Bias-free Parameter Identification of a Fractional Order Model

Oliver Stark, Albertus Johannes Malan, Stefan Krebs and Sören Hohmann

Abstract— **This paper deals with the parameter identification of a fractional system considering a noisy observation of the output signal. The novelty is that the instrumental variable method is applied to the modulating function method applied to a fractional system. A simulated output signal which is not correlated to noise is required as the instrumental variable. Because all known simulation algorithms only consider zero initial conditions, the simulated output signal converges against the true output signal in an undefined time if the zero initial conditions are penalized. Therefore, an algorithm is extended with the short-memory principle. The benefit is that after a fixed time the error between the simulated and true output signal is small and can be used as the instrumental variable. Considering this extension of the simulation algorithms, it is shown that a consistent estimation is yield with the instrumental variable method.**

I. INTRODUCTION

The fractional calculus has been more and more in the focus of research because complex systems or memory effects can be described more efficiently than using the classic calculus (see [1]). One field of research deals with the parameter identification due to the fact that new mathematical problems come along with the fractional systems. Several approaches are developed in frequency domain (see e.g. [2]) as well as in time domain (see e.g. [16, 3, 4, 5, 6]).

In [16, 3], the modulating function method is transferred to fractional systems and it is used to identify the parameters. The benefit is that no derivative of the measured signals has to be calculated. It can be shown that the parameter identification is robust against high frequency sinusoidal noise, but not in general against a noisy observation of the output signal.

In [4], the problem of the initialization function is investigated. It is shown if the modulating function method is used and, in addition, if a special property is fulfilled by the modulating function, no initialization function will be needed anymore. The resulting bias is also calculated if Gaussian white noise is assumed and the least squares method is applied for the parameter identification.

In [5, 6], the modulating function method combined with the least squares method is also used. But, in [5], a prediction of the future noise and, in [6], information about properties of the noise is used to revise the standard least squares method to eliminate the bias.

In this paper, an approach for bias-free parameter identification based on the instrumental variable method is proposed so no information about the noise has to be considered. Therefore, an existing algorithm will be extended with the short-memory principle. So, it is possible to calculate the output signal of the fractional system after a fixed, predetermined time with only a small error compared to the true output signal. The simulated output signal will be used as the instrumental variable like in the classical case.

This paper is structured as follows: In Section II, the basis of the fractional calculus, the considered structure of the fractional system and the modulating function are provided. The main parts are described in Section III and Section IV. First, the existing algorithm which is extended with the short-memory principle, in a second step, is described. Afterwards, the consistent parameter identification using the instrumental variable method within the modulating function method is shown. A numerical example in Section V completes this paper.

O. Stark, A. J. Malan, S. Krebs and S. Hohmann are with Faculty of Electrical Engineering and Information Technology, Institute of Control Systems, Karlsruhe Institute of Technology, 76131 Karlsruhe, Germany oliver.stark@kit.edu

II. PRELIMINARIES

A. Fundamentals of Fractional Calculus

In this section, the left-sided Riemann-Liouville and the right-sided Caputo definition of the uninitialized and initialized fractional derivative used in this paper are described. Also, the right-sided Grünwald-Letnikov definition and the important relation between the derivatives are given. But, first, the fractional integration is presented because the fractional integration is inevitable to define the fractional derivatives.

Through this paper, the following notations are used: $f : [a,t] \to \mathbb{R}$, $h : [t,b] \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}^+$ where f and h are arbitrary continuous functions with $f(t) = 0,\ \forall t \leq a$ and $h(t) = 0,\ \forall t \geq b$. Also, $a \leq e < t < g \leq b$ holds. In addition, $\lfloor \alpha \rfloor$ describes the floor function and denotes the biggest integer smaller or equal to α.

All uninitialized operators given in this section are defined in e.g. [7, 8, 9]. The initialized fractional operators can be obtained from [10, 11] and [12] whose notation is used. The lower or rather upper bound is marked at the operator in the left or rather right lower corner. The fractional order is given in the right upper corner.

Definition 1: Uninitialized Fractional Integration.

$$ {}_e i_t^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_e^t \frac{f(\nu)}{(t-\nu)^{1-\alpha}} \mathrm{d}\nu \tag{1}$$

where Γ is the Gamma function.

Definition 2: Uninitialized Right-Sided Fractional Integration.

$$ {}_t i_g^{\alpha} h(t) := \frac{1}{\Gamma(\alpha)} \int_t^g \frac{h(\nu)}{(\nu-t)^{1-\alpha}} \mathrm{d}\nu \tag{2}$$

Definition 3: Uninitialized Riemann-Liouville Derivative.

$$ {}_e\mathrm{d}RL_t^\alpha f(t) := \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\lfloor\alpha\rfloor+1} \left[{}_e i_t^{\lfloor\alpha\rfloor+1-\alpha} f(t) \right] \quad (3) $$

where ${}_e i_t^{\lfloor\alpha\rfloor+1-\alpha} f(t)$ is the uninitialized fractional integral (1).

Definition 4: Initialized Riemann-Liouville Derivative.

$$ {}_e\mathrm{D}RL_t^\alpha f(t) := {}_e\mathrm{d}RL_t^\alpha f(t) + \varphi_{\mathrm{RL},L}(f,\alpha,a,e,t) \quad (4) $$

where $\varphi_{\mathrm{RL},L}(f,\alpha,a,e,t)$ is the initialization function of the Riemann-Liouville derivative.

Definition 5: Terminal Initialization Function of the Initialized Riemann-Liouville Derivative.

$$ \varphi_{\mathrm{RL},L}(f,\alpha,a,e,t) := \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\lfloor\alpha\rfloor+1} \left[{}_a i_e^{\lfloor\alpha\rfloor+1-\alpha} f(t) \right] \quad (5) $$

Definition 6: Uninitialized Right-Sided Caputo Derivative.

$$ {}_t\mathrm{d}C_g^\alpha h(t) := (-1)^{\lfloor\alpha\rfloor+1} \, {}_t i_g^{\lfloor\alpha\rfloor+1-\alpha} \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\lfloor\alpha\rfloor+1} h(t) \right] \quad (6) $$

where ${}_t i_g^{\lfloor\alpha\rfloor+1-\alpha} h(t)$ is the uninitialized right-sided fractional integral (2).

