A priori estimates for the derivative nonlinear Schrödinger equation

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August 1, 2020

Abstract

We prove low regularity a priori estimates for the derivative nonlinear Schrödinger equation in Besov spaces with positive regularity index conditional upon small L^2-norm. This covers the full subcritical range. We use the power series expansion of the perturbation determinant introduced by Killip–Vişan–Zhang for completely integrable PDE. This makes it possible to derive low regularity conservation laws from the perturbation determinant.

1   Introduction

In this note the following derivative nonlinear Schrödinger equation (*dNLS*) is considered

$$\begin{cases} i\partial_t q + \partial_{xx} q + i\partial_x(|q|^2 q) & = 0 \quad (t,x) \in \mathbb{R} \times \mathbb{K}, \\ \qquad\qquad q(0) & = q_0 \in H^s(\mathbb{K}), \end{cases} \tag{1}$$

where $\mathbb{K} \in \{\mathbb{R}, \mathbb{T} = (\mathbb{R}/(2\pi\mathbb{Z}))\}$. In the seventies (1) was proposed as a model in plasma physics in [31, 23, 24].

In the following let $\mathcal{S}(\mathbb{R})$ denote the Schwartz functions on the line and $\mathcal{S}(\mathbb{T})$ smooth functions on the circle. Here we prove a priori estimates

$$\sup_{t\in\mathbb{R}} \|q(t)\|_{H^s} \lesssim_s \|q_0\|_{H^s}, \quad 0 < s < \frac{1}{2},$$

1

where $q \in C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{K}))$ is a smooth global solution to (1), which is also rapidly decaying in the line case, *conditional upon small L^2-norm.* These estimates are the key to extend local solutions globally in time. Local well-posedness, i.e., existence, uniqueness and continuous dependence locally in time, in $H^{1/2}$ was proved by Takaoka [34] on the real line and Herr [11] on the circle. They proved local well-posedness via the contraction mapping principle, that is *perturbatively.* Furthermore, they showed that the data-to-solution mapping fails to be C^3 below $H^{1/2}$ in either geometry, respectively. Moreover, Biagioni–Linares [2] showed that the data-to-solution mapping even fails to be locally uniformly continuous on the real line below $H^{1/2}$. Thus, the results on local well-posedness in $H^{1/2}$ are the limit of proving local well-posedness via fixed point arguments. However, on the real line (1) admits the scaling symmetry

$$q(t,x) \to \lambda^{-1/2} q(\lambda^{-2}t, \lambda^{-1}x), \tag{2}$$

which distinguishes L^2 as scaling critical space. Hence, we still expect a milder form of local well-posedness in H^s for $0 \le s < 1/2$. By short-time Fourier restriction, Guo [8] proved a priori estimates for $s > 1/4$ on the real line, which the second author [33] extended to periodic boundary conditions. Moreover, Grünrock [6] showed local well-posedness on the real line in Fourier Lebesgue spaces, which scale like H^s, $s > 0$. Deng *et al.* [5] recently extended this to periodic boundary conditions; see also the previous work [7].

Less is known about global well-posedness. Conserved quantities of the flow include the mass, i.e., the L^2-norm,

$$M[q] = \int_{\mathbb{K}} |q|^2 dx,$$

the momentum, related with the $H^{1/2}$-norm,

$$P[q] = \int_{\mathbb{K}} \text{Im}(\bar{q}q_x) - \frac{1}{2}|q|^4 dx,$$

and the energy, related with the H^1-norm,

$$E[q] = \int_{\mathbb{K}} |q_x|^2 - \frac{3}{2}|q|^2 \text{Im}(\bar{q}q_x) + \frac{1}{2}|q|^6 dx.$$

A local well-posedness result in L^2 seems to be very difficult due to the scaling criticality. On the other hand, it is not straight-forward to use the other quantities to prove a global result due to lack of definiteness. The remedy in previous works was to impose a smallness condition on the L^2-norm and use the sharp Gagliardo-Nirenberg inequality.

