On Stability of Sequence-Based LQG Control

Jörg Fischer[1], Maxim Dolgov[1], and Uwe D. Hanebeck[1]

***Abstract*— Sequence-based control is a well-established method applied in Networked Control Systems (NCS) to mitigate the effect of time-varying transmission delays and stochastic packet losses. The idea of this method is that the controller sends sequences of predicted control inputs to the actuator that can be applied in case a future transmission fails. In this paper, the stability properties of sequence-based LQG controllers are analyzed in terms of the boundedness of the long run average costs. On the one hand, we derive sufficient conditions, each for the boundedness and unboundedness of the costs. On the other hand, we give bounds on the minimal length of the control input sequence needed to stabilize a system.**

I. INTRODUCTION

The research area of Networked Control Systems (NCS) investigates control systems whose components are connected via digital data networks. Controller design for such systems is challenging as the data networks cannot only introduce time-varying sampling times but also stochastic transmission delays and packet losses into the control loop [1]. These network-induced effects can strongly degrade system performance and destabilize the control loop [2].

Therefore, a plethora of techniques and control methods have been proposed in the literature to analyze and ensure the stability of systems subject to network-induced effects (see [3], [4] for a survey). Most of the approaches deal with the case that the controller sends one control input per data transmission to the actuator (see, e.g., [2], [5], [6], [7], [8], [9]). It has been shown, however, that the stability of a system can be significantly improved if the controller sends additionally to the current control input also control inputs applicable at future time steps [10]. The "predicted" future control inputs can be applied by the actuator in case a future transmission is delayed or lost. This control method was first mentioned in [11] and is, among others, known as sequence-based control.

In this paper, we investigate the stability properties of such a sequence-based controller. An interesting question is, e.g., whether there exists a minimal sequence length that guarantees stability of the closed-loop system. In literature, stability results for sequence-based controllers are available for constrained systems where the state is directly accessible [12], [13], the system is assumed to be undisturbed [10], [14], [15], [16], [17], the disturbances are supposed to be bounded [18], or where packet drops can occur but no time delays [19]. In the context of unconstrained Linear Quadratic Gaussian (LQG) control, the work [20] and [21] derive stability conditions for TCP-like data connections (see Chapter II for a definition).

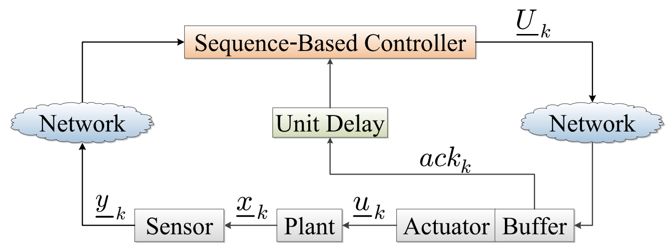

Fig. 1. Considered system setup: A linear plant is controlled and observed over a network. To mitigate stochastic time delays and packet losses in the network connection between controller and actuator, a sequence-based controller sends sequences of control inputs to the actuator. It is assumed that the actuator acknowledges successfully received data packets within one time step due to the TCP-like protocol.

In this paper, we consider the setup depicted in Fig. 1 and extend the former results [20] and [21] on stability of sequence-based LQG controllers. In this context, stability is considered in the sense of boundedness of the long run average costs. The work is based on our previous work [22], where we derived the optimal sequence-based LQG controller for the considered system.

One contribution of this paper is that we derive a sufficient condition for the boundedness of the long run average costs in the sequence-based LQG setup. The condition relaxes the stability conditions derived in [20] and [21] and, furthermore, directly considers stochastic transmission delays. In addition, we determine bounds for the minimum length of the control sequence for which the long run average costs are bounded.

A. Outline

In the following section, we describe the system setup and the sequence-based control method in more detail. In Sec. III, results derived in our previous work on the optimal sequence-based LQG control problem are summarized. The main result of this paper is stated in Sec. IV and a numerical example presented in Sec. V.

B. Notation

Throughout the paper, vector-valued quantities are underlined ($\underline{a}$) and matrices are denoted by bold face capital letters ($\mathbf{A}$). Furthermore, the notation a_k refers to the quantity a at time step k. The identity matrix is denoted by $\mathbf{I}$, a matrix consisting only of zeros by $\mathbf{0}$, the expectation operator by $\mathrm{E}\{\cdot\}$,

This work is supported by the German Science Foundation (DFG) within the priority programme 1305: Control Theory of Digitally Networked Dynamical Systems.

[1]The authors are with the Intelligent Sensor-Actuator-Systems Laboratory (ISAS), Institute for Anthropomatics, Karlsruhe Institute of Technology (KIT), Germany. Email: Joerg.Fischer@kit.edu, Maxim.Dolgov@kit.edu, Uwe.Hanebeck@ieee.org

the trace operator by $\mathrm{tr}(\cdot)$, the Moore-Penrose pseudoinverse of a matrix $\mathbf{A}$ by $\mathbf{A}^{\dagger}$, the set of all eigenvalues of $\mathbf{A}$ by $\mathrm{eig}(\mathbf{A})$, and the set of all natural numbers including and excluding zero by $\mathbb{N}_0$ and $\mathbb{N}_{>0}$, respectively. Furthermore, a sequence of $N+1$ matrices $(\mathbf{A}^0, \mathbf{A}^1, \cdots, \mathbf{A}^N)$ is denoted by $\mathbf{A}^{0:N}$ and $\mathbf{A}^{0:N} > 0$ means that all matrices of the sequence are positive definite.

II. SYSTEM SETUP

The system setup is depicted in Fig. 1. We assume that all components of the NCS are time-triggered, synchronized, and have identical cycle times. Plant and sensor are given by

$$\begin{aligned} \underline{x}_{k+1} &= \mathbf{A}\underline{x}_k + \mathbf{B}\underline{u}_k + \underline{w}_k \ , \\ \underline{y}_k &= \mathbf{C}\underline{x}_k + \underline{v}_k \ , \end{aligned} \tag{1}$$

where $\underline{x}_k \in \mathbb{R}^m$, $\underline{u}_k \in \mathbb{R}^n$, and $\underline{y}_k \in \mathbb{R}^q$ denote the plant state, the control input applied by the actuator, and the measured output, respectively. The terms $\underline{w}_k \in \mathbb{R}^n$ and $\underline{v}_k \in \mathbb{R}^q$ represent mutually independent, zero-mean, white Gaussian random processes with finite second moments and covariance matrices $\mathbf{W}$ and $\mathbf{V}$. The initial state $\underline{x}_0$ is Gaussian distributed with mean $\overline{\underline{x}}_0$ and finite covariance matrix $\mathbf{P}_0$.

Controller and actuator, as well as sensor and controller, are connected via a network that transmits data in time-stamped packets. The data packets are subject to stochastic time delays and packet losses described by mutually independent white stationary random processes with known characteristics. The probability that a packet is delayed by $i \in \mathbb{N}_0$ time steps is denoted by q_i^{CA} for the controller-actuator network and by q_i^{SC} for the sensor-controller connection. Packet losses occur with probability q_∞^{CA} and q_∞^{SE}, respectively. In addition, the controller-actuator network provides acknowledgments for successfully transmitted data packets. The acknowledgments are supposed to arrive at the controller within the same time step as the data packet was successfully transmitted to the actuator. In literature, such a network is often referred to as a TCP-like network.

