Nonlinear Decentralized Data Fusion with Generalized Inverse Covariance Intersection

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***Abstract*—Decentralized data fusion is a challenging task even for linear estimation problems. Nonlinear estimation renders data fusion even more difficult as dependencies among the nonlinear estimates require complicated parameterizations. It is nearly impossible to reconstruct or keep track of dependencies. Therefore, conservative approaches have become a popular solution to nonlinear data fusion. As a generalization of Covariance Intersection, exponential mixture densities have been widely applied for nonlinear fusion. However, this approach inherits the conservativeness of Covariance Intersection. For this reason, the less conservative fusion rule Inverse Covariance Intersection is studied in this paper and also generalized to nonlinear data fusion. This generalization employs a conservative approximation of the common information shared by the estimates to be fused. This bound of the common information is subtracted from the fusion result. In doing so, less conservative fusion results can be attained as an empirical analysis demonstrates.**

***Index Terms*—inverse covariance intersection, decentralized data fusion, nonlinear estimation**

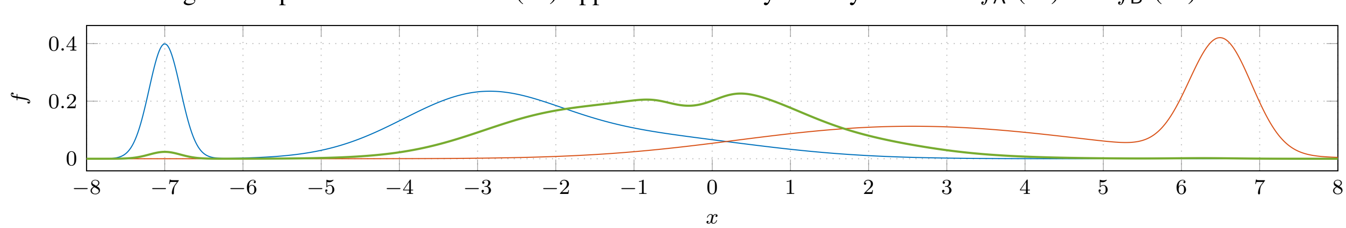
I. Introduction

With the advances in cheap integrated circuits, communication, and sensor technology, wireless sensor networks have matured over the last years. To leverage local processing power and storage capacity, estimation and data fusion algorithms can be implemented on the nodes of the network. An in-network processing of the accrued sensor data avoids many disadvantages associated with completely centralized network architectures [1], [2] but also comes with additional challenges that need to be addressed. In particular, the information acquired on each node is not independent of the other nodes [3]. This renders decentralized data fusion particularly challenging as dependencies can neither easily be kept track of nor can be reconstructed efficiently. Although such techniques are being developed [4]–[6], they often require additional knowledge. In general, decentralized data fusion forces us to find a trade-off between optimistic and pessimistic fusion results, i.e., between inconsistent and conservative results.

A well-known concept in decentralized data fusion is *Covariance Intersection* (CI) [7], [8], which provides consistent fusion results irrespective of the underlying correlations. Although optimality and tightness have been proven [9], [10], CI often gives too conservative results in typical application scenarios, which has stimulated further research on alternative approaches. In [11], a specific parameterization of correlations has been proposed that is based on unknown common information, which is assumed to be shared by the estimates to be fused and is studied further in [12]. By computing a conservative upper bound on the common information, a consistent estimate can be obtained. Such a bound has been proposed in [13], which is tight for all possible candidates of common information. In particular, this bound corresponds to the intersection of inverse covariance ellipsoids, which explains its name *Inverse Covariance Intersection* (ICI). The ICI fusion rule can also be formulated in the joint state space as a bound on the joint error covariance matrix [14], which also proves useful to derive other possible fusion gains [15]. ICI is a less conservative fusion rule than CI and is, as such, tailored to specific correlation structures, which are further studied in [16]. Promising results have already been demonstrated in distributed tracking applications [17]–[20], indicating that ICI has been becoming a viable alternative.

In this paper, we shift our focus away from linear decentralized data fusion to nonlinear fusion problems. As indicated in Fig. 1, we are concerned with the task of fusing two conditional

Fig. 1: Proposed fusion method (—) applied to arbitrary density functions f_{A} (—) and f_{B} (—).



probability densities (blue and red). This case typically renders decentralized data fusion far more difficult as a feasible, general representation and parameterization of dependent information is missing. It is well known that naïve fusion, i.e., simple multiplication and renormalization, leads to overoptimistic, erroneous fusion results. However, it was early discovered that CI can be generalized to arbitrary estimates [21], [22], for which different names have been proposed: *Chernoff fusion* [23], *normalized weighted geometric mean* [24], or *exponential mixture densities* (EMD) [25]. The EMD fusion rule has, for instance, been applied to distributed target tracking [26], [27]. Typical density representations that have been used for decentralized processing and fusion are Gaussian mixtures [28]–[33] or particle representations [34], [35]. As an alternative, a projection-based method using pseudo-Gaussian densities has been proposed in [36]. Also, the weighting parameter, which is required for both CI and EMD, is more challenging to determine in the nonlinear case [37] than in the linear case. However, the EMD rule inherits the conservativeness of CI, i.e., it is often too pessimistic, which is an incentive to study alternatives in the nonlinear case as well.