Wu [38] observed in the line case that combining several conserved quantities improves the L^2-threshold, which can be derived from the energy (cf. [37]). Mosincat–Oh carried out the corresponding argument on the torus [26]. Additionally making use of the I-method (cf. [4, 22]), Guo–Wu [9] proved global well-posedness in $H^{1/2}(\mathbb{R})$ for $\|u_0\|_{L^2} < 4\pi$, and Mosincat [25] proved global well-posedness in $H^{1/2}(\mathbb{T})$ under the same L^2-smallness condition. The question of global well-posedness for arbitrary L^2-norm is still open. Noteworthy, Nahmod *et al.* [27] proved a probabilistic global well-posedness result in Fourier Lebesgue spaces scaling like $H^{1/2-\varepsilon}(\mathbb{T})$. On the half-line and intervals endowed with Dirichlet boundary conditions, Wu [37] and Tan [36] showed the existence of finite-time blow-up solutions.

Kaup–Newell [15] already observed shortly after the proposal of (1) that it admits a Lax pair with operator

$$L(t; q) = \begin{pmatrix} \partial + i\kappa^2 & -\kappa q \\ -\kappa \bar{q} & \partial - i\kappa^2 \end{pmatrix}. \tag{3}$$

Consequently, there are infinitely many conserved quantities of the flow. However, to the best of the authors' knowledge, there are no works using the complete integrability for solutions in *unweighted* L^2-based Sobolev space, i.e., without imposing additional spatial decay. In particular, there are no results for periodic boundary conditions making use of the complete integrability.

Via inverse scattering, Lee [19, 20] proved global existence and uniqueness for certain initial data in $\mathcal{S}(\mathbb{R})$. Later, Liu [21] considered (1) with initial data in weighted Sobolev spaces $H^{2,2}(\mathbb{R})$ and proved global well-posedness via inverse scattering. See the subsequent works [14, 12, 13] due to Jenkins *et al.* for results addressing soliton resolution in weighted Sobolev spaces. Recently, Pelinovsky–Shimabukuro [30] proved global well-posedness in $H^{1,1}(\mathbb{R}) \cap H^2(\mathbb{R})$ without L^2-smallness condition, but assumptions on the Kaup–Newell spectral problem; see also [29, 32].

There are major technical difficulties to apply inverse scattering techniques in unweighted Sobolev spaces, e.g., on the line the decay of the data is typically insufficient for classical arguments. For the nonlinear Schrödinger equation on the line, Koch–Tataru [18] were able to use the transmission coefficient to obtain almost conserved H^s-energies for all $s > -\frac{1}{2}$. Killip–Vişan–Zhang [17] pointed out a power series representation for the determinant

$$\log \det \left(\begin{bmatrix} (-\partial + \tilde{\kappa})^{-1} & 0 \\ 0 & (-\partial - \tilde{\kappa})^{-1} \end{bmatrix} \begin{bmatrix} -\partial + \tilde{\kappa} & iq \\ \mp i\bar{q} & -\partial - \tilde{\kappa} \end{bmatrix} \right),$$

given by

$$\sum_{l=1}^{\infty} \frac{(\mp 1)^{l-1}}{l} \operatorname{tr}\left\{\left[(\tilde{\kappa}-\partial)^{-1/2}q(\tilde{\kappa}+\partial)^{-1}\bar{q}(\tilde{\kappa}-\partial)^{-1/2}\right]^l\right\},$$

which works in either geometry. Killip *et al.* [17] showed that it is conserved for NLS and mKdV by term-by-term differentiation. This led to low regularity conservation laws and corresponding a priori estimates in either geometry. Talbut [35] used the same approach to show low regularity conservation laws for the Benjamin-Ono equation.

Motivated by these results, we show that the determinant

$$\log \det\Big(\begin{bmatrix} (\partial + i\kappa^2)^{-1} & 0 \\ 0 & (\partial - i\kappa^2)^{-1} \end{bmatrix} \begin{bmatrix} \partial + i\kappa^2 & -\kappa q \\ -\kappa\bar{q} & \partial - i\kappa^2 \end{bmatrix}\Big),$$

given by

$$\sum_{l=1}^{\infty} \frac{(-1)^l i^{l+1}\tilde{\kappa}^l}{l} \operatorname{tr}\left\{\left[(\partial-\tilde{\kappa})^{-1/2}q(\partial+\tilde{\kappa})^{-1}\bar{q}(\partial-\tilde{\kappa})^{-1/2}\right]^l\right\}, \tag{4}$$

where we formally set $\tilde{\kappa} = -i\kappa^2$ (we drop the tilde later on), is conserved for solutions of (1). This yields the following theorem:

Theorem 1.1. *Let $u \in C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{K}))$ be a smooth solution to (1). For any $0 < s < 1/2$, $r \in [1, \infty]$, there is $c = c(s, r) < 1$ such that*

$$\|q(t)\|_{B_{r,2}^s} \lesssim \|q(0)\|_{B_{r,2}^s} \tag{5}$$

provided that $\|q(0)\|_2 \le c$.

