# NONLINEAR ESTIMATES FOR TRAVELING WAVE SOLUTIONS OF REACTION DIFFUSION EQUATIONS AND THEIR APPLICATIONS TO MATHEMATICAL ECOLOGY

Li-Chang Hung*

Department of Mathematics, National Taiwan University
Taipei, Taiwan

Xian Liao

Institute for Analysis, Karlsruhe Institute for Technology
Karlsruhe, Germany

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Abstract. In this paper we will establish nonlinear a priori lower and upper bounds for the solutions to a large class of equations which arise from the study of traveling wave solutions of reaction-diffusion equations, and we will apply our nonlinear bounds to the Lotka-Volterra system of two competing species as examples. The idea used in a series of papers [2, 3, 4, 5, 6, 7] where the linear N-barrier maximum principle was established will also be used in the proof.

## 1. Introduction

The present paper is devoted to *nonlinear* a priori upper and lower bounds for the solutions $u_i = u_i(x) : \mathbb{R} \mapsto [0, \infty)$, $i = 1, \cdots, n$ to the following boundary value problem of $n$ equations

$$\begin{cases} d_i\,(u_i)_{xx} + \theta\,(u_i)_x + u_i^{l_i}\, f_i(u_1, u_2, \cdots, u_n) = 0, \quad x \in \mathbb{R}, \quad i = 1, 2, \cdots, n, \\ (u_1, u_2, \cdots, u_n)(-\infty) = \mathbf{e}_-, \quad (u_1, u_2, \cdots, u_n)(\infty) = \mathbf{e}_+. \end{cases} \tag{1}$$

In the above, $d_i$, $l_i > 0$, $\theta \in \mathbb{R}$ are parameters, $f_i \in C^0([0, \infty)^n)$ are given functions and the boundary values $\mathbf{e}_-, \mathbf{e}_+$ take value in the following constant equilibria set

$$\Big\{ (u_1, \cdots, u_n) \;\Big|\; u_i^{l_i}\, f_i(u_1, \cdots, u_n) = 0, \quad u_i \geq 0, \quad \forall i = 1, \cdots, n \Big\}. \tag{2}$$

Equations (1) arise from the study of traveling waves solutions of reaction-diffusion equations (see [16, 18]). A series of papers [2, 3, 4, 5, 6, 7] by Hung *et al.* have been contributed to the *linear* (N-barrier) maximum principle for the $n$ equations (1), and in particular the lower and upper bounds for any linear combination of the solutions

$$\sum_{i=1}^{n} \alpha_i\, u_i(x), \quad \forall (\alpha_1, \cdots, \alpha_n)$$

have been established in terms of the parameters $d_i, l_i, \theta$ in (1).

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* Corresponding author.

Here we aim to derive *nonlinear* estimates for the polynomials of the solutions:

$$\prod_{i=1}^{n}(u_i(x)+k_i)^{\alpha_i}, \quad \forall(\alpha_1,\cdots,\alpha_n)$$

for some $k_i \geq 0$, which is related to the diversity indices of the species in ecology: $D^q = (\sum_{i=1}^{n}(u_i)^q)^{1/(1-q)}$, $q\in[1,\infty)$. Observe that when either $\mathbf{e}_+ = (0,\cdots,0)$ or $\mathbf{e}_- = (0,\cdots,0)$, the trivial lower bound of $\prod_{i=1}^{n}(u_i(x))^{\alpha_i}$ is 0. For $k_i > 0$ the following lower bound for the upper solutions of (1) holds.