Remark 1 *The TCP-like network does not reflect a real TCP/IP connection since the acknowledgments are assumed to arrive without time delay. In some cases, a TCP-like network can be implemented by , e.g., priorizing the acknowledgments. Furthermore, TCP-like connections constitute an upper performance bound for real UDP/IP and TCP/IP connections for which no analytic solutions are available.*

To mitigate the network-induced effects, the controller not only sends a single control input to the actuator, but also control inputs for future $N-1$ time steps (with $N \in \mathbb{N}_{>0}$) within the same data packet. When such a control sequence is received by the actuator, it is stored in a buffer if it contains the most recent information (indicated by the time stamps) or discarded otherwise. At every time step, the actuator applies the appropriate control input of the buffered sequence to the plant. When the buffer is empty, the actuator applies a default control input denoted by $\underline{u}_k^d$.

For the rest of the paper, a control sequence generated by the controller at time step k is denoted by the vector $\underline{U}_k$. The entries of such a sequence of length N are given by

$$\underline{U}_k = \begin{bmatrix} \underline{u}_{k|k}^\top & \underline{u}_{k+1|k}^\top & \cdots & \underline{u}_{k+N-1|k}^\top \end{bmatrix}^\top \ , \tag{2}$$

with the index $\underline{u}_{k+i|k}$ ($i \in \{0, 1, ..., N-1\}$) specifying that a control input is applicable at time step $k+i$ and was generated at time step k.

III. OPTIMAL SEQUENCE-BASED CONTROL

The stability analysis is based on our previous work [22] where we derived the optimal sequence-based LQG controller for the system setup described in Sec. II. In the following, we briefly summarize the obtained results. To that end, we introduce the augmented state

$$\underline{\xi}_k = \begin{pmatrix} \underline{x}_k \\ [\underline{u}_{k|k-1}^\top \ \cdots \ \underline{u}_{k+N-2|k-1}^\top]^\top \\ [\underline{u}_{k|k-2}^\top \ \cdots \ \underline{u}_{k+N-3|k-2}^\top]^\top \\ \vdots \\ \underline{u}_{k|k-N-1} \end{pmatrix} \tag{3}$$

that contains the plant state and all control inputs of the formerly sent sequences $\underline{U}_{k-1}, \cdots, \underline{U}_{k-N-1}$ that still could be applied by the actuator. Setting $\underline{u}_k^d = 0$, the augmented state evolves according to

$$\underline{\xi}_{k+1} = \underbrace{\begin{bmatrix} \mathbf{A} & \mathbf{B}\cdot\mathbf{H}(\theta_k) \\ \mathbf{0} & \mathbf{F} \end{bmatrix}}_{=\ \widehat{\mathbf{A}}(\theta_k)} \underline{\xi}_k + \underbrace{\begin{bmatrix} \mathbf{B}\cdot\mathbf{J}(\theta_k) \\ \mathbf{G} \end{bmatrix}}_{\widehat{\mathbf{B}}(\theta_k)} \underline{U}_k + \underbrace{\begin{pmatrix} \underline{w}_k \\ \mathbf{0} \end{pmatrix}}_{\widehat{\underline{w}}_k} \ ,$$

with

$$\mathbf{F} = \begin{array}{l} \#columns: \ \overbrace{\quad}^{n} \ \overbrace{\qquad}^{n(N-2)} \ \overbrace{\quad}^{n} \ \overbrace{\qquad}^{n(N-3)} \ \cdots \ \overbrace{\quad}^{n} \\ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} \end{bmatrix} \end{array} \begin{array}{l} \#rows: \\ \}n(N-1) \\ \}n(N-2) \\ \}n(N-3) \\ \\ \}n \end{array} \ ,$$

$$\mathbf{G} = \begin{array}{l} \#columns: \ \overbrace{\quad}^{n} \ \overbrace{\qquad}^{n(N-1)} \\ \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \end{array} \begin{array}{l} \#rows: \\ \}n(N-1) \\ \}\frac{n(N-1)(N-2)}{2} \end{array} \ ,$$

$$\begin{array}{l} \#columns: \qquad n \qquad\quad n(N-1) \\ \mathbf{J}(\theta_k) = \Big[\overbrace{\delta_{(\theta_k,0)}\mathbf{I}} \quad \overbrace{\ \mathbf{0}\ } \Big], \end{array} \ \delta_{(\theta_k,i)} = \begin{cases} 1 \ , & \text{if} \quad \theta_k = i \\ 0 \ , & \text{if} \quad \theta_k \neq i \end{cases} \ ,$$

$$\begin{array}{l} \#columns: \qquad n \qquad n(N-2) \qquad n \qquad n(N-3) \qquad\qquad n \\ \mathbf{H}(\theta_k) = \Big[\overbrace{\delta_{(\theta_k,1)}\mathbf{I}} \ \ \overbrace{\ \mathbf{0}\ } \ \ \overbrace{\delta_{(\theta_k,2)}\mathbf{I}} \ \ \overbrace{\ \mathbf{0}\ } \ \cdots \ \overbrace{\delta_{(\theta_k,N-1)}\mathbf{I}} \Big], \end{array}$$

where $\theta_k \in \mathbb{J}$ with $\mathbb{J} = \{0, \cdots, N\}$ is the state of the Markov chain that describes the *age* of the buffered control input, i.e., how many time steps ago the buffered sequence was generated. For the transition probabilities

$p_{ji} = \mathrm{P}\left(\theta_k = i | \theta_{k-1} = j\right)$ of this Markov chain, it holds

$$p_{ji} = \begin{cases} 0 & \text{for } i \geq j+2 \ , \\ 1 - \sum\limits_{r=0}^{i} q_r^{CA} & \text{for } i = j+1 \ , \\ q_i^{CA} & \text{for } i \leq j < N \ , \\ 1 - \sum\limits_{r=0}^{N-1} q_r^{CA} & \text{for } i = j = N \ , \end{cases} \tag{4}$$

where q_r^{CA} is the known probability that a sequence is delayed for $r \in \mathbb{N}_0$ time steps. Defining $\widehat{\mathbf{A}}^{0:N} = \left(\widehat{\mathbf{A}}(0), \ldots, \widehat{\mathbf{A}}(N)\right)$, $\widehat{\mathbf{B}}^{0:N} = \left(\widehat{\mathbf{B}}(0), \ldots, \widehat{\mathbf{B}}(N)\right)$, $\widehat{\mathbf{Q}}^{0:N} = \left(\widehat{\mathbf{Q}}(0), \ldots, \widehat{\mathbf{Q}}(N)\right)$, and $\widehat{\mathbf{R}}^{0:N} = \left(\widehat{\mathbf{R}}(0), \ldots, \widehat{\mathbf{R}}(N)\right)$, the main results of [22] are stated in the following Theorem.

Theorem 1 *[22] Consider the problem of finding an admissible control law with given sequence length $N-1$ according to (2) that minimizes the LQG cost*

$$J_T = \mathrm{E}\left\{ C_T + \sum_{k=0}^{T-1} C_k \middle| \underline{U}_{0:T-1}, \overline{\underline{\xi}}_0, \mathbf{P}_0, \theta_0 \right\}, \tag{5}$$

$$\text{with} \quad C_T = \underline{x}_T^\top \mathbf{Q} \underline{x}_T \ , \quad C_k = \underline{x}_k^\top \mathbf{Q} \underline{x}_k + \underline{u}_k^\top \mathbf{R} \underline{u}_k \ ,$$
$$T \in \mathbb{N}_{>0} \ , \quad \mathbf{Q} \geq 0 \ , \quad \mathbf{R} > 0 \ ,$$

subject to the system setup described in Sec. II. Then,

a) as in standard LQG control, the separation principle holds, i.e., the optimal control law can be separated into 1) an estimator that calculates the minimum mean squared error (MMSE) estimate of the augmented state $\mathrm{E}\{\underline{\xi}_k | \mathcal{I}_k\}$, where $\mathcal{I}_k$ represents the information available to the controller, and 2) into an optimal state feedback controller with feedback matrix $\mathbf{L}_k$,

