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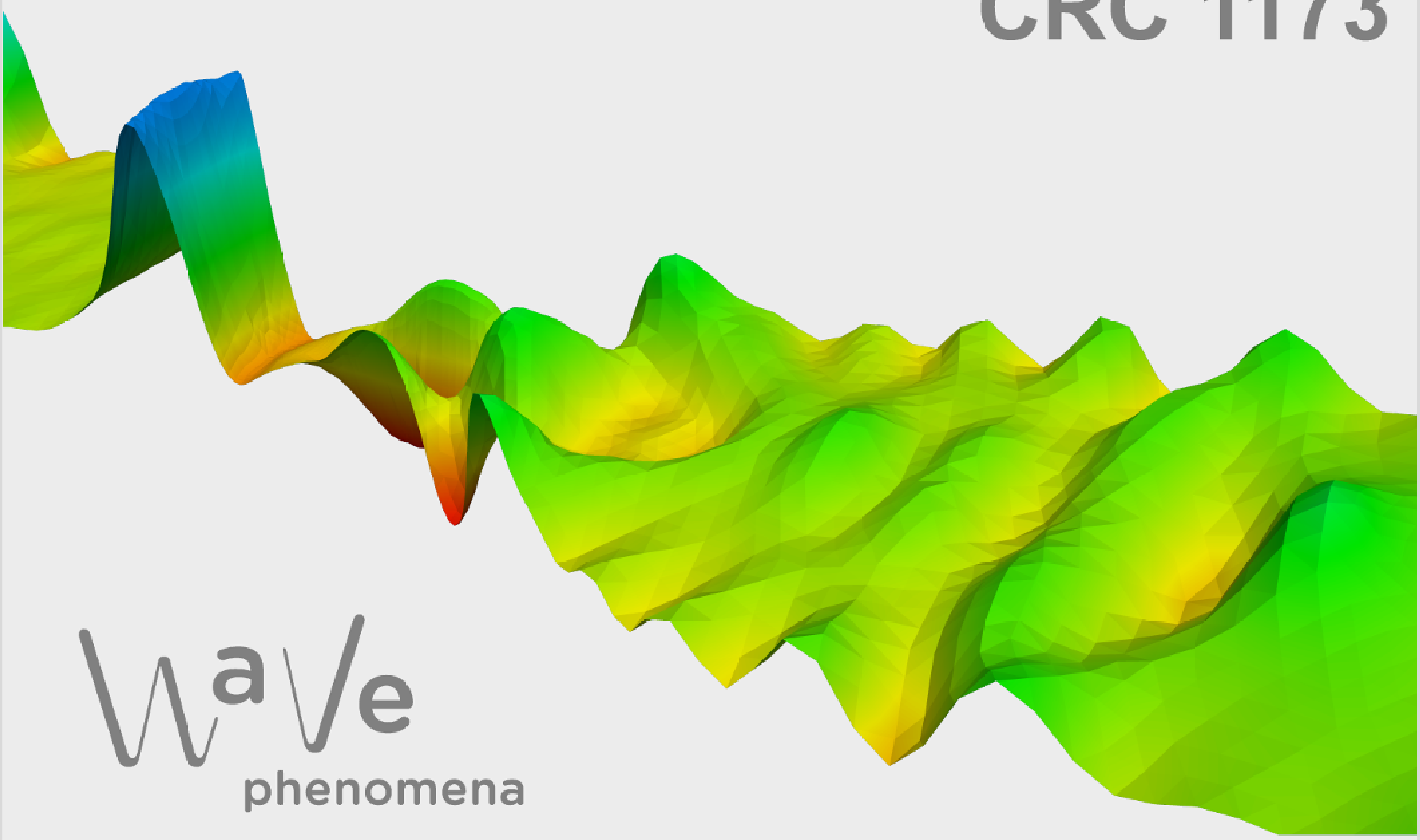

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Application of the Second Domain Derivative in Inverse Electromagnetic Scattering

Felix Hagemann[*] and Frank Hettlich[†]

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Abstract

We consider the inverse scattering problem of reconstructing a perfect conductor from the far field pattern of a scattered time harmonic electromagnetic wave generated by one incident plane wave. In view of iterative regularization schemes for the severely ill-posed problem the first and the second domain derivative of the far field pattern with respect to variations of the domain are established. Charaterizations of the derivatives by boundary value problems allow for an application of second degree regularization methods to the inverse problem. A numerical implementation based on integral equations is presented and its performance is illustrated by a selection of examples.

1 Introduction

A challenging class of inverse problems in scattering theory is the identification of scattering objects by the knowledge of far field patterns of scattered waves (see [3]). Of course, we must distinguish theoretically and numerically the inverse problem, if the response to any or at least to many incident fields is known, or the problem, if only a few far field patterns are given. In this work we are going to consider the extreme situation of the reconstruction of the shape of a perfect conductor just from the knowledge of the far field pattern of one scattered time harmonic electromagnetic wave.

Derivative based iterative regularization schemes are known to be suitable numerical approaches for this class of problems. Thus, we focus on linearization of the far field pattern with respect to variations of the shape of the scattering object. The derivative is given by the far field pattern of the so called domain derivative of the scattered wave. These domain derivatives are well established for most of the usually considered boundary value problems (see [12] and references cited therein). Furthermore, in case of acoustic scattering problems several numerical implementations are documented. Presumably according to the computational effort, there are only a few results for the full vector valued electromagnetic inverse scattering problem. In [8] we recently presented an approach based on boundary integral

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equations in case of electromagnetic scattering. We will extend on these results, mainly by showing the existence and a characterization of the second domain derivative. This gives rise to an application of second order regularization schemes (see [11, 13]). Moreover, in view of convergence of iterative regularization methods we examine local identifiability at least in case of constant expanding or shrinking of the obstacle.

After this introduction we collect notations on the scattering problem and describe its weak formulation for later use. The next chapter is on the first domain derivative of the scattering problem. Although the derivative is already established, we will present it in some detail in preparation of the following investigations for the second derivative. Some remarks based on the characterization of the domain derivative illuminates the challenging question on injectivity of the derivative operator. With these preparations we devote the following chapter to the second domain derivative. It is shown that such a derivative exists and can be characterized again by an electromagnetic boundary value problem. Finally, based on these characterizations we explain and discuss in the last chapter the regularized Halley method applied to the inverse problem and present its numerical performance by some examples.

Some results of this paper, e.g. Theorem 4.5, are part of one of the authors' Ph.D. thesis [7].

2 The Scattering Problem

Let us assume a bounded scattering obstacle $D \subset \mathbb{R}^3$ with smooth boundary and simply connected complement $\mathbb{R}^3 \setminus D$. The object is surrounded by a homogeneous, linear, isotropic medium with constant electric permittivity $\varepsilon_0 > 0$ and constant magnetic permittivity $\mu_0 > 0$, for instance vacuum. At frequency $\omega > 0$, the time harmonic Maxwell system for the electric field E and the magnetic field H then reads as

$$\operatorname{curl} E - \mathrm{i}k\, H = 0, \quad \operatorname{curl} H + \mathrm{i}k\, E = 0, \tag{2.1}$$

with wavenumber $k = \omega\sqrt{\varepsilon_0\mu_0}$. Given an incident plane wave, $E^i(x) = p\,\mathrm{e}^{\mathrm{i}kd\cdot x}$, $H^i(x) = (d \times p)\mathrm{e}^{\mathrm{i}kd\cdot x}$ for $x \in \mathbb{R}^3$ with complex polarisation $p \in \mathbb{C}^3$ and direction $d \in \mathbb{S}^2$ satisfying $p \cdot d = 0$ the scatterer gives rise to a radiating scattered field (E^s, H^s), a solution of the Maxwell system (2.1) in $\mathbb{R}^3 \setminus \overline{D}$, which satisfies the Silver-Müller radiation condition

$$\lim_{|x|\to\infty} \Big[H^s(x) \times x - |x|\, E^s(x) \Big] = 0.$$

The interaction of the perfect conductor D with the incident wave can be formulated as a boundary condition for the total field $E = E^s + E^i$ and is given by

$$\nu \times E = 0 \quad \text{on } \partial D,$$

where ν denotes the outwards directed normal vector to ∂D.

The following investigations require a variational formulation of the scattering problem. Thus, we choose $R > 0$ large enough such that $\overline{D} \subset B_R(0)$, where $B_R(0)$ denotes the open ball of radius R centered in the origin, and introduce the bounded computational domain $\Omega = B_R(0) \setminus \overline{D}$. In order to derive the weak formulation, let (E, H) be a pair of reasonable smooth solutions of the scattering problem and let V denote a test function with $\nu \times V = 0$ an ∂D. By partial integration and the Maxwell system (2.1) we arrive at

$$\int_\Omega \big(\operatorname{curl} E \cdot \overline{\operatorname{curl} V} - k^2\, E \cdot \overline{V}\big)\mathrm{d}x + \mathrm{i}k \int_{\partial B_R(0)} \Lambda(\nu \times E) \cdot \overline{V} \mathrm{d}s$$
$$= \int_{\partial B_R(0)} \big(\mathrm{i}k\, \Lambda(\nu \times E^i) - \nu \times \operatorname{curl} E^i\big) \cdot \overline{V} \mathrm{d}s. \tag{2.2}$$

To ensure that a solution $E^s = E - E^i$ of (2.2) can be extended to a radiating solution of the Maxwell system in $\mathbb{R}^3 \setminus \overline{D}$ we have introduced on the artificial boundary $\partial B_R(0)$ the Calderon operator Λ, which maps $\nu \times \varphi$ onto $\nu \times H^s$, where (E^s, H^s) denote the unique radiating solution of

$$\operatorname{curl} E^s - \mathrm{i}k\, H^s = 0\,, \qquad \operatorname{curl} H^s + \mathrm{i}k\, E^s = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{B_R(0)}$$
$$\nu \times E^s = \nu \times \varphi \qquad \text{on } \partial B_R(0)\,.$$

Equation 2.2 is considered in the Sobolev space $H(\operatorname{curl}, \Omega) = \{E \in L^2(\Omega, \mathbb{C}^3) : \operatorname{curl} E \in L^2(\Omega, \mathbb{C}^3)\}$. Then, boundary integrals on ∂D and $\partial B_R(0)$ exists in the sense of the dual pairing $\langle \cdot, \cdot \rangle_{\partial B_R(0)}$ between the range spaces $H^{-\frac{1}{2}}(\operatorname{Div}, \partial\Omega)$ and $H^{-\frac{1}{2}}(\operatorname{Curl}, \partial\Omega)$ of the traces $\gamma_t \varphi = \varphi \times \nu$ and $\gamma_T \varphi = \nu \times (\varphi \times \nu)$ for $\varphi \in H(\operatorname{curl}, \Omega)$. Then, $\gamma_t E = \nu \times E = 0$ in $H^{-\frac{1}{2}}(\operatorname{Div}, \partial D)$ and we incorporate the boundary condition by the closed subspace

$$H_{\mathrm{pc}}(\Omega) = \{E \in H(\operatorname{curl}, \Omega) : \gamma_t E = 0\}$$

Since, the Calderon operator is extendable to a bounded operator $\Lambda : H^{-\frac{1}{2}}(\operatorname{Div}, \partial B_R(0)) \to H^{-\frac{1}{2}}(\operatorname{Div}, \partial B_R(0))$ (see [18]), we finally can define the bounded sesquilinear form $\mathcal{A} : H_{\mathrm{pc}}(\Omega) \times H_{\mathrm{pc}}(\Omega) \to \mathbb{C}$ and the antilinear map $\ell : H_{\mathrm{pc}}(\Omega) \to \mathbb{C}$ such that (2.2) reads as

$$\mathcal{A}(E, V) = \ell(V). \tag{2.3}$$

A weak solution of the scattering problem is then given by a function $E \in H_{\mathrm{pc}}(\Omega)$ such that $\mathcal{A}(E, V) = \ell(V)$ holds for all $V \in H_{\mathrm{pc}}(\Omega)$. Assuming D to be a Lipschitz domain it is known that for any $\ell \in H_{pc}(\Omega)^*$ there exists a unique solution $E \in H_{pc}(\Omega)$ of (2.3) for all $V \in H_{\mathrm{pc}}(\Omega)$, and it exists $c > 0$ such that $\|E\|_{H(\operatorname{curl},\Omega)} \leqslant c\, \|\ell\|_{H^*_{pc}}$ (see [18, Theorem 10.7]).

Due to the radiation condition, the scattered field E^s in $\mathbb{R}^3 \setminus \overline{D}$ has the asymptotic behavior

$$E^s(x) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{4\pi|x|} \Big[E_\infty(\widehat{x}) + \mathcal{O}\Big(\frac{1}{|x|}\Big)\Big]\,, \quad |x| \to \infty\,.$$

E_∞ is called the (electric) far field pattern and is an analytic tangential vector field on the unit sphere $\mathbb{S}^2$. This motivates the definition of the non-linear boundary to far field operator $\mathbf{F}$, which maps the boundary ∂D onto the far field pattern of E^s, i.e.,

$$\mathbf{F}(\partial D) = E_\infty. \tag{2.4}$$

Of course, $\mathbf{F}$ depends also on the incident field (E^i, H^i) and the wavenumber k, which we assume to be fixed and known. The domain of $\mathbf{F}$ is given by a class of admissible boundaries, for which there is a unique solution of the scattering problem. Thus, the inverse obstacle problem under consideration is given by the inversion of equation (2.4), i.e., for a given far field pattern $E_\infty \in L^2_t(\mathbb{S}^2)$ we look for the scatterer $D \subset \mathbb{R}^3$.

It is known that the far field pattern uniquely determines the solution of the scattering problem, but, nevertheless, the whole inverse obstacle problem is severely ill-posed. For some more details on inverse electromagnetic scattering we refer to [3], where, for instance, uniqueness of the inverse problem is shown in the sense that if for a fixed wave number k the far field patterns of two objects for all incident plane waves coincide, the scattering objects must be identical. Such a result is not known in case of just one incident field.

3 Linearization of the Inverse Problem

In solving for the nonlinear equation (2.4), obviously a linearization is useful. Thus, a derivative of the far field pattern with respect to variations of the boundary of the scattering obstacle D is of specific interest. This, leads to the concept of a domain derivative, which is well established in electromagnetic scattering (see [20, 16, 4, 10, 12]). For convenience to the reader and in preparation of the next section we present the variational approach in some detail, following very closely [10], where penetrable scattering objects are considered.

A perturbation of the scatterer is described by a vector field $h \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ with compact support. Given a set $D \subset \mathbb{R}^3$, we denote by D_h the corresponding perturbed set $D_h = \{x + h(x) : x \in D\}$. If the C^1-norm of h is sufficiently small, the transformation $x \mapsto \varphi(x) = x + h(x)$ is a diffeomorphism. Throughout, we assume that a perturbation h does not change the artificial boundary $\partial B_R(0)$. Thus, without loss of generality we have $h \in C^1_0(B_R(0), \mathbb{R}^3)$. For functions $f : D_h \to \mathbb{R}^d$, $d \in \{1, 3\}$, on the perturbed set D_h we define the corresponding function $\widetilde{f} : D \to \mathbb{R}^d$ on the unperturbed set by $\widetilde{f}(x) = f(\varphi(x))$.

Let $E \in H_{\mathrm{pc}}(\Omega)$ be the weak solution of the scattering problem (2.2) and let $E_h \in H_{\mathrm{pc}}(\Omega_h)$ denote the solution of the scattering problem with respect to a perturbed scatterer D_h, i.e.

$$\int_{\Omega_h} \Big(\operatorname{curl} E_h \cdot \overline{\operatorname{curl} V_h} - k^2\, E_h \cdot \overline{V_h} \Big) \mathrm{d}x - \mathrm{i}k \langle \Lambda \gamma_t E_H, \gamma_T V_H \rangle_{\partial B_R(0)} = \ell(V_h) \tag{3.1}$$

for all $V_h \in H_{\mathrm{pc}}(\Omega_h)$. Note the same right hand sides of the weak formulations (2.2) and (3.1), since the boundary integral on the artificial boundary does not change. According to different sets of definition, E_h has to be transformed. We use the curl conserving transformation $E_h \mapsto \widehat{E_h}$, given by

$$\widehat{E_h}(x) = J_\varphi^\top(x) \widetilde{E_h}(x) = (I + J_h^\top(x)) E_h(x + h(x))\,,$$

where J_φ denotes the Jacobian of φ (see [18, Section 3.9]). Then $E_h \in H_{\mathrm{pc}}(\Omega_h)$ implies $\widehat{E_h} \in H_{\mathrm{pc}}(\Omega)$, what can be seen from the formula $\operatorname{curl}_\sim \widetilde{E_h} = \frac{1}{\det J_\varphi} J_\varphi \operatorname{curl} \widehat{E_h}$, where $\operatorname{curl}_\sim$ denotes the curl operator with respect to the untransformed coordinates, and from $\langle \gamma_t \widehat{E_h}, \gamma_T \widehat{V_h} \rangle_{\partial\Omega} = \langle \gamma_t E_h, \gamma_T V_h \rangle_{\partial\Omega_h}$, which follows by partial integration.

By the transformation we arrive for $\widehat{E_h} \in H_{\mathrm{pc}}(\Omega)$ at

$$\int_\Omega \Big(\operatorname{curl} \widehat{E_h}^\top \frac{J_\varphi^\top J_\varphi}{\det J_\varphi} \overline{\operatorname{curl} V} - k^2\, \widehat{E_h}^\top \det(J_\varphi) J_\varphi^{-1} J_\varphi^{-\top} \overline{V} \Big) \mathrm{d}x - \mathrm{i}k\, \langle \Lambda \gamma_t \widehat{E_h}, \gamma_T V \rangle_{\partial B_R(0)} = \ell(V) \tag{3.2}$$

for all $V \in H_{\mathrm{pc}}(\Omega)$. We define the bounded sesquilinear form $\mathcal{A}_h : H_{\mathrm{pc}}(\Omega) \times H_{\mathrm{pc}}(\Omega) \to \mathbb{C}$ such that (3.2) reads as $\mathcal{A}_h(\widehat{E_h}, V) = \ell(V)$.

Straight forward calculations yields the asymptotic behavior

$$\frac{J_\varphi^\top J_\varphi}{\det J_\varphi} = (1 - \operatorname{div} h)I + J_h + J_h^\top + \mathcal{O}(\|h\|_{C^1}^2)\,, \tag{3.3}$$

$$\det J_\varphi\, J_\varphi^{-1} J_\varphi^{-\top} = (1 + \operatorname{div} h)I - J_h - J_h^\top + \mathcal{O}(\|h\|_{C^1}^2). \tag{3.4}$$

for $\|h\| \to 0$. Now as a first step we can show continuity of the solution with respect to the perturbation h.

