Gauge Coupling β Functions to Four-Loop Order in the Standard Model

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We compute the β functions of the three standard model gauge couplings to four-loop order in the modified minimal subtraction scheme. At this order a proper definition of γ_5 in $D = 4 - 2\epsilon$ space-time dimensions is required; however, in our calculation we determine the γ_5-dependent terms by exploiting relations with β function coefficients at lower loop orders.

Introduction.—β functions are fundamental quantities of quantum field theories. They are important ingredients of the renormalization group equations and determine the energy dependence of the couplings. The perturbative coefficients that are currently available enter into a variety of applications, among which is the running of the standard model (SM) couplings from the electroweak scale to the scale where the coupling of the quartic terms in the scalar potential turns negative and the vacuum becomes unstable [1–3]. A precise running of the coupling constants is also needed in the context of the prediction of Higgs boson masses within the minimal supersymmetric extension of the SM (MSSM). In the approach discussed, e.g., in Ref. [4], all SM quantities are evolved to the supersymmetric scale, which is usually of the order of a few TeV, where the matching between the SM and the MSSM is performed.

The gauge structure of the SM of particle physics is given by SU(3) × SU(2) × U(1), and thus there are three gauge couplings. In this Letter we compute their β functions to four-loop accuracy, with the only approximation that the Yukawa couplings of the first and second generations are set to zero. For our calculation we adapt the widely used $\overline{\text{MS}}$ scheme. Furthermore, since the β functions are mass independent, we can work in the unbroken phase of the SM in which all particles are massless.

Within the SM a number of correction terms to the various β functions are available. The discovery of asymptotic freedom in non-Abelian gauge theories [5,6] prompted the computation of two-loop corrections within the strong sector of the SM, which became available shortly afterwards [7–10]. Three- and four-loop corrections have been

computed in [11,12] and [13,14], respectively, and recently even the five-loop term became available [15–18].

Two-loop corrections to the β functions of all couplings of the SM can be found in Refs. [19–22], and the three-loop corrections to all gauge coupling β functions have been computed in [23–25]. The three-loop gauge coupling β function for a general quantum field theory based on a single gauge group has been computed in [26]. The three-loop Yukawa coupling β functions have been considered in [27–30] and the scalar self coupling β functions in [31–33]. At four-loop order partial results are available; in [34,35] the scalar self coupling β function and in [36,37] the top quark Yukawa contributions to the QCD β function have been computed.

In the approximation that the Yukawa couplings of the first and second generation fermions are neglected, the SM has seven couplings. Their β functions are defined as

$$\mu^2 \frac{d}{d\mu^2} \frac{\alpha_i}{\pi} = \beta_i(\{\alpha_j\}, \epsilon), \qquad (1)$$

with $i = 1, ..., 7$, where $d = 4 - 2\epsilon$ is the space-time dimension, μ is the renormalization scale and $\{\alpha_j\}$ denotes dependence on all seven couplings. α_1, α_2, and α_3 are the three gauge couplings, which we define using a SU(5)-like normalization

$$\alpha_1 = \frac{5}{3} \frac{\alpha_{\text{QED}}}{\cos^2\theta_W}, \qquad \alpha_2 = \frac{\alpha_{\text{QED}}}{\sin^2\theta_W}, \qquad \alpha_3 = \alpha_s, \quad (2)$$

where α_{QED} is the fine structure constant, θ_W is the weak mixing angle and α_s is the strong coupling constant. In order to fix the Yukawa couplings, we provide the corresponding part of the Lagrange density,

$$\mathcal{L} \supset y_t \bar{Q}_L(i\tau_2 \Phi^*) t_R + y_b \bar{Q}_L \Phi b_R + y_\tau \bar{L}_L \Phi \tau_R + \text{H.c.}, \quad (3)$$

where τ_2 is the second Pauli matrix, Q_L and L_L are the third generation left-handed quark and lepton doublets, Φ the Higgs doublet and t_R, b_R, τ_R are the right-handed top, bottom, and τ fields. We use the coupling factors y_i to define the third-generation Yukawa couplings as

$$\alpha_4 = \frac{y_t^2}{4\pi}, \qquad \alpha_5 = \frac{y_b^2}{4\pi}, \qquad \alpha_6 = \frac{y_\tau^2}{4\pi}. \qquad (4)$$