b) the optimal control law is linear in the MMSE estimate of the augmented state, i.e., $\underline{U}_k = \mathbf{L}_k \, \mathrm{E}\{\underline{\xi}_k | \mathcal{I}_k\}$,

c) the feedback matrix $\mathbf{L}_k$ explicitly depends on the acknowledgment signal θ_{k-1} of the controller-actuator network so that $\mathbf{L}_k = \mathbf{L}_k^{\theta_{k-1}}$, and

d) the feedback matrix $\mathbf{L}_k^j$ is given for all $j \in \mathbb{J}$ by

$$\mathbf{L}_k^j = -\left[\sum_{i=0}^{N} p_{ji} \left(\widehat{\mathbf{R}}^i + (\widehat{\mathbf{B}}^i)^\top \mathbf{K}_{k+1}^i \widehat{\mathbf{B}}^i\right)\right]^\dagger \times \left[\sum_{i=0}^{N} p_{ji} (\widehat{\mathbf{B}}^i)^\top \mathbf{K}_{k+1}^i \widehat{\mathbf{A}}^i\right] \ , \tag{6}$$

where the matrices $\mathbf{K}_{k+1}^0, \ldots, \mathbf{K}_{k+1}^N$ are obtained by the Riccati-like recursion evolving backwards in time

$$\begin{aligned} \mathbf{K}_k^j \quad &= \left[\sum_{i=0}^{N} p_{ji} \left(\widehat{\mathbf{Q}}^i + (\widehat{\mathbf{A}}^i)^\top \mathbf{K}_{k+1}^i \widehat{\mathbf{A}}^i\right)\right] \\ &- \left[\sum_{i=0}^{N} p_{ji} (\widehat{\mathbf{A}}^i)^\top \mathbf{K}_{k+1}^i \widehat{\mathbf{B}}^i\right] \\ &\times \left[\sum_{i=0}^{N} p_{ji} \left(\widehat{\mathbf{R}}^i + (\widehat{\mathbf{B}}^i)^\top \mathbf{K}_{k+1}^i \widehat{\mathbf{B}}^i\right)\right]^\dagger \\ &\times \left[\sum_{i=0}^{N} p_{ji} (\widehat{\mathbf{B}}^i)^\top \mathbf{K}_{k+1}^i \widehat{\mathbf{A}}^i\right] \ , \end{aligned} \tag{7}$$

$$\widehat{\mathbf{Q}}(i) = \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}(i)^\top \mathbf{R} \mathbf{H}(i) \end{bmatrix}, \quad \widehat{\mathbf{R}}(i) = \mathbf{J}(i)^\top \mathbf{R} \mathbf{J}(i),$$

that is initialized with $\mathbf{K}_T^j = \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$.

Proof: The proof is given in [22]. ∎

Remark 2 *As shown in [23], the MMSE estimate $\mathrm{E}\{\underline{\xi}_k | \mathcal{I}_k\}$ is obtained by a time-varying Kalman filter that buffers received measurements to incorporate delayed measurements. For an optimal estimate, the length of the buffer, N_B, has to be chosen so that*

$$N_B = \max\{\mathrm{i} \in \mathbb{N} : \mathrm{q}_\mathrm{i}^{\mathrm{SE}} > 0\} \ . \tag{8}$$

In practice, N_B has to be limited leading to a potentially suboptimal filter. Theorem 1 and the following stability analysis, however, also hold if N_B is not optimally chosen.

IV. Stability Analysis

In this section, we analyze the stability properties of the optimal sequence-based LQG controller given in Theorem 1. For this purpose, we introduce the long run average costs

$$J_\infty = \lim_{T \to \infty} \frac{1}{T} J_T \ , \tag{9}$$

used to evaluate the stability of the infinite-horizon optimal sequence-based controller. The controlled system is said to be stable if the associated long run average costs (9) are bounded, i.e., if there exist a $\overline{J}$ such that for all initial conditions $J_\infty \leq \overline{J}$.

Remark 3 *In literature, also other stability criteria are investigated such as mean square stability (MSS). General results for MSS of MJLS are given, e.g., in [24]. These results, however, cannot be directly applied as we do not assume a special structure of the estimator and controller gain. Furthermore, we consider the case where the mode of the associated MJLS is only available with a time delay and the weighting matrices are only positive semidefinite.*

In the following stability analysis, we use the operator

$$\begin{aligned} &g^j\left(\mathbf{X}^{0:N}\right) = \\ &\left[\sum_{i=0}^{N} p_{ji} \left(\widehat{\mathbf{Q}}^i + (\widehat{\mathbf{A}}^i)^\top \mathbf{X}^i \widehat{\mathbf{A}}^i\right)\right] - \left[\sum_{i=0}^{N} p_{ji} (\widehat{\mathbf{A}}^i)^\top \mathbf{X}^i \widehat{\mathbf{B}}^i\right] \\ &\times \left[\sum_{i=0}^{N} p_{ji} \left(\widehat{\mathbf{R}}^i + (\widehat{\mathbf{B}}^i)^\top \mathbf{X}^i \widehat{\mathbf{B}}^i\right)\right]^\dagger \left[\sum_{i=0}^{N} p_{ji} (\widehat{\mathbf{B}}^i)^\top \mathbf{X}^i \widehat{\mathbf{A}}^i\right] \end{aligned} \tag{10}$$

that maps a sequence of $N+1$ square matrices $\mathbf{X}^{0:N} = \left(\mathbf{X}^0, \ldots, \mathbf{X}^N\right)$ to a matrix with the same dimension as $\mathbf{X}^j$. Furthermore, we introduce the operator

$$g\left(\mathbf{X}^{0:N}\right) = \left(g^0\left(\mathbf{X}^{0:N}\right), \ldots, g^N\left(\mathbf{X}^{0:N}\right)\right) \ , \tag{11}$$

that maps a sequence of $N+1$ square matrices to a sequence of $N+1$ matrices with the same dimension. The main results are given in the following theorems.

Theorem 2 *The long run average costs* (9) *are upper bounded for all initial condition* (ξ_0, θ_0) *if and only if*

a) the control related costs described by the sequence $\mathbf{X}_{k+1}^{0:N} = g\big(\mathbf{X}_k^{0:N}\big)$ *are upper bounded and*
b) the expected estimation error covariance

$$\mathrm{E}\Big\{\Big(\underline{\xi}_k - \mathrm{E}\Big\{\underline{\xi}_k\Big|\mathcal{I}_k\Big\}\Big)\Big(\underline{\xi}_k - \mathrm{E}\Big\{\underline{\xi}_k\Big|\mathcal{I}_k\Big\}\Big)^\top\Big|\mathcal{I}_0\Big\} \quad (12)$$

is upper bounded.

