrm{i}|y|\nu_2}}{\nu_2}\right)\beta_f\right)(x)\cdot f(x,y)\,\mathrm{d}(x,y)\right)\\
=&\operatorname{Re}\left(\int_{\mathbb{R}^n} \frac{\mathcal{F}_n^+ f(\xi,\eta)\cdot\overline{\mathcal{F}_n f(\xi,\eta)}}{|\xi|^2+\eta^2-\mu_1^2-\mathrm{i}0}+\frac{\mathcal{F}_n^- f(\xi,\eta)\cdot\overline{\mathcal{F}_n f(\xi,\eta)}}{|\xi|^2+\eta^2-\mu_2^2-\mathrm{i}0}\,\mathrm{d}(\xi,\eta)\right)\\
&+\frac{1}{2}\operatorname{Re}\left(\int_{\mathbb{R}^n}\operatorname{sign}(y)\left(\mathrm{e}^{\mathrm{i}|y|\nu_1(\xi)}-\mathrm{e}^{\mathrm{i}|y|\nu_2(\xi)}\right)\alpha_f(\xi)\cdot\overline{\left(\mathcal{F}_{n-1}f(\cdot,y)\right)(\xi)}\,\mathrm{d}(\xi,y)\right)\\
&-\frac{1}{2}\operatorname{Re}\left(\mathrm{i}\int_{\mathbb{R}^n}\left(\frac{\mathrm{e}^{\mathrm{i}|y|\nu_1(\xi)}}{\nu_1(\xi)}-\frac{\mathrm{e}^{\mathrm{i}|y|\nu_2(\xi)}}{\nu_2(\xi)}\right)\beta_f(\xi)\cdot\overline{\left(\mathcal{F}_{n-1}f(\cdot,y)\right)(\xi)}\,\mathrm{d}(\xi,y)\right).
\end{aligned}
$$

Inserting (25) we get $\mathcal{F}_n^+ f(\xi,\eta)=\mathcal{F}_n f(\xi,\eta)=\frac{\hat{w}(\xi)}{1+\mathrm{i}\eta}, \mathcal{F}_n^- f\equiv 0$ as well as

$$
\begin{aligned}
\begin{pmatrix}\alpha_f(\xi)\\ \beta_f(\xi)\end{pmatrix} &\overset{(20)}{=} \frac{\sqrt{2\pi}}{\nu_1(\xi)+\nu_2(\xi)}\begin{pmatrix}\mathrm{i} & \mathrm{i}\\ \nu_2(\xi) & -\nu_1(\xi)\end{pmatrix}\begin{pmatrix}\mathcal{F}_n^+ f(\xi,-\nu_1(\xi))\\ \mathcal{F}_n^- f(\xi,\nu_2(\xi))\end{pmatrix}\\
&= \frac{\sqrt{2\pi}}{\nu_1(\xi)+\nu_2(\xi)}\frac{\hat{w}_1(\xi)}{1-\mathrm{i}\nu_1(\xi)}\begin{pmatrix}\mathrm{i}\\ \nu_2(\xi)\end{pmatrix}.
\end{aligned}
$$

Notice in particular that, by choice of $w$, we have $|\xi|^2+\eta^2\geq|\xi|^2>\mu_2^2>\mu_1^2$ for all $\xi\in\operatorname{supp}(\hat{w})$ and hence also for all $(\xi,\eta)\in\operatorname{supp}(\mathcal{F}_n^+ f)=\operatorname{supp}(\mathcal{F}_n f)$. Thus, there is no singularity in the former integral, and in the latter integrals we may use $\nu_j(\xi)=\mathrm{i}\sqrt{|\xi|^2-\mu_j^2}$ for $j=1,2$, see (13). This implies

$$
\begin{aligned}
&\int_{\mathbb{R}^n} f(Kf)\,\mathrm{d}(x,y)\\
&=\int_{\mathbb{R}^n}\frac{|\mathcal{F}_n f(\xi,\eta)|^2}{|\xi|^2+\eta^2-\mu_1^2}\,\mathrm{d}(\xi,\eta)\\
&\quad+\frac{1}{2}\operatorname{Re}\left(\int_{\mathbb{R}^{n-1}\times(0,\infty)}\left(\mathrm{e}^{\mathrm{i}y\nu_1(\xi)}-\mathrm{e}^{\mathrm{i}y\nu_2(\xi)}\right)\frac{\mathrm{i}\sqrt{2\pi}}{\nu_1(\xi)+\nu_2(\xi)}\frac{\hat{w}(\xi)}{1-\mathrm{i}\nu_1(\xi)}\cdot\overline{\hat{w}(\xi)}\mathrm{e}^{-y}\,\mathrm{d}(\xi,y)\right)\\
&\quad-\frac{1}{2}\operatorname{Re}\left(\mathrm{i}\int_{\mathbb{R}^{n-1}\times(0,\infty)}\left(\frac{\mathrm{e}^{\mathrm{i}y\nu_1(\xi)}}{\nu_1(\xi)}-\frac{\mathrm{e}^{\mathrm{i}y\nu_2(\xi)}}{\nu_2(\xi)}\right)\frac{\sqrt{2\pi}\nu_2(\xi)}{\nu_1(\xi)+\nu_2(\xi)}\frac{\hat{w}(\xi)}{1-\mathrm{i}\nu_1(\xi)}\cdot\overline{\hat{w}(\xi)}\mathrm{e}^{-y}\,\mathrm{d}(\xi,y)\right)\\
&=\int_{\mathbb{R}^n}\frac{|\hat{w}(\xi)|^2(1+\eta^2)^{-1}}{|\xi|^2+\eta^2-\mu_1^2}\,\mathrm{d}(\xi,\eta)\\
&\quad+\sqrt{\frac{\pi}{2}}\operatorname{Re}\left(\mathrm{i}\int_{\mathbb{R}^{n-1}}\frac{|\hat{w}(\xi)|^2(\nu_1(\xi)-\nu_2(\xi))}{(\nu_1(\xi)+\nu_2(\xi))\nu_1(\xi)(1-\mathrm{i}\nu_1(\xi))}\left(\int_0^\infty \mathrm{e}^{(\mathrm{i}\nu_1(\xi)-1)y}\,\mathrm{d}y\right)\mathrm{d}\xi\right)\\
&=\int_{\mathbb{R}^n}\frac{|\hat{w}(\xi)|^2(1+\eta^2)^{-1}}{|\xi|^2+\eta^2-\mu_1^2}\,\mathrm{d}(\xi,\eta)
\end{aligned}
$$