Definition 7: Initialized Right-Sided Caputo Derivative.

$$ {}_t\mathrm{D}C_g^\alpha h(t) := {}_t\mathrm{d}C_g^\alpha h(t) + \varphi_{\mathrm{C},R}(h,\alpha,g,b,t) \quad (7) $$

where $\varphi_{\mathrm{C},R}(h,\alpha,g,b,t)$ is the initialization function of the right-sided Caputo derivative.

Definition 8: Terminal Initialization Function of the Initialized Right-Sided Caputo Derivative.

$$ \varphi_{\mathrm{C},R}(h,\alpha,g,b,t) := (-1)^{\lfloor\alpha\rfloor+1} \, {}_g i_b^{\lfloor\alpha\rfloor+1-\alpha} \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\lfloor\alpha\rfloor+1} h(t) \right]. \quad (8) $$

Remark 1: The order of the fractional integral and integer derivative of the Caputo derivative is changed compared to the definition of the Riemann-Liouville derivative.

Considering the uninitialized case, the link between the Riemann-Liouville and the Caputo derivative given in [13] is

$$ {}_t\mathrm{d}RL_g^\alpha h(t) = {}_t\mathrm{d}C_g^\alpha h(t) + \sum_{j=0}^{\lfloor\alpha\rfloor} {}_t\mathrm{d}RL_g^\alpha \frac{h^{(j)}(t)\big|_{t=g}}{j!} (g-t)^g. \quad (9) $$

Remark 2: So, if $f^{(j)}(t)\big|_{t=g} = 0 \;\; \forall j \in \{0,1,\ldots,\lfloor\alpha\rfloor\}$ holds, it follows

$$ {}_t\mathrm{d}RL_g^\alpha h(t) = {}_t\mathrm{d}C_g^\alpha h(t) \quad (10) $$

Definition 9: Uninitialized Grünwald-Letnikov Derivative.

$$ {}_a\mathrm{d}GL_t^\alpha f(t) := \lim_{T_s\to 0} \frac{1}{T_s^\alpha} \sum_{l=0}^{\lfloor \frac{t-a}{T_s} \rfloor} (-1)^j \binom{\alpha}{j} f(t-jT_s) \quad (11) $$

where T_s is the sampling time.

Definition 10: Uninitialized Right-Sided Grünwald-Letnikov Derivative.

$$ {}_t\mathrm{d}GL_b^\alpha h(t) := \lim_{T_s\to 0} \frac{1}{T_s^\alpha} \sum_{l=0}^{\lfloor \frac{b-t}{T_s} \rfloor} (-1)^l \binom{\alpha}{l} h(t+jT_s). \quad (12) $$

Remark 3: The Grünwald-Letnikov derivative is equivalent to the Riemann-Liouville derivative under some suitable conditions (see [13, 14]).

Considering (11) and (12), as time goes by, $t \gg a$ or rather $t \ll b$, more and more summands have to be taken into account. The influence of the functional values is decreasing closer to the limits a or rather g due to the binomial coefficient. So, after a certain time these values can be neglected and only the recent past of the function have to be considered (see [7]). This principle is called short-memory principle and within a fixed memory length is used to take the recent past into account.

Definition 11: Short-Memory Principle.

$$ {}_e\mathrm{d}GL_t^\alpha f(t) \approx {}_{t-L\cdot T_s}\mathrm{d}GL_t^\alpha f(t) \quad \text{or rather} \quad (13) $$
$$ {}_t\mathrm{d}GL_g^\alpha h(t) \approx {}_t\mathrm{d}GL_{t+L\cdot T_s}^\alpha h(t) \quad (14) $$

where $L \in \mathbb{N}$ is the fixed memory length.

Definition 12: Relation Between The Frequency And Time Domain.

If the Laplace transform $\mathcal{L}f(s)$ and $\mathcal{L}[{}_e\mathrm{D}_t^k f(t)](s)$ exists and $\lim_{t\to\infty} {}_e\mathrm{D}_t^j f(t) = 0$ for $j = 0,1,...,k-1$ holds true, then the relation

$$ \mathcal{L}[{}_e\mathrm{D}_t^k f(t)](s) = s^k \mathcal{L}f(s) - \sum_{j=0}^{k-1} s^{k-j-1} \, {}_e\mathrm{D}_t^j f(0) \quad (15) $$

exists [13].

Remark 4: The operator ${}_e\mathrm{D}_t^j$ notates that the Riemann-Liouville as well as the Grünwald-Letnikov derivative can be used.

B. Fractional Order Models

In this paper, the investigated system is initially described by the transfer function

$$ \frac{Y(s)}{U(s)} = \frac{\sum_{k=0}^m b_k s^{\beta_k}}{\sum_{i=0}^n a_i s^{\alpha_i}} = \frac{B(s)}{A(s)}. \quad (16) $$

Using (15) and zero initial conditions at $t = a$ leads to the non-commensurable fractional system

$$ \sum_{i=0}^n a_i \, {}_a\mathrm{d}RL_t^{\alpha_i} \tilde{y}(t) = \sum_{k=0}^m b_k \, {}_a\mathrm{d}RL_t^{\beta_k} u(t) \quad (17) $$

in time domain. If it is assumed that the system is not at rest when the parameter identification is done the initialized derivatives have to be used and, therefore, the system is

$$ \sum_{i=0}^n a_i \, {}_e\mathrm{D}RL_t^{\alpha_i} \tilde{y}(t) = \sum_{k=0}^m b_k \, {}_e\mathrm{D}RL_t^{\beta_k} u(t) \quad (18) $$

where $a_i, b_k \in \mathbb{R}$ are unknown parameters collected in the vector $\underline{p} = [a_{n-1},...,a_0,b_m,...,b_0]^\top$. The fractional orders $\alpha_i, \beta_k \in \mathbb{R}^+$ are assumed to be ordered $0 \le \alpha_0 < \cdots < \alpha_n$, $0 \le \beta_0 < \cdots < \beta_m$ as well as $\alpha_n \ge \beta_m$. The number of unknown parameters $n, m \in \mathbb{N}$ and the input signal $u(t)$ are known and $\tilde{y}(t)$ is a noisy observation of $y(t)$:

$$ \tilde{y}(t) = y(t) + \varrho + \epsilon(t) \quad (19) $$

where $y(t)$ is the noiseless output signal, ϱ is a constant bias and $\epsilon(t)$ is assumed to be Gaussian white noise.