Specific types of dependencies in nonlinear estimation have been considered, for instance, in [38] to treat common process noise, which encompasses a generalization of the federated Kalman filter. Also, the channel filter, which is used to keep track of common information shared by the estimates to be fused, has been generalized to arbitrary probability densities [39]–[41] and can also be formulated in the information form [42] by means of a log-density representation. For the linear case, the channel filter bears the following relationship to ICI: While the channel filter explicitly stores common information and subtracts it from the fusion results to avoid double counting, ICI computes a conservative but tight bound on all possible candidates of common information and subtracts this bound from the fusion results. The goal of this paper is to generalize ICI in a similar way as the channel filter has been transferred to the nonlinear case. In doing so, we attain a conservative fusion rule that can be employed to fuse nonlinear estimates under unknown dependencies while conservativeness can be reduced as compared to the EMD fusion rule.

In the following, we briefly review CI and its generalization in Sec. II. A review of ICI is provided in Sec. III. The parameterization of the common information used by ICI is then formulated in terms of the corresponding probability densities. This representation of a bound on possible common information is then removed from the naïve fusion result. Properties of the proposed fusion rule are studied empirically by means of examples in Sec. IV. In Sec. V, we discuss the initial results on generalized ICI and present an overview of the next steps.

Notation

An underlined variable $\underline{x} \in \mathbb{R}^n$ denotes a real-valued vector. Lowercase boldface letters $\underline{\mathbf{x}}$ are used for random quantities. Matrices are written in uppercase boldface letters $\mathbf{C} \in \mathbb{R}^{n\times n}$, and $\mathbf{C}^{-1}$ and $\mathbf{C}^{\mathrm{T}}$ are its inverse and transpose, respectively. $\mathbf{C} \geq \mathbf{C}'$ implies that the difference $\mathbf{C} - \mathbf{C}'$ is positive semi-definite. The notation $(\hat{\underline{\mathbf{x}}}_{\mathsf{A}}, \mathbf{C}_{\mathsf{A}})$ is used for an estimate $\hat{\underline{\mathbf{x}}}_{\mathsf{A}}$ of $\underline{\mathbf{x}}$ computed by agent A with the error covariance matrix $\mathbf{C}_{\mathsf{A}} = \mathrm{E}[\tilde{\underline{\mathbf{x}}}_{\mathsf{A}}\, \tilde{\underline{\mathbf{x}}}_{\mathsf{A}}^{\mathrm{T}}]$, where $\tilde{\underline{\mathbf{x}}}_{\mathsf{A}} = \underline{\mathbf{x}} - \hat{\underline{\mathbf{x}}}_{\mathsf{A}}$ is the estimation error. A conditional probability density function for the state $\underline{\mathbf{x}}$ is denoted as $f_{\mathsf{A}}(\underline{x}) = f(\underline{x}|\mathcal{Z}_{\mathsf{A}})$, where $\mathcal{Z}_{\mathsf{A}}$ is the set of measurements processed by agent A. The Gaussian density function with mean $\hat{\underline{m}}$ and covariance matrix $\mathbf{C}$ is denoted by $f(\underline{x}) = \mathcal{N}(\underline{x}; \hat{\underline{m}}, \mathbf{C})$.

II. Review of Covariance Intersection and Its Generalization

Decentralized data fusion is the key enabler for cooperative estimation tasks in networks of autonomous systems. With Covariance Intersection (CI), we already have the most flexible tool for data fusion that guarantees consistency in any case. In this section, we provide an overview of CI and its generalization to the nonlinear case, which is abbreviated by NCI.

A. Covariance Intersection

Instead of striving for an optimal fusion result, a universal fusion strategy is to conservatively bound missing or discarded cross-covariance information; as a consequence, this information does not need to be maintained or reconstructed. In this respect, CI [7], [8] is probably the most well-known example, which provides the fusion result

$$\begin{aligned}\hat{\underline{\mathbf{x}}}_{\mathrm{CI}} &= \mathbf{C}_{\mathrm{CI}}\Big(\omega \mathbf{C}_{\mathsf{A}}^{-1}\hat{\underline{\mathbf{x}}}_{\mathsf{A}} + (1-\omega)\mathbf{C}_{\mathsf{B}}^{-1}\hat{\underline{\mathbf{x}}}_{\mathsf{B}}\Big) \\ &= \mathbf{K}_{\mathrm{CI}}\,\hat{\underline{\mathbf{x}}}_{\mathsf{A}} + \mathbf{L}_{\mathrm{CI}}\,\hat{\underline{\mathbf{x}}}_{\mathsf{B}}\end{aligned} \tag{1a}$$

with covariance matrix

$$\mathbf{C}_{\mathrm{CI}} = \Big(\omega \mathbf{C}_{\mathsf{A}}^{-1} + (1-\omega)\mathbf{C}_{\mathsf{B}}\Big)^{-1} \tag{1b}$$

and $\omega \in [0,1]$ for the estimates $(\hat{\underline{\mathbf{x}}}_{\mathsf{A}}, \mathbf{C}_{\mathsf{A}})$ and $(\hat{\underline{\mathbf{x}}}_{\mathsf{B}}, \mathbf{C}_{\mathsf{B}})$. Hence, the gains in (1a) are given by $\mathbf{K}_{\mathrm{CI}} = (1-\omega)\mathbf{C}_{\mathrm{CI}}\mathbf{C}_{\mathsf{A}}^{-1}$ and $\mathbf{L}_{\mathrm{CI}} = \omega\mathbf{C}_{\mathrm{CI}}\mathbf{C}_{\mathsf{B}}^{-1}$. CI provides *consistent* fusion results, i.e., $\mathbf{C}_{\mathrm{CI}} \geq \mathrm{E}[\tilde{\underline{\mathbf{x}}}_{\mathrm{CI}}\tilde{\underline{\mathbf{x}}}_{\mathrm{CI}}^{\mathrm{T}}]$ with $\tilde{\underline{\mathbf{x}}}_{\mathrm{CI}} = \underline{\mathbf{x}} - \hat{\underline{\mathbf{x}}}_{\mathrm{CI}}$, given that the estimates $(\hat{\underline{\mathbf{x}}}_{\mathsf{A}}, \mathbf{C}_{\mathsf{A}})$ and $(\hat{\underline{\mathbf{x}}}_{\mathsf{B}}, \mathbf{C}_{\mathsf{B}})$ to be fused are consistent. Consistency implies that the reported covariance matrix is an upper bound of the actual error covariance matrix. In order to generalize CI to the nonlinear case, the estimate (1) is interpreted as the parameters of a Gaussian density, which is discussed in the following subsection.