$$
\begin{aligned}
&+\sqrt{\frac{\pi}{2}}\,\mathrm{Re}\left(\mathrm{i}\int_{\mathbb{R}^{n-1}} \frac{|\hat{w}(\xi)|^2(\nu_1(\xi)-\nu_2(\xi))}{(\nu_1(\xi)+\nu_2(\xi))(1-\mathrm{i}\nu_1(\xi))^2\nu_1(\xi)}\,\mathrm{d}\xi\right) \\
&= \int_{\mathbb{R}^n} \frac{|\hat{w}(\xi)|^2(1+\eta^2)^{-1}}{|\xi|^2+\eta^2-\mu_1^2}\,\mathrm{d}(\xi,\eta) \\
&\quad+\sqrt{\frac{\pi}{2}}\int_{\mathbb{R}^{n-1}} \frac{|\hat{w}(\xi)|^2(|\nu_1(\xi)|-|\nu_2(\xi)|)}{(|\nu_1(\xi)|+|\nu_2(\xi)|)(1+|\nu_1(\xi)|)^2|\nu_1(\xi)|}\,\mathrm{d}\xi \\
&> 0
\end{aligned}
$$

because both terms are positive. As a consequence, we have

$$\int_{\mathbb{R}^n} \left(\Gamma^{\frac{1}{q}} v_\delta\right)\left[K\left(\Gamma^{\frac{1}{q}} v_\delta\right)\right]\,\mathrm{d}(x,y) > 0$$

provided that $\delta > 0$ is small enough. This gives the result. □

## 4. Proof of Proposition 4

In this section we prove Proposition 4 dealing with the small frequency parts $w_{j,\varepsilon}$ of the solutions of the perturbed Helmholtz equation. In order to avoid heavy notation we carry out the estimates for $w_j = \lim_{\varepsilon\searrow 0} w_{j,\varepsilon}$ in detail and briefly discuss the necessary modifications afterwards. We recall from (20),(21) the formulas ($j=1,2$)

$$
\begin{aligned}
w_j(x,y) &= \frac{(-1)^j}{2}\mathcal{F}_{n-1}^{-1}\left(1_{|\cdot|\le\mu_j}\mathrm{e}^{\mathrm{i}|y|\nu_j(\cdot)}m_{j,y}(\cdot)\right)(x),\\
m_{j,y}(\xi) &= \mathrm{i}\beta_f(\xi)\nu_j(\xi)^{-1} - \mathrm{sign}(y)\alpha_f(\xi), \qquad (26)\\
\begin{pmatrix}\alpha_f(\xi)\\ \beta_f(\xi)\end{pmatrix} &= \frac{\sqrt{2\pi}}{\nu_1(\xi)+\nu_2(\xi)}\begin{pmatrix}\mathrm{i} & \mathrm{i}\\ \nu_2(\xi) & -\nu_1(\xi)\end{pmatrix}\begin{pmatrix}\mathcal{F}_n^+ f(\xi,-\nu_1(\xi))\\ \mathcal{F}_n^- f(\xi,\nu_2(\xi))\end{pmatrix}.
\end{aligned}
$$

We actually prove the following slightly more general result.

**Proposition 4** (generalized version). *Let $n\in\mathbb{N}, n\ge 2$. For $p',q > \frac{2(n+2)}{n}$ such that $\frac{1}{p}-\frac{1}{q}\ge\frac{2}{n+1}$ and $\frac{1}{p}-\frac{2(n+1)}{3(n-1)}\frac{1}{q} > \frac{1}{3}$ we have*

$$\|w_1\|_{L^q(\mathbb{R}^n)} + \|w_2\|_{L^q(\mathbb{R}^n)} + \sup_{0<|\varepsilon|\le 1}\left(\|w_{1,\varepsilon}\|_{L^q(\mathbb{R}^n)} + \|w_{2,\varepsilon}\|_{L^q(\mathbb{R}^n)}\right) \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

*In particular, if $n\in\{2,3,4\}$ then this is true whenever $p',q>\frac{2(n+2)}{n}$ and $\frac{1}{p}-\frac{1}{q}\ge\frac{2}{n+1}$.*

*Proof.* In line with our explications above, we first show the estimates for $w_j$. For every fixed $x\in\mathbb{R}^{n-1}, y\in\mathbb{R}$ we have according to (26)

$$
\begin{aligned}
w_j(x,y) &= \frac{(-1)^j}{2(2\pi)^{\frac{n-1}{2}}}\int_{|\xi|\le\mu_j} \mathrm{e}^{\mathrm{i}(x\cdot\xi+|y|\nu_j(\xi))}m_{j,y}(\xi)\,\mathrm{d}\xi\\
&= \frac{(-1)^j}{2(2\pi)^{\frac{n-1}{2}}}\int_{\mathbb{S}^{n-1}_{\mu_j}} \mathrm{e}^{\mathrm{i}(x\cdot\xi+|y|\xi_n)}m_{j,y}(\xi)(1+|\nabla\nu_j(\xi)|^2)^{-\frac{1}{2}}1_{(0,\infty)}(\xi_n)\,\mathrm{d}\sigma_{\mu_j}(\xi,\xi_n)\\
&= \mathcal{F}_n^{-1}(b_j\,\mathrm{d}\sigma_{\mu_j})(x,|y|) + \mathrm{sign}(y)\mathcal{F}_n^{-1}(a_j\,\mathrm{d}\sigma_{\mu_j})(x,|y|)
\end{aligned}
$$

where the densities $a_j, b_j : \mathbb{S}^{n-1}_{\mu_j} \to \mathbb{C}$ are given by

$$a_j(\xi,\xi_n) = -(-1)^j \sqrt{\frac{\pi}{2}}(1+|\nabla\nu_j(\xi)|^2)^{-\frac{1}{2}} 1_{(0,\infty)}(\xi_n)\cdot\alpha_f(\xi),$$
$$b_j(\xi,\xi_n) = \mathrm{i}(-1)^j \sqrt{\frac{\pi}{2}}(1+|\nabla\nu_j(\xi)|^2)^{-\frac{1}{2}}\nu_j(\xi)^{-1} 1_{(0,\infty)}(\xi_n)\cdot\beta_f(\xi).$$

Since $q > \frac{2(n+2)}{n}$ we may apply Theorem 4 and obtain for $s := \left(\frac{n-1}{n+1}\, q\right)'$

$$\begin{aligned}\|w_j\|_{L^q(\mathbb{R}^n)} &\lesssim \|\mathcal{F}_n^{-1}(a_j\,\mathrm{d}\sigma_{\mu_j})\|_{L^q(\mathbb{R}^n)} + \|\mathcal{F}_n^{-1}(b_j\,\mathrm{d}\sigma_{\mu_j})\|_{L^q(\mathbb{R}^n)}\\ &\lesssim \|a_j\|_{L^s(\mathbb{S}^{n-1}_{\mu_j})} + \|b_j\|_{L^s(\mathbb{S}^{n-1}_{\mu_j})}\\ &\overset{(14)}{\lesssim} \|\alpha_f\nu_j\|_{L^s(\mathbb{S}^{n-1}_{\mu_j})} + \|\beta_f\|_{L^s(\mathbb{S}^{n-1}_{\mu_j})}.\end{aligned}$$