**Proposition 1** (**Lower bound**). *Suppose that* $(u_i(x))_{i=1}^n \in (C^2(\mathbb{R}))^n$ *with* $u_i(x) \geq 0$, $\forall i = 1,\cdots,n$ *is an upper solution of* (1)*:*

$$(3)\qquad \begin{cases} d_i(u_i)_{xx} + \theta(u_i)_x + u_i^{l_i} f_i(u_1,u_2,\cdots,u_n) \leq 0, \quad x\in\mathbb{R}, \quad i=1,2,\cdots,n, \\ (u_1,u_2,\cdots,u_n)(-\infty) = \boldsymbol{e}_-, \quad (u_1,u_2,\cdots,u_n)(\infty) = \boldsymbol{e}_+. \end{cases}$$

*and that there exist* $(\underline{u}_i)_{i=1}^n \in (\mathbb{R}^+)^n$ *such that*

$$(4)\qquad \begin{array}{l} f_i(u_1,\cdots,u_n) \geq 0, \text{ for all} \\ (u_1,\cdots,u_n) \in \underline{\mathcal{R}} := \left\{(u_i)_{i=1}^n \in ([0,\infty))^n \mid \sum_{i=1}^n \frac{u_i}{\underline{u}_i} \leq 1\right\}. \end{array}$$

*Then we have for any* $(\alpha_i)_{i=1}^n \in (\mathbb{R}^+)^n$,

$$(5)\qquad \prod_{i=1}^{n}(u_i(x)+k_i)^{d_i\alpha_i} \geq e^{\lambda_1}, \quad x\in\mathbb{R},$$

*where*

$$(6a)\qquad \lambda_1 = \min_{1\leq j\leq n}\Big(\eta\, d_j + \sum_{i=1,i\neq j}^{n}\alpha_i(d_i-d_j)\ln k_i\Big),$$

$$(6b)\qquad \eta = \min_{1\leq j\leq n}\frac{1}{d_j}\Big(\lambda_2 - \sum_{i=1,i\neq j}^{n}\alpha_i(d_i-d_j)\ln k_i\Big),$$

$$(6c)\qquad \lambda_2 = \min_{1\leq j\leq n}\Big(\alpha_j d_j\ln(\underline{u}_j+k_j) + \sum_{i=1,i\neq j}^{n}\alpha_i d_i\ln k_i\Big).$$

**Remark 1** (Equal diffusion)**.** When $d_i = d$ for all $i=1,2,\cdots,n$, then

$$\lambda_1 = \min_{1\leq j\leq n}\Big(\alpha_j\ln(\underline{u}_j+k_j) + \sum_{i=1,i\neq j}^{n}\alpha_i\ln k_i\Big)d = \lambda_2 = d\eta,$$

and the lower bound (5) becomes

$$\prod_{i=1}^{n}(u_i(x)+k_i)^{\alpha_i} \geq \min_{1\leq j\leq n}\Big((\underline{u}_j+k_j)^{\alpha_j}\prod_{i\neq j}k_i^{\alpha_i}\Big), \quad x\in\mathbb{R}.$$

If furthermore $\alpha_i = \alpha$, $\forall i=1,\cdots,n$, then the inequality of arithmetic and geometric averages yields

$$\sum_{i=1}^{n}(u_i+k_i)^{\alpha} \geq n\Big(\prod_{i=1}^{n}(u_i+k_i)^{\alpha}\Big)^{\frac{1}{n}} \geq n\min_{1\leq j\leq n}\Big((\underline{u}_j+k_j)^{\alpha}\prod_{i\neq j}k_i^{\alpha}\Big)^{\frac{1}{n}}.$$

On the other hand, we can find a upper bound of $\prod_{i=1}^{n}(u_i(x))^{\alpha_i}$ for the lower solutions of (1).

