SL$(2, q)$-Unitals

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Erst kommen die Tränen,
dann kommt der Erfolg.

— Bastian Schweinsteiger

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1 Introduction

Unitals of order n are incidence structures consisting of $n^3 + 1$ points such that each block is incident with $n + 1$ points and such that there are unique joining blocks. In the language of designs, a unital of order n is a 2-$(n^3 + 1, n + 1, 1)$ design.
Unitals were first constructed by Bose [4] as a series of balanced Kirkman arrangements (i. e. resolvable Steiner systems) with certain parameters. Bose constructed the geometry given by the points of the curve with equation $X^{q+1} + Y^{q+1} + Z^{q+1} = 0$ in the projective plane $\mathrm{PG}(2, q^2)$ and the lines of $\mathrm{PG}(2, q^2)$ containing more than one point of the curve. In the resulting geometry (today known as classical or Hermitian unital), he found a resolution, i. e. a partition of the set of $q^2(q^2 - q + 1)$ lines into q^2 subsets of $q^2 - q + 1$ lines each.
Since then, unitals were considered as subgeometries of projective planes – in particular, occurring as geometries on sets of absolute points of suitable polarities –, but they were also studied in their own right as abstract designs, irrespective of an ambient plane.

In this thesis, we consider a special construction of unitals which is due to Grundhöfer, Stroppel and Van Maldeghem [9]. Inspired by the action of the special unitary group of degree 2 on the classical unital, they give a general construction for unitals of prime power order q, where the points outside one block are given by the elements of the special linear group of degree 2 over the finite field $\mathbb{F}_q$, which we denote by $\mathrm{SL}(2, q)$. Their construction is justified by the fact that they found by it a unital of order 4 which is not isomorphic to the classical unital. We will call the unitals given by this construction $\mathrm{SL}(2, q)$-unitals.

In Chapter 2 we will be concerned with the special linear group $\mathrm{SL}(2, q)$ and its subgroups of orders q and $q + 1$, since they play a crucial role in the construction of $\mathrm{SL}(2, q)$-unitals.
In Chapter 3 we consider unitals as incidence structures and introduce the construction of $\mathrm{SL}(2, q)$-unitals. We will deal a lot with *affine* unitals, which arise from unitals by removing one block (and all the points on it) and can be completed to unitals via a parallelism on the short blocks. In any affine $\mathrm{SL}(2, q)$-unital there are at least two parallelisms – called “flat” $\flat$ and “natural” $\natural$ –, leading to non-isomorphic completions if $q \geq 3$.

We are interested in automorphism groups of (affine) $\mathrm{SL}(2,q)$-unitals, considered in Chapter 4. We use the isomorphism between the geometry of the short blocks of affine $\mathrm{SL}(2,q)$-unitals and a hyperplane complement of the classical generalized quadrangle $\mathrm{Q}(4,q)$ to determine all possible automorphisms of affine $\mathrm{SL}(2,q)$-unitals. Chapter 4 also includes a proof that in any $\mathrm{SL}(2,q)$-unital with parallelism $\flat$, there is one block fixed by the full automorphism group. In Section 4.3, we collect some known classes of unitals where no block is fixed by the full automorphism group.
In Chapter 5 we introduce a new class of parallelisms, which occurs in every affine $\mathrm{SL}(2,q)$-unital of odd order. Further, we consider *translations*, i. e. automorphisms of unitals fixing every block through one point (the so-called center). We determine all possible translations with center on the block at infinity of $\mathrm{SL}(2,q)$-unitals with the new parallelism and with the parallelism $\flat$.
Chapter 6 contains results where we use our knowledge on automorphism groups and GAP [7] – a system for computational discrete algebra – to search for new affine $\mathrm{SL}(2,q)$-unitals and for further parallelisms. In addition to the non-existence of affine $\mathrm{SL}(2,q)$-unitals under certain conditions, we find three new affine $\mathrm{SL}(2,q)$-unitals of order 8 and several parallelisms for order 4, leading to twelve new $\mathrm{SL}(2,q)$-unitals of order 4. To compute the full automorphism groups and to check isomorphisms between the new unitals, we use the GAP package UnitalSZ by Nagy and Mezőfi [20].
For the computer searches, we explain all considerations made before using GAP and subsequently present the results. If you are interested in the specific implementation, you may find the code in the GitHub repository

https://github.com/moehve/SL2q-Unitals_GAP.git

We conclude this thesis with some open questions about $\mathrm{SL}(2,q)$-unitals.

2 About $\mathrm{SL}(2, q)$

Throughout this thesis, p will be a prime and $q \coloneqq p^e$ a p-power. The finite field of order q will be denoted by $\mathbb{F}_q$.

2.1 Linear and Unitary Groups of Degree 2 over Finite Fields

We will deal a lot with the special linear group of degree 2 over the field $\mathbb{F}_q$, which we denote by $\mathrm{SL}(2, q)$. Sometimes it will be convenient to consider a special unitary group of degree 2 rather than the special linear group, which is why we will show certain isomorphisms between linear and unitary groups of degree 2.

Consider the quadratic field extension $\mathbb{F}_{q^2}/\mathbb{F}_q$ and the unique involutory field automorphism

$$\bar{\cdot} \colon \mathbb{F}_{q^2} \to \mathbb{F}_{q^2},\ x \mapsto \overline{x} \coloneqq x^q,$$

with fixed field $\mathbb{F}_q$. Let $h \colon \mathbb{F}_{q^2}^2 \times \mathbb{F}_{q^2}^2 \to \mathbb{F}_{q^2}$ be a **Hermitian form**, i. e. a sesquilinear form with $h(x, y) = \overline{h(y, x)}$ for all x, $y \in \mathbb{F}_{q^2}^2$. Let M be the Gram matrix of h, i. e.

$$h(x, y) = xM\overline{y}^\top.$$

We call h **nondegenerate** if $\det M \neq 0$. Then we define the **unitary group** with respect to h by

$$\begin{aligned} \mathrm{U}(h) &\coloneqq \{A \in \mathrm{GL}(2, q^2) \mid \forall\, x, y \in \mathbb{F}_{q^2}^2 : h(xA, yA) = h(x, y)\} \\ &= \{A \in \mathrm{GL}(2, q^2) \mid AM\overline{A}^\top = M\}, \end{aligned}$$

the **special unitary group** with respect to h as

$$\mathrm{SU}(h) \coloneqq \mathrm{U}(h) \cap \mathrm{SL}(2, q^2)$$

and the **group of similitudes** of h as

$$\begin{aligned}\mathrm{GU}(h) &:= \{A \in \mathrm{GL}(2, q^2) \mid \exists\, \mu_A \in \mathbb{F}_{q^2}\ \forall\, x, y \in \mathbb{F}_{q^2}^2 : h(xA, yA) = \mu_A \cdot h(x, y)\} \\ &= \{A \in \mathrm{GL}(2, q^2) \mid \exists\, \mu_A \in \mathbb{F}_{q^2} : AM\overline{A}^\top = \mu_A M\}.\end{aligned}$$

The factor μ_A is called **factor of similitude** for $A \in \mathrm{GU}(h)$.

If h' is another nondegenerate Hermitian form on $\mathbb{F}_{q^2}^2$ and M' its Gram matrix, we call h and h' **equivalent** if there exists $T \in \mathrm{GL}(2, q^2)$ with $M' = TM\overline{T}^\top$. Then,

$$\mathrm{U}(h) = T^{-1}\mathrm{U}(h')T,\ \mathrm{SU}(h) = T^{-1}\,\mathrm{SU}(h')T \text{ and } \mathrm{GU}(h) = T^{-1}\mathrm{GU}(h')T.$$

Since $\mathbb{F}_{q^2}$ is a finite field, all nondegenerate Hermitian forms on $\mathbb{F}_{q^2}^2$ are equivalent (see e. g. [8, Corollary 10.4]) and thus all resulting (special) unitary groups (groups of similitudes) are conjugate.

Consider the (nondegenerate) Hermitian form

$$\begin{gathered} f\colon \mathbb{F}_{q^2}^2 \times \mathbb{F}_{q^2}^2 \to \mathbb{F}_{q^2}, \\ ((x_1, x_2), (y_1, y_2)) \mapsto s x_1 \overline{y_2} - s x_2 \overline{y_1}, \end{gathered}$$

where $s \in \mathbb{F}_{q^2}^\times$ with trace $\mathrm{Tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(s) = s + \overline{s} = 0$. Note that the Gram matrix of f is $M_f := \left(\begin{smallmatrix} 0 & s \\ -s & 0 \end{smallmatrix}\right)$ and that $M_f = \overline{M_f}^\top$. We show certain isomorphisms between (projective) unitary and linear groups, where the proof of the following theorem is adapted from Stroppel (unpublished).

Theorem 2.1. *Let* $Z := \mathbb{F}_{q^2}^\times \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ *denote the center of* $\mathrm{GL}(2, q^2)$ *and let* f *be the Hermitian form defined above. Then*

(a) $\mathrm{GU}(f) = Z \cdot \mathrm{GL}(2, q)$,

(b) $\mathrm{PGU}(f) \cong \mathrm{PGL}(2, q)$,

(c) $\mathrm{SU}(f) = \mathrm{SL}(2, q)$.

Proof. Compute first that for any $A \in \mathrm{GL}(2, q^2)$, we have

$$AM_f A^\top = \det A \cdot M_f. \qquad (\star)$$

Now let $A \in \mathrm{GL}(2, q)$ and $c \in \mathbb{F}_{q^2}^\times$ and compute

$$(cA)M_f(\overline{cA})^\top = c\overline{c} \cdot AM_f A^\top \overset{(\star)}{=} c\overline{c} \cdot \det A \cdot M_f.$$

Hence, $cA \in \mathrm{GU}(f)$. Conversely, let $A \in \mathrm{GU}(f)$ with factor of similitude μ_A. Then

$$\overline{\mu_A} M_f = (\overline{\mu_A M_f})^\top = (\overline{A M_f \overline{A}^\top})^\top = A M_f \overline{A}^\top = \mu_A M_f$$

and hence $\mu_A = \overline{\mu_A}$, i.e. $\mu_A \in \mathbb{F}_q$. Comparing determinants, we get $\mu_A^2 = \det A \cdot \overline{\det A}$ and hence the norm $\mathrm{N}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\det A \cdot \mu_A^{-1}) = 1$. Hilbert's Satz 90 (see e.g. [14, Theorem 4.31]) gives the existence of $u \in \mathbb{F}_{q^2}$ such that $\det A \cdot \mu_A^{-1} = \overline{u} \cdot u^{-1}$. We compute

$$M_f \overline{A}^\top = \mu_A A^{-1} M_f \overset{(\star)}{=} \mu_A (\det A)^{-1} M_f A^\top = u \overline{u}^{-1} M_f A^\top$$

and get $uA = (\overline{u} M_f^{-1} M_f \overline{A}^\top)^\top = \overline{uA}$. Hence, $uA \in \mathrm{GL}(2,q)$ and (a) follows. The statement (b) follows directly from (a) with the second isomorphism theorem.

Considering (c), we see that if $A \in \mathrm{SL}(2,q)$ then $A \in \mathrm{GU}(f)$ with $\mu_A = \det A = 1$ and hence $A \in \mathrm{SU}(f)$. If $A \in \mathrm{SU}(f)$, then we compute as above

$$M_f \overline{A}^\top = \mu_A (\det A)^{-1} M_f A^\top = M_f A^\top$$

and hence $A = \overline{A}$ and $A \in \mathrm{SL}(2,q)$. □

The Hermitian form f appeared to be convenient to show isomorphisms between those unitary and linear groups. But since the isomorphism type of the considered unitary groups does not depend on the choice of the nondegenerate Hermitian form in our setting, we get the following

Corollary 2.2. *Let* $h\colon \mathbb{F}_{q^2}^2 \times \mathbb{F}_{q^2}^2 \to \mathbb{F}_{q^2}$ *be a nondegenerate Hermitian form. Then* $\mathrm{PGU}(h) \cong \mathrm{PGL}(2,q)$, $\mathrm{SU}(h) \cong \mathrm{SL}(2,q)$ *and* $\mathrm{PSU}(h) \cong \mathrm{PSL}(2,q)$. □

For further considerations, we will usually choose our Hermitian form h to be

$$\begin{aligned} h\colon \mathbb{F}_{q^2}^2 \times \mathbb{F}_{q^2}^2 &\to \mathbb{F}_{q^2}, \\ ((x_1,x_2),(y_1,y_2)) &\mapsto x_1\overline{y_1} - x_2\overline{y_2}, \end{aligned}$$

with Gram matrix $M := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The special unitary group with respect to h is

$$\begin{aligned} \mathrm{SU}(h) &= \{A \in \mathrm{GL}(2,q^2) \mid AM\overline{A}^\top = M \text{ and } \det A = 1\} \\ &= \{\left(\begin{smallmatrix} x & y \\ \overline{y} & \overline{x} \end{smallmatrix}\right) \mid x,y \in \mathbb{F}_{q^2} \text{ and } x\overline{x} - y\overline{y} = 1\}. \end{aligned}$$

2.2 Subgroups of $\mathrm{SL}(2,q)$ of Orders q and $q+1$

While considering $\mathrm{SL}(2,q)$-unitals, we will be interested in certain subgroups of $\mathrm{SL}(2,q)$, namely in the Sylow p-subgroups of $\mathrm{SL}(2,q)$ and in subgroups of order $q+1$. The Sylow p-subgroups of $\mathrm{SL}(2,q)$ are well known to be characterized by their unique fixed point in the action of $\mathrm{SL}(2,q)$ on the projective line $\mathrm{PG}(1,q)$ (see e. g. [13, Satz II.8.2]). They have order q and there are $q+1$ Sylow p-subgroups, any two of which have trivial intersection.

We will put some more effort into investigating subgroups of order $q+1$. If q is even, then $\mathrm{SL}(2,q) = \mathrm{PSL}(2,q)$. If q is odd, then $q+1$ is even and hence each subgroup of $\mathrm{SL}(2,q)$ of order $q+1$ contains the unique involution $-\mathbb{1}$. Thus, the quotient of any such subgroup in $\mathrm{PSL}(2,q)$ is of order $\frac{1}{2}(q+1)$. We state the possible isomorphism types of subgroups of $\mathrm{PSL}(2,q)$ with order $q+1$ if q is even and with order $\frac{1}{2}(q+1)$ if q is odd.

Lemma 2.3. *Let $\tilde{S} \leq \mathrm{PSL}(2,q)$ be a subgroup of order $\frac{1}{k}(q+1)$ where $k = (2, q+1)$. Then:*

(a) *For $q \not\equiv 3 \mod 4$, the group $\tilde{S}$ is cyclic.*

(b) *For $q \equiv 3 \mod 4$, there are the following possibilities:*
   - (i) *$\tilde{S}$ is cyclic.*
   - (ii) *$\tilde{S}$ is a dihedral group (and $q \geq 7$).*
   - (iii) *$\tilde{S} \cong A_4$ for $q = 23$ or $\tilde{S} \cong S_4$ for $q = 47$.*

For each q, there exists exactly one conjugacy class of cyclic subgroups of order $\frac{1}{k}(q+1)$. For $q \equiv 3 \mod 4$, $q \geq 7$, there are two conjugacy classes of groups of type (b)(ii)*, which are fused under* $\mathrm{PGL}(2,q)$*. For $q = 23$ and $q = 47$, respectively, there are two conjugacy classes of groups of type* (b)(iii)*, which are also fused under* $\mathrm{PGL}(2,q)$*.*

Proof. Using Dickson's list of subgroups of $\mathrm{PSL}(2,q)$ (see e. g. [13, Hauptsatz II.8.27]), we see that there are no other possibilities for $\tilde{S}$. From [13, Satz II.8.5] we know that there is exactly one conjugacy class of cyclic subgroups of $\mathrm{PSL}(2,q)$ of order $\frac{1}{k}(q+1)$.

Now let $q \equiv 3 \mod 4$ and $q \geq 7$ (note that a dihedral group of order 2 is cyclic). The statement for the exceptional cases of type (b)(iii) can easily be computed using GAP. For type (b)(ii), we use the isomorphisms $\mathrm{PGL}(2,q) \cong \mathrm{PGU}(h)$ and $\mathrm{PSL}(2,q) \cong \mathrm{PSU}(h)$, where h is our Hermitian form with Gram matrix $M = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Let x be a generator of $\mathbb{S}_{\mathbb{F}_{q^2}} := \{y \in \mathbb{F}_{q^2} \mid y\overline{y} = 1\}$ (note that $\mathbb{S}_{\mathbb{F}_{q^2}}$ is cyclic of order $q+1$), let $b \in \mathbb{F}_{q^2}$ with $b\overline{b} = -1$ and consider

$$A := \left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right),\ B := \left(\begin{smallmatrix} 0 & b \\ \overline{b} & 0 \end{smallmatrix}\right) \text{ and the subgroup } D_2 := \langle [A], [B] \rangle \leq \mathrm{PGU}(h).$$

In $\mathrm{PGU}(h)$, we have $\mathrm{ord}([A]) = q+1$, $\mathrm{ord}([B]) = 2$ and $[B^{-1}AB] = [A^{-1}]$, i. e. D_2 is a dihedral subgroup of $\mathrm{PGU}(h)$ of order $2(q+1)$. Consider the subgroup

$$D_1 := \langle [A^2], [B] \rangle \leq D_2.$$

Then, $\mathrm{ord}([A^2]) = \frac{1}{2}(q+1)$ (recall that q is odd) and D_1 is a dihedral group of order $q+1$. In fact, D_1 is a subgroup of $\mathrm{PSU}(h)$ since $\det B = 1$ and $[A^2] = [\left(\begin{smallmatrix} 1 & 0 \\ 0 & x^2 \end{smallmatrix}\right)] = [\left(\begin{smallmatrix} \overline{x} & 0 \\ 0 & x \end{smallmatrix}\right)]$ and $\det \left(\begin{smallmatrix} \overline{x} & 0 \\ 0 & x \end{smallmatrix}\right) = 1$. Since $\frac{1}{2}(q+1)$ is even and $q \geq 7$, the dihedral group D_1 has two dihedral subgroups of order $\frac{1}{2}(q+1)$, which are not conjugate in D_1, namely

$$D := \langle [A^4], [B] \rangle \text{ and } D' := \langle [A^4], [BA^2] \rangle.$$

But D and D' are conjugate in D_2 since $[A^{-1}BA] = [BB^{-1}A^{-1}BA] = [BA^2]$.

Every dihedral subgroup of $\mathrm{PSL}(2,q)$ of order $\frac{1}{2}(q+1)$ is contained in a dihedral subgroup of order $q+1$, namely the normalizer of its cyclic subgroup of order $\frac{1}{4}(q+1)$ (see [13, Satz II.8.4/5]). But the partition of $\mathrm{PSL}(2,q)$ into conjugacy classes of subgroups in [13, Satz II.8.5] also implies that there is only one conjugacy class of dihedral subgroups of $\mathrm{PSL}(2,q)$ of order $q+1$, and hence every such subgroup is a conjugate of (the image of) D_1. We have already seen that the two dihedral subgroups of D_1 of order $\frac{1}{2}(q+1)$ are conjugate in $\mathrm{PGU}(h) \cong \mathrm{PGL}(2,q)$. □

We use Lemma 2.3 to determine all subgroups of $\mathrm{SL}(2,q)$ of order $q+1$.

Definition 2.4. Let $k \in \mathbb{N}$. The **generalized quaternion group** of order $4k$ is given by the presentation

$$\langle A, B \mid A^{2k} = 1, A^k = B^2, B^{-1}AB = A^{-1} \rangle.$$

The generalized quaternion group of order 4 is cyclic and the generalized quaternion group of order 8 is the quaternion group.

Proposition 2.5. *Let $S \leq \mathrm{SL}(2,q)$ be a subgroup of order $q+1$. Then:*

(a) *For $q \not\equiv 3 \mod 4$, the group S is cyclic.*

(b) *For* $q \equiv 3 \mod 4$*, there are the following possibilities:*

(i) S *is cyclic.*

(ii) S *is a generalized quaternion group and the quotient* $S/\{\pm\mathbb{1}\}$ *in* $\mathrm{PSL}(2,q)$ *is a dihedral group of order* $\frac{1}{2}(q+1)$*.*

(iii) *The quotient* $S/\{\pm\mathbb{1}\}$ *in* $\mathrm{PSL}(2,q)$ *is isomorphic to* A_4 *for* $q=23$ *or isomorphic to* S_4 *for* $q=47$*.*

For each q*, there exists exactly one conjugacy class of cyclic subgroups of order* $q+1$*. For* $q \equiv 3 \mod 4$*,* $q \geq 7$*, there are two conjugacy classes of groups of type* (b)(ii)*, which are fused under* $\mathrm{PGL}(2,q) \leq \mathrm{Aut}(\mathrm{SL}(2,q))$*. For* $q=23$ *and* $q=47$*, respectively, there are two conjugacy classes of groups of type* (b)(iii)*, which are also fused under* $\mathrm{PGL}(2,q) \leq \mathrm{Aut}(\mathrm{SL}(2,q))$*.*

Proof. Since S contains the central involution $-\mathbb{1}$ if q is odd, S is completely determined by its quotient in $\mathrm{PSL}(2,q)$. Again, we use the isomorphisms $\mathrm{PSL}(2,q) \cong \mathrm{PSU}(h)$ and $\mathrm{SL}(2,q) \cong \mathrm{SU}(h)$.

Let first $\tilde{S} := \{[\left(\begin{smallmatrix} x & 0 \\ 0 & \overline{x} \end{smallmatrix}\right)] \mid x\overline{x} = 1\} \leq \mathrm{PSU}(h)$ be a cyclic subgroup of order $\frac{1}{k}(q+1)$, where $k = (2, q+1)$. Then, $S := \{\left(\begin{smallmatrix} x & 0 \\ 0 & \overline{x} \end{smallmatrix}\right) \mid x\overline{x} = 1\} \leq \mathrm{SU}(h)$ is a cyclic subgroup of order $q+1$ with $S/\{\pm\mathbb{1}\} = \tilde{S}$.

Now let $q \equiv 3 \mod 4$, let x be a generator of $\mathbb{S}_{\mathbb{F}_{q^2}}$ and let $b \in \mathbb{F}_{q^2}$ with $b\overline{b} = -1$. Let further $A := \left(\begin{smallmatrix} x^2 & 0 \\ 0 & \overline{x}^2 \end{smallmatrix}\right)$, $B := \left(\begin{smallmatrix} 0 & b \\ \overline{b} & 0 \end{smallmatrix}\right)$ and let $\tilde{S} := \langle [A], [B] \rangle \leq \mathrm{PSU}(h)$ as in the proof of Lemma 2.3 be a dihedral subgroup of order $\frac{1}{2}(q+1)$. Then the order of A equals $\frac{1}{2}(q+1)$, $A^{\frac{1}{4}(q+1)} = B^2 = -\mathbb{1}$ and $B^{-1}AB = A^{-1}$. Hence, $S := \langle A, B \rangle \leq \mathrm{SU}(h)$ is a generalized quaternion group of order $q+1$ with $S/\{\pm\mathbb{1}\} = \tilde{S}$.

The statement concerning the conjugacy classes follows from the corresponding statement for the subgroups of $\mathrm{PSL}(2,q)$. □

Remark 2.6. *The proposition shows that, up to the exceptional cases in* (b)(iii)*, each subgroup of* $\mathrm{SU}(h) \cong \mathrm{SL}(2,q)$ *of order* $q+1$ *is conjugate* (*in* $\mathrm{PGU}(h) \cong \mathrm{PGL}(2,q)$) *to one of the following subgroups.*

(a) *A cyclic subgroup of* $\mathrm{SU}(h)$ *of order* $q+1$ *is given by* $\{\left(\begin{smallmatrix} x & 0 \\ 0 & \overline{x} \end{smallmatrix}\right) \mid x\overline{x} = 1\}$*.*

(b) *For* $q \equiv 3 \mod 4$*, a generalized quaternion subgroup of* $\mathrm{SU}(h)$ *of order* $q+1$ *is given by*

$$\langle A, B \rangle \text{ with } A := \left(\begin{smallmatrix} x^2 & 0 \\ 0 & \overline{x}^2 \end{smallmatrix}\right) \text{ and } B := \left(\begin{smallmatrix} 0 & b \\ \overline{b} & 0 \end{smallmatrix}\right),$$

where x *generates* $\mathbb{S}_{\mathbb{F}_{q^2}}$ *and* $b\overline{b} = -1$*.*

Using GAP, we find that in SL$(2, 23)$, any subgroup of type (b)(iii) is isomorphic to SL$(2, 3)$. In SL$(2, 47)$, any subgroup S of type (b)(iii) is isomorphic to a non-split extension

$$1 \to C_2 \hookrightarrow S \twoheadrightarrow S_4 \to 1$$

or equally to a non-split extension

$$1 \to \mathrm{SL}(2, 3) \hookrightarrow S \twoheadrightarrow C_2 \to 1.$$

A cyclic or generalized quaternion subgroup $S \leq \mathrm{SL}(2, q)$ of order $q + 1$ may be chosen as described in the following

Remark 2.7.

(a) *Let* $d \in \mathbb{F}_q^\times$ *such that* $X^2 - tX + d$ *has no root in* $\mathbb{F}_q$*, where* $t = 1$ *for* q *even and* $t = 0$ *for* q *odd. Then*

$$C := \{\left(\begin{smallmatrix} a & b \\ -db & a+tb \end{smallmatrix}\right) \mid a^2 + tab + db^2 = 1\}$$

is a cyclic subgroup of SL$(2, q)$ *of order* $q + 1$.

(b) *For* $q \equiv 3 \mod 4$*, let* $M := \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \in \mathrm{SL}(2, q)$ *be of order* $q + 1$ *and let* $A := M^2$*. Let* $x, y \in \mathbb{F}_q$ *with* $x^2 + y^2 = -1$ *and let* $B := \left(\begin{smallmatrix} x & y \\ y & -x \end{smallmatrix}\right)$*. Then* $S := \langle A, B \rangle$ *is a generalized quaternion group of order* $q + 1$*.*

Proof. Consider the extension field $\mathbb{F}_{q^2} \supseteq \mathbb{F}_q$ as $\mathbb{F}_{q^2} = \mathbb{F}_q(u)$ with $u^2 - tu + d = 0$. Consider further the isomorphism

$$\mathbb{F}_{q^2} \to \{\left(\begin{smallmatrix} a & b \\ -db & a+tb \end{smallmatrix}\right) \mid a, b \in \mathbb{F}_q\}, \quad a + ub \mapsto \left(\begin{smallmatrix} a & b \\ -db & a+tb \end{smallmatrix}\right),$$

where $\det \left(\begin{smallmatrix} a & b \\ -db & a+tb \end{smallmatrix}\right)$ equals the norm $\mathrm{N}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(a + ub)$. Then, C corresponds to $\mathbb{S}_{\mathbb{F}_{q^2}}$ – the group of elements with norm equal to 1 in $\mathbb{F}_{q^2}$ – and is hence cyclic of order $q + 1$.

For $q \equiv 3 \mod 4$, note first that M is a generator of C and hence $\operatorname{ord} A = \frac{1}{2}(q + 1)$ and $A^{\frac{1}{4}(q+1)} = -\mathbb{1}$. Since in a finite field, each element can be written as sum of two squares, there exist $x, y \in \mathbb{F}_q$ with $x^2 + y^2 = -1$. Compute $\det B = 1$, $B^2 = -\mathbb{1}$ and $B^{-1}MB = M^{-1}$ (and thus also $B^{-1}AB = A^{-1}$). Hence, $S = \langle A, B \rangle$ is a subgroup of SL$(2, q)$ and a generalized quaternion group of order $q + 1$. □

We want to compute the (isomorphism type of the) stabilizer in the full automorphism group of SL$(2, q)$ for each of the possible subgroups S of order $q + 1$. We will do this in Theorem 2.11, after some preliminary work.

Lemma 2.8. *Let* $m > 2$ *and let*

$$S \coloneqq \langle A, B \mid A^{2m} = 1, A^m = B^2, B^{-1}AB = A^{-1}\rangle$$

be the generalized quaternion group of order $4m$. *Then every automorphism of* S *stabilizes the cyclic subgroup* $\langle A\rangle$.

Proof. Let α be an automorphism of S and assume that $A \cdot \alpha = A^kB$ for some $k \in \{0, \dots, 2m-1\}$. Then $A^4 \cdot \alpha = (A^kB)^4 = (A^kBA^k)B(A^kBA^k)B = B^4 = 1$ and hence $A^4 = 1$, a contradiction for $m > 2$. □

Lemma 2.9. *For* $\pm\mathbb{1} \neq A \in \mathrm{SL}(2, q)$*:*

(a) *A has eigenvalue* $1 \iff \mathrm{tr}(A) = 2 \iff \mathrm{ord}(A) = p$.

(b) *For p odd: A has eigenvalue* $-1 \iff \mathrm{tr}(A) = -2 \iff \mathrm{ord}(A) = 2p$.

Proof. (a) If 1 is an eigenvalue of $\mathbb{1} \neq A \in \mathrm{SL}(2, q)$, then $\det(A) = 1$ implies that the second eigenvalue also equals 1. Thus, A is a conjugate of $\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right)$ with $x \in \mathbb{F}_q^\times$ and hence $\mathrm{tr}(A) = 2$ and $\mathrm{ord}(A) = p$. If $\mathrm{tr}(A) = 2$, then the characteristic polynomial $\chi_A(X) = X^2 - 2X + 1$ has root 1. If $\mathrm{ord}(A) = p$, then A solves $0 = X^p - 1 = (X-1)^p$ and hence $\det(A - \mathbb{1}) = 0$.

(b) If -1 is an eigenvalue of $-\mathbb{1} \neq A \in \mathrm{SL}(2, q)$, then A is a conjugate of $\left(\begin{smallmatrix} -1 & 0 \\ x & -1 \end{smallmatrix}\right)$ with $x \in \mathbb{F}_q^\times$ and hence $\mathrm{tr}(A) = -2$ and $\mathrm{ord}(A) = 2p$ (since p odd). If $\mathrm{tr}(A) = -2$, then $\chi_A(X) = X^2 + 2X + 1$ has root -1. If $\mathrm{ord}(A) = 2p$, then A solves $0 = X^{2p} - 1 = (X^2-1)^p = ((X-1)(X+1))^p$ and hence A has eigenvalue 1 or -1. Eigenvalue 1 is not possible because of (a) and thus A has eigenvalue -1. □

In the following lemma, we exclude the case $q = 3$ and $k = 2$, since then $S = \langle -\mathbb{1}\rangle$ and

$$\#\,\mathrm{Aut}(\mathrm{SL}(2,3))_{\langle -\mathbb{1}\rangle} = \#\,\mathrm{PGL}(2,3) = 24 > 2 \cdot 1 \cdot 4.$$

Lemma 2.10. *Let* $S \coloneqq \langle s\rangle \leq \mathrm{SL}(2, q)$ *be cyclic of order* $\frac{1}{k}(q+1)$ *where either* $k = 1$ *or* $q \equiv 3 \mod 4$, $q \geq 7$ *and* $k = 2$. *Then*

$$\#\,\mathrm{Aut}(\mathrm{SL}(2, q))_S \leq 2e(q+1).$$

Proof. Consider the additive span $\langle S\rangle_+ \subseteq \mathbb{F}_q^{2\times 2}$ and the centralizer $\mathrm{C}_{\mathbb{F}_q^{2\times 2}}(S)$. The linear representation

$$S \to \mathrm{GL}(\mathbb{F}_q^2)$$

is irreducible, for assume there is an 1-dimensional S-invariant subspace $U \leq \mathbb{F}_q^2$. Then every vector in $U \setminus \{0\}$ is an eigenvector of s with eigenvalue $\lambda \in \mathbb{F}_q^\times$. But $\mathrm{ord}(s) = \frac{q+1}{k}$, i. e. $\mathrm{ord}(\lambda) \mid (\frac{q+1}{k}, q-1)$ and hence $\lambda = \pm 1$, contrary to Lemma 2.9. We thus know by Schur's lemma (see e. g. [25, Theorem 1.2]) that $\mathrm{C}_{\mathbb{F}_q^{2\times 2}}(S)$ is a skew field and, since it is finite, a field (see e. g. [29, p. 1]).

We show that $\langle S\rangle_+$ is a subfield of $\mathrm{C}_{\mathbb{F}_q^{2\times 2}}(S)$: Since S is cyclic and hence commutative, $\langle S\rangle_+ \subseteq \mathrm{C}_{\mathbb{F}_q^{2\times 2}}(S)$. Further, $\langle S\rangle_+$ is closed under addition and (since S is a group) under multiplication. Since $-\mathbb{1} \in S$, we have $-A \in \langle S\rangle_+$ for each $A \in \langle S\rangle_+$. Let $0 \neq A \in \langle S\rangle_+$. We know $\langle S\rangle_+$ to be an $\mathbb{F}_p$-vector space of dimension $\leq d := \dim_{\mathbb{F}_p}(\mathbb{F}_q^{2\times 2})$ and hence there exist $c_i \in \mathbb{F}_p$ (not all zero) such that $0 = \sum_{i=0}^d c_i A^i$. Assume $c_0 \neq 0$, for otherwise we may multiply by A^{-1} (recall that $\mathrm{C}_{\mathbb{F}_q^{2\times 2}}(S)$ is a field). Thus

$$\mathbb{1} = A \cdot \sum_{i=1}^{d} (-\frac{c_i}{c_0}) A^{i-1}$$

with $\sum_{i=1}^d (-\frac{c_i}{c_0}) A^{i-1} \in \langle S\rangle_+$.