Notice that in the last line $\nu_j$ is considered as a map on $\mathbb{S}^{n-1}_{\mu_j}$ by putting $\nu_j(\xi,\xi_n) = \nu_j(\xi)$ for $(\xi,\xi_n)\in\mathbb{S}^{n-1}_{\mu_j}$. Plugging in the formulas for $\alpha_f, \beta_f$ from (26) and using

$$|\nu_1(\xi)+\nu_2(\xi)| \gtrsim 1, \quad |\nu_1(\xi)|+|\nu_2(\xi)| \lesssim 1, \quad |\nu_j(\xi)|^s(1+|\nabla\nu_j(\xi)|^2)^{\frac{1}{2}} \lesssim 1 \qquad (|\xi|\le\mu_j),$$

cf. (13),(14), we continue the above chain of estimates as follows:

$$\begin{aligned}\|w_j\|_{L^q(\mathbb{R}^n)} &\overset{(26)}{\lesssim} \|\nu_j(\cdot)\mathcal{F}_n^+ f(\cdot,-\nu_1(\cdot))\|_{L^s(\mathbb{S}^{n-1}_{\mu_j})} + \|\nu_j(\cdot)\mathcal{F}_n^- f(\cdot,\nu_2(\cdot))\|_{L^s(\mathbb{S}^{n-1}_{\mu_j})}\\ &\quad + \|\nu_2(\cdot)\mathcal{F}_n^+ f(\cdot,-\nu_1(\cdot))\|_{L^s(\mathbb{S}^{n-1}_{\mu_j})} + \|\nu_1(\cdot)\mathcal{F}_n^- f(\cdot,\nu_2(\cdot))\|_{L^s(\mathbb{S}^{n-1}_{\mu_j})}\\ &\lesssim \left(\int_{|\xi|\le\mu_j} (|\nu_j(\xi)|^s+|\nu_2(\xi)|^s)|\mathcal{F}_n^+ f(\xi,-\nu_1(\xi))|^s(1+|\nabla\nu_j(\xi)|^2)^{\frac{1}{2}}\,\mathrm{d}\xi\right)^{\frac{1}{s}}\\ &\quad + \left(\int_{|\xi|\le\mu_j} (|\nu_j(\xi)|^s+|\nu_1(\xi)|^s)|\mathcal{F}_n^- f(\xi,\nu_2(\xi))|^s(1+|\nabla\nu_j(\xi)|^2)^{\frac{1}{2}}\,\mathrm{d}\xi\right)^{\frac{1}{s}}\\ &\lesssim \left(\int_{|\xi|\le\mu_1} |\mathcal{F}_n^+ f(\xi,-|\nu_1(\xi)|)|^s(1+|\nabla\nu_1(\xi)|^2)^{\frac{1}{2}}\,\mathrm{d}\xi\right)^{\frac{1}{s}}\\ &\quad + \left(\int_{|\xi|\le\mu_2} |\mathcal{F}_n^- f(\xi,|\nu_2(\xi)|)|^s(1+|\nabla\nu_2(\xi)|^2)^{\frac{1}{2}}\,\mathrm{d}\xi\right)^{\frac{1}{s}}\\ &\quad + \left(\int_{\mu_1\le|\xi|\le\mu_2} |\nu_2(\xi)|^s|\mathcal{F}_n^+ f(\xi,-\mathrm{i}|\nu_1(\xi)|)|^s(1+|\nabla\nu_2(\xi)|^2)^{\frac{1}{2}}\,\mathrm{d}\xi\right)^{\frac{1}{s}}\\ &\overset{(13)}{\lesssim} \|\mathcal{F}_n^+ f\|_{L^s(\mathbb{S}^{n-1}_{\mu_1})} + \|\mathcal{F}_n^- f\|_{L^s(\mathbb{S}^{n-1}_{\mu_2})} + \left(\int_{\mu_1\le|\xi|\le\mu_2} |\mathcal{F}_n^+ f(\xi,-\mathrm{i}|\nu_1(\xi)|)|^s\,\mathrm{d}\xi\right)^{\frac{1}{s}}.\end{aligned}$$

Since $\frac{1}{p}-\frac{1}{q}\ge\frac{2}{n+1}$ implies $s'\ge\left(\frac{n-1}{n+1}\,p'\right)'$, Theorem 4 applies and we get

$$\|\mathcal{F}_n^+ f\|_{L^s(\mathbb{S}^{n-1}_{\mu_1})} + \|\mathcal{F}_n^- f\|_{L^s(\mathbb{S}^{n-1}_{\mu_2})} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

Notice that $\mathcal{F}_n^{\pm} f = \mathcal{F}_n(f_\pm)$ where $f_\pm(x,y) = f(x,y)1_{(0,\infty)}(\pm y)$. So it remains to estimate the last integral. From Minkowski's inequality, Hölder's inequality and the Hausdorff-Young inequality we obtain

