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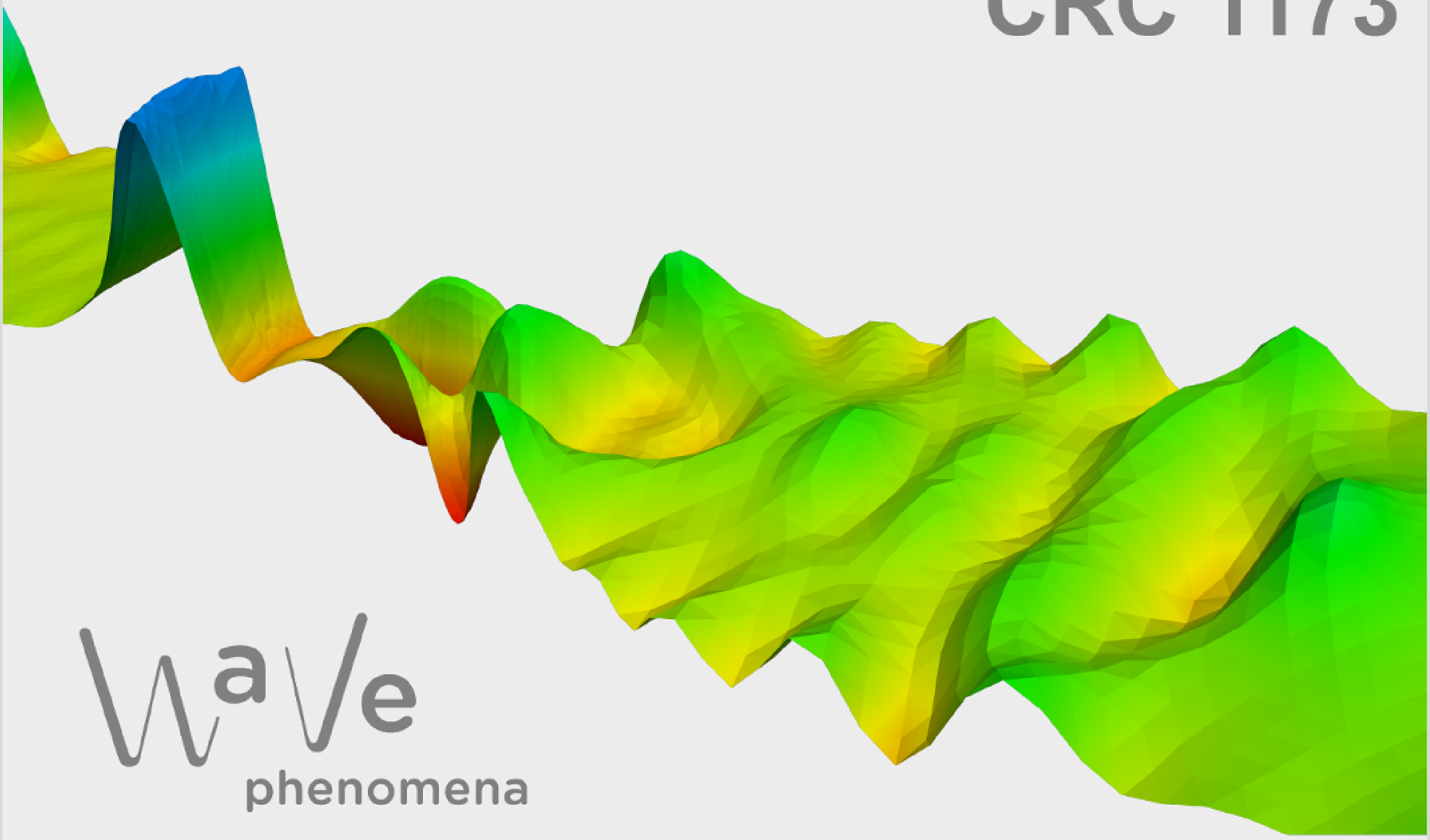

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FINITE ELEMENT DISCRETIZATION OF SEMILINEAR ACOUSTIC WAVE EQUATIONS WITH KINETIC BOUNDARY CONDITIONS*

MARLIS HOCHBRUCK† AND JAN LEIBOLD†

Abstract.
We consider isoparametric finite element discretizations of semilinear acoustic wave equations with kinetic boundary conditions and derive a corresponding error bound as our main result. The difficulty is that such problems are stated on domains with curved boundaries and this renders the discretizations nonconforming. Our approach is to provide a unified error analysis for nonconforming space discretizations for semilinear wave equations. In particular, we introduce a general, abstract framework for nonconforming space discretizations in which we derive a-priori error bounds in terms of interpolation, data and conformity errors. The theory applies to a large class of problems and discretizations that fit into the abstract framework.

The error bound for wave equations with kinetic boundary conditions is obtained from the general theory by inserting known interpolation and geometric error bounds into the abstract error result of the unified error analysis.

Key words. wave equation, dynamic boundary conditions, nonconforming space discretization, error analysis, a-priori error bounds, semilinear evolution equations, operator semigroups, isoparametric finite elements

AMS subject classifications. 65M12, 65M15, 65J08, 65M60

1. Introduction. The aim of this paper is to introduce and analyze finite element discretizations of semilinear acoustic wave equations with kinetic boundary conditions. Kinetic boundary conditions serve as an effective model for the interaction of waves with obstacles or boundaries that are covered by materials with distinctive elastic or damping properties where the wave length is large compared to the width of the boundary layer, see e.g. [11, Section 3.2]. We refer to [13, 14] and references therein for more information and analytical results about these equations.

Kinetic boundary conditions are a special case of dynamic boundary conditions that contain tangential derivatives and are intrinsically posed on domains with (piecewise) smooth and therefore possibly curved boundaries. Hence, most methods are applied on an approximated domain rendering the approximation nonconforming. This makes the error analysis much more involved. Such problems were addressed in [7, 6] where an unified error analysis (UEA) was introduced that allows to analyze nonconforming space discretizations of *linear* wave equations in a systematic way. The UEA yields an abstract error result that can be used to prove convergence rates for specific equations and discretizations by plugging in geometric and interpolation error results. Finite element discretizations of linear wave equations with linear kinetic boundary conditions are only specific examples fitting into the abstract framework. Several others are discussed in [7, 6].

To analyze semilinear wave equations with kinetic boundary conditions we present an extension of the UEA for linear problems [7, 6]. The main difficulty in discretizing and analyzing semilinear problems compared to linear ones is the discretization of the nonlinear term. This has to be done in such a way that the discretization preserves the Lipschitz continuity of the nonlinearity with a Lipschitz constant that is independent of the underlying mesh. Additionally it has to be shown, that the discretization error has the correct order of convergence.

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To the best of our knowledge this is the first error analysis for semilinear wave equations with kinetic boundary conditions. In [9] semilinear parabolic problems with dynamic boundary conditions were analyzed, but the techniques do not apply to the hyperbolic case.

The paper is organized as follows: In Section 2 we introduce semilinear acoustic wave equations with kinetic boundary conditions and their space discretization with isoparametric finite elements. Furthermore, we state the main result of the paper, namely a space discretization error bound of order p in the energy norm for order p isoparametric elements. In Section 3 we present the unified error analysis and prove abstract error bounds for nonconforming space discretizations of semilinear evolution equations. These bounds are used in Section 4 to provide error estimates for the discretizations of semilinear wave equations with kinetic boundary conditions. Finally, in Section 5 we conclude with a numerical experiment.

2. Wave equations with kinetic boundary conditions: problem statement.

In this section we introduce wave equations with kinetic boundary conditions. After formulating the equations in a suitable analytical setting, we present a finite element space discretization and the main error result which will be proven in Section 4. Wave equations with kinetic boundary conditions were already studied in [6] in the linear case.

2.1. Formulation of the equations.

Let $\Omega \subset \mathbb{R}^d, d \geq 2$, be a bounded domain with smooth boundary $\Gamma = \partial\Omega$. With Δ_Γ we denote the Laplace-Beltrami operator on Γ and with n the outer normal vector.

For semilinear wave equations with kinetic boundary conditions we seek $u\colon [0,T] \times \overline{\Omega} \to \mathbb{R}$ satisfying

$$\left\{\begin{aligned} u_{tt} + (\alpha_\Omega + \beta_\Omega \cdot \nabla)\, u_t - \Delta u &= \widetilde{f}_\Omega(t, \mathbf{x}, u), && \text{in } (0,T) \times \Omega, \\ u_{tt} + \partial_n u + (\alpha_\Gamma + \beta_\Gamma \cdot \nabla_\Gamma)\, u_t - \Delta_\Gamma u &= \widetilde{f}_\Gamma(t, \mathbf{x}, u), && \text{in } (0,T) \times \partial\Omega, \\ u(0,\mathbf{x}) = u^0(\mathbf{x}), \quad u_t(0,\mathbf{x}) &= v^0(\mathbf{x}), && \text{in } \overline{\Omega}. \end{aligned}\right. \tag{2.1}$$

More general problems, for instance, containing material parameters can be found in [6]. For the sake of presentation we omit these here and focus on the additional difficulties caused by the nonlinearity.

We require the following assumptions throughout the rest of the paper:

ASSUMPTION 2.1.

