Distributed Estimation with Partially Overlapping States based on Deterministic Sample-based Fusion

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***Abstract*— Distributing workload between sensor nodes is a practical solution to monitor large-scale phenomena. In doing so, the system can be split into smaller subsystems that can be estimated and controlled more easily. While current state-of-the-art fusion methods for distributed estimation assume the fusion of estimates referring to the full dimension of the state, little effort has been made to account for the fusion of unequal state vectors referring to smaller subsystems of the full system. In this paper, a novel method to fuse overlapping state vectors using a deterministic sample-based fusion method is proposed. These deterministic samples can be used to account for the correlated and uncorrelated noise terms and are therefore able to reconstruct the joint covariance matrix in a distributed fashion. The performance of the proposed fusion method is compared to other state-of-the-art methods.**

I. Introduction

Systems monitoring large-scale phenomena often need to distribute the necessary estimation to several instances to reduce computation costs. In doing this, the estimation problem is partitioned into smaller, possibly overlapping subsystems (see Figure 1). Possible applications include traffic regulation, power systems, or economics [1].

Distributed estimation is an advancing field and has gained much attention in recent years [2]. Although there are many methods, as the information filter [3], that allow an efficient calculation in a centralized fashion, a decentralized calculation is a more natural approach for applications with multiple spatially distributed sensors. Since no single point of failure is introduced, distributed estimation also proves to be more robust, modular, and scalable [4]. These advantages come with the need for more complex algorithms to account for the constraints which these methods force upon the fusion algorithm and to merge local estimates into one consistent result [5]–[7].

If the cross-covariances between estimates are known or can be reconstructed, an optimal fusion of two estimates using the well-known Bar-Shalom/Campo formulas [6] can be executed. This can be further generalized for the fusion of multiple sensor nodes [8]. Unfortunately, the cross-covariances between state estimates usually remain unknown, therefore creating the need to bound the possible cross-covariances. Covariance Intersection [9] does not make any assumptions about the underlying correlations between states estimates and always yields a consistent, yet conservative result. Since the resulting estimate can be overly pessimistic

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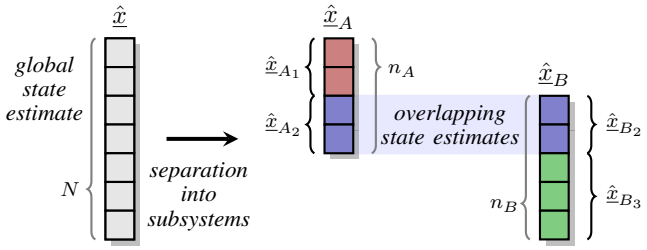


Fig. 1: Separation of the global state estimate $\hat{\underline{x}}$ into two overlapping subsystems A and B with state estimates $\underline{x}_A$ and $\underline{x}_B$ overlapping in section A_2, B_2.

in some applications, other approaches such as Ellipsoidal Intersection [10] or Inverse Covariance Intersection [11] have been developed to find a smaller bound for the cross-covariances. Since all of these approaches tend to over- or underestimate the uncertainty, efforts have been made to find an optimal fusion result by reconstructing the cross-covariances between state estimates. In [12], the common past invariant Ensemble KF (CPI-EnKF) has been proposed to enable cooperative localization of vehicles. A method to reconstruct the cross-covariances using random samplers is described in [13]. Recently, a distributed estimation method that utilizes a set of deterministic samples to reconstruct the cross-covariances for optimal fusion [14] has been proposed. This was further developed to keep the number of samples constant to limit computational and communication costs to perform a suboptimal fusion [15].

Distributing the workload to several sensor nodes observing only a limited number of state estimates instead of the complete state space is further reducing computational costs and enables proper estimation and control. Although many methods for distributed estimation have been designed for fusing estimates of the same state dimension, little effort has been made to fuse partially overlapping state estimates belonging to subsystems of the full system model. Consensus methods [16] can be used to find a common global result, but they lack the ability to take uncertainties into account and therefore do not minimize the error covariance of the resulting state estimate. Model decomposition for distributed Kalman filtering was discussed in [17] and [18]. In [19], an empirical method has been proposed to fuse partial state estimates. Further, in [20] the problem of fusion for partially overlapping state vectors was formulated as a weighted least-squares problem. A method to utilize covariance intersection was proposed to find a conservative solution to the fusion problem, and the result is compared to the optimal fusion

result. Since covariance intersection does not offer a proper way to weigh state estimates of unequal state dimension so that the overall error is minimized, the authors in [21] proposed a method for calculating weight matrices that scale the fused estimate to achieve a more accurate estimate. Yet, the problem of finding an optimal fusion result in a distributed fashion without sharing information about how state estimates correlate remains unsolved. By utilizing the advantages of the Sample-based Fusion method [14], these challenges can inherently be addressed. In this paper, a fusion method using deterministic samples to account for the partial correlations that originate from the overlapping state vectors is proposed. These samples provide the means to optimally or sub-optimally reconstruct the cross-covariances between several distributed sensor nodes and are used to perform optimal fusion.

The paper is structured as follows. Section III recapitulates the problem of partially overlapping state vectors and the Sample-based Fusion method. In Section IV, a method to use deterministic samples to reconstruct the cross-covariances between partially overlapping state vectors is proposed. Section V shows the performance of the proposed method using a small three-dimensional system model and a large scale evaluation example featuring the heat distribution in a rod. Afterwards, the results of the proposed method and further research objectives are discussed in Section VI. Finally, in Section VII the paper is summarized.

II. Preliminaries

Underlined variables $\underline{x}$ denote vector-valued functions, and lowercase boldface letters $\underline{\boldsymbol{x}}$ are used for random quantities. Matrices are written in uppercase boldface letters $\mathbf{P}$. The notation $\hat{\underline{x}}$ is used for the mean of a random variable, an estimate of uncertain quality or an observation. The matrix $\mathbf{I}$ is the identity matrix of the appropriate dimension. By $\{p^{(m)}\}_{m=1}^{M}$, we denote a sample set with a number of M sample vectors.