$$\begin{aligned}
&\left(\int_{\mu_1\leq|\xi|\leq\mu_2} |\mathcal{F}_n^+ f(\xi,-\mathrm{i}|\nu_1(\xi)|)|^s \,\mathrm{d}\xi\right)^{\frac{1}{s}} \\
&\overset{(13)}{=} \left(\int_{\mu_1\leq|\xi|\leq\mu_2} \left|\int_0^\infty (\mathcal{F}_{n-1} f(\cdot,z))(\xi)\mathrm{e}^{-z\sqrt{|\xi|^2-\mu_1^2}}\,\mathrm{d}z\right|^s \mathrm{d}\xi\right)^{\frac{1}{s}} \\
&\lesssim \int_0^\infty \left(\int_{\mu_1\leq|\xi|\leq\mu_2} |(\mathcal{F}_{n-1} f(\cdot,z))(\xi)|^s \mathrm{e}^{-sz\sqrt{|\xi|^2-\mu_1^2}}\,\mathrm{d}\xi\right)^{\frac{1}{s}} \mathrm{d}z \\
&\lesssim \int_0^\infty \|\mathcal{F}_{n-1} f(\cdot,z)\|_{L^{p'}(\mathbb{R}^{n-1})} \left(\int_{\mu_1\leq|\xi|\leq\mu_2} \mathrm{e}^{-\frac{sp}{s+p-sp}z\sqrt{|\xi|^2-\mu_1^2}}\,\mathrm{d}\xi\right)^{\frac{s+p-sp}{sp}} \mathrm{d}z \\
&\lesssim \int_0^\infty \|f(\cdot,z)\|_{L^p(\mathbb{R}^{n-1})} \left(\int_{\mu_1}^{\mu_2} \mathrm{e}^{-\frac{sp}{s+p-sp}z\sqrt{r^2-\mu_1^2}} r^{n-2}\,\mathrm{d}r\right)^{\frac{s+p-sp}{sp}} \mathrm{d}z \\
&\lesssim \int_0^\infty \|f(\cdot,z)\|_{L^p(\mathbb{R}^{n-1})} \left(\int_0^{\sqrt{\mu_2^2-\mu_1^2}} \mathrm{e}^{-\frac{sp}{s+p-sp}z\rho} \rho\,\mathrm{d}\rho\right)^{\frac{s+p-sp}{sp}} \mathrm{d}z \\
&\lesssim \int_0^\infty \|f(\cdot,z)\|_{L^p(\mathbb{R}^{n-1})} (1+z)^{-\frac{2(s+p-sp)}{sp}}\,\mathrm{d}z \\
&\lesssim \|f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Here the last estimate follows from Hölder's inequality and $-\frac{2(s+p-sp)}{sp}\cdot\frac{p}{p-1} < -1$. Notice that the latter inequality is equivalent to $\left(\frac{n-1}{n+1}q\right)' = s < \frac{2}{3}p'$ and thus to $\frac{1}{p} - \frac{2(n+1)}{3(n-1)}\frac{1}{q} > \frac{1}{3}$. In the special case $n \in \{2,3,4\}$ this condition is automatically satisfied because $q > \frac{2(n+2)}{n}$ implies

$$\frac{1}{p} - \frac{2(n+1)}{3(n-1)}\frac{1}{q} - \frac{1}{3} \geq \frac{2}{n+1} - \frac{1}{3} + \left(1 - \frac{2(n+1)}{3(n-1)}\right)\frac{1}{q} = \frac{5-n}{3(n+1)}\left(1 - \frac{n+1}{n-1}\frac{1}{q}\right) > 0.$$

This proves the estimate for $w_j$.

Now we indicate the necessary modifications to get the corresponding uniform estimates for $w_{1,\varepsilon}, w_{2,\varepsilon}$ with respect to $\varepsilon \in (0,1]$. From (18) we recall

$$\begin{aligned}
w_{j,\varepsilon}(x,y) &= \frac{(-1)^j}{2}\mathcal{F}_{n-1}^{-1}\left(1_{|\cdot|\leq\mu_j}\mathrm{e}^{\mathrm{i}|y|\nu_{j,\varepsilon}(\cdot)} m_{j,\varepsilon,y}(\cdot)\right)(x), \\
m_{j,\varepsilon,y}(\xi) &= \mathrm{i}\beta_{f,\varepsilon}(\xi)\nu_{j,\varepsilon}(\xi)^{-1} - \mathrm{sign}(y)\alpha_{f,\varepsilon}(\xi).
\end{aligned}$$

In order to use the same estimates as above, we consider some smooth closed compact hypersurface $\mathbb{S}^{n-1}_{\mu_j,\varepsilon}$ with $\{(\xi, \mathrm{Re}(\nu_{j,\varepsilon}(\xi))) : |\xi| \leq \mu_j\} = \mathbb{S}^{n-1}_{\mu_j,\varepsilon} \cap \{\xi_n \geq \sqrt{\varepsilon/2}\}$ and with the property that its Gaussian curvature has a positive lower bound independent of $\varepsilon$. Denoting by $\sigma_{\mu_j,\varepsilon}$

its surface measure we can proceed as above and obtain

$$w_{j,\varepsilon}(x,y) = \mathcal{F}_n^{-1}(b_{j,\varepsilon}\,\mathrm{d}\sigma_{\mu_{j,\varepsilon}})(x,|y|) + \mathrm{sign}(y)\mathcal{F}_n^{-1}(a_{j,\varepsilon}\,\mathrm{d}\sigma_{\mu_{j,\varepsilon}})(x,|y|)$$

where

$$\begin{aligned} a_{j,\varepsilon}(\xi,\xi_n) &= -(-1)^j\sqrt{\frac{\pi}{2}}(1+|\nabla\,\mathrm{Re}(\nu_{j,\varepsilon}(\xi))|^2)^{-\frac{1}{2}}\mathrm{e}^{-\,\mathrm{Im}(\nu_{j,\varepsilon}(\xi))|y|}1_{(\sqrt{\varepsilon/2},\infty)}(\xi_n)\cdot\alpha_{f,\varepsilon}(\xi),\\ b_{j,\varepsilon}(\xi,\xi_n) &= \mathrm{i}(-1)^j\sqrt{\frac{\pi}{2}}(1+|\nabla\,\mathrm{Re}(\nu_{j,\varepsilon}(\xi))|^2)^{-\frac{1}{2}}\mathrm{e}^{-\,\mathrm{Im}(\nu_{j,\varepsilon}(\xi))|y|}\nu_{j,\varepsilon}(\xi)^{-1}1_{(\sqrt{\varepsilon/2},\infty)}(\xi_n)\cdot\beta_{f,\varepsilon}(\xi). \end{aligned}$$

Replacing $a_j, b_j, \nu_j, \sigma_{\mu_j}$ by $a_{j,\varepsilon}, b_{j,\varepsilon}, \mathrm{Re}(\nu_{j,\varepsilon}), \sigma_{\mu_{j,\varepsilon}}$, respectively, all of the above estimates hold uniformly with respect to $\varepsilon$, which yields the result. Notice that the factor $\mathrm{e}^{-\,\mathrm{Im}(\nu_{j,\varepsilon}(\xi))|y|}$ has a damping (absorptive) effect due to $\mathrm{Im}(\nu_{j,\varepsilon}(\xi))>0$ and may simply be bounded by 1 in all subsequent estimates involving $a_{j,\varepsilon}, b_{j,\varepsilon}$. The same is true for the indicator functions acting on $\xi_n$. □

## 5. Proof of Proposition 5

**Proposition 5.** Let $n\in\mathbb{N}, n\geq 3$ and assume $1<p<\frac{2(n+1)}{n+3}$, $\frac{2}{q}\leq\frac{1}{p}-\frac{1}{q}\leq\frac{2}{n}$. Then we have

$$\|W_1\|_{L^q(\mathbb{R}^n)} + \|W_2\|_{L^q(\mathbb{R}^n)} + \sup_{0<|\varepsilon|\leq 1}\Big(\|W_{1,\varepsilon}\|_{L^q(\mathbb{R}^n)} + \|W_{2,\varepsilon}\|_{L^q(\mathbb{R}^n)}\Big) \lesssim \|f\|_{L^p(\mathbb{R}^n)}\,.$$