(a) *The nonlinearities satisfy*
  (i) $\widetilde{f}_\Omega \in C^1([0,T] \times \overline{\Omega} \times \mathbb{R}; \mathbb{R})$,
  (ii) $\widetilde{f}_\Gamma \in C^1([0,T] \times \Gamma \times \mathbb{R}; \mathbb{R})$,

and the following growth condition: There exist

$$\zeta_\Omega \begin{cases} < \infty, & d = 2, \\ \leq \frac{d}{d-2}, & d \geq 3, \end{cases} \quad \textit{and} \quad \zeta_\Gamma \begin{cases} < \infty, & d = 2,3, \\ \leq \frac{d-1}{d-3}, & d \geq 4, \end{cases} \tag{2.2}$$

such that for all $(t, \mathbf{x}, u) \in [0,T] \times \Omega \times \mathbb{R}$

$$\begin{aligned} |\widetilde{f}_\Omega(t,\mathbf{x},u)| &\leq C(1 + |u|^{\zeta_\Omega}), \\ |\nabla \widetilde{f}_\Omega(t,\mathbf{x},u)| &\leq C(1 + |u|^{\zeta_\Omega - 1}), \end{aligned} \tag{2.3}$$

and for all $(t, \mathbf{x}, u) \in [0,T] \times \Gamma \times \mathbb{R}$

$$\begin{aligned} |\widetilde{f}_\Gamma(t,\mathbf{x},u)| &\leq C(1 + |u|^{\zeta_\Gamma}), \\ |\nabla \widetilde{f}_\Gamma(t,\mathbf{x},u)| &\leq C(1 + |u|^{\zeta_\Gamma - 1}) \end{aligned}$$

hold true.

(b) *The coefficients* $\alpha_\Omega \in C(\overline{\Omega}), \beta_\Omega \in C^1(\overline{\Omega})^d, \alpha_\Gamma \in C(\Gamma)$ *and* $\beta_\Gamma \in C^1(\Gamma)^d$ *are non-negative and satisfy*

$$\alpha_\Omega - \frac{1}{2}\operatorname{div}\beta_\Omega \geq 0 \quad \textit{in } \Omega, \qquad \alpha_\Gamma + \frac{1}{2}(\beta_\Omega \cdot n - \operatorname{div}_\Gamma \beta_\Gamma) \geq 0 \quad \textit{on } \Gamma.$$

Because of (2.2) we have by the Sobolev embedding theorem, cf., e.g., [1, Theorem 4.12]

$$H^1(\Omega) \hookrightarrow L^{2\zeta_\Omega}(\Omega) \quad \text{and} \quad H^1(\Gamma) \hookrightarrow L^{2\zeta_\Gamma}(\Gamma). \tag{2.4}$$

We continue by presenting the weak formulation of (2.1). For this we define

$$\begin{aligned} H &\coloneqq L^2(\Omega) \times L^2(\Gamma), \\ V &\coloneqq H^1(\Omega;\Gamma) \coloneqq \{v \in H^1(\Omega) \mid \gamma(v) \in H^1(\Gamma)\} \subset H^1(\Omega) \times H^1(\Gamma), \end{aligned}$$

where γ denotes the trace operator. It can be proven, that V is a Hilbert space (cf. [8, Lemma 2.5]) which is densely embedded in H. Further we define bilinear forms $m\colon H \times H \to \mathbb{R}$, $b\colon V \times H \to \mathbb{R}$, and $a\colon V \times V \to \mathbb{R}$ via

$$\begin{aligned} m(v,\varphi) &= \int_\Omega v\varphi \,\mathrm{d}\mathbf{x} + \int_\Gamma v\varphi \,\mathrm{d}s, \\ b(v,\varphi) &= \int_\Omega (\alpha_\Omega v + \beta_\Omega \cdot \nabla v)\,\varphi \,\mathrm{d}\mathbf{x} + \int_\Gamma (\alpha_\Gamma v + \beta_\Gamma \cdot \nabla_\Gamma v)\,\varphi \,\mathrm{d}s, \\ a(v,\varphi) &= \int_\Omega \nabla v \nabla \varphi \,\mathrm{d}\mathbf{x} + \int_\Gamma \nabla_\Gamma v \nabla_\Gamma \varphi \,\mathrm{d}s, \end{aligned}$$

and the nonlinear function $f\colon [0,T] \times V \to H$ via

$$m\big(f(t,v),\varphi\big) = \int_\Omega \Big(\widetilde{f}_\Omega(t,\cdot,v(\cdot))\Big)\,\varphi \,\mathrm{d}\mathbf{x} + \int_\Gamma \Big(\widetilde{f}_\Gamma(t,\cdot,v(\cdot))\Big)\,\varphi \,\mathrm{d}s. \tag{2.5}$$

The weak formulation of (2.1) is a special case of the more general variational problem

$$\begin{aligned} m(u'',\varphi) + b(u',\varphi) + a(u,\varphi) &= m\big(f(t,u,u'),\varphi\big) \quad \text{for all } \varphi \in V, t \in (0,T], \\ u(0) = u^0, \qquad u'(0) &= v^0, \end{aligned} \tag{2.6}$$

where f does not depend on u'. The bilinear forms and the nonlinearity satisfy the following more general assumption.

ASSUMPTION 2.2.

(a) *The bilinear form* m *is a scalar product on* H *with induced norm* $\|\cdot\|_m$.

(b) $a\colon V \times V \to \mathbb{R}$ *is a symmetric bilinear form and there exists a constant* $c_G \geq 0$ *s.t.*

$$\widetilde{a} \coloneqq a + c_G m$$

is a scalar product on V *with induced norm* $\|\cdot\|_{\widetilde{a}}$.

(c) *The bilinear form* $b\colon V \times H \to \mathbb{R}$ *is continuous and there exists a* $\beta_{\text{qm}} \geq 0$ *s.t.*

$$b(v,v) + \beta_{\text{qm}}\|v\|_m^2 \geq 0 \quad \textit{for all } v \in V.$$

(d) *The nonlinearity* f *satisfies* $f \in C^1([0,T] \times V \times H; H)$ *and is locally Lipschitz-continuous on* $V \times H$ *with Lipschitz-constant* $L_{T,M}$*, i.e., for all* $t \in [0,T]$ *and* $x, y \in V \times H$ *with* $\|x\|_{V\times H}, \|y\|_{V\times H} \leq M$*:*

$$\|f(t,x) - f(t,y)\|_X \leq L_{T,M}\|x - y\|_{V\times H}.$$

In [6] was shown that for (2.1) we have $c_G = 1$ and $\beta_{\text{qm}} = 0$. The Lipschitz-continuity of f was proven in [10, Lemma 4.2], more general results can be found in [5].

We will see in Section 4 that under Assumption 2.2, (2.6) is (locally) well-posed.

2.2. Space discretization. To discretize (2.1) in space, we use the bulk-surface finite element method presented in [4]. This discretization was also considered in [6] for linear problems. The additional difficulty here is the discretization of the nonlinearity.

We start by giving a short summary of the bulk-surface finite element method, cf. [4, 6] for more details.

Bulk-surface finite element method. Let $\mathcal{T}_h$ be a consistent quasi-uniform mesh of isoparametric elements K of degree p with mesh width h. The discretized domain and its boundary are denoted by

$$\Omega_h \coloneqq \bigcup_{K \in \mathcal{T}_h} K \approx \Omega \quad \text{and} \quad \Gamma_h \coloneqq \partial\Omega_h.$$

We define the bulk and the surface finite element space of order $p \geq 1$ via

$$\begin{aligned} V_{h,p}^{\Omega} &\coloneqq \big\{ v_h \in C(\Omega_h) \mid v_h\big|_K = \widehat{v}_h \circ (F_K)^{-1} \text{ with } \widehat{v}_h \in \mathbb{P}_p(\widehat{K}) \text{ for all } K \in \mathcal{T}_h \big\}, \\ V_{h,p}^{\Gamma} &\coloneqq \big\{ \vartheta_h \in C(\Gamma_h) \mid \vartheta_h = v_h\big|_{\Gamma_h} \text{ with } v_h \in V_{h,p}^{\Omega} \big\}. \end{aligned}$$

Here $\mathbb{P}_p(\widehat{K})$ denotes the space of polynomial of degree p on a reference triangle $\widehat{K}$ and F_K is a transformation from $\widehat{K}$ to K. Since this discretization is nonconforming due to $\Omega_h \neq \Omega$, we need a lift operator to relate the analytical and the numerical solution. In [4] an elementwise smooth homeomorphism $G_h \colon \Omega_h \to \Omega$ with

$$G_h\big|_K \in C^{p+1}(K), \qquad \text{for all } p \leq k \text{ and } K \in \mathcal{T}_h$$

is constructed. This allows us to define lifted versions of $v_h \in V_{h,p}^{\Omega}$ and $\vartheta_h \in V_{h,p}^{\Gamma}$ as

$$v_h^{\ell} \coloneqq v_h \circ G_h^{-1} \qquad \text{and} \qquad \vartheta_h^{\ell} \coloneqq \vartheta_h \circ G_h^{-1}. \tag{2.7}$$

The mapping G_h is constructed in such a way, that $G_h(a_i) = a_i$, $i = 1, \ldots, N = \dim V_h$, where $a_1, \ldots, a_N \in \Omega_h$ are the nodes corresponding to the finite element discretization. This implies $v_h^{\ell}(a_i) = v_h(a_i)$ for $i = 1, \ldots, N$ and for all $v_h \in V_{h,p}^{\Omega}$. Furthermore, it was shown in [4] that there exist constants $c_{\Omega,\Omega_h}, c_{\Gamma,\Gamma_h}, C_{\Omega,\Omega_h}, C_{\Gamma,\Gamma_h} > 0$ independent of h s.t. for all $v_h \in V_{h,p}^{\Omega}$, $\vartheta_h \in V_{h,p}^{\Gamma}$ the following norm equivalences

$$\begin{aligned} c_{\Omega,\Omega_h} \|v_h\|_{L^2(\Omega_h)} &\leq \|v_h^{\ell}\|_{L^2(\Omega)} &&\leq C_{\Omega,\Omega_h} \|v_h\|_{L^2(\Omega_h)}, \\ c_{\Omega,\Omega_h} \|\nabla v_h\|_{L^2(\Omega_h)} &\leq \|\nabla v_h^{\ell}\|_{L^2(\Omega)} &&\leq C_{\Omega,\Omega_h} \|\nabla v_h\|_{L^2(\Omega_h)}, \\ c_{\Gamma,\Gamma_h} \|\vartheta_h\|_{L^2(\Gamma_h)} &\leq \|\vartheta_h^{\ell}\|_{L^2(\Gamma)} &&\leq C_{\Gamma,\Gamma_h} \|\vartheta_h\|_{L^2(\Gamma_h)}, \\ c_{\Gamma,\Gamma_h} \|\nabla_\Gamma \vartheta_h\|_{L^2(\Gamma_h)} &\leq \|\nabla_\Gamma \vartheta_h^{\ell}\|_{L^2(\Gamma)} &&\leq C_{\Gamma,\Gamma_h} \|\nabla_\Gamma \vartheta_h\|_{L^2(\Gamma_h)} \end{aligned} \tag{2.8}$$

holds true.