Proof: For any symmetric random matrix $\mathbf{S}$ and zero-mean random vector $\underline{x}$ that are stochastically independent of each other, we have

$$\mathrm{E}\left\{\underline{x}^\top \mathbf{S}\underline{x}\right\} = \mathrm{tr}\left(\mathrm{E}\left\{\mathbf{S}\right\}\mathrm{E}\left\{\underline{x}\underline{x}^\top\right\}\right) \ . \quad (13)$$

Using this fact and combining (22), (24), and (32) of [22], it follows for the minimal expected cumulated costs

$$\begin{aligned} J_T = {} & \mathrm{tr}\left(\mathrm{E}\left\{\mathbf{K}_0^{\theta_0}\Big|\mathcal{I}_0\right\}\overline{\mathbf{P}}_0\right) + \sum_{k=0}^{T-1}\mathrm{tr}\left(\mathrm{E}\left\{\mathbf{K}_{k+1}^{\theta_k}\Big|\mathcal{I}_0\right\}\mathbf{W}\right) \\ & + \sum_{k=0}^{T-1}\mathrm{tr}\Bigg(\mathrm{E}\left\{\widehat{\mathbf{Q}}^{\theta_k} + (\widehat{\mathbf{A}}^{\theta_k})^\top \mathbf{K}_{k+1}^{\theta_k}\widehat{\mathbf{A}}^{\theta_k} - \mathbf{K}_k^{\theta_{k-1}}\Big|\mathcal{I}_0\right\} \\ & \times \mathrm{E}\left\{\left(\underline{\xi}_k - \mathrm{E}\left\{\underline{\xi}_k|\mathcal{I}_k\right\}\right)\left(\underline{\xi}_k - \mathrm{E}\left\{\underline{\xi}_k|\mathcal{I}_k\right\}\right)^\top\Big|\mathcal{I}_0\right\}\Bigg). \end{aligned}$$

The term $\mathcal{I}_k$ represents the information set available to the estimator at time step k and contains the information about the initial condition as well as all received measurements, acknowledgment signals, and sent control sequences.

Since the matrices $\widehat{\mathbf{Q}}^{\theta_k}$ and $\widehat{\mathbf{A}}^{\theta_k}$ are bounded and $\mathbf{P}_0$ and $\mathbf{W}$ are supposed to have finite second moments, the long run average costs (9) are bounded for every initial condition if and only if the sequence $\mathbf{X}_{k+1}^{0:N} = g\big(\mathbf{X}_k^{0:N}\big)$ and

$$\mathrm{E}\left\{\left(\underline{\xi}_k - \mathrm{E}\left\{\underline{\xi}_k|\mathcal{I}_k\right\}\right)\left(\underline{\xi}_k - \mathrm{E}\left\{\underline{\xi}_k|\mathcal{I}_k\right\}\right)^\top\Big|\mathcal{I}_0\right\}$$

are bounded, which concludes the proof. ■

In the following, we give sufficient conditions for the boundedness of 2b) and 2a) in Theorems 3 and 4, respectively. The boundedness of the expected error covariance matrix (12) has already been investigated ([23], [25]) so that Theorem 3 summarizes these results without proof.

Theorem 3 *[23] Assume that* $(\mathbf{A}, \mathbf{C})$ *is observable,* $(\mathbf{A}, \mathbf{W}^{1/2})$ *is controllable, and* $\mathbf{V} > 0$*. It holds,*

a) if $\max|\mathrm{eig}\left(\mathbf{A}\right)| < 1$*, then* (12) *is bounded,*
b) if $\max|\mathrm{eig}\left(\mathbf{A}\right)| \geq 1$*, then* (12) *is*

$$\begin{aligned} \textit{unbounded if} \quad & p_{arr}^{SC} \leq 1 - \frac{1}{\max|\mathrm{eig}\left(\mathbf{A}\right)|^2} \\ \textit{and bounded if} \quad & p_{arr}^{SC} > p_{crit}^{SC} \ , \end{aligned}$$

where $p_{arr}^{SC} = \sum_{i=0}^{N_B} q_i^{SC}$ *is the probability that a measurement sent over the network can be processed by a Kalman filter with a buffer of length* N_B *(see Remark 2) and* p_{crit}^{SC} *can be computed by the solution of the quasi-convex optimization problem* $p_{crit}^{SC} = \arg\min_p \Psi_p\left(\mathbf{Y}, \mathbf{Z}\right) > 0$ *with constraint* $0 \leq \mathbf{Y} \leq \mathbf{I}$ *and*

$$\Psi_p\left(\mathbf{Y}, \mathbf{Z}\right) = \begin{bmatrix} \mathbf{Y} & \sqrt{p}\left(\mathbf{Y}\mathbf{A} + \mathbf{Z}\mathbf{C}\right) & \sqrt{1-p}\mathbf{Y}\mathbf{A} \\ (*)^\top & \mathbf{Y} & \mathbf{0} \\ (*)^\top & \mathbf{0} & \mathbf{Y} \end{bmatrix}.$$

Proof: The proof is given in [23]. ■

Theorem 4 *Consider the sequence* $\mathbf{X}_{k+1}^{0:N} = g\big(\mathbf{X}_k^{0:N}\big)$ *with* $g(\cdot)$ *defined in* (11)*. Then,*

a) the sequence is bounded for any initial condition $\mathbf{X}_0^{0:N} \geq 0$ *if there exist* $N+1$ *matrices* $\widehat{\mathbf{L}}^{0:N}$ *and* $N+1$ *positive definite matrices* $\mathbf{X}^{0:N}$ *such that*

$$\mathbf{X}^j > \sum_{i=0}^{N} p_{ji}\left(\widehat{\mathbf{A}}^i + \widehat{\mathbf{B}}^i\widehat{\mathbf{L}}^j\right)^\top \mathbf{X}^i\left(\widehat{\mathbf{A}}^i + \widehat{\mathbf{B}}^i\widehat{\mathbf{L}}^j\right), \quad (14)$$

b) if the sequence converges, it converges to the positive semidefinite fixed point

$$\overline{\mathbf{K}}^{0:N} = \left(g^0\left(\overline{\mathbf{K}}^{0:N}\right), \ldots, g^N\left(\overline{\mathbf{K}}^{0:N}\right)\right) \ ,$$

c) the condition in a) is equivalent to the existence of $N+1$ *matrices* $\mathbf{Y}^{0:N}$ *and* $\mathbf{Z}^{0:N}$ *such that* $\mathbf{\Theta}_N\big(\mathbf{Y}^{0:N}, \mathbf{Z}^{0:N}\big) > 0$ *and* $0 < \mathbf{Y}^{0:N} < \mathbf{I}$*, where*

$$\mathbf{\Theta}_N\big(\mathbf{Y}^{0:N}, \mathbf{Z}^{0:N}\big) = \begin{bmatrix} \mathbf{\Theta}^0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{\Theta}^1 & & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{\Theta}^N \end{bmatrix},$$

with

$$\mathbf{\Theta}^j = \begin{bmatrix} \mathbf{Y}^j & \Sigma(j,0) & \cdots & \Sigma(j,N) \\ \Sigma(j,0)^\top & \mathbf{Y}^0 & & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \Sigma(j,N)^\top & \mathbf{0} & \cdots & \mathbf{Y}^N \end{bmatrix},$$

$$\Sigma(j,i) = \sqrt{p_{ji}}\left(\mathbf{Y}^j\big(\widehat{\mathbf{A}}^i\big)^\top + \mathbf{Z}^j\big(\widehat{\mathbf{B}}^i\big)^\top\right) \ ,$$

d) the sequence is unbounded for all initial conditions $\mathbf{X}_0^{0:N} \geq 0$ *if* $(\mathbf{A}, \mathbf{Q}^{1/2})$ *is observable and*

$$p_{NN} \cdot \ \max|\mathrm{eig}\left(\mathbf{A}\right)|^2 > 1 \ , \quad (15)$$

where $p_{NN} = 1 - \sum_{r=0}^{N-1} q_r^{CA}$ *as defined in* (4)*.*

Proof: The proof is given in the appendix. ■

Now, we discuss some special cases and implications of Theorems 2 - 4, starting with the case that no time delays and packet losses occur in the network connections. Then, the sequence-based controller reduces to the standard LQG controller and condition 4a) gives

$$\mathbf{X}^0 > \left(\widehat{\mathbf{A}}^0 + \widehat{\mathbf{B}}^0\widehat{\mathbf{L}}^0\right)^\top \mathbf{X}^0\left(\widehat{\mathbf{A}}^0 + \widehat{\mathbf{B}}^0\widehat{\mathbf{L}}^0\right) \ . \quad (16)$$

According to Lyapunov theory, if this inequality has a solution, all eigenvalues of $(\widehat{\mathbf{A}}^0 + \widehat{\mathbf{B}}^0\widehat{\mathbf{L}}^0)$ are strictly smaller than one justifying that the control related costs 2a) are bounded if 4a) holds. Furthermore, it can be seen that (16) has always a solution if $(\widehat{\mathbf{A}}^0, \widehat{\mathbf{B}}^0)$ is stabilizable since this implies that there exists an $\widehat{\mathbf{L}}^0$ such that $\max|\mathrm{eig}(\widehat{\mathbf{A}}^0 + \widehat{\mathbf{B}}^0\widehat{\mathbf{L}}^0)| < 1$. It can be shown that the stabilizability of $(\widehat{\mathbf{A}}^0, \widehat{\mathbf{B}}^0)$ is equivalent to the stabilizability of $(\mathbf{A}, \mathbf{B})$, so that condition 4a) reduces to this fundamental assumption of LQG control.