*Proof.* Again we concentrate on the estimates for $W_1, W_2$ since the corresponding modifications for $W_{1,\varepsilon}, W_{2,\varepsilon}$ are purely notational. We recall that $W_1, W_2$ are given by

$$W_j(x,y) := \frac{(-1)^j}{2}\mathcal{F}_{n-1}^{-1}\left(1_{|\cdot|\geq\mu_j}\mathrm{e}^{\mathrm{i}|y|\nu_j(\cdot)}m_{j,y}(\cdot)\right)(x) \qquad (x\in\mathbb{R}^{n-1},\, y\in\mathbb{R},\, j\in\{1,2\})$$

for $m_{j,y}$ as in (26). The Hausdorff-Young inequality and $\nu_j(\xi)=\mathrm{i}\sqrt{|\xi|^2-\mu_j^2}$ for $|\xi|\geq\mu_j$ (cf. (13)) imply

$$\begin{aligned} \|W_j\|_{L^q(\mathbb{R}^n)} &\lesssim \left(\int_{\mathbb{R}}\left\|\mathcal{F}_{n-1}^{-1}\left(1_{|\cdot|\geq\mu_j}\mathrm{e}^{\mathrm{i}|y|\nu_j(\cdot)}m_{j,y}(\cdot)\right)\right\|_{L^q(\mathbb{R}^{n-1})}^q\,\mathrm{d}y\right)^{\frac{1}{q}}\\ &\lesssim \left(\int_{\mathbb{R}}\left\|1_{|\cdot|\geq\mu_j}\mathrm{e}^{-|y|\sqrt{|\xi|^2-\mu_j^2}}m_{j,y}(\cdot)\right\|_{L^{q'}(\mathbb{R}^{n-1})}^q\,\mathrm{d}y\right)^{\frac{1}{q}}\\ &\lesssim \left(\int_{\mathbb{R}}\left(\int_{|\xi|\geq\mu_j}\mathrm{e}^{-q'|y|\sqrt{|\xi|^2-\mu_j^2}}|m_{j,y}(\xi)|^{q'}\,\mathrm{d}\xi\right)^{\frac{q}{q'}}\,\mathrm{d}y\right)^{\frac{1}{q}}. \end{aligned}$$

Here $m_{j,y}(\xi)=\mathrm{i}\beta_f(\xi)\nu_j(\xi)^{-1}-\mathrm{sign}(y)\alpha_f(\xi)$ and hence by (20)

$$\begin{aligned} |m_{j,y}(\xi)| &\lesssim |\alpha_f(\xi)|+|\beta_f(\xi)||\nu_j(\xi)|^{-1}\\ &\lesssim (1+|\xi|)^{-1}\left(\frac{|\nu_2(\xi)|+|\nu_j(\xi)|}{|\nu_j(\xi)|}\mathcal{F}_n^{+}f(\xi,-\nu_1(\xi))|+\frac{|\nu_1(\xi)|+|\nu_j(\xi)|}{|\nu_j(\xi)|}|\mathcal{F}_n^{-}f(\xi,\nu_2(\xi))|\right). \end{aligned}$$

We discuss the case $j = 1$ in detail and concentrate on the first term, which is responsible for the most restrictive conditions on $p$ and $q$, see (29) and (30) below.

*First (more singular) term for $j = 1$.*

In view of $(1+|\xi|)^{-1}(|\nu_2(\xi)|+|\nu_1(\xi)|) \lesssim 1$ (cf. (14)) we have to prove

$$A := \left(\int_0^\infty \left(\int_{|\xi|\geq\mu_1} \mathrm{e}^{-q'y\sqrt{|\xi|^2-\mu_1^2}} \left|\frac{\mathcal{F}_n^+ f(\xi,-\nu_1(\xi))}{\nu_1(\xi)}\right|^{q'} \mathrm{d}\xi\right)^{\frac{q}{q'}} \mathrm{d}y\right)^{\frac{1}{q}} \lesssim \|f\|_{L^p(\mathbb{R}^n)}. \tag{27}$$

Minkowski's inequality in integral form and $\nu_1(\xi) = \mathrm{i}\sqrt{|\xi|^2-\mu_1}$ for $|\xi| \geq \mu_1$ (cf. (13)) yield

$$\begin{aligned} A &= \left(\int_0^\infty \left(\int_{|\xi|\geq\mu_1} \left|\int_0^\infty \frac{\mathcal{F}_{n-1}[f(\,\cdot\,,z)](\xi)}{\sqrt{|\xi|^2-\mu_1^2}} \mathrm{e}^{-z\sqrt{|\xi|^2-\mu_1^2}}\,\mathrm{d}z\right|^{q'} \mathrm{e}^{-q'y\sqrt{|\xi|^2-\mu_1^2}}\,\mathrm{d}\xi\right)^{\frac{q}{q'}} \mathrm{d}y\right)^{\frac{1}{q}} \\ &\leq \left(\int_0^\infty \left(\int_0^\infty \left(\int_{|\xi|\geq\mu_1} \left|\frac{\mathcal{F}_{n-1}[f(\,\cdot\,,z)](\xi)}{\sqrt{|\xi|^2-\mu_1^2}}\right|^{q'} \mathrm{e}^{-q'(z+y)\sqrt{|\xi|^2-\mu_1^2}}\,\mathrm{d}\xi\right)^{\frac{1}{q'}} \mathrm{d}z\right)^{q} \mathrm{d}y\right)^{\frac{1}{q}} \\ &\leq \left(\int_0^\infty \left(\int_0^\infty\left(\int_{\mu_1}^\infty \left[\int_{\mathbb{S}^{n-2}} |\mathcal{F}_{n-1}[f(\,\cdot\,,z)](r\omega)|^{q'}\,\mathrm{d}\sigma(\omega)\right] \frac{r^{n-2}\,\mathrm{e}^{-q'(z+y)(r^2-\mu_1^2)^{\frac{1}{2}}}}{(r^2-\mu_1^2)^{\frac{q'}{2}}}\,\mathrm{d}r\right)^{\frac{1}{q'}} \mathrm{d}z\right)^{q} \mathrm{d}y\right)^{\frac{1}{q}}. \end{aligned}$$

As already announced, the key idea in this proof is to use Fourier restriction theory in order to handle the integral with respect to the angular variable $\omega$. We use Tao's Fourier Restriction Inequality from Theorem 4 saying that for any fixed $z, r > 0$ we have

$$\int_{\mathbb{S}^{n-2}} |\mathcal{F}_{n-1}[f(\,\cdot\,,z)](r\omega)|^{q'}\,\mathrm{d}\sigma(\omega) \lesssim r^{-(n-1)\frac{q'}{p'}}\ \|f(\,\cdot\,,z)\|_{L^p(\mathbb{R}^{n-1})}^{q'} \tag{28}$$

since our assumptions imply $n \geq 3$ as well as

$$q \geq \left(\frac{n-2}{n}\,p'\right)' \text{ and } p' > \frac{2(n+1)}{n-1}. \tag{29}$$