Finally, we provide the quartic term of the scalar potential, which fixes α_7:

$$\mathcal{L} \supset -(4\pi\alpha_7)(\Phi^\dagger\Phi)^2. \qquad (5)$$

The β functions are obtained from the renormalization constants using the formula (see, e.g., [23,24])

$$\beta_i = -\left[\epsilon\frac{\alpha_i}{\pi} + \frac{\alpha_i}{Z_{\alpha_i}}\sum_{j=1,j\neq i}^{7}\frac{\partial Z_{\alpha_i}}{\partial\alpha_j}\beta_j\right]\left(1 + \frac{\alpha_i}{Z_{\alpha_i}}\frac{\partial Z_{\alpha_i}}{\partial\alpha_i}\right)^{-1}, \qquad (6)$$

where the renormalization constants are obtained from the relations between the bare and renormalized couplings,

$$\alpha_i^0 = \mu^{2\epsilon}Z_{\alpha_i}(\{\alpha_j\}, \epsilon)\alpha_i. \qquad (7)$$

Note that the Yukawa and self couplings enter the gauge coupling renormalization constants for the first time at two- and three-loop order, respectively. Thus, from Eq. (6) one learns that the four-loop gauge coupling β functions require the knowledge of the two-loop Yukawa coupling β functions and one-loop β function for α_7.

Weyl consistency conditions.—As we will discuss below, the computation of the renormalization constants can be reduced to the evaluation of massless four-loop two-point functions. Although methods for this have been available for a few years, to date the four-loop corrections to the β functions in the electroweak sector have not been computed. The main reason for this is connected to traces containing an odd number of γ_5 matrices: whereas at three-loop order a seminaïve treatment is possible, a proper treatment is (in principle) required at four loops. The classes of diagrams that might require such a treatment need to have at least two (open or closed) fermion lines with sufficiently many vertices. In our case, only the diagram classes shown in Fig. 1 satisfy this criterion. For massless fermions the diagrams in the top row are zero, since all traces involve an odd number of gamma matrices. Furthermore, in the left diagram in the second row the dangerous contributions cancel due to anomaly cancellations within the SM. This leaves only the class of diagrams with two fermion loops that are connected by one vector and two scalar bosons. In Refs. [36,37] such diagrams have been considered for the case where the gauge boson is a
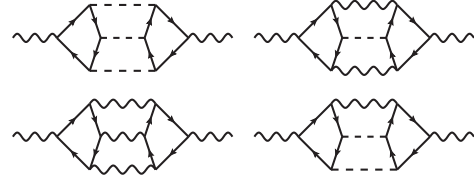

FIG. 1. Representative four-loop diagrams for classes that might involve nontrivial γ_5 contributions. Wavy, dashed, and straight lines represent gauge bosons, scalar bosons, and fermions, respectively.

gluon. In order to treat the problematic traces, the cyclicity of the traces was abandoned, and different results were obtained depending on what starting point was used to write down the traces.