Assumption 1: Noise.

$$\mathrm{E}\{\epsilon(t)\} = 0 \text{ and} \tag{20}$$

$$\mathrm{E}\{\epsilon(t)\epsilon(t-\tau)\} = \sigma^2 \cdot \delta(\tau) \tag{21}$$

where E describes the expected value, σ^2 is the value of the variance and $\delta(t)$ is the Dirac delta function.

Regarding the input and output signal, the following assumption is also made.

Assumption 2: Bounded Signals.
The input and output signal are assumed to be bounded.

The last assumption regarding the system is stated below.

Assumption 3: Standardization.
Without loss of generality, it is assumed that $a_n = 1$ in the fractional system (18).

Remark 5: The investigated system is not at rest. It follows that ${}_e\mathrm{D}RL_t^{\alpha} f(t) \neq {}_e\mathrm{d}RL_t^{\alpha} f(t)$ and the initialization function (5) has to be taken into account.

C. Modulating Function Method

The modulating function method was first presented in [15] and, meanwhile, the modulating function method is transferred to fractional order models (see e.g. [16, 4]). The idea is that the derivatives of the signals are swapped to a, at first, arbitrary and continuously differentiable function using the integration by parts. Each signal of system (18) is multiplied with this function and every product is integrated over $t \in [e, g]$. Applying the integration by parts leads to the result that the derivatives are swapped but with the drawback that boundary terms arise. In [15], two properties

$$(P1) \;:\; \gamma(t) \in \mathcal{C}^{\alpha_n}([e,g])$$

$$(P2) \;:\; \gamma^{(v)}(e) = \gamma^{(v)}(g) = 0 \;\forall v = 0, 1, \ldots, \lfloor \alpha_n \rfloor + 1.$$

are stated which eliminates the boundary terms. The functions which fulfills these properties are called modulating functions. An overview of some modulating functions are given in [17].

If the modulating function additionally fulfill

$$(P3) \;:\; \begin{array}{l} {}_t\mathrm{d}C_g^{\alpha_i}\gamma(t) = 0 \quad \forall i = 0, 1, ..., n \\ {}_t\mathrm{d}C_g^{\beta_k}\gamma(t) = 0 \quad \forall k = 0, 1, ..., m \end{array}$$

where $t \in [a, e]$, the initialization function need not to be considered (see [4]).

If the modulating function also has the property

$$(P4) \;:\; \begin{array}{l} \displaystyle\int_e^g {}_t\mathrm{d}C_g^{\alpha_i}\gamma(t)\mathrm{d}t = 0 \quad \forall i = 0, 1, ..., n \\ \displaystyle\int_e^g {}_t\mathrm{d}C_g^{\beta_k}\gamma(t)\mathrm{d}t = 0 \quad \forall k = 0, 1, ..., m \end{array}$$

an unknown bias will be eliminated (see [18]).

Lemma 1: Applied Modulating Function Method.
Applying the modulating function method to the system (18) results in

$$\begin{aligned} &\sum_{i=0}^{n} a_i \int_e^g {}_t\mathrm{d}C_g^{\alpha_i}\gamma(t)\,\tilde{y}(t)\,\mathrm{d}t = \\ &\quad \sum_{k=0}^{m} b_k \int_e^g {}_t\mathrm{d}C_g^{\beta_k}\gamma(t)\,u(t)\,\mathrm{d}t \end{aligned} \tag{22}$$

assuming that the modulating function also fulfills $(P3)$.

Proof: The proof can be found in [4]. ■

Lemma 2: Replacing Caputo derivative.
The uninitialized right-sided Caputo derivative can be replaced by the uninitialized right-sided Grünwald-Letnikov derivative.

Proof: Because of $(P2)$, the link between the unitialized Caputo and Riemann-Liouville derivative (9) simplifies to

$${}_t\mathrm{d}RL_g^{\alpha}h(t) = {}_t\mathrm{d}C_g^{\alpha}h(t). \tag{23}$$

In [14], the equivalence between the Riemann-Liouville and Grünwald-Letnikov derivative is shown. ■

In the following, the system

$$\begin{aligned} &\sum_{i=0}^{n} a_i \int_e^g {}_t\mathrm{d}GL_g^{\alpha_i}\gamma(t)\,\tilde{y}(t)\,\mathrm{d}t = \\ &\quad \sum_{k=0}^{m} b_k \int_e^g {}_t\mathrm{d}GL_g^{\beta_k}\gamma(t)\,u(t) \end{aligned} \tag{24}$$

will be considered unless it is stated specifically.

D. Definition of a Spline-Type Modulating Function

Definition 13: Spline-Type Modulating Function.
In [19], a spline-type modulating function is described which is a weighted sequence of impulses integrated afterwards

$$\gamma_{\varsigma,\iota}(t) = \int_e^g \overbrace{\cdots}^{o=\varsigma-\iota} \int_e^g \sum_{\nu=0}^{\varsigma} (-1)^{\nu} \binom{\varsigma}{\nu} \delta(\nu T_0 - t + e)\,\mathrm{d}t^o \tag{25}$$

where ς is the order of the modulating function and $T_0 = \frac{T}{\varsigma}$ depends on the identification time T.

III. CLOSED-FORM SOLUTION CONSIDERING SHORT-MEMORY PRINCIPLE

A. Origin Closed-Form Solution

Starting point of developing the algorithm is the transfer function (16). In [22], the developing is described in detail and, in this paper, the most important steps are just given in the following.