Remark 1.2. We focus on regularities, for which global results were previously unknown. It appears feasible to cover higher regularities following [17, Section 3].

In follow-up works to [17], Killip–Vişan showed sharp global well-posedness for the KdV equation [16] and later on with Bringmann for the fifth order KdV equation [3]. Sharp global well-posedness for NLS and mKdV on the real line was shown by Harrop-Griffiths–Killip–Vişan [10]. It is an interesting question to determine whether (1) is within the thrust of these works.

Outline of the paper. In Section 2 we show that (4) is a conserved quantity. In Section 3 we derive low regularity conservation laws, yielding a priori estimates and finishing the proof of the main result.

4

2 The perturbation determinant

In the following we consider

$$\alpha(\kappa; q) = \operatorname{Re} \sum_{l \geq 1} \frac{(-1)^l i^{l+1} \kappa^l}{l} \operatorname{tr}((\kappa - \partial)^{-1} q (\kappa + \partial)^{-1} \bar{q})^l). \tag{6}$$

To ensure that α converges geometrically and we can differentiate term by term, we recall that for $\kappa > 0$ [17, Eq. (45)]

$$\|(\kappa - \partial)^{-1/2} q (\kappa + \partial)^{-1/2}\|_{\mathfrak{I}_2(\mathbb{R})}^2 \approx \int \log\left(4 + \frac{\xi^2}{\kappa^2}\right) \frac{|\hat{q}(\xi)|^2}{\sqrt{4\kappa^2 + \xi^2}} d\xi. \tag{7}$$

Since

$$\int \log\left(4 + \frac{\xi^2}{\kappa^2}\right) \frac{|\hat{q}(\xi)|^2}{\sqrt{4\kappa^2 + \xi^2}} d\xi \leq \sup_\xi \left[\frac{\log\left(4 + \frac{\xi^2}{\kappa^2}\right)}{\sqrt{4\kappa^2 + \xi^2}}\right] \int |\hat{q}(\xi)|^2 d\xi \lesssim \frac{1}{\kappa} \|q\|_{L^2}^2,$$

we conclude the L^2-part of the following:

Lemma 2.1. *Let $q \in C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{K}))$. Suppose that $\|q\|_{L^2} \leq c \ll 1$ small enough and $\kappa > 0$, or $\kappa \gg \|q\|_{H^s}^{1/a}$ for $s > 0$ and $a = \min(1/4, s)$. Then, α defined in* (6) *converges geometrically.*

The H^s-part of the above Lemma follows from the fact that $\mathfrak{I}_p \hookrightarrow \mathfrak{I}_q$ for $p < q$ and the estimate

$$\|(\kappa - \partial)^{-1/2} q (\kappa + \partial)^{-1/2}\|_{\mathfrak{I}_p} \lesssim \kappa^{-1/2+1/p} \|q\|_{L^p} \lesssim \kappa^{-s} \|q\|_{H^s}$$

for $s = 1/2 - 1/p$, $2 \leq p < \infty$, which is shown in Section 3.

Next, we show that α is conserved by term-by-term differentiation.