Theorem 3.1. *Let $E \in H_{\mathrm{pc}}(\Omega)$ be the solution of (2.2) and $\widehat{E_h} \in H_{\mathrm{pc}}(\Omega)$ of (3.2). Then,*

$$\lim_{\|h\|_{C^1} \to 0} \|E - \widehat{E_h}\|_{H(\mathrm{curl},\Omega)} = 0.$$

Proof. Let $A, A_h : H_{\mathrm{pc}}(\Omega) \to H_{\mathrm{pc}}(\Omega)$ denote the bounded linear operators defined by

$$\langle AE, V\rangle_{H(\mathrm{curl},\Omega)} = \mathcal{A}(E,V), \quad \langle A_h E, V\rangle_{H(\mathrm{curl},\Omega)} = \mathcal{A}_h(E,V)$$

and $L \in H(\mathrm{curl},\Omega)$ such that $\ell(V) = \langle L, V\rangle_{H(\mathrm{curl},\Omega)}$. We calculate

$$\begin{aligned}
&\|(A_h - A)V\|_{H(\mathrm{curl},\Omega)}^2 = \mathcal{A}_h(V, (A_h - A)V) - \mathcal{A}(V, (A_h - A)V) \\
&= \int_\Omega \Big[\operatorname{curl} V^\top \Big(\frac{J_\varphi^\top J_\varphi}{\det J_\varphi} - I\Big)\overline{\operatorname{curl}(A_h - A)V} - k^2\, V^\top(\det J_\varphi J_\varphi^{-1} J_\varphi^{-\top} - I)\overline{(A_h - A)V}\Big]\,\mathrm{d}x,
\end{aligned}$$

and by (3.3) and (3.4) we obtain

$$\|(A_h - A)V\|_{H(\mathrm{curl},\Omega)}^2 \leqslant C\|h\|_{C^1}\|V\|_{H(\mathrm{curl},\Omega)}\|(A_h - A)V\|_{H(\mathrm{curl},\Omega)}\,.$$

Therefore, $\|A_h - A\| \to 0$ as $\|h\|_{C^1} \to 0$. The operator A possesses a bounded inverse (see again [18, Theorem 10.7]). Thus, a perturbation argument (see [17, Theorem 10.1]) yields $\|\widehat{E_h} - E\|_{H(\mathrm{curl},\Omega)} \to 0$ as $\|h\|_{C^1} \to 0$. □

Looking closely at the linearizations of the coefficients in the weak formulation (3.2), we can prove differentiability.

Theorem 3.2. *Let $E \in H_{\mathrm{pc}}(\Omega)$ be the solution of (2.2) and $\widehat{E_h} \in H_{\mathrm{pc}}(\Omega)$ of (3.2). Then there exists a function $W \in H_{\mathrm{pc}}(\Omega)$, depending linearly on $h \in C_0^1(B_R(0), \mathbb{R}^3)$, such that*

$$\lim_{\|h\|_{C^1} \to 0} \|\widehat{E_h} - E - W\|_{H(\mathrm{curl},\Omega)} = 0.$$

Proof. We define $W \in H_{\mathrm{pc}}(\Omega)$ as the solution of

$$\mathcal{A}(W,V) = \int_\Omega \Big[\operatorname{curl} E^\top(\operatorname{div}(h)\, I - J_h - J_h^\top)\overline{\operatorname{curl} V} + k^2\, E^\top(\operatorname{div}(h)\, I - J_h - J_h^\top)\overline{V}\Big]\,\mathrm{d}x$$

for all $V \in H_{\mathrm{pc}}(\Omega)$. Then, from

$$\mathcal{A}(\widehat{E_h} - E - W, V) \;=\; \mathcal{A}(\widehat{E_h}, V) - \mathcal{A}_h(\widehat{E_h}, V) - \mathcal{A}(W, V)$$

$$
\begin{aligned}
&= \int_\Omega \operatorname{curl} \widehat{E_h}^\top \Big(I - \frac{J_\varphi^\top J_\varphi}{\det J_\varphi} - (\operatorname{div}(h)\, I - J_h - J_h^\top) \Big) \overline{\operatorname{curl} V} \,\mathrm{d}x \\
&\quad - k^2 \int_\Omega \widehat{E_h}^\top \Big(I - J_\varphi^{-1} J_\varphi^{-\top} \det J_\varphi + (\operatorname{div}(h)\, I - J_h - J_h^\top) \Big) \overline{V} \,\mathrm{d}x \\
&\quad + \int_\Omega \operatorname{curl}(\widehat{E_h} - E)^\top \big(\operatorname{div}(h)\, I - J_h - J_h^\top \big) \overline{\operatorname{curl} V} + k^2 (\widehat{E_h} - E)^\top \big(\operatorname{div}(h)\, I - J_h - J_h^\top \big) \overline{V} \,\mathrm{d}x
\end{aligned}
$$

for any $V \in H(\operatorname{curl}, \Omega)$ we conclude with (3.3) and (3.4)

$$
\frac{1}{\|h\|_{C^1}} \mathcal{A}(\widehat{E_h} - E - W, V) \leqslant C \big(\|\widehat{E_h}\|_{H(\operatorname{curl},\Omega)} \mathcal{O}(\|h\|_{C^1}) + \|\widehat{E_h} - E\|_{H(\operatorname{curl},\Omega)} \big) \|V\|_{H(\operatorname{curl},\Omega)} \to 0,
$$

as $h \to 0$ in C^1, which implies $\lim_{\|h\|_{C^1} \to 0} \|\widehat{E_h} - E - W\|_{H(\operatorname{curl},\Omega)} = 0$ by a perturbation argument as in the proof of Theorem 3.1. ☐

The function $W \in H_{\mathrm{pc}}(\Omega)$ is called material derivative of E and it is no solution of the homogeneous Maxwell's equations. Note that W depends on values of h in the neighborhood of ∂D, which is not adequate in view of perturbations of the boundary ∂D. A formal Taylor expansion motivates to consider the domain derivative $E' = W - J_h^\top E - J_E h$, which leads to the desired derivative of the operator $\mathbf{F}$.

We introduce the notation $V_\nu = V \cdot \nu$ and $V_\tau = \nu \times (V \times \nu)$ for the decomposition $V = V_\nu \nu + V_\tau$ of a vector field on ∂D. Furthermore, we introduce the surface gradient $\operatorname{Grad}_{\partial D} : H^{\frac12}(\partial D) \to H^{-\frac12}(\operatorname{Curl}, \partial D)$, given for smooth functions u by $\operatorname{Grad}_{\partial D} u = \nabla u - \frac{\partial u}{\partial \nu} \nu$, the vector surface rotation $\overrightarrow{\operatorname{Curl}}_{\partial D} : H^{\frac12}(\partial D) \to H^{-\frac12}(\operatorname{Div}, \partial D)$, given by $\overrightarrow{\operatorname{Curl}}_{\partial D} u = \operatorname{Grad}_{\partial D} u \times \nu$, and the surface divergence $\operatorname{Div}_{\partial D} : H^{-\frac12}(\operatorname{Div}, \partial D) \to H^{-\frac12}(\partial D)$, which is defined for a smooth function V by $\operatorname{Div}_{\partial D} V = \operatorname{div} V - \nu \cdot J_V \nu$. Note that the surface divergence satisfies

$$
\operatorname{Div}_{\partial D}(V \times \nu) = \operatorname{curl} V \cdot \nu \tag{3.5}
$$

and is coupled by the duality

$$
\int_{\partial D} u \operatorname{Div}_{\partial D}(V) \,\mathrm{d}s = - \int_{\partial D} V \cdot \operatorname{Grad}_{\partial D}(u) \,\mathrm{d}s \,.
$$

With these notations a representation of the domain derivative can be shown.

Theorem 3.3. *Let ∂D be of class C^2. In the setting of Theorem 3.2, define $E' = W - J_h^\top E - J_E h \in H(\operatorname{curl}, \Omega)$. E' can be uniquely extended to the radiating weak solution of Maxwell's equations*

$$
\operatorname{curl} E' - \mathrm{i}k\, H' = 0, \quad \operatorname{curl} H' + \mathrm{i}k\, E' = 0
$$

in $\mathbb{R}^3 \setminus \overline{D}$ with boundary condition

$$
\nu \times E' = \overrightarrow{\operatorname{Curl}}_{\partial D}(h_\nu E_\nu) - \mathrm{i}k\, h_\nu\, H_\tau \quad \text{on } \partial D.
$$

Proof. By the regularity of ∂D we have $E \in H^1(\Omega, \mathbb{C}^3)$ and therefore $E' = W - J_h^\top E - J_E h \in L^2(\Omega, \mathbb{C}^3)$ (see [1]). Some basic vector calculus shows

$$
\operatorname{curl}(J_h^\top E + J_E h) = \operatorname{curl}\big((J_E - J_E^\top) h + \nabla (h^\top E) \big) = \operatorname{curl}(\operatorname{curl} E \times h)
$$

$$= \operatorname{div}(h)\operatorname{curl} E + J_{\operatorname{curl} E} h - J_h \operatorname{curl} E = \mathrm{i}k \operatorname{div} h\, H + \mathrm{i}k J_H h - \mathrm{i}k J_h H\,, \quad (3.6)$$

which in particular implies $\operatorname{curl} E' \in L^2(\Omega, \mathbb{C}^3)$ and therefore $E' \in H(\operatorname{curl}, \Omega)$. Additionally, by $\nu \times W = 0$ on ∂D, we find

$$\nu \times E' = -\nu \times (J_E h + J_h^\top E) = -\nu \times (\operatorname{curl} E \times h) - \nu \times \nabla(h^\top E)$$

in $H^{-\frac{1}{2}}(\operatorname{Div}, \partial D)$. From $\nu \times E = 0$ we obtain $-\nu \times \nabla(h^\top E) = \overrightarrow{\operatorname{Curl}}_{\partial D}(h_\nu E_\nu)$. Furthermore, with $\nu \times (\operatorname{curl} E \times h) = \mathrm{i}k\,(h_\nu H_\tau + H_\nu h_\tau)$ and $H_\nu = 0$, which follows by Maxwell's equations and (3.5), we conclude the stated boundary condition for E'.

It remains to show, that E' is a radiating solution to Maxwell's equations, which will be achieved by showing $\mathcal{A}(E', V) = 0$ for any $V \in H_{\mathrm{pc}}(\Omega)$. We have

$$\begin{aligned} \mathcal{A}(E', V) &= \mathcal{A}(W, V) - \mathcal{A}(J_h^\top E + J_E h, V) \\ &= \int_\Omega \Big(\operatorname{curl} E^\top (\operatorname{div} h\, I - J_h - J_h^\top) \overline{\operatorname{curl} V} + k^2 E^\top (\operatorname{div} h\, I - J_h - J_h^\top) \overline{V} \Big) \,\mathrm{d}x \\ &\quad - \int_\Omega \Big(\operatorname{curl}(J_h^\top E + J_E h)^\top \overline{\operatorname{curl} V} - k^2 (J_h^\top E + J_E h)^\top \overline{V} \Big) \,\mathrm{d}x. \end{aligned}$$

Using again (3.6), we find

$$\mathcal{A}(E', V) = \int_\Omega \Big(k^2 (J_E h + \operatorname{div} h\, E - J_h E)^\top \overline{V} - (J_{\operatorname{curl} E} h + J_h^\top \operatorname{curl} E)^\top \overline{\operatorname{curl} V} \Big) \,\mathrm{d}x$$

From $\operatorname{div} E = 0$ in $\mathbb{R}^3 \setminus \overline{D}$, we conclude $\operatorname{curl}(E \times h) = \operatorname{div}(h)\, E + J_E h - J_h E$ and Maxwell's equations yield

$$\begin{aligned} J_{\operatorname{curl} E} h + J_h^\top E &= (J_{\operatorname{curl} E} - J_{\operatorname{curl} E}^\top) + J_{\operatorname{curl} E}^\top h + J_h^\top E \\ &= \operatorname{curl}^2 E \times h + \nabla(h^\top \operatorname{curl} E) = k^2 (E \times h) + \nabla(h^\top \operatorname{curl} E). \end{aligned}$$

Together with $\operatorname{div}\big((E \times h) \times \overline{V}\big) = \operatorname{curl}(E \times h)^\top \overline{V} - (E \times h)^\top \overline{\operatorname{curl} V}$, we finally arrive at

$$\mathcal{A}(E', V) = k^2 \int_\Omega \operatorname{div}\big((E \times h) \times \overline{V}\big) \,\mathrm{d}x - \int_\Omega \nabla(h^\top \operatorname{curl} E)^\top \overline{\operatorname{curl} V} \,\mathrm{d}x\,.$$

Since $\operatorname{div}\operatorname{curl} = 0$, we obtain by the divergence theorem

$$\begin{aligned} \mathcal{A}(E', V) &= \int_\Omega \operatorname{div} \Big[k^2 (E \times h) \times \overline{V} - (h^\top \operatorname{curl} E) \overline{\operatorname{curl} V} \Big] \mathrm{d}x \\ &= \int_{\partial D} \big((h^\top \operatorname{curl} E)(\nu^\top \overline{\operatorname{curl} V}) - k^2 \nu^\top ((E \times h) \times \overline{V}) \big) \mathrm{d}s\,. \end{aligned}$$

Note that no boundary integrals on $\partial B_R(0)$ occur, since h is compactly supported in $B_R(0)$. The first term vanishes since $\nu^\top \overline{\operatorname{curl} V} = \operatorname{Div}_{\partial D}(\overline{V} \times \nu) = 0$ for $V \in H_{\mathrm{pc}}(\Omega)$ and for the second term we compute

$$\big((E \times h) \times \overline{V}\big) \cdot \nu = (E \cdot \overline{V})(h \cdot \nu) - (\overline{V} \cdot h)(E \cdot \nu) = (h \times E) \cdot (\nu \times \overline{V}) = 0\,.$$

Thus, we have $\mathcal{A}(E', V) = 0$, which finishes the proof. $\square$

Before establishing the second domain derivative in the next section let us consider the linearization of the operator $\mathbf{F}$, which by the previous result is given by its Fréchet derivative $\mathbf{F}'[\partial D]h = E'_\infty$, if we specify a linear space of admissible boundaries. In general solving an ill-posed nonlinear equation by iterative regularization schemes based on its derivative requires some additional conditions on the operator $\mathbf{F}$. A quite general one is the tangential cone condition, which can be described by the existence of a constant $c > 0$ such that locally $\|\mathbf{F}(y) - \mathbf{F}(x) - \mathbf{F}'[x](y-x)\| \leq c\|y-x\| \, \|\mathbf{F}(x) - \mathbf{F}(y)\|$ holds. It ensures to some extent the equivalence of local ill-posedness of a nonlinear equation and ill-posedness of its linearization (see [14]). To our knowledge the validity of such a condition is an essential open problem in any inverse obstacle scattering problem so far. Here, we just remark on injectivity of $\mathbf{F}'$, a necessary consequence from the cone condition, which is a severe problem in itself.

Corollary 3.4. *Let the wave number $-k^2$ be no eigenvalue of the Laplace-Beltrami operator $\Delta_{\partial D}$ on the boundary of D. Furthermore, let $h \in (C^1(\partial D))^3$ be a vector field with constant normal component $h_\nu = c \in \mathbb{R}$ on ∂D. Then $\mathbf{F}'[\partial D]h = 0$ on $\mathbb{S}^2$ implies $h_\nu = 0$.*

Proof. A vanishing far field pattern $\mathbf{F}'[\partial D]h = 0$ implies $E' = 0$ in $\mathbb{R}^3 \setminus \overline{D}$ (see [3]). Since h_ν is constant we obtain from Theorem 3.3

$$h_\nu \left(\overrightarrow{\operatorname{Curl}}_{\partial D}(E_\nu) - \mathrm{i}k\nu \times (H \times \nu) \right) = 0 \quad \text{on } \partial D\,. \tag{3.7}$$

Assuming $h_\nu \neq 0$, a rotation and taking the surface divergence yields

$$\begin{aligned} 0 &= \operatorname{Div}_{\partial D} \left((\overrightarrow{\operatorname{Curl}}_{\partial D}(E_\nu)) \times \nu - \mathrm{i}k(\nu \times (H \times \nu)) \times \nu \right) \\ &= -\operatorname{Div}_{\partial D} \left(\operatorname{Grad}_{\partial D} E_\nu\right) - \mathrm{i}k \operatorname{Div}_{\partial D}(H \times \nu) \;=\; -\Delta_{\partial D} E_\nu - k^2 E_\nu\,, \end{aligned}$$

where we have used $\overrightarrow{\operatorname{Curl}}_{\partial D}(E_\nu) = \operatorname{Grad}_{\partial D}(E_\nu) \times \nu$ and $\operatorname{Div}_{\partial D}(H \times \nu) = \nu \cdot \operatorname{curl} H = -\mathrm{i}kE_\nu$. Since $-k^2$ is no eigenvalue of the Laplace-Beltrami operator, we obtain $E_\nu = 0$ on ∂D. Furthermore, by (3.7) we have $\nu \times (H \times \nu) = 0$ on ∂D. Applying the Stratton-Chu representation of E^i in D and of the radiating solution E^s in $\mathbb{R}^3 \setminus \overline{D}$ (see [15]) we obtain from vanishing boundary values $\nu \times E = 0$ and $\nu \times (H \times \nu) = 0$ on ∂D of the total field the contradiction

$$E^i(x) = \frac{1}{\mathrm{i}k} \operatorname{curl}^2 \int_{\partial D} (\nu(y) \times H(y)) \Phi(x,y) \, \mathrm{d}s(y) - \operatorname{curl} \int_{\partial D} (\nu(y) \times E(y)) \Phi(x,y) \, \mathrm{d}s(y) \;=\; 0$$

for any $x \in D$. Thus we conclude $h_\nu = 0$. $\square$

Excluding eigenvalues of the Laplace-Beltrami operator seems to be necessary for injectivity of $\mathbf{F}'$. This can be seen from scattering by a ball $B_\rho(0)$ of radius $\rho > 0$, as it was already observed by H. Haddar and R. Kress in [6]. Since, if we consider an incident field

$$E^i(x) = \sqrt{n(n+1)} \frac{j_n(kr)}{r} Y_n^m(\hat{x})\hat{x} + \frac{j_n(kr) + krj_n'(kr)}{r} U_n^m(\hat{x})$$

in spherical coordinates, $x = r\hat{x}$, with spherical surface harmonics Y_n^m, $U_n^m = \frac{1}{\sqrt{n(n+1)}} \operatorname{Grad}_{\mathbb{S}^2} Y_n^m$, $V_n^m = \hat{x} \times U_n^m$ and Bessel- and Hankel-functions j_n, $h_n^{(1)}$ for a positive integer $n \in \mathbb{N}$ and

$m \in \{-n, \ldots, n\}$ (see [15], some calculations show for the total field $E = E^i + E^s$ on the boundary

$$\begin{aligned}
&\overrightarrow{\mathrm{Curl}}_{\partial D}(E_\nu) - \mathrm{i}k\big(\nu \times (H \times \nu)\big) \\
&= \left(k^2 - \frac{n(n+1)}{\rho^2}\right)\left[(j_n(k\rho) + \frac{j_n(k\rho) + k\rho j_n'(k\rho)}{h_n^{(1)}(k\rho) + k\rho h_n^{(1)}(k\rho)} h_n^{(1)}(k\rho)\right] V_n^m(\hat{x})\,.
\end{aligned}$$

Thus, if $k^2 = \frac{n(n+1)}{\rho^2}$ for $n \in \mathbb{N}$, i.e., if $-k^2$ is an eigenvalue of the Laplace-Beltrami operator on $\partial B_\rho(0)$, there are nontrivial incident fields leading to a vanishing boundary condition (3.7) and therefore to a vanishing domain derivative.