Using this we obtain

$$\begin{aligned} A &\overset{(28)}{\lesssim} \left(\int_0^\infty \left(\int_0^\infty \left(\int_{\mu_1}^\infty r^{-(n-1)\frac{q'}{p'}}\ \|f(\,\cdot\,,z)\|_{L^p(\mathbb{R}^{n-1})}^{q'}\ \frac{r^{n-2}\,\mathrm{e}^{-q'(z+y)(r^2-\mu_1^2)^{\frac{1}{2}}}}{(r^2-\mu_1^2)^{\frac{q'}{2}}}\,\mathrm{d}r\right)^{\frac{1}{q'}} \mathrm{d}z\right)^{q} \mathrm{d}y\right)^{\frac{1}{q}} \\ &= \left(\int_0^\infty \left(\int_0^\infty \left(\int_{\mu_1}^\infty \frac{r^{n-2-(n-1)\frac{q'}{p'}}}{(r^2-\mu_1^2)^{\frac{q'}{2}}}\ \mathrm{e}^{-q'(z+y)(r^2-\mu_1^2)^{\frac{1}{2}}}\,\mathrm{d}r\right)^{\frac{1}{q'}} \|f(\,\cdot\,,z)\|_{L^p(\mathbb{R}^{n-1})}\,\mathrm{d}z\right)^{q} \mathrm{d}y\right)^{\frac{1}{q}} \end{aligned}$$

$$\leq \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} K_A(z+y)\, F(z)\,\mathrm{d}z \right)^q \mathrm{d}y \right)^{\frac{1}{q}}$$

where

$$K_A(w) := 1_{(0,\infty)}(w) \cdot \left( \int_{\mu_1}^{\infty} \frac{r^{n-2-(n-1)\frac{q'}{p'}}}{(r^2-\mu_1^2)^{\frac{q'}{2}}}\ \mathrm{e}^{-q'w(r^2-\mu_1^2)^{\frac{1}{2}}}\,\mathrm{d}r \right)^{\frac{1}{q'}},$$
$$F(w) := 1_{(0,\infty)}(w)\ \|f(\,\cdot\,,w)\|_{L^p(\mathbb{R}^{n-1})}.$$

In the Appendix we will prove

$$K_A \in L^{\frac{pq}{pq+p-q},\infty}(\mathbb{R}) \qquad \text{if} \quad 1 < p < \frac{2(n+1)}{n+3},\ \frac{2}{q} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{n}. \tag{30}$$

So the weak-space version of Young's convolution inequality (cf. [17, Theorem 1.4.25]) yields

$$A \lesssim \|K_A * F(-\,\cdot\,)\|_{L^q(\mathbb{R})} \lesssim \|F\|_{L^p(\mathbb{R})} = \left( \int_0^\infty \|f(\,\cdot\,,w)\|_{L^p(\mathbb{R}^{n-1})}^p\,\mathrm{d}w \right)^{\frac{1}{p}} = \|f\|_{L^p(\mathbb{R}^n)}.$$

<u>*Second term for $j=1$.*</u>

In place of (27), we have to demonstrate that our assumptions also imply

$$\left( \int_0^\infty \left( \int_{|\xi|\geq\mu_1} \mathrm{e}^{-q'y\sqrt{|\xi|^2-\mu_1^2}} \left| \frac{\mathcal{F}_n^- f(\xi,\nu_2(\xi))}{1+|\xi|} \right|^{q'} \mathrm{d}\xi \right)^{\frac{q}{q'}} \mathrm{d}y \right)^{\frac{1}{q}} \lesssim \|f\|_{L^p(\mathbb{R}^n)}. \tag{31}$$

By definition of $\nu_2(\xi)$, see (13), the estimate (31) follows once we have proved

$$\left( \int_0^\infty \left( \int_{|\xi|\geq\mu_2} \mathrm{e}^{-q'y\sqrt{|\xi|^2-\mu_1^2}} \left| \frac{\mathcal{F}_n^- f(\xi,\mathrm{i}\sqrt{|\xi|^2-\mu_2^2})}{1+|\xi|} \right|^{q'} \mathrm{d}\xi \right)^{\frac{q}{q'}} \mathrm{d}y \right)^{\frac{1}{q}} \lesssim \|f\|_{L^p(\mathbb{R}^n)},$$

$$\left( \int_0^\infty \left( \int_{\mu_1\leq|\xi|\leq\mu_2} \mathrm{e}^{-q'y\sqrt{|\xi|^2-\mu_1^2}} \left| \frac{\mathcal{F}_n^- f(\xi,\sqrt{\mu_2^2-|\xi|^2})}{1+|\xi|} \right|^{q'} \mathrm{d}\xi \right)^{\frac{q}{q'}} \mathrm{d}y \right)^{\frac{1}{q}} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

The first integral can be treated just as (27) with $\mu_2$ in place of $\mu_1$ because $|\xi| \geq \mu_2$ and $\mu_1 \leq \mu_2$ implies $\sqrt{|\xi|^2-\mu_1^2} \geq \sqrt{|\xi|^2-\mu_2^2}$. The second term will be interpreted as a surface integral over a sphere and can then be estimated using Fourier Restriction Theory:

$$\left( \int_0^\infty \left( \int_{\mu_1\leq|\xi|\leq\mu_2} \mathrm{e}^{-q'y\sqrt{|\xi|^2-\mu_1^2}} \left| \frac{\mathcal{F}_n^- f(\xi,\sqrt{\mu_2^2-|\xi|^2})}{1+|\xi|} \right|^{q'} \mathrm{d}\xi \right)^{\frac{q}{q'}} \mathrm{d}y \right)^{\frac{1}{q}}$$

$$\leq \left( \int_{|\xi|\leq\mu_2} \left| \mathcal{F}_n^- f(\xi, \sqrt{\mu_2^2 - |\xi|^2}) \right|^2 \, \mathrm{d}\xi \right)^{\frac{1}{2}} \left( \int_0^\infty \left( \int_{\mu_1\leq|\xi|\leq\mu_2} \mathrm{e}^{-\frac{2q'}{2-q'} y \sqrt{|\xi|^2-\mu_1^2}} \, \mathrm{d}\xi \right)^{\frac{q(2-q')}{2q'}} \mathrm{d}y \right)^{\frac{1}{q}}.$$

The finiteness of the iterated integral will be checked in the Appendix. Applying the Stein-Tomas Theorem (Theorem 3) we thus get

$$\left( \int_{|\xi|\leq\mu_2} \left| \mathcal{F}_n^- f(\xi, \sqrt{\mu_2^2 - |\xi|^2}) \right|^2 \, \mathrm{d}\xi \right)^{\frac{1}{2}} \lesssim \left\| \mathcal{F}_n^- f \right\|_{L^2(\mathbb{S}^{n-1}_{\mu_2})} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

Notice that this theorem applies since our assumption (29) implies $1 \leq p < \frac{2(n+1)}{n+3}$. Hence, we finally conclude $\|W_1\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$.