Proposition 2.2. *Let $q \in C^\infty(\mathbb{R}; \mathcal{S})$ be a smooth global solution to* (1) *with $\|q(0)\|_2 \leq c \ll 1$. Then,*

$$\frac{d}{dt} \alpha(\kappa; q) = 0.$$

Remark 2.3. *In Section 3 we see that $\alpha(\kappa; q)$ converges* without smallness assumption *on the L^2-norm, but provided that κ is sufficiently large. However, we are not able to show bounds for the $B_{r,2}^s$-norm without L^2-smallness assumption.*

Proof. In the following we omit taking the real part in (4) and will thus show that both real and imaginary part are conserved. Consider

$$\sum_{l=1}^{\infty} \frac{(-i)^{l+1}\kappa^l}{l} \operatorname{tr}(\underbrace{(\partial - \kappa)^{-1}}_{R_-} q \underbrace{(\partial + \kappa)^{-1}}_{R_+} \bar{q})^l = \sum_{l \geq 1} \alpha_l.$$

We note similar to the considerations from [17, Section 4]:

$$\begin{aligned} (|q|^2 q)_x &= (\partial - \kappa)(|q|^2 q) - (|q|^2 q)(\partial + \kappa) + 2\kappa(|q|^2 q) \\ (|q|^2 \bar{q})_x &= (\partial + \kappa)(|q|^2 \bar{q}) - (|q|^2 \bar{q})(\partial - \kappa) - 2\kappa(|q|^2 \bar{q}), \end{aligned} \tag{8}$$

and furthermore,

$$\begin{aligned} q_{xx} &= q(\partial^2 - 2\kappa\partial - \kappa^2) + (\partial^2 + 2\kappa\partial - \kappa^2)q + 2(\kappa - \partial)q(\kappa + \partial), \\ \bar{q}_{xx} &= (\partial^2 - 2\kappa\partial - \kappa^2)\bar{q} + \bar{q}(\partial^2 + 2\kappa\partial - \kappa^2) + 2(\kappa + \partial)\bar{q}(\kappa - \partial). \end{aligned} \tag{9}$$

Differentiating term-by-term, we find two terms $\frac{d}{dt}\alpha_l = A_l + B_l$, which are given by

$$\begin{aligned} A_l &= -(-i)^{l+1}\kappa^l \operatorname{tr}((R_- q R_+ \bar{q})^{l-1}[R_-(|q|^2 q)_x R_+ \bar{q} + R_- q R_+(|q|^2 \bar{q})_x] \\ B_l &= (-i)^{l+1}\kappa^l \operatorname{tr}((R_- q R_+ \bar{q})^{l-1}[R_- i q_{xx} R_+ \bar{q} - i R_- q R_+ \bar{q}_{xx}]). \end{aligned}$$

We show that $A_l + B_{l+1} = 0$. Since $B_1 \equiv 0$ (cf. [17]), this yields the claim.

We plug in (8) in A_l to find

$$\begin{aligned} A_l = -(-i)^{l+1}\kappa^l \operatorname{tr}((R_- q R_+ \bar{q})^{l-1}[|q|^2 q R_+ \bar{q} - R_-|q|^4 + 2\kappa R_-|q|^2 q R_+ \bar{q} \\ + [R_-|q|^4 - R_- q R_+ |q|^2 \bar{q} R_-^{-1} - 2\kappa R_- q R_+ |q|^2 \bar{q}]. \end{aligned} \tag{10}$$

The first term from the first line is cancelled by the second term from the second line by cycling the trace, and the second term in the first line is cancelled by the first term in the second line. Only the third terms remain. Plugging in (9) in B_{l+1}, we note that the first and second terms from (9) cancel each other because constant coefficient differential operators commute, and it remains

$$\begin{aligned} B_{l+1} = (-i)^{l+1} 2\kappa\kappa^l \operatorname{tr}((R_- q R_+ \bar{q})^l \\ [R_-(\kappa - \partial)q(\kappa + \partial)R_+ \bar{q} - R_- q R_+(\kappa + \partial)\bar{q}(\kappa - \partial)]). \end{aligned} \tag{11}$$

The first term in (11) is cancelled by the third term in the second line of (10) and the second term in (11) is cancelled by the third term in the first line of (10). This finishes the proof. $\qquad \square$

3 Conservation of Besov norms with positive regularity index

In the following, we want to construct Besov norms from the leading term of $\alpha(q;\kappa)$. Set

$$w(\xi,\kappa) = \frac{\kappa^2}{\xi^2 + 4\kappa^2} - \frac{(\kappa/2)^2}{\xi^2 + \kappa^2} = \frac{3\kappa^2\xi^2}{4(\xi^2 + \kappa^2)(\xi^2 + 4\kappa^2)}$$

and

$$\|f\|_{Z_r^s} = \Big(\sum_{N \in 2^{\mathbb{N}}} N^{rs} \langle f, w(-i\partial_x, N)f\rangle^{r/2} \Big)^{1/r}. \tag{12}$$

The Z_r^s–norm consists of homogeneous components, which can be linked to the perturbation determinant. We will use the identities from [17]:

$$\|f\|_{B_{r,2}^s} \lesssim_s \|f\|_{H^{-1}} + \|f\|_{Z_r^s}, \tag{13}$$

$$\|f\|_{Z_r^s} \lesssim \|f\|_{B_{r,2}^s}. \tag{14}$$

Consequently, it suffices to control the Z_r^s-norm to infer about the Besov norms.