In the literature one finds various prescriptions for the treatment of γ_5 in D dimensions, see, e.g., Refs. [38–42]. Many of these have been successfully applied in various calculations either in pure QCD or at lower loop order. In our opinion there is no practical prescription that can be applied at fourth order in perturbation theory. However, very recently in Refs. [43,44] Weyl consistency conditions [45–48] have been used in order to establish, with the help of "Osborn's equation," relations between coefficients of the general four-loop gauge, three-loop Yukawa, and two-loop scalar β functions. Osborn's equation shows that the β functions of a general theory are related to a particular scalar function of the couplings, $\tilde{A}$, which at renormalization group fixed points reduces to the coefficient of the Euler density in the trace anomaly. Since all β functions are related (by extension) through $\tilde{A}$, it has been observed that Osborn's equation implies constraints on the β functions, even in the absence of the explicit calculation of any quantity entering the equation. By parametrizing each term as a polynomial in the couplings (respecting flavor symmetries) and manipulating the resulting algebraic relations, one obtains a number of nontrivial constraints between various coefficients of the β functions at different loop orders, referred to as consistency conditions. It was realized in [43,44] that these relations fix all nontrivial γ_5 contributions to the four-loop gauge coupling β function in terms of known coefficients of the three-loop Yukawa β function. In particular, the results of [43,44] could resolve the ambiguity of the four-loop top Yukawa contribution to the β function of the strong coupling constant, which was identified in [36,37].

This observation fixes the outline for our computation; we decompose the β functions into color structures of the three gauge groups. We then perform an explicit computation of those parts of the renormalization constants that do not involve traces with an odd number of γ_5 matrices and fix the remaining parts using the results obtained in [43,44]. In addition, in Refs. [43,44] many further relations have been established that demonstrate the consistency between

predictions derived from Osborn's equation and explicit computations.

Calculation.—For the computation of the gauge coupling renormalization constants, one can, in principle, use any vertex that contains the respective coupling at tree level. The renormalization constant is then obtained by

$$Z_{g_i} = \frac{Z_{\text{vert}}}{\Pi_k \sqrt{Z_{k,\text{wf}}}},\qquad(8)$$

where Z_{vert} stands for the renormalization constant of the vertex and $Z_{k,\text{wf}}$ for the wave function renormalization constants (k runs over all external particles). For the SU(2) and SU(3) gauge groups, it is advantageous to choose the ghost-gauge-boson vertices since one has to deal with fewer diagrams, amounting to $\mathcal{O}(350,000)$ for SU(2) and $\mathcal{O}(200,000)$ for SU(3) at four-loop order. For the U(1) gauge group, it is sufficient to consider the gauge boson propagator renormalization constant for which $\mathcal{O}(200,000)$ four-loop diagrams have to be computed. Sample Feynman diagrams for the various Green's functions we consider are shown in Fig. 2.

Our calculation is based on a well-tested setup, which uses QGRAF [49] for the generation of the amplitudes and Q2E and EXP [50–52] for the mapping to integral families and generation of FORM [53] code. We use COLOR [54] for the computation of the SU(2) and SU(3) color factors. Before the computation we combine diagrams with the same color structure and integral family to form so-called superdiagrams, which guarantees possible cancellations at earlier stages of the calculation. This reduces the computational effort required.

In the unbroken phase of the SM, all particles are massless, and thus all two-point Green's functions lead to massless propagator-type integrals up to four loops. Furthermore, one may set one of the external momenta of the three-point Green's functions to zero. This is possible since the diagrams are logarithmically divergent, and thus the ultraviolet divergences, which must be computed to obtain the renormalization constants in the $\overline{\text{MS}}$ scheme, are independent of kinematic quantities. Consequently, one only needs to compute massless propagator-type integrals
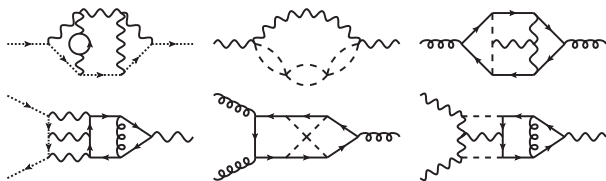

FIG. 2. Sample Feynman diagrams contributing to the Green's functions that have been used for our calculation of the gauge coupling renormalization constants. Solid, dashed, dotted, curly, and wavy lines denote fermions, scalar bosons, ghosts, gluons, and electroweak gauge bosons, respectively.

up to four loops; this also holds for vertex corrections. For this task we use the program FORCER [55].