Based on Theorem 3.3 we can characterize the adjoint operator $(\mathbf{F}')^*$, which is of specific interest for iterative regularization schemes, and adds to comparable results for the exterior Dirichlet problem in acoustic scattering (see [9]). We specify the investigations to the case of starlike domains, which also will be considered in the numerical tests below. Without changing notation we consider the operator $\mathbf{F} : r \mapsto E'_\infty$ for $r \in C^2(\mathbb{S}^2)$ and a starlike parametrized boundary $\partial D = \{y = r(\hat{y})\hat{y} : \hat{y} \in \mathbb{S}^2\}$. Analogously, a variation is given by $h(y) = \tilde{h}(\hat{y})\hat{y}$.

Corollary 3.5. *The L^2-adjoint operator of $\mathbf{F}'[r] : C^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2, \mathbb{C}^3)$ is given by*

$$(\mathbf{F}'[r])^* A(\hat{y}) = k^2 r^2(\hat{y})\Big(\overline{H_\tau(y) \cdot (\mathcal{H}_A(y))_\tau(y) - E_\nu(y)(\mathcal{E}_A(y))_\nu}\Big)\,,$$

where $\mathcal{E}_A$, $\mathcal{H}_A$ denote the total fields of the scattering problem with incident field given by the Herglotz wave function $\mathcal{E}_A^i(y) = \int_{\mathbb{S}^2} \overline{A(\hat{x})} \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\, \mathrm{d}s_{\hat{x}}$ for a tangential field $A \in L^2(\mathbb{S}^2, \mathbb{C}^3)$.

Proof. We introduce the notation $E^s(y; a, \hat{x})$ for a solution of the scattering problem with incident field $E^i(y; a, \hat{x}) = a\mathrm{e}^{\mathrm{i}k\hat{x}\cdot y}$ and $H^i(y; a, \hat{x}) = \hat{x} \times E^i(y; a, \hat{x})$ with direction $\hat{x}$ and polarization $a \perp \hat{x}$. From the integral representation of the far field pattern of the radiating solution E' (see [3]) we compute

$$\begin{aligned}
a \cdot E'_\infty(\hat{x}) &= \mathrm{i}k\, a \cdot \left(\hat{x} \times \int_{\partial D} [\nu \times E'_\tau + (\nu \times H'_\tau) \times \hat{x}]\, \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\, \mathrm{d}s\right) \\
&= \mathrm{i}k \int_{\partial D} (\nu \times E'_\tau) \cdot H^i(\cdot; a, -\hat{x})\, \mathrm{d}s + \mathrm{i}k \int_{\partial D} (\nu \times H'_\tau) \cdot E^i(\cdot; a, -\hat{x})\, \mathrm{d}s
\end{aligned}$$

for any $\hat{x} \in \mathbb{S}^2$ and $\hat{x} \perp a \in \mathbb{C}^3$. Applying Green's vector formula in the exterior of D together with the Silver-Müller radiation conditions implies

$$\int_{\partial D} (\nu \times \mathrm{curl}\, E') \cdot E^s(\cdot; a, -\hat{x})\, \mathrm{d}s = \int_{\partial D} (\nu \times \mathrm{curl}\, E^s(\cdot; a, -\hat{x})) \cdot E'\, \mathrm{d}s$$

Thus, we conclude by $\nu \times E^s(.; a, -\hat{x}) = -\nu \times E^i(.; a, -\hat{x})$ on ∂D that

$$\begin{aligned}
a \cdot E'_\infty(\hat{x}) &= \mathrm{i}k \int_{\partial D} E'_\tau \cdot (H^i(\cdot; a, -\hat{x}) \times \nu)\, \mathrm{d}s + \mathrm{i}k \int_{\partial D} H'_\tau \cdot (\nu \times E^s(\cdot; a, -\hat{x}))\, \mathrm{d}s \\
&= -\mathrm{i}k \int_{\partial D} E'_\tau \cdot (\nu \times H^i(\cdot; a, -\hat{x}))\, \mathrm{d}s - \mathrm{i}k \int_{\partial D} E'_\tau \cdot (\nu \times H^s(\cdot; a, -\hat{x}))\, \mathrm{d}s \\
&= -\mathrm{i}k \int_{\partial D} E'_\tau \cdot (\nu \times H(\cdot; a, -\hat{x}))\, \mathrm{d}s\,.
\end{aligned}$$

Using this representation with polarizations given by a tangential field $A \in L^2_t(\mathbb{S}^2, \mathbb{C}^3)$ and substituting the boundary condition from Theorem 3.3 yields

$$\begin{aligned}
\langle \mathbf{F}'[\partial D]h, A\rangle_{L^2(\mathbb{S}^2)} &= -\mathrm{i}k \int_{\mathbb{S}^2}\int_{\partial D} E'_\tau \cdot (\nu \times H(\cdot;\overline{A(\hat{x})}, -\hat{x})\,\mathrm{d}s\,\mathrm{d}s(\hat{x}) \\
&= \mathrm{i}k \int_{\partial D} (\nu \times E') \cdot \left(\int_{S^2} H(\cdot;\overline{A(\hat{x})}, -\hat{x})\,\mathrm{d}s(\hat{x})\right)\mathrm{d}s \\
&= \mathrm{i}k \int_{\partial D} \left(\overrightarrow{\mathrm{Curl}}_{\partial D}(h_\nu E_\nu) - \mathrm{i}k\, h_\nu H_\tau\right) \cdot \mathcal{H}_A\,\mathrm{d}s \\
&= \mathrm{i}k \int_{\partial D} \mathrm{Grad}_{\partial D}(h_\nu E_\nu) \cdot (\nu \times \mathcal{H}_A) - \mathrm{i}k\, h_\nu H_\tau \cdot \mathcal{H}_A\,\mathrm{d}s \\
&= \int_{\partial D} \Big[-\mathrm{i}k\, h_\nu E_\nu\ \mathrm{Div}_{\partial D}(\nu \times \mathcal{H}_A) + k^2 h_\nu H_\tau \cdot \mathcal{H}_A\Big]\,\mathrm{d}s \\
&= \int_{\partial D} h_\nu \Big[E_\nu(\mathrm{i}k\nu \cdot \mathrm{curl}\,\mathcal{H}_A) + k^2 H_\tau \cdot \mathcal{H}_A\Big]\,\mathrm{d}s \\
&= k^2 \int_{\partial D} h_\nu \big[-E_\nu(\mathcal{E}_A)_\nu + H_\tau \cdot \mathcal{H}_A\big]\,\mathrm{d}s\,,
\end{aligned}$$

where we have denoted as before the solution of the actual scattering problem by $E = E(\cdot;p,d)$ and $H = H(\cdot;p,d)$. If ∂D is starlike parametrized by $y = r(\hat{y})\hat{y}$ and the variation is given by $h(y) = \tilde{h}(\hat{y})\hat{y}$ we can calculate h_ν explicitly and arrive at

$$\langle \mathbf{F}'[r])h, A\rangle_{L^2(\mathbb{S}^2)} = k^2 \int_{\mathbb{S}^2} \tilde{h}(\hat{y})\, r^2(\hat{y}) \Big[-E_\nu(y)(\mathcal{E}_A(y))_\nu + H_\tau(y) \cdot (\mathcal{H}_A(y))_\tau\Big]\,\mathrm{d}s(\hat{y})\,,$$

which shows the assertion. □

4 The Second Domain Derivative

We continue in proving the scattered wave to be twice differentiable with respect to the boundary. If we use two small perturbations $h_1, h_2 \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ with compact support in $B_R(0)$ to perturb the boundary, we arrive at

$$(\partial D_{h_2})_{h_1} = \{y = \varphi_1(\varphi_2(x)) = x + h_2(x) + h_1(x + h_2(x)) : x \in \partial D\},$$

which is not symmetric with respect to the variations h_1 and h_2. But we expect a second derivative to be symmetric, see [5, Chapter VIII.12]. The perturbation becomes symmetric, if we replace h_1 by $h_1 \circ \varphi_2^{-1}$. This motivates our goal: Finding a radiating solution of Maxwell's equations E'', depending bilinearly on h_1 and h_2, being symmetric in h_1 and h_2, and satisfying

$$\lim_{\|h_2\|_{C^1}\to 0} \frac{1}{\|h_2\|_{C^1}} \sup_{\|h_1\|_{C^1}=1} \|E'_{h_1\circ\varphi_2^{-1}}[\partial D_{h_2}] - E'_{h_1}[\partial D] - E''\|_{H(\mathrm{curl},\Omega)} = 0,$$

where $E'_h[\partial D]$ denotes the domain derivative with respect to the variation h at the scatterer D. The Taylor expansion $h_1 \circ \varphi_2^{-1} = h_1 - J_{h_1}h_2 + \mathcal{O}(\|h_2\|^2_{C^1})$ leads to

$$E'' = (E'_1)'_2 - E'_h \tag{4.1}$$

with $(E_1')_2'$ being the domain derivative with respect to the variation h_2 of the domain derivative with respect to the variation h_1. The second term E_h' is the domain derivative from Theorem 3.3 with respect to the variation $h = J_{h_1} h_2$.

We are going to prove that the second domain derivative is given by (4.1) and present a characterization of E'' as a radiating solution to Maxwell's equations. Similar to the first derivative, we start by showing existence of the material derivative of the material derivative.

Let $W_i \in H(\mathrm{curl}, \Omega)$ denote the material derivative with boundary ∂D and with respect to the perturbation h_i, $i = 1, 2$. W_i is the solution of

$$\mathcal{A}(W_i, V) = \int_\Omega \Big(\mathrm{curl}\, E^\top A_i \overline{\mathrm{curl}\, V} + k^2\, E^\top A_i \overline{V} \Big)\, \mathrm{d}x, \quad \text{for all } V \in H(\mathrm{curl}, \Omega)\,, \tag{4.2}$$

where we introduced the abbreviation A_i for the symmetric matrix $A_i = \mathrm{div}\, h_i\, I - J_{h_i} - J_{h_i}^\top$. Let $\widetilde{W}_1 \in H_{\mathrm{pc}}(\Omega_{h_2})$ denote the solution of (4.2) with Ω being replaced by Ω_{h_2}. Again, we define $\widehat{W}_{1,h_2} = J_{\varphi_2}^\top \widetilde{W}_1 \in H_{\mathrm{pc}}(\Omega)$. Then, $\widehat{W}_{1,h_2}$ solves

$$\begin{aligned}\int_\Omega \Big(\mathrm{curl}\, \widehat{W}_{1,h_2}^\top \Big(\frac{J_{\varphi_2}^\top J_{\varphi_2}}{\det J_{\varphi_2}} \Big) \overline{\mathrm{curl}\, V} - k^2\, \widehat{W}_{1,h_2}^\top \big(\det(J_{\varphi_2}) J_{\varphi_2}^{-1} J_{\varphi_2}^{-\top} \big) \overline{V} \Big) \mathrm{d}x + \mathrm{i}k\, \langle \Lambda(\nu \times \widehat{W}_{1,h_2}), V \rangle_{\partial B_R(0)} \\ = \int_\Omega \Big(\mathrm{curl}\, \widehat{E}_{h_2} \Big(\frac{J_{\varphi_2}^\top \widetilde{A}_1 J_{\varphi_2}}{\det J_{\varphi_2}} \Big) \overline{\mathrm{curl}\, V} + k^2\, \widehat{E}_{h_2} \Big(\det J_{\varphi_2} J_{\varphi_2}^{-1} \widetilde{A}_1 J_{\varphi_2}^{-\top} \Big) \overline{V} \Big) \mathrm{d}x\end{aligned} \tag{4.3}$$

for all $V \in H_{pc}(\Omega)$. In the next lemma, we provide the linearization of the new matrices.

Lemma 4.1. *Let $A \in C^1(\mathbb{R}^3, \mathbb{R}^{3\times3})$ and $\varphi(x) = x + h(x)$ with $h \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ sufficiently small. Then we have*

$$\begin{aligned} \frac{\det J_\varphi^\top \widetilde{A} J_\varphi}{\det J_\varphi} &= A + J_h^\top A + A J_h - \mathrm{div}\, h\, A + A'(h) + \mathcal{O}(\|h\|_{C^1}^2), \\ \det J_\varphi J_\varphi^{-1} \widetilde{A} J_\varphi^{-\top} &= A - J_h A - A J_h^\top + \mathrm{div}\, h\, A + A'(h) + \mathcal{O}(\|h\|_{C^1}^2), \end{aligned}$$

where the matrix $A'(h) \in C(\mathbb{R}^3, \mathbb{R}^{3\times3})$ is given by $(A'(h))_{ij} = (\nabla A_{ij})^\top h$, $i, j = 1, \ldots, 3$.

Proof. The linearizations follow from (3.3), (3.4) and the Taylor expansion of the coefficients $A_{ij}(x + h(x))$. □

As a first step we prove that the material derivative W_1 depends continuously on perturbations h_2.

Theorem 4.2. *Let $W_1 \in H_{\mathrm{pc}}(\Omega)$ be the solution of (4.2) with $i = 1$ and $\widehat{W}_{1,h_2} \in H_{\mathrm{pc}}(\Omega)$ a solution of (4.3). Then we have*

$$\lim_{\|h_2\|_{C^1} \to 0} \|W_1 - \widehat{W}_{1,h_2}\|_{H(\mathrm{curl},\Omega)} = 0.$$

Proof. Let $\ell_{h_2}(V)$ denote the right hand side of (4.2) with $i = 1$ and let ℓ_{h_2,h_1} denote the right hand side of (4.3). Recall the notation $\mathcal{A}_{h_2}$ for the sesquilinear form, such that the left hand side of (4.3) is given by $\mathcal{A}_{h_2}(\widehat{W}_{1,h_2}, V)$. Then we have

$$\mathcal{A}(\widehat{W}_{1,h_2} - W_1, V) = \mathcal{A}(\widehat{W}_{1,h_2}, V) - \mathcal{A}_{h_2}(\widehat{W}_{1,h_2}, V) + \ell_{h_1,h_2}(V) - \ell_{h_2}(V). \tag{4.4}$$

Adding and subtracting the integral

$$\int_\Omega \Big(\operatorname{curl} \widehat{E}_{h_2}^\top A_1 \overline{\operatorname{curl} V} + k^2\, \widehat{E}_{h_2}^\top A_1 \overline{V} \Big)\, \mathrm{d}x$$

leads to

$$\begin{aligned}
\mathcal{A}(\widehat{W}_{1,h_2} - W_1, V) &= \int_\Omega \Big[\operatorname{curl} \widehat{W}_{1,h_2}^\top \Big(I - \frac{J_{\varphi_2}^\top J_{\varphi_2}}{\det J_{\varphi_2}} \Big) \overline{\operatorname{curl} V} - k^2\, \widehat{W}_{1,h_2}^\top \Big(I - \det(J_{\varphi_2}) J_{\varphi_2}^{-1} J_{\varphi_2}^{-\top} \Big) \overline{V} \Big]\, \mathrm{d}x \\
&+ \int_\Omega \Big[\operatorname{curl} \widehat{E}_{h_2}^\top \Big(\frac{J_{\varphi_2}^\top \widetilde{A_1} J_{\varphi_2}}{\det J_{\varphi_2}} - A_1 \Big) \overline{\operatorname{curl} V} + k^2\, \widehat{E}_{h_2} \Big(\det(J_{\varphi_2}) J_{\varphi_2}^{-1} \widetilde{A_1} J_{\varphi_2}^{-\top} - A_1 \Big) \overline{V} \Big]\, \mathrm{d}x \\
&+ \int_\Omega \Big(\operatorname{curl} (\widehat{E}_{h_2} - E)^\top A_1 \overline{\operatorname{curl} V} + k^2\, (\widehat{E}_{h_2} - E)^\top A_1 \overline{V} \Big)\, \mathrm{d}x.
\end{aligned}$$

With Lemma 4.1 and Theorem 3.1 we conclude $\mathcal{A}(\widehat{W}_{1,h_2} - W_1, V) \to 0$ as $h_2 \to 0$ in C^1, which finishes the proof. □

As before, we consider the linearizations and prove differentiability.

Theorem 4.3. *Let $W_1 \in H_{\mathrm{pc}}(\Omega)$ be the solution of (4.2) with $i = 1$ and $\widehat{W}_{1,h_2} \in H_{\mathrm{pc}}(\Omega)$ of (4.3). Then there exists a function $W_1' \in H_{\mathrm{pc}}(\Omega)$, depending linearly and continuously on $h_2 \in C^1$, such that*

$$\lim_{\|h_2\|_{C^1} \to 0} \frac{1}{\|h_2\|_{C^1}} \|\widehat{W}_{1,h_2} - W_1 - W_1'\|_{H(\mathrm{curl},\Omega)} = 0.$$

Proof. Motivated by (4.4) we define $W_1' \in H_{\mathrm{pc}}(\Omega)$ as the solution of

$$\begin{aligned}
\mathcal{A}(W_1', V) = &\int_\Omega \Big(\operatorname{curl} W_1^\top A_2 \overline{\operatorname{curl} V} + k^2\, W_1^\top A_2 \overline{V} + \operatorname{curl} W_2^\top A_1 \overline{\operatorname{curl} V} + k^2\, W_2^\top A_1 \overline{V} \Big) \mathrm{d}x \\
&+ \int_\Omega \operatorname{curl} E^\top \big(J_{h_2}^\top A_1 + A_1 J_{h_2} - \operatorname{div} h_2\, A_1 + A_1'(h_2) \big) \overline{\operatorname{curl} V} \mathrm{d}x \\
&+ k^2 \int_\Omega E^\top \big(- J_{h_2} A_1 - A_1 J_{h_2}^\top + \operatorname{div} h_2\, A_1 + A_1'(h_2) \big) \overline{V} \mathrm{d}x\,, \quad \text{for all } V \in H_{\mathrm{pc}}(\Omega)\,.
\end{aligned}$$