<u>*All terms for $j = 2$.*</u>

Here one has to show

$$\left( \int_0^\infty \left( \int_{|\xi|\geq\mu_2} \mathrm{e}^{-q'y\sqrt{|\xi|^2-\mu_2^2}} \left| \frac{\mathcal{F}_n^+ f(\xi, -\nu_1(\xi))}{1+|\xi|} \right|^{q'} \mathrm{d}\xi \right)^{\frac{q}{q'}} \mathrm{d}y \right)^{\frac{1}{q}} \lesssim \|f\|_{L^p(\mathbb{R}^n)},$$

$$\left( \int_0^\infty \left( \int_{|\xi|\geq\mu_2} \mathrm{e}^{-q'y\sqrt{|\xi|^2-\mu_2^2}} \left| \frac{\mathcal{F}_n^- f(\xi, \nu_2(\xi))}{\nu_2(\xi)} \right|^{q'} \mathrm{d}\xi \right)^{\frac{q}{q'}} \mathrm{d}y \right)^{\frac{1}{q}} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

The second (most singular) term can be estimated in exactly the same way as the most singular term for $j = 1$, replacing $\mu_1, \nu_1$ in the proof above by $\mu_2, \nu_2$, respectively. The estimate for the first is obtained as the one for (31); it is even easier to prove due to the smaller region of integration and $\mathrm{e}^{-z\sqrt{|\xi|^2-\mu_1^2}} \leq \mathrm{e}^{-z\sqrt{|\xi|^2-\mu_2^2}}$ for all $|\xi| \geq \mu_2$. Notice that it is not necessary to split the integral as we have done for (31). Once again we find $\|W_2\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$. □

## 6. Proof of Proposition 6

**Proposition 6.** Let $n = 2$ and $p, q' > 1, \frac{2}{3} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{n}$ or $p = 1, 3 < q < \infty$ or $q = \infty, 1 < p < \frac{3}{2}$. Then we have

$$\|W_1\|_{L^q(\mathbb{R}^n)} + \|W_2\|_{L^q(\mathbb{R}^n)} + \sup_{0<|\varepsilon|\leq 1} \left( \|W_{1,\varepsilon}\|_{L^q(\mathbb{R}^n)} + \|W_{2,\varepsilon}\|_{L^q(\mathbb{R}^n)} \right) \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

*Proof.* In order to avoid redundance, we only show how to estimate the most singular term for $W_1$. So we prove the estimate (27) for $n = 2$, namely

$$B := \left( \int_0^\infty \left( \int_{|\xi|\geq\mu_1} \mathrm{e}^{-q'y\sqrt{|\xi|^2-\mu_1^2}} \left| \frac{\mathcal{F}_n^+ f(\xi, -\nu_1(\xi))}{\nu_1(\xi)} \right|^{q'} \mathrm{d}\xi \right)^{\frac{q}{q'}} \mathrm{d}y \right)^{\frac{1}{q}} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

Proceeding as in the proof of Proposition 6 one finds with the aid of Minkowski's inequality

$$B = \left( \int_0^\infty \left( \int_{|\xi|\geq\mu_1} \left| \int_0^\infty \frac{\mathcal{F}_{n-1}[f(\,\cdot\,,z)](\xi)}{\sqrt{|\xi|^2-\mu_1^2}} \mathrm{e}^{-z\sqrt{|\xi|^2-\mu_1^2}} \,\mathrm{d}z \right|^{q'} \mathrm{e}^{-q'y\sqrt{|\xi|^2-\mu_1^2}} \,\mathrm{d}\xi \right)^{\frac{q}{q'}} \mathrm{d}y \right)^{\frac{1}{q}}$$

$$\leq \left( \int_0^\infty \left( \int_0^\infty \left( \int_{|\xi|\geq\mu_1} \left| \frac{\mathcal{F}_{n-1}[f(\,\cdot\,,z)](\xi)}{\sqrt{|\xi|^2-\mu_1^2}} \right|^{q'} \mathrm{e}^{-q'(z+y)\sqrt{|\xi|^2-\mu_1^2}} \,\mathrm{d}\xi \right)^{\frac{1}{q'}} \mathrm{d}z \right)^{q} \mathrm{d}y \right)^{\frac{1}{q}}.$$

Using first Hölder's inequality and then the Hausdorff-Young inequality we obtain

$$B \lesssim \left( \int_0^\infty \left( \int_0^\infty \|\mathcal{F}_{n-1}(f(\cdot,z))\|_{L^{p'}(\mathbb{R}^n)} \left( \int_{|\xi|\geq\mu_1} \frac{\mathrm{e}^{-\frac{pq}{q-p}(z+y)\sqrt{|\xi|^2-\mu_1^2}}}{(\sqrt{|\xi|^2-\mu_1^2})^{\frac{pq}{q-p}}} \,\mathrm{d}\xi \right)^{\frac{q-p}{pq}} \mathrm{d}z \right)^{q} \mathrm{d}y \right)^{\frac{1}{q}}$$

$$\lesssim \left( \int_0^\infty \left( \int_0^\infty \|f(\cdot,z)\|_{L^p(\mathbb{R}^n)} \underbrace{\left( \int_{\mu_1}^\infty \frac{r^{n-2}}{(r^2-\mu_1^2)^{\frac{pq}{2(q-p)}}} \mathrm{e}^{-\frac{pq}{q-p}(z+y)(r^2-\mu_1^2)^{\frac{1}{2}}} \,\mathrm{d}r \right)^{\frac{q-p}{pq}}}_{=:K_B(z+y)} \mathrm{d}z \right)^{q} \mathrm{d}y \right)^{\frac{1}{q}}.$$

In the Appendix we verify

$$\begin{aligned} &K_B \in L^{\frac{pq}{pq-q+p},\infty}(\mathbb{R}) && \text{if } \frac{2}{3} \leq \frac{1}{p} - \frac{1}{q} \leq 1, \\ &K_B \in L^{\frac{pq}{pq-q+p}}(\mathbb{R}) && \text{if } p=1, 3<q<\infty \text{ or } 1<p<\frac{3}{2}, q=\infty. \end{aligned} \tag{32}$$