In our calculation we use an anticommuting γ_5, with $\gamma_5^2 = 1$, and set traces with an odd number of γ_5 occurrences to zero. These contributions to the β functions (and thus to the gauge coupling renormalization constants) are reconstructed using the approach described in the previous section. Note that the ghost-gauge-boson vertices do not suffer from ambiguities related to γ_5. This allows us to reconstruct the nontrivial γ_5 contributions to the renormalization constants of the gauge boson wave functions.

We retain full dependence on all three gauge parameters during the calculation. Whereas the renormalization constants for the vertices and wave functions still depend on the gauge parameters, the dependence drops out in the renormalization constants of the gauge couplings. This serves as a welcome check of our calculation.

As a further strong check we use the triple gauge boson vertices for the SU(2) and SU(3) gauge bosons to recompute the gauge coupling renormalization constants, and find agreement. Furthermore, we verify by explicit calculation that U(1) loop corrections to the triple gauge boson vertex vanish after all bare Feynman diagrams are added. The calculation of these Green's functions are significantly more costly than our default choice; for this reason we fix the gauge parameters to the Feynman gauge for these Green's functions only.

We have performed several cross-checks of our four-loop expressions with results available in the literature. The pure gauge-fermion parts of β_2 and β_3 agree with the findings for a general Yang-Mills theory with fermions in the fundamental representation [13,14]. Furthermore, the contributions to β_3 involving only the strong gauge coupling, the top quark Yukawa coupling and the quartic scalar coupling agree with [36,37].

Finally, the Weyl consistency conditions from Refs. [43,44] represent powerful cross-checks on various coefficients in the β functions, via their relation to the general result. The parametrization of the general four-loop gauge β function (valid for all renormalizable four-dimensional quantum field theories) has 202 coefficients. As mentioned above, the four calculated in [43] determine all contributions from traces over an odd number of γ_5 matrices, and are used directly in the computation of the β functions. While we do not yet have a complete determination of the other 198, matching the general result to our SM calculation does uniquely fix 80; comparison with the full set of 261 consistency conditions in [44] (which also involve coefficients of the general three-loop Yukawa β function) verifies these 80, and fixes another 28. Crucially, we find that these 108 coefficients (and by extension our four-loop computation) are indeed consistent with all Weyl consistency conditions, providing highly nontrivial corroboration.

Results.—Our final results for the gauge coupling β functions contain the full dependence on the gauge and Higgs self couplings and the third generation Yukawa couplings. The analytic results are available in a computer-readable format from [56] for numerical values of the SU(2) and SU(3) Casimir invariants. Due to space restrictions we reproduce below the results for vanishing bottom and tau Yukawa couplings, $\alpha_5 = \alpha_6 = 0$. They are given by