III. Formulation of Problem

We will assume a discrete-time time-variant linear stochastic dynamic system

$$\underline{x}_{k+1} = \mathbf{A}_k \underline{x}_k + \mathbf{B}_k \underline{u}_k + \underline{w}_k \text{ with } \underline{w}_k \sim \mathcal{N}\big(\underline{0}, \mathbf{Q}_k\big) \quad (1)$$

with state matrix $\mathbf{A}_k$, state vector $\underline{x}_k$, input matrix $\mathbf{B}_k$, input vector $\underline{u}_k$, and zero-mean white Gaussian system noise $\underline{w}_k$ with covariance matrix $\mathbf{Q}_k$. A number of L sensor nodes receive information about the observed system through a noisy linear measurement model

$$\underline{y}_k^{(i)} = \mathbf{C}_k^{(i)} \underline{x}_k + \underline{v}_k^{(i)} \text{ with } \underline{v}_k^{(i)} \sim \mathcal{N}\big(\underline{0}, \mathbf{R}_k^{(i)}\big)\,, \quad (2)$$

for sensor node $i \in \{1, \ldots, L\}$ with $\mathbf{C}_k^{(i)}$ describing the observation model and sensor noise $\underline{v}_k^{(i)}$ with covariance $\mathbf{R}_k^{(i)}$. The system is estimated using a linear Kalman Filter.

A. Optimal Fusion

In order to enhance the accuracy of the global state estimate, the local state estimates and covariance matrices can be fused together. The following formulas for the fusion of multiple estimates [22] are a generalization of the Bar-Shalom/Campo formulas [23] that are widely used in multisensor tracking applications. The local state estimates can be rewritten as a joint estimate vector

$$\hat{\underline{m}}_{k|k} = \left[\big(\hat{\underline{x}}_{k|k}^{(1)}\big)^{\mathrm{T}} \quad , \ldots , \quad \big(\hat{\underline{x}}_{k|k}^{(L)}\big)^{\mathrm{T}} \right]^{\mathrm{T}} .$$

The joint covariance matrix can be written as

$$\mathbf{J}_{k|k} = \begin{bmatrix} \mathbf{P}_{k|k}^{(1)} & \mathbf{P}_{k|k}^{(1,2)} & \ldots & \mathbf{P}_{k|k}^{(1,L)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{P}_{k|k}^{(L,1)} & \mathbf{P}_{k|k}^{(L,2)} & \ldots & \mathbf{P}_{k|k}^{(L)} \end{bmatrix}, \quad (3)$$

where

$$\mathbf{P}_{k|k}^{(i,j)} = E[(\hat{\underline{x}}_{k|k}^{(i)} - \underline{x}_k)(\hat{\underline{x}}_{k|k}^{(j)} - \underline{x}_k)^{\mathrm{T}}] \quad (4)$$

is a local covariance or cross-covariance matrix for $(i) = (i,i)$ or $(i,j), i \neq j$, respectively. The fused covariance matrix and state estimate can be calculated according to

$$\mathbf{P}_{k|k} = \Big(\mathbf{H}^{\mathrm{T}} \big(\mathbf{J}_{k|k}\big)^{-1} \mathbf{H}\Big)^{-1}, \quad (5)$$

$$\hat{\underline{x}}_{k|k} = \mathbf{P}_{k|k} \mathbf{H}^{\mathrm{T}} \big(\mathbf{J}_{k|k}\big)^{-1} \hat{\underline{m}}_{k|k}\,. \quad (6)$$

with matrix $\mathbf{H} = [\mathbf{I}, \ldots, \mathbf{I}]^{\mathrm{T}}$ containing the identity matrix $\mathbf{I}$ of the system state dimension and thus reflecting that all estimates contain the same state estimates.

While the vector $\hat{\underline{m}}_{k|k}$ can be obtained easily from the local state estimates as well as the entries on the main diagonal referring to the local covariance matrices, the cross-covariances $\mathbf{P}_{k|k}^{(i,j)}$ with $i \neq j$ on the other hand usually remain unknown. They originate from the dependencies between state estimates, namely double counting of common measurement information as well as common system noise and prior information. When discussing correlations, we will focus on the problem of common process noise and common prior information.

The incorporation of the common process noise into the cross-covariances (4) can be seen during the prediction step

$$\mathbf{P}_{k+1|k}^{(i,j)} = \mathbf{A}_k \mathbf{P}_{k|k}^{(i,j)} \mathbf{A}_k^{\mathrm{T}} + \mathbf{Q}_{k+1}\,. \quad (7)$$

The cross-covariances is additionally altered by the filtering step by

$$\mathbf{P}_{k+1|k+1}^{(i,j)} = \big(\mathbf{I} - \mathbf{K}_k^{(i)} \mathbf{C}_k^{(i)}\big) \mathbf{P}_{k+1|k}^{(i,j)} \big(\mathbf{I} - \mathbf{K}_k^{(j)} \mathbf{C}_k^{(j)}\big)^{\mathrm{T}}\,, \quad (8)$$

where $\mathbf{K}_k^{(i)}$ is the Kalman gain of the local filter at node i. As clearly visible from equation (7) and (8), bookkeeping of the executed filtering steps is necessary to obtain the cross-covariances between state estimates. This is not only expensive, but also proves to be nearly impossible with increasing complexity of topology and number of sensor nodes. It also requires frequent communication and is prone to packet loss. Therefore, other strategies need to be considered to obtain optimal fusion results.

