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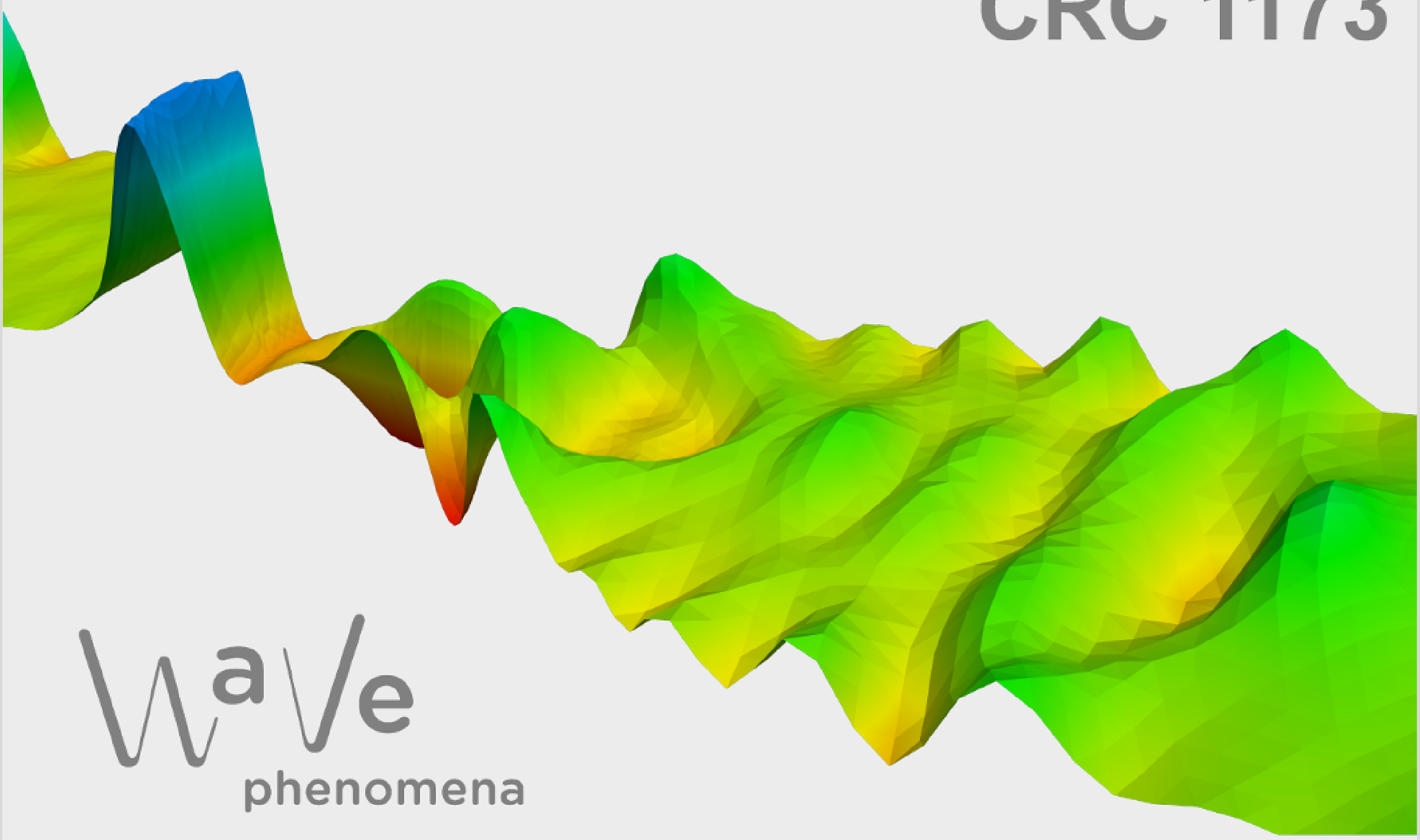

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Abstract The Maxwell equations in the unbounded three dimensional space are coupled to the Landau–Lifshitz–Gilbert equation on a (not necessarily convex) bounded domain. A weak formulation of the whole coupled system is derived based on the boundary integral formulation of the exterior Maxwell equations. We show existence of a weak solution and uniqueness of the Maxwell part of the weak solution. A numerical algorithm is proposed based on finite elements and boundary elements as spatial discretisation and using the backward Euler method and convolution quadratures for the interior domain and the boundary, respectively. Well-posedness and convergence of the numerical algorithm are shown, under minimal assumptions on the regularity of solutions. Numerical experiments illustrate and expand on the theoretical results.

Keywords Maxwell–Landau–Lifshitz–Gilbert system · Maxwell equations · linear scheme · ferromagnetism · transparent boundary conditions · boundary elements · convolution quadratures · convergence

Mathematics Subject Classification (2010) 35Q61 · 65M12 · 65M38 · 65M60

1 Introduction

This work deals with the numerical approximation of the system coupling the Maxwell equations in the whole unbounded 3D space coupled to the Landau–Lifshitz–Gilbert equation (LLG) on a bounded domain. The Maxwell equations on the external domain are transformed to the boundary using the transmission conditions and boundary integral equations. The proposed algorithm uses finite elements/backward Euler method in the interior domain and boundary elements/convolution quadrature method on the boundary. We prove convergence of the proposed algorithm.

The LLG equations serve as an important practical tool and as a valid model of micromagnetic phenomena occurring in, e.g., magnetic sensors, recording heads, and magneto-resistive storage device [23, 30, 38]. Classical results concerning existence and non-uniqueness of solutions can be found in [5, 41]. In a ferro-magnetic material, magnetization is created or affected by external electro-magnetic fields. It

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is therefore necessary to augment the LLG equations with the Maxwell system; see e.g. [19,29,41]. Existence, regularity and local uniqueness for the MLLG equations are studied in [18,17].

While in many applications, the quasi-static approximation of the Maxwell system, i.e., the eddy-current equations yield sufficiently accurate results, recent breakthroughs in ultrafast magnetism require the full Maxwell system to be modeled correctly. In this emerging field of research, femtosecond laser pulses are used to switch the magnetization of ferromagnetic materials in order to improve the speed, density, and stability of magnetic hard drives, with possible implications for the field of spintronics [24].

Numerical approximation methods are known for many variants of simpler versions of the MLLG system, i.e., for the LLG, ELLG (eddy-current LLG) equations [2,4,10,11,19,31,32] (the list is not exhaustive), and even with the full Maxwell system on *bounded domains* [7,8].

Originating from the seminal work [2], the recent works [31,32] consider a similar numeric integrator for a bounded domain. While the numerical integrator of [32] treated LLG and eddy current simultaneously per time step, [31] adapted an idea of [8] and decoupled the time-steps for LLG and the eddy current equation. The recent work [21] considers a finite element/boundary element coupling discretization for the ELLG system and even derives strong error estimates.

The present work studies the full MLLG equations on the whole *unbounded* space $\mathbb{R}^3$. Since the Landau–Lifshitz–Gilbert equation is considered on a bounded domain, the exterior Maxwell equations are transformed to the boundary of the domain, using the imposed transmission conditions for the electric and magnetic fields and boundary integral equations. This is inspired by the work [28], which derived the analogous coupling for the Maxwell equations. The proposed numerical algorithm is built on the tangent plane scheme introduced in [2] for the spatial discretisation and the backward Euler method in time for the LLG equation, finite elements and the backward Euler method for the Maxwell equations (in the interior), and uses boundary elements and convolution quadratures non-local integral equations on the boundary.

The discretization of the Maxwell equations on the whole space via finite element/boundary element coupling has the advantage that there are minimal restrictions on the shape of the interior domain, (in particular, no convexity is needed), as opposed to other methods such as non-local boundary conditions on balls [25,26], local absorbing boundary conditions [20,27], perfectly matched layers [13].

The heart of the work is to show that convolution quadrature coupled to the non-linear LLG equations can be reformulated in a weak sense with minimal assumptions on the regularity of the data (see also [35] where convolution quadrature is analysed in the time-domain in a variational setting). This inspires a numerical algorithm which is shown to converge towards a weak solution in a weak sense. Based on recent strong convergence results [21,22,1], the authors are confident that also the present algorithm exhibits strong convergence behaviour in case of more regular solutions.

The remainder of the work is structured as follows: In Section 2, we derive the boundary integral equations necessary to reformulate the exterior part of the Maxwell system. We also derive the weak form and show uniqueness of a part of the solution. In Section 3 we propose a numerical algorithm of which we show convergence towards a weak solution in Section 4. Some numerical experiments in Section 5 conclude the work.

1.1 The Maxwell–Landau–Lifshitz–Gilbert system

Let $\Omega \subset \mathbb{R}^3$ be a bounded, open and Lipschitz domain with piecewise smooth boundary, which is not necessarily convex. By $\mathbb{S}^2$ we denote the unit sphere in $\mathbb{R}^3$, and by $T > 0$ we denote the final time. By Ω^c we denote the complement of $\overline{\Omega}$, and the space-time cylinders are denoted by $\Omega_T := (0,T) \times \Omega$ and $\Omega_T^c := (0,T) \times \overline{\Omega}^c$. We will often refer to Ω as the interior domain, and to Ω^c as the exterior domain. We seek a magnetization

$$m : [0,T] \times \Omega \to \mathbb{S}^2$$

and electric and magnetic fields

$$E, H : [0,T] \times \left(\mathbb{R}^3 \backslash \partial\Omega\right) \to \mathbb{R}^3$$

that satisfy the Maxwell–Landau–Lifshitz–Gilbert (MLLG) equations, written immediately as a coupled interior–exterior system satisfying: in the interior domain

$$\partial_t m - \alpha m \times \partial_t m + C_e m \times \Delta m = -m \times H \qquad \text{in } \Omega_T \tag{1.1a}$$

$$\varepsilon \partial_t E - \nabla \times H + \sigma E = -J \qquad \text{in } \Omega_T, \tag{1.1b}$$

$$\mu \partial_t H + \nabla \times E = -\mu \partial_t m \qquad \text{in } \Omega_T, \tag{1.1c}$$

and in the exterior domain

$$\varepsilon_0 \partial_t E - \nabla \times H = 0 \qquad \text{in } \Omega_T^c, \tag{1.1d}$$

$$\mu_0 \partial_t H + \nabla \times E = 0 \qquad \text{in } \Omega_T^c, \tag{1.1e}$$

with the transmission conditions (for n being the outward pointing normal vector to $\partial\Omega$, and γ and γ^c denoting the trace operator in Ω and Ω^c, respectively)

$$\gamma E \times n = \gamma^c E \times n \quad \text{and} \quad \gamma H \times n = \gamma^c H \times n \quad \text{on } [0,T] \times \partial\Omega, \tag{1.1f}$$

the boundary condition for the magnetization

$$\partial_n m = 0 \quad \text{on } [0,T] \times \partial\Omega, \tag{1.1g}$$

and the initial conditions

$$m(0,\cdot) = m^0, \quad E(0,\cdot) = E^0, \quad H(0,\cdot) = H^0 \quad \text{in } \Omega, \tag{1.1h}$$

and

$$E(0,\cdot) = 0, \quad H(0,\cdot) = 0 \quad \text{in } \Omega^c. \tag{1.1i}$$

We assume the given initial data satisfies

$$|m^0| = 1, \quad \operatorname{div}(H^0 + m^0) = 0, \quad \operatorname{div}(E^0) = 0 \quad \text{in } \Omega, \qquad \text{and}$$
$$\operatorname{div} J(t,\cdot) = 0 \quad \text{in } \Omega \text{ for all } t \in [0,T],$$

therefore we have $|m(t,\cdot)| = 1$ and $\operatorname{div}(H(t,\cdot) + m(t,\cdot)) = \operatorname{div} E(t,\cdot) = 0$ in Ω and $\operatorname{div}(H) = \operatorname{div}(E) = 0$ in Ω^c for all time $0 \le t \le T$.

The applied current density $J : (0,T) \times \Omega \to \mathbb{R}^3$, the electric and magnetic permeability matrices $\varepsilon, \mu : \Omega \to \mathbb{R}^{3\times3}$ and the conductivity of the ferromagnetic domain $\sigma : \Omega \to \mathbb{R}^{3\times3}$ are considered given data. The damping parameter α and the exchange constant C_e are positive constants. Outside of the domain Ω, the material parameters are assumed to be scalar and constant:

$$\mu = \mu_0, \quad \varepsilon = \varepsilon_0, \quad \sigma = 0.$$

As the Maxwell equations are formulated on the whole space $\mathbb{R}^3$, we are not able to apply a standard finite element discretization to discretize the problem in space. As in [28], we transform the Maxwell equations in Ω^c into a boundary integral equation on the boundary $\Gamma := \partial\Omega$. The main innovation in the present work is that we derive a rigorous weak form of the problem and show existence of solutions with minimal regularity assumptions.

2 Boundary Integral Equations and Weak Solutions

The goal of this section is to reformulate the Maxwell problem into a coupled problem of differential and integral equations, to define a corresponding weak solution and to study their properties concerning equivalence and uniqueness.

2.1 Sobolev Spaces

We shortly repeat the definitions of the most important function spaces required in the following. Recall that $\Omega \subset \mathbb{R}^3$ is a Lipschitz domain, and $T > 0$. We define the standard L^2-space of square integrable functions

$$L^2(\Omega) := L^2(\Omega, \mathbb{R}^3) := \Big\{ v : \Omega \to \mathbb{R}^3 \,\Big|\, v \text{ measuarable and } \int_\Omega |v(x)|^2 \, \mathrm{d}x < \infty \Big\},$$

where we denote the $L^2(\Omega)$ product by $[\,\cdot\,,\,\cdot\,]_\Omega$ and

$$\begin{aligned} H(\mathrm{curl}, \Omega) &:= \big\{ v \in L^2(\Omega) \,\big|\, \nabla \times v \in L^2(\Omega) \big\}, \\ H(\mathrm{curl}, \Omega_T) &:= \big\{ v \in L^2(\Omega_T) \,\big|\, \nabla_x \times v \in L^2(\Omega_T) \big\}, \\ H^1(\mathrm{curl}, \Omega_T) &:= \big\{ v \in L^2(\Omega_T) \,\big|\, \partial_t v, \nabla_x \times v \in L^2(\Omega_T) \big\}. \end{aligned}$$

The spaces are equipped with their natural norms. We define the space $H^k([0,T])$ of k- times weakly differentiable functions and we furthermore define additional initial conditions in the sense

$$H^k_{0,*}([0,T]) := \big\{ \varphi \in H^k([0,T]) \,\big|\, \varphi(0) = \cdots = \partial_t^{k-1}\varphi(0) = 0 \big\}$$

and

$$H^k_{*,0}([0,T]) := \big\{ \varphi \in H^k([0,T]) \,\big|\, \varphi(T) = \cdots = \partial_t^{k-1}\varphi(T) = 0 \big\}.$$

The latter definitions are also used for Hilbert space valued functions and in this case we write $H^k_{0,*}([0,T], X)$, for a Hilbert space X.

2.2 The Trace Space for Boundary Integral Formulation

We define the tangential trace for $w \in C(\overline{\Omega})$ as

$$\gamma_T w := w|_{\partial\Omega} \times n,$$

where n is the outward pointing normal vector on $\partial\Omega$. Note that this definition can be extended continuously to $H(\mathrm{curl}, \Omega)$.

For the boundary integral formulation we require a particular trace space from [15], for more details we refer to [28, Section 2.1]. We keep the formal definition short and focus on the properties.

Definition 1 (Trace space, [15]) *The trace space is given by*

$$\mathcal{H}_\Gamma := \big\{ w \in \gamma_T(H^1(\Omega))' \,\big|\, \mathrm{div}_\Gamma w \in H^{-1/2}(\Gamma) \big\}$$

with the norm

$$\|w\|^2_{\mathcal{H}_\Gamma} = \|w\|^2_{\gamma_T(H^1(\Omega))'} + \|\mathrm{div}_\Gamma w\|^2_{H^{-1/2}(\Gamma)}.$$

The following properties hold true.

- *The trace operator $\gamma_T : H(\mathrm{curl}, \Omega) \to \mathcal{H}_\Gamma$ is continuous and surjective, see [15, Theorem 4.1].*
- *The anti-symmetric pairing*

$$\langle w, v \rangle_\Gamma := \int_\Gamma (w \times n) \cdot v \, \mathrm{d}\sigma = \int_\Gamma -(w \times v) \cdot n \, \mathrm{d}\sigma$$

  for $w, v \in L^2(\Gamma)^3$ can be extended to a continuous, anti-symmetric bilinear form on $\mathcal{H}_\Gamma$. The boundary space $\mathcal{H}_\Gamma$ is its own dual with respect to $\langle \cdot, \cdot \rangle_\Gamma$, see [16, Theorem 2].
- *For $w, v \in H(\mathrm{curl}, \Omega)$ the Green's formula holds $[\nabla \times v, w]_\Omega - [v, \nabla \times w]_\Omega = -\langle \gamma_T v, \gamma_T w \rangle_\Gamma$.*

Note $\langle \cdot, \cdot \rangle_\Gamma$ is not the Hilbert space scalar product on $\mathcal{H}_\Gamma$, however we may define the corresponding adjoint T^* of an operator $T \colon \mathcal{H}_\Gamma \to \mathcal{H}_\Gamma$ as well as weak convergence with respect to $\langle \cdot, \cdot \rangle_\Gamma$ (which coincides with ordinary weak convergence in $\mathcal{H}_\Gamma$).

2.3 Reformulation of the System

In this section, using the approach of [28, Section 2 and 4.2], we transform the interior–exterior Maxwell equation with the transmission conditions into a boundary integral equation on the boundary $\Gamma := \partial\Omega$ coupled to the Maxwell equation in Ω. At the end of the section, using this coupled formulation the whole MLLG system is rewritten.

Let us consider only the Maxwell equation (1.1b)–(1.1e) coupled by the transmission conditions (1.1f). We start with a formal derivation, and return to the precise smoothness requirements later in Section 2.5.

The interior problem reads as

$$\begin{aligned} \varepsilon\partial_t E - \nabla\times H + \sigma E &= -J \qquad \text{in } \Omega_T,\\ \mu\partial_t H + \nabla\times E &= -\mu\partial_t m \qquad \text{in } \Omega_T, \end{aligned}$$

while the exterior problem reads as

$$\begin{aligned} \varepsilon_0\partial_t E^c - \nabla\times H^c &= 0 \qquad \text{in } \Omega_T^c,\\ \mu_0\partial_t H^c + \nabla\times E^c &= 0 \qquad \text{in } \Omega_T^c, \end{aligned}$$

coupled by the transmission conditions

$$\gamma_T E = \gamma_T^c E^c \quad \text{and} \quad \gamma_T H = \gamma_T^c H^c \quad \text{on } [0,T]\times\Gamma,$$

where the tangential trace operator for the exterior domain is given by $\gamma_T^c u = \gamma^c u \times n$, and with the same initial data as in (1.1).

We set $U := \mathcal{L}(E^c)$, where $\mathcal{L}$ is the Laplace transform. With the properties of the Laplace transform from Example 39 in the Appendix, we have $\mathcal{L}(\partial_t^2 E^c)(s) = s^2\mathcal{L}(E^c)(s)$ as well as

$$\left(\partial_t^{-1}E^c\right)(t) := \int_0^t E^c(r)\,\mathrm{d}r = \mathcal{L}^{-1}\left(\frac{1}{s}U(s)\right)(t).$$

For a fixed $s\in\mathbb{C}$, the time-harmonic equation corresponding to the exterior problem reads as

$$\varepsilon_0\mu_0 s^2 U + \nabla\times\big(\nabla\times U\big) = 0 \qquad \text{in } \Omega^c. \tag{2.1}$$

We use the average

$$\{\!\{\gamma_T u\}\!\} := \big(\gamma_T u + \gamma_T^c u\big)/2.$$

The electric single layer potential is given, for $x\in\mathbb{R}^3\setminus\Gamma$, by

$$\left(\mathcal{S}(s)\varphi\right)(x) := s\int_\Gamma G(s,x-y)\varphi(y)\mathrm{d}y - s^{-1}\frac{1}{\epsilon_0\mu_0}\nabla\int_\Gamma G(s,x-y)\mathrm{div}_\Gamma\varphi(y)\mathrm{d}y$$

and the electric double layer potential, for $x\in\mathbb{R}^3\setminus\Gamma$,

$$(\mathcal{D}(s)\varphi)(x) = \nabla\times\int_\Gamma G(s,x-y)\varphi(y)\,\mathrm{d}y,$$

see [16] for more details, where the fundamental solution $G(s,z)$ is given for $z\in\mathbb{R}^3\backslash\{0\}$, as

$$G(s,z) = \frac{e^{-s\sqrt{\varepsilon_0\mu_0}|z|}}{4\pi|z|}.$$

We use the the Calderon operator $B(s)$, cf. [28, Section 3],

$$B(s) := \mu_0^{-1}\begin{pmatrix}(i\sqrt{\mu_0\varepsilon_0})^{-1}V(s) & K(s)\\ -K(s) & -i\sqrt{\mu_0\varepsilon_0}V(s)\end{pmatrix} \tag{2.2}$$

with the boundary integral operators

$$\begin{aligned} V(s) &= i\sqrt{\mu_0\varepsilon_0}\{\!\{\gamma_T\circ\mathcal{S}(s)\}\!\} = (i\sqrt{\mu_0\varepsilon_0})^{-1}\{\!\{\gamma_N\circ\mathcal{D}(s)\}\!\},\\ K(s) &= \{\!\{\gamma_T\circ\mathcal{D}(s)\}\!\} = \{\!\{\gamma_N\circ\mathcal{S}(s)\}\!\}. \end{aligned} \tag{2.3}$$

Following [28, Section 2.3], the solution then has the representation

$$U = \mathcal{S}(s)\varphi + \mathcal{D}(s)\psi, \qquad x \in \mathbb{R}^3 \setminus \Gamma, \tag{2.4}$$

where boundary densities are given by

$$\varphi = [\![\gamma_N U]\!] = s^{-1}[\![\gamma_T(\nabla \times U)]\!] \quad \text{and} \quad \psi = [\![\gamma_T U]\!], \tag{2.5}$$

where $[\![\gamma v]\!] = \gamma v - \gamma^c v$ denotes the jumps in the boundary traces (for both trace operators).