B. Exponential Mixture Densities

The exponential mixture density (EMD) fusion rule has been derived from the observation that (1) can directly be expressed in terms of the corresponding probability density functions [21], [22]. More precisely, the estimates $(\hat{\underline{\mathbf{x}}}_{\mathsf{A}}, \mathbf{C}_{\mathsf{A}})$ and $(\hat{\underline{\mathbf{x}}}_{\mathsf{B}}, \mathbf{C}_{\mathsf{B}})$ given the measurement histories $\mathcal{Z}_{\mathsf{A}}$ and $\mathcal{Z}_{\mathsf{B}}$ are seen as the parameters of the density functions $f_{\mathsf{A}}(\underline{x}) = \mathcal{N}(\underline{x}; \hat{\underline{\mathbf{x}}}_{\mathsf{A}}, \mathbf{C}_{\mathsf{A}})$ and $f_{\mathsf{B}}(\underline{x}) = \mathcal{N}(\underline{x}; \hat{\underline{\mathbf{x}}}_{\mathsf{B}}, \mathbf{C}_{\mathsf{B}})$, respectively. By rewriting (1) in terms of the corresponding densities, we obtain the EMD

$$\begin{aligned}f_{\mathrm{CI}}(\underline{x}) &\propto \mathcal{N}(\underline{x}; \hat{\underline{\mathbf{x}}}_{\mathsf{A}}, \tfrac{1}{\omega}\mathbf{C}_{\mathsf{A}}) \cdot \mathcal{N}(\underline{x}; \hat{\underline{\mathbf{x}}}_{\mathsf{B}}, \tfrac{1}{(1-\omega)}\mathbf{C}_{\mathsf{B}}) \\ &\propto \mathcal{N}(\underline{x}; \hat{\underline{\mathbf{x}}}_{\mathsf{A}}, \mathbf{C}_{\mathsf{A}})^{\omega} \cdot \mathcal{N}(\underline{x}; \hat{\underline{\mathbf{x}}}_{\mathsf{B}}, \mathbf{C}_{\mathsf{B}})^{(1-\omega)}\ .\end{aligned} \tag{2}$$

This combination of two densities can directly be generalized to arbitrary density functions f_{A} and f_{B} according to

$$f_{\mathrm{NCI}}(\underline{x}) = \frac{f_{\mathsf{A}}^{\omega}(\underline{x}) \cdot f_{\mathsf{B}}^{(1-\omega)}(\underline{x})}{\int f_{\mathsf{A}}^{\omega}(\underline{x}) \cdot f_{\mathsf{B}}^{(1-\omega)}(\underline{x})\, d\,\underline{x}} \ ,$$

which corresponds to naïve fusion applied to the *inflated* densities f_{A}^{ω} and $f_{\mathsf{B}}^{(1-\omega)}$. We denote this EMD density as *Nonlinear Covariance Intersection* (NCI) in the remainder of this paper.

While in the linear case a clear understanding of conservativeness can be developed, conservativeness in the nonlinear case remains a topic for ongoing research [24], [43]. However, empirical studies [28] as well as various applications [26], [27] underpin effectiveness and conservativeness of NCI. Fig. 2 presents an example and gives an impression of conservativeness—f_{NCI} has less concentrated probability as compared with $f_{\text{naïve}}$. It is now to be studied whether a corresponding generalization can be found for ICI, which is already shown in selfsame figure. As in the linear case, we strive to reduce conservatism.

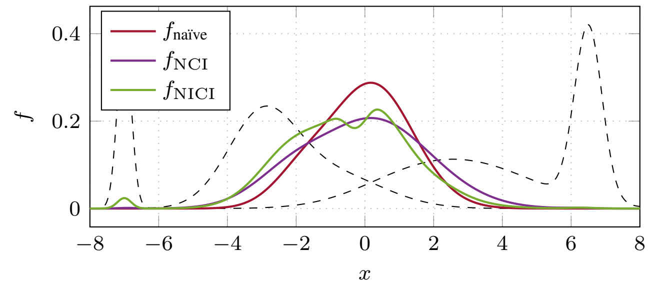


Fig. 2: For the densities f_{A} and f_{B} (dashed) in Fig. 1, the results of different fusion methods are shown.

III. Nonlinear Inverse Covariance Intersection

The ICI fusion rule has been proposed as an alternative to CI with the aim of achieving less conservative fusion results. We first discuss the idea behind ICI and the algorithm. The employed bound on unknown common information is then generalized to the nonlinear case, i.e., to arbitrary density functions. The resulting approximation of common information is then exploited to define nonlinear ICI, as illustrated in Fig. 2.