Assuming there are no time delays but only data losses in the network connections, i.e., $q_\infty^{SC} = 1 - q_0^{SC}$ and $q_\infty^{SC} = 1 - q_0^{SC}$, the setup reduces for $N = 0$ to the one investigated in [26]. The authors have shown, under the additional assumption $(\mathbf{A}, \mathbf{B})$ is controllable and $(\mathbf{A}, \mathbf{Q}^{1/2})$ is observable, that the corresponding conditions 4a) and 4c) are not only sufficient but also necessary and that the fixed point is strictly positive definite. For $N \geq 0$, the conditions in Theorem 3 and 4 are similar to the ones derived in [20]. However, we are able to drop the assumption on the steady state distribution of the Markov chain in Prop. 3 of [20].

An interesting implication of 4b) is that if the long run average costs are bounded, the gain of the optimal sequence-based infinite-horizon LQG controller converges to

$$\mathbf{L}_\infty^j = \left[\sum_{i=0}^{N} p_{ji} \left(\widehat{\mathbf{R}}^i + (\widehat{\mathbf{B}}^i)^\top \mathbf{K}_\infty^i \widehat{\mathbf{B}}^i \right) \right]^\dagger \times \left[\sum_{i=0}^{N} p_{ji} (\widehat{\mathbf{B}}^i)^\top \mathbf{K}_\infty^i \widehat{\mathbf{A}}^i \right] , \tag{17}$$

with $\mathbf{K}_\infty^{0:N} = g\big(\mathbf{K}_\infty^{0:N}\big)$. This is a very useful property for practical implementation where resources might be limited.

Finally, we state some important observations regarding the length of the control sequence.

Corollary 1 *If condition 4a) is satisfied, the minimal sequence length N_{crit} guaranteeing boundedness of the long run average costs satisfies $N_{min} \geq N_{crit} \geq N_{max}$, where*

$$N_{min} = \min_n \left\{ n \in \mathbb{N}_0 : \sum_{r=0}^{n} q_r^{CA} \geq 1 - \frac{1}{\max |\mathrm{eig}\,(\mathbf{A})\,|^2} \right\} ,$$

with $\max |\mathrm{eig}\,(\mathbf{A})\,| \neq 0$, and N_{max} can be obtained as the solution of the optimization problem

$$N_{max} = \arg\min_N \mathbf{\Theta}_N (\mathbf{Y}, \mathbf{Z}) > 0, \tag{18}$$

with constraints $0 < \mathbf{Y}^{0:N} < \mathbf{I}$.

Proof: This directly follows from 4c) and 4d). ■

In the next section, we demonstrate the applicability of the derived stability criteria by a numerical example.

V. SIMULATION

The conditions of Theorem 3 on the boundedness of the error covariance matrix (12) have already been evaluated in [23]. Therefore, we focus on demonstrating the results obtained in Theorem 4 and Corollary 1 and consider a directly observable plant with direct connection between sensor and controller. The system matrices are chosen as

$$\mathbf{A} = \begin{bmatrix} 0.5 & 0 \\ 1 & 1.5 \end{bmatrix} , \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} , \quad \mathbf{C} = \mathbf{I} ,$$

where $\mathbf{A}$ has eigenvalues 0.5 and 1.5. The covariances and initial condition are set to

$$\mathbf{W} = \mathbf{I} , \quad \mathbf{V} = \mathbf{0} , \quad \underline{\overline{x}}_0 = \begin{bmatrix} 10 & 10 \end{bmatrix} , \quad \mathbf{P}_0 = \mathbf{I} .$$

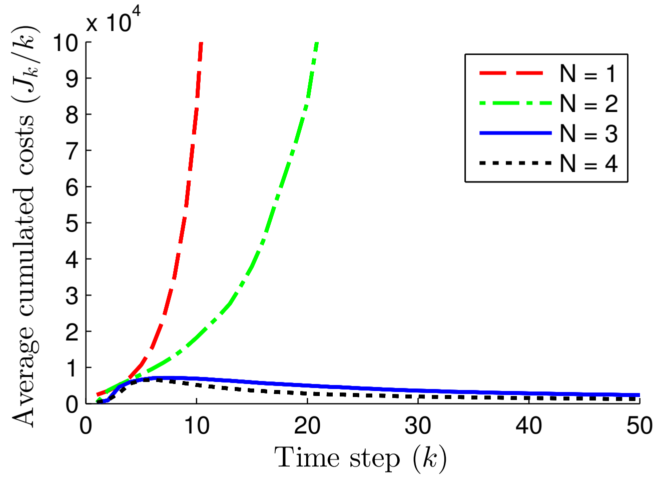

Fig. 2. Comparison of the averaged cumulated LQG costs (5) for different control sequence lengths N.

We assume that a packet sent from the controller to the actuator suffers a delay of 0, 1, 2, or 3 time steps with probability 0.25, each, i.e., $q_0 = q_1 = q_2 = q_3 = 0.25$. Based on this network characteristics, the controller gains $\mathbf{L}_k^{0:N}$ are computed according to Theorem 1 for different control sequence lengths $N = \{1, \ldots, 4\}$, where we choose the weighting matrices to $\mathbf{Q} = \mathbf{I}$ and $\mathbf{R} = 10 \cdot \mathbf{I}$.

For this system, the optimization problem in 4c) turns out to be unfeasible for $N = \{1, 2\}$ and feasible for $N = \{3, 4\}$. Consequently, the condition in Corollary 1 gives that the upper bound on the minimal sequence length N_{crit} for which the long run average costs are bounded is $N_{max} = 3$. Furthermore, $N_{min} = 3$ since $1 - 1/\max|\mathrm{eig}(\mathbf{A})|^2 \approx 0.55$ and $\sum_{r=0}^{1} q_r^{CA} = 0.5$ and $\sum_{r=0}^{2} q_r^{CA} = 0.75$. This indicates that the costs are unbounded for $N \leq 2$. Together, we have that $3 \leq N_{crit} \leq 3$ and therefore $N_{crit} = 3$.

To evaluate this theoretical result, we conduct 100000 Monte Carlo simulation runs over 50 time steps for several control sequence lengths N. The average costs $\frac{J_k}{k}$ are calculated over all simulation runs, with J_k as in (5), and plotted against the time step (Fig. 2). The exponential increase in the average costs for $N = \{1, 2\}$ indicates that the long run average costs are indeed unbounded. For $N = \{3, 4\}$ the average costs converge and, consequently, are bounded. Finally, the figure verifies that $N_{crit} = 3$.