By the proof of Lemma 3.1 in [28], the Calderon operator an the boundary densities satisfy, via (2.3),

$$B(s)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \frac{1}{\mu_0}\begin{pmatrix}\{\!\{\gamma_T U\}\!\}\\ -\{\!\{\gamma_N U\}\!\}\end{pmatrix}. \tag{2.6}$$

Using the inverse Laplace transform we define the time-domain versions of the integral operators: $\mathcal{S}(\partial_t)$, $\mathcal{D}(\partial_t)$, $B(\partial_t)$, given by

$$B(\partial_t)u := \mathcal{L}^{-1}(B(s)\mathcal{L}u) := \mathcal{L}^{-1}\big(s \mapsto B(s)\mathcal{L}(u)(s)\big).$$

Using the above time-domain integral operators the solution of the exterior problem is given as

$$E^c = \mathcal{S}(\partial_t)\varphi + \mathcal{D}(\partial_t)\psi, \tag{2.7}$$

with boundary densities

$$\varphi = \partial_t^{-1}[\![\gamma_T(\nabla \times E^c)]\!] = -\partial_t^{-1}\gamma_T^c(\nabla \times E^c) \quad \text{and} \quad \psi = [\![\gamma_T E^c]\!] = -\gamma_T^c E^c,$$

which satisfy the equation (2.6), that is

$$B(\partial_t)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \frac{1}{2\mu_0}\begin{pmatrix}\gamma_T^c E^c\\ -\partial_t^{-1}\gamma_T^c(\nabla \times E^c)\end{pmatrix}.$$

where for both the jumps and the averages we have used that $\gamma_T(\nabla \times E^c) = 0$ and $\gamma_T E^c = 0$. We rewrite the right-hand side of the above equality further. By rewriting the second equation of the exterior problem $\mu_0 H^c = -\partial_t^{-1}\nabla \times E^c$, then taking the external traces yields $-\partial_t^{-1}\gamma_T^c(\nabla \times E^c) = \mu_0\gamma_T^c H^c$. Using this formula together with the transmission boundary conditions we obtain $\varphi = \partial_t^{-1}\gamma_T^c(\nabla \times E^c) = \mu_0\gamma_T H$ and $\psi = -\gamma_T E$, and hence

$$B(\partial_t)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \frac{1}{2\mu_0}\begin{pmatrix}\gamma_T^c E^c\\ -\partial_t^{-1}\gamma_T^c(\nabla \times E^c)\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\mu_0^{-1}\gamma_T E\\ \gamma_T H\end{pmatrix}.$$

Therefore the coupled Maxwell–Landau–Lifshitz–Gilbert equation (1.1) is rewritten into the following equation system *only* in the interior domain Ω and on its boundary Γ. Find the functions m, E and $H : [0,T] \times \Omega \to \mathbb{R}^3$ in the interior domain and φ and $\psi : [0,T] \times \Gamma \to \mathbb{R}^3$ on the boundary which satisfy the following coupled system: in the interior domain

$$\partial_t m - \alpha m \times \partial_t m + C_e m \times \Delta m = -m \times H \qquad \text{in } \Omega_T, \tag{2.8a}$$

$$\varepsilon\partial_t E - \nabla \times H + \sigma E = -J \qquad \text{in } \Omega_T, \tag{2.8b}$$

$$\mu\partial_t H + \nabla \times E = -\mu\partial_t m \qquad \text{in } \Omega_T, \tag{2.8c}$$

coupled to the boundary integral equations

$$B(\partial_t)\begin{pmatrix}\mu_0\gamma_T H\\ -\gamma_T E\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\mu_0^{-1}\gamma_T E\\ \gamma_T H\end{pmatrix} \qquad \text{on } [0,T] \times \partial\Omega, \tag{2.8d}$$

and where m satisfies the boundary conditions

$$\partial_n m = 0 \qquad \text{on } [0,T] \times \partial\Omega, \tag{2.8e}$$

with the same initial conditions for the problems in Ω as in (1.1).

Note that the expression in (2.7) is representation formula for the exterior solution, while the boundary integral equation (2.8d) is a compatibility condition for the interior boundary data $\gamma_T E$ and $\gamma_T H$. Consistency with the interior solution of (1.1) demands $\gamma_T E(0,x) = 0$ and $\gamma_T H(0,x) = 0$ for all $x \in \Gamma$.

2.4 The Calderon Operator

We look at the properties of the Calderon operator $B(s)$ in more detail. An important property that will play a crucial role later on, is the coercivity of the Calderon operator.

Lemma 2 (Coercivity Lemma, [28, Lemma 3.1]) *There exists $\beta > 0$ such that the Calderon operator (2.2) satisfies*

$$\Re \left\langle \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, B(s) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle_{\Gamma} \geq \beta \mathrm{m}(s) \left((\varepsilon_0 \mu_0)^{-1} \|s^{-1}\varphi\|^2_{\mathcal{H}_\Gamma} + \|s^{-1}\psi\|^2_{\mathcal{H}_\Gamma} \right)$$

for $\Re s > 0$ and all $\varphi, \psi \in \mathcal{H}_\Gamma$, with $\mathrm{m}(s) = \min(1, |s|^2 \varepsilon_0 \mu_0) \Re s$.

Lemma 3 ([28, Lemma 2.3]) *For $\Re s \geq \epsilon > 0$ the Calderon operator (2.2)*

$$B(s) : (\mathcal{H}_\Gamma)^2 \to (\mathcal{H}_\Gamma)^2$$

satisfies

$$\|B(s)\phi\|_{\mathcal{H}_\Gamma} \leq C(\epsilon) \left\| s^2 \phi \right\|_{\mathcal{H}_\Gamma}$$

for $\phi \in (\mathcal{H}_\Gamma)^2$.

The properties of the Laplace transform (Lemma 43) show for a family of suitably bounded operators $A(s)$

$$\mathcal{L}^{-1} \left(A(s) \mathcal{L}\phi \right) = \mathcal{L}^{-1}(A(s)) * \phi, \tag{2.9}$$

where $*$ denotes convolution. As we only have $\|B(s)\| \leq Cs^2$, we cannot conclude that $\mathcal{L}^{-1}(B(s))$ exists, however for $m > 3$,

$$B_m(t) := \mathcal{L}^{-1}(s \mapsto B(s) s^{-m})(t) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} e^{st} s^{-m} B(s) \mathrm{d}s$$

exists for all $t \geq 0$ and is a continuous and bounded function on $[0, T]$.
For $m \in \mathbb{N}$ and $\phi \in H^m_{0,*}([0,\infty), \mathcal{H}^2_\Gamma)$

$$\mathcal{L}^{-1}(s \mapsto s^m \mathcal{L}(\phi)(s)) = \partial_t^m \phi.$$

Therefore it holds for $m \in \mathbb{N}$, $m > 3$ with (2.9)

$$\begin{aligned}
\mathcal{L}^{-1}(B(s)\mathcal{L}(\phi)(s))(t) &= \left(\mathcal{L}^{-1}(s \mapsto B(s) s^{-m}) * \mathcal{L}^{-1}(s \mapsto \mathcal{L}(\phi)(s) s^m) \right)(t) \\
&= \int_0^t B_m(r) (\partial_t^m \phi)(t - r) \ \mathrm{d}r \\
&= -B_m(t) \partial_t^{m-1} \phi(0) + \partial_t \int_0^t B_m(r) \partial_t^{m-1} (\phi(t - r)) \ \mathrm{d}r \\
&= \cdots = \partial_t^m \int_0^t B_m(r) \phi(t - r) \ \mathrm{d}r,
\end{aligned}$$

which says

$$B(\partial_t)\phi = \partial_t^m (B_m * \phi). \tag{2.10}$$

For the following considerations, we fix some $m \in \mathbb{N}$ with $m > 3$. Moreover, we consider the operator

$$\partial_t^{-1} v(t) := \mathcal{L}^{-1} \left(s^{-1} \mathcal{L}(v) \right)(t) = \int_0^t v(s) \ \mathrm{d}s = (1 * v)(t).$$

Note that this operator commutes with $B(\partial_t)$ (for ϕ as above) in the sense

$$\partial_t^{-1} \partial_t^m B_m * \phi = \partial_t^m \partial_t^{-1} B_m * \phi = \partial_t^m B_m * \partial_t^{-1} \phi.$$

2.5 Definition of Weak Solutions

We multiply the LLG equation (1.1a) with a smooth test function ρ and use $\partial_n m = 0$ on Γ to obtain

$$
\begin{aligned}
[\Delta m \times m, \rho]_\Omega &= [\nabla m \times \rho, \nabla m]_\Omega - [\nabla m \times m, \nabla \rho]_\Omega + [m \times \rho, \partial_n m]_\Gamma \\
&= -[\nabla m \times m, \nabla \rho]_\Omega .
\end{aligned}
$$

We arrive at the following definition.

Definition 4 *We consider a solution of the MLLG equations, i.e.* (m, E, H) *that satisfies*

- $m \in H^1(\Omega_T)$ *with* $|m| = 1$ *almost everywhere,* $m(0, \cdot) = m^0$ *in the sense of traces, and for all* $\rho \in C^\infty(\Omega_T)$ *we have*

$$
[\partial_t m, \rho]_{\Omega_T} - \alpha [m \times \partial_t m, \rho]_{\Omega_T} = -C_e [\nabla m \times m, \nabla \rho]_{\Omega_T} + [H \times m, \rho]_{\Omega_T} .
$$

- $E, H \in L^2(\Omega_T)$ *such that* $\partial_t^{-1} E, \partial_t^{-1} H \in H(\mathrm{curl}, \Omega_T)$ *and*

$$
\begin{aligned}
\varepsilon(E - E^0) - \nabla \times (\partial_t^{-1} H) + \sigma \partial_t^{-1} E &= -\partial_t^{-1} J \qquad \text{in } L^2(\Omega_T), \\
\mu(H - H^0) + \nabla \times (\partial_t^{-1} E) &= -\mu(m - m^0) \qquad \text{in } L^2(\Omega_T)
\end{aligned}
$$

as well as $B_m * \binom{\mu_0 \gamma_T(\partial_t^{-1} H)}{-\gamma_T(\partial_t^{-1} E)} \in H^m_{0,*}([0, T], \mathcal{H}_\Gamma)$ *with*

$$
\partial_t^m B_m * \begin{pmatrix} \mu_0 \gamma_T(\partial_t^{-1} H) \\ -\gamma_T(\partial_t^{-1} E) \end{pmatrix} = \frac{1}{2\mu_0} \begin{pmatrix} \gamma_T(\partial_t^{-1} E) \\ \mu_0 \gamma_T(\partial_t^{-1} H) \end{pmatrix} \qquad \text{in } L^2([0, T], \mathcal{H}_\Gamma).
$$

For the following convergence analysis, we require an alternative definition of weak solutions for which we introduce the notation

$$
\langle \varphi, \psi \rangle_{\Gamma_T} := \int_0^T \langle \varphi, \psi \rangle_\Gamma \, \mathrm{d}t
$$

for suitable functions φ, ψ.

Definition 5 *The functions* $(m, E, H, \tilde{\varphi}, \tilde{\psi})$ *are a weak solution of the MLLG equation if:*

- $m \in H^1(\Omega_T)$ *with* $|m| = 1$ *almost everywhere,* $E, H \in L^2(\Omega_T)$ *such that* $\partial_t^{-1} E, \partial_t^{-1} H \in H(\mathrm{curl}, \Omega_T)$ *and* $\tilde{\varphi}, \tilde{\psi} \in L^2([0, T], \mathcal{H}_\Gamma)$.
- *For all* $\rho \in C^\infty(\Omega_T)$, *all* $\zeta_E, \zeta_H \in C^\infty(\Omega_T)$ *with* $\zeta_E(T) = \zeta_H(T) = 0$ *and all* $v, w \in \gamma_T(C^\infty(\Omega_T)) \cap H^{m+1}_{*,0}([0, T], \mathcal{H}_\Gamma)$ *we have*

$$
\begin{aligned}
[\partial_t m, \rho]_{\Omega_T} - \alpha [m \times \partial_t m, \rho]_{\Omega_T} &= -C_e [\nabla m \times m, \nabla \rho]_{\Omega_T} + [H \times m, \rho]_{\Omega_T}, \\
-[\varepsilon E, \partial_t \zeta_E]_{\Omega_T} - [\varepsilon E^0, \zeta_E(0, \cdot)]_\Omega &= -[\nabla \times (\partial_t^{-1} H), \partial_t \zeta_E]_{\Omega_T} - [\sigma E + J, \zeta_E]_{\Omega_T}, \\
-[\mu H, \partial_t \zeta_H]_{\Omega_T} - [\mu H^0, \zeta_H(0, \cdot)]_\Omega &= [\nabla \times (\partial_t^{-1} E), \partial_t \zeta_H]_{\Omega_T} - [\mu \partial_t m, \zeta_H]_{\Omega_T}, \\
(-1)^{m+1} \left\langle \partial_t^{m+1} \begin{pmatrix} v \\ w \end{pmatrix}, B_m * \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right\rangle_{\Gamma_T} &= -\frac{1}{2\mu_0} \left\langle \begin{pmatrix} \partial_t v \\ \partial_t w \end{pmatrix}, \begin{pmatrix} 2\gamma_T(\partial_t^{-1} E) + \tilde{\psi} \\ 2\mu_0 \gamma_T(\partial_t^{-1} H) - \tilde{\varphi} \end{pmatrix} \right\rangle_{\Gamma_T} .
\end{aligned}
\tag{2.11}
$$

- *It holds* $m(0, \cdot) = m^0$ *in the sense of traces.*

2.6 Equivalence and Uniqueness

Theorem 6 *If* (m, E, H) *is a solution in the sense of Definition 4, then*

$$
(m, E, H, \mu_0 \gamma_T \partial_t^{-1} H, -\gamma_T \partial_t^{-1} E)
$$

is a solution in the sense of Definition 5. If $(m, E, H, \tilde{\varphi}, \tilde{\psi})$ *is a solution in the sense of Definition 5, then* (m, E, H) *is a solution in the sense of Definition 4.*

Proof Step 1: Let (m, E, H) be a solution in the sense of Definition 4. We multiply the Maxwell part of Definition 4 with the respective test functions of Definition 5. Integration by parts in time gives the equations stated in Definition 5. We introduce the variable $\widetilde{\varphi} = \mu_0 \gamma_T \partial_t^{-1} H$ for the tangential trace of H as well as $\widetilde{\psi} = -\gamma_T \partial_t^{-1} E$ for the tangential trace of E. For $b = \partial_t \binom{v}{w} \in H^m_{*,0}([0,T], \mathcal{H}^2_\Gamma)$ we integrate by parts m times in time to obtain with $a = \binom{\widetilde{\varphi}}{\widetilde{\psi}}$

$$\begin{aligned}\langle b, \partial_t^m (B_m * a)\rangle_{\Gamma_T} &= -\left\langle \partial_t b, \partial_t^{m-1}(B_m * a)\right\rangle_{\Gamma_T} + \left[\left\langle b, \partial_t^{m-1}(B_m * a)\right\rangle_\Gamma\right]_0^T \\ &= -\left\langle \partial_t b, \partial_t^{m-1}(B_m * a)\right\rangle_{\Gamma_T} = \cdots = (-1)^m \left\langle \partial_t^m b, (B_m * a)\right\rangle_{\Gamma_T}.\end{aligned}$$

Thus we have a solution in the sense of Definition 5.

Step 2: Now let $(m, E, H, \widetilde{\varphi}, \widetilde{\psi})$ be a solution in the sense of Definition 5. The interior Maxwell parts of the Definition 5 and Definition 4 are equivalent via integration by parts in time (note that in Definition 4 all the terms of the interior Maxwell part are zero at $t = 0$). We will prove below that the operator

$$Q(\partial_t) := \left(\frac{1}{2\mu_0}\begin{pmatrix} 0 & -\partial_t^{-m} \\ \partial_t^{-m} & 0 \end{pmatrix} + B_m *\right)$$

is almost a projection in the sense

$$Q(\partial_t)\begin{pmatrix} v \\ w \end{pmatrix} = \frac{1}{\mu_0}\partial_t^{-m}\begin{pmatrix} -w \\ v \end{pmatrix} \quad \text{for all} \quad \frac{1}{\mu_0}\begin{pmatrix} -w \\ v \end{pmatrix} = Q(\partial_t)\begin{pmatrix} v' \\ w' \end{pmatrix}. \tag{2.12}$$

Integration by parts in the last equation of Definition 5 shows that $\frac{1}{\mu_0}\partial_t^{-m}\binom{\gamma_T \partial_t^{-1} E}{\mu_0 \gamma_T \partial_t^{-1} H}$ is in the range of $Q(\partial_t)$. Hence (2.12) implies

$$\begin{aligned}\frac{1}{\mu_0}\partial_t^{-2m}\begin{pmatrix} \gamma_T \partial_t^{-1} E \\ \mu_0 \gamma_T \partial_t^{-1} H \end{pmatrix} &= Q(\partial_t)\partial_t^{-m}\begin{pmatrix} \mu_0 \gamma_T \partial_t^{-1} H \\ -\gamma_T \partial_t^{-1} E \end{pmatrix} \\ &= \frac{1}{2\mu_0}\partial_t^{-2m}\begin{pmatrix} \gamma_T \partial_t^{-1} E \\ \mu_0 \gamma_T \partial_t^{-1} H \end{pmatrix} + B_m * \partial_t^{-m}\begin{pmatrix} \mu_0 \gamma_T \partial_t^{-1} H \\ -\gamma_T \partial_t^{-1} E \end{pmatrix}.\end{aligned}$$

Differentiation in time leads to the boundary integral equation in Definition 4.
It remains to show (2.12). To that end, we use the definition of Q and obtain for $v, w \in L^2([0,T], \mathcal{H}_\Gamma)$ and $\omega := \sqrt{\mu_0 \varepsilon_0}$

$$\begin{aligned}Q(\partial_t)\begin{pmatrix} v \\ w \end{pmatrix} &= \mathcal{L}^{-1}\left(\frac{1}{2\mu_0}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} s^{-m} + s^{-m} B(s)\right)\begin{pmatrix} \mathcal{L}v(s) \\ \mathcal{L}w(s) \end{pmatrix} \\ &= \mathcal{L}^{-1}\left(\frac{1}{2\mu_0}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} s^{-m} + s^{-m}\mu_0^{-1}\begin{pmatrix} (i\omega)^{-1} V(s) & K(s) \\ -K(s) & -i\omega V(s) \end{pmatrix}\right)\begin{pmatrix} \mathcal{L}v(s) \\ \mathcal{L}w(s) \end{pmatrix} \\ &= \mathcal{L}^{-1}\frac{1}{\mu_0 s^m}\left(\frac{1}{2}\begin{pmatrix} 0 & -1 \\ i\omega & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -i\omega & 0 \end{pmatrix}\begin{pmatrix} K(s) & V(s) \\ V(s) & K(s) \end{pmatrix}\right)\begin{pmatrix} (i\omega)^{-1}\mathcal{L}v(s) \\ \mathcal{L}w(s) \end{pmatrix} \\ &= \frac{1}{\mu_0}\begin{pmatrix} 0 & -1 \\ i\omega & 0 \end{pmatrix}\mathcal{L}^{-1}\frac{1}{s^m}\left(\frac{1}{2}\mathrm{Id} - \begin{pmatrix} K(s) & V(s) \\ V(s) & K(s) \end{pmatrix}\right)\begin{pmatrix} (i\omega)^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathcal{L}v(s) \\ \mathcal{L}w(s) \end{pmatrix}.\end{aligned}$$

In [16, Equation (35)] it is shown that

$$\frac{1}{2}\mathrm{Id} - \begin{pmatrix} K(s) & V(s) \\ V(s) & K(s) \end{pmatrix}$$

is a projection. Hence, the above together with Lemma 47 and Lemma 49 conclude (2.12). This concludes the proof. □

Theorem 7 *The interior Maxwell part of a solution in the sense of Definition 5 is unique, i.e. if there is an m such that $(m, E_1, H_1, \tilde{\varphi}_1, \tilde{\psi}_1)$ and $(m, E_2, H_2, \tilde{\varphi}_2, \tilde{\psi}_2)$, both solutions in the sense of Definition 5, then it holds*

$$(E_1, H_1) = (E_2, H_2).$$

Proof Assume, that there exist two solutions in the sense of Definition 5. By Theorem 6, we have that $(m, E_1, H_1, \mu_0\gamma_T\partial_t^{-1}H_1, -\gamma_T\partial_t^{-1}E_1)$ and $(m, E_2, H_2, \mu_0\gamma_T\partial_t^{-1}H_2, -\gamma_T\partial_t^{-1}E_2)$ are solutions in the sense of Definition 5. The difference $U := \partial_t^{-1}(E^1 - E^2)$, $V := \partial_t^{-1}(H^1 - H^2)$ fulfills

$$(U, V) \in H^1(\mathrm{curl}, \Omega_T) \times H^1(\mathrm{curl}, \Omega_T)$$

and for all

$$\zeta_E, \zeta_H \in C^\infty(\Omega_T) \text{ with } \zeta_E(T) = \zeta_H(T) = 0$$

and all

$$v, w \in \gamma_T(C^\infty(\Omega_T)) \cap H^{m+1}_{*,0}([0,T], \mathcal{H}_\Gamma)$$

it holds

$$\begin{aligned}
&[\varepsilon\partial_t U, \partial_t\zeta_E]_{\Omega_T} + [\mu\partial_t V, \partial_t\zeta_H]_{\Omega_T} + (-1)^{m+1}\left\langle \begin{pmatrix}\partial_t^{m+1}v\\ \partial_t^{m+1}w\end{pmatrix}, B_m * \begin{pmatrix}\mu_0\gamma_T V\\ -\gamma_T U\end{pmatrix}\right\rangle_{\Gamma_T} \\
&\quad = [\nabla\times V, \partial_t\zeta_E]_{\Omega_T} - [\sigma U, \partial_t\zeta_E]_{\Omega_T} - [\nabla\times U, \partial_t\zeta_H]_{\Omega_T} - \frac{1}{2\mu_0}\left\langle \begin{pmatrix}\partial_t v\\ \partial_t w\end{pmatrix}, \begin{pmatrix}\gamma_T U\\ \mu_0\gamma_T V\end{pmatrix}\right\rangle_{\Gamma_T}.
\end{aligned} \tag{2.13}$$