With $I_{h,\Omega} \colon C(\overline{\Omega}) \to V_{h,p}^{\Omega}$ and $I_{h,\Gamma} \colon C(\Gamma) \to V_{h,p}^{\Gamma}$ we denote the nodal interpolation operator in Ω and on Γ, respectively. The interpolation operators satisfy

$$\begin{aligned} \|v - (I_{h,\Omega} v)^{\ell}\|_{L^2(\Omega)} + h\|v - (I_{h,\Omega} v)^{\ell}\|_{H^1(\Omega)} &\leq C h^{r+1} \|v\|_{H^{r+1}(\Omega)}, \\ \|\vartheta - (I_{h,\Gamma} \vartheta)^{\ell}\|_{L^2(\Gamma)} + h\|\vartheta - (I_{h,\Gamma} \vartheta)^{\ell}\|_{H^1(\Gamma)} &\leq C h^{r+1} \|\vartheta\|_{H^{r+1}(\Gamma)}, \end{aligned} \tag{2.9}$$

for all $v \in H^{r+1}(\Omega)$ and $\vartheta \in H^{r+1}(\Gamma)$ with $1 \leq r \leq p$, cf. [4, Prop. 5.4]. By construction, the nodes on the surface coincide with the bulk nodes and therefore we have

$$\gamma(I_{h,\Omega} v) = I_{h,\Gamma} \gamma(v) \qquad \text{for all } v \in C(\overline{\Omega}).$$

The semidiscretized equation. As finite element space we choose $V_h = V_{h,p}^{\Omega}$. The discretizations $m_h, b_h, a_h \colon V_h \times V_h \to \mathbb{R}$ of m, b, and a are defined via

$$\begin{aligned} m_h(v_h, \varphi_h) &\coloneqq \int_{\Omega_h} v_h \varphi_h \,\mathrm{d}\mathbf{x} + \int_{\Gamma_h} v_h \varphi_h \,\mathrm{d}s, \\ b_h(v_h, \varphi_h) &\coloneqq \int_{\Omega_h} (I_{h,\Omega}\alpha_\Omega v_h + I_{h,\Omega}\beta_\Omega \cdot \nabla v_h)\, \varphi_h \,\mathrm{d}\mathbf{x} \\ &\quad + \int_{\Gamma_h} (I_{h,\Gamma}\alpha_\Gamma v_h + I_{h,\Gamma}\beta_\Gamma \cdot \nabla_\Gamma v_h)\, \varphi_h \,\mathrm{d}s, \\ a_h(v_h, \varphi_h) &\coloneqq \int_{\Omega_h} \nabla v_h \nabla \varphi_h \,\mathrm{d}\mathbf{x} + \int_{\Gamma_h} \nabla_{\Gamma_h} u_h \nabla_{\Gamma_h} \varphi_h \,\mathrm{d}s, \end{aligned}$$

and we discretize the nonlinearity $f_h \colon [0,T] \times V_h \to H_h$ via

$$\begin{aligned} m_h\big(f_h(t, v_h), \varphi_h\big) &\coloneqq \int_{\Omega_h} I_{h,\Omega} \widetilde{f}_\Omega\big(t, \cdot, v_h^\ell(\cdot)\big)(\mathbf{x}) \varphi_h(\mathbf{x}) \,\mathrm{d}\mathbf{x} \\ &\quad + \int_{\Gamma_h} I_{h,\Gamma} \widetilde{f}_\Gamma\big(t, \cdot, v_h^\ell(\cdot)\big)(\mathbf{x}) \varphi_h(\mathbf{x}) \,\mathrm{d}s \end{aligned} \tag{2.10}$$

for all $\varphi_h \in V_h$.

REMARK 2.3. The nodal interpolation only requires function evaluations in the nodes $a_1, \dots, a_N$. Since these are invariant under the lift operator, the computation of v_h^ℓ is not necessary. It is only needed for the definition of f_h since the interpolation operator acts on functions over Ω.

The discretized version of (2.6) is then given as a special case of

$$\begin{aligned} m_h(u_h'', \varphi_h) + b_h(u_h', \varphi_h) + a_h(u_h, \varphi_h) &= m_h\big(f_h(t, u_h, u_h'), \varphi_h\big), \\ u_h(0) = u_h^0, \qquad u_h'(0) &= v_h^0. \end{aligned} \tag{2.11}$$

The discrete quantities then satisfy similar assumptions as their continuous counterparts:

ASSUMPTION 2.4.

(a) *The bilinear form $a_h \colon V_h \times V_h \to \mathbb{R}$ is symmetric and there exists a constant $\widehat{c}_G \geq 0$ s.t.*

$$\widetilde{a}_h \coloneqq a_h + \widehat{c}_G m_h$$

is a scalar product on V_h with induced norm $\|\cdot\|_{\widetilde{a}_h}$.

(b) *The bilinear form m_h is also a scalar product on V_h. We denote V_h equipped with this scalar product m_h by H_h and the induced norm by $\|\cdot\|_{m_h}$*

(c) *The bilinear form $b_h \colon V_h \times H_h \to \mathbb{R}$ is bounded independent of h and there exists a $\widehat{\beta}_{\text{qm}} \geq 0$ s.t.*

$$b_h(v_h, v_h) + \widehat{\beta}_{\text{qm}} \|v_h\|_{m_h}^2 \geq 0 \quad \textit{for all } v_h \in V_h.$$

(d) *The nonlinearity $f_h \colon [0,T] \times V_h \times H_h \to H_h$ is locally Lipschitz-continuous on $V_h \times H_h$ with constant $\widehat{L}_{T,M}$.*

(e) *There exists a constant $\widehat{C}_{H,V} > 0$ s.t. $\|v_h\|_{m_h} \leq \widehat{C}_{H,V} \|v_h\|_{\widetilde{a}_h}$ for all $v_h \in V_h$.*

All constants in this assumption should be independent of h.

REMARK 2.5. In a finite dimensional space, all norms are equivalent. The crucial point in the last assumption is, that the constants are independent of h, which corresponds to the continuous embedding $V \hookrightarrow H$.

In our specific example we have $c_G = 1, \widehat{c}_{\text{qm}} = 0, \widehat{C}_{H,V} = 1$, cf. [6]. The Lipschitz-continuity of f_h is proven in the following lemma.

LEMMA 2.6. *The discretized nonlinearity $f_h \colon [0,T] \times V_h \to H_h$ defined in (2.10) is locally Lipschitz-continuous on V_h with Lipschitz constant*

$$\widehat{L}_{T,M} = C\left(\sigma(\Omega)^{\frac{\zeta_\Omega - 1}{2\zeta_\Omega}} + \sigma(\Gamma)^{\frac{\zeta_\Gamma - 1}{2\zeta_\Gamma}} + 2M^{\zeta_\Omega - 1} + 2M^{\zeta_\Gamma - 1}\right),$$

i.e., for all $u_h, v_h \in V_h$ with $\|u_h\|_{\widetilde{a}_h}, \|v_h\|_{\widetilde{a}_h} \leq M$, and for all $t \in [0,T]$,

$$\|f_h(t,u_h) - f_h(t,v_h)\|_{m_h} \leq \widehat{L}_{T,M}\|u_h - v_h\|_{\widetilde{a}_h}.$$

The constant C is independent of h, $\sigma(\Omega)$ and $\sigma(\Gamma)$ denote the measure of Ω and Γ, respectively, and ζ_Ω and ζ_Γ are defined in (2.2).

Proof. Let $M > 0, t < T$, and $u_h, v_h \in V_h$ s.t. $\|u_h\|_{\widetilde{a}_h}, \|v_h\|_{\widetilde{a}_h} < M$. With the definition of f_h in (2.10) and the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}\|f_h(t,u_h) - f_h(t,v_h)\|_{m_h} &= \sup_{\|\varphi_h\|_{m_h}=1} m_h\big(f_h(t,u_h) - f_h(t,v_h), \varphi_h\big)\\ &\leq \|I_{h,\Omega}\widetilde{f}_\Omega(t,\cdot,u_h^\ell(\cdot)) - I_{h,\Omega}\widetilde{f}_\Omega(t,\cdot,v_h^\ell(\cdot))\|_{L^2(\Omega_h)}\\ &\quad + \|I_{h,\Gamma}\widetilde{f}_\Gamma(t,\cdot,u_h^\ell(\cdot)) - I_{h,\Gamma}\widetilde{f}_\Gamma(t,\cdot,v_h^\ell(\cdot))\|_{L^2(\Gamma_h)}.\end{aligned}$$

In the following we show a bound for the first term. The second one can be bounded analogously.