Lemma 3: Closed-Form Solution.
The closed-form solution using the Grünwald-Letnikov derivative is

$$y(t) = \frac{1}{\sum\limits_{i=0}^{n} \frac{a_i}{T_s^{\alpha_i}}} \left(x(t) - \sum_{i=0}^{n} \frac{a_i}{T_s^{\alpha_i}} \sum_{l=1}^{\lfloor \frac{t-a}{T_s} \rfloor} (-1)^l \binom{\alpha_i}{l} y(t - l \cdot T_s) \right) \tag{26}$$

where

$$x(t) = \sum_{k=0}^{m} \frac{b_k}{T_s^{\beta_k}} \sum_{l=0}^{\lfloor \frac{t-a}{T_s} \rfloor} (-1)^l \binom{\beta_k}{l} u(t - l \cdot T_s). \tag{27}$$

Proof: The transfer function (16) can be rewritten as

$$\sum_{i=0}^{n} a_i s^{\alpha_i} Y(s) = \sum_{k=0}^{m} b_k s^{\beta_k} U(s). \tag{28}$$

First, the right side of this equation is set equal to $X(s)$ and (15) with zero initial condition and the Grünwald-Letnikov derivative is applied

$$x(t) = \sum_{k=0}^{m} b_k \, {}_a\mathrm{d}GL_t^{\beta_k} u(t). \tag{29}$$

Second, (15) is applied on the left side of (28) again with zero initial condition and the Grünwald-Letnikov derivative and the right side is replaced by (29) which leads to

$$\sum_{i=0}^{n} a_i \, {}_a\mathrm{d}GL_t^{\alpha_i} = x(t). \tag{30}$$

Using (11), it follows

$$\sum_{i=0}^{n} \frac{a_i}{T_s^{\alpha_i}} \sum_{l=0}^{\lfloor \frac{t-a}{T_s} \rfloor} (-1)^l \binom{\alpha_i}{l} y(t - l \cdot T_s) = x(t) \tag{31}$$

and evaluating the inner sum for $l = 0$ and rearranging everything results in (26). ■

Remark 6: In [20], a recursive formulation to calculate the binomial coefficient

$$\begin{aligned} \binom{\alpha}{0} &= 1 \\ \binom{\alpha}{j} &= \left(1 - \frac{\alpha+1}{j}\right)\binom{\alpha}{j-1} \quad \text{for } j = 1, 2, \ldots. \end{aligned} \tag{32}$$

is given which is numerically more robust.

Remark 7: This approach is used in the FOMCON toolbox (see [21]) and the approach which can not be extended with the short-memory principle in the FOTF toolbox (see [22]).

B. IMPLEMENTATION OF THE SHORT-MEMORY PRINCIPLE

In this section, the closed-form solution is extended with the short-memory principle.

Lemma 4: Closed-Form Solution Considering Short-Memory Principle.
The closed-form solution considering the short-memory principle using the Grünwald-Letnikov derivative is

$$y(t) = \frac{1}{\sum_{i=0}^{n} \frac{a_i}{T_s^{\alpha_i}}} \left(x(t) - \sum_{i=0}^{n} \frac{a_i}{T_s^{\alpha_i}} \sum_{l=1}^{L} (-1)^l \binom{\alpha_i}{l} y(t - l \cdot T_s) \right) \tag{33}$$

where

$$x(t) = \sum_{k=0}^{m} \frac{b_k}{T_s^{\beta_k}} \sum_{l=0}^{L} (-1)^l \binom{\beta_k}{l} u(t - l \cdot T_s). \tag{34}$$

Proof: The proof is analogous to the proof of the closed-form solution except that in (29) and (30) the short-memory principle (13) is used. ■

Remark 8: The binomial coefficient is calculated using (32).

The benefit of using the short-memory principle is shown by calculating the closed-form solution without and with the short-memory principle of a system which is not at rest. Therefore, the following system is considered:

$$\begin{aligned} {}_e\mathrm{D}RL_t^{\alpha_2} y(t) + a_1 \, {}_e\mathrm{D}RL_t^{\alpha_1} y(t) + a_0 y(t) &= \\ b_1 \, {}_e\mathrm{D}RL_t^{\beta_2} u(t) + b_0 \, {}_e\mathrm{D}RL_t^{\beta_1} u(t) & \end{aligned} \tag{35}$$

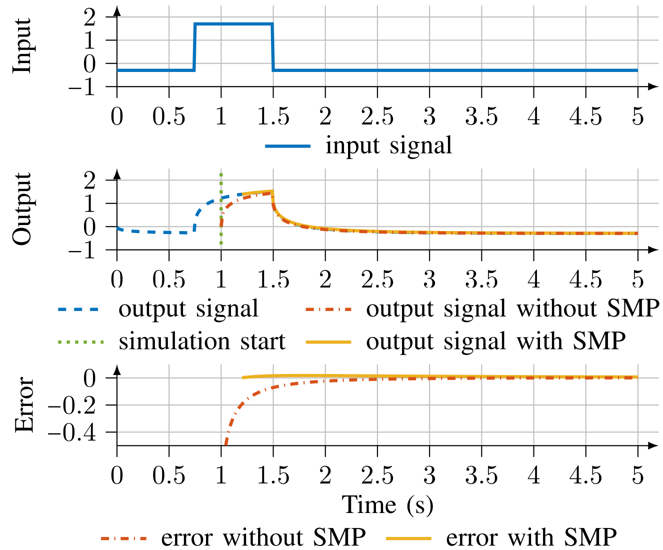

Fig. 1. Comparison of the closed-form solution without and with the short-memory principle.

where $a_1 = 2$, $a_0 = 3$, $b_1 = 1$ and $b_0 = 4$ as well as $\alpha_2 = 0.8$, $\alpha_1 = 0.5$, $\beta_2 = 0.5$ and $\beta_1 = 0.1$. The sampling time is $T_s = 0.01\,\mathrm{s}$. In the top picture of Fig. 1, the input signal, a single pulse between $t = 0.75\,\mathrm{s}$ and $t = 1.5\,\mathrm{s}$, is shown.