Moreover it is $U(0) = 0$ and $V(0) = 0$ in $L^2(\Omega)$ in the sense of traces. By a density/limit argument, since all quantities are bounded in $L^2(\Omega_T)$ or $L^2([0,T], \mathcal{H}_\Gamma)$, respectively, we are able to test with

$$\partial_t\zeta_E := (\overline{\partial}_t)^{-m}\hat{\zeta}_E, \qquad \partial_t\zeta_H := (\overline{\partial}_t)^{-m}\hat{\zeta}_H, \qquad \partial_t v := (\overline{\partial}_t)^{-m}\hat{v}, \qquad \partial_t w := (\overline{\partial}_t)^{-m}\hat{w}, \tag{2.14}$$

where

$$(\overline{\partial}_t)^{-1}g(s) := \int_s^T g(r)\,\mathrm{d}r$$

for

$$(\hat{\zeta}_E, \hat{\zeta}_H, \hat{v}, \hat{w}) \in L^2(\Omega_T) \times L^2(\Omega_T) \times L^2([0,T], \mathcal{H}_\Gamma) \times L^2([0,T], \mathcal{H}_\Gamma).$$

For $g \in L^2([0,T])$ it holds $(\overline{\partial}_t)^{-m}g \in H^m_{*,0}([0,T])$ and it holds for $f \in L^2(0,T)$

$$\begin{aligned}
[f, \overline{\partial}_t^{-1}g]_{(0,T)} &= \int_0^T f(s)\int_s^T g(r)\,\mathrm{d}r\,\mathrm{d}s \\
&= \int_0^T\int_0^r f(s)g(r)\,\mathrm{d}s\,\mathrm{d}r \\
&= [\partial_t^{-1}f, g]_{(0,T)}.
\end{aligned} \tag{2.15}$$

We test (2.13) according to (2.14) with

$$\hat{\zeta}_E := \mathbb{1}_{[0,r]}\partial_t^{-m}U, \qquad \hat{\zeta}_H := \mathbb{1}_{[0,r]}\partial_t^{-m}V, \qquad \hat{v} := -\mu_0\mathbb{1}_{[0,r]}\partial_t^{-m}\gamma_T V, \qquad \hat{w} := \mathbb{1}_{[0,r]}\partial_t^{-m}\gamma_T U$$

for arbitrary $0 \le r \le T$ and obtain for

$$\tilde{U} := \partial_t^{-m}U, \qquad \tilde{V} := \partial_t^{-m}V$$

that

$$\begin{aligned}
&[\varepsilon\partial_t\tilde{U}, \tilde{U}]_{\Omega_r} + [\mu\partial_t\tilde{V}, \tilde{V}]_{\Omega_r} + \left\langle \begin{pmatrix}\mu_0\gamma_T\tilde{V}\\ -\gamma_T\tilde{U}\end{pmatrix}, B_m * \begin{pmatrix}\mu_0\gamma_T V\\ -\gamma_T U\end{pmatrix}\right\rangle_{\Gamma_r} \\
&\quad = [\nabla\times V, \tilde{U}]_{\Omega_r} - [\sigma\tilde{U}, \tilde{U}]_{\Omega_r} - [\nabla\times\tilde{U}, \tilde{V}]_{\Omega_r} + \frac{1}{2\mu_0}\left\langle \begin{pmatrix}\mu_0\gamma_T\tilde{V}\\ -\gamma_T\tilde{U}\end{pmatrix}, \begin{pmatrix}\gamma_T\tilde{U}\\ \mu_0\gamma_T\tilde{V}\end{pmatrix}\right\rangle_{\Gamma_r} \\
&\quad = -[\sigma\tilde{U}, \tilde{U}]_{\Omega_r} + [\nabla\times\tilde{V}, \tilde{U}]_{\Omega_r} - [\nabla\times\tilde{U}, \tilde{V}]_{\Omega_r} + \langle\gamma_T\tilde{V}, \gamma_T\tilde{U}\rangle_{\Gamma_r} \\
&\quad = -[\sigma\tilde{U}, \tilde{U}]_{\Omega_r}.
\end{aligned}$$

By (2.10) and Lemma 2, Lemma 50 and similar arguments like in Lemma 17 below (i.e. considering the limit $\sigma_0 \to 0$ in Lemma 50) we have

$$\left\langle \begin{pmatrix} \mu_0\gamma_T\tilde{V} \\ -\gamma_T\tilde{U} \end{pmatrix}, B_m * \begin{pmatrix} \mu_0\gamma_T V \\ -\gamma_T U \end{pmatrix} \right\rangle_{\Gamma_r} = \left\langle \begin{pmatrix} \mu_0\gamma_T\tilde{V} \\ -\gamma_T\tilde{U} \end{pmatrix}, B(\partial_t) \begin{pmatrix} \mu_0\gamma_T\tilde{V} \\ -\gamma_T\tilde{U} \end{pmatrix} \right\rangle_{\Gamma_r} \geq 0$$

and therefore

$$\begin{aligned} 0 \leq \varepsilon\|\tilde{U}(r)\|_\Omega^2 + \mu\|\tilde{V}(r)\|_\Omega^2 &\leq [\varepsilon\partial_t\tilde{U}, \tilde{U}]_{\Omega_r} + [\mu\partial_t\tilde{V}, \tilde{V}]_{\Omega_r} \\ &+ \left\langle \begin{pmatrix} \tilde{\psi} \\ -\gamma_T(\tilde{U}) \end{pmatrix}, B_m * \begin{pmatrix} \psi \\ -\gamma_T U \end{pmatrix} \right\rangle_{\Gamma_r} + [\sigma\tilde{U}, \tilde{U}]_{\Omega_r} = 0. \end{aligned}$$

Thus we have $\tilde{U} = U = \tilde{V} = V = 0$, which gives the desired result. □

Remark 8 *The uniqueness in m is unclear or not expected in the literature. The uniqueness with respect to $\tilde{\varphi}, \tilde{\psi}$ is also not true, as we ask that the projection on suitable exterior data applied to $\tilde{\varphi}, \tilde{\psi}$ gives $\gamma_T\partial_t^{-1}H, \gamma_T\partial_t^{-1}E$. The projection on suitable exterior data is not injective, so the variables $\tilde{\varphi}, \tilde{\psi}$ are only unique up to an difference of elements in the kernel of the projection, so by suitable interior data.*
However, with any solution $(m, E, H, \tilde{\varphi}, \tilde{\psi})$ in the sense of Definition 5, we have that also the functions $(m, E, H, \mu_0\gamma_T\partial_t^{-1}H, -\gamma_T\partial_t^{-1}E)$ form a solution. Hence, in this sense, the last four components are unique.

3 Discrete Approximation

To formulate an algorithm to approximate the solution of the MLLG system, we reformulate the LLG equation once more. By applying $m \times \cdot$ to (1.1a) and using

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$$

for $a, b, c \in \mathbb{R}^3$, we obtain

$$\alpha\partial_t m + m \times \partial_t m = C_e\Delta m + H - (m \cdot (C_e\Delta m + H))m.$$

It suffices to multiply this with a test function ρ that is orthogonal to m. Therefore, using integration by parts and $\partial_n m = 0$ on Γ for all $t \in [0, T]$, we obtain

$$[\alpha\partial_t m, \rho]_\Omega + [m \times \partial_t m, \rho]_\Omega = -[C_e\nabla m, \nabla\rho]_\Omega + [H, \rho]_\Omega.$$

The approximations are based on the symmetric formulation of the Maxwell part, obtained by applying Green's formula to the half of the curl operators, cf. [28, equation (4.4)], with the variables $\varphi := \mu_0\gamma_T H$ and $\psi := -\gamma_T E$

$$\begin{aligned} [\alpha\partial_t m, \rho]_\Omega + [m \times \partial_t m, \rho]_\Omega &= -[C_e\nabla m, \nabla\rho]_\Omega + [H, \rho]_\Omega, \\ [\varepsilon\partial_t E, \zeta_E]_\Omega &= \frac{1}{2}[\nabla \times H, \zeta_E]_\Omega + \frac{1}{2}[H, \nabla \times \zeta_E]_\Omega \\ &\quad - \frac{1}{2\mu_0}\langle\varphi, \gamma_T\zeta_E\rangle_\Gamma - [\sigma E + J, \zeta_E]_\Omega, \\ [\mu\partial_t H, \zeta_H]_\Omega &= -\frac{1}{2}[\nabla \times E, \zeta_H]_\Omega - \frac{1}{2}[E, \nabla \times \zeta_H]_\Omega \\ &\quad - \frac{1}{2}\langle\psi, \gamma_T\zeta_H\rangle_\Gamma - [\mu\partial_t m, \zeta_H]_\Omega, \\ \left\langle \begin{pmatrix} v_\varphi \\ v_\psi \end{pmatrix}, B(\partial_t)\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle_\Gamma &= \frac{1}{2}\left\langle \begin{pmatrix} v_\varphi \\ v_\psi \end{pmatrix}, \begin{pmatrix} \mu_0^{-1}\gamma_T E \\ \gamma_T H \end{pmatrix} \right\rangle_\Gamma. \end{aligned} \tag{3.1}$$

This non-standard symmetrised weak formulation for Maxwell's equations will prove to be extremely useful, in the same way as in [28]. The analogous formulation was first used for acoustic wave equations in [**?**], and then in [9].

3.1 Preliminaries

For time discretization we use a constant time step size $\tau := T/N$ for $N \in \mathbb{N}$ to approximate the solution on the time points $0 = t_0, \dots, t_n = T$, $t_j = \tau j$. We assume $\tau \le \tau_0$ for a $\tau_0 > 0$.
For spatial discretization (cf. [12]), let $\mathcal{T}_h$ be a regular triangulation of the polyhedral bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ into compact tetrahedra. By $\mathcal{S}^1(\mathcal{T}_h)$ we denote the standard $\mathcal{P}^1$-FEM space of globally continuous and piecewise affine functions from Ω to $\mathbb{R}^3$

$$\mathcal{S}^1(\mathcal{T}_h) := \{\phi_h \in C(\overline{\Omega}, \mathbb{R}^3) \mid \phi_{h|K} \in \mathcal{P}^1(K) \text{ for all } K \in \mathcal{T}_h\}.$$

By $\mathcal{N}_h$ we denote the set of nodes of the triangulation $\mathcal{T}_h$. As we have $|m(t,x)| = 1$ almost everywhere, we define the discrete space for the magnetization by

$$\mathcal{M}_h := \{\phi_h \in \mathcal{S}^1(\mathcal{T}_h) \mid |\phi_h(x)| = 1 \text{ for all } x \in \mathcal{N}_h\}.$$

By $|m(t,x)| = 1$ we get $\partial_t m(t,x) \cdot m(t,x) = 0$ and therefore we define the ansatz space for the time derivative of the magnetization

$$\mathcal{K}_{m_h} := \{\phi_h \in \mathcal{S}^1(\mathcal{T}_h) \mid m_h(x) \cdot \phi_h(x) = 0 \text{ for all } x \in \mathcal{N}_h\}$$

for any $m_h \in \mathcal{M}_h$. We define the nodal interpolation operator for $u \in C(\Omega)$ (or $u \in H^{3/2+\epsilon}$)

$$\Pi_h^{\cdot} u := \sum_{\gamma \in \mathcal{N}_h} u(\gamma)\phi_\gamma,$$

where ϕ_γ for $\gamma \in \mathcal{N}_h$ is the elementwise linear hat function with $\phi_\gamma(\gamma') = \delta_{\gamma,\gamma'}$ for all $\gamma' \in \mathcal{N}_h$.
To discretize the Maxwell system in the interior, we use a Nédélec conforming ansatz space (cf. [36]),

$$\mathcal{X}_h := \{\phi_h \in H(\mathrm{curl}, \Omega) \mid \phi_{h|K} \in \mathcal{P}^1_{skw}(K) \text{ for all } K \in \mathcal{T}_h\},$$

where

$$\mathcal{P}^1_{skw}(K) := \{v : K \to \mathbb{R}^3, v(x) = a + Bx \mid a \in \mathbb{R}^3, B \in \mathbb{R}^{3\times 3}, B^T = -B\}.$$

We define interpolation $\Pi_h^{\nabla\times} : C(\Omega) \to \mathcal{X}_h$ by

$$\int_e u(s) \cdot \tau(s) \, \mathrm{d}s = \int_e (\Pi_h^{\nabla\times} u)(s) \cdot \tau(s) \, \mathrm{d}s$$

for all edges e of the triangulation and corresponding tangential vector τ.

From [14,36] we recall the following error estimates for the above interpolation operators. For the trace variant in combination with the fact that $\gamma_T : H(\mathrm{curl}, \Omega) \to \mathcal{H}_\Gamma$ is bounded.

Lemma 9 *The following approximation properties hold true for sufficiently smooth functions*

$$\begin{aligned}
&\|\phi - \Pi_h^{\cdot}\phi\|_{L^2(\Omega)} + h\|\nabla(\phi - \Pi_h^{\cdot}\phi)\|_{L^2(\Omega)} \le Ch^2\|\phi\|_{H^2(\Omega)},\\
&\|\phi - \Pi_h^{\nabla\times}\phi\|_{L^2(\Omega)} + \|\nabla \times (\phi - \Pi_h^{\nabla\times}\phi)\|_{L^2(\Omega)} \le Ch(\|\phi\|_{H^1(\Omega)} + \|\nabla \times \phi\|_{H^1(\Omega)}),\\
&\|\gamma_T(\phi - \Pi_h^{\nabla\times}\phi)\|_{\mathcal{H}_\Gamma} \le Ch(\|\phi\|_{H^1(\Omega)} + \|\nabla \times \phi\|_{H^1(\Omega)}).
\end{aligned}$$

3.2 Algorithm

We approximate the solution of the Maxwell system by the following algorithm, based on the symmetrised weak formulation (3.1):

Algorithm 10 Input: Discretized initial data m_h^0, H_h^0, E_h^0, $\varphi_h^0 = 0$, $\psi_h^0 = 0$, parameter $\theta \in [0,1]$.
For $j = 0, 1, 2, \dots, N-1$ we compute

– For given m_h^j, H_h^j we compute the unique solution $w_h^j \in \mathcal{K}_{m_h^j}$ such that we have for all $\rho_h \in \mathcal{K}_{m_h^j}$

$$\alpha[w_h^j, \rho_h]_\Omega + \left[m_h^j \times w_h^j, \rho_h\right]_\Omega = -C_e \left[\nabla(m_h^j + \theta\tau w_h^j), \nabla\rho_h\right]_\Omega + \left[H_h^j, \rho_h\right]_\Omega . \qquad (3.2)$$

– We compute $E_h^{j+1}, H_h^{j+1} \in \mathcal{X}_h$ and $\varphi_h^{j+1}, \psi_h^{j+1} \in \gamma_T(\mathcal{X}_h)$ such that we have for all $\zeta_E, \zeta_H \in \mathcal{X}_h$ and $v_\varphi, v_\psi \in \gamma_T(\mathcal{X}_h)$

$$\begin{aligned}[\varepsilon\partial_t^\tau E_h^{j+1}, \zeta_E]_\Omega &= \frac{1}{2}[\nabla \times H_h^{j+1}, \zeta_E]_\Omega + \frac{1}{2}[H_h^{j+1}, \nabla \times \zeta_E]_\Omega \\ &\quad - \frac{1}{2\mu_0}\langle \varphi_h^{j+1}, \gamma_T\zeta_E\rangle_\Gamma - [\sigma E_h^{j+1} + J^{j+1}, \zeta_E]_\Omega,\end{aligned} \tag{3.3}$$

$$\begin{aligned}[\mu\partial_t^\tau H_h^{j+1}, \zeta_H]_\Omega &= -\frac{1}{2}[\nabla \times E_h^{j+1}, \zeta_H]_\Omega - \frac{1}{2}[E_h^{j+1}, \nabla \times \zeta_H]_\Omega \\ &\quad - \frac{1}{2}\langle \psi_h^{j+1}, \gamma_T\zeta_H\rangle_\Gamma - [\mu w_h^j, \zeta_H]_\Omega,\end{aligned} \tag{3.4}$$

$$\left\langle \begin{pmatrix} v_\varphi \\ v_\psi \end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix}\right)(t_{j+1}) \right\rangle_\Gamma = \frac{1}{2}\left(\langle v_\varphi, \mu_0^{-1}\gamma_T E_h^{j+1}\rangle_\Gamma + \langle v_\psi, \gamma_T H_h^{j+1}\rangle_\Gamma\right). \tag{3.5}$$

– Define m_h^{j+1} by

$$m_h^{j+1}(z) := \frac{m_h^j(z) + \tau w_h^j(z)}{|m_h^j(z) + \tau w_h^j(z)|} \qquad \text{for all nodes } z \in \mathcal{N}_h.$$

Output: Sequence of approximations $m_h^j, E_h^j, H_h^j, \varphi_h^j, \psi_h^j$.

In the algorithm, we use the first-order backward difference formula

$$\partial_t^\tau G^{j+1} := \frac{G^{j+1} - G^j}{\tau} \tag{3.6}$$

for $G \in \{E, H\}$. To discretize the convolution $B(\partial_t)w(t)$ we use convolution quadrature (CQ)

$$\left(B(\partial_t^\tau)w\right)((j+1)\tau) := \sum_{l=0}^{j+1} B_{j+1-l}^\tau w(l\tau), \tag{3.7}$$

where the weights B_n are defined as the coefficients of

$$B\left(\frac{\delta(\zeta)}{\tau}\right) = \sum_{n=0}^{\infty} B_n^\tau \zeta^n, \tag{3.8}$$

where, in the present paper, we use the generating polynomial $\delta(\zeta) = 1 - \zeta$, corresponding to the first-order backward difference formula (3.6). For more details on convolution quadratures we refer to [**?**,33, 34].

Remark 11 *We use the first order convolution quadrature based on $\delta(\zeta) = 1 - \zeta$, because in this case $\partial_t^\tau \varphi$ and $(\partial_t^\tau)^{-1}\phi$ can be expressed in a simple and clear way. By the Neumann series formula we have for $|\zeta| < 1$*

$$\frac{1}{1-\zeta} = \sum_{n=0}^{\infty} \zeta^n$$

and for the first order convolution quadrature scheme, we obtain for a sequence $(\varphi^j)_j$

$$\left((\partial_t^\tau)^{-1}\varphi\right)(t_n) = \sum_{j=0}^{n} \tau\varphi^j.$$

Similarly we see that

$$(\partial_t^\tau \varphi)(t_n) = \frac{\varphi^n - \varphi^{n-1}}{\tau}$$

which gives consistent notation with regard to (3.6).

For a sequence of space-dependent functions $(G_h^j)_j$, $G_h^j : \Omega \to \mathbb{R}$ we define the space- and time-dependent functions $G^-_{\tau,h}, G_{\tau,h}, G^+_{\tau,h} : [0,T] \times \Omega \to \mathbb{R}$. For $t \in [t_j, t_j+1)$ and $x \in \Omega$ we define the interval-wise constant functions

$$G^-_{\tau,h}(t,x) := (G_h^j)^-_{\tau,h}(t,x) := G_h^j(x), \qquad G^+_{\tau,h}(t,x) := (G_h^j)^+_{\tau,h}(t,x) := G_h^{j+1}(x),$$

and the interval-wise linear function

$$G_{\tau,h}(t,x) := (G_h^j)_{\tau,h}(t,x) := \frac{t_{j+1}-t}{\tau} G_h^j(x) + \frac{t-t_j}{\tau} G_h^{j+1}(x).$$

Theorem 12 *Algorithm 10 is well defined in the sense, that for every $j \geq 0$, there exist unique approximations m_h^{j+1}, E_h^{j+1}, H_h^{j+1}, φ_h^{j+1}, ψ_h^{j+1} that satisfy (3.2)–(3.5).*

Proof The proof that the tangent plane scheme, (3.2), is well-defined can be found in [3].

For the Maxwell part, we define the bilinear form $a(\cdot,\cdot)$ on $\mathcal{X}_h \times \mathcal{X}_h \times \gamma_T(\mathcal{X}_h) \times \gamma_T(\mathcal{X}_h)$ by

$$\begin{aligned}
&a((\Phi,\Psi,\Theta,\Upsilon),(\phi,\psi,\theta,\upsilon)) \\
&\quad := 1/\tau[\varepsilon\Phi,\phi]_\Omega + 1/\tau[\mu\Psi,\psi]_\Omega + \left\langle \begin{pmatrix}\theta\\ \upsilon\end{pmatrix}, B_0^\tau \begin{pmatrix}\Theta\\ \Upsilon\end{pmatrix}\right\rangle_\Gamma + [\sigma\Phi,\phi]_\Omega \\
&\qquad - [\Psi,\nabla\times\phi]_\Omega/2 - [\nabla\times\Psi,\phi]_\Omega/2 + [\Phi,\nabla\times\psi]_\Omega/2 + [\nabla\times\Phi,\psi]_\Omega/2 \\
&\qquad + \frac{1}{2}\langle\Upsilon,\gamma_T\psi\rangle_\Gamma + \frac{1}{2\mu_0}\langle\Theta,\gamma_T\phi\rangle_\Gamma - \langle\theta,\mu_0^{-1}\gamma_T\Phi\rangle_\Gamma/2 - \langle\upsilon,\gamma_T\Psi\rangle_\Gamma/2
\end{aligned}$$

and the linear functional $L^j(\cdot)$ on $\mathcal{X}_h \times \mathcal{X}_h \times \gamma_T(\mathcal{X}_h) \times \gamma_T(\mathcal{X}_h)$ by

$$\begin{aligned}
L^j(\phi,\psi,\theta,\upsilon) &:= 1/\tau[\varepsilon E_h^j,\phi]_\Omega + 1/\tau[\mu H_h^j,\psi]_\Omega - [J^{j+1},\phi]_\Omega - \mu[w_h^i,\phi]_\Omega \\
&\quad - \left\langle \begin{pmatrix}\theta\\ \upsilon\end{pmatrix}, \sum_{l=0}^{j} B^\tau_{j+1-l}\begin{pmatrix}\varphi_h^l\\ \psi_h^l\end{pmatrix}\right\rangle_\Gamma .
\end{aligned}$$

The equations (3.3)–(3.5) are equivalent to

$$a((E_h^{j+1}, H_h^{j+1}, \varphi_h^{j+1}, \phi_h^{j+1}),(\phi,\psi,\theta,\upsilon)) = L^j(\phi,\psi,\theta,\upsilon)$$

for all $(\phi,\psi,\theta,\upsilon) \in \mathcal{X}_h \times \mathcal{X}_h \times \gamma_T(\mathcal{X}_h) \times \gamma_T(\mathcal{X}_h)$. Next, we aim to show that the bilinear form $a(\cdot,\cdot)$ is positive definite on $\mathcal{X}_h \times \mathcal{X}_h \times \gamma_T(\mathcal{X}_h) \times \gamma_T(\mathcal{X}_h)$. We have $B_0^\tau = B(\tau^{-1})$ and by Lemma 2 for all $\zeta \in \mathcal{H}_\Gamma \times \mathcal{H}_\Gamma$ and $s > 0$

$$\langle\zeta, B(s)\zeta\rangle_\Gamma \geq C(s,\mu_0,\varepsilon_0)\|\zeta\|^2_{\mathcal{H}_\Gamma}.$$

Therefore it is

$$\begin{aligned}
&a((\Phi,\Psi,\Theta,\Upsilon),(\Phi,\Psi,\Theta,\Upsilon)) \\
&\quad = 1/\tau[\varepsilon\Phi,\Phi]_\Omega + 1/\tau[\mu\Psi,\Psi]_\Omega + \left\langle \begin{pmatrix}\Theta\\ \Upsilon\end{pmatrix}, B_0^\tau \begin{pmatrix}\Theta\\ \Upsilon\end{pmatrix}\right\rangle_\Gamma + [\sigma\Phi,\Phi]_\Omega \\
&\quad \geq C(\tau,\mu,\varepsilon)(\|\Phi\|^2_\Omega + \|\Psi\|^2_\Omega + \|\Theta\|^2_{\mathcal{H}_\Gamma} + \|\Upsilon\|^2_{\mathcal{H}_\Gamma})
\end{aligned}$$

positive definite, for arbitrary $\tau > 0$, which yields the desired result. □

dNote that since we use an implicit time discretization method and the problem is linear, no smallness conditions are required on τ.