This proof is more involved than for the continuous nonlinearity since the appearing interpolation operator is not continuous with respect to L^2. To work around this problem, we use discrete L^q-norms defined via

$$\|\!|v_h|\!\|_q := h^{\frac{d}{q}}\left(\sum_{i=1}^N |v_h(a_i)|^q\right)^{\frac{1}{q}}. \tag{2.12}$$

Because of the scaling with $h^{\frac{d}{q}}$ and the mesh regularity we have that the norm $\|\!|\cdot|\!\|_q$ is equivalent to $\|\cdot\|_{L^q(\Omega_h)}$ on $V_{h,p}^\Omega$ for all $q \in [2,\infty)$ with constants independent of h. This is well known for $q = 2$. The generalization to $q \neq 2$ is straightforward, cf. [10, Lemma 5.2]. Combining this with the Sobolev embedding theorem (2.4) (with Ω_h instead of Ω), we have

$$\|\!|v_h|\!\|_{2\zeta_\Omega} \lesssim C\|v_h\|_{L^{2\zeta_\Omega}(\Omega_h)} \lesssim C\|v_h\|_{H^1(\Omega_h)} \qquad \text{for all } v_h \in V_h \tag{2.13}$$

with constants C independent of h.

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The definition of the discrete norms, the growth conditions (2.3), and (2.13) yield

$$
\begin{aligned}
&\|I_{h,\Omega}\widetilde{f}_\Omega(t,\cdot,u_h^\ell(\cdot)) - I_{h,\Omega}\widetilde{f}_\Omega(t,\cdot,v_h^\ell(\cdot))\|_{L^2(\Omega_h)}^2 \\
&\le C |\!|\!| I_{h,\Omega}\widetilde{f}_\Omega(t,\cdot,u_h(\cdot)) - I_{h,\Omega}\widetilde{f}_\Omega(t,\cdot,v_h(\cdot)) |\!|\!|_2^2 \\
&= Ch^d \left(\sum_{i=1}^N \left| \widetilde{f}_\Omega(t,a_i,u_h(a_i)) - \widetilde{f}_\Omega(t,a_i,v_h(a_i)) \right|^2 \right) \\
&= Ch^d \left(\sum_{i=1}^N \left| \big(u_h(a_i) - v_h(a_i)\big) \int_0^1 \partial_3 \widetilde{f}_\Omega\big(t,a_i,v_h(a_i) + s(u_h(a_i)-v_h(a_i))\big)\,\mathrm{d}s \right|^2 \right) \\
&\le Ch^{\frac{d(\zeta_\Omega-1)}{\zeta_\Omega}} \left(\sum_{i=1}^N \left(1 + (|u_h(a_i)| + |v_h(a_i)|)^{\zeta_\Omega-1}\right)^{\frac{2\zeta_\Omega}{\zeta_\Omega-1}} \right)^{\frac{\zeta_\Omega-1}{\zeta_\Omega}} |\!|\!| u_h - v_h |\!|\!|_{2\zeta_\Omega}^2 \\
&\le C \left(|\!|\!| 1 |\!|\!|_{\frac{2\zeta_\Omega}{\zeta_\Omega-1}} + |\!|\!| u_h |\!|\!|_{2\zeta_\Omega}^{\zeta_\Omega-1} + |\!|\!| v_h |\!|\!|_{2\zeta_\Omega}^{\zeta_\Omega-1} \right)^2 |\!|\!| u_h - v_h |\!|\!|_{2\zeta_\Omega}^2 \\
&\le C \left(\|1\|_{L^{\frac{2\zeta_\Omega}{\zeta_\Omega-1}}(\Omega_h)} + \|u_h\|_{L^{2\zeta_\Omega}(\Omega_h)}^{\zeta_\Omega-1} + \|v_h\|_{L^{2\zeta_\Omega}(\Omega_h)}^{\zeta_\Omega-1} \right) \|u_h - v_h\|_{H^1(\Omega_h)}^2 \\
&\le C \left(\sigma(\Omega)^{\frac{\zeta_\Omega-1}{2\zeta_\Omega}} + 2M^{\zeta_\Omega-1} \right) \|u_h - v_h\|_{H^1(\Omega_h)}^2,
\end{aligned}
$$

where we additionally used the bound

$$\sigma(\Omega_h) \le C\sigma(\Omega)$$

which is satisfied independent of h. □

2.3. Main result. We can now state the main result of the paper, namely the error bound for the bulk-surface discretization of wave equations with kinetic boundary conditions. The proof will be done in Section 4.

THEOREM 2.7. *Let* $\Gamma \in C^{p+1}, \alpha_\Omega \in H^p(\overline{\Omega}), \beta_\Omega \in H^p(\overline{\Omega})^d, \alpha_\Gamma \in H^p(\Gamma)$, *and* $\beta_\Gamma \in H^p(\Gamma)^d$. *Furthermore let* u *be a solution of* (2.1) *on* $[0,T]$ *with*

(a) $u \in C^2\big([0,T]; H^2(\Omega;\Gamma)\big) \cap L^\infty\big([0,T]; H^{\max\{4,p+2\}}(\Omega;\Gamma)\big)$,
(b) $u' \in L^\infty\big([0,T]; H^{p+1}(\Omega;\Gamma)\big)$, *and*
(c) $u'' \in L^\infty\big([0,T]; H^p(\Omega;\Gamma)\big)$.

Then there exist $h^*, M > 0$ *s.t. for all* $h < h^*$, *the solution* u_h *of* (2.11) *exists on* $[0,T]$ *and satisfies the error bound*

$$\|u_h^\ell(t) - u(t)\|_{H^1(\Omega;\Gamma)} + \|(u_h')^\ell(t) - u'(t)\|_{L^2(\Omega)\times L^2(\Gamma)} \le C\mathrm{e}^{(\widehat{L}_{T,M}+\frac{1}{2})t}(1+t)h^p \tag{2.14}$$

with $\widehat{L}_{T,M}$ *from Lemma* 2.6 *and a constant* C *independent of* h *and* t.

3. Unified error analysis (UEA) for nonconforming discretizations. In this section we present the UEA for a general class of nonconforming space discretizations of semilinear wave equations in time-domain. It is a tool that provides a priori error bounds in terms of interpolation, data and conformity errors of the method. These bounds can be used to derive convergence rates for a large class of problems in a simple, systematic and modular way. The idea is to treat wave equations abstractly as evolution equations in Hilbert spaces and their space discretizations as differential equations in finite dimensional Hilbert spaces and to perform the error analysis in this abstract setting.

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Here we briefly recall the setting used in [7, 6] and extend it to the semilinear case. As in [7] we start by proving an error bound for discretizations of first order evolution equations in Section 3.1 and then use this result to prove error bounds for second-order equations in Section 3.2. This result will then be used in the next section to prove Theorem 2.7.

More applications of the unified error analysis can be found in [7].

3.1. Semilinear evolution equations with monotone operators. We start by stating an abstract evolution equation and introduce a general space discretization afterwards.

The continuous problem. Let X be a Hilbert space with scalar product p. We consider the evolution equation

$$\begin{aligned} x'(t) + Sx(t) &= g(t, x(t)), \qquad t \in (0,T], \\ x(0) &= x^0. \end{aligned} \tag{3.1}$$

ASSUMPTION 3.1.

(a) *The linear operator $S\colon D(S) \to X$ is the generator of a C_0-semigroup with*

$$\left\| \mathrm{e}^{-tS} \right\|_{X \leftarrow X} \leq \mathrm{e}^{c_{\mathrm{qm}} t}. \tag{3.2}$$

(b) *The nonlinearity $g \in C^1([0,T] \times X; X)$ is locally Lipschitz continuous w.r.t. the second component.*

The following classical well-posedness result can be found in [12], for example:

LEMMA 3.2. *If Assumption 3.1 holds true, then (3.1) is locally well-posed, i.e., for every $x^0 \in X$ there exists $t^*\big(x^0\big) > 0$ s.t. for all $T < t^*\big(x^0\big)$, (3.1) has a unique solution*

$$x \in C^1\big([0,T]; X\big) \cap C\big([0,T]; D(S)\big).$$

Abstract space discretization. We consider a general space discretization of (3.1) and show an abstract error result for a large class of equations and discretizations.

Let X_h be a finite dimensional Hilbert space with scalar product p_h. In this space we seek the numerical approximation x_h. Furthermore let $S_h \in \mathcal{L}(X_h, X_h)$ and $g_h\colon [0,T] \times X_h \to X_h$ be discretizations of S and g, respectively. Similar to their continuous counterparts we require that S_h and g_h satisfy Assumption 3.1 with X_h instead of X and constants $\widehat{c}_{\mathrm{qm}}, \widehat{L}_{T,M}$ independent of h.

Then the discretized version of the evolution equation (3.1) is given by

$$\begin{aligned} x_h'(t) + S_h x_h(t) &= g_h(t, x_h(t)), \qquad t \in (0,T], \\ x_h(0) &= x_h^0. \end{aligned} \tag{3.3}$$

Due to the Picard–Lindelöf theorem, (3.3) is locally well-posed and we denote the maximal existence time of the solution by $t_h^*(x_h^0)$.

Error analysis. Our framework allows us to treat nonconforming space discretizations, where $X_h \nsubseteq X$. To relate the continuous and discrete quantities we therefore assume that there exists a *lift operator* $\mathcal{L}_h\colon X_h \to X$ which satisfies

$$\|\mathcal{L}_h y_h\|_X \leq C_X \|y_h\|_{X_h} \qquad \text{for all } y_h \in X_h \tag{3.4}$$

with C_X independent of h. We then define the lifted discrete space

$$X_h^\ell \coloneqq \mathcal{L}_h(X_h) \subset X.$$

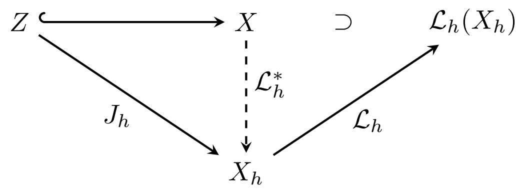

FIG. 3.1. *Overview of spaces and operators, cf. [7].*

Let $\mathcal{L}_h^* \colon X \to X_h$ be the adjoint of the lift operator, i.e.,

$$p_h\big(\mathcal{L}_h^* y, y_h\big) = p\big(y, \mathcal{L}_h y_h\big) \qquad \text{for all } y \in X, y_h \in X_h.$$

Furthermore, for a Hilbert space Z which is densely and continuously embedded in X we make use of a *reference operator* $J_h \in \mathcal{L}(Z, X_h)$ satisfying

$$\|J_h\|_{X_h \leftarrow Z} \leq C_J \tag{3.5}$$

with a constant C_J independent of h. The reference operator should satisfy $\mathcal{L}_h J_h z \approx z$ for all $z \in Z$ and could, e.g., be an interpolation or a projection operator. Figure 3.1 illustrates the operators between the spaces.