**Proposition 2** (**Upper bound**). *Suppose that* $(u_i(x))_{i=1}^n \in (C^2(\mathbb{R}))^n$ *with* $u_i(x) \geq 0$ $\forall i = 1, \cdots, n$ *is a lower solution of* (1)*:*

$$\begin{cases} d_i(u_i)_{xx} + \theta(u_i)_x + u_i^{l_i} f_i(u_1, u_2, \cdots, u_n) \geq 0, \quad x \in \mathbb{R}, \quad i = 1, 2, \cdots, n, \\ (u_1, u_2, \cdots, u_n)(-\infty) = \boldsymbol{e}_-, \quad (u_1, u_2, \cdots, u_n)(\infty) = \boldsymbol{e}_+, \end{cases} \tag{7}$$

*and there exist* $\bar{u}_i > 0, i = 1, \cdots, n,$ *such that*

$$\begin{array}{l} f_i(u_1, \cdots, u_n) \leq 0, \ \text{for all} \\ (u_1, \cdots, u_n) \in \bar{\mathcal{R}} := \left\{(u_i)_{i=1}^n \in ([0,\infty))^n \mid \sum_{i=1}^n \frac{u_i}{\bar{u}_i} \geq 1\right\}. \end{array} \tag{8}$$

*Then we have for any* $m_i \geq 1$ *and* $\alpha_i > 0$ $(i = 1, 2, \cdots, n)$

$$\sum_{i=1}^{n} \alpha_i (u_i(x))^{m_i} \leq \left(\max_{1\leq i\leq n} \alpha_i (\bar{u}_i)^{m_i}\right) \frac{\max\limits_{1\leq i\leq n} d_i}{\min\limits_{1\leq i\leq n} d_i}, \quad x \in \mathbb{R}, \tag{9}$$

*and hence*

$$\prod_{i=1}^{n} (u_i(x))^{m_i/n} \leq \frac{\max\limits_{1\leq i\leq n} \alpha_i\, \bar{u}_i^{m_i}}{n\left(\prod\limits_{i=1}^{n}\alpha_i\right)^{1/n}} \frac{\max\limits_{1\leq i\leq n} d_i}{\min\limits_{1\leq i\leq n} d_i}, \quad x \in \mathbb{R}. \tag{10}$$

*In particular, when* $\alpha_i = \alpha$ *for all* $i = 1, \cdots, n,$ (10) *becomes*

$$\prod_{i=1}^{n} (u_i(x))^{m_i/n} \leq \frac{\max\limits_{1\leq i\leq n} \bar{u}_i^{m_i}}{n} \frac{\max\limits_{1\leq i\leq n} d_i}{\min\limits_{1\leq i\leq n} d_i}, \quad x \in \mathbb{R}. \tag{11}$$

The remainder of this paper is organized as follows. Section 2 is devoted to the proofs of Proposition 1 and Proposition 2. As an example to illustrate our main result, we use the Lotka-Volterra system of two competing species to conclude with Section 2.

## 2. Proofs of Proposition 1 and Proposition 2

*Proof of Proposition 1.* We first rewrite the inequality $d_i(u_i)'' + \theta(u_i)' + u_i^{l_i} f_i \leq 0$ in (3). If $u(x) \geq 0$, then for any $k > 0$, a straightforward calculation gives

$$(\ln(u(x)+k))' = \frac{u'(x)}{u(x)+k},$$
$$(\ln(u(x)+k))'' = \frac{u''(x)}{u(x)+k} - \frac{(u'(x))^2}{(u(x)+k)^2}.$$

Hence we divide the inequality by $u_i + k_i > 0$ with $k_i > 0$ to arrive at

$$d_i(\ln(u_i+k_i))'' + d_i \frac{((u_i)')^2}{(u_i+k_i)^2} + \theta\,(\ln(u_i+k_i))' + \frac{u_i^{l_i}}{u_i+k_i} f_i \leq 0.$$

Thus $(U_i)_{i=1}^n := (\ln(u_i+k_i))_{i=1}^n$ satisfies the following inequalities:

$$d_i U_i'' + \theta\, U_i' + \frac{u_i^{l_i}}{u_i+k_i} f_i \leq 0, \quad i = 1, \cdots, n. \tag{12}$$