A. Inverse Covariance Intersection

Before arbitrary density functions are studied, this section provides a brief summary of ICI for the linear case. Given two consistent estimates $(\hat{\underline{\mathbf{x}}}_{\mathsf{A}}, \mathbf{C}_{\mathsf{A}})$ and $(\hat{\underline{\mathbf{x}}}_{\mathsf{B}}, \mathbf{C}_{\mathsf{B}})$, *Inverse Covariance Intersection* (ICI) provides the fusion result $(\hat{\underline{\mathbf{x}}}_{\mathrm{ICI}}, \mathbf{C}_{\mathrm{ICI}})$ with

$$\hat{\underline{\mathbf{x}}}_{\mathrm{ICI}} = \mathbf{K}_{\mathrm{ICI}}\, \hat{\underline{\mathbf{x}}}_{\mathsf{A}} + \mathbf{L}_{\mathrm{ICI}}\, \hat{\underline{\mathbf{x}}}_{\mathsf{B}} \tag{3a}$$

and

$$\mathbf{C}_{\mathrm{ICI}}^{-1} = \mathbf{C}_{\mathsf{A}}^{-1} + \mathbf{C}_{\mathsf{B}}^{-1} - \big(\omega \mathbf{C}_{\mathsf{A}} + (1-\omega)\mathbf{C}_{\mathsf{B}}\big)^{-1} \tag{3b}$$

for $\omega \in [0, 1]$. The gains in (3a) are given by

$$\mathbf{K}_{\mathrm{ICI}} = \mathbf{C}_{\mathrm{ICI}} \cdot \big(\mathbf{C}_{\mathsf{A}}^{-1} - \omega(\omega \mathbf{C}_{\mathsf{A}} + (1-\omega)\mathbf{C}_{\mathsf{B}})^{-1}\big) \ , \tag{4a}$$

$$\mathbf{L}_{\mathrm{ICI}} = \mathbf{C}_{\mathrm{ICI}} \cdot \big(\mathbf{C}_{\mathsf{B}}^{-1} - (1-\omega)(\omega \mathbf{C}_{\mathsf{A}} + (1-\omega)\mathbf{C}_{\mathsf{B}})^{-1}\big) \ . \tag{4b}$$

The covariance matrix (3b) is a conservative bound on the actual error covariance matrix, i.e.,

$$\widetilde{\mathbf{C}}_{\mathrm{ICI}} = \mathrm{E}\left[(\hat{\underline{\mathbf{x}}}_{\mathrm{ICI}} - \underline{\mathbf{x}})(\hat{\underline{\mathbf{x}}}_{\mathrm{ICI}} - \underline{\mathbf{x}})^{\mathrm{T}}\right] \leq \mathbf{C}_{\mathrm{ICI}}$$

for each $\omega \in [0, 1]$. A simple MATLAB implementation can be downloaded from https://github.com/KIT-ISAS/ICI.

ICI is a novel approach to treat unknown correlations between the estimates to be fused. Since ICI is tailored to a specific correlation structure, less conservative bounds on the fused error covariance matrix are provided than CI can compute. More precisely, it has been shown in [13] that (3b) is smaller than (1b) for each $\omega \in [0, 1]$, i.e., $\mathbf{C}_{\mathrm{ICI}}(\omega) \leq \mathbf{C}_{\mathrm{CI}}(1-\omega)$. The optimization of the weights is, e.g., studied in [44] and approximate closed-form solutions are provided therein.

In order to reformulate the fusion formulas (3) in terms of Gaussian densities, we rearrange the formulas by employing the information form. The ICI estimate can be written as

$$\mathbf{C}_{\mathrm{ICI}}^{-1}\, \hat{\underline{\mathbf{x}}}_{\mathrm{ICI}} = \mathbf{C}_{\mathsf{A}}^{-1}\, \hat{\underline{\mathbf{x}}}_{\mathsf{A}} + \mathbf{C}_{\mathsf{B}}^{-1}\, \hat{\underline{\mathbf{x}}}_{\mathsf{B}} - \mathbf{\Gamma}_{\mathrm{ICI}}^{-1}\, \hat{\underline{\gamma}}_{\mathrm{ICI}} \ , \tag{5a}$$

and

$$\mathbf{C}_{\mathrm{ICI}}^{-1} = \mathbf{C}_{\mathsf{A}}^{-1} + \mathbf{C}_{\mathsf{B}}^{-1} - \mathbf{\Gamma}_{\mathrm{ICI}}^{-1} \tag{5b}$$

in the information form, where $\hat{\underline{\gamma}}_{\mathrm{ICI}}$ and $\mathbf{\Gamma}_{\mathrm{ICI}}$ are given by

$$\hat{\underline{\gamma}}_{\mathrm{ICI}} = \omega\, \hat{\underline{\mathbf{x}}}_{\mathsf{A}} + (1-\omega)\, \hat{\underline{\mathbf{x}}}_{\mathsf{B}} \tag{6a}$$

and

$$\mathbf{\Gamma}_{\mathrm{ICI}} = \omega\, \mathbf{C}_{\mathsf{A}} + (1-\omega)\, \mathbf{C}_{\mathsf{B}} \ , \tag{6b}$$

respectively. The information form (5) implies that the naïve fusion result is computed, and the estimate $(\hat{\underline{\gamma}}_{\mathrm{ICI}}, \mathbf{\Gamma}_{\mathrm{ICI}})$ is then subtracted from it. Hence, it encompasses a form of the channel filter where a bound $(\hat{\underline{\gamma}}_{\mathrm{ICI}}, \mathbf{\Gamma}_{\mathrm{ICI}})$ on all possible common estimates is subtracted from the fusion result to prevent double counting of information. This bound tightly circumscribes the set of all possible common estimates shared by $(\hat{\underline{\mathbf{x}}}_{\mathsf{A}}, \mathbf{C}_{\mathsf{A}})$ and $(\hat{\underline{\mathbf{x}}}_{\mathsf{B}}, \mathbf{C}_{\mathsf{B}})$. Details on this bound and tightness can be found in [13]. In the following, we denote the common estimate to be removed from the naïve fusion result as *common information*, which is not to be confused with the information form.