B. Optimal Deterministic Sample-based Fusion

The deterministic Sample-based Fusion first described in [14] is a method to reconstruct the cross-covariances between state estimates using a set of deterministic samples. The construction of the sample-based representation bears resemblance to the unscented transform [24]. However, in this paper, samples are used to represent the joint error covariance matrix. The samples for the Sample-based Fusion are created using the simple deterministic spherical simplex sampling method described in [25]. This basic sample set will be referred to as the identity set $\{p^{(m)}\}_{m=1}^{M}$ with the special properties

$$\sum_{m=1}^{M} \underline{p}^{(m)} = \underline{0}\,, \ \sum_{m=1}^{M} \underline{p}^{(m)} \big(\underline{p}^{(m)}\big)^{\mathrm{T}} = \mathbf{I}_{D\times D}\,.$$

The sample set is of size $M = D + 1 = N + \mathcal{T} \times W + 1$ where N is the dimension of the system, W the dimension of the system noise, and $\mathcal{T}$ accounts for the user defined number of processing steps that will be encapsulated in the samples.

The deterministic sample set that will be created to reconstruct the joint covariance matrix is a square root representation of the cross-covariances. The square root decomposition of the cross-covariances equation (7) can be rewritten as

$$\mathbf{P}_{k+1|k}^{(i,j)} = \mathbf{A}_k \sqrt{\mathbf{P}_{k|k}^{(i,j)}} \Big(\sqrt{\mathbf{P}_{k|k}^{(i,j)}}\Big)^{\mathrm{T}} \mathbf{A}_k^{\mathrm{T}} + \sqrt{\mathbf{Q}_{k+1}} (\sqrt{\mathbf{Q}_{k+1}})^{\mathrm{T}},$$

where the prior cross-correlation $\mathbf{P}_{k|k}^{(i,j)}$ between node i and j is decomposed into

$$\mathbf{P}_{k|k}^{(i,j)} = \sqrt{\mathbf{P}_{k|k}^{(i,j)}} \Big(\sqrt{\mathbf{P}_{k|k}^{(i,j)}}\Big)^{\mathrm{T}}$$

and the system noise matrix $\mathbf{Q}_{k+1}$ is decomposed into $\sqrt{\mathbf{Q}_{k+1}}(\sqrt{\mathbf{Q}_{k+1}})^{\mathrm{T}}$, respectively. All square root terms obtained by the Cholesky decomposition until the time horizon $\mathcal{T}$ can be rewritten into the block diagonal matrix

$$\mathbf{\Sigma}_k = \mathrm{diag}\Big(\sqrt{\mathbf{P}_{k|k}^{(i,j)}}, \sqrt{\mathbf{Q}_{k+1}}, \ldots, \sqrt{\mathbf{Q}_{k+\mathcal{T}}}\Big)\,.$$

The identity set $\{\underline{p}^{(m)}\}_{m=1}^{M}$ is then weighted with the matrix $\mathbf{\Sigma}_k$

$$\begin{aligned} \underline{d}_k^{(m)} &= \mathbf{\Sigma}_k \underline{p}^{(m)}\,, \ \ \forall m = 1, \ldots, M \\ &= \Big[(\underline{s}_{k|k}^{(i,m)})^{\mathrm{T}}, \quad (\underline{w}_{k+1}^{(m)})^{\mathrm{T}} \quad , \ldots, \quad (\underline{w}_{k+P}^{(m)})^{\mathrm{T}}\Big]^{\mathrm{T}}. \end{aligned} \quad (9)$$

By doing so, a set of weighted samples $\{\underline{d}_k^{(m)}\}_{m=1}^{M}$ is created. The first sample set $\{\underline{s}_{k|k}^{(i,m)}\}^{\mathrm{T}}$ accounts for the prior cross-covariances between the state estimates $\mathbf{P}_{k|k}^{(i,j)}$ and the remaining entries account for the system noise matrices $\mathbf{Q}_{k+1}, \ldots, \mathbf{Q}_{k+P}$. It should be noted that the samples are created in state space, but represent covariances. Due to the underlying simplex structure all entries of $\{\underline{d}_k^{(m)}\}_{m=1}^{M}$ refer to the same state space, but they are uncorrelated to each other. When multiplying sample sets that correlate, the underlying covariances can be reconstructed according to

$$\begin{aligned} \mathbf{P}_{k|k}^{(i,j)} &= \sum_{m=1}^{M} \underline{s}_{k|k}^{(i,m)} \big(\underline{s}_{k|k}^{(i,m)}\big)^{\mathrm{T}}, \\ \mathbf{Q}_{k+\tau'} &= \sum_{m=1}^{M} \underline{w}_{k+\tau'}^{(i,m)} \big(\underline{w}_{k+\tau'}^{(i,m)}\big)^{\mathrm{T}}, \end{aligned}$$

where τ' is an arbitrary processing step with $0 < \tau' \leq \mathcal{T}$. In contrast, when two sample sets that are uncorrelated are multiplied, we observe that they result is the zero matrix

$$\sum_{m=1}^{M} \underline{s}_{k|k}^{(i,m)} \big(\underline{w}_{k+\tau'}^{(m)}\big)^{\mathrm{T}} = \sum_{m=1}^{M} \underline{w}_{k+\tau'}^{(m)} \big(\underline{w}_{k+\tau''}^{(m)}\big)^{\mathrm{T}} = \underline{\mathbf{0}}\,,$$

with the arbitrary processing steps τ' and τ'' and $\tau' \neq \tau''$. To encapsulate the processing steps of the local Kalman filters into the sample set $\{\underline{s}_{k|k-1}^{(i,m)}\}_{m=1}^{M}$, the sample set is altered during the prediction step and the uncorrelated noise set $\{\underline{w}_k^{(m)}\}_{m=1}^{M}$ for the current time step is added

$$\underline{s}_{k|k-1}^{(i,m)} = \mathbf{A}_k \underline{s}_{k-1|k-1}^{(i,m)} + \underline{w}_k^{(m)}\,, \ \ \forall m = 1, \ldots, M\,. \quad (10)$$