Now we know that $\langle S\rangle_+$ and $\mathrm{C}_{\mathbb{F}_q^{2\times 2}}(S)$ are fields of characteristic p with

$$\#\langle S\rangle_+ \leq \#\mathrm{C}_{\mathbb{F}_q^{2\times 2}}(S).$$

For $k=1$, we have $\#\langle S\rangle_+ \geq \#S > q$. For $k=2$ (and $q \equiv 3 \mod 4$, $q \geq 7$), assume $\#\langle S\rangle_+$ to be a p-power strictly less than q. Then

$$\frac{q+1}{2} \leq \#\langle S\rangle_+ \leq \frac{q}{p} < \frac{q}{2},$$

a contradiction. Hence $\#\langle S\rangle_+ \geq q$. Assume $\langle S\rangle_+ \cong \mathbb{F}_q$ (and still $k=2$). Then $\frac{q+1}{2} = \mathrm{ord}(s) \mid q-1$, but $(\frac{q+1}{2}, q-1) = 2$ and $\frac{q+1}{2} \neq 2$, since $q \geq 7$. Thus, for both $k=1$ and $k=2$, we have

$$q < \#\langle S\rangle_+ \leq \#\mathrm{C}_{\mathbb{F}_q^{2\times 2}}(S).$$

Further, we have $\#\mathrm{C}_{\mathbb{F}_q^{2\times 2}}(S) \leq q^2$, for consider the action

$$\mathrm{C}_{\mathbb{F}_q^{2\times 2}}(S) \curvearrowright \mathbb{F}_q^2.$$

Let $0 \neq v \in \mathbb{F}_q^2$ and $A \in \mathrm{C}_{\mathbb{F}_q^{2\times 2}}(S)$ with $vA = v$. Then $A - \mathbb{1}$ has eigenvalue 0 and thus, since $\mathrm{C}_{\mathbb{F}_q^{2\times 2}}(S)$ is a field, $A = \mathbb{1}$. Hence, the stabilizers of the action are trivial and we get

$$\#\mathrm{C}_{\mathbb{F}_q^{2\times 2}}(S) \leq \#\mathbb{F}_q^2 = q^2.$$

Further, $\mathrm{C}_{\mathbb{F}_q^{2\times 2}}(S)$ contains $\mathbb{F}_q \cdot \mathbb{1}$ as a subfield, which forces $\#\mathrm{C}_{\mathbb{F}_q^{2\times 2}}(S)$ to be a q-power and hence $\#\mathrm{C}_{\mathbb{F}_q^{2\times 2}}(S) = q^2$. Likewise, $\#\mathrm{C}_{\mathbb{F}_q^{2\times 2}}(S)$ is a $\#\langle S\rangle_+$-power and since $\#\langle S\rangle_+ > q$, we have $\#\langle S\rangle_+ = q^2$ and

$$\langle S\rangle_+ = \mathrm{C}_{\mathbb{F}_q^{2\times 2}}(S) \cong \mathbb{F}_{q^2}.$$

Now we may compute $\#\operatorname{Aut}(\mathrm{SL}(2,q))_S$. See e. g. [21, Theorem 5.6.6] for the fact that the automorphisms of $\mathrm{SL}(2,q)$ are exactly the semilinear projective transformations, i. e.

$$\operatorname{Aut}(\mathrm{SL}(2,q)) = \mathrm{P\Gamma L}(2,q) = \mathrm{PGL}(2,q) \rtimes \operatorname{Aut}(\mathbb{F}_q),$$

where $\mathrm{PGL}(2,q)$ acts via conjugation and field automorphisms act entrywise on a matrix. This action obviously induces ring automorphisms if applied to $\mathbb{F}_q^{2\times 2}$. Hence, there is a homomorphism

$$\varphi\colon \mathrm{P}(\mathrm{N}_{\Gamma\mathrm{L}(2,q)}(S)) \to \operatorname{Aut}(\langle S\rangle_+) = \operatorname{Aut}(\mathbb{F}_{q^2}).$$

We compute

$$\ker\varphi = \mathrm{P}(\mathrm{C}_{\Gamma\mathrm{L}(2,q)}(\langle S\rangle_+)) = \mathrm{P}(\mathrm{C}_{\Gamma\mathrm{L}(2,q)}(S)).$$

Since $\mathbb{F}_q \cdot \mathbb{1} \leq \mathrm{C}_{\mathbb{F}_q^{2\times 2}}(S) = \langle S\rangle_+$, we have $\mathrm{C}_{\Gamma\mathrm{L}(2,q)}(\langle S\rangle_+) \leq \mathrm{C}_{\Gamma\mathrm{L}(2,q)}(\mathbb{F}_q \cdot \mathbb{1})$. But no non-linear semilinear map centralizes $\mathbb{F}_q \cdot \mathbb{1}$ (while all linear ones do) and hence $\mathrm{C}_{\Gamma\mathrm{L}(2,q)}(\mathbb{F}_q \cdot \mathbb{1}) = \mathrm{GL}(2,q)$. Thus we get

$$\ker\varphi = \mathrm{P}(\mathrm{C}_{\mathrm{GL}(2,q)}(\langle S\rangle_+)) = \mathrm{P}(\mathrm{C}_{\mathrm{GL}(2,q)}(S)).$$

Since $\mathrm{C}_{\mathbb{F}_q^{2\times 2}}(S)$ is a field, $\mathrm{C}_{\mathrm{GL}(2,q)}(S) = \mathrm{C}_{\mathbb{F}_q^{2\times 2}}(S)^\times \geq \mathbb{F}_q^\times \cdot \mathbb{1}$ and hence

$$\#\ker\varphi = \#\mathrm{P}(\mathrm{C}_{\mathrm{GL}(2,q)}(S)) = \frac{q^2-1}{q-1} = q+1.$$

Finally,

$$\begin{aligned}\#\operatorname{Aut}(\mathrm{SL}(2,q))_S &= \#\mathrm{P}(\mathrm{N}_{\Gamma\mathrm{L}(2,q)}(S)) = \#\ker\varphi \cdot \#\operatorname{im}\varphi \\ &\leq (q+1)\cdot \#\operatorname{Aut}(\mathbb{F}_{q^2}) = 2e(q+1).\end{aligned}$$ □

Theorem 2.11.

(a) *For* $C \leq \mathrm{SL}(2,q)$ *cyclic of order* $q+1$*, we have*

$$\mathrm{Aut}(\mathrm{SL}(2,q))_C \cong C_{q+1} \rtimes C_{2e}.$$

(b) *For* $q \equiv 3 \mod 4$*,* $q > 7$ *and* $S \leq \mathrm{SL}(2,q)$ *generalized quaternion of order* $q+1$*, we have*

$$\#\,\mathrm{Aut}(\mathrm{SL}(2,q))_S = e(q+1)$$

and $\mathrm{Aut}(\mathrm{SL}(2,q))_S$ *is conjugate to a subgroup of* $\mathrm{Aut}(\mathrm{SL}(2,q))_C$.

(c) *For* $q \in \{23, 47\}$ *and* $S \leq \mathrm{SL}(2,q)$ *of type* (b)(iii) *in* Proposition 2.5 *or* $q = 7$ *and* $S \leq \mathrm{SL}(2,q)$ *a quaternion group, we have*

$$\mathrm{Aut}(\mathrm{SL}(2,q))_S \cong S_4.$$

Proof. Let $C \leq \mathrm{SL}(2,q)$ be cyclic of order $q+1$. In $\mathrm{SU}(h) \cong \mathrm{SL}(2,q)$, we can choose

$$C \coloneqq \{\left(\begin{smallmatrix} x & 0 \\ 0 & \overline{x} \end{smallmatrix}\right) \mid x\overline{x} = 1\},$$

according to Remark 2.6. Let $\varphi\colon \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}$, $x \mapsto x^p$, be the Frobenius automorphism of order $2e$. Then φ induces an automorphism of $\mathrm{SU}(h)$ of order $2e$ by $\left(\begin{smallmatrix} x & y \\ \overline{y} & \overline{x} \end{smallmatrix}\right) \cdot \varphi \coloneqq \left(\begin{smallmatrix} x^p & y^p \\ \overline{y}^p & \overline{x}^p \end{smallmatrix}\right)$. Obviously, this automorphism stabilizes C. Now consider the subgroup

$$U \coloneqq \{[\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)] \mid a\overline{a} = 1\} \leq \mathrm{PU}(h).$$

The group C is fixed pointwise under conjugation by U. Further, U is cyclic of order $q+1$, is normalized by $\langle\varphi\rangle$ and has trivial intersection with $\langle\varphi\rangle$ as automorphism groups of $\mathrm{SU}(h)$. Hence,

$$\mathrm{Aut}(\mathrm{SL}(2,q))_C \geq C_{q+1} \rtimes C_{2e}$$

and the statement follows with Lemma 2.10.

Now let $q \equiv 3 \mod 4$, $q > 7$, and let $S \coloneqq \langle A, B\rangle \leq \mathrm{SL}(2,q)$ be a generalized quaternion subgroup of order $q+1 > 8$. According to Remark 2.6 and again by using the isomorphism $\mathrm{SU}(h) \cong \mathrm{SL}(2,q)$ we may choose

$$A \coloneqq \left(\begin{smallmatrix} x^2 & 0 \\ 0 & \overline{x}^2 \end{smallmatrix}\right) \text{ and } B \coloneqq \left(\begin{smallmatrix} 0 & b \\ \overline{b} & 0 \end{smallmatrix}\right),$$

where x generates $\mathbb{S}_{\mathbb{F}_{q^2}}$ and $b\overline{b} = -1$. From Lemma 2.10 and the above, we know

$$\mathrm{Aut}(\mathrm{SL}(2,q))_{\langle A \rangle} \cong \mathrm{Aut}(\mathrm{SL}(2,q))_C \cong U \rtimes \langle \varphi \rangle.$$

Since all automorphisms of a generalized quaternion group of order greater than 8 stabilize the subgroup $\langle A \rangle$ (see Lemma 2.8), we have

$$\mathrm{Aut}(\mathrm{SL}(2,q))_S \leq \mathrm{Aut}(\mathrm{SL}(2,q))_{\langle A \rangle}.$$

Let $[\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)] \in U$ and compute

$$\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)^{-1} B \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \overline{a} \end{smallmatrix}\right) B \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right) = \begin{pmatrix} 0 & ab \\ \overline{a}\overline{b} & 0 \end{pmatrix} = \left(\begin{smallmatrix} a & 0 \\ 0 & \overline{a} \end{smallmatrix}\right) B.$$

This is in S exactly if a is an even power of x. Now let $1 \leq l \leq 2e-1$. Since $q \equiv 3 \mod 4$, we have $p \equiv 3 \mod 4$ and hence $p^l - 1$ is even for every $l \in \{1, \ldots, 2e-1\}$ and $\frac{1}{2}(p^l - 1)$ is even exactly if l is even. Thus,

$$\mathrm{N}(b^{p^l-1}) := \mathrm{N}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(b^{p^l-1}) = (b\overline{b})^{p^l-1} = (-1)^{p^l-1} = 1$$

for every $l \in \{1, \ldots, 2e-1\}$ and $\mathrm{N}(b^{\frac{1}{2}(p^l-1)}) = 1$ exactly if l is even. Hence, b^{p^l-1} is a power of x, say $b^{p^l-1} = x^m$, where $m \equiv l \mod 2$. Compute

$$\begin{aligned}
\left(\begin{pmatrix} 1 & 0 \\ 0 & x^k \end{pmatrix}^{-1} B \begin{pmatrix} 1 & 0 \\ 0 & x^k \end{pmatrix}\right) \cdot \varphi^l &= \left(\begin{pmatrix} x^k & 0 \\ 0 & \overline{x}^k \end{pmatrix} B\right) \cdot \varphi^l \\
&= \begin{pmatrix} x^{kp^l} & 0 \\ 0 & \overline{x}^{kp^l} \end{pmatrix} \begin{pmatrix} 0 & b^{p^l} \\ \overline{b}^{p^l} & 0 \end{pmatrix} \\
&= \begin{pmatrix} x^{kp^l} & 0 \\ 0 & \overline{x}^{kp^l} \end{pmatrix} \begin{pmatrix} b^{p^l-1} & 0 \\ 0 & \overline{b}^{p^l-1} \end{pmatrix} B \\
&= \begin{pmatrix} x^{kp^l+m} & 0 \\ 0 & \overline{x}^{kp^l+m} \end{pmatrix} B.
\end{aligned}$$

This is in S exactly if $kp^l + m$ is even, i. e. exactly if

$$0 \equiv kp^l + m \equiv k + m \equiv k + l \mod 2.$$

Hence, for each $\varphi^l \in \langle \varphi \rangle$ $(0 \leq l \leq 2e-1)$, there are exactly $\frac{1}{2}(q+1)$ elements $u := [\begin{pmatrix} 1 & 0 \\ 0 & x^k \end{pmatrix}] \in U$ such that $(u^{-1}Bu) \cdot \varphi^l \in S$, and we get

$$\# \, \mathrm{Aut}(\mathrm{SL}(2,q))_S = 2e \cdot \frac{1}{2}(q+1) = e(q+1).$$

The cases in (c) are obtained by direct computation, e. g. using GAP. □

Remark 2.12. *Since*

$$\left(\begin{smallmatrix}1&0\\0&a\end{smallmatrix}\right)\cdot\varphi^e=\left(\begin{smallmatrix}1&0\\0&a^q\end{smallmatrix}\right)=\left(\begin{smallmatrix}1&0\\0&\overline{a}\end{smallmatrix}\right)=\left(\begin{smallmatrix}1&0\\0&a\end{smallmatrix}\right)^{-1}$$

for $a\overline{a}=1$*, we see that* $U\rtimes\langle\varphi^e\rangle$ *is a dihedral group of order* $2(q+1)$*. In particular, if* $e=1$*, then* $\mathrm{Aut}(\mathrm{SL}(2,q))_C$ *is a dihedral group of order* $2(q+1)$*.*

Corollary 2.13. *Let* $S\leq\mathrm{SL}(2,q)$ *be a subgroup of order* $q+1$*. Then the stabilizer* $\mathrm{Aut}(\mathrm{SL}(2,q))_S$ *is solvable.* □

3 Unitals

We will first consider (affine) unitals as abstract incidence structures, before we introduce the construction of (affine) $\mathrm{SL}(2,q)$-unitals.

3.1 Unitals as Incidence Structures

Definition 3.1. An **incidence structure** is a triple $\mathcal{I} = (\mathcal{P}, \mathcal{B}, I)$, where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and $I \subseteq \mathcal{P} \times \mathcal{B}$. The elements of $\mathcal{P}$ are called **points**, the elements of $\mathcal{B}$ are called **blocks** and I is called the **incidence relation**. We call a point x **incident** with a block B if $(x, B) \in I$.

We will often use geometric language and say that a point *lies on* a block or a block *goes through* a point if they are incident. We will also say that two points (blocks) *are joined* (*meet / intersect*) if there is a block (point) incident with both of them.
The set of points incident with a block B will be denoted by $\mathcal{P}_B$ and the set of blocks incident with a point x will be denoted by $\mathcal{B}_x$.

Definition 3.2. Let $\mathcal{I} = (\mathcal{P}, \mathcal{B}, I)$ and $\mathcal{I}' = (\mathcal{P}', \mathcal{B}', I')$ be incidence structures. An **isomorphism** α between $\mathcal{I}$ and $\mathcal{I}'$ is a bijective map $\alpha\colon \mathcal{P} \cup \mathcal{B} \to \mathcal{P}' \cup \mathcal{B}'$, where

(i) $\mathcal{P} \cdot \alpha = \mathcal{P}'$ and $\mathcal{B} \cdot \alpha = \mathcal{B}'$,
(ii) for every $(x, B) \in \mathcal{P} \times \mathcal{B} : (x, B) \in I \iff (x \cdot \alpha, B \cdot \alpha) \in I'$.

An isomorphism between $\mathcal{I}$ and $\mathcal{I}$ is called an **automorphism** and we denote by $\mathrm{Aut}(\mathcal{I})$ the **full automorphism group** of $\mathcal{I}$.

Definition 3.3. Let $n \in \mathbb{N}$. An incidence structure $\mathbb{U} = (\mathcal{P}, \mathcal{B}, I)$ is called a **unital of order *n*** if:

(U1) There are $n^3 + 1$ points.

(U2) Each block is incident with $n + 1$ points.

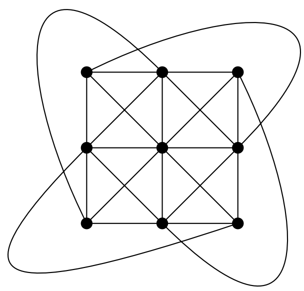
(U3) For any two points there is exactly one block incident with both of them.

Simple counting yields that in a unital of order n there are n^2 blocks through each point and $n^2(n^2-n+1)$ blocks in total.

Remark 3.4. *We will often consider an automorphism α of a unital as bijection on the point set such that for each block B there is a block B' with $\mathcal{P}_B \cdot \alpha = \mathcal{P}_{B'}$.*

Example 3.5. (a) The unital of order 1 consists of two points on one block.

(b) There is only one isomorphism type of unitals of order 2. These are isomorphic to the affine plane over the field $\mathbb{F}_3$.



In the construction of SL$(2,q)$-unitals (see Section 3.2.1), there is one block playing a special role as "block at infinity". For this reason, we will deal a lot with *affine* unitals, meaning there is "one block missing". Note that there are many constructions of unitals (embedded in projective planes), where there is considered only one *point* at infinity and hence a unital of order n has n^3 affine points. However, in this thesis, an affine unital will always be a *block*-affine unital.

Definition 3.6. Let $\mathbb{U} = (\mathcal{P}, \mathcal{B}, I)$ be a unital of order n and let $B \in \mathcal{B}$ be a block of $\mathbb{U}$. The **affine part** ${}^B\mathbb{U} = ({}^B\mathcal{P}, {}^B\mathcal{B}, {}^BI)$ of $\mathbb{U}$ is the incidence structure obtained from $\mathbb{U}$ by removing B and all the points incident with B, i. e.

$$ {}^B\mathcal{P} := \mathcal{P} \setminus \mathcal{P}_B,\ {}^B\mathcal{B} := \mathcal{B} \setminus \{B\},\ {}^BI := I|_{{}^B\mathcal{P} \times {}^B\mathcal{B}}. $$

Definition 3.7. Let $n \in \mathbb{N}$, $n \geq 2$. An incidence structure $\mathbb{U} = (\mathcal{P}, \mathcal{B}, I)$ is called an **affine unital of order n** if:

(AU1) There are $n^3 - n = (n-1)n(n+1)$ points.

(AU2) Each block is incident with either n or $n+1$ points. The blocks incident with n points will be called **short blocks** and the blocks incident with $n+1$ points will be called **long blocks**.

(AU3) Each point is incident with n^2 blocks.

(AU4) For any two points there is exactly one block incident with both of them.

(AU5) There exists a **parallelism** on the short blocks, meaning a partition of the set of all short blocks into $n+1$ parallel classes of size n^2-1 such that the blocks of each parallel class are pairwise non-intersecting.

The number of short blocks indicated in Axiom (AU5) follows indeed from Axioms (AU1)–(AU4), as shown in the following

Proposition 3.8. *Axioms* (AU1)–(AU4) *yield: In any affine unital of order n, there are $n^2(n^2-n+1)-1$ blocks. Through each point go $n+1$ short blocks and there are $(n^2-1)(n+1)$ short blocks in total.*

Proof. Let x be a point of an affine unital, k the number of short blocks through x and l the number of long blocks through x. Then, by (AU3), $k+l=n^2$. With (AU1) and (AU4), we get

$$n^3-n-1=k(n-1)+ln=(k+l)n-k=n^3-k$$

and hence $k=n+1$.
The total number of short blocks is given by

$$\frac{(n^3-n)k}{n}=\frac{(n^3-n)(n+1)}{n}=(n^2-1)(n+1)$$

and the total number of blocks by

$$(n^2-1)(n+1)+\frac{(n^3-n)(n^2-n-1)}{n+1}=n^2(n^2-n+1)-1. \qquad \square$$

It is obvious from the definitions that the affine part ${}^B\mathbb{U}$ of a unital $\mathbb{U}$ satisfies the axioms of an affine unital. A parallelism as required in (AU5) is given by

$$\pi_B := \{\mathcal{B}_x \setminus \{B\} \mid x \in B\}.$$

The completion of an affine unital to a unital is also possible.

Proposition 3.9. *Let* $\mathbb{U} = (\mathcal{P}, \mathcal{B}, I)$ *be an affine unital of order* n *and let* $\pi := \{\mathcal{B}_1, \ldots, \mathcal{B}_{n+1}\}$ *be a parallelism as required in* (AU5)*. Set*

$$\mathcal{P}^\pi := \mathcal{P} \cup \pi, \quad \mathcal{B}^\pi := \mathcal{B} \cup \{[\infty]^\pi\} \quad \text{and}$$
$$I^\pi := I \cup \{(\mathcal{B}_i, B) \mid 1 \leq i \leq n+1, B \in \mathcal{B}_i\} \cup \{(\mathcal{B}_i, [\infty]^\pi) \mid 1 \leq i \leq n+1\}.$$

Then $\mathbb{U}^\pi = (\mathcal{P}^\pi, \mathcal{B}^\pi, I^\pi)$ *is a unital of order* n*, the* **π-closure** *of* $\mathbb{U}$*. We call the block* $[\infty]^\pi$ *the* **block at infinity** *and the points* $\mathcal{B}_i$*,* $1 \leq i \leq n+1$*, the* **points at infinity**.

Proof. The axioms (U1) and (U2) are obviously satisfied by $\mathbb{U}^\pi$. Any two affine points are still joined by a unique block. Any two points at infinity are joined exactly by the block at infinity. Consider an affine point $x \in \mathcal{P}$ and a point at infinity $\mathcal{B}_i \in \pi$. Since there are $n+1$ short blocks through x and the blocks of each parallel class in π are pairwise non-intersecting, there is exactly one (short) block $B \in \mathcal{B}_i$ incident with x. □

It is clear from the construction that ${}^{[\infty]^\pi}(\mathbb{U}^\pi) = \mathbb{U}$ holds for each affine unital $\mathbb{U}$ with parallelism π. If $\mathbb{U}$ is a unital and B a block of $\mathbb{U}$, then $({}^B\mathbb{U})^{\pi_B}$ is isomorphic to $\mathbb{U}$ via $[\infty]^{\pi_B} \mapsto B$.

For the completion of an affine unital to a unital, we used the parallelism required in (AU5). The existence of such a parallelism must explicitly be required, as is shown by the following

Example 3.10 ((AU5) is independent of (AU1)–(AU4))**.** Consider the incidence structure $\mathcal{I}$ with 24 points, 62 blocks and incidences as indicated in Figure 3.1, where each column represents one block. For example, if we label the points with $1, 2, \ldots, 24$, the first block is incident with the points 1, 2, 3 and 4, the second block with the points 1, 5, 6 and 7 and so on. Then, $\mathcal{I}$ satisfies axioms (AU1)–(AU4) with $n = 3$, but there is no parallelism on the short blocks as required in (AU5). Indeed, there is no set of 8 pairwise non-intersecting short blocks in $\mathcal{I}$, which can easily be computed using GAP, or with some patience also manually.
The incidence structure $\mathcal{I}$ can be obtained from an affine unital by slight changes of incidences. Consider the unital `BBTAbstractUnital(1)` in the GAP-package UnitalSZ [20]. This is the first unital in a library of unitals constructed by Betten, Betten and Tonchev [3]. By removing a suitable block and all the points on it, we obtain the affine unital pictured in Figure 3.2. The only changes compared to $\mathcal{I}$ are the ones marked with white circles.

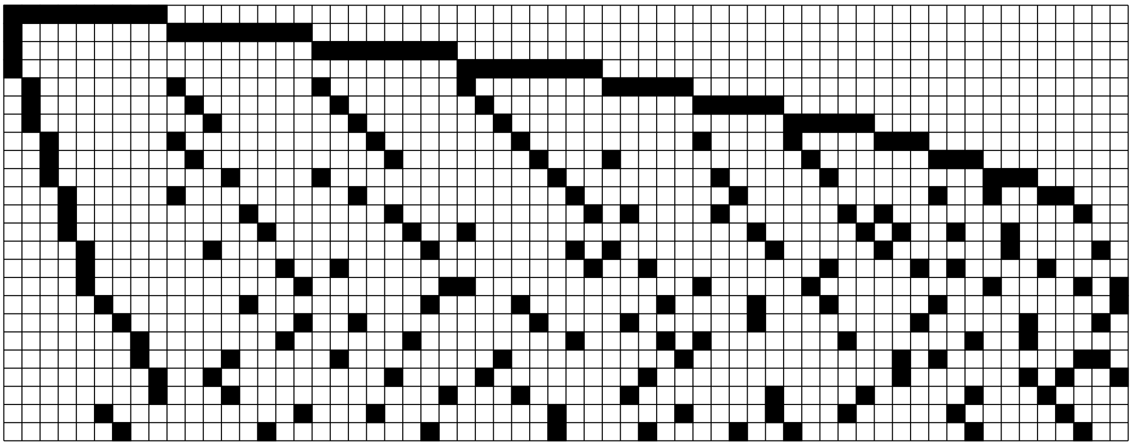


Figure 3.1: An incidence structure satisfying axioms (AU1)–(AU4), but not (AU5)

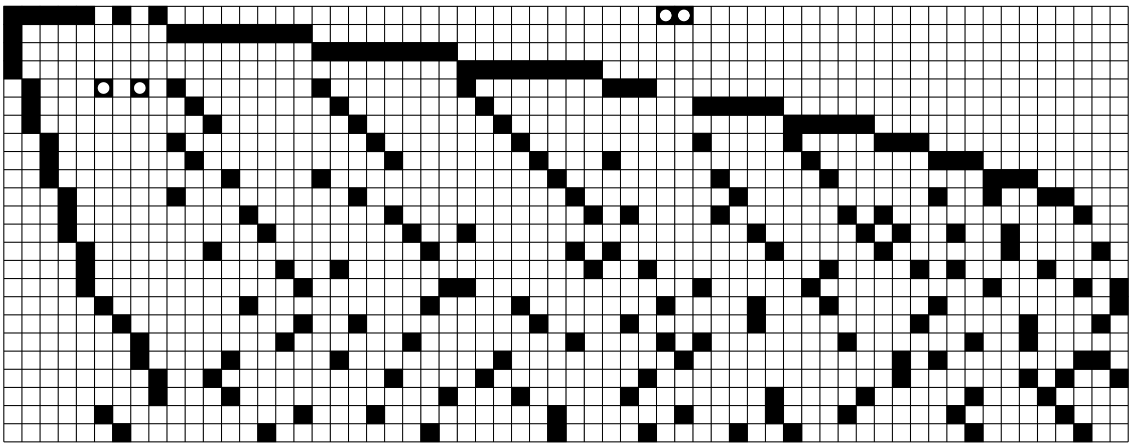


Figure 3.2: An affine part of a unital constructed by Betten, Betten and Tonchev

Note that though we must require the existence of a parallelism in the definition of an affine unital, this parallelism does not have to be unique. It is therefore not convenient to require that isomorphisms of affine unitals fix certain parallelisms and we will only ask them to be isomorphisms of the incidence structures.

It might still be useful to have a notion of two parallelisms being equivalent.

Definition 3.11. Let $\mathbb{U}$ be an affine unital and let π and π' be two parallelisms of $\mathbb{U}$. We call π and π' **equivalent** if there is an automorphism of $\mathbb{U}$ which maps π to π'.

There is a strong connection of two parallelisms being equivalent and the corresponding closures being isomorphic:

Proposition 3.12. *Let $\mathbb{U} = (\mathcal{P}, \mathcal{B}, I)$ be an affine unital of order n and let π and π' be equivalent parallelisms of $\mathbb{U}$. Every automorphism $\alpha \in \operatorname{Aut}(\mathbb{U})$ with $\pi \cdot \alpha = \pi'$ extends to a unique isomorphism $\tilde{\alpha}\colon \mathbb{U}^\pi \to \mathbb{U}^{\pi'}$ with $\tilde{\alpha}|_{\mathbb{U}} = \alpha$.*

Proof. Let $\pi := \{\mathcal{B}_1, \dots, \mathcal{B}_{n+1}\}$ and $\pi' := \{\mathcal{B}'_1, \dots, \mathcal{B}'_{n+1}\}$ with $\mathcal{B}_i \cdot \alpha = \mathcal{B}'_i$ $(1 \leq i \leq n+1)$ as subsets of the block set $\mathcal{B}$. We set

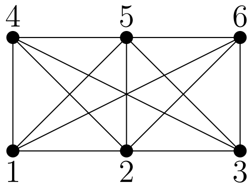
$$\tilde{\alpha}\colon \begin{cases} x \mapsto x \cdot \alpha, & x \in \mathcal{P}, \\ \mathcal{B}_i \mapsto \mathcal{B}'_i, & 1 \leq i \leq n+1, \\ B \mapsto B \cdot \alpha, & B \in \mathcal{B}, \\ [\infty]^\pi \mapsto [\infty]^{\pi'}. \end{cases}$$

Then $\tilde{\alpha}$ is obviously an isomorphism of unitals $\mathbb{U}^\pi \to \mathbb{U}^{\pi'}$ with $\tilde{\alpha}|_{\mathbb{U}} = \alpha$. It is the only way to extend α, since there is only one non-affine block and the points $\mathcal{B}_i$ and $\mathcal{B}'_i$, respectively, are the only non-affine points incident with the short blocks. □

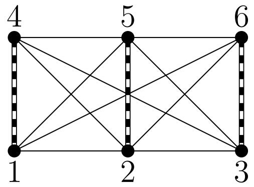
Since each automorphism of the π-closure of any affine unital $\mathbb{U}$ that stabilizes the block $[\infty]^\pi$ obviously induces an automorphism of $\mathbb{U}$ that fixes π, we get the following

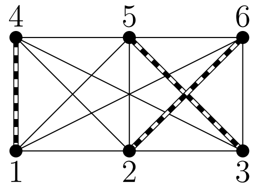
Corollary 3.13. *Let $\mathbb{U}$ be an affine unital with parallelism π. The automorphisms of the π-closure $\mathbb{U}^\pi$ fixing the block $[\infty]^\pi$ correspond exactly to the automorphisms of $\mathbb{U}$ fixing the parallelism π.* □

Example 3.14. An affine unital of order 2 has six points. Through each point go four blocks, three of which are short blocks, i. e. incident with two points. Hence, the two long blocks of the affine unital are non-intersecting and there is only one possible isomorphism type of affine unitals of order 2:

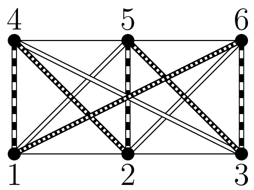


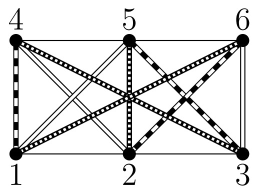
We still have to find a parallelism. First we notice that there are two possible parallel classes containing the block 14:





It is easily seen from the pictures that each of these parallel classes is contained in a unique parallelism:





It is also obvious that interchanging the points 2 and 3 yields an automorphism of the affine unital that interchanges the two parallelisms, which are thus equivalent.
Now we may complete the affine unital to a unital of order 2 by adding a block at infinity and three points at infinity – labeled with the parallel classes ▬▬, ═══ and ▬▬. Incidences are as in Proposition 3.9. Since the two parallelisms are equivalent, this is one way to see that there is only one isomorphism type of unitals of order 2.

3.2 SL$(2, q)$-Unitals

We are now ready to consider the construction of those unitals which we are mainly interested in, namely SL$(2, q)$-unitals. As mentioned in the introduction, the construction of those unitals is due to Grundhöfer, Stroppel and Van Maldeghem [9] and motivated by the action of the special unitary group of degree 2 on the classical unital. We choose a slightly different approach by using the notion of affine unitals.

3.2.1 Construction of $\mathrm{SL}(2,q)$-Unitals

Let $S \leq \mathrm{SL}(2,q)$ be a subgroup of order $q+1$ and let $T \leq \mathrm{SL}(2,q)$ be a Sylow p-subgroup. Recall that T has order q (and thus trivial intersection with S), that any two conjugates $T^h := h^{-1}Th$, $h \in \mathrm{SL}(2,q)$, have trivial intersection unless they coincide and that there are $q+1$ conjugates of T.
Consider a collection $\mathcal{D}$ of subsets of $\mathrm{SL}(2,q)$ such that each set $D \in \mathcal{D}$ contains $\mathbb{1}$, that $\#D = q+1$ for each $D \in \mathcal{D}$, and the following properties hold:

(Q) For each $D \in \mathcal{D}$, the set

$$D^* := \{xy^{-1} \mid x, y \in D,\ x \neq y\}$$

contains $q(q+1)$ elements, i. e. the map

$$(D \times D) \setminus \{(x,x) \mid x \in D\} \to \mathrm{SL}(2,q), \quad (x,y) \mapsto xy^{-1},$$

is injective.