$$
\begin{aligned}
\beta_1 ={}& \frac{\alpha_1^2}{(4\pi)^2}\left(\frac{82}{5}\right) + \frac{\alpha_1^2}{(4\pi)^3}\left(\frac{398\alpha_1}{25} + \frac{54\alpha_2}{5} + \frac{176\alpha_3}{5} - \frac{34\alpha_4}{5}\right) + \frac{\alpha_1^2}{(4\pi)^4}\left(-\frac{388613\alpha_1^2}{6000} + \frac{123\alpha_1\alpha_2}{40} - \frac{548\alpha_1\alpha_3}{75}\right.\\
&+ \frac{789\alpha_2^2}{16} - \frac{12\alpha_2\alpha_3}{5} + \frac{1188\alpha_3^2}{5} - \frac{2827\alpha_1\alpha_4}{200} - \frac{471\alpha_2\alpha_4}{8} - \frac{116\alpha_3\alpha_4}{5} + \frac{189\alpha_4^2}{4} + \frac{54\alpha_1\alpha_7}{25} + \frac{18\alpha_2\alpha_7}{5} - \frac{36\alpha_7^2}{5}\right)\\
&+ \frac{\alpha_1^2}{(4\pi)^5}\left[-\alpha_1^3\left(\frac{143035709}{1080000} + \frac{1638851\zeta_3}{5625}\right) - \alpha_1^2\alpha_2\left(\frac{3819731}{24000} - \frac{16529\zeta_3}{125}\right) - \alpha_1^2\alpha_3\left(\frac{3629273}{6750} - \frac{720304\zeta_3}{1125}\right)\right.\\
&+ \alpha_1\alpha_2^2\left(\frac{572059}{14400} - \frac{6751\zeta_3}{75}\right) - \frac{69\alpha_1\alpha_2\alpha_3}{25} + \alpha_1\alpha_3^2\left(\frac{333556}{675} - \frac{274624\zeta_3}{225}\right) - \alpha_2^3\left(\frac{117923}{2880} + \frac{3109\zeta_3}{5}\right)\\
&- \alpha_2^2\alpha_3\left(\frac{41971}{90} - \frac{7472\zeta_3}{15}\right) - \alpha_2\alpha_3^2\left(\frac{1748}{3} - \frac{2944\zeta_3}{5}\right) + \alpha_3^3\left(\frac{6116}{15} - \frac{18560\zeta_3}{9}\right) + \alpha_1^2\alpha_4\left(\frac{8978897}{72000} + \frac{2598\zeta_3}{125}\right)\\
&- \alpha_1\alpha_2\alpha_4\left(\frac{42841}{800} + \frac{1122\zeta_3}{25}\right) - \alpha_1\alpha_3\alpha_4\left(\frac{2012}{75} - \frac{408\zeta_3}{25}\right) - \alpha_2^2\alpha_4\left(\frac{439841}{960} - \frac{616\zeta_3}{5}\right) + \alpha_2\alpha_3\alpha_4\left(\frac{1468}{5} - \frac{1896\zeta_3}{5}\right)\\
&- \alpha_3^2\alpha_4\left(\frac{11462}{45} - \frac{3184\zeta_3}{5}\right) + \alpha_1\alpha_4^2\left(\frac{29059}{160} - \frac{357\zeta_3}{25}\right) + \alpha_2\alpha_4^2\left(\frac{71463}{160} + \frac{639\zeta_3}{5}\right) + \alpha_3\alpha_4^2\left(\frac{1429}{5} - 240\zeta_3\right)\\
&- \alpha_4^3\left(\frac{13653}{40} + \frac{102\zeta_3}{5}\right) + \frac{3627\alpha_1^2\alpha_7}{500} + \frac{1917\alpha_1\alpha_2\alpha_7}{50} + \frac{889\alpha_2^2\alpha_7}{20} - \frac{1926\alpha_1\alpha_4\alpha_7}{25} - \frac{162\alpha_2\alpha_4\alpha_7}{5} - \frac{474\alpha_4^2\alpha_7}{5}\\