So Young's convolution inequality implies $B \lesssim \|f\|_{L^p(\mathbb{R}^n)}$. □

## Appendix

**From the proof of Proposition 5 - Proof of (30).**

We have to estimate the function

$$K_A(w) = 1_{(0,\infty)}(w) \left( \int_{\mu_1}^\infty \frac{r^{n-2-(n-1)\frac{q'}{p'}}}{(r^2-\mu_1^2)^{\frac{q'}{2}}} \;\mathrm{e}^{-q'w(r^2-\mu_1^2)^{\frac{1}{2}}} \,\mathrm{d}r \right)^{\frac{1}{q'}}$$

$$= 1_{(0,\infty)}(w) \left( \int_0^\infty \varrho^{1-q'} (\varrho^2+\mu_1^2)^{\frac{n-3}{2}-\frac{(n-1)q'}{2p'}} \;\mathrm{e}^{-q'w\varrho} \,\mathrm{d}\varrho \right)^{\frac{1}{q'}}.$$

Notice that the integral is finite since we assumed $q \geq 3p > 3$ in (30), hence $1-q' > -1$. For some fixed $c \in (0,q')$ we find for all $w>0$

$$\begin{aligned} K_A(w) &\lesssim \left(\int_0^1 \varrho^{1-q'}\ \mathrm{e}^{-q'w\varrho}\,\mathrm{d}\varrho + \int_1^\infty \varrho^{n-2-q'-\frac{q'}{p'}(n-1)}\ \mathrm{e}^{-q'w\varrho}\,\mathrm{d}\varrho\right)^{\frac{1}{q'}} \\ &\lesssim \left((1+w)^{-2+q'} + w^{1-n+q'+(n-1)\frac{q'}{p'}}e^{-cw}\right)^{\frac{1}{q'}} \\ &\lesssim 1_{(0,1]}(w)w^{1+(n-1)\left(\frac{1}{p'}-\frac{1}{q'}\right)} + 1_{(1,\infty)}(w)w^{-\frac{2}{q'}+1} \\ &\lesssim 1_{(0,1]}(w)w^{1-(n-1)\left(\frac{1}{p}-\frac{1}{q}\right)} + 1_{(1,\infty)}(w)w^{-1+\frac{2}{q}}. \end{aligned}$$

Our assumptions from Proposition 5 on $p,q$ imply

$$-1+\frac{2}{q} \leq \frac{1}{p}-\frac{1}{q}-1 \leq 1-(n-1)\left(\frac{1}{p}-\frac{1}{q}\right)$$

and thus the claim (30) follows from $|K_A(w)| \lesssim |w|^{\frac{1}{p}-\frac{1}{q}-1}$.

**From the proof of Proposition 5 - Second term.**
We estimate the integral

$$\begin{aligned} \left(\int_0^\infty \left(\int_{\mu_1\leq|\xi|\leq\mu_2} \mathrm{e}^{-\frac{2q'}{2-q'}y\sqrt{|\xi|^2-\mu_1^2}}\,\mathrm{d}\xi\right)^{\frac{q(2-q')}{2q'}} \mathrm{d}y\right)^{\frac{1}{q}} &\lesssim \left(\int_0^\infty \left(\int_{\mu_1}^{\mu_2} \mathrm{e}^{-y\sqrt{r^2-\mu_1^2}}\,\mathrm{d}r\right)^{\frac{q-2}{2}} \mathrm{d}y\right)^{\frac{1}{q}} \\ &\lesssim \left(\int_0^\infty \left(\int_0^1 \mathrm{e}^{-y\varrho}\,\varrho\,\mathrm{d}\varrho\right)^{\frac{q-2}{2}} \mathrm{d}y\right)^{\frac{1}{q}} \\ &\lesssim \left(\int_0^\infty (1+y)^{2-q}\,\mathrm{d}y\right)^{\frac{1}{q}} \end{aligned}$$

and this is finite because of $q>3$ as we have seen above.

**From the proof of Proposition 6 - Proof of** (32)**.**

We have to estimate the function

$$\begin{aligned} K_B(w) &= 1_{(0,\infty)}(w)\left(\int_{\mu_1}^\infty \frac{r^{n-2}}{(r^2-\mu_1^2)^{\frac{pq}{2(q-p)}}}\mathrm{e}^{-\frac{pq}{q-p}w(r^2-\mu_1^2)^{\frac{1}{2}}}\,\mathrm{d}r\right)^{\frac{q-p}{pq}} \\ &= 1_{(0,\infty)}(w)\left(\int_0^\infty \mathrm{e}^{-\frac{pq}{q-p}w\varrho}\varrho^{1-\frac{pq}{q-p}}(\varrho^2+\mu_1^2)^{\frac{n-3}{2}}\,\mathrm{d}\varrho\right)^{\frac{q-p}{pq}}. \end{aligned}$$

For any fixed $c \in (0, \frac{pq}{q-p})$ and $w > 0$ we have

$$\begin{aligned} |K_B(w)| &\lesssim \left( \int_0^1 \mathrm{e}^{-\frac{pq}{q-p} w\varrho} \varrho^{1-\frac{pq}{q-p}} \,\mathrm{d}\varrho + \int_1^\infty \mathrm{e}^{-\frac{pq}{q-p} w\varrho} \varrho^{n-2-\frac{pq}{q-p}} \,\mathrm{d}\varrho \right)^{\frac{q-p}{pq}} \\ &\lesssim \left( (1+w)^{-2+\frac{pq}{q-p}} + w^{-n+1+\frac{pq}{q-p}} \mathrm{e}^{-cw} \right)^{\frac{q-p}{pq}} \\ &\lesssim w^{1+(n-1)\left(\frac{1}{q}-\frac{1}{p}\right)} 1_{(0,1]}(w) + w^{1-\frac{2}{p}+\frac{2}{q}} 1_{[1,\infty)}(w). \end{aligned}$$

The assumptions of Proposition 6 on $p, q$ imply

$$1 + (n-1)\left(\frac{1}{q} - \frac{1}{p}\right) \geq \frac{1}{p} - \frac{1}{q} - 1 \geq 1 - \frac{2}{p} + \frac{2}{q}. \tag{33}$$

and thus $|K_B(w)| \lesssim |w|^{\frac{1}{p}-\frac{1}{q}-1}$, whence $K_B \in L^{\frac{pq}{pq-q+p},\infty}(\mathbb{R})$. Moreover, in case $p = 1, 3 < q < \infty$ or $1 < p < \frac{3}{2}, q = \infty$ both inequalities in (33) are strict (recall $n = 2$) and we can even conclude $K_B \in L^{\frac{pq}{pq-q+p}}(\mathbb{R})$. This is precisely the claim of (32).

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## Conflict of interest

The authors declare that they have no conflict of interest.

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