4 Convergence

In this section we will state and prove the main results of this paper, namely, the weak convergence of a (sub)sequence of approximations obtained by Algorithm 10.

We require the following natural assumptions:

Assumption 13

- *The triangulations $\mathcal{T}_h$ are uniformly shape regular and satisfy the angle condition*

$$\int_\Omega \nabla\zeta(x) \cdot \nabla\xi(x) \, \mathrm{d}x \leq 0$$

for all linear basis functions $\zeta, \xi \in \mathcal{S}^1(\mathcal{T}_h)$ with $\xi \neq \zeta$ (cf. [12, (5.1)–(5.7)]).
- *$J^\pm_{\tau,h} \rightharpoonup J$ in $L^2(\Omega_T)$.*
- *$E^0_h \rightharpoonup E^0$ and $H^0_h \rightharpoonup H^0$ in $L^2(\Omega)$.*
- *$m^0_h \rightharpoonup m^0$ in $H^1(\Omega)$.*

Remark 14 *The angle condition gives despite the normalization step in Algorithm 10*

$$\|\nabla m_h^{j+1}\|_\Omega \leq \|\nabla(m_h^j + \tau w_h^j)\|_\Omega,$$

cf. [12, Remark 5.1]. The angle condition is fulfilled, if all dihedral angles of the tetrahedral mesh are smaller or equal than $90°$.

Remark 15 *All results in this section are formulated for scalar and constant material parameters $\varepsilon, \mu \in \mathbb{R}_+$, but hold with similar arguments for symmetric, coercive and bounded material tensors*

$$\varepsilon, \mu : \Omega \to \mathbb{R}^{3\times 3}$$

and bounded, positive $\sigma : \Omega \to \mathbb{R}^{3\times 3}$.

From [28] we recall the coercivity property of the CQ time discretization of the time-dependent Calderon operator $B(\partial_t^\tau)$, which is the CQ time discrete time-domain variant of Lemma 2. We will use this result at a later point in the convergence proof.

Lemma 16 ([9, Lemma 2.3]) *It holds for $0 < \rho < 1$, $0 < \tau \leq 1$ and sequences $(\varphi(t_i))_{i=0}^\infty$ and $(\psi(t_i))_{i=0}^\infty$ in $\mathcal{H}_\Gamma$ (with only finite many nonzero entries)*

$$\sum_{n=0}^\infty \rho^{2n} \Re \left\langle \begin{pmatrix} \varphi^n \\ \psi^n \end{pmatrix}, B\left(\partial_t^\tau\right) \begin{pmatrix} \varphi \\ \psi \end{pmatrix}(t_n) \right\rangle_\Gamma \geq C \min\left(\frac{1-\rho}{\tau}, \left(\frac{1-\rho}{\tau}\right)^3 \right) \sum_{n=0}^\infty \rho^{2n} \left(\|(\partial_t^\tau)^{-1}\varphi(t_n)\|^2_{\mathcal{H}_\Gamma} + \|(\partial_t^\tau)^{-1}\psi(t_n)\|^2_{\mathcal{H}_\Gamma} \right).$$

The constant $C > 0$ depends on ε_0, μ_0 and $\beta > 0$ of Lemma 2.

Proof For $0 < \rho < 1$ and $|\xi| \leq \rho$ we have

$$\left| \frac{\delta(\zeta)}{\tau} \right| \geq \Re\left(\frac{\delta(\zeta)}{\tau} \right) = \Re\left(\frac{1-\zeta}{\tau} \right) \geq \frac{1-\rho}{\tau} > 0.$$

Therefore we have for $\varphi, \psi \in \mathcal{H}_\Gamma$ by Lemma 2

$$\Re \left\langle \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, B\left(\frac{\delta(\zeta)}{\tau}\right) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle_\Gamma \geq C \min\left(\frac{1-\rho}{\tau}, \left(\frac{1-\rho}{\tau}\right)^3 \right) \left(\|(\tfrac{\delta(\zeta)}{\tau})^{-1}\varphi\|^2_{\mathcal{H}_\Gamma} + \|(\tfrac{\delta(\zeta)}{\tau})^{-1}\psi\|^2_{\mathcal{H}_\Gamma} \right)$$

for $|\xi| \leq \rho$. Now the assertion follows by the time-discrete operator-valued Herglotz theorem [9, Lemma 2.1]. □

Lemma 17 ([28, Lemma 5.3]) *It holds for $0<\tau\le 1$ and $t_j \le T$ for arbitrary sequences $(\varphi(t_i))_{i=0}^j$ and $(\psi(t_i))_{i=0}^j$ in $\mathcal{H}_\Gamma$ that*

$$\tau \sum_{i=0}^{j} e^{-2t_i/T} \left\langle \begin{pmatrix} \varphi(t_i) \\ \psi(t_i) \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right)(t_i) \right\rangle_\Gamma \geq$$
$$C\tau \sum_{i=0}^{j} e^{-2t_i/T} \left(\|(\partial_t^\tau)^{-1}\varphi(t_i)\|_{\mathcal{H}_\Gamma}^2 + \|(\partial_t^\tau)^{-1}\psi(t_i)\|_{\mathcal{H}_\Gamma}^2 \right),$$

where the constant $C>0$ depends on T, ε_0, μ_0, and on $\beta>0$ and $\mathrm{m}(T^{-1})$ of Lemma 2.

Furthermore, there also holds

$$\sum_{i=0}^{j} \left\langle \begin{pmatrix} \varphi(t_i) \\ \psi(t_i) \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right)(t_i) \right\rangle_\Gamma \geq 0.$$

Proof The second assertion follows directly by letting $\rho \to 1$ for fixed τ in Lemma 16. □

Lemma 18 (Discrete Gronwall Lemma, cf. [12, Lemma 5.3]) *Let $(a_i)_{i\in\mathbb{N}_0}$ be a sequence of positive real numbers, $b, C>0$ constants and $j\in\mathbb{N}$. If we have*

$$a_i \le b + C\sum_{k=0}^{i-1} a_k$$

for $i=0,\dots,j$, then it holds

$$a_i \le b e^{Ci}$$

for all $0\le i\le j$.

Lemma 19 *The approximations stay bounded for $\theta \ge 1/2$, i.e. we have for $j \ge 0$*

$$\mathcal{E}_h^j := \frac{\mu}{2}\|H_h^j\|_\Omega^2 + \frac{\varepsilon}{2}\|E_h^j\|_\Omega^2 + \mu\frac{C_e}{2}\|\nabla m_h^j\|_\Omega^2 \le C_1$$

and additionally

$$\begin{aligned}\sum_{i=1}^{j}\|H_h^i - H_h^{i-1}\|_\Omega^2 + \sum_{i=1}^{j}\|E_h^i - E_h^{i-1}\|_\Omega^2 + \tau\sum_{i=1}^{j}\|w_h^{i-1}\|_\Omega^2 \\ + \sum_{i=1}^{j}\tau^2(\theta - 1/2)\|\nabla w_h^{i-1}\|_\Omega^2 + \tau \underbrace{\sum_{i=1}^{j}\left\langle \begin{pmatrix} \varphi_h^i \\ \psi_h^i \end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix}\right)(t_i)\right\rangle_\Gamma}_{\ge 0} \le C_2.\end{aligned} \tag{4.1}$$

The constants C_1 and C_2 depend on T, τ_0, α, ε, μ, J and $\mathcal{E}_h^0$, but do not depend on h and τ.

Proof For simplicity we omit the subscript $_h$ and write $E^{j+1}, H^{j+1}, \dots$ instead of $E_h^{j+1}, H_h^{j+1}, \dots$. We test in Algorithm 10 with $\zeta_E = E^{j+1}$, $\zeta_H = H^{j+1}$, $v_\varphi = \varphi^{j+1}$ and $v_\psi = \psi^{j+1}$ and add up the three last equations to obtain

$$\begin{aligned}\varepsilon[\partial_t^\tau E^{j+1}, E^{j+1}]_\Omega + \mu[\partial_t^\tau H^{j+1}, H^{j+1}]_\Omega + \left\langle \begin{pmatrix} \varphi^{j+1} \\ \psi^{j+1} \end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix} \varphi \\ \psi \end{pmatrix}\right)(t_{j+1})\right\rangle_\Gamma \\ = -[\sigma E^{j+1} + J^{j+1}, E^{j+1}]_\Omega - \mu[w^j, H^{j+1}]_\Omega.\end{aligned}$$

Thus we have for all $i \ge 1$ (rewrite the above equation for $i := j+1$)

$$\begin{aligned}\frac{\varepsilon}{\tau}[E^i - E^{i-1}, E^i]_\Omega + \frac{\mu}{\tau}[H^i - H^{i-1}, H^i]_\Omega + \left\langle \begin{pmatrix} \varphi^{i} \\ \psi^{i} \end{pmatrix}, \left(B(\partial_t^\tau)\begin{pmatrix} \varphi \\ \psi \end{pmatrix}\right)(t_i)\right\rangle_\Gamma \\ = -[\sigma E^i + J^i, E^i]_\Omega - \mu[w^{i-1}, H^i]_\Omega.\end{aligned} \tag{4.2}$$

To treat the terms $[E^i - E^{i-1}, E^i]_\Omega$ and $[H^i - H^{i-1}, H^i]_\Omega$ we repeat Abel's summation by parts: For $u_i \in \mathbb{R}^n$ and $j \geq i \geq 1$, there holds by the third binomial formula and telescoping summation

$$\sum_{i=1}^{j} (u_i - u_{i-1}) \cdot u_i = \frac{1}{2} \sum_{i=1}^{j} |u_i - u_{i-1}|^2 + \frac{1}{2}|u_j|^2 - \frac{1}{2}|u_0|^2.$$

Summing up the equations (4.2) for $i = 1, \ldots, j$, multiplying by τ and applying Abel's summation by parts to the respective terms we obtain

$$\begin{aligned} &\frac{\mu}{2}\left(\|H^j\|_\Omega^2 - \|H^0\|_\Omega^2 + \sum_{i=1}^{j} \|H^i - H^{i-1}\|_\Omega^2 \right) + \frac{\varepsilon}{2}\left(\|E^j\|_\Omega^2 - \|E^0\|_\Omega^2 + \sum_{i=1}^{j} \|E^i - E^{i-1}\|_\Omega^2 \right) \\ &\quad + \tau \sum_{i=1}^{j} \left\langle \begin{pmatrix} \varphi^i \\ \psi^i \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right)(t_i) \right\rangle_\Gamma \\ &\qquad = -\tau\sigma \sum_{i=1}^{j} \|E^i\|_\Omega^2 - \tau \sum_{i=1}^{j} [J^i, E^i]_\Omega - \tau \sum_{i=1}^{j} \mu [w^{i-1}, H^i]_\Omega. \end{aligned} \tag{4.3}$$

We test in Algorithm 10 with $\rho = w^j$ for $j = i-1$ and receive (again with $j = i-1$)

$$\alpha[w^{i-1}, w^{i-1}]_\Omega = -C_e[\nabla(m^{i-1} + \theta\tau w^{i-1}), \nabla w^{i-1}]_\Omega + [H^{i-1}, w^{i-1}]_\Omega.$$

By the mesh condition (Remark 14) we have $\|\nabla m^i\|_\Omega \leq \|\nabla(m^{i-1} + \tau w^{i-1})\|_\Omega$ and therefore we get

$$\begin{aligned} \|\nabla m^i\|_\Omega^2 &\leq \|\nabla m^{i-1}\|_\Omega^2 + 2\tau[\nabla m^{i-1}, \nabla w^{i-1}]_\Omega + \tau^2 \|\nabla w^{i-1}\|_\Omega^2 \\ &= \|\nabla m^{i-1}\|_\Omega^2 + \frac{2\tau}{C_e}(-\alpha\|w^{i-1}\|_\Omega^2 + [H^{i-1}, w^{i-1}]_\Omega) - \tau^2(2\theta - 1)\|\nabla w^{i-1}\|_\Omega^2. \end{aligned}$$

We rewrite this as

$$\begin{aligned} &\mu\frac{C_e}{2}\|\nabla m^i\|_\Omega^2 + \alpha\tau\mu\|w^{i-1}\|_\Omega^2 + C_e\mu\tau^2(\theta - 1/2)\|\nabla w^{i-1}\|_\Omega^2 \\ &\qquad \leq \mu\frac{C_e}{2}\|\nabla m^{i-1}\|_\Omega^2 + \mu\tau[H^{i-1}, w^{i-1}]_\Omega, \end{aligned}$$

sum up from $i = 1, \ldots, j$ to get

$$\begin{aligned} &\mu\frac{C_e}{2}\|\nabla m^j\|_\Omega^2 + \tau\alpha\mu \sum_{i=1}^{j} \|w^{i-1}\|_\Omega^2 + C_e\mu\tau^2(\theta - 1/2) \sum_{i=1}^{j} \|\nabla w^{i-1}\|_\Omega^2 \\ &\qquad \leq \mu\frac{C_e}{2}\|\nabla m^0\|_\Omega^2 + \mu\tau \sum_{i=1}^{j} [H^{i-1}, w^{i-1}]_\Omega, \end{aligned}$$

and add it to (4.3) to receive

$$\begin{aligned} &\frac{\mu}{2}\left(\|H^j\|_\Omega^2 + \sum_{i=1}^{j} \|H^i - H^{i-1}\|_\Omega^2 \right) + \frac{\varepsilon}{2}\left(\|E^j\|_\Omega^2 + \sum_{i=1}^{j} \|E^i - E^{i-1}\|_\Omega^2 \right) + \tau\sigma \sum_{i=1}^{j} \|E^i\|_\Omega^2 \\ &\quad + \mu\frac{C_e}{2}\|\nabla m^j\|_\Omega^2 + \tau \sum_{i=1}^{j} \mu\alpha\|w^{i-1}\|_\Omega^2 + \sum_{i=1}^{j} C_e\mu\tau^2(\theta - 1/2)\|\nabla w^{i-1}\|_\Omega^2 \\ &\quad + \tau \sum_{i=1}^{j} \left\langle \begin{pmatrix} \varphi^i \\ \psi^i \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right)(t_i) \right\rangle_\Gamma \\ &\quad \leq \frac{\mu}{2}\|H^0\|_\Omega^2 + \frac{\varepsilon}{2}\|E^0\|_\Omega^2 + \mu\frac{C_e}{2}\|\nabla m^0\|_\Omega^2 + \tau \sum_{i=1}^{j} \left(-[J^i, E^i]_\Omega \right) + \mu\tau \sum_{i=1}^{j} [H^{i-1} - H^i, w^{i-1}]_\Omega. \end{aligned}$$

We estimate the right-hand side with Cauchy–Schwartz for arbitrary $\delta_1, \delta_2 > 0$

$$\begin{aligned}
&\tau \sum_{i=1}^{j} \left(-[J^i, E^i]_\Omega\right) + \mu\tau \sum_{i=1}^{j} [H^{i-1} - H^i, w^{i-1}]_\Omega \\
&\leq \sum_{i=1}^{j} \frac{\tau}{2\delta_1} \|J^i\|_\Omega^2 + \sum_{i=1}^{j} \frac{\tau\delta_1}{2} \|E^i\|_\Omega^2 + \sum_{i=1}^{j} \frac{\mu\tau}{2\delta_2} \|H^i - H^{i-1}\|_\Omega^2 + \sum_{i=1}^{j} \frac{\mu\tau\delta_2}{2} \|w^{i-1}\|_\Omega^2.
\end{aligned}$$

As $\sigma = 0$ is possible, the terms $\sum_{i=1}^{j} \frac{\tau\delta_1}{2} \|E^i\|_\Omega^2$ on the right-hand side cannot be absorbed by the respective terms on the left-hand side. Therefore we use

$$\sum_{i=1}^{j} \frac{\tau\delta_1}{2} \|E^i\|_\Omega^2 \leq \sum_{i=1}^{j} \tau\delta_1 \|E^i - E^{i-1}\|_\Omega^2 + \sum_{i=1}^{j} \tau\delta_1 \|E^{i-1}\|_\Omega^2$$

and obtain with

$$\mathcal{E}^j := \frac{\mu}{2} \|H^j\|_\Omega^2 + \frac{\varepsilon}{2} \|E^j\|_\Omega^2 + \mu \frac{C_e}{2} \|\nabla m^j\|_\Omega^2$$

that

$$\begin{aligned}
&\mathcal{E}^j + \frac{\mu}{2}\left(1 - \frac{\tau}{\delta_2}\right) \sum_{i=1}^{j} \|H^i - H^{i-1}\|_\Omega^2 + \left(\frac{\varepsilon}{2} - \tau\delta_1\right) \sum_{i=1}^{j} \|E^i - E^{i-1}\|_\Omega^2 + \tau\sigma \sum_{i=1}^{j} \|E^i\|_\Omega^2 \\
&\quad + \tau \sum_{i=1}^{j} \mu(\alpha - \delta_2/2) \|w^{i-1}\|_\Omega^2 + \sum_{i=1}^{j} C_e \mu \tau^2 (\theta - 1/2) \|\nabla w^{i-1}\|_\Omega^2 \\
&\quad + \tau \sum_{i=1}^{j} \left\langle \begin{pmatrix} \varphi^i \\ \psi^i \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi \\ \psi \end{pmatrix}\right)(t_i) \right\rangle_\Gamma \qquad (4.4) \\
&\quad \leq \mathcal{E}^0 + \sum_{i=1}^{j} \frac{\tau}{2\delta_1} \|J^i\|_\Omega^2 + \sum_{i=1}^{j} \tau\delta_1 \|E^{i-1}\|_\Omega^2 \\
&\quad \leq \mathcal{E}^0 + \sum_{i=1}^{j} \frac{\tau}{2\delta_1} \|J^i\|_\Omega^2 + \frac{2\delta_1}{\varepsilon} \tau \sum_{i=1}^{j} \mathcal{E}^{i-1}.
\end{aligned}$$

We have to ensure

$$\left(1 - \frac{\tau}{\delta_2}\right) > 0, \qquad \left(\frac{\varepsilon}{2} - \tau\delta_1\right) > 0 \quad \text{and} \quad (\alpha - \delta_2/2) > 0,$$

which is possible for $\delta_1, \delta_2 = O(1)$ and for small enough $\tau > 0$.
Moreover it holds (cf. Lemma 17)

$$\sum_{i=1}^{j} \left\langle \begin{pmatrix} \varphi^i \\ \psi^i \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi \\ \psi \end{pmatrix}\right)(t_i) \right\rangle_\Gamma \geq 0.$$

Thus equation (4.4) can be simplified to

$$\mathcal{E}^i \leq C + c\tau \sum_{i=1}^{j} \mathcal{E}^{i-1}$$

and the discrete Gronwall Lemma (Lemma 18) gives $\mathcal{E}^i < \tilde{C}$ for $i \leq j$. Thus we have

$$\frac{2\delta_1}{\varepsilon} \tau \sum_{i=1}^{j} \mathcal{E}^{i-1} < \hat{C},$$

what concludes the assertion. □

The following lemma provides energy bounds for the quantities on the boundary. It is a a modification of Lemma 19 with the missing factors $e^{-t_i/T}$ that show up in Lemma 17.

Lemma 20 *We now are able to deduce*

$$\tau \sum_{i=1}^{j} \Big(\|(\partial_t^\tau)^{-1} \varphi(t_i, \cdot)\|_{\mathcal{H}_\Gamma}^2 + \|(\partial_t^\tau)^{-1} \psi(t_i, \cdot)\|_{\mathcal{H}_\Gamma}^2 \Big) \leq C$$

for a constant $C > 0$ depending on $T, \varepsilon, \varepsilon_0, \mu, \mu_0$, β, τ_0, α, J and $\mathcal{E}_h^0$, but independent of h and τ. Thus $((\partial_t^\tau)^{-1}\varphi_h)_{\tau,h}$, $((\partial_t^\tau)^{-1}\varphi_h)_{\tau,h}^{\pm}$, $((\partial_t^\tau)^{-1}\psi_h)_{\tau,h}$ and $((\partial_t^\tau)^{-1}\psi_h)_{\tau,h}^{\pm}$ are bounded in $L^2([0,T], \mathcal{H}_\Gamma)$.