B. Generalization to the Nonlinear Case

For the nonlinear case, we are going to generalize the bound (6) on common information to arbitrary densities. If the common information is known, the optimal result is

$$f_{\mathrm{fus}}(\underline{x}) = C_{\mathrm{fus}} \cdot \frac{f_{\mathsf{A}}(\underline{x}) \cdot f_{\mathsf{B}}(\underline{x})}{f_{\mathsf{A}\cap\mathsf{B}}(\underline{x})} \ , \tag{7}$$

where $f_{\mathsf{A}\cap\mathsf{B}}(\underline{x}) = f(\underline{x}|\mathcal{Z}_{\mathsf{A}} \cap \mathcal{Z}_{\mathsf{B}})$ denotes the common information and $C_{\mathrm{fus}}^{-1} = \int f_{\mathrm{fus}}(\underline{x})\, d\,\underline{x}$ is the normalization constant. The intuition behind (7) is that the estimates f_{A} and f_{B} can be written [45] as

$$f_{\mathsf{A}}(\underline{x}) \propto f_{\mathsf{A}\setminus\mathsf{B}}(\underline{x}) \cdot f_{\mathsf{A}\cap\mathsf{B}}(\underline{x}) \ ,$$
$$f_{\mathsf{B}}(\underline{x}) \propto f_{\mathsf{B}\setminus\mathsf{A}}(\underline{x}) \cdot f_{\mathsf{A}\cap\mathsf{B}}(\underline{x}) \ ,$$

which explains why (7) prevents double counting of $f_{\mathsf{A}\cap\mathsf{B}}$. For ICI, the density $f_{\mathsf{A}\cap\mathsf{B}}$ will be replaced by a conservative approximation, denoted by f_γ.

1) Proposed Bound on Common Information: In order to derive a nonlinear variant of ICI, we start with studying the required bound on common information for the corresponding Gaussian densities

$$f_{\mathsf{A}} = \mathcal{N}\left(\underline{x}; \hat{\underline{\mathbf{x}}}_{\mathsf{A}}, \mathbf{C}_{\mathsf{A}}\right) \ , \tag{8a}$$
$$f_{\mathsf{B}} = \mathcal{N}\left(\underline{x}; \hat{\underline{\mathbf{x}}}_{\mathsf{B}}, \mathbf{C}_{\mathsf{B}}\right) \ . \tag{8b}$$

The parameters $\hat{\underline{\gamma}}_{\text{ICI}}$ and $\mathbf{\Gamma}_{\text{ICI}}$ in (6) correspond to the sum of the two artificial estimates $\left(\omega\,\hat{\underline{\mathbf{x}}}_{\mathsf{A}}, \omega\,\mathbf{C}_{\mathsf{A}}\right)$ and $\left((1-\omega)\,\hat{\underline{\mathbf{x}}}_{\mathsf{B}}, (1-\omega)\,\mathbf{C}_{\mathsf{B}}\right)$. Following the considerations in Sec. II, we strive to express the Gaussian representation $\mathcal{N}\left(\underline{x}, \hat{\underline{\gamma}}_{\text{ICI}}, \mathbf{\Gamma}_{\text{ICI}}\right)$ of the common estimate in terms of the Gaussian densities (8). For this purpose, we first compute the densities $\mathcal{N}\left(\underline{x}; \omega\,\hat{\underline{\mathbf{x}}}_{\mathsf{A}}, \omega\,\mathbf{C}_{\mathsf{A}}\right)$ and $\mathcal{N}\left(\underline{x}; (1-\omega)\,\hat{\underline{\mathbf{x}}}_{\mathsf{B}}, (1-\omega)\,\mathbf{C}_{\mathsf{B}}\right)$ from the original densities f_{A} and f_{B}. After that, these densities are combined into a bound on the common information. The computations for this are explained in the following paragraph and are illustrated Fig. 3. On the left side of each figure, Gaussian densities are shown while, on the right side, the derived calculations are already applied to Gaussian mixture densities.

The required transformations [46] of the densities f_{A} and f_{B} can be carried out in three steps:

a) Contraction: The weighting of the means corresponds to a contraction of the densities according to

$$\mathrm{T}_\omega f_{\mathsf{A}}(\underline{x}) = \tfrac{1}{|\omega|} f\left(\tfrac{1}{\omega}\underline{x}\right) = \mathcal{N}\left(\underline{x}; \omega\,\hat{\underline{\mathbf{x}}}_{\mathsf{A}}, \omega^2\,\mathbf{C}_{\mathsf{A}}\right)$$

and

$$\begin{aligned}\mathrm{T}_{(1-\omega)} f_{\mathsf{B}}(\underline{x}) &= \tfrac{1}{|(1-\omega)|} f\left(\tfrac{1}{(1-\omega)}\underline{x}\right) \\ &= \mathcal{N}\left(\underline{x}; (1-\omega)\,\hat{\underline{\mathbf{x}}}_{\mathsf{B}}, (1-\omega)^2\,\mathbf{C}_{\mathsf{B}}\right) \ ,\end{aligned}$$

as shown in Fig. 3(b). It can be seen that the covariance matrices are modified by the square of the weights. The next step adjusts the covariance matrices.