Finally we define linear remainder operator

$$R_h \coloneqq \mathcal{L}_h^* S - S_h J_h \colon D(S) \cap Z \to X_h$$

and the nonlinear remainder operator $r_h \colon [0,T] \times Z \to X_h$ via

$$r_h(t,z) \coloneqq \mathcal{L}_h^* g(t,z) - g_h(t, J_h z). \tag{3.6}$$

If the solution of the discretized equation (3.3) is bounded, we can state an error bound in terms of the approximation error of x^0, of x and x', and of the remainder operators R_h, r_h.

We make the following regularity assumption on the continuous solution:

ASSUMPTION 3.3. *The solution x of (3.1) satisfy $x \in C^1\big([0, t^*(x^0)); Z\big)$.*

THEOREM 3.4. *Let Assumption 3.3 be satisfied, $T < \min\Big\{t^*\big(x^0\big), t_h^*\big(x_h^0\big)\Big\}$, and*

$$M_h = \max\Big\{C_J \|x\|_{L^\infty([0,T];Z)}, \|x_h\|_{L^\infty([0,T];X_h)}\Big\}.$$

Then, for all $t \in [0,T]$, the lifted discrete solution of (3.3) satisfies the error bound

$$\|\mathcal{L}_h x_h(t) - x(t)\|_X \leq C\mathrm{e}^{(\widehat{L}_{M_h} + \widehat{c}_{\mathrm{qm}})t} E_h(t) + \|(\mathrm{I} - \mathcal{L}_h J_h)x(t)\|_X \tag{3.7}$$

with

$$\begin{aligned} E_h(t) = \Big\|x_h^0 - J_h x^0\Big\|_{X_h} &+ t\big\|(\mathcal{L}_h^* - J_h)x'\big\|_{L^\infty([0,T];X_h)} \\ &+ t\|R_h x\|_{L^\infty([0,T];X_h)} + t\|r_h(\cdot, x(\cdot))\|_{L^\infty([0,T];X_h)}. \end{aligned}$$

Proof. The proof consists of four steps.

(I) Splitting of the error: We split the error into

$$\mathcal{L}_h x_h - x = \mathcal{L}_h e_h + e_{J_h}$$

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with the discrete error

$$e_h = x_h - J_h x \in X_h$$

and the reference error

$$e_{J_h} = (\mathcal{L}_h J_h - \mathrm{I})x.$$

This splitting yields

$$\|\mathcal{L}_h x_h - x\|_X \le C_X \|e_h\|_{X_h} + \|(\mathcal{L}_h J_h - \mathrm{I})x\|_X \tag{3.8}$$

where the second term only depends on the choice of the reference and the lift operator.

(II) Derivation of an evolution equation for the error e_h: Since $x \in C^1([0,T], Z)$ and $J_h \in \mathcal{L}(Z, X_h)$ we have $e_h \in C^1([0,T]; X_h)$ and

$$e_h' = x_h' - J_h x' = (x_h' - \mathcal{L}_h^* x') + (\mathcal{L}_h^* - J_h)x'.$$

Using the continuous and the discrete equations (3.1) and (3.3) we can rewrite the first term as

$$\begin{aligned} x_h' - \mathcal{L}_h^* x' &= -S_h x_h + g_h(\cdot, x_h) - \mathcal{L}_h^*(-Sx + g(\cdot, x)) \\ &= -S_h e_h + g_h(\cdot, x_h) - \mathcal{L}_h^* g(\cdot, x) + (\mathcal{L}_h^* S - S_h J_h)x. \end{aligned}$$

So, we end up with the following equation for the discrete error:

$$\begin{aligned} e_h' + S_h e_h &= (\mathcal{L}_h^* - J_h)x' + R_h x + g_h(\cdot, x_h) - \mathcal{L}_h^* g(\cdot, x) \\ &= (\mathcal{L}_h^* - J_h)x' + R_h x - r_h(\cdot, x) + g_h(\cdot, x_h) - g_h(\cdot, J_h x) \\ &=: d_h. \end{aligned}$$

Hence e_h satisfies a linear evolution equation in X_h.

(III) Stability: By the variation-of-constants formula we have

$$\begin{aligned} \|e_h(t)\|_{X_h} &\le \|\mathrm{e}^{-tS_h} e_h(0)\|_{X_h} + \int_0^t \|\mathrm{e}^{-(t-s)S_h} d_h(s)\|_{X_h}\,\mathrm{d}s \\ &\le \mathrm{e}^{\widehat{c}_{\mathrm{qm}}t}\|e_h(0)\|_{X_h} + \mathrm{e}^{\widehat{c}_{\mathrm{qm}}t}\int_0^t \mathrm{e}^{-\widehat{c}_{\mathrm{qm}}s}\|d_h(s)\|_{X_h}\,\mathrm{d}s. \end{aligned} \tag{3.9}$$

Using the Lipschitz-continuity of g_h and the definition of the nonlinear remainder (3.6), we are able to bound the defect d_g by

$$\begin{aligned} \|d_h(s)\|_{X_h} \le \|(\mathcal{L}_h^* - J_h)x'(s)\|_{X_h} &+ \|R_h x(s)\|_{X_h} \\ &+ \|r_h(s, x(s))\|_{X_h} + \widehat{L}_{T,M_h}\|e_h(s)\|_{X_h}. \end{aligned} \tag{3.10}$$

(IV) Abstract error estimate: Inserting (3.10) into (3.9) yields

$$\begin{aligned} \mathrm{e}^{-\widehat{c}_{\mathrm{qm}}t}\|e_h(t)\|_{X_h} &\le \|e_h(0)\|_{X_h} + t\|(\mathcal{L}_h^* - J_h)x'\|_{L^\infty([0,T];X_h)} + t\|R_h x\|_{L^\infty([0,T];X_h)} \\ &\quad + t\|r_h(\cdot, x(\cdot))\|_{L^\infty([0,T];X_h)} + \widehat{L}_{T,M_h}\int_0^t \mathrm{e}^{-\widehat{c}_{\mathrm{qm}}s}\|e_h(s)\|_{X_h}\,\mathrm{d}s \\ &= E_h(t) + \widehat{L}_{T,M_h}\int_0^t \mathrm{e}^{-\widehat{c}_{\mathrm{qm}}s}\|e_h(s)\|_{X_h}\,\mathrm{d}s. \end{aligned}$$

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With a Gronwall Lemma we finally obtain

$$\|e_h(t)\|_{X_h} \le \mathrm{e}^{(\widehat{L}_{T,M_h}+\widehat{c}_{\mathrm{qm}})t} E_h(t).$$

Together with (3.8) this proves the error bound (3.7). ∎

The following corollary shows under additional consistency assumptions, that the discretized equation remains bounded for sufficiently small h, and that the discrete solution converges to the continuous one.

COROLLARY 3.5. *Let Assumption 3.3 be satisfied and $T < t^*\big(x^0\big)$. Moreover, assume that*

$$\lim_{h\to 0} E_h(t) \to 0 \quad \text{for all } t \in [0,T].$$

Then there exists $h^* > 0$, such that x_h exists for all $h < h^*$ on $[0,T]$ with

$$\|x_h\|_{L^\infty([0,T];X_h)} \le M := 2C_J \|x\|_{L^\infty([0,T];Z)}.$$

Furthermore the error bound (3.7) holds true with $M_h = M$.

If additionally

$$\lim_{h\to 0} \|(\mathrm{I} - \mathcal{L}_h J_h)x(t)\|_X \to 0 \quad \text{for all } t \in [0,T]$$

holds true, then the lifted numerical solution converges, i.e.,

$$\|\mathcal{L}_h x_h(t) - x(t)\|_X \overset{h\to 0}{\longrightarrow} 0, \qquad t \in [0,T].$$

Proof. We only have to show that x_h exists for all $h < h^*$ on $[0,T]$ with

$$\|x_h\|_{L^\infty([0,T];X_h)} \le M.$$

The other assertion then follows immediately from Theorem 3.4.

We define

$$T_h := \sup\Big\{ t \in \big(0, t_h^*\big(x_h^0\big)\big) \mid \|x_h\|_{L^\infty([0,t];X_h)} \le M \Big\}$$

as the maximal time, for which the discrete solution stays bounded by M. Clearly we have $T_h < t_h^*(x_h^0)$ and further by Theorem 3.4, for all $t \le \min\{T, T_h\}$

$$\begin{aligned} \|x_h(t)\|_{X_h} &\le \|x_h(t) - J_h x(t)\|_{X_h} + \|J_h x(t)\|_{X_h} \\ &\le \|e_h(t)\|_{X_h} + \frac{M}{2} \\ &\le C\mathrm{e}^{(\widehat{L}_{T,M}+\widehat{c}_{\mathrm{qm}})t} E_h(t) + \frac{M}{2} \overset{h\to 0}{\longrightarrow} \frac{M}{2}. \end{aligned}$$

Hence there exists a $h^* > 0$ s.t. $\|x_h(t)\|_{X_h} \le \frac{3}{4}M$ for all $h < h^*$ and $t \le \min\{T, T_h\}$. Since x_h is continuous and by the definition of T_h we thus get

$$t_h^*(x_h^0) > T_h > T \qquad \text{and} \qquad \|x_h\|_{L^\infty([0,T];X_h)} < M$$

for all $h < h^*$. ∎

3.2. Second-order semilinear wave-type equations. Next, we apply the results of Section 3.1 to general second-order wave equations. Again, we start by stating the framework.