First, the output signal of system (35) is calculated with zero initial conditions at $t = 0\,\mathrm{s}$ using (26) which is displayed as a solid blue line in Fig. 1. Next, the calculation of the output signal begins at $t = 1\,\mathrm{s}$ marked as a dotted green line. It is assumed that no past data of $t \in [0, 1.5]$ are known. At first, the closed-form solution is calculated using also (26) and shown as a dash-dotted red line in the middle picture of Fig. 1. Because the zero initial condition which is penalized in this case, but which is assumed in (26), the output signal starts at 0 and converges against the true output signal after $t = 2\,\mathrm{s}$. Afterwards, (33) is used to calculate the closed-form solution. A memory length of $L = 20$ or rather $T_{\mathrm{SMP}} = 0.2\,\mathrm{s}$ is used. Because, also in this case, no past data are available, the calculated output signal begins at $t = 1.2\,\mathrm{s}$. In the middle picture of Fig. 1, this output signal considering the short-memory principle is shown as a dashed yellow line. No obvious divergence is between the closed-form solution with zero initial condition starting at $t = 0\,\mathrm{s}$ and the one with the short-memory principle. The error is also shown in the lowest picture of Fig. 1. Therefore, it can be concluded that after the time corresponding to the memory length the error is small.

IV. INSTRUMENTAL VARIABLE METHOD USING THE MODULATING FUNCTION METHOD

In e.g. [4, 5, 6], it is shown that using the least squares method for parameter identification the parameters can not be estimated consistently. In this section, the refined instrumental variable method is described for the modulating function method. Because the undisturbed output signal has to be estimated analogous to the integer order case, an auxiliary model with the latest identified parameters is set up (see e.g. [23]). This model is used to simulate the output signal which is obtained applying the closed-form solution with the short-memory principle (33). In opposite to the closed-form solution with zero initial condition, this approach leads to a small error and the output data can be used after the time corresponding to the memory length.

Before the consist estimation can be shown, some variables have to be introduced: $q \geq n+m+1$, $t_\tau = e + \tau\Delta T$ and $t'_\tau = t_\tau + T$ where ΔT is the shifting time, T the identification time, $t_0 = e$ the start time of the identification and $\tau \in 0,\ldots,q-1$.

Lemma 5: To estimate the unknown parameters, the linear system

$$\underline{Y} = \underline{M}\underline{p} \tag{36}$$

has to be solved where $\underline{p}$ are the unknown parameters, $\underline{Y} := [y_m(0), y_m(1), \cdots, y_m(q-1)]^\top$ describes the α_n-th fractional derivative of the modulated output signal

$$y_m(\tau) := \int_{t_\tau}^{t'_\tau} \tilde{y}(t)\,{}_t\mathrm{d}GL_{t'_\tau}^{\alpha_n}\gamma(t-\tau\Delta T)\mathrm{d}t. \tag{37}$$

$\underline{M} := [\underline{m}^\top(0), \underline{m}^\top(1), \cdots, \underline{m}^\top(q-1)]^\top$ where $\underline{m}^\top(\tau) \in \mathbb{R}^{(n+m+1)\times 1}$ given in (40) are all other modulated derivatives of the output and input signal.

Proof: The proof can be found in [4]. ■

The resulting equation error using the modulating function method is

$$e(\tau) = y_m(\tau) - \underline{m}^\top(\tau)\,\underline{p} \tag{38}$$

where $y_m(\tau)$ is given in (37) and $\underline{m}^\top(\tau)$ is given in (40) (see [23]).

Then, the estimation equation of the instrumental variable method is given by

$$\underline{p} = \left(\underline{W}^\top\underline{M}\right)^{-1}\underline{W}^\top\underline{Y} \tag{39}$$

where $\underline{W} := [\underline{w}^\top(0), \underline{w}^\top(1), \cdots, \underline{w}^\top(q-1)]^\top$ and $\underline{w}^\top(\tau) \in \mathbb{R}^{(n+m+1)\times 1}$ are the instrumental variables given in (41) (see [23]).

In [23], conditions of a consistent estimation for the instrumental variable methods are given. These conditions are given in the following lemma using the definitions above.

Lemma 6: The parameters can be estimated consistently in the mean square if the conditions

- The number of parameters (n and m) has to be known.
- The input signal $u(t)$ has to be known exactly.
- The mean value of the equation error is zero $\mathrm{E}\{e(\tau)\} = 0$.
- The equation error $e(\tau)$ is not correlated with the instrumental variables $w(\tau)$.

are satisfied.

Proof: The proof is structured in four parts corresponding to the conditions. First, considering that the system is modeled using white box or gray box modeling the number of parameters is known. If black box modeling is used, the number of parameters have to be fixed in this case and, afterwards, it is known.

Second, the input signal is specified for the identification. Therefore, the used input signal is known exactly.

Third, inserting the true parameter values in (40) and assuming that the modulating function fulfills $(P4)$ leads to the following equation error

$$\begin{aligned} e(\tau) = &\int_{t_\tau}^{t'_\tau} \epsilon(t)\,{}_t\mathrm{d}GL_{t'_\tau}^{\alpha_n}\gamma(t-\tau\Delta T)\mathrm{d}t + \ldots + \\ &+ a_0\int_{t_\tau}^{t'_\tau} \epsilon(t)\,{}_t\mathrm{d}GL_{t'_\tau}^{\alpha_0}\gamma(t-\tau\Delta T)\mathrm{d}t \end{aligned} \tag{42}$$

Using a numerical integration and rewriting everything as a sum results in

$$\begin{aligned} e(\tau) = &\sum_{i=0}^{n} a_i T_s \sum_{s=0}^{\lfloor \frac{t'_\tau - t_\tau}{T_s} \rfloor} P(s)\epsilon(t_s)\,{}_t\mathrm{d}GL_{t'_\tau}^{\alpha_i}\gamma(t-\tau\Delta T)\Big|_{t=t_s} + \\ &\sum_{i=0}^{n} a_i\mathcal{F}_i \end{aligned} \tag{43}$$

where $\alpha_n = 1$, $t_s = t_\tau + sT_s$, $P(s)$ are the weights of the numerical integration and $\mathcal{F}_i$ the numerical errors of the integral approximation. For the calculation of the expected value, it is assumed that the noise $\epsilon(t)$ is uncorrelated to the modulating function $\gamma(t)$ and, hence, (44) results. Because of Assum. 1, the first part of (44) vanishes. Therefore, the mean value of the equation error is zero if T_s tends to 0.