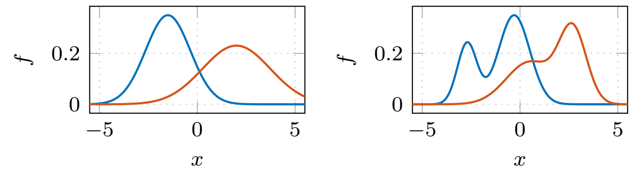
b) Inflation: As for NCI, the results are then inflated according to

$$\begin{aligned}\mathrm{T}_\omega f_{\mathsf{A}}^\omega(\underline{x}) &= C_{\mathsf{A}} \cdot \left(f\left(\tfrac{1}{\omega}\underline{x}\right)\right)^\omega \\ &= C_{\mathsf{A}} \cdot \left(\mathcal{N}\left(\underline{x}; \omega\,\hat{\underline{\mathbf{x}}}_{\mathsf{A}}, \omega^2\,\mathbf{C}_{\mathsf{A}}\right)\right)^\omega \\ &= \mathcal{N}\left(\underline{x}; \omega\,\hat{\underline{\mathbf{x}}}_{\mathsf{A}}, \omega\,\mathbf{C}_{\mathsf{A}}\right)\end{aligned}$$

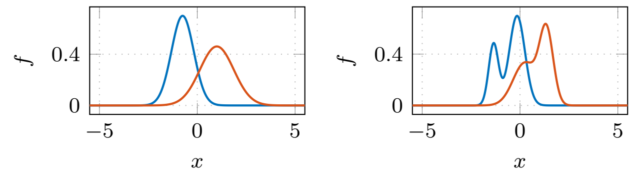
and

$$\begin{aligned}\mathrm{T}_{(1-\omega)} f_{\mathsf{B}}^{(1-\omega)}(\underline{x}) &= C_{\mathsf{B}} \cdot \left(f\left(\tfrac{1}{(1-\omega)}\underline{x}\right)\right)^{(1-\omega)} \\ &= \mathcal{N}\left(\underline{x}; (1-\omega)\,\hat{\underline{\mathbf{x}}}_{\mathsf{B}}, (1-\omega)\,\mathbf{C}_{\mathsf{B}}\right) \ ,\end{aligned}$$

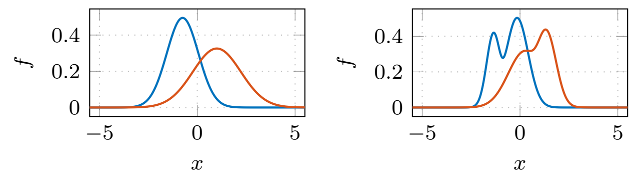
where C_{A} and C_{B} are the corresponding renormalization constants. The results are illustrated in Fig. 3(c). It is worth noticing that these first two steps are interchangeable.



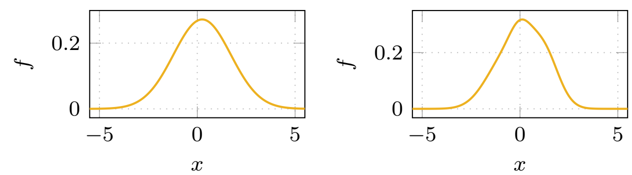
(a) Densities f_{A} and f_{B} to be fused.



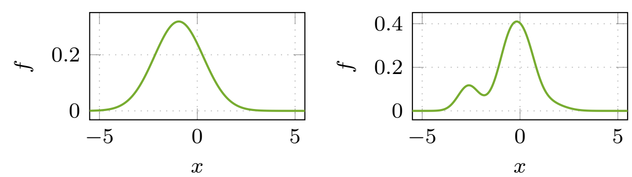
(b) Contraction of the densities: $\mathrm{T}_\omega f_{\mathsf{A}}$ and $\mathrm{T}_{(1-\omega)} f_{\mathsf{B}}$.



(c) Inflation of the densities: $\mathrm{T}_\omega f_{\mathsf{A}}^\omega$ and $\mathrm{T}_{(1-\omega)} f_{\mathsf{B}}^{(1-\omega)}$.



(d) Convolution to obtain common information f_γ.



(e) Fusion result f_{NICI} by removing common information.

Fig. 3: Illustration of the steps for the fusion of densities f_{A} (—) and f_{B} (—) with NICI and $\omega = 0.5$. Left: f_{A} and f_{B} are Gaussian densities. Right: f_{A} and f_{B} are Gaussian mixture densities.

c) Convolution: The final step corresponds to the sum (6). Hence, the combination of the transformed densities becomes the convolution

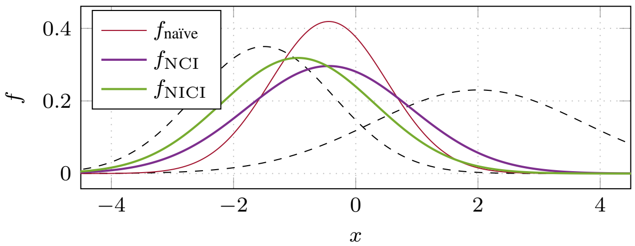
$$\begin{aligned}f_\gamma(\underline{x}) &= \left(\mathrm{T}_\omega f_{\mathsf{A}}^\omega * \mathrm{T}_{(1-\omega)} f_{\mathsf{B}}^{(1-\omega)}\right)(\underline{x}) \\ &= \int \mathrm{T}_\omega f_{\mathsf{A}}^\omega(\underline{\xi}) \cdot \mathrm{T}_{(1-\omega)} f_{\mathsf{B}}^{(1-\omega)}(\underline{x} - \underline{\xi})\, d\,\underline{\xi} \ ,\end{aligned} \tag{9}$$

which yields the proposed bound on common information shared by f_{A} and f_{B}.