Remark 3.1. In the Z_r^s-quantities introduced in [17], there is an additional parameter κ_0. One might hope that this flexibility helps to obtain a result for arbitrary initial data. However, κ_0 enters with a positive exponent into the estimates. This reflects indeed the relation of κ_0 with rescaling and the L^2-criticality of (1). To keep things simple, we choose $\kappa_0 = 1$.

3.1 The line case

To analyze the growth of the Z_r^s-norm, we link the multiplier from above with the first term of α. We recall the following identity on the real line:

Lemma 3.2 ([17, Lemma 4.2]). *For $\kappa > 0$ and $q \in \mathcal{S}$, we find*

$$\mathrm{Re}\big(\kappa \, \mathrm{tr}\,((\kappa - \partial)^{-1} q(\kappa + \partial)^{-1}q)\big) = \int \frac{2\kappa^2|\hat{q}(\xi)|^2}{\xi^2 + 4\kappa^2}.$$

This yields

$$\langle f, w(-i\partial_x, N)f\rangle = \int \frac{N^2}{\xi^2 + 4N^2}|\hat{f}(\xi)|^2 d\xi - \int \frac{(N/2)^2}{\xi^2 + N^2}|\hat{f}(\xi)|^2 d\xi$$

$$= \frac{1}{2}[\alpha_1(N, f) - \alpha_1(N/2, f)].$$

Set $A = (\kappa - \partial)^{-1/2} q (\kappa + \partial)^{-1/2} \kappa^{1/2}$. Firstly, we argue that we can gain powers of κ by estimating q in H^s-norms. Note that

$$\|A\|_{\mathfrak{I}_2} \lesssim \|q\|_2, \qquad \|A\|_{\mathfrak{I}_\infty} \lesssim \kappa^{-1/2} \|q\|_\infty.$$

The first identity follows from (7), and the second from viewing q as a multiplication operator in L^2. Interpolating the estimates (cf. [1, Proposition I.1]) and using Sobolev embedding, we find

$$\|A\|_{\mathfrak{I}_p} \lesssim \kappa^{-1/2+1/p} \|q\|_p \lesssim \kappa^{-s} \|q\|_{H^s} \text{ for } s = 1/2 - 1/p, \quad 2 \leq p < \infty.$$

Let $0 < s' < 1/4$ in the following and set $s' = \frac{1}{2} - \frac{1}{p_{s'}}$. By Hölder's inequality and embeddings for Schatten spaces, we find

$$|\mathrm{tr}(A^4)| \leq \|A\|_{\mathfrak{I}_4}^4 \lesssim \|A\|_{\mathfrak{I}_{p_{s'}}}^4 \lesssim \kappa^{-4s'} \|q\|_{H^{s'}}^4.$$

Similarly for the higher order terms $l \geq 3$, we find

$$|\mathrm{tr}(A^{2l})| \leq \|A\|_{\mathfrak{I}_{2l}}^{2l} \lesssim \|A\|_{\mathfrak{I}_{p_{s'}}}^{2l} \lesssim \kappa^{-2ls'} \|q\|_{H^{s'}}^{2l}.$$

Note that this implies that the series (6) converges for $q \in H^s$, $s > 0$ by choosing $\kappa \gg \|q\|_{H^s}^{1/a}$ for $a = \min(1/4, s)$ as claimed in Remark 2.3.