Fourth, it is shown that under the assumption that an estimation of the undisturbed output signal is given the equation error and the instrumental variables are not correlated. Starting point is the correlation coefficient

$$\rho_{e,w} := \frac{\mathrm{E}\{(e(\tau) - \mathrm{E}\{e(\tau)\})(\underline{w}^\top(\tau) - \mathrm{E}\{\underline{w}^\top(\tau)\})\}}{\sigma_e\sigma_w} \tag{45}$$

where σ_e is the standard deviation of the equation error and σ_w of the instrumental variables. Using the result above and the fact that $\mathrm{E}\{\underline{w}^\top(\tau)\}$ takes on a value, it leads to

$$\rho_{e,w} := \frac{\mathrm{E}\{e(\tau)\,\underline{w}^\top(\tau)\}}{\sigma_e\sigma_w}. \tag{46}$$

Because of the linearity, each element of $\underline{w}$ can be investigated individually and either the numerator has to become 0 or the denominator has to tend to infinity. Only the numerator is evaluated and here, exemplary,

$$w_1(\tau) = -\int_{t_\tau}^{t'_\tau} y_s(t)\,{}_t\mathrm{d}GL_{t'_\tau}^{\alpha_{n-1}}\gamma(t-\tau\Delta T)\mathrm{d}t \tag{47}$$

is used. The investigated expected value is given in (50). Multiplying all parts, using the linearity of the expected value and assuming that T_s tends to zero leads to (51). Again, the assumption that the modulating function is uncorrelated to the other signals is used and it results in (52). Because the simulated signal $y_s(t)$ is not correlated to the noise and Assum. 1 is made,

$$\rho_{e,w1} = 0 \tag{48}$$

$$\underline{m}^{\top}(\tau) := \Bigg[-\int_{t_\tau}^{t_\tau'} \tilde{y}(t)\, {}_t\mathrm{d}GL_{t_\tau'}^{\alpha_{n-1}}\gamma(t-\tau\Delta T)\mathrm{d}t, \dots, -\int_{t_\tau}^{t_\tau'} \tilde{y}(t)\, {}_t\mathrm{d}GL_{t_\tau'}^{\alpha_0}\gamma(t-\tau\Delta T)\mathrm{d}t, \int_{t_\tau}^{t_\tau'} u(t)\, {}_t\mathrm{d}GL_{t_\tau'}^{\beta_m}\gamma(t-\tau\Delta T)\mathrm{d}t, \dots, \int_{t_\tau}^{t_\tau'} u(t)\, {}_t\mathrm{d}GL_{t_\tau'}^{\beta_0}\gamma(t-\tau\Delta T)\mathrm{d}t \Bigg] \tag{40}$$

$$\underline{w}^{\top}(\tau) := \Bigg[-\int_{t_\tau}^{t_\tau'} y_s(t)\, {}_t\mathrm{d}GL_{t_\tau'}^{\alpha_{n-1}}\gamma(t-\tau\Delta T)\mathrm{d}t, \dots, -\int_{t_\tau}^{t_\tau'} y_s(t)\, {}_t\mathrm{d}GL_{t_\tau'}^{\alpha_0}\gamma(t-\tau\Delta T)\mathrm{d}t, \int_{t_\tau}^{t_\tau'} u(t)\, {}_t\mathrm{d}GL_{t_\tau'}^{\beta_m}\gamma(t-\tau\Delta T)\mathrm{d}t, \dots, \int_{t_\tau}^{t_\tau'} u(t)\, {}_t\mathrm{d}GL_{t_\tau'}^{\beta_0}\gamma(t-\tau\Delta T)\mathrm{d}t \Bigg] \tag{41}$$

where $y_s(t)$ is the simulated output signal using (33).

$$\mathrm{E}\{e(\tau)\} = \sum_{i=0}^{n} a_i T_s \sum_{s=0}^{\lfloor \frac{t_\tau' - t_\tau}{T_s} \rfloor} P(s)\mathrm{E}\{\epsilon(t_s)\}\,\mathrm{E}\left\{ {}_t\mathrm{d}GL_{t_\tau'}^{\alpha_i}\gamma(t-\tau\Delta T)\Big|_{t=t_s} \right\} + \sum_{i=0}^{n} a_i \mathrm{E}\{\mathcal{F}_i\} \tag{44}$$

follows. This is also true for all other elements of $\underline{w}^{\top}(t)$ because the result does not depend on the fractional order and the input signal $u(t)$ is known exactly. Therefore,

$$\rho_{e,w} = 0 \tag{49}$$

and it follows directly that the equation error is not correlated with the instrumental variable. ■

V. EXAMPLE

The system (35) which was used to demonstrate the short-memory principle in Section III-B is also the example system for the parameter identification:

$$\begin{aligned} {}_e\mathrm{D}RL_t^{\alpha_2}y(t) + a_1\, {}_e\mathrm{D}RL_t^{\alpha_1}y(t) + a_0 y(t) = \\ b_1\, {}_e\mathrm{D}RL_t^{\beta_2}u(t) + b_0\, {}_e\mathrm{D}RL_t^{\beta_1}u(t) \end{aligned} \tag{53}$$

where $a_1 = 2$, $a_0 = 3$, $b_1 = 1$ and $b_0 = 4$ as well as $\alpha_2 = 0.8$, $\alpha_1 = 0.5$, $\beta_2 = 0.5$ and $\beta_1 = 0.1$. The chosen modulating function is the spline-type modulating (25) with $\varsigma = 20$ and $\nu = 9$. The identification time is set to $T = 20\,\mathrm{s}$ and the modulating function is shifted by $\Delta T = 6\,\mathrm{s}$. The duration of the simulation is $T_g = 170\,\mathrm{s}$ with a sampling time of $T_s = 0.01\,\mathrm{s}$. Even though the simulation starts at $a = 0\,\mathrm{s}$, the begin of the identification is at $e = 15\,\mathrm{s}$. The Gaussian white noise which leads to a signal-to-noise ratio of SNR = $38.4\,\mathrm{dB}$ superimpose the output signal.

In Fig. 2, the pseudo random binary signal which is used as the input signal and the output signal are shown. The original output signal is displayed as a dashed blue course and the noisy observation which is used for the parameter identification as a solid red course. The begin of the identification is marked at $e = 15\,\mathrm{s}$.