(P) The system consisting of $S \setminus \{\mathbb{1}\}$, all conjugates of $T \setminus \{\mathbb{1}\}$ and all sets D^* with $D \in \mathcal{D}$ forms a partition of $\mathrm{SL}(2,q) \setminus \{\mathbb{1}\}$.

We will see in the example in Section 3.2.2 that for each prime power q there exists at least one such collection $\mathcal{D}$.

Set

$$\begin{aligned}\mathcal{P} &:= \mathrm{SL}(2,q),\\ \mathcal{B} &:= \{Sg \mid g \in \mathrm{SL}(2,q)\} \cup \{T^h g \mid h, g \in \mathrm{SL}(2,q)\} \cup \{Dg \mid D \in \mathcal{D}, g \in \mathrm{SL}(2,q)\}\end{aligned}$$

and let the incidence relation $I \subseteq \mathcal{P} \times \mathcal{B}$ be containment.

Then we call the incidence structure $\mathbb{U}_{S,\mathcal{D}} := (\mathcal{P}, \mathcal{B}, I)$ an **affine SL(2, q)-unital**.

Although the $D \in \mathcal{D}$ are not subgroups, we will call the sets Dg (right) cosets of D. We collect some properties of affine $\mathrm{SL}(2,q)$-unitals and show that they are indeed affine unitals of order q.

Proposition 3.15. *Let $\mathbb{U}_{S,\mathcal{D}}$ be an affine* $\mathrm{SL}(2,q)$*-unital. Then:*

(a) $\#\mathcal{D} = q-2$.

(b) *We denote* $\hat{D} \coloneqq \{Dd^{-1} \mid d \in D\}$*. For each* $E \in \hat{D}$*, the set*

$$\tilde{\mathcal{D}} \coloneqq (\mathcal{D} \setminus \{D\}) \cup \{E\}$$

satisfies all the properties required for an affine SL$(2,q)$*-unital and* $\mathbb{U}_{S,\mathcal{D}} = \mathbb{U}_{S,\tilde{\mathcal{D}}}$.

(c) *The blocks through* $\mathbb{1}$ *are exactly* S*, all conjugates of* T *and all elements of the* $\hat{D}$*,* $D \in \mathcal{D}$*.*

(d) SL$(2,q)$ *acts as a group of automorphisms on* $\mathbb{U}_{S,\mathcal{D}}$ *via multiplication from the right. The action is regular on the point set and we may identify each element* $x \in \mathrm{SL}(2,q)$ *with the point* $\mathbb{1} \cdot x$*.*

(e) $\mathbb{U}_{S,\mathcal{D}}$ *is an affine unital of order* q*.*

Proof. (a) The size of $\mathcal{D}$ can easily be computed using properties (P) and (Q) and the fact that the order of SL$(2,q)$ equals $(q-1)q(q+1)$.

(b) Let $E \in \hat{D}$. Then $\mathbb{1} \in E$, the set E contains $q+1$ elements and $E^* = D^*$. Since the sets of right cosets of D and E obviously coincide, the statement follows.

(c) It is clear that in the set of all right cosets of S and all right cosets of the Sylow p-subgroups exactly the groups themselves contain $\mathbb{1}$. Now let $D \in \mathcal{D}$ and $g \in \mathrm{SL}(2,q)$. Then $\mathbb{1} \in Dg$ exactly if $g^{-1} \in D$ and hence $Dg \in \hat{D}$.

(d) Clear from the definition of points and blocks.

(e) The number of points of $\mathbb{U}_{S,\mathcal{D}}$ equals $\#\,\mathrm{SL}(2,q) = (q-1)q(q+1)$. Each block of type Sg or Dg, respectively, is incident with $q+1$ points and each block $T^h g$ is incident with q points. For the number of blocks incident with a given point, we consider first the blocks through $\mathbb{1}$. Incident with $\mathbb{1}$ are

$$1 + (q+1) + (q-2)(q+1) = q^2$$

blocks. Because of the transitive action of SL$(2,q)$ on the points, there is the same number of blocks through each point. For the same reason, we must only show that each point different from $\mathbb{1}$ is joined uniquely to $\mathbb{1}$. But this is clear from the partition property (P). It remains to find a parallelism on the short blocks $T^h g$ with $q+1$ parallel classes of size q^2-1. One possibility is obviously given by partitioning the short blocks into the $q+1$ sets of right cosets (each of size $\frac{(q-1)q(q+1)}{q} = q^2-1$) of the Sylow p-subgroups. □

Definition 3.16. Let $\mathbb{U}_{S,\mathcal{D}}$ be an affine $\mathrm{SL}(2,q)$-unital.

(a) We call the sets $\hat{D} := \{Dd^{-1} \mid d \in D\}$, $D \in \mathcal{D}$, the **hats** of $\mathbb{U}_{S,\mathcal{D}}$.

(b) We call the blocks Dg, $D \in \mathcal{D}$ and $g \in \mathrm{SL}(2,q)$, the **arcuate blocks** of $\mathbb{U}_{S,\mathcal{D}}$.

The following considerations justify the name "arcuate blocks": Consider $\mathrm{SL}(2,q)$ as subset of the 4-dimensional affine space $\mathrm{AG}(4,q) \cong \mathbb{F}_q^{2\times 2}$. The elements of the Sylow p-subgroup $T := \{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \mid x \in \mathbb{F}_q\}$ are exactly the points of the affine line

$$\{\mathbb{1} + \lambda \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \mid \lambda \in \mathbb{F}_q\}$$

and the elements of each conjugate T^h, $h \in \mathrm{SL}(2,q)$, are exactly the points of the affine line $\{\mathbb{1} + \lambda \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)^h \mid \lambda \in \mathbb{F}_q\}$. Further, any block B of an affine $\mathrm{SL}(2,q)$-unital is given by the point set of a line in $\mathrm{AG}(4,q)$ exactly if this holds for each coset Bg, $g \in \mathrm{SL}(2,q)$. Indeed, in any affine $\mathrm{SL}(2,q)$-unital, the short blocks are the only blocks such that three points lie on the same line in $\mathrm{AG}(4,q)$, as shown in Proposition 3.18.

Lemma 3.17. *Let $g, h \in \mathrm{SL}(2,q) \setminus \{\mathbb{1}\}$ with $g \neq h$ and assume there exists $\lambda \in \mathbb{F}_q$ such that*

$$g - \mathbb{1} = \lambda(h - \mathbb{1}).$$

Then, $\mathrm{ord}(g) = \mathrm{ord}(h) = p$.

Proof. Let $g := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $h := \left(\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right) \in \mathrm{SL}(2,q)$ and let $\lambda \in \mathbb{F}_q$ such that $g - \mathbb{1} = \lambda(h - \mathbb{1})$. Since $\det g = \det h = 1$, we get

$$\begin{aligned} 0 &= ad - bc - 1 \\ &= (\lambda(w-1)+1)\cdot(\lambda(z-1)+1) - \lambda^2 xy - 1 \\ &= \lambda^2(wz - xy - w - z + 1) + \lambda(w - 1 + z - 1) + 1 - 1 \\ &= \lambda^2(2 - w - z) + \lambda(w + z - 2) \\ &= (\mathrm{tr}(h) - 2)\lambda(1-\lambda). \end{aligned}$$

If $\lambda = 0$, then $g = \mathbb{1}$ and if $\lambda = 1$, then $g = h$. Hence, $\mathrm{tr}(h) = 2$ and thus $\mathrm{ord}(h) = p$, see Lemma 2.9. Further, we get

$$\mathrm{tr}(g) = \mathrm{tr}(\lambda(h - \mathbb{1})) + \mathrm{tr}(\mathbb{1}) = \lambda(\mathrm{tr}(h) - 2) + 2 = 2$$

and hence $\mathrm{ord}(g) = p$. □

Proposition 3.18. *Let* $\mathbb{U} := \mathbb{U}_{S,\mathcal{D}}$ *be an affine* SL$(2, q)$*-unital and let* B *be a long block through* $\mathbb{1}$ *in* $\mathbb{U}$*. Let further* g*,* h *and* k *be three different points on* B*. Then there exists no* $\lambda \in \mathbb{F}_q$ *such that* $g - k = \lambda(h - k)$*.*

Proof. Let $\lambda \in \mathbb{F}_q$ and assume $g - k = \lambda(h - k)$. Then $gk^{-1} - \mathbb{1} = \lambda(hk^{-1} - \mathbb{1})$. Since g, h and k are three different points, we obtain with Lemma 3.17 that

$$\mathrm{ord}(gk^{-1}) = \mathrm{ord}(hk^{-1}) = p.$$

Assume $B = S$. Then $\mathrm{ord}(gk^{-1})$ divides $\#S = q + 1$ and is hence different from p. Thus, $B = D \in \mathcal{D}$. But then gk^{-1} is contained in $Dk^{-1} \in \hat{D}$ and $\mathrm{ord}(gk^{-1}) \neq p$ because of the partition property (P). □

Considered in the affine space AG$(4, q)$, the long blocks are hence not lines, but *partial ovoids*, in the sense that no three points are collinear. This justifies the name "arcuate". Note that the block S (and its cosets) will *not* be called arcuate because of the special role of S as the long block through $\mathbb{1}$ which is a subgroup of SL$(2, q)$.

We are now ready to define SL$(2, q)$-unitals as closures of affine SL$(2, q)$-unitals.

Definition 3.19. Let $\mathbb{U}_{S,\mathcal{D}}$ be an affine SL$(2, q)$-unital and let π be a parallelism of $\mathbb{U}_{S,\mathcal{D}}$. We call the π-closure of $\mathbb{U}_{S,\mathcal{D}}$ an **SL$(2, q)$-(π-)unital** and denote it by $\mathbb{U}^{\pi}_{S,\mathcal{D}}$.

Let $\mathfrak{P}$ be the set of Sylow p-subgroups of SL$(2, q)$. The short blocks through the point $\mathbb{1}$ in any affine SL$(2, q)$-unital $\mathbb{U}_{S,\mathcal{D}}$ are exactly the elements of $\mathfrak{P}$. Since they intersect in one point, any two of them are contained in different parallel classes, irrespective of the considered parallelism π. We may thus label the parallel classes in π – and hence also the points at infinity in any SL$(2, q)$-unital $\mathbb{U}^{\pi}_{S,\mathcal{D}}$ – by the Sylow p-subgroups of SL$(2, q)$.

There are two parallelisms of affine SL$(2, q)$-unitals to which we will pay more attention, since they exist for every order q and since they are preserved under right multiplication by SL$(2, q)$.
One possible such parallelism – the one mentioned in the proof of Proposition 3.15 – is the partition of the short blocks into the $q + 1$ sets of right cosets of the Sylow p-subgroups. Multiplication from the right by SL$(2, q)$ fixes each parallel class. We will call this parallelism "flat" and denote it by the musical sign ♭:

$$\flat := \{T \cdot \mathrm{SL}(2, q) \mid T \in \mathfrak{P}\}.$$

The notion "flat" indicates that this parallelism is, though the most obvious, not the one having the "nicest" properties. The parallelism which will turn out to be more "natural"

– and is hence denoted by the musical sign $\natural$ – is given by the partition of the short blocks into the $q+1$ sets of *left* cosets of the Sylow p-subgroups:

$$\natural := \{\mathrm{SL}(2,q) \cdot T \mid T \in \mathfrak{P}\}.$$

Each left coset gT of a Sylow p-subgroup T equals the right coset $T^{g^{-1}}g$ and is hence a (short) block in any affine $\mathrm{SL}(2,q)$-unital. Right multiplication with $h \in \mathrm{SL}(2,q)$ maps a left coset gT to the left coset ghT^h. The action of $\mathrm{SL}(2,q)$ is thus conjugation on the (labels of the) parallel classes of $\natural$.

The two parallelisms $\flat$ and $\natural$ are in fact the only two parallelisms in any affine $\mathrm{SL}(2,q)$-unital that are preserved under right multiplication by $\mathrm{SL}(2,q)$, as shown in Proposition 3.22. Recall first the following well-known lemma about normalizers of Sylow p-subgroups (see e. g. [14, p. 81]).

Lemma 3.20. *Let G be a finite group, P a Sylow p-subgroup of G and H a subgroup of order p^j contained in the normalizer* $\mathrm{N}(P)$*. Then $H \leq P$.* □

Lemma 3.21. *Any two different Sylow p-subgroups of* $\mathrm{SL}(2,q)$ *generate* $\mathrm{SL}(2,q)$*.*

Proof. Let first $T_1 := \{\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right) \mid x \in \mathbb{F}_q\}$ and $T_2 := \{\left(\begin{smallmatrix}1 & 0\\ y & 1\end{smallmatrix}\right) \mid y \in \mathbb{F}_q\}$. We show $\langle T_1 \cup T_2\rangle = \mathrm{SL}(2,q)$. Let $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}(2,q)$ and assume without restriction $c \neq 0$, for else we have $d \neq 0$ and consider $\left(\begin{smallmatrix}a+b & b\\ d & d\end{smallmatrix}\right) = \left(\begin{smallmatrix}a & b\\ 0 & d\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)$. Compute

$$\left(\begin{smallmatrix}1 & \frac{1}{c}(a-1)\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & 0\\ c & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & \frac{1}{c}(d-1)\\ 0 & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix}a & \frac{1}{c}(a-1)\\ c & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & \frac{1}{c}(d-1)\\ 0 & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix}a & \frac{1}{c}(ad-a+a-1)\\ c & d-1+1\end{smallmatrix}\right) = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right),$$

since $ad - bc = 1$. Thus, $\langle T_1 \cup T_2\rangle = \mathrm{SL}(2,q)$.
Now let $P, Q \in \mathfrak{P}$, $P \neq Q$, and consider the set $A := \{P^x \mid x \in Q\}$. Assume $P^x = P^y$ with $x, y \in Q$. Then $xy^{-1} \in Q \cap \mathrm{N}(P)$ and hence $x = y$, according to Lemma 3.20. Obviously, $Q \notin A$ and we get $A \cup \{Q\} = \mathfrak{P}$. Thus, for any two different Sylow p-subgroups P and Q of $\mathrm{SL}(2,q)$, we have

$$\mathrm{SL}(2,q) \geq \langle P \cup Q\rangle \geq \langle \textstyle\bigcup(A \cup \{Q\})\rangle = \langle \textstyle\bigcup \mathfrak{P}\rangle \geq \langle T_1 \cup T_2\rangle = \mathrm{SL}(2,q).$$ □

Proposition 3.22. *Let $\mathbb{U}_{S,\mathcal{D}}$ be an affine* $\mathrm{SL}(2,q)$*-unital with parallelism π such that right multiplication by* $\mathrm{SL}(2,q)$ *preserves π. Then $\pi \in \{\flat, \natural\}$.*

Proof. Let $T \in \mathfrak{P}$ and let $[T]$ denote the parallel class of π containing the short block T. Since right multiplication with $t \in T$ fixes the block T, it fixes also the parallel class $[T]$. We distinguish two cases.

Assume first that $[T]$ contains exactly the q^2-1 left cosets of T. Then, for each $h \in \mathrm{SL}(2,q)$,

$$[T] \cdot h = \{gT \mid g \in \mathrm{SL}(2,q)\} \cdot h = \{gTh \mid g \in \mathrm{SL}(2,q)\} = \{ghT^h \mid g \in \mathrm{SL}(2,q)\},$$

which equals the set of left cosets of T^h and must be a parallel class of π. Hence, each parallel class of π is the complete set of left cosets of some Sylow p-subgroup, meaning $\pi = \natural$.

Assume now that $[T]$ contains at least one short block $T^h g = gT^{hg}$ which is not a left coset of T, i. e. $T^{hg} \neq T$. Then, right multiplication with T as well as right multiplication with T^{hg} fixes a block in $[T]$ and hence also the parallel class $[T]$. Since $\langle T \cup T^{hg} \rangle = \mathrm{SL}(2,q)$, according to Lemma 3.21, right multiplication with $\mathrm{SL}(2,q)$ stabilizes $[T]$ and $[T]$ equals the set of right cosets of T. The same reasoning works for each parallel class and hence $\pi = \flat$. □

3.2.2 Example: The Classical Unital

We will now see that for each prime power $q = p^e$, the classical (Hermitian) unital of order q is an $\mathrm{SL}(2,q)$-unital. Since the construction of $\mathrm{SL}(2,q)$-unitals was inspired by the action of the special unitary group of degree 2 (isomorphic to $\mathrm{SL}(2,q)$) on the classical unital of order q (see [9]), this should not come as a big surprise.

Consider the Hermitian form

$$H \colon \mathbb{F}_{q^2}^3 \times \mathbb{F}_{q^2}^3 \to \mathbb{F}_{q^2},$$
$$((x_1,x_2,x_3),(y_1,y_2,y_3)) \mapsto x_1\overline{y_1} - x_2\overline{y_2} - x_3\overline{y_3},$$

where $\bar{\cdot} \colon \mathbb{F}_{q^2} \to \mathbb{F}_{q^2},\ x \mapsto \overline{x} := x^q$, denotes the involutory field automorphism with fixed field $\mathbb{F}_q$.

Then the **classical unital of order q** is defined[1] as $\mathbb{U} = (\mathcal{P}, \mathcal{B}, I)$ with

$$\mathcal{P} := \{[v] := \mathbb{F}_{q^2} \cdot v \mid v \in \mathbb{F}_{q^2}^3 \setminus \{0\} \text{ and } H(v,v) = 0\},$$

[1] In terms of projective geometry, the points of the unital are those points of the classical projective plane $\mathrm{PG}(2,q^2)$ that satisfy the equation

$$X^{q+1} - Y^{q+1} - Z^{q+1} = 0.$$

The blocks of the unital are the projective lines in $\mathrm{PG}(2,q^2)$ which contain more than one point of the unital. For a survey of unitals embedded in finite projective planes, see the monograph of Barwick and Ebert [2].

$$\mathcal{B} := \{W \le \mathbb{F}_{q^2}^3 \mid \dim W = 2 \text{ and } \#\{P \in \mathcal{P} \mid P \le W\} > 1\} \text{ and}$$
$$I := \{(P,B) \in \mathcal{P} \times \mathcal{B} \mid P \le B\}.$$

This incidence structure is indeed a unital, as was already shown by Bose in [4, Sections 4 and 5] or can also be looked up in [2, Chapter 2].

We fix the block $B := \langle (1,0,0),(0,1,0)\rangle$ and consider the affine part ${}^B\mathbb{U}$. (Note that $\#\{P \in \mathcal{P} \mid P \le B\} = q+1$ and hence $B \in \mathcal{B}$.) The point set of ${}^B\mathbb{U}$ is

$$\begin{aligned} {}^B\mathcal{P} &= \{[x,y,1] \mid x,y \in \mathbb{F}_{q^2},\ x\overline{x} - y\overline{y} - 1 = 0\} \\ &\cong \{(x,y) \in \mathbb{F}_{q^2}^2 \mid x\overline{x} - y\overline{y} = 1\}. \end{aligned}$$

Let h as in Chapter 2 denote the Hermitian form

$$\begin{aligned} h\colon \mathbb{F}_{q^2}^2 \times \mathbb{F}_{q^2}^2 &\to \mathbb{F}_{q^2}, \\ ((x_1,x_2),(y_1,y_2)) &\mapsto x_1\overline{y_1} - x_2\overline{y_2}, \end{aligned}$$

and $M := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ its Gram matrix. The special unitary group with respect to h is

$$\begin{aligned} \mathrm{SU}(h) &= \{\left(\begin{smallmatrix} x & y \\ \overline{y} & \overline{x} \end{smallmatrix}\right) \mid x,y \in \mathbb{F}_{q^2} \text{ and } x\overline{x} - y\overline{y} = 1\} \\ &\cong {}^B\mathcal{P}, \end{aligned}$$

where multiplication on ${}^B\mathcal{P}$ is given by $(x,y)\cdot(w,z) := (xw + y\overline{z}, xz + y\overline{w})$. In particular, ${}^B\mathcal{P}$ is closed under right multiplication by $\mathrm{SU}(h)$.

We fix $\mathbb{1} := (1,0) \in {}^B\mathcal{P}$ and describe the blocks through $\mathbb{1}$, i. e. the affine lines of $\mathbb{F}_{q^2}^2$ through $(1,0)$ which contain more than one point $P \in {}^B\mathcal{P}$:

1. The line $(1,0) + \mathbb{F}_{q^2}(0,1)$ meets ${}^B\mathcal{P}$ only in $\mathbb{1}$, it is no block of, but a tangent to the unital.

2. The intersection of $(1,0) + \mathbb{F}_{q^2}(1,0)$ with ${}^B\mathcal{P}$ is

$$C' := \{(x,0) \mid x\overline{x} = 1\}.$$

   Note that $C' \le \mathrm{SU}(h)$ equals the cyclic subgroup of order $q+1$ which is given in Remark 2.6 (a).

3. Let $s \in \mathbb{F}_{q^2},\ s\overline{s} = 1$. The intersection of $(1,0) + \mathbb{F}_{q^2}(1,s)$ with ${}^B\mathcal{P}$ is

$$T'_s := \{(1+x, xs) \mid x + \overline{x} = 0\}.$$

The sets T'_s, $s\overline{s}=1$, are exactly the $q+1$ Sylow p-subgroups of $\mathrm{SU}(h)$.

4. Let $t \in \mathbb{F}_{q^2}$, $t\overline{t} \notin \{0,1\}$. The intersection of $(1,0)+\mathbb{F}_{q^2}(1,t)$ with ${}^B\mathcal{P}$ is

$$H'_t \coloneqq \{(1+x, xt) \mid x+\overline{x} = (t\overline{t}-1)x\overline{x}\}.$$

Lemma 3.23. *The sets* H'_t, $t\overline{t} \notin \{0,1\}$, *satisfy property (Q) and*

$$\hat{H}'_t \coloneqq \{H'_t \cdot d^{-1} \mid d \in H'_t\} = \{H'_c \mid c\overline{c} = t\overline{t}\}.$$

Proof. Let $d \coloneqq (1+y, yt) \in H'_t \setminus \{(1,0)\}$ and $(1+x,xt) \in H'_t$. Compute

$$(1+x,xt)\cdot d^{-1} = (1+\frac{\overline{y}}{y}(y-x), t(x-y)),$$

using $d^{-1} = (1+\overline{y}, -yt)$ and $(1-t\overline{t}) = -(y+\overline{y})(y\overline{y})^{-1}$. Let $c \coloneqq -y\overline{y}^{-1}t$. Then $c\overline{c} = t\overline{t}$ and $(1+x,xt)\cdot d^{-1} \in H'_c$.
Now let $(1+x,xt),(1+w,wt) \in H'_t$ and $(1+y,yt),(1+z,zt) \in H'_t \setminus \{(1,0)\}$ and assume $(1+x,xt)\cdot(1+y,yt)^{-1} = (1+w,wt)\cdot(1+z,zt)^{-1}$. Then $x-y = w-z$ and $\overline{y}y^{-1} = \overline{z}z^{-1}$. With $\overline{y} = (t\overline{t}-1)y\overline{y}-y$ and $\overline{z} = (t\overline{t}-1)z\overline{z}-z$, we get $y=z$ and hence $x=w$. A similar computation holds for $y=0$.
Since $\#\{H'_c \mid c\overline{c} = t\overline{t}\} = q+1$ and $\#H'_c = q+1$ for all c with $c\overline{c} \notin \{0,1\}$, the lemma follows. □

Choose an arbitrary set $\mathcal{H}' = \{H'_1, \ldots, H'_{q-2}\}$ of representatives of the $\hat{H}'_t$. Then C', the groups T'_s and the sets $(H'_i)^*$, $i \in \{1,\ldots,q-2\}$, satisfy property (P), since they intersect only in $\mathbb{1}$ and have appropriate cardinalities. Right multiplication by $\mathrm{SU}(h)$ induces affine maps on $\mathbb{F}_{q^2}^2$ and fixes thus the block set of the affine classical unital ${}^B\mathbb{U}$.

In the classical unital $\mathbb{U}$, each block T'_s, $s\overline{s}=1$, meets the block B in the point $[1,s,0] \in \mathcal{P}$. Since

$$(a,b)\cdot(1,s) = (a+b\overline{s}, as+b) = (a+b\overline{s})(1,s)$$

for each $(a,b) \in \mathrm{SU}(h)$, each left coset of T'_s meets the block B in $[1,s,0]$ as well.

Let $\psi\colon\ \mathrm{SU}(h) \to \mathrm{SL}(2,q)$ be an isomorphism and set

$$C \coloneqq C'\cdot\psi \quad \text{and} \quad \mathcal{H} \coloneqq \{H_1,\ldots,H_{q-2}\} \text{ with } H_i \coloneqq H'_i\cdot\psi.$$

Then it follows from the above that $\mathbb{U}_{C,\mathcal{H}}$ is an affine $\mathrm{SL}(2,q)$-unital such that the corresponding affine unital in $\mathrm{SU}(h)$ coincides with the affine classical unital ${}^B\mathbb{U}$ and such that the corresponding unital to the $\natural$-closure $\mathbb{U}^{\natural}_{C,\mathcal{H}}$ coincides with the classical unital $\mathbb{U}$. Independent of the specific choice of ψ, we will call $\mathbb{U}_{C,\mathcal{H}}$ *the* **classical affine SL$(2,q)$-unital** and $\mathbb{U}^{\natural}_{C,\mathcal{H}}$ *the* **classical SL$(2,q)$-unital**. Note that since there is only one conjugacy class of cyclic subgroups of order $q+1$ in $\mathrm{SL}(2,q)$, we may choose ψ such that C equals the cyclic subgroup given in Remark 2.7.

In [9, Example 5.5], Grundhöfer, Stroppel and Van Maldeghem state that the unital $\mathbb{U}^{\flat}_{C,\mathcal{H}}$ coincides with a unital described by Grüning, that can be embedded in a Hall plane and in its dual, see [11].

There are hence at least two non-isomorphic $\mathrm{SL}(2,q)$-unitals $\mathbb{U}^{\pi}_{S,\mathcal{D}}$ for each prime power $q \geq 3$, namely the classical and the Grüning unital, which differ only in the parallelism π, i. e. in the incidences of the points at infinity with affine short blocks. Further variations could be obtained by changing the group S for $q \equiv 3 \mod 4$ or the set $\mathcal{D}$. Changing S or $\mathcal{D}$ does indeed not affect the parallelism π, so the choice of a pair $(S,\mathcal{D})$ such that $\mathbb{U}_{S,\mathcal{D}}$ is an affine $\mathrm{SL}(2,q)$-unital is independent of the choice of a parallelism on the short blocks to obtain an $\mathrm{SL}(2,q)$-unital $\mathbb{U}^{\pi}_{S,\mathcal{D}}$.

In [9], Grundhöfer, Stroppel and Van Maldeghem present an affine $\mathrm{SL}(2,4)$-unital $\mathbb{U}_{C,\mathcal{E}}$ which is not isomorphic to the classical affine $\mathrm{SL}(2,4)$-unital.

4 Automorphism Groups

We are interested in automorphisms of (affine) $\mathrm{SL}(2, q)$-unitals. Since any automorphism of affine $\mathrm{SL}(2, q)$-unitals maps short blocks to short blocks, we will first consider the incidence structure $\mathfrak{S}$ given by the points and short blocks of an affine $\mathrm{SL}(2, q)$-unital. Note that the short blocks are the same in every affine $\mathrm{SL}(2, q)$-unital, independent of the choice of S and the set of arcuate blocks $\mathcal{D}$.

4.1 Automorphisms of the Geometry of Short Blocks

There is a strong connection between $\mathfrak{S}$ and a hyperplane complement of the classical generalized quadrangle $\mathrm{Q}(4, q)$. We will first give the definition of a generalized quadrangle.

Definition 4.1. Let $s, t \in \mathbb{N}$. A **generalized quadrangle** $Q := (\mathcal{P}, \mathcal{B}, I)$ **of order** $\boldsymbol{(s, t)}$ is an incidence structure satisfying the following axioms:

(Q1) Each block is incident with $s + 1$ points.

(Q2) Each point is incident with $t + 1$ blocks.

(Q3) For any two points there is at most one block incident with both of them.

(Q4) For each non-incident pair of a point p and a block B, there is exactly one block through p that intersects B.

Following Payne and Thas [24, p. 1] and unlike Van Maldeghem [27, pp. 5 sqq.], we do not require a generalized quadrangle to be thick (meaning $s, t \geq 2$).

Let $\mathfrak{Q}$ be a quadric of Witt index 2 in the projective space $\mathrm{PG}(d, q)$, i. e. $\mathfrak{Q}$ contains lines, but no planes of $\mathrm{PG}(d, q)$. The points and lines (with the natural incidence relation) of $\mathfrak{Q}$ form a generalized quadrangle (see e. g. [27, Section 2.3]), which is usually called a **classical (orthogonal) quadrangle**. Over the finite field of order q, there are up to similitudes and change of basis only three non-degenerate quadratic forms

$$\mathbb{F}_q^{d+1} \to \mathbb{F}_q$$

of Witt index 2, one for each $d \in \{3,4,5\}$ (see e. g. [1, Section III.6]). There are hence for each q only three isomorphism types of projective quadrics of Witt index 2 in $\mathrm{PG}(d,q)$ and we may denote *the* classical orthogonal quadrangle in $\mathrm{PG}(d,q)$ (with $d \in \{3,4,5\}$) by $\mathrm{Q}(d,q)$.

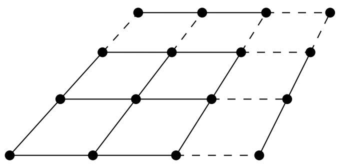
We are mainly interested in the classical generalized quadrangle $\mathrm{Q}(4,q)$. This quadrangle is given by the projective quadric with equation

$$x_1x_3 + x_2x_4 + x_5^2 = 0$$

in the projective space $\mathrm{PG}(4,q)$ and has order (q,q). The intersection of $\mathrm{Q}(4,q)$ with the hyperplane $x_5 = 0$ of $\mathrm{PG}(4,q)$ is a geometric hyperplane of the generalized quadrangle, i. e. each block of the quadrangle either lies completely in the hyperplane or intersects it in exactly one point. This geometric hyperplane is also a subquadrangle, isomorphic to the classical generalized quadrangle $\mathrm{Q}(3,q)$, which is given by the projective quadric with equation

$$x_1x_3 + x_2x_4 = 0$$

in the projective space $\mathrm{PG}(3,q)$. The generalized quadrangle $\mathrm{Q}(3,q)$ is a grid; it consists of $(q+1)^2$ points, each on 2 blocks, and each block is incident with $q+1$ points.



Note that every hyperplane in $\mathrm{Q}(4,q)$ with the structure of a $(q+1)\times(q+1)$-grid is given by a hyperbolic quadric in $\mathrm{PG}(3,q)$, since it is spanned by two non-intersecting blocks. Since any two hyperbolic quadratic forms in $\mathrm{PG}(3,q)$ are equivalent, any two such hyperplanes are isometric. Using Witt's Extension Theorem (see e. g. [28, Theorems 6.4 and 7.6]), we conclude that the full automorphism group of $\mathrm{Q}(4,q)$ acts transitively on the set of all hyperplanes isomorphic to $\mathrm{Q}(3,q)$.

Let $\mathfrak{P} = \{T_1,\dots,T_{q+1}\}$ be the set of Sylow p-subgroups of $G := \mathrm{SL}(2,q)$. Following Payne and Thas in [24, Section 10.7], we consider the incidence structure $Q_{\mathfrak{P}} := (\mathcal{P}_{\mathfrak{P}}, \mathcal{B}_{\mathfrak{P}}, I_{\mathfrak{P}})$, constructed as follows.

$\mathcal{P}_\mathfrak{P}$ consists of two different kinds of points:

(a) Elements of G.

(b) Right cosets of the normalizers $\mathrm{N}_G(T_i)$, $1 \le i \le q+1$.

Note that there are $(q-1)q(q+1)$ points of type (a) and $(q+1)^2$ of type (b), since the number of right cosets of each $\mathrm{N}_G(T_i)$ equals $q+1$.

$\mathcal{B}_\mathfrak{P}$ consists of three kinds of blocks:

(i) Right cosets of the groups T_i, $1 \le i \le q+1$.

(ii) Sets $M_i := \{\mathrm{N}_G(T_i)g \mid g \in G\}$, $1 \le i \le q+1$.

(iii) Sets $L_i := \{g\,\mathrm{N}_G(T_i) \mid g \in G\}$, $1 \le i \le q+1$.

Each block of type (i) is incident with the points of type (a) contained in it and with the point of type (b) containing it. The blocks of type (ii) and (iii) are incident with the points of type (b) contained in it. Note that each left coset $g\,\mathrm{N}_G(T_i)$ equals the right coset $(\mathrm{N}_G(T_i))^{g^{-1}}g = \mathrm{N}_G(T_i^{g^{-1}})g$.