Afterwards, the measurement update is performed using the Kalman filter gain $\mathbf{K}_k^{(i)}$, yielding

$$\underline{s}_{k|k}^{(i,m)} = \big(\mathbf{I} - \mathbf{K}_k^{(i)} \mathbf{H}_k^{(i)}\big) \mathbf{s}_{k|k-1}^{(i)}\,, \ \ \forall m = 1, \ldots, M\,. \quad (11)$$

In case no measurement is available, the prediction is kept, i.e., $\underline{\hat{x}}_{k|k}^{(i)} = \underline{\hat{x}}_{k|k-1}^{(i)}$, $\mathbf{P}_{k|k}^{(i)} = \mathbf{P}_{k|k-1}^{(i)}$ and $\underline{s}_{k|k}^{(i)} = \underline{s}_{k|k-1}^{(i)}$. After $\mathcal{T}$ or fewer time steps, the cross-covariances can be reconstructed using

$$\mathbf{P}_{k+\mathcal{T}|k+\mathcal{T}}^{i,j} = \sum_{m=1}^{M} \underline{s}_{k+\mathcal{T}|k+\mathcal{T}}^{(i,m)} \big(\underline{s}_{k+\mathcal{T}|k+\mathcal{T}}^{(j,m)}\big)^{\mathrm{T}}. \quad (12)$$

These cross-covariances can be reconstructed between all nodes and be processed together with the local covariance matrices into the joint covariance matrix in order to enable the usage of the optimal fusion algorithm as described in Section III-A. The fused state estimate $\underline{\hat{x}}_{k+\mathcal{T}}$ and fused covariance matrix $\mathbf{P}_{k+\mathcal{T}}$ can be used to reinitialize the local state estimates and covariance matrices. Since the fused result is more accurate, this will reduce the local estimation error and prevent the local filter from diverging, e.g., if the local state space is not completely observable [26].

C. Optimal Fusion of Partially Overlapping State Vectors

The optimal fusion algorithm has been described for equal state vectors. When estimating smaller subsystems of a large-scale phenomenon, these fusion algorithms do not have the ability to merge partially overlapping state estimates into one fusion result. In order to better illustrate the problem of overlapping local state estimates we will assume a system containing two local sensor nodes A and B with state estimates $\underline{\hat{x}}_A = [\underline{\hat{x}}_{A1}^{\mathrm{T}}, \underline{\hat{x}}_{A2}^{\mathrm{T}}]^{\mathrm{T}}$ and $\underline{\hat{x}}_B = [\underline{\hat{x}}_{B2}^{\mathrm{T}}, \underline{\hat{x}}_{B3}^{\mathrm{T}}]^{\mathrm{T}}$. Further, these state estimates overlap at sections $\underline{\hat{x}}_{A2}$ and $\underline{\hat{x}}_{B2}$. This example is illustrated in Figure 1.

The global system (see equations (1) and (2)) is separated into subsystems, so that the local system model can be rewritten as

$$\underline{x}_{k+1}^{(i)} = \mathbf{A}_k^{(i)} \underline{x}_k^{(i)} + \mathbf{B}_k^{(i)} \underline{u}_k^{(i)} + \underline{w}_k^{(i)} \text{ with } \underline{w}_k^{(i)} \sim \mathcal{N}\left(\underline{0}, \mathbf{Q}_k^{(i)}\right),$$

where the dimension of the local subsystem $n^{(i)}$ is smaller than the dimension N of the global system. Each state estimate can be regarded as an observation [20] with matrix $\mathbf{H}$ (see equation (5)), which shows how the local state estimates map into the global state space

$$\begin{bmatrix} \hat{\underline{x}}_{A_1} \\ \hat{\underline{x}}_{A_2} \\ \hat{\underline{x}}_{B_2} \\ \hat{\underline{x}}_{B_3} \end{bmatrix} = \mathbf{H} \begin{bmatrix} \underline{\boldsymbol{x}}_1 \\ \underline{\boldsymbol{x}}_2 \\ \underline{\boldsymbol{x}}_3 \end{bmatrix} + \tilde{\underline{x}}, \tag{13}$$

where the matrix $\mathbf{H}$ is defined as

$$\mathbf{H} := \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{matrix} \} \mathbf{H}^A \\ \\ \} \mathbf{H}^B \end{matrix}.$$

This means the matrices $\mathbf{H}^A$ and $\mathbf{H}^B$ select the partial estimates $\hat{\underline{x}}_A$ and $\hat{\underline{x}}_B$, respectively, from the global estimate. The measurement noise $\tilde{\underline{x}}$ has the covariance matrix

$$\begin{aligned} \mathbf{J} &= \begin{bmatrix} \mathbf{P}^A & \mathbf{P}^{A,B} \\ \mathbf{P}^{B,A} & \mathbf{P}^B \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{P}^{A_1,A_1} & \mathbf{P}^{A_1,A_2} & \mathbf{P}^{A_1,B_2} & \mathbf{P}^{A_1,B_3} \\ \mathbf{P}^{A_2,A_1} & \mathbf{P}^{A_2,A_2} & \mathbf{P}^{A_2,B_2} & \mathbf{P}^{A_2,B_3} \\ \mathbf{P}^{B_2,A_1} & \mathbf{P}^{B_2,A_2} & \mathbf{P}^{B_2,B_2} & \mathbf{P}^{B_2,B_3} \\ \mathbf{P}^{B_3,A_1} & \mathbf{P}^{B_3,A_2} & \mathbf{P}^{B_3,B_2} & \mathbf{P}^{B_3,B_3} \end{bmatrix}, \end{aligned} \tag{14}$$

which is the joint covariance matrix from equation (3) rewritten for the overlapping state vector problem and where the time-index k is omitted for simplicity. This joint covariance matrix necessary to obtain (6) can be calculated recursively. The joint process noise distributed over the system can be described by

$$\mathbf{Q}_k^{\mathrm{J}} = \mathbf{H}\mathbf{Q}_k\mathbf{H}^{\mathrm{T}},$$

where $\mathbf{Q}_k$ denotes the system noise distributed over the global model and matrix $\mathbf{H}$ from (13) contains the information on how the state estimates are distributed between the local estimators. It should be noted that the way how the system noise is distributed very much depends on the separation of the global system and the used process noise assumption. It can be useful to inflate the process noise where the systems are separated into the subsystems to account for the missing model information. However, the focus of this paper is not the optimal separation of the global system into the local subsystems, but the optimal fusion of the distributed state estimates. Therefore, for simplicity the separation of the global system is not further discussed.