The continuous integrals arising in the modulating function method are approximated using the trapezoidal rule with uniform grid. The parameters are identified using the described instrumental variable method in matrix form (39). The parameter identification result is compared to the least squares method which is not bias-free considering a noisy observation of the output signal (see e.g. [4, 5, 6]). In Fig. 3, the evolution of the estimated parameters $\underline{p} = [a_1, a_0, b_1, b_0]^{\top}$ is shown. In contrast to the parameters identified using the lest squares approach, the identified parameters using the described instrumental variable method converge against the true values despite the output signal is superimposed by Gaussian white noise. It should be noted that in this case the convergence against the true values needs more than ten times of iterations than in the noise-free case. In this case after four iterations the true values are reached (see [4]). In the noisy case, 45 iterations have to be done.

It has also to be mentioned that the first four iteration leads to the same parameters because not enough data have been collected so the system can not be simulated. Due to this fact no instrumental variable can be set up and the standard least-squares method is used till this iteration. From the fifth iteration, the described instrumental variable method is applied and the results differ from the standard least-squares method.

VI. CONCLUSIONS

This paper focuses on the parameter identification of a non-commensurable fractional system considering a noisy observation of the output signal. An assumption is that the investigated system need not to be at rest because it would be a conservative assumption regarding real world application (see [4]). So that the measured signal need not to be derived and the initialization function of the system does not have to be taken into account, the modulation function method is applied using the spline-type modulating function. In contrast to other known approaches (see e.g. [5, 6]), neither a property of the noise is used nor its future influence is estimated to eliminate the bias, but the instrumental variable is extended to the modulating function method applied to a non-commensurable fractional system.

$$
\begin{aligned}
\rho_{e,w1} = \mathrm{E}\Bigg\{\Bigg(&\Bigg(T_s \sum_{s_1=0}^{\lfloor \frac{t'_\tau - t_\tau}{T_s} \rfloor} P(s_1)\epsilon\left(t_{s1}\right)\ {}_t\mathrm{d}GL_{t'_\tau}^{\alpha_i}\gamma(t-\tau\Delta T)\Big|_{t=t_{s1}} + \mathcal{F}_\epsilon\Bigg) \\
&\Bigg(T_s \sum_{s_2=0}^{\lfloor \frac{t'_\tau - t_\tau}{T_s} \rfloor} P(s_2)y_s\left(t_{s2}\right)\ {}_t\mathrm{d}GL_{t'_\tau}^{\alpha_{n-1}}\gamma(t-\tau\Delta T)\Big|_{t=t_{s2}} + \mathcal{F}_y\Bigg)\Bigg)\Bigg\}
\end{aligned}
\tag{50}
$$

$$
= \mathrm{E}\left\{T_s^2 \sum_{s_1=0}^{\lfloor \frac{t'_\tau - t_\tau}{T_s} \rfloor}\left(P(s_1)\epsilon\left(t_{s1}\right)\ {}_t\mathrm{d}GL_{t'_\tau}^{\alpha_i}\gamma(t-\tau\Delta T)\Big|_{t=t_{s1}} \sum_{s_2=0}^{\lfloor \frac{t'_\tau - t_\tau}{T_s} \rfloor} P(s_2)y_s\left(t_{s2}\right)\ {}_t\mathrm{d}GL_{t'_\tau}^{\alpha_{n-1}}\gamma(t-\tau\Delta T)\Big|_{t=t_{s2}}\right)\right\} \tag{51}
$$

$$
= T_s^2 \sum_{s_1=0}^{\lfloor \frac{t'_\tau - t_\tau}{T_s} \rfloor}\sum_{s_2=0}^{\lfloor \frac{t'_\tau - t_\tau}{T_s} \rfloor} P(s_1)P(s_2)\mathrm{E}\left\{\epsilon\left(t_{s1}\right)y_s\left(t_{s2}\right)\right\}\mathrm{E}\left\{{}_t\mathrm{d}GL_{t'_\tau}^{\alpha_i}\gamma(t-\tau\Delta T)\Big|_{t=t_{s1}}\ {}_t\mathrm{d}GL_{t'_\tau}^{\alpha_{n-1}}\gamma(t-\tau\Delta T)\Big|_{t=t_{s2}}\right\} \tag{52}
$$

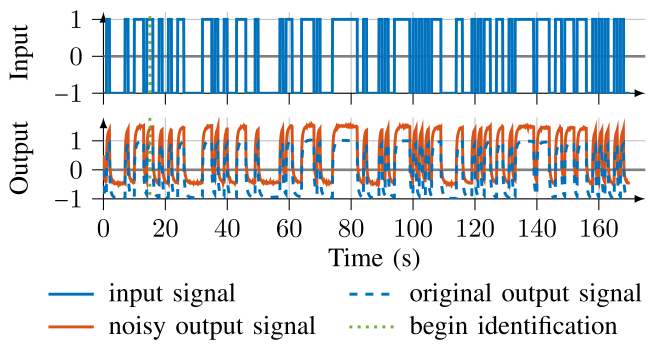

Fig. 2. Input and output signals with an unknown for parameter estimation

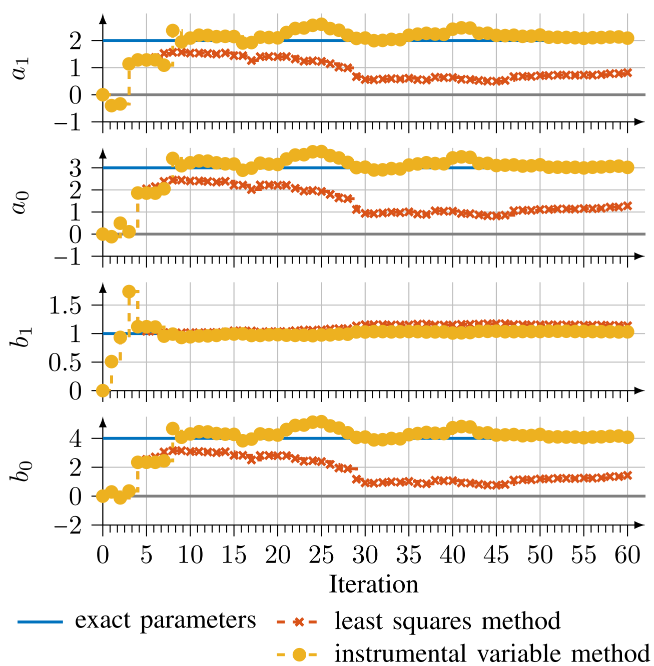

Fig. 3. Evolution of the estimated parameters using the least-squares method and the presented instrumental variable method for fractional systems

To set up the instrumental variable, it is recommended to use a noise-free simulation of the output signal (see [23]). The existing algorithms are considering zero initial conditions. Therefore, an existing algorithm to evaluate the output signal of a fractional system is extended with the short-memory principle. This enables the online simulation of the fractional system with a fixed time on which the simulation is close to the true value. In addition, the presented approach is valid for a numerical implementation because the Grünwald-Letnikov derivative underlies all calculation.