The incidence structure $Q_\mathfrak{P}$ is a span-symmetric generalized quadrangle of order (q,q) (see [24, 10.7.8]) and hence isomorphic to $\mathrm{Q}(4,q)$, see [26, Theorem 1.1]. The restriction to the points of type (b) and the blocks of type (ii) and (iii) is a geometric hyperplane $H_\mathfrak{P}$ of $Q_\mathfrak{P}$ and a $(q+1)\times(q+1)$-grid with two block spreads $\{M_i \mid 1 \le i \le q+1\}$ and $\{L_i \mid 1 \le i \le q+1\}$.
Two different blocks T_ig and T_ih meet exactly (in a point of type (b)) if $gh^{-1} \in \mathrm{N}_G(T_i)$. The blocks of type (i) incident to a point $\mathrm{N}_G(T_i)$ of type (b) are thus exactly the right cosets of T_i which are left cosets of the same T_j. Hence, the blocks of type (i) meeting the block M_i are exactly the right cosets of T_i and the blocks of type (i) meeting the block L_i are exactly the left cosets of T_i.

Obviously, the hyperplane complement $Q_\mathfrak{P} \setminus H_\mathfrak{P}$ coincides with the incidence structure $\mathfrak{S}$ given by the points and short blocks of an affine $\mathrm{SL}(2,q)$-unital. The parallel classes of the parallelism $\natural$ coincide with the sets of blocks in $Q_\mathfrak{P}$ meeting the same block L_i and the parallel classes of the parallelism $\flat$ coincide with the sets of blocks in $Q_\mathfrak{P}$ meeting the same block M_i.

We are interested in the full automorphism group of the hyperplane complement $Q_\mathfrak{P} \setminus H_\mathfrak{P}$. Consider the quadratic form

$$f\colon \mathbb{F}_q^5 \to \mathbb{F}_q,\ x = (x_1,\dots,x_5) \mapsto x_1x_3 + x_2x_4 + x_5^2,$$

and the quadratic space $V := (\mathbb{F}_q^5, f)$. The automorphisms of $\mathrm{Q}(4,q)$ are exactly the semilinear projective similitudes of V, see e. g. [12, Theorem 8.1.5]. Hence,

$$\mathrm{Aut}(Q_{\mathfrak{P}}) \cong \mathrm{Aut}(\mathrm{Q}(4,q)) = \mathrm{P\Gamma O}(f) = \mathrm{PGO}(f) \rtimes \mathrm{Aut}(\mathbb{F}_q).$$

In any thick classical generalized quadrangle, the automorphisms of a hyperplane complement are exactly the automorphisms of the quadrangle stabilizing the hyperplane (see [23, Lemma 2.3]).
Let H be the hyperplane of $\mathrm{Q}(4,q)$ induced by the projective hyperplane $x_5 = 0$ of $\mathrm{PG}(4,q)$. Clearly H is stabilized by the automorphisms of $\mathrm{Q}(4,q)$ induced by field automorphisms. Let

$$M_f := \left(\begin{array}{cc|cc|c} \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{\mathbb{1}} & \begin{array}{c}0\\0\end{array} \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & \begin{array}{c}0\\0\end{array} \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

be a matrix describing f. The polar form of f is defined by

$$\mathbb{F}_q^5 \times \mathbb{F}_q^5 \to \mathbb{F}_q,\ (x,y) \mapsto f(x+y) - f(x) - f(y),$$

and has Gram matrix $M := M_f + M_f^\top$. Let $[A] \in \mathrm{PGO}(f)$, i. e. $AMA^\top = \mu_A M$ with $\mu_A \in \mathbb{F}_q$, and assume $[A]$ stabilizes H. Then $[A]$ stabilizes also the orthogonal complement of H with respect to the polar form of f and hence A is of the form

$$A = \left(\begin{array}{c|c} X & 0 \\ \hline \mathbb{0} & a \end{array}\right)$$

with $X \in \mathbb{F}_q^{4\times 4}$ and $a \in \mathbb{F}_q$. The factor of similitude for A is $\mu_A = a^2$. Choose the representative $\tilde{A} := a^{-1}A \in [A]$. Then

$$\tilde{A} = \left(\begin{array}{c|c} \tilde{X} & 0 \\ \hline \mathbb{0} & 1 \end{array}\right)$$

with $\tilde{X}{=}a^{-1}X$ and hence $\tilde{X} \in \mathrm{O}(\tilde{f})$, where $\tilde{f}$ is the quadratic form

$$\tilde{f} \colon \mathbb{F}_q^4 \to \mathbb{F}_q,\ x = (x_1, \ldots, x_4) \mapsto x_1x_3 + x_2x_4,$$

and $\mathrm{O}(\tilde{f}) = \{M \in \mathrm{GO}(\tilde{f}) \mid \mu_M = 1\}$ is the orthogonal group. We thus have

$$\mathrm{Aut}(\mathrm{Q}(4,q))_H \cong \mathrm{O}(\tilde{f}) \rtimes \mathrm{Aut}(\mathbb{F}_q).$$

Since $\mathrm{Aut}(\mathrm{Q}(4,q))$ acts transitively on the set of hyperplanes isomorphic to $\mathrm{Q}(3,q)$, we have

$$\mathrm{Aut}(\mathfrak{S}) = \mathrm{Aut}(Q_{\mathfrak{P}} \setminus H_{\mathfrak{P}}) \cong \mathrm{Aut}(\mathrm{Q}(4,q))_H \cong \mathrm{O}(\tilde{f}) \rtimes \mathrm{Aut}(\mathbb{F}_q).$$

We use this knowledge of the isomorphism type of $\mathrm{Aut}(\mathfrak{S})$ to specify the action of $\mathrm{Aut}(\mathfrak{S})$ on the incidence structure $\mathfrak{S}$. We already know that $\mathrm{SL}(2,q)$ acts via right multiplication as a group of automorphisms on $\mathfrak{S}$. Applying automorphisms of $\mathrm{SL}(2,q)$ to the point set of $\mathfrak{S}$ also induces automorphisms of $\mathfrak{S}$ and so does inversion, since the set of short blocks of any affine $\mathrm{SL}(2,q)$-unital is given by the set of *all* right cosets of the Sylow p-subgroups of $\mathrm{SL}(2,q)$, and every left coset of a Sylow p-subgroup T is a right coset of a conjugate of T.

Let $R = \{\rho_h \mid h \in \mathrm{SL}(2,q)\} \leq \mathrm{Aut}(\mathfrak{S})$ denote the subgroup given by right multiplication of $\mathrm{SL}(2,q)$, where $\rho_h \in R$ acts on $\mathfrak{S}$ by right multiplication with $h \in \mathrm{SL}(2,q)$. Let further $\mathfrak{A} \leq \mathrm{Aut}(\mathfrak{S})$ denote the subgroup given by automorphisms of $\mathrm{SL}(2,q)$ and $I \leq \mathrm{Aut}(\mathfrak{S})$ the cyclic subgroup of order 2 given by inversion. For those automorphisms $\alpha \in \mathfrak{A}$ which are given by conjugation with $a \in \mathrm{GL}(2,q)$, we write $\alpha = \gamma_a$.

Lemma 4.2. $\mathrm{Aut}(\mathfrak{S}) = (\mathfrak{A} \times I) \cdot R$.

Proof. From the above, the product $\mathfrak{A} \cdot I \cdot R$ is a subset of $\mathrm{Aut}(\mathfrak{S})$. The product $\mathfrak{A} \cdot I$ is direct (and in particular a subgroup of $\mathrm{Aut}(\mathfrak{S})$), since $\mathfrak{A}$ and I commute and have trivial intersection. The group R has trivial intersection with $\mathfrak{A} \times I$, since $\mathfrak{A}$ and I fix the point $\mathbb{1}$, while all non-trivial automorphisms in R have no fixed point. Hence we may compute

$$\begin{aligned}
\#((\mathfrak{A} \times I) \cdot R) &= \#\mathfrak{A} \cdot \#I \cdot \#R \\
&= \#\,\mathrm{P\Gamma L}(2,q) \cdot 2 \cdot \#\,\mathrm{SL}(2,q) \\
&= e(q-1)q(q+1) \cdot 2 \cdot (q-1)q(q+1) \\
&= 2e(q-1)^2 q^2 (q+1)^2.
\end{aligned}$$

From the above as well, we know $\mathrm{Aut}(\mathfrak{S})$ to be isomorphic to $\mathrm{O}(\tilde{f}) \ltimes \mathrm{Aut}(\mathbb{F}_q)$. But $\#\mathrm{O}(\tilde{f}) = 2(q-1)^2 q^2 (q+1)^2$, see [28, Theorems 6.21 and 7.23 with $\nu = 2$ and $\delta = 0$], and the lemma follows. $\square$

Lemma 4.3. $\mathfrak{A}$ *normalizes* R *in the full automorphism group of* $\mathfrak{S}$.

Proof. Let $\alpha \in \mathfrak{A}$, $\rho_h \in R$ and let $x \in \mathrm{SL}(2, q)$ be a point of $\mathfrak{S}$. Then

$$x \cdot (\alpha^{-1}\rho_h\alpha) = ((x \cdot \alpha^{-1})h) \cdot \alpha = x(h \cdot \alpha) = x \cdot \rho_{h\cdot\alpha}. \qquad \square$$

Since $\mathfrak{A}$ and R have trivial intersection, we thus know the product of $\mathfrak{A}$ and R to be semidirect.

4.2 Automorphisms of (Affine) $\mathrm{SL}(2, q)$-Unitals

We use the knowledge of the full automorphism group of the incidence structure $\mathfrak{S}$ of the short blocks to compute automorphism groups of (affine) $\mathrm{SL}(2, q)$-unitals. At first, we show that inversion can never be involved in an isomorphism between affine $\mathrm{SL}(2, q)$-unitals.

Theorem 4.4. *Let* $\psi\colon \mathbb{U}_{S,\mathcal{D}} \to \mathbb{U}_{S',\mathcal{D}'}$ *be an isomorphism of affine* $\mathrm{SL}(2, q)$*-unitals. Then* $\psi = \alpha\rho_h$ *with* $\rho_h \in R$ *and* $\alpha \in \mathfrak{A}$ *such that* $S \cdot \alpha = S'$.

Proof. An isomorphism of affine $\mathrm{SL}(2, q)$-unitals maps short blocks to short blocks and is hence an automorphism of $\mathfrak{S}$, since the incidence structure of the short blocks is the same in every affine $\mathrm{SL}(2, q)$-unital. We thus know $\psi \in \mathrm{Aut}(\mathfrak{S}) = (\mathfrak{A} \times I) \cdot R$. Let $\psi = \alpha i \rho_h$ with $i \in I$ inversion or identity. Then

$$S \cdot \psi = S \cdot (\alpha i \rho_h) = (S \cdot \alpha)h,$$

since S is a group and α an automorphism of $\mathrm{SL}(2, q)$. Since right multiplication by $\mathrm{SL}(2, q)$ induces automorphisms in every affine $\mathrm{SL}(2, q)$-unital, $S \cdot \alpha$ is a block in $\mathbb{U}_{S',\mathcal{D}'}$ that contains $\mathbb{1}$ and is a subgroup of $\mathrm{SL}(2, q)$ of order $q + 1$. The only block with these properties is S' and hence $S \cdot \alpha = S'$. Assume i to be inversion and choose $g \in \mathrm{SL}(2, q)$ with $g \cdot \alpha \notin \mathrm{N}(S')$. Then

$$(Sg) \cdot \psi = (Sg) \cdot (\alpha i \rho_h) = ((S \cdot \alpha)(g \cdot \alpha))^{-1}h = (S')^{g\cdot\alpha}(g \cdot \alpha)^{-1}h.$$

Sg is a block of $\mathbb{U}_{S,\mathcal{D}}$ and hence $(S')^{g\cdot\alpha}$ is a block of $\mathbb{U}_{S',\mathcal{D}'}$ with $\mathbb{1}$ in $(S')^{g\cdot\alpha}$ and $(S')^{g\cdot\alpha} \leq \mathrm{SL}(2, q)$ is a subgroup of order $q + 1$. But $(S')^{g\cdot\alpha} \neq S'$ because of the choice of g, a contradiction. $\square$

Lemma 4.3 and Theorem 4.4 yield:

Corollary 4.5. *Let* $\mathbb{U}_{S,\mathcal{D}}$ *be an affine* SL$(2,q)$*-unital. Then*

$$\mathrm{Aut}(\mathbb{U}_{S,\mathcal{D}}) \leq \mathfrak{A}_S \ltimes R. \qquad \square$$

Proposition 4.6. $\mathrm{Aut}(\mathbb{U}_{C,\mathcal{H}}) = \mathfrak{A}_C \ltimes R$*.*

Proof. We use the description of the classical affine SL$(2,q)$-unital in $\mathrm{SU}(h) \cong \mathrm{SL}(2,q)$, as given in Section 3.2.2. According to the proof of Theorem 2.11, $\mathrm{Aut}(\mathrm{SU}(h))_{C'} = \langle U, \varphi \rangle$ with

$$U := \{\gamma_u \mid u = \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right) \text{ with } a \in \mathbb{F}_{q^2}, a\overline{a} = 1\}$$

and $\varphi\colon \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}$ the Frobenius automorphism of order $2e$. It remains to show that $\langle U, \varphi \rangle$ fixes the set $\mathcal{H}'$. But an easy computation shows that

$$H'_t \cdot \varphi = H'_{t^p} \text{ and } \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)^{-1} H'_t \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right) = H'_{at}$$

with $\hat{H'}_{at} = \hat{H}'_t$. $\square$

We have already seen that right multiplication by SL$(2,q)$ preserves each of the two parallelisms $\flat$ and $\natural$. Since those two parallelisms consist of the sets of right resp. left cosets of the Sylow p-subgroups, they are obviously also preserved under automorphisms of SL$(2,q)$. With Corollary 3.13, we get the following

Corollary 4.7. *Let* $\pi \in \{\flat, \natural\}$ *and let* $\mathbb{U}_{S,\mathcal{D}}$ *be an affine* SL$(2,q)$*-unital. Then*

$$\mathrm{Aut}(\mathbb{U}^{\pi}_{S,\mathcal{D}})_{[\infty]} = \mathrm{Aut}(\mathbb{U}_{S,\mathcal{D}})$$

and in particular $\mathrm{Aut}(\mathbb{U}^{\pi}_{C,\mathcal{H}})_{[\infty]} = \mathfrak{A}_C \ltimes R$*.* $\square$

We will now have a closer look at SL$(2,q)$-unitals with parallelism $\flat$ or $\natural$.

Definition 4.8. A **translation** with **center** c of a unital $\mathbb{U}$ is an automorphism of $\mathbb{U}$ that fixes the point c and each block through c.
We call c a **translation center** if the group of all translations with center c acts transitively on the set of points different from c on any block through c.

Lemma 4.9. *Let* $\mathbb{U}_{S,\mathcal{D}}$ *be an affine* SL$(2,q)$*-unital. The Sylow* p*-subgroups of* SL$(2,q)$ *act via right multiplication as translation groups on* $\mathbb{U}^{\natural}_{S,\mathcal{D}}$ *with translation centers on* $[\infty]$*. For each* $T \in \mathfrak{P}$*, the group* $R_T := \{\rho_t \mid t \in T\}$ *is the full group of translations with center* T*.*

Proof. Each group R_T, $T \in \mathfrak{P}$, acts as a group of translations with center $T \in [\infty]$ on $\mathbb{U}^\natural_{S,\mathcal{D}}$, since R_T fixes each block $T^{g^{-1}}g = gT$ through the point T. The point T is a translation center, since R_T acts transitively on the points different from T on each block gT. Now the group $G_{[c]}$ of all translations with center c of a unital $\mathbb{U}$ acts semiregularly on $\mathbb{U} \setminus \{c\}$, see [10, Theorem 1.3]. Thus, if c is a translation center, then $G_{[c]}$ acts regularly on the set of points different from c on any block through c and the order of $G_{[c]}$ equals the order of $\mathbb{U}$. □

We use this statement on translations and a theorem of Grundhöfer, Stroppel and Van Maldeghem to show that the block $[\infty]$ is fixed by every automorphism in any non-classical SL$(2,q)$-$\natural$-unital.

Proposition 4.10. *Let* $\mathbb{U}^\natural_{S,\mathcal{D}}$ *be a non-classical* SL$(2,q)$-$\natural$-*unital. Then*

$$\mathrm{Aut}(\mathbb{U}^\natural_{S,\mathcal{D}}) = \mathrm{Aut}(\mathbb{U}^\natural_{S,\mathcal{D}})_{[\infty]}.$$

Proof. Assume that there is an automorphism of $\mathbb{U}^\natural_{S,\mathcal{D}}$ not fixing $[\infty]$. Since all points on $[\infty]$ are translation centers (see Lemma 4.9), there are thus three non-collinear translation centers of $\mathbb{U}_{S,\mathcal{D}}$. Then $\mathbb{U}_{S,\mathcal{D}}$ is the classical unital, as is shown in [10]. □

Grüning showed that in each Grüning unital $\mathbb{U}^\flat_{C,\mathcal{H}}$, the block $[\infty]$ is fixed by every automorphism (see [11, Lemma 5.5]). We will show this for every SL$(2,q)$-$\flat$-unital $\mathbb{U}^\flat_{S,\mathcal{D}}$ of order $q \geq 3$, independent of the group S and the set $\mathcal{D}$. Note that there is only one isomorphism type of unitals of order 2 (see e. g. Example 3.14). These are represented by the classical unital and admit hence a 2-transitive automorphism group, which implies that no block is fixed by the full automorphism group.

Definition 4.11. Let G be a group. The **commutator series** $(G^{(n)})_{n\in\mathbb{N}_0}$ of G is defined by

$$G^{(0)} := G, \quad G^{(k)} := [G^{(k-1)}, G^{(k-1)}] \text{ for } k \geq 1.$$

The **stable commutator** $\bigcap_{n\in\mathbb{N}_0} G^{(n)}$ will be denoted by $G^{(\omega)}$.

Theorem 4.12. *Let* $\mathbb{U}^\flat_{S,\mathcal{D}}$ *be an* SL$(2,q)$-$\flat$-*unital of order* $q \geq 3$. *Then every automorphism of* $\mathbb{U}^\flat_{S,\mathcal{D}}$ *fixes the block* $[\infty]$.

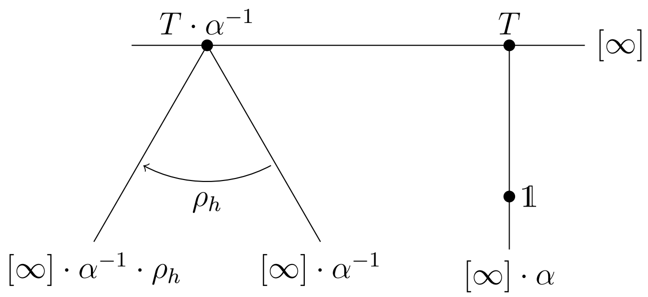
Proof. We exclude the small cases $q = 3$ and $q = 4$ first: For $q = 3$, there is only one isomorphism type of affine SL$(2,q)$-unitals (see [9, Theorem 3.3]), namely the classical affine SL$(2,q)$-unital. Hence, its $\flat$-closure is the Grüning unital and the theorem holds.

For $q = 4$, an exhaustive computer search using GAP showed that there are only two isomorphism types of affine $\mathrm{SL}(2,q)$-unitals (see Chapter 6) and none of their $\flat$-closures admits automorphisms which move the block $[\infty]$.

Recall that right multiplication by $\mathrm{SL}(2,q)$ fixes each parallel class of the parallelism $\flat$ and that hence the group $R \leq \mathrm{Aut}(\mathbb{U}^\flat_{S,\mathcal{D}})_{[\infty]}$ fixes each point on $[\infty]$.
Assume existence of an automorphism $\alpha \in \mathrm{Aut}(\mathbb{U}^\flat_{S,\mathcal{D}})$ with $[\infty] \cdot \alpha \neq [\infty]$. Since R acts transitively on the affine part of $\mathbb{U}^\flat_{S,\mathcal{D}}$, assume without restriction $\mathbb{1} \in [\infty] \cdot \alpha$. We distinguish two cases.

Case 1: The block $[\infty] \cdot \alpha$ intersects $[\infty]$, i. e. $[\infty] \cdot \alpha = T$ for $T \leq \mathrm{SL}(2,q)$ a Sylow p-subgroup.

Assume first that α does not fix T as a point. Then $[\infty] \cdot \alpha^{-1}$ meets $[\infty]$ in the point $T \cdot \alpha^{-1} \neq T$. Choose $h \in \mathrm{SL}(2,q)$ such that $([\infty] \cdot \alpha^{-1})h \neq [\infty] \cdot \alpha^{-1}$. Then the automorphism $\alpha^{-1}\rho_h\alpha$ moves $[\infty]$ and we know $T \cdot (\alpha^{-1}\rho_h\alpha) = T$ as a point, since $T \cdot \alpha^{-1} \in [\infty]$.



We may thus assume without restriction that α fixes T as a point. The group

$$R_T := \{\rho_t \in R \mid t \in T\} \leq R$$

acts regularly on $\mathcal{P}_T \setminus \{T\}$ and trivially on $[\infty]$. Hence, the group of automorphisms $\alpha R_T \alpha^{-1}$ acts regularly on $[\infty] \setminus \{T\}$ and trivially on $[\infty] \cdot \alpha^{-1}$. In particular, an affine point is fixed and we have

$$q = \#\alpha R_T \alpha^{-1} \mid \#\,\mathrm{Aut}(\mathbb{U}^\flat_{S,\mathcal{D}})_{[\infty],\mathbb{1}} \mid \#\,\mathrm{Aut}(\mathrm{SL}(2,q))_S.$$

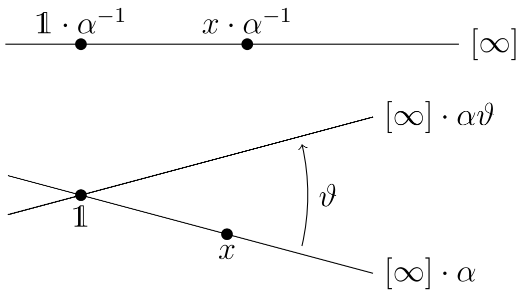
According to Theorem 2.11, this implies $q \mid 2e(q+1)$ or $q \mid e(q+1)$, since $24 = \#S_4$ is not divided by 7, 23 or 47, respectively. Now $(q, q+1) = 1$ and $q = p^e > e$, as can easily be shown by induction on e. Hence, $q = p^e \mid 2e$. Since $p^e > e$, this implies $p = 2$ and $2^{e-1} \mid e$. Again, induction shows that $2^{e-1} > e$ if $e \geq 3$. It remains $q \in \{2, 4\}$, but we are only interested in $q \geq 3$ and have already excluded the case $q = 4$.

Case 2: The block $[\infty]\cdot\beta$ does not meet $[\infty]$ for any automorphism $\beta \in \mathrm{Aut}(\mathbb{U}^\flat_{S,\mathcal{D}})$ with $[\infty]\cdot\beta \neq [\infty]$.

Recall the assumption $\mathbb{1} \in [\infty]\cdot\alpha$. Since $[\infty]\cdot\alpha$ does not intersect $[\infty]$, we know $[\infty]\cdot\alpha \neq T$ for any Sylow p-subgroup $T \leq \mathrm{SL}(2,q)$.
We show that no two blocks in the orbit $[\infty]\cdot\mathrm{Aut}(\mathbb{U}^\flat_{S,\mathcal{D}})$ may intersect in one point: Let $\vartheta \in \mathrm{Aut}(\mathbb{U}^\flat_{S,\mathcal{D}})$ such that $[\infty]\cdot\alpha \neq [\infty]\cdot\alpha\vartheta$ and assume without restriction that the two blocks $[\infty]\cdot\alpha$ and $[\infty]\cdot\alpha\vartheta$ intersect in the point $\mathbb{1}$. Then the automorphism $\alpha\vartheta\alpha^{-1}$ moves the block $[\infty]$. Let $x \coloneqq \mathbb{1}\cdot\vartheta^{-1} \in [\infty]\cdot\alpha$. Note that, other than indicated in the picture, x does not have to be different from $\mathbb{1}$. Then $(x\cdot\alpha^{-1}) \in [\infty]$ and

$$(x\cdot\alpha^{-1})\cdot(\alpha\vartheta\alpha^{-1}) = \mathbb{1}\cdot\alpha^{-1} \in [\infty].$$

Hence, the automorphism $\alpha\vartheta\alpha^{-1}$ moves the block $[\infty]$ to a block intersecting $[\infty]$, a contradiction to the requirement.



For any arcuate block D through $\mathbb{1}$ and $\mathbb{1} \neq d \in D$, it holds that Dd^{-1} is a block different from D such that D and Dd^{-1} intersect in the point $\mathbb{1}$. Since right multiplication by d^{-1} is an automorphism of $\mathbb{U}^\flat_{S,\mathcal{D}}$, the block $[\infty]\cdot\alpha$ may thus be none of the arcuate blocks. Hence, $[\infty]\cdot\alpha = S$. Since $R \leq \mathrm{Aut}(\mathbb{U}^\flat_{S,\mathcal{D}})$ and every affine point is contained in a right coset of S, we have

$$\mathcal{O}_{[\infty]} \coloneqq [\infty]\cdot\mathrm{Aut}(\mathbb{U}^\flat_{S,\mathcal{D}}) = \{Sg \mid g \in \mathrm{SL}(2,q)\} \cup \{[\infty]\}.$$

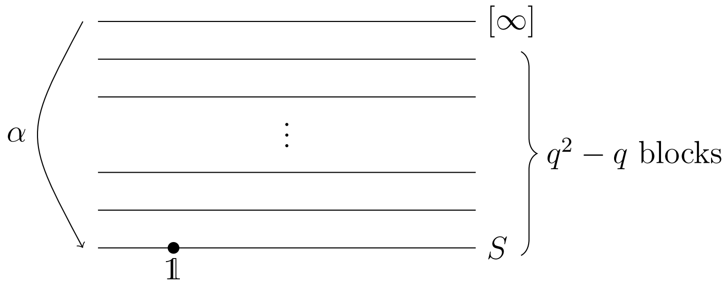


Table 4.1: Degrees of affine 2-transitive groups

Degree	$\boldsymbol{G_0}$	**Condition**
f^d	$\mathrm{SL}(d,f) \leq G_0 \leq \Gamma\mathrm{L}(d,f)$	
f^{2d}		
f^6		
f		$f = 5^2, 7^2, 11^2, 23^2, 3^4, 11^2, 19^2, 29^2, 59^2, 2^4, 2^6, 3^6$

Table 4.2: Degrees of almost simple 2-transitive groups

Degree	$\boldsymbol{N}$	**Condition**
n	A_n	$n \geq 5$
$(f^d-1)/(f-1)$	$\mathrm{PSL}(d,f)$	$d \geq 2$, $(d,f) \neq (2,2),(2,3)$
$2^{2d-1} \pm 2^{d-1}$		$d \geq 3$
f^3+1		
f^2+1		
f		$f = 11, 12, 15, 22, 23, 24, 28, 176, 276$

The group of automorphisms R fixes $[\infty]$ and acts transitively on the q^2-q right cosets of S and hence $\mathrm{Aut}(\mathbb{U}^\flat_{S,\mathcal{D}})$ acts 2-transitively on $\mathcal{O}_{[\infty]}$. Since

$$\#\mathcal{O}_{[\infty]} = q^2 - q + 1,$$

the group $\mathrm{Aut}(\mathbb{U}^\flat_{S,\mathcal{D}})$ must be a 2-transitive group on q^2-q+1 elements and contain $R \cong \mathrm{SL}(2,q)$. We use the classification of finite 2-transitive groups to reach a contradiction.

There are several sources for the classification of finite 2-transitive groups, of which the most convenient one seems to be two lists by Cameron, see [5, Tables 7.3 and 7.4]. In the majority of cases, the degree of the action already yields a contradiction, why we only copied the information needed in Tables 4.1 and 4.2. In both tables, f is a prime power. In the list of affine 2-transitive groups, G_0 denotes the stabilizer of one point in the 2-transitive action. In the list of almost simple 2-transitive groups, N denotes the minimal normal subgroup of the 2-transitive group G. Indeed, there is one action of degree $9^2 = 3^4$ missing in Cameron's list of affine 2-transitive groups, where $\mathrm{SL}(2,5)$ is a normal subgroup of G_0, see e. g. Liebeck's proof of Hering's classification of affine 2-transitive groups [18, Appendix 1]. But since we are only interested in the degree in this case, the missing action will not bother us.

Let $\Gamma_{[\infty]} = \mathrm{Aut}(\mathbb{U}^{\flat}_{S,\mathcal{D}})_{[\infty]}$ denote the stabilizer of the point $[\infty]$ in our required 2-transitive action. Then

$$\mathrm{SL}(2,q) \cong R \leq \Gamma_{[\infty]} \leq \mathfrak{A}_S \ltimes R \cong \mathrm{Aut}(\mathrm{SL}(2,q))_S \ltimes \mathrm{SL}(2,q).$$

In particular, we know that the stable commutator $\Gamma^{(\omega)}_{[\infty]}$ equals $\mathrm{SL}(2,q)$ for $q \geq 4$, since $\mathrm{Aut}(\mathrm{SL}(2,q))_S$ is solvable for each choice of S, see Corollary 2.13. In the first affine 2-transitive action in Table 4.1, the stable commutator $G_0^{(\omega)}$ equals the stable commutator of $\mathrm{SL}(d,f)$, which is trivial or $\mathrm{SL}(d,f)$. Since all isomorphisms between finite (projective) special linear groups are known (see e. g. [13, Satz 6.14]), we conclude $d = 2$ and $f = q$. But this is not possible because of our required degree $q^2 - q + 1 \neq q^2$ and we may thus exclude the first affine 2-transitive action.
Every other affine 2-transitive action is indeed quite easy to exclude, because the degree is always a square and our degree $q^2 - q + 1$ is never a square, since

$$(q-1)^2 < q^2 - q + 1 < q^2.$$

Now we look at the list of almost simple 2-transitive groups. The point stabilizer of the action of A_n is A_{n-1}, which is far bigger than $\Gamma_{[\infty]}$ for $n = q^2 - q + 1$.
The second entry in Table 4.2 is the action on the points or hyperplanes of a projective space. For $d \geq 3$ and $(d-1,f) \notin \{(2,2),(2,3)\}$, the stable commutator of the point stabilizer of this action is the special affine group

$$\mathrm{ASL}(d-1,f) = \mathrm{SL}(d-1,f) \ltimes \mathbb{F}_f^{d-1}.$$

Hence, $\mathrm{ASL}(d-1,f) \cong \mathrm{SL}(2,q)$ $(q \geq 4)$. But $\mathrm{ASL}(d-1,f)$ contains the normal subgroup $\mathbb{F}_f^{d-1}$ of order $f^{d-1} > 4$, while the highest possible order of a normal subgroup of $\mathrm{SL}(2,q)$ is 2 for $q \geq 4$. For $(d-1,f) \in \{(2,2),(2,3)\}$ the degree of the action would be 7 resp. 13 and hence $q = 3$ or $q = 4$, which we already excluded in the beginning. It remains the case $d = 2$. Then the degree is $f + 1$ and hence $f = q^2 - q = q(q-1)$. But since q and $q - 1$ have gcd 1, the product $q(q-1)$ cannot be a prime power. Hence we may exclude the second entry in the list of almost simple 2-transitive actions.

For the remaining cases, considering the degree will suffice. At first we notice that q^2-q+1 is always odd, which excludes the degrees $2^{2d-1} \pm 2^{d-1}$ $(d \geq 3)$ and $12, 22, 24, 28, 176, 276$. Then we just showed that $q^2 - q$ cannot be a prime power f^d. Finally, it is easy to see that $q^2 - q = q(q-1) \notin \{10, 14, 22\}$ for any prime power q.

We thus showed that $\mathrm{Aut}(\mathbb{U}^{\flat}_{S,\mathcal{D}})$ cannot act 2-transitively on $\mathcal{O}_{[\infty]}$, which completes our proof. □

4.3 Non-Existence of Isomorphisms

We use the results about automorphisms of $\mathrm{SL}(2,q)$-unitals to show that $\mathrm{SL}(2,q)$-unitals with parallelism $\flat$ or $\natural$ are not isomorphic to unitals of certain known types.

At first, we show the "uniqueness" of the classical $\mathrm{SL}(2,q)$-unital $\mathbb{U}^{\natural}_{C,\mathcal{H}}$, in the sense that $S = C$, $\mathcal{D} = \mathcal{H}$ and $\pi = \natural$ is essentially (up to conjugation and choice of the representatives in the sets $\hat{H}$, $H \in \mathcal{H}$) the only possibility such that $\mathbb{U}^{\pi}_{S,\mathcal{D}}$ is isomorphic to $\mathbb{U}_{C,\mathcal{H}}$ if $q \geq 3$.