The continuous problem. Let $V \overset{d}{\hookrightarrow} H$ be Hilbert spaces. We consider the variational differential equation (2.6) as a prototype for weak formulations of second-order wave equations and assume that Assumption 2.2 holds true. By the dense embedding of the Hilbert spaces there exists a constant $C_{H,V} > 0$ s.t.

$$\|v\|_m \leq C_{H,V} \|v\|_{\widetilde{a}} \quad \text{for all } v \in V.$$

In order to reformulate the problem as an evolution equation on H we define operators $A\colon D(A) \to H$ and $B\colon V \to H$ corresponding to a and b via

$$\begin{aligned} m\big(Av, w\big) &= a\big(v, w\big), \qquad \text{for all } v \in D(A), w \in V, \\ m\big(Bv, w\big) &= b\big(v, w\big), \qquad \text{for all } v \in V, w \in H, \end{aligned}$$

with

$$D(A) = \big\{ v \in V \mid \exists C = C(v) > 0 \text{ s.t.} \, \forall w \in V : |a\big(v, w\big)| \leq C \|w\|_m \big\}.$$

(2.6) then reads: Find $u \in C^2\big([0,T]; H\big) \cap C^1\big([0,T]; V\big) \cap C\big([0,T]; D\big(A\big)\big)$ s.t.

$$u''(t) + Bu'(t) + Au(t) = f(t, u(t), u'(t)), \qquad u(0) = u^0, \quad u'(0) = v^0. \tag{3.11}$$

By construction a solution of (3.11) is also a solution of (2.6).

First-order formulation. To analyze the well-posedness and space discretizations of (3.11) we want to apply the theory of Section 3.1 and therefore rewrite (3.11) as a first-order equation. Let $u' = v$ and define

$$x = \begin{bmatrix} u \\ v \end{bmatrix}, \quad S = \begin{bmatrix} 0 & -\mathrm{I} \\ A & B \end{bmatrix}, \quad g\,(t, x) = \begin{bmatrix} 0 \\ f(t, u, v) \end{bmatrix}, \quad x^0 = \begin{bmatrix} u^0 \\ v^0 \end{bmatrix}.$$

The linear operator S is defined on its domain $D(S) = D(A) \times V$. With $X = V \times H$, (3.11) is equivalent to (3.1).

LEMMA 3.6. *The operator $-S$ is the generator of a C_0-semigroup on $X = V \times H$ which satisfies* (3.2) *with constant $c_{\mathrm{qm}} = \frac{1}{2} c_G C_{H,V} + \beta_{\mathrm{qm}}$.*

Proof. This follows from a combination of Lemma 4.2 (with $\alpha = 1$), Lemma 2.3, and Theorem 2.4 in [7]. □

Since $f \in C^1([0,T] \times V \times H; H)$ implies $g \in C^1([0,T] \times X; X)$, the problem (3.11) is locally well-posed by Lemma 3.2. We denote the maximal existence time by $t^*(u^0, v^0)$.

Space discretization. Let V_h be a finite dimensional vector space. We consider (2.11) as a space discretization of (2.6) and assume that Assumption 2.4 is satisfied.

To reformulate (2.11) as an evolution equation we define $A_h, B_h \in \mathcal{L}(V_h; V_h)$ via

$$m_h\big(A_h v_h, \varphi_h\big) = a_h\big(v_h, \varphi_h\big), \quad m_h\big(B_h v_h, \varphi_h\big) = b_h\big(v_h, \varphi_h\big) \quad \text{for all } v_h, \varphi_h \in V_h.$$

Then, (2.11) is equivalent to

$$\begin{aligned} u_h''(t) + B_h u_h'(t) + A_h u_h(t) &= f_h(t, u_h(t), u_h'(t)), \\ u_h(0) = u_h^0, \qquad u_h'(0) &= v_h^0. \end{aligned} \tag{3.12}$$

Analogously to the continuous case we can rewrite this as a first-order equation. With the Hilbert space $X_h = V_h \times H_h$ and

$$x_h(t) = \begin{bmatrix} u_h(t) \\ v_h(t) \end{bmatrix}, \quad S_h = \begin{bmatrix} 0 & -\mathrm{I} \\ A_h & B_h \end{bmatrix}, \quad g_h\,(t, x_h(t)) = \begin{bmatrix} 0 \\ f_h(t, u_h(t), v_h(t)) \end{bmatrix},$$

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(3.12) has the form (3.3). Similarly to Lemma 3.6 we obtain that $-S_h$ is the generator of a C_0-semigroup on X_h which satisfies (3.2) with constant $\widehat{c}_{\mathrm{qm}} = \frac{1}{2}\widehat{c}_G \widehat{C}_{H,V} + \widehat{\beta}_{\mathrm{qm}}$ independent of h.

Due to the Picard–Lindelöf theorem, (3.12) is locally well-posed and we denote the maximal existence time of the solution by $t_h^*(u_h^0, v_h^0)$.

Error analysis. To apply the error result from Section 3.1 we have to specify the operators occurring there.

We assume that there exists a lift operator $\mathcal{L}_h^V \in \mathcal{L}(V_h; V)$ satisfying

$$\|\mathcal{L}_h^V v_h\|_m \le C_H \|v_h\|_{m_h}, \qquad \|\mathcal{L}_h^V v_h\|_{\widetilde{a}} \le C_V \|v_h\|_{\widetilde{a}_h}, \tag{3.13}$$

for all $v_h \in V_h$ with constants $C_H, C_V > 0$ independent of h. Using this, we define the first-order lift operator $\mathcal{L}_h \colon X_h \to X$ by

$$\mathcal{L}_h \begin{bmatrix} v_h \\ w_h \end{bmatrix} := \begin{bmatrix} \mathcal{L}_h^V v_h \\ \mathcal{L}_h^V w_h \end{bmatrix}.$$

Note that one lift operator $\mathcal{L}_h^V$ is sufficient since $V \hookrightarrow H$, but we have to distinguish adjoints $\mathcal{L}_h^{V*} \colon V \to V_h$ and $\mathcal{L}_h^{H*} \colon H \to H_h$ w.r.t. the scalar products in V and H. They are defined via

$$\begin{aligned} m_h\big(\mathcal{L}_h^{H*} v, w_h\big) &= m\big(v, \mathcal{L}_h^V w_h\big) && \text{for all } v \in H, w_h \in H_h, \\ \widetilde{a}_h\big(\mathcal{L}_h^{V*} v, w_h\big) &= \widetilde{a}\big(v, \mathcal{L}_h^V w_h\big) && \text{for all } v \in V, w_h \in V_h. \end{aligned}$$

Let $Z^V \overset{d}{\hookrightarrow} V$ be a subspace of V and $I_h \in \mathcal{L}(Z^V; V_h)$ be an interpolation operator satisfying

$$\|I_h\|_{H_h \leftarrow Z^V} \le C_I \tag{3.14}$$

with $C_I > 0$ independent of h. We define the first-order reference operator $J_h \colon Z \to X_h$ by

$$J_h \begin{bmatrix} v \\ w \end{bmatrix} := \begin{bmatrix} \mathcal{L}_h^{V*} v \\ I_h w \end{bmatrix},$$

on $Z = V \times Z^V \overset{d}{\hookrightarrow} X$.

REMARK 3.7. We used I_h instead of $\mathcal{L}_h^{H*}$ in the second component of the reference operator because the adjoint lift operator only leads to suboptimal error bounds.

By (3.13) and (3.14), conditions (3.4) and (3.5) are satisfied with $C_X = \max\{C_V, C_H\}$ and $C_J = \max\{C_V, C_I\}$.

For $v_h, w_h \in V_h$, the errors in the scalar products are defined via

$$\begin{aligned} \Delta m\big(v_h, w_h\big) &:= m\big(\mathcal{L}_h^V v_h, \mathcal{L}_h^V w_h\big) - m_h\big(v_h, w_h\big), \\ \Delta \widetilde{a}\big(v_h, w_h\big) &:= \widetilde{a}\big(\mathcal{L}_h^V v_h, \mathcal{L}_h^V w_h\big) - \widetilde{a}_h\big(v_h, w_h\big), \end{aligned}$$

and for $z = (u, v) \in Z$, the linear and nonlinear remainder term are given by

$$\begin{aligned} R_h z = \left(\mathcal{L}_h^* S - S_h J_h\right) z &= \begin{bmatrix} -(\mathcal{L}_h^{V*} - I_h) v \\ \mathcal{L}_h^{H*}(Au + Bv) - (A_h \mathcal{L}_h^{V*} u + B_h I_h v) \end{bmatrix}, \\ r_h(t, z) = \mathcal{L}_h^* g(t, z) - g_h(t, J_h z) &= \begin{bmatrix} 0 \\ \mathcal{L}_h^{H*} f(t, u, v) - f_h(t, \mathcal{L}_h^{V*} u, I_h v) \end{bmatrix}. \end{aligned}$$

To obtain an error bound for the semi discretization from Theorem 3.4 we have to bound the remainder terms. The nonlinear one is obviously bounded by

$$\|r_h(t,z)\|_{X_h} = \|\mathcal{L}_h^{H*} f(t,u,v) - f_h(t,\mathcal{L}_h^{V*}u, I_h v)\|_{m_h}, \qquad z=(u,v)\in Z. \tag{3.15}$$

For the linear one we get

$$\begin{aligned}\left\| R_h \begin{bmatrix} u \\ v \end{bmatrix} \right\|_{X_h} \le C\Big(&\max_{\|\varphi_h\|_{\widetilde{a}_h}=1} \big|\Delta\widetilde{a}\big(I_h v,\varphi_h\big)\big| + \max_{\|\varphi_h\|_{\widetilde{a}_h}=1}\big|\Delta\widetilde{a}\big(I_h u,\varphi_h\big)\big| \\ &+ \max_{\|\psi_h\|_{m_h}=1}\big|\Delta m\big(I_h u,\psi_h\big)\big| + \|(\mathrm{I}-\mathcal{L}_h^V I_h)u\|_{\widetilde{a}} \\ &+ \|\big(\mathrm{I}-\mathcal{L}_h^V I_h\big)\,v\|_{\widetilde{a}} + \max_{\|\psi_h\|_{m_h}=1}\big|b\big(v,\mathcal{L}_h^V\psi_h\big) - b_h\big(I_h v,\psi_h\big)\big|\Big),\end{aligned} \tag{3.16}$$

i.e., it can be bounded against errors in the bilinear forms and interpolation errors. The bound (3.16) is proven in Lemma 4.7 in [7] (with our choice of the reference operator J_h). Proving the final error bound requires a sufficiently regular solution.