For any $(\alpha_i)_{i=1}^n \in (\mathbb{R}^+)^n$, let

$$p(x) = \sum_{i=1}^n \alpha_i U_i, \quad q(x) = \sum_{i=1}^n \alpha_i\, d_i U_i,$$

then the above inequality (12) reads as

$$q'' + \theta p' + F \le 0, \quad F := \sum_{i=1}^n \frac{\alpha_i\, u_i^{l_i}}{u_i + k_i}\, f_i(u_1, \cdots, u_n). \tag{13}$$

We are going to derive a lower bound for

$$q = \sum_{i=1}^n \alpha_i d_i U_i = \sum_{i=1}^n \alpha_i\, d_i \ln(u_i(x) + k_i),$$

and hence a lower bound for $\prod_{i=1}^n (u_i + k_i)^{d_i \alpha_i}$. The idea is similar as in the papers [2, 3, 4, 5, 6, 7], namely we are going to determine three parameters

$$\lambda_1, \quad \eta, \quad \lambda_2$$

to construct an N-barrier consisting of three hypersurfaces

$$Q_1 := \{(u_i)_{i=1}^n \,|\, q = \lambda_1\}, \quad P := \{(u_i)_{i=1}^n \,|\, p = \eta\}, \quad Q_2 := \{(u_i)_{i=1}^n \,|\, q = \lambda_2\},$$

such that the following inclusion relations hold:

$$\mathcal{Q}_1 := \{(u_i)_{i=1}^n \in ([0,\infty))^n \,|\, q \le \lambda_1\} \subset \mathcal{P} := \{(u_i)_{i=1}^n \in ([0,\infty))^n \,|\, p \le \eta\}$$

$$\subset \mathcal{Q}_2 := \{(u_i)_{i=1}^n \in ([0,\infty))^n \,|\, q \le \lambda_2\} \subset \underline{\mathcal{R}} = \Big\{(u_i)_{i=1}^n \in ([0,\infty))^n \,\Big|\, \sum_{i=1}^n \frac{u_i}{\underline{u}_i} \le 1\Big\}.$$

It will turn out that if $\lambda_1$, $\eta$, and $\lambda_2$ are given respectively by (6a), (6b), and (6c), then $\lambda_1$ determines a lower bound of $q(x)$: $q(x) \ge \lambda_1$, which is exactly (5).

More precisely, we follow the steps as in [2, 3, 4, 5, 6, 7] to determine $\lambda_2$, $\eta$, $\lambda_1$ such that the above inclusion relations $\mathcal{Q}_1 \subset \mathcal{P} \subset \mathcal{Q}_2 \subset \underline{\mathcal{R}}$ hold:

(*i*) **<u>Determine $\lambda_2$</u>** The hypersurface $Q_2$ intersects the $u_j$-axis: $\{(u_i)_{i=1}^n \,|\, u_i = 0,\ \forall i \ne j\}$ at the point

$$\Big\{(u_i)_{i=1}^n \,|\, u_i = 0,\ \forall i \ne j, \quad u_{2,j} = e^{\frac{\lambda_2 - \sum_{i=1, i\ne j}^n \alpha_i\, d_i \ln k_i}{\alpha_j\, d_j}} - k_j\Big\}.$$

If $u_{2,j} \le \underline{u}_j$, $\forall j = 1, \cdots, n$, then by the monotonicity of the function $\ln(\cdot + k)$, $\mathcal{Q}_2 \subset \underline{\mathcal{R}}$. That is, $\mathcal{Q}_2 \subset \underline{\mathcal{R}}$ if $\lambda_2$ is chosen as in (6c):

$$\lambda_2 = \min_{1\le j\le n} \Big(\alpha_j d_j \ln(\underline{u}_j + k_j) + \sum_{i=1, i\ne j}^n \alpha_i d_i \ln k_i\Big).$$