As before, we consider the difference $\mathcal{A}(\widehat{W}_{1,h_2} - W_1 - W_1', V)$. We add and subtract the following integrals

$$\begin{aligned}
I_1 &= \int_\Omega \Big(\widehat{W}_{1,h_2}^\top A_2 \overline{\operatorname{curl} V} - k^2\, \widehat{W}_{1,h_2} A_2 \overline{V} \Big) \mathrm{d}x, \\
I_2 &= \int_\Omega \Big(\operatorname{curl} \widehat{E}_{h_2}^\top A_1 \overline{\operatorname{curl} V} + k^2\, \widehat{E}_{h_2}^\top A_1 \overline{V} \Big) \mathrm{d}x, \\
I_3 &= \int_\Omega \operatorname{curl} \widehat{E}_{h_2}^\top \Big(J_{h_2}^\top A_1 + A_1 J_{h_2} - \operatorname{div} h_2\, A_1 + A_1'(h_2) \Big) \overline{\operatorname{curl} V} \mathrm{d}x, \\
I_4 &= k^2 \int_\Omega \widehat{E}_{h_2}^\top \Big(- J_{h_2} A_1 - A_1 J_{h_2}^\top + \operatorname{div} h_2\, A_1 + A_1'(h_2) \Big) \overline{V} \mathrm{d}x,
\end{aligned}$$

i.e. we consider

$$
\begin{aligned}
&\mathcal{A}(\widehat{W}_{1,h_2} - W_1 - W_1', V) = \mathcal{A}(\widehat{W}_{1,h_2} - W_1 - W_1', V) + \sum_{k=1}^{4} I_k - \sum_{l=1}^{4} I_l \\
&= \int_\Omega \Big[\operatorname{curl} \widehat{W}_{1,h_2}^\top \Big(I - \frac{J_{\varphi_2}^\top J_{\varphi_2}}{\det J_{\varphi_2}} - A_2 \Big) \overline{\operatorname{curl} V} - k^2\, \widehat{W}_{1,h_2}^\top \Big(I - \det J_{\varphi_2} J_{\varphi_2}^{-1} J_{\varphi_2}^{-\top} + A_2 \Big) \overline{V} \Big] \mathrm{d}x \\
&\quad + \int_\Omega \Big[\operatorname{curl}(\widehat{W}_{1,h_2} - W_1)^\top A_2 \overline{\operatorname{curl} V} - k^2\, (\widehat{W}_{1,h_2} - W_1)^\top A_2 \overline{V} \Big) \mathrm{d}x \\
&\quad + \int_\Omega \operatorname{curl} \widehat{E}_{h_2}^\top \Big(\frac{J_{\varphi_2}^\top \widetilde{A_1} J_{\varphi_2}}{\det J_{\varphi_2}} - A_1 - J_{h_2}^\top A_1 - A_1 J_{h_2} + \operatorname{div} h_2\, I - A_1'(h_2) \Big) \overline{\operatorname{curl} V} \mathrm{d}x \\
&\quad + \int_\Omega \widehat{E}_{h_2}^\top \Big(\det J_{\varphi_2} J_{\varphi_2}^{-1} \widetilde{A_1} J_{\varphi_2}^{-\top} - A_1 + J_{h_2} A_1 + A_1 J_{h_2}^\top - \operatorname{div} h_2\, I - A_1'(h_2) \Big) \overline{V} \mathrm{d}x \\
&\quad + \int_\Omega \Big[\operatorname{curl}(\widehat{E}_{h_2} - E - W_2)^\top A_1 \overline{\operatorname{curl} V} + k^2\, (\widehat{E}_{h_2} - E - W_2)^\top A_1 \overline{V} \Big] \mathrm{d}x \\
&\quad + \int_\Omega \operatorname{curl}(\widehat{E}_{h_2} - E)^\top \Big(J_{h_2}^\top A_1 + A_1 J_{h_2} - \operatorname{div} h_2\, A_1 + A_1'(h_2) \Big) \overline{\operatorname{curl} V} \mathrm{d}x \\
&\quad + k^2 \int_\Omega (\widehat{E}_{h_2} - E)^\top \Big(- J_{h_2} A_1 - A_1 J_{h_2}^\top + \operatorname{div} h_2\, A_1 + A_1'(h_2) \Big) \overline{V} \mathrm{d}x.
\end{aligned}
$$

This leads to the estimate

$$
\begin{aligned}
&\mathcal{A}(\widehat{W}_{1,h_2} - W_1 - W_1', V) \\
&\leqslant C \|V\|_{H(\operatorname{curl},\Omega)} \Big(\|\widehat{W}_{1,h_2}\|_{H(\operatorname{curl},\Omega)} \|h_2\|_{C^1}^2 + \|\widehat{W}_{1,h_2} - W_1\|_{H(\operatorname{curl},\Omega)} \|h_2\|_{C^1} \\
&\qquad + \|\widehat{E}_{h_2} - E - W_2\|_{H(\operatorname{curl},\Omega)} + \|\widehat{E}_{h_2}\|_{H(\operatorname{curl},\Omega)} \|h_2\|_{C^1}^2 + \|\widehat{E}_{h_2} - E\|_{H(\operatorname{curl},\Omega)} \|h_2\|_{C^1}^2 \Big)
\end{aligned}
$$

for all $V \in H(\operatorname{curl}, \Omega)$. Again by a perturbation argument, we conclude

$$
\lim_{\|h_2\|_{C^1} \to 0} \|\widehat{W}_{1,h_2} - W_1 - W_1'\|_{H(\operatorname{curl},\Omega)} = 0.
$$

□

Since $W_1' \in H_{\mathrm{pc}}(\Omega)$ is the material derivative with respect to h_2 of the material derivative with respect to h_1, it contains by linearity the domain derivative with respect to h_2 of the domain derivative with respect to h_1, which we denoted by $(E_1')_2'$ before. To calculate it, we consider the formal Taylor expansion

$$
\widehat{W}_{1,h_2}(x) = (I + J_{h_2}^\top(x)) \big(W_1(x) + J_{W_1}(x) h_2(x) + \frac{\mathrm{d}}{\mathrm{d}h_2} W_1(x) + \mathcal{O}(\|h_2\|_{C^1}^2) \big).
$$

With the decomposition $W_1 = E_1' + J_{h_1}^\top E + J_E h_1$ we formally conclude

$$
\frac{\mathrm{d}}{\mathrm{d}h_2} W_1 = (E_1')_2' + J_{h_1}^\top E_2' + J_{E_2'} h_1,
$$

and, furthermore, with $W_1' = \frac{\mathrm{d}}{\mathrm{d}h_2} \widehat{W}_{1,h_2}$ the Ansatz

$$
(E_1')_2' = W_1' - J_{h_2}^\top W_1 - J_{W_1} h_2 - J_{h_1}^\top E_2' - J_{E_2'} h_1
$$

is motivated. Similarly to the first domain derivative, we need higher regularity of the solution and therefore higher regularity of the boundary, to ensure the Ansatz to be well defined.

Theorem 4.4. *Let ∂D be of class C^3. In the setting of the Theorem 4.3, let*

$$(E_1')_2' = W_1' - J_{h_2}^\top W_1 - J_{W_1} h_2 - J_{h_1}^\top E_2' - J_{E_2'} h_1.$$

Then $(E_1')_2' \in H(\operatorname{curl}, \Omega)$. $(E_1')_2'$ can be uniquely extended to a radiating solution of Maxwell's equations.

Proof. see Appendix. □

In order to give a characterization of the second domain derivative $E'' = (E_1')_2' - E_h'$ with $h = J_{h_1} h_2$, we need to introduce the symmetric curvature operator $\mathcal{R} : \partial D \to \mathbb{R}^{3\times 3}$, which acts on the tangential plane and is given by $\mathcal{R}(x) = J_\nu(x)$, $x \in \partial D$. Furthermore we define the mean curvature $\kappa : \partial D \to \mathbb{R}$ by $\kappa = \frac{1}{2} \operatorname{div} \nu$. Note, that these definitions require differentiable extensions of the normal vector field ν in a neighborhood of ∂D which is constant in the direction of ν, see [19]. We state the main result of this paper.

Theorem 4.5. *Let ∂D be of class C^3. The second domain derivative E'' is a radiating solution of Maxwell's equations, satisfying the inhomogeneous boundary condition*

$$\begin{aligned}
\nu \times E'' = \sum_{i\neq j=1}^{2} &\Big[\overrightarrow{\operatorname{Curl}}_{\partial D}\big(h_{i,\nu} E_{j,\nu}' - E_\nu h_{i,\tau}^\top \operatorname{Grad}_{\partial D} h_{j,\nu}\big) - \mathrm{i}k \operatorname{Div}_{\partial D}(h_{j,\nu} H_\tau) h_{i,\tau} - \mathrm{i}k\, h_{i,\nu} H_{j,\tau}'\Big] \\
&- \mathrm{i}k (h_{2,\tau}^\top \mathcal{R} h_{1,\tau}) H_\tau + \overrightarrow{\operatorname{Curl}}_{\partial D}\big[\big((h_{2,\tau}^\top \mathcal{R} h_{1,\tau}) - 2\kappa h_{1,\nu} h_{2,\nu}\big) E_\nu\big] \\
&+ 2\mathrm{i}k\, h_{1,\nu} h_{2,\nu} \big(\mathcal{R} - \kappa\big) H_\tau + \mathrm{i}k \sum_{i\neq j=1}^{2} h_{i,\tau}^\top (\nu \times H) \overrightarrow{\operatorname{Curl}}_{\partial D} h_{j,\nu}.
\end{aligned}$$

Proof. Since $E'' = (E_1')_2' - E_h'$ with $h = J_{h_1} h_2$, the boundary values of E'' are given by $\nu \times E'' = \nu \times (E_1')_2' - \nu \times E_h'$. From Theorem 3.3 we know $\nu \times E_h' = \overrightarrow{\operatorname{Curl}}_{\partial D}\big((\nu^\top J_{h_1} h_2) E_\nu\big) - \mathrm{i}k (\nu^\top J_{h_1} h_2) H_\tau$. We calculate

$$\begin{aligned}
\nu^\top J_{h_1} h_2 &= (\nu^\top J_{h_1} \nu) h_{2,\nu} + \nu^\top J_{h_1} h_{2,\tau} \\
&= h_{2,\nu} \nu^\top (\nabla h_{1,\nu} - J_\nu^\top h_1) + h_{2,\tau}^\top (J_{h_1}^\top \nu + J_\nu h_1 - J_\nu^\top h_1) \\
&= h_{2,\nu} \frac{\partial h_{1,\nu}}{\partial \nu} + h_{2,\tau}^\top \operatorname{Grad}_{\partial D} h_{1,\nu} - h_{2,\tau}^\top \mathcal{R} h_{1,\tau}
\end{aligned}$$

since $\mathcal{R}$ is acting on the tangential plane. With $W_1' \in H_{\mathrm{pc}}(\Omega)$ the boundary values of $(E_1')_2'$ are given by $\nu \times (E_1')_2' = \nu \times \big[- J_{h_1}^\top E_2' - J_{E_2'} h_1 - J_{h_2}^\top W_1 - J_{W_1} h_2\big]$. We use the decomposition of the material derivative $W_i = E_i' + J_{h_i}' + J_E h_i$, for $i = 1, 2$ to find similarly as before

$$\begin{aligned}
\nu \times (E_1')_2' = -\nu \times &\Big[\operatorname{Grad}_{\partial D}(h_1^\top E_2' + h_2^\top E_1') + \operatorname{curl} E_2' \times h_1 + \operatorname{curl} E_1' \times h_2\Big] \\
&- \nu \times \Big[\operatorname{Grad}_{\partial D} \big(h_2^\top (\nabla (h_1 \top E) + \operatorname{curl} E \times h_1)\big) + \operatorname{curl}(\operatorname{curl} E \times h_1) \times h_2\Big].
\end{aligned}$$

As seen before, we have

$$\operatorname{curl}(\operatorname{curl} E \times h_1) \times h_2 = (A_1 \operatorname{curl} E) \times h_2 + k^2 (E \times h_1) \times h_2 + \nabla (h_1^\top \operatorname{curl} E) \times h_2.$$

From the boundary condition $\nu \times E = 0$ on ∂D we conclude

$$\begin{aligned} h_2^\top \nabla (h_1^\top E) &= h_{2,\tau} \operatorname{Grad}_{\partial D} (h_1^\top E) + h_{2,\nu} \frac{\partial h_1^\top E}{\partial \nu} \\ &= h_{2,\tau}^\top \operatorname{Grad}_{\partial D} (h_{1,\nu} E_\nu) + h_{2,\nu} \frac{\partial h_{1,\nu} E_\nu}{\partial \nu} + h_{2,\nu} \frac{\partial h_{1,\tau}^\top E_\tau}{\partial \nu}. \end{aligned}$$

We gather some identities, namely $\nu \times \big((E \times h_1) \times h_2 \big) = E_\nu h_{2,\nu} (\nu \times h_1)$ and

$$\nu \times \big(\operatorname{Grad}_{\partial D} (h_1^\top \operatorname{curl} E) \times h_2 \big) = \mathrm{i} k h_{2,\nu} \operatorname{Grad}_{\partial D} (h_{1,\tau}^\top H_\tau) - \mathrm{i} k \frac{\partial h_{1,\tau}^\top H_\tau}{\partial \nu} h_{2,\tau}\,,$$

as well as $\nu \times \big((A_1 \operatorname{curl} E) \times h_2 \big) = \mathrm{i} k h_{2,\nu} A_1 H_\tau - \mathrm{i} k (\nu^\top A_1 H_\tau) h_{2,\tau}$, and finally

$$\nu \times \operatorname{Grad}_{\partial D} \big(h_2^\top (\operatorname{curl} E \times h_1) \big) = \overrightarrow{\operatorname{Curl}}_{\partial D} \big(h_{2,\nu} \operatorname{curl} E^\top (\nu \times h_1) - h_{1,\nu} (\nu \times h_2)^\top \operatorname{curl} E \big).$$

Combining and substituting these into $\nu \times E'' = \nu \times (E_1')_2' + \nu \times E_h'$ yields the boundary condition

$$\begin{aligned} \nu \times E'' = & \overrightarrow{\operatorname{Curl}}_{\partial D} \big(h_1^\top E_2' + h_2^\top E_1' \big) - \nu \times \big(\operatorname{curl} E_2' \times h_1 + \operatorname{curl} E_1' \times h_2 \big) \\ & - \mathrm{i} k (h_{2,\tau} \mathcal{R} h_{1,\tau} H_\tau + \overrightarrow{\operatorname{Curl}}_{\partial D} \big[(h_{2,\tau} \mathcal{R} h_{1,\tau} E_\nu + h_{1,\nu} h_{2,\nu} \frac{\partial E_\nu}{\partial \nu} \big] \\ & + \overrightarrow{\operatorname{Curl}}_{\partial D} \big[h_{1,\nu} h_{2,\tau}^\top \operatorname{Grad}_{\partial D} E_\nu + h_{2,\nu} h_{1,\tau}^\top \frac{\partial E_\tau}{\partial \nu} \big] \\ & - \overrightarrow{\operatorname{Curl}}_{\partial D} \big[h_{2,\nu} \operatorname{curl} E^\top (\nu \times h_1) - h_{1,\nu} (\nu \times h_2)^\top \operatorname{curl} E \big] \\ & - \mathrm{i} k \big(h_{2,\nu} A_1 H_\tau - (\nu^\top A_1 H_\tau) h_{2,\tau} \big) - k^2 E_\nu h_{2,\nu} (\nu \times h_1) \\ & - \mathrm{i} k h_{2,\nu} \operatorname{Grad}_{\partial D} (h_{1,\tau}^\top H_\tau) + \mathrm{i} k h_{2,\tau} \frac{\partial h_{1,\tau}^\top H_\tau}{\partial \nu} + \mathrm{i} k \Big(h_{2,\nu} \frac{\partial h_{1,\nu}}{\partial \nu} + h_{2,\tau}^\top \operatorname{Grad}_{\partial D} h_{1,\nu} \Big) H_\tau. \end{aligned}$$

For any vector field F, we have on the boundary ∂D

$$\frac{\partial F_\tau}{\partial \nu} = \operatorname{curl} F \times \nu + \operatorname{Grad}_{\partial D} F_\nu - \mathcal{R} F_\tau \tag{4.5}$$

(see (5.4.50) in [19]), and

$$\frac{\partial F_\nu}{\partial \nu} - \operatorname{div} F = - \operatorname{Div}_{\partial D} F_\tau - 2\kappa F_\nu\,. \tag{4.6}$$

For the tangential part of the curl we get

$$(\operatorname{curl} F)_\tau = \overrightarrow{\operatorname{Curl}}_{\partial D} F_\nu + \big(\mathcal{R} - 2\kappa - \frac{\partial}{\partial \nu} \big) (F \times \nu) \tag{4.7}$$

(see Theorem 2.5.20 in [19]). With equation (4.5) we conclude

$$(\nu^\top A_1 H_\tau) + \frac{\partial h_{1,\tau}^\top H_\tau}{\partial \nu} = - H_\tau^\top \operatorname{Grad}_{\partial D} h_{1,\nu}.$$

Furthermore, by $\operatorname{div}(H) = 0$ and $H_\nu = 0$, we have

$$\big(A_1 H\big)_\tau = \operatorname{curl}(H \times h_1)_\tau - \operatorname{Grad}_{\partial D}(h_{1,\tau}^\top H_\tau) - \mathrm{i}k E_\nu (\nu \times h_1).$$

With equation (4.7) we obtain

$$\begin{aligned}
\mathrm{i}k h_{2,\nu} \operatorname{curl}(H \times h_1)_\tau = h_{2,\nu} \overrightarrow{\operatorname{Curl}}_{\partial D}\big(\operatorname{curl} E^\top (h_1 \times \nu)\big) \\
- \mathrm{i}k h_{1,\nu} h_{2,\nu} \big(\mathcal{R} - 2\kappa\big) H_\tau + \mathrm{i}k h_{1,\nu} h_{2,\nu} \frac{\partial H_\tau}{\partial \nu} + \mathrm{i}k h_{2,\nu} \frac{\partial h_{1,\nu}}{\partial \nu} H_\tau.
\end{aligned}$$

Thus, we arrive at

$$\begin{aligned}
\nu \times E'' = {} & \overrightarrow{\operatorname{Curl}}_{\partial D}\big(h_1^\top E_2' + h_2^\top E_1'\big) - \nu \times \Big(\operatorname{curl} E_2' \times h_1 + \operatorname{curl} E_1' \times h_2\Big) \\
& - \mathrm{i}k \big(h_{2,\tau}^\top \mathcal{R} h_{1,\tau}\big) H_\tau + \overrightarrow{\operatorname{Curl}}_{\partial D}\Big[\big(h_{2,\tau}^\top \mathcal{R} h_{1,\tau}\big) E_\nu + h_{1,\nu}\, h_{2,\nu} \frac{\partial E_\nu}{\partial \nu}\Big] \\
& + \mathrm{i}k h_{1,\nu}\, h_{2,\nu} \Big(\mathcal{R} - 2\kappa - \frac{\partial}{\partial \nu}\Big) H_\tau \\
& + \overrightarrow{\operatorname{Curl}}_{\partial D}\Big[h_{1,\nu}\, h_{2,\tau}^\top \operatorname{Grad}_{\partial D} E_\nu + h_{2,\nu}\, h_{1,\tau}^\top \frac{\partial E_\tau}{\partial \nu}\Big] \\
& - \overrightarrow{\operatorname{Curl}}_{\partial D}\Big[h_{2,\nu} \operatorname{curl} E^\top (\nu \times h_1) - h_{1,\nu} (\nu \times h_2)^\top \operatorname{curl} E\Big] \\
& - h_{2,\nu} \overrightarrow{\operatorname{Curl}}_{\partial D}\big(\operatorname{curl} E^\top (h_1 \times \nu)\big) + \mathrm{i}k \Big(\operatorname{Grad}_{\partial D}(h_{1,\nu}) \times (H_\tau \times h_{2,\tau})\Big).
\end{aligned}$$

By (4.6) we obtain

$$\operatorname{Grad}_{\partial D} E_\nu = \frac{\partial E_\tau}{\partial \nu} - \operatorname{curl} E \times \nu.$$

Furthermore, it holds

$$\begin{aligned}
& \Big[\operatorname{Grad}_{\partial D} h_{1,\nu} \times (H_\tau \times h_{2,\tau})\Big]_\tau = \Big[\big(\nu \times (\operatorname{Grad}_{\partial D}(h_{1,\nu}) \times \nu)\big) \times (H_\tau \times h_{2,\tau})\Big]_\tau \\
& = \Big[\big(\operatorname{Grad}_{\partial D}(h_{1,\nu}) \times \nu\big)\nu \cdot (H_\tau \times h_{2,\tau}) - \nu \big(\operatorname{Grad}_{\partial D}(h_{1,\nu}) \times \nu\big)^\top (H_\tau \times h_{2,\tau})\Big]_\tau \\
& = \frac{1}{\mathrm{i}k} \operatorname{curl} E^\top (\nu \times h_2) \overrightarrow{\operatorname{Curl}}_{\partial D}(h_{1,\nu}).
\end{aligned}$$