&\left.\left. - \frac{1269\alpha_1\alpha_7^2}{25} - \frac{981\alpha_2\alpha_7^2}{5} + \frac{1188\alpha_4\alpha_7^2}{5} + \frac{624\alpha_7^3}{5}\right.\right],
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
\beta_2 ={}& \frac{\alpha_2^2}{(4\pi)^2}\left(-\frac{38}{3}\right) + \frac{\alpha_2^2}{(4\pi)^3}\left(\frac{18\alpha_1}{5} + \frac{70\alpha_2}{3} + 48\alpha_3 - 6\alpha_4\right) + \frac{\alpha_2^2}{(4\pi)^4}\left(-\frac{5597\alpha_1^2}{400} + \frac{873\alpha_1\alpha_2}{40} - \frac{4\alpha_1\alpha_3}{5}\right.\\
&+ \frac{324953\alpha_2^2}{432} + 156\alpha_2\alpha_3 + 324\alpha_3^2 - \frac{593\alpha_1\alpha_4}{40} - \frac{729\alpha_2\alpha_4}{8} - 28\alpha_3\alpha_4 + \frac{147\alpha_4^2}{4} + \frac{6\alpha_1\alpha_7}{5} + 6\alpha_2\alpha_7 - 12\alpha_7^2\right)\\
&+ \frac{\alpha_2^2}{(4\pi)^5}\left[-\alpha_1^3\left(\frac{6418229}{72000} - \frac{21173\zeta_3}{375}\right) - \alpha_1^2\alpha_2\left(\frac{787709}{4800} - \frac{659\zeta_3}{25}\right) - \alpha_1^2\alpha_3\left(\frac{52297}{450} - \frac{2032\zeta_3}{15}\right) + \frac{161\alpha_1\alpha_2\alpha_3}{5}\right.\\
&- \alpha_1\alpha_2^2\left(\frac{375767}{2880} - \frac{4631\zeta_3}{15}\right) - \alpha_1\alpha_3^2\left(\frac{1748}{9} - \frac{2944\zeta_3}{15}\right) + \alpha_2^3\left(\frac{124660945}{15552} - \frac{78803\zeta_3}{9}\right) - \alpha_2^2\alpha_3\left(\frac{72881}{18} - \frac{16432\zeta_3}{3}\right)\\
&+ \alpha_2\alpha_3^2\left(\frac{10348}{3} - 2560\zeta_3\right) + \alpha_3^3\left(\frac{1028}{3} - \frac{7040\zeta_3}{3}\right) + \alpha_1^2\alpha_4\left(\frac{465089}{4800} - \frac{498\zeta_3}{25}\right) - \alpha_1\alpha_2\alpha_4\left(\frac{102497}{480} + 28\zeta_3\right)\\
&+ \alpha_1\alpha_3\alpha_4\left(\frac{796}{15} - \frac{376\zeta_3}{5}\right) - \alpha_2^2\alpha_4\left(\frac{500665}{576} - \frac{478\zeta_3}{3}\right) - \alpha_2\alpha_3\alpha_4\left(\frac{1444}{3} + 56\zeta_3\right) - \alpha_3^2\alpha_4\left(\frac{614}{3} - 336\zeta_3\right)\\
&+ \alpha_1\alpha_4^2\left(\frac{3161}{32} + \frac{153\zeta_3}{5}\right) + \alpha_2\alpha_4^2\left(\frac{30213}{32} - 63\zeta_3\right) + \alpha_3\alpha_4^2(239 - 144\zeta_3) - \alpha_4^3\left(\frac{2143}{8} + 18\zeta_3\right) + \frac{457\alpha_1^2\alpha_7}{100}\\
&\left.\left. + \frac{69\alpha_1\alpha_2\alpha_7}{2} + \frac{2905\alpha_2^2\alpha_7}{12} - \frac{54\alpha_1\alpha_4\alpha_7}{5} - 150\alpha_2\alpha_4\alpha_7 - 78\alpha_4^2\alpha_7 - \frac{327\alpha_1\alpha_7^2}{5} - 363\alpha_2\alpha_7^2 + 300\alpha_4\alpha_7^2 + 208\alpha_7^3\right.\right],
\end{aligned}
\tag{10}
$$