(*ii*) **<u>Determine $\eta$</u>** As above, the hypersurface $P$ intersects the $u_j$-axis at

$$\Big\{(u_i)_{i=1}^n \,|\, u_i = 0,\ \forall i \ne j, \quad u_{0,j} = e^{\frac{\eta - \sum_{i=1, i\ne j}^n \alpha_i\ \ln k_i}{\alpha_j}} - k_j\Big\}.$$

If $u_{0,j} \le u_{2,j}$, $\forall j = 1, \cdots, n$, then $\mathcal{P} \subset \mathcal{Q}_2$ and the hypersurface $Q_2$ is above the hypersurface $P$. That is, $\mathcal{P} \subset \mathcal{Q}_2$ if $\eta$ is chosen as in (6b):

$$\eta = \min_{1\le j\le n} \frac{1}{d_j}\Big(\lambda_2 - \sum_{i=1, i\ne j}^n \alpha_i (d_i - d_j) \ln k_i\Big).$$

(*iii*) **Determine $\lambda_1$** Replacing $\lambda_2$ by $\lambda_1$ in (i), the $u_j$-intercept of the hypersurface $Q_1$ is given by $u_{1,j} = e^{\frac{\lambda_1 - \sum_{i=1,i\neq j}^{n} \alpha_i\, d_i \ln k_i}{\alpha_j\, d_j}} - k_j$. Hence if we take $\lambda_1$ as in (6a):

$$\lambda_1 = \min_{1\le j\le n} \Big( \eta\, d_j + \sum_{i=1,i\neq j}^{n} \alpha_i (d_i - d_j) \ln k_i \Big),$$

then $u_{1,j} \le u_{0,j}$, $\forall j = 1, \cdots, n$ and hence $\mathcal{Q}_1 \subset \mathcal{P}$.

We now show $q(x) \ge \lambda_1$, $x \in \mathbb{R}$ by a contradiction argument. Suppose by contradiction that there exists $z \in \mathbb{R}$ such that $q(z) < \lambda_1$. Since $u_i(x) \in C^2(\mathbb{R})$ $(i = 1, \cdots, n)$ and $(u_1, u_2, \cdots, u_n)(\pm\infty) = \mathbf{e}_\pm$, we may assume $\min_{x\in\mathbb{R}} q(x) = q(z)$. We denote respectively by $z_2$ and $z_1$ the first points at which the solution trajectory $\{(u_i(x))_{i=1}^n \,|\, x \in \mathbb{R}\}$ intersects the hypersurface $Q_2$ when $x$ moves from $z$ towards $\infty$ and $-\infty$. For the case where $\theta \le 0$, we integrate (13) with respect to $x$ from $z_1$ to $z$ and obtain

$$q'(z) - q'(z_1) + \theta\,(p(z) - p(z_1)) + \int_{z_1}^{z} F(u_1(x), \cdots, u_n(x))\,dx \le 0. \tag{14}$$

We also have the following facts from the construction of the hypersurfaces $Q_1, Q_2, P$:

- $q'(z) = 0$ because of $\min_{x\in\mathbb{R}} q(x) = q(z)$;
- $q(z_1) = \lambda_2$ because of $(u_i(z_1))_{i=1}^n \in Q_2$.
- $q'(z_1) < 0$ because $z_1$ is the first point for $q(x)$ taking the value $\lambda_2$ when $x$ moves from $z$ to $-\infty$, such that $q(z_1 + \delta) < \lambda_2$ for $z - z_1 > \delta > 0$;
- $p(z) < \eta$ since $(u_i(z))_{i=1}^n$ is below the hypersurface $P$;
- $p(z_1) > \eta$ since $(u_i(z_1))_{i=1}^n$ is above the hypersurface $P$;
- $F(u_1(x), \cdots, u_n(x)) = \displaystyle\sum_{i=1}^{n} \frac{\alpha_i\, u_i^{l_i}}{u_i + k_i}\, f_i(u_1, \cdots, u_n) \ge 0$, $\forall x \in [z_1, z]$. Indeed, since $(u_i(z_1))_{i=1}^n \in Q_2 \subset \mathcal{Q}_2 \subset \mathcal{R}$ and $(u_i(z))_{i=1}^n \in \mathcal{Q}_1 \subset \mathcal{R}$, we derive that $F(u_1(x), \cdots, u_n(x))|_{x\in[z_1,z]} \ge 0$ by the hypothesis (4).