By the product rule we finally arrive at a symmetric characterization, i.e.,

$$\begin{aligned}
\nu \times E'' = {} & \overrightarrow{\operatorname{Curl}}_{\partial D}\big(h_1^\top E_2' + h_2^\top E_1'\big) - \nu \times \Big(\operatorname{curl} E_2' \times h_1 + \operatorname{curl} E_1' \times h_2\Big) \\
& - \mathrm{i}k \big(h_{2,\tau}^\top \mathcal{R} h_{1,\tau}\big) H_\tau + \overrightarrow{\operatorname{Curl}}_{\partial D}\Big[\big(h_{2,\tau}^\top \mathcal{R} h_{1,\tau}\big) E_\nu + h_{1,\nu}\, h_{2,\nu} \frac{\partial E_\nu}{\partial \nu}\Big] \\
& + \mathrm{i}k h_{1,\nu}\, h_{2,\nu} \Big(\mathcal{R} - 2\kappa - \frac{\partial}{\partial \nu}\Big) H_\tau + \overrightarrow{\operatorname{Curl}}_{\partial D}\Big[h_{1,\nu} h_{2,\tau}^\top \frac{\partial E_\tau}{\partial \nu} + h_{2,\nu}\, h_{1,\tau}^\top \frac{\partial E_\tau}{\partial \nu}\Big] \\
& - \operatorname{curl} E^\top (\nu \times h_1) \overrightarrow{\operatorname{Curl}}_{\partial D}(h_{2,\nu}) - \operatorname{curl} E^\top (\nu \times h_2) \overrightarrow{\operatorname{Curl}}_{\partial D}(h_{1,\nu}).
\end{aligned}$$

From (4.5) and from (4.6) we see $\frac{\partial H_\tau}{\partial \nu} = -\mathcal{R} H_\tau$, and $\frac{\partial E_\nu}{\partial \nu} = -2\kappa E_\nu$ and conclude

$$\nu \times E'' = \overrightarrow{\operatorname{Curl}}_{\partial D}\big(h_1^\top E_2' + h_2^\top E_1'\big) - \nu \times \Big(\operatorname{curl} E_2' \times h_1 + \operatorname{curl} E_1' \times h_2\Big)$$

$$
\begin{aligned}
&- \mathrm{i}k\big(h_{2,\tau}^\top \mathcal{R} h_{1,\tau}\big) H_\tau + \overrightarrow{\operatorname{Curl}}_{\partial D}\Big[\Big(\big(h_{2,\tau}^\top \mathcal{R} h_{1,\tau}\big) - 2\kappa h_{1,\nu}\, h_{2,\nu}\Big) E_\nu\Big] \\
&+ 2\mathrm{i}k h_{1,\nu}\, h_{2,\nu}\big(\mathcal{R} - \kappa\big) H_\tau \\
&+ \overrightarrow{\operatorname{Curl}}_{\partial D}\Big[h_{1,\nu} h_{2,\tau}^\top \frac{\partial E_\tau}{\partial \nu} + h_{2,\nu}\, h_{1,\tau}^\top \frac{\partial E_\tau}{\partial \nu}\Big] + \mathrm{i}k \sum_{i\neq j=1}^{s} h_{i,\tau}^\top (\nu \times H) \overrightarrow{\operatorname{Curl}}_{\partial D}(h_{j,\nu}).
\end{aligned}
$$

We use the boundary condition $\nu \times E_i' = \overrightarrow{\operatorname{Curl}}_{\partial D}(h_{i,\nu} E_\nu) - \mathrm{i}k h_{i,\nu} H_\tau$ of the domain derivative, $i = 1, 2$, which leads to $E_{i,\tau}' = -\operatorname{Grad}_{\partial D}(h_{i,\nu} E_\nu) + \mathrm{i}k h_{i,\nu}(\nu \times H)$, and obtain

$$
\begin{aligned}
\overrightarrow{\operatorname{Curl}}_{\partial D}(h_i^\top E_j') &= \overrightarrow{\operatorname{Curl}}_{\partial D}(h_{i,\nu} E_{j,\nu}' + h_{i,\tau}^\top E_{j,\tau}^\top) \\
&= \overrightarrow{\operatorname{Curl}}_{\partial D}\big(h_{i,\nu} E_{j,\nu}' - h_{i,\tau}^\top \operatorname{Grad}_{\partial D}(h_{j,\nu} E_\nu) + \mathrm{i}k h_{j,\nu} h_{i,\tau}^\top (\nu \times H)\big)\,.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
-\nu \times (\operatorname{curl} E_j' \times h_i) &= \operatorname{curl} E_{j,\nu}' h_{i,\tau} - h_{i,\nu} \operatorname{curl} E_{j,\tau}' \\
&= -\operatorname{Div}_{\partial D}(\nu \times E_j') h_{i,\tau} - \mathrm{i}k h_{i,\nu} H_{j,\tau}' = -\mathrm{i}k \operatorname{Div}_{\partial D}(h_{j,\nu} H_\tau) h_{i,\tau} - \mathrm{i}k h_{i,\nu} H_{j,\tau}'.
\end{aligned}
$$

Together with $\frac{\partial E_\tau}{\partial \nu} = \mathrm{i}k(H \times \nu) + \operatorname{Grad}_{\partial D} E_\nu$ we arrive at

$$
\begin{aligned}
\nu \times E'' = \sum_{i\neq j=1}^{2} &\Big[\overrightarrow{\operatorname{Curl}}_{\partial D}\big(h_{i,\nu} E_{j,\nu}' - E_\nu h_{i,\tau}^\top \operatorname{Grad}_{\partial D} h_{j,\nu}\big) - \mathrm{i}k \operatorname{Div}_{\partial D}(h_{j,\nu} H_\tau) h_{i,\tau} - \mathrm{i}k\, h_{i,\nu} H_{j,\tau}'\Big] \\
&- \mathrm{i}k(h_{2,\tau}^\top \mathcal{R} h_{1,\tau}) H_\tau + \overrightarrow{\operatorname{Curl}}_{\partial D}\big[\big((h_{2,\tau}^\top \mathcal{R} h_{1,\tau}) - 2\kappa h_{1,\nu} h_{2,\nu}\big) E_\nu\big] \\
&+ 2\mathrm{i}k\, h_{1,\nu} h_{2,\nu}\big(\mathcal{R} - \kappa\big) H_\tau + \mathrm{i}k \sum_{i\neq j=1}^{2} h_{i,\tau}^\top (\nu \times H) \overrightarrow{\operatorname{Curl}}_{\partial D} h_{j,\nu},
\end{aligned}
$$

as stated in the theorem. □

We do not claim that the characterization is the most elegant or shortest way to describe the boundary condition of Theorem 4.4. But it shows its symmetry with respect to h_1 and h_2. Note that the boundary condition of the second domain derivatives requires both the solution (E, H) as well as the first domain derivatives (E_i', H_i') to be sufficiently smooth in order to be well posed.

5 A Second Degree Method

Recall the boundary to far field operator defined by $\mathbf{F}(\partial D) = E_\infty \in L_t^2(\mathbb{S}^2)$. From the previous section, we know $\mathbf{F}$ to be twice differentiable where the derivatives are given by the far field patterns of the domain derivatives, i.e. $\mathbf{F}'[\partial D]h = E_\infty'$, $\mathbf{F}''[\partial D](h_1, h_2) = E_\infty''$. To solve the equation

$$
\mathbf{F}(\partial D) = E_\infty
$$

for a given $E_\infty \in L_t^2(\mathbb{S}^2)$, we apply a Newton type method. Choosing a starting guess ∂D^0 a classical Newton step consists in solving the linear equation

$$
\mathbf{F}'[\partial D^i]\hat{h} = E_\infty - \mathbf{F}(\partial D^i). \tag{5.1}
$$

Due to ill-posedness the equation (5.1) has to be regularized in order to ensure solvability. Applying Tikhonov regularization, we consider the uniquely solvable equation

$$\big(\mathbf{F}'[\partial D^i]^*\mathbf{F}'[\partial D^i] + \alpha_1\mathbf{I}\big)\hat{h} = \mathbf{F}'[\partial D^i]^*\big(E_\infty - \mathbf{F}(\partial D^i)\big) \tag{5.2}$$

with some chosen regularization parameter $\alpha_1 > 0$. Thus the regularized Newton scheme uses an update of the boundary by $\partial D^{i+1} = \partial D^i_{\widehat{h}}$. For more details on iterative regularization methods we refer to [14].

For the second degree method, we modify this approach and use $\widehat{h}$ just as a predictor to linearize the quadratic approximation $\mathbf{F}(\partial D_h) \approx \mathbf{F}(\partial D) + \mathbf{F}'[\partial D]h + \frac{1}{2}\mathbf{F}''[\partial D](h,h)$. Thus, the corrector step becomes

$$\mathbf{F}'[\partial D^i]h + \frac{1}{2}\mathbf{F}''[\partial D^i](\widehat{h},h) = E_\infty - \mathbf{F}(\partial D^i).$$

Again, we apply a Tikhonov regularization. Defining the linear operator $\mathbf{T}$ by $\mathbf{T}h = \mathbf{F}'[\partial D^i]h + \frac{1}{2}\mathbf{F}''[\partial D^i](\widehat{h},h)$ the corrector step solves for h by

$$\big(\mathbf{T}^*\mathbf{T} + \alpha_2\mathbf{I}\big)h = \mathbf{T}^*(E_\infty - \mathbf{F}(\partial D^i)\,, \tag{5.3}$$

again with some regularization parameter $\alpha_2 > 0$. Then the so called Halley method is given by an update of the boundary by $\partial D^{i+1} = \partial D^i_h$.

To obtain a regularization scheme for the full non-linear problem it is known that we have to add a stopping condition. Therefore, we stop the iteration if the residual $r_i = \|\mathbf{F}(\partial D^i) - E_\infty\|_{L^2(\mathbb{S}^2)}/\|E_\infty\|_{L^2(\mathbb{S}^2)}$ falls below a chosen threshold. The regularized second degree method also called regularized Halley method is introduced in [11] and [13], where regularizing properties are shown under certain assumptions, mainly the tangential cone condition mentioned in chapter three.

However, we consider a numerical implementation of the method, which requires the choice of a set $\mathcal{Y}$ of admissible boundaries as an open set of a normed space $\mathcal{X}$. Then, the domain derivatives become Frechet derivatives. We have chosen $\mathcal{Y}$ to be the set of star shaped domains with center in the origin and boundary of class C^∞, discretized in the same way as in [8, 7] by spherical harmonics Y_n^m in the following way: First, we identify the boundary $\partial D \in \mathcal{Y}$ by the positive smooth function $r : \mathbb{S}^2 \to \mathbb{R}$, such every $x \in \partial D$ is given in spherical coordinates by $x = r(d)d$ for some $d \in \mathbb{S}^2$, i.e. we choose the open set $\mathcal{Y} = \{r \in C^\infty(\mathbb{S}^2) : r > 0\}$ in the space $\mathcal{X} = C^\infty(\mathbb{S}^2)$ as the domain of $\mathbf{F}$. To discretize $\mathcal{Y}$, we choose the finite dimensional subspace $\mathcal{X}_N \subset \mathcal{X}$, using the real and imaginary part of spherical harmonics Y_n^m up to the degree $N \in \mathbb{N}$, i.e.,

$$\mathcal{X}_N = \{r \in C^\infty(\mathbb{S}^2) : r = \sum_{n=0}^{N}\sum_{m=0}^{n} \alpha_n^m \operatorname{Re} Y_n^m + \sum_{n=1}^{N}\sum_{m=1}^{n} \beta_n^m \operatorname{Im} Y_n^m\}$$

with $\dim \mathcal{X}_N = (N+1)^2$. The discretized set of admissible boundaries $\mathcal{Y}_N$ consists of the elements $r \in \mathcal{X}_N$ with $r(d) > 0$ for $d \in \mathbb{S}^2$. Fixing $M \in \mathbb{N}$ evaluation points $\widehat{x}_1, \ldots \widehat{x}_M \in \mathbb{S}^2$ for the far field patterns, we discretize $\mathbf{F}(\partial D) = E_\infty$ by $(E_\infty(\widehat{x}_1), \ldots, E_\infty(\widehat{x}_M)) \in \mathbb{C}^{3\times M}$, Using the linearity of the domain derivative $\mathbf{F}'[\partial D]$ and the notation $E'_\infty(\hat{x}; h)$ for the domain derivative with respect to the perturbation h, evaluated at $\hat{x} \in \mathbb{S}^2$ and the ordered basis

$$\mathcal{B} = \{\operatorname{Re} Y_0^0,\, \operatorname{Re} Y_1^0,\, \operatorname{Re} Y_1^1, \ldots,\, \operatorname{Re} Y_N^N,\, \operatorname{Im} Y_1^1, \ldots,\, \operatorname{Im} Y_N^N\},$$

we arrive at the representation matrix for the discretized operator

$$\mathbf{F}'[\partial D] : \mathbb{R}^{(N+1)^2} \to \mathbb{C}^{3\times M}, \quad \big(\mathbf{F}'[\partial D]\big)_{ijk} = \big(E'_\infty(\widehat{x}_j; h_k)\big)_i$$

with $i = 1, \ldots, (N+1)^2$, $j = 1, 2, 3$ and $k = 1, \ldots, M$, where h_k denotes the $k-$th element of $\mathcal{B}$. The discretization of the identity $\mathbf{I}$ is given by the identity matrix $I_{(N+1)^2}$. We observed a better performance of our scheme by using a different penalty matrix $\mathbf{J}$ instead, which punishes the curvature of the boundary. Such a matrix $\mathbf{J}$ is for example given by the diagonal matrix with entries $(\mathbf{J})_{kk} = 1+\lambda(k)$, $k = 1, \ldots, (N+1)^2$. Here, $\lambda(k)$ is the corresponding eigenvalue of the spherical harmonic Y_n^m with respect to the Laplace Beltrami operator $\Delta_{\mathbb{S}^2} = \operatorname{Div}_{\mathbb{S}^2} \operatorname{Grad}_{\mathbb{S}^2}$, associated to the $k-$th basis element of $\mathcal{B}$. For the predictor $\widehat{h} = (\alpha_h, \beta_h) \in \mathbb{R}^{(N+1)^2}$ we solve the discretized version of equation (5.2) with $\mathbf{I}$ replaced by $\mathbf{J}$. In general, a solution of this equation is complex-valued, so we discard the imaginary part.

Let $E''_\infty(\widehat{x}; h, \widehat{h})$ denote the far field pattern of the second domain derivative with respect to the perturbations $h_1 = h$ and $h_2 = \widehat{h}$, evaluated at $\widehat{x} \in \mathbb{S}^2$. Then the representation matrix for the discretized operator $\mathbf{T} = \mathbf{F}'[\partial D^i] + \frac{1}{2}\mathbf{F}''[\partial D^i](\widehat{h}, \cdot)$ is given by

$$\mathbf{T} : \mathbb{R}^{(N+1)^2} \to \mathbb{C}^{3\times M}, \quad \big(\mathbf{T}\big)_{ijk} = \big(E'_\infty(\widehat{x}_j; h_k)\big)_i + \frac{1}{2}\big(E''_\infty(\widehat{x}_j; h_k, \widehat{h})\big)_i$$

with again $i = 1, \ldots, (N+1)^2$, $j = 1, 2, 3$ and $k = 1, \ldots, M$, where h_k denotes the $k-$th element of $\mathcal{B}$. The corrector h is then given by the real part of the solution of the discretized version of $\big(\mathbf{T}^*\mathbf{T} + \alpha_2\mathbf{J}\big)h = \mathbf{T}^*\big(E_\infty - \mathbf{F}(\partial D^i)\big)$. Full discretization requires the numerical evaluation of $\mathbf{F}(\partial D)$, $\mathbf{F}'[\partial D]$ and $\mathbf{F}''[\partial D]$. Looking closely at the boundary conditions for the first and second domain derivative, we identify the traces of the solutions (E, H) and (E', H') and some terms involving surface differential operators to these traces. We therefore chose an integral equation approach for the full discretization. Our implementations were carried out in the open source Galerkin boundary element methods library BEMPP (`https://bempp.com`). For an overview of the library, see [21]. We will shortly present the tools needed to formulate the scattering from a perfect conductor as an integral equation. Let $\Phi = \frac{1}{4\pi}\frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x-y|}$ denote the fundamental solution of the three-dimensional Helmholtz equation $\Delta u + k^2 u = 0$. We define the electric potential

$$\mathcal{E}\varphi(x) = \mathrm{i}k \int_{\partial D} \varphi(y)\Phi(x,y)\mathrm{d}s(y) - \frac{1}{\mathrm{i}k}\nabla \int_{\partial D} \operatorname{Div}_{\partial D} \varphi(y)\Phi(x,y)\mathrm{d}s(y)$$

and the magnetic potential

$$\mathcal{H}\varphi(x) = \operatorname{curl} \int_{\partial D} \varphi(y)\Phi(x,y)\mathrm{d}s(y),$$

which are bounded operators from $H^{-\frac{1}{2}}(\operatorname{Div}, \partial D)$ to $H_{\mathrm{loc}}(\operatorname{curl}^2, \mathbb{R}^3 \setminus \partial D)$. For any radiating solution $E \in H_{\mathrm{loc}}(\operatorname{curl}^2, \mathbb{R}^3 \setminus \overline{D})$ of Maxwell's equations, we have the Stratton-Chu representation formula $E(x) = -\mathcal{H}\gamma_T E(x) - \mathcal{E}\gamma_N E(x)$, where we introduced the Neumann trace $\gamma_N\varphi = \frac{1}{\mathrm{i}k}\gamma_t \operatorname{curl}\varphi$. The potentials satisfy the following jump conditions on the boundary ∂D,

$$[\gamma_t\mathcal{E}\varphi]_\pm = [\gamma_N\mathcal{H}\varphi]_\pm = 0 \quad \text{and} \quad [\gamma_N\mathcal{E}\varphi]_\pm = [\gamma_t\mathcal{H}\varphi]_\pm = -\varphi \quad \text{for } \varphi \in H^{-\frac{1}{2}}(\operatorname{Div}, \partial D)\,.$$

By the mean of the interior and exterior traces of the potentials, we arrive at the electric boundary operator $\mathbf{E}$ and magnetic boundary operator $\mathbf{H}$, both bounded linear operators from $H^{-\frac{1}{2}}(\mathrm{Div}, \partial D)$ to $H^{-\frac{1}{2}}(\mathrm{Div}, \partial D)$. These operators satisfy

$$\gamma_t \mathcal{E} = \mathbf{E}\,, \quad \gamma_N \mathcal{E} = -\frac{1}{2}\mathbf{I} + \mathbf{H}\,, \quad \gamma_t \mathcal{H} = -\frac{1}{2}\mathbf{I} + \mathbf{H}\,, \quad \text{and} \quad \gamma_N \mathcal{H} = -\mathbf{E}\,.$$

Let E be a radiating solution to Maxwell's equations, satisfying a Dirichlet boundary condition $\gamma_t E = -F$ for some right hand side F, in our case the scattered field E^s with $F = \gamma_t E^i$ or the domain derivatives E', E'' with the right hand sides presented in Theorems 3.3 and 4.5. In each case, we make the Ansatz $E(x) = -\mathcal{E}\lambda(x), \quad x \in \mathbb{R}^3 \setminus \overline{D}$ for some density $\lambda \in H^{-\frac{1}{2}}(\mathrm{Div}, \partial D)$. Then, by the trace and $\gamma_t \mathcal{E} = \mathbf{E}$ we arrive at the indirect electric field equation (EFIE)

$$\mathbf{E}\lambda = F.$$

Assuming k to be no interior eigenvalue of D, the EFIE is uniquely solvable for any right hand side (see [2]). The major challenge arises from calculating the boundary conditions for the domain derivatives. Recall the boundary condition $\nu \times E' = \mathrm{Grad}_{\partial D}(h_\nu E_\nu) \times \nu - \mathrm{i}k h_\nu H_\tau$ of E'. Numerically calculating the boundary condition requires access to the discrete version of the surface gradient Grad, the rotation operator R, defined by $R\gamma_T\varphi = \gamma_t\varphi$, the magnetic trace $H_\tau = (\nu \times (H \times \nu))$, and the normal component of the electric field E_ν. Furthermore, we have to calculate discrete products of discretizations for the product $h_\nu E_\nu$ and $h_\nu H_\tau$. From (3.5) we conclude $E_\nu = -\frac{1}{\mathrm{i}k}\,\mathrm{Div}_{\partial D} H \times \nu$, i.e. we can calculate the normal component of E by applying the surface divergence to $H \times \nu$. Considering

$$\int_{\partial D} \gamma_T\varphi \cdot \gamma_t\psi \,\mathrm{d}s = -\int_{\partial D} \gamma_t\varphi \cdot \gamma_T\psi \,\mathrm{d}s,$$

we see, that the negative dual pairing $-\langle \gamma_t\varphi, \gamma_T\psi\rangle_{\partial D}$ between $H^{-\frac{1}{2}}(\mathrm{Div}, \partial D)$ and its dual space $H^{-\frac{1}{2}}(\mathrm{Curl}, \partial D)$ can be seen as the weak formulation for the rotation operator R.