The prediction of the joint covariance matrix (14) can be calculated by

$$\mathbf{J}_{k|k-1} = \mathbf{A}_k^{\mathrm{J}} \mathbf{J}_{k-1|k-1} (\mathbf{A}_k^{\mathrm{J}})^{\mathrm{T}} + \mathbf{Q}_k^{\mathrm{J}}.$$

The joint system matrix $\mathbf{A}_k^{\mathrm{J}}$ is the block diagonal matrix of the local system matrices

$$\mathbf{A}_k^{\mathrm{J}} = \mathrm{diag}\left(\mathbf{A}_k^{(1)}, \ldots, \mathbf{A}_k^{(L)}\right).$$

The measurement update can be incorporated into the joint covariance matrix

$$\mathbf{J}_{k|k} = \mathbf{L}_k^{\mathrm{J}} \mathbf{J}_{k|k-1} (\mathbf{L}_k^{\mathrm{J}})^{\mathrm{T}} + \mathbf{R}_k^{\mathrm{J}}$$

with the joint measurement matrix $\mathbf{R}^{\mathrm{J}}$ and the joint gain $\mathbf{L}_k^{\mathrm{J}}$

$$\begin{aligned} \mathbf{R}^{\mathrm{J}} &= \mathrm{diag}\left(\mathbf{K}_k^{(1)} \mathbf{R}_k^{(1)} (\mathbf{K}_k^{(1)})^T, \ldots, \mathbf{K}_k^{(L)} \mathbf{R}_k^{(L)} (\mathbf{K}_k^{(L)})^T\right), \\ \mathbf{L}_k^{\mathrm{J}} &= \mathrm{diag}\left(\mathbf{I} - \mathbf{K}_k^{(1)} \mathbf{C}_k^{(1)}, \ldots, \mathbf{I} - \mathbf{K}_k^{(L)} \mathbf{C}_k^{(L)}\right). \end{aligned}$$

This joint covariance matrix can be calculated recursively after each time step to calculate the current cross-covariances between the state estimates. In a distributed network this would mean to constantly communicate all processing steps. Especially under restricted communication bandwidth and package loss or delay this is not a feasible solution to keep track of the cross-covariances.

IV. Optimal Deterministic Sample-based Fusion for Partially Overlapping State Vectors

By studying the derived recursive analytical calculation of the joint covariance matrix, the diagonal structure of the systems becomes evident. This structure can be used to derive a distributed notation of the joint covariance matrix

$$\begin{aligned} \mathbf{P}_{k|k-1}^{(i,j)} &= \mathbf{A}_k^{(i)} \mathbf{P}_{k-1|k-1}^{(i,j)} (\mathbf{A}_k^{(j)})^{\mathrm{T}} + \mathbf{H}_k^{(i)} \mathbf{Q}_k (\mathbf{H}_k^{(j)})^{\mathrm{T}}, \\ \mathbf{P}_{k|k}^{(i,j)} &= (\mathbf{I} - \mathbf{K}_k^{(i)} \mathbf{C}_k^{(i)}) \mathbf{P}_{k|k-1}^{(i,j)} (\mathbf{I} - \mathbf{K}_k^{(j)} \mathbf{C}_k^{(j)})^{\mathrm{T}}, \text{with } i \neq j. \end{aligned}$$

The common process noise is determined by the overlapping parts of the states. Therefore, the state estimates contain an independent part where they do not overlap and a dependent part which results in a cross-covariance term. To model this behaviour, the deterministic samples also need to contain independent and dependent sections.

The concept of correlated and uncorrelated samples for fusion has been discussed by [13] before. In the Sample-based Fusion, this is achieved by the underlying simplex form. It is the same mechanism that leads to independence between correlation terms and process noise terms.

A. Sample-based Fusion for Overlapping State Vectors

In order to reconstruct the cross-covariances between the overlapping state estimates of two sensor nodes, we need to create sample sets with dependent and independent sections. In the above example of two partial estimates, each local estimator uses the corresponding selection matrix $\mathbf{H}^A$ or $\mathbf{H}^B$ to generate its local sample set. The block diagonal matrix for node A is

$$\boldsymbol{\Sigma}_k^A = \mathrm{diag}\left(\mathbf{H}^A \sqrt{\mathbf{P}_{k|k}^{(i,j)}}, \mathbf{H}^A \sqrt{\mathbf{Q}_{k+1}}, \ldots, \mathbf{H}^A \sqrt{\mathbf{Q}_{k+\mathcal{T}}}\right),$$

which is employed to obtain the local reduced samples

$$\underline{d}_k^{(A)} = \boldsymbol{\Sigma}_k^A \underline{p}^{(m)}, \quad \forall m = 1, \ldots, M$$

by means of (9). The number M of samples remains the same as for the full state space representation (9), but the dimension of each sample is reduced according to the dimension of the partial estimate $\underline{\hat{x}}_A$. The gist of this construction is that different matrices $\mathbf{H}^A$ and $\mathbf{H}^B$ select the same entries of each $\mathbf{\Sigma}_k^A \underline{p}^{(m)}$ and $\mathbf{\Sigma}_k^B \underline{p}^{(m)}$ for the overlapping parts and exclusive entries for the non-overlapping parts. These local samples are then processed through the local time update (10) and measurement update (11). The corresponding local samples are generated for node B or, more generally, for each node $i = 1, \ldots, L$ in the network. As a result, different local sample sets $\{\underline{s}_{k|k}^{(i,m)}\}_{m=1}^{M}$ are obtained that encode correlations among overlapping parts of the local state estimates and are uncorrelated for the non-overlapping parts. The cross-covariance matrix for each pair of local estimates can then be computed by (12).