We hence have the following inequality from the above facts when $\theta \le 0$

$$q'(z) - q'(z_1) + \theta\,(p(z) - p(z_1)) + \int_{z_1}^{z} F(u_1(x), \cdots, u_n(x))\,dx > 0,$$

which contradicts (14). Therefore when $\theta \le 0$, $q(x) \ge \lambda_1$ for $x \in \mathbb{R}$. For the case where $\theta \ge 0$, we simply integrate (13) with respect to $x$ from $z$ to $z_2$ to arrive at

$$q'(z_2) - q'(z) + \theta\,(p(z_2) - p(z)) + \int_{z}^{z_2} F(u_1(x), \cdots, u_n(x))\,dx \le 0.$$

Then we apply the facts that $q'(z_2) > 0$, $q'(z) = 0$, $p(z_2) > \eta$, $p(z) < \eta$ and $F(u_1(x), \cdots, u_n(x))|_{x\in[z,z_2]} \ge 0$, as well as a similar contradiction argument as above, to derive $q \ge \lambda_1$.

□

*Proof of Proposition 2.* We prove Proposition 2 in a similar manner to the proof of Proposition 1. We first rewrite the inequality $d_i(u_i)'' + \theta(u_i)' + u_i^{l_i} f_i \ge 0$ in (7). A straightforward calculation shows

$$(u^m)' = m\, u^{m-1} u',$$

$$(u^m)'' = m\left((m-1)\, u^{m-2} (u')^2 + u^{m-1} u''(x)\right).$$

Hence we multiply the inequality by $m_i\, u^{m_i-1}(x)$ to arrive at

$$d_i({u_i}^{m_i})'' - d_i\, m_i(m_i-1)\, {u_i}^{m_i-2}({u_i}')^2 + \theta\, ({u_i}^{m_i})' + m_i\, {u_i}^{m_i-1} u_i^{l_i}\, f_i \geq 0.$$

For notational simplicity, we will adopt the same notations as in the proof of Proposition 1. Since $u_i \geq 0$, $\forall i = 1, \cdots, n$, for any $(m_i)_{i=1}^n \in ([1,\infty))^n$, the vector field $(U_i)_{i=1}^n := (u_i^{m_i})_{i=1}^n$ satisfies the following inequalities

$$d_i U_i'' + \theta\, U_i' + m_i\, {u_i}^{m_i-1} u_i^{l_i}\, f_i \geq 0, \quad \forall i = 1, \cdots, n. \tag{15}$$

For any $(\alpha_i)_{i=1}^n \in (\mathbb{R}^+)^n$, $p(x) = \sum_{i=1}^n \alpha_i\, U_i$ and $q(x) = \sum_{i=1}^n \alpha_i\, d_i U_i$ satisfy

$$q'' + \theta\, p' + F \geq 0, \quad F := \sum_{i=1}^n \alpha_i\, m_i\, {u_i}^{m_i-1} u_i^{l_i}\, f_i(u_1, u_2, \cdots, u_n). \tag{16}$$