Since we use the Ansatz $E^s = -\mathcal{E}\lambda$, the tangential trace of the electric field is given by

$$H^s \times \nu = \gamma_N E^s = \Big(\frac{1}{2}\mathbf{I} - \mathbf{H}\Big)\lambda.$$

For the discrete product $f \cdot_d g = \sum_i \alpha_i \phi_i$ of two functions f and g in a chosen basis of functions ϕ_i, we calculate the L^2 projection of the product onto the bases functions ϕ_i, i.e. we solve the linear system

$$\int_{\partial D} \phi_j(x) \cdot (f(x)g(x))\mathrm{d}s(x) = \alpha_i \int_{\partial D} \phi_i(x) \cdot \phi_j(x)\mathrm{d}s(x), \quad j = 1, \ldots.$$

Note that we chose a basis of scalar functions for the product $h_\nu E_\nu$ and a basis of vector valued functions for the product $h_\nu H_\tau$. For details on the above described implementations and the code of the actual implementations of the first domain derivative and its use to solve an inverse problem, we refer to [8, 7] and the tutorials on the homepage of BEMPP (`https://bempp.com`).

Lets consider now the boundary condition for the second domain derivative E'', given by

$$\nu \times E'' = \sum_{i \neq j = 1}^{2} \Big[\overrightarrow{\mathrm{Curl}}_{\partial D}\big(h_{i,\nu}E'_{j,\nu} - E_\nu h_{i,\tau}^\top \,\mathrm{Grad}_{\partial D}\, h_{j,\nu}\big) - \mathrm{i}k\,\mathrm{Div}_{\partial D}(h_{j,\nu}H_\tau)h_{i,\tau} - \mathrm{i}k\, h_{i,\nu}H'_{j,\tau}\Big]$$

$$- \mathrm{i}k(h_{2,\tau}^\top \mathcal{R} h_{1,\tau}) H_\tau + \overrightarrow{\mathrm{Curl}}_{\partial D}\big[\big((h_{2,\tau}^\top \mathcal{R} h_{1,\tau}) - 2\kappa h_{1,\nu} h_{2,\nu}\big) E_\nu\big]$$
$$+ 2\mathrm{i}k\, h_{1,\nu} h_{2,\nu} \big(\mathcal{R} - \kappa\big) H_\tau + \mathrm{i}k \sum_{i \neq j = 1}^{2} h_{i,\tau}^\top (\nu \times H) \overrightarrow{\mathrm{Curl}}_{\partial D} h_{j,\nu}.$$

Note, that we have formulated the boundary condition in a way, we can use the same tools as before. We only have to consider additionally a discretization of the curvature operator $\mathcal{R}$ and of the mean curvature κ. The discrete scalar product of two functions is another special case of the discrete product $\cdot_d$ described above. Recall the definitions

$$\mathcal{R} = J_\nu \quad \text{and} \quad \kappa = \frac{1}{2} \operatorname{div} \nu.$$

We have $\frac{\partial \nu}{\partial \nu} = 0$ and $\mathcal{R}$ acts only on the tangential plane. Since $\mathcal{R} = \mathcal{R}^\top$, we arrive at

$$\mathcal{R} F = \begin{pmatrix} F \cdot \mathrm{Grad}_{\partial D}\, \nu_1 \\ F \cdot \mathrm{Grad}_{\partial D}\, \nu_2 \\ F \cdot \mathrm{Grad}_{\partial D}\, \nu_1 \end{pmatrix},$$

which is in every component a discrete product of functions. Having calculated each component, we use again L^2 projections to calculate $\mathcal{R}F$ for a given vector field F. For the mean curvature κ, we use the relation $-\Delta_{\partial D} x_i = 2\kappa\, \nu_i$, $i \in \{1, 2, 3\}$, of κ and the Laplace-Beltrami operator (see equation (2.5.212) in [19] with $u = x_i$), to calculate

$$-\frac{1}{2} \sum_{i=1}^{3} \nu_i \Delta_{\partial D} x_i = \kappa\, |\nu|^2 = \kappa.$$

The left hand side can again be implemented by using the discrete product of functions $\cdot_d$ and applications of the surface gradient and the surface divergence.

Now, since we know how to realize the boundary conditions, we present actual reconstructions using the second degree method. We successfully ran reconstructions for exact and also noisy data. As in [8, 7], where we considered a regularized Newton scheme as described by equation (5.2), we consider the following shapes:

- A rounded cuboid, implicitly given by

$$\left(\frac{x_1}{r_1}\right)^n + \left(\frac{x_2}{r_2}\right)^n + \left(\frac{x_3}{r_3}\right)^n = d^n$$

  with some exponent $n \in \mathbb{N}$, a positive radius $d > 0$ and side lengths $r_1, r_2, r_3 > 0$.

- A peanut-shaped object with parametrization

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{d}{2} \sin(\theta) \cos(\varphi) R(\cos(\theta)) \\ \frac{d}{2} \sin(\theta) \sin(\varphi) R(\cos(\theta)) \\ \frac{d}{2} \cos(\theta) \end{pmatrix}, \qquad \varphi \in [0, 2\pi],\ \theta \in [0, \pi],$$

  with diameter $d > 0$ and radial function $R : [-1, 1] \to \mathbb{R}$, given by $R(z) = -\frac{2}{5} \cos(\pi\, z/2)$.

Note, that the characterizations of the second domain derivative requires a smooth boundary. We therefore chose the rounded cuboid to challenge our reconstructions with an object close to the non-smooth cuboid. Additionally, we show the peanut as an example for a non-convex object with positive and negative curvature κ. In order to cancel any positive effects due to symmetry, we applied a translation such that the center of the rounded cuboid and the peanut-shaped object does not coincide with the center of the star shaped reconstructions. Furthermore, we consider the plane wave, given by

$$\begin{pmatrix} E^i \\ H^i \end{pmatrix}(x) = \begin{pmatrix} p \\ d \times p \end{pmatrix} \mathrm{e}^{\mathrm{i}kd\cdot x}, \quad x \in \mathbb{R}^3,$$

with polarization $p = (\mathrm{i}-1\,,\ 2\,,\ -1-\frac{1}{3}\mathrm{i})^\top \in \mathbb{C}^3$ and direction $d = \frac{1}{\sqrt{14}}(1\,,\ 2\,,\ 3)^\top \in \mathbb{S}^2$ such that the direction of the plane wave does not coincide with the symmetries of the considered shapes. In each experiment, the wavenumber is $k = 1.4$. We calculate the exact data $E_\infty = \mathbf{F}(\partial D)$ by picking 168 evaluation points on the unit sphere $\mathbb{S}^2$, i.e. $E_\infty \in \mathbb{C}^{3\times 168}$. Note, that due to the offset of the exact data, we avoid an inverse crime, since the exact data is being calculated by using meshes unrelated to those used in the reconstructions. We additionally use finer meshes for the calculation of the exact data.

In the case of noisy data of level $\delta \geq 0$, we multiply every element of the far field matrix $E_\infty \in \mathbb{C}^{3\times 168}$ by a random complex number $1+\delta\lambda_1 \mathrm{e}^{2\pi\mathrm{i}\lambda_2}$, where λ_1, λ_2 are uniformly distributed random numbers on $(0, 1)$. In our experiments with noisy data, we have chosen $\delta = 0.2$ which results in 10% relative error in comparison with the exact data. The regularization parameters α_1, α_2 were chosen by experience.

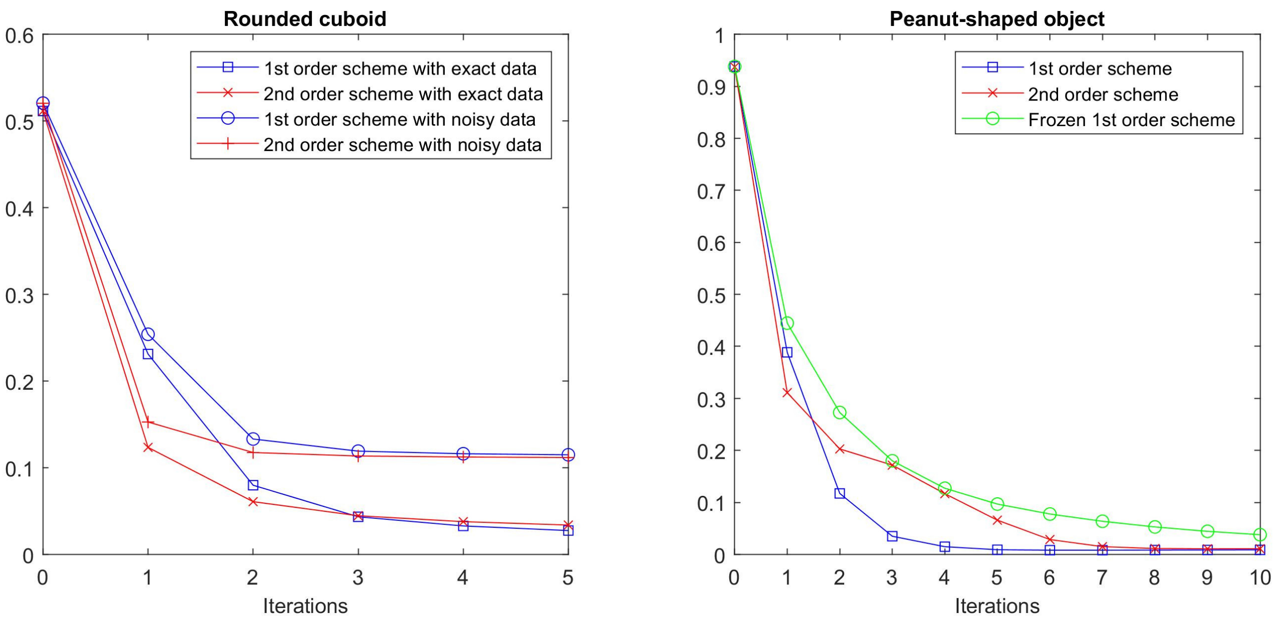

Figure 1: The residuals during the reconstructions of the rounded cuboid with exact data and 10 % noise and the peanut-shaped object with exact data.

In most of our experiments we observed a behavior of the residual as shown on the left side in figure 1 for the rounded cuboid. It occurs a significant decrease in the first few iteration steps which then slows down rapidly. It holds for a wide interval of regularization parameters

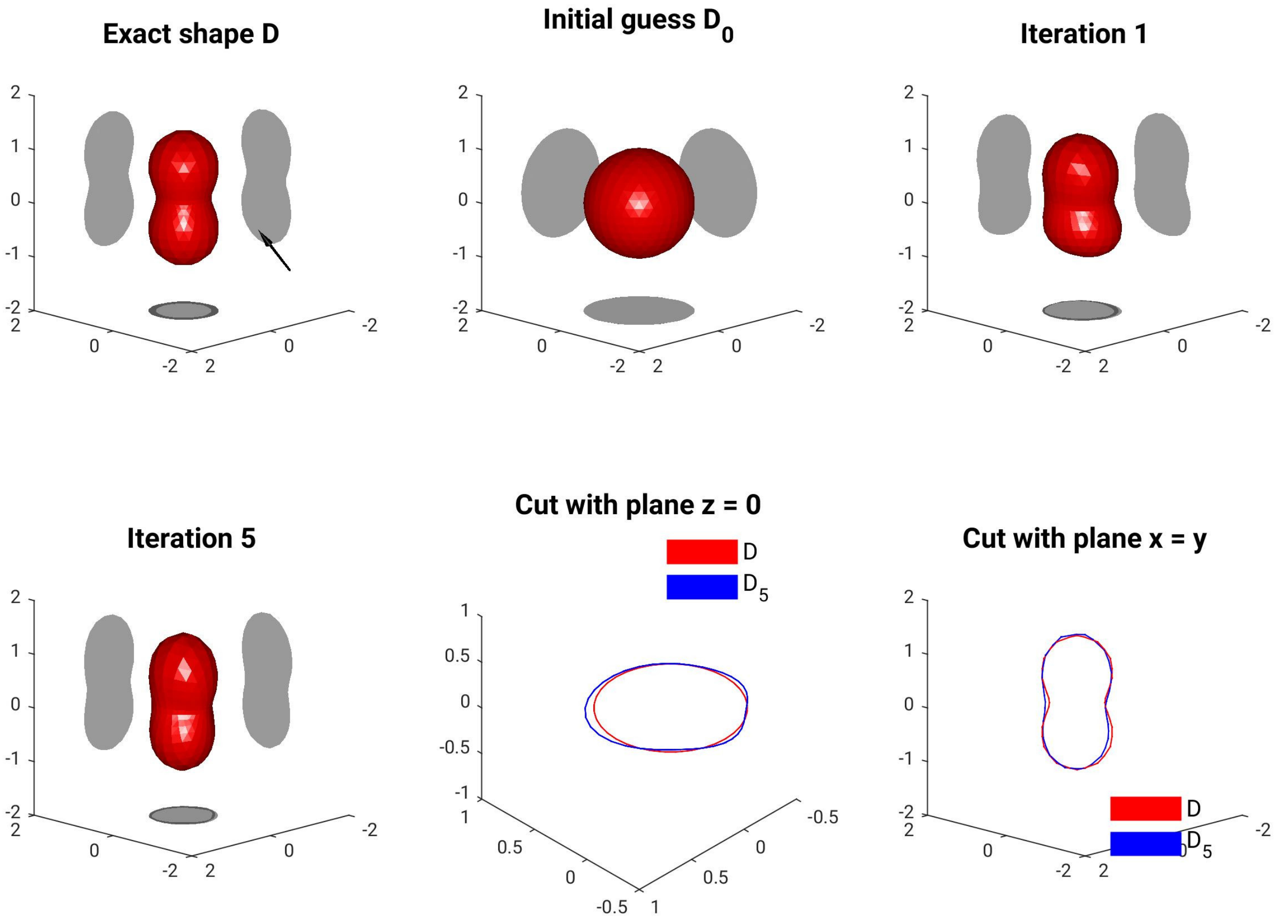

Figure 2: Reconstruction of the peanut-shaped object with exact data using the second degree method.

α_1 and α_2, leading to a comparable qualitative behavior. As seen for the rounded cuboid usually the second degree method shows a faster decrease in the first one to three iteration steps. But finally both methods, the iterative regularized Newton method and the second degree method, lead to similar residual errors in case of noise free data as well as in case of noisy data. Only this general observation was not confirmed in the case of the peanut shaped object as it can be seen on the right hand side of figure 1. While again the first iteration of the second degree method shows an improvement in comparison to the Newton type method, this is no longer true for the following iterations. Again, the final residuals are again similar, but after significantly more iterations. Note, that the performance of the second degree method is still slightly better than the performance of a frozen Newton scheme, where we fix the first derivative in the Newton iteration from the initial step.

In comparing the reconstructions of both approaches some differences are remarkable. In case of noise free data both the Halley as well as the Newton scheme lead to reasonable, similar reconstructions, as shown in figure 2. The only difference are a few less iteration steps required in the Halley methods, which, of course, are payed for by more computational effort for the second derivative in each step. But, additionally, we observed a more stable performance with respect to the choice of the regularization parameter. The range of possible parameters α_1 within the Newton scheme leading to reasonable results is significantly smaller then for the

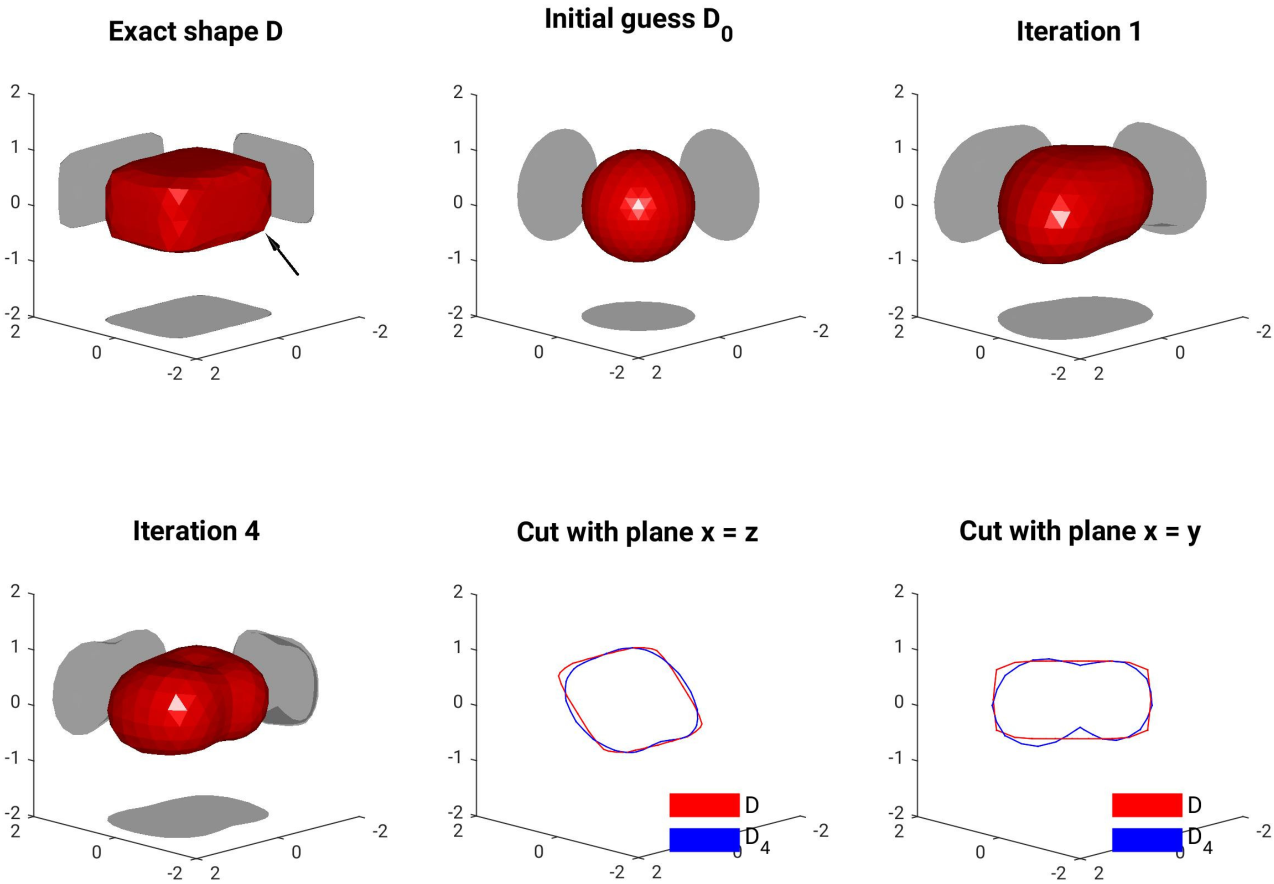

Figure 3: Reconstruction of the rounded cuboid using the Newton scheme with 10 % noise on the given data.

second degree approach.