VI. CONCLUSIONS

In this paper, we presented results on the boundedness of the long run average costs of sequence-based LQG controllers for NCS with time-varying transmission delays and stochastic packet losses. We showed that the costs are bounded if the estimation error covariance is bounded and a Riccati-like recursion associated with the controller costs converges. For the former, we summarized sufficient conditions given in the literature and for latter derived sufficient conditions in form of a LMI feasibility problem. Finally, we derived a procedure to calculate the critical sequence length that guarantees boundedness of the long run average costs.

VII. APPENDIX

The proof of Theorem 4 is based on [25] and [20], where the boundedness of the expected estimation error covariance and the long run average costs in the case of packet losses only were studied. The incorporation of time delays into the stability analysis is not straightforward as control sequences arriving at the actuator with a time delay do not have to be applied starting with the first control input of that sequence. Therefore, when generating a control sequence, the controller does not only know which control inputs will be applied directly before a control input of the currently computed sequence is applied (if any is applied at all), but it is also unknown which control input of the currently computed sequence will be applied. This is a fundamental difference to [20] and major source of difficulty of the stability analysis.

Before we can prove the assertion of Theorem 4, we introduce the operators

$$\Phi^j(\widehat{\mathbf{L}}^j, \mathbf{X}^{0:N}) = \sum_{i=0}^{N} p_{ji}\widehat{\mathbf{Q}}^i + \sum_{i=0}^{N} p_{ji}(\widehat{\mathbf{L}}^j)^\top \widehat{\mathbf{R}}^i \widehat{\mathbf{L}}^j + \sum_{i=0}^{N} p_{ji}\left(\widehat{\mathbf{A}}^i + \widehat{\mathbf{B}}^i\widehat{\mathbf{L}}^j\right)^\top \mathbf{X}^i \left(\widehat{\mathbf{A}}^i + \widehat{\mathbf{B}}^i\widehat{\mathbf{L}}^j\right) , \tag{19}$$

$$L_\Phi^j(\mathbf{X}^{0:N}) = -\left[\sum_{i=0}^{N} p_{ji}\left(\widehat{\mathbf{R}}^i + (\widehat{\mathbf{B}}^i)^\top \mathbf{X}^i \widehat{\mathbf{B}}^i\right)\right]^\dagger \times \left[\sum_{i=0}^{N} p_{ji}(\widehat{\mathbf{B}}^i)^\top \mathbf{X}^i \widehat{\mathbf{A}}^i\right], \tag{20}$$

$$\text{and} \quad h(\mathbf{X}) = \Phi^N\left(L_\Phi^N(\mathbf{0},\ldots,\mathbf{0},\mathbf{X}), (\mathbf{0},\ldots,\mathbf{0},\mathbf{X})\right) . \tag{21}$$

The following Lemmas state some useful properties of $\Phi^j(\cdot)$ and $h(\cdot)$ that will be used to constitute an upper and lower bound for the sequence $\mathbf{X}_{k+1}^{0:N} = g\left(\mathbf{X}_k^{0:N}\right)$.

Lemma 1 *The following facts are true:*

a) $\arg\min_{\widehat{\mathbf{L}}^j} \Phi^j(\widehat{\mathbf{L}}^j, \mathbf{X}^{0:N}) = L_\Phi^j(\mathbf{X}^{0:N})$,
b) $\min_{\widehat{\mathbf{L}}^j} \Phi^j(\widehat{\mathbf{L}}^j, \mathbf{X}^{0:N}) = \Phi^j(L_\Phi^j(\mathbf{X}^{0:N}), \mathbf{X}^{0:N}) = g^j\left(\mathbf{X}^{0:N}\right)$,
c) $g^j\left(\mathbf{X}^{0:N}\right) \le \Phi^j(\mathbf{L}^j, \mathbf{X}^{0:N})$, $\forall\, \mathbf{L}^j$,
d) *if* $\mathbf{X}^{0:N} \ge \mathbf{Y}^{0:N}$*, then* $g^j\left(\mathbf{X}^{0:N}\right) \ge g^j\left(\mathbf{Y}^{0:N}\right)$,
e) *if* $\mathbf{X}^{0:N} \ge 0$ *and* $\mathbf{X}^N \ge \mathbf{Y}$*, then* $g^N\left(\mathbf{X}^{0:N}\right) \ge h(\mathbf{Y})$,
f) *if* $\mathbf{X} \ge \mathbf{Y}$*, then* $h(\mathbf{X}) \ge h(\mathbf{Y})$.

Proof:

a) Since $\Phi^j(\widehat{\mathbf{L}}^j, \mathbf{X}^{0:N})$ is convex and quadratic in $\widehat{\mathbf{L}}^j$ and $\mathbf{X}^{0:N}, \widehat{\mathbf{R}}^{0:N} \ge 0$, it holds for the minimizer of (19) that

$$\frac{\mathrm{d}\Phi^j(\widehat{\mathbf{L}}^j, \mathbf{X}^{0:N})}{\mathrm{d}\widehat{\mathbf{L}}^j} = 2\cdot\left(\sum_{i=0}^{N} p_{ji}\widehat{\mathbf{R}}^i\right)\widehat{\mathbf{L}}^j + 2\cdot\sum_{i=0}^{N} p_{ji}\left((\widehat{\mathbf{B}}^i)^\top \mathbf{X}^i \widehat{\mathbf{B}}^i \widehat{\mathbf{L}}^j + (\widehat{\mathbf{B}}^i)^\top \mathbf{X}^i \widehat{\mathbf{A}}^i\right) \stackrel{!}{=} 0 .$$

Solving for $\widehat{\mathbf{L}}^j$ gives $\widehat{\mathbf{L}}^j = L_\Phi^j(\mathbf{X}^{0:N})$.
b) The fact follows from Lemma 1a) and substitution.
c) This is a direct implication of Lemma 1b).
d) $$g^j\left(\mathbf{Y}^{0:N}\right) = \Phi^j(L_\Phi^j(\mathbf{Y}^{0:N}), \mathbf{Y}^{0:N}) \le \Phi^j(L_\Phi^j(\mathbf{X}^{0:N}), \mathbf{Y}^{0:N}) \le \Phi^j(L_\Phi^j(\mathbf{X}^{0:N}), \mathbf{X}^{0:N}) = g^j\left(\mathbf{X}^{0:N}\right) .$$
e) With $\mathbf{X}^{0:N} \ge (\mathbf{0},\ldots,\mathbf{0},\mathbf{Y})$ it follows from Lemma 1d)
$$g^N\left(\mathbf{X}^{0:N}\right) \ge g^N(\mathbf{0},\ldots,\mathbf{0},\mathbf{Y}) = \Phi^N(L_\Phi^N(\mathbf{0},\ldots,\mathbf{0},\mathbf{Y}), (\mathbf{0},\ldots,\mathbf{0},\mathbf{Y})) = h(\mathbf{Y}) .$$
f) The fact follows from Lemma 1d) with $\mathbf{X}^{0:N} = (\mathbf{0},\ldots,\mathbf{0},\mathbf{X})$ and $\mathbf{Y}^{0:N} = (\mathbf{0},\ldots,\mathbf{0},\mathbf{Y})$. ■