B. Reinitialization

After the fusion has been executed, there are two different approaches to continue with the state estimation process. The sample creation proposed in [14] can be defined recursively so that the nodes could continue creating samples to account for the cross-correlation. The fusion result could also be used to reinitialize the nodes and restart the sample generation from the beginning. In the first approach, the number of samples may grow infinitely and therefore has to be restricted in order to prevent increasing communication and calculation efforts. As proposed in [15], this can be prevented by only communicating a limited set of samples to account for the latest filtering steps.

The fused state estimate $\underline{x}_{k+\mathcal{T}}$ and covariance matrix $\mathbf{P}_{k+\mathcal{T}}$ that were fused at time horizon $\mathcal{T}$ can be used to reinitialize the local estimators. Since this fused covariance matrix contains the full correlation of the state, it is necessary to create a fully correlated sample set with system dimension N. The identity sample set is weighted with the Cholesky decomposition of the covariance matrix $\mathbf{P}_{k+\mathcal{T}}$. Hence, it now contains the full correlations. Since the subsystems do not contain the full correlation, but a subset defined by the selection matrix $\mathbf{H}^{(i)}$, this sample has to be reduced to state dimension $n^{(i)}$ before proceeding.

C. Sample Creation

To obtain the local sample sets for the fusion of overlapping state vectors, the simple deterministic spherical simplex sampling method can be adapted. This section proposes a recursive creation of samples so that the time horizon until the fusion $\mathcal{T}$ does not need to be known from the beginning, but samples can be created as long as necessary.

As discussed in Section IV-B, the first sample set $\{\underline{s}_{k|k}^{(i,m)}\}_{m=1}^{M}$ has to be created with the dimension N of the global state space in order to include the full correlation between states. Afterwards, the sample set will be reduced to the section relevant to the local sensor node. The necessary sample creation can be obtained from [14]. Alternatively the sample creation of Algorithm 1 can be adapted, so that the position $\mathrm{pos}(\hat{x}_1)$ of state $\hat{x}_1$ is 1, the dimension of the local system $n^{(i)}$ is N and the current processing step τ is zero as will be explained in the following section.

Algorithm 1 Recursive Sample Creation

```
function CreateSampleSet(processing step τ, dimension of global system N, dimension of local system n^(i), position pos(x̂_1^(i)) of the first entry of local state estimate x̲̂^(i))
    for k = 0, 1, 2, ... n^(i) do
        d = N · τ + pos(x̂_1^(i)) + k
        c = 1/√(d · (d + 1))
        for m = 0, 1, 2, ... d do
            w_τ(k, m, τ) = c
        end for
        w_τ(k, d + 1, τ) = −d · c
    end for
    return w_τ
end function
```

To obtain the samples dedicated to the local process noise, the adapted sample creation in Algorithm 1 is used. The sample creation starts at the position of the first entry $\hat{x}_1^{(i)}$ of the local state estimate $\underline{\hat{x}}^{(i)}$ in reference to the global system space and ends after length $n^{(i)}$ of the state estimate where the state space is no longer occupied. As an example, if the first entry of estimate $\underline{\hat{x}}^{(i)}$ of node i starts at state $\hat{x}_3$ of the global state estimate, then the position $\mathrm{pos}(\hat{x}_1^{(i)})$ of $\underline{\hat{x}}^{(i)}$ is 3. The created sample $\{\underline{w}_{k+\tau}^{(m)}\}_{m=1}^{M}$ contains the process noise for processing step τ and will be included into the sample set $\{\underline{s}_{k+\tau|k+\tau}^{(i,m)}\}_{m=1}^{M}$ which will then contain the complete correlation of the processing step. Analogously to equations (10) and (11), the time-update is performed with

$$\underline{s}_{k+\tau|k+\tau-1}^{(i,m)} = \mathbf{A}_{k+\tau} \underline{s}_{k+\tau-1|k+\tau-1}^{(i,m)} + \underline{w}_{k+\tau}^{(m)} ,$$

and the measurement step is executed with

$$\underline{s}_{k+\tau|k+\tau}^{(i,m)} = \left(\mathbf{I} - \mathbf{K}_{k+\tau}^{(i)} \mathbf{C}_{k+\tau}^{(i)}\right) \mathbf{s}_{k+\tau|k+\tau-1}^{(i)} ,$$

for all $m = 1, \ldots, M$. Since the size of the sample set increases with the number of processing steps, it increases linearly. The size M of the samples set is $M = D + 1 = N \times \mathcal{T} + \mathrm{pos}(x_1) + n^{(i)} + 1$.

V. Evaluation

In this section, two evaluation examples are discussed. First, a small-scale example of a constant acceleration model demonstrates the performance of the method. Second, a large scale scenario of a heat conduction in a rod that is modeled with 100 states is evaluated. In both examples, the deterministic Sample-based Fusion method is compared with other state-of-the-art methods.