We are going to show the upper bound $q \leq \lambda_1$ by employing the N-barrier method as in the proof of Proposition 1. That is, we are going to construct the three hyperellipsoids

$$Q_1 := \{(u_i)_{i=1}^n \,|\, q = \lambda_1\}, \quad P := \{(u_i)_{i=1}^n \,|\, p = \eta\}, \quad Q_2 := \{(u_i)_{i=1}^n \,|\, q = \lambda_2\},$$

such that the following inclusion relations hold:

$$\mathcal{Q}_1 := \{(u_i)_{i=1}^n \in ([0,\infty))^n \,|\, q \geq \lambda_1\} \supset \mathcal{P} := \{(u_i)_{i=1}^n \in ([0,\infty))^n \,|\, p \geq \eta\}$$

$$\supset \mathcal{Q}_2 := \{(u_i)_{i=1}^n \in ([0,\infty))^n \,|\, q \geq \lambda_2\} \supset \bar{\mathcal{R}} = \Big\{(u_i)_{i=1}^n \in ([0,\infty))^n \,\Big|\, \sum_{i=1}^n \frac{u_i}{\bar{u}_i} \geq 1\Big\},$$

and the upper bound $q \leq \lambda_1$ follows by a contradiction argument. More precisely, we take

$$\lambda_2 = \max_{1\leq i\leq n} \alpha_i\, d_i(\bar{u}_i)^{m_i}, \tag{17}$$

such that the $u_j$-intercept of the hyperellipsoid $Q_2$

$$u_{2,j} = \left(\frac{\lambda_2}{\alpha_j\, d_j}\right)^{1/m_i} \geq \bar{u}_j, \quad j = 1, 2, \cdots, n.$$

Then we take

$$\eta = \frac{\lambda_2}{\min\limits_{1\leq i\leq n} d_i}, \tag{18}$$

such that the $u_j$-intercept of the hyperellipsoid $P$

$$u_{0,j} = \left(\frac{\eta}{\alpha_j}\right)^{1/m_i} \geq u_{2,j}, \quad j = 1, 2, \cdots, n.$$

Finally we take

$$\lambda_1 = \eta \max_{1\leq i\leq n} d_i \tag{19}$$

such that the $u_j$-intercept of the hyperellipsoid $Q_1$

$$u_{1,j} = \left(\frac{\lambda_1}{\alpha_j\, d_j}\right)^{1/m_i} \geq u_{0,j}, \quad j = 1, 2, \cdots, n.$$

Combining (17), (18), and (19), we have

$$\lambda_1 = \left(\max_{1\leq i\leq n} \alpha_i\, d_i(\bar{u}_i)^{m_i}\right) \frac{\max\limits_{1\leq i\leq n} d_i}{\min\limits_{1\leq i\leq n} d_i}. \tag{20}$$

We follow exactly the same contradiction argument to prove $q(x) \leq \lambda_1$ for $x \in \mathbb{R}$ as in the proof of Proposition 1, which is omitted here. Since $(\alpha_i)_{i=1}^n \in (\mathbb{R}^+)^n$ is arbitrary, $q(x) = \sum_{i=1}^n \alpha_i\, d_i(u_i(x))^{m_i} \leq \lambda_1$ implies the upper bound (9). Now we use the inequality of arithmetic and geometric means to obtain

$$\sum_{i=1}^n \alpha_i (u_i(x))^{m_i} \geq n\Big(\prod_{i=1}^n \alpha_i (u_i(x))^{m_i}\Big)^{\frac{1}{n}} \geq n\Big(\prod_{i=1}^n \alpha_i\Big)^{\frac{1}{n}} \prod_{i=1}^n (u_i(x))^{\frac{m_i}{n}}, \tag{21}$$

which together with (9) yields (10).

□

The construction of the N-barrier for the case $n = 2$ is illustrated in the following example, which provides an intuitive idea of the construction of the N-barrier in multi-species cases.