Theorem 4.13. *Let $q \geq 3$ and let $\mathbb{U}^{\pi}_{S,\mathcal{D}}$ be an $\mathrm{SL}(2,q)$-unital of order q. If $\mathbb{U}^{\pi}_{S,\mathcal{D}}$ is isomorphic to the classical unital $\mathbb{U}^{\natural}_{C,\mathcal{H}}$, then $\pi = \natural$, the group S is cyclic and $\mathcal{D}$ is conjugate to a set of arcuate blocks through $\mathbb{1}$ in $\mathbb{U}_{C,\mathcal{H}}$.*

Proof. Since the full automorphism group of the classical unital acts transitively on the set of blocks, we know the affine $\mathrm{SL}(2,q)$-unitals $\mathbb{U}_{S,\mathcal{D}}$ and $\mathbb{U}_{C,\mathcal{H}}$ to be isomorphic. Since in each affine $\mathrm{SL}(2,q)$-unital the group of automorphisms R acts transitively on the set of points, we know the point stabilizers $\mathrm{Aut}(\mathbb{U}_{S,\mathcal{D}})_{\mathbb{1}}$ and $\mathrm{Aut}(\mathbb{U}_{C,\mathcal{H}})_{\mathbb{1}}$ to be isomorphic. Hence, according to Corollary 4.5 and Proposition 4.6,

$$\mathfrak{A}_C = \mathrm{Aut}(\mathbb{U}_{C,\mathcal{H}})_{\mathbb{1}} \cong \mathrm{Aut}(\mathbb{U}_{S,\mathcal{D}})_{\mathbb{1}} \leq \mathfrak{A}_S.$$

Theorem 2.11 shows that this is only possible for $\mathfrak{A}_S \cong \mathfrak{A}_C$ and S cyclic. Since there is only one conjugacy class of cyclic subgroups of order $q+1$ of $\mathrm{SL}(2,q)$, the group S must be a conjugate of C and we may assume $S = C$. According to Theorem 4.4, any isomorphism $\mathbb{U}_{C,\mathcal{H}} \to \mathbb{U}_{C,\mathcal{D}}$ must be contained in $\mathfrak{A}_C \ltimes R$ and is thus an automorphism of $\mathbb{U}_{C,\mathcal{H}}$. Hence, $\mathbb{U}_{C,\mathcal{D}} = \mathbb{U}_{C,\mathcal{H}}$ and $\mathcal{D}$ is a set of arcuate blocks through $\mathbb{1}$ in $\mathbb{U}_{C,\mathcal{H}}$. In particular, the sets of arcuate blocks through $\mathbb{1}$, namely the sets of hats $\{\hat{D} \mid D \in \mathcal{D}\}$ and $\{\hat{H} \mid H \in \mathcal{H}\}$ coincide and we may choose $\mathcal{D} = \mathcal{H}$.
Assume $\mathbb{U}^{\pi}_{C,\mathcal{H}} \cong \mathbb{U}^{\natural}_{C,\mathcal{H}}$. As above,

$$\mathfrak{A}_C \ltimes R = \mathrm{Aut}(\mathbb{U}^{\natural}_{C,\mathcal{H}})_{[\infty]} \cong \mathrm{Aut}(\mathbb{U}^{\pi}_{C,\mathcal{H}})_{[\infty]} \leq \mathfrak{A}_C \ltimes R.$$

In particular, R stabilizes π and thus $\pi \in \{\flat, \natural\}$ (see Proposition 3.22). But as shown in Theorem 4.12, the block at infinity is fixed by every automorphism in any $\mathrm{SL}(2,q)$-$\flat$-unital of order $q \geq 3$, while it can be moved in $\mathbb{U}^{\natural}_{C,\mathcal{H}}$. □

In their monograph on unitals in projective planes [2], Barwick and Ebert introduce several kinds of unitals that can be embedded in projective planes. Particular attention is paid to (non-classical) unitals arising from Buekenhout's construction in the desarguesian plane $\mathrm{PG}(2, q^2)$. Barwick and Ebert distinguish orthogonal-Buekenhout-Metz unitals and Buekenhout-Tits unitals, where an orthogonal-Buekenhout-Metz unital arises from an elliptic cone in $\mathrm{PG}(4, q)$ and a Buekenhout-Tits unital from an ovoidal cone with base a Tits ovoid (see [2, chapter 4] for details).

Lemma 4.14. *In any orthogonal-Buekenhout-Metz unital and any Buekenhout-Tits unital of order q, no block is fixed by the full automorphism group.*

Proof. According to [2, Theorems 4.12 and 4.23], any orthogonal-Buekenhout-Metz unital of order q admits a group of automorphisms (induced by automorphisms of $\mathrm{PG}(2, q^2)$) which fixes one point and acts transitively on the remaining points. Hence, there is no block fixed by the full automorphism group.
According to [2, Theorem 4.31], any Buekenhout-Tits unital of order q admits a group of automorphisms (also induced by automorphisms of $\mathrm{PG}(2, q^2)$) that fixes one point and acts in q orbits of size q^2 on the remaining points. Since each block is incident with $q + 1$ points (and $q + 1 < q^2$ for $q \geq 2$), there is no block fixed by the full automorphism group. □

In [15], Knarr and Stroppel investigate unitals arising from a unitary polarity in (not necessary desarguesian) shift planes introduced by Coulter and Matthews. From their theorem on automorphisms of those unitals, we get the following

Lemma 4.15. *In any unital of order q arising from a unitary polarity in a Coulter-Matthews plane, no block is fixed by the full automorphism group.*

Proof. According to [15, Theorems 5.2 and 5.6], such a unital admits a group of automorphisms with three point orbits of length 1, q^2 and $q^2(q - 1)$, respectively. Again, since each block is incident with $q + 1$ points, the statement follows. □

Another class of unitals, where no block is fixed by the full automorphism group, are the Ree unitals. These unitals of order $q = 3^e$, $e \geq 3$ odd, are described by Lüneburg (see [19]), where the points are given by the Sylow 3-subgroups and the blocks by the involutions of the Ree group $\mathrm{R}(q)$ of order $(q^3 + 1)q^3(q - 1)$. Lüneburg shows that the action of the Ree group on the Ree unital is transitive on the blocks (see [19, 1.]).

In [6], Ganley determines automorphisms of unitals which are given by the absolute points and non-absolute lines of a unitary polarity in a Dickson semifield plane of odd order q^2. He shows that those unitals admit a group of automorphisms which fixes one point and acts transitively on the remaining points, see [6, Lemma 3].

In [16], Knarr and Stroppel introduce unitals given by a polarity in planes over (not necessarily commutative) semifields. They show that those unitals also admit a group of automorphisms which fixes one point and acts transitively on the remaining points, see [16, Lemma 3.3].

We summarize these considerations and the Results 4.10 and 4.12 in the following

Theorem 4.16. *Let* $\mathbb{U}$ *be a non-classical unital, where* $\mathrm{Aut}(\mathbb{U})$ *fixes one block. Then* $\mathbb{U}$ *does not belong to one of the following classes of unitals:*

1. *Orthogonal-Buekenhout-Metz and Buekenhout-Tits unitals in the desarguesian plane* $\mathrm{PG}(2, q^2)$*,*
2. *unitals arising from a unitary polarity in a Coulter-Matthews plane,*
3. *Ree unitals,*
4. *unitals arising from a unitary polarity in a Dickson semifield plane of odd order as described in* [6]*,*
5. *unitals arising from a polarity in a semifield plane as described in* [16]*.*

In particular, this holds for every non-classical $\mathrm{SL}(2, q)$*-unital* $\mathbb{U}^{\pi}_{S,\mathcal{D}}$ *with* $\pi \in \{\flat, \natural\}$*.* □

5 Parallelisms and Translations

We will now regard parallelisms on the short blocks of affine $\mathrm{SL}(2,q)$-unitals, independent of the choice of S and $\mathcal{D}$. Recall that the short blocks of any affine $\mathrm{SL}(2,q)$-unital are given by all right cosets of the $q+1$ Sylow p-subgroups of $\mathrm{SL}(2,q)$. Recall also the definitions of the two obvious parallelisms

$$\flat := \{T \cdot \mathrm{SL}(2,q) \mid T \in \mathfrak{P}\} \quad \text{and} \quad \natural := \{\mathrm{SL}(2,q) \cdot T \mid T \in \mathfrak{P}\},$$

where $\mathfrak{P}$ denotes the set of all Sylow p-subgroups of $\mathrm{SL}(2,q)$.

Although we consider parallelisms on the short blocks independently, we are of course interested in their stabilizers in $\mathfrak{A}_S \ltimes R$, the greatest possible automorphism group of any affine $\mathrm{SL}(2,q)$-unital $\mathbb{U}_{S,\mathcal{D}}$. For any subgroup $S \leq \mathrm{SL}(2,q)$ of order $q+1$, the parallelisms $\flat$ and $\natural$ are both stabilized by $\mathfrak{A}_S \ltimes R$. This implies that in any affine $\mathrm{SL}(2,q)$-unital, no other parallelism is equivalent to $\flat$ or $\natural$, respectively (recall Definition 3.11).

5.1 A Class of Parallelisms for Odd Order

For each odd q, there is at least one class of parallelisms apart from $\flat$ and $\natural$. Let q be odd throughout this section.

Let T be a fixed Sylow p-subgroup of $\mathrm{SL}(2,q)$, namely

$$T := \{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \mid x \in \mathbb{F}_q\}.$$

The normalizer of T in $\mathrm{GL}(2,q)$ is the set of upper triangular matrices. Let $\mathbb{F}_q^{\times,\square}$ denote the set of all squares in $\mathbb{F}_q^{\times}$ and $\mathbb{F}_q^{\times,\not\square}$ the set of all non-squares in $\mathbb{F}_q^{\times}$. Note that $\#\mathbb{F}_q^{\times,\square} = \#\mathbb{F}_q^{\times,\not\square} = \frac{1}{2}(q-1)$, since q is odd. Let

$${}_{\square}\mathrm{SL} := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,q) \mid c \in \mathbb{F}_q^{\times,\square}\} \text{ and } {}_{\not\square}\mathrm{SL} := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,q) \mid c \in \mathbb{F}_q^{\times,\not\square}\}.$$

Let further

$$A := \{Tg \mid g \in \mathrm{N}_{\mathrm{SL}(2,q)}(T)\} \cup \{Tg \mid g \in {}_{\square}\mathrm{SL}\} \cup \{gT \mid g \in {}_{\not\square}\mathrm{SL}\},$$

$$A' \coloneqq \{Tg \mid g \in \mathrm{N}_{\mathrm{SL}(2,q)}(T)\} \cup \{Tg \mid g \in {}_{\boxslash}\mathrm{SL}\} \cup \{gT \mid g \in {}_{\square}\mathrm{SL}\}$$

and

$$\pi^{\square} \coloneqq \{A^h \mid h \in \mathrm{SL}(2,q)\}, \quad {}^{\square}\pi \coloneqq \{A'^h \mid h \in \mathrm{SL}(2,q)\}.$$

Note that in A, the representatives of *right* cosets of T are contained in ${}_{\square}\mathrm{SL}$ (hence the name $\pi^{\square}$), while in A', the representatives of *left* cosets of T are contained in ${}_{\square}\mathrm{SL}$ (hence the name ${}^{\square}\pi$). We show the following

Proposition 5.1. *For odd q, the sets $\pi^{\square}$ and ${}^{\square}\pi$ are parallelisms on the short blocks of any affine $\mathrm{SL}(2,q)$-unital. With $v \in \mathbb{F}_q^{\times,\boxslash}$, conjugation by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & v \end{smallmatrix}\right)$ maps $\pi^{\square}$ to ${}^{\square}\pi$.*

Proof. Since

$$\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \text{ and } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & * \\ c & * \end{smallmatrix}\right),$$

the cosets in A have no common points. Further,

$$\begin{aligned} \#A &= \#\{Tg \mid g \in \mathrm{N}_{\mathrm{SL}(2,q)}(T)\} + \#\{Tg \mid g \in {}_{\square}\mathrm{SL}\} + \#\{gT \mid g \in {}_{\boxslash}\mathrm{SL}\} \\ &= q - 1 + \frac{q-1}{2} \cdot q + \frac{q-1}{2} \cdot q \\ &= q^2 - 1 \end{aligned}$$

and hence A is a set of $q^2 - 1$ short blocks of which no two meet.

Let $h \in \mathrm{N}(T) \coloneqq \mathrm{N}_{\mathrm{SL}(2,q)}(T)$. Then h is of the form $\left(\begin{smallmatrix} s & t \\ 0 & s^{-1} \end{smallmatrix}\right)$ with $s \neq 0$ and we have

$$h^{-1} \left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right) h = \begin{pmatrix} s^{-1} & -t \\ 0 & s \end{pmatrix} \left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right) \left(\begin{smallmatrix} s & t \\ 0 & s^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} * & * \\ cs^2 & * \end{smallmatrix}\right)$$

with $cs^2 \in \mathbb{F}_q^{\times,\square}$ exactly if $c \in \mathbb{F}_q^{\times,\square}$ and $cs^2 = 0$ exactly if $c = 0$. Hence, conjugation by h stabilizes A.

Now let $h \notin \mathrm{N}(T)$ and $Tg \in A$. Then $(Tg)^h = T^h g^h$ is no right coset of T. Compute $T^h g^h = g^h (T^h)^{g^h} = g^h T^{gh}$ and assume $gh \in \mathrm{N}(T)$. Then $h = g^{-1} n$ for an $n \in \mathrm{N}(T)$ and we have $g^h = g^{g^{-1}n} = g^n$. But since $g^n \in {}_{\square}\mathrm{SL}$ exactly if $g \in {}_{\square}\mathrm{SL}$ (as shown above), the coset $(Tg)^h$ is not contained in A. A similar consideration shows that $(gT)^h$ is never contained in A when $gT \in A$ and $h \notin \mathrm{N}(T)$.

Finally, let $g, h \in \mathrm{SL}(2,q)$ and assume $A^h \cap A^g \neq \varnothing$. Then $A^{hg^{-1}} \cap A \neq \varnothing$ and we get $hg^{-1} \in \mathrm{N}(T)$, $A^{hg^{-1}} = A$ and hence $A^h = A^g$. Thus, $\pi^{\square} = \{A^h \mid h \in \mathrm{SL}(2,q)\}$ is indeed a parallelism on the short blocks of any affine $\mathrm{SL}(2,q)$-unital with $q+1$ parallel classes with $q^2 - 1$ short blocks each.

Now let $v \in \mathbb{F}_q^{\times,\boxslash}$ and $f := \left(\begin{smallmatrix} 1 & 0 \\ 0 & v \end{smallmatrix}\right)$. Then $f \in \mathrm{N}_{\mathrm{GL}(2,q)}(T)$ and $g^f \in \mathrm{N}(T)$ exactly if $g \in \mathrm{N}(T)$. Further,

$$f^{-1} \left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right) f = \begin{pmatrix} 1 & 0 \\ 0 & v^{-1} \end{pmatrix} \left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & v \end{smallmatrix}\right) = \left(\begin{smallmatrix} * & * \\ cv^{-1} & * \end{smallmatrix}\right)$$

and hence $g^f \in {}_{\square}\mathrm{SL}$ exactly if $g \in {}_{\boxslash}\mathrm{SL}$. Thus, $A^f = A'$. For each $A^h \in \pi^{\square}$, we get

$$(A^h)^f = A^{hf} = (A^f)^{(h^f)} = A'^{(h^f)} \in {}^{\square}\pi.$$

Hence, we obtain ${}^{\square}\pi$ from $\pi^{\square}$ via conjugation by f and ${}^{\square}\pi$ is a parallelism, since $\pi^{\square}$ is a parallelism. $\square$

Since we are interested in determining automorphisms of our $\mathrm{SL}(2,q)$-unitals, we are interested in the stabilizer of the parallelism $\pi^{\square}$ in the group of possible automorphisms of affine $\mathrm{SL}(2,q)$-unitals.

Theorem 5.2. *Let* $c := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. *The stabilizer of* $\pi^{\square}$ *in the group* $\mathfrak{A} \ltimes R$ *equals*

(a) $\mathrm{P\Sigma L}(2,q) \times \langle \rho_{-\mathbb{1}} \rangle$ *if* $q \equiv 1 \mod 4$ *and*

(b) $\mathrm{P\Sigma L}(2,q) \rtimes \langle \gamma_c \cdot \rho_{-\mathbb{1}} \rangle$ *if* $q \equiv 3 \mod 4$,

where $\mathrm{P\Sigma L}(2,q) := \mathrm{PSL}(2,q) \rtimes \mathrm{Aut}(\mathbb{F}_q)$.

Proof. Note first that the Frobenius automorphism φ stabilizes $\mathbb{F}_q^{\times,\square}$, the group T and its normalizer and hence A and A'. Thus, for each $A^h \in \pi^{\square}$, we have

$$A^h \cdot \varphi = (A \cdot \varphi)^{h \cdot \varphi} = A^{h \cdot \varphi} \in \pi^{\square}$$

and φ stabilizes $\pi^{\square}$ and equally ${}^{\square}\pi$.

The action of $\mathrm{PSL}(2,q)$ obviously stabilizes $\pi^{\square}$ and ${}^{\square}\pi$ by construction. Since the index of $\mathrm{PSL}(2,q)$ in $\mathrm{PGL}(2,q)$ equals 2, the orbit of $\pi^{\square}$ under the action of $\mathrm{PGL}(2,q)$ has length 1 or 2. From Proposition 5.1, we know that conjugation by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & v \end{smallmatrix}\right)$ ($v \in \mathbb{F}_q^{\times,\boxslash}$) maps $\pi^{\square}$ to ${}^{\square}\pi$, and hence the stabilizer of $\pi^{\square}$ in the action of $\mathrm{PGL}(2,q)$ equals $\mathrm{PSL}(2,q)$ and conjugation by any element in $\mathrm{PGL}(2,q) \setminus \mathrm{PSL}(2,q)$ interchanges $\pi^{\square}$ and ${}^{\square}\pi$.

Since we know now that $\mathrm{P\Gamma L}(2,q)$ stabilizes $\{\pi^{\square}, {}^{\square}\pi\}$, we need to find those elements in R which also stabilize the set of these two parallelisms. Let $\rho_g \in R$ and assume $\pi^{\square} \cdot \rho_g \in \{\pi^{\square}, {}^{\square}\pi\}$. Since for each $A^h \in \pi^{\square}$, it holds that ρ_g maps the set of right cosets of T^h in A^h onto a set of right cosets of T^h, it must then also map the set of left cosets of T^h in A^h onto a set of left cosets of T^h. Hence, g is contained in the normalizer of every Sylow p-subgroup of $\mathrm{SL}(2,q)$ and thus $g \in \{\pm\mathbb{1}\}$. We see immediately that

$$\pi^{\square} \cdot \rho_{-\mathbb{1}} = \begin{cases} \pi^{\square} & \text{if } -1 \in \mathbb{F}_q^{\times,\square}, \\ {}^{\square}\pi & \text{if } -1 \in \mathbb{F}_q^{\times,\not\square}. \end{cases}$$

If $q \equiv 1 \mod 4$, then $-1 \in \mathbb{F}_q^{\times,\square}$ and $\rho_{-\mathbb{1}}$ stabilizes $\pi^{\square}$. Since $\rho_{-\mathbb{1}}$ does not fix $\mathbb{1}$ – while every automorphism in $\mathfrak{A}$ does – and since $\rho_{-\mathbb{1}}$ commutes with every automorphism in $\mathfrak{A}$, statement (a) follows.
If $q \equiv 3 \mod 4$, then both $\rho_{-\mathbb{1}}$ and conjugation by c interchange $\pi^{\square}$ and ${}^{\square}\pi$ and hence the product stabilizes $\pi^{\square}$. Again, $\gamma_c \cdot \rho_{-\mathbb{1}}$ does not fix $\mathbb{1}$ – while every automorphism in $\mathfrak{A}$ does – and hence the product $\mathrm{P\Sigma L}(2,q) \cdot \langle \gamma_c \cdot \rho_{-\mathbb{1}} \rangle$ is semidirect, since $\rho_{-\mathbb{1}}$ commutes with every automorphism in $\mathfrak{A}$ and since γ_c normalizes $\mathrm{P\Sigma L}(2,q) \leq \mathfrak{A}$. □

Corollary 5.3. *Let q be odd. The stabilizer of $[\infty]$ in the full automorphism group of the $\mathrm{SL}(2,q)$-unital $\mathbb{U}_{C,\mathcal{H}}^{\pi^{\square}}$ has order $2e(q+1)$.*

Proof. According to Corollary 3.13, the stabilizer of $[\infty]$ in $\mathrm{Aut}(\mathbb{U}_{C,\mathcal{H}}^{\pi^{\square}})$ equals the subgroup of $\mathrm{Aut}(\mathbb{U}_{C,\mathcal{H}}) = \mathfrak{A}_C \ltimes R$ leaving $\pi^{\square}$ invariant.
Since q is odd, the index $[\mathrm{P}(\mathrm{N}_{\mathrm{GL}(2,q)}(C)) : \mathrm{P}(\mathrm{N}_{\mathrm{SL}(2,q)}(C))]$ equals 2. For $q \equiv 1 \mod 4$, $\rho_{-\mathbb{1}}$ is as element of R an automorphism of $\mathbb{U}_{C,\mathcal{H}}$. For $q \equiv 3 \mod 4$, we may choose C as in Remark 2.7 to be $\{\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \mid a^2 + b^2 = 1\}$, since $\pi^{\square}$ is invariant under conjugation with $\mathrm{SL}(2,q)$. Then, c normalizes C and $\gamma_c \cdot \rho_{-\mathbb{1}}$ is hence an automorphism of $\mathbb{U}_{C,\mathcal{H}}$. Thus, for every odd prime power q,

$$\#\,\mathrm{Aut}(\mathbb{U}_{C,\mathcal{H}}^{\pi^{\square}}) = \frac{\#\mathfrak{A}_C}{2} \cdot 2 = 2e(q+1). \qquad \square$$

5.2 Translations

Recall that a translation with center c of a unital is an automorphism that fixes the point c and each block through c. We have already seen in Lemma 4.9 that the Sylow p-subgroups of $\mathrm{SL}(2,q)$ act via right multiplication as translation groups on any $\mathrm{SL}(2,q)$-$\natural$-unital. The possible translations with center on $[\infty]$ in any $\mathrm{SL}(2,q)$-π-unital depend on the parallelism π. We will determine all translations with center on $[\infty]$ of $\mathrm{SL}(2,q)$-π-unitals with $\pi \in \{\flat, \pi^{\square}\}$.

Recall that we may label the points at infinity of any $\mathrm{SL}(2,q)$-π-unital with the Sylow p-subgroups, such that each (affine short) block $T \in \mathfrak{P}$ through $\mathbb{1}$ is incident with the point $T \in [\infty]$.

Lemma 5.4. *Let $\mathbb{U}$ be a unital and let B be a block of $\mathbb{U}$ such that B is fixed by $\mathrm{Aut}(\mathbb{U})$. Then the center of any translation of $\mathbb{U}$ lies on B.*

Proof. In [10, Theorem 1.3], Grundhöfer, Stroppel and Van Maldeghem show that each translation of a unital does not fix any block apart from the ones through its center. □

Lemma 5.5. *Let $\mathbb{U}^{\pi}_{S,\mathcal{D}}$ be an $\mathrm{SL}(2,q)$-unital and let τ be a translation of $\mathbb{U}^{\pi}_{S,\mathcal{D}}$ with center $T \in [\infty]$. Then, $\tau = \alpha\rho_t$ with $\alpha \in \mathfrak{A}_T \cap \mathfrak{A}_S$ and $t \in T$.*

Proof. We know $\tau \in \mathrm{Aut}(\mathbb{U}^{\pi}_{S,\mathcal{D}})_{[\infty]} \leq \mathfrak{A}_S \ltimes R$. Hence, $\tau = \alpha \cdot \rho_h$ with $\alpha \in \mathfrak{A}_S$. Since τ fixes the short block T, we have

$$T = T \cdot \tau = (T \cdot \alpha)h$$

and thus α stabilizes T and we have $h \in T$. □

Lemma 5.6. *Let $T \in \mathfrak{P}$ and let $\mathbb{U}^{\pi}_{S,\mathcal{D}}$ be an $\mathrm{SL}(2,q)$-unital such that every block in $\{Tg \mid g \in \mathrm{N}(T)\}$ is incident with the point $T \in [\infty]$. Let further $\tau = \alpha\rho_t$ as in* Lemma 5.5 *be a translation of $\mathbb{U}^{\pi}_{S,\mathcal{D}}$ with center T. Then one of the following cases occurs:*

(i) $\alpha = \mathrm{id}$,

(ii) *$p = 2$ and α is given by conjugation with some $a \in T$.*

Proof. Note first that right multiplication by any element $t \in T$ stabilizes each block $Tg = gT$ with $g \in \mathrm{N}(T)$.
Assume without restriction $T = \{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \mid x \in \mathbb{F}_q\}$ and let $g \in \mathrm{N}(T)$. Then, $g = \left(\begin{smallmatrix} b^{-1} & c \\ 0 & b \end{smallmatrix}\right)$ and the coset Tg is given by

$$Tg = \{\left(\begin{smallmatrix} b^{-1} & x \\ 0 & b \end{smallmatrix}\right) \mid x \in \mathbb{F}_q\}.$$

Let $[a] \in \mathrm{P}(\mathrm{N}_{\mathrm{GL}(2,q)}(T))$. Then we may choose $a = \left(\begin{smallmatrix} 1 & y \\ 0 & d \end{smallmatrix}\right)$ and for each Tg with $g \in \mathrm{N}(T)$, we have $(Tg)^a = Tg$. Applying a power φ^d of the Frobenius automorphism φ stabilizes the block $Tg = \{\left(\begin{smallmatrix} b^{-1} & x \\ 0 & b \end{smallmatrix}\right) \mid x \in \mathbb{F}_q\}$ exactly if $b^{(p^d)} = b$. According to Lemma 5.5, $\alpha = \gamma_a \cdot \varphi^d$ with $a \in \mathrm{N}_{\mathrm{GL}(2,q)}(T)$. Thus, if $\tau = \alpha\rho_t$ stabilizes each block Tg, $g \in \mathrm{N}(T)$, then $\alpha = \gamma_a$ with $a \in \mathrm{N}_{\mathrm{GL}(2,q)}(T)$.

Let $G_{[T]}$ denote the group of translations of $\mathbb{U}^{\pi}_{S,\mathcal{D}}$ with center T. According to [10, Theorem 1.3], $G_{[T]}$ acts semiregularly on the points different from T on each block through T and hence $G_{[T]}$ is a p-group. Let $R_T := \{\rho_t \mid t \in T\}$ and let $\operatorname{Aut}(\mathbb{U}_{S,\mathcal{D}})_T$ be the stabilizer of the short block T in the full automorphism group of the affine $\mathrm{SL}(2,q)$-unital $\mathbb{U}_{S,\mathcal{D}}$. Since R_T is normal in $\operatorname{Aut}(\mathbb{U}_{S,\mathcal{D}})_T$, we have the projection

$$\mathrm{pr}\colon \operatorname{Aut}(\mathbb{U}_{S,\mathcal{D}})_T \to \operatorname{Aut}(\mathbb{U}_{S,\mathcal{D}})_T / R_T.$$

Let $\tau = \gamma_a \rho_t \in G_{[T]} \leq \operatorname{Aut}(\mathbb{U}_{S,\mathcal{D}})_T$ and consider $\tau R_T = \gamma_a R_T$ in the group $A := \operatorname{im}(\mathrm{pr}|_{G_{[T]}})$. Since the intersection of $\langle \gamma_a \rangle$ and R_T is trivial, the order of $\gamma_a R_T$ in A equals the order of γ_a. As homomorphic image of a p-group, A is a p-group and hence the order of γ_a is a p-power.
Since $\alpha = \gamma_a$ also has to stabilize S, the order of γ_a divides $\#\mathrm{P}(\mathrm{N}_{\mathrm{GL}(2,q)}(S))$. According to Theorem 2.11, this implies $\operatorname{ord}(\gamma_a) \in \{1,2\}$. Thus, $\alpha = \mathrm{id}$ or $\operatorname{ord}(\gamma_a) = 2 = p$. If $\operatorname{ord}(\gamma_a) = 2 = p$, then $\mathrm{PGL}(2,q) \cong \mathrm{SL}(2,q)$ and we may hence choose $a \in \mathrm{SL}(2,q)$ with $\operatorname{ord}(a) = 2$. Since a normalizes T, it follows $a \in T$ (see Lemma 3.20). □

Theorem 5.7. *Let $\mathbb{U} := \mathbb{U}^{\pi}_{S,\mathcal{D}}$ be an $\mathrm{SL}(2,q)$-π-unital.*

(a) *If $\pi = \pi^{\square}$, then $\mathbb{U}$ admits no non-trivial translation with center on $[\infty]$.*

(b) *If $\pi = \flat$ and $q \geq 3$, then:*

  (i) *For $p = 2$, every non-trivial translation is given by left multiplication with an involution contained in $\mathrm{N}(S)$. For each Sylow 2-subgroup $T \in \mathfrak{P}$, the normalizer $\mathrm{N}(S)$ contains exactly one non-trivial element of T.*

  (ii) *For q odd, $\mathbb{U}$ does not admit any non-trivial translation.*

Proof. If $\pi = \pi^{\square}$, then q is odd and any non-trivial translation with center $T \in [\infty]$ must be given by right multiplication with $\mathbb{1} \neq t \in T$, according to Lemmas 5.5 and 5.6. But no right coset Tg is fixed under right multiplication with $t \in T$, unless $g \in \mathrm{N}(T)$ or $t = \mathbb{1}$.

If $\pi = \flat$ and $q \geq 3$, then $\operatorname{Aut}(\mathbb{U})$ fixes the block $[\infty]$ (see Theorem 4.12) and hence the center of any translation of $\mathbb{U}$ lies on $[\infty]$ (see Lemma 5.4). Let $\tau = \alpha\rho_t$ be a non-trivial translation of $\mathbb{U}$ with center $T \in [\infty]$. As for $\pi = \pi^{\square}$, right multiplication with $\mathbb{1} \neq t \in T$ does not fix any block Tg with $g \notin \mathrm{N}(T)$. Hence, α is not the identity and we get $p = 2$ and α is given by conjugation with $a \in T$. We need to show $a = t$. Let $g \notin \mathrm{N}(T)$ and assume $(Tg)^a t = Tg$. Since $a \in T$, this is the case exactly if $gatg^{-1} \in T$. Since $g \notin \mathrm{N}(T)$, this implies $at = \mathbb{1}$ and hence $a = t$ (recall $p = 2$). Since a must normalize S and $t = t^{-1}$, the translation τ is indeed given by left multiplication with $t \in \mathrm{N}(S) \cap T$.

For $p = 2$, the normalizer $\mathrm{N}(S)$ is a dihedral group of order $2(q+1)$ containing S as normal subgroup of order $q+1$. Hence, there are $q+1$ involutions in $\mathrm{N}(S)$, contained in one coset of S. For any $T \in \mathfrak{P}$, the intersection of S and T is trivial and hence no two non-trivial elements of T are contained in $\mathrm{N}(S)$. □

Corollary 5.8. *For q even, the Grüning unital $\mathbb{U}^{\flat}_{C,\mathcal{H}}$ admits exactly $q+1$ non-trivial translations, each of order 2.* □

6 Computer Results

We will search for $\mathrm{SL}(2,q)$-unitals using GAP [7]. As mentioned in the introduction, you may find the code files in the GitHub repository

https://github.com/moehve/SL2q-Unitals_GAP.git

The particular files for the different searches will be named (and in the non-print version also linked) in the text.

For the construction of an $\mathrm{SL}(2,q)$-unital, we need to

1. choose a group $S \leq \mathrm{SL}(2,q)$ of order $q+1$,
2. find a set of arcuate blocks $\mathcal{D}$ satisfying properties (P) and (Q) and
3. choose a parallelism on the short blocks.

Finding the set $\mathcal{D}$ depends on the choice of the group S, while the parallelism on the short blocks is independent of S and $\mathcal{D}$. We may thus treat those subjects independently.

6.1 Search for Arcuate Blocks

We are first interested in finding a set of arcuate blocks $\mathcal{D}$ for a given group S. Since applying automorphisms of $\mathrm{SL}(2,q)$ preserves the isomorphism type of any affine $\mathrm{SL}(2,q)$-unital, we may fix S up to conjugation by $\mathrm{GL}(2,q)$. For each prime power q, we may choose $S = C$ to be cyclic (unique up to conjugation) and $\mathcal{D} := \mathcal{H}$ the classical set of arcuate blocks. Then we obtain the classical affine unital $\mathbb{U}_{C,\mathcal{H}}$.
For $q = 2$, there is only one isomorphism type of affine unitals (cf. Example 3.14) and for $q = 3$, there is only one isomorphism type of affine $\mathrm{SL}(2,q)$-unitals (see [9, Theorem 3.3]), while there exist many other affine unitals of order 3. As already mentioned, Grundhöfer, Stroppel and Van Maldeghem have found a non-classical affine $\mathrm{SL}(2,4)$-unital, see [9]. Our aim is to find more non-classical affine $\mathrm{SL}(2,q)$-unitals or to show that they do not exist under certain conditions.