The choice of the memory length and the freely chosen parameters of the modulating function have great influence on the parameter identification results which have to be investigated carefully. Also, it can not be assumed that after the minimum number of iterations the instrumental variable method is converged. So, the convergence behavior of the proposed instrumental variable method using the modulating function method within should be a topic of future works.

REFERENCES

[1] A. S. V. R. Kanth and N. Garg. "Computational Simulations for Solving a Class of Fractional Models via Caputo-Fabrizio Fractional Derivative". In: *Procedia Computer Science*. The 6th International Conference on Smart Computing and Communications 125 (2018), pp. 476–482.

[2] D. Valério and I. Tejado. "Identifying a Non-Commensurable Fractional Transfer Function from a Frequency Response". In: *Signal Processing*. Special Issue on Ad Hoc Microphone Arrays and Wireless Acoustic Sensor Networks 107 (2015), pp. 254–264.

[3] D.-Y. Liu, T. Laleg-Kirati, O. Gibaru, and W. Perruquetti. "Identification of Fractional Order Systems Using Modulating Functions Method". In: *American Control Conference (ACC)*. 2013, pp. 1679–1684.

[4] O. Stark, M. Kupper, S. Krebs, and S. Hohmann. "Online Parameter Identification of a Fractional Order Model". In: *57th IEEE Conference on Decision and Control (CDC)*. 2018, pp. 2303–2309.

[5] Y. Dai, Y. Wei, Y. Hu, and Y. Wang. “Modulating Function-Based Identification for Fractional Order Systems”. In: *Neurocomputing* 173 (2016), pp. 1959–1966.

[6] Z. Gao. “Modulating Function-Based System Identification for a Fractional-Order System with a Time Delay Involving Measurement Noise Using Least-Squares Method”. In: *International Journal of Systems Science* 48 (2017), pp. 1460–1471.

[7] I. Podlubny. *Fractional Differential Equations*. Cambridge: Academic Press, 1998.

[8] S. G. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional Integrals and Derivatives: Theory and Applications*. 1st ed. Switzerland; Philadelphia, Pa., USA: CRC Press, 1993.

[9] S. Pooseh, R. Almeida, and D. F. M. Torres. “Discrete Direct Methods in the Fractional Calculus of Variations”. In: *Computers & Mathematics with Applications*. Fractional Differentiation and Its Applications 66 (2013), pp. 668–676.

[10] C. F. Lorenzo and T. T. Hartley. “Initialization, Conceptualization, and Application in the Generalized (Fractional) Calculus”. In: *Critical Reviews in Biomedical Engineering* 35 (2007), pp. 447–553.

[11] C. F. Lorenzo and T. T. Hartley. *Initialized Fractional Calculus*. Tech. rep. Cleveland: NASA Glenn Research Center, 2000.

[12] M. Eckert, L. Kölsch, and S. Hohmann. “Fractional Algebraic Identification of the Distribution of Relaxation Times of Battery Cells”. In: *54th IEEE Conference on Decision and Control (CDC)*. Osaka, Japan, 2015, pp. 2101–2108.

[13] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and Applications of Fractional Differential Equations*. 1st ed. Vol. 204. Amsterdam: North-Holland Mathematics Studies, 2006.

[14] K. Diethelm. *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*. Lecture Notes in Mathematics. Berlin Heidelberg: Springer-Verlag, 2010.

[15] M. Shinbrot. *On the Analysis of Linear and Nonlinear Dynamical Systems from Transient-Response Data*. Tech. rep. Moffett Field: National Advisory Committee for Aeronautics, 1954.

[16] M. Eckert, M. Kupper, and S. Hohmann. “Functional Fractional Calculus for System Identification of Battery Cells”. In: *at – Automatisierungstechnik* 62 (2014), pp. 272–281.

[17] H. A. Preisig and D. W. T. Rippin. “Theory and Application of the Modulating Function Method—I. Review and Theory of the Method and Theory of the Spline-Type Modulating Functions”. In: *Computers & Chemical Engineering*. An International Journal of Computer Applications in Chemical Engineering 17 (1993), pp. 1–16.

[18] X. Wei, D.-Y. Liu, D. Boutat, H.-R. Liu, and Y.-Q. Li. “Modulating Functions-Based Fractional Order Differentiator for Fractional Order Linear Systems with a Biased Output”. In: *IFAC-PapersOnLine*. 20th IFAC World Congress 50 (2017), pp. 12859–12865.

[19] V. Maletinsky. “”l-i-P” Identifikation kontinuierlicher dynamischer Prozesse”. PhD thesis. ETH Zurich, 1978.

[20] D. Xue. *Fractional-Order Control Systems, Fundamentals and Numerical Implementations*. Berlin, Boston: De Gruyter, 2017.

[21] A. Tepljakov. *FOMCON Toolbox for MATLAB*. `https://de.mathworks.com/matlabcentral/fileexchange/66323-fomcon-toolbox-for-matlab`. 2018.

[22] D. Xue. *FOTF Toolbox*. `https://de.mathworks.com/matlabcentral/fileexchange/60874-fotf-toolbox`. 2017.

[23] R. Isermann and M. Münchhof. *Identification of Dynamic Systems: An Introduction with Applications*. Berlin Heidelberg: Springer-Verlag, 2011.