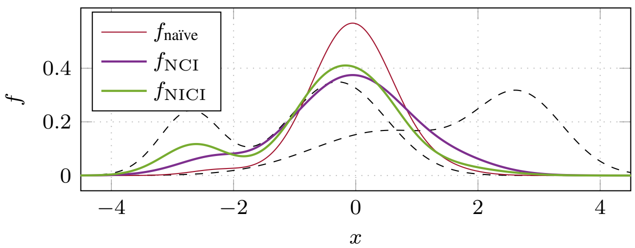
By construction, this result is equivalent to (6) for Gaussian densities, i.e.,

$$f_\gamma(\underline{x}) = \mathcal{N}\left(\underline{x}; \hat{\underline{\gamma}}_{\text{ICI}}, \mathbf{\Gamma}_{\text{ICI}}\right) \ ,$$

which corresponds to the density in Fig. 3(d) on the left side. However, the result (9) generalizes the representation

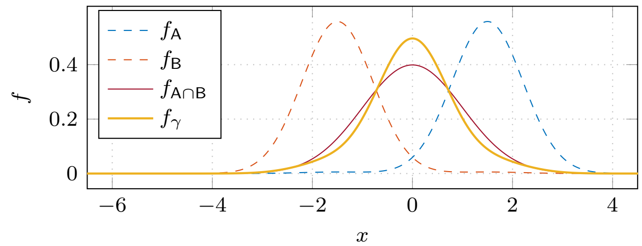


(a) Left part of Fig 3(e), i.e., Gaussian densities.

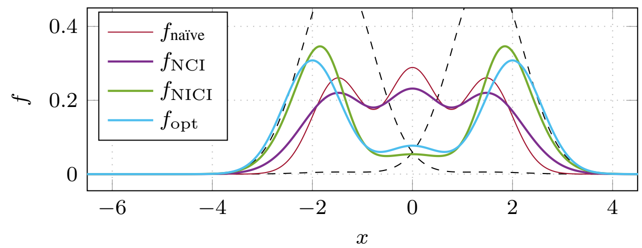


(b) Right part of Fig 3(e), i.e., Gaussian mixtures.

Fig. 4: The results shown in Fig 3(e) are compared with naïve fusion and NCI.



(a) Initial densities and common information.



(b) Fusion results. The original densities are indicated by dashed lines.

Fig. 5: Comparison of fusion results.

of common information to arbitrary densities, as illustrated on the right side of the figure. By inflating the densities f_{A} and f_{B}, they are implicitly assumed to be independent, as it is done for NCI. Here, this assumption of independence is exploited to compute the bound (9).

2) Nonlinear Inverse Covariance Intersection: In the following, we propose to employ f_γ, i.e., the combination (9), as a bound on common information for arbitrary probability density functions f_{A} and f_{B}. In doing so, this bound can directly be exploited to generalize ICI: With the derived density representation (9) of the bound, Nonlinear Inverse Covariance Intersection (NICI) can be computed by

$$f_{\text{NICI}}(\underline{x}) = C_{\text{NICI}} \cdot \frac{f_{\mathsf{A}}(\underline{x}) \cdot f_{\mathsf{B}}(\underline{x})}{f_\gamma(\underline{x})} \ , \tag{10}$$

where C_{NICI} is the corresponding normalization factor. In the special case of Gaussian estimates (8), the fusion rule (10) still reduces to

$$f_{\text{NICI}}(\underline{x}) = \mathcal{N}\left(\underline{x}; \underline{\hat{\mathbf{x}}}_{\text{ICI}}, \mathbf{C}_{\text{ICI}}\right) \ ,$$

i.e., we obtain mean and covariance matrix of the linear ICI fusion result (5). Fig. 3(e) shows NICI fusion results for Gaussian densities on the left side and for Gaussian mixtures on the right side.

The NICI fusion rule corresponds to (7) where $f_{\mathsf{A}\cap\mathsf{B}}$ has been replaced by f_γ as a conservative substitute for the unknown common information. As for NCI, the notion of conservativeness is rather vague and still requires further research. Therefore, we discuss conservativeness by scrutinizing examples in the following section.

IV. Examples for Nonlinear ICI

A first example has already been illustrated in Fig. 1 and is compared with naïve fusion and NCI in Fig. 2. While naïve fusion concentrates probability mass around zero, both NCI and NICI provide rather flat probability densities. Fig. 4 shows another comparison, where the results in Fig. 3(e) are plotted together with the naïve fusion result and the NCI result. In the Gaussian case, the densities correspond to the estimates (1) and (3) for CI and ICI, respectively. Fig. 4(a) also shows that the maximum of f_{NICI} lies slightly above f_{NCI}, which reveals that NICI is less conservative. In [13], consistency of ICI with respect to the mean squared error has been proven. Fig. 4(b) illustrates the fusion results for Gaussian mixture densities. f_{NICI} differs from f_{NCI} in that it preserves the mode on the left side, which corresponds to the Gaussian mixture component with the lowest variance. The weights in these and the following examples have been set to $\omega = 0.5$. In order to reveal and discuss specific differences between NCI and NICI, further examples are studied in the following.