6.1.1 Orders 4 and 5

We first consider the small cases $q = 4$ and $q = 5$. Since $q \not\equiv 3 \mod 4$ in both cases, we may choose $S = C$ as an arbitrary cyclic subgroup of $\mathrm{SL}(2, q)$ of order $q + 1$. We consider the set

$$M := \mathrm{SL}(2, q) \setminus (C \cup (\bigcup \mathfrak{P})),$$

where $\mathfrak{P}$ denotes the set of all Sylow p-subgroups of $\mathrm{SL}(2, q)$, and use GAP[1] to search M for all sets $\mathcal{D}$ of $q - 2$ subsets of size q (note that $\mathbb{1} \notin M$) such that

$$M = \bigcup_{D \in \mathcal{D}} (D \cup \{\mathbb{1}\})^*.$$

Up to choosing representatives of the $\hat{D}$, there is only one such set for $q = 5$ and this set yields the classical affine unital. For $q = 4$, we obtain six sets of two arcuate blocks each. Since five of those six sets form one orbit under conjugation by C, while the other one is invariant (up to choosing representatives of the $\hat{D}$) under $\mathfrak{A}_C$, those six sets yield two isomorphism types of affine unitals. The one invariant under $\mathfrak{A}_C$ yields the classical affine unital while the other ones yield the affine unital presented in [9]. The result of the computer search matches the fact that the full automorphism group of the non-classical affine $\mathrm{SL}(2, 4)$-unital has index 5 in $\mathfrak{A}_C \ltimes R$, see [9, Theorem 4.1].

We summarize the results for orders 4 and 5 in the following

Theorem 6.1 (by exhaustive computer search)**.**

(a) *For $q = 4$, the only two isomorphism types of affine $\mathrm{SL}(2, q)$-unitals are represented by the classical affine unital and the one described in* [9].

(b) *For $q = 5$, the classical affine unital represents the only isomorphism type of affine $\mathrm{SL}(2, q)$-unitals.* □

For higher orders, the cost of an exhaustive search increases immensely, whence it is reasonable to search under certain restrictions.

For the following considerations, it will be useful to write the arcuate blocks in a certain way. Let $D = \{\mathbb{1}, d_2, \ldots, d_{q+1}\}$ be an arcuate block through $\mathbb{1}$. Consider a table $\mathcal{T} = \mathcal{T}_D$ with $q + 1$ rows and $q + 1$ columns, where the entries in the first column are the elements of D with $\mathcal{T}_{11} = \mathbb{1}$ and the entry in the i-th row and j-th column is $\mathcal{T}_{ij} = \mathcal{T}_{i1} \cdot \mathcal{T}_{j1}^{-1}$. Then the columns of $\mathcal{T}$ correspond to the blocks in $\hat{D}$, each diagonal entry $\mathcal{T}_{ii}$ equals $\mathbb{1}$ and $\mathcal{T}_{ij} = \mathcal{T}_{ji}^{-1}$ for all $i, j \in \{1, \ldots, q + 1\}$.

[1] The corresponding files are `order4_exhaustive` and `order5_exhaustive`.

$$\mathcal{T}_D : \quad \boxed{\begin{array}{ccccc} \mathbb{1} & d_2^{-1} & d_3^{-1} & \cdots & d_{q+1}^{-1} \\ d_2 & \mathbb{1} & d_2 d_3^{-1} & \cdots & d_2 d_{q+1}^{-1} \\ d_3 & d_3 d_2^{-1} & \mathbb{1} & \cdots & d_3 d_{q+1}^{-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{q+1} & d_{q+1} d_2^{-1} & d_{q+1} d_3^{-1} & \cdots & \mathbb{1} \end{array}}$$

For any arcuate block $E = \{e_1, \dots, e_{q+1}\}$, we write $E^{-1} := \{e_1^{-1}, \dots, e_{q+1}^{-1}\}$. Then the rows of $\mathcal{T}_D$ correspond to the sets E^{-1} with $E \in \hat{D}$.

6.1.2 Order 7

We consider the case $q = 7$. Now the group S can by cyclic or quaternion and we first choose $S = C$ to be cyclic. Since $7 \equiv 3 \mod 4$, we know that -1 is not a square in $\mathbb{F}_7$. According to Remark 2.7, we can hence choose C to be

$$\{\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \mid a, b \in \mathbb{F}_7,\ a^2 + b^2 = 1\}.$$

A generator of C is given by $g := \left(\begin{smallmatrix} -2 & -2 \\ 2 & -2 \end{smallmatrix}\right)$. Let $c := \left(\begin{smallmatrix} 3 & 2 \\ -2 & 3 \end{smallmatrix}\right)$ and $f := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and note that $c^2 = g$. Then

$$\mathfrak{A}_C = \mathrm{Aut}(\mathrm{SL}(2,7))_C = \langle \gamma_c \rangle \rtimes \langle \gamma_f \rangle \cong C_8 \rtimes C_2 \cong D_8,$$

where D_8 is the dihedral group of order 16. (Recall that we denote by γ_a the automorphism given by conjugation with a.) Our aim is to search for a set $\mathcal{D}$ such that the affine $\mathrm{SL}(2,7)$-unital $\mathbb{U}_{C,\mathcal{D}}$ admits certain subgroups of $\mathfrak{A}_C$ as groups of automorphisms. Consider the subgroups

$$U := \langle \gamma_{c^4} \rangle \cong \langle [\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)] \rangle \cong C_2,$$

$$F := \langle \gamma_f \rangle \cong \langle [\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)] \rangle \cong C_2 \text{ and}$$

$$V := \langle \gamma_{cf} \rangle \cong \langle [\left(\begin{smallmatrix} 2 & 3 \\ 3 & -2 \end{smallmatrix}\right)] \rangle \cong C_2.$$

These are – spoken in terms of dihedral groups – the subgroups given by the involutory rotation and by representatives of the two conjugacy classes of reflections, respectively. Hence, every other representative of a conjugacy class of non-trivial subgroups of $\mathfrak{A}_C$ contains (an $\mathfrak{A}_C$-conjugate of) one of these as a subgroup.

We start our observations with the subgroup $U \leq \mathfrak{A}_C$. Assume $\mathbb{U}_{C,\mathcal{D}}$ to be an affine SL(2, 7)-unital on which $u := c^4$ acts (via conjugation) as an automorphism. Let

$$M := \mathrm{SL}(2,7) \setminus (C \cup (\bigcup \mathfrak{P})),$$

where $\mathfrak{P}$ denotes the set of all Sylow 7-subgroups of SL(2, 7). Recall $\#\mathcal{D} = q - 2 = 5$, where each of the five hats $\hat{D}$, $D \in \mathcal{D}$, contains $q + 1 = 8$ blocks. Let

$$A := \{x \in M \mid x^u = x\} \quad \text{and} \quad B := \{x \in M \mid x^u = x^{-1}\}$$

and compute $\#A = 0$ and $\#B = 40$. If x with $x^u = x^{-1}$ is contained in an arcuate block $D \in \mathcal{D}$, then $D^u = D \cdot x^{-1} \in \hat{D}$, since γ_u is an automorphism and $D \cdot x^{-1}$ is the unique block of $\mathbb{U}_{C,\mathcal{D}}$ containing x^{-1} and $\mathbb{1}$. Each arcuate block through $\mathbb{1}$ contains at most one x with $x^u = x^{-1}$, since: Assume $x, y \in D$ with $x^u = x^{-1}$ and $y^u = y^{-1}$. Then $D \cdot x^{-1} = D^u = D \cdot y^{-1}$ and property (Q) yields $x = y$. Since there are $5 \cdot 8 = 40$ arcuate blocks through $\mathbb{1}$, each of them contains exactly one point x with $x^u = x^{-1}$, and γ_u acts on each hat $\hat{D}$ in four orbits of length two.

Let $D \in \mathcal{D}$ and let $a, a^{-1}, b, b^{-1}, c, c^{-1}, d, d^{-1}$ be the eight elements of B in the blocks in $\hat{D}$. Consider the table $\mathcal{T} = \mathcal{T}_D$ and assume without restriction

$$a = \mathcal{T}_{21},\ b = \mathcal{T}_{43},\ c = \mathcal{T}_{65} \text{ and } d = \mathcal{T}_{87}, \text{ i. e.}$$

$$\mathcal{T}_D: \quad \begin{array}{|cccccccc|} \hline \mathbb{1} & a^{-1} & * & * & * & * & * & * \\ a & \mathbb{1} & * & * & * & * & * & * \\ * & * & \mathbb{1} & b^{-1} & * & * & * & * \\ * & * & b & \mathbb{1} & * & * & * & * \\ * & * & * & * & \mathbb{1} & c^{-1} & * & * \\ * & * & * & * & c & \mathbb{1} & * & * \\ * & * & * & * & * & * & \mathbb{1} & d^{-1} \\ * & * & * & * & * & * & d & \mathbb{1} \\ \hline \end{array}$$

Then we know

$$D = \{\mathbb{1}, a, w, bw, x, cx, y, dy\}$$

with w, x and y not in B and $w^u = bwa^{-1}$, $x^u = cxa^{-1}$ and $y^u = dya^{-1}$, since γ_u acts on the rows of $\mathcal{T}_D$ as it does on the columns.

We are now ready to check with GAP[2] for each 4-element subset $\{a, b, c, d\}$ of B and

[2] The corresponding file is `order7_cyc_automU`.

every 3-element subset $\{w, x, y\}$ of $M \setminus B$ whether $D \coloneqq \{\mathbb{1}, a, w, bw, x, cx, y, dy\}$ satisfies the above conditions and property (Q) and if $D^* = (\bigcup \hat{D}) \setminus \{\mathbb{1}\}$ is contained in M.

In the resulting list of possible arcuate blocks through $\mathbb{1}$, we search for sets $\mathcal{D}$ of five blocks such that the D^* with $D \in \mathcal{D}$ have pairwise empty intersection[3]. The result are all possible sets of arcuate blocks $\mathcal{D}$ such that $\mathbb{U}_{C,\mathcal{D}}$ is an affine $\mathrm{SL}(2,7)$-unital with automorphism γ_u.

The computer search showed that the classical affine unital of order 7 is the only possible affine $\mathrm{SL}(2,7)$-unital with automorphism γ_u.

Next, we consider the subgroup $F \leq \mathfrak{A}_C$ and assume $\mathbb{U}_{C,\mathcal{D}}$ to be an affine $\mathrm{SL}(2,7)$-unital with automorphism γ_f. Let M be as above and let

$$A \coloneqq \{x \in M \mid x^f = x\} \quad \text{and} \quad B \coloneqq \{x \in M \mid x^f = x^{-1}\}.$$

We compute $\#A = 4$ and $\#B = 36$. Again, each arcuate block through $\mathbb{1}$ contains at most one point $x \in B$. Assume that $D \in \mathcal{D}$ contains two elements $a, b \in A$. Then D^* contains six distinct elements $a, a^{-1}, b, b^{-1}, ba^{-1}$ and ab^{-1} all contained in M and invariant under conjugation by f. This is a contradiction to $\#A = 4$. Each arcuate block containing an element of A is fixed under conjugation by f while each arcuate block containing an element of B is not. Hence, each arcuate block through $\mathbb{1}$ contains either exactly one element of B or exactly one element of A.

Let $D \in \mathcal{D}$. If D^* contains no element of A, then the list of possible such arcuate blocks can be computed as above. If D^* contains an element $a \in A$, assume without restriction $a \in D$. Then $D^f = D$ and

$$D = \{\mathbb{1}, a, w, w^f, x, x^f, y, y^f\}$$

with $w, x, y \in M \setminus (A \cup B)$. All such blocks can be computed with GAP[4] and we conclude the search as above.

We obtain a similar result: The classical affine unital of order 7 is the only possible affine $\mathrm{SL}(2,7)$-unital with automorphism γ_f.

[3] The corresponding file is `order7_conclusion`. This file is also used to conclude the searches with groups of automorphisms F and V and with the group of automorphisms U where S is a quaternion group.

[4] The corresponding file is `order7_cyc_automF`.

Finally, we consider the subgroup $V \leq \mathfrak{A}_C$ and assume $\mathbb{U}_{C,\mathcal{D}}$ to be an affine $\mathrm{SL}(2,7)$-unital on which $v := c \cdot f$ acts as an automorphism. Let M be as above and let

$$A := \{x \in M \mid x^v = x\} \quad \text{and} \quad B := \{x \in M \mid x^v = x^{-1}\}.$$

We compute $\#A = 6$ and $\#B = 34$. Since the order of γ_v as automorphism of $\mathbb{U}_{S,\mathcal{D}}$ is 2, the orbits of the action of γ_v on the arcuate blocks have length 1 or 2, respectively. If γ_v fixes an arcuate block $D \in \mathcal{D}$, there is at least one element $d \in D$ with $d^v = d$, since D contains an odd number of elements different from $\mathbb{1}$. Let $D \in \mathcal{D}$ with $D^v \neq D$. If $D^v \notin \hat{D}$, then D^* must not contain any element of B – a contradiction, since $\#B = 34 > 32 = 4 \cdot 8$ and each arcuate block through $\mathbb{1}$ contains at most one element of B. Hence, $D^v = D \cdot d^{-1}$ for some $d \in D$. But then we also know $((D \cdot d^{-1})^{-1})^v = D^{-1}$ and since $d \in (D \cap (D \cdot d^{-1})^{-1}) \setminus \{\mathbb{1}\}$, we have $d^v \in (D \cdot d^{-1} \cap D^{-1}) \setminus \{\mathbb{1}\} = \{d^{-1}\}$. Hence, each arcuate block through $\mathbb{1}$ not fixed by γ_v contains at least one element of B and we obtain again that each arcuate block through $\mathbb{1}$ contains either exactly one element of B or exactly one element of A.

We conclude the search as above[5] and obtain again: The classical affine unital of order 7 is the only possible affine $\mathrm{SL}(2,7)$-unital with automorphism γ_v.

Having considered all three conjugacy classes of minimal subgroups of $\mathfrak{A}_C$, we are able to state the following

Theorem 6.2 (by exhaustive computer search)**.** *Let C be a cyclic subgroup of $\mathrm{SL}(2,7)$ and let $\mathbb{U}_{C,\mathcal{D}}$ be an affine $\mathrm{SL}(2,7)$-unital with non-trivial stabilizer of $\mathbb{1}$ in its full automorphism group. Then $\mathbb{U}_{C,\mathcal{D}}$ is isomorphic to the classical affine unital $\mathbb{U}_{C,\mathcal{H}}$.* □

Next, we choose $S \leq \mathrm{SL}(2,7)$ to be a quaternion group. Since $g^2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $2^2 + 3^2 = -1$, we can choose S as in Remark 2.7 as

$$S := \langle \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 3 \\ 3 & -2 \end{smallmatrix}\right) \rangle.$$

Let $u := g^2$, $f := \left(\begin{smallmatrix} 3 & 3 \\ -1 & -3 \end{smallmatrix}\right)$ and $v := \left(\begin{smallmatrix} 0 & 2 \\ 3 & 1 \end{smallmatrix}\right)$. Then

$$\mathfrak{A}_S = \mathrm{Aut}(\mathrm{SL}(2,7))_S = \langle \gamma_u, \gamma_f, \gamma_v \rangle \cong S_4.$$

Consider the subgroups

[5] The corresponding files are `order7_cyc_automV` and `order7_conclusion`.

$$U := \langle \gamma_u \rangle \cong \langle [\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)] \rangle \cong C_2,$$

$$F := \langle \gamma_f \rangle \cong \langle [\left(\begin{smallmatrix} 3 & 3 \\ -1 & -3 \end{smallmatrix}\right)] \rangle \cong C_2 \text{ and}$$

$$V := \langle \gamma_v \rangle \cong \langle [\left(\begin{smallmatrix} 0 & 2 \\ 3 & 1 \end{smallmatrix}\right)] \rangle \cong C_3$$

of $\mathfrak{A}_S$. Every other representative of a conjugacy class of non-trivial subgroups of $\mathfrak{A}_S$ contains (an $\mathfrak{A}_S$-conjugate of) one of these as a subgroup.

Let again

$$M := \mathrm{SL}(2,7) \setminus (S \cup (\bigcup \mathfrak{P})),$$

where $\mathfrak{P}$ denotes the set of all Sylow 7-subgroups of $\mathrm{SL}(2,7)$. Let

$$A := \{x \in M \mid x^u = x\} \quad \text{and} \quad B := \{x \in M \mid x^u = x^{-1}\}$$

and compute $\#A = 4$ and $\#B = 36$. With the same considerations as above and computation by GAP[6], we obtain that there is no affine $\mathrm{SL}(2,7)$-unital $\mathbb{U}_{S,\mathcal{D}}$ with automorphism γ_u.

We continue with the group F and let

$$A := \{x \in M \mid x^f = x\} \quad \text{and} \quad B := \{x \in M \mid x^f = x^{-1}\}.$$

Then $\#A = 6$ and $\#B = 38$. Since $\#B = 38$ and each arcuate block through $\mathbb{1}$ with orbit length 2 under conjugation by f contains exactly one element of B, there are exactly two arcuate blocks stabilized by γ_f. But then we may only have two fixed elements under conjugation by f, a contradiction. There can thus be no affine $\mathrm{SL}(2,7)$-unital $\mathbb{U}_{S,\mathcal{D}}$ with automorphism γ_f.

Last, we consider the group $V \cong C_3$ and assume $\mathbb{U}_{S,\mathcal{D}}$ to be an affine $\mathrm{SL}(2,7)$-unital with automorphism γ_v. Let

$$A := \{x \in M \mid x^v = x\}$$

and compute $\#A = 4$. The possible orbit lengths of the action of γ_v on the arcuate blocks of $\mathbb{U}_{S,\mathcal{D}}$ are 1 and 3. If γ_v stabilizes $\hat{D}$ for $D \in \mathcal{D}$, then there are at least two fixed blocks in $\hat{D}$, since $\#\hat{D} = 8$. But since the number of fixed elements under conjugation by v is $4 < 6$, the number of fixed blocks in $\hat{D}$ is at most 2. Thus, γ_v acts on the set of hats $\{\hat{D} \mid D \in \mathcal{D}\}$ with two fixed hats and one orbit of length 3.

[6] The corresponding files are `order7_quat_automU` and `order7_conclusion`.

We search via GAP[7] for all pairs of two arcuate blocks D_1 and D_2 through $\mathbb{1}$ with one fixed element each and such that $D_1^* \cap D_2^* = \varnothing$. For each such pair of blocks, we search for an arcuate block D through $\mathbb{1}$ such that $D^* \cup (D^v)^* \cup (D^{v^2})^* = M \setminus (D_1^* \cup D_2^*)$. We find that there is no affine unital $\mathbb{U}_{S,\mathcal{D}}$ with automorphism γ_v.

Again, having considered all three conjugacy classes of minimal subgroups of $\mathfrak{A}_S$, we state the following

Theorem 6.3 (by exhaustive computer search)**.** *Let S be a quaternion subgroup of* $\mathrm{SL}(2,7)$*. Then there is no affine* $\mathrm{SL}(2,7)$*-unital $\mathbb{U}_{S,\mathcal{D}}$ with non-trivial stabilizer of $\mathbb{1}$ in its full automorphism group.* □

For order 7, there were no new unitals appearing throughout the search. This will be different for order 8.

6.1.3 Order 8

We consider the case $q = 8$. Let $\mathbb{F}_8^\times = \langle z \rangle$, with $z^3 = z + 1$. Then, $X^2 + X + 1$ has no root in $\mathbb{F}_8$. Let further φ be the Frobenius automorphism

$$\varphi\colon \mathbb{F}_8 \to \mathbb{F}_8, \quad x \mapsto x^2,$$

of order 3. Since $q = 8$ is even, any subgroup $S \le \mathrm{SL}(2,8)$ of order 9 must be cyclic and we may hence choose $S = C$ as in Remark 2.7 to be

$$\{\left(\begin{smallmatrix} a & b \\ b & a+b \end{smallmatrix}\right) \mid a, b \in \mathbb{F}_8,\ a^2 + ab + b^2 = 1\}.$$

A generator of C is given by $g := \left(\begin{smallmatrix} z^2 & z^4 \\ z^4 & z \end{smallmatrix}\right)$. Let $f := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then

$$\mathfrak{A}_C = \mathrm{Aut}(\mathrm{SL}(2,8))_C = \langle \gamma_g \rangle \rtimes \langle \gamma_f \cdot \varphi \rangle \cong C_9 \rtimes C_6.$$

Again, our aim is to search for $\mathrm{SL}(2,8)$-unitals which admit certain subgroups of $\mathfrak{A}_C$ as groups of automorphisms. Consider the subgroups

$F := \langle \gamma_f \rangle \cong \langle \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle \cong C_2,$

$U := \langle \gamma_{g^3} \rangle \cong \langle \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle \cong C_3$ and

$L := \langle \varphi \rangle \cong C_3.$

Every other representative of a conjugacy class of non-trivial subgroups of $\mathfrak{A}_C$ contains (an $\mathfrak{A}_C$-conjugate of) one of these as a subgroup.

[7] The corresponding file is `order7_quat_automV`.

Example 6.4 (The classical affine unital of order 8)**.** Let

$$\begin{aligned}
&H_1 := \{\mathbb{1}, \left(\begin{smallmatrix} z^5 & 1 \\ z^5 & z^6 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^4 & z^2 \\ 1 & z^2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & z \\ z^6 & z^5 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^3 & z^6 \\ z^4 & z^5 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^3 & z \\ z^6 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & z^2 \\ 1 & z^6 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^4 & 1 \\ z^5 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^5 & 0 \\ 0 & z^2 \end{smallmatrix}\right)\}, \\
&H_2 := H_1 \cdot \varphi,\ H_3 := H_1 \cdot \varphi^2, \\
&H_4 := \{\mathbb{1}, \left(\begin{smallmatrix} z^5 & 0 \\ z^6 & z^2 \end{smallmatrix}\right), \left(\begin{smallmatrix} z & z^6 \\ z^4 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & z^5 \\ z^6 & z^5 \end{smallmatrix}\right), \left(\begin{smallmatrix} z & z^4 \\ 0 & z^6 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^5 & z^2 \\ z^4 & z^4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & z^2 \\ z^5 & z^5 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & z^4 \\ z^3 & z^4 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^2 & z^5 \\ z^5 & z^6 \end{smallmatrix}\right)\}, \\
&H_5 := H_4 \cdot \varphi,\ H_6 := H_4 \cdot \varphi^2
\end{aligned}$$

and $\mathcal{H} := \{H_1, \dots, H_6\}$. Then $\mathbb{U}_{C,\mathcal{H}}$ is the classical affine unital of order 8. As indicated, φ acts on the set of hats $\{\hat{H} \mid H \in \mathcal{H}\}$ in two orbits of length 3. Conjugation by g stabilizes each $\hat{H}$ and acts transitively on the blocks of each $\hat{H}$. Conjugation by f also stabilizes each $\hat{H}$ but fixes exactly one block per $\hat{H}$.

We start our observations with the subgroup $F \leq \mathfrak{A}_C$. Assume $\mathbb{U}_{C,\mathcal{D}}$ to be an affine $\mathrm{SL}(2,8)$-unital on which f acts (via conjugation) as an automorphism. Let

$$M := \mathrm{SL}(2,8) \setminus (C \cup (\bigcup \mathfrak{P})),$$

where $\mathfrak{P}$ denotes the set of all Sylow 2-subgroups of $\mathrm{SL}(2,8)$. Recall $\#\mathcal{D} = q - 2 = 6$, where each of the six hats $\hat{D}$, $D \in \mathcal{D}$, contains $q + 1 = 9$ blocks. Let

$$A := \{x \in M \mid x^f = x\} \quad \text{and} \quad B := \{x \in M \mid x^f = x^{-1}\}$$

and compute $\#A = 0$ and $\#B = 48$. As above, each arcuate block through $\mathbb{1}$ contains at most one point $x \in B$. Since $\#B = 48 > 45 = 5 \cdot 9$, each D^* contains at least one $x \in B$ and the action of γ_f fixes every $\hat{D}$. But since every $\hat{D}$ contains 9 blocks and γ_f has order 2, there is at least one fixed block in each $\hat{D}$. The number of elements in B yields that there is exactly one fixed block in each $\hat{D}$. Since there is no fixed element under γ_f in M and γ_f has order 2, each fixed arcuate block through $\mathbb{1}$ is of the form

$$D = \{\mathbb{1}, a, b, c, d, a^f, b^f, c^f, d^f\}$$

with $a, b, c, d \in M \setminus B$. Again, we compute all possible such blocks with GAP[8] and search in the resulting list for a set $\mathcal{D}$ of six blocks such that the sets D^*, $D \in \mathcal{D}$, have pairwise empty intersection[9].

[8] The corresponding file is order8_automF.
[9] The corresponding file is order8_conclusion. This file is also used to conclude the search with group of automorphisms U.

We obtain as expected the classical affine unital, but there is indeed one more affine $\mathrm{SL}(2,8)$-unital $\mathbb{U}_{C,\mathcal{D}}$ with automorphism γ_f, introduced in the following

Theorem 6.5 (Weihnachtsunital)**.** *Let* $C := \langle g \rangle = \langle \left(\begin{smallmatrix} z^2 & z^4 \\ z^4 & z \end{smallmatrix}\right) \rangle$ *as above and let*

$$\begin{aligned}
D_1 &:= \{\mathbb{1}, \left(\begin{smallmatrix} z^5 & 1 \\ z^5 & z^6 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^4 & z^2 \\ 1 & z^2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & z \\ z^6 & z^2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & z^4 \\ z^2 & z^2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & z \\ z^6 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & z^2 \\ 1 & z^6 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^4 & 1 \\ z^5 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^5 & 0 \\ 0 & z^2 \end{smallmatrix}\right)\},\\
D_2 &:= D_1 \cdot \varphi,\ D_3 := D_1 \cdot \varphi^2,\\
D_4 &:= \{\mathbb{1}, \left(\begin{smallmatrix} z^5 & 0 \\ z^6 & z^2 \end{smallmatrix}\right), \left(\begin{smallmatrix} z & z^6 \\ z^4 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & z \\ z^6 & z^5 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^4 & 0 \\ z^2 & z^3 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^5 & z^2 \\ z^3 & z^6 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & z^2 \\ z^5 & z^5 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & z^4 \\ z^3 & z^4 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^2 & z^5 \\ z^5 & z^6 \end{smallmatrix}\right)\},\\
D_5 &:= D_4 \cdot \varphi,\ D_6 := D_4 \cdot \varphi^2
\end{aligned}$$

and $\mathcal{D} := \{D_1, \ldots, D_6\}$. *Then* $\mathbb{WU} := \mathbb{U}_{C,\mathcal{D}}$ *is an affine* $\mathrm{SL}(2,8)$*-unital and we call it* **Weihnachtsunital**[10]. *The stabilizer of* $\mathbb{1}$ *in* $\mathrm{Aut}(\mathbb{WU})$ *is*

$$\mathrm{Aut}(\mathbb{WU})_{\mathbb{1}} = U \rtimes (F \times L) = \langle \gamma_{g^3} \rangle \rtimes \langle \gamma_f \cdot \varphi \rangle \cong C_3 \rtimes C_6$$

and the full automorphism group

$$\mathrm{Aut}(\mathbb{WU}) = \mathrm{Aut}(\mathbb{WU})_{\mathbb{1}} \ltimes R$$

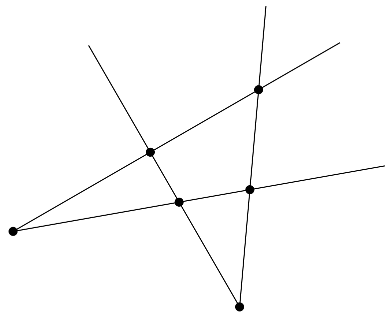
has index 3 *in* $\mathrm{Aut}(\mathbb{U}_{C,\mathcal{H}}) = \mathfrak{A}_C \ltimes R$.

Proof. The proof is basically computation. Note that the given description already uses the automorphism $\varphi \in \mathrm{Aut}(\mathbb{WU})_{\mathbb{1}}$. As implemented in the search, conjugation by f stabilizes each hat with exactly one fixed block per hat. Conjugation by the generator g of C does not induce an automorphism of $\mathbb{WU}$, but conjugation by g^3 yields an automorphism of $\mathbb{WU}$ such that each hat is fixed. $\square$

Remark 6.6. *As indicated in the proof of the theorem, the action of* $\mathrm{Aut}(\mathbb{WU})_{\mathbb{1}} \leq \mathfrak{A}_C$ *on the set of hats of* $\mathbb{WU}$ *is the same as in the classical affine* $\mathrm{SL}(2,8)$*-unital* $\mathbb{U}_{C,\mathcal{H}}$.

Having computed the full automorphism group of $\mathrm{Aut}(\mathbb{WU})$, we know in particular that the Weihnachtsunital is not isomorphic to the classical affine $\mathrm{SL}(2,8)$-unital $\mathbb{U}_{C,\mathcal{H}}$. Another way to see that $\mathbb{WU}$ is not isomorphic to $\mathbb{U}_{C,\mathcal{H}}$ is via O'Nan configurations. An O'Nan configuration consists of four distinct blocks meeting in six distinct points:

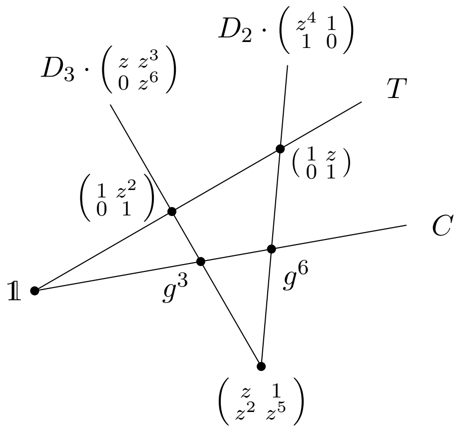
[10] The Weihnachtsunital was discovered around Christmas 2017, so you might guess why it is called like this.



O'Nan observed that classical unitals do not contain such configurations (see [22, p. 507]).

Remark 6.7. *In $\mathbb{WU}$, there are lots of O'Nan configurations, e. g.*

$$\begin{aligned}
C &= \{\mathbb{1}, g, g^2, g^3, g^4, g^5, g^6, g^7, g^8\},\\
T &\coloneqq \{\mathbb{1}, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & z^2 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & z^3 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & z^4 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & z^5 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & z^6 \\ 0 & 1 \end{smallmatrix}\right)\},\\
D_2 \cdot \left(\begin{smallmatrix} z^4 & 1 \\ 1 & 0 \end{smallmatrix}\right) &= \{\left(\begin{smallmatrix} z^4 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & z^3 \\ z^4 & z^3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^2 & 0 \\ z & z^5 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^2 & 1 \\ z^2 & z^4 \end{smallmatrix}\right), \left(\begin{smallmatrix} z & 1 \\ z^2 & z^5 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^4 & z \\ 0 & z^3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & z^3 \\ z^4 & 0 \end{smallmatrix}\right)\},\\
D_3 \cdot \left(\begin{smallmatrix} z & z^3 \\ 0 & z^6 \end{smallmatrix}\right) &= \{\left(\begin{smallmatrix} z & z^3 \\ 0 & z^6 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^3 & z^4 \\ z & z \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & z^3 \\ z^4 & z^2 \end{smallmatrix}\right), \left(\begin{smallmatrix} z & 1 \\ z^2 & z^5 \end{smallmatrix}\right), \left(\begin{smallmatrix} z & 0 \\ z^4 & z^6 \end{smallmatrix}\right), \left(\begin{smallmatrix} z & z \\ z & z^5 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^3 & z \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & z^2 \\ 0 & 1 \end{smallmatrix}\right)\}.
\end{aligned}$$



We continue our search with the group $U = \langle \gamma_u \rangle$, $u \coloneqq g^3$, and assume $\mathbb{U}_{C,\mathcal{D}}$ to be an affine $\mathrm{SL}(2,8)$-unital with automorphism γ_u. Again,

$$M \coloneqq \mathrm{SL}(2,8) \setminus (C \cup (\bigcup \mathfrak{P})) \text{ and } A \coloneqq \{x \in M \mid x^u = x\}.$$

We compute $\#A = 0$ and conclude that U does not fix any arcuate block, since there are eight non-trivial elements in each arcuate block through $\mathbb{1}$ and γ_u has order 3. Assume that γ_u fixes each hat $\hat{D}$ with $D \in \mathcal{D}$ (as it does on $\mathbb{U}_{C,\mathcal{H}}$ and $\mathbb{WU}$). Then γ_u acts on $\hat{D}$ in three orbits of length 3. We look at the table $\mathcal{T}_D$ and assume without restriction that

γ_u permutes columns (and rows) 1–3, 4–6 and 7–9 of $\mathcal{T}_D$. Then

$$\mathcal{T}_D : \quad \begin{array}{|ccc|ccc|ccc|} \hline \mathbb{1} & * & a^{u^2} & * & * & * & * & * & * \\ a & \mathbb{1} & * & * & * & * & * & * & * \\ * & a^u & \mathbb{1} & * & * & * & * & * & * \\ \hline * & * & * & \mathbb{1} & * & b^{u^2} & * & * & * \\ * & * & * & b & \mathbb{1} & * & * & * & * \\ * & * & * & * & b^u & \mathbb{1} & * & * & * \\ \hline * & * & * & * & * & * & \mathbb{1} & * & c^{u^2} \\ * & * & * & * & * & * & c & \mathbb{1} & * \\ * & * & * & * & * & * & * & c^u & \mathbb{1} \\ \hline \end{array}$$

and we get $a^{u^2} = (a^u a)^{-1}$, $b^{u^2} = (b^u b)^{-1}$ and $c^{u^2} = (c^u c)^{-1}$. Computing with GAP, we find exactly $18 = 6 \cdot 3$ three-element subsets of M of the form $\{x, x^u, x^{u^2}\}$ with $x^{u^2} = (x^u x)^{-1}$. We may thus search for arcuate blocks of the form

$$D = \{\mathbb{1}, a, a^u a, x, bx, b^u bx, y, cy, c^u cy\},$$

where a, b and c satisfy the described property while x and y do not[11]. In the resulting list of arcuate blocks, we search as above for a set $\mathcal{D}$ of six blocks such that the sets D^*, $D \in \mathcal{D}$, have pairwise empty intersection.