Proof The proof works analogously as the one of Lemma 19, by inserting the missing factors $e^{-t_i/T}$. We test in Algorithm 10 with $\zeta_E = E^{j+1}$, $\zeta_H = H^{j+1}$, $v_\varphi = \varphi^{j+1}$ and $v_\psi = \psi^{j+1}$ and add the three last equations to obtain

$$\begin{aligned} \varepsilon[\partial_t^\tau E^{j+1}, E^{j+1}]_\Omega + \mu[\partial_t^\tau H^{j+1}, H^{j+1}]_\Omega + \left\langle \begin{pmatrix} \varphi^{j+1} \\ \psi^{j+1} \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right)(t_{j+1}) \right\rangle_\Gamma \\ = -[\sigma E^{j+1} + J^{j+1}, E^{j+1}]_\Omega - \mu[w^j, H^{j+1}]_\Omega. \end{aligned}$$

By rewriting the above equation for $i := j+1$, multiplying it by $e^{-2t_i/T}$, and by using the abbreviations

$$\tilde{E}^i := e^{-t_i/T} E^i, \quad \tilde{H}^i := e^{-t_i/T} H^i, \quad \tilde{w}^i := e^{-t_i/T} w^i, \text{ and } \tilde{J}^i := e^{-t_i/T} J^i,$$

we have for all $i \geq 1$

$$\begin{aligned} \frac{\varepsilon}{\tau}[\tilde{E}^i - e^{-\tau/T}\tilde{E}^{i-1}, \tilde{E}^i]_\Omega + \frac{\mu}{\tau}[\tilde{H}^i - e^{-\tau/T}\tilde{H}^{i-1}, \tilde{H}^i]_\Omega + e^{-2t_i/T} \left\langle \begin{pmatrix} \varphi^i \\ \psi^i \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right)(t_i) \right\rangle_\Gamma \\ = -[\sigma\tilde{E}^i + \tilde{J}^i, \tilde{E}^i]_\Omega - \mu[\tilde{w}^{i-1}, e^{-\tau/T}\tilde{H}^i]_\Omega. \end{aligned} \tag{4.5}$$

To treat the terms $[\tilde{E}^i - e^{-\tau/T}\tilde{E}^{i-1}, \tilde{E}^i]_\Omega$ and $[\tilde{H}^i - e^{-\tau/T}\tilde{H}^{i-1}, \tilde{H}^i]_\Omega$ we modify Abel's summation by parts. For $u_i \in \mathbb{R}^n$ and $j \geq i \geq 1$, there holds

$$\begin{aligned} \sum_{i=1}^{j} (u_i - e^{-\tau/T} u_{i-1}) \cdot u_i &= \frac{1}{2} \sum_{i=1}^{j} |u_i - e^{-\tau/T} u_{i-1}|^2 + \frac{1}{2} \sum_{i=1}^{j} |u_i|^2 - e^{-2\tau/T} |u_{i-1}|^2 \\ &\geq \frac{1}{2} \sum_{i=1}^{j} |u_i - e^{-\tau/T} u_{i-1}|^2 + \frac{1}{2} \sum_{i=1}^{j} |u_i|^2 - |u_{i-1}|^2 \\ &= \frac{1}{2} \sum_{i=1}^{j} |u_i - e^{-\tau/T} u_{i-1}|^2 + \frac{1}{2}|u_j|^2 - \frac{1}{2}|u_0|^2. \end{aligned}$$

Summing up the equations (4.5) for $i = 1, \ldots, j$, multiplying by τ and applying the modified summation by parts to $\tilde{E}^i = e^{-t_i/T} E^i$ and $\tilde{H}^i = e^{-t_i/T} H^i$ we obtain

$$\begin{aligned} &\frac{\mu}{2} \left(\|\tilde{H}^j\|_\Omega^2 - \|\tilde{H}^0\|_\Omega^2 + \sum_{i=1}^{j} \|\tilde{H}^i - e^{-\tau/T}\tilde{H}^{i-1}\|_\Omega^2 \right) + \frac{\varepsilon}{2} \left(\|\tilde{E}^j\|_\Omega^2 - \|\tilde{E}^0\|_\Omega^2 + \sum_{i=1}^{j} \|\tilde{E}^i - e^{-\tau/T}\tilde{E}^{i-1}\|_\Omega^2 \right) \\ &+ \tau \sum_{i=1}^{j} e^{-2t_i/T} \left\langle \begin{pmatrix} \varphi^i \\ \psi^i \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right)(t_i) \right\rangle_\Gamma + \tau\sigma \sum_{i=1}^{j} \|\tilde{E}^i\|_\Omega^2 \\ &\leq -\tau \sum_{i=1}^{j} [\tilde{J}^i, \tilde{E}^i]_\Omega - \tau \sum_{i=1}^{j} \mu[\tilde{w}^{i-1}, e^{-\tau/T}\tilde{H}^i]_\Omega \\ &\leq \left(\tau \sum_{i=1}^{j} \|\tilde{J}^i\|_\Omega^2 \right)^{1/2} \left(\tau \sum_{i=1}^{j} \|\tilde{E}^i\|_\Omega^2 \right)^{1/2} + \left(\tau \sum_{i=1}^{j} \|\tilde{H}^i\|_\Omega^2 \right)^{1/2} \left(\tau \sum_{i=1}^{j} \|\mu\tilde{w}^i\|_\Omega^2 \right)^{1/2}. \end{aligned} \tag{4.6}$$

By Assumptions 13 and Lemma 19, we have

$$\tau\sum_{i=1}^{j}\|\tilde{E}^i\|_{\Omega}^2+\tau\sum_{i=1}^{j}\|\tilde{H}^i\|_{\Omega}^2+\tau\sum_{i=1}^{j}\|\tilde{J}^i\|_{\Omega}^2+\tau\sum_{i=1}^{j}\|\tilde{w}^i\|_{\Omega}^2\le C.$$

As all other terms on the left-hand side of (4.6) are positive and/or bounded, we have

$$\tau\sum_{i=0}^{j}e^{-2i\tau/T}\left\langle\begin{pmatrix}\varphi^i\\ \psi^i\end{pmatrix},\left(B(\partial_t^\tau)\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right)(t_i)\right\rangle_{\Gamma}\le C,$$

and therefore, by Lemma 17 for some constants $c, C>0$

$$\begin{aligned}\tau\sum_{i=0}^{j}\|(\partial_t^\tau)^{-1}\varphi^i(t_i,\cdot)\|_{\mathcal{H}_\Gamma}^2+\|(\partial_t^\tau)^{-1}\psi(t_i,\cdot)\|_{\mathcal{H}_\Gamma}^2&\\ \le c\tau\sum_{i=0}^{j}e^{-2i\tau/T}\left\langle\begin{pmatrix}\varphi^i\\ \psi^i\end{pmatrix},\left(B(\partial_t^\tau)\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right)(t_i)\right\rangle_{\Gamma}&\le C\end{aligned}$$

which yields the assertion. □

Let P_h be the L^2-orthogonal projection onto the closed (because finite-dimensional) subspace $\mathcal{X}_h$, i.e.

$$\mathrm{P}_h: L^2(\Omega)\to\mathcal{X}_h$$

is linear and it holds for every $v\in L^2(\Omega)$

$$[(1-\mathrm{P}_h)v,\xi_h]_\Omega=0 \text{ for all } \xi_h\in\mathcal{X}_h.$$

We define for $\xi\in L^2(\Omega_T)$

$$\begin{aligned}f_{\tau,h}(\xi)&:=\frac{1}{2}[(\nabla\times(\partial_t^\tau)^{-1}(H_h^{j+1})_j)_{\tau,h}^-,\mathrm{P}_h\xi]_{\Omega_T}+\frac{1}{2}[((\partial_t^\tau)^{-1}(H_h^{j+1})_j)_{\tau,h}^-,\nabla\times(\mathrm{P}_h\xi)]_{\Omega_T}\\&\quad-\frac{1}{2\mu_0}\left\langle((\partial_t^\tau)^{-1}(\varphi_h^{j+1})_j)_{\tau,h}^-,\gamma_T(\mathrm{P}_h\xi)\right\rangle_{\Gamma_T},\\ g_{\tau,h}(\xi)&:=\frac{1}{2}[(\nabla\times(\partial_t^\tau)^{-1}(E_h^{j+1})_j)_{\tau,h}^-,\mathrm{P}_h\xi]_{\Omega_T}+\frac{1}{2}[((\partial_t^\tau)^{-1}(E_h^{j+1})_j)_{\tau,h}^-,\nabla\times(\mathrm{P}_h\xi)]_{\Omega_T}\\&\quad+\frac{1}{2\mu_0}\left\langle((\partial_t^\tau)^{-1}(\psi_h^{j+1})_j)_{\tau,h}^-,\gamma_T(\mathrm{P}_h\xi)\right\rangle_{\Gamma_T}.\end{aligned}$$

Lemma 21 *For $\xi\in L^2(\Omega_T)$ it holds*

$$|f_{\tau,h}(\xi)|\le C\|\xi\|_{\Omega_T}$$

and

$$|g_{\tau,h}(\xi)|\le C\|\xi\|_{\Omega_T}$$

The constants do not depend on h or τ. We identify $f_{\tau,h}\in L^2(\Omega_T)$ by $f_{\tau,h}(\xi)=[f_{\tau,h},\xi]_{\Omega_T}$ and $g_{\tau,h}\in L^2(\Omega_T)$ by $g_{\tau,h}(\xi)=[g_{\tau,h},\xi]_{\Omega_T}$.

Proof We test equation (3.3) by $\zeta_h\in\mathcal{X}_h$, multiply by τ and sum over $j=0,\dots,k$ to obtain

$$\begin{aligned}[\varepsilon E_h^{k+1},\zeta_h]_\Omega-[\varepsilon E_h^0,\zeta_h]_\Omega&=\frac{1}{2}[(\nabla\times(\partial_t^\tau)^{-1}(H_h^{j+1})_j)(t_k),\zeta_h]_\Omega+\frac{1}{2}[((\partial_t^\tau)^{-1}(H_h^{j+1})_j)(t_k),\nabla\times\zeta_h]_\Omega\\&\quad-\frac{1}{2\mu_0}\langle((\partial_t^\tau)^{-1}(\varphi_h^{j+1})_j)(t_k),\gamma_T\zeta_h\rangle_\Gamma+[((\partial_t^\tau)^{-1}(\sigma E^{j+1}+J^{j+1})_j)(t_k),\zeta_h]_\Omega\end{aligned}$$

For ζ_h we insert $\mathrm{P}_h\xi(t)$, integrate over $[t_k,t_{k+1}]$, sum up from $k=0,\dots,N-1$ and obtain

$$f_{\tau,h}(\xi)=[E_{\tau,h}^+-E_h^0,\varepsilon\mathrm{P}_h\xi]_{\Omega_T}-[((\partial_t^\tau)^{-1}(\sigma E^{j+1}+J^{j+1})_j)_{\tau,h}^-,\mathrm{P}_h\xi]_{\Omega_T}.$$

With Lemma 19 and Assumption 13 we have

$$\|E_h^{k+1}\|_\Omega + \|E_h^0\|_\Omega + \|(\partial_t^\tau)^{-1}(\sigma E^{j+1} + J^{j+1})_j(t_k)\|_\Omega \le C$$

and as P_h is an L^2 orthogonal projection and therefore bounded, we have

$$|f_{\tau,h}(\xi)| \le C\|\mathrm{P}_h\xi\|_{\Omega_T} \le C\|\xi\|_{\Omega_T},$$

which concludes the first assertion. The second one follows similarly by using

$$\|(\partial_t^\tau)^{-1}w_h(t_j)\|_\Omega \le C,$$

which is again a consequence of Lemma 19. □

Due to the boundedness of the quantities, we are now able to extract weakly convergent subsequences.

Lemma 22 (cf. [12, Lemma 5.5, Lemma 5.6]) *There exist functions*

$$(m, H, E, \tilde{\varphi}, \tilde{\psi}) \in H^1(\Omega_T, \mathbb{S}^2) \times L^2(\Omega_T) \times L^2(\Omega_T) \times L^2([0,T], \mathcal{H}_\Gamma) \times L^2([0,T], \mathcal{H}_\Gamma)$$

such that

$$\begin{aligned}
m_{\tau,h} &\overset{\text{sub}}{\rightharpoonup} m && \text{in } H^1(\Omega_T),\\
m_{\tau,h}, m_{\tau,h}^{\pm} &\overset{\text{sub}}{\rightharpoonup} m && \text{in } L^2([0,T], H^1(\Omega)),\\
m_{\tau,h}, m_{\tau,h}^{\pm} &\overset{\text{sub}}{\to} m && \text{in } L^2(\Omega_T),\\
w_{\tau,h}^{-} &\overset{\text{sub}}{\rightharpoonup} \partial_t m && \text{in } L^2(\Omega_T),\\
H_{\tau,h}, H_{\tau,h}^{\pm} &\overset{\text{sub}}{\rightharpoonup} H && \text{in } L^2(\Omega_T),\\
E_{\tau,h}, E_{\tau,h}^{\pm} &\overset{\text{sub}}{\rightharpoonup} E && \text{in } L^2(\Omega_T),\\
((\partial_t^\tau)^{-1}\varphi_h)_{\tau,h}, ((\partial_t^\tau)^{-1}\varphi_h)_{\tau,h}^{\pm} &\overset{\text{sub}}{\rightharpoonup} \tilde{\varphi} && \text{in } L^2([0,T], \mathcal{H}_\Gamma) \text{ w.r.t to } \langle\cdot,\cdot\rangle_{\Gamma_T},\\
((\partial_t^\tau)^{-1}\psi_h)_{\tau,h}, ((\partial_t^\tau)^{-1}\psi_h)_{\tau,h}^{\pm} &\overset{\text{sub}}{\rightharpoonup} \tilde{\psi} && \text{in } L^2([0,T], \mathcal{H}_\Gamma) \text{ w.r.t to } \langle\cdot,\cdot\rangle_{\Gamma_T},
\end{aligned}$$

where the subsequences are successively constructed, i.e., for arbitrary time step sizes $\tau \to 0$ and mesh sizes $h \to 0$ there exist subindices τ_l, h_l for which the above convergence properties are satisfied simultaneously.

Proof The proof for the LLG part works analogous as in [12, Lemma 5.5, Lemma 5.6].
By the uniform boundedness of the approximations in the respective Hilbert spaces (cf. Lemma 19 and Lemma 20) and uniqueness of weak limits, we have the existence of limit functions and the weak convergence of a (fixed) subsequence

$$(E_{\tau,h}, H_{\tau,h}, ((\partial_t^\tau)^{-1}\varphi_h)_{\tau,h}, ((\partial_t^\tau)^{-1}\psi_h)_{\tau,h}) \rightharpoonup (E, H, \tilde{\varphi}, \tilde{\psi}) \in L^2(\Omega_T)^2 \times L^2([0,T], \mathcal{H}_\Gamma)^2.$$

It remains to show that the $(E_{\tau,h}^{\pm}, H_{\tau,h}^{\pm}, ((\partial_t^\tau)^{-1}\varphi_h)_{\tau,h}^{\pm}, ((\partial_t^\tau)^{-1}\psi_h)_{\tau,h}^{\pm})$ converge to the same limit functions. We show exemplary that $E_{\tau,h}^{-}$ converges to the same limit function E. The proof can then be adapted for $E_{\tau,h}^{+}, H_{\tau,h}^{\pm}$ and the functions on the boundary.
It holds for $w \in C_0^1(\Omega_T)$

$$\begin{aligned}
[E_{\tau,h} - E_{\tau,h}^{-}, w]_{\Omega_T} &= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} [E_h^j + \frac{t-t_j}{\tau}\left(E_h^{j+1} - E_h^j\right) - E_h^j, w(t)]_\Omega \,\mathrm{d}t\\
&= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \frac{t-t_j}{\tau}[E_h^{j+1} - E_h^j, w(t)]_\Omega \,\mathrm{d}t\\
&= \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \frac{t-t_j}{\tau}[E_h^{j+1} - E_h^j, w(t_j)]_\Omega \,\mathrm{d}t\\
&\quad + \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \frac{t-t_j}{\tau}[E_h^{j+1} - E_h^j, w(t) - w(t_j)]_\Omega \,\mathrm{d}t.
\end{aligned}$$

By $w(T) = w(0) = 0$ we see

$$\sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \frac{t-t_j}{\tau}[E_h^{j+1} - E_h^j, w(t_j)]_\Omega \, \mathrm{d}t = \frac{\tau}{2} \sum_{j=0}^{N-1} [E_h^{j+1} - E_h^j, w(t_j)]_\Omega$$
$$= -\frac{\tau}{2} \sum_{j=0}^{N-1} [E_h^{j+1}, w(t_{j+1}) - w(t_j)]_\Omega.$$

Therefore we have by the boundedness of $E^{\pm}_{\tau,h}$

$$|[E_{\tau,h} - E^-_{\tau,h}, w]_{\Omega_T}| \le \frac{1}{2} \left(\tau \sum_{j=0}^{N-1} \|E_h^{j+1}\|^2_\Omega \right)^{1/2} \left(\tau \sum_{j=0}^{N-1} \|w(t_{j+1}) - w(t_j)\|^2_\Omega \right)^{1/2}$$
$$+ \left(\tau \sum_{j=0}^{N-1} \|E_h^j - E_h^{j+1}\|^2_\Omega \right)^{1/2} \left(\tau \sum_{j=0}^{N-1} \| \int_{t_j}^{t_{j+1}} \frac{w(t) - w(t_j)}{\tau} \, \mathrm{d}t \|^2_\Omega \right)^{1/2}$$
$$\lesssim \max_{j=0,\dots,N-1} \max_{t \in [t_j, t_{j+1}]} \|w(t) - w(t_j)\|_\Omega$$
$$\lesssim \tau \to 0.$$

As $C_0^1(\Omega_T)$ is dense in $L^2(\Omega_T)$, and

$$\|E^-_{\tau,h}\|_{\Omega_T} \le C < \infty,$$

it holds $E^-_{\tau,h} \rightharpoonup E$. □

Theorem 23 *There exists a subsequence such that*

$$f_{\tau,h} \overset{\text{sub}}{\rightharpoonup} (\nabla \times \partial_t^{-1} H) \qquad \textit{in } L^2(\Omega_T),$$
$$g_{\tau,h} \overset{\text{sub}}{\rightharpoonup} (\nabla \times \partial_t^{-1} E) \qquad \textit{in } L^2(\Omega_T).$$

For smooth enough ξ, it holds for $\partial_t \xi^+_{\tau,h} := (\partial_t^\tau \Pi_h^{\nabla\times} \xi)^+_{\tau,h} \to \partial_t \xi$ in $H(\mathrm{curl}, \Omega_T)$ and

$$\frac{1}{2} \left\langle (\gamma_T((\partial_t^\tau)^{-1}(H_h^{j+1})_j))^-_{\tau,h}, \gamma_T(\partial_t \xi^+_{\tau,h}) \right\rangle_{\Gamma_T} \overset{\text{sub}}{\to} \langle \gamma_T(\partial_t^{-1} H), \gamma_T(\partial_t \xi) \rangle_{\Gamma_T} - \frac{1}{2\mu_0} \langle \tilde{\varphi}, \gamma_T(\partial_t \xi) \rangle_{\Gamma_T},$$
$$\frac{1}{2} \left\langle (\gamma_T((\partial_t^\tau)^{-1}(E_h^{j+1})_j))^-_{\tau,h}, \gamma_T(\partial_t \xi^+_{\tau,h}) \right\rangle_{\Gamma_T} \overset{\text{sub}}{\to} \langle \gamma_T(\partial_t^{-1} E), \gamma_T(\partial_t \xi) \rangle_{\Gamma_T} + \frac{1}{2} \langle \tilde{\psi}, \gamma_T(\partial_t \xi) \rangle_{\Gamma_T}.$$

Proof As $f_{\tau,h}$ is bounded by Lemma 21, there exists a weakly convergent subsequence, such that $f_{\tau,h} \rightharpoonup f$ in $L^2(\Omega_T)$. Now we show that $f = \nabla \times (\partial_t^{-1} H)$. Let $\zeta \in C_0^\infty(\Omega_T)$ and especially $\gamma_T \zeta = 0$. It holds $\Pi_h^{\nabla\times} \zeta \to \zeta$ in $L^2(\Omega_T)$ (cf. Lemma 9). Therefore we have

$$[f_{\tau,h}, \Pi_h^{\nabla\times} \zeta]_{\Omega_T} \to [f, \zeta]_{\Omega_T}.$$

Moreover we have $\mathrm{P}_h \Pi_h^{\nabla\times} \zeta = \Pi_h^{\nabla\times} \zeta$, $\nabla \times \Pi_h^{\nabla\times} \zeta \to \nabla \times \zeta$ in $L^2(\Omega_T)$ (cf. Lemma 9), $\gamma_T \Pi_h^{\nabla\times} \zeta = 0$ (cf. [36, Lemma 5.35]) and $((\partial_t^\tau)^{-1}(H_h^{j+1})_j)^-_{\tau,h} \rightharpoonup \partial_t^{-1} H$ and therefore we have

$$[f_{\tau,h}, \Pi_h^{\nabla\times} \zeta]_{\Omega_T} = [((\partial_t^\tau)^{-1}(H_h^{j+1})_j)^-_{\tau,h}, \nabla \times \Pi_h^{\nabla\times} \zeta]_{\Omega_T}$$
$$\to [\partial_t^{-1} H, \nabla \times \zeta]_{\Omega_T},$$

which concludes $f = \nabla \times (\partial_t^{-1} H)$.
Now let ξ be sufficiently smooth. We have $((\partial_t^\tau)^{-1}(\varphi_h^{j+1})_j)^-_{\tau,h} \rightharpoonup \tilde{\varphi}$ as well as $\gamma_T \Pi_h^{\nabla\times} \xi \to \gamma_T \xi$ in

$L^2([0,T],\mathcal{H}_\Gamma)$ ($\gamma_T : H(\text{curl},\Omega) \to \mathcal{H}_\Gamma$ is continuous) and therefore we obtain

$$\begin{aligned}
&\frac{1}{2}[(\nabla\times(\partial_t^\tau)^{-1}(H_h^{j+1})_j)^-_{\tau,h}, \Pi_h^{\nabla\times}\xi]_{\Omega_T} \\
&\qquad = [f_{\tau,h}, \Pi_h^{\nabla\times}\xi]_{\Omega_T} - \frac{1}{2}[((\partial_t^\tau)^{-1}(H_h^{j+1})_j)^-_{\tau,h}, \nabla\times(\mathrm{P}_h\Pi_h^{\nabla\times}\xi)]_{\Omega_T} \\
&\qquad\quad + \frac{1}{2\mu_0}\left\langle((\partial_t^\tau)^{-1}(\varphi_h^{j+1})_j)^-_{\tau,h}, \gamma_T(\mathrm{P}_h\Pi_h^{\nabla\times}\xi)\right\rangle_{\Gamma_T} \\
&\qquad \to [\nabla\times\partial_t^{-1}H,\xi]_{\Omega_T} - \frac{1}{2}[\partial_t^{-1}H,\nabla\times\xi]_{\Omega_T} + \frac{1}{2\mu_0}\langle\tilde{\varphi},\gamma_T\xi\rangle_{\Gamma_T}.
\end{aligned}$$

Furthermore we have

$$\begin{aligned}
&-\frac{1}{2}\langle(\gamma_T((\partial_t^\tau)^{-1}(H_h^{j+1})_j))^-_{\tau,h}, \gamma_T(\Pi_h^{\nabla\times}\xi)\rangle_{\Gamma_T} \\
&\qquad = \frac{1}{2}[(\nabla\times(\partial_t^\tau)^{-1}(H_h^{j+1})_j)^-_{\tau,h}, \Pi_h^{\nabla\times}\xi]_{\Omega_T} \\
&\qquad\quad - \frac{1}{2}[((\partial_t^\tau)^{-1}(H_h^{j+1})_j)^-_{\tau,h}, \nabla\times\Pi_h^{\nabla\times}\xi]_{\Omega_T} \\
&\qquad \to [\nabla\times\partial_t^{-1}H,\xi]_{\Omega_T} - [\partial_t^{-1}H,\nabla\times\xi]_{\Omega_T} + \frac{1}{2\mu_0}\langle\tilde{\varphi},\gamma_T\xi\rangle_{\Gamma_T} \\
&\qquad = -\left\langle\gamma_T(\partial_t^{-1}H),\gamma_T\xi\right\rangle_{\Gamma_T} + \frac{1}{2\mu_0}\langle\tilde{\varphi},\gamma_T\xi\rangle_{\Gamma_T}.
\end{aligned}$$

The statement of the theorem now is shown by replacing ξ through $\partial_t^\tau\xi$ and using $(\partial_t^\tau\Pi_h^{\nabla\times}\xi)^+_{\tau,h} \to \partial_t\xi$. Similar considerations for $g_{\tau,h}$ and $(\gamma_T((\partial_t^\tau)^{-1}(E_h^{j+1})_j))^-_{\tau,j}$ conclude the assertion. □

Remark 24 *Even for arbitrary smooth functions with non vanishing boundary, we are* ***not*** *able to show* $\tilde{\varphi} = \mu_0\gamma_T(\partial_t^{-1}H)$ *and therefore also* ***not***

$$[(\nabla\times(\partial_t^\tau)^{-1}(H_h^{j+1})_j)^-_{\tau,h}, \Pi_h^{\nabla\times}\xi]_{\Omega_T} \to [\nabla\times\partial_t^{-1}H,\xi]_{\Omega_T}$$

and ***not***

$$\left\langle(\gamma_T((\partial_t^\tau)^{-1}(H_h^{j+1})_j))^-_{\tau,h}, \gamma_T(\Pi_h^{\nabla\times}\xi)\right\rangle_{\Gamma_T} \to \left\langle\gamma_T(\partial_t^{-1}H),\gamma_T\xi\right\rangle_{\Gamma_T}.$$

But we will see, that we have convergence to a solution in the sense of Definition 5, thus E, H solve the MLLG equations in the interior and their boundary values are suitable exterior data. The projection of $\tilde{\varphi}$, $\tilde{\psi}$ on suitable exterior data gives $\mu_0\gamma_T H$, $\gamma_T E$.