A more significant effect in the reconstructions occurs in case of noisy data. Due to regularization the iteration has to be stopped if the residual error is becoming to small. It is seen also in the acoustic case that the iterated shapes start to deteriorate if a receding cusp is developed. This effect is regularized slightly by choosing the matrix $\mathbf{J}$ instead of $\mathbf{I}$ in the Tikhonov regularization, but if it occurs, in general the iteration process can not compensate on it and the iteration must be stopped. Of course, by a larger regularization parameter we can avoid the effect but then reconstructions become worse, close to the initial guess. Here we observed the main advantage of considering the second degree method, since it turned out that the method reaches frequently the stopping level before such cusps occur. This can be seen from the figures 3 and 4 showing reconstructions from noisy data with the iterative regularized Newton method and with the Halley method.

Finally, we consider the performance of the schemes in the case, where we know that the tangential cone condition fails. Thus we consider a ball, where the radius is chosen such that the wavenumber is an eigenvalue of the Laplace-Beltrami operator. From Corollary 3.4 we have seen that injectivity of $\mathbf{F}'$ does not hold if we illuminate the ball by an incident field generated by a vector surface harmonics. Especially $\mathbf{F}'h = 0$ if h_ν is constant. Thus the iteration schemes can not just expand or shrink the size of the ball. Exactly this was observed, using such a ball as an initial guess both the Newton as well as the second degree scheme slow

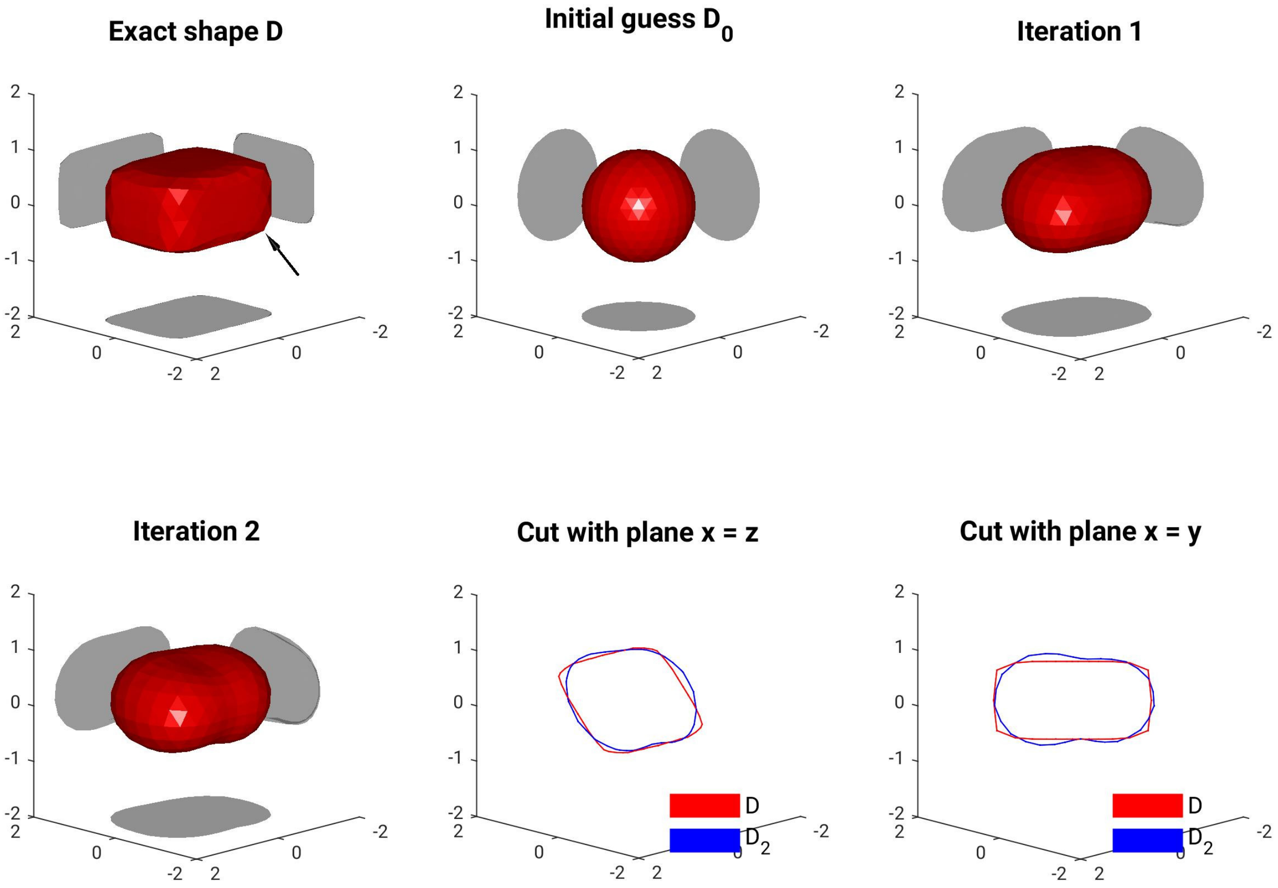

Figure 4: Reconstruction of the rounded cuboid using the second degree method with 10 % noise on the given data.

down and do not reach as good reconstructions as in non critical cases (see figure 5).

As a conclusion from these numerical investigations we can state a slightly more stable performance of the Halley method compared to an iterative regularized Newton approach as it was already observed for the acoustic case (see [11]), but by the prize of a higher computational effort in each iteration step. Additionally, the last observations in case of non injective domain derivatives confirm that further research is required in understanding the performance of iterative regularization schemes in inverse obstacle scattering.

Appendix

Proof of Theorem 4.4.

Proof. Let $(E_1')_2'$ be defined as in the theorem. The regularity of the boundary implies $E, H \in H^2(\Omega, \mathbb{C}^3)$ and $E', H' \in H^1(\Omega, \mathbb{C}^3)$, see [1, Corollary 2.15]. We conclude, similar as before $(E_1')_2' \in H(\mathrm{curl}, \Omega)$. We will again show $\mathcal{A}\big((E_1')_2', V\big) = 0$ for all $V \in H_{\mathrm{pc}}(\Omega)$. For $i = 1, 2$ we have

$$W_i = E_i' + J_{h_i}^\top E + J_E h_i = E_i' + \nabla(h_i^\top E) + \mathrm{curl}\, E \times h_i. \tag{5.4}$$

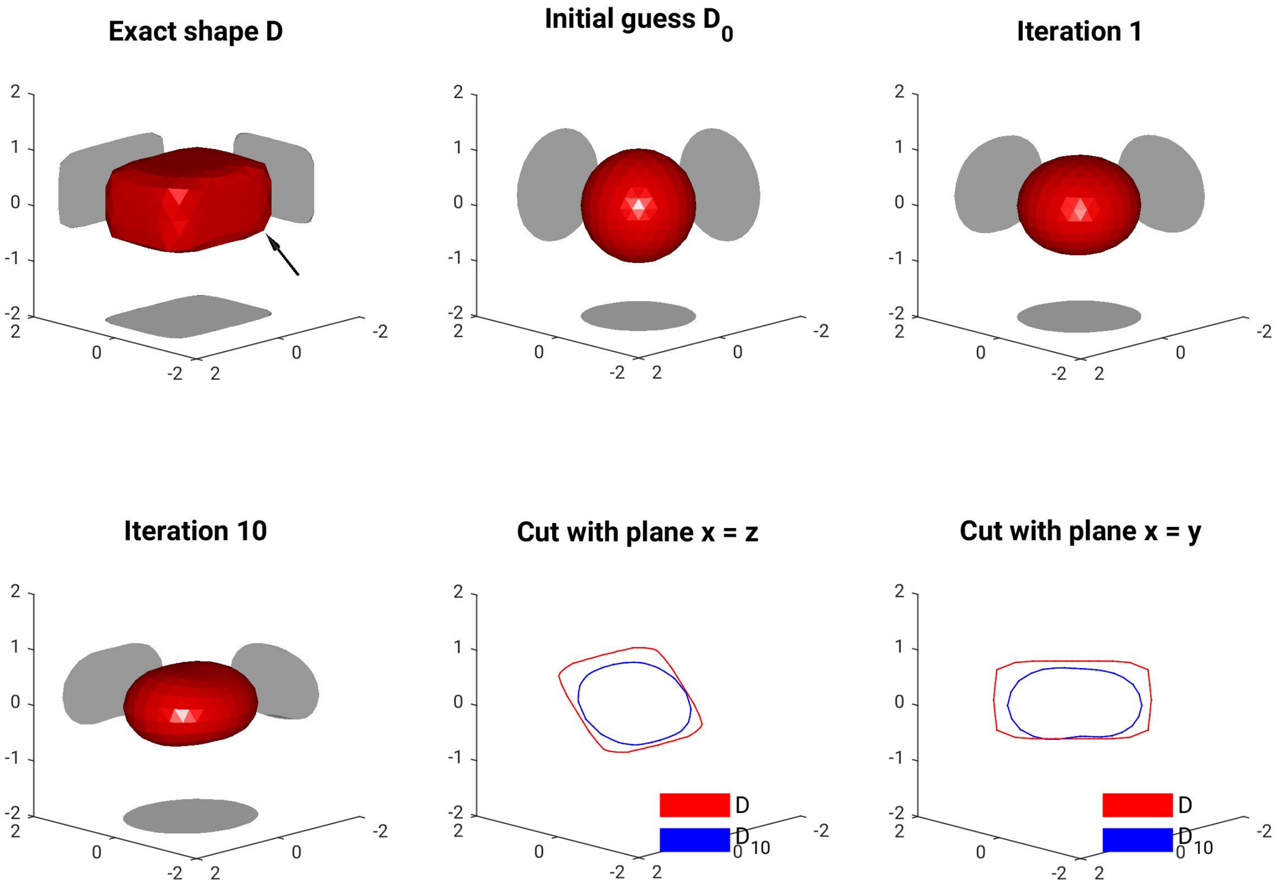

Figure 5: Reconstruction of the rounded cuboid using the regularized Newton scheme in the setting, where $\mathbf{F}'[D_0]$ is **not** injective.

The curl of the material derivative is then given by

$$\begin{aligned}\operatorname{curl} W_i &= \operatorname{curl} E_i' + \operatorname{curl}\big(\operatorname{curl} E \times h_i\big)\\ &= \operatorname{curl} E_i' + \operatorname{curl} E \operatorname{div} h_i + (J_{\operatorname{curl} E} - J_{\operatorname{curl} E}^\top)h_i\\ &\quad + (J_{\operatorname{curl} E}^\top h_i + J_{h_i}^\top \operatorname{curl} E) - (J_{h_i} - J_{h_i}^\top)\operatorname{curl} E\\ &= \operatorname{curl} E_i' + A_i \operatorname{curl} E + k^2(E \times h_i) + \nabla(h_i^\top \operatorname{curl} E).\end{aligned}$$

Note from the proof of Theorem 3.3, that for any divergence free solution $F \in H^1(\Omega, \mathbb{C}^3)$ of Maxwell's equations and $V \in H_{\mathrm{pc}}(\Omega)$ we have for $i = 1, 2$

$$\int_\Omega \Big(\operatorname{curl} F^\top A_i \overline{V} + k^2\, F^\top A_i \overline{V}\Big)\mathrm{d}x - \mathcal{A}(J_{h_i}^\top F + J_F h_i, V) = 0.$$

We apply the identity for $F = E_i'$, $i = 1, 2$. Let us continue by eliminating the terms involving the material derivative in (5.4). We obtain

$$J_{h_2} W_1 + J_{W_1} h_2 = J_{h_2}^\top E_1' + J_{E_1'} h_2 + J_{h_2}^\top J_{h_1}^\top E + J_{h_2}^\top J_E h_1 + J_{J_{h_1}^\top E + J_E h_1} h_2.$$

In order to deal with the Jacobian of the Jacobians, we write

$$J_{h_2}^\top J_E h_1 + J_{J_{h_1}^\top E + J_E h_1} h_2 = \nabla\Big(h_2^\top\big(h_1^\top E\big) + \operatorname{curl} E \times h_1\big)\Big) + \operatorname{curl}\big(\operatorname{curl} E \times h_1\big) \times h_2$$

$$= \nabla\Big(h_2^\top\big(h_1^\top E) + \operatorname{curl} E \times h_1)\Big) + \big(A_1 \operatorname{curl} E) \times h_2 + k^2(E \times h_1) \times h_2 + \nabla(h_1^\top \operatorname{curl} E) \times h_2.$$

Let us consider now

$$\begin{aligned}
\mathcal{A}\big((E_1')_2', V\big) =& \mathcal{A}(W_1, V) - \mathcal{A}(J_{h_1}^\top E_2' + J_{E_2'} h_1 - J_{h_2}^\top W_1 + J_{W_1} h_2, V) \\
=& \int_\Omega \Big(\operatorname{curl} W_1^\top A_2 \overline{\operatorname{curl} V} + k^2\, W_1^\top A_2 \overline{V} + \operatorname{curl} W_2^\top A_1 \overline{\operatorname{curl} V} + k^2\, W_2^\top A_1 \overline{V}\Big) \mathrm{d}x \\
&+ \int_\Omega \operatorname{curl} E^\top \Big(J_{h_2}^\top A_1 + A_1 J_{h_2} - \operatorname{div} h_2\, A_1 + A_1'(h_2)\Big) \overline{\operatorname{curl} V} \mathrm{d}x \\
&+ k^2 \int_\Omega E^\top \Big(- J_{h_2} A_1 - A_1 J_{h_2}^\top + \operatorname{div} h_2\, A_1 + A_1'(h_2)\Big) \mathrm{d}x \\
&- \mathcal{A}(J_{h_1}^\top E_2' + J_{E_2'} + J_{h_2}^\top E_1' + J_{E_1'} h_2 + J_{h_2}^\top J_{h_1}^\top E + J_{h_2}^\top J_E h_1 + J_{J_{h_1}^\top E + J_E h_1} h_2, V).
\end{aligned}$$

With the previous calculations, we get

$$\begin{aligned}
\mathcal{A}\big((E_1')_2', V\big) =& \int_\Omega \Big(\operatorname{curl} E_1'^\top A_2 \overline{\operatorname{curl} V} + k^2\, E_1'^\top A_2 \overline{V}\Big) \mathrm{d}x - \mathcal{A}(J_{h_2}^\top E_1' + J_{E_1'} h_2, V) \\
&+ \int_\Omega \Big(\operatorname{curl} E_2'^\top A_1 \overline{\operatorname{curl} V} + k^2\, E_2'^\top A_1 \overline{V}\Big) \mathrm{d}x - \mathcal{A}(J_{h_1}^\top E_2' + J_{E_2'} h_1, V) \\
&+ \int_\Omega \Big(A_2 \operatorname{curl} E + k^2 (E \times h_2) + \nabla(h_2^\top \operatorname{curl} E)\Big) A_1 \overline{\operatorname{curl} V} \mathrm{d}x \\
&+ \int_\Omega \Big(A_1 \operatorname{curl} E + k^2 (E \times h_1) + \nabla(h_1^\top \operatorname{curl} E)\Big) A_2 \overline{\operatorname{curl} V} \mathrm{d}x \\
&+ k^2 \int_\Omega \Big[\big(\operatorname{curl} E \times h_2 + \nabla(h_2^\top E)\big) A_1 \overline{V} + \big(\operatorname{curl} E \times h_1 + \nabla(h_1^\top E)\big)^\top A_2 \overline{V} \mathrm{d}x \\
&+ \int_\Omega \operatorname{curl} E^\top \big(J_{h_2}^\top A_1 + A_1 J_{h_2} - \operatorname{div} h_2\, A_1 + A_1'(h_2)\big) \overline{\operatorname{curl} V} \mathrm{d}x \\
&+ k^2 \int_\Omega E^\top \big(- J_{h_2} A_1 - A_1 J_{h_2}^\top + \operatorname{div} h_2\, A_1 + A_1'(h_2)\big) \overline{V} \mathrm{d}x \\
&- \int_\Omega \operatorname{curl} \Big[(A_1 \operatorname{curl} E) \times h_2 + k^2 (E \times h_1) \times h_2 + \nabla(h_1^\top \operatorname{curl} E) \times h_2 \Big]^\top \overline{\operatorname{curl} V} \mathrm{d}x \\
&+ k^2 \int_\Omega \Big[\nabla\big(h_2^\top (\nabla(h_1^\top E) + \operatorname{curl} E \times h_1)\big) + (A_1 \operatorname{curl} E) \times h_2 \\
&\qquad + k^2 (E \times h_1) \times h_2 + \nabla(h_1^\top \operatorname{curl} E) \times h_2 \Big]^\top \overline{V} \mathrm{d}x.
\end{aligned}$$