Since γ_u acts on the classical affine unital as well as on the Weihnachtsunital in this way, both of them appear in the results of the search. But we obtain indeed two more affine SL(2, 8)-unitals with automorphism γ_u, described in the following

Theorem 6.8 (Osterunital and Pfingstunital[12]). *Let* $C := \langle g \rangle = \langle \left(\begin{smallmatrix} z^2 & z^4 \\ z^4 & z \end{smallmatrix}\right) \rangle$ *as above.*

(a) *Let*

$$D_1 := \{\mathbb{1}, \left(\begin{smallmatrix} z^5 & 1 \\ z^5 & z^6 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^4 & z^2 \\ 1 & z^2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & z \\ z^6 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & z \\ z^6 & z^2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & z^4 \\ z^2 & z^2 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^3 & z^5 \\ z^3 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^5 & z^4 \\ z^2 & z^4 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^2 & 0 \\ 0 & z^5 \end{smallmatrix}\right)\},$$
$$D_2 := D_1^g,\ D_3 := D_1^{g^2},$$
$$D_4 := \{\mathbb{1}, \left(\begin{smallmatrix} z^5 & 0 \\ z^6 & z^2 \end{smallmatrix}\right), \left(\begin{smallmatrix} z & z^6 \\ z^4 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^5 & z^2 \\ z^5 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^3 & z^4 \\ z^6 & z^5 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ z^3 & z \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & z \\ 1 & z^3 \end{smallmatrix}\right), \left(\begin{smallmatrix} z & z^2 \\ 1 & z^5 \end{smallmatrix}\right), \left(\begin{smallmatrix} z & 0 \\ z & z^6 \end{smallmatrix}\right)\},$$
$$D_5 := D_4^g,\ D_6 := D_4^{g^2}$$

and $\mathcal{D} := \{D_1, \ldots, D_6\}$. *Then* $\mathbb{OU} := \mathbb{U}_{C,\mathcal{D}}$ *is an affine* SL(2, 8)*-unital and we call it* **Osterunital**.

[11] The corresponding file is order8_automU.

[12] The Osterunital and the Pfingstunital were discovered in 2018, you might guess the approximate dates.

(b) *Let* $f = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *as above and let*

$$\begin{gathered} D'_1 \coloneqq D_1,\ D'_2 \coloneqq D_2,\ D'_3 \coloneqq D_3, \\ D'_4 \coloneqq D_4^f,\ D'_5 \coloneqq (D'_4)^g,\ D'_6 \coloneqq (D'_4)^{g^2} \end{gathered}$$

and $\mathcal{D}' \coloneqq \{D'_1, \ldots, D'_6\}$. *Then* $\mathbb{PU} \coloneqq \mathbb{U}_{C,\mathcal{D}'}$ *is an affine* $\mathrm{SL}(2,8)$*-unital and we call it* **Pfingstunital**.

The full stabilizers of $\mathbb{1}$ *in* $\mathrm{Aut}(\mathbb{OU})$ *and* $\mathrm{Aut}(\mathbb{PU})$*, respectively, are*

$$\mathrm{Aut}(\mathbb{OU})_{\mathbb{1}} = \mathrm{Aut}(\mathbb{PU})_{\mathbb{1}} = C \rtimes L = \langle \gamma_g \rangle \rtimes \langle \varphi \rangle \cong C_9 \rtimes C_3$$

and the full automorphism groups

$$\mathrm{Aut}(\mathbb{OU}) = \mathrm{Aut}(\mathbb{PU}) = (C \rtimes L) \ltimes R$$

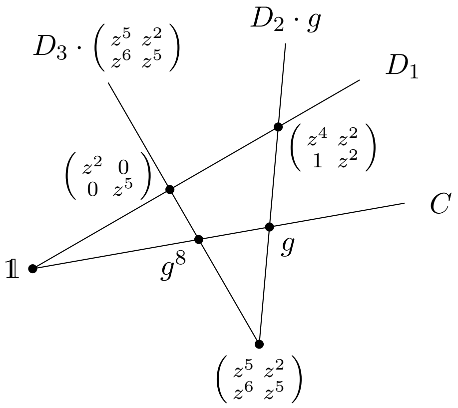
have index 2 *in* $\mathrm{Aut}(\mathbb{U}_{C,\mathcal{H}})$.

Proof. Again this is basically computation. The given description already uses the automorphism γ_g in both $\mathrm{Aut}(\mathbb{OU})_{\mathbb{1}}$ and $\mathrm{Aut}(\mathbb{PU})_{\mathbb{1}}$. The Frobenius automorphism φ acts as automorphism on $\mathbb{OU}$ as well as on $\mathbb{PU}$ in the same way as it does on $\mathbb{U}_{C,\mathcal{H}}$ and $\mathbb{WU}$. The orbits of φ in $\mathcal{D}$ are $\{D_1, D_2, D_3\}$ and $\{D_4, D_5, D_6\}$ and its orbits in $\mathcal{D}'$ are $\{D'_1, D'_2, D'_3\}$ and $\{D'_4, D'_5, D'_6\}$. Conjugation by f induces no automorphism on neither $\mathbb{OU}$ nor $\mathbb{PU}$. □

Remark 6.9. *Other than in the Weihnachtsunital, there is a difference between the action of* $\mathrm{Aut}(\mathbb{OU})_{\mathbb{1}} = \mathrm{Aut}(\mathbb{PU})_{\mathbb{1}} \leq \mathfrak{A}_C$ *on the set of hats of the Oster- and Pfingstunital, respectively, and its action on the set of hats of the classical affine* $\mathrm{SL}(2,8)$*-unital* $\mathbb{U}_{C,\mathcal{H}}$. *In* $\mathbb{U}_{C,\mathcal{H}}$*, conjugation by* g *fixes every hat, while on* $\mathbb{OU}$ *and* $\mathbb{PU}$ *it acts on the set of hats in two orbits of length* 3.

Remark 6.10. *As in the Weihnachtsunital, there are also many O'Nan configurations in* $\mathbb{OU}$ *and* $\mathbb{PU}$*, e. g.*

$$\begin{aligned} C &= \{\mathbb{1}, g, g^2, g^3, g^4, g^5, g^6, g^7, g^8\}, \\ D_1 &= \{\mathbb{1}, \left(\begin{smallmatrix} z^5 & 1 \\ z^5 & z^6 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^4 & z^2 \\ 1 & z^2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & z \\ z^6 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & z \\ z^6 & z^2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & z^4 \\ z^2 & z^2 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^3 & z^5 \\ z^3 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^5 & z^4 \\ z^2 & z^4 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^2 & 0 \\ 0 & z^5 \end{smallmatrix}\right)\}, \\ D_2 \cdot g &= \{g, \left(\begin{smallmatrix} z^4 & z^2 \\ 1 & z^2 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^3 & 0 \\ z^3 & z^4 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^5 & z^5 \\ z^3 & z^5 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^6 & 0 \\ 1 & z \end{smallmatrix}\right), \left(\begin{smallmatrix} z^6 & z \\ z^3 & z^6 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^5 & z^2 \\ z^6 & z^5 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & z^3 \\ z^4 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^2 & z^3 \\ 0 & z^5 \end{smallmatrix}\right)\}, \\ D_3 \cdot \left(\begin{smallmatrix} z^5 & z^2 \\ z^6 & z^5 \end{smallmatrix}\right) &= \{\left(\begin{smallmatrix} z^5 & z^2 \\ z^6 & z^5 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^2 & 0 \\ 0 & z^5 \end{smallmatrix}\right), \left(\begin{smallmatrix} z & z^2 \\ z^3 & z^3 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^2 & 1 \\ z^6 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^6 & z \\ z^5 & z^3 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^4 & z^3 \\ z^4 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} z & z^4 \\ z^4 & z^2 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^3 & z^2 \\ z^5 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^3 & z \\ z^4 & z \end{smallmatrix}\right)\}. \end{aligned}$$



Although they look quite similar, the Osterunital and the Pfingstunital are not isomorphic, as shown in the following

Proposition 6.11. *There is no isomorphism between* $\mathbb{OU}$ *and* $\mathbb{PU}$*.*

Proof. According to Theorem 4.4, any isomorphism between $\mathbb{OU}$ and $\mathbb{PU}$ must be contained in $\mathfrak{A}_C \ltimes R$. But since the index of $\mathrm{Aut}(\mathbb{OU})$ in $\mathfrak{A}_C \ltimes R$ equals 2 and computation shows that D_1^f is no block of $\mathbb{PU}$, the statement follows. □

In particular, the Oster- and Pfingstunital are two non-isomorphic affine $\mathrm{SL}(2,q)$-unitals with the same full automorphism group.

We conclude the search with the group $L = \langle \varphi \rangle$ and assume $\mathbb{U}_{C,\mathcal{D}}$ to be an affine $\mathrm{SL}(2,8)$-unital with group of automorphisms L. As for γ_u, we assume φ to act on the set of hats of $\mathbb{U}_{C,\mathcal{D}}$ as it does in $\mathbb{U}_{C,\mathcal{H}}$ and in $\mathbb{WU}$, namely with two orbits of length 3. We search with GAP[13] for every arcuate block D through $\mathbb{1}$ such that D^*, $(D \cdot \varphi)^*$ and $(D \cdot \varphi^2)^*$ have pairwise empty intersection and for each of those blocks we search in $M' := M \setminus (D^* \cup (D \cdot \varphi)^* \cup (D \cdot \varphi^2)^*)$ for an arcuate block D_2 with

$$D_2^* \cup (D_2 \cdot \varphi)^* \cup (D_2 \cdot \varphi^2)^* = M'.$$

Doing so, we find (as expected) all affine unitals $\mathbb{U}_{C,\mathcal{H}}$, $\mathbb{WU}$, $\mathbb{OU}$ and $\mathbb{PU}$. Other affine unitals are not found.

[13] The corresponding file is `order8_automL`.

We summarize the results in the following

Theorem 6.12 (by exhaustive computer search)**.** *Let* $C := \langle g \rangle = \langle \left(\begin{smallmatrix} z^2 & z^4 \\ z^4 & z \end{smallmatrix}\right) \rangle$ *and let* $\mathbb{U}_{C,\mathcal{D}}$ *be an affine* $\mathrm{SL}(2,8)$*-unital.*

(a) *If conjugation by* $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *induces an automorphism on* $\mathbb{U}_{C,\mathcal{D}}$*, then* $\mathbb{U}_{C,\mathcal{D}}$ *is isomorphic to either the classical affine* $\mathrm{SL}(2,8)$*-unital or the Weihnachtsunital.*

(b) *If conjugation by* $g^3 = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *induces an automorphism on* $\mathbb{U}_{C,\mathcal{D}}$ *such that each hat is fixed, then* $\mathbb{U}_{C,\mathcal{D}}$ *is isomorphic to either the classical affine* $\mathrm{SL}(2,8)$*-unital, the Weihnachtsunital, the Osterunital or the Pfingstunital.*

(c) *If the Frobenius automorphism* φ *acts as automorphism on* $\mathbb{U}_{C,\mathcal{D}}$ *such that it permutes the hats in two orbits of length* 3*, then* $\mathbb{U}_{C,\mathcal{D}}$ *is isomorphic to either the classical affine* $\mathrm{SL}(2,8)$*-unital, the Weihnachtsunital, the Osterunital or the Pfingstunital.* □

6.2 Search for Parallelisms

We will now search with GAP[14] for parallelisms on the short blocks of affine $\mathrm{SL}(2,q)$-unitals, i. e. partitions of the set of all right cosets of the Sylow p-subgroups of $\mathrm{SL}(2,q)$ into $q+1$ sets of q^2-1 pairwise disjoint cosets each. We already know for each q the parallelisms $\flat$ and $\natural$ and for each odd q the parallelism $\pi^\square$ (introduced in Section 5.1).

6.2.1 Orders 3 and 5

For $q \in \{3,5\}$, we find by an exhaustive search $(q-1)q(q+1)+2$ parallelisms on the short blocks. As seen in Section 6.1, for both $q=3$ and $q=5$, the classical affine $\mathrm{SL}(2,q)$-unital $\mathbb{U}_{C,\mathcal{H}}$ is up to isomorphisms the only affine $\mathrm{SL}(2,q)$-unital. The orbit lengths of the action of $\mathrm{Aut}(\mathbb{U}_{C,\mathcal{H}}) = \mathfrak{A}_C \ltimes R$ on the sets of parallelisms are the following:

$q=3$	$q=5$
1	1
1	1
24	120

Hence, there are up to equivalence exactly three parallelisms for both $q=3$ and $q=5$; namely $\flat$ and $\natural$ (which are both stabilized by $\mathfrak{A}_C \ltimes R$) and $\pi^\square$. Note that the orbit lengths 24 and 120 equal $(q-1)q(q+1)$, which matches the fact that the stabilizer of $\pi^\square$ in $\mathfrak{A}_C \ltimes R$ has order $2e(q+1)$, see Corollary 5.3.

[14] The corresponding file is parallelisms_search for $q \in \{3,4,5\}$.

We summarize the results in the following

Theorem 6.13 (by exhaustive computer search)**.** *For $q \in \{3,5\}$, there are exactly three isomorphism types of* $\mathrm{SL}(2,q)$*-unitals each, represented by* $\mathbb{U}^{\natural}_{C,\mathcal{H}}$*,* $\mathbb{U}^{\flat}_{C,\mathcal{H}}$ *and* $\mathbb{U}^{\pi^{\square}}_{C,\mathcal{H}}$*.* □

6.2.2 Order 4

Let $\mathbb{F}_4^{\times} = \langle z \rangle$, with $z^2 = z+1$. Then, $X^2 + X + z$ has no root in $\mathbb{F}_4$. Let φ be the Frobenius automorphism

$$\varphi \colon \mathbb{F}_4 \to \mathbb{F}_4, \quad x \mapsto x^2,$$

of order 2. We choose $S = C$ as in Remark 2.7 to be

$$\{\left(\begin{smallmatrix} a & b \\ zb & a+b \end{smallmatrix}\right) \mid a, b \in \mathbb{F}_4,\ a^2 + ab + zb^2 = 1\}.$$

A generator of C is given by $g := \left(\begin{smallmatrix} 0 & z \\ z^2 & z \end{smallmatrix}\right)$. Let $f := \left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right)$. Then

$$\mathfrak{A}_C = \mathrm{Aut}(\mathrm{SL}(2,4))_C = \langle \gamma_g \rangle \rtimes \langle \gamma_f \cdot \varphi \rangle \cong C_5 \rtimes C_4.$$

Example 6.14 (Affine $\mathrm{SL}(2,4)$-unitals)**.** Let $\mathcal{H} := \{H_1, H_2\}$ and $\mathcal{E} := \{E_1, E_2\}$ with

$$\begin{aligned} H_1 &:= \{\mathbb{1}, \left(\begin{smallmatrix} z^2 & 0 \\ 0 & z \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & z \end{smallmatrix}\right), \left(\begin{smallmatrix} z^2 & z^2 \\ z^2 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)\}, \\ H_2 &:= H_1^f \cdot \varphi, \\ E_1 &:= \{\mathbb{1}, \left(\begin{smallmatrix} z^2 & z^2 \\ 0 & z \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & z \end{smallmatrix}\right), \left(\begin{smallmatrix} z^2 & 0 \\ z^2 & z \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)\}, \\ E_2 &:= E_1^f \cdot \varphi. \end{aligned}$$

Then, $\mathbb{U}_{C,\mathcal{H}}$ equals the classical affine unital and $\mathbb{U}_{C,\mathcal{E}}$ equals the non-classical affine $\mathrm{SL}(2,4)$-unital presented in [9]. The full automorphism group of $\mathbb{U}_{C,\mathcal{H}}$ equals $\mathfrak{A}_C \ltimes R$ and the full automorphism group of $\mathbb{U}_{C,\mathcal{E}}$ equals $\langle \gamma_f \cdot \varphi \rangle \ltimes R$, see [9, Theorem 4.1].

For order 4, an exhaustive computer search yields 182 parallelisms on the short blocks. We consider the actions of the automorphism groups

$$\mathrm{Aut}(\mathfrak{S}) = (\mathfrak{A} \times I) \cdot R, \quad \mathrm{Aut}(\mathbb{U}_{C,\mathcal{H}}) = \mathfrak{A}_C \ltimes R \quad \text{and} \quad \mathrm{Aut}(\mathbb{U}_{C,\mathcal{E}}) = \langle \gamma_f \cdot \varphi \rangle \ltimes R$$

on the set of 182 parallelisms found by GAP. The orbit lengths of these actions are listed in Table 6.1.

Table 6.1: Orbit lengths on the set of 182 parallelisms for order 4

$\mathrm{Aut}(\mathfrak{S})$	$\mathrm{Aut}(\mathbb{U}_{C,\mathcal{H}})$	$\mathrm{Aut}(\mathbb{U}_{C,\mathcal{E}})$
2	1	1
	1	1
60	30	24
		6
	25	20
		5
	5	5
120	60	60
	60	60

The orbit of length 2 under the action of $\mathrm{Aut}(\mathfrak{S})$ contains the parallelisms $\flat$ and $\natural$, which are mapped to each other by inversion but are stabilized under the action of $\mathfrak{A} \ltimes R$.
The orbit of length 60 under the action of $\mathrm{Aut}(\mathfrak{S})$ contains parallelisms, where one parallel class is given by the set of all right (resp. left) cosets of one Sylow 2-subgroup. The remaining four parallel classes – consisting of $q^2 - 1 = 15$ blocks each – contain for a set of three Sylow 2-subgroups five right (resp. left) cosets each. This structure is indicated in Figure 6.1, where each of the small lines represents one short block and where blocks with the same dash pattern belong to the same parallel class. The five rows correspond to the sets of left cosets and the five columns to the sets of right cosets of the Sylow 2-subgroups.
The parallel classes of the parallelisms in the orbit of length 120 under the action of $\mathrm{Aut}(\mathfrak{S})$ contain eleven right (resp. left) cosets of one Sylow 2-subgroup each, while the remaining four blocks are given by left (resp. right) cosets of one (possibly different) Sylow 2-subgroup. This structure is indicated in Figure 6.2.

Let $\pi_1, \dots, \pi_7$ be representatives of the orbits of length greater than 1 under the action of $\mathrm{Aut}(\mathbb{U}_{C,\mathcal{E}})$ such that the orbits of π_1 and π_2 and the orbits of π_3 and π_4 coincide under the action of $\mathrm{Aut}(\mathbb{U}_{C,\mathcal{H}})$. Let the set of Sylow 2-subgroups of $\mathrm{SL}(2,4)$ be $\mathfrak{P} := \{T_1, \dots, T_5\}$ with

$$T_1 := \{\mathbb{1}, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & z^2 \\ 0 & 1 \end{smallmatrix}\right)\},$$
$$T_2 := \{\mathbb{1}, \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ z^2 & 1 \end{smallmatrix}\right)\},$$
$$T_3 := \{\mathbb{1}, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^2 & z \\ z & z^2 \end{smallmatrix}\right), \left(\begin{smallmatrix} z & z^2 \\ z^2 & z \end{smallmatrix}\right)\},$$
$$T_4 := \{\mathbb{1}, \left(\begin{smallmatrix} 0 & z \\ z^2 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^2 & z^2 \\ 1 & z^2 \end{smallmatrix}\right), \left(\begin{smallmatrix} z & 1 \\ z & z \end{smallmatrix}\right)\},$$

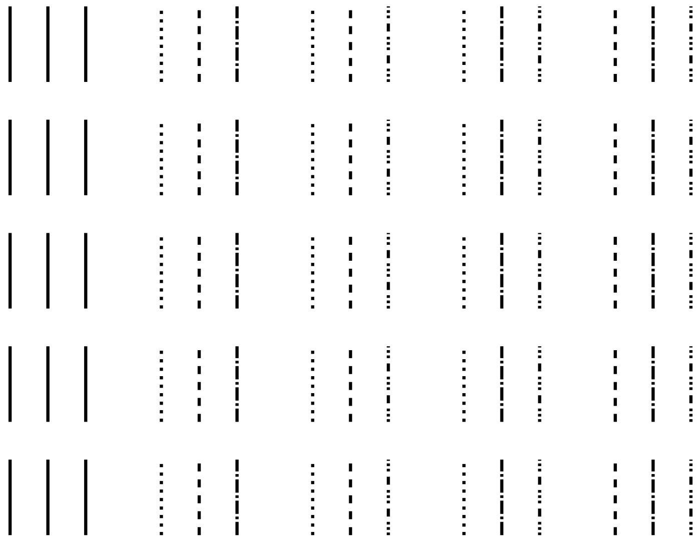


Figure 6.1: Structure of parallelisms in the orbit of length 60 under Aut($\mathfrak{S}$)

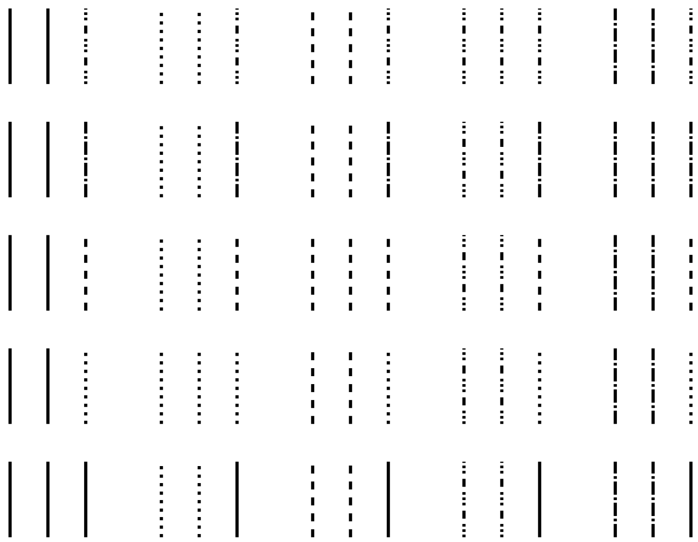


Figure 6.2: Structure of parallelisms in the orbit of length 120 under Aut($\mathfrak{S}$)

$$T_5 := \{\mathbb{1}, \left(\begin{smallmatrix} 0 & z^2 \\ z & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} z^2 & 1 \\ z^2 & z^2 \end{smallmatrix}\right), \left(\begin{smallmatrix} z & z \\ 1 & z \end{smallmatrix}\right)\},$$

and see Table 6.2 for the chosen parallelisms. Then there are twelve pairwise non-isomorphic SL(2, 4)-unitals

$$\mathbb{U}^{\pi_2}_{C,\mathcal{H}},\ \mathbb{U}^{\pi_4}_{C,\mathcal{H}},\ \mathbb{U}^{\pi_5}_{C,\mathcal{H}},\ \mathbb{U}^{\pi_6}_{C,\mathcal{H}},\ \mathbb{U}^{\pi_7}_{C,\mathcal{H}},\ \mathbb{U}^{\pi_1}_{C,\mathcal{E}},\ \mathbb{U}^{\pi_2}_{C,\mathcal{E}},\ \mathbb{U}^{\pi_3}_{C,\mathcal{E}},\ \mathbb{U}^{\pi_4}_{C,\mathcal{E}},\ \mathbb{U}^{\pi_5}_{C,\mathcal{E}},\ \mathbb{U}^{\pi_6}_{C,\mathcal{E}},\ \mathbb{U}^{\pi_7}_{C,\mathcal{E}}.$$

Since those parallelisms for order 4 were found during the Leonids meteor shower in November 2018, we will call the twelve resulting unitals **Leonids unitals**.

Knowing all affine SL(2, 4)-unitals and all parallelisms on the short blocks for order 4, we state the following

Theorem 6.15 (by exhaustive computer search)**.** *There are exactly* 16 *isomorphism types of* SL(2, 4)*-unitals, represented by* $\mathbb{U}^{\natural}_{C,\mathcal{H}}$, $\mathbb{U}^{\flat}_{C,\mathcal{H}}$, $\mathbb{U}^{\natural}_{C,\mathcal{E}}$, $\mathbb{U}^{\flat}_{C,\mathcal{E}}$ *and the twelve Leonids unitals.* □

Recall that the stabilizer of the block at infinity in any SL(2, q)-unital $\mathbb{U}^{\pi}_{S,\mathcal{D}}$ equals the group of those automorphisms of the affine unital $\mathbb{U}_{S,\mathcal{D}}$, which stabilize the parallelism π. We compute with GAP that in each Leonids unital there are indeed no automorphisms moving the block at infinity, and we are thus able to compute their full automorphism groups as subgroups of Aut($\mathbb{U}_{C,\mathcal{H}}$) or Aut($\mathbb{U}_{C,\mathcal{E}}$), respectively. Let

$$b := \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \text{ and } c := \left(\begin{smallmatrix} z & 0 \\ 0 & z^2 \end{smallmatrix}\right)$$

and see Table 6.3 for the full automorphism groups of the Leonids unitals.
Since in each Leonids unital, there is always at least one block fixed by its full automorphism group, no Leonids unital is isomorphic to a unital of the classes treated in Theorem 4.16. In [17], Krčadinac, Nakić and Pavčević introduce a method for the construction of designs with prescribed automorphism groups. Applying this method with some selected automorphism groups, they find 1777 non-isomorphic unitals of order 4. These unitals are contained in a library shipped with the GAP package UnitalSZ [20]. Checking isomorphisms with GAP, we find that none of the Leonids unitals is isomorphic to a unital in this list of 1777 unitals of order 4.

Remark 6.16. *Regarding the orders of the full automorphism groups of the Leonids unitals in* Table 6.3*, we see that the order of* Aut($\mathbb{U}^{\pi_5}_{C,\mathcal{H}}$) *is notably greater than the other orders. Indeed, although R is not contained in its full automorphism group, the unital* $\mathbb{U}^{\pi_5}_{C,\mathcal{H}}$ *admits a group of automorphisms which acts regularly on the affine points, namely* $(\langle \rho_b, \rho_c \rangle \times \langle \gamma_{g^{-1}} \cdot \rho_g \rangle) \cong A_4 \times C_5$.

Table 6.2: Representatives of the orbits of length greater than 1 of parallelisms under the action of $\mathrm{Aut}(\mathbb{U}_{C,\mathcal{E}})$

$\pi_1 :=$ $\{\{T_3 g \mid g \in \mathrm{SL}(2,q)\},$
$\{T_1\begin{pmatrix}z&0\\0&z^2\end{pmatrix}, T_1\begin{pmatrix}0&z^2\\z&0\end{pmatrix}, T_1\begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}, T_1\begin{pmatrix}0&z\\z^2&1\end{pmatrix}, T_1\begin{pmatrix}0&z^2\\z&1\end{pmatrix}, T_2\begin{pmatrix}0&1\\1&0\end{pmatrix}, T_2\begin{pmatrix}z^2&0\\0&z\end{pmatrix}, T_2\begin{pmatrix}1&1\\0&1\end{pmatrix}, T_2\begin{pmatrix}1&z\\0&1\end{pmatrix}, T_2\begin{pmatrix}z^2&z\\0&z\end{pmatrix}, T_4\begin{pmatrix}z&0\\1&z^2\end{pmatrix}, T_4\begin{pmatrix}0&z^2\\z&z\end{pmatrix}, T_4\begin{pmatrix}0&z\\z^2&z\end{pmatrix}, T_4, T_4\begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}\},$
$\{T_1\begin{pmatrix}z^2&0\\0&z\end{pmatrix}, T_1\begin{pmatrix}0&1\\1&0\end{pmatrix}, T_1\begin{pmatrix}0&z^2\\z&z\end{pmatrix}, T_1\begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}, T_1\begin{pmatrix}0&1\\1&z^2\end{pmatrix}, T_4\begin{pmatrix}1&0\\z^2&1\end{pmatrix}, T_4\begin{pmatrix}0&z\\z^2&1\end{pmatrix}, T_4\begin{pmatrix}0&1\\1&z\end{pmatrix}, T_4\begin{pmatrix}0&1\\1&1\end{pmatrix}, T_4\begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}, T_5\begin{pmatrix}z&0\\z^2&z^2\end{pmatrix}, T_5\begin{pmatrix}0&z^2\\z&1\end{pmatrix}, T_5\begin{pmatrix}0&z\\z^2&0\end{pmatrix}, T_5\begin{pmatrix}0&z\\z^2&z\end{pmatrix}, T_5\},$
$\{T_2\begin{pmatrix}0&z\\z^2&0\end{pmatrix}, T_2, T_2\begin{pmatrix}z&z\\0&z^2\end{pmatrix}, T_2\begin{pmatrix}z&z^2\\0&z^2\end{pmatrix}, T_2\begin{pmatrix}1&z^2\\0&1\end{pmatrix}, T_4\begin{pmatrix}z^2&0\\z&z\end{pmatrix}, T_4\begin{pmatrix}0&1\\1&z^2\end{pmatrix}, T_4\begin{pmatrix}0&z^2\\z&0\end{pmatrix}, T_4\begin{pmatrix}0&z^2\\z&1\end{pmatrix}, T_4\begin{pmatrix}0&1\\1&0\end{pmatrix}, T_5\begin{pmatrix}z^2&0\\1&z\end{pmatrix}, T_5\begin{pmatrix}0&1\\1&z\end{pmatrix}, T_5\begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}, T_5\begin{pmatrix}0&z^2\\z&z\end{pmatrix}, T_5\begin{pmatrix}0&1\\1&1\end{pmatrix}\},$
$\{T_1, T_1\begin{pmatrix}0&z\\z^2&0\end{pmatrix}, T_1\begin{pmatrix}0&1\\1&1\end{pmatrix}, T_1\begin{pmatrix}0&1\\1&z\end{pmatrix}, T_1\begin{pmatrix}0&z\\z^2&z\end{pmatrix}, T_2\begin{pmatrix}0&z^2\\z&0\end{pmatrix}, T_2\begin{pmatrix}z&0\\0&z^2\end{pmatrix}, T_2\begin{pmatrix}z^2&z^2\\0&z\end{pmatrix}, T_2\begin{pmatrix}z^2&1\\0&z\end{pmatrix}, T_2\begin{pmatrix}z&1\\0&z^2\end{pmatrix}, T_5\begin{pmatrix}1&0\\z&1\end{pmatrix}, T_5\begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}, T_5\begin{pmatrix}0&1\\1&z^2\end{pmatrix}, T_5\begin{pmatrix}0&1\\1&0\end{pmatrix}, T_5\begin{pmatrix}0&z\\z^2&1\end{pmatrix}\}\}$