Lemma 25 (Discrete Integration by Parts) *For $N \in \mathbb{N}$ and sequences $(a^j)_{j=0,\ldots,N}$, $(b^j)_{j=0,\ldots,N}$ it holds*

$$\begin{aligned}
[(\partial_t^\tau a)^+_\tau, b^-_\tau]_{[0,T]} &= a^N b^N - a^0 b^0 - [a^+_\tau, (\partial_t^\tau b)^+_\tau]_{[0,T]}, \\
[a^+_\tau, b^-_\tau]_{[0,T]} &= ((\partial_t^\tau)^{-1}(a^{k+1})_k)(t_{N-1})b^N - [((\partial_t^\tau)^{-1}(a^{k+1})_k)^-_\tau, (\partial_t^\tau b)^+_\tau]_{[0,T]}.
\end{aligned}$$

Proof It holds

$$\begin{aligned}
[(\partial_t^\tau a)^+_\tau, b^-_\tau]_{[0,T]} + [a^+_\tau, (\partial_t^\tau b)^+_\tau]_{[0,T]} &= \tau\sum_{j=0}^{N-1}\frac{a^{j+1}-a^j}{\tau}b^j + \tau\sum_{j=0}^{N-1}a^{j+1}\frac{b^{j+1}-b^j}{\tau} \\
&= \sum_{j=0}^{N-1} a^{j+1}b^{j+1} - a^j b^j \\
&= a^N b^N - a^0 b^0.
\end{aligned}$$

The second assertion can be shown similarly, by setting $c^0 := 0$, $c^j := ((\partial_t^\tau)^{-1}(a^{k+1})_k)(t_{j-1}) = \tau \sum_{k=0}^{j-1} a^{k+1}$ for $j = 1, \dots, N$ and using $(c^{j+1} - c^j)/\tau = a^{j+1}$ for $j = 0, \dots, N-1$:

$$\begin{aligned}
[a_\tau^+, b_\tau^-]_{[0,T]} + [((\partial_t^\tau)^{-1}(a^{k+1})_k)_\tau^-, (\partial_t^\tau b)_\tau^+]_{[0,T]} &= \tau \sum_{j=0}^{N-1} \frac{c^{j+1} - c^j}{\tau} b^j + \tau \sum_{j=0}^{N-1} c^{j+1} \frac{b^{j+1} - b^j}{\tau} \\
&= c^N b^N - c^0 b^0 \\
&= ((\partial_t^\tau)^{-1}(a^{k+1})_k)(t_{N-1}) b^N.
\end{aligned}$$

□

Theorem 26 *Let $(m_{\tau,h}, E_{\tau,h}, H_{\tau,h}, \varphi_{\tau,h}, \psi_{\tau,h})$ be the approximations obtained by Algorithm 10 and assume that $\theta \in (1/2, 1]$ and the validity of Assumption 13. Then there exists for any $(\tau, h) \to 0$ a subsequence of $(m_{\tau,h}, E_{\tau,h}, H_{\tau,h}, \varphi_{\tau,h}, \psi_{\tau,h})$, such that*

$$(m_{\tau,h}, E_{\tau,h}, H_{\tau,h}, ((\partial_t^\tau)^{-1}\varphi_h)_{\tau,h}, ((\partial_t^\tau)^{-1}\psi_h)_{\tau,h})$$

converges weakly in

$$H^1(\Omega_T) \times L^2(\Omega_T)^2 \times L^2([0,T], \mathcal{H}_\Gamma)^2$$

to a weak solution of the MLLG system in the sense of Definition 5.

Proof We have to show that the weak limit functions are a weak solution in the sense of Definition 5. We choose arbitrary test functions

$$\rho \in C^\infty(\Omega_T), \quad \zeta_H, \zeta_E \in C^\infty(\Omega_T)$$

with $\zeta_H(T) = \zeta_E(T) = 0$ and

$$v, w \in \gamma_T(C^\infty(\Omega_T))$$

with $v(T) = \partial_t v(T) = \dots = \partial_t^m v(T) = 0 = w(T) = \dots = \partial_t^m w(T)$. As discrete test functions we take

$$\rho_h(t, \cdot) := \Pi_h^{\cdot}(m_{\tau,h}^- \times \rho),$$

$$\zeta_{E,h}(t, \cdot) := \Pi_h^{\nabla\times} \zeta_E(t, \cdot), \qquad \zeta_{H,h}(t, \cdot) := \Pi_h^{\nabla\times} \zeta_H(t, \cdot),$$

and

$$v_h(t, \cdot) := \gamma_T(\Pi^{\nabla\times} \hat{v})(t, \cdot) \text{ and } w_h(t, \cdot) := \gamma_T(\Pi^{\nabla\times} \hat{w})(t, \cdot).$$

Here $\hat{v}, \hat{w} \in C^\infty(\Omega_T)$ with $\gamma_T \hat{v} = v$ and $\gamma_T \hat{w} = w$.
The proof that the limit functions satisfy the LLG equation can be found in [12, Proof of Theorem 5.2] or [2]. There, the authors show that the approximations converge to a weak solution and that it holds $m(0, \cdot) = m_0$ in the sense of traces.
We only look at the first one of the Maxwell equations, the second one can be treated analogously. For simplicity we write ζ instead of ζ_H. By testing with $\zeta_h(t_k)$ and summing up from $k = 0 \dots, N-1$, Algorithm 10 gives

$$\begin{aligned}
\mu[(\partial_t^\tau H)_{\tau,h}^+, \zeta_{\tau,h}^-]_{\Omega_T} &= -\frac{1}{2}[\nabla \times E_{\tau,h}^+, \zeta_{\tau,h}^-]_{\Omega_T} - \frac{1}{2}[E_{\tau,h}^+, \nabla \times \zeta_{\tau,h}^-]_{\Omega_T} \\
&\quad - \frac{1}{2\mu}\langle \psi_{\tau,h}^+, \gamma_T \zeta_{\tau,h}^- \rangle_{\Gamma_T} - \mu[w_{\tau,h}^-, \zeta_{\tau,h}^-]_{\Omega_T}.
\end{aligned}$$

We consider each of the terms separately. By discrete integration by parts and $\zeta(T, \cdot) = 0$ we obtain

$$\begin{aligned}
\mu[(\partial_t^\tau H)_{\tau,h}^+, \zeta_{\tau,h}^-]_{\Omega_T} &= -\mu[H_{\tau,h}^+, (\partial_t^\tau \zeta)_{\tau,h}^+]_{\Omega_T} - \mu[H_h^0, \zeta_h(0, \cdot)]_\Omega \\
&\to -\mu[H, \partial_t \zeta]_{\Omega_T} - \mu[H^0, \zeta(0, \cdot)]_\Omega,
\end{aligned}$$

where we used the weak convergence of $H_h^0 \rightharpoonup H^0$ (cf. Assumption 13), $H_{\tau,h}^+ \rightharpoonup H$ (cf. Theorem 22), $\zeta_h(0,\cdot) \to \zeta(0,\cdot)$ in $L^2(\Omega)$ and $(\partial_t^\tau \zeta)_{\tau,h}^+ \to \partial_t \zeta$ in $L^2(\Omega_T)$, as ζ is smooth. We have by $\zeta(T) = 0$, discrete integration by parts and Theorem 23 that

$$
\begin{aligned}
-\frac{1}{2}[\nabla \times E_{\tau,h}^+, \zeta_{\tau,h}^-]_{\Omega_T} - \frac{1}{2}[E_{\tau,h}^+, \nabla \times \zeta_{\tau,h}^-]_{\Omega_T} - \frac{1}{2\mu_0}\langle \psi_{\tau,h}^+, \gamma_T \zeta_{\tau,h}^- \rangle_{\Gamma_T} \\
&= \frac{1}{2}[(\nabla \times (\partial_t^\tau)^{-1}(E_h^{j+1})_j)_{\tau,h}^-, (\partial_t^\tau \zeta_h)_{\tau,h}^+]_{\Omega_T} \\
&\quad + \frac{1}{2}[((\partial_t^\tau)^{-1}(E_h^{j+1})_j)_{\tau,h}^-, \nabla \times (\partial_t^\tau \zeta_h)_{\tau,h}^+]_{\Omega_T} \\
&\quad + \frac{1}{2\mu_0}\langle ((\partial_t^\tau)^{-1}(\psi_h^{j+1})_j)_{\tau,h}^-, (\gamma_T \partial_t^\tau \zeta_h)_{\tau,h}^+ \rangle_{\Gamma_T} \\
&= g_{\tau,h}((\partial_t^\tau \zeta_h)_{\tau,h}^+) \\
&\to [\nabla \times \partial_t^{-1} E, \partial_t \zeta]_{\Omega_T}.
\end{aligned}
$$

The remaining term is a straightforward application of Lemma 22

$$
-\mu[w_{\tau,h}^-, \zeta_{\tau,h}^-]_{\Omega_T} \to -\mu[w, \zeta]_{\Omega_T}.
$$

For the boundary equation, Algorithm 10 gives by testing with $\tau v_h(t_{k+1}), \tau w_h(t_{k+1})$ and summation from $k = 0$ to $k = N - 1$

$$
\left\langle \begin{pmatrix} v_{\tau,h}^+ \\ w_{\tau,h}^+ \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix} \right)_{\tau,h}^+ \right\rangle_{\Gamma_T} = \frac{1}{2}\left(\langle v_{\tau,h}^+, \mu_0^{-1} \gamma_T E_{\tau,h}^+ \rangle_{\Gamma_T} + \langle w_{\tau,h}^+, \gamma_T H_{\tau,h}^+ \rangle_{\Gamma_T} \right).
$$

With discrete integration by parts like above, we see with Theorem 23 that

$$
\langle v_{\tau,h}^+, \gamma_T E_{\tau,h}^+ \rangle_{\Gamma_T} \to -\langle \partial_t v, 2\gamma_T \partial_t^{-1} E + \tilde{\psi} \rangle_{\Gamma_T}
$$

and

$$
\langle w_{\tau,h}^+, \mu_0 \gamma_T H_{\tau,h}^+ \rangle_{\Gamma_T} \to -\langle \partial_t w, 2\mu_0 \gamma_T \partial_t^{-1} H - \tilde{\varphi} \rangle_{\Gamma_T}.
$$

We now consider the term on the left-hand side $\left\langle \binom{v_{\tau,h}^+}{w_{\tau,h}^+}, \left(B(\partial_t^\tau) \binom{\varphi_h}{\psi_h} \right)_{\tau,h}^+ \right\rangle_{\Gamma_T}$. The strategy is to bring $B(\partial_t^\tau)$ from the approximations to the test functions. By setting $v_h^j := v_h(t_j)$, the $\langle \cdot, \cdot \rangle_\Gamma$-adjoint B^* of B, $\bar{v}_h^j := v_h^{N-j}$ and by using $\psi_h^0 = \varphi_h^0 = 0$ we have

$$
\begin{aligned}
X_h^\tau &:= \left\langle \begin{pmatrix} v_{\tau,h}^+ \\ w_{\tau,h}^+ \end{pmatrix}, \left(B(\partial_t^\tau) \begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix} \right)_{\tau,h}^+ \right\rangle_{\Gamma_T} \\
&= \tau \sum_{j=1}^{N} \left\langle \begin{pmatrix} v_h^j \\ w_h^j \end{pmatrix}, \sum_{k=0}^{j} B_{j-k} \begin{pmatrix} \varphi_h^k \\ \psi_h^k \end{pmatrix} \right\rangle_\Gamma \\
&= \tau \sum_{k=0}^{N} \left\langle \sum_{j=0}^{N-\max(1,k)} B_{N-j-k}^* \begin{pmatrix} \bar{v}_h^j \\ \bar{w}_h^j \end{pmatrix}, \begin{pmatrix} \varphi_h^k \\ \psi_h^k \end{pmatrix} \right\rangle_\Gamma \\
&= \tau \sum_{k=1}^{N} \left\langle B^*(\partial_t^\tau) \begin{pmatrix} \bar{v} \\ \bar{w} \end{pmatrix} (T - t_k), \begin{pmatrix} \varphi_h^k \\ \psi_h^k \end{pmatrix} \right\rangle_\Gamma.
\end{aligned}
$$

Now we integrate by parts and obtain by using $v_h(T) = w_h(T) = 0$

$$
\begin{aligned}
X_h^\tau &= \sum_{k=1}^N \left\langle B^*(\partial_t^\tau)\begin{pmatrix}\bar{v}_h\\ \bar{w}_h\end{pmatrix}(T-t_k), (\partial_t^\tau)^{-1}\begin{pmatrix}\varphi_h\\ \psi_h\end{pmatrix}(t_k) - (\partial_t^\tau)^{-1}\begin{pmatrix}\varphi_h\\ \psi_h\end{pmatrix}(t_{k-1})\right\rangle_\Gamma \\
&= \sum_{k=1}^N \left\langle B^*(\partial_t^\tau)\begin{pmatrix}\bar{v}_h\\ \bar{w}_h\end{pmatrix}(T-t_k) - B^*(\partial_t^\tau)\begin{pmatrix}\bar{v}_h\\ \bar{w}_h\end{pmatrix}(T-t_{k+1}), (\partial_t^\tau)^{-1}\begin{pmatrix}\varphi_h\\ \psi_h\end{pmatrix}(t_k)\right\rangle_\Gamma \\
&= \tau\sum_{k=1}^N \left\langle B^*(\partial_t^\tau)\begin{pmatrix}\partial_t^\tau\bar{v}_h\\ \partial_t^\tau\bar{w}_h\end{pmatrix}(T-t_k), (\partial_t^\tau)^{-1}\begin{pmatrix}\varphi_h\\ \psi_h\end{pmatrix}(t_k)\right\rangle_\Gamma \\
&= \tau\sum_{k=1}^N \left\langle (B^*(\partial_t^\tau)\partial_t^\tau)\begin{pmatrix}\bar{v}_h\\ \bar{w}_h\end{pmatrix}(T-t_k), (\partial_t^\tau)^{-1}\begin{pmatrix}\varphi_h\\ \psi_h\end{pmatrix}(t_k)\right\rangle_\Gamma .
\end{aligned}
$$

Here we additionally used

$$
(B^*(\partial_t^\tau)\partial_t^\tau)(\phi^j)(t_k) := (B^*(s)s)(\partial_t^\tau)(\phi^j)(t_k) = (B^*(\partial_t^\tau))(\partial_t^\tau\phi^j)(t_k).
$$

In this situation, we are able to apply the weak convergence result Lemma 22 for the approximations and convolution quadrature convergence results of [34] on the smooth test functions. We apply a operator valued version of [33, Theorem 3.2], as done e.g. in [28] and [9]. Due to $\|B^*(s)s\|_{L(\mathcal{H}_\Gamma)} \le C s^3$ for $\Re s \ge \sigma > 0$ and $\overline{v}(0) = v(T) = 0$, $\partial_t\overline{v}(0) = -\partial_t v(T) = 0, \ldots, \partial_t^m\overline{v}(0) = 0$ and similarly $\overline{w}(0) = \cdots = \partial_t^m\overline{w}(0) = 0$ we have

$$
(B^*(\partial_t^\tau)\partial_t^\tau)\begin{pmatrix}\bar{v}\\ \bar{w}\end{pmatrix}(T-t_k) \to (B^*(\partial_t)\partial_t)\begin{pmatrix}\bar{v}\\ \bar{w}\end{pmatrix}(T-t_k)
$$

uniformly in $0 \le t_k \le T$, $t_k = \tau k$, $k \ge 1$. By the pointwise convergence and the boundedness of the first derivative of $B^*(\partial_t)\partial_t\binom{\bar{v}}{\bar{w}}(T-\cdot)$, the convergence holds

$$
B^*(\partial_t^\tau)\partial_t^\tau\begin{pmatrix}\bar{v}\\ \bar{w}\end{pmatrix}(T-\cdot)^+ \to B^*(\partial_t)\partial_t\begin{pmatrix}\bar{v}\\ \bar{w}\end{pmatrix}(T-\cdot) \text{ in } L^2([0,T],\mathcal{H}_\Gamma).
$$

Moreover we have by the discrete Herglotz theorem (Theorem 51)

$$
\begin{aligned}
&\tau\sum_{k=1}^N \left\| (B^*(\partial_t^\tau)\partial_t^\tau)\begin{pmatrix}\bar{v}_h\\ \bar{w}_h\end{pmatrix}(T-t_k) - (B^*(\partial_t^\tau)\partial_t^\tau)\begin{pmatrix}\bar{v}\\ \bar{w}\end{pmatrix}(T-t_k)\right\|_{\mathcal{H}_\Gamma}^2 \\
&\qquad \le C\tau\sum_{k=1}^N \|(\partial_t^\tau)^3(\bar{v}_h-\bar{v})(T-t_k)\|_{\mathcal{H}_\Gamma}^2 + \|(\partial_t^\tau)^3(\bar{w}_h-\bar{w})(T-t_k)\|_{\mathcal{H}_\Gamma}^2 \to 0
\end{aligned}
$$

for $(\tau, h) \to 0$. So we obtain

$$
X_h^\tau \to \left\langle (B^*(\partial_t)\partial_t)\begin{pmatrix}\bar{v}\\ \bar{w}\end{pmatrix}(T-\cdot), \begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix}\right\rangle_{\Gamma_T} =: X.
$$

Now, in the continuous expression, we bring the operator $B(\partial_t)$ back on $\binom{\tilde{\varphi}}{\tilde{\psi}}$ and we obtain

$$
\begin{aligned}
X &= \int_0^T \left\langle \partial_x^{m+1} \int_0^x \mathcal{L}^{-1}(B^*(r)r^{-m})(s) \begin{pmatrix} \overline{v} \\ \overline{w} \end{pmatrix} (x-s)\ \mathrm{d}s \Big|_{x=T-t}, \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} (t) \right\rangle_\Gamma \mathrm{d}t \\
&= \int_0^T \left\langle \partial_x^{m} \int_0^x B_m^*(s)\partial_x \begin{pmatrix} \overline{v} \\ \overline{w} \end{pmatrix} (x-s)\ \mathrm{d}s \Big|_{x=T-t}, \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} (t) \right\rangle_\Gamma \mathrm{d}t \\
&= (-1)^{m+1} \int_0^T \left\langle \int_0^{T-t} B_m^*(s) \left(\partial_t^{m+1} \begin{pmatrix} v \\ w \end{pmatrix} \right) (T-(T-t-s))\ \mathrm{d}s, \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} (t) \right\rangle_\Gamma \mathrm{d}t \\
&= (-1)^{m+1} \int_0^T \left\langle \int_t^{T} B_m^*(s-t) \partial_t^{m+1} \begin{pmatrix} v \\ w \end{pmatrix} (s)\ \mathrm{d}s, \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} (t) \right\rangle_\Gamma \mathrm{d}t \\
&= (-1)^{m+1} \int_0^T \left\langle \partial_t^{m+1} \begin{pmatrix} v \\ w \end{pmatrix} (s), \int_0^s B_m(s-t) \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} (t)\ \mathrm{d}t \right\rangle_\Gamma \mathrm{d}s \\
&= (-1)^{m+1} \int_0^T \left\langle \partial_t^{m+1} \begin{pmatrix} v \\ w \end{pmatrix} (s), \left(B_m * \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right) (s) \right\rangle_\Gamma \mathrm{d}s.
\end{aligned}
$$

This is exactly the term that shows up in the formulation of our weak solution in Definition 5.