A. Small-scale Example of Constant Acceleration Model

In this section, a simple three-dimensional model is utilized to show the performance of the proposed method. The linear discrete-time stochastic global system model is given by

$$\underline{x}_{k+1} = \mathbf{A} \underline{x}_k + \underline{w}_k \text{ with } \underline{w}_k \sim \mathcal{N}(\underline{0}, \mathbf{Q}) \text{ and}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0.2 & 0 \\ -0.2 & 1 & 0.1 \\ 0 & -0.5 & 0.8 \end{bmatrix}, \mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Further, the system is segmented into two subsystems where subsystem 1 is observing states $\hat{x}_1$ and $\hat{x}_2$, and subsystem 2 observing states $\hat{x}_2$ and $\hat{x}_3$. Therefore, both subsystems are overlapping at state $\hat{x}_2$. The local system models of subsystem 1 and 2 are given by

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0.2 \\ -0.2 & 1 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 1 & 0.1 \\ -0.2 & 0.8 \end{bmatrix},$$

$$\mathbf{Q}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{Q}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The global measurement model is given by

$$\underline{y}_k = \mathbf{C}\underline{x}_k + \underline{v}_k \text{ with } \underline{v}_k \sim \mathcal{N}(\underline{0}, \mathbf{R}) \text{ and}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{R} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

According to the before mentioned separation into subsystem 1 and 2, the local measurement models are given by

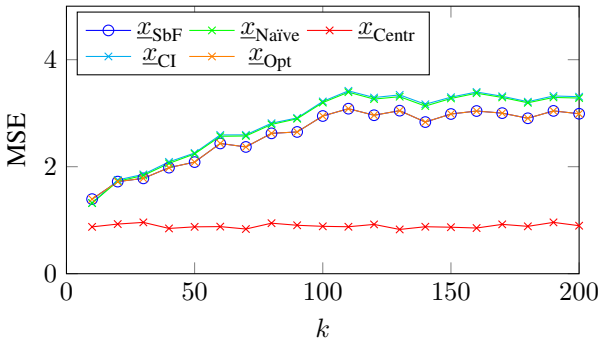
$$\mathbf{C}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \mathbf{R}_1 = 0.1$$
$$\mathbf{C}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \mathbf{R}_2 = 0.1$$

The fusion step is executed every 10 time steps. After the fusion result is obtained it is used to reinitialize the local subsystems to enhance the accuracy and prevent the filter from diverging.

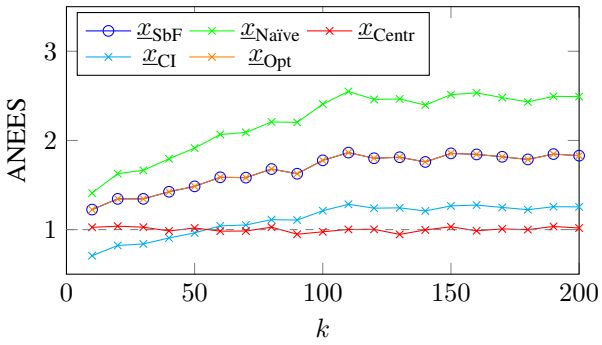
The system is simulated with 1000 Monte Carlo runs over 200 time steps. Figure 2 shows the comparison between different fusion methods for this evaluation example. In addition to the proposed Sample-based Fusion method (SbF), the optimal fusion obtained by bookkeeping of the cross-covariances (Opt) (see Section III-C), naïve fusion (Naïve) where the cross-covariances are completely neglected and covariance intersection (CI) [20] are evaluated. Also using the full system representation, a central Kalman filter is simulated to compare the fusion methods to the optimal filter result with full system knowledge. Figure 2(a) shows the mean squared error (MSE). The central Kalman filter outperforms the other fusion methods, as expected. The difference between the central Kalman filter and the fusion methods is large since the missing model information was not compensated, for e.g. by increasing the process noise. The proposed sample-based method performs identically to the optimal fusion method as intended. The fusion results of covariance intersection and naïve fusion show a slightly higher MSE. The fusion algorithms are also compared in terms of the average normalized estimation error squared (ANEES) [27]

$$\bar{\epsilon} = \frac{1}{N n_{\text{MCR}}} \sum_{i=1}^{n_{\text{MCR}}} \epsilon_i = \frac{1}{N n_{\text{MCR}}} \sum_{i=1}^{n_{\text{MCR}}} (\hat{\underline{x}}_i - \underline{x}_i)^{\mathrm{T}} \mathbf{P}_i^{-1} (\hat{\underline{x}}_i - \underline{x}_i),$$

where N is the dimension of the global system and n_{MCR} is the number of Monte Carlo runs. The ANEES can be



(a) Mean Squared Error (MSE) from 1000 test runs.



(b) Average normalized estimation error squared (ANEES) from 1000 test runs.

Fig. 2: Comparison of Sample-based Fusion (SbF), covariance intersection (CI), naïve fusion, Optimal fusion (Opt) and the central Kalman filter (Centr).

used for testing if an estimator is credible and if it is optimistic or pessimistic. The value of the ANEES should be close to 1, where a value over 1 is ranking the estimator optimistic and a value under 1 pessimistic, respectively. In figure 2(b) it can be seen that the naïve fusion results in a much higher ANEES then the other fusion method, since the error is underestimated due to the missing cross-covariances. Covariance intersection is performing more conservative than the Sample-based Fusion method or the optimal fusion. The performance of the Sample-based Fusion is identical to the optimal fusion and is more conservative than the naïve fusion and less conservative then covariance intersection.