A. Mode Preservation

To analyze properties of NICI, we consider two estimates f_{A} and f_{B} that share the common information

$$f_{\mathsf{A}\cap\mathsf{B}}(x) = \mathcal{N}(x; 0, 1)$$

and have the form

$$f_{\mathsf{A}}(x) \propto f_{\mathsf{A}\cap\mathsf{B}}(x) \cdot \Big(0.01 \cdot \mathcal{N}(x; -3, 1) + 0.99 \cdot \mathcal{N}(x; 3, 1)\Big) \ ,$$

and

$$f_{\mathsf{B}}(x) \propto f_{\mathsf{A}\cap\mathsf{B}}(x) \cdot \Big(0.99 \cdot \mathcal{N}(x; -3, 1) + 0.01 \cdot \mathcal{N}(x; 3, 1)\Big) \ .$$

Hence, the exclusive information is a Gaussian mixture with different weightings. Fig. 5(a) shows the initial densities and the common information. Also, the NICI bound f_γ on common information is plotted, i.e., NICI replaces $f_{\mathsf{A}\cap\mathsf{B}}$ by f_γ which has lower uncertainty. This underpins the idea of ICI to

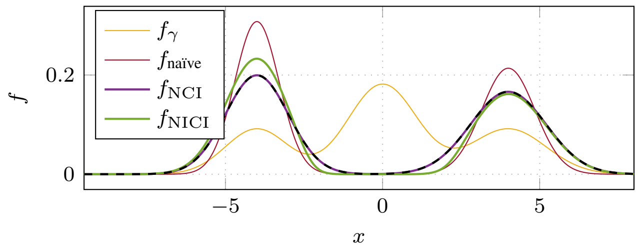


Fig. 6: Densities to be fused are the same.

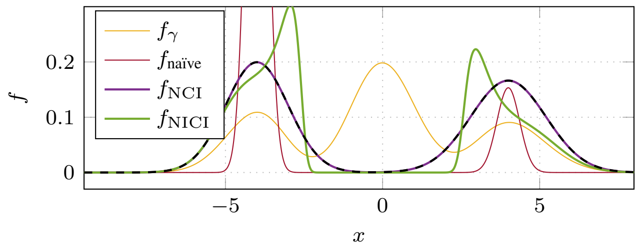


Fig. 7: The same densities are fused multiple times.

compute the maximum possible information shared by the estimates. Fig. 5(b) presents the results of the different fusion methods. It can be seen, that f_{NICI} is close to the optimal fusion result

$$f_{\text{opt}}(x) = C_{\text{opt}} \cdot \frac{f_{\mathsf{A}}(x) \cdot f_{\mathsf{B}}(x)}{f_{\mathsf{A}\cap\mathsf{B}}(x)}$$

while f_{NCI} does not preserve the modes and rather resembles the naïve fusion result $f_{\text{naïve}}$. The following example further studies the bound on common information employed by NICI.

B. Fusion of Equal Densities

In the following example, we consider the special case that the densities to be fused are equal. In Fig. 6, the considered densities

$$f_{\mathsf{A}}(x) = f_{\mathsf{B}}(x) = 0.5 \cdot \mathcal{N}(x; -4, 1) + 0.5 \cdot \mathcal{N}(x; 4, 1.2)$$

are indicated by the dashed line. Naïve fusion treats these densities like independent information, which can be far too overconfident. For NCI, it can easily be seen from (2) that the result is equal to the input densities, i.e., $f_{\text{NCI}}(x) = f_{\mathsf{A}}(x) = f_{\mathsf{B}}(x)$. Hence, NCI implicitly assumes that both densities are reported by the same source and are fully dependent. Interestingly, NICI still leads to an update but is less confident than naïve fusion. This is in stark contrast to the linear Gaussian case, where ICI also leaves equal input estimates unaltered.

The reason for the different behavior of NICI can be seen in how the common information f_γ is determined, which is also depicted in Fig. 6. The density f_γ has its largest values between the modes of $f_{\mathsf{A}}(x)$. The intuition behind this result is that NICI strives at maximizing the common information between the estimates. The *maximum* common information f_γ between the equal densities $f_{\mathsf{A}}(x)$ and $f_{\mathsf{B}}(x)$ is not the function itself but a more concentrated density. The effect of removing f_γ becomes more apparent when the same densities are fused multiple times. Fig. 7 provides the results after applying the same fusion methods 10 times. Naïve fusion yields more overconfident results, and NCI still provides the original density. For NICI, probability between the modes vanishes as the common information is removed in each of the multiple fusion steps. As f_γ is concentrated around zero, the modes of f_{NICI} are pushed apart. However, f_{NICI} still preserves much probability at the modes of $f_{\mathsf{A}}(x)$ on the left and right side; only the between-modes area vanishes. These properties of the proposed fusion rule are to be further analyzed in future work.

V. Conclusions

In this paper, we propose a novel method for nonlinear decentralized data fusion, which encompasses a generalization of Inverse Covariance Intersection. Key to this concept is the computation of an approximation of the underlying common information shared by the estimates to be fused. The derived common information can then be removed from the fusion result in order to prevent double counting. In the linear case, this fusion rule preserves consistency and is less conservative than CI. However, in the nonlinear case, these properties are difficult to prove, and conservative fusion is an ongoing research area [24], [47], [48]. Similarly, the definition of maximum common information has to be studied in more detail. While ICI maximizes the common information shared by the estimates to compute a bound, an adequate notion of maximum common information still needs to be found for the nonlinear case. Despite these research questions, we believe that NICI has the potential for being a viable alternative to the NCI fusion rule. In particular, the preservation of modes is an appealing property. Further research will also focus on defining proper criteria to optimize the weights.

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