□

5 Numerical Experiments

We implement Algorithm 10 in Fenics and Bempp with some minor changes detailed in the following.

5.1 Notation

For a finite dimensional space $\mathcal{V}_h$ and a function $E_h \in \mathcal{V}_h$, we denote by $\boldsymbol{E}(\mathcal{V}_h)$ the vector of coefficients with respect to the basis used in BEMPP for $\mathcal{V}_h$.
For the spaces inside of the domain Ω, we abbreviate the linear finite element space $\mathcal{S}^1(\mathcal{T}_h)$ by $S1$ and the first order Nédélec space $\mathcal{X}_h$ by $N1$.
For the spaces on the boundary Γ (cf. [40]), we abbreviate the Raviart–Thomas space by RT, the Rao–Wilton–Glisson space by RWG, the scaled Nédélec space by SNC, the Buffa–Christiansen space by BC and the rotated Buffa–Christiansen by RBC. We use the same abbreviations for the corresponding spaces on the baricentrically refined grid that are used for computational reasons (mathematically, that are the same spaces).
For a linear operator $F: DS \to RS$ with domain space DS, range space RS and dual space to the range space $DSRS$, we denote by $F_{DS\to RS}$ the discrete strong form and by ${}_{DSRS}F_{DS}$ the discrete weak form (Details about the operator concept in BEMPP can be found in [40] and the corresponding online tutorial.). For $\varphi_i(X_h)$ being the i-th basis function of X_h, it holds

$$({}_{DSRS}F_{DS})_{ij} = \int_\Gamma \phi_i(DSRS) \cdot F(\phi_j(DS))\ \mathrm{d}s$$

and

$$F_{DS\to RS} = ({}_{DSRS}Id_{RS})^{-1}\ {}_{DSRS}F_{DS}.$$

For a sequence $(\phi^j)_{j\in\mathbb{N}_0}$ we will use the notation $\phi_{|\phi^j=0} := (\phi^0, \dots, \phi^{j-1}, 0, \phi^{j+1}, \dots)$.

5.2 Tangent plane scheme

Using a saddle point approach, we seek $(w_h^j, \lambda_h) \in S^1(\mathcal{T}_h, \mathbb{R}^3) \times \mathcal{S}^1(\mathcal{T}_h, \mathbb{R})$ such that for all $(\rho_h, \xi_h) \in S^1(\mathcal{T}_h, \mathbb{R}^3) \times \mathcal{S}^1(\mathcal{T}_h, \mathbb{R})$

$$
\begin{aligned}
\alpha[w_h^j, \rho_h]_\Omega + \left[m_h^j \times w_h^j, \rho_h\right]_\Omega &= -C_e \left[\nabla(m_h^j + \theta\tau w_h^j), \nabla\rho_h\right]_\Omega + \left[H_h^j, \rho_h\right]_\Omega, \\
&\quad + [\rho_h \cdot m_h^j, \lambda_h]_\Omega + [w_h^j \cdot m_h^j, \xi_h]_\Omega.
\end{aligned}
$$

We update and normalize by computing $m_h^{j+1}(z) := \frac{m_h^j(z)+\tau w_h^j(z)}{|m_h^j(z)+\tau w_h^j(z)|}$, thus projecting the outcome to $S^1(\mathcal{T}_h, \mathbb{R}^3)$. There are other possibilities to implement the tangent plane scheme, i.e., one could directly parametrize the tangent space. For simplicity however, we stick with the present approach.

5.3 Convolution Quadrature

As in [33, formula (3.10)], we approximate the convolution weight operators B_n with the trapezoidal rule

$$B_n \approx \frac{\rho^{-n}}{L} \sum_{l=0}^{L-1} B(\delta(\zeta_l)/\tau) e^{-2\pi i n l/L}, \quad n = 0, \dots, N, \tag{5.1}$$

with $L = 2N$ or $L = N$ evaluation points $\zeta_l = \rho e^{2\pi i l/L}$, $l = 0, \dots, L-1$ and radius of integration $\rho = \mathrm{tol}^{1/(2N)}$. We compute $B_0 = B(\delta(0)/\tau)$ exactly.

5.4 BEMPP operators

We denote the tangential trace mapping by $(\gamma_T)_{N1\to RT}$ and it holds for $E_h \in \mathcal{X}_h$

$$(\boldsymbol{\gamma_T E})(RT) = (\gamma_T)_{N1\to RT} \cdot \boldsymbol{E}(N1).$$

The Calderon operator is implemented in BEMPP and it holds

$$\hat{B}(k) = \begin{pmatrix} \hat{D} & \hat{E} \\ \hat{F} & \hat{G} \end{pmatrix}(k),$$

where

$$\begin{array}{l|l} \hat{D} : RWG \xrightarrow{RBC} RWG & \hat{E} : BC \xrightarrow{RBC} RWG \\ \hline \hat{F} : RWG \xrightarrow{SNC} BC & \hat{G} : BC \xrightarrow{SNC} BC \end{array}.$$

By rescaling this operator, we can express the Calderon operator used in this paper as

$$B(s) = \frac{-1}{\mu_0} \begin{pmatrix} \mu_0^{-1}\sqrt{\frac{\mu_0}{\varepsilon_0}}\hat{E} & -\hat{D} \\ \hat{G} & -\mu_0\sqrt{\frac{\varepsilon_0}{\mu_0}}\hat{F} \end{pmatrix} (i\sqrt{\mu_0\varepsilon_0}s).$$

5.5 Implementation

We express the anti symmetric pairing $\langle\cdot,\cdot\rangle_\Gamma$ as $\langle\zeta,\xi\rangle_\Gamma = [\zeta \times n, \xi]_\Gamma$ and build up the respective terms for rotated basis functions with respect to the L^2-product $[\cdot,\cdot]_\Gamma$. In contrast to Algorithm 10 the trace variable φ and the test function v_φ are given with respect to Buffa–Christiansen elements instead of RT-functions. This is due to stability reasons and the lack of preconditioning for this particular problem. Attention has to be paid to the correct sign of the discretized terms, e.g. it holds

$$\langle\varphi_h, \gamma_T\zeta\rangle_\Gamma = [\varphi_h \times n, \gamma_T\zeta]_\Gamma = -\boldsymbol{\varphi}(BC) \cdot {}_{RBC}Id_{BRT} \cdot (\gamma_T)_{N1\to RT} \cdot \boldsymbol{\zeta}(NC).$$

We summarize and build up the full system. We define the mass matrices

$$M_0 := {}_{NC}Id_{NC}, \quad M_1 := {}_{RBC}Id_{BRT}(\gamma_T)_{N1\to RT}, \quad M_2 := {}_{SNC}Id_{BRT}(\gamma_T)_{N1\to RT},$$

and the symmetric, discrete differential operator

$$D := \frac{1}{2}\, {}_{NC}\nabla\times_{NC} + \frac{1}{2}({}_{NC}\nabla\times_{NC})^T$$

and Calderon sub operators

$$
\begin{aligned}
B_{1,1} &:= \frac{-1}{\mu_0^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,{}_{RBC}\hat{E}_{BC}(i\sqrt{\mu_0\varepsilon_0}\delta(0)/\tau)\delta(0)^{-m},\\
B_{1,2} &:= \frac{1}{\mu_0}\,{}_{RBC}\hat{D}_{RWG}(i\sqrt{\mu_0\varepsilon_0}\delta(0)/\tau)\delta(0)^{-m},\\
B_{2,1} &:= \frac{-1}{\mu_0}\,{}_{SNC}\hat{G}_{BC}(i\sqrt{\mu_0\varepsilon_0}\delta(0)/\tau)\delta(0)^{-m},\\
B_{2,2} &:= \sqrt{\frac{\varepsilon_0}{\mu_0}}\,{}_{SNC}\hat{F}_{RWG}(i\sqrt{\mu_0\varepsilon_0}\delta(0)/\tau)\delta(0)^{-m}.
\end{aligned}
$$

The overall discretization matrix then is

$$
Lhs := \begin{pmatrix} (\frac{\varepsilon_0}{\tau}+\sigma)M_0 & -D & \frac{-1}{2\mu_0}M_1^T & 0\\ D & \frac{\mu_0}{\tau}M_0 & 0 & \frac{-1}{2}M_2^T\\ \frac{1}{2\mu_0}M_1 & 0 & -B_{1,1} & -B_{1,2}\\ 0 & \frac{1}{2}M_2 & -B_{2,1} & -B_{2,2}\end{pmatrix}
$$

with right-hand side

$$
Rhs^i := \begin{pmatrix} \frac{\varepsilon_0}{\tau}\,{}_{N1}Id_{N1}\boldsymbol{E^i}(N1) - {}_{N1}Id_{S1}\boldsymbol{J^{i+1}}(S1)\\ \frac{\mu_0}{\tau}\,{}_{N1}Id_{N1}\boldsymbol{H^i}(N1) - {}_{N1}Id_{S1}\boldsymbol{w^i}(S1)\\ {}_{\binom{RBC}{SNC}}B_{\binom{BC}{RWG}}(\tilde{\partial}_t^\tau)\begin{pmatrix}\boldsymbol{\varphi}(BC)\\ \boldsymbol{\psi}(RWG)\end{pmatrix}_{|\binom{\boldsymbol{\varphi}(BC)}{\boldsymbol{\psi}(RWG)}^{i+1}=0}(t_{i+1})\end{pmatrix}
$$

and the system to solve in the i-th time step is

$$
Lhs\begin{pmatrix}\boldsymbol{E^{i+1}}(\mathcal{X}_h)\\ \boldsymbol{H^{i+1}}(\mathcal{X}_h)\\ \boldsymbol{\varphi^{i+1}}(BC)\\ \boldsymbol{\psi^{i+1}}(RWG)\end{pmatrix} = Rhs^i. \tag{5.2}
$$

5.6 Numerical Results

We consider a simple example on the three dimensional unite cube

$$\Omega = [0,1]^3,$$

where we choose the observation time and the material parameters as

$$T = 0.125,\quad \varepsilon=\varepsilon_0 = 1.1,\quad \mu=\mu_0=1.2,\quad \sigma=1.3,\quad \alpha=1.4,\quad C_e=1.5,$$

as well as the initial and input data

$$
m^0=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad E^0=\begin{pmatrix}0\\0\\0\end{pmatrix},\quad H^0=\begin{pmatrix}0\\0\\0\end{pmatrix},\quad \varphi^0=\begin{pmatrix}0\\0\\0\end{pmatrix},\quad \psi^0=\begin{pmatrix}0\\0\\0\end{pmatrix},\quad J(t)=(1-t/T)\begin{pmatrix}100\\0\\0\end{pmatrix}.
$$

Finally, the tolerance for the iterative solver (GMRES), the implicity parameter for the tangent plane scheme and the convolution quadrature parameters are set to

$$\text{tolgmres} = 10^{-8},\quad \theta=1.0,\quad \rho_N = \text{tolgmres}^{1/(2N)},\quad L=N.$$

As discretized initial data and input data we use L^2-projections to the respective spaces. We look at the time discretization error on a fixed coarse mesh. We compare the approximations to a reference solution computed on a fine time-grid.

We use time step sizes $\tau_i = T\cdot 2^{-i}$, for $i=0,\dots,8$ and the reference solution is computed with $\tau_{\text{ref}} = \min(\tau_i)/2$.

We compute the maximum L^2-error as

$$\mathrm{err}_i = \max_{j=0,\ldots,N_i} \|E_h^i - E_h^{\mathrm{ref}}(t_i)\|_{\Omega}$$

and obtain first order convergence results for E, H, φ and ψ.

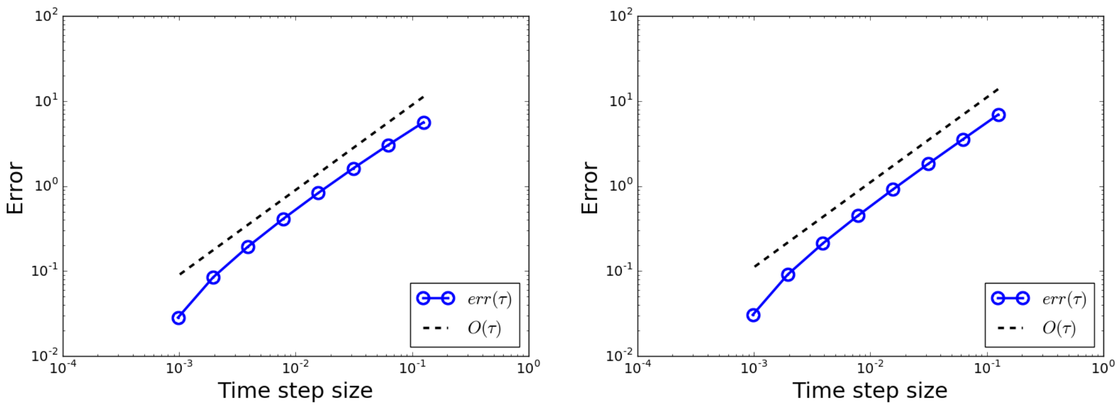

Fig. 1: Temporal convergence plot for E (left) and H (right).

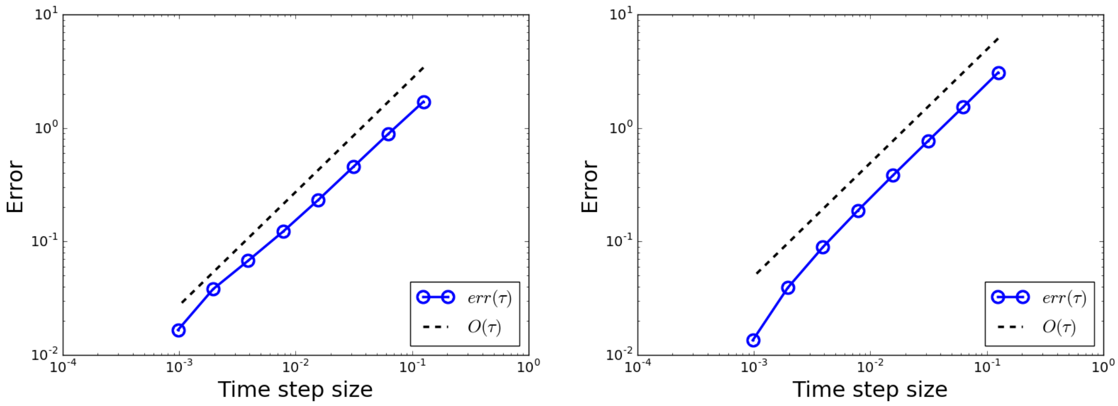

Fig. 2: Temporal convergence plot for φ (left) and ψ (right).

Especially in this experiment, the convergence rate for the magnetization is slightly higher than 1.

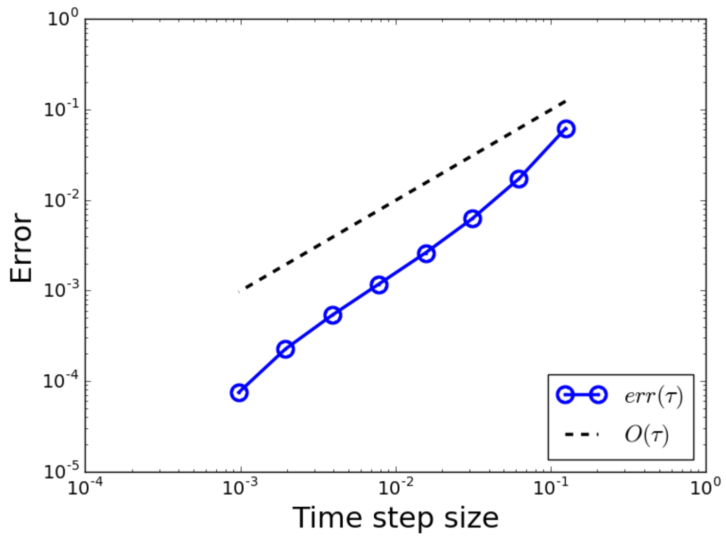

Fig. 3: Temporal convergence plot for m.

The observed convergence order for small τ is higher than 1, as then the approximation is already near to the reference solution.

A Properties of the Laplace Transform

In the following, we list certain properties of the Laplace transform on non-smooth functions. While the results are not surprising, we were not able to find the precise results in the literature. The Laplace transform of a function $u\colon [0,\infty) \to \mathbb{R}$ is defined as

$$(\mathcal{L}u)(s) := \int_0^\infty u(t)e^{-st} \,\mathrm{d}t \text{ for } s \in \mathbb{C}$$

and the inverse Laplace transform for $U\colon \{\Re(s) > \sigma_0\} \to \mathbb{C}$ as

$$(\mathcal{L}^{-1}U)(t) := \frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}} e^{st}U(s)\,\mathrm{d}s \text{ for } t \in [0,\infty) \text{ for a } \sigma > \sigma_0.$$

We see that the inverse Laplace transform is a priori not uniquely defined. It turns out that the choice of σ does not matter for certain function classes and hence the definition is valid. We require the following well-known property of the Fourier transform

Theorem 27 **([39, Chapter 9])** *The Fourier transform*

$$\mathcal{F} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R}) \cap C(\mathbb{R}), (\mathcal{F}f)(x) := \int_{\mathbb{R}} f(\xi)e^{-ix\xi}\,\mathrm{d}\xi \tag{A.1}$$

can be extended to a continuous and continuously invertible operator

$$\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

The inverse operator is given as the extension of

$$\mathcal{F}^{-1} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R}) \cap C(\mathbb{R}), (\mathcal{F}^{-1}f)(x) := \frac{1}{2\pi}\int_{\mathbb{R}} f(\xi)e^{ix\xi}\,\mathrm{d}\xi.$$

The similarities between the two transforms are expressed in the identity

$$(\mathcal{L}u)(\sigma + i\tau) = \mathcal{F}(u(\cdot)e^{-\sigma\cdot})(\tau) \quad \text{for all } \sigma, \tau \in \mathbb{R},$$

where we extended u by zero on $(-\infty, 0)$. This allows us to define a useful domain of definition for the Laplace transform.

Definition 28 *For*

$$u \in L^2_*[0,\infty) := \big\{u : [0,\infty) \to \mathbb{R} \,\big|\, e^{-c\,\cdot}\,u(\,\cdot\,) \in L^2[0,\infty) \text{ for a } c \in \mathbb{R}\big\},$$

we define the Laplace transform for $s \in \mathbb{C}$, $\Re s \geq c$, (where $c \in \mathbb{R}$ such that $e^{-c\,\cdot}\,u(\,\cdot\,) \in L^2[0,\infty)$) as

$$\mathcal{L}u(s) := \mathcal{F}(u(\,\cdot\,)\mathbb{1}_{[0,\infty)}(\,\cdot\,)e^{-\Re s\,\cdot})(\Im s). \tag{A.2}$$

We summarize the properties and the welldefinedness of the inverse Laplace transform.

Definition 29 *For functions in the Hardy space*

$$U \in \mathcal{H} := \big\{U \,\big|\, \text{ For a } \sigma_0 \in \mathbb{R},\ U : \{\Re s > \sigma_0\} \to \mathbb{C} \text{ is analytic and } \sup_{\sigma > \sigma_0} \int_{\sigma + i\mathbb{R}} |U(s)|^2 \text{ d}s < \infty\big\},$$

we define the inverse Laplace transform as

$$(\mathcal{L}^{-1}U)(t) := e^{\sigma t}\mathcal{F}^{-1}(U(\sigma + i\,\cdot\,))(t)$$

for a $\sigma > \sigma_0$*.*

Theorem 30 (cf. [37, Theorem V]) *For* $U \in \mathcal{H}$ *there exists exactly one* $u \in L^2_*(\mathbb{R}_+)$*, such that* $U = \mathcal{L}u$*. The inverse* u *is given through* $\mathcal{L}^{-1}U = u$*.*

In the following, for a Hilbert space X, we want to generalize the definitions to Hilbert space valued functions $[0,\infty) \ni t \mapsto u(t) \in X$. For a family of operators $B(s) : X \to X$, we will define the corresponding convolution operator, with domain spaces living on $([0,\infty), X)$ and on $([0,T], X)$, respectively. This is done in a component-wise definition by using an orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of X.

Definition 31 *For*

$$u \in L^2_*([0,\infty), X) := \big\{u : [0,\infty) \to X \,\big|\, e^{-c\,\cdot}\, u(\,\cdot\,) \in L^2([0,\infty), X) \text{ for a } c \in \mathbb{R}\big\},$$

we define the Laplace transform for $s \in \mathbb{C}$*,* $\Re s \geq c$*, (where* $c \in \mathbb{R}$ *such that* $e^{-c\,\cdot}\, u(\,\cdot\,) \in L^2([0,\infty), X)$*) as*

$$\mathcal{L}u(s) := \sum_{j=1}^{\infty} \mathcal{L}([e_j, u]_X)(s) e_j \tag{A.3}$$

Definition 32 *For* $c \in \mathbb{R}$*, we define the spaces*

$$L^2_c([0,\infty), X) := \big\{u : [0,\infty) \to X \,\big|\, e^{-c\,\cdot}\, u(\,\cdot\,) \in L^2([0,\infty), X)\big\},$$

equipped with the norm $\|u\|_{L^2_c([0,\infty),X)} := \|e^{-c\,\cdot}\, u\|_{L^2([0,\infty),X)}$*.*

Definition 33 *For functions in the Hardy space*

$$U \in \mathcal{H} := \big\{U \,\big|\, \text{ For a } \sigma_0 \in \mathbb{R},\ U : \{\Re s > \sigma_0\} \to X \text{ is holomorph and } \sup_{\sigma > \sigma_0} \int_{\sigma + i\mathbb{R}} \|U(s)\|^2_X \text{ d}s < \infty\big\},$$

we define the inverse Laplace transform as

$$\mathcal{L}^{-1}U := \sum_{j\in\mathbb{N}} \mathcal{L}^{-1}([e_j, U]_X) e_j.$$

Definition 34 *For* $\sigma_0 \in \mathbb{R}$*, we define the space*

$$\mathcal{H}(\sigma_0) := \big\{U \,\big|\, U : \{\Re s > \sigma_0\} \to X \text{ is holomorphic and } \sup_{\sigma > \sigma_0} \int_{\sigma + i\mathbb{R}} \|U(s)\|^2_X \text{ d}s < \infty\big\},$$

equipped with the norm

$$\|u\|^2_{\mathcal{H}(\sigma_0)} := \sup_{\sigma > \sigma_0} \int_{\sigma + i\mathbb{R}} \|U(s)\|^2_X \text{ d}s.$$

Theorem 35 (cf. [6, Theorem 1.8.3]) *For* $U \in \mathcal{H}(\sigma_0)$ *the inverse Laplace transform is well defined and* $\mathcal{L}^{-1}U \in L^2_{\sigma_0}([0,\infty), X)$*. There exists exactly one* $u \in L^2_*([0,\infty), X)$*, such that* $U = \mathcal{L}u$ *and* u *is given through*

$$\mathcal{L}^{-1}U = u.$$

Theorem 36 (Plancherel's Formula, cf. [6, Theorem 1.8.2]) *It holds for* $u, v \in L^2_c([0,\infty), X)$ *for all* $\sigma \geq c$

$$\int_0^{\infty} e^{-2\sigma t}[u(t), v(t)]_X \text{ d}t = \frac{1}{2\pi} \int_{\sigma + i\mathbb{R}} [\mathcal{L}u(s), \mathcal{L}v(s)]_X \text{ d}s,$$

especially we have

$$\|\mathcal{L}u\|_{\mathcal{H}(\sigma)} = \sqrt{2\pi}\|u\|_{L^2_\sigma[0,\infty),X}.$$

This gives a one to one identity through the Laplace transform between $L^2_*([0,\infty), X)$ *and* $\mathcal{H}$ *and between* $L^2_c([0,\infty), X)$ *and* $\mathcal{H}(c)$ *for* $c \in \mathbb{R}$*. Instead of the Hilbert space scalar product* $[\cdot,\cdot]_X$*, the result also holds for any continuous and sesquilinear product on* $X \times X$*.*

We denote by $L(X)$ the linear, bounded operators $X \to X$. For a function $B : \{\Re s > \sigma_0\} \to L(X)$ we want to define $B(\partial_t)f$ as $\mathcal{L}^{-1}(B(s)\mathcal{L}(f)(s))$.
The following definition is very general and not practical and will be refined in the following.