Lemma 2 *Consider the operators*

$$\mathcal{L}^j\left(\mathbf{Y}^{0:N}\right) = \sum_{i=0}^{N} p_{ji}\left(\widehat{\mathbf{A}}^i + \widehat{\mathbf{B}}^i\mathbf{L}^j\right)^\top \mathbf{Y}^i \left(\widehat{\mathbf{A}}^i + \widehat{\mathbf{B}}^i\mathbf{L}^j\right)$$

$$\text{and} \quad \mathcal{L}\left(\mathbf{Y}^{0:N}\right) = \left(\mathcal{L}^0\left(\mathbf{Y}^{0:N}\right),\ldots,\mathcal{L}^N\left(\mathbf{Y}^{0:N}\right)\right) ,$$

and assume that there exist matrices $\overline{\mathbf{Y}}^{0:N} > 0$ *such that* $\overline{\mathbf{Y}}^{0:N} > \mathcal{L}\left(\overline{\mathbf{Y}}^{0:N}\right)$*, then*

a) *it holds for the sequence* $\mathbf{M}_{k+1}^{0:N} = \mathcal{L}\left(\mathbf{M}_k^{0:N}\right)$ *initialized with* $\mathbf{M}_0^{0:N} \ge 0$ *that* $\lim_{k\to\infty} \mathbf{M}_k^{0:N} = 0$*, and*
b) *the sequence* $\mathbf{M}_{k+1}^{0:N} = \mathcal{L}\left(\mathbf{M}_k^{0:N}\right) + (\mathbf{S}^0,\cdots,\mathbf{S}^N)$ *initialized with* $\mathbf{M}_0^{0:N} \ge 0$ *is bounded for all* $\mathbf{S}^{0:N} \ge 0$.

Proof:

a) Choose $0 \le m$ such that $\mathbf{M}_0^{0:N} \le m\overline{\mathbf{Y}}^{0:N}$. Furthermore, choose $0 < r < 1$ such that $\mathcal{L}\left(\overline{\mathbf{Y}}^{0:N}\right) < r\overline{\mathbf{Y}}^{0:N}$ and consider the sequence $\mathbf{N}_{k+1}^{0:N} = \mathcal{L}\left(\mathbf{N}_k^{0:N}\right)$ initialized with $\mathbf{N}_0^{0:N} = m\overline{\mathbf{Y}}^{0:N}$. Then,
$$0 \le \mathbf{M}_{k+1}^{0:N} \le \mathbf{N}_{k+1}^{0:N} \le mr^{(k+1)}\overline{\mathbf{Y}}^{0:N}$$
since 1) $\mathbf{Y}^{0:N} \ge 0$ implies that $\mathcal{L}\left(\mathbf{Y}^{0:N}\right) \ge 0$ and 2) if $\mathbf{Y}^{0:N} \ge \mathbf{X}^{0:N}$ then $\mathcal{L}\left(\mathbf{Y}^{0:N}\right) \ge \mathcal{L}\left(\mathbf{X}^{0:N}\right)$. Taking the limit $k\to\infty$ justifies the proposition.
b) Choose $0 \le s$ such that $\mathbf{S}^{0:N} \le s\overline{\mathbf{Y}}^{0:N}$. Consider the sequence $\mathbf{S}_{k+1}^{0:N} = \mathcal{L}\left(\mathbf{S}_k^{0:N}\right)$ initialized with $\mathbf{S}_0^{0:N} = \mathbf{S}^{0:N}$ and the sequence $\mathbf{N}_{k+1}^{0:N} = \mathcal{L}\left(\mathbf{N}_k^{0:N}\right)$ initialized with $\mathbf{N}_0^{0:N} = \mathbf{M}_0^{0:N}$. Then $\mathbf{M}_{k+1}^{0:N} = \mathbf{N}_{k+1}^{0:N} + \sum_{t=0}^{k}\mathbf{S}_t^{0:N}$ and it follows by Lemma 2a) with $0 \le m_N, m_U$ and $0 < r_N, r_U < 1$ that
$$\mathbf{M}_{k+1}^{0:N} \le m_N r_N^{(k+1)}\overline{\mathbf{Y}}^{0:N} + \sum_{t=0}^{k} m_U r_U^{(t)}\overline{\mathbf{Y}}^{0:N} \le \left(m_N + \frac{m_U}{1-r}\right)\overline{\mathbf{Y}}^{0:N} .$$ ■

With Lemmas 1 and 2 we are ready to prove Theorem 4.

Proof: Theorem 4

a) Using Lemma 1c), we have
$$\mathbf{X}_{k+1}^{0:N} = g\left(\mathbf{X}_k^{0:N}\right) \le \Phi(\widehat{\mathbf{L}}^{0:N}, \mathbf{X}_k^{0:N}) = \mathcal{L}\left(\mathbf{X}_k^{0:N}\right) + (\mathbf{S}^0,\ldots,\mathbf{S}^N) ,$$
where
$$\mathbf{S}^j = \sum_{i=0}^{N} p_{ji}\left(\widehat{\mathbf{Q}}^i + (\widehat{\mathbf{L}}^j)^\top \widehat{\mathbf{R}}^i \widehat{\mathbf{L}}^j\right) ,$$

$$\Phi(\widehat{\mathbf{L}}^{0:N}, \mathbf{X}_k^{0:N}) = (\Phi^0(\widehat{\mathbf{L}}^0, \mathbf{X}_k^{0:N}), \ldots, \Phi^N(\widehat{\mathbf{L}}^N, \mathbf{X}_k^{0:N})\ ,$$

and $\mathcal{L}\left(\mathbf{X}^{0:N}\right)$ as defined in Lemma 2. By the assumption $\mathbf{X}^{0:N} > \mathcal{L}\left(\mathbf{X}^{0:N}\right)$, the condition of Lemma 2 is satisfied and according to Lemma 2b) the sequence $\mathbf{X}_{k+1}^{0:N} = g\left(\mathbf{X}_k^{0:N}\right)$ is bounded.

b) Consider the sequence $\mathbf{X}_{k+1}^{0:N} = g\left(\mathbf{X}_k^{0:N}\right)$ initialized with $\mathbf{X}_0^{0:N} = \mathbf{0}$. Since this sequence is monotonically increasing (Lemma 1d) and bounded from above (Theorem 4a), the sequence converges. As $g(\cdot)$ is a continuous function, the limit has to be the fixed point $\overline{\mathbf{K}}^{0:N} = g\left(\overline{\mathbf{K}}^{0:N}\right)$. The uniqueness of the solution and the convergence for sequences initialized with $\mathbf{X}_0^{0:N} \geq \overline{\mathbf{K}}^{0:N}$ can be proved similar to Theorem 1 in [25].

c) The Linear Matrix Inequality (LMI) can be derived by applying the Schur complement on (14) and introducing the new variables $\mathbf{Y}^j = (\mathbf{X}^j)^{-1}$ and $\mathbf{Z}^j = (\mathbf{X}^j)^{-1} \cdot \widehat{\mathbf{L}}^j$.

d) Consider the sequences $\mathbf{X}_{k+1}^{0:N} = g\left(\mathbf{X}_k^{0:N}\right)$ and $\mathbf{N}_{k+1} = h(\mathbf{N}_k)$ initialized with $\mathbf{X}_0^{0:N} = \mathbf{0}$ and $\mathbf{N}_0 = \mathbf{0}$. Then, $\mathbf{X}_1^{0:N} \geq 0$ and $\mathbf{X}_1^N \geq \mathbf{N}_1$ and it follows from 1d) that $\mathbf{X}_k^N \geq \mathbf{N}_k$, i.e., that $\mathbf{X}_k^N$ is lower bounded by $\mathbf{N}_k$.
If $\mathbf{N}_{k+1} = h(\mathbf{N}_k)$ converges, $\overline{\mathbf{N}} = \lim_{k\to\infty} \mathbf{N}_k$ has to be a fixed point of $h(\cdot)$ since $h(\cdot)$ is a continuous operator. Using (21), the fixed point equation $\overline{\mathbf{N}} = h(\overline{\mathbf{N}})$ is given by $\overline{\mathbf{N}} = \hat{\mathbf{Q}} + \hat{\mathbf{A}}^\top \overline{\mathbf{N}} \hat{\mathbf{A}}$, with

$$\hat{\mathbf{Q}} = \sum_{i=0}^{N} p_{ji} \widehat{\mathbf{Q}}^i + p_{NN} \overline{\mathbf{L}}^\top \widehat{\mathbf{R}}^i \overline{\mathbf{L}}\ ,$$
$$\hat{\mathbf{A}} = \sqrt{p_{NN}} \left(\widehat{\mathbf{A}}^N + \widehat{\mathbf{B}}^N \overline{\mathbf{L}}\right).$$