Furthermore, we can estimate $|\alpha - \alpha_1|$ favorably:

$$\Big| \sum_{l \geq 2} \alpha_l(\kappa, q(t)) \Big| \lesssim \kappa^{-4s'} \|q(t)\|_{H^{s'}}^4.$$

This gives by the embedding $B_r^{s,2} \hookrightarrow H^{s'}$ for $s > s'$ and $r \in [1, \infty]$

$$\langle q(t), w(-i\partial_x, N) q(t) \rangle$$
$$\lesssim \langle q(0), w(-i\partial_x, N) q(0) \rangle + N^{-4s'} [\|q(t)\|_{H^{s'}}^4 + \|q(0)\|_{H^{s'}}^4]$$
$$\lesssim \langle q(0), w(-i\partial_x, N) q(0) \rangle + N^{-4s'} [\|q(t)\|_{B_r^{s,2}}^4 + \|q(0)\|_{B_r^{s,2}}^4].$$

Raising the estimate to the power $r/2$, multiplying with N^{rs}, and carrying out the dyadic sums over $N \in 2^{\mathbb{N}_0}$, we find

$$\|q(t)\|_{Z_r^s}^r \lesssim \|q(0)\|_{Z_r^s}^r + [\|q(t)\|_{B_r^{s,2}}^{2r} + \|q(0)\|_{B_r^{s,2}}^{2r}]$$

provided that we choose $s' < s < 2s'$. This can be satisfied for $0 < s < 1/2$. By (14) and L^2-conservation, we arrive at

$$\|q(t)\|_{Z_r^s} \leq C_{r,s}(\|q(0)\|_{Z_r^s} + \|q(0)\|_{L^2}^2 + [\|q(t)\|_{Z_r^s}^2 + \|q(0)\|_{Z_r^s}^2]) \tag{15}$$

8

with $C_{r,s} \geq 1$. The above display can be bootstrapped. Suppose that

$$\max(\|q(0)\|_{Z_r^s}, \|q(0)\|_{L^2}) \leq \varepsilon \ll 1,$$

where ε is chosen below as $\varepsilon = \varepsilon(C_{r,s})$. We prove that for any $t \in \mathbb{R}$

$$\sup_{t \in \mathbb{R}} \|q(t)\|_{Z_r^s} \leq 2C_{r,s}\varepsilon. \tag{16}$$

For this purpose, let I denote the maximal interval containing the origin such that (16) holds for any $t \in I$. I is non-empty and closed due to continuity of $\|q(t)\|_{Z_r^s}$. Furthermore, I is open: For $t \in I$ (15) yields

$$\|q(t)\|_{Z_r^s} \leq C_{r,s}(\varepsilon + 2\varepsilon^2 + (2C_{r,s}\varepsilon^2)) \leq (3/2)C_{r,s}\varepsilon$$

by choosing $\varepsilon = C_{r,s}/8$. We conclude that $I = \mathbb{R}$.

This finishes the proof for small initial data. The assumption $\|q(0)\|_{Z_r^s} \leq \varepsilon$ can be salvaged for arbitrary initial data by rescaling, leaving us with the assumption $\|q(0)\|_2 \leq \varepsilon$. The proof of Theorem 1.1 is complete in the line case.

3.2 The circle case

We shall rescale the circle, too, to accomplish smallness of the homogeneous norms. In [17], this was not necessary due to more freedom in the parameter κ. We use the conventions from [28].
Given $\lambda \geq 1$, let $\mathbb{T}_\lambda = \mathbb{R}/(2\pi\lambda\mathbb{Z})$ and set

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi\lambda} f(x)e^{-ix\xi}dx \text{ and } f(x) = \frac{1}{\sqrt{2\pi\lambda}} \sum_{\xi \in \mathbb{Z}/\lambda} \hat{f}(\xi)e^{ix\xi}$$

for $f \in L^1(\mathbb{T}_\lambda, \mathbb{C})$, where $\xi \in \mathbb{Z}_\lambda = \lambda^{-1}\mathbb{Z}$. The guideline for the conventions is that Plancherel's theorem remains true:

$$\|f\|_{L^2(\mathbb{T}_\lambda)} = \|\hat{f}\|_{L^2(\mathbb{Z}_\lambda, (d\xi)_\lambda)},$$

where $(d\xi)_\lambda$ denotes the normalized counting measure on $\mathbb{Z}_\lambda$:

$$\int_{\mathbb{Z}_\lambda} f(\xi)(d\xi)_\lambda = \frac{1}{\lambda} \sum_{\xi \in \mathbb{Z}_\lambda} f(\xi).$$