$$\beta_3 = \frac{\alpha_3^2}{(4\pi)^2}(-28) + \frac{\alpha_3^2}{(4\pi)^3}\left(\frac{22\alpha_1}{5} + 18\alpha_2 - 104\alpha_3 - 8\alpha_4\right) + \frac{\alpha_3^2}{(4\pi)^4}\left(-\frac{523\alpha_1^2}{30} - \frac{3\alpha_1\alpha_2}{10} + \frac{109\alpha_2^2}{2} + \frac{308\alpha_1\alpha_3}{15}\right.$$

$$\left. + 84\alpha_2\alpha_3 + 130\alpha_3^2 - \frac{101\alpha_1\alpha_4}{10} - \frac{93\alpha_2\alpha_4}{2} - 160\alpha_3\alpha_4 + 60\alpha_4^2\right)$$

$$+ \frac{\alpha_3^2}{(4\pi)^5}\left[-\alpha_1^3\left(\frac{6085099}{54000} - \frac{17473\zeta_3}{225}\right) - \alpha_1^2\alpha_2\left(\frac{46951}{1200} - \frac{973\zeta_3}{25}\right) - \alpha_1^2\alpha_3\left(\frac{35542}{135} - \frac{902\zeta_3}{9}\right) + \frac{69\alpha_1\alpha_2\alpha_3}{5}\right.$$

$$- \alpha_1\alpha_2^2\left(\frac{37597}{720} - \frac{691\zeta_3}{15}\right) - \alpha_1\alpha_3^2\left(\frac{57739}{135} - \frac{32476\zeta_3}{45}\right) - \alpha_2^3\left(\frac{176815}{432} + 935\zeta_3\right) + \alpha_2^2\alpha_3\left(\frac{3812}{9} - \frac{950\zeta_3}{3}\right)$$

$$- \alpha_2\alpha_3^2\left(\frac{5969}{3} - 3476\zeta_3\right) + \alpha_3^3\left(\frac{127118}{9} - \frac{179792\zeta_3}{9}\right) + \alpha_1^2\alpha_4\left(\frac{362287}{3600} - \frac{19\zeta_3}{25}\right) + \alpha_1\alpha_2\alpha_4\left(\frac{77}{40} - 54\zeta_3\right)$$

$$- \alpha_1\alpha_3\alpha_4\left(\frac{1283}{15} + \frac{32\zeta_3}{5}\right) - \alpha_2^2\alpha_4\left(\frac{12887}{48} - 117\zeta_3\right) - \alpha_2\alpha_3\alpha_4(473 + 288\zeta_3) - \alpha_3^2\alpha_4\left(\frac{26836}{9} - 1088\zeta_3\right)$$

$$\left. + \alpha_1\alpha_4^2\left(\frac{3641}{40} + \frac{42\zeta_3}{5}\right) + \alpha_2\alpha_4^2\left(\frac{3201}{8} + 90\zeta_3\right) + \alpha_3\alpha_4^2(1708 - 384\zeta_3) - \alpha_4^3(423 + 24\zeta_3) - 120\alpha_4^2\alpha_7 + 144\alpha_4\alpha_7^2\right],$$

$$(11)$$

where ζ_3 is the Riemann zeta function evaluated at argument 3. It has been observed in [36,37] that at four-loop order the top quark Yukawa corrections amount to 7% of the corrections to β_3. It is interesting to note that the remaining terms, computed in this Letter, cancel much of this contribution such that at the scale $\mu = M_Z$, about 99% of the four-loop coefficient is provided by the pure QCD contribution. At three loops this is not the case; here the remaining terms cancel about 40% of the pure QCD contribution and thus have a significant effect on the value. For this reason, the complete four-loop contribution to β_3 provides a large correction compared to the three-loop contributions. We find that the four-loop contributions to β_1, β_2, and β_3 amount to 8, 5, and 127% of the three-loop contributions.

Summary.—We compute analytic expressions for the four-loop gauge coupling β functions in the SM, which require a consistent treatment of γ_5 in $D = 4 - 2\epsilon$ space-time dimensions. We circumvent this problem by exploiting the findings of Refs. [43,44], which fix the relevant terms through relations with known, unambiguous, lower-order results. Our calculation neglects the Yukawa contributions from the first and second generations, which are numerically small; their inclusion would not pose any practical problem. The calculation performed in this letter represents the highest full-SM loop calculation of phenomenologically relevant quantities to date.

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