B. Large-scale Example of Heat Conduction in a Rod

In this section, we will discuss the example of heat conduction in a rod. Hence, we will recapitulate the example considered in [19] and [20]. As pictured in figure 3(a), the temperature of each of the 100 segments is calculated in a distributed fashion by five local estimators with overlapping state vectors. The initial temperature of the rod is set to 300 K. The rod is then heated at segment 50 with 15 W and at segments 30 and 70 with -10 W. The discretized system model can be described by $x_{k+1,n} = 0.17x_{k,n-1} + 0.66x_{k,n} + 0.17x_{k,n+1} + w_{k,n}$, where k denotes the time steps and n for the n-th segment of the rod. Every segment of the rod

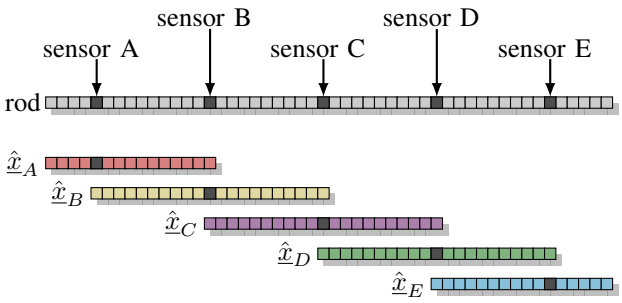
is affected by normally distributed noise $w_{k,n} \sim \mathcal{N}(0, 30)$. Where state estimates were separated into the subsystems, the process noise of the edge segment is inflated by 1.66 to $\mathcal{N}(0, 50)$, which is the local process noise at this segment plus the influence of the process noise from the neighboring segments and can be calculated from the system equation. This accounts for the influence of neighboring segments that are not included in the local state vector. Each of the local estimators features one sensor with variance 0.01 measuring the temperature of a single segment. These sensors can be found at segment 10, 30, 50, 70, and 90. The local measurement models can be described by

$$z_k^A = x_{k,10} + v_k^A\,,\; z_k^B = x_{k,30} + v_k^B\,,\; z_k^C = x_{k,50} + v_k^C\,,$$
$$z_k^D = x_{k,70} + v_k^D\,,\; z_k^E = x_{k,90} + v_k^E$$

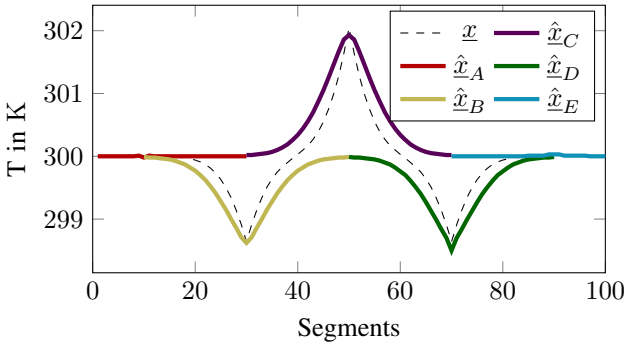
The system is initialized with a mean of 300 K for all segments and covariance matrix $\mathbf{P}_0 = 30\,\mathbf{I}$. A central fusion center is assumed that merges the local state estimates at time step $k = 100$ together. Analogously to the example in [20] the local state estimates are not reinitialized with the fused estimate and covariance matrix. Figure 3(b) shows how the distributed estimation results differ from the global result due to the limited information every local state estimate has about the global system. In Figure 4 the result from the optimal fusion and the results obtained by the proposed sample-fusion method and covariance intersection are compared. It can be seen, that the Sample-based Fusion, as well as the covariance intersection, are very close to the optimal result.

VI. Results and Discussion

The Sample-based Fusion is able to reconstruct the cross-covariances between state estimates in a distributed fashion. The evaluation examples show, that it is also possible to use the proposed overlapping deterministic sample structure to reconstruct the cross-covariances of partially overlapping state estimates. This can be valuable in applications such as surveillance, where for e.g. several objects may be seen by a varying number of networked cameras. Because of the straight-forward processing of the sample set, the method can be easily extended to non-linear filters, e.g., the EKF [28] and UKF [29]. The evaluation examples also show that a lot of care has to be taken when separating a global system in several subsystems. An additional hypothesis on the observability can be determined to help design a proper distributed setup. The precision of the Sample-based Fusion comes with the need to communicate the set of deterministic samples each node produces. The sample size grows linearly with the number of processing steps of the local filters and is dependent on the dimension of the global system. This can lead to an increased communication overhead when observing systems with large dimension or when communication between nodes is very rare. Further, the processing of the sample set and the reconstruction of the cross-covariances leads to additional computational costs. The number of processing steps included in the sample set can be kept constant [15], but future research should investigate how the remaining cross-covariances can be bounded to ensure that the fusion results stay consistent.



(a) Separation of the global system into the local subsystems.



(b) Overlapping state estimates of nodes A to E and comparison to the real system state $\hat{\underline{x}}$.

Fig. 3: Evaluation example of a 100 segment long rod estimated in an overlapping fashion.

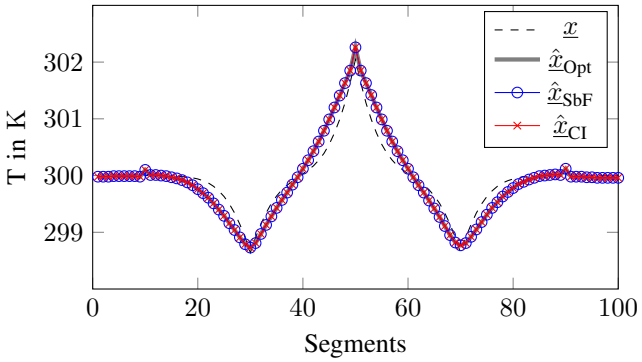


Fig. 4: Comparison of central Kalman filter and optimally fused state estimate

VII. Conclusion

In this paper, a novel method to fuse correlated and uncorrelated state estimates of partially overlapping state estimates using a set of deterministic partially overlapping samples has been proposed. The size of the sample set grows linearly with the number of processing steps included and depends on the dimension of the global system. The performance of the method was demonstrated using a small three-dimensional evaluation example and a large state estimate example featuring the temperature distribution in a heated and cooled rod. It has been shown that the proposed

deterministic sample fusion provides the means to calculate the cross-covariances between unequal state vectors under common process noise and prior information.

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