$\pi_2 :=$ $\{\{T_2 g \mid g \in \mathrm{SL}(2,q)\},$
$\{T_1\begin{pmatrix}z&0\\0&z^2\end{pmatrix}, T_1\begin{pmatrix}0&z^2\\z&0\end{pmatrix}, T_1\begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}, T_1\begin{pmatrix}0&z\\z^2&1\end{pmatrix}, T_1\begin{pmatrix}0&z^2\\z&1\end{pmatrix}, T_3\begin{pmatrix}1&0\\1&1\end{pmatrix}, T_3\begin{pmatrix}0&z\\z^2&z\end{pmatrix}, T_3, T_3\begin{pmatrix}0&1\\1&z^2\end{pmatrix}, T_3\begin{pmatrix}0&z\\z^2&0\end{pmatrix}, T_5\begin{pmatrix}z^2&0\\1&z\end{pmatrix}, T_5\begin{pmatrix}0&1\\1&z\end{pmatrix}, T_5\begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}, T_5\begin{pmatrix}0&z^2\\z&z\end{pmatrix}, T_5\begin{pmatrix}0&1\\1&1\end{pmatrix}\},$
$\{T_1\begin{pmatrix}z^2&0\\0&z\end{pmatrix}, T_1\begin{pmatrix}0&1\\1&0\end{pmatrix}, T_1\begin{pmatrix}0&z^2\\z&z\end{pmatrix}, T_1\begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}, T_1\begin{pmatrix}0&1\\1&z^2\end{pmatrix}, T_4\begin{pmatrix}1&0\\z^2&1\end{pmatrix}, T_4\begin{pmatrix}0&z\\z^2&1\end{pmatrix}, T_4\begin{pmatrix}0&1\\1&z\end{pmatrix}, T_4\begin{pmatrix}0&1\\1&1\end{pmatrix}, T_4\begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}, T_5\begin{pmatrix}z&0\\z^2&z^2\end{pmatrix}, T_5\begin{pmatrix}0&z^2\\z&1\end{pmatrix}, T_5\begin{pmatrix}0&z\\z^2&0\end{pmatrix}, T_5\begin{pmatrix}0&z\\z^2&z\end{pmatrix}, T_5\},$
$\{T_1, T_1\begin{pmatrix}0&z\\z^2&0\end{pmatrix}, T_1\begin{pmatrix}0&1\\1&1\end{pmatrix}, T_1\begin{pmatrix}0&1\\1&z\end{pmatrix}, T_1\begin{pmatrix}0&z\\z^2&z\end{pmatrix}, T_3\begin{pmatrix}z&0\\z&z^2\end{pmatrix}, T_3\begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}, T_3\begin{pmatrix}0&z\\z^2&1\end{pmatrix}, T_3\begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}, T_3\begin{pmatrix}0&z^2\\z&z\end{pmatrix}, T_4\begin{pmatrix}z^2&0\\z&0\end{pmatrix}, T_4\begin{pmatrix}0&1\\1&z^2\end{pmatrix}, T_4\begin{pmatrix}0&z^2\\z&0\end{pmatrix}, T_4\begin{pmatrix}0&z^2\\z&1\end{pmatrix}, T_4\begin{pmatrix}0&1\\1&0\end{pmatrix}\},$
$\{T_3\begin{pmatrix}z^2&0\\z^2&z\end{pmatrix}, T_3\begin{pmatrix}0&1\\1&1\end{pmatrix}, T_3\begin{pmatrix}0&z^2\\z&1\end{pmatrix}, T_3\begin{pmatrix}0&z^2\\z&0\end{pmatrix}, T_3\begin{pmatrix}0&1\\1&z\end{pmatrix}, T_4\begin{pmatrix}z&0\\1&z^2\end{pmatrix}, T_4\begin{pmatrix}0&z^2\\z&z\end{pmatrix}, T_4\begin{pmatrix}0&z\\z^2&z\end{pmatrix}, T_4, T_4\begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}, T_5\begin{pmatrix}1&0\\z&1\end{pmatrix}, T_5\begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}, T_5\begin{pmatrix}0&1\\1&z^2\end{pmatrix}, T_5\begin{pmatrix}0&1\\1&0\end{pmatrix}, T_5\begin{pmatrix}0&z\\z^2&1\end{pmatrix}\}\}$

$\pi_3 :=$ $\{\{gT_2 \mid g \in \mathrm{SL}(2,q)\},$
$\{\begin{pmatrix}z^2&0\\0&z\end{pmatrix}T_1, \begin{pmatrix}0&1\\1&0\end{pmatrix}T_1, \begin{pmatrix}1&0\\1&1\end{pmatrix}T_1, \begin{pmatrix}z^2&0\\z&z\end{pmatrix}T_1, \begin{pmatrix}1&0\\z&1\end{pmatrix}T_1, \begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}T_3, \begin{pmatrix}z&z\\0&z^2\end{pmatrix}T_3, \begin{pmatrix}0&z^2\\z&1\end{pmatrix}T_3, \begin{pmatrix}0&z\\z^2&z\end{pmatrix}T_3, \begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}T_3, \begin{pmatrix}0&z\\z^2&1\end{pmatrix}T_4, \begin{pmatrix}z&z^2\\0&z^2\end{pmatrix}T_4, \begin{pmatrix}0&z^2\\z&0\end{pmatrix}T_4, T_4, \begin{pmatrix}0&z^2\\z&z\end{pmatrix}T_4\},$
$\{\begin{pmatrix}z&0\\0&z^2\end{pmatrix}T_1, \begin{pmatrix}0&z\\z^2&0\end{pmatrix}T_1, \begin{pmatrix}z^2&0\\z^2&z\end{pmatrix}T_1, \begin{pmatrix}z&0\\1&z^2\end{pmatrix}T_1, \begin{pmatrix}z^2&0\\1&z\end{pmatrix}T_1, \begin{pmatrix}0&z^2\\z&z\end{pmatrix}T_3, \begin{pmatrix}1&1\\0&1\end{pmatrix}T_3, T_3, \begin{pmatrix}0&z^2\\z&0\end{pmatrix}T_3, \begin{pmatrix}0&1\\1&z^2\end{pmatrix}T_3, \begin{pmatrix}0&z^2\\z&1\end{pmatrix}T_5, \begin{pmatrix}1&z^2\\0&1\end{pmatrix}T_5, \begin{pmatrix}0&1\\1&z\end{pmatrix}T_5, \begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}T_5, \begin{pmatrix}0&1\\1&1\end{pmatrix}T_5\},$
$\{T_1, \begin{pmatrix}0&z^2\\z&0\end{pmatrix}T_1, \begin{pmatrix}z&0\\z&z^2\end{pmatrix}T_1, \begin{pmatrix}1&0\\z^2&1\end{pmatrix}T_1, \begin{pmatrix}z&0\\z^2&z^2\end{pmatrix}T_1, \begin{pmatrix}0&1\\1&z\end{pmatrix}T_4, \begin{pmatrix}z^2&1\\0&z\end{pmatrix}T_4, \begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}T_4, \begin{pmatrix}0&1\\1&1\end{pmatrix}T_4, \begin{pmatrix}0&z\\z^2&z\end{pmatrix}T_4, \begin{pmatrix}0&1\\1&z^2\end{pmatrix}T_5, \begin{pmatrix}z^2&z\\0&z\end{pmatrix}T_5, \begin{pmatrix}0&z\\z^2&0\end{pmatrix}T_5, \begin{pmatrix}0&1\\1&0\end{pmatrix}T_5, \begin{pmatrix}0&z\\z^2&1\end{pmatrix}T_5\},$
$\{\begin{pmatrix}0&1\\1&1\end{pmatrix}T_3, \begin{pmatrix}z^2&z^2\\0&z\end{pmatrix}T_3, \begin{pmatrix}0&z\\z^2&1\end{pmatrix}T_3, \begin{pmatrix}0&1\\1&z\end{pmatrix}T_3, \begin{pmatrix}0&z\\z^2&0\end{pmatrix}T_3, \begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}T_4, \begin{pmatrix}1&z\\0&1\end{pmatrix}T_4, \begin{pmatrix}0&1\\1&z^2\end{pmatrix}T_4, \begin{pmatrix}0&z^2\\z&1\end{pmatrix}T_4, \begin{pmatrix}0&1\\1&0\end{pmatrix}T_4, \begin{pmatrix}0&z\\z^2&z\end{pmatrix}T_5, \begin{pmatrix}z&1\\0&z^2\end{pmatrix}T_5, \begin{pmatrix}0&z^2\\z&z\end{pmatrix}T_5, \begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}T_5, T_5\}\}$

$\pi_4 :=$ $\{\{gT_2 \mid g \in \mathrm{SL}(2,q)\},$
$\{\begin{pmatrix}z^2&0\\0&z\end{pmatrix}T_1, \begin{pmatrix}0&1\\1&0\end{pmatrix}T_1, \begin{pmatrix}z^2&0\\z^2&z\end{pmatrix}T_1, \begin{pmatrix}z&0\\1&z^2\end{pmatrix}T_1, \begin{pmatrix}z&0\\z^2&z^2\end{pmatrix}T_1, \begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}T_4, \begin{pmatrix}1&z\\0&1\end{pmatrix}T_4, \begin{pmatrix}0&z^2\\z&0\end{pmatrix}T_4, T_4, \begin{pmatrix}0&z\\z^2&z\end{pmatrix}T_4, \begin{pmatrix}0&z^2\\z&1\end{pmatrix}T_5, \begin{pmatrix}1&z^2\\0&1\end{pmatrix}T_5, \begin{pmatrix}0&z^2\\z&z\end{pmatrix}T_5, \begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}T_5, \begin{pmatrix}0&z\\z^2&1\end{pmatrix}T_5\},$
$\{\begin{pmatrix}z&0\\0&z^2\end{pmatrix}T_1, \begin{pmatrix}0&z\\z^2&0\end{pmatrix}T_1, \begin{pmatrix}z&0\\z&z^2\end{pmatrix}T_1, \begin{pmatrix}1&0\\z^2&1\end{pmatrix}T_1, \begin{pmatrix}1&0\\z&1\end{pmatrix}T_1, \begin{pmatrix}0&1\\1&1\end{pmatrix}T_3, \begin{pmatrix}z^2&z^2\\0&z\end{pmatrix}T_3, T_3, \begin{pmatrix}0&z^2\\z&0\end{pmatrix}T_3, \begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}T_3, \begin{pmatrix}0&1\\1&z\end{pmatrix}T_4, \begin{pmatrix}z^2&1\\0&z\end{pmatrix}T_4, \begin{pmatrix}0&1\\1&z^2\end{pmatrix}T_4, \begin{pmatrix}0&z^2\\z&1\end{pmatrix}T_4, \begin{pmatrix}0&z^2\\z&z\end{pmatrix}T_4\},$
$\{T_1, \begin{pmatrix}0&z^2\\z&0\end{pmatrix}T_1, \begin{pmatrix}1&0\\1&1\end{pmatrix}T_1, \begin{pmatrix}z^2&0\\z&z\end{pmatrix}T_1, \begin{pmatrix}z^2&0\\1&z\end{pmatrix}T_1, \begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}T_3, \begin{pmatrix}z&z\\0&z^2\end{pmatrix}T_3, \begin{pmatrix}0&z\\z^2&1\end{pmatrix}T_3, \begin{pmatrix}0&1\\1&z\end{pmatrix}T_3, \begin{pmatrix}0&1\\1&z^2\end{pmatrix}T_3, \begin{pmatrix}0&z\\z^2&z\end{pmatrix}T_5, \begin{pmatrix}z&1\\0&z^2\end{pmatrix}T_5, \begin{pmatrix}0&z\\z^2&0\end{pmatrix}T_5, \begin{pmatrix}0&1\\1&0\end{pmatrix}T_5, \begin{pmatrix}0&1\\1&1\end{pmatrix}T_5\},$
$\{\begin{pmatrix}0&z^2\\z&z\end{pmatrix}T_3, \begin{pmatrix}1&1\\0&1\end{pmatrix}T_3, \begin{pmatrix}0&z^2\\z&1\end{pmatrix}T_3, \begin{pmatrix}0&z\\z^2&z\end{pmatrix}T_3, \begin{pmatrix}0&z\\z^2&0\end{pmatrix}T_3, \begin{pmatrix}0&z\\z^2&1\end{pmatrix}T_4, \begin{pmatrix}z&z^2\\0&z^2\end{pmatrix}T_4, \begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}T_4, \begin{pmatrix}0&1\\1&1\end{pmatrix}T_4, \begin{pmatrix}0&1\\1&0\end{pmatrix}T_4, \begin{pmatrix}0&1\\1&z^2\end{pmatrix}T_5, \begin{pmatrix}z^2&z\\0&z\end{pmatrix}T_5, \begin{pmatrix}0&1\\1&z\end{pmatrix}T_5, \begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}T_5, T_5\}\}$

$\pi_5 :=$ $\{\{gT_2 \mid g \in \mathrm{SL}(2,q)\},$
$\{\begin{pmatrix}z&0\\0&z^2\end{pmatrix}T_1, \begin{pmatrix}0&1\\1&0\end{pmatrix}T_1, \begin{pmatrix}z&0\\z&z^2\end{pmatrix}T_1, \begin{pmatrix}z^2&0\\z&z\end{pmatrix}T_1, \begin{pmatrix}z^2&0\\1&z\end{pmatrix}T_1, \begin{pmatrix}0&z\\z^2&1\end{pmatrix}T_4, \begin{pmatrix}1&z\\0&1\end{pmatrix}T_4, \begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}T_4, \begin{pmatrix}0&z^2\\z&1\end{pmatrix}T_4, \begin{pmatrix}0&z^2\\z&z\end{pmatrix}T_4, \begin{pmatrix}0&z\\z^2&z\end{pmatrix}T_5, \begin{pmatrix}1&z^2\\0&1\end{pmatrix}T_5, \begin{pmatrix}0&z\\z^2&0\end{pmatrix}T_5, \begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}T_5, T_5\},$
$\{\begin{pmatrix}z^2&0\\0&z\end{pmatrix}T_1, \begin{pmatrix}0&z^2\\z&0\end{pmatrix}T_1, \begin{pmatrix}z^2&0\\z^2&z\end{pmatrix}T_1, \begin{pmatrix}1&0\\z^2&1\end{pmatrix}T_1, \begin{pmatrix}1&0\\z&1\end{pmatrix}T_1, \begin{pmatrix}0&1\\1&1\end{pmatrix}T_3, \begin{pmatrix}z&z\\0&z^2\end{pmatrix}T_3, T_3, \begin{pmatrix}0&z\\z^2&z\end{pmatrix}T_3, \begin{pmatrix}0&z\\z^2&0\end{pmatrix}T_3, \begin{pmatrix}0&1\\1&z^2\end{pmatrix}T_5, \begin{pmatrix}z&1\\0&z^2\end{pmatrix}T_5, \begin{pmatrix}0&1\\1&z\end{pmatrix}T_5, \begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}T_5, \begin{pmatrix}0&z\\z^2&1\end{pmatrix}T_5\},$
$\{\begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}T_3, \begin{pmatrix}1&1\\0&1\end{pmatrix}T_3, \begin{pmatrix}0&z\\z^2&1\end{pmatrix}T_3, \begin{pmatrix}0&z^2\\z&0\end{pmatrix}T_3, \begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}T_3, \begin{pmatrix}0&1\\1&z\end{pmatrix}T_4, \begin{pmatrix}z&z^2\\0&z^2\end{pmatrix}T_4, \begin{pmatrix}0&1\\1&z^2\end{pmatrix}T_4, T_4, \begin{pmatrix}0&z\\z^2&z\end{pmatrix}T_4, \begin{pmatrix}0&z^2\\z&1\end{pmatrix}T_5, \begin{pmatrix}z^2&z\\0&z\end{pmatrix}T_5, \begin{pmatrix}0&z^2\\z&z\end{pmatrix}T_5, \begin{pmatrix}0&1\\1&0\end{pmatrix}T_5, \begin{pmatrix}0&1\\1&1\end{pmatrix}T_5\},$
$\{T_1, \begin{pmatrix}0&z\\z^2&0\end{pmatrix}T_1, \begin{pmatrix}1&0\\1&1\end{pmatrix}T_1, \begin{pmatrix}z&0\\1&z^2\end{pmatrix}T_1, \begin{pmatrix}z&0\\z^2&z^2\end{pmatrix}T_1, \begin{pmatrix}0&z^2\\z&z\end{pmatrix}T_3, \begin{pmatrix}z^2&z^2\\0&z\end{pmatrix}T_3, \begin{pmatrix}0&z^2\\z&1\end{pmatrix}T_3, \begin{pmatrix}0&1\\1&z\end{pmatrix}T_3, \begin{pmatrix}0&1\\1&z^2\end{pmatrix}T_3, \begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}T_4, \begin{pmatrix}z^2&1\\0&z\end{pmatrix}T_4, \begin{pmatrix}0&z^2\\z&0\end{pmatrix}T_4, \begin{pmatrix}0&1\\1&1\end{pmatrix}T_4, \begin{pmatrix}0&1\\1&0\end{pmatrix}T_4\}\}$

$\pi_6 :=$ $\{\{T_4\begin{pmatrix}1&0\\z^2&1\end{pmatrix}, T_4\begin{pmatrix}z&0\\1&z^2\end{pmatrix}, T_4\begin{pmatrix}z^2&0\\z&z\end{pmatrix}, T_4\begin{pmatrix}0&z\\z^2&1\end{pmatrix}, T_4\begin{pmatrix}0&z^2\\z&z\end{pmatrix}, T_4\begin{pmatrix}0&z\\z^2&z\end{pmatrix}, T_4\begin{pmatrix}0&z^2\\z&0\end{pmatrix}, T_4, T_4\begin{pmatrix}0&z^2\\z&1\end{pmatrix}, T_4\begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}, T_4\begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}, \begin{pmatrix}z&0\\0&z^2\end{pmatrix}T_1, \begin{pmatrix}0&1\\1&0\end{pmatrix}T_1, \begin{pmatrix}z^2&0\\z^2&z\end{pmatrix}T_1, \begin{pmatrix}1&0\\z&1\end{pmatrix}T_1\},$
$\{T_3\begin{pmatrix}1&0\\1&1\end{pmatrix}, T_3\begin{pmatrix}z&0\\z&z^2\end{pmatrix}, T_3\begin{pmatrix}0&z\\z^2&z\end{pmatrix}, T_3\begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}, T_3, T_3\begin{pmatrix}0&z^2\\z&1\end{pmatrix}, T_3\begin{pmatrix}0&1\\1&z^2\end{pmatrix}, T_3\begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}, T_3\begin{pmatrix}0&z^2\\z&0\end{pmatrix}, T_3\begin{pmatrix}0&1\\1&z\end{pmatrix}, T_3\begin{pmatrix}0&z\\z^2&0\end{pmatrix}, \begin{pmatrix}0&z\\z^2&1\end{pmatrix}T_4, \begin{pmatrix}z^2&1\\0&z\end{pmatrix}T_4, \begin{pmatrix}0&1\\1&1\end{pmatrix}T_4, \begin{pmatrix}0&z^2\\z&z\end{pmatrix}T_4\},$
$\{T_2\begin{pmatrix}0&z\\z^2&0\end{pmatrix}, T_2\begin{pmatrix}0&z^2\\z&0\end{pmatrix}, T_2, T_2\begin{pmatrix}z&0\\0&z^2\end{pmatrix}, T_2\begin{pmatrix}z^2&0\\0&z\end{pmatrix}, T_2\begin{pmatrix}z&z\\0&z^2\end{pmatrix}, T_2\begin{pmatrix}z^2&z^2\\0&z\end{pmatrix}, T_2\begin{pmatrix}1&z\\0&1\end{pmatrix}, T_2\begin{pmatrix}z&z^2\\0&z^2\end{pmatrix}, T_2\begin{pmatrix}1&z^2\\0&1\end{pmatrix}, T_2\begin{pmatrix}z^2&z\\0&z\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}T_2, \begin{pmatrix}0&1\\1&1\end{pmatrix}T_2, \begin{pmatrix}0&1\\1&z^2\end{pmatrix}T_2, \begin{pmatrix}0&1\\1&z\end{pmatrix}T_2\},$
$\{T_1, T_1\begin{pmatrix}z^2&0\\0&z\end{pmatrix}, T_1\begin{pmatrix}0&z\\z^2&0\end{pmatrix}, T_1\begin{pmatrix}0&z^2\\z&0\end{pmatrix}, T_1\begin{pmatrix}0&1\\1&1\end{pmatrix}, T_1\begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}, T_1\begin{pmatrix}0&1\\1&z\end{pmatrix}, T_1\begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}, T_1\begin{pmatrix}0&1\\1&z^2\end{pmatrix}, T_1\begin{pmatrix}0&z\\z^2&z\end{pmatrix}, T_1\begin{pmatrix}0&z^2\\z&1\end{pmatrix}, \begin{pmatrix}z&1\\0&z^2\end{pmatrix}T_5, \begin{pmatrix}0&z^2\\z&z\end{pmatrix}T_5, \begin{pmatrix}0&1\\1&0\end{pmatrix}T_5, \begin{pmatrix}0&z\\z^2&1\end{pmatrix}T_5\},$
$\{T_5\begin{pmatrix}z&0\\z^2&z^2\end{pmatrix}, T_5\begin{pmatrix}z^2&0\\1&z\end{pmatrix}, T_5\begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}, T_5\begin{pmatrix}0&z^2\\z&1\end{pmatrix}, T_5\begin{pmatrix}0&1\\1&z^2\end{pmatrix}, T_5\begin{pmatrix}0&z\\z^2&0\end{pmatrix}, T_5\begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}, T_5\begin{pmatrix}0&1\\1&0\end{pmatrix}, T_5\begin{pmatrix}0&z\\z^2&z\end{pmatrix}, T_5\begin{pmatrix}0&1\\1&1\end{pmatrix}, T_5, \begin{pmatrix}0&z^2\\z&z\end{pmatrix}T_3, \begin{pmatrix}1&1\\0&1\end{pmatrix}T_3, \begin{pmatrix}0&z\\z^2&1\end{pmatrix}T_3, \begin{pmatrix}0&1\\1&z\end{pmatrix}T_3\}\}$

$\pi_7 :=$ $\{\{\begin{pmatrix}0&z\\z^2&0\end{pmatrix}T_2, \begin{pmatrix}0&z^2\\z&0\end{pmatrix}T_2, T_2, \begin{pmatrix}z&0\\0&z^2\end{pmatrix}T_2, \begin{pmatrix}z^2&0\\0&z\end{pmatrix}T_2, \begin{pmatrix}0&z\\z^2&z\end{pmatrix}T_2, \begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}T_2, \begin{pmatrix}0&z\\z^2&1\end{pmatrix}T_2, \begin{pmatrix}0&z^2\\z&z\end{pmatrix}T_2, \begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}T_2, \begin{pmatrix}0&z^2\\z&1\end{pmatrix}T_2, T_2\begin{pmatrix}0&1\\1&0\end{pmatrix}, T_2\begin{pmatrix}1&1\\0&1\end{pmatrix}, T_2\begin{pmatrix}z^2&1\\0&z\end{pmatrix}, T_2\begin{pmatrix}z&1\\0&z^2\end{pmatrix}\},$
$\{\begin{pmatrix}0&1\\1&1\end{pmatrix}T_3, \begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}T_3, \begin{pmatrix}z&z\\0&z^2\end{pmatrix}T_3, \begin{pmatrix}z^2&z^2\\0&z\end{pmatrix}T_3, T_3, \begin{pmatrix}0&z^2\\z&1\end{pmatrix}T_3, \begin{pmatrix}0&z\\z^2&z\end{pmatrix}T_3, \begin{pmatrix}0&z^2\\z&0\end{pmatrix}T_3, \begin{pmatrix}0&1\\1&z^2\end{pmatrix}T_3, \begin{pmatrix}0&z\\z^2&0\end{pmatrix}T_3, \begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}T_3, T_5\begin{pmatrix}1&0\\z&1\end{pmatrix}, T_5\begin{pmatrix}0&1\\1&z\end{pmatrix}, T_5\begin{pmatrix}0&z^2\\z&z\end{pmatrix}, T_5\begin{pmatrix}0&z\\z^2&1\end{pmatrix}\},$
$\{\begin{pmatrix}0&1\\1&z^2\end{pmatrix}T_5, \begin{pmatrix}0&z\\z^2&z\end{pmatrix}T_5, \begin{pmatrix}0&z^2\\z&1\end{pmatrix}T_5, \begin{pmatrix}1&z^2\\0&1\end{pmatrix}T_5, \begin{pmatrix}z^2&z\\0&z\end{pmatrix}T_5, \begin{pmatrix}0&1\\1&z\end{pmatrix}T_5, \begin{pmatrix}0&z\\z^2&0\end{pmatrix}T_5, \begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}T_5, \begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}T_5, \begin{pmatrix}0&1\\1&1\end{pmatrix}T_5, T_5, T_1\begin{pmatrix}z&0\\0&z^2\end{pmatrix}, T_1\begin{pmatrix}0&1\\1&0\end{pmatrix}, T_1\begin{pmatrix}0&z^2\\z&z\end{pmatrix}, T_1\begin{pmatrix}0&z\\z^2&1\end{pmatrix}\},$
$\{T_1, \begin{pmatrix}z^2&0\\0&z\end{pmatrix}T_1, \begin{pmatrix}0&z\\z^2&0\end{pmatrix}T_1, \begin{pmatrix}0&z^2\\z&0\end{pmatrix}T_1, \begin{pmatrix}1&0\\1&1\end{pmatrix}T_1, \begin{pmatrix}z&0\\z&z^2\end{pmatrix}T_1, \begin{pmatrix}1&0\\z^2&1\end{pmatrix}T_1, \begin{pmatrix}z&0\\1&z^2\end{pmatrix}T_1, \begin{pmatrix}z^2&0\\z&z\end{pmatrix}T_1, \begin{pmatrix}z&0\\z^2&z^2\end{pmatrix}T_1, \begin{pmatrix}z^2&0\\1&z\end{pmatrix}T_1, T_4\begin{pmatrix}0&1\\1&z^2\end{pmatrix}, T_4\begin{pmatrix}0&1\\1&z\end{pmatrix}, T_4\begin{pmatrix}0&1\\1&1\end{pmatrix}, T_4\begin{pmatrix}0&1\\1&0\end{pmatrix}\},$
$\{\begin{pmatrix}0&1\\1&z\end{pmatrix}T_4, \begin{pmatrix}0&z^2\\z&z^2\end{pmatrix}T_4, \begin{pmatrix}1&z\\0&1\end{pmatrix}T_4, \begin{pmatrix}z&z^2\\0&z^2\end{pmatrix}T_4, \begin{pmatrix}0&1\\1&z^2\end{pmatrix}T_4, \begin{pmatrix}0&z\\z^2&z^2\end{pmatrix}T_4, \begin{pmatrix}0&z^2\\z&0\end{pmatrix}T_4, T_4, \begin{pmatrix}0&z^2\\z&1\end{pmatrix}T_4, \begin{pmatrix}0&1\\1&0\end{pmatrix}T_4, \begin{pmatrix}0&z\\z^2&z\end{pmatrix}T_4, T_3\begin{pmatrix}z^2&0\\z^2&z\end{pmatrix}, T_3\begin{pmatrix}0&1\\1&1\end{pmatrix}, T_3\begin{pmatrix}0&z\\z^2&1\end{pmatrix}, T_3\begin{pmatrix}0&z^2\\z&z\end{pmatrix}\}\}$

Table 6.3: Full automorphism groups of the Leonids unitals

$\mathbb{U}$	$\mathrm{Aut}(\mathbb{U})$		$\#\,\mathrm{Aut}(\mathbb{U})$
$\mathbb{U}^{\pi_2}_{C,\mathcal{H}}$	$\langle\gamma_f\cdot\varphi\rangle\ltimes\langle\rho_g,\rho_b\rangle$	$\cong C_4\ltimes D_5$	40
$\mathbb{U}^{\pi_4}_{C,\mathcal{H}}$	$\langle\gamma_f\cdot\varphi\rangle\ltimes\langle\rho_b,\rho_c\rangle$	$\cong C_4\ltimes A_4$	48
$\mathbb{U}^{\pi_5}_{C,\mathcal{H}}$	$\langle\gamma_f\cdot\varphi\rangle\ltimes(\langle\rho_b,\rho_c\rangle\times\langle\gamma_{g^{-1}}\cdot\rho_g\rangle)$	$\cong C_4\ltimes(A_4\times C_5)$	240
$\mathbb{U}^{\pi_6}_{C,\mathcal{H}}$	$\langle\gamma_g\cdot\rho_g\rangle\rtimes\langle\gamma_f\cdot\varphi\rangle$	$\cong C_5\rtimes C_4$	20
$\mathbb{U}^{\pi_7}_{C,\mathcal{H}}$	$\langle\gamma_g\cdot\rho_g\rangle\rtimes\langle\gamma_f\cdot\varphi\rangle$	$\cong C_5\rtimes C_4$	20
$\mathbb{U}^{\pi_1}_{C,\mathcal{E}}$	$\langle\rho_g,\rho_b\rangle$	$\cong D_5$	10
$\mathbb{U}^{\pi_2}_{C,\mathcal{E}}$	$\langle\gamma_f\cdot\varphi\rangle\ltimes\langle\rho_g,\rho_b\rangle$	$\cong C_4\ltimes D_5$	40
$\mathbb{U}^{\pi_3}_{C,\mathcal{E}}$	$\langle\rho_b,\rho_c\rangle$	$\cong A_4$	12
$\mathbb{U}^{\pi_4}_{C,\mathcal{E}}$	$\langle\gamma_f\cdot\varphi\rangle\ltimes\langle\rho_b,\rho_c\rangle$	$\cong C_4\ltimes A_4$	48
$\mathbb{U}^{\pi_5}_{C,\mathcal{E}}$	$\langle\gamma_f\cdot\varphi\rangle\ltimes\langle\rho_b,\rho_c\rangle$	$\cong C_4\ltimes A_4$	48
$\mathbb{U}^{\pi_6}_{C,\mathcal{E}}$	$\langle\gamma_f\cdot\varphi\rangle$	$\cong C_4$	4
$\mathbb{U}^{\pi_7}_{C,\mathcal{E}}$	$\langle\gamma_f\cdot\varphi\rangle$	$\cong C_4$	4

We conclude this chapter with a description of all translations of the Leonids unitals. Let again $f := \left(\begin{smallmatrix}1&0\\z&1\end{smallmatrix}\right)$, $b := \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ and $c := \begin{pmatrix}z&0\\0&z^2\end{pmatrix}$ and note that $(\gamma_f\cdot\varphi)^2 = \gamma_{f(f\cdot\varphi)} = \gamma_b$. With $T_2 := \{\left(\begin{smallmatrix}1&0\\x&1\end{smallmatrix}\right) \mid x\in\mathbb{F}_4\}$ as above, we have $b\in T_2$ and $\gamma_b\cdot\rho_b$ equals left multiplication with b.

Proposition 6.17. *All translations of the Leonids unitals are the following:*

(i) $\mathbb{U}^{\pi_2}_{C,\mathcal{H}}$ *and* $\mathbb{U}^{\pi_2}_{C,\mathcal{E}}$ *admit one translation of order* 2 *each.*

(ii) $\mathbb{U}^{\pi_4}_{C,\mathcal{H}}$, $\mathbb{U}^{\pi_5}_{C,\mathcal{H}}$, $\mathbb{U}^{\pi_3}_{C,\mathcal{E}}$, $\mathbb{U}^{\pi_4}_{C,\mathcal{E}}$ *and* $\mathbb{U}^{\pi_5}_{C,\mathcal{E}}$, *respectively, admit a translation group of order* 4 *with translation center.*

Proof. One parallel class in π_2 is given by the set of all right cosets of T_2 (see Table 6.2). Since both $\mathrm{Aut}(\mathbb{U}^{\pi_2}_{C,\mathcal{H}})$ and $\mathrm{Aut}(\mathbb{U}^{\pi_2}_{C,\mathcal{E}})$ contain $\gamma_b\cdot\rho_b$ (see Table 6.3), left multiplication by b induces a translation of order 2 with center T_2 on $\mathbb{U}^{\pi_2}_{C,\mathcal{H}}$ and $\mathbb{U}^{\pi_2}_{C,\mathcal{E}}$, respectively.
In π_3 as well as in π_4 and π_5, one parallel class is given by the set of all left cosets of T_2 (see Table 6.2). Since $T_2 = \{\mathbb{1}, b, b^c, b^{c^2}\}$, the group $R_{T_2} := \{\rho_t \mid t\in T_2\}$ is a group of automorphisms of $\mathbb{U}^{\pi_4}_{C,\mathcal{H}}$, $\mathbb{U}^{\pi_5}_{C,\mathcal{H}}$, $\mathbb{U}^{\pi_3}_{C,\mathcal{E}}$, $\mathbb{U}^{\pi_4}_{C,\mathcal{E}}$ and $\mathbb{U}^{\pi_5}_{C,\mathcal{E}}$, respectively (see Table 6.3). On each of those unitals, R_{T_2} acts as translation group of order 4 with translation center $T_2\in[\infty]$.
Computation (e. g. with GAP) shows that there are no other translations in any of the Leonids unitals. □

7 Open Problems

While working on $\mathrm{SL}(2,q)$-unitals, there occurred still unsolved questions, some of which we list here (in no particular order).

1. Is every affine $\mathrm{SL}(2,q)$-unital with automorphism group $\mathfrak{A}_C \ltimes R$ isomorphic to the classical affine $\mathrm{SL}(2,q)$-unital?
2. Is the block at infinity fixed by the full automorphism group in every non-classical $\mathrm{SL}(2,q)$-unital?
3. Is there a non-classical affine $\mathrm{SL}(2,q)$-unital with odd order?
4. If yes, is there an affine $\mathrm{SL}(2,q)$-unital $\mathbb{U}_{S,\mathcal{D}}$, where S is non-cyclic?
5. In the non-classical affine $\mathrm{SL}(2,q)$-unitals $\mathbb{U}_{C,\mathcal{E}}$, $\mathbb{WU}$, $\mathbb{OU}$ and $\mathbb{PU}$, can the set of arcuate blocks through $\mathbb{1}$ be described in a way that extends to higher orders?
6. Can the parallelisms on the short blocks for order 4 be described in a way that extends to higher orders?

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Figure A.1: A pig wearing a nice shirt and saying thank you to all the people who supported me in some way or another