ASSUMPTION 3.8. *The solution u of (3.11) satisfies $u \in C^2\big([0,t^*(u^0,v^0)); Z^V\big)$.*

The following two results are direct consequences of Theorem 3.4 and Corollary 3.5.

THEOREM 3.9. *Let Assumption 3.8 be satisfied and $T < \min\Big\{t^*\big(u^0,v^0\big), t_h^*\big(u_h^0,v_h^0\big)\Big\}$. Then, for all $t\in[0,T]$ the lifted semidiscrete solution $\mathcal{L}_h^V u_h$ of (3.12) satisfies the error bound*

$$\|\mathcal{L}_h^V u_h(t) - u(t)\|_{\widetilde{a}} + \|\mathcal{L}_h^V u_h'(t) - u'(t)\|_m \le C\mathrm{e}^{(\widehat{L}_{T,M_h}+\widehat{c}_{\mathrm{qm}})t}(1+t)\sum_{i=1}^{5} E_i. \tag{3.17}$$

with a constant C that is independent of h and t. The other constants are given by $\widehat{c}_{\mathrm{qm}} = \frac{1}{2}\widehat{c}_G\widehat{C}_{H,V} + \widehat{\beta}_{\mathrm{qm}}$,

$$M_h = \max\left\{ \max\Big\{C_V, C_I\Big\} \left\| \begin{bmatrix} u \\ u' \end{bmatrix} \right\|_{L^\infty([0,T];V\times Z^V)}, \left\| \begin{bmatrix} u_h \\ u_h' \end{bmatrix} \right\|_{L^\infty([0,T];V_h\times H_h)} \right\},$$

and

$$\begin{aligned} E_1 &:= \|u_h^0 - \mathcal{L}_h^{V*}u^0\|_{\widetilde{a}_h} + \|v_h^0 - I_h v^0\|_{m_h}, \\ E_2 &:= \|\mathcal{L}_h^{H*} f(\cdot,u(\cdot),u'(\cdot)) - f_h(\cdot,\mathcal{L}_h^{V*}u(\cdot), I_h u'(\cdot))\|_{L^\infty([0,T];H_h)}, \\ E_3 &:= \|(\mathrm{I}-\mathcal{L}_h^V I_h)u\|_{L^\infty([0,T];V)} + \|(\mathrm{I}-\mathcal{L}_h^V I_h)u'\|_{L^\infty([0,T];V)} \\ &\quad + \|(\mathrm{I}-\mathcal{L}_h^V I_h)u''\|_{L^\infty([0,T];H)}, \\ E_4 &:= \Big\| \max_{\|\varphi_h\|_{\widetilde{a}_h}=1} \Delta\widetilde{a}\big(I_h u,\varphi_h\big)\Big\|_{L^\infty(0,t)} + \Big\| \max_{\|\psi_h\|_{m_h}=1} \Delta m\big(I_h u,\psi_h\big)\Big\|_{L^\infty(0,t)} \\ &\quad + \Big\| \max_{\|\varphi_h\|_{\widetilde{a}_h}=1} \Delta\widetilde{a}\big(I_h u',\varphi_h\big)\Big\|_{L^\infty(0,t)} + \Big\| \max_{\|\psi_h\|_{m_h}=1} \Delta m\big(I_h u'',\psi_h\big)\Big\|_{L^\infty(0,t)}, \\ E_5 &:= \Big\| \max_{\|\psi_h\|_{m_h}=1} \big|b\big(u',\mathcal{L}_h^V\psi_h\big) - b_h\big(I_h u',\psi_h\big)\big| \Big\|_{L^\infty(0,t)}. \end{aligned}$$

If $E_i \to 0$, $i=1,\dots,5$, we can conclude convergence.

COROLLARY 3.10. *Let Assumption 3.8 be satisfied, $T < t^*\big(u^0, v^0\big)$, and*

$$M := 2\max\Big\{C_V, C_I\Big\}\left\|\begin{bmatrix} u \\ u' \end{bmatrix}\right\|_{L^\infty([0,T];V\times Z^V)}.$$

Further let $E_i \overset{h\to 0}{\longrightarrow} 0$ for $i = 1,\dots,5$. Then there exists $h^* > 0$, s.t. u_h exists in $[0,T]$ for all $h < h^*$ with

$$\left\|\begin{bmatrix} u_h \\ u_h' \end{bmatrix}\right\|_{L^\infty([0,T];V_h\times H_h)} \le M.$$

Additionally the error bound (3.17) holds true with $M_h = M$ and the lifted semidiscrete solution converges, i.e.,

$$\lim_{h\to 0}\|\mathcal{L}_h^V u_h(t) - u(t)\|_{\widetilde{a}} + \|\mathcal{L}_h^V u_h'(t) - u'(t)\|_m = 0, \qquad t \in [0,T].$$

Proof of Theorem 3.9. We apply Theorem 3.4. Recall that we have

$$C_X = \max\{C_V, C_H\}, \qquad C_J = \max\{C_V, C_I\}, \qquad \widehat{c}_{\mathrm{qm}} = \frac{1}{2}\widehat{c}_G\widehat{C}_{H,V} + \widehat{\beta}_{\mathrm{qm}}.$$

As in the proof of Theorem 4.8 in [7] we obtain (3.17) by applying the error estimate (3.7) and using (3.15) and (3.16). ∎

Corollary 3.10 follows directly from Corollary 3.5.

4. Proof of Theorem 2.7.

For the proof we use the results of Section 3.2.

Proof of Theorem 2.7. In Section 2 we already showed, that the weak formulation of wave equations with kinetic boundary conditions (2.6) as well as their discretizations with the bulk-surface FEM fit into the general setting presented in Section 3.2.

We define the space

$$Z^V := H^2(\Omega;\Gamma) \overset{d}{\hookrightarrow} V = H^1(\Omega;\Gamma),$$

the interpolation operator $I_h := I_{h,\Omega}$, and the lift operator via

$$\mathcal{L}_h^V v := v^\ell$$

with v^ℓ given in (2.7). By (2.8) we have $\mathcal{L}_h \in \mathcal{L}(V_h; V)$. Moreover, (3.13) is satisfied and I_h satisfies (3.14) by (2.9).

Hence, all assumptions of Corollary 3.10 are satisfied. It remains to bound the error terms by $\mathcal{O}(h^p)$ to obtain the desired error bound. In [6, Theorem 7.4] it was shown that

$$E_1, E_3, E_4, E_5 \le Ch^p,$$

so that we only have to study the nonlinear error term. By Lemma (2.6) we have

$$\begin{aligned} E_2 &= \|\mathcal{L}_h^{H*} f(\cdot,u) - f_h(\cdot,\mathcal{L}_h^{V*}u)\|_{L^\infty([0,T];H_h)} \\ &\le \|\mathcal{L}_h^{H*} f(\cdot,u) - f_h(\cdot,I_h u)\|_{L^\infty([0,T];H_h)} + \|f_h(\cdot,I_h u) - f_h(\cdot,\mathcal{L}_h^{V*}u)\|_{L^\infty([0,T];H_h)} \\ &\le \|\mathcal{L}_h^{H*} f(\cdot,u) - f_h(\cdot,I_h u)\|_{L^\infty([0,T];H_h)} + \widehat{L}_{T,M}\|(I_h - \mathcal{L}_h^{V*})u\|_{L^\infty([0,T];H_h)}. \end{aligned}$$

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The second term is of order h^{p+1} and for the first we obtain by definition of f and f_h

$$\begin{aligned}
&\|\mathcal{L}_h^{H*} f(t,u) - f_h(t, I_h u)\|_{m_h} \\
&= \sup_{\|\varphi_h\|_{m_h}=1} m_h\big(\mathcal{L}_h^{H*} f(t,u) - f_h(t, I_h u), \varphi_h\big) \\
&= \sup_{\|\varphi_h\|_{m_h}=1} \Big(m\big(f(t,u), \mathcal{L}_h^{V}\varphi_h\big) - m_h\big(f_h(t, I_h u), \varphi_h\big)\Big) \\
&= \sup_{\|\varphi_h\|_{m_h}=1} \bigg(\Big(\int_\Omega \widetilde{f}_\Omega(t,\mathbf{x},u(\mathbf{x}))\varphi_h^\ell(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_{\Omega_h} I_{h,\Omega}\widetilde{f}_\Omega(t,\cdot,(I_{h,\Omega}u)^\ell(\cdot))(\mathbf{x})\varphi_h(\mathbf{x})\,\mathrm{d}\mathbf{x} \\
&\qquad + \int_\Gamma \widetilde{f}_\Gamma(t,\mathbf{x},\gamma(u)(\mathbf{x}))\varphi_h^\ell(\mathbf{x})\,\mathrm{d}s - \int_{\Gamma_h} I_{h,\Gamma}\widetilde{f}_\Gamma(t,\cdot,(I_{h,\Gamma}\gamma(u))^\ell(\cdot))(\mathbf{x})\varphi_h(\mathbf{x})\,\mathrm{d}s \bigg).
\end{aligned}$$