For further basic Fourier analysis identities on $\mathbb{T}_\lambda$, we refer to [4, Section 2]. In [28] the following identities were computed:

9

Lemma 3.3. *Let $\lambda \geq 1$ and $n \geq 1$. Then, we have*

$$\|(\kappa - \partial)^{-1/2} u (\kappa + \partial)^{-1/2}\|^2_{\mathfrak{I}_2(\mathbb{T}_\lambda)} \sim \int_{\mathbb{Z}_\lambda} \log\big(4 + \frac{\xi^2}{\kappa^2}\big) \frac{|\hat{u}(\xi)|^2}{\sqrt{4\kappa^2 + \xi^2}} (d\xi)_\lambda,$$

$$\|(\kappa + \partial)^{-1/2} \bar{u} (\kappa - \partial)^{-1/2}\|^2_{\mathfrak{I}_2(\mathbb{T}_\lambda)} \sim \int_{\mathbb{Z}_\lambda} \log\big(4 + \frac{\xi^2}{\kappa^2}\big) \frac{|\hat{u}(\xi)|^2}{\sqrt{4\kappa^2 + \xi^2}} (d\xi)_\lambda$$

for any smooth function u on $\mathbb{T}_\lambda$.

For the leading term in (4) we find:

Lemma 3.4. *Let $\kappa \geq 1$ and $\lambda \geq 1$. Then, we have*

$$\mathrm{Re}\, tr\big(\kappa(\kappa - \partial)^{-1} u (\kappa + \partial)^{-1} \bar{u}\big) = \frac{1 + e^{-2\pi\lambda\kappa}}{1 - e^{-2\pi\lambda\kappa}} \int_{\mathbb{Z}_\lambda} \frac{2\kappa^2 |\hat{u}(\xi)|^2}{4\kappa^2 + \xi^2} (d\xi)_\lambda$$

for any smooth function u on $\mathbb{T}_\lambda$.

Set $C(\lambda, N) = (1 + e^{-2\pi\lambda N})/(1 - e^{-2\pi\lambda N})$. Clearly, $C(\lambda, N) \sim 1$ for $\lambda N \geq 1$. With w defined as above, we find

$$\langle f, w(-i\partial_x, N)f\rangle = \frac{1}{2}\Big[\frac{\alpha_1(N, f)}{C(\lambda, N)} - \frac{\alpha_1(N/2, f)}{C(\lambda, N/2)}\Big].$$

Suppose that $q \in C^\infty(\mathbb{R} \times \mathbb{T})$ is a solution to (1). Let q_λ denote the rescaled solution to (1):

$$q_\lambda : \mathbb{R} \times \mathbb{T}_\lambda \to \mathbb{C}, \quad q_\lambda(t, x) = \lambda^{-1/2} q(\lambda^{-2}t, \lambda^{-1}x).$$

With the conventions introduced above, the identities from Lemmas 3.3 and 3.4 allow for the same error estimates in the real line case uniformly for $\lambda \geq 1$. We arrive at

$$\|q_\lambda(t)\|_{Z_r^s} \lesssim_{r,s} \|q_\lambda(0)\|_{Z_r^s} + [\|q_\lambda(t)\|^2_{B_r^{s,2}} + \|q_\lambda(0)\|^2_{B_r^{s,2}}].$$

By L^2-conservation and estimating the $B_r^{s,2}$-norm in terms of the Z_r^s-norm:

$$\|q_\lambda(t)\|_{Z_r^s} \lesssim_{r,s} \|q_\lambda(0)\|_{Z_r^s} + \|q_\lambda(0)\|^2_{L_\lambda^2} + [\|q_\lambda(t)\|^2_{Z_r^s} + \|q_\lambda(0)\|^2_{Z_r^s}].$$

Smallness of the Z_r^s-norm can be accomplished by taking $\lambda \to \infty$. The L_λ^2-norm is scaling critical:

$$\|q_\lambda(0)\|_{L_\lambda^2} = \|q(0)\|_{L^2}.$$

The continuity argument given in the line case proves global a priori estimates in the circle case for small L^2-norm of the initial data. The proof of Theorem 1.1 is complete.

Acknowledgements

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