The observability of $(\widehat{\mathbf{A}}^N, (\sum_{i=0}^N p_{ji} \widehat{\mathbf{Q}}^i)^{1/2})$ and $(\hat{\mathbf{A}}, \hat{\mathbf{Q}}^{1/2})$ follows from the assumption that $(\mathbf{A}, \mathbf{Q}^{1/2})$ is observable, what can be proved by using the Belovich-Popov-Hautus test. In addition, since $\hat{\mathbf{Q}} \geq 0$, it follows according to Lyapunov theory that there exists no positive semidefinite solution to the fixed point equation if $\max|\mathrm{eig}(\hat{\mathbf{A}})| > 1$. Noting that $\mathrm{eig}(\mathbf{A}) \subset \mathrm{eig}(\hat{\mathbf{A}})$, it holds that the sequence $\mathbf{N}_k$ has no fixed point if $\sqrt{p_{NN}} \max|\mathrm{eig}(\mathbf{A})| > 1$. Under this condition, the sequence $\mathbf{N}_k$ diverges because, first, it does not converge to a fixed point and, second, it is increasing monotonically (Lemma 1f). Noting that $\mathbf{X}_k^N \geq \mathbf{N}_k$, concludes the proof.

e) The assertions follows directly from Theorem 4c). ■

REFERENCES

[1] P. Antsaklis and J. Baillieul, "Special Issue on Technology of Networked Control Systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 5–8, 2007.

[2] W. Zhang, M. S. Branicky, and S. M. Phillips, "Stability of Networked Control Systems," *IEEE Control Systems Magazine*, vol. 21, no. 1, pp. 84–99, 2001.

[3] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, "A Survey of Recent Results in Networked Control Systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 138–162, 2007.

[4] L. Zhang, H. Gao, and O. Kaynak, "Network-Induced Constraints in Networked Control Systems - A Survey," *IEEE Transactions on Industrial Informatics*, vol. 9, no. 1, pp. 403–416, 2013.

[5] D. Něsić and A. R. Teel, "Input-Output Stability Properties of Networked Control Systems," *IEEE Transactions on Automatic Control*, vol. 49, no. 10, pp. 1650–1667, 2004.

[6] W. P. M. H. Heemels, A. R. Teel, N. Van de Wouw, and D. Něsić, "Networked Control Systems with Communication Constraints: Tradeoffs between Transmission Intervals, Delays and Performance," *IEEE Transactions on Automatic Control*, vol. 55, no. 8, pp. 1781–1796, 2010.

[7] P. Seiler and R. Sengupta, "An H_∞ Approach to Networked Control," *IEEE Transactions on Automatic Control*, vol. 50, no. 3, pp. 356–364, 2005.

[8] Y. Yang, Z. Wang, Y. S. Hung, and M. Gani, "H_∞ Control for Networked Systems with Random Communication Delays," *IEEE Transactions on Automatic Control*, vol. 51, no. 3, pp. 511–518, 2006.

[9] M. C. F. Donkers, W. P. M. H. Heemels, D. Bernardini, A. Bemporad, and V. Shneer, "Stability Analysis of Stochastic Networked Control Systems," *Automatica*, vol. 48, no. 5, pp. 917–925, May 2012.

[10] G. P. Liu, "Predictive Controller Design of Networked Systems with Communication Delays and Data Loss," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 57, no. 6, pp. 481–485, Jun 2010.

[11] A. Bemporad, "Predictive Control of Teleoperated Constrained Systems with Unbounded Communication Delays," in *Proceedings of the IEEE Conference on Decision and Control*, vol. 2, 1998, pp. 2133–2138.

[12] D. E. Quevedo and D. Nešić, "Input-to-State Stability of Packetized Predictive Control over Unreliable Networks Affected by Packet-Dropouts," *IEEE Transactions on Automatic Control*, vol. 56, no. 2, pp. 370–375, Feb 2011.

[13] M. Reble, D. E. Quevedo, and F. Allgöwer, "Control over Erasure Channels: Stochastic Stability and Performance of Packetized Unconstrained Model Predictive Control," *International Journal of Robust and Nonlinear Control*, vol. 23, no. 10, pp. 1151–1167, Jul 2013.

[14] P. L. Tang and C. W. de Silva, "Stability Validation of a Constrained Model Predictive Networked Control System with Future Input Buffering," *International Journal of Control*, vol. 80, no. 12, pp. 1954–1970, 2007.

[15] L. Greco, A. Chaillet, and A. Bicchi, "Exploiting Packet Size in Uncertain Nonlinear Networked Control Systems," *Automatica*, vol. 48, no. 11, pp. 2801–2811, Nov 2012.

[16] R. Findeisen and P. Varutti, "Stabilizing Nonlinear Predictive Control over Nondeterministic Communication Networks," in *Nonlinear Model Predictive Control*, ser. Lecture Notes in Control and Information Sciences. Springer Berlin Heidelberg, 2009, vol. 384, pp. 167–179.

[17] I. G. Polushin, P. X. Liu, and C.-H. Lung, "On the Model-Based Approach to Nonlinear Networked Control Systems," *Automatica*, vol. 44, no. 9, pp. 2409 – 2414, 2008.

[18] G. Pin and T. Parisini, "Networked Predictive Control of Uncertain Constrained Nonlinear Systems: Recursive Feasibility and Input-to-State Stability Analysis," *IEEE Transactions on Automatic Control*, vol. 56, no. 1, pp. 72–87, 2011.

[19] W.-J. Ma and V. Gupta, "Input-to-state Stability of Hybrid Systems with Receding Horizon Control in the Presence of Packet Dropouts," *Automatica*, vol. 48, no. 8, pp. 1920–1923, Aug 2012.

[20] V. Gupta, B. Sinopoli, S. Adlakha, A. Goldsmith, and R. Murray, "Receding Horizon Networked Control," in *In Proceedings of the Annual Allerton Conference*, 2006, pp. 169–176.

[21] M. Moayedi, Y. K. Foo, and Y. C. Soh, "LQG Control for Networked Control Systems with Random Packet Delays and Dropouts via Multiple Predictive-Input Control Packets," in *Preprints of the IFAC World Congress*, vol. 18, no. 1, 2011, pp. 72–77.

[22] J. Fischer, A. Hekler, M. Dolgov, and U. D. Hanebeck, "Optimal Sequence-Based LQG Control over TCP-like Networks Subject to Random Transmission Delays and Packet Losses," in *Proceedings of the American Control Conference*, Jun. 2013, pp. 1543–1549.

[23] L. Schenato, "Optimal Estimation in Networked Control Systems Subject to Random Delay and Packet Drop," *IEEE Transactions on Automatic Control*, vol. 53, no. 5, pp. 1311–1317, Jun 2008.

[24] O. do Valle Costa, M. Fragoso, and R. Marques, *Discrete-Time Markov Jump Linear Systems*. Springer Verlag, 2005.

[25] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. Jordan, and S. S. Sastry, "Kalman Filtering with Intermittent Observations," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1453–1464, Sep 2004.

[26] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. S. Sastry, "Foundations of Control and Estimation over Lossy Networks," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 163–187, Jan 2007.