Definition 37 *For a function $B(s) : \{\Re s > \sigma_0\} \to L(X)$ for a $\sigma_0 \in \mathbb{R}$ and $f \in L^2_*([0,\infty), X)$, such that*

$$B(s)\mathcal{L}f \in \mathcal{H}, \tag{A.4}$$

we say that $B(\partial_t)f$ exists and we define $B(\partial_t)f$ as

$$B(\partial_t)f := \mathcal{L}^{-1}(B(s)\mathcal{L}(f)(s)). \tag{A.5}$$

Definition 38 *We define for $m \in \mathbb{N}$ the exponentially weighted spaces of m-times weakly differentiable functions with zero condition at $t = 0$*

$$H^m_{0,*}([0,\infty), X) := \{\phi : [0,\infty) \to X \,|\, e^{-c\,\cdot}\phi \in H^m_0([0,\infty), X) \text{ for a } c \in \mathbb{R}\}.$$

Furthermore, we define for fixed damping parameter $c \in \mathbb{R}$

$$H^m_{0,c}([0,\infty), X) := \{\phi : [0,\infty) \to X \,|\, e^{-c\,\cdot}\phi \in H^m_0([0,\infty), X)\}.$$

We equip the latter spaces with the norm $\|u\|_{H^m_{0,c}([0,\infty),X)} := \|e^{-c\,\cdot}u\|_{H^m([0,\infty),X)}$.

Example 39 *a) For the operator $B(s) = s$, $f \in H^1_{0,*}([0,\infty), X)$, it holds $s(\mathcal{L}f)(s) \in \mathcal{H}$ and we have*

$$B(\partial_t)f = \partial_t f.$$

Thus the Laplace differential operator ∂_t coincides with the weak derivative ∂_t, if f is weakly differentiable and $f(0) = 0$.
b) For the operator $B(s) = s^{-1}$, $f \in L^2_*([0,\infty), X)$ it holds $s^{-1}(\mathcal{L}f)(s) \in \mathcal{H}$ and we have

$$B(\partial_t) = \partial_t^{-1}f := \int_0^t f(\tau)\ \mathrm{d}\tau.$$

Thus the Laplace differential operator ∂_t^{-1} coincides with the integration over time $\int_0^t\ \mathrm{d}\tau$.

Proof b) Let $f \in L^2_\sigma([0,\infty, X))$, then $s^{-1}\mathcal{L}f(s) \in \mathcal{H}(\max(\sigma,\varepsilon))$ for $\varepsilon > 0$. Furthermore it holds for $\Re s > \max(\sigma,\varepsilon)$ that $r \mapsto \frac{1}{\Re s}e^{-\Re s r}f(r) \in L^1([0,\infty), X)$ and therefore we have by Fubini's Theorem

$$\begin{aligned}
\mathcal{L}(\partial_t^{-1}f)(s) &= \int_0^\infty e^{-st}\int_0^t f(r)\ \mathrm{d}r\ \mathrm{d}t \\
&= \int_0^\infty\int_0^\infty \mathbb{1}_{r\le t}e^{-st}f(r)\ \mathrm{d}r\ \mathrm{d}t \\
&= \int_0^\infty\int_r^\infty e^{-st}\ \mathrm{d}t f(r)\ \mathrm{d}r \\
&= \int_0^\infty \frac{1}{s}e^{-sr}f(r)\ \mathrm{d}r \\
&= \frac{1}{s}\mathcal{L}f(s).
\end{aligned}$$

As $s^{-1}\mathcal{L}f(s) \in \mathcal{H}$, it holds $\partial_t^{-1}f = \mathcal{L}^{-1}s^{-1}\mathcal{L}f(s)$.
a) Let $f \in H^1_{0,\sigma}([0,\infty), X)$ for $\sigma \in \mathbb{R}$. It is $\partial_t^{-1}\partial_t f = f$ and therefore by *b)* for $\Re s \ge \max(\sigma,\varepsilon) > 0$

$$\frac{1}{s}\mathcal{L}(\partial_t f)(s) = \mathcal{L}f(s).$$

As $\mathcal{L}(\partial_t f) \in \mathcal{H}$, it holds $\partial_t f = \mathcal{L}^{-1}(s\mathcal{L}f)$. □

With Example 39 we are able to state concrete conditions for the existence in Definition 37.

Lemma 40 *If there exists in the situation of Definition 37 an $m \in \mathbb{N}_0$, $\sigma_1 \in \mathbb{R}$ and a constant $C > 0$, such that B is holomorphic inside of its definition regime and*

$$\|B(s)\|_{L(X)} \le C|s|^m \text{ for all } \Re s > \sigma_1,$$

then $B(\partial_t)f$ exists for every $f \in H^m_{0,*}([0,\infty), X)$ and it holds

$$B(\partial_t)f = \partial_t^m\mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}f) = \mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}\partial_t^m f).$$

We can define $B(\partial_t)$ as a continuous operator for $\sigma_2 \in \mathbb{R}$

$$B(\partial_t) : H^m_{0,\sigma_2}([0,\infty), X) \to L^2_{\max(\sigma_1,\sigma_2)}([0,\infty), X).$$

Proof The proof follows from Example 39 and Plancherel's formula. For $\phi \in H^m_{0,\sigma_2}([0,\infty),X)$ it holds

$$\begin{aligned}\|B(\partial_t)\phi\|_{L^2_{\max(\sigma_1,\sigma_2)}([0,\infty),X)} &= \frac{1}{\sqrt{2\pi}}\|B(s)\mathcal{L}\phi\|_{\mathcal{H}(\max(\sigma_1,\sigma_2))}\\ &\le C\|s^m\mathcal{L}\phi\|_{\mathcal{H}(\max(\sigma_1,\sigma_2))}\\ &= C\|\partial_t^m\phi\|_{L^2_{\max(\sigma_1,\sigma_2)}([0,\infty),X)}\\ &\le C\|\phi\|_{H^m_{0,\sigma_2}([0,\infty),X)}.\end{aligned}$$

□

Definition 41 *We define for $m \in \mathbb{N}_0$*

$$\mathcal{H}_m := \{B \mid \text{There exists a } \sigma_0 \in \mathbb{R} \text{ such that } B : \{\Re s > \sigma_0\} \to L(X) \text{ is holomorphic and } \|B(s)\|_{L(X)} \le C|s|^m \text{ for all } \Re s > \sigma_0\}$$

and for $\sigma_0 \in \mathbb{R}$

$$\mathcal{H}_m(\sigma_0) := \{B : \{\Re s > \sigma_0\} \to L(X) \text{ holomorphic} \mid \|B(s)\|_{L(X)} \le C|s|^m \text{ for all } \Re s > \sigma_0\}.$$

We call $B \in \mathcal{H}_0$ a smoothing operator.

Definition 42 *For a family of bounded linear operators $A(t) : X \to X, t \in [0,\infty)$ we define the convolution with $b(t) \in X$ as*

$$(A * b)(t) := \int_0^t A(\tau)b(t-\tau)\,\mathrm{d}\tau := \sum_{i\in\mathbb{N}}\left(\sum_{k\in\mathbb{N}}\int_0^t [e_i, A(\tau)e_k]_X[e_k, b(t-\tau)]_X\,\mathrm{d}\tau\right)e_i.$$

Similarly we define the inverse Laplace transform of an operator family $B(s) : X \to X$, $s \in \Re s > \sigma_0$ entry-wise as

$$(\mathcal{L}^{-1}B)(t)b := \sum_{i\in\mathbb{N}}\left(\sum_{k\in\mathbb{N}}\mathcal{L}^{-1}([e_i, B(\cdot)e_k]_X)(s)[e_k, b]_X\right)e_i.$$

We want to apply the (inverse) Laplace transform to operators, $B(s) : X \to X$ and convolute with functions $f(t) \in X$. The difference comparing to the scalar case is now, with the induced norm, L(X) is no Hilbert space, but only a Banach space. Plancherel's Formula does not hold in general.

Lemma 43 *For $B \in L^1(\sigma_0 + i\mathbb{R}, L(X)) \cap \mathcal{H}(\sigma_0)$ the convolution with the inverse Laplace transform gives for every $\delta > 0$ a welldefined and continuous operator*

$$\mathcal{L}^{-1}B* : L^2_{\sigma_0}([0,\infty),X) \to L^2_{\sigma_0+\delta}([0,\infty),X)$$

and it holds

$$\|\mathcal{L}^{-1}B * u\|_{L^2_{\sigma_0+\delta}([0,\infty),X)} \le C(\delta)\|B\|_{L^1(\sigma_0+i\mathbb{R},L(X))}\|u\|_{L^2_{\sigma_0}([0,\infty),X)}.$$

Proof The proof can be shown by combining Hölder's inequality

$$\|\mathcal{L}^{-1}B * u\|_{L^2_{\sigma_0+\delta}([0,\infty),X)} \le C(\delta)\|e^{-(\sigma_0+\delta/2)(\cdot)}\mathcal{L}^{-1}B * u\|_{L^\infty([0,\infty),X)},$$

Young's inequality for convolution

$$\begin{aligned}&\|e^{-(\sigma_0+\delta/2)(\cdot)}\mathcal{L}^{-1}B * u\|_{L^\infty([0,\infty),X)}\\ &\qquad\le \|e^{-(\sigma_0+\delta/2)(\cdot)}\mathcal{L}^{-1}B\|_{L^\infty([0,\infty),L(X))}\|e^{-(\sigma_0+\delta/2)(\cdot)}u\|_{L^1([0,\infty),X)},\end{aligned}$$

the estimates for the inverse Laplace transform which follow from the equivalence with the Fourier transform,

$$\|e^{-(\sigma_0+\delta/2)(\cdot)}\mathcal{L}^{-1}B\|_{L^\infty([0,\infty),L(X))} \le \frac{1}{2\pi}\|B\|_{L^1(\sigma_0+i\mathbb{R},L(X))},$$

and again Hölder's inequality

$$\|e^{-(\sigma_0+\delta/2)(\cdot)}u\|_{L^1([0,\infty),X)} \le C(\delta)\|u\|_{L^2_{\sigma_0}([0,\infty),X)}.$$

□

Lemma 44 *Under the assumptions of Lemma 40, every $\epsilon > 0$ satisfies*

$$B(s)s^{-(m+2)} \in \mathcal{H}(\max(\epsilon,\sigma_1)) \cap L^1(\max(\epsilon,\sigma_1) + i\mathbb{R}, L(X)).$$

Thus $\mathcal{L}^{-1}(B(s)s^{-(m+2)})$ is continuous and we have for $f \in H^m_{0,*}([0,\infty),X)$

$$B(\partial_t)f = \partial_t^{m+2}\mathcal{L}^{-1}(B(s)s^{-m+2}) * f$$

and for $f \in H^{(m+2)}_{0,*}([0,\infty),X)$

$$B(\partial_t)f = \mathcal{L}^{-1}(B(s)s^{-m+2}) * \partial_t^{m+2}f.$$

Proof The proof follows from Example 39, Lemma 43 and the Laplace-convolution-identity

$$\mathcal{L}(b)(s)\mathcal{L}(f)(s) = \mathcal{L}(b * f)(s)$$

for sufficiently bounded functions b, f. □

As we will mainly work on bounded time intervals, we want to define the Laplace transform and Laplace differential operators for functions on $[0,T]$, e.g., for $f \in L^2([0,T],X)$. The Laplace transform can easily be defined by extending f by zero outside of $[0,T]$ and the following results can be found also in [33, Section 2.1].

Definition 45 (cf. [33, (2.2)]) *Let $B(s) \in L(X)$ be a family of operators and $f \in L^2([0,T],X)$. We extend f by zero to $[0,\infty)$. Whenever there is an $m \in N_0$ such that*

$$B(s)s^{-m}\mathcal{L}f \in \mathcal{H} \text{ and } \mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}f) \in H^m([0,T],X),$$

we say that $B(\partial_t)f$ exists and we set

$$B(\partial_t)f := \partial_t^m \mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}f).$$

We call the mapping $B(\partial_t)$ causal, if for every $T > 0$ and for every f, such that $B(\partial_t)f$ exists, $B(\partial_t)f$ does not depend on an arbitrarily chosen extension of f in $L^2_*([0,\infty),X)$.

Note that this is another definition of $B(\partial_t)$, that does not coincide with Definition 37 in general.

Definition 46 (cf. [33, (2.5)]) *We define for $m \in \mathbb{N}$ the space of $m-$times weakly differentiable functions with initial condition zero as*

$$H^m_{0,*}([0,T],X) := \{f \in H^m([0,T],X) \mid f(0) = \cdots = f^{(m-1)}(0) = 0\}.$$

With the induced norm

$$\|\cdot\|_{H^m_{0,*}([0,T],X)} := \|\cdot\|_{H^m([0,T],X)} = \sqrt{\langle \cdot, \cdot \rangle_{H^m([0,T],X)}},$$

this is a Hilbert space.
Attention, the sub index $0,*$ in $H^m_{0,*}([0,T],X)$ has the meaning 0 at $t=0$ and arbitrary value at $t=T$ and we also define

$$H^m_{*,0}([0,T],X) := \{f \in H^m([0,T],X) \mid f(T) = \cdots = f^{(m-1)}(T) = 0\}.$$

Lemma 47 (cf. [33, Lemma 2.1]) *Let $m \in \mathbb{N}_0$. For*

$$B \in \mathcal{H}_m,$$

$B(\partial_t)f$ exists for every $f \in H^m_{0,*}([0,T],X)$ and it holds $\mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}f) \in H^m_{0,*}([0,T],X)$ and

$$B(\partial_t)f = \mathbb{1}_{[0,T]}\mathcal{L}^{-1}(B(s)s^{-m}\mathcal{L}(\partial_t^m f))$$

We can define $B(\partial_t)$ as a continuous operator

$$B(\partial_t) : H^m_{0,*}([0,T],X) \to L^2([0,T],X).$$

Every $B \in \mathcal{H}_m$ is causal and for every sufficiently smooth extension $\tilde{f}$ of f on $[0,\infty)$ it holds

$$B(\partial_t)f = \mathbb{1}_{[0,T]}\mathcal{L}^{-1}(B(s)\mathcal{L}\tilde{f}).$$

Lemma 48 (cf. [33, (2.1)]) *Under the assumptions of Lemma 47, $\mathcal{L}^{-1}(B(s)s^{-(m+2)})$ is continuous and we have for $f \in H^m_{0,*}([0,T],X)$*

$$B(\partial_t)f = \partial_t^{m+2}\mathcal{L}^{-1}(B(s)s^{-m+2}) * f$$

and for $f \in H^{(m+2)}_{0,*}([0,T],X)$

$$B(\partial_t)f = \mathcal{L}^{-1}(B(s)s^{-m+2}) * \partial_t^{m+2} f.$$

Lemma 49 (cf. [33, (2.2)]) *For $A \in \mathcal{H}_m(\sigma_1)$, $B \in \mathcal{H}_n(\sigma_2)$, $AB \in \mathcal{H}_p$, $f \in H^{\max(m,n,p)}_{0,*}([0,T],X)$ it holds*

$$(AB)(\partial_t)f = A(\partial_t)B(\partial_t)f$$

and if $A(s)B(s) = B(s)A(s)$ on a line $\sigma + i\mathbb{R}$ with $\sigma > \max(\sigma_1,\sigma_2)$ it holds

$$(AB)(\partial_t)f = (BA)(\partial_t)f = B(\partial_t)A(\partial_t)f.$$

Theorem 50 (Herglotz Theorem on $[0,T]$, cf. [9, Lemma 2.2]) *Let $B, R \in \mathcal{H}_m(\sigma_0)$ for $\sigma_0 \in \mathbb{R}$. Let $a(\cdot,\cdot) : X \times X \to \mathbb{C}$ sesquilinear and continuous. If there exists a $c > 0$ such that for all $w \in X$, all $\Re s > \sigma_0$*

$$\Re a(w, B(s)w) \geq c\|R(s)w\|_X^2,$$

then it holds for all $w \in H^m_{0,*}([0,T],X)$, for all $\sigma \geq \sigma_0$

$$\int_0^T e^{-2\sigma t}\Re a(w(t), B(\partial_t)w(t))\, \mathrm{d}t \geq ce^{-2\sigma T}\|R(\partial_t)w\|^2_{L^2([0,T],X)}.$$

Proof The assertion can be shown as in the scalar case by a discrete Herglotz theorem (cf. [28, Lemma 2.1]) and the convergence of convolution quadrature. □

Theorem 51 (Discrete Herglotz Theorem on $[0,T]$, cf. [9, Lemma 2.1, Lemma 2.3]) *Let $B \in \mathcal{H}_m(\sigma_0)$ for $\sigma_0 \in \mathbb{R}_+$. For $N \in \mathbb{N}$ sufficiently large and a sequence $(w^n)_{n=0,\dots,N} \subset X$, it holds*

$$\tau \sum_{j=0}^{N} \|(B(\partial_t^\tau)w)(t_j)\|^2 \le C\tau \sum_{j=0}^{N} \|((\partial_t^\tau)^m w)(t_j)\|^2.$$

The constant C depends on σ_0, T and B, but not on τ.

Proof We extend w to a sequence $(w^n)_{n\in\mathbb{N}}$ such that, $((\partial_t^\tau)^m w)(t_j) = 0$ for all $j > N$. This is always possible by an iterative procedure, as we can write $((\partial_t^\tau)^m w)(t_{k+1}) = w^{k+1}/\tau^m - f((w^n)_{n\le k})$, where $f((w^n)_{n\le k})$ does not depend on w^{k+1}. Now we compute iteratively w^{N+1}, such that $((\partial_t^\tau)^m w)(t_{N+1}) = 0$, w^{N+2} such that $((\partial_t^\tau)^m w)(t_{N+2}) = 0, \dots$. Now we define the finite sequence $w_M^j := w^j$ for $j = 0, \dots, M$ and $w_M^j = 0$, $j > M$. As in Lemma 17 we have for $\rho = e^{-2\sigma_0\tau}$, $|\zeta| < \rho$ and sufficiently small τ

$$\Re\left(\frac{\delta(\zeta)}{\tau}\right) \ge \frac{1 - e^{-2\sigma_0\tau}}{\tau} = \int_0^{2\sigma_0} e^{-\tau r}\, \mathrm{d}r \ge 2\sigma_0 e^{-2\tau\sigma_0} > \sigma_0.$$

With similar arguments as in [9, Lemma 2.1, Lemma 2.3] we obtain

$$\tau \sum_{j=0}^{\infty} e^{-4\sigma_0 t_j} \|(B(\partial_t^\tau)w)(t_j)\|^2 \le C\tau \sum_{j=0}^{\infty} e^{-4\sigma_0 t_j} \|((\partial_t^\tau)^m w_M)(t_j)\|^2.$$

For $j \ge M$, it is $w^j \le Ct_j^m$ (this can be shown by discrete integration) and therefore

$$\tau \sum_{j=0}^{\infty} e^{-4\sigma_0 t_j} \|((\partial_t^\tau)^m w_M)(t_j)\|^2 \le \tau \sum_{j=0}^{N} e^{-4\sigma_0 t_j} \|((\partial_t^\tau)^m w_M)(t_j)\|^2 + C(\tau,m) e^{-4\sigma_0 t_M} t_M^m$$

and the limit $M \to \infty$ exists on the right-hand side. We obtain by discrete causality (i.e., $B(\partial_t^\tau)w(t_j)$ is independent of w^n, $n > j$) for $M > N$

$$\begin{aligned} \tau \sum_{j=0}^{N} e^{-4\sigma_0 t_j} \|(B(\partial_t^\tau)w)(t_j)\|^2 &= \tau \sum_{j=0}^{N} e^{-4\sigma_0 t_j} \|(B(\partial_t^\tau)w_M)(t_j)\|^2 \\ &\le \tau \sum_{j=0}^{\infty} e^{-4\sigma_0 t_j} \|(B(\partial_t^\tau)w_M)(t_j)\|^2. \end{aligned}$$

Combining the previous estimates for the limit $M \to \infty$ gives

$$\tau \sum_{j=0}^{N} e^{-4\sigma_0 t_j} \|(B(\partial_t^\tau)w)(t_j)\|^2 \le C\tau \sum_{j=0}^{N} e^{-4\sigma_0 t_j} \|((\partial_t^\tau)^m w)(t_j)\|^2.$$

Now the bounds $e^{-4\sigma_0 T} \le e^{-4\sigma_0 t_j} \le 1$ yield the assertion. □

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