To illustrate Proposition 2 for the case $n = 2$, we use the Lotka-Volterra system of two competing species coupled with Dirichlet boundary conditions:

$$\begin{cases} d_1 u_{xx} + \theta\, u_x + u\,(1 - u - a_1\, v) = 0, & x \in \mathbb{R}, \\ d_2 v_{xx} + \theta\, v_x + \lambda\, v\,(1 - a_2\, u - v) = 0, & x \in \mathbb{R}, \\ (u, v)(-\infty) = \mathbf{e}_i, \quad (u, v)(+\infty) = \mathbf{e}_j, \end{cases} \tag{22}$$

where $a_1$, $a_2$, $\lambda > 0$ are constants. In (22), the constant equilibria are $\mathbf{e}_1 = (0, 0)$, $\mathbf{e}_2 = (1, 0)$, $\mathbf{e}_3 = (0, 1)$ and $\mathbf{e}_4 = (u^*, v^*)$, where $(u^*, v^*) = \left(\dfrac{1 - a_1}{1 - a_1\, a_2}, \dfrac{1 - a_2}{1 - a_1\, a_2}\right)$ is the intersection of the two straight lines $1 - u - a_1\, v = 0$ and $1 - a_2\, u - v = 0$ whenever it exists. We call the solution $(u(x), v(x))$ of (22) an $(\mathbf{e}_i, \mathbf{e}_j)$-wave.

Tang and Fife ([17]), and Ahmad and Lazer ([1]) established the existence of the $(\mathbf{e}_1, \mathbf{e}_4)$-waves. Kan-on ([10, 11]), Fei and Carr ([8]), Leung, Hou and Li ([15]), and Leung and Feng ([14]) proved the existence of $(\mathbf{e}_2, \mathbf{e}_3)$-waves using different approaches. $(\mathbf{e}_2, \mathbf{e}_4)$-waves were studied for instance, by Kanel and Zhou ([13]), Kanel ([12]), and Hou and Leung ([9]).

For the above-mentioned $(\mathbf{e}_1, \mathbf{e}_4)$-waves, $(\mathbf{e}_2, \mathbf{e}_3)$-waves, and $(\mathbf{e}_2, \mathbf{e}_4)$-waves, we show by Proposition 2 that an upper bound of $u(x)v(x)$ exists for *all* of these waves (see (24) below). To this end, letting

$$\bar{u} = \max\left(1, \frac{1}{a_2}\right),$$
$$\bar{v} = \max\left(1, \frac{1}{a_1}\right),$$

we see that (8) in Proposition 2 is satisfied. According to (10), letting $\alpha_1 = \alpha_2 = m_1 = m_2 = 1$ leads to

$$\sqrt{u(x)v(x)} \leq \frac{1}{2} \max\,(\bar{u}, \bar{v})\, \frac{\max\,(d_1, d_2)}{\min\,(d_1, d_2)}, \quad x \in \mathbb{R} \tag{23}$$

or

$$u(x)v(x) \leq \frac{1}{4}\, (\max\,(\bar{u}, \bar{v}))^2 \left(\frac{\max\,(d_1, d_2)}{\min\,(d_1, d_2)}\right)^2, \quad x \in \mathbb{R}. \tag{24}$$

For the equal diffusion case $d_1 = d_2 = 1$ with the bistable condition $a_1, a_2 > 1$ For the bistable condition $a_1, a_2 > 1$ and the equal diffusion case $d_1 = d_2 = 1$, (24) is simplified to

$$u(x)v(x) \le \frac{1}{4}, \quad x \in \mathbb{R}. \tag{25}$$

If we further consider the boundary conditions in the $(\mathbf{e}_2, \mathbf{e}_4)$-waves (also $(\mathbf{e}_3, \mathbf{e}_4)$-waves) or the $(\mathbf{e}_4, \mathbf{e}_4)$-waves, the upper bound given by (25) is optimal for the case $a := a_1 = a_2 > 1$ since as $a \to 1^+$, we have

$$(u^*, v^*) = \left(\frac{1-a_1}{1-a_1\,a_2}, \frac{1-a_2}{1-a_1\,a_2}\right) = \left(\frac{1}{1+a}, \frac{1}{1+a}\right) \to (1/2, 1/2)\,. \tag{26}$$

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*Email address*: lichang.hung@gmail.com
*Email address*: xian.liao@kit.edu