Let $\varphi_h \in V_h$ with $\|\varphi_h\|_{m_h} = 1$. For the error in Ω we obtain

$$\begin{aligned}
&\int_\Omega \widetilde{f}_\Omega(t,\mathbf{x},u(\mathbf{x}))\varphi_h^\ell(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_{\Omega_h} I_{h,\Omega}\widetilde{f}_\Omega(t,\cdot,(I_{h,\Omega}u)^\ell(\cdot))(\mathbf{x})\varphi_h(\mathbf{x})\,\mathrm{d}\mathbf{x} \\
&\quad = \int_\Omega \widetilde{f}_\Omega(t,\mathbf{x},u(\mathbf{x}))\varphi_h^\ell(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_{\Omega_h} I_{h,\Omega}\widetilde{f}_\Omega(t,\cdot,u(\cdot))(\mathbf{x})\varphi_h(\mathbf{x})\,\mathrm{d}\mathbf{x} \\
&\quad = \int_\Omega \widetilde{f}_\Omega(t,\mathbf{x},u(\mathbf{x}))\varphi_h^\ell(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_\Omega \Big(I_{h,\Omega}\widetilde{f}_\Omega(t,\cdot,u(\cdot))\Big)^\ell(\mathbf{x})\varphi_h^\ell(\mathbf{x})\,\mathrm{d}\mathbf{x} \\
&\qquad + \int_\Omega \Big(I_{h,\Omega}\widetilde{f}_\Omega(t,\cdot,u(\cdot))\Big)^\ell(\mathbf{x})\varphi_h^\ell(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_{\Omega_h} I_{h,\Omega}\widetilde{f}_\Omega(t,\cdot,u(\cdot))(\mathbf{x})\varphi_h(\mathbf{x})\,\mathrm{d}\mathbf{x},
\end{aligned}$$

where we used the definition of the nodal interpolation in the first step: The inner interpolation can be omitted, since the outer interpolation only depends on the function values at the nodes a_i which are invariant under the inner interpolation.

For the first term we obtain with (2.8), (2.9), and $\|\varphi_h\|_{L^2(\Omega)} \le \|\varphi_h\|_{m_h} = 1$

$$\begin{aligned}
&\int_\Omega \widetilde{f}_\Omega(t,\mathbf{x},u(\mathbf{x}))\varphi_h^\ell(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_\Omega \Big(I_{h,\Omega}\widetilde{f}_\Omega(t,\cdot,u(\cdot))\Big)^\ell(\mathbf{x})\varphi_h^\ell(\mathbf{x})\,\mathrm{d}\mathbf{x} \\
&\qquad \le \Big\|\widetilde{f}_\Omega(t,\cdot,u(\cdot)) - \Big(I_{h,\Omega}\widetilde{f}_\Omega(t,\cdot,u(\cdot))\Big)^\ell\Big\|_{L^2(\Omega)} \|\varphi_h^\ell\|_{L^2(\Omega)} \\
&\qquad \le C_{\Omega,\Omega_h} C h^p \big\|\widetilde{f}_\Omega(t,\cdot,u(\cdot))\big\|_{H^p(\Omega)} \\
&\qquad \le C h^p \Big(\|u_{tt}\|_{H^p(\Omega)} + \|\nabla u_t\|_{H^p(\Omega)} + \|\Delta u\|_{H^p(\Omega)}\Big).
\end{aligned}$$

In the last step we used the differential equation (2.1).

Since $I_{h,\Omega}\widetilde{f}_\Omega\big(t,\cdot,u(\cdot)\big) \in V_{h,p}^\Omega$ we can bound the second term with the estimate (5.10) from [7] by

$$\begin{aligned}
&\int_\Omega \Big(I_{h,\Omega}\widetilde{f}_\Omega(t,\cdot,u(\cdot))\Big)^\ell(\mathbf{x})\varphi_h^\ell(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_{\Omega_h} I_{h,\Omega}\widetilde{f}_\Omega(t,\cdot,u(\cdot))(\mathbf{x})\varphi_h(\mathbf{x})\,\mathrm{d}\mathbf{x} \\
&\qquad \le C h^p \big\|I_{h,\Omega}\widetilde{f}_\Omega(t,\cdot,u(\cdot))\big\|_{L^2(\Omega_h)} \|\varphi_h\|_{L^2(\Omega_h)} \\
&\qquad \le C h^p \big\|\widetilde{f}_\Omega(t,\cdot,u(\cdot))\big\|_{H^2(\Omega)} \\
&\qquad \le C h^p \Big(\|u_{tt}\|_{H^2(\Omega)} + \|\nabla u_t\|_{H^2(\Omega)} + \|\Delta u\|_{H^2(\Omega)}\Big).
\end{aligned}$$

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Here we also used $I_{h,\Omega} \in \mathcal{L}(H^2(\Omega); L^2(\Omega_h))$ and the differential equation (2.1).

The error term on Γ can be bounded analogously and we obtain

$$E_2 \leq Ch^p \Big(\|u''(t)\|_{H^p(\Omega;\Gamma)} + \|u(t)\|_{H^{\max\{4,p+2\}}(\Omega;\Gamma)} \Big).$$

The only additional term that has to be bounded in this case is

$$\|\partial_n u\|_{H^p(\Omega)} \leq \|u\|_{H^{p+2}(\Omega)}.$$

This completes the proof. □

5. Numerical examples. In this section we illustrate Theorem 2.7 numerically.

We choose $\Omega = B(0,1) \subset \mathbb{R}^2$ as two dimensional unit sphere and

$$u(t,\mathbf{x}) = \sin(2\pi t)\mathbf{x}_1\mathbf{x}_2.$$

Furthermore we set

$$\widetilde{f}_\Omega(t,\mathbf{x},u) = |u|u + \eta_\Omega(t,\mathbf{x}),$$
$$\widetilde{f}_\Gamma(t,\mathbf{x},u) = \Big(|u|^2 - 1\Big)\, u + \eta_\Gamma(t,\mathbf{x}).$$

Then, u solves the semilinear wave equation with kinetic boundary conditions

$$\begin{aligned} u_{tt} + (1+\mathbf{x}\cdot\nabla)u_t - \Delta u &= |u|u + \eta_\Omega(t,\mathbf{x}), && (0,T)\times\Omega, \\ u_{tt} + \partial_n u - \Delta_\Gamma u &= |u|^2 u + \eta_\Gamma(t,\mathbf{x}), && (0,T)\times\partial\Omega, \\ u(0,\mathbf{x}) = 0, \quad u_t(0,\mathbf{x}) &= 2\pi\mathbf{x}_1\mathbf{x}_2, && \text{in } \overline{\Omega}, \end{aligned}$$

with

$$\eta_\Omega(t,\mathbf{x}) = -\left(4\pi^2 + |\sin(2\pi t)\mathbf{x}_1\mathbf{x}_2|\right)\sin(2\pi t)\mathbf{x}_1\mathbf{x}_2 + 6\pi\cos(2\pi t)\mathbf{x}_1\mathbf{x}_2,$$
$$\eta_\Gamma(t,\mathbf{x}) = \left(7 - 4\pi^2\right)\sin(2\pi t)\mathbf{x}_1\mathbf{x}_2 - \left(\sin(2\pi t)\mathbf{x}_1\mathbf{x}_2\right)^3.$$

We implemented the bulk-surface FEM by using the `C++` finite element-library `deal.II` [2, 3]. with discrete initial values $u_h^0 = I_{h,\Omega}u^0$ and $v_h^0 = I_{h,\Omega}v^0$. For time integration we applied the Crank-Nicolson scheme with sufficiently small step size, such that the time integration error is negligible. The codes are available from the authors on request.

In Figure 5.1 the error

$$E_h(t) := \|u_h(t) - u(t)|_{\Omega_h}\|_{H^1(\Omega_h;\Gamma_h)} + \|u_h'(t) - u'(t)|_{\Omega_h}\|_{L^2(\Omega_h)\times L^2(\Gamma_h)}$$

is plotted against the mesh width h for the discretization of the test example with isoparametric elements of order $p=1$ and $p=2$ and $t=0.8$. We evaluated the integrals with a quadrature rule of degree $2p$. The restriction of u to Ω_h is possible since for convex domains $\Omega_h \subset \Omega$.

The error behaves as predicted by Theorem 2.7.

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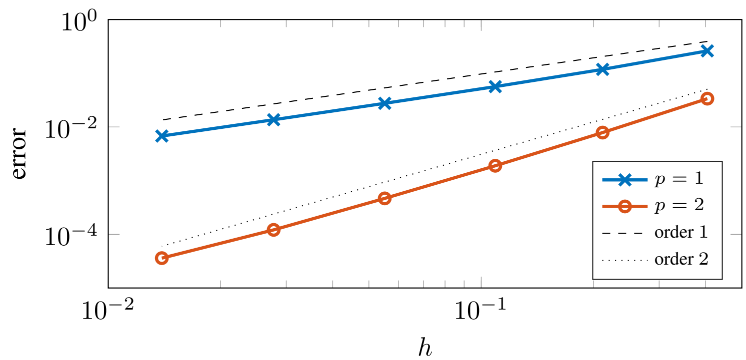

FIG. 5.1. *Error $E_h(0.8)$ for the test example.*

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