Recall the definition of the symmetric matrix $A_i = \operatorname{div} h_i\, I - J_{h_i} - J_{h_i}^\top$. We have

$$A_2 A_1 + A_1 A_2 = 2 \operatorname{div} h_2\, A_1 - J_{h_2} A_1 - J_{h_2}^\top A_1 - A_1 J_{h_2} - A_1 J_{h_2}^\top$$

and therefore

$$\begin{aligned}
\mathcal{A}\big((E_1')_2', V\big) =& k^2 \int_\Omega E^\top \big(- J_{h_2} A_1 - A_1 J_{h_2}^\top + \operatorname{div} h_2\, A_1 + A_1'(h_2)\big) \overline{V} \mathrm{d}x \\
&+ \int_\Omega \operatorname{curl} E^\top \big(- J_{h_2} A_1 - A_1 J_{h_2}^\top + \operatorname{div} h_2\, A_1 + A_1'(h_2)\big) \overline{\operatorname{curl} V} \mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega \Big(k^2(E\times h_2)+\nabla(h_2^\top \operatorname{curl} E)\Big)^\top A_1\overline{\operatorname{curl} V}\,\mathrm{d}x\\
&+\int_\Omega \Big(k^2(E\times h_1+\nabla(h_1^\top \operatorname{curl} E)\Big)^\top A_2\overline{\operatorname{curl} V}\,\mathrm{d}x\\
&+k^2\int_\Omega \Big[\big(\operatorname{curl} E\times h_2+\nabla(h_2^\top E)\big)^\top A_1+\big(\operatorname{curl} E\times h_1+\nabla(h_1^\top E)\big)^\top A_2\Big]\overline{V}\,\mathrm{d}x\\
&-\int_\Omega \operatorname{curl}\Big[(A_1\operatorname{curl} E)\times h_2+k^2(E\times h_1)\times h_2+\nabla(h_1^\top \operatorname{curl} E)\times h_2\Big]^\top\overline{\operatorname{curl} V}\,\mathrm{d}x\\
&+k^2\int_\Omega \Big[\nabla\big(h_2^\top(\nabla(h_1^\top E)+\operatorname{curl} E\times h_1)\big)+(A_1\operatorname{curl} E)\times h_2\\
&\qquad+k^2(E\times h_1)\times h_2+\nabla(h_1^\top \operatorname{curl} E)\times h_2\Big]^\top\overline{V}\,\mathrm{d}x.
\end{aligned}\tag{5.5}
$$

Considering vector fields E, V, h and a symmetric matrix A, elementary calculus yields the following identities:

$$
\begin{aligned}
\operatorname{div}\big((h^\top E)V\big)&=(h^\top E)\operatorname{div} V+V^\top J_h^\top E+V^\top J_E^\top h,\\
\operatorname{div}\big((h^\top V)AE\big)&=(h^\top V)\operatorname{div}(AE)+E^\top AJ_h^\top V+E^\top AJ_V^\top h,\\
\operatorname{div}\big((E^\top AV)h\big)&=E^\top AV\operatorname{div} h+h^\top J_E^\top AV+E^\top A'(h)V+E^\top AJ_V h.
\end{aligned}
$$

With these identities, we conclude

$$
\begin{aligned}
&\operatorname{curl} E^\top\big(-J_{h_2}A_1-A_1J_{h_2}^\top+\operatorname{div} h_2\,A_1+A_1'(h_2)\big)\overline{\operatorname{curl} V}\\
&=-\operatorname{div}\big[(h_2^\top \operatorname{curl} E)A_1\overline{\operatorname{curl} V}+(h_2^\top\overline{\operatorname{curl} V})A_1\operatorname{curl} E-(\operatorname{curl} E^\top A_1\overline{\operatorname{curl} V})h_2\big]\\
&\quad+(h_2^\top\overline{\operatorname{curl} V})\operatorname{div}(A_1E)+(h_2^\top \operatorname{curl} E)\operatorname{div}(A_1\overline{\operatorname{curl} V})\\
&\quad+(A_1\operatorname{curl} E)^\top(\overline{\operatorname{curl}\operatorname{curl} V}\times h_2)-k^2(E\times h_2)^\top A_1\overline{\operatorname{curl} V},
\end{aligned}\tag{5.6}
$$

since $\operatorname{curl}\operatorname{curl} E=k^2E$. Similarly, we have

$$
\begin{aligned}
&E^\top\big(-J_{h_2}A_1-A_1J_{h_2}^\top+\operatorname{div} h_2\,A_1+A_1'(h_2)\big)\overline{V}\\
&=\operatorname{div}\big[(h_2^\top E)A_1\overline{V}+(h_2^\top\overline{V})A_1E+(E^\top A_1\overline{V})h_2\big]+(h_2\top\overline{V})\operatorname{div}(A_1E)\\
&\quad+(h_2^\top E)\operatorname{div}(A_1\overline{V})^\top(\overline{\operatorname{curl} V}\times h_2)-(\operatorname{curl} E\times h_2)^\top A_1\overline{V}.
\end{aligned}\tag{5.7}
$$

We combine (5.6), (5.7) and (5.5) to get

$$
\begin{aligned}
\mathcal{A}\big((E_1')_2',V\big)&=\int_\Omega \operatorname{div}\big[(\operatorname{curl} E^\top A_1\overline{\operatorname{curl} V})h_2-(h_2^\top\overline{\operatorname{curl} V})A_1\operatorname{curl} E\big]\mathrm{d}x\\
&+\int_\Omega\Big((h_2^\top\overline{\operatorname{curl} V})\operatorname{div}(A_1\operatorname{curl} E)-(A_1\operatorname{curl} E)^\top(\overline{\operatorname{curl}\operatorname{curl} V}\times h_2)\Big)\mathrm{d}x\\
&+k^2\int_\Omega \operatorname{div}\big[(E^\top A_1\overline{V})h_2-(h_2^\top\overline{V})A_1E\big]\mathrm{d}x+k^2\int_\Omega\big(h_2^\top\overline{V})\operatorname{div}(A_1E)A_1E(\overline{\operatorname{curl} V}\times h_2)\big)\mathrm{d}x\\
&+\int_\Omega\big(k^2(E\times h_1)+\nabla(h_1^\top \operatorname{curl} E)\big)^\top A_2\overline{\operatorname{curl} V}+k^2\int_\Omega\big(\operatorname{curl} E\times h_1+\nabla(h_1^\top E)\big)^\top A_2\overline{V}\,\mathrm{d}x\\
&-\int_\Omega \operatorname{curl}\Big[(A_1\operatorname{curl} E)\times h_2+k^2(E\times h_1)\times h_2+\nabla(h_1^\top \operatorname{curl} E)\times h_2\Big]^\top\overline{\operatorname{curl} V}\,\mathrm{d}x
\end{aligned}
$$

$$+ k^2 \int_\Omega \Big[\nabla\big(h_2^\top(\nabla(h_1^\top E) + \operatorname{curl} E \times h_1)\big) + (A_1 \operatorname{curl} E) \times h_2$$
$$+ k^2 (E \times h_1) \times h_2 + \nabla(h_1^\top \operatorname{curl} E) \times h_2\Big]^\top \overline{V} \mathrm{d}x.$$

By the divergence theorem, we have

$$\begin{aligned}
&\int_\Omega \operatorname{div}\big[(\operatorname{curl} E^\top A_1 \overline{\operatorname{curl} V}) h_2 - (h_2^\top \overline{\operatorname{curl} V}) A_1 \operatorname{curl} E\big] \mathrm{d}x \\
&= - \int_{\partial D} \big(h_{2,\nu}(\operatorname{curl} E^\top A_1 \overline{\operatorname{curl} V} - (\nu^\top A_1 \operatorname{curl} E)(h_2^\top \overline{\operatorname{curl} V})\big) \mathrm{d}s,
\end{aligned}$$

since h_1, h_2 are compactly supported in Ω. Application of the partial integration formula leads to

$$\begin{aligned}
&\int_\Omega \Big(- (A_1 \operatorname{curl} E)^\top (\overline{\operatorname{curl}\operatorname{curl} V} \times h_2) - \operatorname{curl}\big((A_1 \operatorname{curl} E) \times h_2\big)^\top \overline{\operatorname{curl} V}\Big) \mathrm{d}x \\
&= \int_\Omega \Big((A_1 \operatorname{curl} E \times h_2)^\top \overline{\operatorname{curl}\operatorname{curl} V} - \operatorname{curl}\big((A_1 \operatorname{curl} E) \times h_2\big)^\top \overline{\operatorname{curl} V}\Big) \mathrm{d}x \\
&= - \int_{\partial D} (\nu \times \overline{\operatorname{curl} V})^\top \big((A_1 \operatorname{curl} E) \times h_2\big) \mathrm{d}s.
\end{aligned}$$

In the last integral, we only need to consider the tangential component of $(A_1 \operatorname{curl} E) \times h_2$, which is given by

$$\big[(A_1 \operatorname{curl} E) \times h_2\big]_\tau = (\nu^\top A_1 \operatorname{curl} E)(\nu \times h_2) - h_{2,\nu}\big(\nu \times (A_1 \operatorname{curl} E)\big).$$

We use again (3.5) to conclude $\nu^\top \overline{\operatorname{curl} V} = -\operatorname{Div}_{\partial D}(\overline{\nu \times V}) = 0$, since $V \in H_{\mathrm{pc}}(\Omega)$, which yields

$$(\nu \times \overline{\operatorname{curl} V})^\top \big((A_1 \operatorname{curl} E) \times h_2\big) = (h_2^\top \overline{\operatorname{curl} V})(\nu^\top A_1 \operatorname{curl} E) - h_{2,\nu}(\operatorname{curl} E^\top A_1 \overline{\operatorname{curl} V}).$$

Finally, we have shown

$$\begin{aligned}
&\int_\Omega \operatorname{div}\big[(\operatorname{curl} E^\top A_1 \overline{\operatorname{curl} V}) h_2 - (h_2^\top \overline{\operatorname{curl} V}) A_1 \operatorname{curl} E\big] \mathrm{d}x \\
&- \int_\Omega \Big((A_1 \operatorname{curl} E)^\top (\overline{\operatorname{curl}\operatorname{curl} V} \times h_2) - \operatorname{curl}\big((A_1 \operatorname{curl} E) \times h_2\big)^\top \overline{\operatorname{curl} V}\Big) \mathrm{d}x \;=\; 0\,.
\end{aligned} \tag{5.8}$$

A second application of the divergence theorem, together with the boundary condition $\nu \times E = \nu \times \overline{V} = 0$ on ∂D leads to

$$\begin{aligned}
&\int_\Omega \operatorname{div}\big[(E^\top A_1 \overline{V}) h_2 - (h_2^\top \overline{V}) A_1 E\big] \mathrm{d}x \\
&= - \int_{\partial D} \Big((\nu^\top A_1 \nu) h_{2,\nu} E_\nu \overline{V_\nu} - (\nu^\top A_1 \nu) h_{2,\nu} E_\nu \overline{V_\nu}\Big) \mathrm{d}s \;=\; 0\,.
\end{aligned} \tag{5.9}$$

Applying (5.8) and (5.9), we have

$$\mathcal{A}\big((E_1')_2', V\big) = \int_\Omega \big((h_2^\top \overline{\operatorname{curl} V}) \operatorname{div}(A_1 \operatorname{curl} E) + k^2 (h_2^\top \overline{V}) \operatorname{div}(A_1 E)\big) \mathrm{d}x$$

$$
\begin{aligned}
&- k^2 \int_\Omega (A_1 E)^\top (\overline{\operatorname{curl} V} \times h_2) \mathrm{d}x + k^2 \int_\Omega \big(\operatorname{curl} E \times h_2 + \nabla(h_1^\top E)\big)^\top A_2 \overline{V} \mathrm{d}x \\
&+ \int_\Omega \big(k^2 (E \times h_1) + \nabla(h_1^\top \operatorname{curl} E)\big)^\top A_2 \overline{\operatorname{curl} V} \mathrm{d}x \\
&- \int_\Omega \operatorname{curl} \big(k^2 (E \times h_1) \times h_2 + \nabla(h_1^\top \operatorname{curl} E) \times h_2\big)^\top \overline{\operatorname{curl} V} \mathrm{d}x \\
&+ k^2 \int_\Omega \big[\nabla\big(h_2^\top (\nabla(h_1^\top E) + \operatorname{curl} E \times h_1)\big) + (A_1 \operatorname{curl} E) \times h_2 \\
&\qquad + k^2 (E \times h_1) \times h_1 + \nabla(h_1^\top \operatorname{curl} E) \times h_2\big]^\top \overline{V} \mathrm{d}x.
\end{aligned}
$$

We consider the term $\operatorname{curl}\big((E \times h_1) \times h_2\big)$. Elementary calculations yields

$$
\begin{aligned}
\operatorname{curl}\big((E \times h_1) \times h_2\big) &= A_2 (E \times h_1) - h_2 \operatorname{div}(E \times h_1) + \nabla(h_2^\top (E \times h_1)) + \operatorname{curl}(E \times h_1) \times h_2 \\
&= A_2 (E \times h_1) - h_2 \operatorname{div}(E \times h_1) + \nabla\big(h_2^\top (E \times h_1)\big) \\
&\quad + (A_1 E) \times h_2 + \nabla(h_1^\top E) \times h_2 + (\operatorname{curl} E \times h_1) \times h_2\,,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
\mathcal{A}\big(E_1')_2', V\big) = &\int_\Omega \big((h_2^\top \overline{\operatorname{curl} V}) \operatorname{div}(A_1 \operatorname{curl} E) + k^2 (h_2^\top \overline{V}) \operatorname{div}(A_1 E)\big) \mathrm{d}x \\
&+ \int_\Omega \big(\nabla(h_1^\top \operatorname{curl} E) + k^2 \operatorname{curl} E \times h_1 + k^2 \nabla(h_1^\top E)\big)^\top A_2 \overline{V} \mathrm{d}x \\
&- \int_\Omega \operatorname{curl}\big(\nabla(h_1^\top \operatorname{curl} E) \times h_2\big)^\top \overline{\operatorname{curl} V} \mathrm{d}x \\
&+ k^2 \int_\Omega \big[\operatorname{div}(E \times h_1) h_2 - \nabla(h_2^\top (E \times h_1)) \\
&\qquad - \nabla(h_1^\top \mathrm{e}) \times h_2 - (\operatorname{curl} E \times h_1) \times h_2\big]^\top \overline{\operatorname{curl} V} \mathrm{d}x \\
&+ k^2 \int_\Omega \big[\nabla(h_2^\top (\nabla(h_1^\top E) + \operatorname{curl} E \times h_1)) + (A_1 \operatorname{curl} E) \times h_2 \\
&\qquad + k^2 (E \times h_1) \times h_2 + \nabla(h_1^\top \operatorname{curl} E) \times h_2\big]^\top \overline{\operatorname{curl} V} \mathrm{d}x.
\end{aligned}
$$

We apply again the divergence theorem to conclude

$$
\begin{aligned}
&\int_\Omega \nabla(h_2^\top (E \times h_1))^\top \overline{\operatorname{curl} V} \mathrm{d}x = - \int_\omega \operatorname{div}\big[\nabla(h_2^\top (E \times h_1)) \times \overline{V}\big] \mathrm{d}x \\
&= \int_{\partial D} \nu^\top \big(\nabla(h_2^\top (E \times h_1)) \times \overline{V}\big) \mathrm{d}x = \int_{\partial D} \nabla(h_2^\top (E \times h_1))^\top (\nu \times \overline{V}) \mathrm{d}s = 0,
\end{aligned}
$$

since $V \in H_{\mathrm{pc}}(\Omega)$. Similarly, we show

$$
\int_\Omega \nabla(h_2^\top \nabla(h_1^\top \operatorname{curl} E))^\top \overline{\operatorname{curl} V} \mathrm{d}x = 0.
$$

The second integral occurs to

$$
\operatorname{curl}\big(\nabla(h_1^\top \operatorname{curl} E) \times h_2\big) = A_2 \nabla(h_1^\top \operatorname{curl} E) - h_2 \Delta(h_1^\top \operatorname{curl} E) + \nabla(h_2^\top \nabla(h_1^\top \operatorname{curl} E)).
$$

Combining all implies

$$
\begin{aligned}
\mathcal{A}\big((E_1')_2', V\big) = &\int_\Omega (h_2^\top \overline{\operatorname{curl} V})(\operatorname{div}(A_1 \operatorname{curl} E) + \Delta(h_1^\top \operatorname{curl} E))\mathrm{d}x \\
&+ k^2 \int_\Omega (h_2^\top \overline{V})(\operatorname{div}(A_1 E) + \Delta(h_1 \top E))\mathrm{d}x + k^2 \int_\Omega (\operatorname{curl} E \times h_1)^\top A_2 \overline{V} \mathrm{d}x \\
&+ k^2 \int_\Omega \big[\nabla(h_2^\top(\operatorname{curl} E \times h_1)) + (A_1 \operatorname{curl} E) \times h_2 \\
&\qquad + k^2 (E \times h_1) \times h_2 + \nabla(h_1^\top \operatorname{curl} E) \times h_2\big]^\top \overline{V} \mathrm{d}x \\
&+ k^2 \int_\Omega \big(\operatorname{div}(E \times h_1) h_2 - (\operatorname{curl} E \times h_1) \times h_2\big)^\top \overline{\operatorname{curl} V} \mathrm{d}x.
\end{aligned}
$$

Considering

$$
\begin{aligned}
\operatorname{curl}\big((\operatorname{curl} E \times h_1) \times h_2\big) =& A_2(\operatorname{curl} E \times h_1) - \operatorname{div}(\operatorname{curl} E \times h_1) h_2 + (A_1 \operatorname{curl} E) \times h_2 \\
&+ k^2 (E \times h_1) \times h_2 + \nabla(h_1^\top \operatorname{curl} E) \times h_2 + \nabla(h_2^\top(\operatorname{curl} E \times h_1))
\end{aligned}
$$

and the partial integration

$$
\int_\Omega \big((\operatorname{curl} E \times h_1) \times h_2\big)^\top \overline{\operatorname{curl} V} \mathrm{d}x = \int_\Omega \operatorname{curl}\big((\operatorname{curl} E \times h_1) \times h_2\big)^\top \overline{V} \mathrm{d}x
$$

we arrive at

$$
\begin{aligned}
\mathcal{A}\big((E_1')_2', V\big) =& k^2 \int_\Omega (h_2^\top \overline{V})\big(\operatorname{div}(A_1 E) + \Delta(h_1^\top E) + \operatorname{div}(\operatorname{curl} E \times h_1)\big)\mathrm{d}x \\
&+ \int_\Omega (h_2^\top \overline{\operatorname{curl} V})\big(\operatorname{div}(A_1 \operatorname{curl} E) + \Delta(h_1^\top \operatorname{curl} E) + k^2 \operatorname{div}(E \times h_1)\big)\mathrm{d}x.
\end{aligned}
$$

Since $\operatorname{div} E = 0$ and $\operatorname{curl}\operatorname{curl} E = k^2 E$ we have, again by elementary calculations, the identities

$$
\begin{aligned}
0 &= \operatorname{div}\big(\operatorname{curl}(E \times h_1)\big) = \operatorname{div}(A_1 E) + \Delta(h_1^\top E) \operatorname{div}(\operatorname{curl} E \times h_1) \\
0 &= \operatorname{div}(\operatorname{curl}(\operatorname{curl} E \times h_1) = \operatorname{div}(A_1 \operatorname{curl} E) + \Delta(h_1^\top \operatorname{curl} E) + k^2 \operatorname{div}(E \times h_1).
\end{aligned}
$$

We finally conclude

$$
\mathcal{A}\big((E_1')_2', V\big) = \mathcal{A}(W_1' - J_{h_1}^\top E_2' - J_{E_2'} h_1 - J_{h_2}^\top E_1 - J_{W_1} h_2, V\big) = 0
$$

for all $V \in H_{\mathrm{pc}}(\Omega)$, i.e., $(E_1')_2'$ is a radiating solution to homogeneous Maxwell's equations. $\square$

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